ELEMENTARY CONFORMAL SURFACE PARAMETRIZATIONS

PARAMETRIZACIONES CONFORMES ELEMENTALES DE SUPERFICIES

by

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A Clara y a Sonia

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Introducción y resumen de resultados y conclusiones

La presente tesis discute algunos nuevos resultados sobre parametrizaciones conformes de superficies, definidas por funciones simples.

Una parametrización de una superficie en $\mathbb{R}^n$ es una función vectorial:

$$\bar{\Phi}(u,v) \equiv \big(x_1(u,v)\,,\, x_2(u,v)\,, \ldots\,, x_n(u,v) \big)\,,$$

cuyo dominio es un abierto del plano uv; se dice que es una *inmersión* si todas sus diferenciales tienen rango 2; en esta tesis no se abordan cuestiones de inyectividad. Las funciones escalares $x_1(u,v)\,,\, x_2(u,v)\,, \ldots\,, x_n(u,v)$ son llamadas las *componentes* de la parametrización. Una parametrización $\bar{\Phi}$ es conforme si conserva los ángulos del plano al espacio. Esto equivale a que $\bar{\Phi}$ satisfaga el siguiente sistema de EDPs:

$$\left.\begin{array}{rcl} \bar{\Phi}_u \cdot \bar{\Phi}_v & = & 0 \\ \|\bar{\Phi}_u\|^2 - \|\bar{\Phi}_v\|^2 & = & 0 \end{array}\right\}$$

al cual nos vamos a referir en esta introducción como *el sistema no lineal,* por razones obvias.

Los resultados principales de esta tesis estudian la existencia de parametrizaciones cuyas funciones componentes son todas polinomios o todas racionales, y que sean *nuevas,* es decir no construibles por los métodos clásicos.

Una herramienta clásica para la construcción sistemática de parametrizaciones conformes en $\mathbb{R}^3$ es la *representación de Weierstrass-Enneper,* que produce una parametrización conforme a partir de cualquier par de funciones holomorfas de la variable compleja $z = u + \mathbf{i}v$. Si partimos de un par de polinomios en la variable z, entonces la parametrización que resulta tiene sus tres componentes polinómicas, además de ser conforme. Este método de construcción se generaliza a $\mathbb{R}^n$ pero tiene una limitación: sólo puede producir parametrizaciones de superficies *minimales*. En particular, nunca vamos a ver en la superficie puntos elípticos ni parabólicos. El primer resultado de esta tesis se demuestra en el capítulo 2 y dice lo siguiente:

Teorema A. *Si una parametrización, de un abierto del plano a $\mathbb{R}^n$, tiene todas sus componentes polinomios, entonces es armónica. Como consecuencia, la superficie así parametrizada es minimal.*

Con componentes polinómicas no hay, pues, ejemplos nuevos: todos pueden obtenerse via la representación de Weierstrass-Enneper.

Para demostrar el teorema **A**, primero se introducen coordenadas polares r, θ en el plano uv, poniendo:

$$u \;=\; r\cos\theta \;\;,\;\; v \;=\; r\operatorname{sen}\theta\,,$$

y entonces $\bar{\Phi}$ es conforme si y sólo si satisface el siguiente sistema:

$$\left.\begin{array}{rcl} r\bar{\Phi}_r\cdot\bar{\Phi}_\theta &=& 0 \\ r^2\bar{\Phi}_r\cdot\bar{\Phi}_r-\bar{\Phi}_\theta\cdot\bar{\Phi}_\theta &=& 0 \end{array}\right\}$$

De estas dos ecuaciones, la segunda por sí sola ya obliga a una $\bar{\Phi}$ polinómica a ser armónica. Sea $\bar{\Phi}$ una parametrización polinómica, de grado m, que cumple la segunda ecuación. Por una parte la función $\phi(r)$ definida por:

$$\phi(r) \;\equiv\; \int_0^{2\pi}\left(r^2\bar{\Phi}_r\cdot\bar{\Phi}_r-\bar{\Phi}_\theta\cdot\bar{\Phi}_\theta\right)d\theta\;,$$

es idénticamente nula y a la vez es un polinomio de grado $2m$ en la variable r:

$$0 \;\equiv\; \phi(r) \;\equiv\; a_0+a_1r^2+a_2r^4+\cdots+a_mr^{2m}\,,$$

lo que implica $0 = a_0 = a_1 = \cdots = a_m$. Por otra parte, se separan las partes homogéneas:

$$\bar{\Phi} \;\equiv\; \bar{\Phi}_0+\bar{\Phi}_1+\cdots+\bar{\Phi}_m\,,$$

pudiendo suponer nula la parte constante después de una conveniente traslación en $\mathbb{R}^3$. Dado $d > 0$, Para el espacio de polinomios homogéneos de grado d en (u, v) se utiliza la *base de Fischer,* formada por los siguientes elementos:

$$r^{2j}\,\mathrm{Re}(z^{d-2j})\ ,\ r^{2j}\,\mathrm{Im}(z^{d-2j}) \quad , \quad 0 \leq j \leq \left[\frac{d}{2}\right] ,$$

y se separa la parte homogénea $\bar{\Phi}_d$ como $\bar{\Phi}_{d,h} + \bar{\Phi}_{d,1}$, siendo $\bar{\Phi}_{d,h}$ la *parte armónica,* generada por los dos polinomios de Fischer con $j = 0$, y $\bar{\Phi}_{d,1}$ la *parte complementaria,* generada por los polinomios de Fischer con $j > 0$. Tenemos identidades para los coeficientes de $\phi(r)$:

$$\begin{aligned}
a_m &= Q_m(\bar{\Phi}_{m,1})\,, \\
a_{m-1} &= Q_{m-1}(\bar{\Phi}_{m-1,1}) + B_{1,1}(\bar{\Phi}_{m,1}\,,\,\bar{\Phi}_{m-2})\,, \\
a_{m-2} &= Q_{m-2}(\bar{\Phi}_{m-2,1}) + B_{2,1}(\bar{\Phi}_{m-1,1}\,,\,\bar{\Phi}_{m-3}) + B_{2,2}(\bar{\Phi}_{m,1}\,,\,\bar{\Phi}_{m-4}) \\
\text{etc.}
\end{aligned}$$

donde las Q_j son formas cuadráticas y las $B_{j,k}$ son formas bilineales.

Expresándolas en las bases de Fischer, se ve que las Q_j son definidas positivas en las partes complementarias $\bar{\Phi}_{d,1}$. Entonces de $a_m = 0$ se deduce que $\bar{\Phi}_{m,1} = 0$, lo que nos permite poner $a_{m-1} = Q_{m-1}(\bar{\Phi}_{m-1,1})$ (ya sin el término cruzado) y de $a_{m-1} = 0$ se sigue $\bar{\Phi}_{m-1,1} = 0$; pero entonces $a_{m-2} = Q_{m-2}(\bar{\Phi}_{m-2,1})$ (ya sin los dos términos cruzados) y de $a_{m-2} = 0$ se obtiene $\bar{\Phi}_{m-2,1} = 0$. Siguiendo así, se llega a que todas las partes complementarias son nulas y por lo tanto $\bar{\Phi} \equiv \bar{\Phi}_{m,h} + \bar{\Phi}_{m-1,h} + \cdots + \bar{\Phi}_{1,h}$ es armónica.

A partir del teorema **A** se demuestra en el capítulo 3 un resultado de rigidez local:

Teorema B. *Dadas dimensiones n, m con $n > m \geq 3$, una aplicación polinómica conforme de un abierto de $\mathbb{R}^m$ a $\mathbb{R}^n$ tiene que tener todas sus componentes de primer grado; o sea que debe ser la restricción a un abierto de $\mathbb{R}^m$ de una aplicación afín. La imagen es, pues, un trozo de m-plano.*

La no existencia de ejemplos *nuevos* polinómicos, expuesta en el teorema **A**, aumenta el interés en los posibles ejemplos racionales, o sea ejemplos de parametrizaciones conformes de superficies cuyas componentes sean todas funciones racionales de las variables (u, v). Pero aquí también se tiene una infinidad de ejemplos *clásicos* que se obtienen de la siguiente manera: se parte de cualquier ejemplo polinómico, obtenido mediente Weierstrass-Enneper, y se lo compone con la inversión respecto de una $(n-1)$-esfera en $\mathbb{R}^n$.

El tercer resultado principal de esta tesis, demostrado en el capítulo 4, dice que sí hay ejemplos racionales auténticamente nuevos:

Teorema C. *Existen ejemplos explícitos de aplicaciones racionales conformes de abiertos de* $\mathbb{R}^2$ *a* $\mathbb{R}^3$ *cuyas imágenes no son superficies de Willmore.*

La *ecuación de Willmore* es una EDP de origen variacional, es satisfecha por todas las superficies minimales y es invariante por transformaciones conformes del espacio. Por lo tanto, una superficie que no sea de Willmore ni es minimal ni es imagen de una minimal por inversiones respecto de esferas en $\mathbb{R}^3$. Luego los ejemplos explícitos del teorema **C** no son construibles por métodos clásicos.

Además de los ejemplos racionales que no producen superficies de Willmore, se exponen en el capítulo 4 otros ejemplos no triviales de parametrizaciones conformes explícitas cuyas componentes son funciones sencillas.

Los ejemplos del teorema **C** se construyen utilizando la *representación espinorial,* que es un método para construir (todas) las parametrizaciones conformes de superficies en $\mathbb{R}^3$ o en $\mathbb{R}^4$. El método en dimensión 3 se expone en el capítulo 4, donde se utiliza para demostrar el teorema **C** y dar otros ejemplos explícitos no racionales. El método en dimensión 4 se expone en el capítulo 5.

A continuación se describen los elementos esenciales de la representación espinorial en $\mathbb{R}^3$. Se definen las variables complejas:

$$z = u + \mathbf{i}v \quad , \quad \bar{z} = u - \mathbf{i}v \, ,$$

y los operadores:

$$\partial_z \;=\; (1/2)(\partial_u - \mathbf{i}\partial_v) \quad , \quad \partial_{\bar z} \;=\; (1/2)(\partial_u + \mathbf{i}\partial_v)\,.$$

Dada una parametrización $\bar\Phi \equiv \big(\, x_1(u,v)\,,\, x_2(u.v)\,,\, x_3(u,v)\,\big)$, y dado un par de funciones (f,g) con valores en $\mathbb{C}$, decimos que (f,g) *es el espinor de* $\bar\Phi$ si se cumple la identidad:

$$\partial_z \bar\Phi \;=\; \big(\, 2fg\,,\, g^2 - f^2\,,\, \mathbf{i}(f^2+g^2)\,\big)\,.$$

Dicho esto, resulta que una parametrización en $\mathbb{R}^3$ es conforme si y sólo si tiene un espinor. Si se parte de funciones f,g analíticas en las variables $z,\bar z$, se puede integrar con respecto a $\bar z$ en el sistema:

$$\left.\begin{array}{rcl} \partial_{\bar z} x_1 & = & 2fg \\ \partial_{\bar z} x_1 & = & g^2 - f^2 \\ \partial_{\bar z} x_1 & = & \mathbf{i}(f^2+g^2) \end{array}\right\}$$

pero no siempre se obtendrá una parametrización conforme en $\mathbb{R}^3$, debido a que las funciones calculadas x_1,x_2,x_3 pueden tomar valores complejos imaginarios. Existen soluciones x_1,x_2,x_3 con todos sus valores en $\mathbb{R}$ si y sólo si el par (f,g) satisface un *sistema de Dirac:*

$$\left.\begin{array}{rcr} f_{\overline z} & = & A\,\overline{g} \\ g_{\overline z} & = & -A\,\overline{f} \end{array}\right\}$$

cuyo potencial A sea una función real. En resumen, la representación espinorial sustituye el sistema no lineal, que hemos dado al principio de esta introducción, por el sistema de Dirac que es lineal. Aprovechando el lenguage de la Física, se puede decir: un par (f,g) es el espinor de una parametrización conforme en $\mathbb{R}^3$ si y sólo si es un *espinor de Dirac con potencial real.*

Descrito el método en dimensión 3, los ejemplos del teorema **C** se construyen de la siguente manera. Se empieza con una parametrización clásica de la esfera: la *estereográfica*

$$\bar\Psi_0 \;\equiv\; \left(\, 1 + \frac{-2}{1+|z|^2}\,,\, \frac{z+\bar z}{1+|z|^2}\,,\, i\frac{z-\bar z}{1+|z|^2} \,\right)\,,$$

cuyo espinor (f_0, g_0) tiene las siguientes componentes:

$$f_0 \;\equiv\; \frac{\bar{z}}{1+|z|^2} \quad , \quad g_0 \;\equiv\; \frac{1}{1+|z|^2} \;,$$

y satisface el sistema de Dirac con potencial $A \equiv 1/(1+|z|^2)$. Se buscan otros espinores de Dirac (f, g) con este mismo potencial, para lo cual se usa el *método de variación de las constantes* pensando formalmente en las EDPs que forman el sistema de Dirac como ecuaciones diferenciales ordinarias en las que $\bar{z}$ fuese la variable independiente. Esta idea se materializa sutituyendo las incógnitas f, g por nuevas incógnitas p, q que guardan con aquellas las siguientes relaciones:

$$f = pf_0 + qg_0 \,,$$
$$g = -\bar{q}f_0 + \bar{p}g_0 \,.$$

Con esta sustitución, el sistema de Dirac se convierte en el siguiente:

$$\left.\begin{array}{rcl} p_{\bar{z}} & = & -\dfrac{1}{\bar{z}}\, q_{\bar{z}} \\ p_z & = & z\, q_z \end{array}\right\}$$

que determina la función p a partir de la función q. En particular, para cada valor constante λ la función $q_\lambda \equiv 1 + \lambda\bar{z}^2$ determina la función $p_\lambda \equiv -2\lambda\bar{z}$. Resulta una familia a un parámetro (f_λ, g_λ) de espinores de Dirac, todos con el mismo potencial real $A \equiv 1/(1+|z|^2)$, que son los espinores de una familia a un parámetro $\bar{\Phi}_\lambda$ de parametrizaciones conformes. El cálculo explícito muestra que las $\bar{\Phi}_\lambda$ son todas racionales (con coeficientes enteros sencillos) y, para $\lambda \neq 0$, no satisfacen la ecuación de Willmore.

Conclusiones. No hay ejemlos, distintos de los clásicos, de parametrizaciones conformes polinómicas de superficies en $\mathbb{R}^n$. Pero, al menos en $\mathbb{R}^3$, sí hay ejemplos racionales no clásicos; algunos explícitos y sencillos.

Las únicas subvariedades de dimensión $m \geq 3$ en $\mathbb{R}^n$ que admiten parametrizaciones conformes polinómicas son los m-planos afines.

Abstract

The current thesis discusses some new results about conformal surface parametrizations defined by very simple functions. The first main result is the following theorem:

Given a conformal parametrization of a 2-dimensional surface in $\mathbb{R}^n$ whose component functions are all polynomial in the parameters, it must be harmonic.
As a first corollary, every surface in $\mathbb{R}^n$ that admits a conformal polynomial parametrization must be a minimal surface.

The second main result consists of interesting explicit examples of rational conformal parametrizations defining surfaces in $\mathbb{R}^3$ that are not Willmore surfaces. The resulting surfaces are thus neither minimal nor inversions of minimal surfaces. The main tool for this construction is the spinorial surface representation: a conformal surface parametrization is obtained by quadratures from a spinor whose components satisfy a Dirac type system of equations with a scalar potential. This is an extension, to arbitrary surfaces in $\mathbb{R}^3$ or in $\mathbb{R}^4$, of the Weierstrass-Enneper formulae.

In addition to the non-Willmore rational examples, some other non-trivial examples of explicit conformal parametrizations are obtained using this method.

A third result establishes rigidity of conformal polynomial parametrizations of m-dimensional submanifolds, with $m \geq 3$, in the euclidean space $\mathbb{R}^n$:

The only conformal polynomial immersions of $\mathbb{R}^m$ into $\mathbb{R}^n$, with $n > m \geq 3$, are the affine ones. The surface must be an m-plane.

Agradecimientos

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Por último, gracias al viejo de la tienda, por dar la vuelta al cartel de la puerta y recordarme que la tienda de caramelos aún seguía abierta.

Madrid
28 de septiembre de 2012

David Pérez Fernández

Preface

This thesis is dedicated to the study of conformal polynomial surface parametrizations in $\mathbb{R}^n$. It is composed of five chapters. In the first one an introduction to conformal parametrizations is made. In this chapter some classical results are exposed. A brief historical review is made and many applications of conformal geometry to theoretical physics and other science fields are enunciated.

In the second chapter, conformal polynomial parametrizations are discussed. First, the conditions for conformal homogeneous polynomial parametrizations are established. After that a general result is established: given a conformal polynomial surface parametrization of any degree it must be harmonic.

An immediate corollary is proved; if one surface admits a conformal polynomial parametrization it must be a minimal surface.

In the third chapter, a general theorem about the rigidity of conformal polynomial parametrizations of m-dimensional submanifolds is proved. The result is consistent with the Liouville conformal theorem and establishes the non-existence of any m-dimensional conformal polynomial parametrization, in $\mathbb{R}^n$, of degree higher than one.

The only surfaces that can be parametrized conformally and by polynomials are m-planes.

The fourth chapter is devoted to the spinorial surface representation. This is a generalization of the Weierstrass-Enneper minimal surface representation that is possible in $\mathbb{R}^3$ and $\mathbb{R}^4$. This surface parametrization is also conformal. The system of equations used to find the spinorial components, for a given surface, is a Dirac type system of equations for a massless particle in an external scalar field.

