Journal des Mathématiques Pures et Appliquées 129 (2019):153-179

DOI: https://doi.org/10.1016/j.matpur.2018.12.006

CONTROLLABILITY AND POSITIVITY CONSTRAINTS IN POPULATION DYNAMICS WITH AGE STRUCTURING AND DIFFUSION

DEBAYAN MAITY, MARIUS TUCSNAK, AND ENRIQUE ZUAZUA

Abstract. In this article, we study the null controllability of a linear system coming from a population dynamics model with age structuring and spatial diffusion (of Lotka-McKendrick type). The control is localized in the space variable as well as with respect to the age. The first novelty we bring in is that the age interval in which the control needs to be active can be arbitrarily small and does not need to contain a neighbourhood of 0. The second one is that we prove that the whole population can be steered into zero in a uniform time, without, as in the existing literature, excluding some interval of low ages. Moreover, we improve the existing estimates of the controllability time and we show that our estimates are sharp, at least when the control is active for very low ages. Finally, we show that the system can be steered between two positive steady states by controls preserving the positivity of the state trajectory. The method of proof, combining final-state observability estimates with the use of characteristics and with L^∞ estimates of the associated semigroup, avoids the explicit use of parabolic Carleman estimates.

Key words. Population dynamics, Null controllability.
AMS subject classifications. 93B03, 93B05, 92D25

1. Introduction and main results

In this article, we study the null-controllability of an infinite dimensional linear system describing the dynamics of a single species age-structured population with spatial diffusion. In these models, going back to Gurtin [9] and generalizing the classical Lotka-McKendrick system, the state space of the system is $H = L^2([0, a_\dagger] \times \Omega)$, where $a_\dagger$ denotes the maximal age an individual can attain and $\Omega \subset \mathbb{R}^n$ (with $n \in \mathbb{N}$ in general but with $n = 3$ for real life applications) is an open bounded set which represents the spatial environment occupied by the individuals. Let $p(t, a, x)$ be the distribution density of individuals with respect to age $a \geqslant 0$ and spatial position $x \in \Omega$ at some time $t \geqslant 0$. Then, according to the above reference, the function p satisfies the degenerate parabolic partial differential equation

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - Lp + \mu(a)p = mv \qquad (t, a, x) \in (0, \infty) \times (0, a_\dagger) \times \Omega, \tag{1.1}$$

where the operator L is defined by

$$Lp = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial p}{\partial x_j}\right), \tag{1.2}$$

Date: October 4, 2018.
Debayan Maity and Marius Tucsnak acknowledge the support of the Agence Nationale de la Recherche - Deutsche Forschungsgemeinschaft (ANR - DFG), project INFIDHEM, ID ANR-16-CE92-0028. The research of Enrique Zuazua was supported by the Advanced Grant DyCon (Dynamical Control) of the European Research Council Executive Agency (ERC), the MTM2014-52347 and MTM2017-92996 Grants of the MINECO (Spain) and the ICON project of the French ANR-16-ACHN-0014.

with $\sigma_{ij} = \sigma_{ji} \in C^2(\overline{\Omega})$, for $1 \leqslant i, j \leqslant n$, and we assume there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^{n} \sigma_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2 \qquad (x \in \overline{\Omega},\ \xi \in \mathbb{R}^n).$$

Moreover, the positive function μ denotes the natural mortality rate of individuals of age a, supposed to be independent of the spatial position x and of time. The control function is v, depending on t, a and x, whereas m is the characteristic function of $(a_1, a_2) \times \omega$, with $0 \leqslant a_1 < a_2 \leqslant a_\dagger$ and $\omega \subset \Omega$ an open set. Thus the control is localized both in age and with respect to the spatial variable. This control process corresponds to harvesting of adding individuals of age between a_1 and a_2 from the spatial domain ω. Note that equation (1.1) is a slight generalization of the one proposed in [9], where the operator L is just the standard Laplacian. We denote by β the positive function describing the fertility rate at age a, supposed to be independent of the spatial position x and of time, so that the density of newly born individuals at the point x at time t is given by

$$p(t, 0, x) = \int_0^{a_\dagger} \beta(a)p(t, a, x)\,\mathrm{d}a \qquad (t, x) \in (0, \infty) \times \Omega. \tag{1.3}$$

We assume that the individuals never leave the set Ω, so that p satisfies the Neumann boundary condition

$$\frac{\partial p}{\partial \nu_L} = \sum_{i,j=1}^{n} \sigma_{ij} \frac{\partial p}{\partial x_j} n_i = 0 \qquad (t, a, x) \in (0, \infty) \times (0, a_\dagger) \times \partial\Omega, \tag{1.4}$$

where n denotes the unit outer normal to $\partial\Omega$. To complete the model, we introduce the initial condition

$$p(0, a, x) = p_0(a, x) \qquad (a, x) \in (0, a_\dagger) \times \Omega. \tag{1.5}$$

We assume that the fertility rate β and the mortality rate μ satisfy the conditions

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta \geqslant 0$ for almost every $a \in (0, a_\dagger)$.
(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu \geqslant 0$ for almost every $a \in (0, a_\dagger)$.
(H3) $\displaystyle\int_0^{a_\dagger} \mu(a)\,\mathrm{d}a = +\infty$.

For more details about the modelling of such system and the biological significance of the hypotheses, we refer to Webb [20].

Theorem 1.1. *Assume that β and μ satisfy the conditions (H1)-(H3) above. Moreover, suppose that the fertility rate β is such that*

$$\beta(a) = 0 \text{ for all } a \in (0, a_b), \tag{1.6}$$

for some $a_b \in (0, a_\dagger)$ and that $a_1 < a_b$. Recall that m is the characteristic function of $(a_1, a_2) \times \omega$, with $0 \leqslant a_1 < a_2 \leqslant a_\dagger$ and that $\omega \subset \Omega$ is an open set. Then for every $\tau > a_1 + a_\dagger - a_2$ and for every $p_0 \in L^2((0, a_\dagger) \times \Omega)$ there exists a control $v \in L^2((0, \tau) \times (a_1, a_2) \times \omega)$ such that the solution p of (1.1)-(1.5) satisfies

$$p(\tau, a, x) = 0 \text{ for all } a \in (0, a_\dagger), x \in \Omega. \tag{1.7}$$

To state our result on controllability with positivity constraints, we first define the concept of *non-negative steady state for* (1.1) - (1.5).

Definition 1.2. *Let $v_s \in L^\infty((0, a_\dagger) \times \Omega)$ be a steady interior control such that*

$$v_s \geqslant 0 \text{ a.e. on } (0, a_\dagger) \times \Omega.$$

A non-negative function $p_s \in L^\infty((0,a_\dagger)\times\Omega)$ *satisfying the equations*

$$\begin{cases} \dfrac{\partial p_s}{\partial a} - Lp_s + \mu(a)p_s = mv_s & (a,x) \in (0,a_\dagger)\times\Omega, \\ \dfrac{\partial p_s}{\partial \nu_L} = 0 & (a,x) \in (0,a_\dagger)\times\partial\Omega, \\ p_s(0,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p_s(a,x)\ \mathrm{d}a, & x \in \Omega, \end{cases} \tag{1.8}$$

is said to be a non-negative steady state for (1.1) - (1.5).

Our second main result can be stated as follows:

Theorem 1.3. *Assume the hypothesis of Theorem 1.1. Let* $p_{s,I}$ *and* $p_{s,F}$ *are two non-negative steady states of the system* (1.1) - (1.5)*. Assume that there exist* $a_* \in (0,a_\dagger)$ *and* $\delta > 0$ *such that*

$$p_{s,I}(a,x), p_{s,F}(a,x) \geqslant \delta \text{ a.e. on } [0,a_*]\times\overline{\Omega}. \tag{1.9}$$

Then there exist $\tau > 0$ *and* $v \in L^\infty((0,\tau)\times(0,a_\dagger)\times\Omega)$ *such that the problem* (1.1) - (1.5) *with*

$$p_0(a,x) = p_{s,I}(a,x)$$

admits a unique solution p *satisfying*

$$p(\tau,a,x) = p_{s,F}(a,x) \text{ for all } (a,x) \in (0,a_\dagger)\times\Omega.$$

Moreover, $p(\tau,a,x) \geqslant 0$ *for a.e.* $(t,a,x) \in (0,\tau)\times(0,a_\dagger)\times\Omega$.

Remark 1.4. *We denote by* $R = \displaystyle\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu(r)\ \mathrm{d}r}\ \mathrm{d}a$ *the reproductive number. It is known that (see, for instance* [2, Theorem 3.1]*)*

- *if* $R < 1$ *then there exists a unique non-negative solution to* (1.8)
- *if* $R = 1$ *and* $v_s \equiv 0$*, then there exists infinitely many solutions to* (1.8) *of the form* $p_s = \alpha e^{-\int_0^a \mu(r)\ \mathrm{d}r}$, $\alpha \in (0,\infty)$.
- *if* $R > 1$ *then there is no non-negative solution to* (1.8).

Consequently, the existence of non-negative steady states satisfying (1.9) *is ensured at least when* $R = 1$ *and* $v_s = 0$*. Another situation where we know that such states exist is when the control is active on all* $[0,a_\dagger]\times\Omega$ *and* $R < 1$ *(see* [2, Theorem 3.1]*).*

On the other hand, one can show that, if p_s *is a non-negative solution to the system* (1.8)*, then*

$$\lim_{a\to a_\dagger} p_s(a,x) = 0 \text{ a.e. } x \in \Omega. \tag{1.10}$$

Therefore we cannot assume that the initial or target non-negative steady states are bounded from below by a strictly positive constant on $[0,a_\dagger]$*, i.e., we cannot take* $a_* = a_\dagger$ *in Theorem 1.3.*

Let us now mention some related works from the literature. The null controllability results of the diffusion free age-dependent population dynamics model were first obtained by Barbu, Iannelli and Martcheva [5]. They proved the state of the system can be steered to any steady state, except for a small interval of ages near zero. Recently, Hegoburu, Magal and Tucsnak [10] proved that this restriction is not necessary, provided individuals do not reproduce at the age close to zero. They also proved there exists controls which preserves the positivity of the state trajectory. However, in the both works, the control is supported in the interval $(0,a_0)$, for some $a_0 < a_\dagger$. Recently, Maity [14] proved that null controllability can be achieved by controls supported in any subinterval of $(0,a_\dagger)$, provided we control before the individuals start to reproduce.

Concerning the models with spatial diffusion, namely for the system (1.1) - (1.5), as far as we know, the first result was obtained by Ainseba and Aniţa [2]. They proved that the system (1.1) - (1.5) can be driven to a steady state in any arbitrary time $\tau > 0$ keeping the positivity of the trajectory, provided the initial data is close to the steady state and the control acts in a spatial subdomain $\omega \subset \Omega$ but for all ages. When control acts in a spatial subdomain and only for small ages, a similar result for a large time was proved by Ainseba and Aniţa [3]. In [1] Ainseba proved null controllability of the system (1.1)-(1.5) except for a small interval of ages near zero, with controls acting everywhere in the ages but in a spatial subdomain. Recently, Hegoburu and Tucsnak [11] proved that the system (1.1) - (1.5) is null controllable for all ages and in any time by controls localized with respect to the spatial variable but active for all ages. Their method is based on Lebeau-Robbiano type strategy, originally developed for the null-controllability of the heat equation. Traore [18] considered a similar model with nonlinear distributions of the newborns. He proved null controllability except for small ages with controls localized in space variable and active for all ages. Martinez et. al [15] considered linearized Croco-type equation, which is similar to the system (1.1) -(1.5), with $\beta = \mu = 0$. They proved regional null controllability of such system.

