TESIS DOCTORAL

High-order perturbation theory of spherical spacetimes with application to vacuum and perfect fluid matter

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Resumen

En esta memoria de tesis doctoral presentamos un formalismo para estudiar perturbaciones a altos órdenes de espaciotiempos esféricos. Proporcionamos un método recursivo para construir variables invariantes *gauge* a cualquier orden perturbativo. En particular, a segundo orden construimos explícitamente los objetos invariantes *gauge* y sus ecuaciones de movimiento. Seguidamente, aplicamos este formalismo general a los espaciotiempos de fondo (*backgrounds*) específicos correspondientes a la solución de Schwarzschild y a un fluido perfecto. También analizamos las condiciones perturbativas de ensamblaje (*matching*) a través de una superficie temporal que separe una estrella de fluido perfecto del vacío. Estas investigaciones tienen como resultado un marco completo para estudiar varios problemas de relevancia astrofísica, desde el acoplo de modos cuasinormales en la fase *ring-down* de una binaria de agujeros negros, hasta la evolución de segundas perturbaciones de una estrella esférica en colapso.

Motivados por la búsqueda de una descripción más sencilla de la radiación gravitatoria, también estudiamos con un tratamiento hamiltoniano las perturbaciones lineales de un espaciotiempo esférico con un contenido material de campo escalar. Tras realizar varias transformaciones canónicas, conseguimos definir, por primera vez para un fondo dinámico, dos variables *master*, en el sentido de que obedecen ecuaciones de evolución sin ligaduras y que la métrica perturbada completa se puede reconstruir a partir de ellas exclusivamente.

Asimismo, durante el transcurso de la realización de esta tesis, hemos diseñado varios módulos de computación algebraica para manejar las complejas ecuaciones del formalismo. Dichos módulos, que están accesibles en internet, se distribuyen libremente, y han sido utilizados ya por otros autores.

Abstract

We present a complete formalism to deal with high-order perturbations of spherical spacetimes. A method to construct gauge-invariant variables at any perturbative order is given. In particular, the gauge-invariant metric perturbations and their equations of motion are explicitly obtained at second order. This general formalism is then applied to a Schwarzschild background and to a perfect-fluid spacetime. The high-order perturbative junction conditions across a timelike surface separating a fluid star from pure vacuum are also analyzed. These investigations give rise to a complete framework for the study of a series of astrophysically relevant scenarios, ranging from quasi-normal mode coupling in the ring-down phase of a binary black hole collision, to the evolution of second-order perturbations of a spherical collapsing star.

Motivated by the search of simpler descriptions of gravitational radiation, we also study linear perturbations of a spherical spacetime containing a real massless scalar field, in a Hamiltonian setting. After several canonical transformations, we succeed in defining, for the first time for a dynamical background, two master variables, in the sense that they obey unconstrained evolution equations and the whole perturbed metric can be reconstructed in terms of them.

During the course of this thesis, several computer algebra tools have been designed to handle the intrincated equations of the formalism. These tools are now freely distributed, and are already being used by other authors.

Chapter 1

Introducción

Hoy en día, la Relatividad General se considera la mejor teoría física diponible para dar cuenta de la interacción gravitacional clásica (en contraposición a la cuántica). Las primeras confirmaciones experimentales de la teoría, ya predichas por Einstein, fueron la deflexión de la luz y el avance del perihelio de Mercurio. No fue hasta el comienzo de la década de 1960 cuando los descubrimientos astronómicos (como los cuásares o los púlsares) ofrecieron nuevas observaciones con las que comprobar la validez de la Relatividad General en el régimen de gravedad débil. Una de las más conocidas es el decrecimiento del periodo orbital del púlsar binario Hulse-Taylor [1], por el que ambos investigadores recibieron el premio Nobel [2, 3]. Esta observación se ajustaba perfectamente a las predicciones teóricas de la Relatividad General y sirvió para excluir varias teorías alternativas [4, 5]. Desde los años ochenta, la atención se ha centrado principalmente en encontrar observaciones en el régimen de campo fuerte que puedan corroborar o contradecir las predicciones de Relatividad General en dos extremos diferentes. Por un lado está el límite asociado con la física de la escala de Planck. Existen procesos con distancias características muy pequeñas que producen interacciones gravitatorias fuertes, hasta cierto régimen en el que se supone que la Relatividad General fallará debido a la fenomenología cuántica. Por otro lado está el límite astrofísico, que involucra objectos muy densos de gran masa, los cuales producen también interacciones gravitacionales fuertes. Se espera que varios fenónemos astrofísicos, tales como colisiones de agujeros negros y/o estrellas de neutrones, emitan suficiente radiación gravitatoria como para ser detectada desde la Tierra en un futuro cercano.

En este contexto astrofísico, la detección de ondas gravitatorias se considera uno de los problemas abiertos más importantes de la física experimental. Aparte de proporcionar un test de la Relatividad General en el régimen de campo fuerte, podría abrir una nueva ventana para la observación astrofísica, dando lugar a una era de astronomía de ondas gravitatorias. Varios detectores de ondas gravitatorias se encuentran actualmente tomando datos (tales como GEO [6], LIGO [7] o VIRGO [8]) y existen misiones espaciales (la más notable LISA [9]) planeadas para los cercanos años 2020.

La señal que se pretende detectar es muy pequeña, y está enterrada bajo varias fuentes de ruido de origen diverso. Se han desarrollado potentes técnicas estadísticas para ayudar en la recuperación de la señal de los datos recogidos. Sin embargo, aún es esencial disponer de antemano de un catálogo de patrones con posibles señales que observar (para poder utilizar, de esta manera, las técnicas de filtro adaptado). Estos patrones solamente pueden obtenerse resolviendo las ecuaciones de Relatividad General: las ecuaciones de Einstein. Estas diez ecuaciones en derivadas parciales forman un sistema, no lineal y acoplado, que resulta muy difícil de resolver. Por consiguiente, no se espera construir un catálogo con soluciones analíticas que describa el perfil de las ondas gravitatorias emitidas en tales acontecimientos astrofísicos. Solamente las situaciones con una gran simetría permiten obtener explícitamente la solución, y sólo unas pocas de ellas son relevantes desde el punto de vista astrofísico [10]. Por lo tanto, como es habitual en física cuando un problema no se puede resolver de manera exacta, se recurre a métodos aproximados. Aquí mencionaremos las tres técnicas que mejor han funcionado hasta el momento, y que se pueden considerar complementarias en el análisis de la dinámica de Relatividad General en este contexto: métodos post-newtonianos, Relatividad Numérica y teoría de perturbaciones.

Los métodos post-newtonianos están basados en la combinación de una aproximación de gravedad débil (es decir, teoría de perturbaciones alrededor del espaciotiempo de Minkowski, también conocida como aproximación post-minkowskiana) y una expansión de las soluciones en serie de potencias del parámetro v/c, donde c es la velocidad de la luz y v una velocidad típica de la materia del problema en cuestión [11]. Por ejemplo, esta aproximación es válida para modelar las ondas gravitatorias generadas por una fuente de movimiento interno lento y auto-gravedad débil. Truncando las mencionadas expansiones a altas potencias de v/c, es posible considerar fuentes altamente relativistas (en la actualidad los resultados a orden seis se utilizan sistemáticamente [12]), y normalmente la aproximación post-newtoniana se comporta mejor de lo esperado. El límite newtoniano ($1/c \to 0$) ya fue considerado por Einstein [13] y por Landau y Lifshitz [14] para derivar la famosa fórmula del cuadrupolo, que permite obtener la potencia total emitida en forma de ondas gravitatorias en función de la derivada temporal del momento cuadrupolar de la fuente. De hecho, esta relación fue suficiente para explicar el decrecimiento del periodo del pulsar binario Hulse-Taylor [15,16].

Actualmente, la Relatividad Numérica se considera, por derecho propio, una rama de la Relatividad General [17]. Cualquier intento de resolver las ecuaciones de Einstein utilizando métodos numéricos se podría incluir en esta área, pero aquí nos referiremos solamente a aquellas simulaciones que traten con las ecuaciones de Einstein completas; quizá bajo cierta reducción de simetría (incluso trabajando con métodos post-newtonianos o teoría de perturbaciones, es habitual tener que integrar algunas otras ecuaciones numéricamente.) Se pueden utilizar diferentes formulaciones de las ecuaciones de Einstein para su discretización en un ordenador, dependiendo de cómo se describa el espaciotiempo. La manera más sencilla de hacerlo está basada en el tratamiento hamiltoniano debido a Arnowitt, Deser y Misner (ADM) [18, 19], en el que se exfolia el espaciotiempo en una familia de superficies espaciales tridimensionales. Otras formulaciones están basadas en el uso de superficies nulas, lo que ofrece un marco más adecuado para tratar la radiación. Hubo ya algunos intentos de resolver numéricamente las ecuaciones de Einstein en dos dimensiones espaciales en los años sesenta y setenta [20, 21]. Sin embargo, en aquella época los ordenadores no tenían la suficiente capacidad para obtener resultados tridimensionales de relevancia, y las formulaciones que se utilizaron no eran matemáticamente consistentes, como se demostró posteriormente. Laa simulaciones de binarias de agujeros negros es uno de los problemas clave en esta área. El desarrollo de lo métodos númericos y analíticos para resolverlo tuvo lugar durante la década de los noventa. En 2005, Pretorius logró por vez primera evolucionar el sistema de una manera estable [22]. Desde entonces, varios grupos de investigación han publicado un extenso número de artículos, documentando interesantes propiedades del mencionado sistema y aportando patrones muy fiables del perfil de las ondas gravitatorias emitidas. Además del caso particular de vacío, se ha invertido un gran esfuerzo en la simulación de espaciotiempos con fluidos, y más recientemente fluidos acoplados al campo electromagnético, para modelar el colapso del núcleo estelar y la colisión de estrellas de neutrones, también fuentes importantes de radiación gravitatoria [23].

La teoría de perturbaciones proporciona otro tratamiento aproximado, permitiendo una descripción en términos de pequeñas desviaciones alrededor de una solución exacta de fondo. En el contexto de la Relatividad General, la teoría de perturbaciones ha desempeñado un papel destacado en el análisis de la estabilidad de ciertas soluciones y en la comprensión de los procesos dinámicos en términos de simples "modos de oscilación", siendo actualmente un complemento natural y eficiente de las simulaciones completas de Relatividad Numérica [24]. En las próximas secciones la describiremos con más detalle.

Las tres técnicas aproximadas tienen sus propios dominios de validez, que normalmente son complementarios y, por lo tanto, la mejor estrategia suele ser una combinación de todos ellos. Por ejemplo, las tres fases de una colisión de agujeros negros, *inspiral*, *merger* y *ring-down*, se pueden describir adecuadamente mediante métodos post-newtonianos, Relatividad Numérica y teoría de perturbaciones, respectivamente. Durante la fase inicial, los agujeros negros se encuentran lejos y la interacción gravitatoria mutua es débil. En consecuencia, esta situación puede describirse prácticamente mediante gravedad newtoniana. Éste es el escenario perfecto para los métodos post-newtonianos. A medida que los agujeros negros se acercan, debido a la pérdida de energía, radiada en forma de ondas gravitatorias, la interacción gravitacional mutua irá aumentando. En este punto, los métodos post-newtonianos necesitan más términos en sus expansiones para seguir las trayectorias con suficiente precisión. Cuando los dos agujeros están muy cerca, las no-linealidades crecen y es necesario recurrir a la Relatividad Numérica completa para trazar las trayectorias. Al final, ambos agujeros se fusionan en un único agujero negro, que sigue radiando ondas gravitatorias hasta convertirse en un agujero negro estacionario de Kerr o Schwarzschild. Durante esta etapa final de la evolución dinámica, el espaciotiempo puede aproximarse mediante la mencionada solución de fondo más una pequeña desviación, lo que puede describirse adecuadamente a través de la teoría de perturbaciones. A continuación, centraremos nuestra atención en la teoría de perturbaciones, describiendo detalladamente su historia y logros en el contexto de la Relatividad General.

1.1 Aplicaciones de la teoría de perturbaciones lineal

La teoría de perturbaciones puede utilizarse para estudiar la estabilidad de las soluciones a las ecuaciones de Einstein. En particular, una considerable cantidad de trabajo se ha dedicado a discutir la estabilidad de los agujeros negros [25] y las soluciones cosmológicas [26], dado el interés físico de estos espaciotiempos.

Además, el análisis perturbativo nos permite comprobar la presencia de inestabilidades *gauge* [27], la violación de la ligaduras [28], y otro tipo de inestabilidades en la implementación numérica de las ecuaciones de Einstein, debido a que los errores numéricos pueden considerarse distorsiones de la solución que se está calculando.

Otra importante aplicación es el estudio de la producción y propagación de ondas en un determinado espaciotiempo curvo. La teoría de perturbaciones proporciona estimaciones de la cantidad de radiación gravitatoria y el perfil de la señal emitida en procesos astrofísicos tales como las oscilaciones de una estrella de neutrones [29], el colapso gravitatorio de una estrella [30], una binaria de cociente de masas extremo (*extreme mass ratio binary*) [31], o una colisión frontal de dos agujeros negros en el límite cercano (*close limit*) [32].

La teoría de perturbaciones también ha sido reconocida como una herramienta muy útil en cosmología. Basándose en los trabajos pioneros debidos a Bardeen [33] y a Kodama y Sasaki [34], se ha utilizado esta teoría para estudiar la evolución de pequeñas inhomogeneidades en el Universo Primitivo, proporcionando una comprensión detallada de la dependencia angular del espectro de potencias en el fondo cósmico de microondas. Estas desviaciones de la homogeneidad han sido medidas por los experimentos COBE [35] y WMAP [36]. La formación de estructuras a grandes escalas también se explica mediante perturbaciones inicialmente pequeñas que han crecido con el paso del tiempo.

Existe una extensa literatura sobre perturbaciones de estrellas, debido a su enorme relevancia astrofísica. Aquí solamente mencionaremos los artículos más importantes en los que hemos basado nuestra extensión a segundo orden de la teoría lineal de perturbaciones de estrellas en colapso. En particular, nos centraremos en perturbaciones alrededor de fondos esféricos, aunque se permitirá que sean estáticos o dinámicos y que tengan diferentes ecuaciones de estado.

En lo que respecta a fondos estáticos, el trabajo de Chandrasekhar a mediados de los años sesenta fue pionero en el análisis de las perturbaciones radiales de estrellas esféricas en Relatividad General [37, 38]. En una serie de artículos [39–43], Thorne y sus colaboradores se ocuparon de las perturbaciones no-radiales de estrellas compuestas por fluido perfecto, estableciendo la base teórica para tratar dicho problema. Esta base se desarrolló y amplió en otras referencias, tales como [44] y [45].

La investigación de fondos no estáticos comenzó a finales de los años setenta, cuando Cunningham *et al.* [46, 47] utilizaron un formalismo invariante *gauge* para evolucionar las perturbaciones no-radiales de polvo en colapso esférico. Seidel y sus colaboradores [48–50] aplicaron el formalismo de Gerlach y Sengupta (GS) [51–54], que es válido para cualquier fondo esféricamente simétrico con cualquier tipo de contenido material, para evolucionar las perturbaciones de un fluido perfecto dependiente del tiempo con una ecuación de estado más genérica. Basándose en estas referencias, Gundlach y Martín-García [55] desarrollaron un marco covariante e invariante *gauge* para analizar una perturbación arbitraria de un fluido perfecto esféricamente simétrico. Dicho marco fue utilizado posteriormente por Harada *et al.* [30] para analizar las perturbaciones axiales del colapso estelar.

En paralelo, la teoría de perturbaciones ha sido una de las herramientas más importantes para el análisis de las propiedades de los agujeros negros. El primer artículo en el que se estudiaron perturbaciones de un agujero negro data de 1957 [56]. En esta época, los agujeros negros no estaban aún aceptados como entidades físicas. La solución de Schwarzschild a las ecuaciones de campo de Einstein era conocida, pero la visión más aceptada era que esta exótica solución, *constituida del campo de Einstein sin masa* [56], existía debido a la gran (e ideal) simetría que se le presuponía. Para analizar la estabilidad de este objeto, Regge y Wheeler (RW) estudiaron pequeñas perturbaciones que se desviaban de la esfericidad. Dichas perturbaciones se clasifican en dos polaridades diferentes (polar y axial), que corresponden a los dos grados de libertad de las ondas gravitatorias. Encontraron una ecuación de onda, que hoy en día se conoce como la ecuación de Regge-Wheeler, para una de las polaridades. Varios años después, Zerilli obtuvo la ecuación de onda correspondiente a la otra polaridad [57]. Estas ecuaciones tienen un potencial que depende de la masa del agujero negro de fondo y del número entero l, que procede de los harmónicos esféricos utilizados para descomponer la dependencia angular de las perturbaciones en cuestión.

Haciendo uso de estas ecuaciones, Vishveshwara encontró una respuesta convincente a la estabilidad de la métrica de Schwarzschild [58]. Definiendo una integral de energía que debe mantenerse constante a lo largo de la evolución, dio una cota para la derivada temporal de la perturbación, excluyendo soluciones que crecieran exponencialmente. Esto parecía indicar la inexistencia de modos inestables. Sin embargo, había algunos vacíos en el razonamiento que posteriormente fueron completados por Kay y Wald [59]. Demostraron que cualquier perturbación con datos iniciales suaves y acotados permanecería acotada puntualmente.

Otras dos propiedades sorprendentes de los agujeros negros que fueron descubiertas en los años setenta con métodos perturbativos fueron los modos cuasinormales y las colas con ley de potencias *power-law tails*. Los modos cuasinormales son soluciones a las ecuaciones de Regge-Wheeler y Zerilli con condiciones de contorno puramente salientes (*outgoing*); es decir, en el horizonte son ondas planas dirigidas hacia el interior del horizonte, mientras que en el infinito nulo son ondas planas dirigidas hacia fuera. Estos modos cuasinormales son los equivalentes a los modos normales de los sistemas mecánicos convencionales en situaciones donde existe disipación de energía (en nuestro caso las ondas gravitatorias son las encargadas de extraer energía del sistema). Estos modos han sido ampliamente estudiados [61], incluso para estrellas y agujeros negros en rotación, y su principal interés estriba en el hecho de que sus frecuencias dependen de la masa y del momento angular del agujero negro (o estrella) que ha sido perturbado. Por lo tanto, a través de la detección de los modos cuasinormales es posible obtener información directa de las propiedades del objeto emisor.

Las *power-law tails* fueron calculadas por primera vez por Price [62, 63]. Dado un radio fijo, a tiempos altos, una perturbación de Schwarzschild con número harmónico l decae como $t^{-(2l+3)}$, independientemente del tipo de pertubación estudiada (escalar, gravitatoria,...). Estas colas están presentes también en los infinitos futuro y nulo, donde decaen siguiendo leyes de potencias particulares [64, 65]. De manera intuitiva, normalmente se interpretan como resultado del *backscattering* de las ondas con el potencial de curvatura efectivo.

El análisis se complica en lo que respecta a los agujeros negros en rotación (Kerr). En el caso sin rotación se utilizan los harmónicos esféricos para descomponer la dependencia angular de las perturbaciones y, a nivel lineal, cada multipolo evoluciona de manera independiente. Pero en el caso con rotación los diferentes multipolos se acoplan. Sin embargo, aún existen ciertas simetrías (axisimetría y estacionariedad) que permiten desacoplar las dependencias temporales y axiales. Utilizando la formulación en tétradas de Newmann-Penrose para las ecuaciones de Einstein, Teukolsky reescribió las ecuaciones de movimiento para las perturbaciones lineales de un agujero negro de Kerr como dos ecuaciones desacopladas [66]. La estabilidad fue probada por Whiting [67], tras complicadas transformaciones de coordenadas, utilizando el mismo método de Vishveshwara para el caso sin rotación [58]. Debido a que no se dispone de una noción invariante de coordenadas de multipolo para el fondo Kerr, las leyes de potencias para las *tails* dependen de la exfoliación [68, 69]. Incluso se ha argumentado que el decaimiento no es universal y que depende del campo perturbativo.

1.2 Teoría de perturbaciones a altos órdenes

1.2.1 Motivación

Existen diferentes razones que justifican la importancia de ir más allá del primer orden en teoría de perturbaciones. Una primera motivación es el deseo de alcanzar mayor precisión en los resultados. Por ejemplo, esto es un punto crucial en la construcción de patrones realistas para la detección de ondas gravitatorias. Una segunda razón es que los cálculos a altos órdenes deberían proporcionar una manera de establecer el rango de aplicabilidad de los resultados de primer orden, estimando erores cuantitativos y dando límites de validez para la aproximación de primer orden. Estos límites de validez serían muy útiles, ya que habitualmente el interés radica no ya en perturbaciones despreciables, sino en situaciones más generales para las que la extrapolación de los resultados perturbativos puede ponerse en duda. Además, a segundo orden y superiores, aparecerá el acoplo entre los modos perturbativos. Dicho acoplo incorporará la no-linealidad de la Relatividad General completa en la teoría aproximada y podría dar lugar a la aparición de escalas propias en ciertos problemas.

Por ejemplo, el acoplo entre los modos de oscilación de un agujero negro con frecuencias ω_1 y ω_2 puede generar sobretonos de frecuencia $\omega_1 \pm \omega_2$ a través de la teoría de perturbaciones a segundo orden, o desplazamientos en la frecuencia a tercer orden [14]. Durante la fase de *ring-down* en la colisión de agujeros negros supermasivos, dichos sobretonos podrían ser detectados por LISA con una buena relación señal-ruido a una distancia de 1Gpc. Sin embargo, las simulaciones numéricas actuales no han podido aún proporcionar indicaciones claras sobre tales sobretonos, ya que su tamaño es similar al del ruido numérico. Éste es un problema para el que la teoría de perturbaciones a segundo orden está perfectamente adaptada.

1.2.2 Historia

Hasta hace poco, la mayoría de las investigaciones perturbativas habían sido llevadas a cabo a primer orden. Por lo que conocemos, la primera aplicación de la teoría de perturbaciones a segundo orden en el contexto de la Relatividad General fue el estudio de perturbaciones cosmológicas alrededor del fondo de Friedmann-Robertson-Walker realizado por Tomita en 1967 [71], y posteriormente el estudio del mismo autor sobre la estabilidad no-lineal de la solución de Schwarzschild [72, 73].

Actualmente, cada vez se proponen más aplicaciones de la teoría de perturbaciones a altos órdenes. En particular, los casos cosmológicos están bajo un estudio intensivo, como demuestran por ejemplo [74–80]. Asimismo, la colisión frontal de dos agujeros negros en el régimen cercano [81] se ha interpretado como una perturbación polar de un único agujero negro [32, 82–85]. Estas investigaciones tuvieron sorprendentes resultados en los que la aproximación a segundo orden seguía de manera muy precisa las trayectorias obtenidas mediante las simulaciones no-lineales completas. Recientemente, también perturbando Schwarzschild, se han definido los modos cuasinormales de segundo orden [86, 87] y se ha deducido un formalismo para tratar las *extreme mass ratio inspirals* [88].

El espaciotiempo de Kerr también ha sido analizado a segundo orden [89], generalizando el formalismo de Teukolsky [66] para perturbaciones lineales. El exponente crítico del escalado del momento angular de un campo escalar se ha predicho utilizando argumentos perturbativos de segundo orden [90]. El problema general del *matching* a través de una superficie también se ha analizado a segundo orden [91].

La teoría de perturbaciones a altos órdenes solamente se ha aplicado a fondos de fluido para modelar las perturbaciones de estrellas en rotación lenta. Para ello, se toma la rotación como una perturbación axial a primer orden de una estrella esférica de fondo. Uno de los primeros trabajos en este contexto es, de nuevo, uno de los artículos de la serie de Cunningham *et al.* [92], donde las perturbaciones de segundo orden de una bola de polvo fueron estudiadas incluyendo el *matching* a través de la superficie estelar para las perturbaciones interiores y exteriores. La misma idea se ha utilizado frecuentemente en fondos estáticos [93, 94], para modelar estrellas estacionarias en rotación lenta. También existen estudios [95, 96] de rotación diferencial, donde es necesario considerar más de una perturbación de primer orden para describir la rotación, pero se suele utilizar la aproximación Cowling, que desprecia todas los acoplos espaciotemporales. Sin la aproximación Cowling sólo se han considerado modos radiales [97].

1.3 Libertad gauge

Uno de los problemas centrales en la teoría de perturbaciones en Relatividad General, heredado de la invariancia bajo difeomorfismos de la teoría completa (pero no equivalente), es el de aislar los grados de libertad físicos de la información dependiente *gauge*. Esto se puede realizar imponiendo condiciones de fijación de *gauge* sobre las perturbaciones, tal y como Regge y Wheeler [56] originalmente hicieron en su estudio de perturbaciones del agujero negro de Schwarzschild. Como ya se ha comentado, ellos y posteriormente Zerilli [57] lograron aislar los dos grados de libertad físicos del campo gravitatorio alrededor de un vacío esférico, tomando convenientes combinaciones lineales de las perturbaciones y sus derivadas radiales. Estas dos variables se desacoplan entre sí debido a sus diferentes comportamientos bajo cambio de paridad: la variable de Regge-Wheeler es axial y la de Zerilli es polar.

Una alternativa a la fijación de *gauge* fue dada por Sachs en 1964 [98], y posteriormente mejorada por Stewart y Walker [99], introduciendo el concepto de invariantes *gauge* para teoría de perturbaciones a primer orden. Estos autores ya obtuvieron el resultado de que la invariancia *gauge*, como ellos la definieron, era bastante restrictiva y sólo aplicable a fondos con gran simetría. Bruni y colaboradores [100] mostraron que este enfoque geométrico se vuelve aún más restrictivo a altos órdenes.

Un tratamiento más útil de la libertad *gauge* para la teoría de perturbaciones en Relatividad General fue elaborado por Moncrief [101] en su estudio hamiltoniano sobre las perturbaciones no-esféricas de Schwarzschild. En este contexto hamiltoniano, las cuatro ligaduras que obedecen las doce variables gravitacionales dinámicas forman los generadores de las transformaciones *gauge*. Moncrief consiguió utilizar esta información para realizar varias transformaciones canónicas que reorganizaran los seis pares canónicos de variables iniciales en dos pares físicos sin ligaduras (equivalentes a las variables de Regge-Wheeler y Zerilli y sus momentos conjugados) y otros cuatro pares, cada uno de los cuales está formado por una variable invariante *gauge* obligada a anularse por las ligaduras y su momento conjugado puro *gauge*. La misma técnica se aplicó a otros fondos esféricos con simetrías adicionales, como Reissner-Nordström [102, 103], Oppenheimer-Snyder [46] o Friedmann-Robertson-Walker [104], pero nunca ha sido utilizada para fondos generales con simetría esférica, permitiendo que la dependencia temporal tenga una considerable relevancia física. La teoría de perturbaciones hamiltoniana se ha utilizado recientemente en Gravedad Cuántica con un fondo cosmológico [105, 106]. Uno de los inconvenientes del enfoque hamiltoniano es que, en principio, está ligado a una foliación particular del espaciotiempo de fondo y, por consiguiente, las propiedades geométricas de las variables invariantes *gauge* bajo transformaciones de coordenadas que involucren el tiempo no son triviales en absoluto.

Gerlach y Sengupta introdujeron un formalismo lagrangiano [51] para estudiar perturbaciones alrededor de espaciotiempos esféricos generales. Éste es un marco altamente geométrico, en el que el significado de las perturbaciones es transparente, y que también permite la construcción de variables invariantes *gauge*. En el caso axial ha sido posible aislar el grado de libertad gravitacional en un única variable escalar *master* que obedece una ecuación de onda y puede acoplarse a cualquier tipo de materia, tanto en el fondo como en las perturbaciones. Este escalar *master* generaliza la variable de Regge-Wheeler al problema axial perturbativo alrededor de simetría esférica para cualquier tipo de materia razonable, y por lo tanto se puede considerar como el marco de trabajo óptimo para un estudio perturbativo. Desafortunadamente, en el caso polar no hay un escalar *master* válido para un fondo general con simetría esférica, aunque existen resultados para algunos casos particulares. Por ejemplo, un escalar *master* de tipo Zerilli fue introducido por Sarbach y Tiglio [107] para un fondo de Schwarzschild, que posteriormente fue generalizado a electrodinámica no-lineal [108]. En las referencias [109, 110] se incluyen las combinaciones invariante *gauge* del tensor energía-momento, pero aún en un fondo de vacío.

Ambos tratamientos de la teoría de perturbaciones métrica son complementarios: el formalismo hamiltoniano ofrece un marco de trabajo mejor para tratar la invariancia *gauge*, mientras que el análisis lagrangiano da una imagen más clara de la estructura geométrica que se está perturbando. Ambos han sido ampliamente utilizados en la literatura.

1.4 La necesidad del álgebra computacional

En general, la teoría de perturbaciones a altos órdenes involucra ecuaciones de complejidad y tamaño creciente. Debido a este hecho, el análisis a cualquiera de esto órdenes ha sido impracticable hasta hace poco, excepto para algunos casos particulares en situaciones con gran simetría. No obstante, las actuales técnicas de álgebra computacional han sido desarrolladas hasta un grado en el que se puede afrontar ya problemas más realistas y genéricos con garantía de éxito.