On the last chapter, some conformal parametrization explicit examples are explained. Interesting non-Willmore surfaces examples of rational conformal parametrizations defining surfaces in $\mathbb{R}^3$ are explained. Also some simple potential integration examples are given and nontrivial non-polynomial conformal surface parametrizations are obtained.

Chapter 1

Introduction to conformal transformations

1.1 A brief history

This section is a short introduction to the history of the study of conformal transformations in geometry and also explain some of the main applications to different fields of the science. The conformal mapping history is explained from the origins, the works related to sphere stereographic projection, the relation to complex analytic functions, the development of projective geometry and the use of radial inversions on the plane and higher dimensional spaces. Finally, polyspheric coordinate linearization of conformal transformations are also recalled.

There is a lot of applications of conformal transformations, one of the outstanding is the resolution of Laplace equation on different science domains. The Laplace equation, $\Delta f = 0$, is invariant under conformal transformations. This equation is very common on physics, specially on dynamic systems governed by potential functions; electromagnetic and gravitational fields, elasticity or fluid dynamics are good examples.

The conformal transformations on the plane are very powerful for domain transformation of boundary conditions for bidimensional problems.

There are other application fields; from navigation and cartography to General Relativity or Quantum Field Theory. Conformal transformations are being used in the study of the asymptotic behavior of solutions. Conformal transformations are also used on conformal invariant quantum field theories.

For a complete review of the history of conformal transformations, the main discoveries and applications, see [7] and [8].

1.1.1 Riemannian Geometry

A map that *conforms to the principle of angle preservation* receive the name of conformal mapping.

The conformal property of a transformation can be described in terms of the jacobian matrix. If the transformation jacobian matrix is a multiple of a matrix rotation, at every point, then the transformation is conformal.

In Riemannian geometry, the Riemann surface metric of a n-dimensional surface, in $\mathcal{M}^n$, is defined by:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

Note 1. *Einstein notation for summation over repeated indexes will be used.*

The notation for surface tangent vectors will be:

$$a = a^\mu \partial_\mu, \quad b = b^\mu \partial_\mu$$

The angle between vectors, at x point, is given by:

$$\alpha = \frac{g_{\mu\nu}(x)a^\mu b^\nu}{\sqrt{g_{\mu\nu}(x)a^\mu a^\nu}\sqrt{g_{\mu\nu}(x)b^\mu b^\nu}} \tag{1.1.1}$$

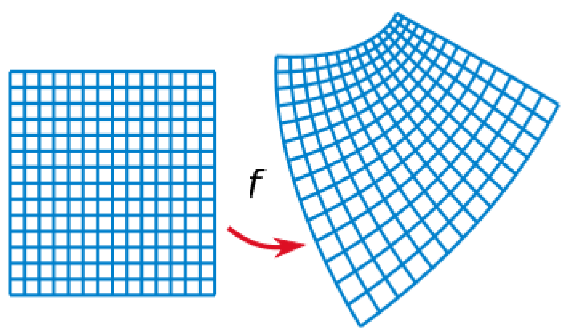
Definition 1. Two manifolds, with the same dimension, are said *conformal* if there exists one map $f : \mathcal{M}^n \to \mathcal{M}'^m$, that maps them conformally:

$$f^* g' = \lambda g \tag{1.1.2}$$

The metric of two conformal surfaces must be the same up to a positive multiplicative factor $\lambda(x)$, sometimes called *conformal scale factor*.

It can be easily checked that this kind of transformations preserves the angles (1.1.1). The angle between two curves, that intersects and are contained on one of the conformally equivalent surfaces, is transformed in to another pair of curves that meet forming the same angle (not necessarily with the same orientation).

A *conformal structure* on the surface is defined by an atlas where all coordinate trasitions are conformal maps between domains in the plane.



1.1.2 Conformal surface parametrizations

A surface parametrization in $\mathbb{R}^n$ will be denoted by $\bar{X}$:

$$\bar{X}(x,y) = \{X_1(x,y), X_2(x,y), \dots, X_n(x,y)\}, \quad \text{with} \quad (x,y) \in \mathbb{R}^2 \tag{1.1.3}$$

A parametrization $\bar{X}$ is said *conformal* if the metric matrix is a diagonal matrix multiplied by a positive function, the conformal scaling factor.

In local coordinates, a surface parametrization is conformal if the following conditions are satisfied:

$$\begin{cases} \bar{X}_x \cdot \bar{X}_x - \bar{X}_y \cdot \bar{X}_y = 0 \\ \bar{X}_x \cdot \bar{X}_y = 0 \end{cases} \tag{1.1.4}$$

In polar coordinates, these conditions are expressed as:

$$\begin{cases} r^2\bar{X}_r \cdot \bar{X}_r - \bar{X}_\theta \cdot \bar{X}_\theta = 0 \\ r\bar{X}_r \cdot \bar{X}_\theta = 0 \end{cases} \tag{1.1.5}$$

Next, some conformal parametrization properties are enumerated:

- Equations (1.1.4) implies that a surface parametrization, as a conformal map from $\mathbb{R}^2$ to the ambient space, preserves the angles. The tangent vectors to the cartesian coordinates on the plane, $\{\partial_u, \partial_v\}$, are transformed by the parametrization conformally into surface tangent vectors, $\{\bar{X}_u, \bar{X}_v\}$, with the same module and orthogonal between them on each surface point.

- The first fundamental form is diagonal and the diagonal matrix elements are identical. In other words, $ds^2 = \lambda(u,v)(du^2 + dv^2)$, where $\lambda(u,v)$ is the conformal scaling factor.

- The uv-Laplacian of the parametrization, $\Delta\bar{X}$, equals $2\lambda HN$, where H is the mean curvature and N the surface normal vector.

For a detailed proof of this and other conformal transformation properties see [9], [11] and [12].

Some classical results related to conformal transformations are recalled:

- *Riemann*: Every simply connected complex plane region, that is not the entire plain, can be conformally transformed into the unit disk.

- *Liouville*: Every conformal transformation of $\mathbb{R}^n$, with $n > 2$, must be a composition of one or more of the following operations: translations, dilations, rigid motions and inversions.

- *Korn-Lichtenstein* theorem: every surface admits conformal parameters locally.
- *Koebe-Poincaré* uniformization theorem: every surface has a global conformal parametrization (although this conformal parametrization can be very complex). More concretely, every Riemann surface admits a conformal covering by one of the following:
  - the Riemann sphere
  - the complex plane
  - the unit disk in the complex plane.

  see [?Gray] for a detailed history of this discovery.

1.1.3 The complex plane

One important class of conformal transformations is generated by complex *holomorphic* and meromorphic functions:

$$f(z) = u + \mathbf{i}v, \quad z = x + \mathbf{i}y$$

The real and imaginary components of a holomorphic function, $f = u + \mathbf{i}v$, must satisfy the *Cauchy-Riemann conditions*:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Thus each component must be harmonic, $\Delta u = \Delta v = 0$. It is related to the fact that the Laplacian operator is invariant under conformal transformations.

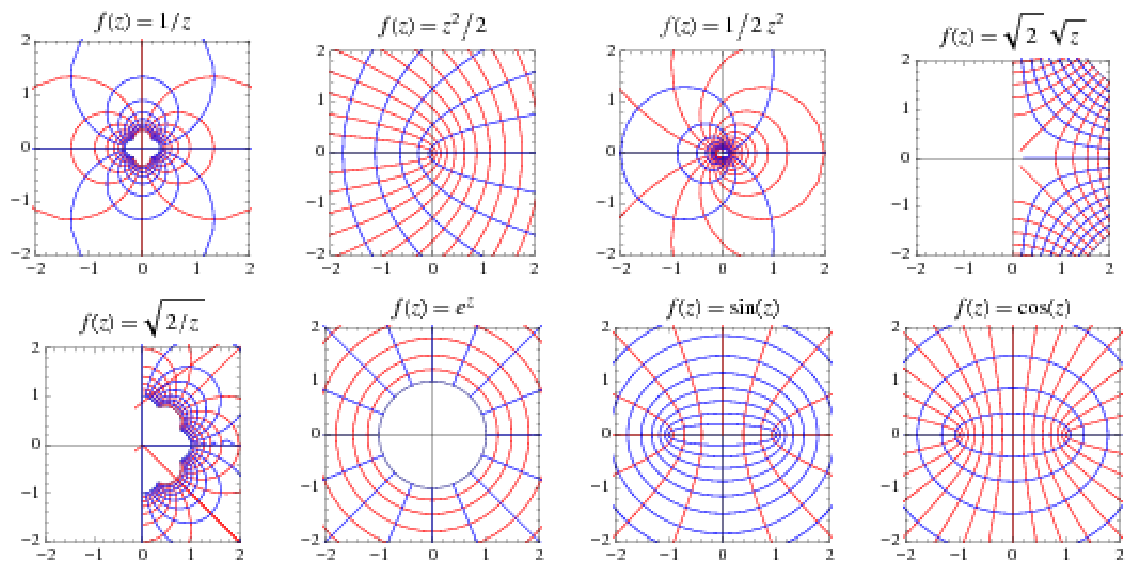
It can be checked that every holomorphic function is a conformal transformation:

$$ds^2 = du^2 + dv^2 = \left|\frac{df}{dz}\right|(dx^2 + dy^2)$$

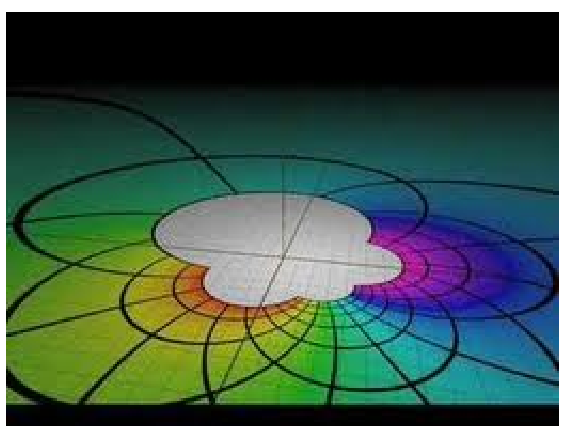
Antiholomorphic functions $f(\bar{z})$, where $\bar{z} = x - \mathbf{i}y$, also provide conformal maps. These functions preserve angles but invert vector basis orientation.

Note 2. *This property is used in the resolution of bidimensional boundary problems for the Laplace equation. Using a conformal transformation, the boundary conditions of the initial problem are transformed into easier ones. This method is commonly used in physics, notably electromagnetism, gravitation and fluid dynamics.*

The next figure shows some simple examples of complex plane conformal transformations:



See the video: http://www.youtube.com/watch?v=JX3VmDgiFnY for a beautiful visual explanation of a distinguished conformal plane transformation group called Möbius group:



The first works on conformal transformations using complex holomorphic functions were done by Euler and Gauss. Later, Riemann, working as a Gauss doctorate student, made a proof of his famous theorem about conformal transformations on the complex plane.

The Riemann theorem about plane conformal transformations, a deep complex analysis result, establish that every simple connected region on the complex plane can be conformally transformed, by a complex bijective map, on the unit circle. Thus every pair of simple regions can be transformed conformal and bijectively on each other.

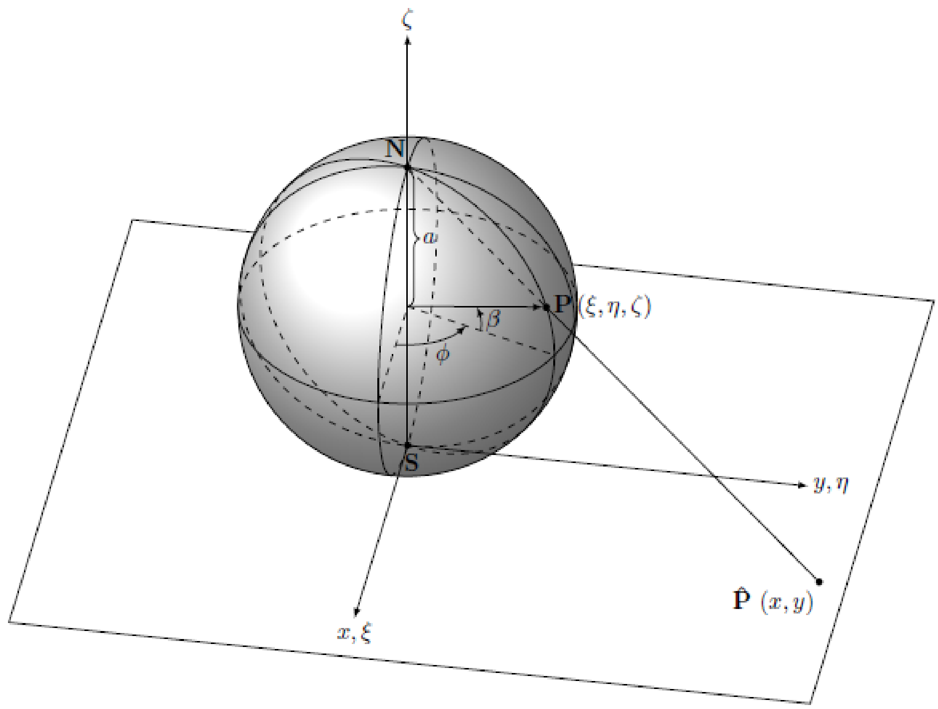
1.1.4 Stereographic projection

There are other simple examples of conformal transformations on $\mathbb{R}^n$ using the composition of radial inversions R and translations T:

$$R : x'^{\nu} = \frac{x^{\nu}}{|x|^2}$$
$$T(b) : x'^{\nu} = x^{\nu} + b^{\nu}$$

where $\nu = 1, \ldots, n$. These examples will be detailed on the next sections, this transformations generate the *special conformal group*, an important subgroup of the conformal group $C(n)$.

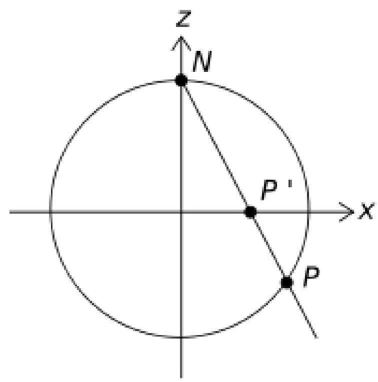
An important conformal map is the *stereographic projection* that maps the sphere S^1 into the plane:



This transformation maps each sphere point P into the plane point $\hat{P}$. The points are joined by a straight line from the north or south pole. The transformation maps circles into straight lines and vice versa.

As can be seen on the figure, the south pole coincides with the plane origin $(x, y) = (0, 0)$ and the north pole of the radix a is $(\xi = 0, \eta = 0, \zeta = 2a) \in \mathbb{R}^3$. The stereographic projection joins plane points $\hat{P}(x, y)$, using straight lines that intersects the spherical surface $P(\xi, \eta, \zeta)$, $\xi^2 + \eta^2 + (\zeta - a)^2 = a^2$.

The next figure shows a section of the stereographic projection:



In local coordinates, the south pole stereographic projection of a sphere, with a radius, is given by:

$$\begin{cases} x = \frac{2a\xi}{2a-\zeta} \\ y = \frac{2a\eta}{2a-\zeta} \quad with \;\; \xi^2 + \eta^2 + (\zeta - a)^2 = a^2 \end{cases}$$

The inverse map is given by:

$$\begin{cases} \xi = \frac{4a^2 x}{4a^2+x^2+y^2} \\ \eta = \frac{4a^2 y}{4a^2+x^2+y^2} \\ \zeta = \frac{2a(x^2+y^2)}{4a^2+x^2+y^2} \end{cases}$$

Using polar coordinates, with latitude and longitude angles, (β, ϕ):

$$\begin{cases} \xi = a \cos\phi \cos\beta \\ \eta = a \sin\phi \cos\beta \\ \zeta - a = a \sin\beta \end{cases}$$

The inverse projection, from sphere to plane:

$$\begin{cases} x = \frac{2a\cos\phi\cos\beta}{1-\sin\beta} \\ y = \frac{2a\sin\phi\cos\beta}{1-\sin\beta} \quad with \;\; \frac{\cos\beta}{1-\sin\beta} = \tan\frac{\beta}{2} + \frac{\pi}{4} \end{cases}$$

Using the polar coordinates, the metric transformation carried out by stereographic projection can be easily calculated:

$$g(x,y) = dx^2 + dy^2 = \frac{4}{(1-\sin\beta)^2} g(\phi, \beta)$$

where $g(\phi, \beta) = a^2[\cos\beta^2 d\phi^2 + d\beta^2]$ is the standard metric in spherical coordinates. As can be seen, it satisfies (1.1.2) and hence the stereographic projection is a conformal transformation.

Apart from the conformal properties, the stereographic projection has other interesting properties. It transforms circles into circles (the straight line can be taken as an infinite radius circle). Every circle contained on the sphere can be obtained by the intersection with a plane:

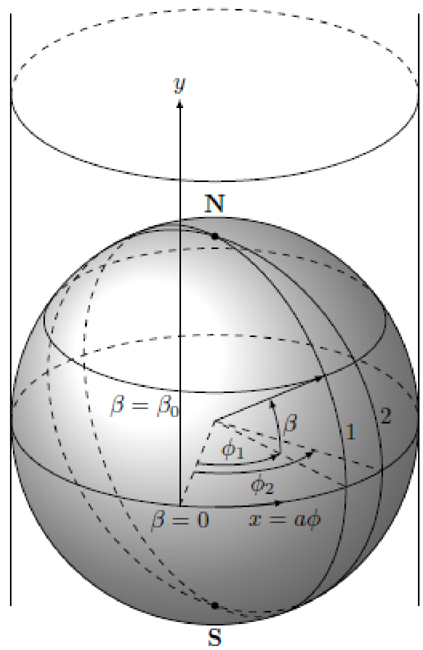
$$c_1\xi + c_2\eta + c_3\zeta + c_0 = 0$$

If the plane contains the north pole, and a sphere contained circle, this can be turned into a straight line on the plane. If the plane contains the north pole and a meridian, this circle is converted into a line that contains the origin.