The main novelties brought in by our paper are:

- We improve the existing estimates on the time necessary to control the system to zero and we show that our global controllability result applies to individuals of all ages, without needing to exclude ages in a neighbourhood of zero.
- We are able to tackle the case of a control which is active for ages $a \in [a_1, a_2]$, with arbitrary $a_1 \in [0, a_\dagger)$ and $a_2 \in (a_1, a_\dagger]$, provided that $\operatorname{supp} \beta \cap [0, a_1] = \emptyset$. Thus, unlike in the existing literature, we do not need to control arbitrarily low ages.
- Unlike most of the approaches in the literature, our methodology does not require adaptations of the existing parabolic Carleman estimates to the adjoint system of (1.1) - (1.5). We just combine characteristics method with existing observability estimates for parabolic equations. Thus our approach applies independently of the method used to derive final-state observability for the associated parabolic system (moment methods, local or global Carleman estimates, Lebeau-Robbiano strategy,...).
- Controllability with positivity constraints is proved, as far as we know for the first time, with a control which is localized both in age and with respect to the space variable. The methodology employed to obtain this result is based on duality and L^∞ estimates for parabolic PDEs.

The remaining part of this work is organized as follows:

- In Section 2 we first recall some basic facts about the Lotka-McKendrick semigroup with diffusion. We next formulate our control problem in a semigroup setting and we define the associated adjoint semigroup.
- In Section 3 we prove the final state observability for the adjoint system and, as a consequence, we obtain the proof of the main result in Theorem 3.2.
- Section 4 is devoted to the proof that controllability between positive steady states can be achieved in sufficiently large time, *i.e.*, to the proof of Theorem 1.3.
- Section 5 is devoted to the description of possible extensions and open questions.

2. Lotka-McKendrick Semigroup with Diffusion

In this section, we provide some basic results on the population semigroup for the linear age structured model with diffusion and its adjoint operator. Most of them were existing in the literature, so we just give the statements and the appropriate references. In some cases, namely when the adjoint operator is involved, we did not find detailed justifications in the existing literature, so, with no claim of originality, we felt necessary to give a more detailed presentation.

We write below equations (1.1)-(1.5) as an abstract control system with input space

$$H = L^2(0, a_\dagger; L^2(\Omega))$$

Before introducing the semigroup generator, we consider the diffusion free population operator

$$A_1 : \mathcal{D}(A_1) \to H,$$

defined by

$$\begin{aligned} \mathcal{D}(A_1) = \Big\{ &\varphi \in H \mid \varphi(\cdot, x) \text{ is locally absolutely continuous on } [0, a_\dagger), \\ &\varphi(0, x) = \int_0^{a_\dagger} \beta(a)\varphi(a, x)\, \mathrm{d}a \text{ for a.e. } x \in \Omega,\ \frac{\partial \varphi}{\partial a} + \mu\varphi \in H \Big\}, \\ A_1\varphi = &-\frac{\partial \varphi}{\partial a} - \mu\varphi, \end{aligned} \tag{2.1}$$

and the diffusion operator $A_2 : \mathcal{D}(A_2) \to H$ defined by

$$\mathcal{D}(A_2) = \Big\{ \varphi \in L^2(0, a_\dagger; H^2(\Omega)) \mid \frac{\partial \varphi}{\partial \nu_L} = 0 \text{ on } \partial\Omega \Big\},$$

$$A_2\varphi = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sigma_{ij}\frac{\partial \varphi}{\partial x_j}\right) \qquad (\varphi \in \mathcal{D}(A_2)). \tag{2.2}$$

We also introduce the input space $U = H$ and the control operator $B \in \mathcal{L}(U, H)$ defined by

$$Bu = mu \qquad (u \in U). \tag{2.3}$$

With the above notation, we rewrite the system (1.1)-(1.5) as:

$$\dot{z}(t) = \mathcal{A}z(t) + Bu(t), \tag{2.4}$$

$$z(0) = p_0, \tag{2.5}$$

where we have set $p(t, \cdot) = z(t)$, $v(t, \cdot) = u(t)$ and the population operator with diffusion $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H$ is defined by

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2), \quad \mathcal{A} = A_1 + A_2. \tag{2.6}$$

The fact that the system we consider is well-posed follows from the following result:

Lemma 2.1. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}$ on H.*

Proof. A proof of this lemma can be found in [12, Theorem 2.8]. □

With the above notation, our main result in Theorem 1.1 can be rephrased to : the pair $(\mathcal{A}, B)$, with $\mathcal{A}$ defined in (2.6) and B defined in (2.3) is null controllable in ant time $\tau > a_1 + \max\{a_1, a_\dagger - a_2\}$. It is well-known that, the null controllability in time τ of $(\mathcal{A}, B)$ is equivalent to the final-state observability in time τ of the pair $(\mathcal{A}^*, B^*)$, where $\mathcal{A}^*$ and B^* are the adjoint operators of $\mathcal{A}$ and B, respectively (see, for instance, [19, Section 11.2]). It is thus important to determine the adjoint of the operator $\mathcal{A}$. To this aim, we introduce an auxiliary unbounded operator $(\mathcal{A}_0, \mathcal{D}(\mathcal{A}_0))$ defined by

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0) = \Big\{ &\varphi \in H \mid \varphi(\cdot, x) \text{ is locally absolutely continuous on } [0, a_\dagger), \varphi \in L^2(0, a_\dagger; H^2(\Omega)), \\ &\lim_{a \to a_\dagger} \varphi(a, x) = 0 \text{ for a.e. } x \in \Omega,\ \frac{\partial \varphi}{\partial \nu_L} = 0 \text{ on } \partial\Omega, \frac{\partial \varphi}{\partial a} - \mu\varphi + L\varphi \in H \Big\}, \\ &\mathcal{A}_0\varphi = \frac{\partial \varphi}{\partial a} - \mu\varphi + L\varphi \qquad (\varphi \in \mathcal{D}(\mathcal{A}_0)). \end{aligned} \tag{2.7}$$

We have the following lemma:

Lemma 2.2. *The operator $(\mathcal{A}_0, \mathcal{D}(\mathcal{A}_0))$ is the infinitesimal generator of a strongly continuous semigroup $\mathbb{T}^0$ on H.*

Proof. For $\varphi \in \mathcal{D}(\mathcal{A}_0)$, we have

$$(\mathcal{A}_0\varphi, \varphi)_H = \lim_{a\to a_\dagger^-} \int_0^a \int_\Omega \left(\frac{\partial\varphi}{\partial a} - \mu\varphi + B\varphi\right)\varphi$$
$$= \lim_{a\to a_\dagger^-} \int_\Omega \frac{\varphi^2(a)}{2} - \int_\Omega \frac{\varphi^2(0)}{2} - \lim_{a\to a_\dagger^-} \int_0^a \int_\Omega \left(\mu\varphi^2 + \sum a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\right) \leqslant 0,$$

thus $\mathcal{A}_0$ is a dissipative operator.

To show that it is $m-$dissipative, we prove below that $I - \mathcal{A}_0$ is onto. To this aim, let $f \in H$ and consider the equation, of unknown $\varphi \in \mathcal{D}(\mathcal{A}_0)$,

$$\varphi - \frac{\partial\varphi}{\partial a} + \mu\varphi - L\varphi = f \text{ in } (0, a_\dagger) \times \Omega,\ \lim_{a\to a_\dagger^-} \varphi(a, x) = 0,\ \frac{\partial\varphi}{\partial\nu_L} = 0 \text{ on } \partial\Omega. \tag{2.8}$$

Denoting

$$\widetilde{\varphi}(a, x) = \exp\left(-a - \int_0^a \mu(r)\, \mathrm{d}r\right)\varphi(a, x) \qquad (a \in (0, a_\dagger),\ x \in \Omega), \tag{2.9}$$

we see that (2.8) is equivalent to the equation, of unknown $\widetilde{\varphi}$,

$$-\frac{\partial\widetilde{\varphi}}{\partial a} - L\widetilde{\varphi} = \widetilde{f} \text{ in } (0, a_\dagger) \times \Omega, \quad \widetilde{\varphi}(a_\dagger, x) = 0 \text{ in } \Omega, \quad \frac{\partial\widetilde{\varphi}}{\partial\nu_L} = 0 \text{ on } \partial\Omega,$$

where $\widetilde{f}(a, x) = \exp\left(-a - \int_0^a \mu(r)\, \mathrm{d}r\right) f(a, x)$. It is easy to see that $\widetilde{f} \in H$. It is easily seen that the above equation has a solution $\widetilde{\varphi} \in L^2(0, a_\dagger; H^2(\Omega)) \cap H^1(0, a_\dagger; L^2(\Omega))$ with

$$\|\widetilde{\varphi}(a, \cdot)\|_{L^2(\Omega)} \leqslant C\left(-a - \int_0^a \mu(r)\, \mathrm{d}r\right)\|f\|_H.$$

The above estimate and (2.9) imply that (2.8) has a solution $\varphi \in \mathcal{D}(\mathcal{A}_0)$, thus $\mathcal{A}_0$ is $m-$disiipative. Hence $\mathcal{A}_0$ generates a C^0 semigroup on H. □

We are now in a position to rigourously construct the adjoint of the unbounded operator $\mathcal{A}$.

Proposition 2.3. *The adjoint of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ in H is defined by*

$$\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}_0), \quad \mathcal{A}^*\psi = \frac{\partial\psi}{\partial a} - \mu\psi + \beta\psi(0, x) + L\psi. \tag{2.10}$$

Proof. We can easily verify that $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{A}^*)$.

To prove the reverse inclusion, we first note that for $\lambda > 0$ large enough, the operator $\lambda I - \mathcal{A}_0$ is boundedly invertible (this follows from the that $\mathcal{A}_0$ is a semigroup generator). For those values of λ we can thus consider the operator $I - \mathcal{F}(\lambda)$, where $\mathcal{F}(\lambda) \in \mathcal{L}(L^2(\Omega))$ is defined by

$$\mathcal{F}(\lambda)g(x) = \left[(\lambda I - \mathcal{A}_0)^{-1}(\beta(a)g(x))\right](0, x). \tag{2.11}$$

We note that for λ large enough the operator $I - \mathcal{F}(\lambda)$ is invertible. Indeed, this follows from the fact that $\lim_{\lambda\to\infty} \|\mathcal{F}(\lambda)\|_{\mathcal{L}(L^2(\Omega),H)} = 0$.