Por lo tanto, para intentar sobrellevar la complejidad de los cálculos, utilizaremos intensivamente herramientas de álgebra computacional. En particular, haremos uso del paquete *xTensor*, que es parte del marco más general llamado *xAct* [111], que actualmente es el manipulador de expresiones tensoriales más rápido para *Mathematica*. Con el paquete *xTensor* se pueden definir variedades que contengan campos tensoriales con simetrías arbitrarias, conexiones de cualquier tipo, métricas y otros objetos. *xTensor* se basa en la notación de índices abstractos de Penrose y tiene un único canonicalizador que simplifica completamente todas las expresiones utilizando eficientes técnicas de teoría de grupos computacional. Implementaremos todas las ecuaciones con estas herramientas, tanto para producirlas como para comprobarlas, y para construir un eficiente marco de trabajo computacional capaz de tratar futuras aplicaciones del formalismo de teoría de perturbaciones a alto orden.

1.5 Objetivos

Este trabajo se divide en dos líneas de investigación principales con sus correspondientes objetivos. Por un lado, construiremos un formalismo general para analizar las perturbaciones a altos órdenes sobre espaciotiempos esféricos y trataremos su aplicación al vacío y a un fluido perfecto. Por otro lado, utilizaremos un tratamiento hamiltoniano para perturbaciones lineales sobre un espaciotiempo esférico, pero dinámico, para buscar variables *master* que obedezcan ecuaciones de movimiento sin ligaduras y que contengan toda la información física del problema.

La primera línea de investigación se puede considerar como una continuación del trabajo debido a Gerlach y Sengupta a primer orden [51, 54] y su aplicación a fondos de fluido perfecto [55, 112]. El formalismo GS está basado en cuatro ingredientes básicos: i/ una descomposición 2+2 del espaciotiempo, que separa las órbitas esféricas S^2 de una variedad general 1+1 Lorentziana M^2; ii/ una descripción covariante en la variedad Lorentziana; iii/ la descomposición de las perturbaciones de S^2 en harmónicos tensoriales; y iv/ el uso de variables perturbativas invariantes *gauge*. La notación covariante es particularmente conveniente: por un lado, permite formular todas las ecuaciones sin escoger coordenadas en M^2, algo que resulta muy útil en fondos dinámicos; por otro lado, extrae todos los factores trigonométricos de las ecuaciones de movimiento, factores que no contienen ninguna información relevante y habitualmente oscurecen la interpretación geométrica del resultado.

Mostraremos que es posible extender el análisis alrededor de espaciotiempos con simetría esférica a cualquier orden en teoría de perturbaciones en Relatividad General. Construiremos un método general y calcularemos las fórmulas exactas para las perturbaciones a cualquier orden de las cantidades geométricas relevantes. Con este objetivo, introduciremos una base de tensores harmónicos bien adaptada, que resulta especialmente apropiada para el estudio de la radiación gravitatoria y consiste esencialmente en la generalización de los harmónicos de Regge-Wheeler-Zerilli (RWZ). Además, para tratar satisfactoriamente las perturbaciones a altos órdenes, necesitaremos derivar expresiones cerradas para el producto de estos harmónicos. También proporcionaremos un procedimiento iterativo para construir cantidades invariantes *gauge* hasta el orden deseado. Obtendremos explícitamente todas las fuentes cuadráticas para los objectos invariantes *gauge* de segundo orden y sus ecuaciones de movimiento.

Aplicaremos este formalismo de segundo orden a los fondos correspondientes al vacío y a un fluido perfecto. En el caso del vacío obtendremos la potencia radiada, hasta cualquier orden en teoría de perturbaciones. A segundo orden, las fuentes que definiremos *ad hoc* para las variables de Regge-Wheeler y de Zerilli no decaerán con el radio al tender al infinito nulo y habrá que regularizarlas. Esta aplicación dará lugar a un formalismo capaz de describir perturbaciones de segundo orden arbitrarias del agujero negro de Schwarzschild. En el caso de fluido perfecto, daremos los términos de fuente que proceden de expresar las perturbaciones del tensor energía-momento en función de las variables del fluido. Simplificaremos las ecuaciones de movimiento utilizando el sistema de referencia (*frame*) que proporciona la velocidad del fluido de fondo. Además, especificaremos las condiciones de *matching* de segundo orden para unir el fluido perfecto y el vacío a través de una supeficie temporal. Aclararemos qué objetos deben ser continuos a cualquier orden perturbativo a través de la mencionada superficie. En realidad, este análisis de las condiciones de *matching* será válido para cualquier espaciotiempo de fondo. De esta forma, esta memoria de tesis doctoral proporciona un formalismo completo y consistente para estudiar perturbaciones de segundo orden generales de una estrella esférica, pero posiblemente dependiente del tiempo, formada por fluido perfecto.

La segunda línea de investigación se puede considerar como una generalización de los trabajos de Moncrief [101, 102] al caso de fondos dinámicos. En sus investigaciones, Moncrief construyó las llamadas variables *master* para los fondos de Schwarzschild y Reissner-Nordström en un marco canónico. Estas variables *master* son invariantes *gauge*, obedecen ecuaciones de movimiento sin ligaduras y contienen toda la información física del problema, en el sentido de que el resto de las perturbaciones se pueden reconstruir en términos de ellas.

También nos restringiremos a fondos esféricos, pero podrán ser altamente dinámicos. La dinámica será introducida utilizando un campo escalar real sin masa, pero podría hacerse igualmente a través de cualquier otro tipo de materia que admita una descripción hamiltoniana. En el caso axial, la solución es el escalar GS [51], previamente encontrado utilizando solamente métodos lagrangianos. Mostraremos cómo el marco hamiltoniano permite una derivación más sistemática de este objecto, y cuál es la relación mutua entre ambos tratamientos. Más importante resultará la aplicación de la mismas técnicas al caso polar, mediante lo que encontraremos una variable de Zerilli para este escenario dinámico. Ésta es la primera vez que se presenta una variable *master* para un caso dinámico. Esta investigación abre el camino para una búsqueda sitemática de variables polares *master*.

Como ya se ha explicado, la computación algebraica resulta necesaria para tratar este tipo de problemas. Por ello, otro de los objetivos principales de este trabajo es construir un sistema de computación algebraica que sea capaz de manejar las ecuaciones que el formalismo de teoría de perturbaciones (a altos órdenes) involucra. En particular escribiremos dos módulos dentro del entorno *xAct* [111], para álgebra computacional de tensores en *Mathematica*, llamados *xPert* y *Harmonics*. El primero permite tratar la teoría de perturbaciones a altos órdenes sobre cualquier fondo. Está basado en una combinación de algoritmos combinatorios adaptados y potentes técnicas de álgebra computacional de tensores. El segundo implementa la simetría esférica a través de diferentes harmónicos esféricos tensoriales. *xPert* ha sido utilizado ya en varios proyectos de investigación, que van desde teoría de perturbaciones en Relatividad General [113, 114] hasta problemas del transporte de la radiación en fondos curvos [115–117] o perturbaciones cosmológicas [118, 119].

1.6 Organización

Esta tesis describe el trabajo contenido en las referencias [113, 114, 120–125] y está organizado en seis partes. En lo que sigue, remarcaremos especialmente las partes de la tesis que contienen trabajo original.

En la Parte I se introducen los conceptos básicos y ecuaciones de la teoría de perturbaciones. En la Sección 3.3 se presentan fórmulas cerradas para la enésima perturbación de las cantidades geométricas de interés (fórmulas que son nuevas, hasta donde nosotros sabemos). Se analizan las transformaciones *gauge* a altos órdenes y se explica la invariancia *gauge*. En la Subsección 4.1.2 introducimos un método para construir objectos invariantes gauge a cualquier orden perturbativo sobre espaciotiempos genéricos.

La Parte II se ocupa del tratamiento de la simetría esférica y está divida en dos capítulos. El análisis y la notación que se utilizará para un espaciotiempo esférico general se explican en el Capítulo 5. En el Capítulo 6 se presenta una revisión de los diferentes tipos de harmónicos esféricos tensoriales junto con sus propiedades más relevantes. También se obtiene la fórmula del producto entre cualquier par de harmónicos. Este es un resultado de particular importancia de esta tesis, que nos permite ir a segundo orden perturbativo y superiores, y está detallado en la segunda parte del Capítulo 6 (Secciones 6.5, 6.6, y 6.7).

En la Parte III presentamos la búsqueda de variables *master* en un espaciotiempo esférico con un campo de materia escalar. El Capítulo 7 recupera el conocido escalar *master* de GS desde un punto de vista canónico. Siguiento las mismas técnicas hamiltonianas, en el

Capítulo 8 damos una generalización de la variable de Zerilli a este escenario dinámico. Ésta es la primera vez que se encuentra una variable polar *master* invariante *gauge* para un fondo dinámico y toda esta parte (Capítulos 7 y 8) es nueva.

Las perturbaciones no esféricas se discuten en la Parte IV. En el Capítulo 9 se realiza la descomposición harmónica de las perturbaciones y se construyen las cantidades invariantes *gauge* a cualquier orden perturbativo. Este es un resultado importante de esta tesis y está contenido en las Subsecciones 9.2.2 y 9.2.3. El Capítulo 10 es un repaso del formalismo GS de primer orden, introduciendo la notación que se utilizará posteriormente, excepto por la Subsección 10.3.2, que completa el análisis realizado en la Parte III. En el Capítulo 11 se presenta, por primera vez en la literatura, un conjunto completo de invariantes *gauge* de segundo orden de manera explícita, así como las fuentes para las ecuaciones de evolución de dichos invariantes y para las ecuaciones de conservación de energía-momento. Todo el material contenido a partir del Capítulo 11 es original exceptuando las cuestiones a primer orden perturbativo.

Las aplicaciones del formalismo GS de segundo orden que hemos desarrollado están contenidas en la Parte V. Más concretamente, en el Capítulo 12 presentamos las ecuaciones de RW y de Zerilli de segundo orden regularizadas. La potencia emitida en forma de ondas gravitatorias también se expresa en términos de las mencionadas variables *master*. El Capítulo 13 desarrolla la aplicación a un fluido perfecto. Las ecuaciones de movimiento de segundo orden se obtienen explícitamente y se simplifican. El Capítulo 14 estudia las condiciones de *matching* perturbativas a altos órdenes y las particulariza para unir los dos espaciotiempos previos: vacío y fluido perfecto.

La Parte VI contiene detalles sobre el procedimiento empleado para implementar nuestros cálculos en *Mathematica*, utilizando el paquete *xTensor*. Consiste de un único capítulo que presenta los dos módulos (*xPert* y *Harmonics*) que hemos construido durante el curso de esta investigación.

Finalmente, se añaden cuatro apéndices. Los dos primeros apéndices explican diferentes aspectos de la definición de las funciones esféricas y la manera de construir la parte simétrica y sin traza de un tensor dado. El Apéndice C presenta las fuentes regularizadas para las ecuaciones de RW y de Zerilli de segundo orden para un caso particular. Por último, el Apéndice D contiene las fuentes correspondientes a las ecuaciones de segundo orden de las perturbaciones del fluido para diferentes números harmónicos.

Chapter 2

Introduction

Nowadays, General Relativity is considered to be the best available theory that describes the classical (non-quantum) gravitational interaction. The first experimental confirmations of the theory, already predicted by Einstein, were the deflection of light and the perihelion advance of Mercury. It was not until the beginning of the 1960s that astronomical discoveries (like quasars or pulsars) provided new observations to confront General Relativity in the weak gravity regime. One of the best known is the decrease in the orbital period of the Hulse-Taylor binary pulsar [1], for which they both received the Nobel prize [2, 3]. These observations agreed very well with the theoretical predictions of General Relativity and excluded several alternative theories [4, 5]. Since the 1980s attention has been mainly focused on finding strong-field regime observations that could corroborate or contradict the predictions of General Relativity in two extremes. On the one hand there is the limit associated with the Planck scale physics. There are processes with very small characteristic distances which produce strong gravitational interactions up to some point where General Relativity is assumed to fail due to quantum phenomenology. On the other hand there is the astrophysical limit, which involves very dense objects with large masses, that will also lead to such strong gravitational interactions. It is expected that several scenarios, like black hole and/or neutron star collisions, will produce enough gravitational radiation emission to be detected from Earth in the near future.

In this astrophysical context, the detection of gravitational waves is considered to be one of the most important open problems in experimental physics. Apart from providing a test of General Relativity in the strong-field regime, it may open a new window for astrophysical observation, giving rise to an era of gravitational-wave astronomy. Several ground-based gravitational wave detectors are already taking data (such as GEO [6], LIGO [7] or VIRGO [8]), and space-missions (most notably LISA [9]) are planned for the early 2020s.

The signal to be detected is very small, and buried under a number of different types of noise. Powerful statistical techniques have been developed to help in the recovery of the signal from the data measured. However, it is still essential to have in advance a catalogue of templates for the possible signals to be observed (*matched filtering* technique). These templates can only be obtained by solving the equations of General Relativity, the Einstein equations. These are a system of ten non-linear coupled partial differential equations and have proven very difficult to solve. There is no hope of constructing a catalogue of exact analytical solutions describing the gravitational waves profiles emitted in such astrophysical scenarios. Only situations with very high symmetry allow us to write down explicitly the solution, and only a few of them are relevant from an astrophysical point of view [10]. Hence, as it is usual in physics, when a physical problem cannot be solved exactly, one resorts to approximate methods. Here we will mention the three techniques that have worked best so far, and which can be considered complementary in the analysis of the dynamics of General Relativity in this context: post-Newtonian methods, Numerical Relativity and perturbation theory.

Post-Newtonian methods are based on the combination of a weak-gravity approximation (i.e., perturbation theory around Minkowski, also known as post-Minkowskian approximation) and an expansion of the solutions of the equations into a power series with parameter v/c, where c is the speed of light and v a typical matter velocity of the problem in question [11]. Therefore, this approximation is valid to model the gravitational waves generated by a source with slow internal motion and weak self-gravitation. By going up to high powers of v/c it is possible to consider relativistic sources (results at order 6 are now used systematically [12]), and actually the post-Newtonian approximations typically behave better than expected. The Newtonian limit ($1/c \to 0$), was already considered by Einstein [13] and Landau and Lifshitz [14] to derive the famous quadrupole formula, which gives the total emitted power in form of gravitational waves in terms of the time derivatives of the quadrupole moment of the source. In fact, this relation was enough to explain the decrease of the period in the Hulse-Taylor binary pulsar [15, 16].

Numerical Relativity is nowadays a branch of General Relativity in its own right [17]. Any attempt at solving Einstein equations using numerical methods could be included in this area, but here we will refer only to those simulations dealing with the full Einstein equations, perhaps under some symmetry reduction. (Even working with post-Newtonian methods or perturbation theory we are typically led to integrate some other equations numerically.) Many different formulations of the Einstein equations can be used for their discretization in a computer, depending on how the spacetime is described. The simplest way to do it is based on the Hamiltonian approach by Arnowitt, Deser and Misner (ADM)

[18, 19], which foliates the spacetime in a family of three-dimensional spacelike surfaces. Other frequent approaches are based on the use of null surfaces, which offer a better way of dealing with radiation. There were some attempts already in 1960s and 70s to solve numerically the Einstein equations in two spatial dimensions [20, 21]. But at that time the computers were not powerful enough to obtain relevant three-dimensional results, and the formulations used were not mathematically consistent, as was later realized. In the problem of simulating a binary black hole, of key importance in this area, progress in both the numerical and the analytic sides took place during the 1990s, until Pretorius [22] in 2005 was able to evolve the system in a stable way for the first time. Since then, a large number of articles by several groups have been published, reporting on interesting features of this system, and providing reliable gravitational wave templates. Apart from the special problem of pure vacuum, much effort has also been invested in the simulation of spacetimes containing fluid matter, and more recently fluid matter coupled to the electromagnetic field, to model stellar core-collapse and the collisions of neutron stars, other important sources of gravitational radiation [23].

Perturbation theory provides another approximate approach, allowing a description in terms of small departures around an exact solution. In the context of General Relativity, perturbation theory plays a prominent role in analyzing the stability of particular solutions, and in understanding dynamical processes in terms of the behaviour of simple "oscillation modes", being today an efficient and natural complement to full Numerical Relativity simulations [24]. We will describe it in depth in the following sections.

The three approximation techniques have their own domains of validity, which are frequently complementary, and so their combined use is usually the best strategy. For example, the three phases of a binary black hole collision, inspiral, merger and ring-down, can be adequately described with post-Newtonian methods, Numerical Relativity and perturbation theory, respectively. During the initial inspiral phase the black holes are far apart and the mutual gravitational interaction is weak. Hence one can almost approximate the picture by Newtonian gravity. This is the perfect arena for post-Newtonian methods. As the black holes get closer due to the energy radiated in form of gravitational waves, their mutual gravitational interaction will be stronger. At this point, the post-Newtonian methods need more terms in their expansion to accurately follow the trajectories. When the two holes are very close, the nonlinearities become very large and this is the phase when one needs full Numerical Relativity to follow the evolution. At the end both holes merge in a unique black hole, which goes on radiating gravitational waves until it becomes a stationary Kerr or a Schwarzschild black hole. During this final part of the dynamical evolution, the spacetime can be approximated by the mentioned known solution plus some small deviations, which can be properly described by perturbation theory. In the following, we will center our attention in perturbation theory, describing more deeply its history and achievements in the context of General Relativity.

2.1 Applications of linear perturbation theory

Perturbation theory can be used to study the stability of solutions of the Einstein equations. A considerable amount of work has been devoted to the discussion of the stability of black-holes [25] and cosmological solutions [26], given the physical interest in these spacetimes.

In addition, perturbative analyses allow us to check the presence of gauge instabilities [27], constraint violations [28], and other type of instabilities in numerical implementations of the Einstein equations, since numerical errors can be considered themselves as distortions of the solution that one is computing.

Another important application is the study of the production and propagation of waves in a certain curved spacetime. Perturbation theory can provide us with estimates of the amount of gravitational radiation and of the signal profiles emitted in astrophysical scenarios like an oscillating neutron star [29], the gravitational collapse of a star [30], an extreme mass-ratio binary [31], or a close limit head-on collision of two black holes [32].

Perturbation theory has been recognized as a powerful tool in cosmology, as well. Following the pioneering works by Bardeen [33] and Kodama and Sasaki [34], it has been used to study the evolution of small inhomogeneities in the early Universe, giving a precise understanding of the angular dependence of the power spectrum in the cosmic microwave background. These departures from homogeneity have been measured by the COBE [35] and WMAP [36] experiments. Large scale structure formation in the Universe is also explained by initial small perturbations that grew with time.

There is extensive literature on perturbations of fluid stars, due to its enormous relevance in astrophysics. Here we will simply mention the most relevant articles on which we have based our extension to second-order perturbation theory of collapsing stars. In particular we will concentrate on perturbations around spherical backgrounds, though they can be static or time-dependent, and can have various equations of state.

On static backgrounds the work by Chandrasekhar [37, 38] pioneered the analysis of radial perturbations of spherical stars in General Relativity, in the middle sixties. In a series of papers, Thorne and collaborators [39–43] dealt with non-radial perturbations of perfect fluid stars, establishing the theoretical basis to treat the problem. This basis was further developed in other references, such as [44] and [45].

Research on non-static backgrounds began in the late seventies, when Cunningham *et al.* [46,47] used a gauge-invariant formalism to evolve non-radial perturbations of a spherical collapsing dust. Seidel and coworkers [48–50] applied the formalism of Gerlach and Sengupta (GS) [51–54], valid for any spherically symmetric background with any matter content, to evolve the perturbations of a time-dependent perfect fluid with a more general equation of state. Building on these references, Gundlach and Martín-García [55] developed a covariant and gauge-invariant framework to analyze an arbitrary perturbation of a spherical perfect fluid. This framework was latter used by Harada *et al.* [30] to analyze axial perturbations of stellar collapse.

In parallel, perturbation theory has been one of the most relevant tools in the analysis of the properties of black holes. The first article that studied perturbations of a black hole dates from 1957 [56]. At this time black holes were not still accepted as physical entities. The Schwarzschild solution to the Einstein field equations was known but the most accepted belief was that this exotic solution, *built out of the mass-free Einstein field* [56], existed due to the high (and ideal) symmetry assumed. In order to analyze the stability of this object, Regge and Wheeler (RW) studied small perturbations departing from sphericity. These perturbations are classified in two different polarities (polar and axial), corresponding to the two degrees of freedom of the gravitational wave. They found a wave equation, which is nowadays known as the Regge-Wheeler equation, for one of the polarities. Several years later, Zerilli wrote down the wave equation corresponding to the other polarity [57]. These equations have a potential that depends on the mass of the background black hole and the integer l, that comes from the spherical harmonic function used to decompose the angular dependence of the perturbation in question.

Making use of these equations, Vishveshwara found a convincing answer to the stability of the Schwarzschild metric [58]. Defining an energy integral that should be kept constant through evolution, he gave a bound for the time derivative of the perturbation, excluding exponentially growing solutions. This seemed to indicate the nonexistence of unstable modes. However, there were some gaps in the reasoning that were later filled by Kay and Wald [59]. They proved that any perturbation with smooth and bounded initial data will remain bounded pointwise.

Another two surprising properties of the black holes that were discovered in the 1970s with perturbative methods were quasinormal modes and power-law tails. The quasinormal modes are solutions to the Regge-Wheeler and Zerilli equations [60] with purely outgoing boundary conditions, that is, at the horizon they are plane waves directed inside the horizon whereas at null infinity they point outwards. These quasinormal modes are the equivalent to the normal modes in standard mechanical systems for situations where there is energy dissipation (in our case the gravitational waves remove energy out of the system). These modes have been widely studied [61], also for stars and rotating black holes, and their main interest lies in the fact that their frecuencies depend on the mass and angular momentum of the black hole (or star) being perturbed. Hence, through the detection of quasinormal modes one could obtain direct information about the properties of the radiating object.

Power-law tails were first computed by Price [62, 63]. At a fixed radius, for large times, a perturbation of Schwarzschild with harmonic label l, decays like $t^{-(2l+3)}$, regardless of the kind of perturbation (scalar, gravitational,...). These tails are also present at both null and future infinities obeying particular power-law decay rates [64, 65]. Intuitively they are usually understood as arising from the backscattering of the waves off the effective curvature potential.

Regarding rotating (Kerr) black holes everything gets more complicated. In the non-rotating case one uses the spherical harmonics to decompose the angular dependence of the perturbations and, at linear level, each multipole evolves independently. But in the rotating case the different multipoles are coupled. However, there are still some symmetries (axisymmetry and stationarity) that allow the decoupling of the time and axial dependencies. Making use of the Newmann-Penrose tetrad formulation of Einstein equations, Teukolsky rewrote the equations of motion for the linear perturbations of a Kerr black hole as two decoupled equations [66]. The stability was proven by Whiting [67] after some intrincate coordinate transformations following the same method as Vishveshwara for the non-rotating case [58]. Since there is no known coordinate-independent notion of multipole on a Kerr background, the power laws for the tails depend on the foliation [68, 69]. Even more, it has been argued that the decay is not universal and depends on the perturbative field.

2.2 High-order perturbation theory

2.2.1 Motivation

Different reasons justify the importance of going beyond first order in perturbation theory. A first motivation is the desire to reach better accuracy in the results. For instance, this is a decisive point in the construction of sufficiently realistic templates for the detection of gravitational waves. A second reason is that high-order calculations would provide a way to establish the range of applicability of the first-order results, estimating the quantitative errors and leading to validation limits for the first-order approximation. These validation limits would be very useful, because one is usually interested not just in negligible perturbations, but in more general situations for which the extrapolation of the perturbative results can be casted into doubt. In addition, at second and higher orders, coupling between perturbative modes will appear. This coupling will bring the nonlinearity of full General Relativity to the approximate theory and could result in the appearance of proper scales of certain problems.

For instance, the coupling among oscillation modes of black holes with frequencies ω_1 and ω_2 can generate overtones with frequency $\omega_1 \pm \omega_2$ through second order perturbation theory, or shifts in the frequency in third order perturbation theory [70]. During the ring-down phase of the merger of supermassive black holes, those overtones could be detected by LISA with good signal-to-noise-ratio at a distance of 1Gpc. However, current numerical simulations have not yet been able to provide clear indications of the presence of such overtones, because their size is approximately that of the numerical noise. This is a problem for which second-order perturbation theory is best suited.

2.2.2 History

Until recently, most perturbative investigations had been carried out at first order. As far as we know, the pioneering application of second-order perturbation theory in a General Relativity context was the study of cosmological perturbations around a Friedmann-Robertson-Walker background by Tomita [71] in 1967, and later the study by the same author of the nonlinear stability of the Schwarzschild solution [72, 73].

At present, more and more applications of high-order perturbation theory are being proposed. In particular, cosmological scenarios are under intensive studies, see for instance [74–80] and references therein. In addition, the head-on collision of two black holes in the close-regime approach [81] has been interpreted as a quadrupolar perturbation of a single black hole [32, 82–85]. These investigations led to surprising results where the second-order approximation follow very accurately the full non-linear numerical simulations. Recently, also perturbing Schwarzschild, second-order quasinormal modes have been defined [86, 87] and a formalism to deal with extreme mass ratio inspirals deduced [88].

The Kerr spacetime has also been analyzed at second order [89], generalizing the Teukolsky formalism [66] for linear perturbations. The critical exponent of angular momentum scaling has been predicted for scalar field collapse using second-order perturbation arguments [90]. Second-order perturbations of the general problem of a matching through a surface have been analyzed in [91].

To our knowledge, high-order perturbation theory has only been applied on fluid backgrounds to model the perturbations of a slowly uniformly rotating star. This is done by taking the rotation as an axial first-order perturbation of the background spherical star. One of the pioneering works in this context is again one paper in the series by Cunningham *et al.* [92], where the second-order nonspherical perturbations of a collapsing ball of dust were studied including the matching of the internal and external perturbations through the surface of the star. The same idea has been applied frequently on static backgrounds [93,94], to model slowly rotating stationary stars. There are also studies [95, 96] of differential rotation, where more than one first-order perturbation is needed to describe the rotation, but they make use of the Cowling approach, which neglects all the spacetime couplings. Without the Cowling approach only radial modes have been considered [97].

2.3 Gauge freedom

A central problem in General Relativity perturbation theory, inherited from the diffeomorphism invariance of the full theory (but not equivalent to it), is that of isolating the physical degrees of freedom from the gauge-dependent information. This can be done by imposing convenient gauge fixing conditions on the perturbations, as Regge and Wheeler [56] originally did in their study of perturbations of a Schwarzschild black hole. As we have already commented, they and later Zerilli [57] succeeded in isolating the two physical degrees of freedom of the gravitational field around spherical vacuum, by taking suitable linear combinations of the perturbations and their radial derivatives. These two variables further decouple due to their different properties under parity inversion: the Regge-Wheeler variable is axial and the Zerilli variable is polar.

An alternative to gauge-fixing was given by Sachs in 1964 [98], later improved by Stewart and Walker [99], introducing the concept of gauge-invariants for first-order perturbation theory. These authors already obtained the result that gauge-invariance, as defined by them, was rather restrictive and only applicable on highly symmetric backgrounds. Bruni and collaborators [100] have shown that this geometrical approach becomes even more restrictive at higher orders.

A more useful treatment of the gauge freedom in General Relativity perturbation theory was pioneered by Moncrief [101] in his Hamiltonian study of the nonspherical perturbations of Schwarzschild. In a Hamiltonian context the four constraints obeyed by the twelve dynamical gravitational variables are the generators of the gauge transformations. Moncrief was able to use this information to perform several canonical transformations which reorganized the original six canonical pairs of variables into two physical unconstrained pairs (equivalent to the Regge-Wheeler and Zerilli variables and their canonical momenta) and another four pairs, each of them composed by a gauge-invariant variable constrained to vanish and its conjugated pure-gauge momentum, without any gauge fixing. The same technique was later applied to other spherical backgrounds with additional symmetries, like Reissner-Nordström [102, 103], Oppenheimer-Snyder [46] or Friedmann-Robertson-Walker [104], but has never been applied to general spherically symmetric backgrounds, possibly highly time dependent. Hamiltonian perturbation theory has also been recently revisited in Quantum Gravity with a cosmological background [105, 106]. A drawback of the Hamiltonian approach is that, in principle, it is tied to a particular foliation of the background spacetime, and hence the geometric properties of the gauge-invariant variables under coordinate transformations involving time are far from obvious.

A Lagrangian formalism was introduced by Gerlach and Sengupta [51] to study perturbations around generic spherical spacetimes. This is a highly geometrical framework, in which the meaning of the perturbations is transparent, and which also allows the construction of gauge-invariant variables. In the axial case it has been possible to isolate the gravitational degree of freedom in a single scalar master variable which obeys a wave equation and can be coupled to any kind of matter, both in the background and the perturbations. This master scalar generalizes the Regge-Wheeler variable to the axial perturbative problem around spherical symmetry for any reasonable matter model, and hence can be considered as the optimal framework for a perturbative study. Unfortunately, in the polar case there is not a master scalar valid for a generic spherical background and any matter model, though there are results for some particular cases. For instance, a master Zerilli scalar has been introduced by Sarbach and Tiglio [107] for a Schwarzschild background, which was later generalized to nonlinear electrodynamics [108]. In references [109, 110] the gauge-invariant combinations of the stress-energy tensor were also included but still on a vacuum background.