The stereographic projection was of great importance on the construction astrolabe or planisphere, a very useful navigation instrument. Some important contributors to this invention and to celestial maps development where done by Hipparchus, Ptolemy, Al-Farghani, Clavius and Raleigh.

Other conformal transformations where used on cartography. It's of special importance the Mercator works. He was one of the first in to use a *conformal cylindrical projection* for a map representation.

He used a cylinder where the represented sphere points are projected. The sphere is contained in the cylinder of the same radius. This projection maps the meridians to parallel straight lines. This map conformally transforms the sphere into a cylinder, see [7] for a complete proof.



The Mercator projection differential analysis was carried by Lambert. Lambert proposed the general problem of a sphere projection to the plane that preserves angles and areas. Later, Euler solved the problem and proved the non existence of projections that simultaneously preserve area and angles.

1.1.5 Group theory and conformal transformations

Apart from the important work of Gauss, on the nineteenth century, there was a great interest on the study of geometrical relations between lines and circles, planes and spheres, given by conformal transformations. Mathematicians like Steiner, Durrande, Bellavitis, Quetelet and Dandelin were among the most outstanding. On this works can be seen the first analytical treatment of radial inversion transformations (also called reciprocal radius transformations).

The stereographic projection idea of transformation between lines and circles is repeated. In this case, one infinite point must be added to make the transformation bijective.

The first mathematician who employs the group theory point of view to analyze conformal transformations was Möbius, who discovered the transformation that receives his name:

$$z' = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$ and $z, a, b, c, d \in \mathbb{C}$. It implies the following relation between the metrics:

$$(dx')^2 + (dy')^2 = \frac{|ad-bc|^2}{|cz+d|^4}[dx^2 + dy^2]$$

where it can be seen that the transformation is conformal, (1.1.2).

If it's normalized, $ad - bc = 1$, the eight real components of the complex parameters a, b, c, d are reduced to six real parameters. This Lie group contains some interesting subgroups:

- Linear transformations:

$$z' = az + b$$

for $a = 0$ it represents a *translation*, $T(b)$, and for $b = 0$ can be obtained a *rotation* or a *dilation*.

- *Inversions*

$$R : z' = \frac{1}{z}$$

- Combinations of translations and inversions:

$$C_2(b) \equiv R \cdot T(b) \cdot R : z' = \frac{z + b|z|^2}{1 + 2(b \cdot z) + |b|^2|z|^2}$$

This subgroup, that also belongs to higher dimensional conformal group, it is known as the *special conformal group*.

Many properties of radial inversions in the plane where studied by he same authors (Durrande, Steiner, Plücker, Möbius) in $\mathbb{R}^3$. Later, Liouville proved that inversions are essentially the main non linear conformal transformations in $\mathbb{R}^n$ space, for $n \geq 3$. Liouville proved that all the conformal transformations of the space $\mathbb{R}^n$, $n \geq 3$, can be obtained as the composition of translations and inversions. The higher dimensional case, $n \geq 3$, is very different from the plane one, where the Riemann theorem establish the possibility of conformally transform every simple connected domain.

1.1.6 Polycyclic coordinates

When projective geometry and stereographic projection was exposed, it was necessary to add one line or imaginary point at infinity to make the map bijective. This fact is related to a topological problem: to map two different kind of surface; a compact one, the sphere, to a non compact one, the plane. The conformal transformations only changes the metric by a multiplicative factor, that could be different from one surface point to another.

On this section, it will be seen how using homogeneous coordinates the non linear conformal transformations can be converted into linear ones, over the sphere.

Darboux, in 1869, [33], discusses how to study the plane and tridimensional space properties using *homogeneous coordinates* on the sphere (S^2 or S^3). He used the stereographic projection.

First, homogeneous coordinates are introduced in $\mathbb{R}^2$:

$$x = \frac{y^1}{k}, \quad y = \frac{y^2}{k}, \quad with \quad (y^1, y^2, k) \neq (0, 0, 0)$$

In $\mathbb{R}^3$, the homogeneous coordinates are defined as:

$$\xi = \frac{\eta^1}{k}, \quad \eta = \frac{\eta^2}{k}, \quad \zeta = \frac{\eta^3}{k} \quad with \quad (\eta^1, \eta^2, \eta^3, k) \neq (0, 0, 0, 0)$$

As could be seen in [7], using this new homogeneous coordinates, the Möbius group transformations can be linearized. The Möbius conformal group is isomorphic to the Lorentz pseudo-orthogonal group, $O(1, 3)/\mathbb{Z}_2$.

These coordinates are known as *tetracyclic coordinates*. The name is related to the fact that every circle contained on the sphere is represented by a plane. When each of the coordinates are made null, four circles on the plane are obtained, this circles defines the point coordinates.

In the $\mathbb{R}^3$ case, Darboux works with five homogeneous coordinates called pentacyclic. Later, the Darboux idea was generalized to *polycyclic coordinates*. On this coordinates the conformal transformation group is related to the $O(1,n)/\mathbb{Z}_2$ group.

1.2 Applications of conformal transformations

Apart from the geometrical and mathematical uses, the conformal transformations has a great number of applications on different fields of physics and engineering. For a complete review of the main applications see [8] and [7].

There is a lot of applications of conformal transformations related to the resolution of Laplace equation on different contexts. On the plane, conformal transformations of the domain can simplify the boundary conditions. The main fields of application are electromagnetism, gravitation, elasticity and fluid dynamics.

At the beginning of the 20th century, Bateman, see [2],and Cunningham, see [1], proved that Maxwell electromagnetic field equations are invariant under conformal transformations. The conformal invariance is a generalization of the Lorentz group invariance found previously by Einstein and Minkowski.

Later, there were several unification attempts of General Relativity and Electromagnetic field theories by Einstein, Weyl, see [3], and Kaluza. The Weyl main idea was to use the conformal invariance on the unified theory. Finally, Einstein dismiss the Weyl ideas because it didn't fit to the physical evidences. The theory developed by Weyl was applied in mathematics and later in physics, but this time in Quantum Field theory, as a reinterpretation of the gauge theories ideas on this framework.

In General Relativity, the conformal transformations of the Minkowski space are the simpler transformations that maintain the event causal order. This kind of transformations are also used to study the asymptotic behavior near the space-time curvature singularities of the gravitational field (for example, to model the neighborhood of a black hole or the immediate time after the big bang).

In Quantum mechanics, there was an extensive work about the conformal invariance of the field equations. In 1936, Dirac studied the conformal invariance of his equations, see [4]. Also in 1936, Dirac proved the Maxwell equations invariance under 15 parameter conformal group using hexaspherical coordinates. He used this coordinates to transform his spinorial relativistic wave equation. Finally, the Weyl and Brouwer works used Clifford algebras to analyze the spinorial representation of the pseudo-orthogonal group and to prove the conformal invariance of Maxwell and Dirac null mass equations (as will be see later, this is the same system of equations used for spinorial conformal surface representation), see [5].

In the late years, there were some advances in the use of conformal symmetries in Quantum Relativistic fields theories, related to supersymmetry models, see Coleman and Mandula, [6]

Beyond the theoretical physic applications there are many other application fields:

- Elasticity theory, vibrant membranes and acoustics.
- Determination of the transmission lines and wavefront of electromagnetic field waves.
- Optics and light propagation on optical fiber.

- Electromagnetic wave diffraction.
- Particle collision models in atomic physics.
- Heat transfer and non linear diffusion problems.

Chapter 2

Conformal polynomial surface parametrizations

2.1 Harmonic and homogeneous polynomials

This section is devoted to some definitions that will be used.

Definition 2. A *conformal surface parametrization* must satisfy the following condition:

$$\begin{cases} \bar{X}_x \cdot \bar{X}_x - \bar{X}_y \cdot \bar{X}_y = 0 \\ \bar{X}_x \cdot \bar{X}_y = 0 \end{cases} \tag{2.1.1}$$

In polar coordinates the condition is:

$$\begin{cases} r^2 \bar{X}_r \cdot \bar{X}_r - \bar{X}_\theta \cdot \bar{X}_\theta = 0 \\ r\bar{X}_r \cdot \bar{X}_\theta = 0 \end{cases} \tag{2.1.2}$$

Definition 3. A polynomial p is said to be *homogeneous* when all the monomial components have the same degree. In other words, p is a homogeneous polynomial of degree k in $\mathbb{R}^n$ if it has the following form:

$$p(x_1, x_2, \ldots, x_n) = \sum_i a_i x_1^{\alpha_1^i} x_2^{\alpha_2^i} \cdots x_n^{\alpha_n^i}$$

where summation contains all combinations that satisfy $\sum_{j=1}^{n} \alpha_j^i = k \quad \forall i$.

The set of n variable *homogeneous polynomials* will be denoted by $\mathcal{P}(\mathbb{R}^n)$ and the set of homogeneous polynomials of degree i by $\mathcal{P}_i(\mathbb{R}^n)$.

Definition 4. A polynomial p is said to be *harmonic* when its Laplacian is null, i.e. $\Delta p = 0$.

$\mathcal{H}(\mathbb{R}^n)$ will denote the *harmonic homogeneous polynomials* set in $\mathbb{R}^n$ and $\mathcal{H}_i(\mathbb{R}^n)$ symbolize the harmonic homogeneous polynomial set of degree i.

Note 3. *It's possible to make a more general definition of the Laplace operator for curved spaces called Laplace-Beltrami*

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j f\right)$$

The operator definition is the composition of the generalized gradient and divergence operators:

$$\operatorname{div} X = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right)$$
$$(\operatorname{grad} f)^i = \partial^i f = g^{ij}\partial_j f$$

The following known result, based on a more general theorem proved by Ernst Fischer in 1917 [35], will be used:

Theorem 4. *Every homogeneous polynomial can be uniquely decomposed as a sum of harmonic homogeneous polynomials multiplied by r^2 powers.*

More explicitly, every m degree homogeneous polynomial $p \in \mathcal{P}_m(\mathbb{R}^n)$, can be decomposed as:

$$p = h_m + r^2 h_{m-2} + \cdots + r^{2s} h_{m-2s}$$

where $s = [\frac{m}{2}]$ ([ ] is the integer part operator), and r^2 is the square of the position vector length in $\mathbb{R}^n$, $r^2 = x_1^2 + \cdots + x_n^2 = |x|^2$, and every h_i are harmonic homogeneous polynomials of degree i, $h_i \in \mathcal{H}_i(\mathbb{R}^n)$.

The proof of this theorem can be seen in [10], theorem 5.7.

The homogeneous polynomial space admits the decomposition in harmonic polynomial spaces given by:

$$\mathcal{P}_m(\mathbb{R}^n) = \mathcal{H}_m(\mathbb{R}^n) \oplus r^2\mathcal{H}_{m-2}(\mathbb{R}^n) \oplus \cdots \oplus r^{2s}\mathcal{H}_{m-2s}(\mathbb{R}^n)$$

where $s = [\frac{m}{2}]$.

In the cited reference [10] can be seen a proof of the following proposition about the harmonic polynomial space dimension:

Proposition 5. *If $m > 2$ then:*

$$dim \ \mathcal{H}_m(\mathbb{R}^n) = \begin{pmatrix} n+m-1 \\ n-1 \end{pmatrix} - \begin{pmatrix} n+m-3 \\ n-1 \end{pmatrix} \tag{2.1.3}$$

In order to study conformal surface parametrizations only the space $\mathcal{H}_m(\mathbb{R}^2)$ of harmonic polynomials in two variables will be used. Following the above proposition, the $\mathcal{H}_m(\mathbb{R}^2)$ harmonic polynomial space is two dimensional.

Note 6. *The harmonic two variables k degree polynomial space, $\mathcal{H}^k(\mathbb{R}^2)$, can be expressed as the real and imaginary part of z^k, where $z \in \mathbb{C}$. A basis of the space $\mathcal{H}^k(\mathbb{R}^2)$ is $\{Re(z^k), Im(z^k)\}$ or:*

$$\{r^k \cos k\theta, r^k \sin k\theta\} \tag{2.1.4}$$

This is known as the Fourier basis and the decomposition exposed in the theorem 4, in the two variable case, is the Fourier series expansion of a homogeneous polynomial.

The $\mathcal{H}^m(\mathbb{R}^2)$ Fourier basis will be denoted by $\{h_1^m, h_2^m\}$.

Note 7. *The next example tries to clarify the vector coefficient notation.*

The standard basis of $\mathcal{H}^i(\mathbb{R}^2)$*,* $i = 1, 2, 3$*, are the next harmonic homogeneous polynomial pairs:*

$$\begin{aligned}\mathcal{H}^1(\mathbb{R}^2) &= \{h_1^1(x,y), h_2^1(x,y)\} = \{x, y\}\\ \mathcal{H}^2(\mathbb{R}^2) &= \{h_1^2(x,y), h_2^2(x,y)\} = \{x^2 - y^2, 2xy\}\\ \mathcal{H}^3(\mathbb{R}^2) &= \{h_1^3(x,y), h_2^3(x,y)\} = \{x^3 - 3xy^2, 3x^2y - y^3\}\end{aligned}$$

The Enneper minimal surface has polynomial components and also is conformal. This surface can be expressed in terms of vector coefficients multiplied by harmonic base elements as:

$$\begin{aligned}&\bar{\psi}(x,y) = \left(x - x^3/3 + xy^2, -y + y^3/3 - x^2y, x^2 - y^2\right) =\\ &= \bar{\lambda}h_1^3(x,y) + \bar{\beta}h_2^3(x,y) + \bar{\gamma}h_1^2(x,y) + \bar{\mu}h_1^1(x,y) + \bar{\nu}h_2^1(x,y)\end{aligned}$$

where:

$$\bar{\lambda} = \begin{pmatrix} -1/3 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\beta} = \begin{pmatrix} 0 \\ -1/3 \\ 0 \end{pmatrix}, \quad \bar{\gamma} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \bar{\nu} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

The notation for the angular component of the k degree Fourier basis elements will be:

$$\bar{f}_k = \bar{v}_k \sin k\theta + \bar{w}_k \cos k\theta$$

where the radial factor, r^k, is deliberately eliminated.

The next definition will also be used:

$$\bar{f}'_k = \bar{v}_k \cos k\theta - \bar{w}_k \sin k\theta = \frac{1}{k}\frac{d\bar{f}_k}{d\theta}$$

The next relations are consequences of the Fourier basis orthogonality properties on the unit circle S_1:

$$\int_0^{2\pi} |\bar{f}_k|^2 d\theta = \int_0^{2\pi} |\bar{f}'_k|^2 d\theta = \pi(|\bar{v}_k|^2 + |\bar{w}_k|^2) \tag{2.1.5}$$

$$\int_0^{2\pi} \bar{f}_k \cdot \bar{f}_i d\theta = \int_0^{2\pi} \bar{f}'_k \cdot \bar{f}'_i d\theta = \pi(|\bar{v}_k|^2 + |\bar{w}_k|^2)\delta_{ki} \tag{2.1.6}$$

$$\int_0^{2\pi} \bar{f}_k \cdot \bar{f}'_i d\theta = 0 \tag{2.1.7}$$

$$\int_0^{2\pi} \bar{f}_k \cdot \bar{g}_k d\theta = \int_0^{2\pi} \bar{f}'_k \cdot \bar{g}'_k d\theta = \pi(\bar{v}_k \cdot \bar{o}_k + \bar{w}_k \cdot \bar{q}_k) \tag{2.1.8}$$

where $\bar{o}_k$ and $\bar{q}_k$ are the vector coefficients of $\bar{g}_k$, the angular component of another harmonic polynomial.

2.2 Homogeneous conformal polynomial parametrizations

As a first step, k degree homogeneous polynomial parametrizations will be studied for a later resolution of the general problem: conformal polynomial surface parametrizations.

Proposition 8. *Every conformal surface parametrization, $\bar{X} : \mathbb{R}^2 \to \mathbb{R}^n$, in $\mathbb{R}^n$ with k degree homogeneous polynomial components must have harmonic components.*

In other words, the surface parametrization in polar coordinates has the form:

$$\bar{X} = r^k(\bar{v}_k \cos k\theta + \bar{w}_k \sin k\theta)$$

or using cartesian coordinates:

$$\bar{X} = \bar{v}_k Re(z^k) + \bar{w}_k Im(z^k), \quad z \in \mathbb{C}$$

Also the extreme vector coefficients must be orthogonal and with identical length:

$$\begin{cases} |\bar{v}_k| = |\bar{w}_k| \\ \bar{v}_k \cdot \bar{w}_k = 0 \end{cases}$$

Proof. The decomposition theorem 4 will be used. In order to simplify calculations, the homogeneous polynomial basis of $\mathbb{R}^2$ will be used as explained on remark 6 (2.1.4).

The conformal parametrization will be expressed on polar coordinates as:

$$\bar{X} = r^k(\bar{f}_k + \bar{f}_{k-2} + \ldots + \bar{f}_{k-2s})$$

where $s = [\frac{k}{2}]$ and the elements $\bar{f}_i$ correspond to the angular part of the harmonic polynomial base elements, following the above notation.