For λ as above, we define $\mathcal{G}_\lambda : H \mapsto H$ defined by by $\mathcal{G}_\lambda f = \varphi_\lambda$ where φ_λ solves

$$\lambda\varphi_\lambda - \frac{\partial\varphi_\lambda}{\partial a} + \mu\varphi_\lambda - L\varphi_\lambda - \beta(a)\varphi_\lambda(0, x) = f \text{ in } (0, a_\dagger) \times \Omega,\ \lim_{a\to a_\dagger^-} \varphi_\lambda(a, x) = 0,\ \frac{\partial\varphi_\lambda}{\partial\nu_L} = 0 \text{ on } \partial\Omega. \tag{2.12}$$

The fact that the operator $\mathcal{G}_\lambda$ is well defined follows from the fact that the unique solution of (2.12) is clearly given by

$$\varphi_\lambda(a,x) = (\lambda I - \mathcal{A}_0)^{-1}\left(f(a,x) + V_{\lambda,f}(a,x)\right), \tag{2.13}$$

where

$$V_{\lambda,f}(a,x) = \beta(a)\left(I - F(\lambda)\right)^{-1}\left(\left[(\lambda I - \mathcal{A}_0)^{-1} f\right](0,x)\right).$$

This means, in particular, that $\varphi_\lambda \in \mathcal{D}(\mathcal{A}_0)$.

We are now in the position to prove the inclusion $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A}_0)$. To this aim, take λ as above and let $\psi \in \mathcal{D}((\lambda I - \mathcal{A})^*)$. Then there exists $f \in H$ such that

$$\int_0^{a_\dagger}\int_\Omega \psi(\lambda I - \mathcal{A})\varphi = \int_0^{a_\dagger}\int_\Omega f\varphi \text{ for all } \varphi \in \mathcal{D}(\mathcal{A}).$$

Let $\eta_\lambda = \mathcal{G}_\lambda f$, with $\mathcal{G}_\lambda$ defined several lines above. Then, using (2.12) and integrating by parts we obtain

$$\int_0^{a_\dagger}\int_\Omega f\varphi = \lim_{a\to a_\dagger^-}\int_0^a\int_\Omega \left(\lambda\eta_\lambda - \frac{\partial\eta_\lambda}{\partial a} + \mu\eta_\lambda - L\eta_\lambda - \beta(a)\eta_\lambda(0,x)\right)\varphi$$
$$= \lim_{a\to a_\dagger^-}\int_\Omega\int_0^a \eta_\lambda\left(\lambda\varphi + \frac{\partial\varphi}{\partial a} + \mu\varphi - L\varphi\right) = \int_0^{a_\dagger}\int_\Omega \eta_\lambda(\lambda I - \mathcal{A})\varphi.$$

Therefore,

$$\int_0^{a_\dagger}\int_\Omega (\psi - \eta_\lambda)(\lambda I - \mathcal{A})\varphi = 0, \text{ for all } \varphi \in \mathcal{D}(\mathcal{A}). \tag{2.14}$$

By choosing $\varphi = (\lambda I - \mathcal{A})^{-1}(\psi - \eta_\lambda)$ we get

$$\int_0^{a_\dagger}\int_\Omega |\psi - \eta_\lambda|^2 = 0.$$

Thus $\psi \in \mathcal{D}(\mathcal{A}_0)$ and ψ solves (2.12). This completes the proof of the proposition. □

3. An Observability Inequality:

As mentioned above, the null-controllability of a pair $(\mathcal{A}, B)$ is equivalent to the final state observability of the pair $(\mathcal{A}^*, B^*)$, see [19, Theorem 11.2.1]. Recall that that final-state observability of $(\mathcal{A}^*, B^*)$ is defined as

Definition 3.1. [19, Definition 6.1.1] *The pair $(\mathcal{A}^*, B^*)$ is final state observable in time τ if there exists a $k_\tau > 0$ such that*

$$\int_0^\tau \|B^*\mathbb{T}_t^* q_0\|_H \;\geqslant\; k_\tau^2\|\mathbb{T}_\tau^* q_0\|^2 \qquad (q_0 \in \mathcal{D}(\mathcal{A}^*)).$$

For $\mathcal{A}$ defined in (2.6) and $q_0 \in H$ we set

$$q(t) = \mathbb{T}_t^* q_0 \qquad (t \geqslant 0),$$

where $\mathbb{T}$ is the semigroup generated by $\mathcal{A}$. According to Proposition 2.3 we have:

$$\begin{cases} \dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - Lq - \beta(a)q(t,0,x) + \mu(a)q = 0, & t \geqslant 0, (a,x) \in (0,a_\dagger)\times\Omega, \\ q(t,a_\dagger,x) = 0, & t \geqslant 0, x \in \Omega, \\ \dfrac{\partial q}{\partial \nu_L} = 0, & t \geqslant 0, (a,x) \in (0,a_\dagger)\times\partial\Omega, \\ q(0,a,x) = q_0(a,x), & (a,x) \in (0,a_\dagger)\times\Omega. \end{cases} \tag{3.1}$$

In view of [19, Theorem 11.2.1], the statement in Theorem 1.1 is equivalent to the following theorem:

Theorem 3.2. *Under the assumption of Theorem 1.1, the pair $(\mathcal{A}^*, B^*)$ is final-state observable for every $\tau > a_1 + a_\dagger - a_2$. In other words, for every $\tau > a_1 + a_\dagger - a_2$ there exists $k_\tau > 0$ such that the solution q of (3.1) satisfies*

$$\int_0^{a_\dagger}\int_\Omega q^2(\tau, a, x)\,\mathrm{d}x\mathrm{d}a \leqslant k_\tau^2 \int_0^\tau \int_{a_1}^{a_2}\int_\omega q^2(t, a, x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}t \qquad (q_0 \in \mathcal{D}(\mathcal{A}^*)). \tag{3.2}$$

Before we begin the proof of the above result, let us briefly describe its main steps. The first one is writing an explicit expression of q. We define

$$V(t, x) = q(t, 0, x), \qquad (t, x) \in (0, \tau) \times \Omega. \tag{3.3}$$

Integrating along the characteristic lines, the solution of (3.1) can be written as

$$q(t) = \begin{cases} \dfrac{\pi(a)}{\pi(a+t)} e^{tA_2} q_0(a+t, \cdot) + \displaystyle\int_0^t \dfrac{\pi(a)}{\pi(a+t-s)} e^{(t-s)A_2}\beta(a+t-s)V(s, \cdot) & t \leqslant a_\dagger - a, \\ \displaystyle\int_{t+a-a_\dagger}^t \dfrac{\pi(a)}{\pi(a+t-s)} e^{(t-s)A_2}\beta(a+t-s)V(s, \cdot)\,\mathrm{d}s & t > a_\dagger - a, \end{cases} \tag{3.4}$$

where $\pi(a) = \mathrm{e}^{-\int_0^a \mu(r)\mathrm{d}r}$. Without loss of generality, we can assume that $a_2 \leqslant a_b$ and let τ be as in Theorem 3.2. We decompose the interval $(0, a_\dagger)$ as

$$(0, a_\dagger) = (0, \widetilde{a}) \cup (\widetilde{a}, a_\dagger),$$

where a_0 is chosen suitably so that $a_0 < a_2$ and $\tau > a_\dagger - a$ for all $a \in (a_0, a_\dagger)$. Now using the expression of q in (3.4) and choosing a_0 suitably, we can show that

$$\int_0^{a_\dagger}\int_\Omega q^2(\tau, a, x)\,\mathrm{d}x\mathrm{d}a = \int_0^{a_0}\int_\Omega q^2(\tau, a, x)\,\mathrm{d}x\mathrm{d}a + \int_{a_0}^{a_\dagger}\int_\Omega q^2(\tau, a, x)\,\mathrm{d}x\mathrm{d}a \tag{3.5}$$

$$\leqslant C\left(\int_0^{a_0}\int_\Omega q^2(\tau, a, x)\,\mathrm{d}x\mathrm{d}a + \int_\eta^\tau\int_\Omega q^2(t, 0, x)\,\mathrm{d}x\mathrm{d}t\right), \tag{3.6}$$

for some $\eta > a_1$ (see proof of Theorem 3.2 for more details).

The second step consists in deriving upper appropriate bounds for each one of the terms in the right-hand side of (3.5). This is accomplished by combining some change of variables using the characteristics of the diffusion free problem with some known observability inequalities for parabolic equations.

To accomplish this program, we first recall the following observability inequality for parabolic equations (see, for instance, Imanuvilov and Fursikov [8]) :

Proposition 3.3. *Let $T > 0$, $0 \leqslant t_0 < \tau$ and $t_1 \in (t_0, T]$. Then for every $w_0 \in L^2(\Omega)$, the solution w of the initial and boundary problem*

$$\begin{cases} \dfrac{\partial w}{\partial s}(s, x) - Lw(s, x) = 0 & ((s, x) \in (t_0, T) \times \Omega), \\ \dfrac{\partial w}{\partial \nu_L} = 0 & ((s, x) \in (t_0, T) \times \partial\Omega), \\ w(t_0, x) = w_0(x), & (x \in \Omega), \end{cases} \tag{3.7}$$

satisfies the estimate

$$\int_\Omega w^2(T,x)\,\mathrm{d}x \leqslant \int_\Omega w^2(t_1,x)\,\mathrm{d}x \leqslant c_1 \mathrm{e}^{\dfrac{c_2}{t_1-t_0}} \int_{t_0}^{t_1}\int_\omega w^2(s,x)\,\mathrm{d}x\,\mathrm{d}s, \tag{3.8}$$

where the constants c_1 *and* c_2 *depend on* L*, on* Ω *and on* τ*.*

In the following two propositions we estimate each of the terms appearing in the right-hand side of (3.5).

Proposition 3.4. *Let us assume the hypothesis of Theorem 1.1 and let* $\tau > a_1$ *and* $a_0 \in (0, a_b)$*. Then there exists a constant* $C > 0$ *such that, for every* $q_0 \in \mathcal{D}(\mathcal{A}^*)$*, the solution* q *of the system* (3.1)*, obeys*

$$\int_0^{a_0}\int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega q^2(t,a,x)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t. \tag{3.9}$$

Proof. First of all, without loss of of generality we can assume that $a_2 \leqslant a_b$ (otherwise we simply observe for small ages). We can also assume that $a_0 > \max\{a_1, a_2-\tau\}$. Since $\beta(a) = 0$ for all $a \in (0, a_b)$, q satisfies

$$\begin{cases} \dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - Lq + \mu(a)q = 0, & t \geqslant 0, (a,x) \in (0,a_b)\times\Omega,\\ \dfrac{\partial q}{\partial \nu_L} = 0 & t \geqslant 0, (a,x) \in (0,a_b)\times\partial\Omega. \end{cases} \tag{3.10}$$

This means that q satisfies the adjoint of a system called “Crocco type” (see [15]), where the authors proved a regional controllability result. We set

$$\widetilde{q}(t,a,x) = q(t,a,x)\,\mathrm{e}^{-\displaystyle\int_0^a \mu(r)\,\mathrm{d}r}. \tag{3.11}$$

Then $\widetilde{q}$ satisfies

$$\begin{cases} \dfrac{\partial \widetilde{q}}{\partial t} - \dfrac{\partial \widetilde{q}}{\partial a} - L\widetilde{q} = 0, & t \geqslant 0, (a,x) \in (0,a_b)\times\Omega,\\ \dfrac{\partial \widetilde{q}}{\partial \nu_L} = 0 & t \geqslant 0, (a,x) \in (0,a_b)\times\partial\Omega. \end{cases} \tag{3.12}$$

The desired conclusion of this Proposition follows as soon as we show that there exists a constant $C_\tau > 0$ such that

$$\int_0^{a_0}\int_\Omega \widetilde{q}^2(\tau,a,x)\,\mathrm{d}x\,\mathrm{d}a \leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t, \tag{3.13}$$

for every $\tau > a_1$. Indeed, using (3.13) we get

$$\begin{aligned} \int_0^{a_0}\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\,\mathrm{d}a &\leqslant \left(\mathrm{e}^{2\displaystyle\int_0^{\tilde{a}} \mu(r)\ \mathrm{d}r}\right)\int_0^{a_0}\int_\Omega \widetilde{q}^2(\tau,a,x)\,\mathrm{d}x\,\mathrm{d}a \\ &\leqslant \left(\mathrm{e}^{2\|\mu\|_{L^1[0,a_0]}}\right)\int_0^{a_\dagger}\int_\Omega \widetilde{q}^2(\tau,a,x)\,\mathrm{d}x\,\mathrm{d}a \leqslant C_\tau\int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t \\ &\leqslant C_\tau\int_0^\tau\int_{a_1}^{a_2}\int_\omega q^2(t,a,x)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t, \end{aligned}$$

where C_τ is a generic constant depending only on τ. We have thus shown that (3.13) implies (3.9). We can thus concentrate on the remaining part of the proof in checking (3.13).

