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Familias de sistemas ortonormales. Transferencia de acotación de operadores.

Memoria presentada para optar al grado de

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por

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ماغاب عن ذاكرة المكان يوماً، ولا أهملتْ جغرافيا الأرض لحظه، أنّ سُنّة الله قد قضت بأنّ المعذّبين مثلي لن ينقذهم الهروب، ولن يُبْعدهم عن عذابهم ألإرتحال أو إختراق الحدود، فما غفلت متوالية العمرِ-ولو مرّه- عن أن يكون المرُ أكبر طياً من طيّاتِها والصعبُ الأكثر قرباً للمستحيل سِدرةَ مُنتهاها. غير أنّكِ كنت العزاء يا أُمي، وزاد المُتعبين ومُنقطعي السَّبيل امثالي، وكنت السَّندَ والعزوَّه يا أُختي، غيرَ أنّكنَّ رحلتُنَّ، فرحلَ معكُنَّ العزاء، وغاب عن الذَّائقه مذاقُ الأمل. لكنِّي الله أسأل أن يُسكِنَكنَّ الفردوس الأعلى وأن لا يقطعَ بي إليكُنَّ سبيلَ الدُّعاء.

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Capítulo 1

Introducción

1.1. Organización de esta memoria

Esta memoria se presenta para optar al grado de Doctor en Ciencias Matemáticas y como tal contiene los resultados obtenidos durante los estudios de doctorado. Estos estudios comenzaron en la primavera de 2002. Claramente el trabajo se ha extendido en el tiempo mucho más de lo razonable. En nuestro caso y después de tantos años, en ocasiones hemos vuelto sobre alguno de los temas investigados al comienzo y en otras hemos analizado más profundamente los logros obtenidos. Para dar una visión más clara de cómo ha sido el trabajo a lo largo de este período hemos decidido presentar una memoria con los siguientes contenidos. Los Capítulos 3, 5 y 7 reflejan exactamente los artículos ya publicados [2], [1] y [3]. El Capítulo 4 contiene una parte nueva que hemos trabajado en los últimos meses. El Capítulo 6 presenta unas figuras aclaratorias del Capítulo 5, mientras que el Capítulo 2 da una visión rápida de los operadores manejados a lo largo de la memoria. Evidentemente esta distribución produce que algunos capítulos estén en inglés y (tal vez un poco más molesto para el lector) que las notaciones no sean uniformes a lo largo de la memoria, pero a cambio creemos que esta presentación refleja bien el trabajo realizado.

1.2. Motivación

Desde la primavera de 2002 han ocurrido muchas cosas en el mundo, en las matemáticas y en la pequeña parcela de ellas que desarrollamos en esta memoria. En aquel momento había un gran volumen de trabajo dedicado a la investigación de resultados y conceptos del Análisis Armónico clásico en el campo de los sistemas ortogonales clásicos. Este tipo de investigación tenía sus antecedentes remotos en artículos de B. Muckenhoupt [63], [64], [65] y B.Muckenhoupt y E. Stein [67]. Estos trabajos fueron publicados a finales de los años 60 del siglo pasado. La idea subyacente era analizar el sustituto de serie conjugada del análisis armónico clásico y estudiar el comportamiento en los espacios L^p. En este sentido las ecuaciones de Cauchy-Riemann y los análogos de algunas propiedades de las funciones armónicas también fueron tratados. Si nos situamos en 2003, el antecedente más cercano en Análisis se encontraba alrededor de la figura de E. Fabes. A comienzos de los años 90, el profesor Fabes dirigió dos tesis doctorales (R.Scotto y W. Urbina) en las que se analizaban las "transformadas de Riesz asociadas a la medida gaussiana", ver [96] y [29]. En ambos trabajos se definían unos operadores llamados "transformadas de Riesz", y se probaban acotaciones paralelas a las transformadas de Riesz clásicas (tipo fuerte (p,p), $p>1$ y débil $(1,1)$). Sin embargo en ningún momento se explicaba la razón del nombre. Junto con los trabajos citados hay que recordar las contribuciones (en los años 80) de P.A. Meyer, [60], R. Gundy [34] y G.Pisier [76] que con pruebas probabilísticas o de transferencia probaron la acotación en $L^p, 1<p<\infty$ de las "transformadas de Riesz gaussianas". Sería injusto no citar aquí que la teoría de sistemas (polinomios) ortogonales tuvo una vida muy activa a lo largo del siglo pasado y que problemas como: convergencia de series, multiplicadores de Fourier, localizaciones de ceros, fórmulas de sumación, fórmulas de recurrencia, etc constituyen todo un área dentro del análisis matemático moderno.

En nuestra opinión las tesis dirigidas por el profesor Fabes fueron un revulsivo dentro del mundillo de los expertos en Análisis Armónico y comenzó un gran florecimiento de un Análisis de Fourier asociado a laplacianos generales. Además se contó con la ayuda inestimable del libro de E. Stein "Topics in Harmonic Analysis Related to the Littlewood-Paley Theory", ver [84]. Este libro apareció en 1970 y ha sido una guía esencial para un gran número de profesionales. A modo de ejemplo las transformadas de Riesz de un operador $\mathcal{L}$ se definen (Stein las define) como

$$R_j = X_j(\mathcal{L})^{-1/2}, \quad 1 \le j \le n. \tag{1.1}$$

Siendo X_j operadores diferenciales de primer orden tales que $\mathcal{L} = \sum_{j=1}^n X_j^* X_j$. $(\mathcal{L})^{-1/2}$ denota la integral fraccionaria del operador $\mathcal{L}$, que puede definirse utilizando el núcleo del calor, ver la Sección 2.2 del Capítulo 2.

Observemos que la definición de transformadas de Riesz contenida en la fórmula (1.1) explica la importancia de estudiar acotaciones $L^p(\Omega)$ de estos operadores. Así por ejemplo la acotación en $L^p(\Omega, d\mu)$ de las (habitualmente llamadas) transformadas de Riesz de segundo orden, $R_{i,j} = X_i X_j \mathcal{L}^{-1}$, proporcionan acotaciones "a priori" en $L^p(\Omega)$ del problema

$$(*) \begin{cases} \mathcal{L}u = f & \text{en} \quad \Omega, \\ \text{con dato } f & \text{en} \quad L^p(\Omega). \end{cases}$$

En efecto, como $u = (\mathcal{L})^{-1} f$, podremos escribir $X_i X_j u = X_i X_j (\mathcal{L})^{-1} f = R_{i,j} f$. Por tanto la acotación en L^p de los operadores $R_{i,j}$ garantiza que la solución u tiene dos derivadas en $L^p(\Omega)$. De un modo análogo puede verse que para estudiar problemas de regularidad (estimaciones de Schauder), la acotación en clases C^α de las transformadas de Riesz y de la integral fraccionaria son esenciales.

Por otra parte si se considera el problema de extensión

$$(**) \begin{cases} (\partial_t + \mathcal{L})u = 0 & \text{en} \quad \Omega \times [0, \infty), \\ u(x, 0) = f(x) & f \in L^p(\Omega). \end{cases}$$

Aparece de manera natural el **operador maximal** $\sup_t |u(x,t)|$ como herramienta para el estudio de la convergencia al dato inicial de la solución de la ecuación $(\partial_t + \mathcal{L})u = 0$.

Intentemos dar una idea general que ha guiado a muchos trabajos relacionados con este Análisis Armónico asociado a laplacianos generales. Como se puede ver en la Sección 2.2 del Capítulo 2, los operadores que se estudian pueden obtenerse a partir del semigrupo del calor, sin más que utilizar ciertas fórmulas relacionadas con la función Γ. En particular los núcleos de los diferentes operadores también pueden obtenerse utilizando estas fórmulas. Por otro lado distintas apelaciones a teoremas generales producen la acotación en algún p del operador que se esté tratando. Así la acotación en L^p de operadores maximales de semigrupos satisfaciendo ciertas condiciones (semigrupos markovianos de difusión) se sigue del teorema maximal de E.Stein (ver [84]), mientras que la acotación (en L^2) de las transformadas de Riesz se sigue o bien del teorema espectral o bien de la ortogonalidad del sistema de autofunciones del operador. Una referencia esencial en la que se ve muy bien las ideas anteriores es la monografía de S. Thangavelu [94]. Sin querer (ni poder) ser exhaustivos podemos citar como trabajos en la línea de investigación que estamos describiendo a los siguientes [82], [69], [57], [56], [73], [74], [59], [38] , [37], [41], [89], [95], [36], [88], [71], [39], [53],[54], [21], [13],[16],[12],[11], [14], [15], [28], [27], [26].

En la mayoría de los casos, una vez que se conoce la acotación en algún $p, 1 \leq p < \infty$, se intenta determinar el núcleo del operador, a veces con fórmulas que involucran integrales. Obtenidos los núcleos se pone en marcha una maquinaria de análisis real que suele ser muy técnica y con gran complejidad de cálculo. La frecuente aparición de funciones especiales conduce a tener casi siempre como referencias básicas los tratados [92], [52] y [9].

La complejidad técnica de la maquinaria utilizada hace que sean muy apreciados los llamados "teoremas de transferencia" que permiten asegurar la validez de un resultado para operadores asociados a un laplaciano generalizado, sabiendo el resultado para operadores asociados a otro. El intento de probar resultados de transferencia entre sistemas ortogonales es muy antiguo. En muchos casos los teoremas se refieren a resultados sobre series en las cuales un sistema reemplaza a otro y se mantienen sus coeficientes. A este tipo de resultados se les denomina teoremas de "transplantación". En la literatura pueden encontrarse muchos trabajos en esta línea de pensamiento, por ejemplo [32], [31], [66], [6], [5], [8], [7] y más recientemente (nuevamente sin poder ser exhaustivos) [33], [30],[23],[62],[72],[47],[22],[49],[46],[10],[61],[86], [50],[48],[20].

Cuando comenzamos esta investigación se acababan de publicar resultados sobre el operador de Hermite ([89], [87])

$$\mathcal{H} = -\Delta + |x|^2,$$

así como sobre el operador de Ornstein-Uhlenbeck ([82], [40],[40],[42])

$$\mathcal{O} = -\Delta + 2x.\nabla.$$

A pesar de que ambos operadores están muy relacionados, ver la sección 2.1.1 en el Capítulo 2, en ningún momento se planteaba la posibilidad de paso de uno a otro. La misma situación se daba con los sistemas de Laguerre, con respecto a los artículos [94] [85] y [36]. Nuestra primera motivación fue explotar la relación existente entre los anteriores sistemas.

1.3. Resultados

Cuando comenzamos con este trabajo, se conocían dos resultados para el operador de Ornstein Uhlenbeck :

(a) El primero de ellos, contenido en [40], caracterizaba los pesos v para los cuales el operador maximal del semigrupo del calor $\sup_{t>0} e^{-t\mathcal{O}}$ era finito en casi todo punto para toda función en $L^p(v(x)d\gamma(x))$, $1 < p < \infty$.

(b) El segundo, en [42], caracterizaba los espacios de Banach B para los cuales la transformada de Riesz $\partial_{x_i}(\mathcal{O})^{-1/2}$ era acotada de $L^p_B(d\gamma)$ en sí mismo, para $1 < p < \infty$ (o de tipo débil $(1,1)$).

Ya hemos destacado antes la importancia de las acotaciones de las transformadas de Riesz, así como el papel fundamental que juega el semigrupo del calor. Entonces nuestra primera pretensión fue obtener los resultados paralelos a los (a) y (b) anteriores en el contexto de el operador de Hermite, $\mathcal{H}_n = -\Delta + |x|^2$. La relación natural que liga a los polinomios y funciones de Hermite, puede formularse mediante una isometría U que se propaga a los operadores, ver Sección 2.3.1, Capítulo 2. Sin embargo esta propagación no es exacta ya que dado un polinomio, f, se verifica

$$U^{-1}e^{-t(\mathcal{H}_n - n)}Uf(x) = e^{-t\mathcal{O}}f(x).$$

Esto hizo que los resultados que se obtuvieron en aquel momento o bien no fueran completamente satisfactorios, como es el caso del Teorema 3.2.1 en el Capítulo 3 o bien precisasen un trabajo extra de comparación como es el caso del Teorema 3.2.3 que necesitó la Proposición 3.3.3 (ambos en el Capítulo 3). Esta investigación fue publicada en el año 2006 en el Glasgow Mathematical Journal número 48, [2]. Como ya hemos explicado antes, dicho artículo es el contenido exacto del Capítulo 3, salvo la bibliografía que se ha incluido en la general de esta memoria.

Con el paso del tiempo el Teorema 3.2.1 de dicho capítulo nos empezó a resultar especialmente frustrante. Si se presenta solo la parte que se refiere a los semigrupos de $\mathcal{H}$, el resultado es el siguiente:

Teorema 1.1. Sea v un peso (función positiva y finita en casi todo punto) tal que satisface $\int_{\mathbb{R}^d} v^{-1}(x)e^{-|x|^2}dx < \infty$. Entonces $\sup_{t>0} e^{-t\mathcal{H}_n}$ es acotado de $L^2(\mathbb{R}^d, v(x)dx)$ en $L^2(\mathbb{R}^d, u(x)dx)$ para algún peso u.

Es decir, la condición sobre el peso era suficiente, pero no se sabía si era necesaria. En particular, utilizando las técnicas habituales de convergencia en casi todo punto, este teorema daba una condición suficiente sobre un peso v para que la solución $e^{-t\mathcal{H}}f$ de la ecuación (del calor) $\partial_t + \mathcal{H} = 0$ tuviese convergencia al dato inicial en casi todo punto y para toda función de $L^2(\mathbb{R}^n, v)$. Obviamente aquí quedaba un problema abierto muy interesante, caracterizar las clases de pesos para las cuales hay convergencia al dato. Por supuesto debería de poner probarse para L^p con $p \neq 2$

Un análisis naíf del problema $\lim_{t\to 0} e^{-t\mathcal{H}}f(x)$ llevaría a la conclusión de que el conjunto de funciones para el cual hay convergencia a.e. x, debería ser el mismo que para el problema $\lim_{t\to 0} e^{t\Delta}f(x)$. Esto parece claro teniendo en cuenta que localmente los operadores $e^{t\Delta}$ y $e^{-t\mathcal{H}}$ tienen un comportamiento similar. Lo que ocurría hasta hace unos meses es que la caracterización de la clase $L^p(v)$ de funciones para la cual se de la convergencia en casi todo punto del operador $e^{t\Delta}$ no existía. Afortunadamente motivados por ciertos problemas de oscilación y variación de operadores los autores en [43] han conseguido caracterizar dicha clase de pesos, ver Teorema 4.6 en el Capítulo 4. Es bien conocido que el semigrupo del calor de Hermite está acotado por el semigrupo del calor clásico. El recíproco no es cierto, pero nosotros (conjuntamente con P.Stinga y J.L. Torrea) hemos conseguido encontrar un recíproco débil suficiente para poder concluir propiedades para $e^{-t\mathcal{O}}$, ver el Lema 4.5 en el Capítulo 4. Más aún, con cierta sorpresa descubrimos que utilizando la isomería (en L^2) U anteriormente citada (Sección 2.3.1) podíamos caracterizar el espacio de convergencia para el caso de Ornstein-Uhlenbeck incluso en $L^p(v(x)\gamma(x)dx)$. El resultado está contenido en los Teoremas 4.2 y 4.3 del Capítulo 4, pero la parte esencial podríamos enunciarla como sigue.

Teorema 1.2. Sea v un peso (función estrictamente positiva y finita en casi todo punto) en $\mathbb{R}^n$ y $1 \le p < \infty$. Dado R, $0 < R < \infty$, consideramos los operadores $\mathcal{O}^*_R f(x) = \sup_{t<R} |e^{-t\mathcal{O}}f(x)|$ y $\mathcal{H}^*_R f(x) = \sup_{t<R} |e^{-t\mathcal{H}}f(x)|$ Las siguientes afirmaciones son equivalentes:

(1) Existe $0 < R < \infty$ tal que

$$e^{-R\mathcal{H}} f(x) < \infty, \ a.e.x$$

y el límite $\lim_{t\to 0} e^{-t\mathcal{H}} f(x)$ existe a.e. x para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(2) Existe $0 < R < \infty$ tal que

$$e^{-R\mathcal{O}} f(x) < \infty, \ a.e.x$$

y el límite $\lim_{t\to 0} e^{-t\mathcal{O}} f(x)$ existe a.e. x para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(3) Existe $0 < R < \infty$ tal que

$$\mathcal{O}^*_R f(x) < \infty,$$

a.e. x, para toda $f \in L^p(\mathbb{R}^n, v(x)\gamma(x)dx)$.

(4) Existe $0 < R < \infty$ tal que

$$\mathcal{H}^*_R f(x) < \infty,$$

a.e. x, para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(5) El peso $v \in D_p^W$. Ver la Definición 4.1 en el Capítulo 4.

Para el caso de las funciones de Laguerre, se podían nuevamente encontrar resultados diversos para sistemas particulares, tanto de funciones como de polinomios, ese es el caso de [36] (para polinomios) y [53], [54], [90], [94] para funciones. En algunos artículos se planteaba de manera superficial el problema de obtener resultados para un sistema conocidos para otros, puede verse el libro [94] donde se hacen débiles alusiones a este tipo de transferencia . En este sentido uno de los resultados mas conocidos era el Teorema de transplantación de Kanjin [45] que trabajaba dentro de los sistemas $\mathcal{L}_k^\alpha$. Sin embargo no había ningún resultado satisfactorio que permitiera pasar de sistema a sistema. En algunos artículos de hecho se hacían los cálculos para cada sistema, ver por ejemplo [85], [90]. Nos propusimos hacer una clarificación de la situación y en colaboración con los profesores R.Macías y C. Segovia (fallecido el 3 de Abril de 2007) publicamos el trabajo [1]. De dicho trabajo queremos destacar tres aspectos:

(1) Es suficiente conocer las acotaciones en $L^p(y^\delta, dy)$, de un operador asociado al semigrupo, para uno de los sistemas de Laguerre. Para el resto se obtienen a través de los cambios de variable desarrollados la Sección 2.1.2 en el Capítulo 2. Este es esencialmente el contenido del Teorema 5.10 del Capítulo 5.

(2) Resultados para funciones de Laguerre también tienen consecuencias para resultados sobre polinomios de Laguerre para el caso L^2. Este es el contenido del Teorema 5.13.

(3) A la vista de los dibujos presentados en aquel momento (ver el Capítulo 5), da la sensación que el comportamiento de los semigrupos es muy diferente para los sistemas $\{\mathcal{L}_k^\alpha\}$ y $\{\varphi_k^\alpha\}$ que para los sistemas $\{\psi_k^\alpha\}$ y $\{\ell_k^\alpha\}$. En particular el comportamiento anómalo de existencia de un intervalo dentro de $(1, \infty)$ para el cual no hay acotación fuerte (p, p), parecería darse solamente para los dos primeros sistemas.

Respecto del punto (1). Parece que las isometrías presentadas en la Sección 2.3 del Capítulo 2 solamente puede ser útiles para el estudio de acotaciones entre espacios L^p con pesos potencia y además solamente dentro del contexto de las funciones de Laguerre (para los polinomios parece ser posible una dirección pero no la contraria). Sin embargo recientemente, [91], estas isometrías han sido utilizadas con éxito para probar desigualdades de Harnack para la potencias fraccionarias de operadores de Laguerre. En este caso la idea de transferir resultados se ha llevado a cabo no sólo entre las funciones de Laguerre si no también entre los polinomios.

Respecto del punto (3). Las figuras que se presentan en el artículo tienen como variables: ordenada $1/p$ y abscisa α, además la potencia del peso es $\delta = 0$. Para visualizar mejor cúal es el comportamiento del semigrupo en el caso $\delta \neq 0$ es mejor presentar figuras en las cuales la ordenada es p y la abscisa es α y δ va tomando distintos valores. Más aún, enseguida se ve que la situación de tener un intervalo de p para el cual el semigrupo no está acotado aparece en todos los sistemas. Ver el Capítulo 6 en el cúal se presentan además las regiones de acotación con secciones verticales y en tres dimensiones. Gracias a la ayuda inestimable (y desinteresada) de Pablo Angulo hemos podido utilizar el software libre SAGE para los bocetos de las figuras.

1.4. Operador de Schrödinger

Nuestra primera pretensión fue caracterizar los espacios de Banach X para los cuales la función $g^{\mathcal{H},q}$ definida en (7.2) en el Capítulo 7 era acotada de $L^p_X(\mathbb{R}^d, dx)$ en $L^p(\mathbb{R}^d, dx)$, así como de $BMO_{\mathcal{H},X}$ en $BMO_{\mathcal{H}}$. Es decir para el operador de Hermite. Dado que para el operador de Ornstein-Uhlenbeck el resultado ya se conocía en los espacios de Lebesgue $L^p(\mathbb{R}^n, \gamma(x)dx)$. La idea era nuevamente utilizar la isometría U definida en (7.5) del Capítulo 2. Después de un tiempo nos dimos cuenta que probablemente era más efectivo atacar el problema para el caso más general de operadores de Scrödinger. Los resultados, recogidos en el Capítulo 7, constituyen un trabajo en colaboración con P.Stinga y J.L. Torrea en Studia Mathematica.

En el año 1995, Z. Shen publica su celebrado trabajo [78]. Con el paso de los años, esta publicación se ha convertido en una referencia esencial en cualquier trabajo sobre el operador de Scrödinger (independiente del tiempo) $\mathcal{L} = -\Delta + V$ donde V es un potencial positivo satisfaciendo una cierta desigualdad de Hölder inversa, ver (7.3) en el Capítulo 7. El propio Shen publicó otros trabajos sobre este operador, ver [79],[80].

Pero además un grupo muy numeroso de investigadores comenzaron a trabajar en una gran variedad de problemas alrededor de este operador. Como ya hemos dicho, nuestra meta era caracterizar los espacios de Banach X para los cuales la función $g^{\mathcal{L},q}$ definida en (7.2) en el Capítulo 7 era acotada de $L^p_X(\mathbb{R}^d, dx)$ en $L^p(\mathbb{R}^d, dx)$, así como de $BMO_{\mathcal{L},X}$ en $BMO_{\mathcal{L}}$. Donde $BMO_{\mathcal{L},X}$ es el espacio BMO asociado al operador $\mathcal{L}$, ver la definición en la sección 7.2 del Capítulo 7. El resultado esperado era que el espacio de Banach satisfaciese la misma condición que para el caso del Laplaciano clásico, es decir que X admitiese una norma equivalente para la cual el espacio fuese $q-$uniformemte convexo. El resultado para operadores que generasen semigrupos de difusión markovianos había sido probado en [55]. Probablemente este resultado puede demostrarse de varias maneras diferentes. Es evidente que todas las razonables pasan por obtener acotaciones de la función $g^{\mathcal{L},q}$ a partir de la $g^{\Delta,q}$ y viceversa. Dentro de la línea general del trabajo de tesis, nuestro interés era dar una prueba que contuviese ideas unificadoras y que pudiesen ser utilizadas para otros operadores. La demostración que encontramos es en efecto muy general y aplicable a otros operadores asociados a $\mathcal{L}$. Puede resumirse en el siguiente Teorema (Remarks 7.13, 7.15 del Capítulo 7).

Teorema 1.3. Sean dos espacios de Banach X_1 X_2. Sea T un operador lineal que envía $C_c^\infty(\mathbb{R}^d; X_1)$ en el espacio de las funciones fuertemente medibles con valores en X_2. Supongamos que T tiene un núcleo que satisface las condiciones de Calderón-Zygmund. Se define el operador "local"

$$T_{\text{loc}} f(x) = T\left(\chi_N(x,\cdot) f(\cdot)\right)(x), \quad x \in \mathbb{R}^d,$$

donde N es la región $|x-y| \leq \rho(x)$ y ρ es la función auxiliar definida en (7.4) Capítulo 7. Entonces se tienen los siguientes resultados:

(1) Si T se extiende acotadamente de $L^p_{X_1}(\mathbb{R}^d)$ en $L^p_{X_2}(\mathbb{R}^d)$ para algún p, $1 < p < \infty$. Entonces T_{loc} se extiende acotadamente de $L^p_{X_1}(\mathbb{R}^d)$ en $L^p_{X_2}(\mathbb{R}^d)$.

(2) Si T se extiende acotadamente de $L^1_{X_1}(\mathbb{R}^d)$ en $L^1_{X_2}(\mathbb{R}^d)$-débil. Entonces T_{loc} se extiende acotadamente de $L^1_{X_1}(\mathbb{R}^d)$ en $L^1_{X_2}(\mathbb{R}^d)$-débil.

(3) Si $\|Tf(x)\|_{X_2} < \infty$ c.t.p. $x \in \mathbb{R}^d$ para toda $f \in L^1_{X_1}(\mathbb{R}^d)$. Lo mismo es cierto en el caso T_{loc}.

(4) Supongamos que T es acotado de $L^p_{X_1}(\mathbb{R}^d)$ en $L^p_{X_2}(\mathbb{R}^d)$ para algún p, $1 < p < \infty$. Además $T1$ puede definirse y $T1 = 0$ Entonces

- T_{loc} es acotado de $BMO_{\mathcal{L},X_1}$ en $BMO_{\mathcal{L},X_2}$, y
- T_{loc} es acotado de $H^1_{\mathcal{L},X_1}$ en $L^1_{X_2}(\mathbb{R}^d)$.

Es decir el operador T_{loc} se comporta como un operador asociado a $\mathcal{L}$.

El Teorema anterior marca la pauta de la estrategia a seguir para transferir resultados de operadores ligados a $\mathcal{L}$ a operadores ligados a Δ. Dado un operador $T^{\mathcal{L}}$ asociado a $\mathcal{L}$ y su análogo T^{Δ} asociado a Δ, se consideran sus partes "local" T_{loc} y "global" T_{glob} (esencialmente $T - T_{\text{loc}}$) . Con el Teorema anterior como herramienta, es claro que sólo necesitaremos buenas acotaciones de los operadores $T^{\mathcal{L}} - T^{\mathcal{L}}_{\text{glob}}$, $T^{\Delta} - T^{\Delta}_{\text{glob}}$ y $T^{\mathcal{L}}_{\text{loc}} - T^{\Delta}_{\text{loc}}$. Estas acotaciones están desarrolladas en el Capítulo 7.

Capítulo 2

Preliminares

2.1. Operadores diferenciales considerados en esta memoria

2.1.1. Polinomios y funciones de Hermite

Operador de Ornstein-Uhlenbeck. Polinomios de Hermite.

Los polinomios de Hermite en una variable pueden definirse mediante la siguiente fórmula de Rodrigues

$$H_k(s) = (-1)^k \, e^{s^2} \frac{d^k e^{-s^2}}{ds^k}, \quad k = 0, 1, 2, \dots, \quad s \in \mathbb{R}.$$

Es sencillo ver que los primeros polinomios de Hermite son

$$H_0(s) = 1, \quad H_1(x) = 2s, \quad H_2(s) = 4s^2 - 2,$$

y en general $H_n(s) = \displaystyle\sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2s)^{n-2k}$, donde $[q]$ denota la parte entera de q. Los polinomios de Hermite son ortogonales con respecto a la medida $d\gamma(s) = \pi^{-1/2} e^{-s^2} ds$. Puede comprobarse que

$$\int_{-\infty}^{\infty} H_n(s) H_m(s) \pi^{-1/2} e^{-s^2} ds = 2^n n! \, \delta_{n,m}.$$

Además son soluciones particulares de la ecuación

$$u'' - 2xu' + 2nu = 0.$$

Dado un multi-índice $\alpha = (\alpha_1, \ldots, \alpha_n)$ el polinomio de Hermite $H_\alpha(x)$, se define como

$$H_\alpha(x) = \Pi_{j=1}^n H_{\alpha_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Utilizando las observaciones anteriores en dimensión uno, es fácil deducir que los polinomios de Hermite normalizados $\tilde{H}_\alpha(x) = (2^{|\alpha|}|\alpha|!)^{-n/2} H_\alpha(x)$ forman un sistema ortonormal completo en $\mathbb{R}^n$ con respecto a la medida $d\gamma(x) = \pi^{-n/2} e^{-|x|^2}$ y además satisfacen la ecuación

$$-\Delta H_\alpha + 2x \cdot \nabla H_\alpha = 2|\alpha| H_\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = (\alpha_1, \ldots . \alpha_n). \tag{2.1}$$

Al operador $-\Delta + 2x \cdot \nabla$ (operador de Ornstein-Uhlenbeck) lo denotaremos a lo largo de este Capítulo por $\mathcal{O}$.

Operador de Hermite (Oscilador Harmónico). Funciones de Hermite

Dado un multi-índice $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotaremos por h_α y la denominaremos "función de hermite" a la función

$$h_\alpha(x) = \pi^{-d/4} e^{-|x|^2/2} H_\alpha(x), \tag{2.2}$$

donde $H_\alpha(x)$ es el correspondiente polinomio de Hermite. Utilizando las observaciones de la subsección anterior es fácil deducir que las funciones $\tilde{h}_\alpha(x) = e^{-|x|^2/2} \tilde{H}_\alpha(x)$ son un sistema ortonormal completo en $L^2(\mathbb{R}^n, dx)$ y además satisfacen la ecuación diferencial

$$-\Delta h_\alpha + |x|^2 h_\alpha = (2|\alpha| + n) h_\alpha.$$

Al operador $-\Delta + |x|^2$ (oscilador armónico) lo denotaremos por $\mathcal{H}_n$, o simplemente $\mathcal{H}$ cuando no haya lugar a confusión.

2.1.2. Polinomios y funciones de Laguerre

Polinomios de Laguerre

Los polinomios de Laguerre de orden $\alpha > -1$ pueden definirse mediante la fórmula de Rodrigues

$$L_k^\alpha(y) = \frac{1}{k!} e^y y^{-\alpha} \frac{d^k}{dy^k}(e^{-y} y^{k+\alpha}), \quad y > 0.$$

Constituyen un sistema ortogonal completo en $L^2((0, \infty), d\gamma_\alpha(y))$ con $d\gamma_\alpha(y) = y^\alpha e^{-y} dy$ y satisfcen la ecuación differencial

$$-y \frac{d^2}{dy^2} L_k^\alpha - (\alpha + 1 - y) \frac{d}{dy} L_k^\alpha = k L_k^\alpha, \quad \alpha > -1.$$

Al operador $-y \frac{d^2}{dy^2} - (\alpha + 1 - y) \frac{d}{dy}$ lo denotaremos por $\mathcal{L}_\alpha$.

Familia $\{\mathcal{L}_k^\alpha\}_k$ de funciones de Laguerre

Las funciones $\mathcal{L}_k^\alpha$, se definen como

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y),$$

donde $\{L_k^\alpha\}_{k=0}^\infty$ denotan los polinomios de Laguerre de tipo α. Estas funciones $\mathcal{L}_k^\alpha$ son autofunciones del operador

$$L_{\alpha,\mathcal{L}} = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y > 0.$$

De hecho

$$L_{\alpha,\mathcal{L}}(\mathcal{L}_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right) \mathcal{L}_k^\alpha.$$

Dado que los polinomios de Laguerre son ortogonales con respecto a la medida $e^{-y}y^\alpha\, dy$, se sigue que la familia $\{\mathcal{L}_k^\alpha\}_k$ es un sistema ortogonal (de hecho ortonormal) en $L^2((0,\infty), dy)$.

Familia $\{\varphi_k^\alpha\}_k$ de funciones de Laguerre

Las funciones φ_k^α se definen como

$$\varphi_k^\alpha(y) = \mathcal{L}_k^\alpha(y^2)(2y)^{1/2},$$

donde $\mathcal{L}_k^\alpha$ son las consideradas en (2.3). Las funciones φ_k^α forman un sistema ortonormal en $L^2((0,\infty), dy)$ y son autofunciones del operador

$$L_{\alpha,\varphi} = \frac{1}{4}\left\{-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}.$$

de hecho

$$L_{\alpha,\varphi}(\varphi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right) \varphi_k^\alpha.$$

Familia $\{\ell_k^\alpha\}_k$ de funciones de Laguerre

El sistema ortonormal $\{\ell_k^\alpha\}_{k=0}^\infty$ en $L^2((0,\infty), d\mu_\alpha(y))$, $d\mu_\alpha(y) = y^\alpha dy$ se define como

$$\ell_k^\alpha(y) = \mathcal{L}_k^\alpha(y) y^{-\alpha/2},$$

donde $\mathcal{L}_k^\alpha$ son las funciones definidas en (2.3). ℓ_k^α son autovectores del operador

$$L_{\alpha,\ell} = -y\frac{d^2}{dy^2} - (\alpha+1)\frac{d}{dy} + \frac{y}{4}.$$

De hecho

$$L_{\alpha,\ell}\ell_k^\alpha = \left(k + \frac{\alpha+1}{2}\right) \ell_k^\alpha.$$

Familia $\{\psi_k^\alpha\}_k$ de funciones de Laguerre

Sea $\{\psi_k^\alpha\}_{k=0}^\infty$ el sistema ortonormal , en $L^2((0,\infty), y^{2\alpha+1}dy)$, dado por $\psi_k^\alpha(y) = \sqrt{2}y^{-\alpha}\mathcal{L}_k^\alpha(y^2)$, donde $\mathcal{L}_k^\alpha(y)$ son las funciones definidas en (2.3). ψ_k^α son autofunciones del operador

$$L_{\alpha,\psi} = -\frac{1}{4}\Big\{\frac{d^2}{dy^2} + \Big(\frac{2\alpha+1}{y}\Big)\frac{d}{dy} - y^2\Big\},$$

de hecho

$$L_{\alpha,\psi}(\psi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\ \psi_k^\alpha.$$

2.1.3. Operador de Schrödinger

En $\mathbb{R}^d$ con $d \geq 3$ consideramos el operador de Schrödinger

$$\mathcal{L} := -\Delta + V.$$

El potencial V se supone no negativo y para algún $s > d/2$, saitsface la desigualdad de Hölder

$$\left(\frac{1}{|B|}\int_B V(y)^s\ dy\right)^{1/s} \leq \frac{C}{|B|}\int_B V(y)\ dy,$$

para toda bola $B \subset \mathbb{R}^d$, la constante C depende de s y de V. El operador $\mathcal{L}$ es simétrico con respecto la medida de Lebesgue en $\mathbb{R}^d$.

2.2. Operadores del Análisis Armónico asociados a un operador diferencial de segundo orden

Los operadores diferenciales considerados en la memoria tienen asociado un "semigrupo del calor" que denotaremos por $e^{-t\mathcal{L}}$ (siendo $\mathcal{L}$ el operador diferencial correspondiente). Además este semigrupo viene determinado por un núcleo, $K_t(x,y)$, no negativo de modo que para funciones suficientemente buenas se tendrá la identidad

$$e^{-t\mathcal{L}}f(x) = \int_\Omega K_t(x,y)f(y)d\mu(y),\ x \in \Omega.$$

Siendo $d\mu$ la medida con respecto a la cual el operador diferencial $\mathcal{L}$ es autoadjunto.

Siguiendo las ideas desarrolladas por E. Stein en su excelente libro, [84], dado un operador diferencial de segundo orden, $\mathcal{L}$, autoadjunto y no negativo consideraremos los siguientes operadores asociados a su semigrupo del calor:

(i) **Operador maximal del semigrupo del calor :**

$$T^*f(x) = \sup_{t>0} |e^{-t\mathcal{L}} f(x)|.$$

(ii) **Operador maximal del semigrupo subordinado de Poisson**:

$$P^*f(x) = \sup_{t>0} |P_t f(x)|.$$

El **semigrupo subordinado de Poisson** se define, siguiendo la fórmula $e^{-\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\beta^2/4u} du$, como

$$e^{-t\sqrt{\mathcal{L}}} f(x) = P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-(t^2/4u)\mathcal{L}} f(x) du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} e^{-s\mathcal{L}} f(x) ds.$$

(iii) **Potenciales de Riesz, (Integrales fraccionarias)** :

$$\mathcal{L}^{-\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-t\mathcal{L}} f(x) dt, \quad \text{con} \quad \sigma > 0,$$

sugeridos por la fórmula, $s^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-ts} dt.$

(iv) **Funciones** g **de Littlewood Paley** :

$$g(f)(x) = \left(\int_0^\infty \left| t \frac{\partial}{\partial t} T_t f(x) \right|^2 \frac{dt}{t} \right)^{1/2},$$

donde T_t puede ser el semigrupo del calor o de Poisson.

(v) **Transformadas de Riesz** :

$$R_j = \partial_j (\mathcal{L})^{-1/2}.$$

Por ∂_j denotamos un operador diferencial de primer order que factoriza al operador $\mathcal{L}$ en el sentido $\mathcal{L} + C = \sum_j \partial_j^* \partial_j$ donde C es una constante y ∂_j^* es el adjunto de ∂_j con respecto a la medida μ.

2.3. Cambios de variable

Es fácil ver que alguno de los sistemas de funciones ortogonales están relacionados por medio de sencillos cambios de variable o simplemente por medio de multiplicación por funciones sencillas. A continuación presentamos los que se utilizan a lo largo de esta memoria.

2.3.1. Polinomios y funciones de Hermite.

Consideramos la isometría

$$U : L^2(\pi^{-d/2}e^{-|x|^2}) \to L^2(dx) \quad \text{dada por } Uf(x) = \pi^{-d/4}e^{-\frac{|x|^2}{2}}f(x).$$

Teniendo en cuenta la relación entre funciones y polinomios de Hermite, ver (2.2), es claro que las funciones de Hermite h_α y los polinomios de Hermite H_α, satisfacen la identidad $h_\alpha = UH_\alpha$. Esta relación se transmite a los semigrupos del calor de ambos operadores. Se tiene la siguiente identidad, ver Capítulo 3

$$\text{Dado un polinomio } f \text{ en } \mathbb{R}^n, \text{entonces } U^{-1}e^{-t(\mathcal{H}_n-n)}Uf(x) = e^{-t\mathcal{O}_n}f(x).$$

La identidad se propaga a los operadores definidos para ambos operadores como puede verse en la Proposición 3.3 del Capítulo 3.

2.3.2. Polinomios y funciones de Laguerre

A continuación exponemos un método general de cambios de variable que será de aplicación para el caso de las funciones de Laguerre. Para ello necesitamos una definición y una observación, ambas muy sencillas y que las exponemos para futuras referencias.

Definición 2.1. Sea $h : \Omega \to \bar{\Omega} \subseteq \mathbb{R}^n$ una aplicación uno a uno y C^∞ en Ω. Denotamos por $|J_{h^{-1}}|$ al jacobiano de la aplicación inversa $h^{-1} : \bar{\Omega} \to \Omega$. Denotamos por W el cambio de variables de $L^2(\bar{\Omega}, M(h^{-1}(\bar{x}))^2\,|J_{h^{-1}}|\,d\eta(\bar{x}))$ en $L^2(\Omega, M(x)^2 d\eta(x))$ dado por

$$(Wf)(x) = f(h(x)), \quad x \in \Omega.$$

Observación 2.2. Sea $M(x) \in C^\infty(\Omega), \Omega \subseteq \mathbb{R}^n$ una función positiva. Es claro que $(Uf)(x) = M(x)f(x)$ define una isometría de $L^2(\Omega, M(x)^2 d\eta(x))$ en $L^2(\Omega, d\eta(x))$. Además si $\{\varphi_k\}_{k\in\mathcal{N}_0^n}$ es un sistema ortonormal en $L^2(\Omega, M(x)^2 d\eta(x))$ entonces $\{U\varphi_k\}_{k\in\mathcal{N}_0^n}$ es también un sistema ortonormal en $L^2(\Omega, d\eta(x))$.