The parametrization, $\bar{X} = r^k(\bar{f}_k + \bar{f}_{k-2} + \ldots + \bar{f}_{k-2s})$, has tangent vectors in polar coordinates:

$$\begin{aligned} \bar{X}_r &= kr^{k-1}(\bar{f}_k + \bar{f}_{k-2} + \ldots + \bar{f}_{k-2s}) \\ \bar{X}_\theta &= r^k\big(k\bar{f}'_k + (k-2)\bar{f}'_{k-2} + \ldots + (k-2s)\bar{f}'_{k-2s}\big) \end{aligned}$$

The first condition of conformal parametrization in polar coordinates (2.1.2) is given by:

$$
\begin{aligned}
&r^2\bar{X}_r\cdot\bar{X}_r-\bar{X}_\theta\cdot\bar{X}_\theta=\\
&k^2\big[|\bar{f}_k|^2+|\bar{f}_{k-2}|^2)+\cdots+|\bar{f}_{k-2s}|^2\big]-\\
&-\big[k^2|\bar{f}_k|^2+(k-2)^2|\bar{f}_{k-2}|^2+\cdots+(k-2s)|\bar{f}_{k-2s}|^2\big]=0
\end{aligned}
$$

If this equation is integrated on the unit circle, S_1:

$$
\begin{aligned}
&\int_{S_1} r^2\bar{X}_r\cdot\bar{X}_r-\bar{X}_\theta\cdot\bar{X}_\theta|_{S_1}d\sigma=\\
&=\int_0^{2\pi} k^2\big[|\bar{f}_k|^2+|\bar{f}_{k-2}|^2)+\cdots+|\bar{f}_{k-2s}|^2\big]d\theta-\\
&-\int_0^{2\pi}\big[k^2|\bar{f}_k|^2+(k-2)^2|\bar{f}_{k-2}|^2+\cdots+(k-2s)|\bar{f}_{k-2s}|^2\big]d\theta=\\
&=k^2\pi\big[(|\bar{v}_k|^2+|\bar{w}_k|^2)+(|\bar{v}_{k-2}|^2+|\bar{w}_{k-2}|^2)+\cdots\\
&\cdots+(|\bar{v}_{k-2s}|^2+|\bar{w}_{k-2s}|^2)\big]-\\
&-\pi\big[k^2(|\bar{v}_k|^2+|\bar{w}_k|^2)+(k-2)^2(|\bar{v}_{k-2}|^2+|\bar{w}_{k-2}|^2)+\cdots\\
&\cdots+(k-2s)^2(|\bar{v}_{k-2s}|^2+|\bar{w}_{k-2s}|^2)\big]=\\
&=\pi[(k^2-(k-2)^2)(|\bar{v}_{k-2}|^2+|\bar{w}_{k-2}|^2)+\cdots\\
&\cdots+(k^2-(k-2s)^2)(|\bar{v}_{k-2s}|^2+|\bar{w}_{k-2s}|^2]=0
\end{aligned}
$$

with $s=[\frac{k}{2}]$ and where orthogonality properties (2.1.5) were used.

It can be simplified to:

$$
\begin{aligned}
&\big[(k^2-(k-2)^2\big](|\bar{v}_{k-2}|^2+|\bar{w}_{k-2}|^2)+\cdots\\
&\cdots+\big[k^2-(k-2s)^2\big](|\bar{v}_{k-2s}|^2+|\bar{w}_{k-2s}|^2)=0\\
&\Rightarrow|\bar{v}_{k-2}|=|\bar{w}_{k-2}|=\ldots=|\bar{v}_{k-2s}|=|\bar{w}_{k-2s}|=0
\end{aligned}
$$

where the coefficients $\big[k^2-(k-2i)^2\big]$ are positive for $i\in(1,[\frac{k}{2}])$.

Definition 5. The highest degree elements, $\bar{f}_k$, will be called the polynomial *principal harmonic components*. This part of the polynomial harmonic decomposition is not multiplied by any r power and corresponds to the harmonic part of the homogeneous polynomial.

The above equation means that all the non principal harmonic components of the polynomial parametrization are null. In other words, only the *principal harmonic component* can be non null. Thus the surface parametrization must be harmonic and has the form:

$$\bar{X} = r^k(\bar{v}_k \cos k\theta + \bar{w}_k \sin k\theta)$$

Using the polar form of the conformal conditions (2.1.2) and the fact that coefficients of the different powers of r must be null, the additional condition is obtained:

$$\begin{cases} |\bar{v}_k| = |\bar{w}_k| \\ \bar{v}_k \cdot \bar{w}_k = 0 \end{cases}$$

□

2.3 Conformal polynomial parametrizations

This section is devoted to the proof of the following result:

Theorem 9. *Every conformal polynomial surface parametrization, immersed in* $\mathbb{R}^n$*, must be harmonic.*

The general form of any conformal polynomial parametrization of k *degree in polar coordinates is:*

$$\bar{X} = \sum_{i=0}^{k} r^i(\bar{v}_i \cos i\theta + \bar{w}_i \sin i\theta)$$

or using cartesian coordinates:

$$\bar{X} = \sum_{i=0}^{k} \bar{v}_i Re(z^i) + \bar{v}_i Im(z^i) = \sum_{i=0}^{k} \bar{v}_i h_1^i + \bar{v}_i h_2^i$$

where $\{h_1^i, h_2^i\}$ *is the basis of* $\mathcal{H}_i(\mathbb{R}^2)$*.*

Also the vector coefficients of maximum degree, $j = k$*, and minimum degree,* $j = 1$*, must satisfy:*

$$\begin{cases} |\bar{v}_j| = |\bar{w}_j| \\ \bar{v}_j \cdot \bar{w}_j = 0 \end{cases}$$

Corollary 10. *Every Riemann surface* M *in* $\mathbb{R}^n$ *that admits a conformal polynomial parametrization must be a minimal surface.*

This follows from the usual formula for mean curvature over a conformal parametrization:

$$H = \frac{1}{2}\Delta\bar{X} \cdot \frac{\bar{N}}{E} = 0 \tag{2.3.1}$$

Note 11. *There is a large infinity of examples which are obtained by applying the Weierstrass-Enneper formulae to holomorphic polynomial data. In fact, the Enneper surface and its higher order analogues are already quite non-trivial surfaces with polynomial conformal parametrizations.*

Proof of the theorem. The proof is decomposed in the following steps:

1. Take a polynomial surface parametrization, in $\mathbb{R}^n$, of maximum degree k:
$$\bar{X}(x,y) = (X_1(x,y), \dots, X_n(x,y))$$

2. The polynomial parametrization splits into the sum of homogeneous components of degrees $1, \dots, k$. The constant terms are neglected because the conformal condition is invariant under surface translations.

   Using the introduced vector notation, the polynomial surface parametrization takes the form:
$$\bar{X} = \bar{P}_k + \bar{P}_{k-1} + \cdots + \bar{P}_1$$
   where $\bar{P}_i$ are the vectors of i degree homogeneous polynomials, $P_i \in \mathcal{P}_i(\mathbb{R}^2)$.

3. The decomposition theorem 4 is applied to each homogeneous component:
$$\begin{aligned}
\bar{P}_k &= r^k\big(\bar{f}_k + \bar{g}_{k-2} + \cdots + \bar{h}_{k-2s_k}\big)\\
\bar{P}_{k-1} &= r^{k-1}\big(\bar{f}_{k-1} + \bar{g}_{k-3} + \cdots + \bar{h}_{k-2s_{k-1}}\big)\\
&\vdots\\
\bar{P}_2 &= r^2\big(\bar{f}_2 + \bar{g}_0\big)\\
\bar{P}_1 &= r\bar{f}_1
\end{aligned}$$
   where $s_k = [\frac{k}{2}]$ and the vectors $\bar{f}_i, \bar{g}_i, \dots, \bar{h}_i$ symbolize the angular component of the i degree homogeneous polynomials. Following the previous polar notation:
$$\begin{aligned}
\bar{f}_i &= \bar{v}_i \sin i\theta + \bar{w}_i \cos i\theta\\
\bar{g}_i &= \bar{o}_i \sin i\theta + \bar{q}_i \cos i\theta\\
&\vdots\\
\bar{h}_i &= \bar{t}_i \sin i\theta + \bar{u}_i \cos i\theta
\end{aligned}$$
   The k degree polynomial parametrization takes the form:
$$\bar{X}(r,\theta) = r^k\big(\bar{f}_k + \bar{g}_{k-2} + \cdots + \bar{h}_{k-2s_k}\big) + r^{k-1}\big(\bar{f}_{k-1} + \bar{g}_{k-3} + \cdots + \bar{h}_{k-2s_{k-1}}\big) + + \ldots + r^2\big(\bar{f}_2 + \bar{g}_0\big) + r\bar{f}_1$$
   As established on the above theorem, the maximum order harmonic terms, $\bar{f}_k, \bar{f}_{k-1}, \dots$ will be called *principal harmonic components* of the harmonic decomposition.

4. The tangent vectors can be expressed in polar coordinates. The $r\bar{X}_r$ is expressed as:

$$r\partial_r(\bar{P}_k) = kr^k[\bar{f}_k + \bar{g}_{k-2} + \cdots + \bar{h}_{k-2s_k}]$$
$$r\partial_r(\bar{P}_{k-1}) = (k-1)r^{k-1}[\bar{f}_{k-1} + \bar{g}_{k-3} + \cdots + \bar{h}_{k-2s_{k-1}}]$$
$$\vdots$$
$$r\partial_r(\bar{P}_2) = 2r^2[\bar{f}_2 + \bar{g}_0]$$
$$r\partial_r(\bar{P}_1) = r\bar{f}_1$$

and $\bar{X}_\theta$:

$$\partial_\theta(\bar{P}_k) = r^k[k\bar{f}'_k + (k-2)\bar{g}'_{k-2} + \cdots + (k-2s_k)\bar{h}'_{k-2s_k}]$$
$$\partial_\theta(\bar{P}_{k-1}) = r^{k-1}[(k-1)\bar{f}'_{k-1} + (k-3)\bar{g}'_{k-3} + \cdots + (k-2s_{k-1})\bar{h}'_{k-2s_{k-1}}]$$
$$\vdots$$
$$\partial_\theta(\bar{P}_2) = r^2[2\bar{f}'_2]$$
$$\partial_\theta(\bar{P}_1) = r\bar{f}'_1$$

These equalities are replaced on the first conformal parametrization condition (2.1.2):

$$\begin{aligned}
&r^{2k}[k^2(\bar{f}_k\cdot\bar{f}_k - \bar{f}'_k\cdot\bar{f}'_k) + k^2\bar{g}_{k-2}\cdot\bar{g}_{k-2} - (k-2)^2\bar{g}'_{k-2}\cdot\bar{g}'_{k-2} + \cdots +\\
&\quad +k^2\bar{h}_{k-2s_k}\cdot\bar{h}_{k-2s_k} - (k-2s_k)^2\bar{h}'_{k-2s_k}\cdot\bar{h}'_{k-2s_k}] +\\
&+r^{2k-2}[(k-1)^2(\bar{f}_{k-1}\cdot\bar{f}_{k-1} - \bar{f}'_{k-1}\cdot\bar{f}'_{k-1}) + (k-1)^2\bar{g}_{k-3}\cdot\bar{g}_{k-3} -\\
&\quad -(k-3)^2\bar{g}'_{k-3}\cdot\bar{g}'_{k-3} + \cdots + (k-1)^2\bar{h}_{k-2s_{k-1}}\cdot\bar{h}_{k-2s_{k-1}} -\\
&\quad -(k-2s_{k-1})^2\bar{h}'_{k-2s_{k-1}}\cdot\bar{h}'_{k-2s_{k-1}} + \ldots +\\
&\quad +k(k-2)\bar{f}_k\cdot\bar{g}_{k-4} - (k-2)^2\bar{f}'_k\cdot\bar{g}'_{k-4} + \ldots] +\\
&\quad \vdots\\
&+r^6[3^2\bar{g}_1\cdot\bar{g}_1 - \bar{g}'_1\cdot\bar{g}'_1 + (4\cdot 2)\bar{g}_2\cdot\bar{f}_2 - (2\cdot 2)\bar{g}'_2\cdot\bar{f}'_2 + 5\bar{h}_1\cdot\bar{f}_1 - \bar{h}'_1\cdot\bar{f}'_1] +\\
&+r^4[2^2\bar{g}_0\cdot\bar{g}_0 - \bar{g}'_0\cdot\bar{g}'_0 + 3\bar{g}_1\cdot\bar{f}_1 - \bar{g}'_1\cdot\bar{f}'_1] + r^2[\bar{f}_1\cdot\bar{f}_1 - \bar{f}'_1\cdot\bar{f}'_1]\\
&+\sum_{i=2}^{k} r^{2i-1}[\ldots] = 0
\end{aligned}$$

The last term groups odd r powers because it will be null when the equation is integrated on the unit circle S_1, by the harmonic polynomial orthogonality properties (2.1.5).

5. Coefficients of different r powers must be null simultaneously, because conformal condition must be satisfied in all the space. The following relations are obtained:

$$k^2(\bar{f}_k \cdot \bar{f}_k - \bar{f}'_k \cdot \bar{f}'_k) + k^2\bar{g}_{k-2} \cdot \bar{g}_{k-2} - (k-2)^2\bar{g}'_{k-2} \cdot \bar{g}'_{k-2} + \cdots +$$
$$+k^2\bar{h}_{k-2s_k} \cdot \bar{h}_{k-2s_k} - (k-2s_k)^2\bar{h}'_{k-2s_k} \cdot \bar{h}'_{k-2s_k} = 0$$

$$(k-1)^2(\bar{f}_{k-1} \cdot \bar{f}_{k-1} - \bar{f}'_{k-1} \cdot \bar{f}'_{k-1}) +$$
$$+(k-1)^2\bar{g}_{k-3} \cdot \bar{g}_{k-3} - (k-3)^2\bar{g}'_{k-3} \cdot \bar{g}'_{k-3} + \cdots$$
$$\cdots + (k-1)^2\bar{h}_{k-2s_{k-1}} \cdot \bar{h}_{k-2s_{k-1}} - (k-2s_{k-1})^2\bar{h}'_{k-2s_{k-1}} \cdot \bar{h}'_{k-2s_{k-1}} + \ldots$$
$$\ldots + k(k-2)\bar{f}_k \cdot \bar{g}_{k-4} - (k-2)^2\bar{f}'_k \cdot \bar{g}'_{k-4} + \cdots = 0$$
$$\vdots$$
$$3^2\bar{g}_1 \cdot \bar{g}_1 - \bar{g}'_1 \cdot \bar{g}'_1 + (4 \cdot 2)\bar{g}_2 \cdot \bar{f}_2 - (2 \cdot 2)\bar{g}'_2 \cdot \bar{f}'_2 + 5\bar{h}_1 \cdot \bar{f}_1 - \bar{h}'_1 \cdot \bar{f}'_1 = 0$$

$$2^2\bar{g}_0 \cdot \bar{g}_0 - \bar{g}'_0 \cdot \bar{g}'_0 + 3\bar{g}_1 \cdot \bar{f}_1 - \bar{g}'_1 \cdot \bar{f}'_1 = 0$$
$$\bar{f}_1 \cdot \bar{f}_1 - \bar{f}'_1 \cdot \bar{f}'_1 = 0$$
$$[\text{Odd } r \text{ powers}] = 0$$

6. Now each equation is integrated on the unit circle.
Using the cited Fourier basis orthogonality properties, (2.1.5), the last term (corresponding to products of principal harmonic components) have identical coefficients $k^2, (k-1)^2, \ldots, 1$:

$$\int_0^{2\pi} \bar{f}_k \cdot \bar{f}_k - \bar{f}'_k \cdot \bar{f}'_k d\theta = \int_0^{2\pi} \bar{f}_{k-1} \cdot \bar{f}_{k-1} - \bar{f}'_{k-1} \cdot \bar{f}'_{k-1} d\theta = 0$$

Harmonic component products of different degree are null (it includes all the odd powers of r) by the Fourier basis orthogonality properties (2.1.5):

$$\int_0^{2\pi} \bar{f}_m \cdot \bar{f}_n d\theta = \int_0^{2\pi} \bar{f}'_m \cdot \bar{f}'_n d\theta = 0$$

with $m \neq n$.