Without loss of generality, let us assume that

$$\tau < a_2, \quad \tau > a_2 - a_1, \quad a_0 \in (a_1, a_2). \tag{3.14}$$

We set $b_0 = a_2 - \tau$ and we split the interval $(0, a_0)$ as follows

$$(0, a_0) = (0, b_0) \cup (b_0, a_1) \cup (a_1, a_0). \tag{3.15}$$

As explained in the introduction, we are going to use Proposition 3.3 along the characteristics. Before doing that, let us explain why we have divided the interval $(0, a_0)$ in the above way. Basically, this division depends on the point in which the trajectory $\gamma(s) = (\tau - s, a + s)$, $s \in [0, \tau]$ (or equivalently the backward characteristics staring from (τ, a)) enters the observation region $(a_1, a_2) \times (0, \tau)$ and exits from the same region (see Fig. 1). More precisely:

- For $a \in (0, b_0)$, the trajectory $\gamma(s)$ enters the observation region for $s = a_1 - a$. As $b_0 + \tau < a_2$, for $s = \tau$, $\gamma(s)$ it hits the line $t = 0$ without leaving the observation region (blue region in Fig. 1).
- For $a \in (b_0, a_1)$, the trajectory $\gamma(s)$ enters the observation domain for $s = a_1 - a$ and exits the observation region for $s = a_2 - a < \tau$ (red region in Fig. 1).
- For $a \in (a_1, a_0)$ the trajectory $\gamma(s)$ starts inside the observation region but it exits the region in time $s = a_2 - a < \tau$ (green region in Fig. 1).

Let us remark that, the choices in (3.14) are made to cover all possible scenarios. Towards the end of the proof of the proposition, we shall explain how to split the interval in other cases.

In the remaining part of the proof we give upper bounds for $\displaystyle\int_I \int_\Omega \widetilde{q}^2(\tau, a, x)\, \mathrm{d}x\, \mathrm{d}a$ where I is successively each one of the intervals appearing in the decomposition (3.15).

Upper bound on $(0, b_0)$:

For a.e $a \in (0, b_0)$, we first set

$$w(s, x) = \widetilde{q}(s, a + \tau - s, x) \qquad (s \in (0, \tau), x \in \Omega).$$

Then w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial s} - Lw = 0, & (s, x) \in (0, \tau) \times \Omega, \\ \dfrac{\partial w}{\partial \nu_L} = 0, & (s, x) \in (0, \tau) \times \partial\Omega, \\ w(0, x) = \widetilde{q}(0, \tau + a, x), & x \in \Omega. \end{cases} \tag{3.16}$$

Applying Proposition 3.3, with $t_0 = 0$, $t_1 = \tau + a - a_1$ and $T = \tau$, we obtain

$$\int_\Omega w^2(\tau, x)\, \mathrm{d}x \leqslant \int_\Omega w^2(\tau + a - a_1, x)\, \mathrm{d}x \leqslant c_1 \mathrm{e}^{\frac{c_2}{\tau + a - a_1}} \int_0^{\tau + a - a_1} \int_\omega w^2(s, x)\, \mathrm{d}x \mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality writes

$$\begin{aligned} \int_\Omega \widetilde{q}^2(\tau, a, x)\, \mathrm{d}x &\leqslant c_1 \mathrm{e}^{\frac{c_2}{\tau + a - a_1}} \int_0^{\tau + a - a_1} \int_\omega \widetilde{q}^2(s, a + \tau - s, x)\, \mathrm{d}x \mathrm{d}s \\ &= c_1 \mathrm{e}^{\frac{c_2}{\tau + a - a_1}} \int_{a_1}^{\tau + a} \int_\omega \widetilde{q}^2(\tau + a - s, s, x)\, \mathrm{d}x \mathrm{d}s. \end{aligned}$$

Integrating with respect to a over $(0, b_0)$ we obtain

$$\int_0^{b_0}\int_\Omega \widetilde{q}^2(\tau,a,x)\,\mathrm{d}x\mathrm{d}a \leqslant c_1 \mathrm{e}^{\dfrac{c_2}{\tau-a_1}} \int_0^{b_0}\int_{a_1}^{\tau+a}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\,\mathrm{d}x\mathrm{d}s\mathrm{d}a$$
$$= C_\tau \int_{a_1}^{a_2}\int_{s-\tau}^{b_0}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}s = C_\tau \int_{a_1}^{a_2}\int_0^{a_2-s}\int_\omega \widetilde{q}^2(r,s,x)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s$$
$$\leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}t. \quad (3.17)$$

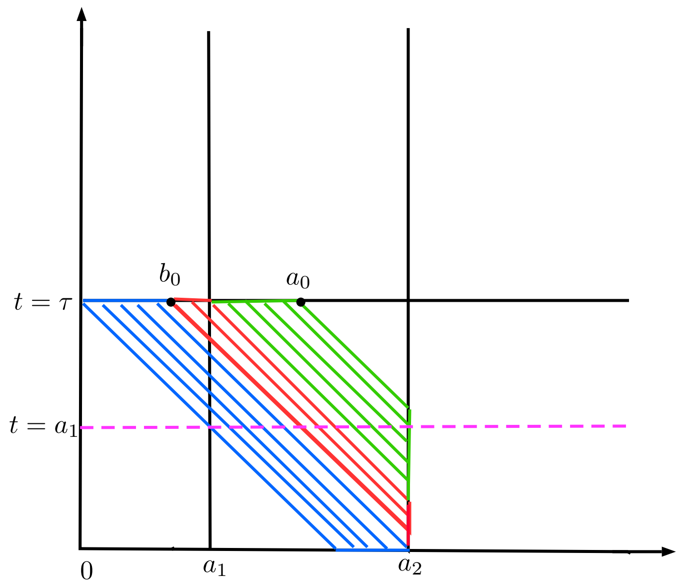

FIGURE 1. An illustration of the choice made in (3.14): Blue region corresponds to the interval $(0,b_0)$, Red corresponds to the interval (b_0,a_1), Green corresponds to the interval (a_1,a_0).

Upper bound on (b_0,a_1):
For a.e. $a\in(b_0,a_1)$, we define

$$w(s,x)=\widetilde{q}(s,a+\tau-s,x) \qquad (s\in(\tau+a-a_2,\tau), x\in\Omega).$$

Then w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial s}-Lw=0, & (s,x)\in(\tau+a-a_2,\tau)\times\Omega,\\ \dfrac{\partial w}{\partial \nu_L}=0, & (s,x)\in(\tau+a-a_2,\tau)\times\partial\Omega,\\ w(\tau+a-a_2,x)=\widetilde{q}(\tau+a-a_2,a_2,x), & x\in\Omega. \end{cases} \quad (3.18)$$

Applying Proposition 3.3 with the choice $t_0=\tau+a-a_2$, $t_1=\tau+a-a_1$ and $T=\tau$, it follows that

$$\int_\Omega w^2(\tau,x)\,\mathrm{d}x \leqslant \int_\Omega w^2(\tau+a-a_1,x)\,\mathrm{d}x \leqslant c_1\mathrm{e}^{\dfrac{c_2}{a_2-a_1}}\int_{\tau+a-a_2}^{\tau+a-a_1}\int_\omega w^2(s,x)\,\mathrm{d}x\mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality becomes

$$\int_\Omega \widetilde{q}^2(\tau,a,x)\ \mathrm{d}x \leqslant c_1 \mathrm{e}^{\overline{\frac{c_2}{a_2-a_1}}} \int_{\tau+a-a_2}^{\tau+a-a_1}\int_\omega \widetilde{q}^2(s,a+\tau-s,x)\ \mathrm{d}x\mathrm{d}s$$
$$= C\int_{a_1}^{a_2}\int_\omega \widetilde{z}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}s.$$

Integrating with respect to a over (b_0,a_1) we get

$$\int_{b_0}^{a_1}\int_\Omega \widetilde{q}^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant C\int_{b_0}^{a_1}\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}s\mathrm{d}a$$
$$= C\int_{a_1}^{a_2}\int_{b_0}^{a_1}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}s = C\int_{a_1}^{a_2}\int_{\tau+b_0-s}^{\tau+a_1-s}\int_\omega \widetilde{q}^2(r,s,x)\ \mathrm{d}x\mathrm{d}r\mathrm{d}s$$
$$\leqslant C\int_{a_1}^{a_2}\int_0^\tau\int_\omega \widetilde{q}^2(r,s,x)\ \mathrm{d}x\mathrm{d}r\mathrm{d}s = C\int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}t. \quad (3.19)$$

Upper bound on (a_1,a_0):
For a.e. $a\in(a_1,a_0)$, we define

$$w(s,x) = \widetilde{q}(s,a+\tau-s,x) \qquad (s\in(\tau+a-a_2,\tau), x\in\Omega).$$

Then w satisfies the system (3.18). Applying Proposition 3.3 with $t_0=\tau+a-a_2$ and $t_1=T=\tau$, we have that

$$\int_\Omega w^2(\tau,x)\ \mathrm{d}x \leqslant c_1 e^{\overline{\frac{c_2}{a_2-a}}}\int_{\tau+a-a_2}^\tau\int_\omega w^2(s,x)\ \mathrm{d}x\mathrm{d}s.$$

In terms of $\widetilde{q}$, the above inequality reads as follows

$$\int_\Omega \widetilde{q}^2(\tau,a,x)\ \mathrm{d}x \leqslant c_1 e^{\overline{\frac{c_2}{a_2-a}}}\int_{\tau+a-a_2}^\tau\int_\omega \widetilde{q}^2(s,a+\tau-s,x)\ \mathrm{d}x\mathrm{d}s$$
$$= c_1 e^{\overline{\frac{c_2}{a_2-a}}}\int_a^{a_2}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}s.$$

Integrating with respect to a over (a_1,a_0) we get

$$\int_{a_1}^{a_0}\int_\Omega \widetilde{q}^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant c_1 e^{\overline{\frac{c_2}{a_2-a_0}}}\int_{a_1}^{a_0}\int_a^{a_2}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}s\mathrm{d}a$$
$$= C_\tau\int_{a_1}^{a_2}\int_{a_1}^{s}\int_\omega \widetilde{q}^2(\tau+a-s,s,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}s = C_\tau\int_{a_1}^{a_2}\int_{\tau+a_1-s}^{\tau}\int_\omega \widetilde{q}^2(r,s,x)\ \mathrm{d}x\mathrm{d}r\mathrm{d}s$$
$$\leqslant C_\tau\int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}t. \quad (3.20)$$

Therefore, combining (3.17), (3.19) and (3.20) we get

$$\int_0^{a_0}\int_\Omega \widetilde{q}^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant C_\tau\int_0^\tau\int_{a_1}^{a_2}\int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t,, \quad (3.21)$$

Let us explain how to split the interval $(0,a_0)$ in other possible cases:

- If $\tau<a_2,\ \tau<a_2-a_1,\ a_0\in(a_2-\tau,a_2)$, then we use $(0,a_0)=(0,a_1)\cup(a_1,b_0)\cup(b_0,a_0)$.
- If $\tau\geqslant a_2,\ a_0\in(a_1,a_2)$ then we use $(0,a_0)=(0,a_1)\cup(a_1,a_0)$.