Both approaches to metric perturbation theory are complementary: the Hamiltonian approach offers a better framework to handle gauge-invariance, while the Lagrangian approach gives a clearer picture of the geometrical structures being perturbed. Both are extensively used in the literature.

2.4 The need for computer algebra

Perturbation theory at second and higher orders involves in general increasingly large and complicated equations. Because of this fact, the analysis at any of these orders has been impracticable until recently, except for particular cases in situations with a high symmetry. However, current computer algebra techniques have been developed to a point in which one can face more realistic, generic problems with a warranty of success.

Hence, to cope with the complexity of the calculations, we will intensively use computer algebra tools. In particular, we will make use of *xTensor*, that is part of the general framework *xAct* [111] and is now the fastest manipulator of tensor expressions for *Mathematica*. With this package, one can define manifolds containing tensor fields with arbitrary symmetry, connections of any type, metrics and other objects. *xTensor* is based on the Penrose abstract-index notation and has a single canonicalizer which fully simplifies all expressions, using efficient techniques of computational group theory. We will encode all the equations with these tools, both to produce and check them, and to construct an efficient computer framework to deal with future applications of the formalism of high-order perturbations.

2.5 Goals

This work can be separated in two major lines of research with corresponding objectives. On the one hand, we will construct a generic framework to analyze second and higher-order perturbations of spherical spacetimes and deal with its application to vacuum and perfect fluid matter. On the other hand, we will use a Hamiltonian framework for linear perturbation theory on a spherical, but dynamical, spacetime to look for master variables that obey unconstrained equations of motion encoding the physical information of the problem.

The first line of research can be considered as a continuation of the work by Gerlach and Sengupta at first order [51, 54] and its application to fluid background [55, 112]. The GS formalism is based on four basic ingredients: i/ a 2+2 decomposition of the spacetime separating the spherical S^2 symmetry orbits from a general 1+1 Lorentzian manifold M^2; ii/ the use of a covariant description on the Lorentzian manifold; iii/ the decomposition of the perturbations in S^2 tensor harmonics; and iv/ the use of gauge-invariant perturbation variables. The use of a covariant notation is particularly convenient: on the one hand, it allows us to formulate all equations without choosing coordinates on M^2, something that becomes very useful on dynamical backgrounds; on the other hand, it removes all trigonometric factors from the equations of motion, factors which do not contain any relevant information and typically obscure the geometrical interpretation of the results.

We will show that it is possible to extend the analysis around spherical spacetimes to all orders in perturbation theory for General Relativity. We will construct a general method and calculate exact formulas for the perturbation of the relevant geometric quantities at any order. With this purpose we will introduce a well-adapted basis of tensor harmonics, which is specially suitable for the study of gravitational radiation and consists essentially of a generalization of the Regge-Wheeler-Zerilli (RWZ) harmonics. In addition, to satisfactorily deal with high-order perturbations, we will need to derive closed expressions for the products of these harmonics. We will also give an iterative procedure to construct gauge-invariant quantities up to the desired order. We will explicitly obtain all first-order quadratic sources for second-order gauge invariant quantities and their equations of motion.

This second-order formalism will be applied to vacuum and perfect fluid backgrounds. In the case of vacuum we will obtain the radiated power up to any order in perturbation theory. At second-order, the sources we will define *ad hoc* for the Regge-Wheeler and Zerilli variables will not decay with radius at null infinity and we will regularize them. This application will lead to a formalism able to describe arbitrary second-order perturbations of a Schwarzschild black hole. In the case of the perfect fluid, we will give the source terms coming from expressing the perturbations of the energy-momentum tensor in terms of the fluid variables. The equations of motion will be simplified by making use of the frame provided by the background fluid velocity. The second-order matching conditions will also be investigated, to match the perfect fluid and vacuum spacetimes across a timelike surface. We will clarify which objects are continuous at any perturbative order through that surface. This analysis of the matching conditions will also be valid for any background spacetime. In summary, this thesis provides a complete and consistent formalism to study generic second-order perturbations of a spherical, but possibly time-dependent, fluid star.

The second subject of research can be considered as a generalization of Moncrief's results [101, 102] to the case of dynamical backgrounds. In his investigations Moncrief constructed the so-called master variables for a Schwarzschild and Reissner-Nordström backgrounds in a canonical framework. These master variables are gauge-invariant, obey unconstrained equations of motion and contain all the physical information of the problem in the sense that the rest of the perturbations can be reconstructed in terms of them.

We will also restrict ourselves to spherical backgrounds, but these can be highly dynamical. For definiteness, the dynamics will be introduced using a real massless scalar field, but could similarly be done through any other matter model admitting a Hamiltonian description. In the axial case, the sought solution is the Gerlach and Sengupta master scalar [51], previously found using the Lagrangian method only. We will show how the Hamiltonian formalism allows a more systematic derivation of this object, and how both approaches mutually relate. Most importantly, we will apply the same techniques to the polar case and find a generalization of the Zerilli variable to this dynamical scenario. This is the first time that a gauge-invariant master variable is presented for a dynamical scenario. This investigation paves the way for a systematic search for master polar variables.

As we have explained, algebraic computation turns out to be necessary to cope with this kind of problems. That is why another important goal of this work is to construct an efficient computer algebra framework that is able to handle the calculations involved in the formalism of (high-order) perturbation theory. In particular we will write two modules inside the general framework *xAct* [111], for tensor computer algebra in *Mathematica*, named *xPert* and *Harmonics*. The first one is for high-order metric perturbation theory on any background. It is based on the combination of adapted combinatorial algorithms and powerful techniques of tensor computer algebra. The later implements the spherical symmetry through different kind of tensor spherical harmonics. *xPert* has been already used in several research projects, ranging from perturbation theory in General Relativity [113, 114] to problems in radiation transport in curved backgrounds [115–117] or cosmological perturbations [118, 119].

2.6 Outline

This thesis describes the work contained in references [113,114,120–125] and is organized in six parts. In the following, we will particularly remark those parts of this thesis containing original work.

Part I introduces the basic concepts and equations of perturbation theory. Closed formulas for the nth-order perturbations of the geometric quantities of interest (formulas that are new in the literature to the best of our knowledge) are presented in Section 3.3. High-order gauge transformations are analyzed and gauge invariance is explained for latter use. We introduce a method to construct gauge invariant objects at any perturbative order in 4.1.2.

Part II develops a number of tools for our later work on spherically symmetric backgrounds, and is divided in two chapters. The notations used for a general spherical spacetime are explained in Chapter 5. A review of the different kinds of tensor spherical harmonics with all their properties is presented in Chapter 6. A product formula for any pair of tensor spherical harmonics is also obtained. This is an important result of this thesis, that allows us to go to second and higher perturbative orders, and it is contained in the second part of Chapter 6 (Sections 6.5, 6.6, and 6.7).

In Part III we present a combination of Hamiltonian and covariant methods to search for master variables on a dynamcial spherical spacetime (for definiteness, we use a scalar field as matter content). Chapter 7 shows that our method reobtains the well-known GS master scalar for the axial sector. Following the same techniques, in Chapter 8 we give a generalization of the Zerilli variable to dynamical scenarios. This is the first time that a gauge-invariant master variable is found for a dynamical background and this whole part (Chapters 7 and 8) is new.

The general theory of nonspherical perturbations is discussed in Part IV. In Chapter 9 we decompose general perturbations in tensor harmonic bases, and then we introduce gauge-invariant combinations of those perturbations for arbitrary perturbative order, never given before. This is an important result of the thesis, and it is contained in Sections 9.2.2 and 9.2.3. Chapter 10 is mainly a review of the first-order GS formalism, introducing the notation used later, except for 10.3.2, which finishes the analysis of Part III. In Chapter 11, for the first time in the literature, a complete set of second-order gauge invariants is explicitly presented, as well as sources for the evolution equations of these invariants and for the energy-momentum conservation equations. All the material included from Chapter 11 on is original with the exception of issues at first order in perturbation theory.

We have applied this general second-order GS formalism to three different problems, as described in Part V. We first study perturbations of vacuum in Chapter 12, presenting the regularized second-order Regge-Wheeler and Zerilli equations. The power emitted in form of gravitational waves is also given in terms of those master variables. Second, Chapter 13 develops the application to perfect-fluid matter. The second-order equations of motion are explicitly obtained and simplified. Third, Chapter 14 studies the high-order perturbative matching conditions and particularize them to match the previous two systems: vacuum and perfect fluid.

Part VI contains details about the procedure employed to implement our calculations in *Mathematica*, using the tensor system *xAct*. It consists of a single chapter that presents the packages (*xPert* and *Harmonics*) we have constructed in the course of this investigation, and which are distributed as free software. *xPert* has been already used by other authors in independent investigations in perturbation theory.

Finally, four appendices are given. The first two appendices explain different aspects of the definition of spherical functions and how to construct the symmetric trace-free part of a given tensor. Appendix C presents the regularized sources for the second-order RW and Zerilli equations for a particular case. Finally, Appendix D contains the sources corresponding to equations for the second-order fluid perturbations for the different harmonic labels.

Part I

High-order perturbation theory

Chapter 3

Perturbation theory in General Relativity

3.1 General considerations

The Einstein equations are a system of ten coupled nonlinear partial differential equations for the metric. Perturbation theory reformulates them as an infinite set of linear partial differential equations for the deviation of the metric with respect to a known solution. This infinite set is organized into hierarchies labelled by power orders of a dimensionless parameter ε. Furthermore, in this hierarchy of equations, the principal parts are always given by the same linear differential operator acting on a perturbative correction of increasing order. In this way, one translates the difficulty from nonlinearity to the infinite number of equations. The key assumption of the perturbative scheme is that one can truncate the problem at a finite order and still obtain an approximate solution to the original system.

As starting point we will suppose a family of spacetimes depending on the dimensionless parameter ε. Each spacetime consists in a four dimensional Lorentzian manifold $\tilde{M}(\varepsilon)$ with a metric $\tilde{g}_{\mu\nu}(\varepsilon)$ and some matter fields which will be described by the stress-energy tensor $\tilde{t}_{\mu\nu}(\varepsilon)$. The objects with $\varepsilon = 0$ will form the background spacetime and be denoted without tilde, that is, $\{\tilde{M}(0), \tilde{g}_{\mu\nu}(0), \tilde{t}_{\mu\nu}(0)\} = \{M, g_{\mu\nu}, t_{\mu\nu}\}$. This background metric $g_{\mu\nu}$ will be assumed to be a known solution of the Einstein equations for the stress-energy tensor $t_{\mu\nu}$.

Our aim is to compare perturbed tensors ($\tilde{T}(\varepsilon)$) with their background counterparts ($\tilde{T}(\varepsilon = 0) = T$). But, since these tensors live in different manifolds and there is no available preferred structure to relate them, we are forced to arbitrarily choose a mapping from $M(\varepsilon)$ to M. This mapping will generate a pull-back (push-forward) between the tangent (cotangent) spaces; in this way, the pull-back of a tensor $\tilde{T}(\varepsilon)$ will be defined on the background manifold M. The freedom in the choice of that mapping is the gauge freedom that will be analyzed in the following chapter. For the moment being, we will not introduce any explicit mapping to keep the notation simple, but from now on $\tilde{T}(\varepsilon)$ has to be understood as the pull-back (push-forward) of that tensor to the background manifold. Hence, all objects considered will be tensors on the manifold M and their indices will be lowered or raised with the background metric $g_{\mu\nu}$ or its inverse $g^{\mu\nu}$.

3.2 Notation

In order to introduce the perturbative hierarchy, now that all tensors are defined in the same manifold, we will Taylor expand the perturbed tensors around their background counterparts. For that, the dependence on ε will be supposed smooth, or at least C^n if we want to work only up to order n in perturbation theory.

$$\tilde{g}_{\mu\nu}(\varepsilon) = g_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \, {}^{\{n\}}h_{\mu\nu}, \tag{3.1}$$

$$\tilde{t}_{\mu\nu}(\varepsilon) = t_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \, {}^{\{n\}}t_{\mu\nu}. \tag{3.2}$$

The perturbations ${}^{\{n\}}h_{\mu\nu}$ and ${}^{\{n\}}t_{\mu\nu}$ are then defined as ε-derivatives evaluated in the background and, of course, are tensors of the background manifold. Hence, we introduce a formal "perturbation" operator Δ,

$$\Delta^n[T] \equiv \left. \frac{d^n\tilde{T}(\varepsilon)}{d\varepsilon^n} \right|_{\varepsilon=0}, \tag{3.3}$$

so that any object $\tilde{T}(\varepsilon)$ can be expanded as

$$\tilde{T}(\varepsilon) = T + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \Delta^n[T]. \tag{3.4}$$

For instance, $\Delta[g_{\mu\nu}] = {}^{\{1\}}h_{\mu\nu}$ and $\Delta[{}^{\{n\}}h_{\mu\nu}] = {}^{\{n+1\}}h_{\mu\nu}$. In this notation, the brackets are intended to avoid confusion with index positioning because, e.g., for a given vector v^μ,

$$\Delta[v_\mu] = \Delta[g_{\mu\nu}v^\nu] = g_{\mu\nu}\Delta[v^\nu] + {}^{\{1\}}h_{\mu\nu}v^\nu \neq g_{\mu\nu}\Delta[v^\nu], \tag{3.5}$$

so the notation Δv_μ might be misleading.

An important variation of the general perturbative formalism explained in this chapter is frequently used in quantum field theory in curved backgrounds. It is usually called 'background field method' and decomposes the full metric $\tilde{g}$ as $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ [126]. Other objects depending nonlinearly on $\tilde{g}$ are later truncated at a given order in powers of h. This can be considered as a particular case of the general formalism used here, in which all but the first perturbation of the covariant metric are defined to be zero: ${}^{\{n\}}h_{\mu\nu} = 0$ for $n \geq 2$. All formulas in this chapter can be translated to the background field method using such a simple restriction.

3.3 Perturbative formulas for different objects

An important consequence of the derivative character of Δ is that it obeys the Leibnitz rule, whose n-th order generalization on the product of m tensors is

$$\Delta^n[T_1 \dots T_m] = \sum_{\{k_i\}} \frac{n!}{k_1! \dots k_m!} \Delta^{k_1}[T_1] \dots \Delta^{k_m}[T_m], \tag{3.6}$$

where the notation $\{k_i\}$ means that the sum extends to all sorted partitions of n in m nonnegative integers (that is, including zero) obeying $k_1 + ... + k_m = n$. For instance, for the case $n = 3$ and $m = 2$, the corresponding partitions would be $\{(3,0),(0,3),(2,1),(1,2)\}$. For the particular but important case of two objects ($m = 2$), the above formula takes the simpler form

$$\Delta^n[T_1 T_2] = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \Delta^k[T_1] \Delta^{n-k}[T_2]. \tag{3.7}$$

The Leibnitz formula (3.6) can be understood as a particularization of the Faà di Bruno formula [127] (the n-th order chain rule), that gives the expansion of $\Delta^n[F(\zeta_1, ..., \zeta_m)]$ for an arbitrary function F of m scalar arguments $\zeta_1, ..., \zeta_m$.

3.3.1 Perturbations of derivatives

There are several types of derivatives which may appear in a perturbative computation. We will study here three of those types: partial, covariant and Lie derivatives.

Partial derivatives are associated to coordinate systems and hence do not change under perturbations of the metric. Therefore, by construction, they commute with the Δ operator: $\Delta^n[T_{,\mu}] = \Delta^n[T]_{,\mu}$ for any tensor field T of any rank.

General covariant derivatives can be perturbed. For instance, the Levi-Civita connection of a metric will change when its associated metric is perturbed. The question arises then about what is the perturbation of the covariant derivative of a tensor. Transforming to partial derivatives and Christoffel symbols, perturbing and coming back to covariant derivatives, we get, for an arbitrary tensor density T of weight w,

$$\begin{aligned}\Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k;\sigma}] &= \Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k}]_{;\sigma} \\ &+ \sum_{i=1}^{m}\{\Delta^n[\Gamma^{\mu_i}{}_{\sigma\alpha}T^{\mu_1\ldots\alpha\ldots\mu_m}{}_{\nu_1\ldots\nu_k}] - \Gamma^{\mu_i}{}_{\sigma\alpha}\Delta^n[T^{\mu_1\ldots\alpha\ldots\mu_m}{}_{\nu_1\ldots\nu_k}]\} \\ &- \sum_{j=1}^{k}\{\Delta^n[\Gamma^{\alpha}{}_{\sigma\nu_j}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\alpha\ldots\nu_k}] - \Gamma^{\alpha}{}_{\sigma\nu_j}\Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\alpha\ldots\nu_k}]\} \\ &- w\ \{\Delta^n[\Gamma^{\alpha}{}_{\sigma\alpha}T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k}] - \Gamma^{\alpha}{}_{\sigma\alpha}\Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k}]\}\,,\end{aligned} \tag{3.8}$$

which can be rewritten applying the Leibnitz rule (3.7) as

$$\begin{aligned}\Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k;\sigma}] &= \Delta^n[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k}]_{;\sigma} + \sum_{l=1}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\Big\{\sum_{i=1}^{m}\Delta^l[\Gamma^{\mu_i}{}_{\sigma\alpha}]\Delta^{n-l}[T^{\mu_1\ldots\alpha\ldots\mu_m}{}_{\nu_1\ldots\nu_k}] \\ &-\sum_{j=1}^{k}\Delta^l[\Gamma^{\alpha}{}_{\sigma\nu_j}]\Delta^{n-l}[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\alpha\ldots\nu_k}] - w\Delta^l[\Gamma^{\alpha}{}_{\sigma\alpha}]\Delta^{n-l}[T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_k}]\Big\}.\end{aligned} \tag{3.9}$$

Note that this expression does not involve the metric directly. That is, it only contains the background connection ∇, and the perturbations of its Christoffel symbols, without assuming that either the background or the perturbed connections derive from a metric. Hence, it can be applied to perturb a manifold without defining a metric on it, for example to the Palatini equations. Nevertheless, for the cases that the connection arises from a metric, in the following subsection we will relate the perturbations of the Christoffel symbols to the perturbations of the mentioned metric.

Finally, the perturbation formula for the Lie derivative along the vector field v of a tensor T of any rank can be computed by intermediate transformation to partial or covariant derivatives,

$$\Delta^n[\mathcal{L}_v T] = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\mathcal{L}_{\Delta^k[v]}\Delta^{n-k}[T]. \tag{3.10}$$

This expression bears an obvious similarity with the Leibnitz rule, reflecting the fact that both v and T are being perturbed but not the Lie structure itself, which is directly given by the differential structure of the manifold and thus remains unperturbed.

3.3.2 Perturbations of the curvature tensors

The expansion for the inverse metric can be obtained by iteration of the matricial identity

$$\tilde{g}^{\mu\nu} \equiv g^{\mu\nu} - g^{\mu\lambda}(\tilde{g}_{\lambda\sigma} - g_{\lambda\sigma})\tilde{g}^{\sigma\nu}, \tag{3.11}$$

or, equivalently, by repeated perturbation of the relation

$$\Delta[g^{\mu\nu}] = -g^{\mu\alpha}\Delta[g_{\alpha\beta}]g^{\beta\nu}. \tag{3.12}$$

Up to third order this leads to

$$\begin{aligned}\tilde{g}^{\mu\nu} &= g^{\mu\nu} - \varepsilon\, {}^{\{1\}}h^{\mu\nu} - \frac{\varepsilon^2}{2}\left({}^{\{2\}}h^{\mu\nu} - 2\, {}^{\{1\}}h^{\mu\alpha}\, {}^{\{1\}}h_{\alpha}{}^{\nu}\right) \\ &- \frac{\varepsilon^3}{6}\left({}^{\{3\}}h^{\mu\nu} - 3\, {}^{\{1\}}h^{\mu\alpha}\, {}^{\{2\}}h_{\alpha}{}^{\nu} - 3\, {}^{\{2\}}h^{\mu\alpha}\, {}^{\{1\}}h_{\alpha}{}^{\nu} + 6\, {}^{\{1\}}h^{\mu\alpha}\, {}^{\{1\}}h_{\alpha\beta}\, {}^{\{1\}}h^{\beta\nu}\right) + O(\varepsilon^4).\end{aligned} \tag{3.13}$$

In order to obtain the general term of that series, let us define $h_{\mu\nu} \equiv \tilde{g}_{\mu\nu} - g_{\mu\nu}$ and temporarily obviate the indices. With this notation, we have

$$\tilde{g}^{-1} = (g+h)^{-1}, \tag{3.14}$$

which can be Taylor expanded around $h = 0$ as

$$\tilde{g}^{-1} = g^{-1}\sum_{m=0}^{\infty}(-1)^m (g^{-1}h)^m. \tag{3.15}$$

Substituting there the decomposition of h in power series of ε and displaying again the indices,

$$\tilde{g}^{\mu\nu} = g^{\mu\alpha}\sum_{m=0}^{\infty}(-1)^m \left(\sum_{k_1=1}^{\infty}\frac{\varepsilon^{k_1}}{k_1!}\, {}^{\{k_1\}}h_{\alpha}{}^{\beta}\right)\left(\sum_{k_2=1}^{\infty}\frac{\varepsilon^{k_2}}{k_2!}\, {}^{\{k_2\}}h_{\beta}{}^{\gamma}\right) \dots \left(\sum_{k_m=1}^{\infty}\frac{\varepsilon^{k_m}}{k_m!}\, {}^{\{k_m\}}h_{\rho}{}^{\nu}\right), \tag{3.16}$$

which can be organized to provide the general term

$$\Delta^n[g^{\mu\nu}] = \sum_{(k_i)}(-1)^m \frac{n!}{k_1! \dots k_m!}\, {}^{\{k_m\}}h^{\mu\alpha}\ {}^{\{k_{m-1}\}}h_{\alpha}{}^{\beta} \dots {}^{\{k_2\}}h_{\tau}{}^{\rho}\ {}^{\{k_1\}}h_{\rho}{}^{\nu}, \tag{3.17}$$

where the notation in round brackets (k_i) stands for extending the sum to the 2^{n-1} sorted partitions of n in $m \leq n$ positive integers (not including zero) $k_1 + ... + k_m = n$. For example, for $n = 3$ there are four partitions $\{(3), (1,2), (2,1)$ and $(1,1,1)\}$, which generate the four terms appearing in the second line of the formula (3.13).

With the expansions of the metric and its inverse at hand, we can obtain the Christoffel symbols in the usual way,

$$\tilde{\Gamma}^{\alpha}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\alpha\beta}(\partial_\mu \tilde{g}_{\nu\beta} + \partial_\nu \tilde{g}_{\mu\beta} - \partial_\beta \tilde{g}_{\mu\nu}). \tag{3.18}$$

Displaying it up to third order, this leads to

$$\begin{aligned}\tilde{\Gamma}^{\alpha}{}_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu} + \varepsilon\ {}^{\{1\}}h^{\alpha}{}_{\mu\nu} + \frac{\varepsilon^2}{2}\left({}^{\{2\}}h^{\alpha}{}_{\mu\nu} - 2\, {}^{\{1\}}h^{\alpha\beta}\, {}^{\{1\}}h_{\beta\mu\nu}\right) \\ &+ \frac{\varepsilon^3}{6}\left({}^{\{3\}}h^{\alpha}{}_{\mu\nu} - 3\, {}^{\{1\}}h^{\alpha\beta}\, {}^{\{2\}}h_{\beta\mu\nu} - 3\, {}^{\{2\}}h^{\alpha\beta}\, {}^{\{1\}}h_{\beta\mu\nu} + 6\, {}^{\{1\}}h^{\alpha\beta}\, {}^{\{1\}}h_{\beta}{}^{\gamma}\, {}^{\{1\}}h_{\gamma\mu\nu}\right) + O(\varepsilon^4),\end{aligned} \tag{3.19}$$

where we have defined the three-indices tensor perturbation

$$ {}^{\{n\}}h_{\alpha\mu\nu} \equiv \frac{1}{2}\left({}^{\{n\}}h_{\alpha\mu;\nu} + {}^{\{n\}}h_{\alpha\nu;\mu} - {}^{\{n\}}h_{\mu\nu;\alpha}\right), \tag{3.20} $$

which is symmetric in its last two indices and satisfies ${}^{\{0\}}h_{\alpha\mu\nu} = 0$. The covariant derivative in (3.20) is that associated with the background metric. Higher-order terms of the expansion can be easily computed noting that

$$ \Delta\left[{}^{\{n\}}h_{\alpha\mu\nu}\right] = {}^{\{n+1\}}h_{\alpha\mu\nu} - {}^{\{n\}}h_{\alpha}{}^{\beta}\, {}^{\{1\}}h_{\beta\mu\nu}, \tag{3.21} $$

which leads to

$$ \Delta^n[\Gamma^\alpha{}_{\mu\nu}] = \sum_{(k_i)} (-1)^{m+1} \frac{n!}{k_1!\dots k_m!}\, {}^{\{k_m\}}h^{\alpha\beta}\ {}^{\{k_{m-1}\}}h_{\beta\gamma} \dots {}^{\{k_2\}}h_{\tau\rho}\ {}^{\{k_1\}}h^{\rho}{}_{\mu\nu}. \tag{3.22} $$

Here, the sum extends again to all sorted partitions of n. For the case $n = 1$ this formula must be understood as $\Delta[\Gamma^\alpha{}_{\mu\nu}] = {}^{\{1\}}h^\alpha{}_{\mu\nu}$. Note that each term in the above expression contains one and only one tensor ${}^{\{k\}}h_{\rho\mu\nu}$, but a variable number of metric perturbations. In order to obtain the general perturbative term for the Riemann tensor, we can start with its definition in terms of the Christoffel symbols

$$ R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu\Gamma^\sigma{}_{\mu\rho} - \partial_\mu\Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\alpha\mu}, \tag{3.23} $$

and use the perturbation operator Δ^n on it. This leads to the general term

$$ \Delta^n[R_{\mu\nu\rho}{}^{\sigma}] = \partial_\nu\left(\Delta^n[\Gamma^\sigma{}_{\mu\rho}]\right) - \sum_{k=0}^{n} \binom{n}{k} \Delta^k[\Gamma^\alpha{}_{\nu\rho}]\Delta^{n-k}[\Gamma^\sigma{}_{\alpha\mu}] - (\mu \leftrightarrow \nu). \tag{3.24} $$

To make transparent the tensorial character of this object, we can rewrite it as

$$ \Delta^n[R_{\mu\nu\rho}{}^{\sigma}] = \nabla_\nu\left(\Delta^n[\Gamma^\sigma{}_{\mu\rho}]\right) - \sum_{k=1}^{n-1} \binom{n}{k} \Delta^k[\Gamma^\alpha{}_{\nu\rho}]\Delta^{n-k}[\Gamma^\sigma{}_{\alpha\mu}] - (\mu \leftrightarrow \nu). \tag{3.25} $$

If the background connection comes from a metric, (3.19–3.22) ensure that the perturbed connection also derives from a metric. Then, the (implicit) triple sum in (3.25) can be rearranged as follows,

$$ \begin{aligned} \Delta^n[R_{\mu\nu\rho}{}^{\sigma}] \;=\;& \sum_{(k_i)} (-1)^m \frac{n!}{k_1!\dots k_m!}\Big[{}^{\{k_m\}}h^{\sigma\lambda_m} \dots {}^{\{k_2\}}h^{\lambda_3\lambda_2}\ {}^{\{k_1\}}h_{\lambda_2\rho\nu;\mu} \\ &+ \sum_{s=2}^{m} {}^{\{k_m\}}h^{\sigma\lambda_m} \dots {}^{\{k_{s+1}\}}h^{\lambda_{s+2}\lambda_{s+1}}\ {}^{\{k_s\}}h_{\lambda_s\lambda_{s+1}\mu}\ {}^{\{k_{s-1}\}}h^{\lambda_s\lambda_{s-1}} \dots {}^{\{k_2\}}h^{\lambda_3\lambda_2}\ {}^{\{k_1\}}h_{\lambda_2\nu\rho}\Big] \\ &- (\mu \leftrightarrow \nu). \end{aligned} \tag{3.26} $$

If we display this formula up to second order we obtain

$$\begin{aligned}\tilde{R}_{\mu\nu\sigma}{}^{\lambda} &= R_{\mu\nu\sigma}{}^{\lambda} + 2\varepsilon\ {}^{\{1\}}h^{\lambda}{}_{\sigma[\mu;\nu]} \\ &+ \varepsilon^2 \left({}^{\{2\}}h^{\lambda}{}_{\sigma[\mu;\nu]} - 2\,{}^{\{1\}}h^{\lambda\alpha}\,{}^{\{1\}}h_{\alpha\sigma[\mu;\nu]} + 2\,{}^{\{1\}}h_{\alpha[\mu}{}^{\lambda}\,{}^{\{1\}}h^{\alpha}{}_{\nu]\sigma} \right) + O(\varepsilon^3).\end{aligned} \tag{3.27}$$

Formula (3.26) is surprisingly simple because all covariant derivatives of the metric perturbations are grouped in terms of the form (3.20). It is clear that the explicit sum in (3.25) contains only that kind of terms; however the derivatives of the perturbations of the Christoffel symbols give rise to isolated covariant derivatives of the metric perturbations. Nonetheless, because of formula (3.9), they can be combined to obtain the displayed result.