The equations can be simplified and takes the form:

$$(k^2-(k-2)^2)\int_0^{2\pi}\bar{g}_{k-2}\cdot\bar{g}_{k-2}d\theta+\cdots+$$
$$+(k^2-(k-2s_k)^2)\int_0^{2\pi}\bar{h}_{k-2s_k}\cdot\bar{h}_{k-2s_k}d\theta=0$$

$$((k-1)^2-(k-3)^2)\int_0^{2\pi}\bar{g}_{k-3}\cdot\bar{g}_{k-3}d\theta+\cdots+$$
$$+((k-1)^2-(k-2s_{k-1})^2)\int_0^{2\pi}\bar{h}_{k-2s_{k-1}}\cdot\bar{h}_{k-2s_{k-1}}d\theta+\ldots+$$
$$+(k(k-2)-(k-2)^2)\int_0^{2\pi}\bar{g}_{k-2}\cdot\bar{f}_{k-2}d\theta+\cdots=0$$

$$\vdots$$

$$8\int_0^{2\pi}\bar{g}_1\cdot\bar{g}_1d\theta+4\int_0^{2\pi}\bar{g}_2\cdot\bar{f}_2d\theta+4\int_0^{2\pi}\bar{h}_1\cdot\bar{f}_1d\theta=0$$
$$4\int_0^{2\pi}\bar{g}_0\cdot\bar{g}_0d\theta+2\int_0^{2\pi}\bar{g}_1\cdot\bar{f}_1d\theta=0$$

There are only two types of products of the same degree:

- Terms corresponding to squares of harmonic components, like:

$$\int_0^{2\pi}\bar{g}_{k-1}\cdot\bar{g}_{k-1}d\theta=\pi(|\bar{o}_k|^2+|\bar{q}_k|^2)$$

- Cross-products of different degree harmonic components, for example:

$$\int_0^{2\pi}\bar{g}_k\cdot\bar{f}_{k-2}d\theta$$

Using the orthogonality properties (2.1.5), the above equations are simplified into:

$$\pi[(k^2-(k-2)^2)(|\bar{o}_{k-2}|^2+|\bar{q}_{k-2}|^2)+\cdots+ \\ +(k^2-(k-2s_k)^2)(|\bar{t}_{k-2s_k}|^2+|\bar{u}_{k-2s_k}|^2)]=0$$

$$((k-1)^2-(k-3)^2)\pi(|\bar{o}_{k-3}|^2+|\bar{q}_{k-3}|^2)+\cdots+ \\ +((k-1)^2-(k-2s_{k-1})^2)\pi(|\bar{t}_{k-2s_{k-1}}|^2+|\bar{u}_{k-2s_{k-1}}|^2)+\ldots+ \\ +(k(k-2)-(k-2)^2)\int_0^{2\pi}\bar{g}_{k-2}\cdot\bar{f}_{k-2}d\theta+\cdots=0$$

$$\vdots$$

$$8\pi(|\bar{o}_1|^2+|\bar{q}_1|^2)+4\int_0^{2\pi}\bar{g}_2\cdot\bar{f}_2d\theta+4\int_0^{2\pi}\bar{h}_1\cdot\bar{f}_1d\theta=0$$

$$4\int_0^{2\pi}\bar{g}_0\cdot\bar{g}_0d\theta+2\int_0^{2\pi}\bar{g}_1\cdot\bar{f}_1d\theta=0$$

7. The second type of terms, the cross-products, are removed gradually. Each equation represents the coefficient of a different power of r. Staring from the last equation, it makes null the square of the terms that appears on the cross-terms of the next r power. This process can be followed on the complete equation sequence.

   On the first equation, the square elements must be null, because all the coefficients are positive and can be deduced:

$$\bar{o}_{k-2}=\bar{q}_{k-2}=0 \quad \text{or} \quad |\bar{g}_{k-2}|=0\Rightarrow\bar{g}_{k-2}=0$$

   In the second equation the cross-product terms vanish because it contains the product of the harmonic components g_{k-2}:

$$\int_0^{2\pi}\bar{g}_{k-2}\cdot\bar{f}_{k-2}d\theta=0$$

   If we remove this cross term, the new equation cancel the harmonic coefficients of the next lower degree equation:

$$\bar{o}_{k-3}=\bar{q}_{k-3}=\cdots=\bar{t}_{k-2s_{k-1}}=\bar{u}_{k-2s_{k-1}}=0$$

8. This procedure is iterated over all the equations. The only nonzero resulting terms are the square terms of the principal harmonic components. The coefficients of the squares of non principal harmonic components are positive (it takes the form $k^2 - (k-i)^2$, with $i < k$ and $k > 1$) so all the quadratic terms must be null simultaneously: $|\bar{g}_i| = \ldots = |\bar{h}_j| = 0$.

$$\begin{aligned}
&\int_0^{2\pi} \bar{g}_i \cdot \bar{g}_i d\theta = \cdots = \int_0^{2\pi} \bar{h}_j \cdot \bar{h}_j d\theta = 0 \Rightarrow \\
&|\bar{o}_i| = |\bar{q}_i| = \cdots = |\bar{t}_i| = |\bar{u}_i| = 0 \Rightarrow \\
&\bar{g}_i = \cdots \bar{h}_j = 0
\end{aligned}$$

   Therefore, all non principal harmonic components are null. The polynomial parametrization can only contain principal harmonic components. In other words, the function components of the polynomial parametrization must be harmonic.

9. As in the homogeneous case, it can be applied the conformal parametrization condition (2.1.2) to the harmonic polynomial parametrization and then r powers can be grouped obtaining additional conditions for vector coefficients. The more simple conditions, for the higher degree, $j = k$, and lower vector coefficients, $j = 1$, are:

$$\begin{cases} |\bar{v}_j| = |\bar{w}_j| \\ \bar{v}_j \cdot \bar{w}_j = 0 \end{cases}$$

□

Note 12. *The above proof only uses the first conformal parametrization condition (2.1.2):*

$$\bar{X}_x \cdot \bar{X}_x - \bar{X}_y \cdot \bar{X}_y = 0$$

It could be thought that the second condition:

$$\bar{X}_x \cdot \bar{X}_y = 0$$

imposes additional restrictions. In general, for non polynomial parametrizations, that is true. In the polynomial case it will be shown that this condition is superfluous.

The conformal polynomial parametrization can be written on the complex plane as:

$$\bar{X}(r,\theta) = \sum_{j=1}^{n} r^j (\bar{v}_j \cos j\theta + \bar{w}_j \sin j\theta) = \sum_{j=1}^{n} \bar{V}_j z^j$$

with $z, \bar{V}_j \in \mathbb{C}$ *y* $\bar{V}_j = \bar{v}_j - \mathbf{i}\bar{w}_j$.

The surface tangent vectors are given by:

$$\begin{aligned}\bar{X}_x &= \bar{X}_z + \bar{X}_{\bar{z}}\\ \bar{X}_y &= i(\bar{X}_z - \bar{X}_{\bar{z}})\end{aligned}$$

The first condition on the complex plane is:

$$\begin{aligned}&\bar{X}_x \cdot \bar{X}_x - \bar{X}_y \cdot \bar{X}_y = 0 \Rightarrow\\ &\Rightarrow \bar{X}_z \cdot \bar{X}_z + \bar{X}_{\bar{z}} \cdot \bar{X}_{\bar{z}} = 0\end{aligned}$$

This condition takes the form:

$$\bar{X}_z \cdot \bar{X}_z = 0$$

The second conformal parametrization condition on the complex plane is:

$$\begin{aligned}&\bar{X}_x \cdot \bar{X}_y = 0 \Rightarrow\\ &i(\bar{X}_z \cdot \bar{X}_z - \bar{X}_{\bar{z}} \cdot \bar{X}_{\bar{z}}) = 0 \Rightarrow\\ &i(\bar{X}_z \cdot \bar{X}_z) = 0\end{aligned}$$

If the parametrization X *is harmonic and polynomial, it can be expressed as a polynomial in* z *variable, see remark 6, i.e., it must be an holomorphic function. When* $\bar{z}$ *derivatives are neglected it can be seen that the two conditions are equivalent.*

Chapter 3

A rigidity result for submanifolds

3.1 Rigid vs. non-rigid: Liouville and Riemann theorems

Using the above theorem 9 a higher dimension generalization will be studied for m-dimensional immersions into $\mathbb{R}^n$.

Liouville theorem establish rigid limitations to any conformal transformation of a region with dimension higher than two.

Theorem 13. *(Liouville) Every conformal transformation of a space region in $\mathbb{R}^n$, with $n > 2$, can be expressed as a composition of one or many of this operations: inversions, translations, rotations and dilations.*

In other words, in analytic form, every conformal map of $\mathbb{R}^n$, with $n > 2$, is the composition of this operations:

- *Translations:*
$$\bar{r}' = \bar{r} + \bar{a}$$
- *Dilations:*
$$\bar{r}' = b\bar{r}$$
- *Rotations:*
$$\bar{r}' = A\bar{r}$$

- *Special conformal transformations:*

$$\bar{r}' = \frac{\bar{r} + \bar{a}r^2}{1 + 2\bar{a} \cdot \bar{r} + a^2 r^2}$$

In a more geometrical form, this transformation is the composition of a inversion, a translation by a vector $\bar{a}$ and a new inversion:

$$\frac{\bar{r}'}{|\bar{r}'|^2} = \frac{\bar{r}}{|\bar{r}|^2} + \bar{a}$$

All this subgroups form the conformal group, that will be denoted by $C(n)$.

A complete proof of the Liouville theorem can be seen in [13] or [12], volume 3.

Liouville theorem express the rigidity of the conformal transformations in the space $\mathbb{R}^n$, for $n > 2$. This result contrast with the $\mathbb{R}^2$ case, where conformal transformations can be much flexible as the Riemann theorem express:

Theorem 14. *(Riemann) Any two simply connected planar domain can be conformally transformed one to the other.*

A proof of this theorem can be seen in [14].

3.2 Rigidity of higher-dimensional polynomial conformal immersions

The theorem 9 idea can be generalized to m-dimensional immersions into $\mathbb{R}^n$. It will be seen that there are rigidity conditions, as restrictive as the established by the Liouville theorem.

In fact, it will be shown that the only conformal polynomial immersions of a m-dimensional submanifold in $\mathbb{R}^n$, with $n > m \geq 3$, must be composed linear polynomials.

Theorem 15. *Every conformal polynomial parametrization of a m-dimensional submanifold, with $n > m \geq 3$, immersed in $\mathbb{R}^n$, must be linear. In other words, the only m-dimensional submanifold in $\mathbb{R}^n$ that admits a conformal polynomial parametrization are m-planes.The surface parametrization must be a affine transformation of the m-dimensional cartesian framework.*

Proof. Let $\bar{\psi}$ be a conformal polynomial parametrization of a m-dimensional submanifold in $\mathbb{R}^n$ and let $\bar{\phi}$ be a conformal polynomial parametrization of a bidimensional surface in $\mathbb{R}^n$. The two conformal polynomial parametrizations are:

$$\begin{cases} \bar{\phi}(x, y) : \mathbb{R}^2 \to \mathbb{R}^m \\ \bar{\psi}(x_1, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}^n \end{cases}$$

The composition of both maps, $\bar{X} : \mathbb{R}^2 \to \mathbb{R}^n$, must be a conformal map too and a polynomial parametrization of a bidimensional surface in $\mathbb{R}^n$. The theorem 9 establish that this parametrization must be harmonic. Thus every component, X_i, of the parametrization:

$$X^i = \psi^i(\phi^1(x, y), \phi^2(x, y), \ldots, \phi^m(x, y))$$

must be harmonic, $\Delta X^i = 0$.

The Laplacian of each component can be calculated explicitly:

$$\begin{aligned}
X^i_x &= \sum_{j=1}^{m} \psi^i_j\Big|_{\bar\phi} \phi^j_x \\
X^i_{xx} &= \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_x\phi^k_x + \sum_{j=1}^{m} \psi^i_j\Big|_{\bar\psi} \phi^j_{xx} \\
X^i_{yy} &= \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_y\phi^k_y + \sum_{j=1}^{m} \psi^i_j\Big|_{\bar\psi} \phi^j_{yy} \\
\Delta X^i &= \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_x\phi^k_x + \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_y\phi^k_y + \sum_{j=1}^{m} \psi^i_j\Big|_{\bar\psi} \Delta\phi^j = \\
&= \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_x\phi^k_x + \sum_{j,k=1}^{m} \psi^i_{jk}\Big|_{\bar\phi} \phi^j_y\phi^k_y \qquad (3.2.1)
\end{aligned}$$

The term $\Delta\phi^j$ can be removed because the components are conformal and polynomial so, by theorem 9, has to be harmonic.

Finally, the Laplacian of the parameter components X^i is:

$$\Delta X^i = \bar\phi_x \cdot Hess(\psi^i) \cdot \bar\phi_x + \bar\phi_y \cdot Hess(\psi^i) \cdot \bar\phi_y \qquad (3.2.2)$$

where $Hess(\psi^i)$ is the hessian matrix of the ψ parametrization i component.

The previous relation must be true for every conformal parametrization $\bar\phi$, of a surface in $\mathbb{R}^m$, used on the composition $\bar X = \bar\psi(\bar\phi(x,y))$. The simpler polynomial parametrization is a linear one:

$$\bar\phi(x,y) = \bar\lambda x + \bar\beta y$$

In order to be conformal it must satisfy:

$$\begin{cases} |\bar\lambda| = |\bar\beta| \\ \bar\lambda \cdot \bar\beta = 0 \end{cases}$$

The relation (3.2.2) must be satisfied at every point $p \in \mathbb{R}^m$:

$$\Delta X^i\Big|_p = \bar\lambda \cdot A \cdot \bar\lambda + \bar\beta \cdot A \cdot \bar\beta = 0 \qquad (3.2.3)$$

with $A \equiv Hess(\psi^i)\Big|_p$, where A is a symmetric matrix because it symbolize the Hessian matrix evaluated at the point p. The later equation is equivalent to the projection of a bilinear form represented by the symmetric matrix A into the plane formed by the vectors $\bar{\lambda}, \bar{\beta}$. The relation must be true for every conformal parametrization ϕ. The pair of vectors $\{\bar{\lambda}, \bar{\beta}\}$ can be chosen to be A matrix eigenvectors, $\{\bar{v_1}, \bar{v_2}\}$.

The only requirement for the vectors $\bar{\lambda}, \bar{\beta}$ is that they have to be orthogonal and of identical size. For example, it can be taken unitary. This condition is fulfilled by the matrix A eigenvectors because A is real and symmetric. The spectral theorem for finite spaces states that every real symmetric matrix can be diagonalized in a orthogonal basis and the eigenvalues must be real numbers.

Using the eigenvectors $\{\bar{v_1}, \bar{v_2}\}$ as $\bar{\lambda}, \bar{\beta}$ in the equation (3.2.3) can be transformed into:

$$\Delta X^i \bigg|_p = \bar{v_1} \cdot A \cdot \bar{v_1} + \bar{v_2} \cdot A \cdot \bar{v_2} = (\lambda_1 + \lambda_2)|\bar{v_1}| = 0 \tag{3.2.4}$$

where λ_i is the eigenvalue associated to the eigenvector v_i.

The same reasoning can be applied to the other m eigenvectors of the A matrix:

$$\lambda_i + \lambda_j = 0 \quad \forall i \neq j \quad i, j = 1, \ldots, m$$

This linear system of equations has only the null solution, for $m > 2$. In other words, the matrix A must be null, or equivalently, $Hess(\psi^i) = 0$ at every point p. All the second order derivatives and the second order cross derivatives of the parametrization components must be null and thus the conformal polynomial parametrization must be linear.

□

Note 16. *The above result is consistent with the Liouville theorem, when the dimension values m, n are the same. In the case $n = m$, the only conformal polynomial transformations, from $\mathbb{R}^m$ to $\mathbb{R}^m$, allowed by the Liouville theorem are the linear ones. These linear transformations corresponds to the composition of rotations, scale transformations and translations, in other words, affine transformations. The special conformal subgroup transformations can not be used because it contains radial inversions that are not polynomial transformations.*

Chapter 4

Spinors and conformal parametrizations

We recall here the Weierstrass-Enneper surface representation, which generates a conformal parametrizations from any pair of holomorphic functions. However, this method only gives parametrizations of minimal surfaces. It is described in the following theorem.

Theorem 17. *Given complex functions f and g, where g is meromorphic and f is analytic, such that wherever g has a pole of order m, f has a zero of order $2m$ (or equivalently, such that the product fg^2 is holomorphic). Then the surface with parametrization (X_1, X_2, X_3) is* minimal*, where the X_k are defined using the real part of a complex integral, as follows:*

$$X_k(\zeta) = Re\left(\int_0^{\zeta} \psi_k(z)\, dz \right) + \psi_k(\bar{z}), \qquad k = 1, 2, 3$$

$$\begin{cases} \psi_1 = 2fg \\ \psi_2 = f^2 - g^2 \\ \psi_3 = \mathbf{i}(f^2 + g^2) \end{cases}$$

where $\psi_k(\bar{z})$ are antiholomorphic functions or constants.This surface parametrization is also conformal.

For example, Enneper's minimal surface can be obtained from $f(z) = 1, g(z) = z$.

The Weierstrass-Enneper representation can be extended to obtain a conformal representation of any surface in $\mathbb{R}^3$ and $\mathbb{R}^4$ (and only for these dimensions, as can be seen in [17]).

The spinorial surface representation in $\mathbb{R}^3$ is detailed in the first section of this chapter. The spinorial representation for surfaces in $\mathbb{R}^4$ is described in the second section.

The method is explained following the lessons by J. Gonzalo. There were some previous works about spinorial surface representation by Pedit, Pinkall, Kusner, Kamberov, Sullivan, Schmitt, Abresch and Kenmotsu. For a complete reference see [20], [19], [21], [22] and [?Kam2].

A parametrization $\bar{\Phi}(u,v)$ is conformal if and only if it fulfills the conditions:

$$\begin{cases} \bar{\Phi}_u \cdot \bar{\Phi}_v = 0 \\ \|\bar{\Phi}_u\|^2 = \|\bar{\Phi}_v\|^2 \end{cases} \tag{4.0.1}$$

Observe that the equations are quadratic in the derivatives of the map, hence a fully nonlinear PDE system. The constructions next described linearize this system if the dimension is three or four.

4.1 Spinorial surface representation in $\mathbb{R}^3$

4.1.1 First linearization

To any parametrization $\bar{\Phi}(u,v)$ in $\mathbb{R}^3$ one associates the complex vector $(z_1, z_2, z_3) = \bar{\Phi}_u - \mathbf{i}\bar{\Phi}_v$, which satisfies the identity:

$$z_1^2 + z_2^2 + z_3^2 = (\|\bar{\Phi}_u\| - \|\bar{\Phi}_v\|) + \mathbf{i}(2\bar{\Phi}_u \cdot \bar{\Phi}_v)\,,$$

therefore $\bar{\Phi}(u,v)$ fulfills (4.0.1) if and only if the associated complex vector is a solution of the second degree Fermat equation:

$$z_1^2 + z_2^2 + z_3^2 = 0\,. \tag{4.1.1}$$

Every triple of numbers that solves this equation can be put in Diophantus form:

$$\begin{cases} z_1 = 2ab \\ z_2 = b^2 - a^2 \\ z_3 = \mathbf{i}(a^2 + b^2) \end{cases}$$

where a, b are arbitrary complex numbers.