Teniendo en cuenta la Definición 2.1 y la Observación 2.2, es claro que si un operador $\mathcal{L}$ tiene un sistema ortonormal de autofunciones, φ_k, en $L^2(\Omega, d\eta)$, entonces el operador

$$\bar{L} := (U \circ W)^{-1} \circ L \circ (U \circ W)$$

tendrá el sistema ortonormal de autofunciones $W^{-1}U^{-1}\varphi_k$ en $L^2(\bar{\Omega}, d\bar{\eta}(\bar{x}))$,donde $\bar{\Omega} = h(\Omega)$ y $d\bar{\eta}(\bar{x}) := M(h^{-1}(\bar{x}))^2\,|J_{h^{-1}}|\,d\eta(\bar{x})$.

El lector puede acudir ahora al Capítulo 5 y observar que las aplicaciones V, W^α y Z^α definidas en la sección 5.3 son casos particulares de composiciones del tipo $U \circ W$.

Capítulo 3

Hermite function expansions versus Hermite polynomial expansions

Colaboración con J.L. Torrea
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3.1. Introduction

We shall work in the space $\mathbb{R}^d$, endowed either with the Lebesgue measure dx or with the Gaussian measure $d\gamma(x) = \pi^{-d/2}e^{-|x|^2}dx$. Consider the system of multidimensional Hermite polynomials

$$H_\alpha(x) = H_{\alpha_1}(x_1) \cdot \dots \cdot H_{\alpha_d}(x_d), \quad x = (x_1, \dots, x_d), \quad \alpha = (\alpha_1, \dots \alpha_d), \alpha_i \in \{0, 1, \dots\},$$

where $H_k(s) = (-1)^k e^{s^2} \dfrac{d^k e^{-s^2}}{ds^k}$, $s \in \mathbb{R}$, denotes the $1-$dimensional kth Hermite polynomial, see [92]. It is well known that the Hermite polynomials are the eigenfunctions of the Ornstein-Uhlenbeck differential operator $\mathcal{L} = -\Delta + 2x \cdot \nabla$, namely

$$\mathcal{L}H_\alpha = 2|\alpha|H_\alpha, \quad |\alpha| = \alpha_1 + \dots + \alpha_d. \tag{3.1}$$

The operator $\mathcal{L}$ is positive and symmetric in $L^2(\mathbb{R}^d, d\gamma(x))$ on the domain $C_c^\infty(\mathbb{R}^d)$. The orthonormalized Hermite polynomials, $\tilde{H}_k = \frac{2^{-k/2}}{\sqrt{k!}} H_k$ form an orthonormal basis for $L^2(d\gamma(x))$.

We shall also consider the system of multidimensional Hermite functions

$$h_\alpha(x) = h_{\alpha_1}(x_1) \cdot \dots \cdot h_{\alpha_d}(x_d), \quad x = (x_1, \dots, x_d), \quad \alpha = (\alpha_1, \dots \alpha_d), \alpha_i \in \{0, 1, \dots\},$$

where $h_k(s) = (\pi^{1/2}2^k k!)^{-1/2}H_k(s)e^{-s^2/2}$ and H_k denotes the 1−dimensional kth Hermite polynomial. It is well known that the Hermite functions are the eigenfunctions of the Hermite differential operator $\mathcal{H} = -\Delta + |x|^2$, namely

$$\mathcal{H}h_\alpha = (2|\alpha| + d)h_\alpha. \tag{3.2}$$

The functions h_α form an orthonormal basis for $L^2(\mathbb{R}^n, dx)$. The operator $\mathcal{H}$ is positive and symmetric in $L^2(\mathbb{R}^d, dx)$ on the domain $C_c^\infty(\mathbb{R}^d)$, see [94].

The Ornstein-Uhlenbeck, $e^{-t\mathcal{L}}$, (respectively the Hermite, $e^{-t\mathcal{H}}$,) semigroup with infinitesimal generator $-\mathcal{L}$ (respectively $-\mathcal{H}$) can be defined in spectral sense. Namely for functions $g \in L^2(d\gamma(x))$ such that $g = \sum c_\alpha \tilde{H}_\alpha$ define $e^{-t\mathcal{L}}g$ the $L^2(e^{-|x|^2}dx)$ function given by $e^{-t\mathcal{L}}g = \sum e^{-2t|\alpha|}c_\alpha \tilde{H}_\alpha$. On the other hand if $f \in L^2(dx)$ such that $f = \sum c_\alpha h_\alpha$ define $e^{-t\mathcal{H}}f$ be the $L^2(dx)$ function given by $e^{-t\mathcal{H}}f = \sum e^{-t(2|\alpha|+d)}c_\alpha h_\alpha$.

B. Muckenhoupt initiated in 1969 the study, in dimension one, of the maximal operator of the Ornstein-Uhlenbeck semigroup, $\sup_t e^{-t\mathcal{L}}$, and also the notion of "conjugate functionrelated to that semigroup, see [63], [64]. $e^{-t\mathcal{L}}$ is a symmetric diffusion semigroup, in the sense of [84], the (L^p, L^p), $1 < p < \infty$, boundedness of the maximal operator is deduced from the general theory developed in [84]. In finite dimension, the proof of the $(1,1)$-weak boundedness for the maximal operator was given in 1982 by P. Sjögren, see [81]. The corresponding result for the Riesz transforms was proved by Fabes, Gutiérrez and Scotto in [29]. They also proved that the Riesz transforms are principal value operators. Due to the relation with the Wiener chaos, proving that the constants appearing in the boundedness are independent of the dimension became an important task. Some research was done in this direction, see [35] and the references in the survey [82]. Finally, weighted inequalities and vector-valued inequalities were studied in [42] and [40].

As for the semigroup $e^{-t\mathcal{H}}$, the main reference is Thangavelu. He proved (in several papers but we refer to his book, [94], and the references there) the $(L^p(dx), L^p(dx))$ and $(L^1(dx), L^{1,\infty}(dx))$ boundedness for the maximal operator of the semigroup. He also proved, by using analogues of the classical conjugate harmonic functions, that the Riesz transforms, see the definition in section 2 formula (3.6) and the comments there, are $(L^p(dx), L^p(dx))$ and $(L^1(dx), L^{1,\infty}(dx))$ bounded. This study was extended in [89] and weighted inequalities for the weights in the A_p-class of Muckenhoupt were proved. For an introduction to the A_p theory see [25]. Neither in [94] nor in [89] the description of the Riesz transforms as principal value operator was considered.

There is a close relation between the semigroups $e^{-t\mathcal{H}}$ and $e^{-t\mathcal{L}}$. This relation, that is determined by the fact $h_k(s) = (\pi^{1/2}2^k k!)^{-1/2}H_k(s)e^{-s^2/2}$, is propagated to the operators defined through the semigroups (maximal operators, Riesz transforms, etc). This kind of correspondence between these operators is sometimes described vaguely (in this case) saying that the operators associated to $\mathcal{H}$ and $\mathcal{L}$ are "unitary equivalent in L^2 ". The purpose of this note is to describe, in a transparent and clear way, this relation and to get, as a consequence, new results for several operators associated either to $\mathcal{L}$ or to $\mathcal{H}$. The relationship between both parts is described in Proposition 5.20 and Theorem 3.3.5. By using these results we can get new weighted inequalities in both sides, see Theorems 3.2.1 and 3.2.5, we also get new descriptions of the Riesz transforms in Hermite function case, see Theorem 3.2.2 and 3.2.3.

The organization of the paper is the following. We present the results in section 3.2. These results shall be proven in section 3.4, with the help of some thecnical results that we present in section 3.3.

3.2. Main results

If f is a linear combination of Hermite functions then $e^{-t\mathcal{H}}f(x) = \int_{\mathbb{R}^n} G_t(x,y)f(y)dy$, where $G_t(x,y)$ is given by

$$\begin{aligned} G_t(x,y) &= \sum_\alpha e^{-t(2|\alpha|+d)}h_\alpha(x)h_\alpha(y) \\ &= (2\pi\sinh 2t)^{-d/2}exp(-\frac{1}{2}|x-y|^2\coth 2t - x\cdot y\tanh t), \qquad (3.3) \end{aligned}$$

see [94], [89]. Clearly $e^{-t(\mathcal{H}-d)}f(x) = \int_{\mathbb{R}^n} e^{td}G_t(x,y)f(y)dy$

We have the following Theorem

Theorem 3.2.1. Let v be a positive measurable function. The following conditions are equivalent:

(i) There exists a positive measurable function u and a constant C such that for every $f \in L^2(v(x)dx)$ we have

$$\sup_t \int_{\mathbb{R}^d} |e^{-t(\mathcal{H}-d)}f(x)|^2u(x)dx \leq C\int_{\mathbb{R}^d}|f(x)|^2v(x)dx.$$

(ii) There exists a positive measurable function u and a constant C such that for every $f \in L^2(v(x)dx)$ we have

$$\int_{\mathbb{R}^d}\sup_t |e^{-t(\mathcal{H}-d)}f(x)|^2u(x)dx \leq C\int_{\mathbb{R}^d}|f(x)|^2v(x)dx.$$

(iii) The function v satisfies $\int_{\mathbb{R}^d} v^{-1}(x)e^{-|x|^2}dx < \infty$.

In particular for a function v satisfying (iii) the operator $\sup_t e^{-t\mathcal{H}}$ maps $L^2(\mathbb{R}^d, v(x)dx)$ into $L^2(\mathbb{R}^d, u(x)dx)$ for some positive u.

It is well known that $\mathcal{T}_t = e^{-t\mathcal{H}}$(respectively $\mathbf{T}_t = e^{-t\mathcal{L}}$) is a diffusion semigroup in $L^p(dx)$, $1 \leq p \leq \infty$ (respectively in $L^p(d\gamma(x))$, $1 \leq p \leq \infty$) in the sense of [84], see details in [82],[94] and [89].

The operators $e^{-t\mathcal{L}}$ and $e^{-t\mathcal{H}}$ are positive,($f(x) \geq 0 \to e^{-t\mathcal{L}} f(x) \geq 0$) , bounded in $L^p(d\gamma(x))$ (respectively $L^p(dx)$) and therefore each $\mathbf{T}_t$ (respectively each $\mathcal{T}_t$) have a straightforward extension to $L^p_B(d\gamma(x))$ (respectively $L^p_B(dx)$) for every Banach space B. Moreover the norm of the extension is the same as the original norm of the operator. By $L^p_B(d\gamma(x))$ we denote the Bochner-Lebesgue space of $B-$valued functions defined in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \|f(x)\|^p_B d\gamma(x) < \infty$. Analogous definitions can be given for $L^p_B(dx)$. Since these extensions are linear, they act in a natural way over the tensor products $B \otimes L^p(d\gamma(x))$ and $B \otimes L^p(dx)$. In particular

$$\mathbf{T}_t(\sum_{i=1}^{n} b_i \varphi_i) = \sum_{i=1}^{n} b_i \mathbf{T}_t \varphi_i, \quad b_i \in B, \varphi_i \in L^p(d\gamma(x)). \tag{3.4}$$

Analogous expressions can be given for $\mathcal{T}_t$.

Let μ be a σ-finite measure in $\mathbb{R}^n$. Let $\{T_t\}$ be a symmetric diffusion semigroup of operators acting on measurable functions on $(\mathbb{R}^n, d\mu)$, with a second order differential operator $-L$, (symmetric in $L^2(d\mu)$) as its infinitesimal generator. In this context, the following operators can be considered; see [84],

Riesz potentials:

$$\text{given} \quad a > 0, \quad (-L)^{-a} f(x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} T_t f(x) dt. \tag{3.5}$$

Riesz transforms:

$$\text{For} \quad 1 \leq i \leq n, \quad R_i f(x) = \text{``}\frac{\partial}{\partial x_i}\text{''} L^{-1/2} f(x). \tag{3.6}$$

Here "$\frac{\partial}{\partial x_i}$" denotes the component of the "gradient" associated to the operator L.

It is easy to check that

$$\mathcal{L} = \sum_{j=1}^{n} \delta_j^* \delta_j, \quad \text{where} \quad \delta_j = \frac{\partial}{\partial x_j} \text{ and } \delta_j^* = -\frac{\partial}{\partial x_j} + 2x_j. \tag{3.7}$$

Observe that δ_j^* is the adjoint operator of δ_j in $L^2(d\gamma(x))-$sense. In a parallel way we have

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^{n} (A_j^* A^* + A_j A_j^*), \text{ where } A_j = \frac{\partial}{\partial x_j} + x_j, \text{ and } A_j^* = -\frac{\partial}{\partial x_j} + x_j. \tag{3.8}$$

Observe that A_j^* and A_j are adjoints in $L^2(dx)-$sense.
Therefore the operator "$\frac{\partial}{\partial x_i}$" will be, δ_i in the case of $\mathcal{L}$, and either A_i or A_i^* in the case of the operator $\mathcal{H}$.
Since 0 is an eigenvalue of $\mathcal{L}$, the negative powers $\mathcal{L}^{-a}$ are not defined for every function in $L^2(\mathbb{R}^n, d\gamma(x))$. Let Π_0 be the orthogonal projection onto the orthogonal complement of the eigenspace correponding to the eigenvalue 0. Then the Riesz transforms for the Ornstein-Uhlenbeck operator are defined as, see [82], $\mathbf{R}_i = \delta_i(\mathcal{L})^{-1/2}\Pi_0$, in particular in defining $\mathbf{R}_i f$ we always can assume that $\int_{\mathbb{R}^n} f(x)d\gamma(x) = 0$. As we said it is known that $\mathbf{R}_i$ are bounded from $L^p(\mathbb{R}^n, d\gamma)$ into itself for p in the range $1 < p < \infty$, and from $L^1(\mathbb{R}^n, d\gamma)$ into $L^{1,\infty}(\mathbb{R}^n, d\gamma)$. They are principal value operators, that is

$$\mathbf{R}_i f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \mathbf{R}_i(x,y)f(y)d\gamma(y), a.e.x, \quad f \in L^1(d\gamma).$$

see the survey [82] and the references there.
In the case of $\mathcal{H}$, the Riesz transforms, due to (3.8), were defined, see [94], as $\mathcal{R}_i^+ = A_i(\mathcal{H})^{-1/2}$ and $\mathcal{R}_i^- = A_i^*(\mathcal{H})^{-1/2}$. For them we shall prove the following

Theorem 3.2.2. The operators $\mathcal{R}_i^\pm$, $i = 1, \ldots, n$ are principal valued operators. That is

$$\mathcal{R}_i^\pm f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \mathcal{R}_j^\pm(x,y)f(y)dy, \quad f = \sum_{\text{finite}} c_\alpha h_\alpha.$$

The operators $\delta_i^*(\mathcal{L})^{-1/2}\Pi_0$, $i = 1, \ldots, n$ are principal value operators over the class of polynomial functions.

We call $\mathcal{R}_i^\pm$ the linear extension of these operators, in the sense described in (3.4), to functions taking values in a Banach space B. We recall that a Banach space is in the UMD class if the Hilbert transform has a bounded extension to $L_B^2(\mathbb{R}, dx)$, see [19] and [18]. We have the following Theorem

Theorem 3.2.3. Let B be a Banach space. The following statements are equivalent:

(i) B is a UMD Banach space.

(ii) $|\{x \in \mathbb{R}^n : \|\mathcal{R}_j^+ f(x)\|_B > \lambda\}| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} \|f(x)\|_B dx$, $1 \leq j \leq n$.

(iii) For every p, $1 < p < \infty$ (and equivalently for some $1 < p < \infty$),

$$\|\mathcal{R}_j^+ f\|_{L_B^p(dx)} \leq C_p \|f\|_{L_B^p(dx)}, \quad 1 \leq j \leq n$$

(iv) $\mathcal{R}_j^+$ maps boundedly L_B^∞ into BMO_B, $1 \leq j \leq n$.

(v) $\mathcal{R}_j^+$ maps H_B^1 into L_B^1, $1 \leq j \leq n$.

In $(ii), (iii), (iv)$ and (v), $\mathcal{R}_j^+$ can be replaced by $\mathcal{R}_j^-$. The constants C and C_p are independent of f but they may depend on the Banach space B.
Moreover if B is a UMD Banach space then, for $1 \leq j \leq n$,

$$\mathcal{R}_j^{\pm} f(x) = \lim_{\varepsilon \to 0} \mathcal{R}_{j,\varepsilon}^{\pm} f(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \mathcal{R}_j^{\pm}(x,y) f(y) dy, \ a.e.x, \ f \in \cup_{1 \leq p \leq \infty} L_B^p(dx).$$

Remark 3.2.4. Observe that in the above Theorem we said that $\mathcal{R}_j^{\pm}$ are defined in $L^\infty(\mathbb{R}^n, dx)$. This is different from the case of the classical euclidean Riesz transforms for which a definition for L^∞ functions has to be given .ªd hoc", see [83]. To justify this fact it is enough to see that for a function $f \in L^\infty$, the limit

$$\lim_{n \to \infty} (\mathcal{R}_j^+(f \chi_{B(0,n)})(x) + \int_{|y|>n} \mathcal{R}_j^+(x,y) f(y) dy)$$

exists a.e. x. In order to prove the existence of this limit we need two ingredients: first the existence of the limit for functions in L^p (this is the case of $f\chi_{B(0,n)}$), second, the bound $\mathcal{R}_j^+(x,y) \leq Ce^{-\frac{|x-y|^2}{c}}$ for $|x-y| > 1$, see Proposition 3.3.4, guarantees the convergence of the second summand.

The Riesz transforms $\mathcal{R}_j^{\pm}$ are Calderón-Zygmund operators with associated Calderón-Zygmund kernels $\mathcal{R}_j^{\pm}(x,y)$, see [89], in the sense that

$$\mathcal{R}_i^{\pm} f(x) = \int_{\mathbb{R}^n} \mathcal{R}_j^{\pm}(x,y) f(y) dy, \quad f \in C_c^\infty(\mathbb{R}^n), \quad x \notin supf.$$

Therefore they are bounded from $L^p(\mathbb{R}^n, \omega(x)dx)$ into itself for p in the range $1 < p < \infty$ and from $L^1(\mathbb{R}^n, \omega(x)dx)$ into $L^{1,\infty}(\mathbb{R}^n, \omega(x)dx)$, where ω is a weight in the Muckenhoupt A_p-class , $1 \leq p < \infty$. We have the following Theorem.

Theorem 3.2.5. Let ω be a weight in Muckenhoupt class A_2. The operators $\mathbf{R}_i$ and $\delta_i^*(\mathcal{L})^{-1/2}\Pi_0$, $i = 1, \ldots, n$ are bounded from $L^2(\mathbb{R}^n, \omega(x)d\gamma(x))$ into itself.

3.3. Technical Lemmas

We define in the following simple Lemma the key operator which shall be the carrier of the results from the polynomial side to the function side and vice-versa.

Lemma 3.3.1. Let B be a Banach space and ω a weight in $\mathbb{R}^n$. The operator U defined by $Uf(x) = f(x)\pi^{d/4}e^{-\frac{|x|^2}{2}}$, is an isometry from $L_B^2(\omega(x)d\gamma(x))$ into $L_B^2(\omega(x)dx)$.

Proof

$$\begin{aligned}\|Uf\|^2_{L^2_B(\omega(x)dx)} &= \int \|Uf(x)\|^2_B \omega(x)dx = \int \|f(x)\|^2_B \pi^{d/2} e^{-|x|^2}\omega(x)dx \\ &= ||f||^2_{L^2_B(\omega(x)d\gamma(x))}\end{aligned}$$

■

Definition 3.3.2. Let B a Banach space . Let H_k be the Hermite polynomials in $\mathbb{R}^d$. Any function f of the form $f(x) = \sum_{finite} b_\alpha H_\alpha, x \in \mathbb{R}^d$, where $b_\alpha \in B$, will be called "B-valued polynomial function".

Proposition 3.3.3. Let B be a Banach space and f be a $B-$valued polynomial function in $\mathbb{R}^n$. We have the following pointwise identities

(i) $A_jUf(x) = U\delta_j f(x), \quad A_j^*Uf(x) = U\delta_j^* f(x)$.

(ii) $(\mathcal{H} - d)Uf(x) = U\mathcal{L}f(x), \quad \mathcal{H}Uf(x) = U(\mathcal{L} + d)f(x)$.

(iii) $e^{-t(\mathcal{H}-d)}Uf(x) = Ue^{-t\mathcal{L}}f(x), \quad e^{-t(\mathcal{H})}Uf(x) = Ue^{-t(\mathcal{L}+d)}f(x)$.

(iv) Let $s > 0$.
If $\int_{\mathbb{R}^n} f(x)d\gamma(x) = 0$ then $(\mathcal{H} - d)^{-s}Uf(x) = U(\mathcal{L})^{-s}f(x)$.
For every f, $(\mathcal{H})^{-s}Uf(x) = U(\mathcal{L} + d)^{-s}f(x)$.

(v) If $\int_{\mathbb{R}^n} f(x)d\gamma(x) = 0$ then $A_i(\mathcal{H} - d)^{-1/2}Uf(x) = U\mathbf{R}_i f(x)$.
For every f we have $\mathcal{R}_i^+Uf(x) = U\delta_i(\mathcal{L} + d)^{-1/2}f(x)$, $i = 1, \dots, n$.

We use the notations in (3.7) and (3.8)

Proof. (i) and (ii) are tedious calculations. By using (3.1), (3.2) we get (iii). By using the definition of Riesz potentials, it is very easy to check that $\mathcal{L}^{-s}H_k = (2|k|)^{-s}H_k$ and $\mathcal{L}_H^{-s}h_k = (2|k| + d)^{-s}h_k$. Observe that a polynomial function f belongs to Π_0 when $\int_{\mathbb{R}^d} f(x)e^{-|x|^2}dx = 0$. Finally, by using (3.7) and (3.8) we get (v). ■

The size of the kernels involved with the Riesz transforms where analyzed in Theorem 3.3 of [89]. In fact the following result is proved there

Proposition 3.3.4. Let f be a finite combination of Hermite functions. Then
(i) There exists a positive kernel L such that

$$|\mathcal{H}^{-1/2}f(x)| \leq \int_{\mathbb{R}^d} L(x, y)|f(y)|dy, \; x \in \mathbb{R}^d.$$

If $d=1$ there exists an ε, $0<\varepsilon<1$ with $L(x,y)\leq C(\frac{1}{|x-y|^{\varepsilon}}\chi_{|x-y|<1}+e^{-\frac{|x-y|^2}{c}}\chi_{|x-y|>1})$.

If $d>1$, there exists a constant C with $L(x,y)\leq C(\frac{1}{|x-y|^{d-1}}\chi_{|x-y|<1}+e^{-\frac{|x-y|^2}{c}}\chi_{|x-y|>1})$.

(ii) There exist constants c, C such that $|\mathcal{R}_j^{\pm}(x,y)|\leq C(\frac{1}{|x-y|^{d}}\chi_{|x-y|<1}+e^{-\frac{|x-y|^2}{c}}\chi_{|x-y|>1})$

The Proposition 3.3.3 suggest us to study the difference $(\mathcal{H})^{-1/2}-(\mathcal{H}-d)^{-1/2}$.

Theorem 3.3.5. There exist kernels $N, L_i, i=1,\ldots,n$ such that for any function f which is a linear combination of Hermite functions, with $\int_{\mathbb{R}^n} f(y)e^{-\frac{y^2}{2}}dy=0$ we have

(i) $\left((\mathcal{H}-d)^{-1/2}-\mathcal{H}^{-1/2}\right)f(x)=\int_{\mathbb{R}^d} N(x,y)f(y)dy,\ x\in\mathbb{R}^d.$

(ii) $\left(A_i(\mathcal{H}-d)^{-1/2}-\mathcal{R}_i^{+}\right)f(x)=\int_{\mathbb{R}^d} L_i^{+}(x,y)f(y)dy,\ x\in\mathbb{R}^d,\quad i=1,\ldots n.$

(iii) $\left(A_i^*(\mathcal{H}-d)^{-1/2}-\mathcal{R}_i^{-}\right)f(x)=\int_{\mathbb{R}^d} L_i^{-}(x,y)f(y)dy,\ x\in\mathbb{R}^d,\quad i=1,\ldots n.$

Moreover there exist a one variable positive decreasing function $\Phi\in L^1(\mathbb{R},dx)$ such that if we denote by M either the kernel N or the kernel $L_i^{\pm}$, $\quad i=1,\ldots,n$, we have $|M(x,y)|\leq C\Phi(|x-y|)$.

Proof. Observe that the change of parameter

$$t=t(s)=\frac{1}{2}\log\frac{1+s}{1-s},\quad 0<s<1,\quad 0<t<\infty$$

produces

$$G_t(x,y)=K_s(x,y)=(\frac{1-s^2}{4\pi s})^{d/2}\exp(-\frac{1}{4}(s|x+y|^2+\frac{1}{s}|x-y|^2)).\tag{3.9}$$

where G_t is the kernel in (3.3). On the other hand, this change of parameter in the formula (3.5), drives us to the expression $L^{-1/2}f=\frac{1}{\Gamma(1/2)}\int_0^1\left(\log\frac{1+s}{1-s}\right)^{-1/2}e^{-t(s)L}f\frac{ds}{1-s^2}$.

Therefore for functions f satisfying $\int_{\mathbb{R}^n} f(y)e^{-\frac{|y|^2}{2}}dy=0$ we have

$$\begin{aligned}\mathcal{H}^{-1/2}f &= \frac{1}{\Gamma(1/2)}\int_0^1\left(\log\frac{1+s}{1-s}\right)^{-1/2}\int_{\mathbb{R}^n}K_s(x,y)f(y)dy\frac{ds}{1-s^2}\\ &= \frac{1}{\Gamma(1/2)}\int_0^1\int_{\mathbb{R}^n}\left(K_s(x,y)-\chi_{(1/2,1)}(s)(\frac{1-s^2}{4\pi s})^{d/2}e^{-\frac{1}{2}(|x|^2+|y|^2)}\right)f(y)dy\\ &\quad\times\left(\log\frac{1+s}{1-s}\right)^{-1/2}\frac{ds}{1-s^2},\end{aligned}$$

where K_s is the kernel defined in (3.9). Analogous considerations can be made with $e^{-t(\mathcal{H}-d)} = e^{td}e^{-t\mathcal{H}}$ and we can write

$$\begin{aligned}
&\left((\mathcal{H}-d)^{-1/2} - \mathcal{H}^{-1/2}\right) f(x) \\
= &\sqrt{\frac{2}{\pi}} \int_0^1 \left\{ (\frac{1+s}{1-s})^{d/2} - 1 \right\} \int_{\mathbb{R}^n} \left(K_s(x,y) - \chi_{(1/2,1)}(s) (\frac{1-s^2}{4\pi s})^{d/2} e^{-\frac{1}{2}(|x|^2+|y|^2)} \right) f(y) dy \\
&\times \left(\log \frac{1+s}{1-s} \right)^{-1/2} \frac{ds}{1-s^2} := N(x,y). \qquad (3.10)
\end{aligned}$$

We shall see that the function $N(x,y)$ just defined satisfies the Theorem. Write

$$N(x,y) = \int_0^{1/2} + \int_{1/2}^1 = I_0 + I_1$$

Observe that for $s \in (0,1/2)$ we have $(\frac{1+s}{1-s})^{d/2} - 1 \sim s$ and $\log \frac{1+s}{1-s} \sim s$ therefore

$$I_0 \leq C \int_0^{1/2} s^{1/2} \frac{1}{s^{d/2}} e^{-\frac{c}{s}|x-y|^2} ds \leq \frac{C}{|x-y|^{d-3}} \int_{c_0|x-y|^2}^{\infty} u^{\frac{d-3}{2}} e^{-u} \frac{du}{u}.$$

Where in the last inequality we have performed the change of variables $u = c\frac{|x-y|^2}{s}$. If $c_0|x-y| > 1$, by using the inequality $z^n e^{-z} \leq Ce^{-z/2}$, we get

$$I_0 \leq \frac{C}{|x-y|^{d-3}} e^{-\frac{|x-y|^2}{c}} \int_c^{\infty} u^{\frac{d-3}{2}} e^{-u/2} \frac{du}{u} \leq Ce^{-\frac{|x-y|^2}{c}}.$$

On the other hand, if $c_0|x-y| < 1$ and $d \geq 4$ we have

$$I_0 \leq \frac{C}{|x-y|^{d-3}} (\int_{c_0|x-y|^2}^1 + \int_1^{\infty}) u^{\frac{d-3}{2}} e^{-u} \frac{du}{u} \leq \frac{C}{|x-y|^{d-3}}.$$

If $c_0|x-y| < 1$ and $d < 4$ we have

$$\begin{aligned}
I_0 &\leq \frac{C}{|x-y|^{d-3}} (\int_{c_0|x-y|^2}^1 + \int_1^{\infty}) u^{\frac{d-3}{2}} e^{-u} \frac{du}{u} \leq \frac{C}{|x-y|^{d-3}} (\int_{c_0|x-y|^2}^1 u^{\frac{d-3}{2}} e^{-u} \frac{du}{u} + C) \\
&\leq C(\int_{c_0|x-y|^2}^1 e^{-u} \frac{du}{u} + C) \leq C(-\log|x-y| + 1).
\end{aligned}$$

Where we have used $(\frac{u}{|x-y|^2})^{d-3} \leq C$, valid for $d \leq 3$.

On the other hand we write $I_1 = I_{11} + I_{12}$ where

$$I_{12} = \sqrt{\frac{2}{\pi}} \int_{1/2}^1 \left[(\frac{1+s}{1-s})^{d/2} K_s(x,y) - (\frac{1+s}{4\pi s})^{d/2} \exp(-\frac{|x|^2+|y|^2}{2}) \right] \left(\log \frac{1+s}{1-s} \right)^{-1/2} \frac{ds}{1-s^2}.$$

Consider the function $\beta(\theta) = \exp(-\frac{1}{4}(\theta|x+y|^2 + \frac{1}{\theta}|x-y|^2)$. Since $\frac{1}{2} < s < 1$, applying the mean value theorem we have

$$| \exp(-\frac{1}{4}(s|x+y|^2 + \frac{1}{s}|x-y|^2) - \exp(-\frac{|x|^2+|y|^2}{2})| \leq Ce^{-\frac{|x-y|^2}{c}}(1-s).$$

Hence

$$I_{12} \leq Ce^{-\frac{|x-y|^2}{c}} \int_{1/2}^{1} (1-s)\left(\log\frac{1+s}{1-s}\right)^{-1/2} \frac{ds}{1-s} \leq Ce^{-\frac{|x-y|^2}{c}}.$$

The case I_{11} is similar. In order to prove part (ii) of the Theorem we consider the kernel

$$A_iN(x,y) = (\frac{\partial}{\partial x_i} - x_i)N(x,y) = \frac{\partial}{\partial x_i}N(x,y) - x_iN(x,y) = N_1(x,y) - N_2(x,y).$$

In order to handle these kernels we follow the procedure we did for N, that is, consider separately the cases $0 \leq s \leq 1/2$ and $1/2 \leq s \leq 1$. We shall estimate first the kernel N_2. We call again I_0 the integral in the range $0 \leq s \leq 1/2$. If $x\cdot y \leq 0$ then $|x| \leq |x-y|$, therefore by using $z^n e^{-z} \leq Ce^{-z/2}$ we have

$$\begin{aligned} I_0 &\leq C\int_0^{1/2} |x-y|s^{1/2}\frac{1}{s^{d/2}}e^{-\frac{|x-y|^2}{cs}}ds \leq C\int_0^{1/2} s\frac{1}{s^{d/2}}e^{-\frac{|x-y|^2}{cs}}ds \\ &\leq \frac{C}{|x-y|^{d-4}}\int_{c|x-y|^2}^{\infty} u^{\frac{d-4}{2}}e^{-u}\frac{du}{u}. \end{aligned}$$

Where in the last inequality we have performed the change of variables $u = c\frac{|x-y|^2}{s}$. If $c|x-y| > 1$, by using the inequality $z^n e^{-z} \leq Ce^{-z/2}$, we get

$$I_0 \leq \frac{C}{|x-y|^{d-4}}e^{-\frac{|x-y|^2}{c}}\int_c^{\infty} u^{\frac{d-4}{2}}e^{-u/2}\frac{du}{u} \leq Ce^{-\frac{|x-y|^2}{c}}.$$

On the other hand, if $c|x-y| < 1$ and $d \geq 5$ we have

$$I_0 \leq \frac{C}{|x-y|^{d-4}}(\int_{c_0|x-y|^2}^{1} + \int_1^{\infty})u^{\frac{d-4}{2}}e^{-u}\frac{du}{u} \leq \frac{C}{|x-y|^{d-4}}.$$

If $c_0|x-y| < 1$ and $d < 5$ we have

$$\begin{aligned} I_0 &\leq \frac{C}{|x-y|^{d-4}}(\int_{c_0|x-y|^2}^{1} + \int_1^{\infty})u^{\frac{d-4}{2}}e^{-u}\frac{du}{u} \leq \frac{C}{|x-y|^{d-4}}(\int_{c_0|x-y|^2}^{1} u^{\frac{d-4}{2}}e^{-u}\frac{du}{u} + C) \\ &\leq C(\int_{c_0|x-y|^2}^{1} e^{-u}\frac{du}{u} + C) \leq C(-\log|x-y| + 1). \end{aligned}$$

where we have used that since $d \leq 4$ then $(\frac{u}{|x-y|^2})^{d-4} \leq C$. If $x \cdot y \geq 0$ then $|x| \leq |x+y|$, therefore we have (we use the term $e^{-s|x+y|^2}$ in K_s and the fact $s^{1/2}|x+y|e^{-s|x+y|^2} \leq C$)

$$I_0 \leq C \int_0^{1/2} \frac{1}{s^{d/2}} e^{-\frac{|x-y|^2}{cs}} ds \leq \frac{C}{|x-y|^{d-2}} \int_{c|x-y|^2}^{\infty} u^{\frac{d-2}{2}} e^{-u} \frac{du}{u}.$$

Where in the last inequality we have performed the change of variables $u = c\frac{|x-y|^2}{s}$. We proceed analogously to the case $x \cdot y \leq 0$. Pasting up the above arguments with the arguments we gave above for the integral I_1 (in the range $1/2 \leq s \leq 1$) for N we get in this case $I_1 \leq Ce^{-\frac{|x-y|^2}{c}}$. This ends the proof for the kernel N_2.

Now we shall analyze the Kernel $N_1 = \frac{\partial}{\partial x_i} N(x,y)$. Observe that

$$\begin{aligned} \frac{\partial}{\partial x_i} N(x,y) &= \sqrt{\frac{2}{\pi}} \int_0^1 \left(\log \frac{1+s}{1-s}\right)^{-1/2} \left\{ (\frac{1+s}{1-s})^{d/2} - 1 \right\} \\ &\times \left[(-\frac{1}{2}(s(x_i+y_i) + \frac{x_i - y_i}{s})) K_s(x,y) + \chi_{[1/2,1]}(s) (\frac{1-s^2}{4\pi s})^{d/2} x_i \exp(-\frac{1}{2}(|x|^2+|y|^2)) \right] \\ &\quad \times \frac{ds}{1-s^2} \qquad (3.11) \\ &= \sqrt{\frac{2}{\pi}} \left(\int_0^{1/2} + \int_0^{1/2} \dots ds \right) = I_1 + I_2. \end{aligned}$$

Observe that $|(-\frac{1}{2}(s(x_i+y_i) + \frac{x_i-y_i}{s}))| \leq (\frac{1}{2}(s|x+y| + \frac{|x-y|}{s}))$. We consider again separately the case $0 \leq s \leq 1/2$ and denoting I_0 the corresponding integral, we have

$$I_0 \leq C \int_0^{1/2} (s|x+y| + \frac{1}{s}|x-y|) s^{1/2} \frac{1}{s^{d/2}} e^{-\frac{1}{4}(s|x+y|^2 + \frac{1}{s}|x-y|^2)} ds. \qquad (3.12)$$

The same arguments used for N_2 can be repeated to get the required bound for N_1 in this case.

As for I_1 we can proceed analogously by considering the function

$$\beta(\theta) = (-\frac{1}{2}(\theta(x_i+y_i) + \frac{1}{\theta}(x_i - y_i))) \exp(-\frac{1}{4}(\theta|x+y|^2 + \frac{1}{\theta}|x-y|^2).$$

Since $\frac{1}{2} < s < 1$, applying the mean value theorem we have

$$\begin{aligned} &|(-\frac{1}{2}(s(x_i+y_i) + \frac{1}{s}(x_i - y_i))) \exp(-\frac{1}{4}(s|x+y|^2 + \frac{1}{s}|x-y|^2) + (x_i \exp(-\frac{|x|^2+|y|^2}{2})| \\ &= |\beta(s) - \beta(1)| \leq Ce^{-\frac{|x-y|^2}{c}}(1-s). \end{aligned}$$

Again the arguments given for N_1 and N are valid in this case. ∎

3.4. Proofs of the main results

We begin this section by presenting the proof of Theorem 3.2.1. This Theorem will be obtained by using the following Theorem, that can be found in [40]

Theorem 3.4.1. Let v be a positive measurable function. The following conditions are equivalent:

(i) There exists a positive measurable function u and a constant C such that for every $f \in L^2(v(x)dx)$ we have
$\sup_t \int_{\mathbb{R}^d} |e^{-t\mathcal{L}} f(x)|^2 u(x) d\gamma(x) \leq C \int_{\mathbb{R}^d} |f(x)|^2 v(x) d\gamma(x).$

(ii) There exists a positive measurable function u and a constant C such that for every $f \in L^2(v(x)d\gamma(x))$ we have
$\int_{\mathbb{R}^d} \sup_t |e^{-t\mathcal{L}} f(x)|^2 u(x) dx \leq C \int_{\mathbb{R}^d} |f(x)|^2 v(x) d\gamma(x).$

(iii) The function v satisfies $\int_{\mathbb{R}^d} v^{-1}(x) d\gamma(x) < \infty.$

Observe that by using Proposition 3.3.3 (iii) and Lemma 3.3.1 we have

$$\begin{aligned} \|e^{-t(\mathcal{H}-d)} f\|_{L^2(u(x)dx)} &= \|U^{-1} e^{-t(\mathcal{H}-d)} f\|_{L^2(u(x)d\gamma(x))} = \|e^{-t(\mathcal{L})} U^{-1} f\|_{L^2(u(x)d\gamma(x))} \\ &\leq C\|U^{-1} f\|_{L^2(v(x)d\gamma(x))} = \|f\|_{L^2(v(x)dx)} \end{aligned}$$

were in the penultimate inequality we have used Theorem 3.4.1. In order to finish the proof of Theorem 3.2.1 observe that for each t and each x we have $e^{-t(\mathcal{H})} f(x) \leq e^{-t(\mathcal{H}-d)} f(x)$

We continue by presenting the **proof of Theorem 3.2.2**. If f is a linear combination of Hermite functions with $\int_{\mathbb{R}^n} f(y) e^{-\frac{|y|^2}{2}} dy = 0$, then $f = Ug$ and $g(y) = U^{-1} f(y) = f(y) e^{\frac{|y|^2}{2}} \pi^{-d/4}$ (where U is the isometry in Lemma 3.3.1 and g is a scalar valued polynomial function with $\int g(y) d\gamma(y) = 0)$. Then as we mention in the introduction, the Riesz transforms associated to the Ornstein-Uhlenbeck differential operator are principal value operators, therefore by using Proposition 3.3.3 (v), we have

$$\begin{aligned} A_j(\mathcal{H}-d)^{-1/2} f(x) &= U\delta_j(\mathcal{L})^{-1/2} g(x) = U\mathbf{R}_j g(x) \\ &= e^{-\frac{|x|^2}{2}} \pi^{d/4} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \mathbf{R}_j(x,y) f(y) e^{\frac{|y|^2}{2}} \pi^{-d/4} d\gamma(y) \\ &= \pi^{-d/2} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \mathbf{R}_j(x,y) f(y) e^{-\frac{|x|^2}{2}} e^{\frac{-|y|^2}{2}} dy. \end{aligned}$$

If h is a linear combination of Hermite functions and $\int_{\mathbb{R}^n} h(y)e^{-|y|^2/2}dy = 0$, then the conclusion of the Theorem for the operator $\mathcal{R}_i^+$, follows from Theorem 3.3.5 (ii). For a general f linear combination of Hermite functions we have $f(x) = h(x) + \pi^{-d/2}e^{-|x|^2/2}(\int_{\mathbb{R}^n} f(y)e^{-|y|^2/2}dy) = h(x) + ch_0(x)$ with $\displaystyle\int_{\mathbb{R}^n} h(y)e^{-|y|^2/2} = 0$ and h_0 is the first Hermite function. Therefore, as $A_i(h_0) = 0$, we have $\mathcal{R}_i^+ f = A_i(\mathcal{H})^{-1/2} f = A_i(\mathcal{H})^{-1/2}h + cA_i(\mathcal{H})^{-1/2}(h_0) = \mathcal{R}_i^+ h$. Then the Theorem follows for $\mathcal{R}_i^+$. Observe that Since $\mathcal{H}^{-1/2}$ is given by an integrable kernel, see Proposition 3.3.4, the operator $x_i\mathcal{H}^{-1/2}$ is a principal value operator. Therefore as $\mathcal{R}_i^- = -\mathcal{R}_i^+ + 2x_i\mathcal{H}^{-1/2}$ we get the desired result for $\mathcal{R}_i^-$. Once we get the conclusion for $\mathcal{R}_i^-$ we use again Theorem 3.3.5 and Proposition 3.3.3 and we obtain the conclusion for $\delta_i^*\mathcal{L}^{-1/2}\Pi_0$. ■

Proof of Theorem 3.2.5.