It is straightforward to compute the general perturbation for the Ricci tensor from (3.25) if we want to express it in terms of the perturbations of the background connection, or from (3.26) if we want it in terms of the perturbations of the metric. As there is no metric contraction involved in the passage from the Riemann to the Ricci tensor, it is clear that

$$\Delta^n[R_{\mu\rho}] = \Delta^n[R_{\mu\nu\rho}{}^{\nu}]. \tag{3.28}$$

The resulting expression is symmetric in μ and ρ owing to the identity

$$H^{\alpha\beta}\,{}^{\{k\}}h_{\alpha\beta\rho;\mu} = H^{\alpha\beta}\,{}^{\{k\}}h_{\alpha\beta\mu;\rho}, \tag{3.29}$$

valid for any symmetric tensor $H^{\alpha\beta}$ such that $H^{\alpha\beta}\,{}^{\{k\}}h_{\beta}{}^{\mu}$ is also symmetric in α and μ.

When considering the Ricci scalar, one has to take into account that there is a metric contraction,

$$R = g^{\alpha\beta}R_{\alpha\beta}, \tag{3.30}$$

which will have a non-zero contribution when the perturbation operator is applied. Using formula (3.7), it is easy to compute the general perturbative term,

$$\Delta^n[R] = \sum_{k=0}^{n} \binom{n}{k} \Delta^k[g^{\alpha\beta}]\Delta^{n-k}[R_{\alpha\beta}]. \tag{3.31}$$

3.3.3 Perturbations of the metric determinant

As will be made explicit in the following section, when making perturbative calculations in a canonical framework, one frequently finds the perturbation of the determinant of the metric. This is a basis-dependent concept, in the sense that what we are computing is the determinant of the components of the metric in a given basis and the result depends on the basis we have chosen. Under a change of basis the determinant changes with a squared Jacobian and it is, hence, a density of weight $+2$.

The determinant of the metric $g_{\alpha\beta}$ of a N-dimensional manifold, can be defined as

$$\det(g_{\alpha\beta}) \equiv \frac{1}{N!}\,\widetilde{\eta}^{\alpha_1\ldots\alpha_N}\,\widetilde{\eta}^{\beta_1\ldots\beta_N}\,g_{\alpha_1\beta_1}\cdots g_{\alpha_N\beta_N}, \tag{3.32}$$

using the upper antisymmetric density $\widetilde{\eta}^{\alpha_1\ldots\alpha_N}$ (the over-tilde denotes weight $+1$), whose components in the chosen basis are $+1$, -1 or 0. This object, as well as its lower counterpart $\underset{\sim}{\eta}_{\alpha_1\ldots\alpha_N}$ of weight -1, stays invariant under the Δ perturbation. Therefore, the Leibnitz rule (3.6) implies

$$\Delta^n[\det(g_{\alpha\beta})] = \frac{1}{N!}\,\widetilde{\eta}^{\alpha_1\ldots\alpha_N}\,\widetilde{\eta}^{\beta_1\ldots\beta_N}\sum_{\{k_i\}}\frac{n!}{k_1!\ldots k_N!}\,{}^{\{k_1\}}h_{\alpha_1\beta_1}\ldots{}^{\{k_N\}}h_{\alpha_N\beta_N}. \tag{3.33}$$

Finally, this can be simplified using the well known relation

$$\widetilde{\eta}^{\alpha_1\ldots\alpha_N}\,\widetilde{\eta}^{\beta_1\ldots\beta_N} = \det(g_{\alpha\beta})\begin{vmatrix} g^{\alpha_1\beta_1} & \cdots & g^{\alpha_1\beta_N} \\ \vdots & & \vdots \\ g^{\alpha_N\beta_1} & \cdots & g^{\alpha_N\beta_N}\end{vmatrix}. \tag{3.34}$$

We conclude that the n-th order perturbation of the determinant of the metric is always the product of the determinant itself times a scalar formed by contraction of metric perturbations. It is interesting to note that such a scalar factor contains the product of at most N metric perturbations, and not n, as we might have anticipated by inspection of the formulas in the previous subsections. This is actually the only place in this chapter in which the dimension of the manifold being perturbed plays a role. With this formula at hand it is straightforward to give the n-th order perturbation of the volume form $\epsilon_{a_1\ldots a_N} \equiv |\det(g)|^{1/2}\,\underset{\sim}{\eta}_{a_1\ldots a_N}$.

It is worth emphasizing that the combinatorial formulas for the perturbations of the curvature tensors at a general order achieved in this section are extremely useful for computational purposes. Essentially, the problem of perturbations is reduced to that of listing the sorted partitions of a given number, which can be done very fast in any computer-algebra system. This fact simply reflects the recursive differential origin of the perturbation process. The formulas in this section have been implemented in the free package *xPert* that will be explained in Chapter 15.

3.4 Equations of motion

We will do perturbations in the context of standard General Relativity. Hence, in order to obtain the perturbed equations of motion one has two different alternatives. It is possible to perturb the Einstein equations directly, which is the most commom approach to perturbation theory in General Relativity. But there is another choice, that is perturbing the Hilbert-Einstein action in a Hamiltonian setting. This last approach will be very illuminating when discussing the gauge freedom in the next chapter. In the following subsections we explain both approaches in more detail.

3.4.1 Covariant framework

In the covariant framework we proceed by perturbing the Einstein equations directly. The formulas from the previous section allow us to obtain the general perturbative term of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}, \tag{3.35}$$

where again Leibnitz formula (3.6) is applied, since the second term of the equation is the product of three objects. It is straightforward to rearrange labels and write

$$\Delta^n[G_{\mu\nu}] = \Delta^n[R_{\mu\nu}] - \frac{1}{2}\sum_{k=0}^{n}\sum_{j=k}^{n} \frac{n!}{k!(j-k)!(n-j)!}\, {}^{\{n-j\}}h_{\mu\nu} \Delta^{j-k}[g^{\alpha\beta}]\Delta^k[R_{\alpha\beta}]. \tag{3.36}$$

It is not worth writing this equation in a more explicit form. The most important point to notice is that, as already explained in the introduction of this chapter, the full Einstein equations are equivalent to an infinite hierarchy of linear differential equations. In order to make this explicit, it is enough to take the Einstein equations and replace the tensors by their expansions in power series of ε, which gives

$$\Delta^n[G_{\mu\nu}] = 8\pi\, {}^{\{n\}}t_{\mu\nu} \quad \text{for all } n. \tag{3.37}$$

In order to close this system of equations, the linearized matter evolution equations are necessary. We can separate the n-th order perturbation of the Einstein tensor into a part that is linear in the n-th perturbation of the metric and a source that will be composed of products of lower order perturbations. Then equation (3.37) can be written, for every n, in a schematic form

$$L[\,{}^{\{n\}}h_{\mu\nu}] + {}^{\{n\}}S_{\mu\nu}[\,{}^{\{1\}}h \ldots {}^{\{n-1\}}h] = 8\pi\, {}^{\{n\}}t_{\mu\nu}, \tag{3.38}$$

where ${}^{\{n\}}S$ is the source term and L is a linear differential operator that only depends on the background geometry. In general it can be written as

$$\begin{aligned} L[\,{}^{\{n\}}h_{\mu\nu}] \;=\; & \frac{1}{2}[g_{\mu\nu}(\,{}^{\{n\}}h^{\alpha\beta}R_{\alpha\beta} + {}^{\{n\}}h^{\beta}{}_{\beta;}{}^{\alpha}{}_{\alpha} - {}^{\{n\}}h^{\alpha\beta}{}_{;\alpha\beta}) - {}^{\{n\}}h_{\mu\nu}R \\ & +2\,{}^{\{n\}}h^{\alpha}{}_{\mu\nu;\alpha} - {}^{\{n\}}h^{\alpha}{}_{\alpha;\mu\nu}]. \end{aligned} \tag{3.39}$$

There are two special cases which deserve attention. If the background is a vacuum ($t_{\mu\nu} = 0$), the background Ricci tensor will cancel out simplifying the expression. Besides that, if we suppose that all the perturbations of the stress-energy tensor are zero ($^{\{n\}}t_{\mu\nu} = 0$) then (3.39) reduces to

$$L[\,^{\{n\}}h_{\mu\nu}] = -\frac{1}{2} g_{\mu\nu}\,^{\{n\}}S^{\alpha}{}_{\alpha}[\,^{\{1\}}h \ldots\,^{\{n-1\}}h] + \,^{\{n\}}h^{\alpha}{}_{\mu\nu;\alpha} - \frac{1}{2}\,^{\{n\}}h^{\alpha}{}_{\alpha;\mu\nu}. \tag{3.40}$$

In order to see this, take the trace of (3.38), obtaining $^{\{n\}}h^{\beta}{}_{\beta;}{}^{\alpha}{}_{\alpha} = \,^{\{n\}}h^{\alpha\beta}{}_{;\alpha\beta} - \,^{\{n\}}S_{\alpha}{}^{\alpha}$. Replacement in (3.39) provides then the commented result.

3.4.2 Canonical framework

In this subsection we provide a very brief summary of the canonical formalism for General Relativity developed by Arnowitt, Deser and Misner [18,19]. We will also analyze how it can be used to obtain the perturbative equations of motion.

Since this is a Hamiltonian approach, we need to specify the matter model to explicitly carry out the calculations. We will suppose a massless scalar matter field Φ. The case of a vacuum is included just by letting Φ vanish. This matter model will be later used in spherical symmetry to make the background dynamical and find a master equation.

Given the four-dimensional spacetime $(M^4, {}^{(4)}g_{\mu\nu})$, we introduce a foliation of three-dimensional spacelike slices Σ_t as level surfaces of the time field $t(x)$. The orthogonal vector u^{μ} defines the projected metric ${}^{(3)}g_{\mu\nu} = {}^{(4)}g_{\mu\nu} + u_{\mu}u_{\nu}$ on the slices. We introduce coordinates (t, x^i) adapted to the foliation, and work with three-dimensional objects. Greek and Latin indices denote four- and three-dimensional tensors respectively. A left-superindex indicates dimensionality when confusion may arise.

The conjugated momentum of the Klein-Gordon field Φ is defined as,

$$\Pi_{\Phi} \equiv -\sqrt{-{}^{(4)}g}\;{}^{(4)}g^{t\mu}\Phi_{,\mu}, \tag{3.41}$$

where ${}^{(4)}g$ denotes the determinant of the four-metric. In terms of this pair of variables the action corresponding to the matter reads,

$$\mathcal{S}_{KG} = -\frac{1}{2}\int dx^4 \sqrt{-{}^{(4)}g}\;{}^{(4)}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} \tag{3.42}$$

$$= \int dt \int_{\Sigma_t} dx^3 \left[\Pi_{\Phi}\Phi_{,t} - \frac{\alpha}{2}\left(\frac{\Pi_{\Phi}^2}{\mu_g} + \mu_g g^{ij}\Phi_{,i}\Phi_{,j}\right) - \beta^i\left(\Pi_{\Phi}\Phi_{,i}\right)\right]. \tag{3.43}$$

Following ADM decomposition, the four-metric is expressed in terms of the lapse function, the shift vector and the spatial metric on the slices

$$\alpha^{-2} \equiv -{}^{(4)}g^{tt}, \qquad \beta_i \equiv {}^{(4)}g_{ti}, \qquad g_{ij} \equiv {}^{(4)}g_{ij}, \tag{3.44}$$

with inverse

$$g^{ij} = {}^{(4)}g^{ij} + \alpha^{-2}\beta^i\beta^j, \tag{3.45}$$

with Latin indices always raised and lowered with g^{ij} and g_{ij}. The gravitational dynamical variables in the ADM Hamiltonian formalism are g_{ij} and their conjugated momenta

$$\Pi^{ij} \equiv \mu_g \left(g^{ij}K^l{}_l - K^{ij}\right), \tag{3.46}$$

where K^{ij} is the extrinsic curvature and $\mu_g \equiv \sqrt{|\det g_{ij}|}$ is the determinant of the metric of the foliation hypersurfaces.

In this way, the complete action of the system, with coupling constant $16\pi G_N = 1$, is given by

$$\mathcal{S} = \mathcal{S}_G + \mathcal{S}_{KG} = \int dt \int_{\Sigma_t} d^3x \left(\Pi^{ij}g_{ij,t} + \Pi_\Phi \Phi_{,t} - \alpha\mathcal{H} - \beta^i\mathcal{H}_i\right). \tag{3.47}$$

Taking the variation of last expression with respect to Lagrange multipliers α and β^i one obtains the constraints of the background spacetime,

$$\mathcal{H} \equiv \frac{1}{\mu_g}\left[\Pi^{ij}\Pi_{ij} - \frac{1}{2}\left(\Pi^l{}_l\right)^2\right] - \mu_g{}^{(3)}R + \frac{1}{2}\left(\frac{\Pi_\Phi^2}{\mu_g} + \mu_g g^{ij}\Phi_{,i}\Phi_{,j}\right) = 0, \tag{3.48}$$

$$\mathcal{H}_i \equiv -2D_j\Pi_i{}^j + \Pi_\Phi\Phi_{,i} = 0, \tag{3.49}$$

where D_j is the covariant derivative associated to g_{ij}. Variation of the action (3.47) with respect to g_{ij}, Π_{ij}, Φ and Π_Φ gives the evolution equations for their corresponding conjugated variables.

In order to obtain the perturbed equations of motion, one can follow the method used by Taub [128] and Moncrief [101] at linear order. Here we generalize that method to any perturbative order. The idea is that if one wants to obtain the equations of motion at order n, it is necessary to construct an effective action quadratic in the mentioned variables. This can be achieved just by perturbing the action (3.47) up to order $2n$. The general term of such a perturbation will be composed by products of factors of order $k \leq 2n$. We will keep only those terms that have at least one of the factors of order n. These are the meaningful terms since the physical equations will be obtained by variation of the resulting action with respect to variables of the mentioned order n. Therefore, the rest of the terms will not contribute. Defining the following shorthand,

$$^{\{n\}}C \equiv \Delta^n[\alpha], \qquad ^{\{n\}}B^i \equiv \Delta^n[\beta^i], \tag{3.50}$$

$$^{\{n\}}h_{ij} \equiv \Delta^n[g_{ij}], \qquad ^{\{n\}}p^{ij} \equiv \Delta^n[\Pi^{ij}], \tag{3.51}$$

$$^{\{n\}}\varphi \equiv \Delta^n[\Phi], \qquad ^{\{n\}}p \equiv \Delta^n[\Pi_\Phi], \tag{3.52}$$

we can schematically write down the effective action,

$$\Delta_n^{2n}[\mathcal{S}] = \frac{(2n)!}{n!^2}\int dx^4 \Big[{}^{\{n\}}p^{ij}\,{}^{\{n\}}h_{ij,t} + {}^{\{n\}}p\,{}^{\{n\}}\varphi_{,t} - {}^{\{n\}}C\Delta^n[\mathcal{H}] - {}^{\{n\}}B^i\Delta^n[\mathcal{H}_i] - \sum_{k=0}^{n-1}\frac{n!^2}{k!(2n-k)!}\left\{ {}^{\{k\}}C\Delta_n^{2n-k}[\mathcal{H}] + {}^{\{k\}}B\Delta_n^{2n-k}[\mathcal{H}_i]\right\}\Big]. \qquad (3.53)$$

The subindex in the operator Δ_n means to keep only those terms which have at least one of the factors of order n. For instance, compare the following expression for the third-order perturbation of the inverse metric with the full perturbation presented in equation (3.13),

$$\Delta_2^3[g^{\mu\nu}] = 3\,{}^{\{1\}}h^{\mu\alpha}\,{}^{\{2\}}h_\alpha{}^\nu + 3\,{}^{\{2\}}h^{\mu\alpha}\,{}^{\{1\}}h_\alpha{}^\nu. \qquad (3.54)$$

Hence, it is sufficient to take variations of the effective action (3.53) with respect to different nth-order variables to obtain the equations of motion at that order. In the next chapter we will analyze the structure of the effective action (3.53) since it will be very illuminating in the context of gauge freedom.

Chapter 4

Gauge freedom

In this chapter we will analyze the problem of recognizing the actual physical degrees of freedom of the perturbations. We will encounter again the two different approaches to analyze this problem: the canonical and covariant frameworks. In General Relativity the former has been used more frequently. Even though, as we will explain below, the latter has its own advantages when dealing with the gauge freedom since it also considers the dynamics of the system. The problem is that, as far as we know, the interpretation of the gauge freedom at second and higher order has not yet been developed in a canonical setting.

4.1 Covariant framework

As we have explained in the introduction to the previous chapter, in perturbation theory we consider a family of spacetimes $\{\tilde{M}(\varepsilon), \tilde{g}_{\mu\nu}(\varepsilon), \tilde{t}_{\mu\nu}\}$ in which associated families of tensor fields $\tilde{T}(\varepsilon)$ are defined. All manifolds $\tilde{M}(\varepsilon)$ are assumed to be diffeomorphic. The main issue in perturbation theory is comparing tensor fields for a given nonzero value of ε with their background counterparts ($\varepsilon = 0$). There exists diffeomorphism invariance on each of the manifolds $\tilde{M}(\varepsilon)$ but, in addition, there is no preferred point-to-point identification mapping between any two such manifolds, so that the comparison of two tensor fields with different values of ε is not an invariantly defined concept. This is the origin of the so-called gauge freedom in perturbation theory [99].

4.1.1 Gauge transformations

Let us call a gauge ϕ_ε a family of point-to-point identification diffeomorphisms from the background manifold M to $\tilde{M}(\varepsilon)$:

$$\phi_\varepsilon : \quad M \quad \longrightarrow \quad \tilde{M}(\varepsilon). \tag{4.1}$$

Given a gauge ϕ_ε we can now pull-back a generic tensor $\tilde{T}(\varepsilon)$ on $\tilde{M}(\varepsilon)$ to a tensor $\phi_\varepsilon^* \tilde{T}(\varepsilon)$ on M. This latter tensor can be compared with the background member T (at each point in M), resulting in a ϕ-dependent concept of what a perturbation means. Assuming smooth dependence of all structures in ε, we can define the perturbative expansion

$$\phi_\varepsilon^* \tilde{T}(\varepsilon) \equiv T + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \Delta_\phi^n[T], \tag{4.2}$$

where all terms of the equation are defined at the same point of the background manifold M. In particular the perturbations $\Delta_\phi^n[T]$ are tensor fields on M. The explicit notation Δ_ϕ (that we will only use in this chapter) stresses the fact that, in general, it is not possible to define a perturbation without explicitly indicating which gauge ϕ is used. For instance, the statement that a perturbation vanishes is generically meaningless unless one specifies the gauge in which this occurs. Note also that the infinite series in equation (4.2) arises from the simultaneous dependence on ε of both $\tilde{T}(\varepsilon)$ and ϕ_ε.

One then has to face the question of how the perturbations $\Delta_\phi^n[T]$ vary under a change of gauge from ϕ_ε to, let's say, ψ_ε while keeping unaltered the family of tensors $\tilde{T}(\varepsilon)$. Such a gauge transformation will be described by a family χ_ε of diffeomorphisms on the background manifold

$$\chi_\varepsilon \equiv \phi_\varepsilon^{-1} \circ \psi_\varepsilon : \quad M \quad \longrightarrow \quad M, \tag{4.3}$$

which clearly satisfy

$$\psi_\varepsilon^* \tilde{T}(\varepsilon) = \chi_\varepsilon^* \, \phi_\varepsilon^* \tilde{T}(\varepsilon). \tag{4.4}$$

We emphasize that χ_ε is not a gauge, but a gauge transformation.

Flows are families of diffeomorphisms Ω_ε that form a group under the composition operation, that is, if $\Omega_{\varepsilon_1} \circ \Omega_{\varepsilon_2} = \Omega_{\varepsilon_1+\varepsilon_2}$. These flows are well known and are used, for example, to define the Lie derivative [129]. In fact, from the very definition of the Lie derivative it is straightforward to find the Taylor expansion for the pullback Ω_ε^* generated by the flow Ω_ε acting on a generic tensor T,

$$\Omega_\varepsilon^* T = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \mathcal{L}_\xi^n T. \tag{4.5}$$

But flows are a very special kind of families of diffeomorphisms, so that the above result is far from being directly useful in our discussion.

The expansion of a general family of diffeomorphisms is much more difficult. In fact there were some partial results up to second-order [130–132], but the formalism to deal with the high-order expansions of families of diffeomorphisms was developed recently in [100]. The most important result from that article is that, given any one-parameter family of diffeomorphisms χ_ε, there always exists an infinite set of flows $\{\Omega^{(1)}_\varepsilon, ..., \Omega^{(m)}_{\varepsilon^m/m!}, ...\}$, such that

$$\chi_\varepsilon = ... \circ \Omega^{(m)}_{\varepsilon^m/m!} \circ ... \circ \Omega^{(2)}_{\varepsilon^2/2} \circ \Omega^{(1)}_\varepsilon. \tag{4.6}$$

In this way, any one-parameter family of diffeomorphisms is equivalent to an infinite set of flows and, hence, to an infinite set of vector fields (the generators of the flows). The special combination of flows appearing in (4.6) is called a family of knight diffeomorphisms. This terminology is inspired in the similarity of their action with the movement of the knight in the chess game.

With relation (4.6) at hand, formula (4.5) turns out to be very useful for our purposes. Applying it repeatedly, we attain a power series expansion for the right-hand side of equation (4.4), so that two gauge choices are related by

$$\psi^*_\varepsilon \tilde{T}(\varepsilon) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} ... \sum_{k_m=0}^{\infty} ... \frac{\varepsilon^{k_1+2k_2+...+mk_m+...}}{2^{k_2}...(m!)^{k_m}...k_1!k_2!...k_m!...}\mathcal{L}^{k_1}_{\{1\}\xi}\mathcal{L}^{k_2}_{\{2\}\xi}...\mathcal{L}^{k_m}_{\{m\}\xi}...\phi^*_\varepsilon \tilde{T}(\varepsilon). \tag{4.7}$$

The vectors $\{ {}^{\{1\}}\xi^\mu, {}^{\{2\}}\xi^\mu, ..., {}^{\{m\}}\xi^\mu, ...\}$ generate the respective flows $\{\Omega^{(1)}_\varepsilon, ..., \Omega^{(m)}_{\varepsilon^m/m!}, ...\}$. Up to third order, relation (4.7) reads explicitly

$$\begin{aligned}\psi^*_\varepsilon \tilde{T}(\varepsilon) &= \phi^*_\varepsilon \tilde{T}(\varepsilon) + \varepsilon \mathcal{L}_{\{1\}\xi}\phi^*_\varepsilon \tilde{T}(\varepsilon) + \varepsilon^2\left(\mathcal{L}_{\{2\}\xi} + \mathcal{L}^2_{\{1\}\xi}\right)\phi^*_\varepsilon \tilde{T}(\varepsilon) \\ &+ \frac{\varepsilon^3}{6}\left(\mathcal{L}_{\{3\}\xi} + 3\mathcal{L}_{\{1\}\xi}\mathcal{L}_{\{2\}\xi} + \mathcal{L}^3_{\{1\}\xi}\right)\phi^*_\varepsilon \tilde{T}(\varepsilon) + O(\varepsilon^4).\end{aligned} \tag{4.8}$$

In order to obtain the action of the gauge transformation at each order, we just replace expansion (4.2) in the above equation and get [133]

$$\Delta^n_\psi[T] = \sum_{m=1}^{n}\frac{n!}{(n-m)!}\sum_{[k_m]}\frac{1}{2^{k_2}...(m!)^{k_m}k_1!...k_m!}\mathcal{L}^{k_1}_{\{1\}\xi}...\mathcal{L}^{k_m}_{\{m\}\xi}\Delta^{n-m}_\phi[T], \tag{4.9}$$

where we have defined $[k_m] = \{(k_1, ..., k_m) \in \mathbb{N}/\sum_{i=1}^m ik_i = m\}$. Expanding this relation up to third order we obtain

$$\Delta_\psi[T] - \Delta_\phi[T] = \mathcal{L}_{\{1\}\xi}T, \tag{4.10}$$

$$\Delta^2_\psi[T] - \Delta^2_\phi[T] = \left(\mathcal{L}_{\{2\}\xi} + \mathcal{L}^2_{\{1\}\xi}\right)T + 2\mathcal{L}_{\{1\}\xi}\Delta[T], \tag{4.11}$$

$$\begin{aligned}\Delta^3_\psi[T] - \Delta^3_\phi[T] &= \left(\mathcal{L}_{\{3\}\xi} + \mathcal{L}^3_{\{1\}\xi} + 3\mathcal{L}_{\{1\}\xi}\mathcal{L}_{\{2\}\xi}\right)T \\ &+ 3\left(\mathcal{L}_{\{2\}\xi} + \mathcal{L}^2_{\{1\}\xi}\right)\Delta[T] + 3\mathcal{L}_{\{1\}\xi}\Delta^2[T].\end{aligned} \tag{4.12}$$

These formulas describe the effect of general gauge transformations on any high-order perturbation of a generic background tensor T. They contain all the information needed to analyze the issue of gauge transformations in perturbation theory.

It is interesting to ask ourselves whether it is possible to perform gauge transformations and combine them working purely at a given order m. In practical applications we can keep terms only up to a finite order ε^n, which projects the full group $\mathcal{G}$ of gauge transformations into a truncated group ${}^{\{n\}}\mathcal{G}$ of nth-order gauge transformations. Each of these transformations is described by a collection of n vector fields $\{{}^{\{1\}}\xi, ..., {}^{\{n\}}\xi\}$, and we will say that it is pure mth-order if all those vectors are zero except for ${}^{\{m\}}\xi$. The important point is that, in general, composition of pure mth-order transformations is not pure mth-order, unless $m = n$. For example, the composition of two generic second-order transformations described by $\{{}^{\{1\}}\bar{\xi}, {}^{\{2\}}\bar{\xi}\}$ and $\{{}^{\{1\}}\hat{\xi}, {}^{\{2\}}\hat{\xi}\}$ is described by the pair

$$\{{}^{\{1\}}\bar{\xi} + {}^{\{1\}}\hat{\xi}, \quad {}^{\{2\}}\bar{\xi} + {}^{\{2\}}\hat{\xi} + [{}^{\{1\}}\bar{\xi}, {}^{\{1\}}\hat{\xi}]\}, \tag{4.13}$$

and hence the subset of pure first-order transformations $\{{}^{\{1\}}\xi, 0\}$ is not a subgroup of ${}^{\{2\}}\mathcal{G}$. In fact, the group ${}^{\{1\}}\mathcal{G}$ is not equivalent to this subset, but only to a truncated form of it, and therefore it is important to distinguish between first-order transformations $\{{}^{\{1\}}\xi\}$ and pure first-order transformations $\{{}^{\{1\}}\xi, 0, ...0\}$. In general, the set ${}^{\{n\}}\mathcal{G}$ of all nth-order gauge transformations is a group, but the subset of all pure mth-order transformations is not. The only exception is the reduced case $m = n$ of transformations of the form $\{0, ..., 0, {}^{\{n\}}\xi\}$, in which only a single linear term in equations (4.9) survives (this includes first-order perturbation theory as the case $m = n = 1$). There are more general subgroups of a given ${}^{\{n\}}\mathcal{G}$, like the subgroup of transformations of the form $\{0, ..., 0, {}^{\{m\}}\xi, {}^{\{m+1\}}\xi, ..., {}^{\{n\}}\xi\}$, but they have less interest for our discussion and will not be considered in this work.

4.1.2 Gauge invariants

Once we have defined the concept of gauge transformation in equation (4.3), we discuss now the associated notion of gauge invariance of a family of tensors $\tilde{T}(\varepsilon)$ under a group of gauge transformations. We will then find the inherited gauge invariance of the perturbations $\Delta^n_\phi[T]$ under the respective truncated version of that group.

The most natural definition of gauge invariance was given by Sachs [98]: A tensor family $\tilde{T}(\varepsilon)$ is identification gauge invariant (IGI) if the pull-back of its members to the background manifold is independent of the gauge, though the result still depends on ε. That is, $\phi^*_\varepsilon\tilde{T}(\varepsilon) = \psi^*_\varepsilon\tilde{T}(\varepsilon)$ for all gauges $\phi_\varepsilon, \psi_\varepsilon$. This can also be interpreted as the invariance under the full group $\mathcal{G}$ of gauge transformations. Perturbatively, a tensor family $\tilde{T}(\varepsilon)$

is IGI up to order n if and only if $\Delta^m_\phi[T] = \Delta^m_\psi[T]$ for all $m \leq n$ and all gauges ϕ, ψ [100]. Again, this is equivalent to the requirement of invariance under the truncated group ${}^{\{n\}}\mathcal{G}$ defined above. This definition turns out to be too restrictive because, as it is well known [98, 99], only perturbations of vanishing tensors, constant scalars, or constant linear combinations of products of delta tensors can be IGI in first-order perturbation theory, since these are the only tensors with zero Lie derivative along every vector field. For higher, nth-order perturbations the problem becomes even worse because, apart from the background quantities, all of the mth-order perturbations with $m < n$ must also be of the form that we have commented [100]. In principle, this restricts to a very narrow physical scenario the possibilities that are left.