In our context, the Fermat equation has to be satisfied at every surface point. It follows that $\bar{\Phi}(u,v)$ is conformal if and only if there exist two complex-valued functions $f(u,v), g(u,v)$ such that the following identity holds:

$$\frac{1}{2}(\bar{\Phi}_u - \mathbf{i}\bar{\Phi}_v) = \big(\, 2fg\,,\, g^2 - f^2\,,\, \mathbf{i}(f^2 + g^2)\,\big)\,. \tag{4.1.2}$$

The left-hand side of the above identity will be rewritten now in a more standard way. For any scalar or vectorial function $B(u,v)$, there are complex coefficients $B_z, B_{\bar{z}}$ that give $dB = B_z dz + B_{\bar{z}} d\bar{z}$, with $z = x + \mathbf{i}y \in \mathbb{C}$. These coefficients are:

$$B_z = \frac{1}{2}(B_u - \mathbf{i}B_v)$$
$$B_{\bar{z}} = \frac{1}{2}(B_u + \mathbf{i}B_v)$$

Inspired by that, the following differential operators are defined:

$$\partial_z = \frac{1}{2}(\partial_u - \mathbf{i}\partial_v)$$
$$\partial_{\bar{z}} = \frac{1}{2}(\partial_u + \mathbf{i}\partial_v)$$

and now equation (4.1.2) can be represented as:

$$\partial_z \bar{\Phi} = \left(2fg \, , \, g^2 - f^2 \, , \, \mathbf{i}(f^2 + g^2) \right) . \tag{4.1.3}$$

Writing $x_1(u,v), x_2(u,v), x_3(u,v)$ for the three components of $\bar{\Phi}(u,v)$, equation (4.1.3) is the same as the following linear PDE system:

$$\begin{cases} \partial_z x_1 = 2fg \\ \partial_z x_2 = g^2 - f^2 \\ \partial_z x_3 = \mathbf{i}(f^2 + g^2) \end{cases} \tag{4.1.4}$$

The only accepted solutions to this system are the real-valued ones and, if they exist, will be called **conformal parametrizations with spinor** (f, g).

The restriction to real-valued solutions for (4.1.4) gives raise to conditions on the spinor components f, g, as the following lemma states.

Lemma 18. *Given a function $\psi(u,v)$, with possible complex values, the following are equivalent conditions:*

- *Locally there exist real functions $x(u,v)$ that satisfy $\partial_z x = \psi$.*
- *The function $\partial_{\bar{z}}\psi$ is real.*

In view of this lemma, the system (4.1.4) has real solutions x_1, x_2, x_3 if and only if the spinor components f, g are, in their turn, solutions to the following system:

$$\begin{cases} Im(\partial_{\bar{z}}(2fg)) = 0 \\ Im(\partial_{\bar{z}}(g^2 - f^2)) = 0 \\ Im(\partial_{\bar{z}}\mathbf{i}(f^2 + g^2))) = 0 \end{cases} \tag{4.1.5}$$

Once f, g satisfy (4.1.5), there is an $\mathbb{R}^3$-valued solution $\bar{\Phi}(u,v)$ to (4.1.3), unique up to constant translation, which is an honest conformal parametrization with (f, g) as spinor.

4.1.2 Linearizing the equations on the spinor

The system (4.1.5) consists of three nonlinear PDE equations with two complex-valued unknown functions. This could seem to be worse that the original system (4.0.1), but in fact the conditions (4.1.5) can be replaced by a linear system on f, g, as established in theorem 19 below.

The only spinors (f, g) considered are those for which the product fg does not vanish on any open set in the uv plane. The reason for this restriction is that if $fg = 0$ on an open set then the formula (4.1.2) implies that $\bar{\Phi}_u$ and $\bar{\Phi}_v$ have null first component and the corresponding surface region is contained in an affine plane $\{x_1 = \text{constant}\}$. If we are content with surfaces without planar regions, then fg can at most vanish on a set with empty interior in the uv plane.

Theorem 19. *Given complex-valued functions $f(u,v), g(u,v)$, with the product fg not vanishing on any open set of the uv plane, the following are equivalent:*

- *(a) There are real-valued functions $x_1(u,v), x_2(u,v), x_3(u,v)$ satisfying (4.1.4).*
- *(b) There is a real-valued function $A(u,v)$ such that f, g satisfy the following linear system:*

$$\left.\begin{array}{rcl} f_{\bar{z}} & = & A\overline{g} \\ g_{\bar{z}} & = & -A\overline{f} \end{array}\right\} \tag{4.1.6}$$

From the physical point of view, the PDE system (4.1.6) is known as the Dirac equation for a relativistic null mass particle in a scalar potential A.

Proof. It is trivial to check that *(b)* implies (4.1.5) and thus also *(a)*. Let us assume that $\Phi(u,v) \equiv \big(\, x_1(u,v)\,,\, x_2(u,v)\,,\, x_3(u,v)\,\big)$ is a real-valued solution to (4.1.3) and

(4.1.4), and deduce that f, g satisfy *(b).*

From (4.1.3) one computes:

$$\frac{1}{8}\Delta\Phi \;\equiv\; \frac{1}{2}\,\Phi_{z\overline{z}} \;=\; \big(f_{\overline{z}}g + fg_{\overline{z}}\,,\, gg_{\overline{z}} - ff_{\overline{z}}\,,\, \mathbf{i}(ff_{\overline{z}} + gg_{\overline{z}})\big)\,. \tag{4.1.7}$$

As Φ is assumed to be $\mathbb{R}^3$-valued, so is its Laplacian $\Delta\Phi$. That is to say, the following functions are all real-valued:

$$f_{\overline{z}}g + fg_{\overline{z}} \;\;,\;\; gg_{\overline{z}} - ff_{\overline{z}} \;\;,\;\; \mathbf{i}(ff_{\overline{z}} + gg_{\overline{z}}\big)\;.$$

Thus the following expression is real and non-negative:

$$(f_{\overline{z}}g + fg_{\overline{z}})^2 + (gg_{\overline{z}} - ff_{\overline{z}})^2 + \big[\,\mathbf{i}(ff_{\overline{z}} + gg_{\overline{z}}\big)\,\big]^2\,,$$

but it is easily seen to be identical with $(f_{\overline{z}}g - fg_{\overline{z}})^2$. Therefore the function $f_{\overline{z}}g - fg_{\overline{z}}$ is real, because its square is real and non-negative.

So $f_{\overline{z}}g + fg_{\overline{z}}$ and $f_{\overline{z}}g - fg_{\overline{z}}$ turn out to be both real, which is equivalent to $f_{\overline{z}}g$ and $fg_{\overline{z}}$ being both real. This, in turn, is equivalent to the existence of two real-valued functions $A_1(u, v), A_2(u, v)$ such that the following system is satisfied:

$$\left.\begin{array}{rcl} f_{\overline{z}} &=& A_1\overline{g} \\ g_{\overline{z}} &=& A_2\overline{f} \end{array}\right\} \tag{4.1.8}$$

On the other hand, the fact that $gg_{\overline{z}} - ff_{\overline{z}}$ and $\mathbf{i}(ff_{\overline{z}} + gg_{\overline{z}})$ are both real is equivalent to $ff_{\overline{z}} = -\overline{gg_{\overline{z}}}$. This last equality is transformed via the system (4.1.8) into the following:

$$f\overline{g}A_1 \;=\; -\overline{g\overline{f}A_2} \;=\; -f\overline{g}A_2\,,$$

that is $(A_1 + A_2)f\overline{g} \equiv 0$. But we are assuming that neither f nor g vanishes on an open set of the uv plane, hence $A_1 + A_2 \equiv 0$ and system (4.1.8) really is system (4.1.6) with $A \equiv A_1$.

□

Now it is known that if f, g satisfy (4.1.6) then there is a conformal parametrization, unique up to constant translation, that is a solution of the system (4.1.3), i.e. it has (f, g) as spinor.

4.1.3 Geometrical meaning of the spinor

Denote by $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ the standard quaternion basis elements. Each vector in $\mathbb{R}^3$ can be identified with a purely imaginary quaternion:

$$\bar{a} = (a_1, a_2, a_3) \in \mathbb{R}^3 \;\longleftrightarrow\; a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\,.$$

For each purely imaginary quaternion $\mathbf{a}$, let $[\mathbf{a}]$ denote the corresponding vector in $\mathbb{R}^3$. Write $\mathbb{H}_0$ for the space of purely imaginary quaternions, then

$$\mathbf{a} \in \mathbb{H}_0 \implies q\mathbf{a}\bar{q} \in \mathbb{H}_0\ ,$$

where q is any quaternion and $\bar{q}$ is the quaternion conjugate of q. Every non-null quaternion q_0 determines a map as follows:

$$\begin{aligned} \mathrm{conj}_{q_0} : \mathbb{H}_0 &\longrightarrow \mathbb{H}_0 \\ \mathbf{a} &\longmapsto q_0\mathbf{a}\bar{q}_0 \end{aligned} \tag{4.1.9}$$

which is a conformal linear transformation in $\mathbb{R}^3 \approx \mathbb{H}_0$ and preserves orientation. In fact, this transformation is the composition of a dilation with factor $q_0\|^2$ and a rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$. This defines a surjective group homomorphism:

$$\mathbb{H} \setminus \{0\} \to \mathbb{R}^+ \times SO(3) \quad , \quad q_0 \longmapsto \mathrm{conj}_{q_0}$$

whose kernel is $\{1, -1\}$. In particular, to each rotation $R \in SO(3)$ there correspond two unitary quaternions $\pm q$ and $S^3 \approx SU(2)$ becomes a double-sheeted covering group for $SO(3)$.

The space $\mathbb{H} \setminus \{0\}$ also serves as double cover for the space of direct conformal bases of $\mathbb{R}^3$. Given a pair $(\bar{v}_2, \bar{v}_3)$ of vectors in $\mathbb{R}^3$, orthogonal and with the same length ℓ, there exists a quaternion $q_0 \in \mathbb{H} \setminus \{0\}$ so that the following equalities are true only for $q = q_0$ or $q = -q_0$:

$$\left.\begin{array}{rcl} \bar{v}_2 & = & [q\mathbf{j}\bar{q}] \\ \bar{v}_3 & = & [q\mathbf{k}\bar{q}] \end{array}\right\} \tag{4.1.10}$$

and the vector $\bar{v}_1 := [q\mathbf{i}\bar{q}]$, independent of the choice in $\pm q_0$, is the unique vector such that the triple $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is a direct conformal basis of $\mathbb{R}^3$, that is

$$[q\mathbf{i}\bar{q}] \;=\; \frac{\bar{v}_2 \times \bar{v}_3}{\ell}\,. \tag{4.1.11}$$

It is convenient to write the right-hand sides of (4.1.10) and (4.1.11) in terms of a pair (a, b) of complex numbers. Begin by grouping the general quaternion in the following way:

$$q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = a_0 + a_1\mathbf{i} + \mathbf{j}(a_2 - a_3\mathbf{i})$$

and thus write it as $q = a' + \mathbf{j}b$, with a' and b complex numbers. We further represent the number a' as $a\mathbf{i}$ and thus use the new number a. Given $a, b \in \mathbb{C}$ and the quaternion $q = \mathbf{i}a + \mathbf{j}b$, the following hold:

$$q\mathbf{i}\bar{q} = \mathbf{i}(|a|^2 - |b|^2) + \mathbf{j}(2\bar{a}b)$$

$$q\mathbf{j}\bar{q} = \mathbf{i}(ab + \bar{a}\bar{b}) + \mathbf{j}(b^2 - \bar{a}^2)$$

$$q\mathbf{k}\bar{q} = (\bar{a}\bar{b} - ab) - \mathbf{k}(\bar{a}^2 + b^2)$$

and also the identities:

$$\mathbf{j}Re(b^2 - a^2) + \mathbf{k}Re\big(\,\mathbf{i}(a^2 + b^2)\,\big) = \mathbf{j}(b^2 - \bar{a}^2)$$

$$\mathbf{j}Im(b^2 - a^2) + \mathbf{k}Im\big(\,\mathbf{i}(a^2 + b^2)\,\big) = \mathbf{k}(\bar{a}^2 + b^2)$$

Using these five relations, one obtains:

$$\begin{aligned} q\mathbf{i}\bar{q} &= \mathbf{i}(|a|^2 - |b|^2) + \mathbf{j}Re(2a\bar{b}) + \mathbf{k}Im(2a\bar{b}) \\ q\mathbf{j}\bar{q} &= \mathbf{i}Re(2ab) + \mathbf{j}Re(b^2 - a^2) + \mathbf{k}Re\big(\,\mathbf{i}(a^2 + b^2)\,\big) \\ q\mathbf{k}\bar{q} &= -\mathbf{i}Im(2ab) - \mathbf{j}Im(b^2 - a^2) - \mathbf{k}Im\big(\,\mathbf{i}(a^2 + b^2)\,\big) \end{aligned}$$

or equivalently:

$$\left.\begin{aligned} [q\mathbf{i}\bar{q}] &= \big(\ |a|^2 - |b|^2\ ,\ Re(2a\bar{b})\ ,\ Im(2a\bar{b})\ \big) \\ [q\mathbf{j}\bar{q}] &= \big(\ Re(2ab)\ ,\ Re(b^2 - a^2)\ ,\ Re\big(\,\mathbf{i}(a^2 + b^2)\big)\ \big) \\ [q\mathbf{k}\bar{q}] &= -\big(\ Im(2ab)\ ,\ Im(b^2 - a^2)\ ,\ Im\big(\,\mathbf{i}(a^2 + b^2)\big)\ \big) \end{aligned}\right\} \tag{4.1.12}$$

Now the system (4.1.10) can be written in a nicer way as follows:

$$\bar{v}_2 - \mathbf{i}\bar{v}_3 = \big(\ 2ab\ ,\ b^2 - a^2\ ,\ \mathbf{i}(a^2 + b^2)\ \big)\,.$$

the obvious analogy of this formula with (4.1.2) yields the following result.

Proposition 20. *Given a conformal surface parametrization $\bar{\Phi}(u,v)$, the identities (4.1.2) and (4.1.3) are equivalent to the following pair of identities:*

$$\frac{1}{2}\bar{\Phi}_u = [Q\mathbf{j}\bar{Q}] \quad , \quad \frac{1}{2}\bar{\Phi}_v = [Q\mathbf{k}\bar{Q}]\,, \tag{4.1.13}$$

where Q is the quaternion-valued function defined by $Q \equiv \mathbf{i}f + \mathbf{j}g$. Then the first fundamental form of the parametrization is $E(du^2{+}dv^2)$, and E satisfies the following:

$$|f|^2 + |g|^2 = \|(1/2)\bar{\Phi}_u\| = \|(1/2)\bar{\Phi}_v\| = (1/2)\sqrt{E}\,. \tag{4.1.14}$$

One also has:

$$[Q\mathbf{i}\bar{Q}] = (|f|^2 + |g|^2)N = (1/2)\sqrt{E}\,N\,, \tag{4.1.15}$$

where N is the choice of unit normal such that $\{EN, \bar{\Phi}_u, \bar{\Phi}_v\}$ is a direct conformal basis of $\mathbb{R}^3$, that is, $N = (\bar{\Phi}_u \times \bar{\Phi}_v)/E^2$.

Finally, the potential A which appears in the Dirac-type system (4.1.6) has a very simple relation with classical geometric quantities. Namely, combining the formulas (4.1.6), (4.1.7), and (4.1.14), together with the identity $\Delta\bar{\Phi} = 2EHN$, one arrives at:

$$A = \frac{1}{2}\sqrt{E}H \tag{4.1.16}$$

4.1.4 First consequences

There is a simple *superposition principle*: fixed any real function $A(u,v)$, the solutions f, g of the system (4.1.6) form a vector space. If $\bar{\Phi}_1(u,v), \ldots, \bar{\Phi}_k(u,v)$ are conformal surface parametrizations with the same Dirac potential A, then spinors that define them can be linearly combined, using any constant coefficients, and a new conformal parametrization will be obtained with the same Dirac potential A. This property will be used in section 5.2.2 to construct non-trivial conformal parametrizations given by simple elementary functions.

The set of all spinors associated to conformal parametrizations is a *ruled* space, in other words, it can be expressed as the union of vector spaces:

$$\bigcup_{A(u,v)} \left\{ \partial_{\bar{z}} f = A\bar{g}, \partial_{\bar{z}} g = -A\bar{f} \right\}$$

Theorem 9, from chapter 2, implies that for a non-null potential, $A \neq 0$, there are no polynomial solutions to the Dirac system (4.1.6). In other words, there is no polynomial spinor surface representation for surfaces in $\mathbb{R}^3$ (and also polynomial wave functions) when the scalar potential A is not null.

4.2 Spinorial representation of $\mathbb{R}^4$ surfaces

The spinorial surface representation, as a Weierstrass-Enneper generalization, can be extended to surfaces in the euclidean space $\mathbb{R}^4$. The next is a brief exposition, for a complete formulation see [17].