Observe that given a function $f \in L^2(\mathbb{R}^n, dx)$ then

$$f(x) = g(x) + \pi^{-1/2}e^{-|x|^2/2}(\int_{\mathbb{R}^n} f(y)e^{-|y|^2/2}dy) = g(x) + P_0(f)(x),$$

and $\displaystyle\int_{\mathbb{R}^d} g(x)e^{-|x|^2/2}dx = 0$. Clearly $\mathcal{R}_j^+ f = \mathcal{R}_j^+ g$ and

$$\mathcal{R}_j^- f = \mathcal{R}_j^- g + cp(x)e^{-|x|^2/2}(\int_{\mathbb{R}^n} f(y)e^{-|y|^2/2}dy), \tag{3.13}$$

where $p(x)$ is a polynomial de degree one in x. As we said in the introduction, see [89], the operators $\mathcal{R}_i^\pm$, $i = 1, \ldots, n$ are bounded in $L^2(\mathbb{R}^n, \omega(x)dx)$ for any weight ω which belongs to the A_2 Muckenhoupt class. In particular

$$\|\mathcal{R}_j^+ g\|^2_{L^2(\mathbb{R}^d,\omega(x)dx)} \leq \|g\|^2_{L^2(\mathbb{R}^d,\omega(x)dx)}$$

It is well known that the Hardy-Littlewood maximal operator M maps $L^2(\mathbb{R}^n, \omega(x)dx)$ into itself, again for $\omega \in A_2$. Therefore Theorem 3.3.5 says that the difference $A_i(\mathcal{H} - d)^{-1/2} - \mathcal{R}_i^+$ maps $L^2(\mathbb{R}^n, \omega(x)dx)$ into itself. Then $A_i(\mathcal{H} - d)^{-1/2}$ maps $L^2(\mathbb{R}^n, \omega(x)dx)$, we get the result for $\mathbf{R}_i$ by using Proposition 3.3.3 and Lemma 3.3.1.

On the other hand it is well known that if ν is a weight which belongs to the A_2 Muckenhoupt class then the measure $\nu(x)dx$ is doubling, that is there exists a constant such that $\int_{\{|x|<2r\}} \nu(x)dx \leq A\int_{\{|x|<r\}} \nu(x)dx$ therefore, for any $\varepsilon > 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \nu(y)e^{-\varepsilon|y|^2}dy &\leq \sum_{j=0}(\int_{2^j<|y|<2^{j+1}} \nu(y)e^{-\varepsilon|y|^2}dy) + \int_{|y|<1} \nu(y)e^{-\varepsilon|y|^2}dy \\
&\leq \sum_{j=0} e^{-\varepsilon 2^{2j}}(\int_{2^j<|y|<2^{j+1}} \nu(y)dy) + \int_{|y|<1} \nu(y)dy \\
&\leq \sum_{j=0} e^{-\varepsilon 2^{2j}} A^j \int_{|y|<1} \nu(y)dy \leq C_\varepsilon(\nu).
\end{aligned}$$

By using $|p(x)|e^{-|x|^2} \leq Ce^{-|x|^2/2}$ and the fact that if $\omega \in A_2$ then $\omega^{-1} \in A_2$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} p(x)^2 \left(\int_{\mathbb{R}^d} f(y)e^{-|y|^2/2}dy\right)^2 e^{-|x|^2}\omega(x)dx \\ \leq\ & C\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(y)|^2\omega(y)dy\right)(\int_{\mathbb{R}^d}\omega(y)^{-1}e^{-|y|^2}dy)e^{-|x|^2/2}\omega(x)dx \\ \leq\ & C\left(\int_{\mathbb{R}^d} |f(y)|^2\omega(y)dy\right)(\int_{\mathbb{R}^d}\omega(y)^{-1}e^{-|y|^2}dy)(\int_{\mathbb{R}^d}\omega(x)e^{-|x|^2/2}dx) \\ \leq\ & C\left(\int_{\mathbb{R}^d} |f(y)|^2\omega(y)dy\right). \end{aligned}$$

Therefore, by using (3.13) and the fact that $\mathcal{R}_i^-$, $i = 1, \dots, n$ are bounded in $L^2(\mathbb{R}^n, \omega(x)dx)$, we have that

$$\|\mathcal{R}_j^- g\|^2_{L^2(\mathbb{R}^d,\omega(x)dx)} \leq \|g\|^2_{L^2(\mathbb{R}^d,\omega(x)dx)}.$$

In order to get th eresult for $\delta^*(\mathcal{L})^{-1/2}\Pi_0$ we can now proceed as with $\mathbf{R}_i$.

Proof of Theorem 3.2.3. We call $\mathbf{R}_i$ the linear extension of these operators, in the sense described in (3.4), to functions taking values in a Banach space B. The following Theorem was proved in [42]

Theorem 3.4.2. The following statements are equivalent: : (i) $\mathbf{R}_i, i = 1, \dots, n$ are $L^p_B(\mathbb{R}^n, d\gamma)$ bounded for every $p, 1 < p < \infty$. (ii) $\mathbf{R}_i, i = 1, \dots, n$ are $L^p_B(\mathbb{R}^n, d\gamma)$ bounded for a particular $p, 1 < p < \infty$. (iii) $\mathbf{R}_i, i = 1, \dots, n$ are bounded from $L^1_B(\mathbb{R}^n, d\gamma)$ into $L^{1,\infty}_B(\mathbb{R}^n, d\gamma)$. (iv) B has the UMD property.

It was proved in [89] that the operators $\mathcal{R}_j^{\pm}$ are Calderón-Zygmund operators with associated kernels $\mathcal{R}_j^{\pm}(x, y)$, in these circumstances it is known that (ii), (iii), (iv), (v) are equivalent, where in (iv) L^∞ has to be substituted by L^∞_c . The proof of this fact consists in adapting the scalar case to this vector valued case. For the scalar case see [44]. These equivalences have as a consequence that any of them is equivalent to the following statement:

$(iii)'$ There exists a constant C_2 such that $\|\mathcal{R}_j^+ f\|_{L^2_B(dx)} \leq C_2\|f\|_{L^2_B(dx)}, \quad 1 \leq j \leq n.$

Since by Theorem 3.3.5, the difference between $\mathcal{R}_j^+$ and $A_j(\mathcal{H} - d)^{-1/2}$ is controlled by a positive operator bounded in L^p, $1 \leq p \leq \infty$, we have that in $(iii)'$ we can replace $\mathcal{R}_j^+$ by $A_j(\mathcal{H} - d)^{-1/2}$. Now by using Lemma 3.3.1 we see that $(iii)'$ is equivalent to

$(iii)''$ There exists a constant C_2 such that $\|\mathbf{R}_j f\|_{L^2_B(d\gamma)} \leq C_2\|f\|_{L^2_B(d\gamma)}, \quad 1 \leq j \leq n.$

But statement $(iii)''$ is equivalent to say that the Banach space B is UMD, see Theorem 3.4.2.

By using Theorem 3.2.2 and the vector valued version of the general theory of Calderón-Zygmund operators, [44], [77], we get

$$\mathcal{R}_j^+ f(x) = \lim_{\varepsilon\to 0} \mathcal{R}_{j,\varepsilon}^+ f(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} \mathcal{R}_j^+(x, y)f(y)dy, \quad a.e.x, \quad f \in \cup_{1\leq p<\infty} L^p_B.$$

Now we can use Remark 3.2.4 and we get the result for L^∞. The theorem for $\mathcal{R}_j^-$ follows by observing that $\mathcal{R}_j^- = -\mathcal{R}_j^+ + 2x_i\mathcal{H}^{-1/2}$. ∎

Capítulo 4

Convergencia al dato inicial de la ecuación del calor: Oscilador Harmónico y Ornstein-Uhlenbeck

4.1. Introducción

Sea $\mathcal{L}$ un operador diferencial positivo de segundo orden en $\mathbb{R}^n$. Consideramos el problema

$$(*)\begin{cases} \partial_t u + \mathcal{L}u = 0 & (x,t) \in \mathbb{R}^n \times [0,\infty), \\ u(x,0) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Es bien conocido que con condiciones muy generales se tiene la convergencia al dato inicial para funciones de $L^p(\mathbb{R}^n, v(x)dx)$, $1 \leq p < \infty$. Es decir

$$\lim_{t\to 0} u(x,t) = f(x) \ a.e.x, \quad f \in L^p(\mathbb{R}^n, dx),\ 1 \leq p < \infty.$$

Una pregunta natural es la siguiente:

¿Existe una clase de pesos v (función estrictamente positiva y finita en casi todo punto) que caracterice la convergencia anterior para funciones f en $L^p(\mathbb{R}^n, v(x)dx)$?

La pregunta ha sido contestada recientemente para el caso del laplaciano clásico en [43]. De hecho se ha caracterizado mediante una nueva clase de pesos D_p^W. Nuestro propósito es intentar contestar a la pregunta anterior para el operador de Hermite $\mathcal{H} = -\Delta + |x|^2$ y también para el operador de Ornstein-Uhlenbeck $\mathcal{O} = -\Delta + 2x \cdot \nabla$.

El estudio de convergencias en casi todo punto siempre lleva detrás un análisis de acotaciones del correspondiente operador maximal. Por tanto en nuestro camino aparecerán los operadores $\sup_t |e^{-t(-\Delta+|x|^2)}f(x)|$ y $\sup_t |e^{-(-\Delta+2x\cdot\nabla)}f(x)|$. Por otro lado, estamos interesados en convergencia cuando $t \to 0$. Parece claro que el problema es local y dado el aspecto de nuestros operadores, podría pensarse que la condición sobre el peso debería de ser cercana a la condición obtenida para el laplaciano clásico. En efecto la situación es así. La clave en el caso del operador $e^{-t(-\Delta+|x|^2)}$ radica en que se puede dar una acotación puntual del núcleo del operador $e^{-t(-\Delta+|x|^2)}$ por el núcleo del operador $e^{t\Delta}$. Recíprocamente puede obtenerse una acotación puntual (no uniforme) de $e^{t\Delta}f(x)$ por la función $e^{-t(-\Delta+|x|^2)}f(x)$, ver el Lema 4.5. Por tanto la convergencia de $e^{t\Delta}f(x)$ es equivalente a la de $e^{-t(-\Delta+|x|^2)}f(x)$. En cuanto al operador de Ornstein-Uhlenbeck, la función U introducida en la Sección 2.3 del Capítulo 3 proporciona la herramienta necesaria para que la convergencia de $e^{-t(-\Delta+2x\cdot\nabla)}f(x)$ sea equivalente a la convergencia de $e^{-t(-\Delta+|x|^2)}f(x)$.

4.2. Resultados sobre convergencia al dato inicial

Comenzamos recordando la definición de una clase de pesos que apareció en [43].

Definición 4.1. Sea $W_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$, $t > 0$. Diremos que el peso v (función estrictamente positiva y finita en casi todo punto de $\mathbb{R}^n$) pertenece a la clase D_p^W, $1 \leq p < \infty$ si existe un $t_0 > 0$ tal que

$$\|W_{t_0} v^{-\frac{1}{p}}\|_{L^{p'}(\mathbb{R}^n,dx)} < \infty.$$

Teorema 4.2. Sea v un peso en $\mathbb{R}^n$ y $1 \leq p < \infty$. Dado R, $0 < R < \infty$, consideramos los operadores $T_R^* f(x) = \sup_{t<R} |e^{-t\mathcal{H}} f(x)|$ y $\tilde{T}_R^* f(x) = \sup_{t<R} |e^{-t(\mathcal{H}-n)} f(x)|$ Las siguientes afirmaciones son equivalentes:

(1) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to \tilde{T}_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)dx)$ en $L^p(\mathbb{R}^n, u(x)dx)$, $\quad 1 < p < \infty$.
Para $p = 1$ de $L^1(\mathbb{R}^n, v(x)dx)$ en $L^1(\mathbb{R}^n, u(x)dx)$-débil.

(2) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to \tilde{T}_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)dx)$ en $L^p(\mathbb{R}^n, u(x)dx)-$débil.

(3) Existe $0 < R < \infty$ tal que

$$e^{-R(\mathcal{H}-n)} f(x) < \infty, \ a.ex$$

y el límite $\lim_{t\to 0} e^{-t(\mathcal{H}-n)} f(x)$ existe a.e. x para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(4) Existe $0 < R < \infty$ such that

$$\tilde{T}_R^* f(x) < \infty,$$

a.e. x, para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(5) En cualquiera de las condiciones anteriores puede sustituirse el operador $\tilde{T}$ por el operador T.

(6) El peso $v \in D_p^W$ (ver la definición 4.1).

Los cambios de variable considerados en los Capítulos 2 y 3 nos permitirán obtener un resultado paralelo para el caso de Ornstein-Uhlenbeck, en concreto obtenemos el siguiente Teorema

Teorema 4.3. Sea v un peso en $\mathbb{R}^n$ y $1 \leq p < \infty$. Dado R, $0 < R < \infty$, consideramos los operadores $\mathcal{O}_R^* f(x) = \sup_{t<R} |e^{-t(\mathcal{O})} f(x)|$ Las siguientes afirmaciones son equivalentes:

(1) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to \mathcal{O}_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)d\gamma(x))$ en $L^p(\mathbb{R}^n, u(x)d\gamma(x))$, $\quad 1 < p < \infty$.
Para $p = 1$ de $L^1(\mathbb{R}^n, v(x)\gamma(x)dx)$ en $L^1(\mathbb{R}^n, u(x)\gamma(x)dx)$-débil.

(2) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to \mathcal{O}_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)d\gamma(x))$ en $L^p(\mathbb{R}^n, u(x)d\gamma(x))-$débil.

(3) Existe $0 < R < \infty$ tal que

$$e^{-R\mathcal{O}} f(x) < \infty, \ a.ex$$

y el límite $\lim_{t\to 0} e^{-t\mathcal{O}} f(x)$ existe a.e. x para toda $f \in L^p(\mathbb{R}^n, v(x)d\gamma(x))$.

(4) Existe $0 < R < \infty$ such that

$$\mathcal{O}_R^* f(x) < \infty,$$

a.e. x, para toda $f \in L^p(\mathbb{R}^n, v(x)d\gamma(x))$.

(5) El peso $v \in D_p^W$ (ver la definición 4.1).

4.3. Demostraciones de los Teoremas 4.2 y 4.3

Consideramos el semigrupo del calor del operador de Hermite $e^{-s\mathcal{H}_n}$. Como se ha visto en el Capítulo 3 (prueba del Teorema 3.5) podemos escribir :

$$e^{-t(s)\mathcal{H}_n}f(x) = \Big(\frac{1-s^2}{4\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\frac{1}{s^{n/2}}e^{-\frac{1}{4}(\frac{|x-y|^2}{s}+s|x+y|^2)}f(y)dy, \quad 0<s<1, \tag{4.1}$$

siendo

$$t(s) = \frac{1}{2}\log\frac{1+s}{1-s}. \tag{4.2}$$

La función $t(s)$ es creciente con $\lim_{s\downarrow 0} t(s) = 0$ y $\lim_{s\uparrow 1} t(s) = \infty$.

Por otro lado es bien conocido que el semigrupo del calor del laplaciano clásico en $\mathbb{R}^n$ puede escribirse como

$$e^{t\Delta}f(x) = W_tf(x) = \int_{\mathbb{R}^n} W_t(x-y)f(y)dy = \Big(\frac{1}{4\pi}\Big)^{n/2}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}e^{-\frac{1}{4}(\frac{|x-y|^2}{t})}f(y)dy. \tag{4.3}$$

Observación 4.4. Teniendo en cuenta (4.1) y (4.3) es claro que, para las funciones que las integrales anteriores sean finitas, se tiene

$$e^{-t(s)\mathcal{H}_n}f(x) \le (1-s^2)^{n/2}W_sf(x), \quad 0<s<1.$$

El siguiente Lemma muestra que puede obtenerse una desigualdad inversa no uniforme. Pero aunque dependerá del punto x y del parámetro s será muy útil para nuestros propósitos.

Lema 4.5. Dados $x \in \mathbb{R}^n$, $0<s<1$ existe $0 < C(x,s) < \infty$ tal que para funciones f positivas

$$\Big(1-s^2\Big)^{n/2} W_{\frac{9s}{9+25s^2}}f(x) \le C(x,s)e^{-t(s)\mathcal{H}_n}f(x).$$

El valor de $C(x,s)$ es $C(x,s) = (34/9)^{n/2}\, e^{\frac{1}{4}\,s\,25|x|^2}$.

Demostración. Distinguiremos dos casos

Caso 1.- $|y| > 4|x|$. Entonces $|x-y| \le |x|+|y| \le \frac{5}{4}|y|$ y además $|y| \le |x-y|+|x| \le |x-y|+\frac{1}{4}|y|$. Por tanto $\frac{3}{4}|y| \le |x-y| \le \frac{5}{4}|y|$. Análogamente se tiene $\frac{3}{4}|y| \le |x+y| \le \frac{5}{4}|y|$. Como consecuencia $|x+y| \le \frac{5}{3}|x-y|$. Por tanto

$$e^{-\frac{1}{4}(\frac{|x-y|^2}{s}+s|x+y|^2)}\chi_{|y|\ge 4|x|} \ge e^{-\frac{1}{4}(\frac{|x-y|^2}{s}+\frac{25}{9}s|x-y|^2)}\chi_{|y|\ge 4|x|} = e^{-\frac{1}{4}(\frac{1}{s}+\frac{25\,s}{9})|x-y|^2}\chi_{|y|\ge 4|x|}$$

Caso 2.- $|y| \le 4|x|$, entonces tenemos $0 \le |x+y| \le 5|x|$. Entonces como $\dfrac{1}{s} \le \dfrac{1}{s} + \dfrac{25\,s}{9}$ tenemos

$$\begin{aligned} e^{-\frac{1}{4}\,s\,25|x|^2}e^{-\frac{1}{4}(\frac{1}{s}+\frac{25\,s}{9})|x-y|^2}\chi_{|y|\le 4|x|} &\le e^{-\frac{1}{4}\,s\,|x+y|^2}e^{-\frac{1}{4}\frac{|x-y|^2}{s}}\chi_{|y|\le 4|x|} \\ &= e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)}\chi_{|y|\le 4|x|}. \end{aligned}$$

Teniendo en cuenta que $\frac{9}{9+25}\,s \leq \frac{9s}{9+25s^2}$, obtenemos

$$e^{-\frac{1}{4}\,s\,25|x|^2}\left(\frac{1}{\frac{9s}{9+25s^2}}\right)^{n/2} e^{-\frac{1}{4}\left(\frac{1}{\frac{9s}{9+25s^2}}\right)|x-y|^2} \leq \left(\frac{1}{\frac{9s}{9+25s^2}}\right)^{n/2} e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)}$$
$$\leq \left(\frac{1}{\frac{9}{9+25}s}\right)^{n/2} e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)} = \left(\frac{34}{9}\right)^{n/2}\frac{1}{s^{n/2}}e^{-\frac{1}{4}\left(\frac{|x-y|^2}{s}+s|x+y|^2\right)}$$

Por lo tanto para funciones f positivas, tendremos

$$\Big(9/34\Big)^{n/2} e^{-\frac{1}{4}\,s\,25|x|^2}(1-s^2)^{n/2}W_{\frac{9s}{9+25s^2}} \leq e^{-t(s)\mathcal{H}_n}f(x).$$

□

Este Lema nos permitirá probar de manera sencilla el Teorema 4.2 de la Introducción. Para ello necesitamos presentar al lector el siguiente resultado que se encuentra en el artículo [41].

Teorema 4.6. Sea v un peso en $\mathbb{R}^n$ y $1 \leq p < \infty$. Dado R, $0 < R < \infty$, consideramos el operador $W_R^*f(x) = \sup_{t<R}|e^{-t(-\Delta)}f(x)|$ Las siguientes afirmaciones son equivalentes:

(i) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to W_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)dx)$ en $L^p(\mathbb{R}^n, u(x)dx)$.
Para $p = 1$ de $L^1(\mathbb{R}^n, v(x)dx)$ en $L^1(\mathbb{R}^n, u(x)dx)$-débil.

(ii) Existe $0 < R < \infty$ y un peso u tal que el operador

$$f \to W_R^* f$$

es acotado de $L^p(\mathbb{R}^n, v(x)dx)$ en $L^p(\mathbb{R}^n, u(x)dx)-$débil.

(iii) Existe $0 < R < \infty$ tal que $e^{R\Delta}f(x) < \infty$ a.e x y el límite $\lim_{t\to 0} e^{t\Delta}f(x)$ existe a.e. x para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(iv) Existe $0 < R < \infty$ tal que

$$W_R^* f(x) < \infty,$$

a.e. x, para toda $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(v) El peso $v \in D_p^W$ (ver la definición 4.1).

Demostración. (Teorema 4.2). El cambio de parámetro (4.2) produce $e^{-t(s)} = \left(\frac{1+s}{1-s}\right)^{1/2}$. Por lo tanto teniendo en cuenta la observación 4.4 tenemos

$$\begin{aligned} e^{-t(s)\mathcal{H}_n} f(x) &\leq e^{-t(s)(\mathcal{H}_n - n)} f(x) \leq e^{nt(s)}(1-s^2)^{n/2} W_s f(x) \qquad (4.4) \\ &= \left(\frac{1+s}{1-s}\right)^{n/2} (1-s^2)^{n/2} W_s f(x) = (1+s)^n W_s f(x) \leq 2^n W_s f(x). \end{aligned}$$

Utilizando ahora el Teorema 4.6 tenemos que (6) $\Longrightarrow$ (1). Las implicaciones (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) son obvias, observar que las funciones continuas con soporte compacto son densas en $L^p(\mathbb{R}^n, v(x)dx)$ y es bien conocido que $\lim_{t\to 0} e^{-t(\mathcal{H}_n - n)} f(x) = f(x)$ en ese caso (ver [94]).

(3) $\Longrightarrow$ (4). Basta probarlo para funciones f no negativas. Sea x tal que $\tilde{T}_R f(x) < \infty$, el lema 4.5 nos garantiza la existencia de un s_R (tal que $R = \frac{1}{2}\log\frac{1+s_R}{1-s_R}$) satisfaciendo

$$W_{\frac{9s_R}{9+25s_R^2}} f(x) < \infty.$$

Por otra parte, dado un $s > 0$ definimos $s^* = \frac{9s}{9+25s^2}$, teniendo en cuenta la observación 4.4 y de nuevo el lema 4.5 podemos escribir

$$\begin{aligned} e^{-t(s^*)(\mathcal{H}_n - n)} f(x) &= e^{t(s^*)n} e^{-t(s^*)(\mathcal{H}_n)} f(x) \leq \left(\frac{1+s^*}{1-s^*}\right)^{n/2} (1-(s^*)^2)^{n/2} W_{s^*} f(x) \\ &= (1+s^*)^n \Big(1-s^2\Big)^{-n/2} \Big(1-s^2\Big)^{n/2} W_{s^*} f(x) \qquad (4.5) \\ &\leq 2^n \Big(1-s^2\Big)^{-n/2} C(x,s) e^{-t(s)\mathcal{H}_n} f(x). \end{aligned}$$

Observemos que la existencia de límite garantiza $\lim_{t\to 0} e^{-t(\mathcal{H}_n - n)} f(x) = \lim_{t\to 0} e^{tn} e^{-t\mathcal{H}_n} f(x)$. Por lo tanto la cadena de desigualdades (4.5) garantiza la existencia de $\lim_{t\to 0} e^{t\Delta} f(x)$. Aplicando el Teorema 4.6 obtenemos que existe un R tal que $W_R^* f(x) < \infty$ a.e. x. Utilizando de nuevo la observación 4.4 obtenemos (4).

(4) $\Longrightarrow$ (5) es obvio.

Probemos finalmente que (5) $\Longrightarrow$ (6). Claramente es suficiente probar que el apartado (4) (para el operador T_R) implica (6). La cadena de desigualdades en (4.5) nos garantiza la existencia de un cierto S_R tal que $\sup_{0<s<S_R} W_s f(x) < \infty$ a.e. x y por lo tanto podemos aplicar nuevamente el Teorema 4.6 obteniendo (6). □

Demostración. (Teorema 4.3). Las implicaciones (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) son obvias.

(3) $\Longrightarrow$ (4). Supongamos que f es no negativa entonces sabemos, ver la Proposición 3.3.3 en el Capítulo 3, que $U^{-1} e^{-t(\mathcal{H}-n)} U f(x) = e^{-t\mathcal{O}} f(x)$. Por lo tanto $\lim_{t\to 0} e^{-t(\mathcal{H}-n)} U f(x)$ existe a.e para toda $f \in L^p(\mathbb{R}^n, v(x)d\gamma(x))$. Es decir $\lim_{t\to 0} e^{-t(\mathcal{H}-n)} g(x)$ existe a.e para toda función $g \in L^p(\mathbb{R}^n, v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})}dx)$ y además

$e^{-R(\mathcal{H}-n)}Uf(x) < \infty$ para un cierto R. Por el Teorema 4.2 sabemos que esto implica la existencia de un R para el que se tiene la finitud en casi todo punto del operador maximal $\sup_{t<R} |e^{-R(\mathcal{H}-n)}g(x)|$ para toda función $g \in L^p(\mathbb{R}^n, v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})}dx$. Por lo tanto tenemos la finitud en casi todo punto del operador maximal $\sup_{t<R} e^{-t\mathcal{O}}f(x)$.

(4) $\implies$ (5). Procediendo como antes vemos que el Teorema 4.2 garantiza que el peso $v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})}$ pertenece a la clase D_p^W. Es decir existe un M tal que $\int_{\mathbb{R}^n} e^{-M|x|^2}(v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})})^{-p'/p}dx < \infty$. Es claro que el valor del parámetro M (que corresponde a $\frac{1}{t}$ en el semigrupo puede elegirse tan grande como se desee. en particular podemos elegirlo $M > -p'\left(-\frac{1}{p}+\frac{1}{2}\right)$ y entonces vemos que el peso v cumple la definición 4.1.

(5) $\implies$ (1). Recíprocamente si el peso cumple la definición 4.1 se cumple que para un cierto $M > 0$ tendremos $\int_{\mathbb{R}^n} e^{-M|x|^2}(v(x))^{-p'/p}dx < \infty$. Eligiendo $M > p'\left(-\frac{1}{p}+\frac{1}{2}\right)$ tenemos que

$$\int_{\mathbb{R}^n} e^{-\left((M-p'(-\frac{1}{p}+\frac{1}{2}))|x|^2\right)}(v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})})^{-p'/p}dx < \infty.$$

En otras palabras el peso $v(x)e^{|x|^2 p(-\frac{1}{p}+\frac{1}{2})}$ cumple la definición 4.1 y por lo tanto se cumple (1) del Teorema 4.2 para el operador maximal $\sup_{t<R} |e^{-R(\mathcal{H}_n-n)}g(x)|$. Procediendo como antes vemos que esto implica la acotación del operador maximal $\sup_{t<R} e^{-t\mathcal{O}}f(x)$ en $L^p(v(x)d\gamma(x)$

□

Capítulo 5

Transfering strong boundedness among Laguerre orthogonal systems.

Colaboración con R. Macías, C. Segovia y J.L. Torrea.
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5.1. Introduction

In this paper we will deal with the Laguerre second order differential operator defined by

$$L_\alpha = -y\frac{d^2}{dy^2} - \frac{d}{dy} + \frac{y}{4} + \frac{\alpha^2}{4y}, \quad y > 0, \tag{5.1}$$

where $\alpha > -1$. Thus L_α is a nonnegative and selfadjoint operator with respect to the Lebesgue measure on $(0, \infty)$. The Laguerre functions, $\mathcal{L}_k^\alpha$, are defined as

$$\mathcal{L}_k^\alpha(y) = \left(\frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)}\right)^{1/2} e^{-y/2} y^{\alpha/2} L_k^\alpha(y), \tag{5.2}$$

where $\{L_k^\alpha\}_{k=0}^\infty$ are the Laguerre polynomials of type α, see [92, p. 100] and [94, p. 7]. These functions $\mathcal{L}_k^\alpha$ are eigenfunctions of L_α. In fact

$$L_\alpha(\mathcal{L}_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\mathcal{L}_k^\alpha. \tag{5.3}$$

Since the Laguerre polynomials are orthogonal with respect to the measure $e^{-y}y^{\alpha}$, it follows that the family $\{\mathcal{L}_k^{\alpha}\}_k$ is orthonormal in $L^2((0,\infty),dy)$.

Besides the orthonormal system $\{\mathcal{L}_k^{\alpha}\}$, some other types of Laguerre functions orthonormal systems like $\{\varphi_k^{\alpha}\},\{\ell_k^{\alpha}\}$ and $\{\psi_k^{\alpha}\}$ (section 5.2), have been considered previously , see for instance [53],[54] [94], [85], [68], [70].

The orthogonality of these systems, with respect to the corresponding measure, is an immediate consequence of the orthogonality of the Laguerre polynomials. Differential operators similar to the operator in 5.1 can be defined, in such a way that the functions in the orthonormal systems become the eigenfunctions of these operators, with the same eigenvalues as in 5.3. This allows to define, in a natural way, semigroups associated to these differential operators that are related through isometries of the L^2 Hilbert spaces corresponding to the different measures that intervene. Therefore these isometries establish relations among operators defined canonically from the semigroups, as for instance the maximal operator and the infinitesimal generator.

Given a factorization of the infinitesimal generator by first order differential operators, following Stein, [84], a notion of derivative can be given. Through the isometries defined above, we can obtain factorizations of the infinitesimal generators of the semigroups we are dealing with. Thus operators arising from these notions of derivatives and the semigroup, as for instance Riesz transforms and $g-$functions, can be transferred from one system to another, see Proposition 5.3.2. Likewise, the L^2 results obtained for one orthonormal system can be transferred to another of the orthonormal systems under consideration.

The isometries relating the different orthonormal systems are "also isometries" on L^p with respect to power weighted measures, see Lemma 5.7, and therefore the results in L^p transfer from system to system. Thus, we conclude that an exhaustive knowledge of the boundedness of the operators, in L^p associated to a particular Laguerre orthonormal system, implies a complete knowledge of the boundedness of the corresponding operators on the other Laguerre orthonormal system, see Theorem 5.10.

Among other results in [70], sufficient conditions on weights are given for the boundedness of the Riesz transforms associated to the Laguerre orthonormal system $\{\varphi_k^{\alpha}\}$, in the case $\alpha > -\frac{1}{2}$. Our Theorem 5.10 contains sufficient conditions that, in the case of Riesz transforms and maximal operators, are also necessary for power weights in the range $\alpha > -1$, for all the Laguerre orthonormal systems mentioned above.

There is also a natural isometry among Laguerre functions and Laguerre polynomials. However, this isometry does not preserve infinitesimal generators, see Lemma 5.12. In the case of the maximal operator we overcome the difficulties by means of a result stated in Lemma 5.15. When dealing with the Riesz transforms we use results that are obtained by transferring similar results for Hermite functions, see Lemma 5.14. With these ingredients, our method works well, allowing new estimates with weights, for the maximal operator and the Riesz transforms associated to Laguerre polynomials, for $\alpha > -1$, see Theorem 5.13. We point out that the Riesz transforms for polynomials were studied in the case $\alpha \geq -1/2$ in [36] and [68].

5.2. Preliminaries

Following Stein, [84], given a second order, non negative and selfadjoint differential operator L, taking $T_t = e^{-tL}$, its heat semigroup, we can introduce

(i) Maximal operator: $T^* f(x) = \sup_{t>0} |T_t f(x)|$.

(ii) Maximal operator of the subordinated Poisson semigroup: $P^* f(x) = \sup_{t>0} |P_t f(x)|$, where P_t is defined by the subordination formula

$$P_t f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t e^{-t^2/4s} T_s f(x) s^{-3/2} ds,$$

(iii) Riesz potentials: $L^{-\sigma} f(x) = \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_0^\infty t^{\sigma-1} T_t f(x) dt$, for $0 < \sigma$, derived from the identity, $s^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-ts} dt$,

(iv) Littlewood-Paley $g-$function: $g(f)(x) = \left(\displaystyle\int_0^\infty \left| t \frac{\partial}{\partial t} T_t f(x) \right|^2 \frac{dt}{t} \right)^{1/2}$, and

(v) Riesz transforms: $R_j = \partial_j (L)^{-1/2}$, where by ∂_j we mean a kind of "derivation"which appears in a factorization of L.

In the context of Laguerre functions $\{\mathcal{L}_k^\alpha\}_k$, $\alpha > -1$, in order to define the Riesz transforms, appropriate first order derivatives were introduced in [41], that is

$$D_\alpha = \sqrt{y} \frac{d}{dy} + \frac{1}{2} \left(\sqrt{y} - \frac{\alpha}{\sqrt{y}} \right). \tag{5.4}$$

The actions on the corresponding Laguerre functions are given by

$$D_\alpha(\mathcal{L}_k^\alpha) = -\sqrt{k}\, \mathcal{L}_{k-1}^{\alpha+1} \qquad \text{and} \qquad (D_\alpha)^*(\mathcal{L}_k^{\alpha+1}) = -\sqrt{k+1}\, \mathcal{L}_{k+1}^\alpha \quad , \tag{5.5}$$

where $(D_\alpha)^* = -\sqrt{y} \dfrac{d}{dy} + \dfrac{1}{2} \big(\sqrt{y} - \dfrac{\alpha+1}{\sqrt{y}} \big)$, is the formal adjoint of D_α with respect to the Lebesgue measure. ¿From these definitions it follows that

$$L_\alpha - (\frac{\alpha+1}{2}) = (D_\alpha)^* D_\alpha.$$

Accordingly, we can define the Riesz transforms for the Laguerre function expansions as

$$R_+^\alpha = D_\alpha (L_\alpha)^{-1/2},\ \alpha > -1 \quad \text{and} \quad R_-^\beta = \quad (D_{\beta-1})^*\,(L^\beta)^{-1/2},\ \beta > 0. \tag{5.6}$$

Hence,

$$R_+^\alpha(\mathcal{L}_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\,\mathcal{L}_{k-1}^{\alpha+1} \quad \text{and} \quad R_-^\beta(\mathcal{L}_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k+\frac{\beta+1}{2}}}\,\mathcal{L}_{k+1}^{\beta-1}.$$

Remark 5.1. The definition of R_-^β only for $\beta > 0$, can be argue firstly by observing that when applying the operator $(D\alpha)^*$ to the function $(L_\alpha)^{-1/2}L_0^\alpha = (\dfrac{\alpha+1}{2})^{-1/2}\Gamma(\alpha+1)^{-1/2}e^{-y/2}y^{\alpha/2}$, we have

$$(D_\alpha)^*(L_\alpha)^{-1/2}L_0^\alpha = (\frac{\alpha+1}{2})^{-1/2}\Gamma(\alpha+1)^{-1/2}\Big(\sqrt{y}-\frac{\alpha+1}{\sqrt{y}}\Big)e^{-y/2}y^{\alpha/2}$$

where the last function belongs to $L^2(dx)$ if and only if $\alpha > 0$.

Secondly, we recall that one of the main interests in studying Riesz transforms lies into their intimate connection with Sobolev spaces. It is easy to check that $R_-^{\alpha+1} \circ R_+^\alpha = T_m$, the multiplier operator associated with the sequence

$$m_k = \frac{k}{\sqrt{(k+\frac{\alpha+1}{2})(k+\frac{\alpha}{2})}}.$$

The boundedness of T_m and $T_{m^{-1}}$ in $L^2((0,\infty), dx)$ is obvious and we may write

$$\|f\|_2 = \|T_{m^{-1}} \circ R_-^{\alpha+1} \circ R_+^\alpha f\|_2 \le C\|R_+^\alpha f\|_2 = C\|D_\alpha(L_\alpha)^{-1/2}f\|_2 \le C\|f\|_2.$$

Therefore

$$\|D_\alpha f\|_2 \sim \|(L_\alpha)^{1/2}f\|_2.$$

This equivalence, jointly with its analogous for $1 < p < \infty$, are the keys to define the Sobolev spaces associated to this Laplacian in terms of the derivative $D_\alpha$and only R_-^β, $\beta > 0$, intervene.

Consequently, it is natural to introduce the Riesz transform vector, $\mathcal{R}^\alpha$, associated to L_α as in [39]

$$\mathcal{R}^\alpha = (R_+^\alpha, R_-^{\alpha+1}) = (D_\alpha(L_\alpha)^{-1/2}, (D_\alpha)^*(L_{\alpha+1})^{-1/2}).$$

Moreover, in [39] it is proved.

Theorem 5.2. *(Riesz transforms Theorem) Let $\alpha > -1$, $1 < p < \infty$ and δ be real numbers. Assume that $-\frac{\alpha}{2}p - 1 < \delta < p - 1 + \frac{\alpha}{2}p$ then the operator $\|\mathcal{R}^\alpha\|$ defined as*

$$\|\mathcal{R}^\alpha\|(f) = (|R_+^\alpha f|^2 + |R_-^{\alpha+1} f|^2)^{1/2},$$

maps $L^p(y^\delta dy)$ boundedly into $L^p(y^\delta dy)$.

We shall also need the following results that can be found in [30].

Theorem 5.3. *(Multiplier Theorem) Let $-1 < \alpha$, $1 < p < \infty$ and $m \in C^\infty[0, \infty)$, such that*

$$|D^\ell m(\xi)| \leq C_\ell (1 + \xi)^{-\ell},\ \xi \geq 0,\ \ell = 0, 1, 2 \dots . \tag{5.7}$$

Consider the operator $T_m f = \sum_{k \geq 0} m(k) < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\alpha$, defined at least for $f \in L^2((0, \infty), dy)$. Then T_m admits a bounded extension to $L^p((0, \infty), y^\delta dy)$ whenever $-\dfrac{\alpha}{2}p - 1 < \delta < p - 1 + \dfrac{\alpha}{2}p$.