Other forms of gauge invariance can be defined using subgroups of $\mathcal{G}$ or ${}^{\{n\}}\mathcal{G}$. For example invariance with respect to the reduced subgroup of pure nth-order transformations has been used in the past [92], by fixing the gauge perturbatively at all orders $1, ..., n-1$, but not at order n.

Only perturbations of highly symmetric backgrounds admit a complete description in terms of IGI variables. Even at first order, significant limitations have been found: Stewart and Walker showed that, for vacuum spacetimes, only backgrounds with Petrov type D are possible [99], which fortunately includes the Kerr spacetime. In cosmology, only perturbations of static Friedman-Robertson-Walker (FRW) backgrounds can be described in terms of IGI variables [134]. For spherical backgrounds with matter, only first-order perturbations with axial polarity admit such a description [135, 136], but not the complementary set of polar perturbations. This latter result is specially relevant for us, because we want to construct high-order gauge invariants of a spherical spacetime and, as we will see in Chapter 11, the polarities mix already at second order: hence there is no hope of getting a purely IGI description in general. Note that vacuum [101] and electro-vacuum [102] spacetimes with spherical symmetry are very special cases (in particular included in the cited result for type D spacetimes), for which the programme of construction of gauge invariants can be further developed [107].

On the other hand, when describing gravitational radiation in a vacuum, the Weyl tensor provides all the relevant geometrical information, and therefore many investigations employ it as the basic object to be perturbed. Furthermore, the Weyl tensor defines a set of principal null directions, so that it becomes natural to decompose it using the Newman-Penrose formalism. The analysis of IGI is then simplified, but an additional type of gauge invariance is introduced, called tetrad gauge invariance, which requires invariance under (the 6-parametric Lorentz group of) transformations among null tetrads [89, 99, 136]. We will not use this approach in this paper, but instead appeal to a different and more general notion of gauge invariance in which one makes use of an additional geometrical structure: a privileged gauge $\overline{\phi}_\varepsilon$. The basic idea is that, given a family of tensors $\tilde{T}(\varepsilon)$, one can select a privileged gauge to extract the physical information contained in this family and express this information in terms of the pull-back of $\tilde{T}(\varepsilon)$ in an arbitrary gauge. In other words, the gauge invariant is defined as the function(al) that provides the value of $\overline{\phi}_\varepsilon^*\tilde{T}(\varepsilon)$ in a generic gauge:

$$\overline{\phi}_\varepsilon^*\tilde{T}(\varepsilon) = F\left[\phi_\varepsilon^*\tilde{T}(\varepsilon)\right]. \tag{4.14}$$

So, the gauge invariant is now supplied by the function(al) F rather than by the family of tensors $\tilde{T}(\varepsilon)$ itself, as was the case for IGI.

This notion of gauge invariants is similar to that of the constants of motion defined in Mechanics by the particular values that the variables of the system take at some fixed instant of time [137], or even to the notion of evolving constants of motion recently introduced in Quantum Gravity [138] (although in that case one ought to consider a family of privileged gauges parameterized by a set of real numbers, rather than just one of them $\overline{\phi}_\varepsilon$). In spite of appearing counterintuitive at first, this notion can be very useful in those cases in which the computations can be carried out to completion, i.e. when one can obtain the explicit expression of the invariants in terms of gauge-dependent quantities, in our context, or in terms of time-dependent variables, in Mechanics. In other words, we need to determine the explicit form of the gauge transformation $F = \chi_\varepsilon^* = \phi_\varepsilon^{-1}\circ\overline{\phi}_\varepsilon$ for arbitrary ϕ_ε. Whether this is possible or not essentially depends on the choice of gauge $\overline{\phi}_\varepsilon$.

In practice, the privileged gauge is defined by imposing some conditions R_ε on the pull-back $\overline{\phi}_\varepsilon^*\tilde{T}(\varepsilon)$ of a particular tensor $\tilde{T}(\varepsilon)$. Therefore, $\overline{\phi}_\varepsilon$ will be characterized as the gauge in which the tensor $\overline{\phi}_\varepsilon^*\tilde{T}(\varepsilon)$ satisfies some specific requirements. For this method to work satisfactorily, this privileged gauge choice has to be rigid. This means that the conditions $R_\varepsilon[\overline{\phi}_\varepsilon^*\tilde{T}(\varepsilon)] = 0$ must fix uniquely the gauge $\overline{\phi}_\varepsilon$, and so any further gauge transformation will violate those conditions.

In perturbation theory, the invariants will then be the combinations ${}^{\{n\}}F[\{\Delta_\phi^m[T]\}]$ obtained by performing a gauge transformation from the perturbations defined on a generic gauge $\Delta_\phi^m[T]$ to those defined in the rigid gauge $\Delta_{\overline{\phi}}^m[T]$. This kind of combination of perturbations have been characteristic of this approach to gauge invariance, starting with the pioneering work of Moncrief [101] for non-spherical perturbations of Schwarzschild, where the Regge-Wheeler gauge was implicitly used as the privileged gauge. His work was later generalized by GS [51] to non-spherical perturbations of any spherical background, also implicitly using the RW gauge. The same procedure has been employed by Bardeen [33], Stewart [134], and many other authors in their study of perturbations of FRW cosmologies. It can also be found in several recent investigations of second-order perturbations of vacuum

[89, 139, 140] or cosmological backgrounds [141].

For instance, the first-order gauge invariants of a generic tensor T will be given by

$$F\left[\Delta_\phi[T]\right] \equiv \Delta_\phi[T] + \mathcal{L}_p T, \tag{4.15}$$

where p^μ is the vector field generating the first-order gauge transformation from ϕ to $\overline{\phi}$, so that this vector contains now information about our choice of privileged gauge $\overline{\phi}$. Again, in practical applications the gauge $\overline{\phi}$ is selected by imposing some rigid conditions R on the perturbations $\Delta_{\bar{\phi}}[\tilde{T}]$ for some specific tensor $\tilde{T}$, and such that no residual freedom is left in the choice of gauge:

$$R\left[\Delta_{\bar{\phi}}[\tilde{T}]\right] = 0. \tag{4.16}$$

In this way we get the equations

$$R\left[\Delta_\phi[\tilde{T}] + \mathcal{L}_p\tilde{T}\right] = 0 \tag{4.17}$$

which must be solved for p^μ in terms of $\Delta_\phi[\tilde{T}]$. Substituting the vector p^μ obtained in this way, expressions (4.15) provide gauge invariants by construction. Note that when $T = \tilde{T}$, some of those expressions (or combinations of them) are trivial identities [equivalent to the requirements (4.17)]. This method for the determination of invariants can be straightforwardly generalized to higher perturbative orders, as we will see in the following case.

Since metric perturbations play a central role in our analysis, we choose the background metric $g_{\mu\nu}$ as the tensor $\tilde{T}$ on which one imposes the conditions to fix the privileged gauge. We introduce the following compact notation for the perturbations of the metric:

$${}^{\{n\}}h_{\mu\nu} \equiv \Delta^n_\phi[g_{\mu\nu}], \tag{4.18}$$

$${}^{\{n\}}\mathcal{K}_{\mu\nu} \equiv \Delta^n_{\overline{\phi}}[g_{\mu\nu}], \tag{4.19}$$

for a generic gauge ϕ and our privileged one $\overline{\phi}$, respectively. At first order we have that expressions (4.15) for the metric become

$$\mathcal{K}_{\mu\nu} \equiv h_{\mu\nu} + \mathcal{L}_p g_{\mu\nu}. \tag{4.20}$$

The vector p^μ is determined by demanding some conditions $R[\mathcal{K}_{\mu\nu}] = 0$ which characterize the gauge $\overline{\phi}$ at first order. Then, the vector p^μ is determined in terms of the components of $h_{\mu\nu}$ by solving the equations

$$R[h_{\mu\nu} + \mathcal{L}_p g_{\mu\nu}] = 0. \tag{4.21}$$

This completes the definition (4.20) of the gauge invariant $\mathcal{K}_{\mu\nu}$ as a function of $h_{\mu\nu}$.

Nonetheless we note that, owing to the presence of the Lie derivative, equations (4.21) contain derivatives of the vector p^μ, so that their solution will involve in general integrals of the metric perturbations. Only when p^μ can be determined explicitly in an amenable way from the metric perturbations we will have a useful form of gauge invariants. This fact will depend on the choice of the privileged gauge. In particular, we will see later that around spherical backgrounds the requirement of getting explicit and non-integral expressions for the harmonic components of the vector p^μ will almost uniquely single out the RW gauge. We also point out that the same vector p^μ, obtained by solving equations (4.21), can now be employed to define the gauge invariants associated with any other tensor T as in equation (4.15). In addition, note that we can still interpret $\mathcal{K}_{\mu\nu}$ as (the value of) the metric perturbations expressed in the rigid gauge $\overline{\phi}$ which satisfies conditions (4.21).

At higher orders, and once a rigid gauge is chosen via some conditions ${}^{\{m\}}R$ for all $m \leq n$, one can obtain the nth-order metric invariants as

$$ {}^{\{n\}}\mathcal{K}_{\mu\nu} \equiv {}^{\{n\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{n\}}p}\, g_{\mu\nu} + {}^{\{n\}}\mathcal{J}_{\mu\nu}. \tag{4.22} $$

Since this equality reflects the effect of a gauge transformation, the source ${}^{\{n\}}\mathcal{J}_{\mu\nu}$ is explicitly given by equation (4.9) and depends on lower-order vectors ${}^{\{m\}}p^\mu$ and perturbations ${}^{\{m\}}h_{\mu\nu}$ with $m < n$, but not on ${}^{\{n\}}p^\mu$. Besides, we remember that the source vanishes at first order (${}^{\{1\}}\mathcal{H}_{\mu\nu} = 0$). On the other hand, the equation that one has to solve iteratively in order to determine the gauge vectors ${}^{\{m\}}p^\mu$, from $m = 1$ to $m = n$, takes now the expression

$$ {}^{\{m\}}R\big[\, {}^{\{m\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{m\}}p}\, g_{\mu\nu} + {}^{\{m\}}\mathcal{J}_{\mu\nu}\big] = 0. \tag{4.23} $$

In particular, when all the conditions ${}^{\{m\}}R$ have the same linear functional dependence on their arguments (for instance because they arise from the perturbative expansion of just one set of exact linear gauge conditions on the metric), equation (4.23) will have the form (4.21) but with the source term ${}^{\{m\}}R[\,{}^{\{m\}}\mathcal{J}_{\mu\nu}]$. Therefore, the solutions of these equations will be constructed essentially in the same way.

Nakamura has suggested a similar approach [142, 143] to construct high-order gauge invariants. He starts from the basic assumption that a splitting equivalent to equation (4.22) is given from the outset, separating the metric perturbation ${}^{\{n\}}h_{\mu\nu}$ into its gauge-invariant part ${}^{\{n\}}\mathcal{K}_{\mu\nu}$ and gauge-variant part (containing the vectors ${}^{\{m\}}p^\mu$), with the vectors ${}^{\{m\}}p^\mu$ satisfying some set of requirements. No proposal is made, however, on how such a splitting can be attained. Our scheme goes beyond that proposal, giving a constructive and general prescription to generate the vectors ${}^{\{m\}}p^\mu$ from the choice of a rigid gauge, in such a way that the requirements imposed on ${}^{\{m\}}p^\mu$ are automatically fulfilled.

After determining the vectors $\{\,{}^{\{1\}}p^\mu(h), ..., {}^{\{n\}}p^\mu(h)\}$, the perturbations of any tensor field, and in particular those of the stress-energy tensor ${}^{\{n\}}t_{\mu\nu}$, can be taken to its gauge-invariant form ${}^{\{n\}}\Psi_{\mu\nu}$ just by applying a gauge transformation parameterized by the above vectors:

$$ {}^{\{n\}}\Psi_{\mu\nu} = \sum_{m=0}^{n} \frac{n!}{(n-m)!} \sum_{[k_m]} \frac{1}{2^{k_2}...(m!)^{k_m} k_1!...k_m!} \mathcal{L}^{k_1}_{\{1\}p}...\mathcal{L}^{k_m}_{\{m\}p} \, {}^{\{n-m\}}t_{\mu\nu}. $$

In this way we will get a tensor ${}^{\{n\}}\Psi_{\mu\nu}(t,h)$ whose dependence on the perturbations ${}^{\{m\}}t_{\mu\nu}$ and ${}^{\{m\}}h_{\mu\nu}$ $(m \leq n)$ will not change when any gauge transformation is applied to them. In Chapter 9 we will use these techniques to compute the metric and matter gauge invariants for perturbations of a spherical background spacetime.

4.2 Canonical framework

In this section we follow again the ADM formalism explained in Subsection (3.4.2) to analyze the issue of gauge freedom from a canonical perspective.

4.2.1 Gauge transformations

In General Relativity the Poisson brackets among the constraints $\mathcal{H}$ and $\mathcal{H}_i$ vanish on shell. In terms of the terminology introduced by Dirac [144–146] they are first-class constraints, and hence generators of gauge transformations on the constraint surface in phase space. The action of a generic transformation (diffeomorphism) for a functional $F(g_{ij}, \Pi^{ij})$ in phase space is parameterized by the lapse α and shift β. The infinitesimal form of this transformation is given by,

$$ F(g_{ij}, \Pi^{kl}) \longrightarrow F(g_{ij}, \Pi^{kl}) + \left\{ F(g_{ij}, \Pi^{kl}), \int_{\Sigma_t} d^3x \left(\alpha\mathcal{H} + \beta^i \mathcal{H}_i\right) \right\}, \tag{4.24} $$

where we have made use of the standard Poisson brackets,

$$ \{g_{ij}(x), \Pi^{kl}(y)\} = \delta^k{}_i \, \delta^l{}_j \, \delta^3(x-y), \tag{4.25} $$

$$ \{\Phi(x), \Pi(y)\} = \delta^3(x-y), \tag{4.26} $$

for any two points (x, y) in the hypersurface Σ_t, namely, a section of constant time.

This identifies the gauge orbits, but in general it is not possible to invert relations (3.48-3.49) and explicitly separate the four gauge-invariant functions that would contain the physical degrees of freedom from the four gauge variables and the four constrained variables in g_{ij} and Π^{ij}. This is exactly the task we plan to do in the perturbative approach.

Since it will be the basis of the following discussion, we display here again the effective action for the perturbative problem at order n (3.53),

$$\begin{aligned}\Delta_n^{2n}[\mathcal{S}] &= \frac{(2n)!}{n!^2}\int dt\int_{\Sigma_t} dx^3\Big[\,{}^{\{n\}}p^{ij}\,{}^{\{n\}}h_{ij,t} + {}^{\{n\}}p\,{}^{\{n\}}\varphi_{,t} - {}^{\{n\}}C\Delta^n[\mathcal{H}] - {}^{\{n\}}B^i\Delta^n[\mathcal{H}_i] \\ &\quad -\sum_{k=0}^{n-1}\frac{n!^2}{k!(2n-k)!}\left\{\,{}^{\{k\}}C\Delta_n^{2n-k}[\mathcal{H}] + {}^{\{k\}}B\Delta_n^{2n-k}[\mathcal{H}_i]\right\}\Big]. \qquad (4.27)\end{aligned}$$

In this action there are three kinds of terms. First we have the kinetic terms containing time derivatives of ${}^{\{n\}}h_{ij}$ and ${}^{\{n\}}\varphi$. Second, the terms in the summation symbol form a Hamiltonian that do not vanish on shell. These terms will generate the evolution equations for the gauge-invariant degrees of freedom. And finally, we have the nth order perturbations of the background constraints that, under a variation of the effective action (4.27) with respect to ${}^{\{n\}}B^i$ and ${}^{\{n\}}C$, give the constraints that must be obeyed by the nth order variables,

$$\Delta^n[\mathcal{H}] = 0, \qquad \Delta^n[\mathcal{H}_i] = 0. \qquad (4.28)$$

These constraints contain the gauge freedom of the system and are the generators of the perturbative gauge transformations. The perturbation of the lapse and the shift $\{\,{}^{\{n\}}C,\,{}^{\{n\}}B^i\}$ parameterize the gauge transformation. They are the equivalent of the four-dimensional gauge vectors ${}^{\{n\}}\xi^\mu$ in the covariant approach of the previous section. In principle one could explicitly obtain the nth order gauge transformation with this approach finding the formula equivalent to (4.9) in the canonical world. But this is not a trivial task and we leave it for future research.

For the rest of this thesis we will use the Hamiltonian approach only in the context of linear perturbation theory. For this case it is possible to go further in the explicit calculations. For instance, in the case $n = 1$ the effective action (4.27) takes the following simpler form,

$$\frac{1}{2}\Delta_1^2\mathcal{S} = \int dt\int_{\Sigma_t} dx^3\left\{p^{ij}h_{ij,t} + p\varphi_{,t} - C\Delta[\mathcal{H}] - B^i\Delta[\mathcal{H}_i] - \frac{\alpha}{2}\Delta_1^2[\mathcal{H}] - \frac{\beta^i}{2}\Delta_1^2[\mathcal{H}_i]\right\}. \quad (4.29)$$

The perturbation of the background constraints are given by,

$$\begin{aligned}\Delta[\mathcal{H}] &= \frac{1}{\mu_g}\left(\Pi_{ij} - \frac{1}{2}g_{ij}\Pi^l{}_l\right)\left(2p^{ij} + 2h^i{}_k\Pi^{kj} - \frac{1}{2}h^k{}_k\Pi^{ij}\right) \\ &+ \mu_g\left({}^{(3)}G^{ij}h_{ij} - D^iD^jh_{ij} + D^jD_jh^i{}_i\right) \\ &+ \frac{1}{4}h^k{}_k\left(-\frac{\Pi_\Phi^2}{\mu_g} + \mu_g\Phi_{,i}\Phi^{,i}\right) + \frac{p\Pi_\Phi}{\mu_g} + \mu_g\varphi_{,i}\Phi^{,i} - \frac{1}{2}\mu_g h^{ij}\Phi_{,i}\Phi_{,j}, \qquad (4.30)\\ \Delta[\mathcal{H}_i] &= -2D_k(h_{ij}\Pi^{jk} + g_{ij}p^{jk}) + \Pi^{jl}D_ih_{jl} + p\Phi_{,i} + \Pi_\Phi\varphi_{,i}, \qquad (4.31)\end{aligned}$$

$$
\begin{aligned}
\Delta_1^2[\mathcal{H}] &= \frac{1}{8\mu_g}\{8p^2 + 8\mu_g^2\,{}^{(3)}G^{ij}(h_{ij}h_k{}^k - 2h_i{}^k h_{jk}) + 16P_{ij}P^{ij} - 8P_i{}^i P_j{}^j \\
&+ 2h^{ij}\left[h^{kl}(8\Pi_{ik}\Pi_{jl} - 4\Pi_{ij}\Pi_{kl}) + h_{ij}\left(\Pi_\Phi^2 - 2\mu_g^{2\,(3)}R + 2\Pi_{kl}\Pi^{kl} - \Pi_k{}^k\Pi_l{}^l\right)\right. \\
&- 8\mu_g^2(D_jD_ih_k{}^k - D_jD_kh_i{}^k - D_kD_jh_i{}^k + D_kD^kh_{ij}) - 8\Pi_k{}^kP_{ij} + 32\Pi_i{}^kP_{jk} \\
&- \left. 8\,\Pi_{ij}P_k{}^k\right] + h_i{}^i\left[h_j{}^j\left(\Pi_\Phi^2 + 2\Pi_{kl}\Pi^{kl} - \Pi_k{}^k\Pi_l{}^l + 2\mu_g^{2\,(3)}R\right) + 8h^{jk}\left(\Pi_{jk}\Pi_l{}^l\right.\right. \\
&- \left.\left. 2\Pi_j{}^l\Pi_{kl}\right) - 8\mu_g^2(D_kD_jh^{jk} - D_kD^kh_j{}^j) - 8\Pi_\Phi p - 16\Pi^{jk}P_{jk} + 8\Pi_j{}^jP_k{}^k\right] \\
&+ \mu_g^2[8D_i\varphi D^i\varphi + h_j{}^jD_i\Phi(8D^i\varphi + h_k{}^kD^i\Phi) + 2h_{jk}D^i\Phi(4h_i{}^kD^j\Phi - h^{jk}D_i\Phi) \\
&+ 4D_jh_k{}^kD^jh_i{}^i - 4h_{ij}(4D^i\varphi + h_k{}^kD^i\Phi)D^j\Phi + 16(D_ih^{ij} - D^jh_i{}^i)D_kh_j{}^k \\
&+ 4(2D_jh_{ik} - 3D_kh_{ij})D^kh^{ij}]\}, \qquad (4.32) \\
\Delta_1^2[\mathcal{H}_i] &= 2p\varphi_{,i} + 2p^{jk}(D_ih_{jk} - 2D_kh_{ij}) - 4h_{ij}D_kp^{jk}, \qquad (4.33)
\end{aligned}
$$

where we have not written explicitly the perturbative order of each object since all of them are of first order. In this linearized situation, a gauge transformation of a first-order functional $F(h_{ij}, p^{ij})$ is given by,

$$
F(h_{ij}, p^{ij}) \longrightarrow \left\{F(h_{ij}, p^{ij}), \int_{\Sigma_t} d^3x\left(C\Delta[\mathcal{H}] + B^i\Delta[\mathcal{H}_i]\right)\right\}. \qquad (4.34)
$$

Comparing this last expression with the formula (4.25) that accounts for the infinitesimal gauge transformation of the background geometry, we can realize again that the origin of both gauge freedoms is different since the generators and parameters of each transformation are not the same and could be independently chosen.

4.2.2 Gauge invariants

In order to explain how to construct gauge invariants in this Hamiltonian setting, we will discuss a general example without considering the perturbations of the matter field (φ, p). The addition of these perturbations to the general procedure is straightforward. Let us suppose that the initial perturbative variables (h_{ij}, p^{ij}) are decomposed into some coordinates or, as we will do in spherical symmetry, in spherical harmonics. Then we will have six pairs of conjugated variables (h_I, p_I), for $I = 1, \ldots, 6$. The idea is to make a canonical transformation from this initial set of perturbative variables to another new set $(\check{h}_I, \check{p}_I)$ with the requirement that,

$$
\Delta[\mathcal{H}_J] = \check{h}_J. \qquad (4.35)
$$

The subindex takes the values $J = 1, 2, 3, 4$ and we have defined, in order to have a compact notation for this section, $\Delta[\mathcal{H}_4] \equiv \Delta[\mathcal{H}]$. In this way, the variables $\check{h}_J$ will be gauge invariant but constrained to vanish by equations (4.28). Whereas their conjugated momenta $\check{p}_J$ will be pure gauge, so that their initial value can be arbitrarily chosen. In addition, the evolution equations for these momenta will contain the free functions B^i and C, since they will be obtained by variation of the action with respect to the linearized constraints (4.35). This permits us to choose also the time derivatives of the pure-gauge momenta $\check{p}_J$. Then, if we are able to follow this procedure completely, we will have isolated the non-physical information (gauge as well as constraints) in the four pairs of conjugated variables $(\check{h}_J, \check{p}_J)$. The remaining variables $[(\check{h}_5, \check{p}_5), (\check{h}_6, \check{p}_6)]$ will be the so-called master variables, which are gauge invariant and obey non-constrained equations of motion. These master variables are the two degrees of freedom of the gravitational wave and contain all physical information of the problem. The initial variables (h_I, p_I) can be reconstructed in terms of the master variables in any gauge just by applying the inverse of the canonical transformation. We will see that examples of such equations are the RW and Zerilli equations [101].

The situation in the linearized theory is simpler than in the general non-linear case, but still only highly symmetric background scenarios allow the construction of gauge-invariant algebraic combinations of perturbations and their derivatives containing the physical information in the linearized approximation. As we will show in Chapters 7 and 8, one of such background scenarios are spherically symmetric spacetimes with a massless scalar matter field.

The gauge invariants constructed with this procedure are of the same kind as the ones explained in the covariant approach. The canonical transformations we make to convert four of the new variables $(\check{h}_J, \check{p}_J)$ into the constraints $(\Delta[\mathcal{H}], \Delta[\mathcal{H}_i])$, are equivalent to the gauge transformation we made in the previous section from a generic to the particular rigid gauge. But there are two great advantages in working in this Hamiltonian setting. On the one hand, the constraints serve as a guideline to choose the most convenient transformations, whereas in the covariant approach one has to guess the rigid gauge. On the other hand, in the canonical approach, apart from constructing gauge invariants, the constraints (as equations of motion) are also automatically solved and one ends up with useful master equations.

In order to clarify the relation between the two approaches let us compare and count the different degrees of freedom. As usual, we define a degree of freedom as a function that obeys a second-order in time equation of motion. In the covariant approach we start with ten degrees of freedom contained in the symmetric tensor $h_{\mu\nu}$. Once we removed the four gauge dependences, that are encoded in the vector ξ^μ, by constructing the gauge invariant objects $\mathcal{K}_{\mu\nu}$, we have six degrees of freedom. There are also six gauge-invariant degrees of freedom in the canonical approach contained in twelve variables; namely, the two canonical pairs $[(\check{h}_5, \check{p}_5), (\check{h}_6, \check{p}_6)]$, the four constraints $\check{h}_J$, the Lagrange multipliers (C, B^i) and their conjugated (vanishing) momenta. The advantage of the Hamiltonian framework is that it also provides the dynamical role of each object. More precisely, (C, B^i) are non-dynamical functions since their momenta vanish, $\check{h}_J$ are constrained to vanish and the canonical pairs $[(\check{h}_5, \check{p}_5), (\check{h}_6, \check{p}_6)]$ describe the two gauge-invariant physical degrees of freedom.

Part II

Spherical symmetry

Chapter 5

Spherical spacetimes

The content of this chapter is well-known and we have included it to fix the notation and make this thesis self-contained. It is divided into two main sections that impose spherical symmetry to the spacetime under consideration, after performing one of two different splittings. The first section deals with a $2+2$ block diagonal splitting of the metric, which separates it into the trivial geometry of the two-sphere and that of its orthogonal part. This splitting is well suited to spherical symmetry and will be used in Part IV of this thesis to generalize the Gerlach and Sengupta formalism to second order. The second section follows the $3+1$ decomposition required by the Hamiltonian or canonical approach to General Relativity that divides the metric into a time and a spatial part. This splitting will be used in Part III to find the master equations for the perturbations of a spherical but dynamical background spacetime. For future convenience, we will analyze different matter models in each splitting: in the $2+2$ splitting we will deal with a vacuum as well as a perfect fluid matter content. Whereas in the $3+1$ splitting we develop the equations for a massless scalar field. The notation will overlap in some cases. For instance, the vector u will be used to denote both the velocity of the perfect fluid and the orthogonal vector to constant time slices in the $3+1$ decomposition. But since both (covariant and canonical) approaches will not be mixed, the meaning of each object should be always clear form the context.

5.1 2+2 splitting of the spacetime

In the following chapters we will consider spherically symmetric spacetimes. This means that its isometry group contains a subgroup isomorphic to the group SO(3), and the orbits of this subgroup (i.e., the sets of points resulting from the action of the subgroup on a given point) are two-dimensional spheres. So those isometries may then be physically interpreted as rotations.

Because of spherical symmetry, M can globally be decomposed as $M^2 \times S^2$, where M^2 is a two-dimensional Lorentzian manifold with boundary, to be associated with the centre of symmetry, and S^2 is the two-sphere. Following Gerlach and Sengupta we choose a coordinate system $x^\mu = (x^A, x^a)$ adapted to this decomposition. Uppercase Latin indices denote arbitrary coordinates on the manifold M^2, x^A, whereas lowercase Latin indices label the spherical coordinates on the sphere, $x^a = (x^2 \equiv \theta, x^3 \equiv \phi)$. The idea is that, in spite of fixing coordinates on the sphere, we are going to work covariantly in the manifold M^2.

The four dimensional metric $g_{\mu\nu}$ will induce a metric on each orbit two-sphere, which, because of the rotational symmetry, must be a multiple of the round (unit Gaussian curvature) metric γ_{ab} and will be completely characterized by the total area $A(x^A)$ of the two-sphere at the point x^A. We introduce the nonnegative scalar $r(x^A)$ on M^2, defined by

$$r^2(x^A) = \frac{A(x^A)}{4\pi}. \tag{5.1}$$

Thus, the induced metric on each sphere is

$$r^2(x^A)\gamma_{ab}dx^a dx^b = r^2(x^A)(d\theta^2 + \sin\theta^2 d\phi^2). \tag{5.2}$$

In this way, the four-dimensional metric $g_{\mu\nu}$ and the stress-energy tensor of the background $t_{\mu\nu}$ can always be written as

$$g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu = g_{AB}(x^D)dx^A dx^B + r^2(x^D)\,\gamma_{ab}(x^d)dx^a dx^b, \tag{5.3}$$

$$t_{\mu\nu}(x^\lambda)dx^\mu dx^\nu = t_{AB}(x^D)dx^A dx^B + \frac{1}{2}r^2(x^D)\,Q(x^D)\,\gamma_{ab}(x^d)dx^a dx^b, \tag{5.4}$$

where Q is a scalar on the manifold M^2 and g_{AB} is an arbitrary Lorentzian metric tensor on M^2. There is no inconsistency in using the same symbol g for both the metrics on M^2 and M because they indeed coincide on M^2. However, the same does not happen with the round metric, since

$$g_{ab} = r^2\gamma_{ab} \qquad \text{and} \qquad g^{ab} = \frac{1}{r^2}\gamma^{ab}. \tag{5.5}$$

Using this decomposition it is possible to express all four-dimensional curvature tensors in terms of the curvature tensors corresponding to g and γ and derivatives of the scalar r. In doing so, it is useful to define a vector field

$$v_A \equiv \frac{r_{,A}}{r} = (\ln r)_{,A} \tag{5.6}$$

to avoid working with explicit logarithms of r.