The grassmannians of two-dimensional planes in $\mathbb{R}^n$ are diffeomorphic to quadratic equations in $\mathbb{C}P^{n-1}$. In a two-dimensional plane an orthogonal basis can be taken with positive orientation, $\bar{u} = (u_1, \ldots, u_n)$, $\bar{v} = (v_1, \ldots, v_n)$, with vectors of the same length $\|\bar{u}\| = \|\bar{v}\|$ and $\bar{u} \cdot \bar{v} = 0$. The vector $\bar{y} = \bar{u} + \mathbf{i}\bar{v} \in \mathbb{C}^n$ can be defined by:

$$|y|^2 = y_1^2 + \cdots + y_n^2 = \left[\bar{u} \cdot \bar{u} - \bar{v} \cdot \bar{v}\right] + 2\mathbf{i}\bar{u} \cdot \bar{v} = 0$$

the plane determines the vector basis up to rotations of the complex vector, $y \to re^{i\varphi}y$. Thus, the grassmannians $\tilde{G}_{n,2}$ of bidimensional planes in $\mathbb{R}^n$ are diffeomorphic to the quadratic equations in $\mathbb{C}P^{n-1}$:

$$y_1^2 + \cdots + y_n^2 = 0, \qquad (y_1 : \cdots : y_n) \in \mathbb{C}P^{n-1}$$

where $(y_1 : \cdots : y_n)$ are the $\mathbb{C}P^{n-1}$ homogeneous coordinates. The grassmannian $G_{n,2}$ is quotient between $\tilde{G}_{n,2}$ and a fixed point inversion $y \to \bar{y}$.

Given a surface immersion:

$$f : \Sigma \to \mathbb{R}^n$$

with a conformal surface parameter z, the surface Gauss map is:

$$\Sigma \to \tilde{G}_{n,2} : \qquad P \to (x_z^1(P) : \cdots : x_z^n(P))$$

where $x^1, \cdots, x^n$ are euclidean coordinates in $\mathbb{R}^n$ y $P \in \Sigma$.

There are only two cases where the grassmannian admits a rational parametrization:

$$\tilde{G}_{3,2} = \mathbb{C}P^1, \quad \tilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1 \tag{4.2.1}$$

just in this cases it's possible to obtain a generalized Weierstrass-Enneper representation.

In the first case, the spinor representation in $\mathbb{R}^3$ can be reviewed. In this case, the grassmannian corresponds to the quadratic equation:

$$y_1^2 + y_2^2 + y_3^2 = 0$$

using Diophantus solution:

$$\begin{cases} y_1 = \frac{\mathbf{i}}{2}(b^2 + a^2) \\ y_2 = \frac{1}{2}(b^2 - a^2) \\ y_3 = ab \end{cases}$$

with $(a : b) \in \mathbb{C}P^1$. In the above section it was explained how the Diophantus solution is related to the rational parametrization (also known as the integer Lagrange representation of the integer solutions of equation: $x^2 + y^2 = z^2$).

Pedit and Pinkall work in the spinorial generalized Weierstrass-Enneper surface representation in $\mathbb{R}^3$ and later on the extension to $\mathbb{R}^4$ immersed surfaces.

With this purpose they identified $\mathbb{C}^2$ with the quaternion space $\mathbb{H}$ in the following way:

$$(z_1, z_2) \to z_1 + \mathbf{j} z_2 = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$$

and they used the next matrix operators:

$$\partial = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \quad \mathbf{j}U = \mathbf{j} \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix} = \begin{pmatrix} 0 & -\bar{U} \\ U & 0 \end{pmatrix}$$

where $\mathbf{j}$ is one of the quaternion base elements and this relations are satisfied:

$$\mathbf{j}^2 = -1$$

$$z\mathbf{j} = \mathbf{j}\bar{z}$$

$$\bar{\partial}\mathbf{j} = \mathbf{j}\partial$$

then the Dirac system takes the form:

$$(\bar{\partial} + \mathbf{j}U)(\psi_1 + \bar{\psi_2}) = (\bar{\partial}\psi_1 - \bar{U}\psi_2) + \mathbf{j}(\partial\psi_2 + U\psi_1) = 0$$

in term of quaternions, the Dirac equation is expressed as:

$$(\bar{\partial} + \mathbf{j}U)(\psi_1 + \bar{\psi_2}\mathbf{j}) = 0$$

In $\mathbb{R}^4$ the bidimensional plane grassmannian is diffeomorphic to the quadratic equation:

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0, \quad y \in \mathbb{C}P^3$$

The new coordinates will be used:

$$\begin{cases} y_1 = \frac{\mathbf{i}}{2}(y_1' + y_2') \\ y_2 = \frac{1}{2}(y_1' - y_2') \\ y_3 = \frac{1}{2}(y_3' + y_4') \\ y_4 = \frac{\mathbf{i}}{2}(y_3' - y_4') \end{cases}$$

In term of this new coordinates $\tilde{G}_{4,2}$ id defined by the equation:

$$y'_1 y'_2 = y'_3 y'_4$$

there exists a diffeomorphism:

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to \tilde{G}_{4,2}$$

given by the Segre map:

$$\begin{cases} y'_1 = a_2 b_2 \\ y'_2 = a_1 b_1 \\ y'_3 = a_2 b_1 \\ y'_4 = a_1 b_2 \end{cases}$$

where $(a_1 : a_2)$ and $(b_1 : b_2)$ are homogeneous coordinates in the two components $\mathbb{C}P^1$.

It can be parametrized in the following way:

$$\begin{cases} a_1 = \varphi_1 \\ a_2 = \bar{\varphi}_2 \\ b_1 = \psi_1 \\ b_2 = \bar{\psi}_2 \end{cases}$$

The situation is different from the three-dimensional case, now the spinorial representation is not unique (not only by a ± 1 factor like in $\mathbb{R}^3$ case). The spinor can be multiplied by ± 1 or can be mapped by a gauge transformation of the next form:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^f \psi_1 \\ e^{\bar{f}} \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-f} \varphi_1 \\ e^{-\bar{f}} \varphi_2 \end{pmatrix} \tag{4.2.2}$$

where f is an arbitrary function.

The surface immersion in $\mathbb{R}^4$ spinorial representation is given by the formulas:

$$x^k = \int (x^k_z dz + \bar{x}^k_z d\bar{z}), \;\; k = 1, \dots, 4$$

where each component can be obtained from:

$$\begin{cases} x^1_z = \frac{\mathbf{i}}{2}(\bar{\varphi}_2\bar{\psi}_2 + \varphi_1\psi_1) \\ x^2_z = \frac{1}{2}(\bar{\varphi}_2\bar{\psi}_2 - \varphi_1\psi_1) \\ x^3_z = \frac{1}{2}(\bar{\varphi}_2\psi_1 + \varphi_1\bar{\psi}_2) \\ x^4_z = \frac{\mathbf{i}}{2}(\bar{\varphi}_2\psi_1 - \varphi_1\bar{\psi}_2) \end{cases} \tag{4.2.3}$$

As in the three-dimensional case, the integrability condition of the PDE system (4.2.3) simplifies to a simpler system. The integrand closed form condition is equivalent to the condition $Im(x^k_{z\bar{z}}) = 0, \quad k = 1, \dots, 4$. This equation can be rewritten as:

$$\begin{cases} (\bar{\varphi}_2\psi_1)_{\bar{z}} = (\bar{\varphi}_1\psi_2)_z \\ (\bar{\varphi}_2\bar{\psi}_2)_{\bar{z}} = -(\bar{\varphi}_1\bar{\psi}_1)_z \end{cases}$$

Not as the three-dimensional case, the general spinor components, φ, ψ, can not be written as a Dirac equation. There are some particular cases where integration condition can be simplified.

Theorem 21. *Let $r : W \to \mathbb{R}^4$ be a surface immersion of a conformal parametrization z and let $G_\psi = (e^{\mathbf{i}\cos\eta} : \sin\eta)$ be one of the components of the associated gaussian map.*

There is another representation ψ of the gaussian map $G_\psi = (\psi_1 : \bar{\psi}_2)$ that satisfies the Dirac equation:

$$\mathcal{D}\psi = 0, \quad \mathcal{D} = \begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar{z}} & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}$$

the other component, φ, satisfies:

$$\check{\mathcal{D}}\varphi = 0, \quad \check{\mathcal{D}} = \begin{pmatrix} 0 & \partial_z \\ -\partial_{\bar{z}} & 0 \end{pmatrix} + \begin{pmatrix} \bar{A} & 0 \\ 0 & A \end{pmatrix}$$

For a given surface, any solution of the above systems is invariant under gauge transformations of the type (4.2.2).

For a complete description of the spinor surface representation in $\mathbb{R}^4$ see [18].

Note 22. *Again the theorem 9 can be applied to prove the non-existence of polynomial solutions for non null potentials,* $A \neq 0$.

Note 23. *The generalized Weierstrass-Enneper representation is also applicable to the representation of three-dimensional and four dimensional Lie groups. This allow to define conformal parametrizations of this groups. It is also possible to define sets of groups elements equivalent to minimal surfaces inside the Lie group. See* [17] *for a complete development of this subject.*

Chapter 5

Non-polynomial examples

5.1 Rational Non-Willmore examples

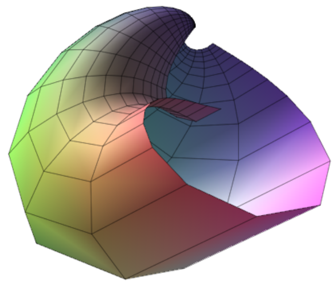
A lot of counterexamples of conformal rational surface parametrizations can be obtained with not everywhere null curvature, i.e., the new surfaces are neither minimal nor harmonic.

Start with the Enneper minimal surface parametrization given by:

$$\bar{X}(u,v) = \left(-\frac{1}{3}u^3 + uv^2 + u, -\frac{1}{3}v^3 + vu^2 + v, u^2 - v^2 \right) \tag{5.1.1}$$

and apply to it a conformal space transformation defined by $S = R \circ T \circ R$, where R is a inversion over a sphere and T is a unitary translation.

A rational conformal parametrization is obtained, whose image is shown here:



It is not a minimal surface. In general, mean curvature is not invariant under conformal transformations.

It has to be stressed that any surface obtained as the image of a minimal surface under a conformal space transformation is a Willmore surface, because the Willmore functional is conformally invariant.

Non-Willmore surfaces

It is therefore natural to ask whether there are rational conformal parametrizations of non-Willmore surfaces. The rest of the present section is devoted to the construction of an explicit example of such a parametrization.

This example can be started with a conformal surface parametrization. One of the simpler conformal surface parametrizations is the stereographic parametrization of the sphere:

$$\psi = \left(\frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2}, \frac{2x}{1+x^2+y^2}\right)$$

using a complex variable as surface parameter:

$$\psi = \left(1 + \frac{-2}{1+|z|^2}, \frac{z+\bar{z}}{1+|z|^2}, i\frac{z-\bar{z}}{1+|z|^2}\right)$$

The surface spinorial representation is:

$$f_0 = \frac{\bar{z}}{1+|z|^2}$$
$$g_0 = \frac{1}{1+|z|^2}$$

It can be checked that is a particular solution of Dirac equation:

$$f_{\bar{z}} = A\bar{g} \tag{5.1.2}$$
$$g_{\bar{z}} = -A\bar{f}$$

for a potential $A = \frac{1}{1+|z|^2}$.

A more general solution can be found by the *variation of constants method*. The following type of generalized solutions are proposed:

$$f = pf_0 + qg_0$$
$$g = -\bar{q}f_0 + \bar{p}g_0$$

replacing it on Dirac equation (5.1.2) a condition for the coefficients is obtained:

$$p_{\bar{z}} = -\phi q_{\bar{z}}$$
$$p_z = \frac{1}{\bar{\phi}} q_z$$

where $\phi = \frac{g_0}{f_0} = \frac{1}{\bar{z}}$.

It can be obtained:

$$p_{\bar{z}} = -\frac{1}{\bar{z}} q_{\bar{z}}$$
$$p_z = z q_z$$

where λ can be taken to be real. To $q = 1 + \lambda\bar{z}^2$ correspond $p = -2\lambda\bar{z}$.

The spinors for the new surfaces obtained by the variation of constants method are:

$$f = \frac{1 - \lambda\bar{z}^2}{1 + z\bar{z}}$$
$$g = -\lambda z - \frac{\lambda z + \bar{z}}{1 + z\bar{z}}$$

In order to find the surface family associated to this spinor family the next relation is used:

$$\partial_z X_1 = 2fg \tag{5.1.3}$$
$$\partial_z X_2 = i(f^2 + g^2)$$
$$\partial_z X_3 = g^2 - f^2$$

it's obtained:

$$\partial_z X_1 = \frac{2(1 - \lambda\bar{z}^2)(-\lambda z - \frac{\lambda z + \bar{z}}{1 + z\bar{z}})}{1 + z\bar{z}}$$
$$\partial_z X_2 = i\left[\left(\frac{1 - \lambda\bar{z}^2}{1 + z\bar{z}}\right)^2 + \left(- \lambda z - \frac{\lambda z + \bar{z}}{1 + z\bar{z}}\right)^2\right]$$
$$\partial_z X_3 = -\left(\frac{1 - \lambda\bar{z}^2}{1 + z\bar{z}}\right)^2 + \left(- \lambda z - \frac{\lambda z + \bar{z}}{1 + z\bar{z}}\right)^2$$

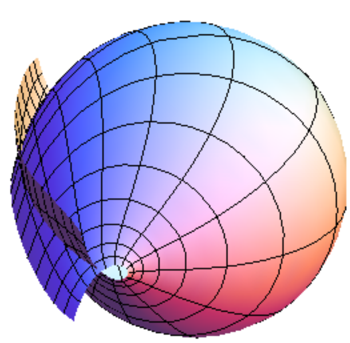
Integrating and adding the next antiholomorphic functions to each of the parametrization components (that is possible because the integration, in (5.1.3), use the z variable) to force parametrization component to be real:

$$\phi_1(\bar{z}) = \frac{2\lambda}{\bar{z}^2}$$
$$\phi_2(\bar{z}) = \frac{i\lambda^2}{\bar{z}^3} + \frac{i}{\bar{z}} - \frac{2i\lambda}{\bar{z}} - 2i\lambda\bar{z} - \frac{1}{3}i\lambda^2\bar{z}^3$$
$$\phi_3(\bar{z}) = \frac{\lambda^2}{\bar{z}^3} - \frac{1+2\lambda}{\bar{z}} + 2\lambda\bar{z} + \frac{1}{3}\lambda^2\bar{z}^3$$

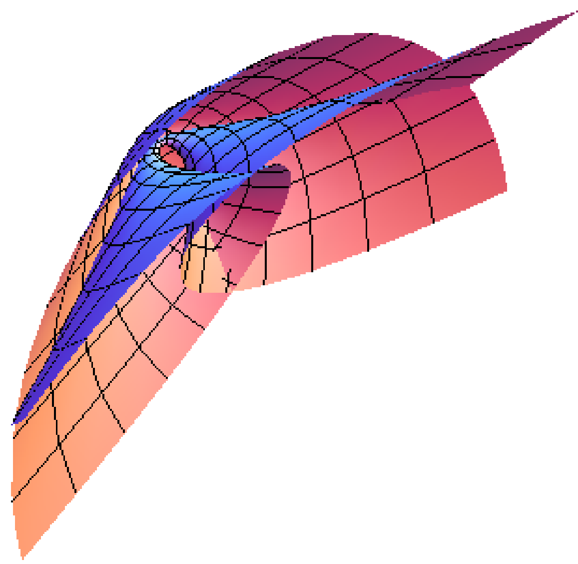
The next equation shows a real and conformal parametrization of the surface family:

$$X_1(x,y) = \frac{2(1+\lambda^2-2\lambda x^2+\lambda^2x^2+\lambda^2x^4+2\lambda y^2+\lambda^2y^2+2\lambda^2x^2y^2+\lambda^2y^4)}{1+x^2+y^2}$$
$$X_2(x,y) = \frac{2y(-3-6\lambda x^2-12\lambda^2x^2-3\lambda^2x^4-6\lambda y^2+4\lambda^2y^2-2\lambda^2x^2y^2+\lambda^2y^4)}{3(1+x^2+y^2)}$$
$$X_3(x,y) = \frac{2x(-3+6\lambda x^2+4\lambda^2x^2+\lambda^2x^4+6\lambda y^2-12\lambda^2y^2-2\lambda^2x^2y^2-3\lambda^2y^4)}{3(1+x^2+y^2)}$$

The $\lambda = 0$ value correspond to the stereographic parametrization of the sphere. When parameter λ changes, from the null value, a conformal deformation of the sphere is carried out. The next images shows this conformal deformation for different λ values. For $\lambda = 0.07$:



For the value $\lambda = 1$:



Non-Willmore test

It can be checked if all the surfaces that belongs to the above family are of Willmore type. In other words, if this surfaces is a extremal of the action:

$$\mathcal{W} = \int_S H^2 - K\, dA \tag{5.1.4}$$

where H is the mean curvature and K is the gaussian curvature.

For a given surface, in order to be an extremal of this functional, it must satisfy the following Euler equation:

$$\Delta H(x,y) + 2H(x,y)(H(x,y)^2 - K(x,y)) = 0 \tag{5.1.5}$$

In this case, the mean curvature is given by:

$$H(x,y) = \frac{1}{\lambda^2x^4 + 4\lambda^2x^2 + 2\lambda^2x^2y^2 + 4\lambda^2y^2 + \lambda^2y^4 + 1 + 2\lambda x^2 - 2\lambda y^2}$$

it can be checked that not all the surfaces that belong to this family are minimal.