This completes the proof of the proposition. □

In the next proposition, we estimate $q(t,0,x)$. More precisely, we prove the following:

Proposition 3.5. *Let us assume the hypothesis of Theorem 1.1 and let $\tau > a_1$ and $\eta \in (a_1, \tau)$. Then there exists a constant $C > 0$ such that, for every $q_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution q of the system (3.1), satisfies*

$$\int_\eta^\tau \int_\Omega q^2(t,0,x)\,\mathrm{d}x\mathrm{d}t \leqslant C \int_0^\tau \int_{a_1}^{a_2} \int_\omega q^2(t,a,x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{3.22}$$

Proof. Let $\widetilde{q}$ be defined as in (3.11). In particular, $\widetilde{q}$ satisfies (3.12). Here also we are going to use Proposition 3.3 along the characteristics. Since we want to estimate $q(t,0,x)$ we need to consider the trajectory $\gamma(s) = (t-s, s)$, $s \leqslant t \leqslant \tau$ (or equivalently the backward characteristic stating from $(t,0)$). If $\tau < a_1$, the trajectory $\gamma(s)$ never reaches the observation region $(0,\tau)\times(a_1,a_2)$ (see Fig. 2). This is why we choose $\tau > a_1$. Without loss of generality, let us assume that

$$\tau > a_b, \quad \eta < a_b \text{ and } a_2 \leqslant a_b.$$

Case 1: For a.e. $t \in (a_b, \tau)$, we define

$$w(s,x) = \widetilde{q}(s, t-s, x), \quad s \in (t-a_b, t), x \in \Omega. \tag{3.23}$$

Then w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial s} - Lw = 0 & ((s,x) \in (t-a_b,t)\times\Omega), \\ \dfrac{\partial w}{\partial \nu_L} = 0 & ((s,x) \in (t-a_b,t)\times\partial\Omega), \\ w(t-a_b, x) = q(t-a_b, a_b, x) & (x \in \Omega). \end{cases} \tag{3.24}$$

Using Proposition 3.3, with $t_0 = t - a_b$, $t_1 = t - a_1$ and $T = t$, we obtain

$$\int_\Omega w^2(t,x)\,\mathrm{d}x \leqslant \int_\Omega w^2(t-a_1,x)\,\mathrm{d}x \leqslant c_1 \mathrm{e}^{\dfrac{c_2}{a_b - a_1}} \int_{t-a_b}^{t-a_1} \int_\omega w^2(s,x)\,\mathrm{d}x\mathrm{d}s.$$

In terms of $\widetilde{q}$ the above inequality reads as

$$\int_\Omega \widetilde{q}^2(t,0,x)\,\mathrm{d}x \leqslant c_1 \mathrm{e}^{\dfrac{c_2}{a_b - a_1}} \int_{t-a_b}^{t-a_1} \int_\omega \widetilde{q}^2(s, t-s, x)\,\mathrm{d}x\mathrm{d}s$$
$$= c_1 \mathrm{e}^{\dfrac{c_2}{a_b - a_1}} \int_{a_1}^{a_b} \int_\omega \widetilde{q}^2(t-s, s, x)\,\mathrm{d}x\mathrm{d}s.$$

Integrating with respect to t over $[a_b, \tau]$ we obtain

$$\begin{aligned} \int_{a_b}^\tau \int_\Omega \widetilde{q}^2(t,0,x)\,\mathrm{d}x\mathrm{d}t &\leqslant c_1 e^{\dfrac{c_2}{a_b - a_1}} \int_{a_b}^\tau \int_{a_1}^{a_b} \int_\omega \widetilde{q}^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &= C \int_{a_1}^{a_b} \int_{a_b}^\tau \int_\omega \widetilde{q}^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s = C \int_{a_1}^{a_b} \int_{a_b - s}^{\tau - s} \int_\omega \widetilde{q}^2(r,s,x)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s \\ &\leqslant C \int_0^\tau \int_{a_1}^{a_b} \int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}t. \end{aligned} \tag{3.25}$$

Case 2: For a.e $t \in (\eta, a_b)$, we define

$$w(s,x) = \widetilde{q}(s, t-s, x) \qquad (s \in (0,t),\ x \in \Omega). \tag{3.26}$$

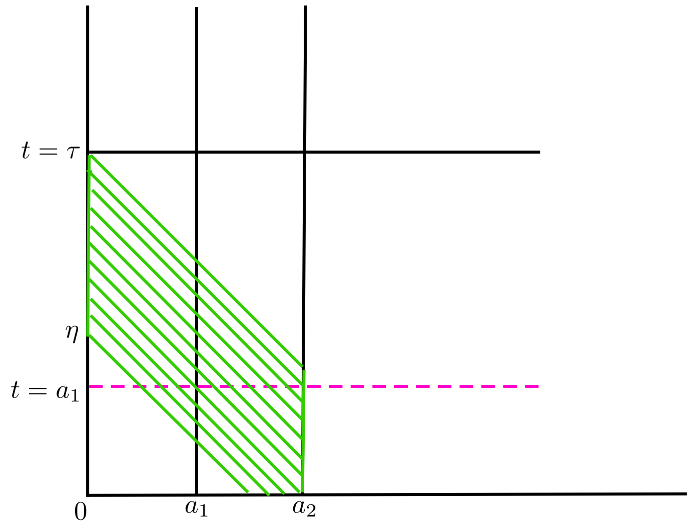

FIGURE 2. An illustration of the estimate of $\widetilde{q}(t,0,x)$. Here we have chosen $a_2 = a_b$. Since $\tau > a_1$ all the backward characteristics starting from $(t,0)$ enters the observation domain (the green region).

Then w satisfies

$$\begin{cases} \dfrac{\partial w}{\partial s} - Lw = 0 & ((s,x) \in (0,t)\times\Omega), \\ \dfrac{\partial w}{\partial \nu_L} = 0 & ((s,x)\in(0,t)\times\partial\Omega), \\ w(0,x) = \widetilde{q}(0,t,x) & (x\in\Omega). \end{cases}$$

By applying Proposition 3.3, with $t_0 = 0$, $t_1 = t - a_1$ and $T = t$, we obtain

$$\int_\Omega w^2(t,x)\,\mathrm{d}x \leqslant \int_\Omega w^2(t-a_1,x)\,\mathrm{d}x \leqslant c_1 \mathrm{e}^{\frac{c_2}{t-a_1}} \int_0^{t-a_1}\int_\omega w^2(s,x)\,\mathrm{d}x\mathrm{d}s.$$

This yields

$$\int_\Omega \widetilde{q}^2(t,0,x)\,\mathrm{d}x \leqslant c_1\mathrm{e}^{\frac{c_2}{t-a_1}} \int_0^{t-a_1}\int_\omega \widetilde{q}^2(s,t-s,x)\,\mathrm{d}x\mathrm{d}s = c_1\mathrm{e}^{\frac{c_2}{t-a_1}}\int_{a_1}^t\int_\omega \widetilde{q}^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}s.$$

Integrating with respect to t over $[\eta, a_b]$ we get

$$\begin{aligned} \int_\eta^{a_b}\int_\Omega \widetilde{q}^2(t,0,x)\,\mathrm{d}x\mathrm{d}t &\leqslant c_1\mathrm{e}^{\frac{c_2}{\eta-a_1}}\int_\eta^{a_b}\int_{a_1}^t\int_\omega \widetilde{q}^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}s\mathrm{d}t \\ &\leqslant C_\eta \int_0^{a_b}\int_{a_1}^t\int_\omega \widetilde{q}^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}s\mathrm{d}t = C\int_{a_1}^{a_b}\int_s^{a_b}\int_\omega q^2(t-s,s,x)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s \\ &= C\int_{a_1}^{a_b}\int_0^{a_b-s}\int_\omega q^2(r,s,x)\,\mathrm{d}x\mathrm{d}r\mathrm{d}s \leqslant C\int_0^\tau\int_{a_1}^{a_b}\int_\omega \widetilde{q}^2(t,a,x)\,\mathrm{d}x\mathrm{d}a\mathrm{d}t. \end{aligned} \tag{3.27}$$

Combining, (3.25) and (3.27) we obtain

$$\int_\eta^T \int_\Omega \widetilde{q}^2(t,0,x)\ \mathrm{d}x\mathrm{d}t \leqslant C \int_0^T \int_{a_1}^{a_2} \int_\omega \widetilde{q}^2(t,a,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}t.$$

Note that, from the definition of $\widetilde{q}$ in (3.11), we have $\widetilde{q}(t,0,x) = q(t,0,x)$. Thus from the above estimate we clearly obtain (3.22). ☐

3.1. **Proof of the first main result.** We are now in a position to prove Theorem 3.2, thus, consequently, our first main result in Theorem 1.1.

Proof of Theorem 3.2. Without loss of generality let us assume that $a_1 < a_\dagger - a_2$ and $a_2 \leqslant a_b$. Let us set

$$\delta = \tau - (a_1 + a_\dagger - a_2).$$

Let us choose $\varepsilon < \delta$ such that

$$a_2 - \varepsilon > \max\{a_1, a_\dagger - \tau\}.$$

Note that such a choice is always possible as $\tau > a_1 + a_\dagger - a_2 \geqslant a_\dagger - a_2$ (see Fig. 3).