The notation used for the covariant derivatives associated to each metric is

$$g_{\mu\nu;\lambda} = 0, \qquad g_{AB|D} = 0, \qquad \gamma_{ab:d} = 0. \tag{5.7}$$

The Levi-Civita tensors ϵ obey the conventions $\epsilon_{ABcd} = r\epsilon_{AB}\epsilon_{cd}$ with ϵ_{01} and ϵ_{23} being positive on M^2 and S^2, respectively. The non-null components of the four dimensional Riemann tensor ${}^{(4)}R_{\mu\nu\rho\lambda}$ are

$$\begin{aligned} {}^{(4)}R_{ABCD} &= \frac{1}{2}(g_{AC}g_{BD} - g_{AD}g_{BC})R, & (5.8)\\ {}^{(4)}R_{AbCd} &= -(v_A v_C + v_{A|C})g_{bd}, & (5.9)\\ {}^{(4)}R_{abcd} &= \left(\frac{1}{r^2} - v_A v^A\right)(g_{ac}g_{bd} - g_{ad}g_{bc}), & (5.10) \end{aligned}$$

where R is the Ricci scalar of g_{AB}.

Taking the appropriate trace we obtain the Ricci tensor that, as the metric, is diagonal by boxes due to the spherical symmetry:

$$\begin{aligned} {}^{(4)}R_{AB} &= \frac{1}{2}Rg_{AB} - 2(v_A v_B + v_{A|B}), & (5.11)\\ {}^{(4)}R_{ab} &= \left(\frac{1}{r^2} - 2v^A v_A + v^A{}_{|A}\right) g_{ab}. & (5.12) \end{aligned}$$

The four-dimensional Ricci scalar is given by

$${}^{(4)}R = R + \frac{2}{r^2} - 6v_A v^A - 4v^A{}_{|A}. \tag{5.13}$$

The Einstein equations for a general spherically symmetric spacetime are

$$\begin{aligned} {}^{(4)}G_{AB} = \left(-\frac{1}{r^2} + 3v^D v_D + 2v^D{}_{|D}\right) g_{AB} - 2(v_A v_B + v_{A|B}) &= 8\pi t_{AB}, & (5.14)\\ {}^{(4)}G_a{}^a = -R + 2v^A v_A + 2v^A{}_{|A} &= 8\pi Q. & (5.15) \end{aligned}$$

The energy-momentum conservation leads to the nontrivial relation

$$Qv_A = \frac{1}{r^2}\left(r^2 t_{AB}\right)^{|B}, \tag{5.16}$$

which can be used to see that (5.15) is a consequence of (5.14). So, all the information of the Einstein equations for a spherical spacetime is contained in (5.14).

It is interesting to study the form of the Weyl tensor in spherical symmetry. If we define

$$\mathcal{U} = -\frac{1}{6}\left(R + \frac{2}{r^2} + 2v^A{}_{|A}\right), \tag{5.17}$$

the non-vanishing components of the four dimensional Weyl tensor are proportional to this scalar,

$$
\begin{aligned}
{}^{(4)}W_{ABCD} &= (g_{AD}g_{BC} - g_{AC}g_{BD})\mathcal{U}, && (5.18)\\
{}^{(4)}W_{AbCd} &= \frac{1}{2}g_{AC}g_{bd}\mathcal{U}, && (5.19)\\
{}^{(4)}W_{abcd} &= (g_{ad}g_{bc} - g_{ac}g_{bd})\mathcal{U}. && (5.20)
\end{aligned}
$$

The object $\mathcal{U}$ is essentially the Newman-Penrose scalar Ψ_2, the only nonzero Newman-Penrose scalar in a spherical spacetime.

5.1.1 Frames on M^2 manifold

The covariant notation on M^2 is very useful to perform abstract calculations, but when we want to implement the formalism numerically it turns out necessary to fix a coordinate system. An intermediate stage between those two cases would be to work with frames. Depending on the physical problem that we are dealing with, it may happen that there exists a preferred direction which fixes the frame in a natural way, for example the velocity field of a fluid. Nevertheless, in the general case we can only use the derivative of the scalar r to define a frame.

We define the orthonormal frame of vectors

$$r^A = \frac{r^{|A}}{f} \qquad \text{and} \qquad t^A = -\epsilon^{AB} r_B, \tag{5.21}$$

where $f = \sqrt{g^{AB} r_{|A} r_{|B}}$, so that

$$r^A r_A = 1, \qquad t^A t_A = -1, \qquad r^A t_A = 0. \tag{5.22}$$

This frame has been chosen so that r^A points to the exterior (larger area). Thus, its component r^1 is positive. On the other hand, the timelike vector field t^A points to the future, with t^0 being positive.

The Hawking mass [147] in spherical symmetry (Misner-Sharp mass [148]) is defined as

$$m(x^A) \equiv \frac{r}{2}(1 - f^2), \tag{5.23}$$

which measures the mass in the interior of the sphere at the point x^A. In particular, the condition $f = 0$, which implies that the Misner-Sharp mass is $\frac{r}{2}$, indicates the appearance of an apparent horizon. The spatial derivative of m is given by

$$r^A m_{|A} = 4\pi r^2 t_{AB} t^A t^B, \tag{5.24}$$

where t_{AB} is the M^2 part of the stress-energy tensor (5.4). This relation shows why the mass m is constant in vacuo.

We can expand the metric and the Levi-Civita tensor as

$$g_{AB} = -t_A t_B + r_A r_B, \tag{5.25}$$
$$\epsilon_{AB} = r_A t_B - r_B t_A. \tag{5.26}$$

The covariant derivative of any tensor field T will be given in terms of derivatives along the frame vectors,

$$T_{|A} = r_A r^B T_{|B} - t_A t^B T_{|B}. \tag{5.27}$$

From the Einstein equation (5.14) it is possible to obtain the covariant derivative of the vector v_A in terms of the vector itself and the stress-energy tensor,

$$v_{A|B} = \frac{1}{2}\left(\frac{1}{r^2} - v_D v^D + 8\pi t_D{}^D\right) g_{AB} - v_A v_B - 8\pi t_{AB}. \tag{5.28}$$

Using this expression it is easy to derive a formula for the covariant derivative of the frame vector r_A,

$$r_{A|B} = -\frac{1}{2} r_A (\ln f^2)_{|B} + \frac{1}{2rf}(1 - f^2 + 8\pi r^2 t_D{}^D) g_{AB} - \frac{8\pi r}{f} t_{AB}. \tag{5.29}$$

It is interesting to note that the traces of the last two equations do not depend explicitly on the energy-matter content,

$$v^A{}_{|A} = \frac{1}{r^2} - 2 v_A v^A, \tag{5.30}$$
$$r^A{}_{|A} = -\frac{1}{2} r^A (\ln f^2)_{|A} + \frac{1}{rf}(1 - f^2). \tag{5.31}$$

In order to obtain a similar formula for the other frame vector t_A, it suffices to take into account that

$$t_{A|B} \equiv (-\epsilon_A{}^D r_D)_{|B} = -\epsilon_A{}^D r_{D|B}. \tag{5.32}$$

We will use the other Einstein equation (5.15) to obtain the Ricci scalar of the manifold M^2,

$$R = \frac{2}{r^2}(1 - f^2) - 8\pi Q, \tag{5.33}$$

which can also be written as

$$R = \frac{2}{r}(f r^A)_{|A} - 8\pi Q. \tag{5.34}$$

As M^2 is a two-dimensional manifold, all its curvature freedom is contained in the Ricci scalar. The Riemann and Ricci tensors are proportional to it,

$$R_{ABCD} = \frac{1}{2}(g_{AC}g_{BD} - g_{AD}g_{BC})R, \tag{5.35}$$

$$R_{AB} = \frac{1}{2}Rg_{AB}. \tag{5.36}$$

Therefore, we have totally characterized the geometry of M^2 in terms of the scalar f and the components Q and t_{AB} of the stress-energy tensor.

5.1.2 Vacuum

According to Birkhoff's theorem [149], any spherically symmetric solution to the vacuum field equations (${}^{(4)}R_{\mu\nu} = 0$) must be the Schwarzschild metric. This theorem has a straightforward generalization to the Einstein-Maxwell system, that is, the unique solution to the vacuum Einstein-Maxwell equations is the Reissner-Nordström metric.

For our purposes, there are two important consequences of this theorem. On the one hand, if we have a spherically symmetric vacuum spacetime we know its exact form. On the other hand, as the application of this theorem is local, we can have some region where $t_{\mu\nu}$ is not zero but respects the spherical symmetry; then, the solution outside that region will be Reissner-Nordström.

When considering Reissner-Nordström the scalar f introduced in the last section is

$$f = \sqrt{1 - \frac{2M}{r} + \frac{q^2}{r^2}}, \tag{5.37}$$

where M and q are the central mass and charge respectively. Whereas the Misner-Sharp mass (5.23) is a combination between these both quantities,

$$m = M - \frac{q^2}{2r}. \tag{5.38}$$

Note that in the particular case of Schwarzschild ($q = 0$) the mass m equals the constant M.

Using formula (5.29) we obtain the covariant derivative of the frame vectors:

$$r_{A|B} = -\frac{M}{r^2 f}t_A t_B + \frac{q^2}{2r^3 f}(t_A t_B + r_A r_B), \tag{5.39}$$

$$t_{A|B} = -\frac{M}{r^2 f}r_A t_B + \frac{q^2}{2r^3 f}(r_A t_B + r_B t_A). \tag{5.40}$$

If we calculate the derivatives of the vectors fr_A and ft_A:

$$(fr_A)_{|B} = \left(\frac{M}{r^2} - \frac{q^2}{2r^3}\right) g_{AB}, \tag{5.41}$$

$$(ft_A)_{|B} = -\left(\frac{M}{r^2} - \frac{q^2}{2r^3}\right) \epsilon_{AB}, \tag{5.42}$$

we can see that the first one is explicitly symmetric, as it must be because it is the second covariant derivative of the scalar field r, and the second one is a Killing vector field. In fact, it is the time-like Killing vector field that renders our spacetime static.

It is also obvious how to compute the Ricci scalar from equation (5.34):

$$R = \frac{2}{r^3}(2Mr - q^2). \tag{5.43}$$

5.1.3 Perfect fluid

In this section we introduce the notation for the background objects when considering a spherical spacetime with a perfect fluid. We will use the same notation as in [55] since the formalism developed in that reference for linear perturbation theory will be generalized to second-order in Chapter 13.

The energy-momentum tensor for a perfect fluid with four-velocity u^μ, total energy density ρ and pressure p is given by

$$t_{\mu\nu} \equiv (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \tag{5.44}$$

Comparing with equation (5.4), this makes $Q = 2p$. We will consider a generic equation of state of the form $p = p(\rho, s)$ with s being the entropy per particle, and define the adiabatic speed of sound and another thermodynamic fluid property

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s, \qquad C \equiv \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho. \tag{5.45}$$

There is no need to consider other thermodynamical quantities.

In spherical symmetry the four-velocity of the fluid takes the form $u_\mu = (u_A, 0)$. This vector defines a unique outwards pointing spacelike unit vector $n_A \equiv -\epsilon_{AB} u^B$ on M^2, with ϵ_{AB} being the antisymmetric tensor on M^2. These two vectors define an orthonormal basis on this manifold, which can be used to decompose all geometrical objects, like

$$g_{AB} = -u_A u_B + n_A n_B, \qquad \epsilon_{AB} = n_A u_B - u_A n_B. \tag{5.46}$$

Using this decomposition, the tensorial component of the energy-momentum tensor (5.4) takes the following form

$$t_{AB} = \rho u_A u_B + p n_A n_B. \tag{5.47}$$

While dealing with the perfect fluid background, that is, in this subsection and in Chapter 13, we will use the following frame derivatives acting on any scalar function ζ,

$$\dot{\zeta} \equiv u^A \zeta_{|A}, \qquad \zeta' \equiv n^A \zeta_{|A}. \tag{5.48}$$

It is not difficult to find that these derivatives obey the following commutation relation,

$$(\dot{\zeta})' - (\zeta')^{\cdot} = \mu\zeta' - \nu\dot{\zeta}, \tag{5.49}$$

where we have defined

$$\mu \equiv u^A{}_{|A}, \qquad \nu \equiv n^A{}_{|A}. \tag{5.50}$$

These "structure" functions are the components of the covariant derivatives of the frame vectors in their own frame,

$$u_{A|B} = n_A(n_B\mu - u_B\nu), \quad n_{A|B} = u_A(n_B\mu - u_B\nu). \tag{5.51}$$

In order to deal only with scalar quantities, we define the following background scalars,

$$\Omega \equiv \ln\rho, \qquad U \equiv u^A v_A, \qquad W \equiv n^A v_A. \tag{5.52}$$

In this notation, the four independent Einstein equations that exist in spherical symmetry, can be written in the following way,

$$U' = W(\mu - U), \tag{5.53}$$

$$\dot{W} = U(\nu - W), \tag{5.54}$$

$$W' = -4\pi\rho - W^2 + U\mu + \frac{m}{r^3}, \tag{5.55}$$

$$\dot{U} = -4\pi p - U^2 + W\nu - \frac{m}{r^3}, \tag{5.56}$$

where in this case, the Misner-Sharp mass (5.23) takes the following form,

$$m = \frac{r}{2}\left[1 + r^2(U^2 - W^2)\right]. \tag{5.57}$$

The system of equations for the background is closed with the equations of motion of the perfect fluid. Conservation of the energy-momentum tensor is equivalent to energy conservation and the Euler equation, respectively,

$$\dot{\Omega} + \left(1 + \frac{p}{\rho}\right)(2U + \mu) = 0, \tag{5.58}$$

$$c_s^2\Omega' + Cs' + \left(1 + \frac{p}{\rho}\right)\nu = 0. \tag{5.59}$$

Finally, a perfect fluid does not dissipate energy and hence the entropy of each fluid element is conserved,

$$\dot{s} = 0. \tag{5.60}$$

Equations (5.53-5.60) fully describe the dynamics of this background spacetime. In Chapter 13, they will be used to simplify the background dependent coefficients that will appear in the perturbed equations.

5.2 3+1 splitting of the spacetime

In order to make explicitly the calculations in the Hamiltonian framework, we define the generic coordinates ($x^0 \equiv t, x^1 \equiv \rho$) on the manifold M^2. Thus, we can explicitly write the background spatial three-dimensional metric as

$$g_{ij}dx^i dx^j = a^2(t,\rho)d\rho^2 + r^2(t,\rho)d\Omega^2. \tag{5.61}$$

The assumption of spherical symmetry forces the lapse to depend solely on (t,ρ) coordinates, $\alpha = \alpha(t,\rho)$, and the shift vector not to have angular components $\beta^i = (\beta(t,\rho),0,0)$. Therefore, the decomposition for the four-dimensional metric is given by

$$g_{\mu\nu}dx^\mu dx^\nu = (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta dt d\rho + g_{ij}dx^i dx^j \tag{5.62}$$

$$= -\alpha^2 dt^2 + a^2(d\rho + \beta dt)^2 + r^2 d\Omega^2, \tag{5.63}$$

which takes the following matricial form,

$$g_{AB} = \begin{pmatrix} -\alpha^2 + a^2\beta^2 & a^2\beta \\ a^2\beta & a^2 \end{pmatrix}, \qquad g^{AB} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta \\ \alpha^{-2}\beta & a^{-2} - \alpha^{-2}\beta^2 \end{pmatrix}. \tag{5.64}$$

The normal vector to the surfaces of constant time Σ_t is $u_\mu = (-\alpha,0,0,0)$ or $u^\mu = \alpha^{-1}(1,-\beta,0,0)$. Its orthogonal, radial vector is $n^\mu = (0,a^{-1},0,0)$ or $n_\mu = a(\beta,1,0,0)$. Hence, in terms of this frame of vectors, the metric and the volume element of M^2 are given by

$$g_{AB} = -u_A u_B + n_A n_B, \qquad \epsilon_{AB} = n_A u_B - u_A n_B. \tag{5.65}$$

In order to deal with more geometrical objects, while working in the Hamiltonian framework (in this section and in Part III of this thesis) we will use the following frame derivatives that act on any scalar field ζ,

$$\dot{\zeta} = u^\mu \zeta_{,\mu} = \frac{\zeta_{,t} - \beta\zeta_{,\rho}}{\alpha}, \qquad \zeta' = n^\mu \zeta_{,\mu} = \frac{\zeta_{,\rho}}{a}. \tag{5.66}$$

5.2.1 Scalar field

We now derive the equations of motion for the case of a massless scalar field, that will be later used to simplify the coefficients of the equations for the perturbations. It is convenient to define the following momentum-like variables, which have a definite tensorial character with respect to changes of the ρ coordinate,

$$\Pi_1 \equiv \frac{a^2\Pi^{\rho\rho}}{\mu_g}, \qquad \Pi_2 \equiv \frac{2r^2\Pi^{\theta\theta}}{\mu_g}, \qquad \Pi_3 \equiv \frac{\Pi_\Phi}{\mu_g}. \tag{5.67}$$

We can write the constraints (3.48-3.49) in terms of these spherical variables,

$$\frac{\mathcal{H}}{\mu_g} = \Pi_1\left(\frac{\Pi_1}{2} - \Pi_2\right) - {}^{(3)}R + \frac{1}{2}\left({\Pi_3}^2 + \Phi'^2\right) = 0, \tag{5.68}$$

$$\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g} = -\frac{2}{r^2}(r^2\Pi_1)' + \frac{2r'}{r}\Pi_2 + \Pi_3\Phi' = 0, \tag{5.69}$$

so that the action (3.47), once integrated in angles, is

$$\frac{1}{4\pi}\mathcal{S} = \int dt \int d\rho\, ar^2 \left[2\Pi_1\frac{a_{,t}}{a} + 2\Pi_2\frac{r_{,t}}{r} + \Pi_3\Phi_{,t} - \alpha\frac{\mathcal{H}}{\mu_g} - \beta\frac{\mathcal{H}_\rho}{\mu_g}\right]. \tag{5.70}$$

The evolution equations can be obtained by a simple variation with respect to different variables:

$$\frac{1}{\alpha}\left[a_{,t} - (\beta a)_{,\rho}\right] = \frac{a}{2}\left(\Pi_1 - \Pi_2\right), \tag{5.71}$$

$$\frac{1}{\alpha}(r_{,t} - \beta r_{,\rho}) = -\frac{r}{2}\Pi_1, \tag{5.72}$$

$$\frac{1}{\alpha}(\Phi_{,t} - \beta\Phi_{,\rho}) = \Pi_3 \tag{5.73}$$

$$\frac{1}{\alpha}\left(\Pi_{1,t} - \beta\Pi_{1,\rho}\right) = \frac{3\Pi_1^2}{4} + \frac{1}{r^2} - \frac{r'}{r}\frac{(\alpha^2 r)'}{\alpha^2 r} + \frac{1}{4}\left(\Pi_3^2 + \Phi'^2\right), \tag{5.74}$$

$$\frac{1}{\alpha}\left(\Pi_{2,t} - \beta\Pi_{2,\rho}\right) = \frac{1}{2}(\Pi_1^2 + \Pi_2^2 - \Pi_1\Pi_2) + \frac{2\alpha' r'}{\alpha r} - \frac{2(\alpha r)''}{\alpha r} + \frac{1}{2}\left(\Pi_3^2 - \Phi'^2\right), \tag{5.75}$$

$$\frac{1}{\alpha}(\Pi_{3,t} - \beta\Pi_{3,\rho}) = \frac{\Pi_3(\Pi_1 + \Pi_2)}{2} + \frac{(\alpha r^2\Phi')'}{\alpha r^2}. \tag{5.76}$$

The restriction to vacuum, choosing Schwarzschild coordinates (t, $r = \rho$), is given by $\Pi^{\mu\nu} = 0$, ${}^{(3)}R = 0$ and $\Phi = 0$, that is, $\Pi_1 = \Pi_2 = \Pi_3 = \Phi = 0$. This simplifies the previous equations severely. In particular, the constraints (5.68) and (5.69) are then trivially obeyed. This was the case for which Moncrief [101] developed the analysis of constructing gauge invariants and finding the master equations. In Part III of this thesis we will generalize his study to the dynamical case.

Chapter 6

Tensor spherical harmonics

Expansions in spherical harmonic are frequently used in many fields of physics. When dealing with scalar fields (Newton's potential in Newtonian gravity, wave-function in Quantum Mechanics...), one can use the scalar spherical harmonics Y_l^m. But when tensor fields are involved, as it happens in General Relativity, one must use tensor spherical harmonics. A vast literature exists about these harmonics, approaching them from different points of view and using different conventions. In this chapter we will briefly review the most common tensor spherical harmonics used in General Relativity and the relationship between them. Special attention will be paid to the Regge-Wheeler-Zerilli spherical harmonics because we will employ them in the rest of the work. We have chosen these harmonics because they have proven to be highly convenient to study gravitational radiation, since they happen to be tangent to the sphere and have a well-defined parity. Other harmonics will be introduced for completeness and as a complementary tool in certain calculations, like in the case of the pure-spin harmonics when obtaining a product formula for the RWZ harmonics.

6.1 Regge-Wheeler-Zerilli harmonics

The scalar spherical harmonics Y_l^m are defined as the eigenfunctions of the Laplacian operator

$$\gamma^{ab}\nabla_a\nabla_b Y_l^m = -l(l+1)Y_l^m \tag{6.1}$$

where γ^{ab} is the inverse of the round metric on S^2 and l is a positive integer. This label l has a geometrical, coordinate-independent, meaning as it is defined by a tensorial equation. After the choice of a fixed z-axis, another label m is provided:

$$\partial_\phi Y_l^m = imY_l^m. \tag{6.2}$$

Each linearly independent spherical harmonic will be completely characterized by the pair $(l,\ m)$. The integer m takes the range $l \geq |m|$, hence for a fixed l there exist $(2l+1)$ independent spherical harmonics.

These harmonics are normalized so that

$$\int d\Omega\ Y_{l'}^{m'} Y_l^{m*} = \delta_{l'l}\,\delta_{m'm}, \tag{6.3}$$

with $*$ denoting complex conjugation and $d\Omega$ being the area element on S^2 ($d\Omega = \sin\theta\, d\theta\, d\phi$).

Some of their most important properties are their behaviour under a parity transformation,

$$Y_l^m(\pi-\theta, \pi+\phi) = (-1)^l Y_l^m(\theta,\phi), \tag{6.4}$$

and under complex conjugation,

$$Y_l^{m*}(\theta,\phi) = Y_l^{-m}(\theta,\phi). \tag{6.5}$$

They provide an orthonormal basis of functions on the sphere S^2, namely, any function $\zeta(\theta,\phi)$ can be expanded as

$$\zeta(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \zeta_l^m Y_l^m(\theta,\phi), \tag{6.6}$$

where the coefficients ζ_l^m are defined as

$$\zeta_l^m = \int d\Omega\ \zeta(\theta,\phi) Y_l^{m*}. \tag{6.7}$$

Starting with these scalar harmonics, Regge and Wheeler [56] defined a basis for the vector and 2-symmetric tensor fields on the sphere as follows.

The basis for the vectors is formed by the vector fields

$$\{Y_l^m{}_{:a}, S_l^m{}_a \equiv \epsilon_{ab}\gamma^{bc} Y_l^m{}_{:c}\}. \tag{6.8}$$

By definition, these vectors are orthogonal and hence independent at every point of the sphere. For the case $l=0$ both vectors are zero because this case respects the spherical symmetry and any non-vanishing vector field tangent to the sphere would break that symmetry.

Inherited from the scalar harmonics, the vectors of this basis have a well defined parity under inversion of axes:

$$Y_l^m{}_{:a}(\pi-\theta,\pi+\phi) = (-1)^l Y_l^m{}_{:a}(\theta,\phi), \tag{6.9}$$

$$S_l^m{}_a(\pi-\theta,\pi+\phi) = (-1)^{l+1} S_l^m{}_a(\theta,\phi). \tag{6.10}$$

This property can easily be checked taking into account that the parity of the metric γ_{ab} and of the Levi-Civita tensor ϵ_{ab} are, respectively, (+1) and (-1), and realizing that covariant derivatives do not change the parity. In this way, it is usual to separate harmonics with momentum l into two families of different *polarity*: on one hand, those with parity $(-1)^l$ will be called *polar*, and on the other hand, those which change sign as $(-1)^{l+1}$ will be called *axial*.

Regge and Wheeler used the adjectives even and odd instead of polar and axial, respectively. Nonetheless, this terminology could be misleading, since it is common to understand odd (even) parity the property of changing (preserving) sign under a parity transformation.

Alternatively, the polar and axial parities are sometimes called electric and magnetic, respectively, because of the properties

$$\epsilon^{ab}(Y_l^m{}_{:a})_{:b} = 0 \longrightarrow \text{irrotational vector,} \tag{6.11}$$

$$\gamma^{ab}(S_l^m{}_a)_{:b} = 0 \longrightarrow \text{solenoidal vector.} \tag{6.12}$$

One must not confuse parity and polarity: whereas all equations must have a well-defined parity at any order in perturbation theory, polarity is only useful in the first-order theory because products of harmonics couple the two polarities, as we will see.

Using the introduced basis, any vector $w_a(\theta,\phi)$ on the sphere can be decomposed as

$$w_a(\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}(\mathcal{W}_l^m Y_l^m{}_{:a} + w_l^m S_l^m{}_a). \tag{6.13}$$

From now on, uppercase and lowercase letters will be employed to denote the respective polar and axial parts of the decomposition. The normalization of these vector harmonics is

$$\int d\Omega\; Y_{l'}^{m'}{}_{:a}\gamma^{ab}Y_l^m{}_{:b}{}^* = l(l+1)\,\delta_{l'l}\,\delta_{m'm}, \tag{6.14}$$

$$\int d\Omega\; S_{l'}^{m'}{}_{a}\gamma^{ab}S_l^m{}_{b}{}^* = l(l+1)\,\delta_{l'l}\,\delta_{m'm}. \tag{6.15}$$

Thus, the coefficients of the harmonic decomposition (6.13) are defined as

$$\mathcal{W}_l^m = \frac{1}{l(l+1)}\int d\Omega\; w_a\gamma^{ab}Y_l^m{}_{:b}{}^*, \tag{6.16}$$

$$w_l^m = \frac{1}{l(l+1)}\int d\Omega\; w_a\gamma^{ab}S_l^m{}_{b}{}^*. \tag{6.17}$$

The basis that Regge and Wheeler used for 2-tensor symmetric fields on the sphere is formed by

$$\{Y_l^m{}_{:ab}, Y_l^m\gamma_{ab}, X_l^m{}_{ab} \equiv \frac{1}{2}(S_l^m{}_{a:b} + S_l^m{}_{b:a})\}. \tag{6.18}$$

The last harmonic is different from that used by Gerlach and Sengupta by a factor of 2. The problem with this basis is that for the case $l = 1$ we have $Y_1^m{}_{:ab} = -Y_1^m \gamma_{ab}$, so that they are linearly dependent. In order to avoid this complication, Zerilli introduced the harmonic

$$Z_l^m{}_{ab} \equiv (Y_l^m{}_{:ab})^{\rm TF} = Y_l^m{}_{:ab} + \frac{l(l+1)}{2}\gamma_{ab}Y_l^m, \tag{6.19}$$

where the superscript TF means the trace-free part. Therefore, the basis for the 2-symmetric tensor fields remain

$$\{Z_l^m{}_{ab}, Y_l^m\gamma_{ab}, X_l^m{}_{ab}\}, \tag{6.20}$$

where $Z_l^m{}_{ab}$ and $X_l^m{}_{ab}$ are trace-free, and zero for the cases $l = 0, 1$. Note that, while $Z_l^m{}_{ab}$ and $Y_l^m\gamma_{ab}$ are polar, $X_l^m{}_{ab}$ is axial. The normalization of these harmonics is:

$$\int d\Omega Z_l^m{}_{ab}\gamma^{ac}\gamma^{bd}Z_{l'}^{m'}{}_{cd}{}^* = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'}, \tag{6.21}$$

$$\int d\Omega X_l^m{}_{ab}\gamma^{ac}\gamma^{bd}X_{l'}^{m'}{}_{cd}{}^* = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'}. \tag{6.22}$$

Therefore any symmetric 2-tensor $T_{ab}(\theta,\phi)$ can be expanded as

$$T_{ab}(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{T}_l^m\gamma_{ab}Y_l^m + \sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(T_l^m Z_l^m{}_{ab} + t_l^m X_l^m{}_{ab}\right), \tag{6.23}$$

where

$$\tilde{T}_l^m = \frac{1}{2}\int d\Omega\gamma^{ab}T_{ab}(\theta,\phi)Y_l^{m*}, \tag{6.24}$$

$$T_l^m = 2\frac{(l-2)!}{(l+2)!}\int d\Omega T_{ab}(\theta,\phi)\gamma^{ac}\gamma^{bd}Z_l^m{}_{cd}{}^*, \tag{6.25}$$

$$t_l^m = 2\frac{(l-2)!}{(l+2)!}\int d\Omega T_{ab}(\theta,\phi)\gamma^{ac}\gamma^{bd}X_l^m{}_{cd}{}^*. \tag{6.26}$$

In the following, the basis of tensor spherical harmonics defined in this section will be called the Regge-Wheeler-Zerilli basis.