The *transplantation operators*, for $\alpha, \beta > -1$ and $f \in L^2(dy)$ are defined by

$$T_\beta^\alpha f = \sum_{k=0}^{\infty} < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\beta.$$

The following Theorem is an important result regarding these operators.

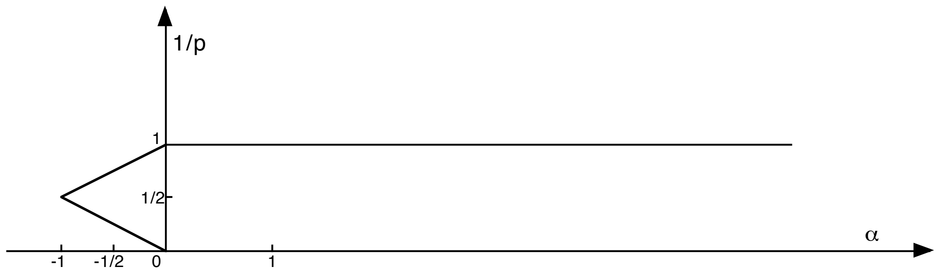
Theorem 5.4. *(Transplantation Theorem) Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T_β^α and T_α^β admit a bounded extension to $L^p((0, \infty), y^\delta dy)$ if and only if $-\dfrac{\alpha}{2}p - 1 < \delta < p - 1 + \dfrac{\alpha}{2}p$.*

For a proof of Theorem 5.3 see [30]. A weaker version was given by Thangavelu in [94]. The Theorem 5.4 was proved by Kanjin, when $\delta = 0$, in [45]. For multiple Laguerre expansions, a result is due to Thangavelu, see [93]. In the weighted case, a more restricted version of Theorem 5.4 was proved by Stempak and Trebels, see [90]. Theorem 5.4 as stated was proved in [30].

Remark 5.5. In the unweighted case, $\delta = 0$, the restriction in p that appears in Theorems 5.2, 5.3 and 5.4 can be rewritten as

$$-\frac{\alpha}{2} - \frac{1}{p} < 0 < 1 - \frac{1}{p} + \frac{\alpha}{2}.$$

Then the region $(\alpha, \frac{1}{p})$, for α and p satisfying the above conditions, can be visualized as



For obvious reasons we call this situation "*pencil phenomenon*" for the system of the Laguerre functions $\{\mathcal{L}_k^\alpha\}$.

As it was announced in the introduction, in addition to the $\mathcal{L}_k^\alpha$ system we shall deal with other orthonormal systems closely related with it.

The Laguerre functions $\{\varphi_k^\alpha\}_{k=0}^\infty$, $\alpha > -1$.
We consider the orthonormal system in $L^2((0,\infty), dy)$ given by

$$\varphi_k^\alpha(y) = \mathcal{L}_k^\alpha(y^2)(2y)^{1/2}, \tag{5.8}$$

where $\mathcal{L}_k^\alpha$ are the functions defined in (5.2). The functions φ_k^α are eigenfunctions of the operator

$$\mathbf{L}_\alpha = \frac{1}{4}\left\{-\frac{d^2}{dy^2} + y^2 + \frac{1}{y^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}.$$

In fact,

$$\mathbf{L}_\alpha(\varphi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\ \varphi_k^\alpha. \tag{5.9}$$

The operator $\mathbf{L}_\alpha$ can be "factorized" as:

$$\mathbf{L}_\alpha - (\frac{\alpha+1}{2}) = (\mathbf{D}_\alpha)^*\mathbf{D}_\alpha,$$

being $\mathbf{D}_\alpha = \frac{1}{2}\left\{\frac{d}{dy} + y - \frac{1}{y}(\alpha + \frac{1}{2})\right\}$ and $\mathbf{D}_\alpha{}^* = \frac{1}{2}\left\{-\frac{d}{dy} + y - \frac{1}{y}(\alpha + \frac{1}{2})\right\}$, where $\mathbf{D}_\alpha{}^*$ is the formal adjoint of $\mathbf{D}_\alpha$ with respect to the Lebesgue measure. Then

$$\mathbf{D}_\alpha(\varphi_k^\alpha) = -\sqrt{k}\varphi_{k-1}^{\alpha+1} \quad \text{and} \quad (\mathbf{D}_{\beta-\mathbf{1}})^*(\varphi_k^\beta) = -\sqrt{k+1}\varphi_{k+1}^{\beta-1}. \tag{5.10}$$

According to [41] the Riesz transforms can be defined as

$$\mathbf{R}_+^\alpha = \mathbf{D}_\alpha(\mathbf{L}_\alpha)^{-1/2},\ \alpha > -1 \quad \text{and} \quad \mathbf{R}_-^\beta = (\mathbf{D}_{\beta-\mathbf{1}})^*(\mathbf{L}_\beta)^{-1/2},\ \beta > 0. \tag{5.11}$$

Hence, $\mathbf{R}_+^\alpha(\varphi_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\,\varphi_{k-1}^{\alpha+1}$ and $\mathbf{R}_-^\beta(\varphi_k^\beta) = -\frac{\sqrt{k+1}}{\sqrt{k+\frac{\beta+1}{2}}}\,\varphi_{k+1}^{\beta-1}$.
Then, the Riesz transform vector is

$$(\mathbf{R}_+^\alpha, \mathbf{R}_-^{\alpha+1}) = \Big(\mathbf{D}_\alpha(\mathbf{L}_\alpha)^{-1/2},\ (\mathbf{D}_\alpha)^*(\mathbf{L}_{\alpha+1})^{-1/2}\Big).$$

The Laguerre functions ℓ_k^α, $\alpha > -1$.

The orthonormal system, $\{\ell_k^\alpha\}_{k=0}^\infty$ in $L^2((0,\infty), d\mu_\alpha(y))$, $d\mu_\alpha(y) = y^\alpha dy$ is given by

$$\ell_k^\alpha(y) = \mathcal{L}_k^\alpha(y)y^{-\alpha/2},$$

where $\mathcal{L}_k^\alpha$ are the functions defined in (5.2). The functions ℓ_k^α are eigenfunctions of the differential operator

$$\mathbb{L}_\alpha = -y\frac{d^2}{dy^2} - (\alpha+1)\frac{d}{dy} + \frac{y}{4}.$$

More explicitly

$$\mathbb{L}_\alpha \ell_k^\alpha = \left(k + \frac{\alpha+1}{2}\right)\ \ell_k^\alpha. \tag{5.12}$$

The operator $\mathbb{L}_\alpha$ can be "factorized" as

$$\mathbb{L}_\alpha - \left(\frac{\alpha+1}{2}\right) = (\mathbb{D}_\alpha)^*\mathbb{D}_\alpha,$$

where $\mathbb{D}_\alpha = \sqrt{y}\frac{d}{dy} + \frac{1}{2}\sqrt{y}$ and $(\mathbb{D}_\alpha)^* = -\sqrt{y}\frac{d}{dy} + \frac{1}{2}\sqrt{y} - \frac{\alpha}{\sqrt{y}} - \frac{1}{2\sqrt{y}}$ is the formal adjoint of $\mathbb{D}_\alpha$ with respect to the measure $d\mu_\alpha$. Furthermore,

$$\mathbb{D}_\alpha \ell_k^\alpha(y) = -\sqrt{k}\sqrt{y}\ell_{k-1}^{\alpha+1}(y). \tag{5.13}$$

It is easy to check that $(\mathbb{D}_\alpha)^*(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot))$ is a function in $L^2(d\mu_\alpha)$, namely, $(\mathbb{D}_\alpha)^*\Big(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot)\Big)(y) = -\sqrt{k}\,\ell_k^\alpha(y)$. On the other hand, the family $\{\sqrt{y}\ell_k^{\alpha+1}(y)\}_k$ is orthonormal with respect to the measure $d\mu_\alpha$. Clearly, $\mathbb{D}_\alpha(\mathbb{D}_\alpha)^*\Big(\sqrt{(\cdot)}\ell_k^{\alpha+1}(\cdot)\Big)(y) = (k+1)\sqrt{y}\ell_k^{\alpha+1}(y)$. In this situation the Riesz transform vector becames

$$(\mathbb{R}_+^\alpha, \mathbb{R}_-^{\alpha+1}) = \Big(\mathbb{D}_\alpha(\mathbb{L}_\alpha)^{-1/2},\ (\mathbb{D}_\alpha)^*\left[\mathbb{D}_\alpha\,(\mathbb{D}_\alpha)^* + \frac{\alpha}{2}\right]^{-1/2}\Big).$$

Hence,

$$\mathbb{R}_+^\alpha(\ell_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\ \sqrt{y}\ell_{k-1}^{\alpha+1}(y) \ \text{ and}$$

$$\mathbb{R}_-^{\alpha+1}(\sqrt{(\cdot)}\ \ell_k^{\alpha+1}(\cdot))(y) = -\frac{\sqrt{k+1}}{(\sqrt{k+1+\frac{\alpha}{2}})}\ \ell_{k+1}^\alpha(y).$$

The Laguerre functions $\{\psi_k^\alpha\}_k$, $\alpha > -1$.
Let $\{\psi_k^\alpha\}_{k=0}^\infty$ be the the orthonormal system, in $L^2((0,\infty), y^{2\alpha+1}dy)$, given by $\psi_k^\alpha(y) = \sqrt{2}y^{-\alpha}\mathcal{L}_k^\alpha(y^2)$, where $\mathcal{L}_k^\alpha(y)$ are the functions defined in (5.2). The functions ψ_k^α are eigenfunctions for the operator $\mathfrak{L}_\alpha = -\frac{1}{4}\Big\{\frac{d^2}{dy^2} + \Big(\frac{2\alpha+1}{y}\Big)\frac{d}{dy} - y^2\Big\}$, in effect

$$\mathfrak{L}_\alpha(\psi_k^\alpha) = \left(k + \frac{\alpha+1}{2}\right)\ \psi_k^\alpha. \tag{5.14}$$

Furthermore, the operator $\mathfrak{L}_\alpha$ can be "factorized" as

$$\mathfrak{L}_\alpha - (\frac{\alpha+1}{2}) = (\mathfrak{D}_\alpha)^*\mathfrak{D}_\alpha,$$

with $\mathfrak{D}_\alpha = \frac{1}{2}\left\{\frac{d}{dy} + y\right\}$ and $(\mathfrak{D}_\alpha)^* = -\frac{1}{2}\left\{\frac{d}{dy} + \frac{(2\alpha+1)}{y} - y\right\}$, $(\mathfrak{D}_\alpha)^*$ turns out to be the adjoint of $\mathfrak{D}_\alpha$ with respect to the measure $d\omega_\alpha(y) = y^{2\alpha+1}dy$. Because of this

$$\mathfrak{D}_\alpha(\psi_k^\alpha)(y) = -\sqrt{k}y\psi_{k-1}^{\alpha+1}(y) \quad \text{and} \quad (\mathfrak{D}_\alpha)^*\Big((\cdot)\psi_{k-1}^{\alpha+1}(\cdot)\Big)(y) = -\sqrt{k}\psi_k^\alpha(y). \tag{5.15}$$

As a consequence, the Riesz transforms become

$$\mathfrak{R}_+^\alpha = \mathfrak{D}_\alpha(\mathfrak{L}_\alpha)^{-1/2}, \text{ and } \ \mathfrak{R}_-^\alpha = (\mathfrak{D}_\alpha)^*\Big[\mathfrak{D}_\alpha\,(\mathfrak{D}_\alpha)^* + \frac{\alpha}{2}\Big]^{-1/2}.$$

Hence,

$$\mathfrak{R}_+^\alpha(\psi_k^\alpha) = -\frac{\sqrt{k}}{\sqrt{k+\frac{\alpha+1}{2}}}\ \sqrt{y}\psi_{k-1}^{\alpha+1}(y) \quad \text{and}$$

$$\mathfrak{R}_-^{\alpha+1}(\sqrt{(\cdot)}\ \psi_k^{\alpha+1}(\cdot))(y) = -\frac{\sqrt{k+1}}{\sqrt{k+1+\frac{\alpha}{2}}}\ \psi_{k+1}^\alpha(y).$$

Notation. The following typographical convention will be used. The font-type of a letter, T, $\mathbf{T}$, $\mathbb{T}$ and $\mathfrak{T}$, will identify the orthonormal system under consideration, namely $\{\mathcal{L}_k^\alpha\}$, $\{\varphi_k^\alpha\}$, $\{\ell_k^\alpha\}$ and $\{\psi_k^\alpha\}$, respectively. For instance, L_α, $\mathbf{L}_\alpha$, $\mathbb{L}_\alpha$, $\mathfrak{L}_\alpha$, are the corresponding differential operators. Lemma 5.6 belongs to the folklore of the subject and it is not difficult to obtain a proof by means of the arguments in Theorem 5.7.1. of [92].

Lemma 5.6. *Let $\alpha > -1$, $1 < p < \infty$, δ, γ and ρ be real numbers, $d\mu_\alpha(y) = y^\alpha dy$ and $d\omega_\alpha(y) = y^{2\alpha+1}dy$.*

(i) The functions $\{\mathcal{L}_k^\alpha\}_k$ are in $L^p((0,\infty), y^\delta dy) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\delta} dy)$, if and only if, $-1-\alpha\frac{p}{2} < \delta < \alpha\frac{p}{2}+(p-1)$. Moreover the set S_α of finite linear combinations of Laguerre functions, $\{\mathcal{L}_k^\alpha\}_k$, is dense in $L^p((0,\infty), y^\delta dy)$ and $L^{p'}((0,\infty), y^{-\frac{p'}{p}\delta} dy)$.

(ii) The functions $\{\varphi_k^\alpha\}_k$ are in $L^p((0,\infty), y^\gamma dy) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\gamma} dy)$, if and only if, $-1-\alpha p - \frac{p}{2} < \gamma < \alpha p + \frac{p}{2} + (p-1)$. Besides the set $\mathbf{S}_\alpha$ of finite linear combinations of Laguerre functions $\{\varphi_k^\alpha\}_k$ is dense in $L^p((0,\infty), y^\gamma dy)$ and $L^{p'}((0,\infty), y^{-\frac{p'}{p}\gamma} dy)$.

(iii) The functions $\{\ell_k^\alpha\}_k$ are in $L^p((0,\infty), y^\rho d\mu_\alpha) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\rho} d\mu_\alpha)$, if and only if, $-1-\alpha < \rho < (\alpha+1)(p-1)$. The set $\mathbb{S}_\alpha$ of finite linear combinations of Laguerre functions $\{\ell_k^\alpha\}_k$ is dense in $L^p((0,\infty), y^\rho dy)$ and $L^{p'}((0,\infty), y^{-\frac{p'}{p}\rho} dy)$.

(iv) The functions $\{\psi_k^\alpha\}_k$ are in $L^p((0,\infty), y^\eta dy) \cap L^{p'}((0,\infty), y^{-\frac{p'}{p}\eta} d\omega_\alpha(y))$, if and only if, $-2(1+\alpha) < \eta < 2(p-1)(\alpha+1)$. In addition the set $\mathfrak{S}_\alpha$ of finite linear combinations of Laguerre functions $\{\psi_k^\alpha\}_k$ is dense in $L^p((0,\infty), y^\eta d\omega_\alpha)$ and $L^{p'}((0,\infty), y^{-\frac{p'}{p}\eta} d\omega_\alpha(y))$.

5.3. The isometries V, W^α and Z^α connecting the different Laguerre function systems

Let V, W^α and Z^α be the operators defined by

$$Vf(y) = (2y)^{1/2} f(y^2) \quad W^\alpha f(y) = y^{-\frac{\alpha}{2}} f(y) \quad \text{and} \quad Z^\alpha f(y) = \sqrt{2} y^{-\alpha} f(y^2),$$

For f a measurable function with domain on $(0,\infty)$. For further reference we state the following Lemma, whose simple proof is left to the reader.

Lemma 5.7. *Let $\alpha > -1$.*

(i) Let $2\delta = \gamma + \frac{p}{2} - 1$, then $\quad \|Vf\|_{L^p(y^\gamma\, dy)} = 2^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(y^\delta\, dy)}$.

(ii) *Let* $\delta = \rho - \alpha(\frac{p}{2} - 1)$, *then* $\|W^\alpha f\|_{L^p(y^\rho\, d\mu_\alpha)} = \|f\|_{L^p(y^\delta\, dy)}$, *where* $d\mu_\alpha(y) = y^\alpha dy$.

(iii) *Let* $\delta = \frac{\eta}{2} - \alpha(\frac{p}{2} - 1)$, *then* $\|Z^\alpha f\|_{L^p(y^\eta\, d\omega_\alpha)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p(y^\delta\, dy)}$, *where* $d\omega_\alpha(y) = y^{2\alpha+1}dy$.

Remark 5.8. Given a Banach space B, and a strongly measurable B-valued function f, we can define the operators

$$V_B f(y) = (2y)^{1/2}\, f(y^2) \quad W_B^\alpha f(y) = y^{-\alpha/2}\, f(y) \quad \text{and} \quad Z_B^\alpha f(y) = 2y^{-\alpha}\, f(y^2).$$

Hence

$$\|V_B\, f(y)\|_B = V(\|f\|_B)(y), \|W_B^\alpha\, f(y)\|_B = W^\alpha(\|f\|_B)(y), \|Z_B^\alpha\, f(y)\|_B = Z^\alpha(\|f\|_B)(y).$$

Therefore, under the conditions of Lemma 5.7, for any Banach space B the identities

$$\|V_B f\|_{L_B^p(y^\gamma\, dy)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L_B^p(y^\delta\, dy)}, \ \|W_B^\alpha f\|_{L_B^p(y^\rho\, d\mu_\alpha)} = \|f\|_{L_B^p(y^\delta\, dy)} \ \text{and}$$

$$\|Z_B^\alpha f\|_{L_B^p(y^\eta\, d\omega_\alpha)} = 2^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L_B^p(y^\delta\, dy)}$$

hold. Moreover if $B = \ell^\infty$ and $f = (f_j)_j$ is a strongly measurable ℓ^∞-valued function then

$$V_{\ell^\infty} f = (V\, f_j)_j, \quad W_{\ell^\infty}^\alpha f = (W^\alpha\, f_j)_j \ \text{and} \quad Z_{\ell^\infty}^\alpha f = (Z^\alpha\, f_j)_j.$$

Analogously, if $B = L^2(\Omega, d\mu)$ and $f(\cdot) = g(\cdot, z)$, $z \in \Omega$, is a $L^2(\Omega, d\mu)$-valued function if follows

$$V_{L^2(\Omega,d\mu)} f(y) = V g(y,z) = g(y^2, z)(2y)^{1/2}, \quad W_{L^2(\Omega,d\mu)}^\alpha f(y) = W^\alpha g(y,z) = g(y,z)\, y^{-\alpha/2}$$

$$\text{and} \quad Z_{L^2(\Omega,d\omega)}^\alpha f(y) = Z^\alpha g(y,z) = \sqrt{2} g(y^2, z)\, y^{-\alpha}.$$

Proposition 5.3.1. Let $1 < p < \infty$, δ, γ and ρ, η be real numbers. Let B_1, B_2 be Banach spaces and T be an operator defined over the set of finite linear combination of Laguerre functions $\{\mathcal{L}_k^\alpha\}_k$.

(i) The operator T has a bounded extension from $L_{B_1}^p((0,\infty), y^\delta dy)$ into $L_{B_2}^p((0,\infty), y^\delta dy)$ if and only if the operator $\mathbf{T} = V_{B_2}\, T\, V_{B_1}^{-1}$ has a bounded extension from $L_{B_1}^p((0,\infty), y^\gamma dy)$ into $L_{B_2}^p((0,\infty), y^\gamma dy)$, where $2\delta = \gamma + \frac{p}{2} - 1$.

(ii) The operator T has a bounded extension from $L_{B_1}^p(y^\delta dy)$ into $L_{B_2}^p(y^\delta dy)$ if and only if the operator $\mathbb{T} = W_{B_2}^\alpha\, T\, (W_{B_1}^\alpha)^{-1}$ has a bounded extension from $L_{B_1}^p(y^\rho\ d\mu_\alpha(y)$ into $L_{B_2}^p(y^\rho\ d\mu_\alpha(y))$, with $\delta = \rho - \alpha(\frac{p}{2} - 1)$ and $d\mu_\alpha(y) = y^\alpha dy$.

(iii) The operator T has a bounded extension from $L^p_{B_1}(y^\delta dy)$ into $L^p_{B_2}(y^\delta dy)$ if and only if the operator $\mathfrak{T} = Z^\alpha_{B_2}\, T\, (Z^\alpha_{B_1})^{-1}$ has a bounded extension from $L^p_{B_1}(y^\eta\; d\omega_\alpha(y)$ into $L^p_{B_2}(y^\eta\; d\omega_\alpha(y))$, if $\delta = \dfrac{\eta}{2} - \alpha(\dfrac{p}{2} - 1)$ and $d\omega_\alpha(y) = y^{2\alpha+1}dy$.

Furthermore, the norms of the operators T, $\mathbf{T}$, $\mathbb{T}$ and $\mathfrak{T}$ coincide.

Proof. Let f be a finite linear combination of the Laguerre functions φ^α_k, then $V^{-1}f$ is a finite linear combination of Laguerre functions $\mathcal{L}^\alpha_k$. By using the relation between the operators $\mathbf{T}$ and T and appropriate changes of variables, it follows

$$\begin{aligned}\int_0^\infty \|\mathbf{T}f(y)\|^p_{B_2}\, y^\gamma dy &= \int_0^\infty \|T\, V^{-1}_{B_1} f(y^2)\|^p_{B_2}\, (2y)^{p/2} y^\gamma dy \\ &= \int_0^\infty \|T\, V^{-1}_{B_1} f(u)\|^p_{B_2}\, 2^{p/2} u^{p/4} u^{\gamma/2} \frac{1}{2} u^{-1/2} du = 2^{p/2-1} \int_0^\infty \|T\, V^{-1}_{B_1} f(u)\|^p_{B_2} u^\delta du \\ &\le 2^{p/2-1} \|T\|^p \int_0^\infty \|V^{-1}_{B_1} f(u)\|^p_{B_1}\, u^\delta du = 2^{p/2-1}\|T\|^p \int_0^\infty \|f(u^{1/2})\|^p_{B_1}\, (2u^{1/2})^{-p/2} u^\delta du \\ &= \|T\|^p \int_0^\infty \|f(y)\|^p_{B_1}\, y^{-p/2} y^{2\delta} y dy = \|T\|^p \int_0^\infty \|f(y)\|^p_{B_1}\, y^\gamma dy.\end{aligned}$$

Then Lemma 5.6 gives (i). The proof of (ii) and (iii) are analogous. ■

Observe that as a byproduct of this proof we get

$$\begin{aligned}\|T\|_{\mathcal{L}(L^p(y^\delta,dy),L^p(y^\delta,dy))} &= \|\mathbf{T}\|_{\mathcal{L}(L^p(y^\gamma,dy),L^p(y^\gamma,dy))} = \|\mathbb{T}\|_{\mathcal{L}(L^p(y^\rho\, d\mu_\alpha),L^p(y^\rho\, d\mu_\alpha))} \\ &= \|\mathfrak{T}\|_{\mathcal{L}(L^p(y^\eta\, d\omega_\alpha),L^p(y^\eta\, d\omega_\alpha))}.\end{aligned}$$

The following Proposition shows how the operators, defined at the beginning of Section 5.2, for different Laguerre function systems are related by means of the isometries V, W^α, Z^α.

Proposition 5.3.2. Let $\alpha > -1$ and let f be a finite linear combination of Laguerre functions $\{\mathcal{L}^\alpha_k\}$. Therefore

(i) $e^{-tL_\alpha} f = V^{-1} e^{-t\mathbf{L}_\alpha} V f = (W^\alpha)^{-1}\, e^{-t\mathbb{L}_\alpha} W^\alpha f = (Z^\alpha)^{-1}\, e^{-t\mathfrak{L}_\alpha} Z^\alpha f$,

(ii) $\sup_t e^{-tL_\alpha} f(y) = V^{-1} \sup_t e^{-t\mathbf{L}_\alpha} V f(y) = (W^\alpha)^{-1} \sup_t e^{-t\mathbb{L}_\alpha} W^\alpha f(y)$ $= (Z^\alpha)^{-1} \sup_t e^{-t\mathfrak{L}_\alpha} Z^\alpha f(y)$,

(iii) Given $s > 0$, $\quad (L_\alpha)^{-s} f = V^{-1}(\mathbf{L}_\alpha)^{-s} V f = (W^\alpha)^{-1} (\mathbb{L}_\alpha)^{-s} W^\alpha f$ $= (Z^\alpha)^{-1} (\mathfrak{L}_\alpha)^{-s} Z^\alpha f$,

(iv) $D_\alpha f = V^{-1}\mathbf{D}_\alpha V f = (W^\alpha)^{-1}\mathbb{D}_\alpha W^\alpha f = (Z^\alpha)^{-1}\mathfrak{D}_\alpha Z^\alpha f$.

(v) $R_+^\alpha f = V^{-1}\mathbf{R}_+^\alpha Vf = (W^\alpha)^{-1}\mathbb{R}_+^\alpha W^\alpha f = (Z^\alpha)^{-1}\mathfrak{R}_+^\alpha Z^\alpha f$ and

(vi) $\frac{\partial}{\partial t}e^{-t\sqrt{L_\alpha}}f = V^{-1}\frac{\partial}{\partial t}e^{-t\sqrt{\mathbf{L}_\alpha}}Vf = (W^\alpha)^{-1}\frac{\partial}{\partial t}e^{-t\sqrt{\mathbb{L}_\alpha}}W^\alpha f, = (Z^\alpha)^{-1}\frac{\partial}{\partial t}e^{-t\sqrt{\mathfrak{L}_\alpha}}Z^\alpha f.$

Besides if g is a finite combination of Laguerre functions $\{\mathcal{L}_k^{\alpha+1}\}_k$, then

$$R_-^{\alpha+1}g = V^{-1}\mathbf{R}_-^{\alpha+1}Vf = (W^\alpha)^{-1}\mathbb{R}_-^{\alpha+1}W^\alpha f = (Z^\alpha)^{-1}\mathfrak{R}_-^{\alpha+1}Z^\alpha f.$$

Proof. Observe that $\mathcal{L}_k^\alpha = V^{-1}\varphi_k^\alpha = (W^\alpha)^{-1}\ell_k^\alpha = (Z^\alpha)^{-1}\psi_k^\alpha$. Hence, in order to prove (i), (ii) and (iii) we just use (5.3), (5.9), (5.12) and (5.14). On the other hand $\mathcal{L}_{k-1}^{\alpha+1} = (W^\alpha)^{-1}(\sqrt{(\cdot)}\ell_{k-1}^{\alpha+1}(\cdot)) = (Z^\alpha)^{-1}((\cdot)\psi_{k-1}^{\alpha+1}(\cdot))$. Thus (5.5), (5.10), (5.13),(5.15) give (iv), (v) and (vi). ■

Theorem 5.9. *Let $\alpha > -1$, $1 < p < \infty$, δ, ρ, η and γ be real numbers, such that $2\delta = \gamma + \frac{p}{2} - 1$, $\delta = \rho + \alpha - \frac{\alpha p}{2}$, and $\delta = \frac{\eta}{2} - \frac{\alpha p}{2} + \alpha$. Let S stand for any of the operators e^{-tL_α}, T^*, P^*, g_α, R_+^α, $R_-^{\alpha+1}$ defined in Section 5.2, then the following conditions are equivalent:*

(i) The operator S has a bounded extension from $L^p((0,\infty), y^\delta dy)$ into itself.

(ii) The operator $\mathbf{S}$ has a bounded extension from $L^p((0,\infty), y^\gamma dy)$ into itself.

(iii) The operator $\mathbb{S}$ has a bounded extension from $L^p((0,\infty), y^\rho d\mu_\alpha)$ into itself.

(iv) The operator $\mathfrak{S}$ has a bounded extension from $L^p((0,\infty), y^\eta d\omega_\alpha)$ into itself.

In addition, the norms of the operators S, $\mathbf{S}$, $\mathbb{S}$ and $\mathfrak{S}$ coincide.

Proof. If S is any of the operators e^{-tL_α}, T^*, P^*, $L^{-1/2}$, R_+^α, R_-^α, the proof follows directly from Propositions 5.3.2 and 5.3.1. When $S = g_\alpha$, we observe that S is bounded from $L^p((0,\infty), y^\delta dy)$ into $L^p((0,\infty), y^\delta dy)$, if and only if the operator $f \to \frac{\partial}{\partial t}e^{-t\sqrt{L_\alpha}}f$ is bounded from $L^p((0,\infty), y^\delta dy)$ into $L^p((0,\infty), y^\delta dy)_{L^2((0,\infty),\frac{dt}{t})}$. Consequently the result follows by using (vi) in Proposition 5.3.2 and Proposition 5.3.1. ■

The Theorem above shows the equivalence of the boundedness of the corresponding operators for the different systems of Laguerre functions. In the next Theorem we prove that they are actually bounded.

Theorem 5.10. *Let $\alpha > -1$, $1 < p < \infty$, δ, ρ, η and γ be real numbers. Let S be any one of the operators e^{-tL_α}, T^*, P^*, g_α, R_+^α, $R_-^{\alpha+1}$ and L^{-s}, $s > 0$. Therefore le*

(i) the operator S has a bounded extension from $L^p((0,\infty), y^\delta dy)$ into itself, for δ such that

$$-1 - \frac{\alpha p}{2} < \delta < \frac{\alpha p}{2} + p - 1,$$

(ii) the operator $\mathbf{S}$ *has a bounded extension from* $L^p((0,\infty), y^\gamma dy)$ *into itself, for* γ *satisfying*

$$-1-\alpha p-\frac{p}{2}<\gamma<\alpha p+\frac{3p}{2}-1.$$

(iii) the operator $\mathbb{S}$ *has a bounded extension from* $L^p((0,\infty), y^\rho d\mu_\alpha)$ *into itself, for* ρ *in the range*

$$-1-\alpha<\rho<(\alpha+1)(p-1).$$

(iv) The operator $\mathfrak{S}$ *has a bounded extension from* $L^p((0,\infty), y^\eta d\omega_\alpha)$ *into itself, for* η

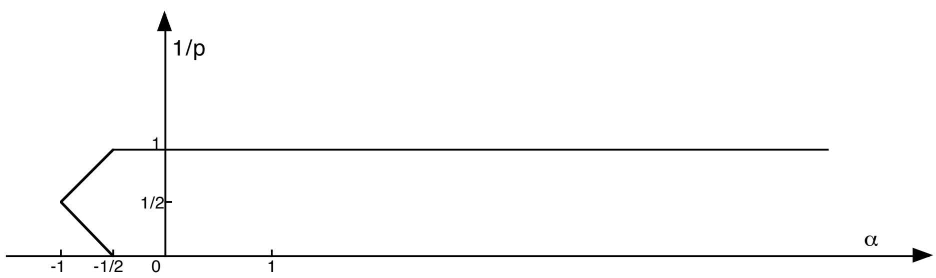
$$-2(1+\alpha)<\eta<2(p-1)(\alpha+1).$$

Proof. The boundedness of the maximal operator, the Riesz transform and the $g-$function in the case (i) was proved in [24], [39], [30], respectively. The boundedness of L^{-s} is due to B. Bongioanni (oral communication). The theorem follows by applying Theorem 5.9. ∎

Remark 5.11. We observe that in the case $\mathbf{S}$, that is for for the system $\{\varphi_k^\alpha\}$, and $\gamma = 0$ we get the condition

$$-\alpha-\frac{1}{2}<\frac{1}{p}<\alpha+\frac{3}{2}.$$

This restriction on α and $\frac{1}{p}$ can be visualized as the shaded region in the following picture



However, in the cases $\mathbb{S}, \mathfrak{S}$ and $\rho = 0, \eta = 0$ we get

$$-(1+\alpha)<0<(p-1)(1+\alpha) \quad \text{and} \quad -2(1+\alpha)<0<2(p-1)(\alpha+1). \quad (5.16)$$

In other words, for the system $\{\varphi_k^\alpha\}_k$ there is a "pencil phenomenon"for $-1<\alpha<-\frac{1}{2}$.

Since the conditions (5.16) are fulfilled for any $\alpha > -1$ and every $1 < p < \infty$, there is not "pencil phenomenon"for the systems $\{\ell_k^\alpha\}_k$ and $\{\psi_k^\alpha\}_k$.

5.4. Application to operators related with Laguerre polynomials

As is well known the one-dimensional Laguerre polynomials of type $\alpha > -1$ can be defined by $L_k^\alpha(y) = \frac{1}{k!}e^y y^{-\alpha}\frac{d^k}{dy^k}(e^{-y}y^{k+\alpha})$. They form a complete orthogonal system in $L^2((0,\infty), d\gamma_\alpha(y))$ where $d\gamma_\alpha(y) = y^\alpha e^{-y}dy$. For a given $\alpha > -1$, the Laguerre differential operator is

$$\Pi_\alpha = -y\frac{d^2}{dy^2} - (\alpha+1-y)\frac{d}{dy}.$$

The polynomials L_k^α satisfy $\Pi_\alpha L_k^\alpha(y) = kL_k^\alpha(y)$.

If $\text{grad}_\alpha f(y) = \sqrt{y}\frac{d}{dy}f(y)$ and $\text{div}_\alpha f(y) = -\sqrt{y}\Big(\frac{d}{dy}f(y) + (\frac{\alpha+1/2}{y} - 1)f(y)\Big)$, then $\Pi_\alpha = \text{div}_\alpha \text{grad}_\alpha$. The following lemma establishes a connection between Laguerre polynomials and Laguerre functions. The proof is simple and we leave it to the reader.

Lemma 5.12. *Let $\Lambda_\alpha f(y) = f(y)y^{-\alpha/2}e^{y/2}$. For any weight ω, The operator Λ_α is an isometry from $L^p((0,\infty), \omega(y)dy)$ into $L^p\Big((0,\infty),\, \omega(y)y^{-\alpha(1-\frac{p}{2})}e^{y(1-\frac{p}{2})}d\gamma_a(y)\Big)$. Moreover, the following identities are satisfied for any polynomial :*

(i) $\Pi_\alpha f = \Lambda_\alpha \circ \Big(L_\alpha - \frac{\alpha+1}{2}\Big) \circ (\Lambda_\alpha)^{-1}f,$

(ii) $\sup_t e^{-t\Pi_\alpha} f = \Lambda_\alpha \circ \sup_t e^{-t(L_\alpha - \frac{\alpha+1}{2})} \circ (\Lambda_\alpha)^{-1}f,$

(iii) $\text{grad}_\alpha f = \Lambda_\alpha \circ D_a \circ (\Lambda_\alpha)^{-1}f$ *and* $\text{div}_\alpha f = \Lambda_\alpha \circ (D_a)^* \circ (\Lambda_\alpha)^{-1}f,$

where D_α, $(D_\alpha)^*$ and L_α are defined in (5.4) and (5.1).

In the setting of Laguerre polynomials the Riesz transforms can be defined by

$$\Re_\alpha = \text{grad}_\alpha(\Pi_a)^{-1/2}, \tag{5.17}$$

see [36]. In that paper it was proved that these operators are bounded in $L^p(d\gamma_a)$ for $1 < p < \infty$ and $\alpha = \frac{n}{2} - 1$, $n = 1, 2, \ldots$. Later on in [68] the boundedness in $L^p(d\gamma_\alpha)$ was proved for any $\alpha \geq -\frac{1}{2}$.

Theorem 5.13. *Let $\alpha > -1$. Let $\mathcal{T}_\alpha$ be either the heat maximal semigroup, $\sup_t e^{-t\Pi_\alpha}$, or the Riesz transform, $\Re_\alpha$, associated to the Laguerre operator Π_α. Then, $\mathcal{T}_\alpha$ is a bounded operator from $L^p(y^\sigma e^{y(1-p/2)}d\gamma_\alpha)$ into itself for $-(1+\alpha) < \sigma < (p-1)(1+\alpha)$.*

In order to prove this theorem we need some known results on Hermite functions. The Hermite polynomials, H_k, are given by the formula $H_k(t) = (-1)^k e^{t^2} \frac{d^k e^{-t^2}}{dt^k}$, $t \in \mathbb{R}$. The Hermite normalized functions with respect to Lebesgue measure turn out to be $h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} H_k(t) e^{-t^2/2}$, $t \in \mathbb{R}$. Given the multiindex $\mu \in \mathbb{Z}_+^n$, the $n-$dimensional Hermite functions h_μ are defined by

$$h_\mu(x) = \Pi_{i=1}^n h_{\mu_i}(x_i), \quad \mu = (\mu_1, \ldots, \mu_n).$$

The second order Hermite differential operator on $\mathbb{R}^n$ is

$$\mathbf{H}_n = -\Delta + |x|^2 = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + |x|^2.$$

This operator is nonegative and selfadjoint with respect to the Lebesgue measure on $\mathbb{R}^n$. Its eigenfunctions are the $n-$dimensional Hermite functions, in fact

$$\mathbf{H}_n(h_\mu) = (2|\mu| + n)h_\mu. \tag{5.18}$$

It is kown that if

$$A_j g(x) = \frac{\partial g}{\partial x_j}(x) + x_j \quad \text{and} \quad A_j^* g(x) = -\frac{\partial g}{\partial x_j}(x) + x_j, \ j \in \{1, \ldots, n\}, \tag{5.19}$$

then $\mathbf{H}_n - n = \sum_{j=1}^n A_j^* A_j$.

For a measurable function $f : (0, \infty) \to \mathbb{R}$ we define the function $\mathcal{W}_\alpha(f) : \mathbb{R}^n \to \mathbb{R}$ as $\mathcal{W}_\alpha(f)(x) = \frac{f(|x|^2)}{|x|^\alpha}$. It is easy to see that

$$\|\mathcal{W}_\alpha f\|_{L^p(\mathbb{R}^n, |x|^\tau dx)} = C_{p,\alpha} \|f\|_{L^p((0,\infty), y^\delta dy)}, \qquad \text{with} \quad \delta = \frac{\tau - \alpha p}{2} + \frac{n}{2} - 1. \tag{5.20}$$

Lemma 5.14. *Let D_α and A_j as in (5.4) and (5.19) respectively. For a function $f : (0, \infty) \to \mathbb{R}$, good enough, we have*

(i) $A_j(\mathcal{W}_\alpha(f))(x) = 2\frac{x_j}{|x|} \mathcal{W}_\alpha(D_\alpha f)(x)$, $j = 1, \ldots, n$.

(ii) If $\alpha = \frac{n}{2} - 1$ then, when f is a finite linear combination of Laguerre functions $\mathcal{L}_k^\alpha$

$$\mathcal{W}_\alpha(L_\alpha f)(x) = \frac{1}{4} \mathbf{H}_n \mathcal{W}_\alpha(f)(x),$$

$$\mathcal{W}_\alpha(e^{-tL_\alpha}f)(x) = e^{-t\frac{1}{4}\mathbf{H}_n}\mathcal{W}_\alpha(f)(x) \ \text{and}$$
$$\mathcal{W}_\alpha((L_\alpha)^{-1/2}f)(x) = (\mathbf{H}_n)^{-1/2}\mathcal{W}_\alpha f(x),$$

hold.