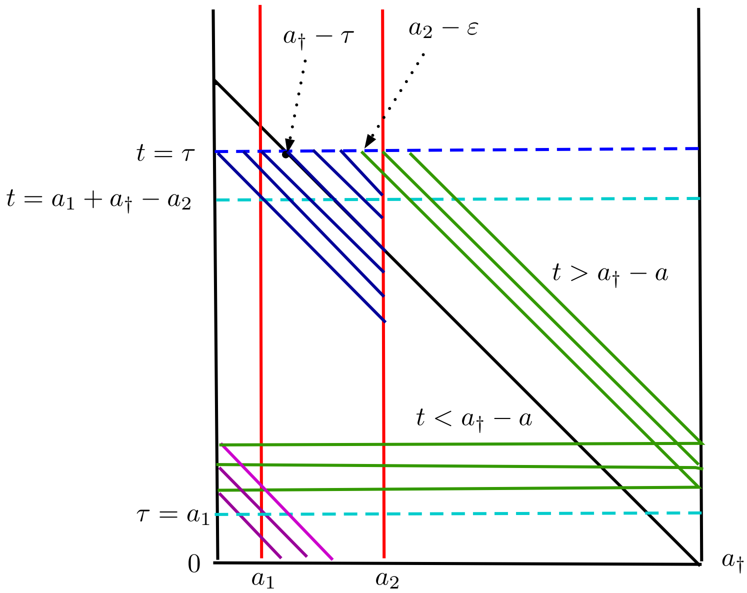

FIGURE 3. An illustration of the final time observability: For $a \in (0, a_2 - \varepsilon)$ (blue region) the backward characteristics enters the observation domain. Thus we have the estimate (3.29). For $a \in (a_2 - \varepsilon, a_\dagger)$, the backward characteristics (green region) hits the line $a = a_\dagger$, gets renewed by the renewal condition $\beta(a)q(t,0,x)$ and then enters the observation domain (purple region). This is obtained in (3.32).

Now

$$\int_0^{a_\dagger} \int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a = \int_0^{a_2-\varepsilon} \int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a + \int_{a_2-\varepsilon}^{a_\dagger} \int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a. \tag{3.28}$$

By applying Proposition 3.4, we have

$$\int_0^{a_2-\varepsilon} \int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant C_\tau \int_0^\tau \int_{a_1}^{a_2} \int_\omega q^2(t,a,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{3.29}$$

Thus the theorem is proved as soon as we show that

$$\int_{a_2-\varepsilon}^{a_\dagger}\int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a \leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega q^2(t,a,x)\ \mathrm{d}x\mathrm{d}a\mathrm{d}t. \tag{3.30}$$

Therefore in the sequel, we concentrate on proving (3.30). We recall that q is given by the formula (3.4). For $a\in(a_2-\varepsilon,a_\dagger)$ we have $\tau+a>a_\dagger$. So the expression of q in (3.4) yields

$$q(\tau,a,\cdot)=\int_{\tau+a-a_\dagger}^{\tau}\frac{\pi(a)}{\pi(a+t-s)}e^{(t-s)A_2}\beta(a+t-s)V(s,\cdot)\ \mathrm{d}s,\qquad a\in(a_2-\varepsilon,a_\dagger). \tag{3.31}$$

Integrating over Ω it is easy to verify that

$$\int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\leqslant C_\tau\int_{\tau+a-a_\dagger}^{\tau}\int_\Omega V^2(s,x)\ \mathrm{d}x\mathrm{d}s.$$

Now integrating with respect to a over $(a_2-\varepsilon,a_\dagger)$ we obtain

$$\int_{a_2-\varepsilon}^{a_\dagger}\int_\Omega q^2(\tau,a,x)\ \mathrm{d}x\mathrm{d}a\leqslant C_\tau\int_{a_1+(\delta-\varepsilon)}^{\tau}\int_\Omega q^2(t,0,x)\ \mathrm{d}x\mathrm{d}t. \tag{3.32}$$

Finally using Proposition 3.5 to the above estimate we get (3.30). This completes the proof of the theorem. □

4. Controls preserving positivity

An important issue in view of applications (namely in population dynamics) is to design controls such that the corresponding state trajectories join two different non-negative stationary states in some time τ, while preserving the positivity of the controlled trajectory for $t\in[0,\tau]$. This type of result has been proved in [10] for the diffusion free Lotka-McKendrick system (in a uniform time) and in Lohéac, Trélat and Zuazua [13] for purely parabolic problems (in a time depending on an appropriate norm of the difference of the two stationary states). We prove below that the situation encountered in the latter case also applies to the problem considered in the present work. An essential ingredient in obtaining this type of result is proving the null controllability of the system by means of L^∞ controls and then "slowly" (s.t. positivity is preserved) driving, the initial state towards the desired target.

We first recall a classical estimate for the semigroup generated by a strictly elliptic operator with Neumann boundary conditions.

Lemma 4.1. *Let A_2 is defined in (2.2). Then the following holds*

$$\|e^{tA_2}\varphi\|_{L^\infty(\Omega)}\leqslant\|\varphi\|_{L^\infty(\Omega)},\qquad \textit{for all } t\geqslant 0, \varphi\in L^\infty(\Omega).$$

Proof. For the proof of this result we refer to Daners [6, Corollary 7.2] or Ouhabaz [17, Corollary 4.10]. □

As a consequence of the above result, we show below that for every $t\geqslant 0$, the restrictions to $L^\infty((0,a_\dagger)\times\Omega)$ of the operator $\mathbb{T}_t$, where $\mathbb{T}$ is the semigroup constructed in Section 2, are bounded on $L^\infty((0,a_\dagger)\times\Omega)$. In PDE terms, we have:

Proposition 4.2. *There exists a constant $C>0$ such that the solution of (1.1) - (1.5) satisfies*

$$\|p\|_{L^\infty((0,\tau)\times(0,a_\dagger)\times\Omega))}\leqslant C\left(\|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}+\|v\|_{L^\infty((0,\tau)\times(0,a_\dagger)\times\Omega)}\right), \tag{4.1}$$

for every $p_0\in L^\infty((0,a_\dagger)\times\Omega)$ and $v\in L^\infty((0,\tau)\times(0,a_\dagger)\times\Omega)$.

Proof. Consider the operator $\mathcal{A}$ defined in (2.6) and the semigroup $\mathbb{T}$ on $H = L^2((0,a_\dagger)\times\Omega)$ generated by $\mathcal{A}$. Integrating along the characteristic lines, we have

$$\mathbb{T}_t p_0 = \begin{cases} \dfrac{\pi(a)}{\pi(a-t)} e^{tA_2} p_0(a-t,x) & (t \leqslant a), \\ \pi(a) e^{tA_2} V(t-a,x) & (t > a), \end{cases} \qquad (p_0 \in H). \tag{4.2}$$

where $\pi(a) = \mathrm{e}^{-\int_0^a \mu(r)\mathrm{d}r}$ and $V(t,x) = \displaystyle\int_0^{a_\dagger} \beta(a)\mathbb{T}_t p_0(a,x)\,\mathrm{d}a$. Moreover, $V(t,x)$ satisfies

$$V(t,x) = \int_0^{\min\{t,a_\dagger\}} \beta(t-s)\pi(t-s)e^{tA_2}V(s,x)\,\mathrm{d}s + \int_{\min\{t,a_\dagger\}}^{a_\dagger} \beta(a)\frac{\pi(a)}{\pi(a-t)}e^{tA_2}p_0(a-t,x)\mathrm{d}a.$$

From the above expression and using Lemma 4.1 we obtain

$$\|\mathbb{T}_t p_0\|_{L^\infty((0,a_\dagger)\times\Omega)} \leqslant C\|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}, \text{ for all } t \in [0,\tau],$$

where the constant C is independent of t. Finally using Duhamel's formula one can easily obtain (4.1). □

We next show that under the assumptions in this section, the observability inequality in Theorem 3.2 can be strengthened to an inequality where the upper bound is the L^1 norm of the observation. More precisely, we have:

Proposition 4.3. *Under the assumption and with the notation in Theorem 3.2, for every $\tau > a_1 + a_\dagger - a_2$, there exists $k_\tau > 0$ such that the solution q of (3.1) satisfies*

$$\int_0^{a_\dagger} \left(\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}a \leqslant k_\tau \int_0^\tau \int_{a_1}^{a_2} \int_\omega |q(t,a,x)|\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t \qquad (q_0 \in H). \tag{4.3}$$

The proof of the above proposition is similar to that of Theorem 3.2. The main idea is the same, i.e., to use observability inequality for parabolic equations along the characteristic lines. The difference is that now we want to observe the L^1 norm of q instead of L^2 norm of q. Thus we can not use Proposition 3.3. Rather we are going to use the below observability inequality for parabolic equations, which is a slight variation of a result from Fernandez-Cara and Zuazua [7, Proposition 3.2].

Proposition 4.4. *Let $\tau > 0$, $0 \leqslant t_0 < \tau$ and $t_1 \in (t_0,\tau]$. Then for every $w_0 \in L^2(\Omega)$, the solution $w(s,x)$ of the Cauchy problem*

$$\begin{cases} \dfrac{\partial w}{\partial s}(s,x) - Lw(s,x) = 0, & (s,x) \in (t_0,T)\times\Omega, \\ \dfrac{\partial w}{\partial \nu_L} = 0, & (s,x) \in (t_0,\tau)\times\partial\Omega, \\ w(t_0,x) = w_0(x), & x \in \Omega, \end{cases} \tag{4.4}$$

satisfies the estimate

$$\int_\Omega w^2(\tau,x)\,\mathrm{d}x \leqslant \int_\Omega w^2(t_1,x)\,\mathrm{d}x \leqslant c_1 e^{c_2/(t_1-t_0)} \left(\int_{t_0}^{t_1}\int_\omega |w(s,x)|\,\mathrm{d}x\mathrm{d}s\right)^2, \tag{4.5}$$

where the constants c_1 and c_2 depend on L, on Ω and on τ.

With the help of the above proposition we obtain:

Proposition 4.5. *Let us assume the hypothesis of Theorem 1.1 and let $\tau > a_1$, $a_0 \in (0,a_b)$ and $\eta \in (a_1,\tau)$. Then there exists $C_\tau > 0$ such that, for every $q_0 \in \mathcal{D}(\mathcal{A}_0)$, the solution q of (3.1), satisfies*

(i)
$$\int_0^{a_0}\left(\int_\Omega z^2(\tau,a,x)\ \mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}a \leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega |z(t,a,x)|\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t. \tag{4.6}$$
(ii)
$$\int_\eta^{\tau}\left(\int_\Omega q^2(t,0,x)\ \mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t \leqslant C_\tau \int_0^\tau\int_{a_1}^{a_2}\int_\omega |q(t,a,x)|\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}t. \tag{4.7}$$

Proof. The proof is similar to the one of Proposition 3.4 and Proposition 3.5. The only difference is that we have to use Proposition 4.4 instead of Proposition 3.3. As the procedure is completely similar we skip the details of the proof. □

We can now prove Proposition 4.3.

Proof of Proposition 4.3. The proof of this theorem is similar to that of Theorem 3.2. Let δ and ε are defined as in the proof of Theorem 3.2. Then
$$\int_0^{a_\dagger}\left(\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}a = \int_0^{a_2-\varepsilon}\left(\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}a + \int_{a_2-\varepsilon}^{a_\dagger}\left(\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}a. \tag{4.8}$$
On the other hand, using the expression of q in (3.4) we obtain
$$\int_{a_2-\varepsilon}^{a_\dagger}\left(\int_\Omega q^2(\tau,a,x)\,\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}a \leqslant C_\tau \int_{a_1+(\delta-\varepsilon)}^{\tau}\left(\int_\Omega q^2(t,0,x)\ \mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t. \tag{4.9}$$
Finally, combining the above two estimates together with Proposition 4.5 we get (4.3). □

In the following theorem we prove the null controllability of the system (1.1) -(1.5) by means of L^∞ controls. Besides the above ingredients, we use a classical duality argument, following closely the methodology in Micu, Roventa and Tucsnak [16, Proposition 2.5].