6.2 Wigner matrices

There exists another basis of harmonics that is extensively used in General Relativity, the so-called scalar spin-weighted harmonics. The adjective scalar could be misleading because tensors can also be expanded in that basis. In fact, very recently, Newman and Silva-Ortigoza [150] defined the tensorial spin-weighted harmonics making use of the Newman-Penrose null basis of vectors in the four-dimensional spacetime [151]. As they showed, these tensorial harmonics are in a one to one correspondence with the scalar ones. The scalar spin-weighted harmonics were defined by Newman and Penrose [151], making use of the edth operator. Later, they were analized by Goldberg et al. [152] and explicitly defined as

$$ {}_sY_l^m(\theta,\phi) \equiv \sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^{(l)}_{-s,m}(0,\theta,\phi), \tag{6.27}$$

where $\mathcal{D}^{(l)}_{m,m'}(\alpha,\beta,\gamma)$ are the Wigner matrices [153] or, in other words, the irreducible representation matrices of the rotation group SO(3). They are defined in Appendix A in terms of the Euler angles (α,β,γ) that parameterize any rotation.

In the particular case $s=0$, we recover the usual spherical harmonics from the above definition, that is, ${}_0Y_l^m = Y_l^m$.

Here we list the most important properties, at least for our purposes, of the Wigner matrices,

$$\mathcal{D}^{(j)}_{m_1 m_2}{}^*(\alpha,\beta,\gamma) = (-1)^{m_1-m_2}\mathcal{D}^{(j)}_{-m_1-m_2}(\alpha,\beta,\gamma), \tag{6.28}$$

$$\int d\mathfrak{A}\,\mathcal{D}^{(j_1)}_{m_1 m_1'}(\alpha,\beta,\gamma)\mathcal{D}^{(j_2)}_{m_2 m_2'}{}^*(\alpha,\beta,\gamma) = \frac{8\pi^2}{2j_1+1}\delta_{j_1 j_2}\delta_{m_1 m_2}\delta_{m_1' m_2'} \tag{6.29}$$

with

$$\int d\mathfrak{A} \equiv \frac{1}{4}\int_0^{4\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{4\pi} d\gamma \tag{6.30}$$

where the integration over α and γ go from 0 to 4π because of the double covering of SO(3) by SU(2). The product between two Wigner matrices with the same Euler angles $R=(\alpha,\beta,\gamma)$ can be decomposed as

$$\mathcal{D}^{(j_1)}_{m_1' m_1}(R)\,\mathcal{D}^{(j_2)}_{m_2' m_2}(R) = \sum_j C^{m_1\,m_2\,m_1+m_2}_{j_1\,j_2\,j} C^{m_1'\,m_2'\,m_1'+m_2'}_{j_1\,j_2\,j}\mathcal{D}^{(j)}_{m_1'+m_2',m_1+m_2}(R), \tag{6.31}$$

where the $C^{m_1\,m_2\,m_1+m_2}_{j_1\,j_2\,j}$ are Clebsch-Gordan coefficients.

6.3 Pure-spin harmonics

The rank-two pure-spin harmonics were defined by Zerilli [154], and earlier by Mathews [155] but only for spin two (the spin refers to the angular momentum of the vectors used to construct them; this will be explained in the next section), embedding the sphere in the 3-dimensional Euclidean space. Here we will only consider the tangent part to the sphere of those harmonics and generalize them to any number of indices (rank).

Following Newman and Penrose [151], we introduce a basis of complex vectors on S^2 composed by

$$m^a = \frac{1}{\sqrt{2}}(e_\theta{}^a + i e_\phi{}^a) \qquad \text{and} \qquad \bar{m}^a = \frac{1}{\sqrt{2}}(e_\theta{}^a - i e_\phi{}^a), \tag{6.32}$$

where $e_\theta{}^a$ and $e_\phi{}^a$ are the unit norm, with respect to the round metric, basis vectors. From these vectors we obtain

$$\bar{m}^a m^b = \frac{1}{2}(\gamma^{ab} + i\epsilon^{ab}), \tag{6.33}$$

a relation which can be inverted to get

$$\gamma^{ab} = m^a \bar{m}^b + \bar{m}^a m^b, \tag{6.34}$$

$$\epsilon^{ab} = i(m^a \bar{m}^b - \bar{m}^a m^b). \tag{6.35}$$

These vectors are null, $\gamma_{ab} m^a m^b = \gamma_{ab} \bar{m}^a \bar{m}^b = 0$, and are normalized so that $\gamma_{ab} m^a \bar{m}^b = 1$.

Symmetric and trace-free (STF) tensors on the unit sphere can easily be constructed from those vectors. Two independent STF tensors of rank s are

$$m^{a_1} ... m^{a_s} \qquad \text{and} \qquad \bar{m}^{a_1} ... \bar{m}^{a_s}, \tag{6.36}$$

because of the null character of the vectors. We then define the pure-spin tensor harmonics with $s \geq 0$ indices on S^2 as

$$\mathcal{Y}_l^{s,m}{}_{a_1...a_s} \equiv (-1)^s k(l,s) \mathcal{D}^{(l)}_{s,m}(0,\theta,\phi) m_{a_1} ... m_{a_s}, \tag{6.37}$$

$$\mathcal{Y}_l^{-s,m}{}_{a_1...a_s} \equiv k(l,s) \mathcal{D}^{(l)}_{-s,m}(0,\theta,\phi) \bar{m}_{a_1} ... \bar{m}_{a_s}, \tag{6.38}$$

with the normalization factor

$$k(l,s) = \sqrt{\frac{(2l+1)(l+s)!}{2^{s+2}\,\pi\,(l-s)!}}, \tag{6.39}$$

different to that used by Zerilli but very convenient for later purposes. It is interesting to see the behaviour of those harmonics under complex conjugation: using (6.28) one obtains

$$\mathcal{Y}_l^{s,m}{}_{a_1...a_s}{}^* = (-1)^m \mathcal{Y}_l^{-s,-m}{}_{a_1...a_s}. \tag{6.40}$$

In particular, when $m = 0$ the two harmonics are conjugate to each other.

Using the normalization of the Wigner matrices (6.29) it is easy to get

$$\int d\Omega\; \mathcal{Y}_l^{\pm s,m a_1...a_s} \mathcal{Y}_{l'}^{\pm s,m'}{}_{a_1...a_s}{}^* = \frac{1}{2^s}\frac{(l+s)!}{(l-s)!}\delta_{ll'}\delta_{mm'}, \tag{6.41}$$

$$\int d\Omega\; \mathcal{Y}_l^{s,m a_1...a_s} \mathcal{Y}_{l'}^{-s,m'}{}_{a_1...a_s}{}^* = 0. \tag{6.42}$$

In this way, a complete basis for any tensor on the sphere is provided by the pure-spin harmonics (6.37-6.38). The basis for a tensor of rank s is composed by the two symmetric trace-free tensors of rank s, $\mathcal{Y}_l^{s,m}{}_{a_1\ldots a_s}$ and $\mathcal{Y}_l^{-s,m}{}_{a_1\ldots a_s}$, and products between the metric γ_{ab} and the Levi-Civita tensor ϵ_{ab} with elements of the basis for $(s-2)$ rank tensors. The simplest way to show this is by noting that the Newman-Penrose vectors, m^a and $\bar{m}^a$, form a basis at generic points. Therefore a complete tensor basis can be formed by all products of s_1 vectors m^a and s_2 vectors $\bar{m}^a$ with $s_1+s_2=s$. Besides, we know from formulas (6.33-6.35) that products between those vectors can be transformed into the metric and the Levi-Civita tensor.

6.4 Pure-orbital harmonics

Firstly, one constructs pure-orbital vector harmonics by composing scalar harmonics of angular momentum l with a set of 3-dimensional vectors $t^m{}_i$ which transform under a representation of spin 1 [156]:

$$\mathcal{O}_l^{j,m}{}_i \equiv \sum_{m'=-1}^{+1} C_{l\ \ \ 1\ \ j}^{m-m'\,m'\,m}\, Y_l^{m-m'}\, t^{m'}{}_i, \tag{6.43}$$

with $j=l-1,l,l+1$ and $|m|\leq j$. The vectors $t^m{}_i$ are defined in terms of a fixed orthonormal Cartesian basis:

$$t^{\pm 1}{}_i = \frac{\mp e_{xi} - i e_{yi}}{\sqrt{2}}, \qquad t^0{}_i = e_{zi}. \tag{6.44}$$

(The index i is an abstract index on the manifold R^3 with Euclidean metric, in which S^2 is embedded.) These vector harmonics $\mathcal{O}_l^{j,m}{}_i$ transform under a representation of total angular momentum j and their Cartesian components are eigenfunctions, with eigenvalue $l(l+1)$, of the S^2-Laplacian (also called orbital angular momentum [157])

$$L^2 \equiv -r^2\vec{\nabla}^2 + \partial_r(r^2\partial_r) = -\gamma^{ab}\nabla_a\nabla_b. \tag{6.45}$$

Pure-orbital vector harmonics are however not transverse to the radial direction,

$$\hat{r}_i \mathcal{O}_l^{j,m}{}_i = -C_{j\,1\,l}^{000}\, Y_j^m. \tag{6.46}$$

Therefore, one must take certain linear combinations cancelling their radial contribution to get the pure-spin harmonics.

Pure-orbital bases for higher-rank tensors can be constructed recursively from the above vector basis. The basis for STF tensors with s indices and well-defined spin s can be built by composition of the bases with s' and $s-s'$ indices (with any $0<s'<s$) as follows:

$$t^m{}_{i_1\dots i_s} = \sum_{m'=-s'}^{s'} C^{m'\,m-m'\,m}_{s'\;\,s-s'\;\,s}\; t^{m'}_{(i_1\dots i_{s'}}t^{m-m'}_{i_{s'+1}\dots i_s)}. \tag{6.47}$$

From this, we construct orbital harmonics with s indices:

$$\mathcal{O}^{j,m}_{l}{}_{i_1\dots i_s} \equiv \sum_{m'=-s}^{+s} C^{m-m'\,m'\,m}_{l\quad\;\; s\;\; j}\, Y^{m-m'}_{l}\, t^{m'}{}_{i_1\dots i_s}, \tag{6.48}$$

which are normalized so that

$$\int d\Omega\ \mathcal{O}^{j,m}_{l}{}_{i_1\dots i_s}\mathcal{O}^{j',m'}_{l'}{}_{i_1\dots i_s}{}^{*} = \delta_{ll'}\delta_{jj'}\delta_{mm'}. \tag{6.49}$$

The pure-orbital harmonics transform under a representation of "total angular momentum" j, with $|m|\le j$, and their Cartesian components are eigenfunctions of the "orbital angular momentum" operator (6.45) with eigenvalue $l(l+1)$. This latter property becomes very useful when solving wave equations in a 3-dimensional setting. Unfortunately, these harmonics are not transverse to the radial direction, a fact that unnecessarily complicates the analysis of the radiation in the far region.

The radial component of the orbital harmonics is

$$\frac{\hat{r}^{i_s}\mathcal{O}^{j,m}_{l}{}_{i_1\dots i_{s-1}i_s}}{\sqrt{(2s+1)(2l+1)}} = \sum_{l'} C^{000}_{l\,1\,l'} W(s-1,1,j,l;s,l')\mathcal{O}^{j,m}_{l'}{}_{i_1\dots i_{s-1}}, \tag{6.50}$$

where W is the Racah coefficient [158]. Note that this relation is a sum with only two contributions, $l'=l\pm1$.

Pure-spin harmonic tensors, with indices on S^2, can be obtained as linear combinations of the pure-orbital ones. We have constructed the tensors $t^m{}_{i_1\dots i_s}$ with well-defined spin s. This means that, under a rotation parameterized by the Euler angles (α,β,γ), they will transform under an irreducible representation of order s,

$$\sum_{m=-s}^{s} \mathcal{D}^{(s)}_{s,m}(\alpha,\beta,\gamma)t^m{}_{i_1\dots i_s}. \tag{6.51}$$

If we consider the rotation $(\alpha=0,\beta=\theta,\gamma=\phi)$ we recover the STF tensors

$$m_{a_1}\dots m_{a_s} = (-1)^s\sum_{m=-s}^{s} \mathcal{D}^{(s)}_{s,m}{}^{*}(0,\theta,\phi)t^m{}_{a_1\dots a_s}, \tag{6.52}$$

$$\bar{m}_{a_1}\dots\bar{m}_{a_s} = \sum_{m=-s}^{s} \mathcal{D}^{(s)}_{-s,m}{}^{*}(0,\theta,\phi)t^m{}_{a_1\dots a_s}. \tag{6.53}$$

Substituting this in the definition of the pure spin harmonics (6.37-6.38) we get

$$\mathcal{Y}_j^{\pm s,m}{}_{a_1\ldots a_s} = k(l,s)\mathcal{D}^{(l)}_{\pm s,m}(0,\theta,\phi)\sum_{m'=-s}^{s}\mathcal{D}^{(s)}_{\pm s,m'}{}^{*}(0,\theta,\phi)\, t^{m'}{}_{a_1\ldots a_s}. \tag{6.54}$$

Using formula (6.31) for the product between Wigner matrices and the relation between those matrices and the scalar harmonics [see equation (6.27) with $s=0$], one can express the pure spin harmonics as a linear combination of the scalar harmonics with constant coefficients

$$\mathcal{Y}_j^{\pm s,m}{}_{a_1\ldots a_s} = \sqrt{\frac{4\pi}{2j+1}}k(j,s)\sum_{l=j-s}^{j+s}\sum_{m'=-s}^{+s} C^{\mp s}_{j}{}^{\pm s}_{s}{}^{0}_{l}\, C^{m-m'}_{l}{}^{m'}_{s}{}^{m}_{j}\, Y_l^{m-m'}\, t^{m'}{}_{i_1\ldots i_s}, \tag{6.55}$$

which can be rearranged to obtain the relation between the pure-spin and the pure-orbital harmonics

$$\mathcal{Y}_j^{\pm s,m}{}_{a_1\ldots a_s} = \sqrt{\frac{4\pi}{2j+1}}k(j,s)\sum_{l=j-s}^{j+s} C^{\mp s}_{j}{}^{\pm s}_{s}{}^{0}_{l}\, \mathcal{O}_l^{j,m}{}_{a_1\ldots a_s}. \tag{6.56}$$

Employing formula (6.50) for the radial projection of the orbital harmonics, it is a simple exercise to check that the radial part of the pure-spin harmonics indeed vanishes. In other words, the pure-spin harmonics are just the projection of the orbital harmonics to the sphere. Note also that, from relation (6.56), the l label of the pure-orbital harmonics is no more well-defined for the pure-spin ones, because they get contributions from states with different values of l.

6.5 Product of harmonics

In Chapter 6 we will expand the metric perturbations ${}^{\{n\}}h_{\mu\nu}$ in tensor harmonics. From the expressions derived in Chapter 2, like equations (3.16), (3.22), and (3.26), it is clear that, for this aim, we need to compute products of several tensor harmonics when working beyond linear perturbation theory. Even though those expressions contain products of many harmonics, the problem can be dealt with recursively because the product of two tensor harmonics can be decomposed as a series of tensor harmonics of adequate rank. In principle, we might conclude that at perturbation order n we need to work with tensor harmonics of rank $2n$ or similar, but the situation turns out to be simpler in General Relativity.

The formalism starts from perturbations of the metric, which contain tensor harmonics on S^2 of rank zero, one or two, and computes the decomposition in harmonics of the perturbations of the Einstein tensor, which also contain harmonics of those ranks. On the other hand, only second (at most) derivatives of the metric perturbations will appear in the perturbations of any curvature tensor, at any order. Finally, as long as we are interested just in perturbations of curvature tensors, only those contractions in equation (3.26) are required. From these three observations we conclude that we only need harmonics with up to four indices (if one works in RW gauge, to be defined below, only three-indices harmonics are required) and formulas for their thirteen products

$$\begin{array}{llll} YY', & & & \\ YY'_{:a}, & Y_{:a}Y'_{:b}, & & \\ YY'_{:ab}, & Y_{:a}Y'_{:bc}, & Y_{:ab}Y'_{:cd}, & \\ YY'_{:abc}, & Y_{:a}Y'_{:bcd}, & Y_{:ab}Y'_{:bcd}, & Y_{:abc}Y'_{:def}, \\ YY'_{:abcd}, & Y_{:a}Y'_{:bcde}, & Y_{:ab}Y'_{:cdef}, & \end{array} \tag{6.57}$$

where the prime denotes that Y and Y' have different labels l and m. Only seven of these products are really independent because using the Leibnitz rule we have relations like

$$Y_{:ab}Y'_{:cd} = (Y_{:a}Y'_{:cd})_{:b} - Y_{:a}Y'_{:cdb}. \tag{6.58}$$

Therefore, computing the expansion formula for the canonical products $YY'_{:a_1...a_n}$ with $n = 0, ..., 6$ would be enough to solve a general problem of non-spherical perturbations in General Relativity.

That method would be, however, rather complicated to program for algebraic computation, because it requires expanding the products of multiple harmonics in a very particular order, and difficult to use in any mathematical proof involving products of harmonics. It is far more interesting and general to follow a different route: we first generalize the RWZ harmonics to an arbitrary number of indices and then find a general formula for the product of any two of them. This has two important advantages: first, it is more efficient and simple for our algebraic code because all cases are considered in a single formula. Second, the formalism is more general: it can be applied to arbitrary matter models, it is possible to perturb objects like derivatives of the Riemann tensor, or it can be used in other problems (for example theories of gravity with more than two derivatives in their basic equations).

6.6 Generalization of Regge-Wheeler-Zerilli harmonics

Complete bases for higher-rank tensors tangent to the sphere can be easily constructed [113]. There always exist two nontrivial symmetric trace-free (STF) tensors

$$Z_l^m{}_{a_1...a_s} \equiv (Y_l^m{}_{:a_1...a_s})^{\rm STF} = -\epsilon_{(a_1}{}^b X_l^m{}_{ba_2...a_s)}, \tag{6.59}$$

$$X_l^m{}_{a_1...a_s} \equiv (S_l^m{}_{a_1:a_2...a_s})^{\rm STF} = \epsilon_{(a_1}{}^b Z_l^m{}_{ba_2...a_s)}, \tag{6.60}$$

valid for $|m| \leq l$ and $1 \leq s \leq l$. In all other cases the harmonics are defined to be identically zero, except for $s = 0$, when $Z_l^m \equiv Y_l^m$. Note that, in fact, we do not need symmetrization on the far right-hand side because the tensors Z and X are traceless. All other objects in the basis can be obtained from products of γ, ϵ and the basis for tensors of order $s-2$. For example the basis for 3-index tensors is given by $Z_l^m{}_{abc}$, $X_l^m{}_{abc}$, and six independent combinations of $\gamma_{ab} Z_l^m{}_c$, $\gamma_{ab} X_l^m{}_c$, $\epsilon_{ab} Z_l^m{}_c$, $\epsilon_{ab} X_l^m{}_c$, and their index-permutations. The general case results from the iteration of the relations (valid for $s \geq 2$):

$$Z_l^m{}_{a_1 \ldots a_s : b} = Z_l^m{}_{a_1 \ldots a_s b} + \frac{(l+s)(l-s+1)}{2} \times \left[\frac{1}{2} \gamma_{(a_1 a_2} Z_l^m{}_{a_3 \ldots a_s) b} - \gamma_{b (a_1} Z_l^m{}_{a_2 \ldots a_s)} \right], \tag{6.61}$$

$$X_l^m{}_{a_1 \ldots a_s : b} = X_l^m{}_{a_1 \ldots a_s b} + \frac{(l+s)(l-s+1)}{2} \times \left[\frac{1}{2} \gamma_{(a_1 a_2} X_l^m{}_{a_3 \ldots a_s) b} - \gamma_{b (a_1} X_l^m{}_{a_2 \ldots a_s)} \right]. \tag{6.62}$$

Appendix B gives a different approach to expand the definitions (6.59) and (6.60).

Remembering the definitions of the scalars $Z_l^m \equiv Y_l^m$ and $X_l^m \equiv 0$, and those of the vectors $Z_l^m{}_a \equiv Y_l^m{}_{:a}$ and $X_l^m{}_a \equiv S_l^m{}_a$, we obtain the three remaining special cases:

$$Z_l^m{}_{:a} = Z_l^m{}_a, \tag{6.63}$$

$$Z_l^m{}_{a:b} = Z_l^m{}_{ab} - \frac{l(l+1)}{2} \gamma_{ab} Z_l^m, \tag{6.64}$$

$$X_l^m{}_{a:b} = X_l^m{}_{ab} - \frac{l(l+1)}{2} \epsilon_{ab} Z_l^m. \tag{6.65}$$

Formulas (6.61–6.65) for the STF tensors Z and X constitute a complete set of simplification rules which allow us to express any derivative of a tensor harmonic field on the sphere in a unique canonical way. Note that ϵ_{ab} appears only in equation (6.65).

Finally, it is important to point out that all these harmonics have a well-defined parity under inversion of axes. This is because, as we have already commented, the parity of γ_{ab}, ϵ_{ab}, and the scalar harmonics Y_l^m is +1, -1 and $(-1)^l$, respectively, and because taking covariant derivatives does not change the parity. In this way, it is now very easy to recognize families of harmonics with different polarity: $Z_l^m{}_{a_1 \ldots a_s}$ are polar and $X_l^m{}_{a_1 \ldots a_s}$ are axial.

The relation between these harmonics and the pure-spin ones is rather simple:

$$Z_l^m{}_{a_1 \ldots a_s} = \mathcal{Y}_l^{s,m}{}_{a_1 \ldots a_s} + \mathcal{Y}_l^{-s,m}{}_{a_1 \ldots a_s}, \tag{6.66}$$

$$-i X_l^m{}_{a_1 \ldots a_s} = \mathcal{Y}_l^{s,m}{}_{a_1 \ldots a_s} - \mathcal{Y}_l^{-s,m}{}_{a_1 \ldots a_s}. \tag{6.67}$$

For the case $s = 0$, one has $Z_l^m = \mathcal{Y}_l^{0,m} = Y_l^m$.

We can invert the previous relations to get (except for the special case $s = 0$)

$$\mathcal{Y}_l^{s,m}{}_{a_1\ldots a_s} = \left(m_{a_1}\bar{m}^b\, Y_l^m{}_{:ba_2\ldots a_s}\right)^{\rm STF}, \qquad (6.68)$$
$$\mathcal{Y}_l^{-s,m}{}_{a_1\ldots a_s} = \left(\bar{m}_{a_1}m^b\, Y_l^m{}_{:ba_2\ldots a_s}\right)^{\rm STF}. \qquad (6.69)$$

In fact we introduced the factor $k(l,s)$ in the definition of the pure-spin harmonic in order to obtain such a simple relation with the RWZ harmonics. To write the normalization of these generalized RWZ harmonics and for future convenience, we adopt the compact notation

$$^{(\pm)}\mathcal{Z}_l^m{}_{a_1\ldots a_s} \equiv \mathcal{Y}_l^{s,m}{}_{a_1\ldots a_s} \pm \mathcal{Y}_l^{-s,m}{}_{a_1\ldots a_s}. \qquad (6.70)$$

It is important to note that

$$^{(+)}\mathcal{Z}_l^m{}_{a_1\ldots a_s} = Z_l^m{}_{a_1\ldots a_s} \quad \text{and} \quad ^{(-)}\mathcal{Z}_l^m{}_{a_1\ldots a_s} = -iX_l^m{}_{a_1\ldots a_s}, \qquad (6.71)$$

for all number of indices s except for the case $s = 0$, which turns out to be $^{(+)}\mathcal{Z}_l^m = 2Y_l^m$ and $^{(-)}\mathcal{Z}_l^m = 0$. With that notation the normalization of these harmonics is

$$\int d\Omega\ ^{(\pm)}\mathcal{Z}_l^{m\,a_1\ldots a_s}\,{}^{(\pm)}\mathcal{Z}_{l'}^{m'}{}_{a_1\ldots a_s}{}^* = \pm\frac{2}{2^s}\frac{(l+s)!}{(l-s)!}\delta_{ll'}\delta_{mm'} \qquad (6.72)$$

We do not orthonormalize these generalized RWZ tensor harmonics because the RWZ harmonics have been frequently used in the literature and we want a generalization containing them.

6.7 Product formula

The product of two scalar harmonics can be expanded in terms of finite sums of scalar harmonics using Clebsch-Gordan coefficients [158]:

$$Y_{l'}^{m'}\, Y_l^m = \sum_{l''=|l'-l|}^{l'+l} E^{0l\ m}_{0l'm'l''}\, Y_{l''}^{m+m'}, \qquad (6.73)$$

where we have defined the symbol

$$E^{0l\ m}_{0l'm'l''} \equiv \sqrt{\frac{(2l+1)(2l'+1)}{4\pi(2l''+1)}}\, C^{m'\,m\,m'+m}_{l'\ l\ \ l''} C^{000}_{l'l\,l''}. \qquad (6.74)$$

Recall that the Clebsch-Gordan coefficient $C^{000}_{l'l\,l''}$ vanishes if $l' + l + l''$ is odd. This fact guarantees that only scalars with parity $(-1)^{l''} = (-1)^{l'+l}$ are present in the expansion.

In this section we will construct a generalization of equation (6.73) valid for any pair of tensor harmonics on the 2-sphere. There are two main routes to find such a formula [113]. The standard route, followed by most books in Quantum Mechanics, is adapted to the Euclidean structure of $\mathbb{R}^3$ and uses the pure-orbital harmonics. The second route is based on pure-spin harmonics. They are adapted to the 2-sphere, and hence are transverse to the radial direction. Besides, they are closely related to the Wigner representation matrices of the rotation group, for which a product formula is well known.

6.7.1 Pure-orbital harmonics

One can obtain the following multiplication rule by using equation (6.73) and the formulas available in the literature for the composition of three angular momenta [156]:

$$\begin{aligned} Y_{l'}^{m'}\,\mathcal{O}_l^{j,m}{}_{i_1\ldots i_s} \;=\; & \sum_{l''=|l'-l|}^{l'+l} \sqrt{\frac{(2l+1)(2l'+1)(2j+1)}{4\pi}}\, C_{l\;l'l''}^{00\,0} \\ & \times \sum_{j'=l''-l}^{l''+l} W(s,j,l'',l';l,j')\, C_{j\;\,l'\;\,j'}^{m m' m+m'}\, \mathcal{O}_{l''}^{j',m+m'}{}_{i_1\ldots i_s}, \end{aligned} \tag{6.75}$$

Employing this formula it is possible to compute the product of any two orbital harmonics using the Leibnitz rule, as explained in Section 6.5.

6.7.2 Pure-spin harmonics

Formula (6.31) provides the following product of pure-spin harmonics with the same sign:

$$\mathcal{Y}_{l'}^{\pm s',m'}{}_{a_1\ldots a_{s'}}\mathcal{Y}_l^{\pm s,m}{}_{b_1\ldots b_s} = \sum_{l''=|l-l'|}^{l'+l} E_{\pm s\;l\;m\;l''}^{\pm s' l' m'}\,\mathcal{Y}_{l''}^{\pm(s'+s),m'+m}{}_{a_1\ldots a_{s'}b_1\ldots b_s}, \tag{6.76}$$

where we have introduced the real coefficients

$$E_{s' l' m' l''}^{s\;l\;m} \equiv \frac{k(l',|s'|)k(l,|s|)}{k(l'',|s+s'|)}\, C_{l'\;\,l\;\,l''}^{m' m\, m'+m}\, C_{l'\;l\;l''}^{s' s\, s'+s}, \tag{6.77}$$

which generalize the coefficients (6.74). These inherit from the Clebsch-Gordan coefficients the symmetry properties

$$E_{-s\;l\;m\;l''}^{-s' l' m'} = E_{s\;\;l\;-m\;l''}^{s'\;\;l'\,-m'} = (-1)^{l'+l-l''}\,E_{s\;l\;m\;l''}^{s' l' m'}, \tag{6.78}$$

$$E_{s\;l\;m\;l''}^{s' l' m'} = E_{s' l' m' l''}^{s\;l\;m}. \tag{6.79}$$

From the fact that $C_{l\ l\ l''}^{mm2m} = 0$ for odd l'' we also get that the E-coefficients vanish for odd l'' if $l = l'$ and either $m = m'$ or $s = s'$.