Proof. Observe that given a function $g : \mathbb{R}^n \to \mathbb{R}$ it follows

$$\begin{aligned}
\frac{\partial}{\partial x_j}\frac{f(|x|^2)}{|x|^\alpha} &= f'(|x|^2)\frac{2x_j}{|x|^\alpha} - \alpha\frac{f(|x|^2)}{|x|^{\alpha+1}}\frac{x_j}{|x|} \\
&= \frac{2x_j}{|x|}\left\{\frac{|x|}{|x|^\alpha}f'(|x|^2) - \frac{1}{2}\Big(\frac{\alpha}{|x|}\Big)\frac{f(|x|^2)}{|x|^\alpha}\right\} \\
&= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) - \frac{1}{2}\Big(\frac{\alpha}{|x|}\Big)\mathcal{W}_\alpha f(x)\right\}.
\end{aligned}$$

Accordingly,

$$\begin{aligned}
A_j(\mathcal{W}_\alpha f)(x) &= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) - \frac{1}{2}\Big(\frac{\alpha}{|x|}\Big)\mathcal{W}_\alpha f(x)\right\} + x_j\mathcal{W}_\alpha f(x) \\
&= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) - \frac{\alpha}{2|x|}\mathcal{W}_\alpha f(x) + \frac{|x|}{2}\mathcal{W}_\alpha f(x)\right\} \\
&= \frac{2x_j}{|x|}\left\{\mathcal{W}_\alpha(\sqrt{(\cdot)}f'(\cdot))(x) + \frac{1}{2}\Big(|x| - \frac{\alpha}{|x|}\Big)\mathcal{W}_\alpha f(x)\right\} \\
&= \frac{2x_j}{|x|}\mathcal{W}_\alpha(D_\alpha f)(x).
\end{aligned}$$

In order to prove (ii), we recall the relationship between the families of Laguerre and Hermite functions given by

$$\mathcal{L}_k^\alpha(|x|^2) = \mathcal{W}_\alpha(\mathcal{L}_k^\alpha)(x)|x|^\alpha = c_k^\alpha\sum_{|r|=k}\frac{a_r}{b_{2r}}h_{2r}(x)|x|^\alpha, \quad x \in \mathbb{R}^n, \quad \alpha = \frac{n}{2} - 1, \qquad (5.21)$$

where $h_s(x)$ are the Hermite functions on $\mathbb{R}^n$, of order $|s|$; c_k^α and b_k are the orthonormalization coefficients given by $c_k^\alpha = (\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)})^{1/2}$, $b_k = (2^k k!\sqrt{\pi})^{-1/2}$ and $b_\mu = \Pi_{i=1}^n b_{\mu_i}$, see [41]. The coefficients a_r are those given in the known formula

$$L_k^\alpha(|x|^2) = \sum_{|r|=k}a_rH_{2r}(x), \quad r = (r_1,\dots,r_n), \quad \alpha = \frac{n}{2} - 1.$$

See [92, formula 5.6.1] in the case $n = 1$ and [36, Lemma 1.1] in the general case. The proof of (ii) is a consequence of formulas (5.18), (5.3) and (5.21). ∎

In order to prove Theorem 5.13 we will need the following

Lemma 5.15. *Let $-1 < \alpha$, $1 < p < \infty$ and δ be real numbers such that $-1 - \dfrac{\alpha p}{2} < \delta < +\dfrac{\alpha p}{2} + (p-1)$. If L_α is defined as in (5.1) then the maximal operator*

$$\sup_t \left| e^{-t(L_\alpha - \frac{\alpha+1}{2})} f(y) \right|$$

is bounded from $L^p((0,\infty), y^\delta dy)$ into itself.

Proof. We have

$$e^{-t(L_\alpha - \frac{\alpha+1}{2})} f(y) = e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y).$$

Thus

$$\begin{aligned}
\sup_t \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| &\leq \sup_{t\leq 1} \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| + \sup_{t>1} \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| \\
&\leq e^{\frac{\alpha+1}{2}} \sup_t \left| e^{-tL_\alpha} f(y) \right| + \sup_{t>1} \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| \\
&= A + B.
\end{aligned}$$

In [24] and [54] it is shown that $\|A\|_{L^p((0,\infty),y^\delta dy)} \leq C\|f\|_{L^p((0,\infty),y^\delta dy)}$. As for B, taken a function f good enough, it follows that

$$\begin{aligned}
\sup_{t\geq 1} \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| &= \sup_{t\geq 1} \left| \int_{[0,\infty)} \sum_k e^{-tk} \mathcal{L}_k^\alpha(y) \mathcal{L}_k^\alpha(z) f(z) dz \right| \\
&\leq \sum_k e^{-k} |\mathcal{L}_k^\alpha(y)| \left| \int_{[0,\infty)} \mathcal{L}_k^\alpha(z) f(z) dz \right|.
\end{aligned}$$

Then by Hölder's inequality we get

$$\sup_{t\geq 1} \left| e^{t\frac{\alpha+1}{2}} e^{-tL_\alpha} f(y) \right| \leq \sum_k e^{-k} |\mathcal{L}_k^\alpha(y)| \; \|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}dy)} \|f\|_{L^p([0,\infty),y^\delta dy)}.$$

Hence by Minkowski's inequality

$$\|B\|_{L^p([0,\infty),y^\delta dy)} \leq \sum_k e^{-k} \|\mathcal{L}_k^\alpha\|_{L^p([0,\infty),y^\delta dy)} \; \|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}dy)} \|f\|_{L^p([0,\infty),y^\delta dy)}.$$

¿From [94, le 1.5.4] we obtain that if $\delta > -1 - \frac{\alpha p}{2}$ then $\|\mathcal{L}_k^\alpha\|_{L^p([0,\infty),y^\delta dy)} \leq Ck^{\theta_1}$, for some $\theta_1 > 0$. Analogously when $\delta < p - 1 + \frac{\alpha p}{2}$, applying the same Lemma we get $\|\mathcal{L}_k^\alpha\|_{L^{p'}([0,\infty),y^{-\delta p'/p}dy)} \leq Ck^{\theta_2}$, for some $\theta_2 > 0$. Therefore

$$\|B\|_{L^p([0,\infty),y^\delta dy)} \leq C\Big(\sum_k e^{-k} k^{\theta_1+\theta_2} \Big) \|f\|_{L^p([0,\infty),y^\delta dy)} \leq C\|f\|_{L^p([0,\infty),y^\delta dy)}.$$

■

Proof of Theorem 5.13.
By Lemma 5.12 we have

$$\|\sup_t e^{-t\mathcal{L}_\alpha} g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} = \|\sup_t e^{-t(L-\frac{\alpha+1}{2})} \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)} dy)}.$$

Observe that

$$-1 - \alpha p/2 < \sigma + \alpha(1-p/2) < p - 1 + \alpha p/2. \tag{5.22}$$

Hence by Lemma 5.15, for $\delta = \sigma + \alpha(1-p/2)$, we have

$$\begin{aligned} \|\sup_t e^{-t(L-\frac{\alpha+1}{2})} \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)} dy)} &\leq C_p \|(\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)} dy)} \\ &= C_p \|g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)}. \end{aligned}$$

This finishes the proof for the maximal operator.

As for the Riesz transform, by Lemma 5.12 and the definition of R_+^α given in (5.6), we get

$$\begin{aligned} \Re_\alpha g &= \operatorname{grad}_\alpha (\mathcal{L}_\alpha)^{-1/2} g = \Lambda_\alpha \circ D_\alpha \circ (L_\alpha - \frac{\alpha+1}{2})^{-1/2} \circ (\Lambda_\alpha)^{-1} g \\ &= \Lambda_\alpha \circ D_\alpha \circ \Big((L_\alpha - \frac{\alpha+1}{2})^{-1/2} - (L_\alpha)^{-1/2} \Big) \circ (\Lambda_\alpha)^{-1} g + \Lambda_\alpha \circ R_+^\alpha \circ (\Lambda_\alpha)^{-1} g = I + II. \end{aligned}$$

Again, by Lemma 5.12, we get $\|II\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} = \|R_+^\alpha \circ (\Lambda_\alpha)^{-1} g\|_{L^p(y^\sigma y^{\alpha(1-p/2)} dy)}$. By (5.22), applying Theorem 5.2 and Lemma 5.12 we have

$$\|II\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} \leq C \|(\Lambda_\alpha)^{-1} g\|_{L^p(y^{\sigma+\alpha(1-p/2)} dy)} = C \|g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)}.$$

To finish the proof for the Riesz transforms it is enough to show

$$\|I\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)} \leq C \|g\|_{L^p(y^\sigma e^{y(1-p/2)} d\gamma_\alpha)}.$$

This is equivalent to show the boundedness of $D_\alpha \circ \Big((L_\alpha - \frac{\alpha+1}{2})^{-1/2} - (L_\alpha)^{-1/2} \Big)$ from $L^p(y^\delta dy)$ into itself, for $-1 - \alpha p/2 < \delta < p - 1 + \alpha p/2$. Moreover

$$D_\alpha \circ \Big((L_\alpha - \frac{\alpha+1}{2})^{-1/2} - (L_\alpha)^{-1/2} \Big) = T_\alpha^\beta \circ D_\beta \circ \tau_m \circ \Big((L_\beta - \frac{\beta+1}{2})^{-1/2} - (L_\beta)^{-1/2} \Big) \circ T_\beta^\alpha,$$

where T_β^α is the transplantation operator, $T_\beta^\alpha(\sum c_\alpha \mathcal{L}_k^\alpha) = \sum c_\alpha \mathcal{L}_k^\beta$, and τ_m is the multiplier operator given by the $\mathcal{C}^\infty$ function

$$m(t) = \frac{\frac{\alpha+1}{2} \sqrt{t + \frac{\beta+1}{2}} \left(\sqrt{t + \frac{\beta+1}{2}} + \sqrt{t} \right)}{\frac{\beta+1}{2} \sqrt{t + \frac{\alpha+1}{2}} \left(\sqrt{t + \frac{\alpha+1}{2}} + \sqrt{t} \right)}.$$

By the Transplantation Theorem 5.4 and the Multiplier Theorem 5.3 we just need to prove the boundedness of the operator $D_\beta \circ \Big((L_\beta - \frac{\beta+1}{2})^{-1/2} - (L_\beta)^{-1/2} \Big)$ from $L^p(y^\delta dy)$ into itself for $-1-\beta p/2 < \delta < p-1+\beta p/2$, for a β bigger than α. Choosing $\beta = \frac{n}{2} - 1$, by Lemma 5.14 we obtain

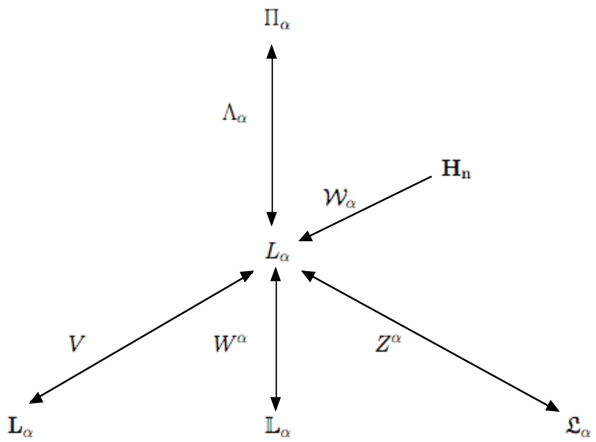
$$\begin{aligned} &\Big| \mathcal{W}_\beta \circ D_\beta \circ \Big((L_\beta - \frac{\beta+1}{2})^{-1/2} - (L_\beta)^{-1/2} \Big) f \Big| \\ &= \Big\{ \sum_{j=1}^{n} \Big(A_j \circ ((\mathbf{H}_n - n)^{-1/2} - (\mathbf{H}_n)^{1/2}) \circ \mathcal{W}_\beta f \Big)^2 \Big\}^{1/2}. \end{aligned}$$

Since $A_j \circ ((\mathbf{H}_n - n)^{-1/2} - (\mathbf{H}_n)^{-1/2})$, $j = 1, \ldots, n$; is bounded in $L^p(\mathbb{R}, |x|^\tau dx)$ for $-1 - \beta p/2 < \frac{\tau - \beta p}{2} + \beta < p - 1 + \beta p/2$, see [2, Theorem 3.5], and $\mathcal{W}_\beta$ is an isometry therefore

$$\| D_\beta \circ \Big((L_\beta - \frac{\beta+1}{2})^{-1/2} - (L_\beta)^{-1/2} \Big) f \Big) \|_{L^p(y^\delta dy)} \leq C_p \| f \|_{L^p(y^\delta dy)}.$$

■

The following diagram illustrates the relations among the different infinitesimal generators and the isometries considered in this paper.



$\alpha > -1$ and for $\mathcal{W}_\alpha$, $\alpha = \frac{n}{2} - 1$.

Acknowledgement. The authors want to thank professor G. Garrigós his helpful commentaries.

Capítulo 6

Regiones de acotación del operador maximal de Laguerre

6.1. Introducción

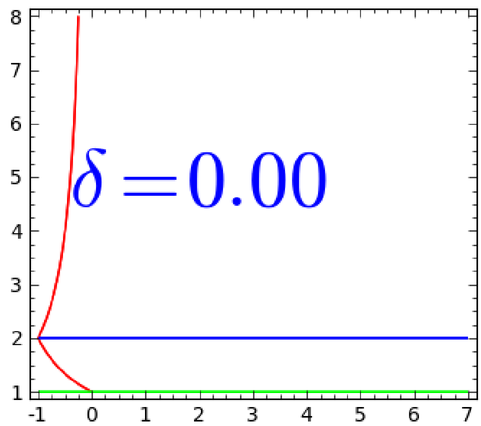
Como ya hemos dicho en el Capítulo 1, las figuras que se presentan en el Capítulo 5 tienen como variables: ordenada $1/p$ y abscisa α, además la potencia del peso es $\delta = 0$. Para visualizar mejor cúal es el comportamiento del semigrupo en el caso $\delta \neq 0$ es mejor presentar figuras tridimensionales en las cuales la coordenada vertical es p, una de las horizontales es α y la otra δ. A continuación presentamos dichas figuras para cada uno de los sistemas de Laguerre considerados en el Capítulo anterior, junto con sus cortes verticales. En la última Sección del Capítulo hemos copiado una sintaxis empleada junto con la figura resultante.

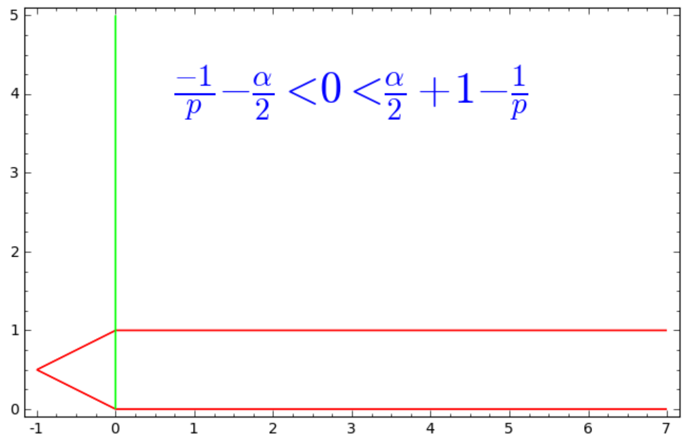
6.2. Familia $\{\mathcal{L}_k^\alpha\}_k$

Recordemos que, ver la Sección 5.3 del Capítulo 5 que la relacin entre p, α, δ que garantiza acotaciones para los operadores manejados es

$$-1 - \frac{\alpha\, p}{2} < \delta < \frac{\alpha\, p}{2} + p - 1 \qquad (6.1)$$

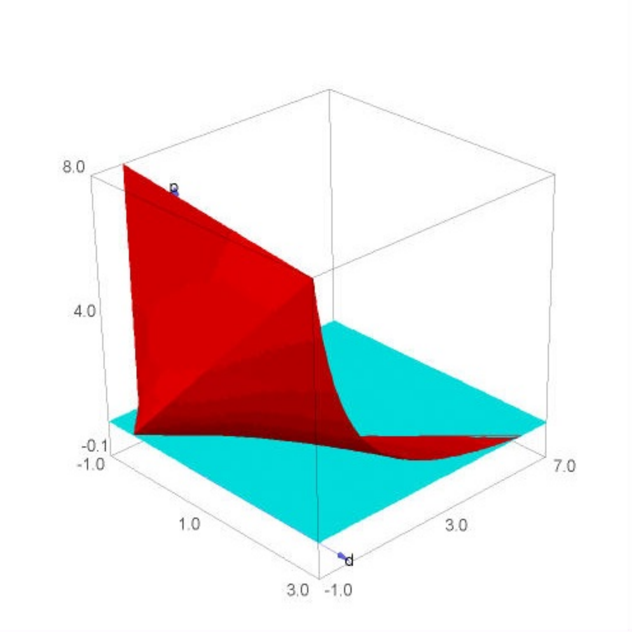
En el caso de $\delta = 0$, ya hemos presentado una figura en el Capítulo 5 para $\delta = 0$ y abscisa α. Esta figura cambia completamente si se toma como ordenada p. En efecto, los dibujos en ambos casos son:

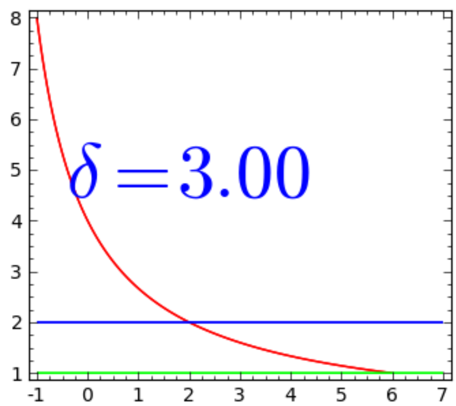


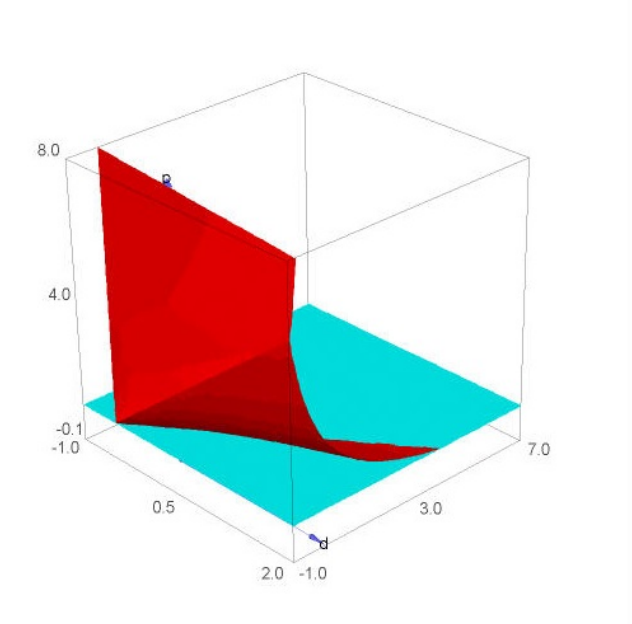


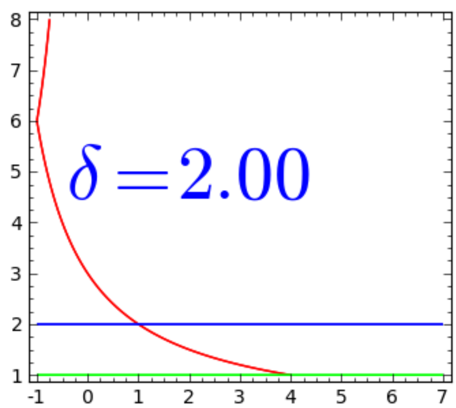
A continuación se presentan las secciones verticales de la zona descrita por la fórmula (6.1) cuando δ recorre los valores 3, 2, 1, 0,3, 0,1, 0, $-0{,}1$, $-0{,}2$.

$$-1 - \frac{\alpha\, p}{2} < \delta < \frac{\alpha\, p}{2} + p - 1$$

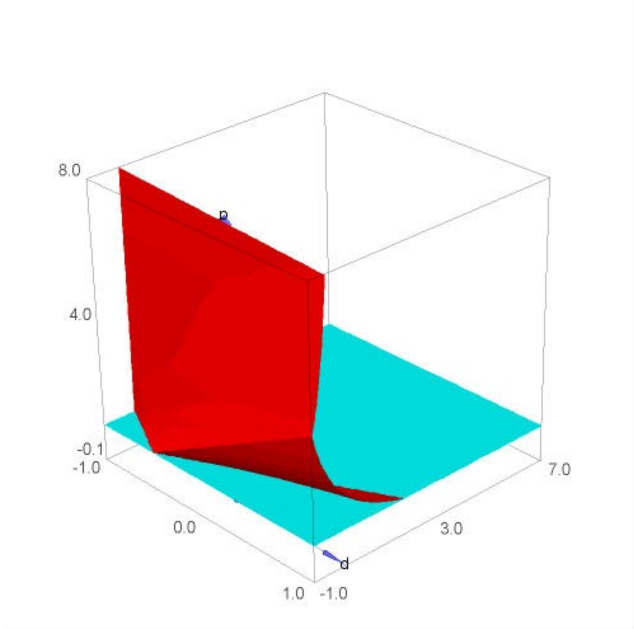


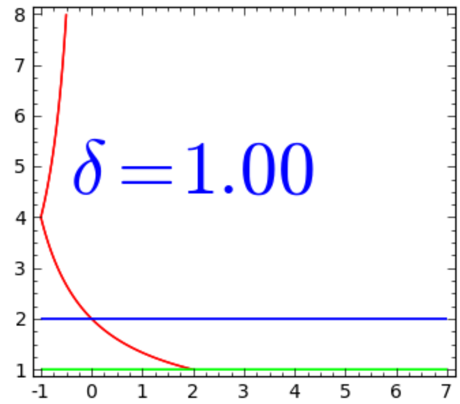


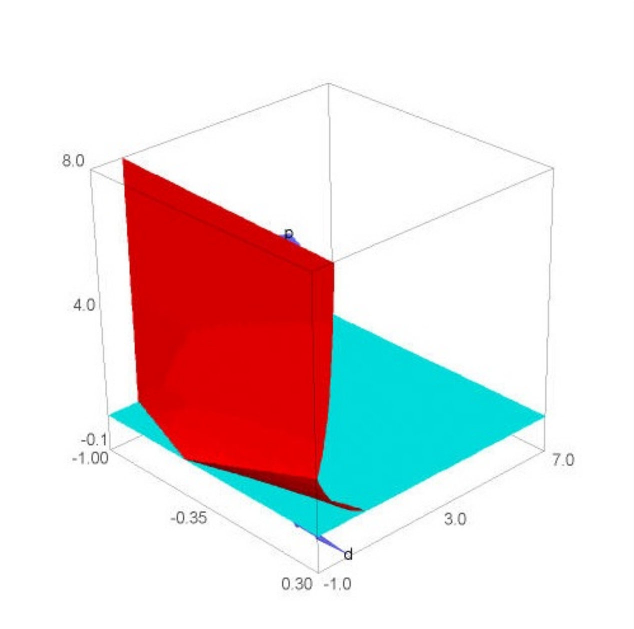


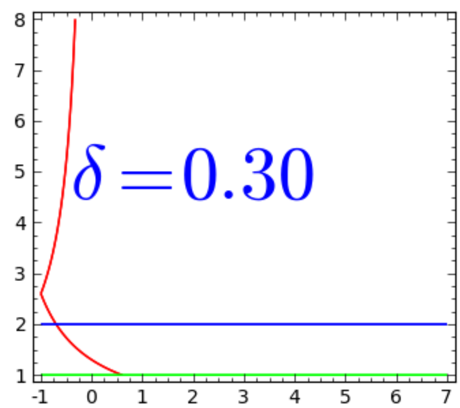


$$-1-\frac{\alpha\ p}{2}<\delta<\frac{\alpha\,p}{2}+p-1$$

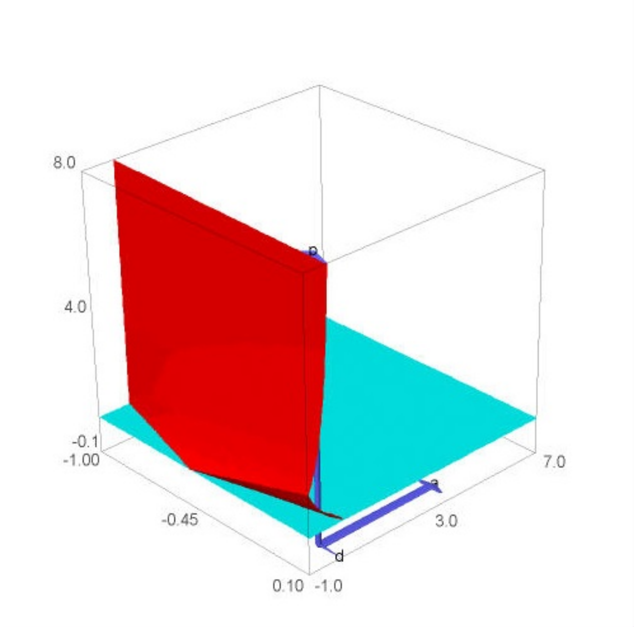


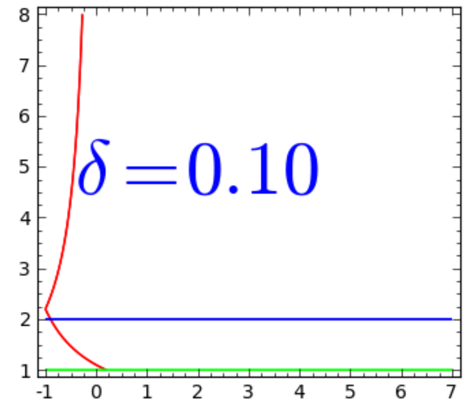


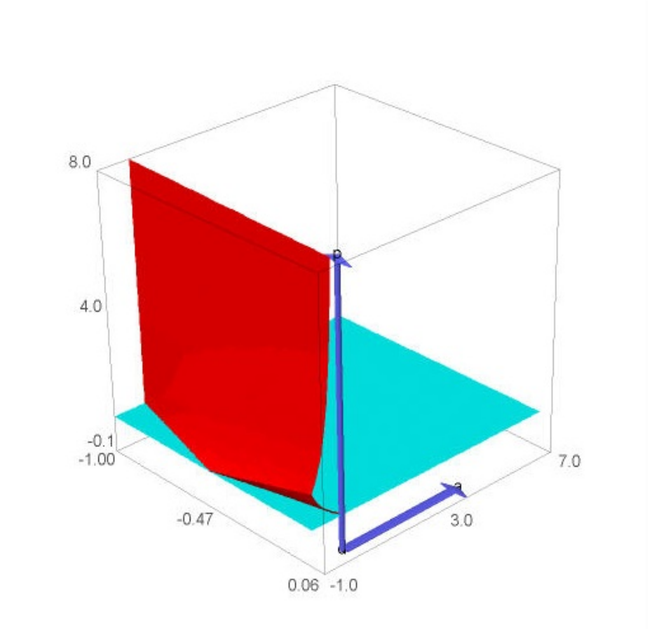


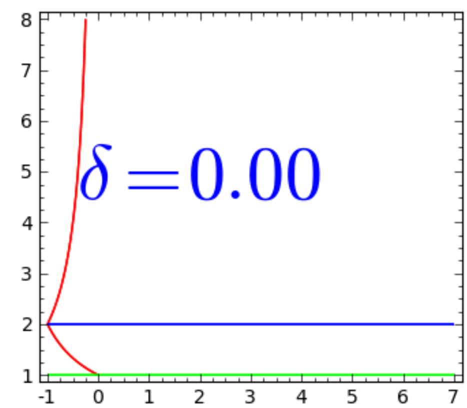


$$-1-\frac{\alpha\ p}{2}<\delta<\frac{\alpha\,p}{2}+p-1$$

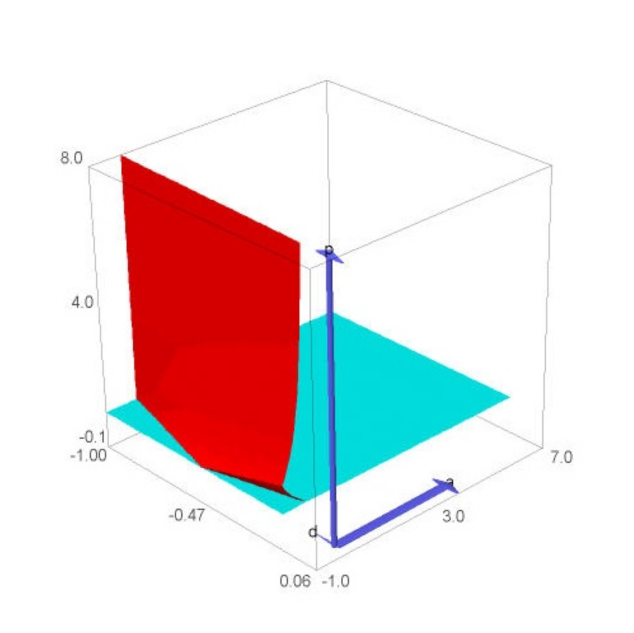


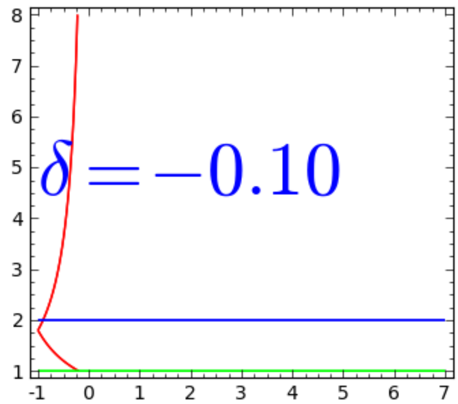


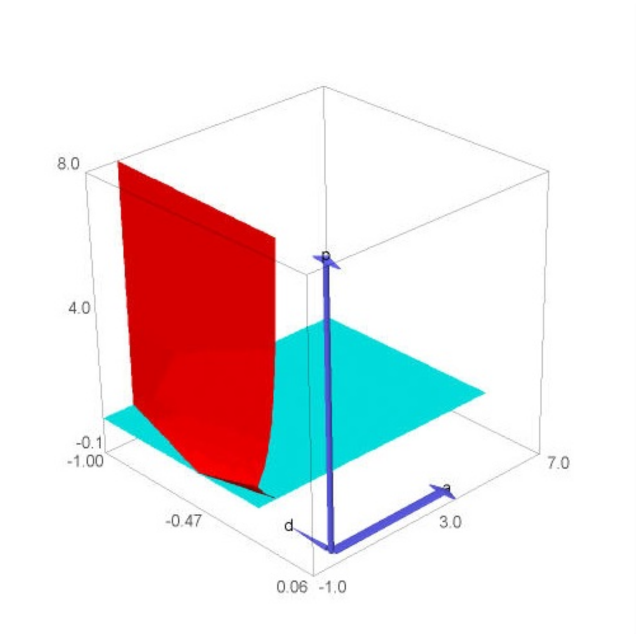


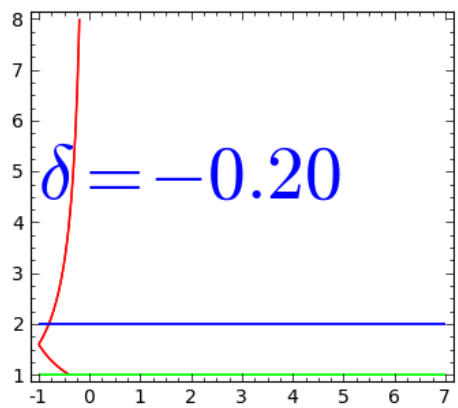


$$-1-\frac{\alpha\, p}{2}<\delta<\frac{\alpha\, p}{2}+p-1$$







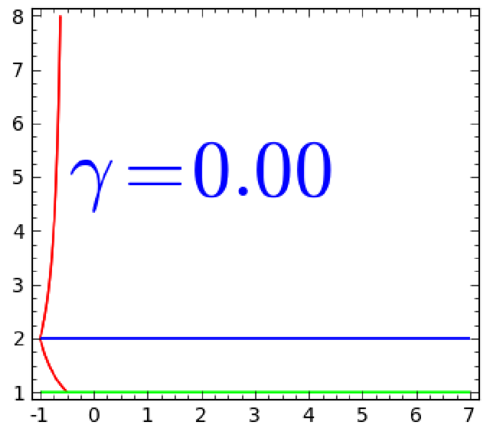


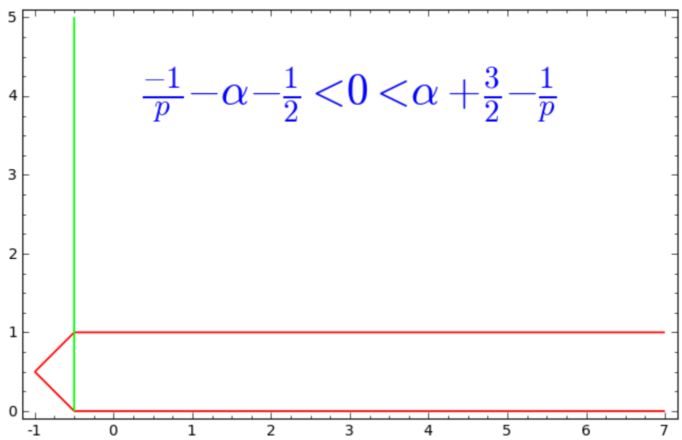
6.3. Familia $\{\varphi_k^\alpha\}_k$

Análogamenta al caso de las funciones $\mathcal{L}_k^\alpha$ también se tiene un arealción entre p, α y la potencia γ del peso, esta es

$$-1 - \alpha p - \frac{p}{2} < \gamma < \alpha p + \frac{3\,p}{2} - 1 \tag{6.2}$$

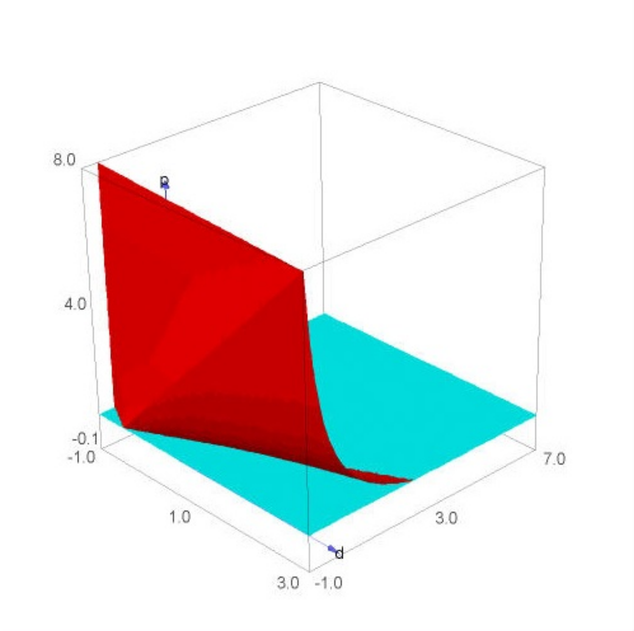
En el caso de $\gamma = 0$ la figura es muy diferente si se toma como ordenada p o $1/p$, de hecho se obtienen los siguientes bocetos.