The gaussian curvature is given by:

$$\begin{aligned}
K(x,y) = -\big(&-1 + 4\lambda^2 + 12\lambda^2x^2y^2 - 2\lambda^2x^4 - 2\lambda^2y^4 - 4\lambda x^2 + 4\lambda y^2 - 12\lambda^3y^2x^4 + \\
&+12\lambda^3y^4x^2 - 24\lambda^4x^2y^2 + 12\lambda^4x^6y^2 + 18\lambda^4x^4y^4 + 12\lambda^4x^2y^6 - 32\lambda^3x^4 + \\
&+32\lambda^3y^4 + 12\lambda^3y^6 - 12\lambda^4x^4 - 12\lambda^4y^4 + 3\lambda^4x^8 + 3\lambda^4y^8 - 12x^6\lambda^3 - 8\lambda^3x^2 + \\
&+8\lambda^3y^2\big)/\big(1 - 4\lambda^2x^2y^2 + 8\lambda^2x^2 + 6\lambda^2x^4 + 8\lambda^2y^2 + 6\lambda^2y^4 + 4\lambda x^2 - \\
&-4\lambda y^2 + 4\lambda^3y^2x^4 - 4\lambda^3y^4x^2 + 32\lambda^4x^2y^2 + 24\lambda^4x^4y^2 + 24\lambda^4x^2y^4 + 4\lambda^4x^6y^2 \\
&+6\lambda^4x^4y^4 + 4\lambda^4x^2y^6 + 16\lambda^3x^4 - 16\lambda^3y^4 - 4\lambda^3y^6 + 16\lambda^4x^4 + 8\lambda^4x^6 + 16\lambda^4y^4 \\
&+8\lambda^4y^6 + \lambda^4x^8 + \lambda^4y^8 + 4x^6\lambda^3\big)^2
\end{aligned}$$

After a long calculation, it can be seen that the left side of the Euler type equation is, in general, not null everywhere. For example, the left side of the equation (5.1.5) evaluated at the point $(x, y) = (0, 0)$ is $-8\lambda^2$. The left side of the Euler equation is null for $\lambda = 0$ value, that represents the sphere.

This example, a conformal sphere deformation, shows new conformal surface parameterizations that are rational. It has been shown that there exists examples of rational surface parametrizations that are neither Willmore surfaces nor minimal surfaces.

5.2 Examples with a simple Dirac potential

In this section, the Dirac system (5.1.2) for some simple A potential values is integrated. First, for a null A potential, later for A constant potential and finally for A dependent of one real variable.

5.2.1 Null potential

This case correspond to surface with null mean curvature $H = 0$, i.e. minimal surfaces.

The Dirac equations (5.1.2) are reduced to:

$$\begin{cases} \partial_{\bar{z}} f = 0 \\ \partial_{\bar{z}} g = 0 \end{cases}$$

These are the Cauchy-Riemann equations for f, g. In other words, $\mathbf{i}f + \mathbf{j}g$ is the spinor of conformal parametrization of a minimal surface if and only if the f, g components are holomorphic functions. We recover the known Weierstrass-Enneper minimal surface representation for any pair of holomorphic functions.

Following theorem 9, polynomial spinorial solutions can exist only for null potential, $A = 0$.

5.2.2 Constant potential

In this case, the Dirac PDE system (5.1.2) can be derived by z and combined to simplify and separate the spinor component equations. The uncouple system for f, g components:

$$\begin{cases} \partial_{z\bar{z}} f = -A^2 f \\ \partial_{z\bar{z}} g = -A^2 g \end{cases} \tag{5.2.1}$$

In other words, it must satisfy the Helmholtz equation:

$$\Delta h(x, y) = kh(x, y)$$

using $\partial_{z\bar{z}} = \frac{1}{4}\Delta$, where Δ is the Laplacian operator.

In order to extract some simple examples, the constant potential can be fixed to $A = 1$. The functions $e^{-\omega_i z} e^{\bar{\omega}_i \bar{z}}$, with $\omega_i \in \mathbb{R}$, are eigenfunctions of the operator $\frac{1}{4}\Delta$, or $\partial_{z\bar{z}}$. Using this functions, any solution of the Helmholtz equation (5.2.1) can be constructed as a linear combination.

Next, two examples will be developed, for one or two Helmholtz eigenfunctions, to obtain two new conformal surface parametrizations.

In the first case, *one eigenfunction component* the calculations are simplified using a unique frequency $\omega_1 = \mathbf{i}$. The spinorial components associated to the surface are:

$$\begin{cases} f = e^{\mathbf{i}(z+\bar{z})} \\ g = -\mathbf{i} e^{-\mathbf{i}(z+\bar{z})} \end{cases}$$

the local coordinates for the conformal parametrization:

$$\Phi_x = \int g^2 - f^2 dz = \frac{e^{-2\mathbf{i}(z+\bar{z})} - e^{2\mathbf{i}(z+\bar{z})}}{2\mathbf{i}} = -\sin(4x)$$
$$\Phi_y = \mathbf{i}\int f^2 + g^2 dz = \frac{e^{2\mathbf{i}(z+\bar{z})} + e^{-2\mathbf{i}(z+\bar{z})}}{2\mathbf{i}} = -\cos(4x)$$
$$\Phi_z = \int 2fg dz = -2\mathbf{i}z + \varphi(\bar{z}) = 2\mathbf{i}(\bar{z} - z) = 4y$$

where a integration constant $\varphi(\bar{z}) = 2\mathbf{i}\bar{z}$, an antiholomorphic function, has been added to make the coordinate function real.

It can be checked that the new surface parametrization is conformal. This surface corresponds to a cylinder. If a complex number, not an imaginary frequency, is chosen, $\omega = a + \mathbf{i}b$, a conformal parametrization of the cone surface can be obtained.

Next, a *two eigenfunction example* will be developed. Using again $A = 1$, and taken two simple frequencies:

$$f = e^{-z}e^{\bar{z}} + ce^{-2z}e^{\frac{\bar{z}}{2}} \qquad c \in \mathbb{R}$$

The other component of the spinor can be obtained using the relation with f from (5.1.2):

$$g = e^{z}e^{-\bar{z}} + \frac{c}{2}e^{-2\bar{z}}e^{\frac{z}{2}}$$

fixing $b = \frac{c}{2}$ the following spinor is obtained:

$$f = e^{-z}e^{\bar{z}} + 2be^{-2z}e^{\frac{\bar{z}}{2}} \qquad b \in \mathbb{R}$$
$$g = e^{z}e^{-\bar{z}} + be^{-2\bar{z}}e^{\frac{z}{2}}$$

The real components $\{x_1(x,y), x_2(x,y), x_3(x,y)\}$ satisfy:

$$\begin{cases} \partial_z x_1 = 2fg \\ \partial_z x_2 = \mathbf{i}(f^2+g^2) \\ \partial_z x_3 = f^2 - g^2 \end{cases}$$

and also $\{x_1(x,y), x_2(x,y), x_3(x,y)\}$ is a conformal parametrization.

After integrating and adding an antiholomorphic function, in the case of being necessary, each component is calculated:

$$x_1 = 2z - 4be^{-\frac{z}{2}}e^{\bar{z}} - 4be^{-z}e^{\frac{\bar{z}}{2}} - \frac{8}{3}b^2e^{-\frac{3z}{2}}e^{-\frac{3\bar{z}}{2}} + \varphi_1(\bar{z}) =$$
$$= (2z + \varphi_1(\bar{z})) - 8bRe(e^{-\frac{z}{2}}e^{-\bar{z}}) - \frac{8}{3}b^2e^{Re(-3z)}$$

$\varphi_1(\bar{z}) = 2\bar{z}$ antoholomorphic function is added to obtain a real parametrization component:

$$x_1 = 4x - 8be^{-\frac{3x}{2}}\cos(\frac{y}{2}) - \frac{8}{3}b^2e^{-3x}$$

The next coordinate function:

$$x_2 = -\mathbf{i}\Big[-\frac{1}{2}e^{-2z}e^{2\bar{z}} + \frac{1}{2}e^{2z}e^{-2\bar{z}} - \frac{4}{3}be^{-3z}e^{3\bar{z}/2} +$$
$$+\frac{4}{3}be^{3z/2}e^{-3\bar{z}} - b^2e^{-4z}e^{\bar{z}} + b^2e^{z}e^{-4\bar{z}}\Big] + \varphi_2(\bar{z}) =$$
$$= Re(-\mathbf{i}e^{-2z+2\bar{z}} - \frac{8}{3}\mathbf{i}e^{-3z+3\bar{z}/2} - 2\mathbf{i}b^2e^{-4z+\bar{z}}) + \varphi_2(\bar{z})$$

it is not necessary to add any antiholomorphic function $\varphi_2(\bar{z})$ because the component is real:

$$x_2 = -\sin 4y + \frac{8}{3}be^{-\frac{3x}{2}}\sin -9y/2 + 2b^2e^{-3x}\sin -5y$$

The last component is:

$$\begin{aligned} x_3 &= -\frac{1}{2}e^{-2z}e^{2\bar{z}} - \frac{1}{2}e^{2z}e^{-2\bar{z}} - \frac{4}{3}be^{-3z}e^{3\bar{z}/2} - \\ &-\frac{4}{3}be^{3z/2}e^{-3\bar{z}} - b^2e^{-4z}e^{\bar{z}} - b^2e^{z}e^{-4\bar{z}} + \varphi_3(\bar{z}) = \\ &= Re(-e^{-2z+2\bar{z}} - \frac{8}{3}be^{-3z+3\bar{z}/2} - 2b^2e^{-4z+\bar{z}}) + \varphi_3(\bar{z}) \end{aligned}$$

again, it's not necessary to add any antiholomorphic function:

$$x_3 = -\cos 4y - \frac{8}{3}be^{-\frac{3x}{2}}\cos -9y/2 - 2b^2e^{-3x}\cos -5y$$

Some variable changes are made to simplify $X_j = -\frac{3}{8}x_j$ and transforming conformally the independent variables (u, v):

$$\begin{cases} u = -\frac{x}{2} \\ v = \frac{y}{2} \end{cases}$$

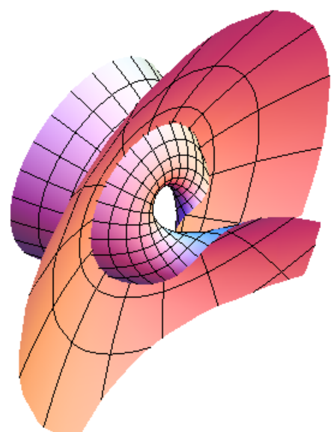
Finally, for each b value, the new conformal surface family is obtained:

$$\bar{\Phi}(u, v) = \big(X_1(u, v), X_2(u, v), X_3(u, v)\big)$$

given by:

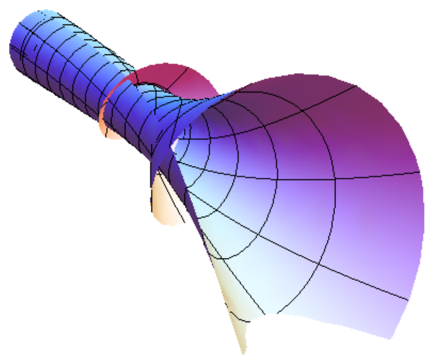
$$\begin{cases} X_1(u, v) = 3u + 3be^{3u}\cos v + b^2e^{6u} \\ X_2(u, v) = \frac{3}{8}\sin 8v + be^{3u}\sin 9v + \frac{3}{4}b^2e^{6u}\sin 10v \\ X_3(u, v) = \frac{3}{8}\cos 8v + be^{3u}\cos 9v + \frac{3}{4}b^2e^{6u}\cos 10v \end{cases}$$

The following are some views of this new conformally parametrized surface, for $b = 1$. Seen from far away, the surface looks like an acute cone with a half line (really a thin tube) attached to the vertex. The first image is a blowup of the region where the thin tube is joined to one of the sheets of the cone.



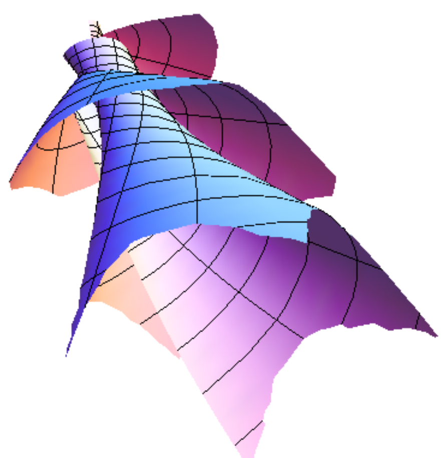
with $u \in (-0.5, 0.1), v \in (-3.5, -2.9)$.

The second image displays part of the thin tube and part of the cone-like region.



for $u \in (-1.1, 0.1), v \in (-3, -1.6)$.

The third image exhibits several sheets of the cone-like part.



with $u \in (-0.5, 0.2), v \in (-3, -1.6)$.

5.2.3 Potential dependent of a real variable

The Dirac equation (5.1.2) can be simplified when the potential $A(z,\bar{z})$ is independent of the imaginary part of z, in other words, $A(x,y) = A(x)$, where $x = Re(z), y = Im(z)$.

As sugested in [17], solutions of the following form can be proposed:

$$\psi(z,\bar{z}) = \varphi(x)e^{\mathbf{i}\frac{y}{2}} \tag{5.2.2}$$

For this kind of solutions the Dirac PDE system is reduced to the *Zakharov-Shabat system*:

$$\mathcal{L}\varphi = 0, \quad \mathcal{L} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} q & -ik \\ -ik & q \end{pmatrix} \right] \tag{5.2.3}$$

where $q(x) = 2A(x)$ and $k = \mathbf{i}/2$.

Note 24. *It is a simple exercise to check the case A constant, for example for $A = 1$, the $Zakharov - Shabat$ equation is reduced to:*

$$\mathcal{L}\varphi = 0, \quad \mathcal{L} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 2 & -\frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \right]$$

equivalent to the following ODE system:

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 2 \\ 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

the characteristic polynomial of the matrix is $\lambda^2 + \frac{15}{4} = 0$ and the pure imaginary eigenvalues are $\lambda = \pm\mathbf{i}\frac{\sqrt{15}}{2}$.

The general solution of the above system is:

$$\varphi_1(x) = c_1\big(A_1 \cos\frac{\sqrt{15}}{2}x + A_2 \sin\frac{\sqrt{15}}{2}x\big) + c_2\big(A_1 \sin\frac{\sqrt{15}}{2}x + A_2 \cos\frac{\sqrt{15}}{2}x\big)$$
$$\varphi_2(x) = c_1\big(B_1 \cos\frac{\sqrt{15}}{2}x - B_2 \sin\frac{\sqrt{15}}{2}x\big) + c_2\big(B_1 \sin\frac{\sqrt{15}}{2}x + B_2 \cos\frac{\sqrt{15}}{2}x\big)$$

where c_i, A_i, B_i are real variables.

The general solution is:

$$\begin{pmatrix} \psi_1(x,y) \\ \psi_2(x,y) \end{pmatrix} = \begin{pmatrix} \varphi_1(x)e^{\mathbf{i}\frac{y}{2}} \\ \varphi_2(x)e^{\mathbf{i}\frac{y}{2}} \end{pmatrix}$$

where it is possible to recover the one term solutions for $A = 1$ calculated above, ie, the spinor associated to the cylinder and the cone surfaces.

The solutions ψ of the form (5.2.2) include revolution surfaces. In this case, y is the rotation angle.

Note 25. *The operator $\mathcal{L}$ is associated to the Korteweg-de Vries hierarchical equation set investigated for soliton deformation equations. These deformations are conformal deformations of revolution surfaces. For more information see [17].*

Notation

- Einstein summation convention is used along the thesis. Repeated index implies summation over all the possible values of the index. For example:

$$g^{\mu\nu}\partial_\mu\partial_\nu = \sum_{\mu=1,\nu=1}^{n} g^{\mu\nu}\partial_\mu\partial_\nu \tag{5.2.4}$$

- $\bar{X}$: symbol is used for a surface parametrization in $\mathbb{R}^n$:
  $\bar{X}(u,v) = \{X_1(u,v), \ldots, X_n(u,v)\}$.
  Surface tangent vectors are also represented by $\bar{X}_u, \bar{X}_v$ or $\partial_z X, \partial_{\bar{z}} X_v$.

- Surface metric or first form is denoted by s. For example: $ds^2 = du^2 + dv^2$.

- $(y_1 : \cdots : y_n)$ is used for projective space coordinates $\mathbb{C}P^{n-1}$.

- $\bigoplus$: symbol is used for space direct sum.

- ∂_i: represents i coordinate tangent vector.

- $\mathcal{H}(\mathbb{R}^n)$: symbolizes the harmonic homogeneous polynomial space in $\mathbb{R}^n$.

- $\mathcal{P}(\mathbb{R}^n)$: denote the space of homogeneous polynomials in $\mathbb{R}^n$. The base elements of the space $\mathcal{H}^m(\mathbb{R}^m)$ are represented by $\{h_1^m, h_2^m, \ldots h_k^m\}$.

- $\triangle$: represents the Laplace operator in $\mathbb{R}^n$.

- The next group notation is used:

$SO(n)$	Special Orthogonal group in $\mathbb{R}^n$
$SU(n)$	Special Unitary group in $\mathbb{C}^n$
$C(n)$	Conformal group
$SCG(n)$	Special Conformal group
$O(1, n)$	Lorentz group

- $|\ |$: denotes the usual vector module.

- **i**, **j**, **k**: represents the quaternion base elements.

- H, K are used for Mean and Gaussian curvature.

- $\mathcal{W}$ denotes the Willmore functional.

- E, F, G and e, f, g correspond to first and second form elements.

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