Theorem 4.6. *With the notation and with the assumption in Theorem 1.1, for every $\tau > a_1 + a_\dagger - a_2$ and for every $p_0 \in L^\infty((0,a_\dagger)\times\Omega)$ there exists a control $v \in L^\infty((0,\tau)\times(a_1,a_2)\times\omega)$ such that the solution p of (1.1)-(1.5) satisfies*
$$p(\tau,a,x) = 0 \text{ for all } a \in (0,a_\dagger), x\in\Omega. \tag{4.10}$$
Moreover, there exists a positive constant K_τ such that for $p_0 \in L^\infty((0,a_\dagger)\times\Omega)$ the control function v and the corresponding state trajectory p satisfy
$$\|p\|_{L^\infty([0,\tau]\times(0,a_\dagger)\times\Omega)} + \|v\|_{L^\infty([0,\tau]\times(0,a_\dagger)\times\Omega)} \leqslant K_\tau \|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}. \tag{4.11}$$

Proof. We first remind the notation in Section 2, considering the pair $(\mathcal{A}, B)$, with $\mathcal{A}$ defined in (2.6) and B defined in (2.3), and denoting by $\mathbb{T}$ the C^0 semigroup on $H = L^2([0;a_\dagger]\times\Omega)$ generated by A.

Consider the subspace $\mathcal{X}$ of $L^1([0,\tau]\times[0,a_\dagger]\times\Omega)$ defined by
$$\mathcal{X} = \left\{B^*\mathbb{T}_t^* q_0 \mid q_0 \in H\right\}.$$
Given $p_0 \in L^\infty((0,a_\dagger)\times\Omega)$ consider the linear functional $\mathcal{F}$ on $\mathcal{X}$ defined by
$$\mathcal{F}\left(B^*\mathbb{T}_t^* q_0\right) = -\int_0^{a_\dagger}\int_\Omega p_0\left(\mathbb{T}_\tau^* q_0\right). \tag{4.12}$$
Using (4.3), it follows $\mathcal{F}$ is well defined and that
$$|\mathcal{F}w| \leqslant k_\tau\sqrt{|\Omega|}\|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}\|w\|_{L^1((0,\tau)\times(0,a_\dagger)\times\Omega)} \qquad (w\in\mathcal{X})$$

By the Hahn-Banach theorem, we can extend the linear functional $\mathcal{F}$ to a bounded linear functional $\widetilde{\mathcal{F}}$ on $L^1([0,\tau]\times[0,a_\dagger]\times\Omega)$ such that

$$|\mathcal{F}w| \leqslant k_\tau\sqrt{|\Omega|}\|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}\|w\|_{L^1((0,\tau)\times(0,a_\dagger)\times\Omega)}, \qquad \left(w\in L^1([0,\tau]\times[0,a_\dagger]\times\Omega)\right).$$

By the Riesz representation theorem there exists $v\in L^\infty((0,\tau)\times(a_1,a_2)\times\omega)$ such that

$$\int_0^\tau\int_0^{a_\dagger}\int_\Omega v(t-\sigma)B^*\mathbb{T}_t^*q_0+\int_0^{a_\dagger}\int_\Omega p_0\left(\mathbb{T}_\tau^*q_0\right)=0 \qquad (q_0\in H).$$

From the above formula it follows that

$$\int_0^\tau\left\langle\mathbb{T}_{\tau-s}Bv(s),q_0\right\rangle_H\,\mathrm{d}s+\left\langle\mathbb{T}_\tau p_0,q_0\right\rangle_H=0 \qquad (q_0\in H), \tag{4.13}$$

which is equivalent to

$$\mathbb{T}_\tau p_0+\int_0^\tau\mathbb{T}_{\tau-s}Bv(s)\,\mathrm{d}s=0.$$

Thus the solution p of (1.1)-(1.5) corresponding to the control v constructed above satisfies (4.10). Moreover, we have that

$$\|v\|_{L^\infty([0,\tau]\times(0,a_\dagger)\times\Omega)}\leqslant C\|p_0\|_{L^\infty((0,a_\dagger)\times\Omega)}.$$

Therefore, using the above estimate and Proposition 4.2 we obtain (4.11). □

Now we are in a position to prove our second main result.

Proof of Theorem 1.3. **Step 1.** Let $p_{s,I}$ and $p_{s,F}$ be two non negative steady states of the system (1.1) - (1.5) and let $v_{s,I}$ and $v_{s,F}$ be the corresponding steady controls. We set

$$p_{s,r}=\left(1-\frac{r}{N}\right)p_{s,I}+\frac{r}{N}p_{s,F}, \quad v_{r,k}=\left(1-\frac{r}{N}\right)v_{s,I}+\frac{r}{N}v_{s,F} \qquad (r=0,1,\cdots,N), \tag{4.14}$$

where $N\in\mathbb{N}$ will be made precise later on. Using (1.9), we have that

$$p_{s,r}(a,x)\geqslant\delta \qquad (\text{ a.e. on } [0,a_*]\times\overline{\Omega},\ r=0,1,\cdots,N). \tag{4.15}$$

Step 2. Without loss of generality let us assume that $a_2<a_*$. Let us fix $\tau_*>a_1+a_\dagger-a_2$ and consider the following control problem for $r\geqslant 1$

$$\begin{cases}\dfrac{\partial\widetilde{p}_r}{\partial t}+\dfrac{\partial\widetilde{p}_r}{\partial a}-L\widetilde{p}_r+\mu(a)\widetilde{p}_r=m\widetilde{v}_r & ((t,a,x)\in(0,\tau_*)\times(0,a_\dagger)\times\Omega),\\ \dfrac{\partial\widetilde{p}_r}{\partial\nu_L}=0 & ((a,x)\in(0,a_\dagger)\times\partial\Omega),\\ \widetilde{p}_r(t,0,x)=\displaystyle\int_0^{a_\dagger}\beta(a)p_r(t,a,x)\,\mathrm{d}a & ((t,x)\in(0,\tau_*)\times\Omega),\\ \widetilde{p}_r(0,a,x)=p_{s,r-1}(a,x)-p_{s,r}(a,x) & ((a,x)\in(0,a_\dagger)\times\Omega).\end{cases} \tag{4.16}$$

By Theorem 4.6, there exits $\widetilde{u}_r\in L^\infty((0,\tau_*)\times(a_1,a_2)\times\omega)$ such that

$$\widetilde{p}_r(\tau_*,a,x)=0 \qquad (a\in(0,a_\dagger),x\in\Omega),$$

and

$$\|\widetilde{p}_r\|_{L^\infty((0,\tau_*)\times(0,a_\dagger)\times\Omega)}\leqslant K_{\tau_*}\|p_{s,r-1}(a,x)-p_{s,r}(a,x)\|_{L^\infty((0,a_\dagger)\times\Omega)}, \tag{4.17}$$

where the constant K_{τ_*} does not depend on r. In particular, if we set

$$p_r=\widetilde{p}_r+p_{s,r}, \quad v_r=\widetilde{v}_r+v_{s,r} \qquad ((t,a,x)\in(0,\tau_*)\times(0,a_\dagger)\times\Omega), \tag{4.18}$$

then p_r satisfies the system (1.1)-(1.4) with

$$p_r(0,a,x) = p_{s,r-1}(a,x), \quad p_r(\tau_*,a,x) = p_{s,r}(a,x) \qquad ((a,x) \in (0,a_\dagger) \times \Omega, r = 1,2,\cdots,N). \tag{4.19}$$

At this point we choose N sufficiently large to have

$$\|p_{s,r-1} - p_{s,r}\|_{L^\infty((0,a_\dagger)\times\Omega)} < \frac{\delta}{K_{\tau_*}},$$

where $\delta > 0$ is the constant appearing in (1.9). Using the above relation, together with (4.15) and (4.17), for a.e. $(t,a,x) \in [0,\tau_*] \times [0,a_*] \times \Omega$ we have

$$p_r(t,a,x) \geqslant \widetilde{p}_r(t,a,x) + p_{s,r}(a,x) \geqslant 0. \tag{4.20}$$

For $(t,a,x) \in (0,\tau_*) \times (a_*,a_\dagger) \times \Omega$, we note that $p_r(t,a,x)$ satisfies

$$\begin{cases} \dfrac{\partial p_r}{\partial t} + \dfrac{\partial p_r}{\partial a} - Lp_r + \mu(a)p_r = 0 & ((t,a,x) \in (0,\tau_*) \times (a_*,a_\dagger) \times \Omega), \\ \dfrac{\partial p_r}{\partial \nu_L} = 0 & ((a,x) \in (a_*,a_\dagger) \times \partial\Omega). \end{cases} \tag{4.21}$$

Moreover,

$$p_r(0,a,x) \geqslant 0 \qquad ((a,x) \in (a_*,a_\dagger) \times \Omega),$$

and

$$p_r(t,a_*,x) \geqslant 0 \qquad ((t,x) \in (0,\tau_*) \times \Omega).$$

Therefore, by a comparison principle (see, for instance, [4, Theorem 4.1.4]) we have

$$p_r(t,a,x) \geqslant 0 \qquad ((t,a,x) \in (0,\tau_*) \times (a_*,a_\dagger) \times \Omega \text{ a.e. }).$$

Combing the above with (4.20), we obtain

$$p_r(t,a,x) \geqslant 0 \qquad ((t,a,x) \in (0,\tau_*) \times (0,a_\dagger) \times \Omega \text{ a.e. }). \tag{4.22}$$

Step 3. We define

$$p(t,a,x) = \begin{cases} p_1(t,\cdot,\cdot) & \text{if } t \in (0,\tau_*) \\ p_2(t-\tau_*,\cdot,\cdot) & \text{if } t \in (\tau_*,2\tau_*) \\ \vdots & \\ p_N(t-(N-1)\tau_*,\cdot,\cdot) & \text{if } t \in \Big((N-1)\tau_*, N\tau_*\Big) \end{cases} \tag{4.23}$$

and

$$v(t,a,x) = \begin{cases} v_1(t,\cdot,\cdot) & \text{if } t \in (0,\tau_*) \\ v_2(t-\tau_*,\cdot,\cdot) & \text{if } t \in (\tau_*,2\tau_*) \\ \vdots & \\ v_N(t-(N-1)\tau_*,\cdot,\cdot) & \text{if } t \in \Big((N-1)\tau_*, N\tau_*\Big). \end{cases} \tag{4.24}$$

Then we can easily verify that (p,v) satisfies the system (1.1) - (1.4). Moreover, the conclusion of Theorem 1.3 holds with $\tau = N\tau_*$. This completes the proof of the theorem. $\square$

5. Comments and extensions

The main result in this section gives lower bounds for the controllability time in Theorem 1.1. We show, in particular, that the controllability time in Theorem 1.1 is sharp in the case $a_1 = 0$. More precisely, we have:

Proposition 5.1. *Under the assumptions of of Theorem 1.1, let $\tau < \max\{a_1, a_\dagger - a_2\}$. Then there exists $q_0 \in \mathcal{D}(\mathcal{A}^*)$ such that, the solution q of (3.1), satisfies*

- $q(t, a, x) = 0$ *for all* $(t, a, x) \in (0, \tau) \times (a_1, a_2) \times \Omega$.
- $\displaystyle\int_0^{a_\dagger} \int_\Omega q^2(\tau, a, x) \mathrm{d}x \mathrm{d}a > 0.$

Proof. Let $(\varphi_j)_{j \geqslant 1}$ be an orthonormal basis of $L^2(\Omega)$ comprising of eigenvectors of the operator $-A_2$ and let $(\lambda_j)_{j \geqslant 1}$ be the corresponding eigenvalues. The solution q of (3.1) writes

$$q(t, a, \cdot) = \sum_{j=1}^{\infty} q^j(t, a)\varphi_j,$$

where

$$\begin{cases} \dfrac{\partial q^j}{\partial t} - \dfrac{\partial q^j}{\partial a} - \beta(a) q^j(t, 0) + (\mu(a) + \lambda_j) q^j = 0, & t \geqslant 0, a \in (0, a_\dagger), \\ q^j(t, a_\dagger) = 0, & t \geqslant 0, \\ q^j(0, a) = q_0^j(a), & a \in (0, a_\dagger) \end{cases} \tag{5.1}$$

and

$$q_0(a, \cdot) = \sum_{j=1}^{\infty} q_0^j(a)\varphi_j.$$

Integrating along the characteristic lines, we obtain the following expression of $q^j(t, a)$

$$q^j(t, a) = \begin{cases} \dfrac{\pi^j(a)}{\pi^j(a+t)} q_0^j(a+t) + \displaystyle\int_0^t \dfrac{\pi^j(a)}{\pi^j(a+t-s)} \beta(a+t-s) V^j(s)\, \mathrm{d}s, & t \leqslant a_\dagger - a, \\ \displaystyle\int_{t+a-a_\dagger}^t \dfrac{\pi^j(a)}{\pi^j(a+t-s)} \beta(a+t-s) V^j(s)\, \mathrm{d}s, & t > a_\dagger - a, \end{cases} \tag{5.2}$$

where

$$V^j(t) = q^j(t, 0) \qquad \text{and} \qquad \pi^j(a) = \exp\left(\lambda_j a + \int_0^a \mu(r)\, \mathrm{d}r\right).$$

Without loss of generality, let us assume that $a_1 > 0$.

Case 1: $\tau < a_1$

Let us choose $\bar{a} \in (\tau, a_1)$ and $\varepsilon > 0$ such that $(\bar{a} - \varepsilon, \bar{a} + \varepsilon) \subset (\tau, a_1)$. Let $q_0^1 \in C_c^\infty(0, a_\dagger)$ such that

$$q_0^1 = 0 \text{ for all } a \in [0, \tau] \cup [a_1, a_\dagger] \quad \text{and} \quad q_0^1 = 1 \text{ for all } a \in (\bar{a} - \varepsilon, \bar{a} + \varepsilon).$$

Since $\beta(a) = 0$ for $a \in [0, a_b)$, using the expression (5.2), we first have $V^1(t) = q^1(t, 0) = 0$ for all $t \in [0, \tau]$. Using this it is easy to see that

$$q^1(t, a) = \begin{cases} \dfrac{\pi^1(a)}{\pi^1(a+t)} & \text{if } a \in (\bar{a} - t - \varepsilon, \bar{a} - t + \varepsilon), t \in [0, \tau] \\ 0 & \text{if } a \geqslant a_1, t \in [0, \tau]. \end{cases}$$

Now set $q_0(a,x) = q_0^1(a)\varphi_1(x)$. Then $q(t,a,x) = 0$ for all $(t,a,x) \in (0,\tau) \times (a_1, a_\dagger) \times \Omega$. Moreover,

$$\int_0^{a_\dagger} \int_\Omega q^2(\tau,a,x)\mathrm{d}x\mathrm{d}a \geqslant \int_{\bar{a}-\tau-\varepsilon}^{\bar{a}-\tau+\varepsilon} (q^1)^2(\tau,a)\ \mathrm{d}a > 0.$$

Case 2: $a_1 \leqslant \tau < a_\dagger - a_2$

Let us choose $\bar{a} \in (a_2+\tau, a_\dagger)$ and $\varepsilon > 0$ such that $(\bar{a}-\varepsilon, \bar{a}+\varepsilon) \subset (a_2+\tau, a_\dagger)$. Let $q_0^1 \in C_c^\infty(0,a_\dagger)$ such that

$$q_0^1 = 0 \text{ for all } a \in [0, a_2+\tau] \quad \text{ and } \quad q_0^1 = 1 \text{ for all } a \in (\bar{a}-\varepsilon, \bar{a}+\varepsilon).$$

As $\beta(a) = 0$, for $a \in [0, a_b)$, using the expression (5.2) we obtain

$$V^1(t,0) = 0 \text{ for all } t \in [0, a_b).$$

Using the above identity and (5.2), we can easily obtain that

$$q^1(t,a) = 0 \text{ for all } t \in (0,a_b), a \in (0, a_2+\tau-t),$$

and

$$q^1(a_b,a) = 0 \text{ for all } a \in [0, a_2+\tau-a_b] \cup [a_\dagger - a_b, a_\dagger]$$
$$q^1(a_b,a) = \frac{\pi^1(a)}{\pi^1(a+t)} \text{ for all } a \in (\bar{a}-a_b-\varepsilon, \bar{a}-a_b+\varepsilon).$$

Next we can calculate q^1 for $(t,a) \in (a_b, 2a_b) \times (0,a_\dagger)$ with $q^1(a_b,\cdot)$ as initial data. Continuing this process, we obtain

$$q^1(t,a) = 0 \text{ for all } t \in (0,\tau), a \in (0, a_2+\tau-t),$$

and

$$q^1(\tau,a) = \frac{\pi^1(a)}{\pi^1(a+t)} \text{ for all } a \in (\bar{a}-\tau-\varepsilon, \bar{a}-\tau+\varepsilon).$$

Then we can proceed as Case 1 to conclude the proof of the proposition. □

As a consequence of the above proposition, the following theorem follows easily:

Theorem 5.2. *Under the assumption of Theorem 1.1, the system (1.1)-(1.5) is not null-controllable in time* $\tau < \max\{a_1, a_\dagger - a_2\}$.

Remark 5.3. *The above theorem shows that the controllability time in Theorem 1.1 is sharp in the case* $a_1 = 0$.

Before ending the paper, we describe some possible extensions to be considered in future work.

First, still in the case $a_1 = 0$, it would be interesting to make precise the dependence of the control cost on a_2. This could allow the extension to the diffusive case of the singular perturbation result obtained in [10], which describes the behaviour of the control problem when $a_2 \to 0$ (direct birth control) in the diffusion free case. Moreover, let us note that the methods in this work are easily adaptable to the case when the mortality and fertility rates depend on the spatial variable x, whereas their adaptation to time dependent mortality and fertility rates seems a more difficult question.

Other possible directions for future extensions of the results and methods in this work concern nonlinear problems (such as considering, for instance, mortality rates depending on the total populations), controllability issues for systems involving competing species or feedback control problems.

Acknowledgement: We would like to thank Nicolas Hegoburu for fruitful discussions which help us to improve the controllability time.

References

[1] B. Ainseba, *Exact and approximate controllability of the age and space population dynamics structured model*, J. Math. Anal. Appl., 275 (2002), pp. 562–574.

[2] B. Ainseba and S. Aniţa, *Local exact controllability of the age-dependent population dynamics with diffusion*, Abstr. Appl. Anal., 6 (2001), pp. 357–368.

[3] B. Ainseba and S. Aniţa, *Internal exact controllability of the linear population dynamics with diffusion*, Electron. J. Differential Equations, (2004), pp. No. 112, 11.

[4] S. Ani¸ta, *Analysis and control of age-dependent population dynamics*, vol. 11 of Mathematical Modelling: Theory and Applications, Kluwer Academic Publishers, Dordrecht, 2000.

[5] V. Barbu, M. Iannelli, and M. Martcheva, *On the controllability of the Lotka-McKendrick model of population dynamics*, J. Math. Anal. Appl., 253 (2001), pp. 142–165.

[6] D. Daners, *Heat kernel estimates for operators with boundary conditions*, Mathematische Nachrichten, 217 (2000), pp. 13–42.

[7] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 17 (2000), pp. 583–616.

[8] A. V. Fursikov and O. Y. Imanuvilov, *Controllability of evolution equations*, vol. 34 of Lecture Notes Series, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[9] M. E. Gurtin, *A system of equations for age-dependent population diffusion*, Journal of Theoretical Biology, 40 (1973), pp. 389–392.

[10] N. Hegoburu, P. Magal, and M. Tucsnak, *Controllability with positivity constraints of the lotka–mckendrick system*, SIAM Journal on Control and Optimization, 56 (2018), pp. 723–750.

[11] N. Hegoburu and M. Tucsnak, *Null controllability of the Lotka-McKendrick system with spatial diffusion*. working paper or preprint, Nov. 2017.

[12] W. Huyer, *Semigroup formulation and approximation of a linear age-dependent population problem with spatial diffusion*, Semigroup Forum, 49 (1994), pp. 99–114.

[13] J. Lohéac, E. Trélat, and E. Zuazua, *Minimal controllability time for the heat equation under unilateral state or control constraints*, Math. Models Methods Appl. Sci., 27 (2017), pp. 1587–1644.

[14] D. Maity, *On the Null Controllability of the Lotka-Mckendrick System*. working paper or preprint, May 2018.

[15] P. Martinez, J.-P. Raymond, and J. Vancostenoble, *Regional null controllability of a linearized Crocco-type equation*, SIAM J. Control Optim., 42 (2003), pp. 709–728.

[16] S. Micu, I. Roventa, and M. Tucsnak, *Time optimal boundary controls for the heat equation*, Journal of Functional Analysis, 263 (2012), pp. 25–49.

[17] E. M. Ouhabaz, *Analysis of heat equations on domains*, vol. 31 of London Mathematical Society Monographs Series, Princeton University Press, Princeton, NJ, 2005.

[18] O. Traore, *Null controllability of a nonlinear population dynamics problem*, Int. J. Math. Math. Sci., (2006), pp. Art. ID 49279, 20.

[19] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009.

[20] G. F. Webb, *Theory of nonlinear age-dependent population dynamics*, vol. 89 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1985.

Institut de Mathématiques, Université de Bordeaux, Bordeaux INP, CNRS F-33400 Talence, France
E-mail address: debayan.maity@u-bordeaux.fr

Institut de Mathématiques, Université de Bordeaux, Bordeaux INP, CNRS F-33400 Talence, France
E-mail address: marius.tucsnak@u-bordeaux.fr

(1) DeustoTech, Fundación Deusto, Avda. Universidades, 24, 48007, Bilbao, Basque Country, Spain., (2) Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain, (3) Facultad de Ingeniería, Universidad de Deusto, Avda. Universidades, 24, 48007, Bilbao, Basque Country, Spain
E-mail address: enrique.zuazua@deusto.es