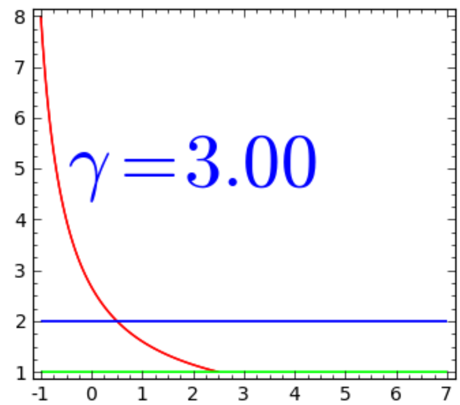


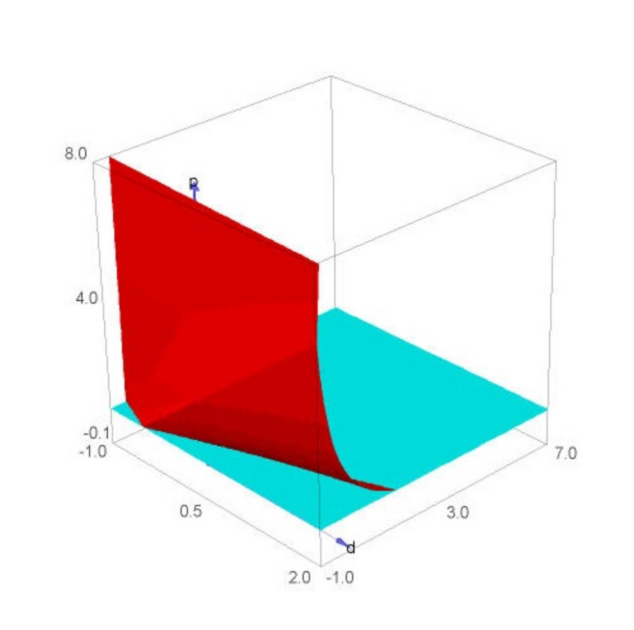


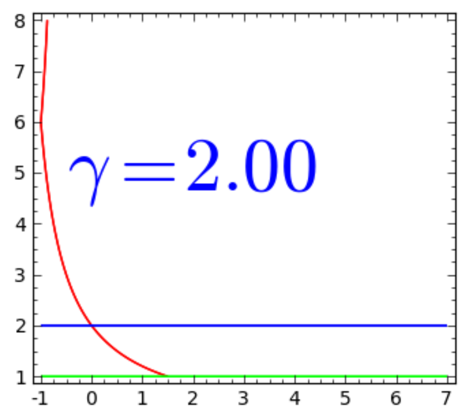
A continuación se presentan las secciones verticales de la zona descrita por la fórmula (6.2) cuando γ recorre los valores 3, 2, 1, 0,3, 0,1, 0, $-0{,}1$, $-0{,}2$.

$$-1-\alpha p-\frac{p}{2}<\gamma<\alpha p+\frac{3\,p}{2}-1$$

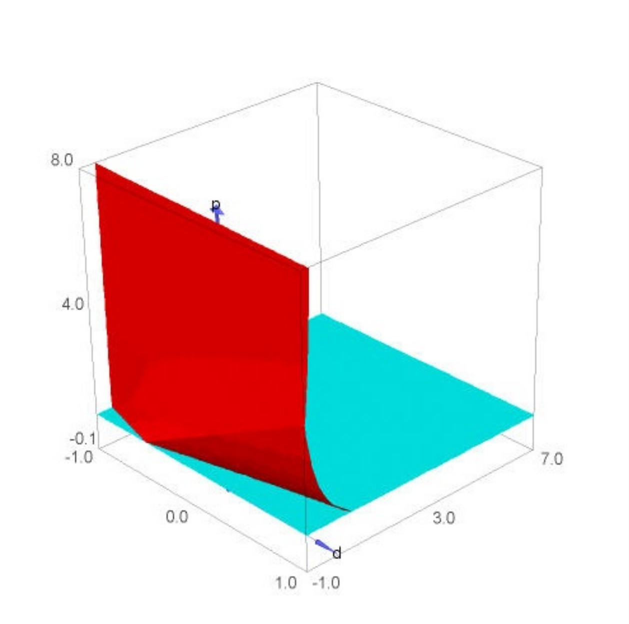


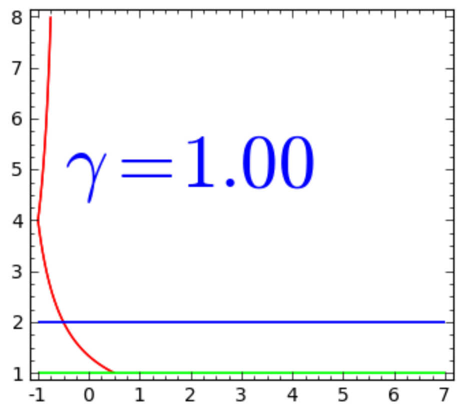


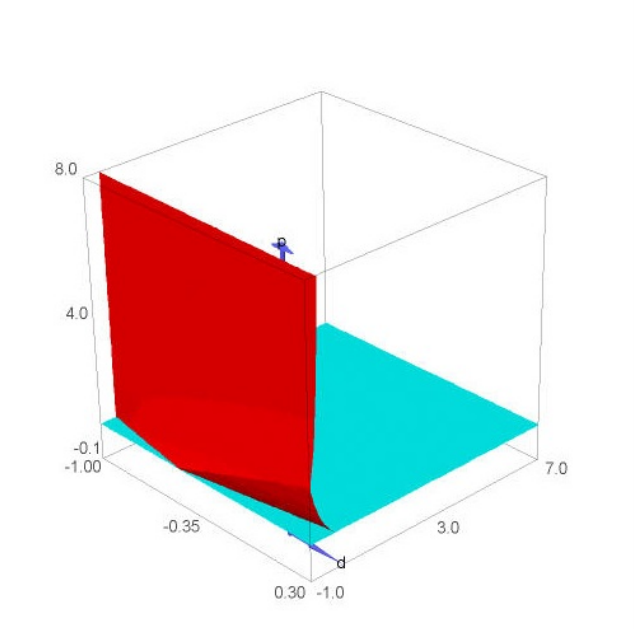


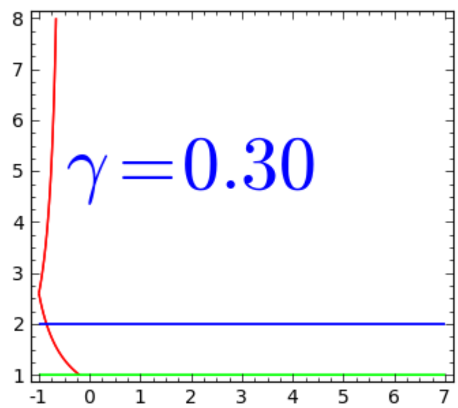


$$-1 - \alpha p - \frac{p}{2} < \gamma < \alpha p + \frac{3\,p}{2} - 1$$

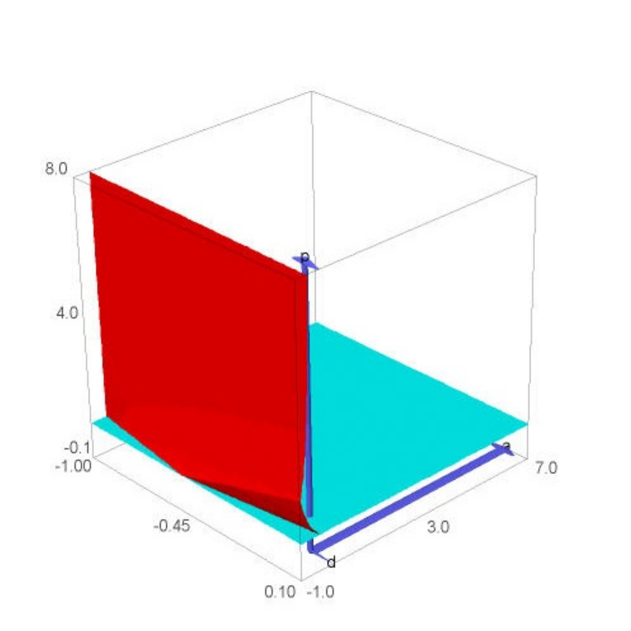


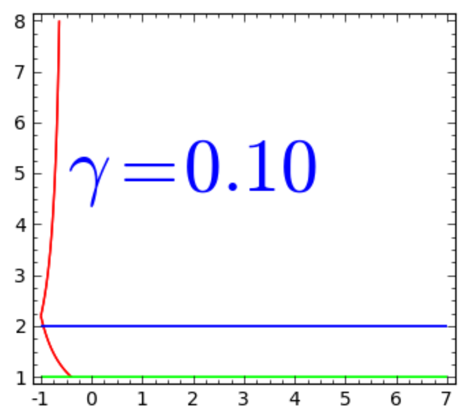


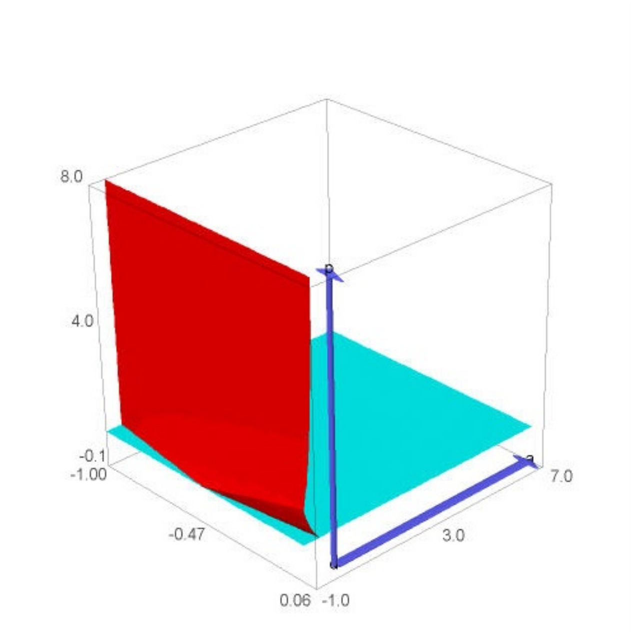


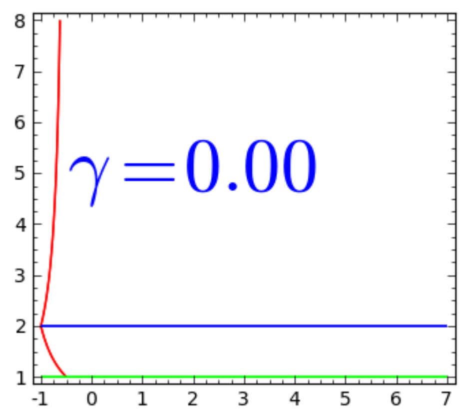


$$-1-\alpha p-\frac{p}{2}<\gamma<\alpha p+\frac{3\,p}{2}-1$$

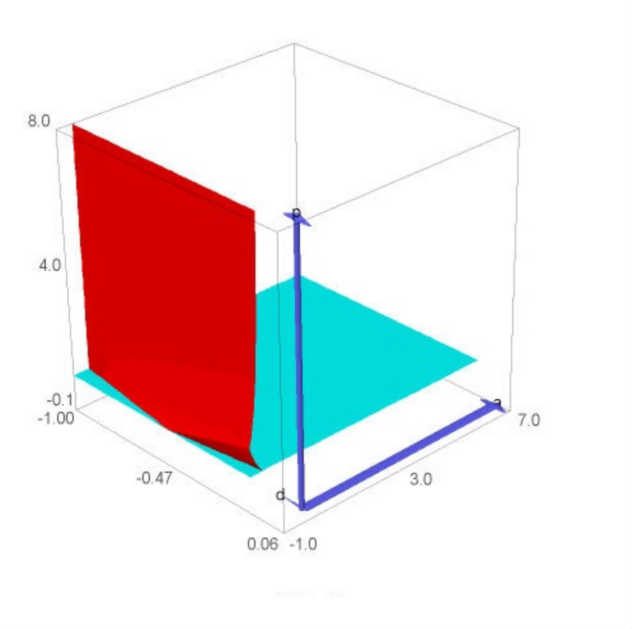


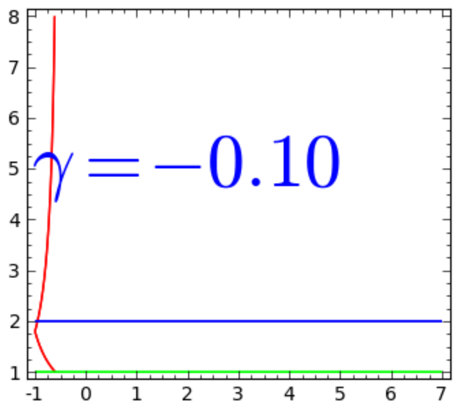


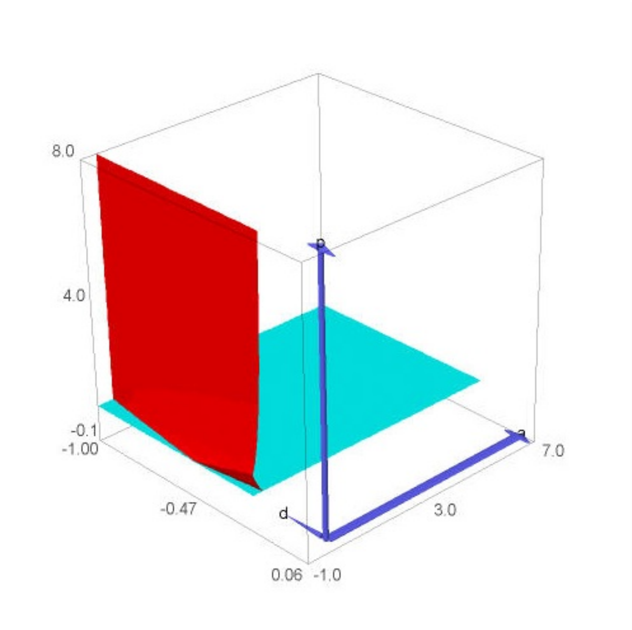


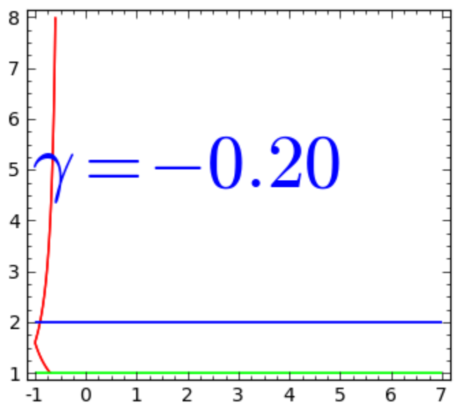


$$-1-\alpha p-\frac{p}{2}<\gamma<\alpha p+\frac{3\,p}{2}-1$$







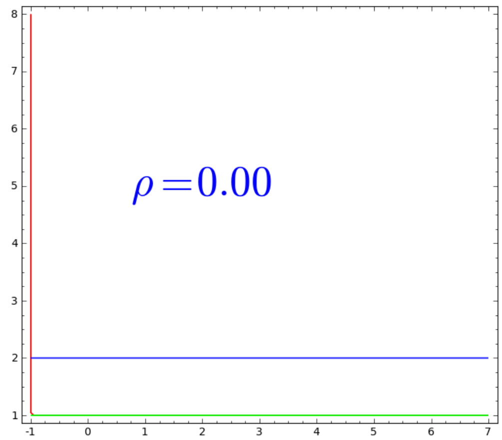


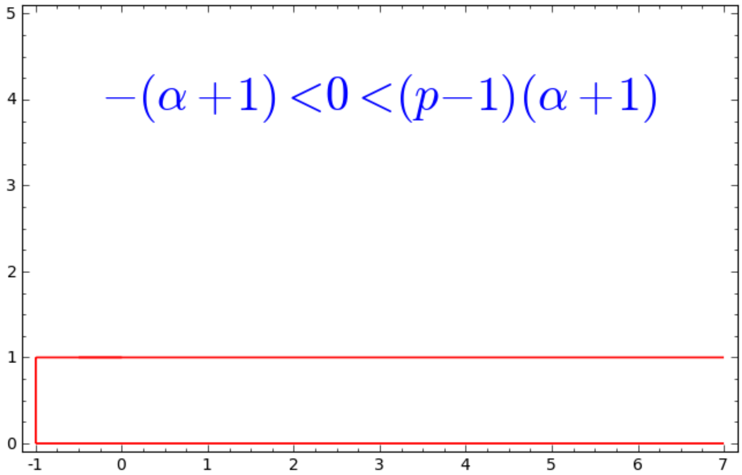
6.4. Familia $\{\ell_k^\alpha\}_k$

Nuevamente hay un rango de p, α y potencias ρ del peso para el cual se tienen acotación, Este es:

$$-1-\alpha < \rho < (\alpha+1)(p-1). \tag{6.3}$$

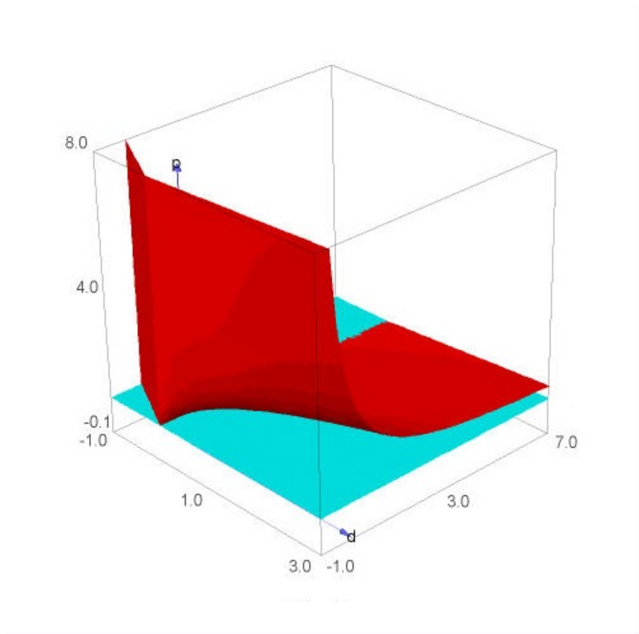
En este caso, para $\rho = 0$ se tiene acotación en todo el intervalo $1 < p < \infty$. Por lo tanto la comparación entre poner como ordenada p o $1/p$ da poca información y las figuras son:

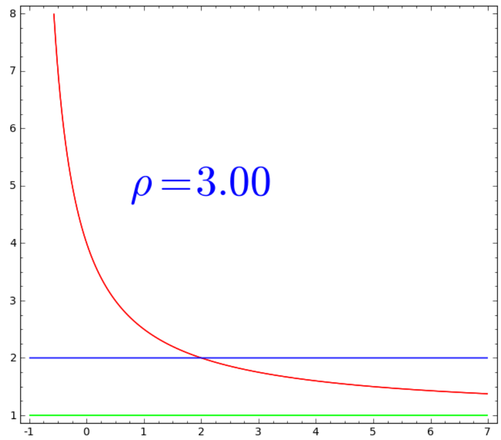


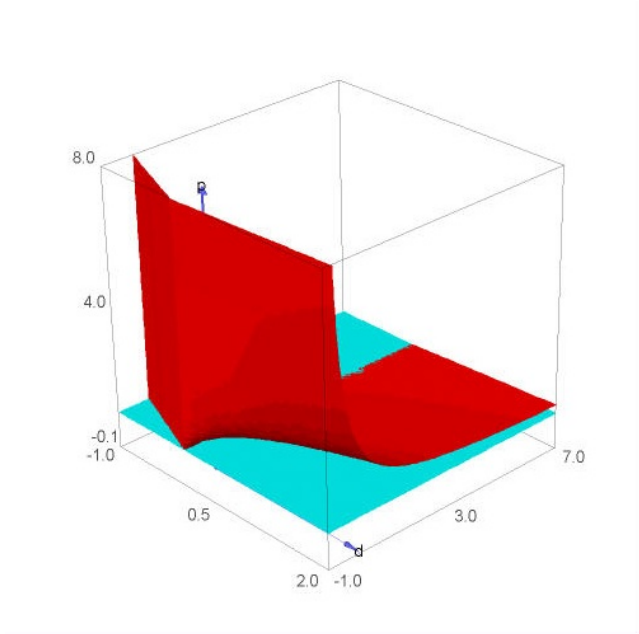


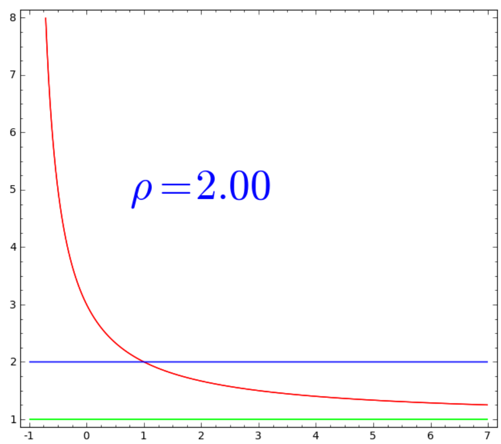
Nuevamente hemos hecho las secciones verticales de la regón limitada por la fórmula (6.3)

$$-1 - \alpha < \rho < (\alpha + 1)(p - 1).$$

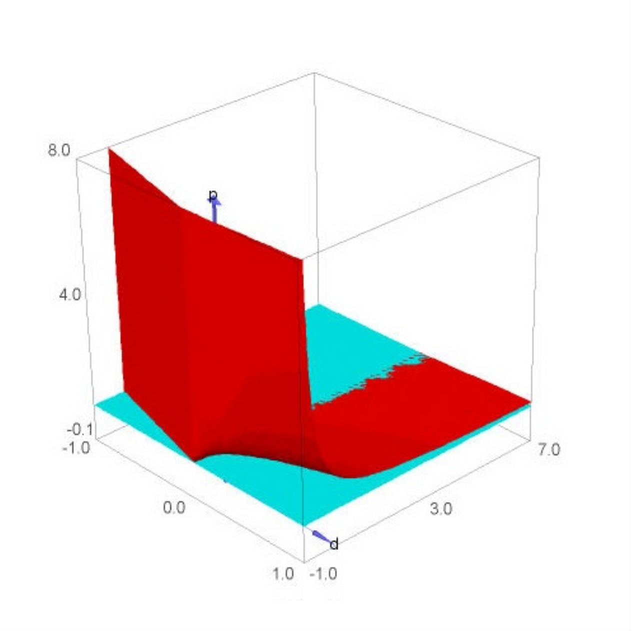


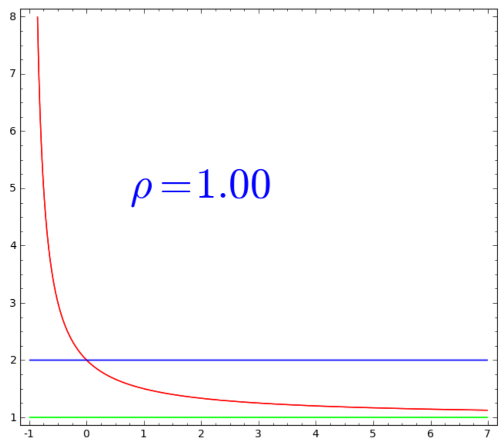


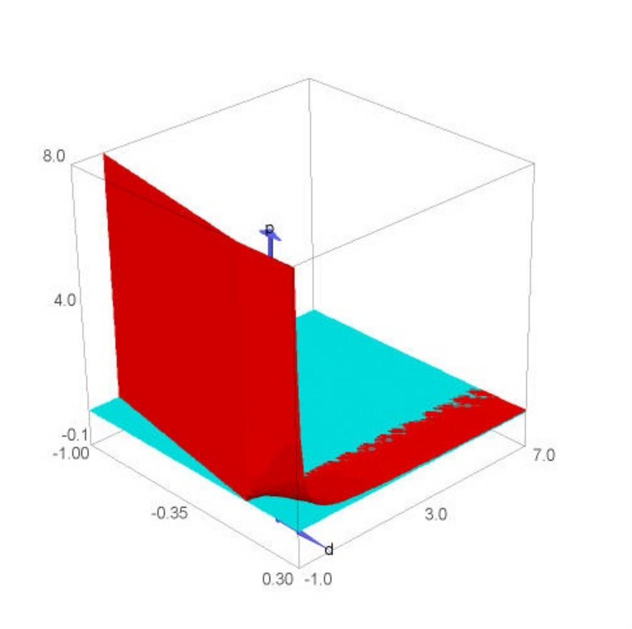


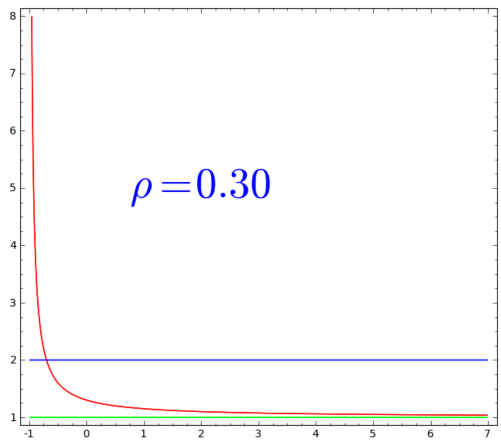


$$-1-\alpha<\rho<(\alpha+1)(p-1)$$

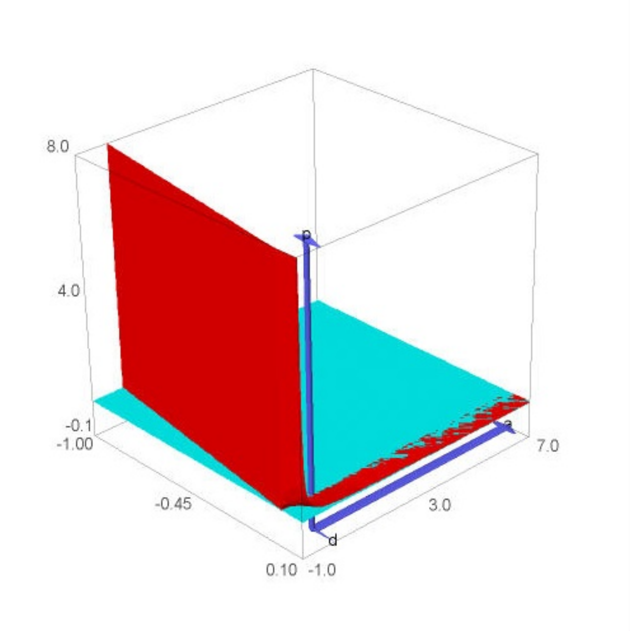


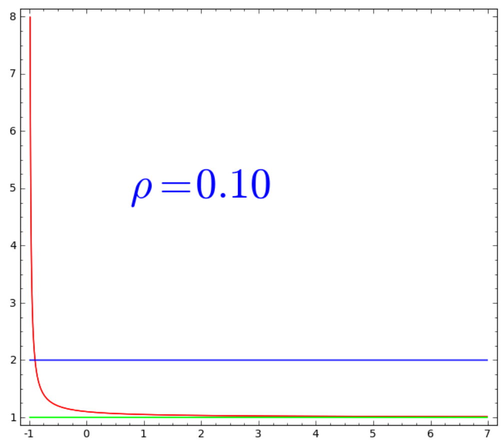


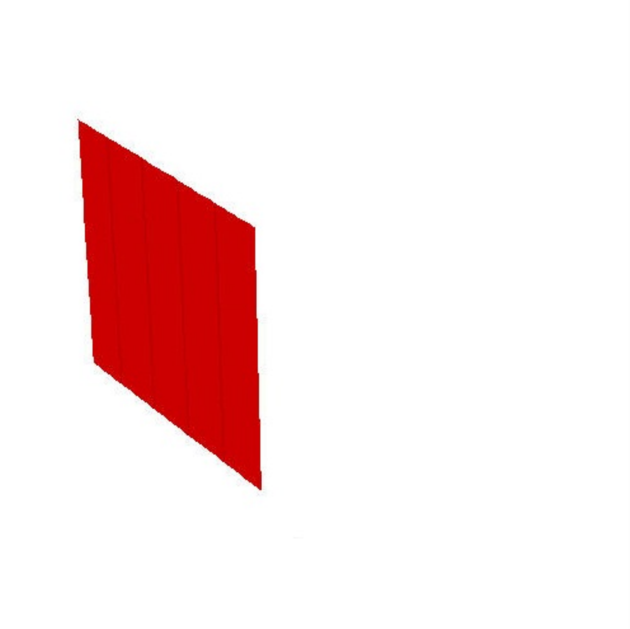


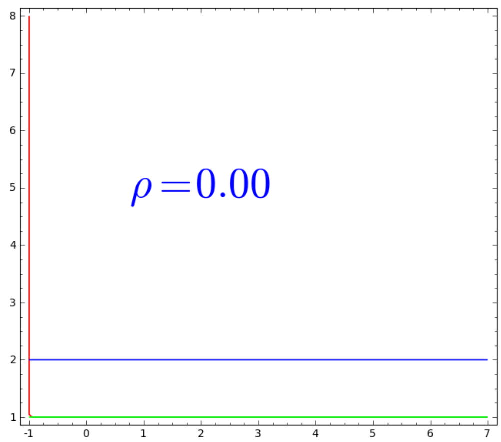


$$-1 - \alpha < \rho < (\alpha + 1)(p - 1)$$

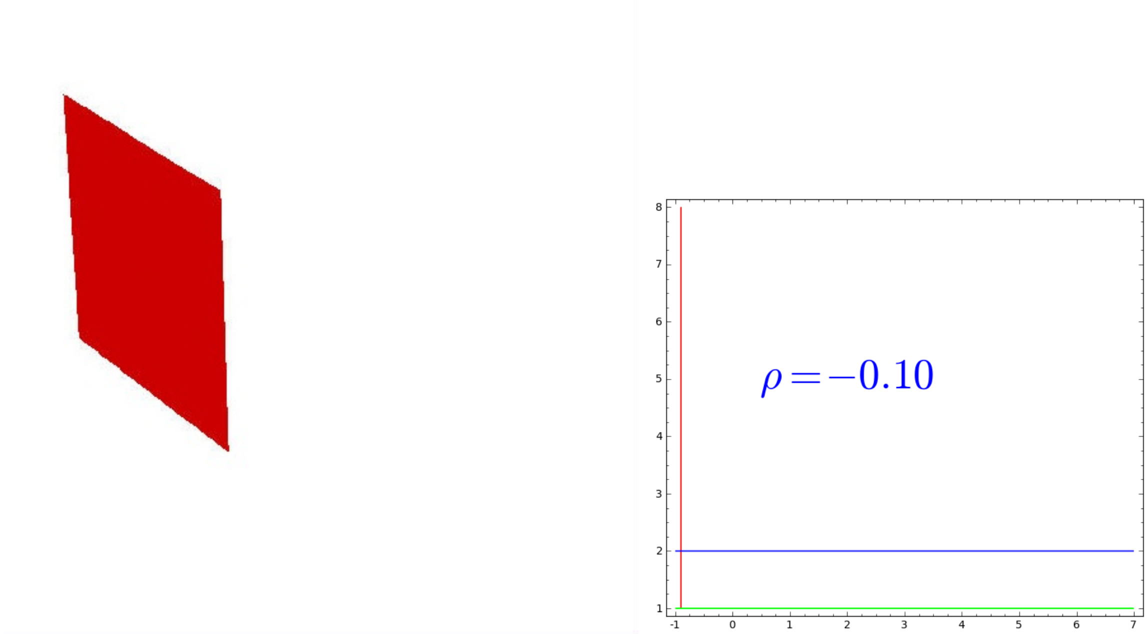


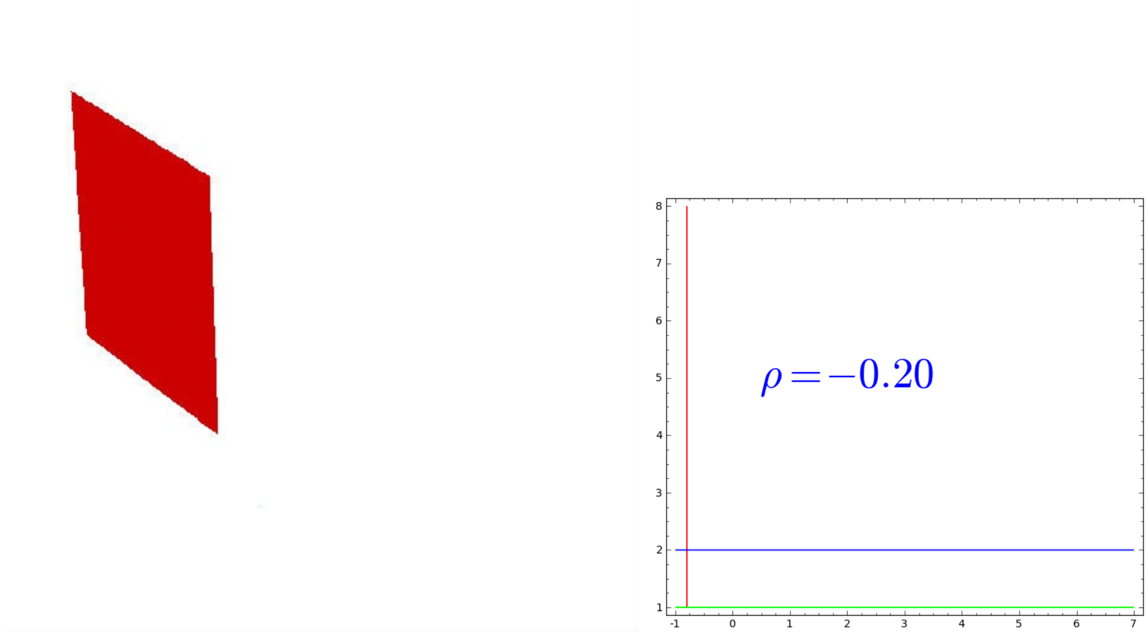






$$-1-\alpha<\rho<(\alpha+1)(p-1)$$



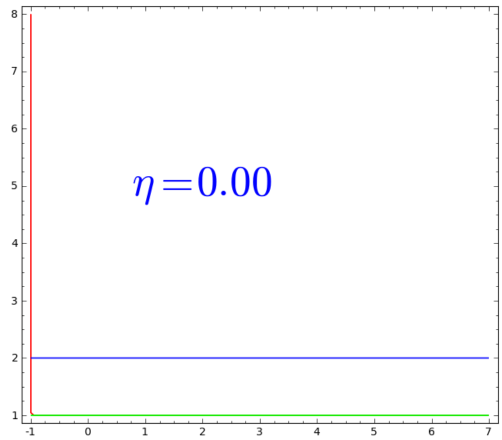


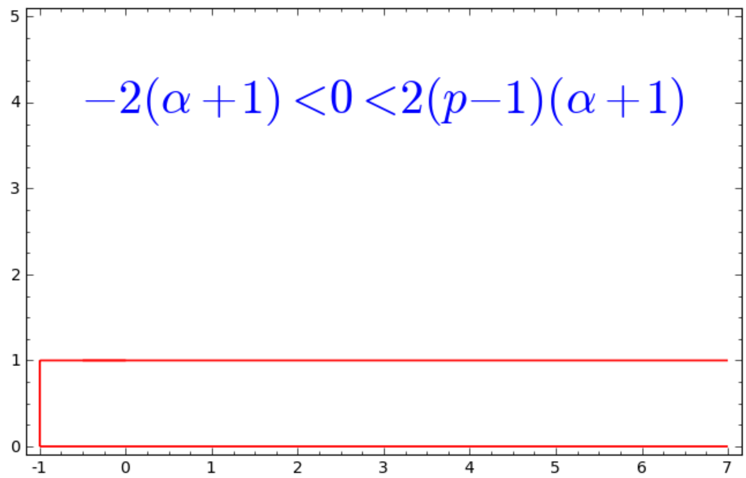
6.5. Familia $\{\psi_k^\alpha\}_k$

Procediendo como en las secciones anteriores tenemos la fórmula

$$-2(1+\alpha) < \eta < 2(\alpha+1)(p-1) \tag{6.4}$$

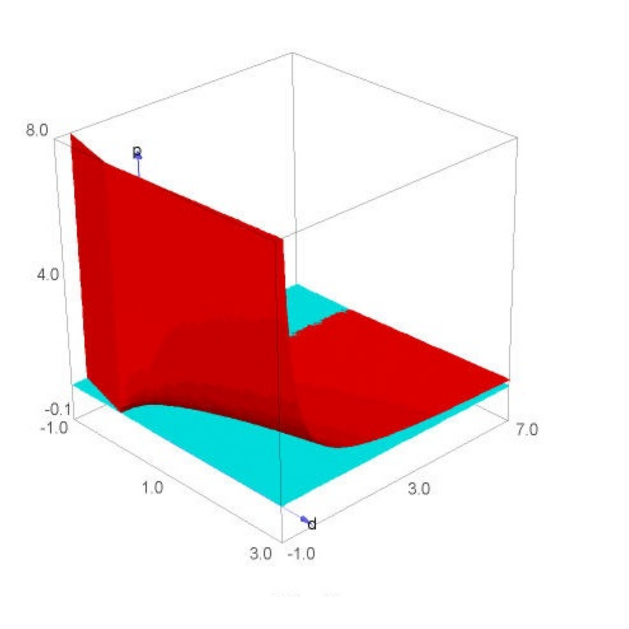
Y en el caso de la potencia $\eta = 0$:

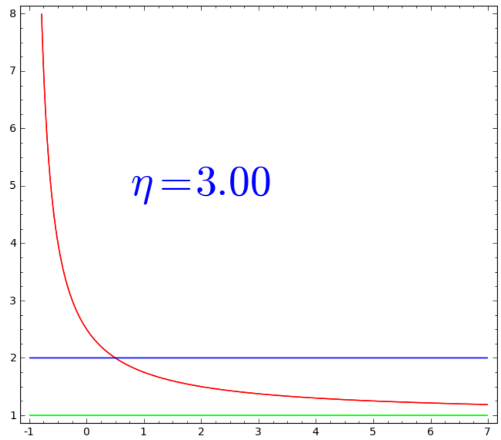


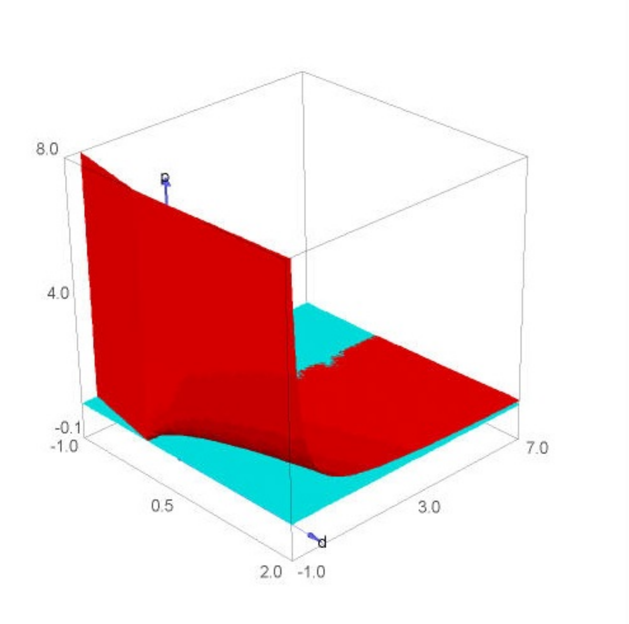


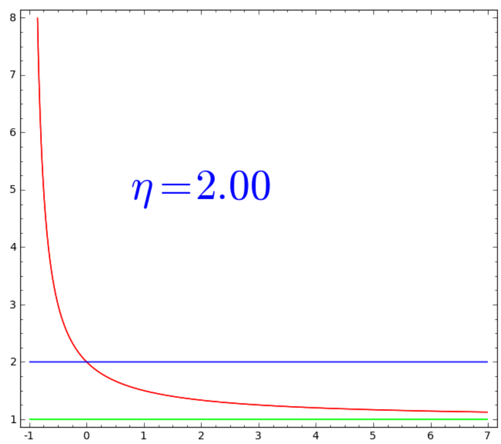
Nuevamente hacemos las secciones verticales de la región definida por la fórmula (6.4)

$$-2(1+\alpha) < \eta < 2(\alpha+1)(p-1)$$

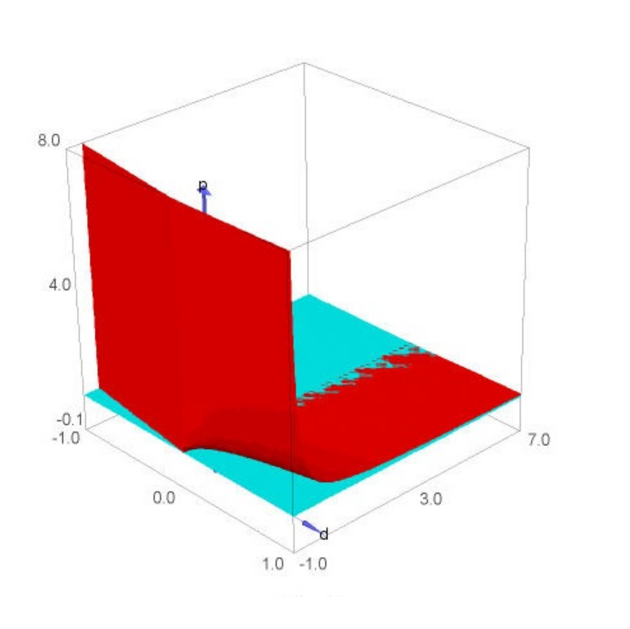


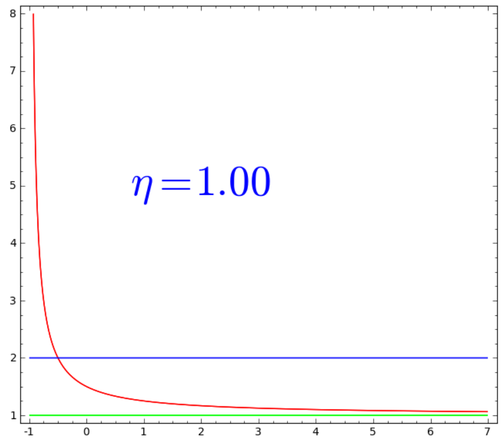


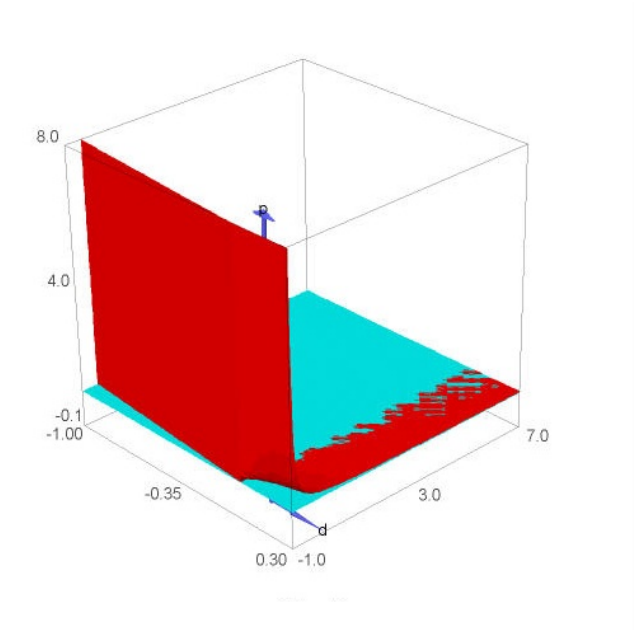


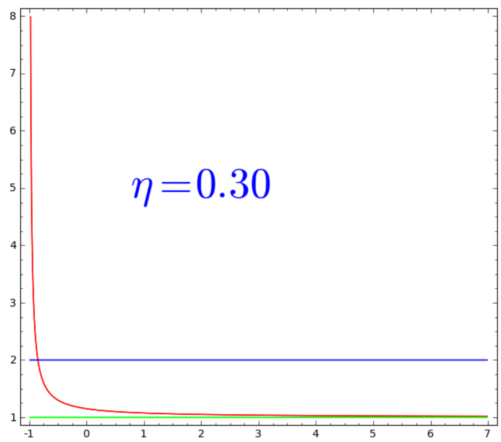


$$-2(1+\alpha) < \eta < 2(\alpha+1)(p-1)$$

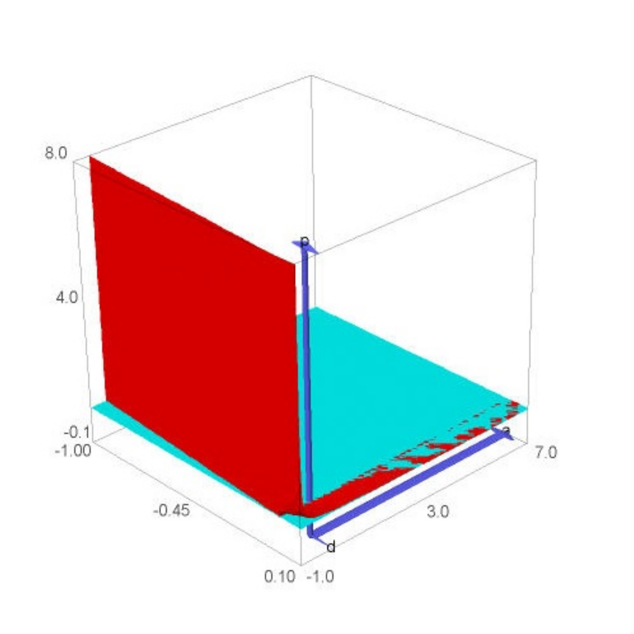


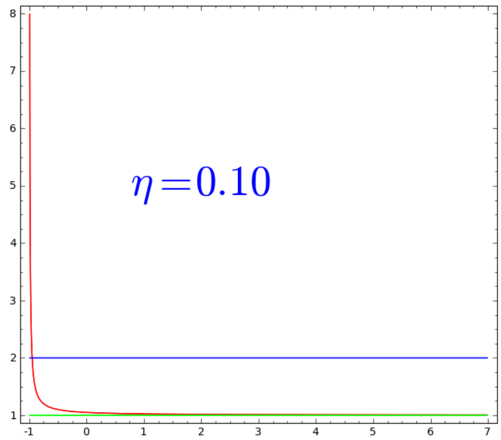


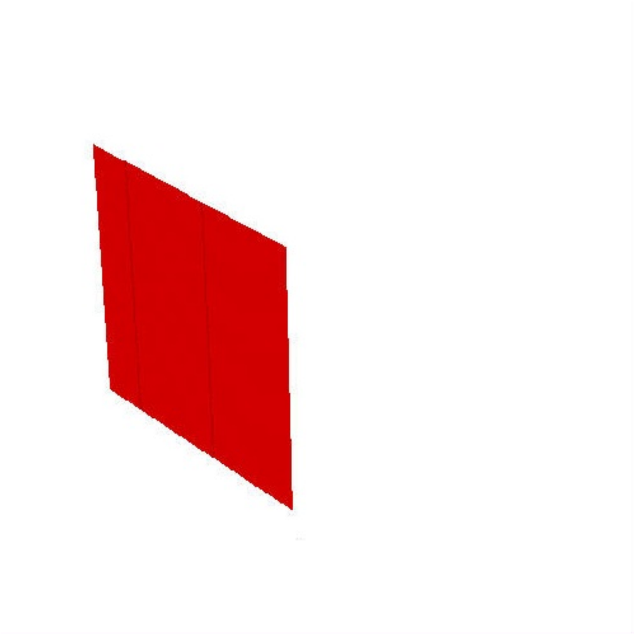


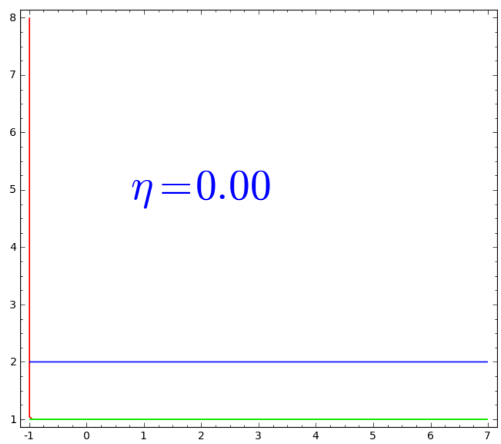


$$-2(1+\alpha) < \eta < 2(\alpha+1)(p-1)$$

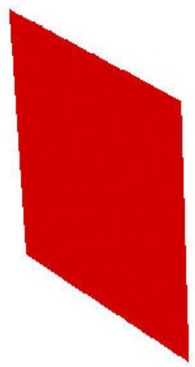


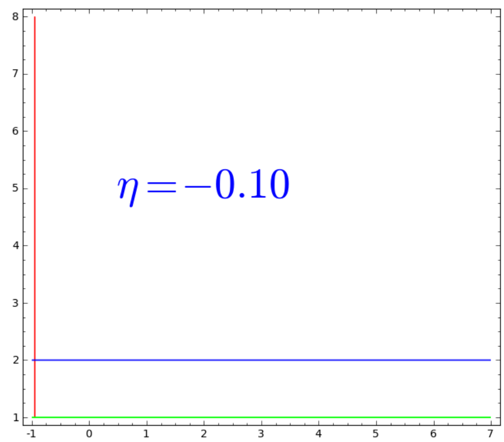


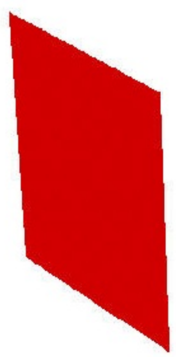


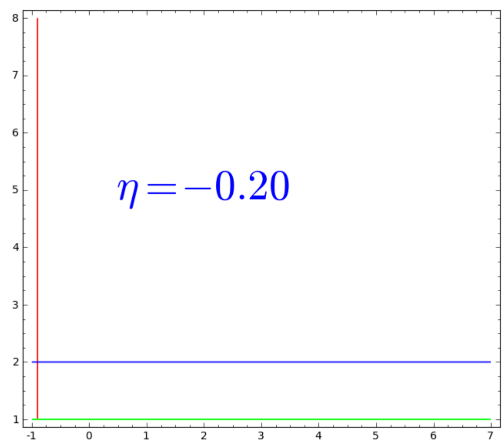


$$-2(1+\alpha) < \eta < 2(\alpha+1)(p-1)$$







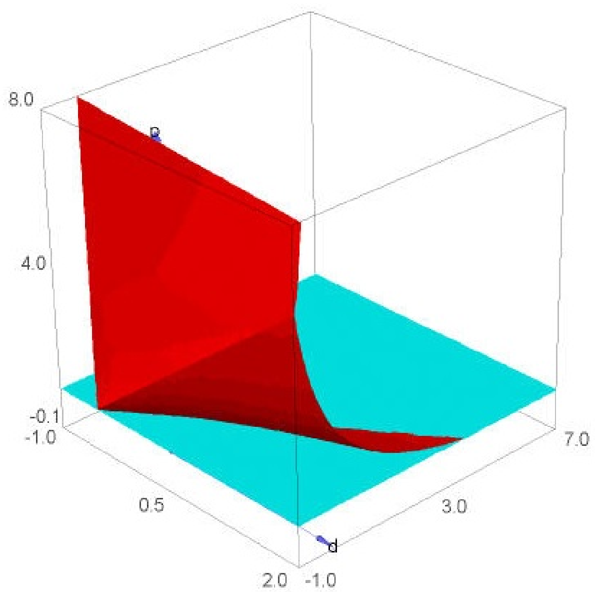


6.6. Código utilizado

A continuación presentamos una de las sintaxis utilizadas en el programa para el dibujo de las figuras. La figura que aparece en esta página es el resultado de dicha sintaxis

```
var('p d a')
g = (
    implicit_plot3d(-a*p/2-1-d,(d,-1,2),(a,-1,7), (p,1,8), color=(1,0,0))
   + implicit_plot3d(p+p*a/2-1-d, (d,-1,2), (a,-1,7), (p,1,8), color=(1,0,0))
   + implicit_plot3d(p-1,(d,-1,2), (a,-1,7), (p,0,8),  color=(0,1,1,0.9))
   + text3d('d', (2,0,0))
   + text3d('a', (0,7,0))
   + text3d('p', (0,0,8))
   + arrow3d((0,0,0), (2,0,0), 1)
   + arrow3d((0,0,0), (0,7,0), 1)
   + arrow3d((0,0,0), (0,0,8), 1)

    )
g.show(frame=True)
```



Capítulo 7

Square functions associated to Schrödinger operators

Colaboración con P.R. Stinga y J.L. Torrea
Publicado en Studia Math. 203 (2011) 171-194

7.1. Introduction

Consider the time independent Schrödinger operator in $\mathbb{R}^d$, $d \geq 3$,

$$\mathcal{L} := -\Delta + V, \tag{7.1}$$

where the nonnegative potential V satisfies a reverse Hölder inequality for some $s > d/2$, see (7.3).

Let X be a Banach space and let $\{\mathcal{P}_t\}_{t>0} = \{e^{-t\sqrt{\mathcal{L}}}\}_{t>0}$ be the (subordinated) Poisson semigroup associated to $\mathcal{L}$, see (7.10). For $2 \leq q < \infty$ consider the generalized square function

$$g^{\mathcal{L},q}f(x) = \left(\int_0^\infty \left\| t\frac{\partial \mathcal{P}_t f(x)}{\partial t}\right\|_X^q \frac{dt}{t}\right)^{1/q} = \|t\partial_t \mathcal{P}_t f(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}, \quad x \in \mathbb{R}^d. \tag{7.2}$$

By using the method described below we prove the following Theorem.

Theorem A. *Let X be a Banach space and $2 \leq q < \infty$. The following statements are equivalent.*

(i) *X admits an equivalent norm for which it is q-uniformly convex.*

(II) *The operator* $g^{\mathcal{L},q}$ *maps* $BMO_{\mathcal{L},X}$ *into* $BMO_{\mathcal{L}}$.

(III) *The operator* $g^{\mathcal{L},q}$ *maps* $L^p_X(\mathbb{R}^d)$ *into* $L^p(\mathbb{R}^d)$, *for any* p *in the range* $1 < p < \infty$.

(IV) *The operator* $g^{\mathcal{L},q}$ *maps* $L^1_X(\mathbb{R}^d)$ *into weak-*$L^1(\mathbb{R}^d)$.

(V) *The operator* $g^{\mathcal{L},q}$ *maps* $H^1_{\mathcal{L},X}$ *into* $L^1(\mathbb{R}^d)$.

(VI) *For every* $f \in L^1_X(\mathbb{R}^d)$, $g^{\mathcal{L},q}f(x) < \infty$ *for almost every* $x \in \mathbb{R}^d$.

In 1995 Z. Shen proved L^p-boundedness of the Riesz transforms associated to the operator $\mathcal{L}$, see [78]. The main idea in that paper is to break the kernels of the operators into "local" and "global" parts (close to the diagonal and far from the diagonal according to a certain distance $\rho(x)$ related to $\mathcal{L}$). Such a paper, a nice and exhaustive piece of mathematics, has became a classic and it has been a source of inspiration for a lot of manuscripts regarding Harmonic Analysis of operators associated to (7.1). However, when these operators are defined with some formula involving the heat semigroup (as in the case of the maximal operator $\sup_{t>0} \left|e^{-t\mathcal{L}}f\right|$ and the square function $\left(\int_0^\infty \left|t\partial_t e^{-t\mathcal{L}}f\right|^2 \frac{dt}{t}\right)^{1/2}$) the word "locally" usually refers to the parameter t of $e^{-t\mathcal{L}}$ being small and controlled in some sense by $\rho(x)$, see Dziubański et al., [26].

Beyond the characterization of q-uniformly convex Banach spaces through boundedness properties of $\mathcal{L}$-square functions, we have another purpose. Namely enlighten the "localization" technique by sharpening the method introduced in [78] in order to avoid the manipulations with the parameter t. At the same time, we get a unified approach to prove H^1, L^p and BMO boundedness results for classical Harmonic Analysis operators associated to $\mathcal{L}$. Observe that, in particular, Theorem A gives an alternative proof of the boundedness of $g^{\mathcal{L},2}$ in the scalar case.

Let us briefly describe the procedure that within the paper is developed in detail for the case of the square function $g^{\mathcal{L},q}$ acting on vector valued functions.

Description of the method. *Let* $\rho(x)$ *be the auxiliary critical radii function determined by the potential* V, *see* (7.4), *and* N *be the region consisting of points* $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ *such that* $|x-y| \le \rho(x)$. *Given a linear operator associated to* $\mathcal{L}$, *that we denote by* $T^{\mathcal{L}}$, *let* T^{Δ} *be the parallel operator associated to the classical Laplacian* $-\Delta$. *Define the localized operator* $T^{\mathcal{L}}_{\text{loc}}f(x) := T^{\mathcal{L}}\left(\chi_N(x,\cdot)f(\cdot)\right)(x)$ *and analogously* $T^{\Delta}_{\text{loc}}f(x)$. *Then* T^{Δ}_{loc} *inherits the* L^p*-boundedness properties from the operator* T^{Δ}. *Even more, if* T^{Δ}_{loc} *is bounded in* L^p *then it is also bounded in* $BMO_{\mathcal{L}}$. *In other words, the operator* T^{Δ}_{loc} *behaves as a natural operator associated to* $\mathcal{L}$. *Now the method finishes by observing that the difference operators* $T^{\Delta}_{\text{loc}} - T^{\mathcal{L}}_{\text{loc}}$ *and* $T^{\mathcal{L}}_{\text{loc}} - T^{\mathcal{L}}$ *are bounded from* L^p *into* L^p *for* $1 \le p \le \infty$ *and from* $BMO_{\mathcal{L}}$ *into* L^∞.

In order to unify the method we consider a “local” part, defined through ρ, where the cutting acts on the heat kernel. This idea allows us to handle any operator defined via a formula involving the heat kernel, as for example Riesz transforms, square functions, etc.

Besides the unification, we believe that our main contribution is to show that the local part (given in terms of the distance ρ) of an operator associated to the standard Laplacian $-\Delta$ shares the natural boundedness properties with the corresponding operators associated to $\mathcal{L}$. We must emphasize how surprising this phenomenon is in the case of boundedness in BMO. See Theorems 7.12 and 7.14 in Section 7.3 for the case of the g-function. The general ideas are summarized in Remarks 7.13 and 7.15. Observe that the localized operator has always a rough kernel, see Remark 7.15, and it is not clear a priori how to prove the necessary smoothness properties in order to get the desired boundedness in BMO.

Here $L^p_X(\mathbb{R}^d)$ denotes the usual L^p-space of Bochner-Lebesgue p-integrable functions on $\mathbb{R}^d$ with values in X. The spaces $H^1_{\mathcal{L},X}$ and $BMO_{\mathcal{L},X}$ are defined in the same way as in the scalar case just by replacing the absolute value of $\mathbb{C}$ by the norm of X, see (7.14) and (7.11). For the definition of q-uniform convexity we refer to Section 7.4. Throughout the paper the letter C denotes a positive constant that may change in each appearance and does not depend on the significant quantities.

The paper is organized as follows. We collect in Section 7.2 the preliminary results already known in the context of Schrödinger operators. Section 7.3 contains the technical results needed for the application of the method. Finally Section 7.4 is devoted to the proof of Theorem A.

Acknowledgements

We are very grateful to the referee for the thorough revision of the original manuscript. His exhaustive and detailed comments, together with his helpful suggestions, certainly helped us to strongly improve the presentation of the paper.

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7.2. Preliminaries

The nonnegative potential V in (7.1) satisfies a reverse Hölder inequality for some $s > d/2$; that is, there exists a constant $C = C(s, V)$ such that

$$\left(\frac{1}{|B|}\int_B V(y)^s\,dy\right)^{1/s} \le \frac{C}{|B|}\int_B V(y)\,dy, \tag{7.3}$$

for all balls $B \subset \mathbb{R}^d$. Associated to this potential, Shen defines in [78] the critical radii function as

$$\rho(x) := \sup\Big\{r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\, dy \leq 1\Big\}, \qquad x \in \mathbb{R}^d. \tag{7.4}$$

Some properties of this function ρ are well known. We are particularly interested in the following.

Lemma 7.1 (see Lemma 1.4 in [78]). *There exist $c > 0$ and $k_0 \geq 1$ so that for all $x, y \in \mathbb{R}^d$*

$$c^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq c\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \tag{7.5}$$

In particular, there exists a positive constant $C_1 < 1$ such that

$$\text{if} \quad |x-y| \leq \rho(x) \quad \text{then} \quad C_1\rho(x) < \rho(y) < C_1^{-1}\rho(x).$$

Lemma 7.2 (see Lemma 2.3 in [27]). *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^d$ such that the family of balls $\{Q_k\}_{k=1}^{\infty}$ defined by $Q_k := B(x_k, \rho(x_k))$ satisfy*

- $\bigcup_k Q_k = \mathbb{R}^d$;
- *There exists $N = N(\rho)$ so that, for every $k \geq 1$,*

$$\operatorname{card}\{j : 2Q_j \cap 2Q_k \neq \emptyset\} \leq N;$$

where for a ball B and a positive number c we denote by cB the ball with the same center as B and radius c times the radius of B.

Let $\{\mathcal{T}_t\}_{t>0}$ be the heat–diffusion semigroup associated to $\mathcal{L}$ acting on X-valued functions:

$$\mathcal{T}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} k_t(x,y) f(y)\, dy, \quad f \in L^2_X(\mathbb{R}^d),\ x \in \mathbb{R}^d,\ t > 0. \tag{7.6}$$

The following Lemmas are known.

Lemma 7.3 (see [28, 51]). *For every $\alpha > 0$ there exists a constant C_α such that*

$$0 \leq k_t(x,y) \leq C_\alpha \frac{1}{t^{d/2}}\, e^{-\frac{|x-y|^2}{5t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-\alpha}, \tag{7.7}$$

for all $x, y \in \mathbb{R}^d$, $t > 0$.

Let

$$h_t(x) := \frac{1}{(4\pi t)^{d/2}}\, e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d,\ t > 0,$$

be the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ in $\mathbb{R}^d$.

Lemma 7.4 (see Proposition 2.16 in [28])**.** *There exists a nonnegative Schwartz class function ω in $\mathbb{R}^d$ such that*

$$|k_t(x,y) - h_t(x-y)| \le \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} \omega_t(x-y), \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{7.8}$$

where $\omega_t(x-y) := t^{-d/2}\omega\left((x-y)/\sqrt{t}\right)$ and

$$\delta := 2 - \frac{d}{s} > 0. \tag{7.9}$$

Given the heat semigroup (7.6), the Poisson semigroup associated to $\mathcal{L}$ is obtained through Bochner's subordination formula, see [83]:

$$\mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x) = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}}\, \mathcal{T}_u f(x)\, du, \quad x \in \mathbb{R}^d,\ t > 0. \tag{7.10}$$

With this we define, for $2 \le q < \infty$, the square function related to $\mathcal{L}$ as in (7.2).

Remark 7.5 (Notational convention)**.** The Poisson semigroup associated to the classical Laplace operator in $\mathbb{R}^d$ will be denoted by $\{P_t\}_{t>0} = \{e^{-t\sqrt{-\Delta}}\}_{t>0}$. Recall that $P_t f(x) = P_t * f(x)$, where

$$P_t(x) = c_d \frac{t}{(t^2 + |x|^2)^{\frac{d+1}{2}}}, \quad x \in \mathbb{R}^d,\ t > 0.$$

The square function considered in (7.2) will be denoted by $g^{\Delta,q} f$ when replacing $\mathcal{P}_t f$ by $P_t f$.

A locally integrable function $f : \mathbb{R}^d \to X$ is in $BMO_{\mathcal{L},X}$ whenever there exists a constant C such that

(i) $\displaystyle \frac{1}{|B|}\int_B \|f(x) - f_B\|_X\, dx \le C$, for every ball B in $\mathbb{R}^d$, and

(ii) $\displaystyle \frac{1}{|B|}\int_B \|f(x)\|_X\, dx \le C$, for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^d$ and $r_0 \ge \rho(x_0)$.

As usual, $f_B := \dfrac{1}{|B|}\displaystyle\int_B f(x)\,dx$, for every ball B in $\mathbb{R}^d$. The norm $\|f\|_{BMO_{\mathcal{L},X}}$ of f is defined as

$$\|f\|_{BMO_{\mathcal{L},X}} = \text{ínf}\,\{C \geq 0 : \text{(i) and (ii) hold}\}\,. \tag{7.11}$$

Let us note that if (ii) is true for some ball B then (i) holds true for the same ball, so we might ask to (i) only for balls with radii smaller than $\rho(x_0)$. By using the classical John-Nirenberg inequality it can be seen that if in (i) and (ii) L^1_X-norms are replaced by L^p_X-norms, for $1 < p < \infty$, then the space $BMO_{\mathcal{L},X}$ does not change and equivalent norms appear, see [26, Corollary 3].

We define the vector-valued atomic Hardy space related to $\mathcal{L}$ following the scalar-valued definition in [27]. A function $a : \mathbb{R}^d \to X$ is an $H^1_{\mathcal{L},X}$-atom associated with a ball $B(x_0, r)$ when $\operatorname{supp} a \subset B(x_0, r)$,

$$\|a\|_{L^\infty_X(\mathbb{R}^d)} \leq \frac{1}{|B(x_0, r)|}, \tag{7.12}$$

and, in addition,

$$\int_{\mathbb{R}^d} a(x)\,dx = 0, \qquad \text{whenever } 0 < r < \rho(x_0). \tag{7.13}$$

An X-valued integrable function f in $\mathbb{R}^d$ belongs to $H^1_{\mathcal{L},X}$ if and only if it can be written as $f = \sum_j \lambda_j a_j$, where a_j are $H^1_{\mathcal{L},X}$-atoms and $\sum_j |\lambda_j| < \infty$. The norm is given by

$$\|f\|_{H^1_{\mathcal{L},X}} = \text{ínf}\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\}. \tag{7.14}$$

In [26] it is shown that $BMO_{\mathcal{L}}$ is the dual space of $H^1_{\mathcal{L}}$.

7.3. Technical Lemmas

As we said in the description of our method, the following region N will play a fundamental role:

$$N := \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \leq \rho(x)\right\}.$$

Given N we define the "global" and "local" parts of the square function defined in (7.2) as

$$\begin{aligned} g^{\mathcal{L},q}_{\text{glob}} f(x) &= g^{\mathcal{L},q}\left(\chi_{N^c}(x,\cdot) f(\cdot)\right)(x) \qquad \text{and} \\ g^{\mathcal{L},q}_{\text{loc}} f(x) &= g^{\mathcal{L},q} f(x) - g^{\mathcal{L},q}_{\text{glob}} f(x). \end{aligned} \tag{7.15}$$

Note that

$$g^{\mathcal{L},q}_{\mathrm{loc}} f(x) \le g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x) \le g^{\mathcal{L},q} f(x) + g^{\mathcal{L},q}_{\mathrm{glob}} f(x), \quad \text{a.e. } x \in \mathbb{R}^d, \tag{7.16}$$

or equivalently,

$$\left| g^{\mathcal{L},q} f(x) - g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x) \right| \le g^{\mathcal{L},q}_{\mathrm{glob}} f(x), \quad \text{a.e. } x \in \mathbb{R}^d. \tag{7.17}$$

Lemma 7.6. *Let X be any Banach space and $\alpha > 0$. Then for any $f \in \bigcup_{1\le p\le\infty} L^p_X(\mathbb{R}^d)$ we have*

$$g^{\mathcal{L},q}_{\mathrm{glob}} f(x) \le C \int_{\mathbb{R}^d} L(x,y)\chi_{N^c}(x,y) \left\| f(y) \right\|_X dy, \quad x \in \mathbb{R}^d,$$

where $L(x,y) = \dfrac{\rho(x)^\alpha}{|x-y|^{d+\alpha}}$, $x, y \in \mathbb{R}^d$.

PROOF: Using Bochner's subordination formula (7.10) it can be checked that for any function h,

$$\left\| \partial_t \mathcal{P}_t h(x) \right\|_X \le C \int_0^\infty \frac{e^{-t^2/(8u)}}{u^{3/2}} \left\| \mathcal{T}_u h(x) \right\|_X du,$$

where we applied the inequality $r^\eta e^{-r} \le C_\eta e^{-r/2}$, valid for $\eta \ge 0$, $r > 0$. Hence, by Minkowski's inequality,

$$\begin{aligned} g^{\mathcal{L},q}_{\mathrm{glob}} f(x) &\le C \int_0^\infty \left\| t \frac{e^{-t^2/(8u)}}{u^{3/2}} \right\|_{L^q((0,\infty),\frac{dt}{t})} \left\| \mathcal{T}_u(\chi_{N^c}(x,\cdot)f(\cdot))(x) \right\|_X du \\ &= C \int_0^\infty \left\| \mathcal{T}_u(\chi_{N^c}(x,\cdot)f(\cdot))(x) \right\|_X \frac{du}{u} \\ &\le C \int_0^\infty \int_{\mathbb{R}^d} k_u(x,y)\chi_{N^c}(x,y) \left\| f(y) \right\|_X dy \, \frac{du}{u}. \end{aligned}$$

¿From (7.7) of Lemma 7.3 and the change of variables $r = \frac{|x-y|^2}{cu}$ we get

$$\begin{aligned} \int_0^\infty k_u(x,y) \, \frac{du}{u} &\le C \int_0^\infty \frac{1}{u^{d/2}} \, e^{-\frac{|x-y|^2}{cu}} \left(\frac{\rho(x)}{\sqrt{u}} \right)^\alpha \frac{du}{u} \\ &= C \frac{\rho(x)^\alpha}{|x-y|^{d+\alpha}} \int_0^\infty r^{\frac{d+\alpha}{2}} e^{-r} \frac{dr}{r}. \end{aligned}$$

Lemma 7.7. *Let X be any Banach space. Then the global operator $g^{\mathcal{L},q}_{\mathrm{glob}}$ maps*

(a) $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for any p, $1 \le p \le \infty$,

(b) $BMO_{\mathcal{L},X}$ into $L^\infty(\mathbb{R}^d)$, and

(c) $H^1_{\mathcal{L},X}$ *into* $L^1(\mathbb{R}^d)$.

Proof: Let $L(x,y)$, $x, y \in \mathbb{R}^d$, be as in Lemma 7.6. Observe that

$$\int_{\mathbb{R}^d} L(x,y)\chi_{N^c}(x,y)\, dy = \rho(x)^\alpha \int_{|x-y|>\rho(x)} \frac{1}{|x-y|^{d+\alpha}}\, dy = C,$$

for all $x \in \mathbb{R}^d$. On the other hand, by Lemma 7.1, there exists a positive number $\varepsilon < 1$ such that

$$\begin{aligned} L(x,y) &\le C \frac{\rho(y)^\alpha}{|x-y|^{d+\alpha}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{\varepsilon\alpha} \\ &\le C \left(\frac{\rho(y)^\alpha}{|x-y|^{d+\alpha}} + \frac{\rho(y)^{(1-\varepsilon)\alpha}}{|x-y|^{d+(1-\varepsilon)\alpha}} \right). \end{aligned} \tag{7.18}$$

Assume that $|x-y| > \rho(x)$. Then we claim that $|x-y| \ge C\rho(y)$ for some positive constant C depending on the constants c and k_0 that appear in Lemma 2.1. Indeed, by Lemma 2.1 and the fact that $\frac{|x-y|}{\rho(x)} \ge 1$ and $\frac{k_0}{k_0+1} \le 1$, we have

$$\begin{aligned} \rho(y) &\le C\rho(x) \left(1 + \left(\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}\right) \le C\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right) \\ &\le C\left(\rho(x) + |x-y|\right) \le 2C\,|x-y|\,. \end{aligned}$$

This together with (7.18) give us $\displaystyle\int_{\mathbb{R}^d} L(x,y)\chi_{N^c}(x,y)\, dx \le C$. Hence the operator given by the kernel $L(x,y)\chi_{N^c}(x,y)$ maps $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for every p, $1 \le p \le \infty$. Using Lemma 7.6 we get (a).

In order to see (b) we observe that for a function f in $BMO_{\mathcal{L},X}$, by Lemma 7.6,

$$\begin{aligned} g^{\mathcal{L},q}_{\mathrm{glob}} f(x) &\le C\rho(x)^\alpha \sum_{j=0}^{\infty} \int_{2^j\rho(x)<|x-y|\le 2^{j+1}\rho(x)} \frac{1}{|x-y|^{d+\alpha}} \left\|f(y)\right\|_X dy \\ &\le C\rho(x)^\alpha \sum_{j=0}^{\infty} \frac{1}{(2^j\rho(x))^{d+\alpha}} \int_{|x-y|\le 2^{j+1}\rho(x)} \left\|f(y)\right\|_X dy \\ &= C \sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}} \frac{1}{(2^{j+1}\rho(x))^d} \int_{|x-y|\le 2^{j+1}\rho(x)} \left\|f(y)\right\|_X dy \\ &\le C \left\|f\right\|_{BMO_{\mathcal{L},X}} \sum_{j=0}^{\infty} \frac{1}{2^{j\alpha}} = C\left\|f\right\|_{BMO_{\mathcal{L},X}}, \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

For (c) just note that $H^1_{\mathcal{L},X} \subset L^1_X(\mathbb{R}^d)$ and then apply (a).

Lemma 7.8. *Let X be any Banach space. Then, for any strongly measurable X-valued function f,*

$$\left|g_{\rm loc}^{\mathcal{L},q} f(x) - g_{\rm loc}^{\Delta,q} f(x)\right| \le C \int_{\mathbb{R}^d} M(x,y)\chi_N(x,y)\, \|f(y)\|_X \, dy, \quad x \in \mathbb{R}^d,$$

where $M(x,y) = \dfrac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}}$, *for* $x, y \in \mathbb{R}^d$, *and* $\delta > 0$ *is given in* (7.9).

Proof: Proceeding as in the proof of Lemma 7.6 it is easy to check that

$$\begin{aligned}&\left|g_{\rm loc}^{\mathcal{L},q} f(x) - g_{\rm loc}^{\Delta,q} f(x)\right| \\ &\le C \int_0^\infty \int_{\mathbb{R}^d} |k_u(x,y) - h_u(x-y)|\, \chi_N(x,y)\, \|f(y)\|_X \, dy \, \frac{du}{u}.\end{aligned}$$

Using (7.8) in Lemma 7.4 and the fact that ω is a rapidly decreasing function,

$$\begin{aligned}&\int_0^\infty |k_u(x,y) - h_u(x-y)| \, \frac{du}{u} \\ &\le C\rho(x)^{-\delta} \int_0^\infty \frac{1}{u^{(d-\delta)/2}} \, \omega\left((x-y)/\sqrt{u}\right) \, \frac{du}{u} \\ &\le C\rho(x)^{-\delta} \left[\frac{1}{|x-y|^{d-\delta+\varepsilon}} \int_0^{|x-y|^2} \left(\frac{|x-y|}{\sqrt{u}}\right)^{d-\delta+\varepsilon} \omega\left((x-y)/\sqrt{u}\right) \frac{du}{u^{1-\varepsilon/2}} \right. \\ &\qquad \left. + \int_{|x-y|^2}^\infty \frac{1}{u^{(d-\delta)/2}} \, \frac{du}{u}\right] \\ &\le C \frac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}} \left[\frac{1}{|x-y|^{\varepsilon}} \int_0^{|x-y|^2} \frac{du}{u^{1-\varepsilon/2}} + 1\right] = C \frac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}}.\end{aligned}$$

Lemma 7.9. *Let X be any Banach space. Then the difference operator $g_{\rm loc}^{\mathcal{L},q} - g_{\rm loc}^{\Delta,q}$ maps*

(a) $L^p_X(\mathbb{R}^d)$ *into* $L^p(\mathbb{R}^d)$ *for any* p, $1 \le p \le \infty$,

(b) $BMO_{\mathcal{L},X}$ *into* $L^\infty(\mathbb{R}^d)$, *and*

(c) $H^1_{\mathcal{L},X}$ *into* $L^1(\mathbb{R}^d)$.

Proof: Let $M(x,y)$, $x, y \in \mathbb{R}^d$, be as in Lemma 7.8. First note that

$$\int_{\mathbb{R}^d} M(x,y)\chi_N(x,y)\, dy = \rho(x)^{-\delta} \int_{|x-y| \le \rho(x)} \frac{1}{|x-y|^{d-\delta}} \, dy = C, \quad x \in \mathbb{R}^d.$$

On the other hand, by Lemma 7.1,

$$\begin{aligned} M(x,y) &\leq \frac{C}{|x-y|^{d-\delta}}\ \rho(y)^{-\delta}\left(1+\frac{|x-y|}{\rho(y)}\right)^{k_0\delta} \\ &\leq C\left(\frac{\rho(y)^{-\delta}}{|x-y|^{d-\delta}}+\frac{\rho(y)^{-(1+k_0)\delta}}{|x-y|^{d-(1+k_0)\delta}}\right), \end{aligned}$$

where $k_0 \geq 1$. This, and the fact that $|x-y| > \rho(x)$ implies $|x-y| > C\rho(y)$ (see the proof of Lemma 7.7), give $\int_{\mathbb{R}^d} M(x,y)\chi_N(x,y)\ dx \leq C$ for all $y \in \mathbb{R}^d$. Applying Lemma 7.8 we conclude (a) and as a consequence we also get (c).

We shall prove (b). Let $f \in BMO_{\mathcal{L},X}$. Then

$$\begin{aligned} &\int_{\mathbb{R}^d} M(x,y)\chi_N(x,y)\,\|f(y)\|_X\,dy \\ &= \sum_{j=0}^{\infty}\int_{2^{-(j+1)}\rho(x)<|x-y|\leq 2^{-j}\rho(x)} \frac{\rho(x)^{-\delta}}{|x-y|^{d-\delta}}\,\|f(y)\|_X\,dy \\ &\leq C\sum_{j=0}^{\infty}\frac{1}{2^{j\delta}}\frac{1}{(2^{-j}\rho(x))^d}\int_{|x-y|\leq 2^{-j}\rho(x)}\|f(y)\|_X\,dy \\ &\leq C\sum_{j=0}^{\infty}\frac{1}{2^{j\delta}}\left[\frac{1}{(2^{-j}\rho(x))^d}\int_{|x-y|\leq 2^{-j}\rho(x)}\left\|f(y)-f_{B(x,2^{-j}\rho(x))}\right\|_X\ dy\right. \\ &\qquad\left.+\sum_{k=0}^{j-1}\left(\left\|f_{B(x,2^{-k}\rho(x))}-f_{B(x,2^{-(k+1)}\rho(x))}\right\|_X\right)+\left\|f_{B(x,\rho(x))}\right\|_X\right] \\ &\leq C\sum_{j=0}^{\infty}\frac{1}{2^{j\delta}}\left[\|f\|_{BMO_X}+j\,\|f\|_{BMO_X}+\|f\|_{BMO_{X,\mathcal{L}}}\right] \\ &\leq C\,\|f\|_{BMO_{\mathcal{L},X}}\sum_{j=0}^{\infty}\frac{(j+2)}{2^{j\delta}} = C\,\|f\|_{BMO_{\mathcal{L},X}}, \end{aligned}$$

for all $x \in \mathbb{R}^d$. To finish use Lemma 7.8.

Lemma 7.10. *Let C_1 be the constant that appears in Lemma 7.1 and $\gamma > 0$. Take $x, y \in \mathbb{R}^d$ such that $|x| < \gamma$ and $|y| < \frac{C_1^2}{2}\rho(0)$. Then there exists a sufficiently large $R = R_\gamma > 0$ for which $\left|\frac{x}{R} - y\right| < \rho\left(\frac{x}{R}\right)$.*

Proof: Lemma 7.1 ensures that $C_1\rho(0) < \rho(y) < C_1^{-1}\rho(0)$. Let $R > 0$ be such that $\left|\frac{x}{R} - y\right| < C_1^2\rho(0)$ (it is enough to take $R > \frac{2\gamma}{C_1^2\rho(0)}$). Hence $\left|\frac{x}{R} - y\right| < C_1\rho(y) < \rho(y)$. Once more using Lemma 7.1 we obtain $\rho(y) < C_1^{-1}\rho\left(\frac{x}{R}\right)$ and therefore $\left|\frac{x}{R} - y\right| < C_1C_1^{-1}\rho\left(\frac{x}{R}\right) = \rho\left(\frac{x}{R}\right)$.

Lemma 7.11. *Let f be a function with compact support. For a real number r denote by f^r the dilation of f defined by $f^r(x) := f(rx)$, $x \in \mathbb{R}^d$. Then for any given $\gamma > 0$ there exists $R > 0$, depending on γ and the support of f, such that*

$$g^{\Delta,q} f(x) = g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot) f^R(\cdot)\right)(\frac{x}{R}), \quad \text{for all } |x| < \gamma.$$

PROOF: The scaling of the classical Poisson semigroup $P_t f^R(x/R) = P_{tR} f(x)$, $R > 0$ (see Remark 7.5), implies that the square function satisfies $g^{\Delta,q} f(x) = g^{\Delta,q} f^R(x/R)$ for all $R > 0$. In order to get the conclusion it is enough to take a sufficiently large R such that the support of f^R is contained in $B(0, \frac{C_1^2}{2}\rho(0))$ and such that Lemma 7.10 can be applied.

The following result establishes that the boundedness in L^p of the square function $g^{\Delta,q}$ related to the Laplacian $-\Delta$ implies the same type of boundedness for the ρ-localized operator $g^{\Delta,q}_{\text{loc}}$. In fact this is a fairly general property: see Remark 7.13 below.

Theorem 7.12. *Assume that $g^{\Delta,q}$ maps $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for some p, $1 < p < \infty$ (resp. $L^1_X(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$). Then the operator $f \longmapsto t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)$, $x \in \mathbb{R}^d$, $t > 0$, maps $L^p_X(\mathbb{R}^d)$ into $L^p_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$ (resp. $L^1_X(\mathbb{R}^d)$ into weak-$L^1_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$). In particular $g^{\Delta,q}_{\text{loc}}$ maps $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ (resp. $L^1_X(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$).*

Moreover, if for every function $f \in L^1_X(\mathbb{R}^d)$ we have $g^{\Delta,q} f(x) < \infty$ for almost all $x \in \mathbb{R}^d$, then $\|t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)\|_{L^q_X((0,\infty),\frac{dt}{t})} < \infty$ for almost all $x \in \mathbb{R}^d$.

PROOF: We shall prove only the boundedness in L^p. We leave to the reader the details of the rest of the proofs.

Let $\{Q_k\}_{k=1}^\infty$ be the covering of $\mathbb{R}^d$ by critical balls whose existence is guaranteed by Lemma 7.2. Consider the auxiliary operator given by

$$f \longmapsto Sf(x) = \sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t(\chi_{2Q_k} f)(x), \quad x \in \mathbb{R}^d,\ t > 0.$$

Then S is a bounded operator from $L^p_X(\mathbb{R}^d)$ into $L^p_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$. Indeed, by using Minkowski's inequality, the finite overlapping of the balls Q_k, the boundedness in L^p of $g^{\Delta,q}$ and once more the finite overlapping of $2Q_k$ we get

$$\begin{aligned}
&\|Sf\|_{L^p_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)} \\
&\leq \left(\int_{\mathbb{R}^d} \Big| \sum_{k\geq 1} \chi_{Q_k}(x) \left\|t\partial_t P_t(\chi_{2Q_k} f)(x)\right\|_{L^q_X((0,\infty),\frac{dt}{t})} \Big|^p dx\right)^{1/p} \\
&\leq \sum_{k\geq 1} \left\|\chi_{Q_k} g^{\Delta,q}(\chi_{2Q_k} f)\right\|_{L^p_X(\mathbb{R}^d)} \leq C \sum_{k\geq 1} \left\|g^{\Delta,q}(\chi_{2Q_k} f)\right\|_{L^p_X(\mathbb{R}^d)} \\
&\leq C \sum_{k\geq 1} \|\chi_{2Q_k} f\|_{L^p_X(\mathbb{R}^d)} \leq C \|f\|_{L^p_X(\mathbb{R}^d)}.
\end{aligned}$$

Recall that for a compactly supported function f in $L^\infty_X(\mathbb{R}^d)$ we have, as in (7.16),

$$g^{\Delta,q}_{\text{loc}} f(x) \leq g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x) = \|t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x)\|_{L^q_X((0,\infty),\frac{dt}{t})},$$

a.e. $x \in \mathbb{R}^d$. Our idea is to compare the operators S and $f \longmapsto t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)$. In order to do that we need some geometrical considerations. Let C_1 be the constant that appears in Lemma 7.1. Consider the set

$$\widetilde{N} = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \frac{C_1}{1+C_1}\,\rho(x)\right\}.$$

It is an exercise to prove that if $(x,y) \in \widetilde{N}$ then, since the family $\{Q_k\}_{k=1}^\infty$ is a covering of $\mathbb{R}^d$, there exists a positive integer k such that $(x,y) \in Q_k \times 2Q_k$. On the other hand, if $(x,y) \in Q_k \times 2Q_k$, then by using Lemma 7.1 we get $|x-y| \leq |x-x_k| + |x_k - y| \leq 3C_1^{-1}\rho(x)$. Observe that it follows from the finite overlapping property of the balls Q_k that

$$\|t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x)\|_X \sim \Big\|\sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x)\Big\|_X,$$

$x \in \mathbb{R}^d$, $t > 0$. The geometrical comments just made ensure that the kernel of the difference operator

$$f \longmapsto \sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x) - Sf(x), \quad x \in \mathbb{R}^d,\ t > 0, \tag{7.19}$$

is supported in the region

$$A := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \frac{C_1}{1+C_1}\,\rho(x) \leq |x-y| \leq 3C_1^{-1}\rho(x)\right\}.$$

Consequently, as

$$\|t\partial_t P_t(x-y)\|_{L^q((0,\infty),\frac{dt}{t})} = \frac{C}{|x-y|^d}, \quad x,y \in \mathbb{R}^d, \tag{7.20}$$

we have

$$\Big\|\sum_{k\geq 1} \chi_{Q_k}(x) t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x) - Sf(x)\Big\|_{L^q((0,\infty),\frac{dt}{t})} \leq C\int_{\mathbb{R}^d} \frac{\chi_A(x,y)}{|x-y|^d}\,\|f(y)\|_X\; dy.$$

Note that

$$\int_{\mathbb{R}^d} \frac{\chi_A(x,y)}{|x-y|^d}\,dy = \int_{\frac{C_1}{1+C_1}\rho(x) \leq |x-y| \leq 3C_1^{-1}\rho(x)} \frac{1}{|x-y|^d}\,dy = C\log\frac{3(1+C_1)}{C_1^2}, \quad x \in \mathbb{R}^d,$$

and, by Lemma 2.1,

$$\int_{\mathbb{R}^d} \frac{\chi_A(x,y)}{|x-y|^d}\, dx \le \int_{\alpha_1\rho(y)\le|x-y|\le\alpha_2\rho(y)} \frac{1}{|x-y|^d}\, dx = C\log\frac{\alpha_2}{\alpha_1}, \quad y\in\mathbb{R}^d,$$

for some constants α_1 and α_2 independent of y. Therefore the operator

$$f \longmapsto \int_{\mathbb{R}^d} \frac{\chi_A(x,y)}{|x-y|^d}\, \|f(y)\|_X\, dy$$

is bounded from L^p_X into L^p for every p, $1\le p<\infty$. Hence we get the conclusion.

Remark 7.13. Consider two Banach spaces X_1 and X_2. Let T be a linear operator that maps $C_c^\infty(\mathbb{R}^d;X_1)$ into X_2-valued strongly measurable functions. Suppose T has an associated kernel which satisfies the standard Calderón-Zygmund estimates. Define the "ρ-localized" operator

$$T_{\mathrm{loc}}f(x) = T\left(\chi_N(x,\cdot)f(\cdot)\right)(x), \quad x\in\mathbb{R}^d,$$

where N is the region determined by $|x-y|\le\rho(x)$ as above. Then:

- Assume T has a bounded extension from $L^p_{X_1}(\mathbb{R}^d)$ into $L^p_{X_2}(\mathbb{R}^d)$ for some p, $1<p<\infty$. Then T_{loc} has a bounded extension from $L^p_{X_1}(\mathbb{R}^d)$ into $L^p_{X_2}(\mathbb{R}^d)$.
- Assume T has a bounded extension from $L^1_{X_1}(\mathbb{R}^d)$ into weak-$L^1_{X_2}(\mathbb{R}^d)$. Then T_{loc} has a bounded extension from $L^1_{X_1}(\mathbb{R}^d)$ into weak-$L^1_{X_2}(\mathbb{R}^d)$.
- Assume that for every function $f\in L^1_{X_1}(\mathbb{R}^d)$ we have $\|Tf(x)\|_{X_2}<\infty$ for almost all $x\in\mathbb{R}^d$. Then the same is true for T_{loc}.

The reader can check the validity of this Remark just by exchanging X by X_1, $L^q_X((0,\infty),\frac{dt}{t})$ by X_2 and $f\longmapsto t\partial_t P_t f(x)$ by $f\longmapsto Tf(x)$ along the lines of the proof of Theorem 7.12 above.

The next Theorem permits us to pass, for ρ-localized operators related to $-\Delta$, from L^p-boundedness to $BMO_{\mathcal{L}}$ and $H^1_{\mathcal{L}}-L^1$ boundedness.

Theorem 7.14. *Let X be a Banach space such that the operator*

$$f\longmapsto Tf(x) = t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x), \quad x\in\mathbb{R}^d,\ t>0,$$

is bounded from $L^p_X(\mathbb{R}^d)$ into $L^p_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$ for some p, $1<p<\infty$. Then T maps $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L},L^q_X((0,\infty),\frac{dt}{t})}$ and $H^1_{\mathcal{L},X}$ into $L^1_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$.

PROOF: **Boundedness from $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L},L^q_X((0,\infty),\frac{dt}{t})}$.** We first analyze the behavior over "small" balls. Consider a ball $B = B(x_0, r_0)$, such that $5r_0 < C_1\rho(x_0)$, where $C_1 < 1$ is the constant that appears in Lemma 7.1. Given a function f we decompose it as

$$f = (f - f_B)\chi_{4B} + (f - f_B)\chi_{(4B)^c} + f_B =: f_1 + f_2 + f_3.$$

Before entering into the concrete proof, we need some small preparation. For $x, z \in B$,

$$Tf(x) - Tf(z) = Tf_1(x) - Tf_1(z) + Tf_2(x) - Tf_2(z) + Tf_3(x) - Tf_3(z).$$

We begin by observing that

$$\begin{aligned} &Tf_2(x) - Tf_2(z) + Tf_3(x) - Tf_3(z) \\ &= \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\, \chi_{|x-y|\le\rho(x)}(y) f_2(y)\, dy \\ &\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y)\right) f_2(y)\, dy \\ &\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right) f_2(y)\, dy \\ &\quad + f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y))\, \chi_{|x-y|\le\rho(x)}(y)\, dy \\ &\quad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y)\right)\, dy \\ &\quad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right)\, dy. \end{aligned}$$

Using Lemma 7.1,

$$\chi_{(4B)^c}(y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right) = \chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y).$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right) f_2(y)\, dy \\ &\qquad = \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right) (f(y) - f_B)\, dy. \end{aligned}$$

As a consequence,

$$\begin{aligned}
&Tf_2(x) - Tf_2(z) + Tf_3(x) - Tf_3(z)\\
&= \int_{\mathbb{R}^d} \left(t\partial_t P_t(x-y) - t\partial_t P_t(z-y)\right) \chi_{|x-y|\le\rho(x)}(y) f_2(y)\, dy\\
&\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y)\right) f_2(y)\, dy\\
&\quad + \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|z-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(z)}(y)\right) f(y)\, dy\\
&\quad + f_B \int_{\mathbb{R}^d} \left(t\partial_t P_t(x-y) - t\partial_t P_t(z-y)\right) \chi_{|x-y|\le\rho(x)}(y)\, dy\\
&\quad + f_B \int_{\mathbb{R}^d} t\partial_t P_t(z-y) \left(\chi_{|x-y|\le\rho(x)}(y) - \chi_{|z-y|\le\rho(x)}(y)\right)\, dy\\
&=: A_1(x,z) + A_2(x,z) + A_3(x,z) + A_4(x,z) + A_5(x,z).
\end{aligned}$$

After these remarks, we can start the actual proof of the boundedness in BMO. We have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \|Tf(x) - (Tf)_B\|_{L^q_X((0,\infty),\frac{dt}{t})}\, dx\\
&\le \frac{2}{|B|}\int_B \|Tf_1(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\, dx\\
&\qquad + \sum_{i=1}^{5} \frac{1}{|B|^2}\int_B\int_B \|A_i(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})}\, dx\, dz.
\end{aligned}$$

By hypothesis T is bounded from $L^p_X(\mathbb{R}^d)$ into $L^p_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$, so

$$\begin{aligned}
\frac{1}{|B|}\int_B \|Tf_1(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\, dx &\le C\left(\frac{1}{|B|}\int_B \|Tf_1(x)\|^p_{L^q_X((0,\infty),\frac{dt}{t})}\, dx\right)^{1/p}\\
&\le C\left(\frac{1}{|B|}\int_{\mathbb{R}^d} \|f_1(x)\|^p_X\, dx\right)^{1/p}\\
&= C\left(\frac{1}{|B|}\int_{4B} \|f(x) - f_B\|^p_X\, dx\right)^{1/p}\\
&\le C\,\|f\|_{BMO_X} \le C\,\|f\|_{BMO_{\mathcal{L},X}},
\end{aligned}$$

where in the penultimate inequality we applied an argument as in (7.21) below. Let us now estimate all the $A_i(x,z)$, $i = 1, \ldots, 5$, for $x, z \in B = B(x_0, r_0)$. By the Mean

Value Theorem and (7.20),

$$
\begin{aligned}
\|A_1(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})} &\leq C\int_{\mathbb{R}^d}\frac{|x-z|}{|x-y|^{d+1}}\|f_2(y)\|_X\,dy\\
&\leq Cr_0\int_{|x_0-y|>4r_0}\frac{1}{|x_0-y|^{d+1}}\|f(y)-f_B\|_X\,dy\\
&= Cr_0\sum_{j=2}^{\infty}\int_{2^jr_0<|x_0-y|\leq 2^{j+1}r_0}\frac{1}{|x_0-y|^{d+1}}\|f(y)-f_B\|_X\,dy\\
&\leq C\sum_{j=0}^{\infty}\frac{1}{2^j}\frac{1}{(2^{j+1}r_0)^d}\int_{|x_0-y|\leq 2^{j+1}r_0}\|f(y)-f_B\|_X\,dy\\
&\leq C\|f\|_{BMO_X}\leq C\|f\|_{BMO_{X,\mathcal{L}}}.
\end{aligned}
$$

Again by (7.20),

$$
\begin{aligned}
&\|A_2(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})}\\
&\leq C\int_{\mathbb{R}^d}\frac{1}{|z-y|^d}\left|\chi_{|x-y|\leq\rho(x)}(y)-\chi_{|z-y|\leq\rho(x)}(y)\right|\|f_2(y)\|_X\,dy.
\end{aligned}
$$

Observe that A_2 will be non zero in the following cases:

(I) $|x-y|\leq\rho(x)$ and $|z-y|>\rho(x)$,

(II) $|x-y|>\rho(x)$ and $|z-y|\leq\rho(x)$.

In the first case $\rho(x)<|z-y|\leq|z-x|+|x-y|<2r_0+|x-y|$ and then $\rho(x)-2r_0<|x-y|\leq\rho(x)$. While in (ii) we have $\rho(x)<|x-y|\leq|x-z|+|z-y|<2r_0+\rho(x)$. On the other hand $|x-y|\sim|z-y|$. Let j_0 and j_1 be nonnegative integers such that $2^{j_0}r_0\leq\rho(x)/2<2^{j_0+1}r_0$ and $2^{j_1}r_0\leq 2\rho(x)<2^{j_1+1}r_0$. Observe that, since $5r_0<\rho(x)$ for all $x\in B(x_0,r_0)$, we have $j_0\geq 1$. The Mean Value Theorem gives $(\rho(x)-2r_0)^d-(\rho(x)+2r_0)^d\leq C\rho(x)^{d-1}r_0$, hence applying Hölder's inequality with some $r\in(1,\infty)$

we get

$$
\begin{aligned}
&\|A_2(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})} \le C \int_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^d} \|f_2(y)\|_X \, dy \\
&\le C \left(\int_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^{dr}} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \rho(x)^{(d-1)/r'} r_0^{1/r'} \\
&\le C \left(\int_{\rho(x)/2<|x-y|<2\rho(x)} \frac{1}{|x-y|^{dr}} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \rho(x)^{(d-1)/r'} r_0^{1/r'} \\
&\le C \left(\sum_{j=j_0}^{j_1} \frac{1}{(2^j r_0)^{(d-1)(r-1)}} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \\
&\quad \times \rho(x)^{(d-1)/r'} \\
&\le C \left(\frac{1}{(2^{j_0} r_0)^{(d-1)(r-1)}} \sum_{j=j_0}^{j_0+2} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \\
&\quad \times \rho(x)^{(d-1)/r'} \\
&\le C \left(\sum_{j=0}^{\infty} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \le C \|f\|_{BMO_X} .
\end{aligned}
$$

Observe that in the penultimate inequality above we pass to the infinite series since j_0 depends on $\rho(x)$ and we want an estimate independent of it. For the last inequality above we first note that, by the triangle inequality and Minkowski's integral inequality,

$$
\begin{aligned}
&\left(\frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_B\|_X^r \, dy \right)^{1/r} \\
&\le \left(\frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \left(\|f(y)-f_{2^{j+2}B}\|_X + \sum_{k=0}^{j+1} \|f_{2^{k+1}B}-f_{2^k B}\|_X \right)^r dy \right)^{1/r} \\
&\le \left(\frac{4^d}{(2^{j+2} r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y)-f_{2^{j+2}B}\|_X^r \, dy \right)^{1/r} + C \sum_{k=0}^{j+1} \|f_{2^{k+1}B}-f_{2^k B}\|_X \\
&\le C(j+3) \|f\|_{BMO_X} . \qquad (7.21)
\end{aligned}
$$

Hence,

$$\sum_{j=0}^{\infty} \frac{1}{2^{j(r-1)}} \frac{1}{(2^j r_0)^d} \int_{|x_0-y|<2^{j+2}r_0} \|f(y) - f_B\|_X^r \, dy$$
$$\leq C \sum_{j=0}^{\infty} \frac{(j+3)^r}{2^{j(r-1)}} \|f\|_{BMO_X}^r = C \|f\|_{BMO_X}^r .$$

Since $x, z \in B$,

$$\begin{aligned}\|A_3(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})} &\leq \int_{C_1\rho(x_0)<|z-y|<C_1^{-1}\rho(x_0)} \frac{1}{|z-y|^d} \|f(y)\|_X \, dy \\ &\leq \frac{C}{\rho(x_0)^d} \int_{|z-y|<C_1^{-1}\rho(x_0)} \|f(y)\|_X \, dy \\ &\leq \frac{C}{\rho(z)^d} \int_{|z-y|<C\rho(z)} \|f(y)\|_X \, dy \leq C \|f\|_{BMO_{\mathcal{L},X}} .\end{aligned}$$

By dominated convergence,

$$\int_{\mathbb{R}^d} \partial_t P_t(x-y) \, dy = \partial_t \int_{\mathbb{R}^d} P_t(x-y) \, dy = \partial_t 1 = 0, \quad x \in \mathbb{R}^d.$$

Therefore,

$$\begin{aligned}&\|A_4(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})} \\ &= \left\| f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y)) \, \chi_{|x-y|\leq\rho(x)}(y) \, dy \right\|_{L^q_X((0,\infty),\frac{dt}{t})} \\ &= \left\| f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y)) \, \chi_{|x-y|\leq\rho(x)}(y) \, dy \right. \\ &\qquad \left. - f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y)) \, dy \right\|_{L^q_X((0,\infty),\frac{dt}{t})} \\ &= \left\| f_B \int_{\mathbb{R}^d} (t\partial_t P_t(x-y) - t\partial_t P_t(z-y)) \, \chi_{|x-y|>\rho(x)}(y) \, dy \right\|_{L^q_X((0,\infty),\frac{dt}{t})} \\ &\leq C \|f_B\|_X \int_{\mathbb{R}^d} \left\| t\partial_t P_t(x-y) - t\partial_t P_t(z-y) \right\|_{L^q((0,\infty)\frac{dt}{t})} \chi_{|x-y|>\rho(x)}(y) \, dy \\ &\leq C \|f_B\|_X \int_{\mathbb{R}^d} \frac{|x-z|}{|x-y|^{d+1}} \chi_{|x-y|>\rho(x)}(y) \, dy \\ &\leq C \|f_B\|_X \frac{r_0}{\rho(x)} \leq C \|f_B\|_X \frac{r_0}{\rho(x_0)}.\end{aligned}$$

As $\|f_B\|_X \le C\left(1+\log\frac{\rho(x_0)}{r_0}\right)\|f\|_{BMO_{\mathcal{L},X}}$ (see [26, Lemma 2]) we get the appropriate bound for A_4. Finally, by using the arguments in A_2,

$$\begin{aligned}\|A_5(x,z)\|_{L^q_X((0,\infty),\frac{dt}{t})} &\le C\,\|f_B\|_X \int_{\rho(x)-2r_0<|x-y|<\rho(x)+2r_0} \frac{1}{|x-y|^d}\,dy \\ &\le C\,\|f\|_{BMO_{\mathcal{L},X}}\left(1+\log\frac{\rho(x_0)}{r_0}\right)\log\left(\frac{\rho(x)+2r_0}{\rho(x)-2r_0}\right)\end{aligned}$$

Since $\frac{r_0}{\rho(x)} < 1/5$, we have $\log\left(\frac{\rho(x)+2r_0}{\rho(x)-2r_0}\right) \sim \frac{r_0}{\rho(x)} \sim \frac{r_0}{\rho(x_0)}$, that gives the desired bound for A_5.

Let us now analyze the behavior over "big" balls. Let $B_1 = B(x_0, k\rho(x_0))$, with $k \ge \frac{C_1}{5}$. Given a function f we decompose it as $f = f_1 + f_2$, where $f_1 = f\chi_{2B_1}$. By Hölder's inequality and the hypothesis,

$$\begin{aligned}&\frac{1}{|B_1|}\int_{B_1}\|t\partial_t P_t\left(\chi_N(x,\cdot)f_1(\cdot)\right)(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\,dx \\ &\le C\left(\frac{1}{|B_1|}\int_{2B_1}\|f(x)\|_X^p\,dx\right)^{1/p} \le C\,\|f\|_{BMO_{\mathcal{L},X}}.\end{aligned}$$

On the other hand, by using Lemma 7.1 and (7.20),

$$\begin{aligned}&\|t\partial_t P_t\left(\chi_N(x,\cdot)f_2(\cdot)\right)(x)\|_{L^q_X((0,\infty),\frac{dt}{t})} \\ &\le \int_{\mathbb{R}^d}\frac{1}{|x-y|^d}\chi_{2k\rho(x_0)\le|x-y|\le\rho(x)}(y)\,\|f(y)\|_X\,dy \\ &\le \frac{C}{\rho(x)^d}\int_{|x-y|\le\rho(x)}\|f(y)\|_X\,dy \le C\,\|f\|_{BMO_{\mathcal{L},X}}, \quad x\in B_1.\end{aligned}$$

This finishes the proof of the BMO boundedness.

Boundedness from $H^1_{\mathcal{L},X}$ into $L^1_{L^q_X((0,\infty),\frac{dt}{t})}(\mathbb{R}^d)$. We begin with the analysis over atoms supported on "small" balls. Let a be an atom with support contained in a ball $\tilde{B} = B(y_0, r_0)$, with $r_0 < \rho(y_0)$. Then

$$\begin{aligned}&\int_{\mathbb{R}^d}\|Ta(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\,dx \\ &= \int_{4\tilde{B}}\|Ta(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\,dx + \int_{(4\tilde{B})^c}\|Ta(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}\,dx \\ &=: A_1 + A_2.\end{aligned}$$

Since T is bounded in L^p, by (7.12) we have

$$\begin{aligned} A_1 &\leq C\left(\int_{4\tilde{B}} \|Ta(x)\|^p_{L^q_X((0,\infty),\frac{dt}{t})}\,dx\right)^{1/p}|\tilde{B}|^{1/p'} \\ &\leq C\left(\int_{\tilde{B}} \|a(x)\|^p_X\,dx\right)^{1/p}|\tilde{B}|^{1/p'} \leq C. \end{aligned}$$

Applying the fact that the atom a has mean zero (7.13) we get

$$\begin{aligned} A_2 &= \int_{(4\tilde{B})^c}\Big\|\int_{\mathbb{R}^d}\big(t\partial_t P_t(x-y)\chi_{|x-y|\leq\rho(x)}(y) - t\partial_t P_t(x-y_0)\chi_{|x-y_0|\leq\rho(x)}(x)\big) \\ &\qquad\qquad \times a(y)\;dy\Big\|_{L^q_X((0,\infty),\frac{dt}{t})}dx \\ &\leq \int_{(4\tilde{B})^c}\Big\|\int_{\mathbb{R}^d}\left(t\partial_t P_t(x-y) - t\partial_t P_t(x-y_0)\right)\chi_{|x-y|\leq\rho(x)}(y) \\ &\qquad\qquad \times a(y)\;dy\Big\|_{L^q_X((0,\infty),\frac{dt}{t})}dx \\ &\quad + \int_{(4\tilde{B})^c}\Big\|\int_{\mathbb{R}^d} t\partial_t P_t(x-y_0)\left(\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|x-y_0|\leq\rho(x)}(x)\right) \\ &\qquad\qquad\quad \times a(y)\;dy\Big\|_{L^q_X((0,\infty),\frac{dt}{t})}dx \\ &\leq C\int_{(4\tilde{B})^c}\int_{\mathbb{R}^d}\frac{|y-y_0|}{|x-y_0|^{d+1}}\|a(y)\|_X\,dy\;dx \\ &\quad + C\int_{(4\tilde{B})^c}\int_{\mathbb{R}^d}\frac{1}{|x-y_0|^d}\left|\chi_{|x-y|\leq\rho(x)}(y) - \chi_{|x-y_0|\leq\rho(x)}(x)\right|\|a(y)\|_X\,dy\;dx \\ &=: C\left(A_{21} + A_{22}\right). \end{aligned}$$

Fubini's Theorem and (7.12) give

$$\begin{aligned} A_{21} &= \int_{\mathbb{R}^d}|y-y_0|\,\|a(y)\|_X\left[\int_{\mathbb{R}^d}\chi_{|x-y_0|\geq 4r_0}(x)\frac{1}{|x-y_0|^{d+1}}\;dx\right]dy \\ &= \frac{C}{r_0}\int_{|y-y_0|<r_0}|y-y_0|\,\|a(y)\|_X\,dy \leq C. \end{aligned}$$

A geometric reasoning parallel to the one developed above for the BMO case gives that in order to $A_{22} \neq 0$ we must have $3r_0 < \rho(x)$, $\rho(x) - r_0 < |x-y| < \rho(x) + r_0$ and, in addition, $|x-y_0| \sim \rho(x) \sim \rho(y_0)$. Therefore, since the atom a is supported in

$\tilde{B} = B(y_0, r_0)$ and is controlled in L^∞ norm by Cr_0^{-d},

$$\begin{aligned} A_{22} &\le \frac{C}{\rho(y_0)^d} \int_{|x-y_0|\le C\rho(y_0)} \int_{\rho(x)-r_0<|x-y|<\rho(x)+r_0} \|a(y)\|_X \, dy \, dx \\ &\le \frac{C}{\rho(y_0)^d} \int_{|x-y_0|\le C\rho(y_0)} \int_{|y-y_0|<r_0} \|a(y)\|_X \, dy \, dx \\ &\le \frac{C}{\rho(y_0)^d} \int_{|x-y_0|\le C\rho(y_0)} dx \le C. \end{aligned}$$

We continue with the analysis over atoms supported on "big" balls. Let a be an atom supported in a ball $\bar{B}(y_0, \gamma\rho(y_0))$, with $\gamma > 1$. We begin by proceeding as in the previous case for A_1. For A_2, since we do not have the cancelation property (7.13), we estimate its size as follows:

$$\begin{aligned} A_2 &= \int_{(4\bar{B})^c} \|Ta(x)\|_{L^q_X((0,\infty)\frac{dt}{t})} \, dx \\ &\le C \int_{(4\bar{B})^c} \int_{\bar{B}} \frac{1}{|x-y|^d} \, \|a(y)\|_X \, \chi_{|x-y|\le\rho(x)}(y) \, dy \, dx. \end{aligned}$$

The domain of integration above is contained in the set defined by the conditions $|x - y_0| \ge 4\gamma\rho(y_0)$, $|y - y_0| < \gamma\rho(y_0)$ and $|x - y| \le \rho(x)$. These conditions imply that $4\gamma\rho(y_0) \le |x - y_0| \le |x - y| + |y - y_0| < |x - y| + \gamma\rho(y_0)$, hence $3\gamma\rho(y_0) \le |x - y|$. Note that, by Lemma 7.1, $\rho(x) \le C\rho(y) \le \bar{C}\gamma\rho(y_0)$. Therefore $3\gamma\rho(y_0) \le |x - y| \le \bar{C}\gamma\rho(y_0)$ and we get

$$A_2 \le C \int_{\bar{B}} \|a(y)\|_X \int_{3\gamma\rho(y_0)\le|x-y|\le\bar{C}\gamma\rho(y_0)} \frac{1}{|x-y|^d} \, dx \, dy \le C.$$

Remark 7.15. Consider two Banach spaces X_1 and X_2. Let T be a linear operator that maps $L^p_{X_1}(\mathbb{R}^d)$ into $L^p_{X_2}(\mathbb{R}^d)$ for some p, $1 < p < \infty$, such that $T1$ can be defined and $T1 = 0$. Assume T has an associated kernel which satisfies the standard estimates of Calderón-Zygmund operators. Define the operator

$$T_{\text{loc}} f(x) = T\left(\chi_N(x, \cdot) f(\cdot)\right)(x), \qquad x \in \mathbb{R}^d.$$

Then:

- T_{loc} is bounded from $BMO_{\mathcal{L},X_1}$ into $BMO_{\mathcal{L},X_2}$, and
- T_{loc} is bounded from $H^1_{\mathcal{L},X_1}$ into $L^1_{X_2}(\mathbb{R}^d)$.

Parallel to Remark 7.13, the reader can check the validity of these claims just by exchanging, along the lines of the proof of Theorem 7.14, X by X_1, $L^q_X((0,\infty), \frac{dt}{t})$ by X_2 and $f \longmapsto t\partial_t P_t f(x)$ by $f \longmapsto Tf(x)$.

7.4. Proof of Theorem A

Given a Banach space X, define the modulus of convexity by

$$\delta_X(\varepsilon) = \text{ínf}\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\right\},$$

for $0 < \varepsilon < 2$. The Banach space X is called q-uniformly convex, $2 \leq q < \infty$, if $\delta_X(\varepsilon) \geq c\varepsilon^q$ for some positive constant c. By Pisier's Renorming Theorem [75], X is q-uniformly convex if and only if X is of martingale cotype q. For martingale cotype the following Theorem holds, see [97] and [55].

Theorem 7.16. *Let X be a Banach space and $2 \leq q < \infty$. The following statements are equivalent.*

(1) X is of martingale cotype q.

(2) The operator $g^{\Delta,q}$ maps $BMO_{c,X}$ into BMO.

(3) The operator $g^{\Delta,q}$ maps $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$, for any p in the range $1 < p < \infty$.

(4) The operator $g^{\Delta,q}$ maps $L^1_X(\mathbb{R}^d)$ into weak-$L^1(\mathbb{R}^d)$.

(5) The operator $g^{\Delta,q}$ maps H^1_X into $L^1(\mathbb{R}^d)$.

(6) For every $f \in L^1_X(\mathbb{R}^d)$, $g^{\Delta,q}f(x) < \infty$ for almost every $x \in \mathbb{R}^d$.

The space H^1_X denotes the atomic Hardy space in $\mathbb{R}^d$. By $BMO_{c,X}$ we mean the set of functions that belong to the classical BMO with values in X and have compact support.

Proof of Theorem A: Observe that hypothesis (i) is equivalent to one of the statements in Theorem 7.16.

$(i) \implies (ii)$. We can apply Theorems 7.12 and 7.14 to get that the operator $f \longmapsto t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)$ maps $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L},L^q_X((0,\infty),\frac{dt}{t})}$. By using Lemma 7.9 we obtain the boundedness from $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L}}$ of the operator $g^{\mathcal{L},q}_{\text{loc}}$. Finally, by Lemma 7.7 (b) we arrive to (ii).

$(i) \implies (iii)$. By Theorem 7.12 and Lemma 7.9 (a) the local operator $g^{\mathcal{L},q}_{\text{loc}}$ is bounded in L^p. Boundedness of the global part follows from Lemma 7.7 (a).

$(i) \implies (iv)$. Theorem 7.12 and Lemma 7.9 (a), together with Lemma 7.7 (a), give the conclusion.

$(i) \implies (v)$. By using Theorems 7.12, 7.14 and 7.9 (c) we see that $g^{\mathcal{L},q}_{\text{loc}}$ maps $H^1_{\mathcal{L},X}$ into $L^1(\mathbb{R}^d)$. Then Lemma 7.7 (c) gives the result.

$(i) \implies (vi)$. Apply Theorem 7.12 and Lemmas 7.9 (a) and 7.7 (a).

$(ii) \implies (i)$. Theorem 7.16 tells us that it is enough to prove the boundedness of $g^{\Delta,q}$ from $BMO_{c,X}$ into BMO. From the hypothesis, Lemma 7.7 (b) and (7.17) we can deduce that the operator $f(x) \longmapsto g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x)$, $x \in \mathbb{R}^d$, is bounded from $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L}}$. On the other hand, the proof of Lemma 7.9 shows that the difference operator $f(x) \longmapsto g^{\mathcal{L},q}(\chi_N(x,\cdot)f(\cdot))(x) - g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x)$ is bounded from $BMO_{\mathcal{L},X}$ into L^∞. Thus the operator $f(x) \longmapsto g^{\Delta,q}(\chi_N(x,\cdot)f(\cdot))(x)$ is bounded from $BMO_{\mathcal{L},X}$ into $BMO_{\mathcal{L}} \subset BMO$. Let f be a function in $BMO_{c,X}$. Given a ball $B(x_0,s)$, by Lemma 7.11 there exists $R>0$ depending on s and the support of f such that $\operatorname{supp} f^R \subset B(0,\frac{\rho(0)}{2})$ (see the proof of Lemma 7.11) and

$$\begin{aligned}
&\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)} g^{\Delta,q}f(x)\,dx \\
&= \frac{1}{|B(x_0,s)|}\int_{B(x_0,s)} g^{\Delta,q}\Big(\chi_N(\frac{x}{R},\cdot)f^R(\cdot)\Big)(\frac{x}{R})\,dx \\
&= \frac{1}{\left|B(\frac{x_0}{R},\frac{s}{R})\right|}\int_{B(\frac{x_0}{R},\frac{s}{R})} g^{\Delta,q}(\chi_N(z,\cdot)f^R(\cdot))(z)\,dz.
\end{aligned}$$

Since R can be arbitrarily large, we fix it in such a way that $(R\rho(0))^{-d}\|f\|_{L^1_X(\mathbb{R}^d)} \le \|f\|_{BMO_X}$. Therefore,

$$\begin{aligned}
&\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)} \left|g^{\Delta,q}f(x) - \left(g^{\Delta,q}\left(\chi_N(z,\cdot)f^R(\cdot)\right)(z)\right)_{B(\frac{x_0}{R},\frac{s}{R})}\right|\,dx \\
&\quad= \frac{1}{\left|B(\frac{x_0}{R},\frac{s}{R})\right|}\int_{B(\frac{x_0}{R},\frac{s}{R})} \Big|g^{\Delta,q}\left(\chi_N(x,\cdot)f^R(\cdot)\right)(x) \\
&\qquad\qquad - \left(g^{\Delta,q}\left(\chi_N(z,\cdot)f^R(\cdot)\right)(z)\right)_{B(\frac{x_0}{R},\frac{s}{R})}\Big|\,dx \\
&\quad\le C\left\|f^R\right\|_{BMO_{\mathcal{L},X}} \le C\|f\|_{BMO_X},
\end{aligned}$$

where for the last inequality above the following argument is applied. Note that to have such an inequality we only have to compare the integral means of f^R with the BMO_X-norm of f. Let $\alpha \ge 1$. If $B(x,\alpha\rho(x))$ does not intersect $B(0,\frac{\rho(0)}{2})$ then $\int_{B(x,\alpha\rho(x))}\left\|f^R(y)\right\|_X\,dy = 0$ and there is nothing to prove. In case $B(x,\alpha\rho(x)) \cap B(0,\frac{\rho(0)}{2}) \neq \emptyset$ then, by Lemma 7.1, $\rho(x) \sim \rho(0)$ and, by the choice of R,

$$\begin{aligned}
\frac{1}{|B(x,\alpha\rho(x))|}\int_{B(x,\alpha\rho(x))}\left\|f^R(y)\right\|_X dy &\le \frac{C_n}{(R\alpha\rho(x))^d}\int_{B(0,R\frac{\rho(0)}{2})}\|f(z)\|_X\,dz \\
&\le \frac{C}{(R\rho(0))^d}\|f\|_{L^1_X(\mathbb{R}^d)} \le C\|f\|_{BMO_X};
\end{aligned}$$

here the constant C is independent of f.

$(iii) \implies (i)$. Lemmas 7.7 (a) and 7.9 (a) assure that $g^{\Delta,q}(\chi_N f)$ is bounded from $L^p_X(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$. Let $f \in L^p_X(\mathbb{R}^d)$ be a function with support contained in a ball $B_M = B(0,M)$, $M>0$. By Lemma 7.11 we can find $R>0$ such that $g^{\Delta,q}f(x) = g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot)f^R(\cdot)\right)(\frac{x}{R})$, for all $|x|<M$. Hence

$$\begin{aligned}\left\|\chi_{B_M} g^{\Delta,q} f\right\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\left|\chi_{B_M}(x) g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot)f^R(\cdot)\right)(\frac{x}{R})\right|^p dx\\ &\le R^d\int_{\mathbb{R}^d}\left|g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot)f^R(\cdot)\right)(\frac{x}{R})\right|^p dx\\ &\le CR^d\int_{\mathbb{R}^d}\left\|f^R(x)\right\|^p_X dx = C\left\|f\right\|^p_{L^p_X(\mathbb{R}^d)}.\end{aligned}$$

As the constant C does not depend on M we can take $M\to\infty$ to get $\left\|g^{\Delta,q}f\right\|_{L^p(\mathbb{R}^d)} \le C\left\|f\right\|_{L^p_X(\mathbb{R}^d)}$.

$(iv) \implies (i)$. We leave this case to the reader.

$(v) \implies (i)$. By Theorem 7.16 it is enough to prove the boundedness of $g^{\Delta,q}$ from H^1_X into $L^1(\mathbb{R}^d)$. Lemmas 7.7 (c) and 7.9 (c) imply that the localized operator $f \longmapsto \|t\partial_t P_t(\chi_N(x,\cdot)f(\cdot))(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}$ is bounded from $H^1_{\mathcal{L},X}$ into $L^1(\mathbb{R}^d)$. Therefore we only have to prove the boundedness over H^1-atoms with cancelation but supported in big balls. Let a be such an atom, namely a function supported in a ball $B(y_0,\gamma\rho(y_0))$ with $\gamma>1$ and $\int_{\mathbb{R}^d}a(y)\,dy=0$. Consider the function $\widetilde{a}^R(x) := R^d a^R(x) = R^d a(Rx)$, $x\in\mathbb{R}^d$, $R>0$. The function $\widetilde{a}^R$ is an atom with support contained in the ball $B(\frac{y_0}{R},\frac{\gamma y_0}{R})$. Given $M>0$, Lemma 7.11 allows us to choose a sufficiently large R such that $g^{\Delta,q}a(x) = g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot)a^R(\cdot)\right)(\frac{x}{R})$, for $|x|<M$. Hence

$$\begin{aligned}\int_{|x|<M}\left|g^{\Delta,q}a(x)\right|\,dx &= \int_{|x|<M}\left|g^{\Delta,q}\left(\chi_N(\frac{x}{R},\cdot)a^R(\cdot)\right)(\frac{x}{R})\right|\,dx\\ &= \int_{|z|<\frac{M}{R}}\left|g^{\Delta,q}(\chi_N(z,\cdot)\widetilde{a}^R(\cdot))(z)\right|\,dz\\ &\le C\left\|\widetilde{a}^R\right\|_{H^1_{\mathcal{L},X}} = C\left\|\widetilde{a}^R\right\|_{H^1_X} \le C,\end{aligned}$$

where C does not depend on M. To conclude take $M\to\infty$.

$(vi) \implies (i)$. We will prove that $g^{\Delta,q}f(x)<\infty$ for almost every $x\in\mathbb{R}^d$, see Theorem 7.16. By Lemma 7.7 (a) we have that $g^{\mathcal{L},q}_{\text{loc}}f(x)<\infty$ for almost all $x\in\mathbb{R}^d$. Hence by Lemma 7.9 (a) we have $g^{\Delta,q}_{\text{loc}}f(x)<\infty$, for almost all $x\in\mathbb{R}^d$. In fact, from the proof of Lemma 7.9 it can be deduced that $\|t\partial_t P_t\left(\chi_N(x,\cdot)f(\cdot)\right)(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}<\infty$, for almost all $x\in\mathbb{R}^d$. The arguments in the proof of Theorem 7.12 can be used to conclude that $\left\|\sum_{k\ge1}\chi_{Q_k}(x)t\partial_t P_t\left(\chi_{2Q_k}f\right)(x)\right\|_{L^q_X((0,\infty),\frac{dt}{t})}$ is finite for almost all $x\in\mathbb{R}^d$.

By the finite overlapping property of the balls Q_k we get the finiteness almost every x of each term $\left\|\chi_{Q_k}(x) t\partial_t P_t\left(\chi_{2Q_k} f\right)(x)\right\|_{L^q_X((0,\infty),\frac{dt}{t})}$. On the other hand, observe that

$$\begin{aligned}
&\left\|\chi_{Q_k}(x) t\partial_t P_t\left((1-\chi_{2Q_k}) f\right)(x)\right\|^q_{L^q_X((0,\infty),\frac{dt}{t})} \\
&\leq C \int_0^\infty \left(\int_{|x-y|>\rho(x_k)} \frac{t\,\|f(y)\|_X}{(t+|x-y|)^{d+1}}\,dy\right)^q \frac{dt}{t} \\
&\leq C\,\|f\|^q_{L^1_X(\mathbb{R}^d)} \int_0^\infty \frac{t^q}{(t+\rho(x_k))^{(d+1)q}}\,\frac{dt}{t} \\
&\leq C_k\,\|f\|^q_{L^1_X(\mathbb{R}^d)}\,.
\end{aligned}$$

Pasting together the last two thoughts we get that for every k and almost every $x \in \mathbb{R}^d$ the norm $\|\chi_{Q_k}(x) t\partial_t P_t f(x)\|_{L^q_X((0,\infty),\frac{dt}{t})}$ is finite. Hence $\|t\partial_t P_t f(x)\|_{L^q_X((0,\infty),\frac{dt}{t})} = g^{\Delta,q} f(x)$ is finite for almost all x.

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