For the remaining products of pure-spin harmonics (those with opposite signs), we obtain (assuming e.g. that $s' \geq s$ without loss of generality)

$$\mathcal{Y}_{l'}^{\mp s',m'}{}_{a_1\ldots a_{s'}}\mathcal{Y}_{l}^{\pm s,m}{}_{b_1\ldots b_s} = \sum_{l''=|l-l'|}^{l'+l} E_{\pm s\ l\ m\ l''}^{\mp s' l' m'}\mathcal{Y}_{l''}^{\mp(s'-s),m'+m}{}_{a_{s+1}\ldots a_{s'}}T^{\pm s}{}_{a_1 b_1\ldots a_s b_s}, \tag{6.80}$$

where the products

$$T^{s}{}_{a_1 b_1 \ldots a_s b_s} \equiv (-1)^s \bar{m}_{a_1} m_{b_1}\ldots\bar{m}_{a_s} m_{b_s}, \tag{6.81}$$

$$T^{-s}{}_{a_1 b_1 \ldots a_s b_s} \equiv (-1)^s m_{a_1} \bar{m}_{b_1}\ldots m_{a_s} \bar{m}_{b_s} \tag{6.82}$$

must be expanded using equation (6.33). We define $T^0 \equiv 1$.

6.7.3 Generalized Regge-Wheeler-Zerilli harmonics

After introducing the tensors $\mathcal{T}^{\pm} \equiv \frac{1}{2}(T^{-s} \pm T^{s})$ and the alternating sign $\epsilon \equiv (-1)^{l+l'-l''}$, the discussion of the previous subsection leads to the following formula (assuming again that $s' \geq s$) valid for the product of any two generalized harmonics

$$\begin{aligned}{}^{(\sigma')}\mathcal{Z}_{l'}^{m'}{}_{a_1\ldots a_{s'}}{}^{(\sigma)}\mathcal{Z}_{l}^{m}{}_{b_1\ldots b_s} &= \sum_{l''=|l'-l|}^{l'+l} E_{s\ l\ m\ l''}^{s' l' m'}\ {}^{(\epsilon\sigma\sigma')}\mathcal{Z}_{l''}^{m'+m}{}_{a_1\ldots a_{s'} b_1\ldots b_s} \\ &+ \sum_{l''=|l'-l|}^{l'+l} \sigma E_{-sl\ m\ l''}^{s'\ l' m'} \left({}^{(\epsilon\sigma\sigma')}\mathcal{Z}_{l''}^{m'+m}{}_{a_{s+1}\ldots a_{s'}} \mathcal{T}^{+}_{a_1 b_1\ldots a_s b_s} + {}^{(-\epsilon\sigma\sigma')}\mathcal{Z}_{l''}^{m'+m}{}_{a_{s+1}\ldots a_{s'}} \mathcal{T}^{-}_{a_1 b_1\ldots a_s b_s}\right),\end{aligned} \tag{6.83}$$

which constitutes the main result of this section. The first sum in this formula is very simple [similar to that in equation (6.76)] and involves only harmonics with $s' + s$ indices. The second sum involves harmonics with $s' - s$ indices and has a more complicated structure in order to include the case of products with scalar harmonics.

Part III

Master equations on a dynamical background

Chapter 7

Axial perturbations

Here we will use the canonical approach explained in previous chapters to obtain gauge invariants and the master equations they obey for the particular case of linear perturbations of a spherical background with a massless scalar field. This part of the research can be considered as the generalization of Moncrief's work [101] for linear perturbations on a Schwarzschild spacetime to a dynamical background.

For this and the following chapter, all perturbative objects will be of first order, and therefore we will not include the explicit $n = 1$ label. The definitions used for background objects are those presented in Section 5.2. In particular, the shorthands $\dot{\zeta}$ and ζ' for frame derivatives acting on any scalar field ζ are defined in (5.66).

7.1 Expansion in harmonics

Following Regge-Wheeler's notation [56] for the metric perturbations and Moncrief's notations [101] for the momentum and shift vector, we expand the perturbative variables in tensor spherical harmonics,

$$h_{ij}dx^i dx^j = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{-2(h_1)_l^m \, d\rho \, X_l^m{}_a \, dx^a + (h_2)_l^m \, X_l^m{}_{ab} \, dx^a dx^b\right\}, \tag{7.1}$$

$$\frac{1}{\mu_g}p_{ij}dx^i dx^j = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{-2(\hat{p}_1)_l^m \, d\rho \, X_l^m{}_a \, dx^a + (\hat{p}_2)_l^m \, X_l^m{}_{ab} \, dx^a dx^b\right\}, \tag{7.2}$$

$$B_i dx^i = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} -(h_0)_l^m \, X_l^m{}_a \, dx^a, \tag{7.3}$$

$$C = 0, \tag{7.4}$$
$$p = 0, \tag{7.5}$$
$$\varphi = 0. \tag{7.6}$$

Because of the lack of axial scalar harmonics, there are no axial perturbations of the three-dimensional scalars α, Φ and Π. That means that the scalar field plays no role from the perturbative point of view, though the background scalar field is still instrumental to allow for a general dynamical spacetime. As we will see, this does not imply any loss of generality. Different (l, m) harmonic components also decouple around spherical symmetry, and so for the rest of the chapter we will drop them from the perturbative variables, assuming that we work with a fixed pair of labels at any time. It is important to note that h_2, $\hat{p}_2$ and h_0 are scalars under changes of the ρ coordinate, but h_1 and $\hat{p}_1$ behave as components of a vector.

The variables (h_1, p_1) and (h_2, p_2) form two pairs of canonically conjugated variables, whose evolution is partially determined by the arbitrary function h_0. For example the evolution equations for the variables h_1 and h_2 can be easily obtained by perturbation of the definition of the momenta (3.46) after introducing the expansions (7.1–7.6)

$$\frac{1}{\alpha}\left(h_{1,t} - (\beta h_1)_{,\rho}\right) = 2\hat{p}_1 + \Pi_1 h_1 + \frac{r^2}{\alpha}\left(\frac{h_0}{r^2}\right)_{,\rho}, \tag{7.7}$$
$$\frac{1}{\alpha}\left(h_{2,t} - \beta h_{2,\rho}\right) = 2\hat{p}_2 + (\Pi_2 - \Pi_1)h_2 - \frac{2h_0}{\alpha}. \tag{7.8}$$

We will later obtain the evolution equations for more convenient momenta variables.

7.2 Effective action

The action functional for the axial perturbations is

$$\frac{1}{2}\left(\Delta^2\mathcal{S}\right)^{\rm axial} = \int dt \int dx^3 \left[p^{ij}h_{ij,t} - B^i\Delta(\mathcal{H}_i) + ...\right]^{\rm axial} \tag{7.9}$$
$$= \int dt \left\{ \int d\rho \ (p_1 h_{1,t} + p_2 h_{2,t}) + H[h_0] + ... \right\}, \tag{7.10}$$

where the dots denote those terms coming from the second variation of the constraints (4.32-4.33), which we do not need to consider in this subsection. The functional H will be defined below in terms of the first variation of the constraint. We have also defined

$$p_1 = \frac{2l(l+1)}{a}\hat{p}_1^*, \qquad p_2 = \frac{a\lambda}{r^2}\hat{p}_2^*, \tag{7.11}$$

with

$$\lambda \equiv \frac{1}{2}\frac{(l+2)!}{(l-2)!}. \tag{7.12}$$

In term of these variables, the perturbed constraint is given by

$$\Delta[\mathcal{H}_a]^{\rm axial} = \frac{X_a \mu_g}{l(l+1)}\left[\frac{(r^2 p_1)_{,\rho}}{ar^2} + 2\frac{p_2}{a} + \lambda\frac{\Pi_2 h_2}{r^2} + \frac{2l(l+1)}{ar^2}\left(\frac{r^2\Pi_1 h_1}{a}\right)_{,\rho}\right] \tag{7.13}$$

which in turn defines the functional

$$H[h_0] \equiv -\int dx^3 B^i \Delta[\mathcal{H}_i]^{\rm axial} \tag{7.14}$$

$$= \int d\rho\left[-r^2\left(\frac{h_0}{r^2}\right)_{,\rho} p_1 + 2h_0 p_2 + \lambda a \Pi_2 \frac{h_0}{r^2} h_2 - \frac{2l(l+1)}{a} r^2 \Pi_1 \left(\frac{h_0}{r^2}\right)_{,\rho} h_1\right]. \tag{7.15}$$

This functional is the generator of gauge transformations, as can be seen in formula (4.34), and commutes with itself on shell,

$$\{\, H[\zeta_1], H[\zeta_2]\,\} = \frac{1}{l(l+1)}\int d\rho\left[r^4(\zeta_{1,\rho}\zeta_2 - \zeta_{2,\rho}\zeta_1)\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g}\right], \tag{7.16}$$

for arbitrary scalar fields ζ_1 and ζ_2.

7.3 Gauge-invariant variables

Following Moncrief [101] we perform two canonical transformations to separate the gauge-invariant information from the pure-gauge content in the canonical pairs (h_1, p_1) and (h_2, p_2). The first canonical transformation constructs the gauge-invariant combination k_1, also a vector component,

$$k_1 \equiv h_1 + \frac{r^2}{2}\left(\frac{h_2}{r^2}\right)_{,\rho}, \qquad k_2 \equiv h_2. \tag{7.17}$$

It induces the following transformation on the momenta,

$$\pi_1 = p_1, \qquad \pi_2 = p_2 + \frac{(r^2 p_1)_{,\rho}}{2r^2}. \tag{7.18}$$

This transformation can be obtained from the generating function

$$G(p_1, p_2, k_1, k_2) = p_1 k_1 + p_2 k_2 - p_1 \frac{r^2}{2}\left(\frac{k_2}{r^2}\right)_{,\rho}, \tag{7.19}$$

by direct variation,

$$h_1 = \frac{\delta G}{\delta p_1}, \qquad h_2 = \frac{\delta G}{\delta p_2}, \qquad \pi_1 = \frac{\delta G}{\delta k_1}, \qquad \pi_2 = \frac{\delta G}{\delta k_2}. \tag{7.20}$$

In terms of the new variables we can write the first variation of the axial constraint as

$$\Delta[\mathcal{H}_a]^{\mathrm{axial}} = \frac{X_a \mu_g}{l(l+1)} \left\{ 2\frac{\pi_2}{a} + \lambda \frac{\Pi_2 k_2}{r^2} + \frac{2l(l+1)}{ar^2} \left[\frac{r^2 \Pi_1}{a} \left(k_1 - \frac{r^2}{2} \left(\frac{k_2}{r^2} \right)_{,\rho} \right) \right]_{,\rho} \right\}, \tag{7.21}$$

which does not contain π_1 and therefore commutes with k_1. That is, k_1 is gauge-invariant, as we had anticipated. This suggests the second canonical transformation:

$$Q_1 \equiv k_1, \qquad Q_2 \equiv k_2, \tag{7.22}$$

with conjugated momenta

$$P_1 \equiv \pi_1 - l(l+1) \frac{r^2 \Pi_1}{a} \left(\frac{k_2}{r^2} \right)_{,\rho}, \tag{7.23}$$

$$P_2 \equiv \pi_2 + \frac{\lambda}{2r^2} a \Pi_2 k_2 + \frac{l(l+1)}{r^2} \left[\frac{r^2 \Pi_1}{a} \left(k_1 - \frac{r^2}{2} \left(\frac{k_2}{r^2} \right)_{,\rho} \right) \right]_{,\rho}. \tag{7.24}$$

This second canonical transformation can be obtained from the generating function

$$G(P_1, P_2, k_1, k_2) = P_1 k_1 + P_2 k_2 + a\, l(l+1) \left\{ \frac{r^2 \Pi_1}{a} \left(\frac{k_2}{r^2} \right)_{,\rho} \frac{k_1}{a} - \Pi_1 \left[\frac{r^2}{2a} \left(\frac{k_2}{r^2} \right)_{,\rho} \right]^2 - \frac{(l-1)(l+2)}{8r^2} \Pi_2 k_2^2 \right\}. \tag{7.25}$$

The first canonical transformation is independent of the dynamical content of the background spacetime, in the sense that it does not contain the background momenta Π_1, Π_2, Π_3. It is actually identical to that of Moncrief [101]. For the sake of clarity, we have separated the influence of the dynamical background into the second canonical transformation, which is trivialized for any static background.

At this point we have isolated the physical information of the axial metric perturbation in the pair (Q_1, P_1) while the (Q_2, P_2) contains the gauge subsystem. P_2 is the generator of gauge transformations,

$$\Delta[\mathcal{H}_a]^{\mathrm{axial}} = \frac{X_a \mu_g}{l(l+1)} \frac{2P_2}{a} \tag{7.26}$$

and hence it is gauge invariant but constrained to vanish. Its conjugate variable Q_2 is gauge-dependent and its time evolution is determined by the arbitrary function h_0, which can be used to set any desired value for $Q_{2,t}$ as we will show in the following section.

7.4 Equations of motion

After replacing the new variables and integrating by parts a number of times, we get the following Jacobi action

$$\frac{1}{2}\left(\Delta_1^2\mathcal{S}\right)^{\mathrm{axial}} = \int dt \int d\rho \left[P_1(Q_{1,t} - (\beta Q_1)_{,\rho}) + P_2(Q_{2,t} - \beta Q_{2,\rho}) + 2P_2 h_0 - \alpha\mathcal{H}^{(1)}\right], \quad (7.27)$$

where we have defined the first-order quadratic Hamiltonian

$$\begin{aligned}\mathcal{H}^{(1)} \equiv\ & \Pi_1(P_1Q_1 - P_2Q_2) + \frac{1}{a\lambda r^2}\left[\left(\frac{r^2P_1}{2} + l(l+1)\frac{r^2\Pi_1}{a}Q_1\right)_{,\rho} - r^2P_2\right]^2 \quad (7.28)\\ &+\frac{aP_1^2}{2l(l+1)} + \frac{l(l+1)}{a}\left[\frac{(l-1)(l+2)}{2r^2} + \frac{\Pi_1(\Pi_2-\Pi_1)}{2} + \dot{\Pi}_1\right]Q_1^2.\end{aligned}$$

The variation of the action with respect to h_0 gives the constraint that must be obeyed by the perturbations. This constraint now takes the simple form

$$P_2 = 0. \quad (7.29)$$

This constraint is conserved in the evolution since variation with respect to Q_2 gives

$$(r^2P_2)_{,t} = (\beta r^2 P_2)_{,\rho}. \quad (7.30)$$

As P_2 is the generator of the gauge transformations, its conjugate variable Q_2 is pure gauge. Its evolution equation comes from taking the variation of the action with respect to P_2,

$$\frac{1}{\alpha}(Q_{2,t} - \beta Q_{2,\rho}) = -2\frac{h_0}{\alpha} - \Pi_1 Q_2 - \frac{1}{a\lambda}\left(r^2P_1 + l(l+1)\frac{2r^2\Pi_1}{a}Q_1\right)_{,\rho}. \quad (7.31)$$

The initial data for Q_2 is gauge, and its evolution is fully determined by the free function h_0. In particular it is possible to choose $Q_2 = 0$ initially and take h_0 so that $Q_2 = 0$ at all times.

We can obtain the physically relevant equations by variation of the action with respect to the variables (Q_1, P_1). This gives rise to a system of two coupled second order equations in ρ-derivatives, whose principal part is, in matricial form,

$$\frac{(l-1)(l+2)}{2r^2}\ \frac{1}{\alpha}\begin{pmatrix}\frac{2l(l+1)}{a}Q_1\\ P_1\end{pmatrix}_{,t} = \begin{pmatrix}-\Pi_1 & -1\\ \Pi_1^2 & \Pi_1\end{pmatrix}\frac{1}{a^2}\begin{pmatrix}\frac{2l(l+1)}{a}Q_1\\ P_1\end{pmatrix}_{,\rho\rho} + \dots \quad (7.32)$$

the dots denoting lower order terms in ρ-derivatives of Q_1 and P_1. We have divided Q_1 by a to make it a scalar under changes of ρ coordinate. This is a second order in time evolution system, as corresponds to a single wave-like degree of freedom, but it apparently has fourth order in ρ-derivatives for generic values of the background variable Π_1. This is false because the 2×2 matrix has always vanishing square, and hence the system has third order at most. Actually it has second order, as can be checked by taking the matrix to its Jordan canonical form. We define the combination

$$\mathcal{Q} \equiv P_1 + l(l+1)\frac{2\Pi_1}{a}Q_1, \tag{7.33}$$

that is a scalar under change of coordinate ρ. Then, the system (7.32) is equivalent to the pair (we now use the dot and prime frame derivatives to simplify the expressions):

$$\begin{aligned}(r^2\mathcal{Q})^{\cdot} &= -2\lambda\frac{Q_1}{a}, & (7.34)\\ \left(-\frac{2\lambda}{r^2}\frac{Q_1}{a}\right)^{\cdot} &= \frac{1}{\alpha}\left[\frac{\alpha}{r^2}(r^2\mathcal{Q})'\right]' + \frac{\Pi_2 - \Pi_1}{2r^2}(r^2\mathcal{Q})^{\cdot} - \frac{(l-1)(l+2)}{r^2}\mathcal{Q}, & (7.35)\end{aligned}$$

which can be clearly combined into a single second order equation for $\mathcal{Q}$, the sought generalization of the Regge-Wheeler equation for dynamical backgrounds.

As will be made explicit in Chapter 10, whereas the physical variables Q_1 and P_1 are not scalars in M^2, the combination $\mathcal{Q}$ is indeed a scalar. There we will show that $\mathcal{Q}$ is the particularization to this case of the GS master variable, which is valid for any spherically symmetric background. Therefore, we have just re-deduced the GS master variable from a canonical point of view. This was the expected result, but it is important to emphasize that the combination of Hamiltonian gauge methods with the imposition of having a scalar field on M^2 has determined uniquely the GS scalar.

When restricting to vacuum the variable $r\mathcal{Q}/\lambda$ is the Cunningham-Price-Moncrief master function [46], which obeys the Regge-Wheeler equation, though it is not immediately related to the Regge-Wheeler variable. Using the gauge $r = \rho, \beta = 0$ in vacuum we have $\Pi_1 = 0$ and hence $\mathcal{Q} = P_1$, while the Regge-Wheeler variable is $Q_1/(a^2 r)$. We have seen that the former is easily generalizable to a dynamical situation as given in (7.33), but not the latter, because it would require dividing by Π_1, which may vanish.

Chapter 8

Polar perturbations

We now reproduce the analysis of the previous chapter in the polar sector. Contrary to the axial case, there is no known master polar scalar which can be used on all spherical backgrounds and with any type of matter. Using again a scalar field as matter model, for which no such master variable was known, we apply the same Hamiltonian techniques and find master variables for the gravitational wave and the scalar field perturbation with the expected properties. The equations obeyed, generalizing the Zerilli equation, are unfortunately too complicated to be given here in full detail, and we are currently working to reexpress them in a manageable way. The results in this chapter are new and pave the way for a polar master equation in any dynamical spacetime.

8.1 Expansion in harmonics

We decompose the polar part of the metric perturbation into spherical harmonics,

$$\begin{aligned} h_{ij}dx^i dx^j &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^2 (H_2)_l^m Y_l^m d\rho^2 + 2(h_1)_l^m d\rho\, Z_l^m{}_a dx^a \\ &+ r^2 \left[K_l^m \gamma_{ab} Y_l^m + G_l^m Z_l^m{}_{ab}\right] dx^a dx^b, \end{aligned} \tag{8.1}$$

$$\begin{aligned} \frac{1}{\mu_g} p_{ij} dx^i dx^j &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^2 (P_H)_l^m Y_l^m d\rho^2 + 2(P_h)_l^m d\rho\, Z_l^m{}_a dx^a \\ &+ r^2 \left[(P_K)_l^m \gamma_{ab} Y_l^m + (P_G)_l^m Z_l^m{}_{ab}\right] dx^a dx^b, \end{aligned} \tag{8.2}$$

$$C = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{-\alpha}{2} (H_0)_l^m Y_l^m, \tag{8.3}$$

$$B_i dx^i = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (H_1)_l^m\, Y_l^m d\rho + (h_0)_l^m\, Z_l^m{}_a dx^a, \quad (8.4)$$

$$\frac{1}{\mu_g}\Delta\Pi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \hat{p}_l^m Y_l^m, \quad (8.5)$$

$$\Delta\Phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \varphi_l^m Y_l^m. \quad (8.6)$$

As we did in the previous chapter, we will remove the (l, m) labels from the harmonic coefficients, since all perturbations are decoupled at linear order. We follow the notations by Regge-Wheeler and Moncrief, and hence introduce geometric factors a^2 to compensate the tensor character of the variables H_2 and P_H with respect to changes in the ρ coordinate. However, additional factors of a should have been included multiplying h_1, P_h and H_1, because these are vector components. We will later need explicit a factors in our formulas to correct this problem.

8.2 Effective action

Introducing decomposition (8.1-8.6) into the effective action (4.27), one obtains

$$\begin{aligned}\frac{1}{2}\left(\Delta^2\mathcal{S}\right)^{\rm polar} &= \int dx^3 \left\{p^{ij}h_{ij} + \Delta[\Pi]\Delta[\Phi_{,t}] - C\Delta[\mathcal{H}] - B^i\Delta[\mathcal{H}_i] + ...\right\}^{\rm polar} \quad (8.7)\\ &= \int d\rho \left\{p_1 h_{1,t} + p_2 H_{2,t} + p_3 K_{Z,t} + p_4 G_{,t} + p\varphi_{,t}\right\} + F_0[-\alpha H_0/2]\\ &+ F_1[H_1] + F_2[h_0] + ..., \quad (8.8)\end{aligned}$$

where the dots stand for terms coming from the second perturbation of the constraints (4.32-4.33), that do not enter the gauge transformations, and the functionals (F_0, F_1, F_2) will be defined below. The conjugate momenta are related to the harmonic coefficients given by the expansions (8.1-8.6) in the following way,

$$p_1 = \frac{2l(l+1)}{a} P_h^*, \quad (8.9)$$

$$p_2 = ar^2 P_H^*, \quad (8.10)$$

$$p_3 = 2ar^2 P_K^*, \quad (8.11)$$

$$p_4 = \lambda ar^2 P_G^*, \quad (8.12)$$

$$p = ar^2 \hat{p}^*. \quad (8.13)$$

The polar harmonic decomposition of the perturbation of the constraints (4.30-4.31) is given by,

$$\begin{aligned}
\Delta[\mathcal{H}] &= \mu_g Y \left\{ H_2 \left[\Pi_1(\Pi_1 - \Pi_2) - \frac{l^2+l+2}{r^2} - \frac{\mathcal{H}}{2\mu_g} \right] - 2{H_2}' \frac{r'}{r} + \frac{p_2}{ar^2}(\Pi_1 - \Pi_2) \right. \\
&\quad + \frac{1}{2} K \left[-\Pi_1^2 - \Pi_3^2 + \Phi'^2 \right] - \frac{p_3}{ar^2}\Pi_1 - \left[{}^{(3)}R + \frac{(l-1)(l+2)}{r^2} \right] K \\
&\quad \left. + \frac{2}{r^3}(r^3 K')' + \Phi'\varphi' + \frac{p}{ar^2}\Pi_3 + \frac{2l(l+1)}{r^3}(ra^{-1}h_1)' - \frac{\lambda}{r^2} G \right\}, \qquad (8.14) \\
\frac{1}{a}\Delta[\mathcal{H}_\rho] &= \mu_g Y \left\{ -\frac{2(a^{-1}p_2)'}{r^2} + \frac{p_1}{r^2} + \frac{2p_3}{ar^2}\frac{r'}{r} + \frac{p}{ar^2}\Phi' - {H_2}'\Pi_1 - \frac{2}{r^2}(r^2\Pi_1)' H_2 \right. \\
&\quad \left. + \frac{l(l+1)}{ar^2}\Pi_2 h_1 + \frac{\Pi_2}{r^2}(r^2 K)' + \Pi_3 \varphi' \right\}, \qquad (8.15) \\
\Delta[\mathcal{H}_a]^{\mathrm{polar}} &= \mu_g Z_a \left\{ -\frac{1}{l(l+1)}\frac{(r^2 p_1)'}{r^2} - \frac{p_3}{ar^2} + \frac{2}{l(l+1)}\frac{p_4}{ar^2} + \frac{(l-1)(l+2)}{2}\Pi_2 G \right. \\
&\quad \left. + \Pi_1 H_2 - \frac{2}{r^2}\left(\frac{r^2 \Pi_1 h_1}{a} \right)' + \Pi_3 \varphi \right\}. \qquad (8.16)
\end{aligned}$$

With these relations at hand, we can write down the three generators of polar gauge transformations,

$$F_0[\zeta] = \int dx^3 \zeta\, Y \Delta[\mathcal{H}], \qquad (8.17)$$

$$F_1[\zeta] = \int dx^3 \zeta\, Y \frac{1}{a}\Delta[\mathcal{H}_\rho], \qquad (8.18)$$

$$F_2[\zeta] = \int dx^3 \zeta\, Z_a \frac{1}{r^2}\gamma^{ab}\Delta[\mathcal{H}_b]^{\mathrm{polar}}, \qquad (8.19)$$

that act on any smooth arbitrary scalar field ζ. It is possible to calculate the Poisson brackets between different generators,

$$\{F_0[\zeta_1], F_0[\zeta_2]\} = \int d\rho\, ar^2(\zeta_1\zeta_2' - \zeta_1'\zeta_2)\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g}, \qquad (8.20)$$

$$\{F_0[\zeta_1], F_1[\zeta_2]\} = \int d\rho\, a\zeta_1 \left(r^2 \zeta_2 \frac{\mathcal{H}}{\mu_g} \right)', \qquad (8.21)$$

$$\{F_0[\zeta_1], F_2[\zeta_2]\} = -l(l+1)\int d\rho\, a\zeta_1\zeta_2 \frac{\mathcal{H}}{\mu_g}, \qquad (8.22)$$

$$\{F_1[\zeta_1], F_1[\zeta_2]\} = \int d\rho\, ar^2(\zeta_1\zeta_2' - \zeta_1'\zeta_2)\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g}, \qquad (8.23)$$

$$\{F_1[\zeta_1], F_2[\zeta_2]\} = 0, \qquad (8.24)$$

$$\{F_2[\zeta_1], F_2[\zeta_2]\} = l(l+1)\int d\rho\, a(\zeta_1\zeta_2' - \zeta_1'\zeta_2)\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g}, \qquad (8.25)$$

all vanishing on-shell, which confirms that they are first-class constraints.

8.3 Gauge invariant variables

Moncrief isolated the Zerilli variable after two canonical transformations on the four pairs (h_1, p_1), (H_2, p_2), (K, p_3), (G, p_4). Here we have the additional pair (φ, p) for the scalar field, and the fact that the background is dynamical also makes the problem harder. We will instead proceed in five steps, to clarify the role of each step and simplify the computations. In particular we will first eliminate the two gauge degrees of freedom related to the momentum constraint (which are rather trivial and very similar to the axial case) and then remove the gauge degree associated with the Hamiltonian constraint, the nontrivial step of this computation.

There are many possible transformations that implement this program, but we would like them to they obey certain minimal criteria. First, they should be algebraic transformations so that they do not involve any integration in the process and can be performed explicitly. And second, they should not require dividing by any background object that could vanish, in particular one of the background momenta (Π_1, Π_2, Π_3). The full transformation we will propose here fulfills the first criterion, but it is unclear whether it also satisfies the second one, as we will explain.

The first canonical transformation is motivated by the Gerlach and Sengupta choice of gauge-invariants, that will be presented in Chapter 9,

$$k_1 = K + \frac{l(l+1)}{2}G - \frac{2r'}{r}\left(a^{-1}h_1 - \frac{r^2}{2}G'\right), \tag{8.26}$$

$$k_2 = H_2 - 2\left(a^{-1}h_1 - \frac{r^2}{2}G'\right)', \tag{8.27}$$

$$k_3 = G, \tag{8.28}$$

$$k_4 = a^{-1}h_1 - \frac{r^2}{2}G', \tag{8.29}$$

$$k_5 = \varphi - \left(a^{-1}h_1 - \frac{r^2}{2}G'\right)\Phi', \tag{8.30}$$

which requires the canonical momenta

$$\pi_1 = p_3, \tag{8.31}$$

$$\pi_2 = p_2, \tag{8.32}$$

$$\pi_3 = p_4 - \frac{l(l+1)}{2}p_3 - \frac{1}{2}a(r^2 p_1)', \tag{8.33}$$

$$\pi_4 = ap_1 - 2a(a^{-1}p_2)' + \frac{2r'}{r}p_3 + p\,\Phi', \tag{8.34}$$

$$\pi_5 = p. \tag{8.35}$$

In terms of these new variables, we can write the perturbed momentum constraints as

$$\begin{aligned}
\frac{1}{a}\Delta[\mathcal{H}_\rho] &= \mu_g Y \left\{ \frac{\pi_4}{ar^2} + \Pi_2\frac{(r^2k_1)'}{r^2} - \Pi_1 k_2' - 2k_2\frac{(r^2\Pi_1)'}{r^2} - \frac{2}{r^2}(r^2\Pi_1 k_4')' + k_4'\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g} \right. \\
&\left. + \frac{l(l+1)}{r^2}\Pi_2\left(k_4 - k_3 r r'\right) + k_4\Pi_3\Phi'' + k_4\frac{\Pi_2}{r^2}(r^2)'' + \Pi_3 k_5' \right\}, \qquad (8.36)\\
\Delta[\mathcal{H}_a]^{\rm polar} &= \mu_g Z_a \left\{ \frac{2\pi_3}{l(l+1)ar^2} + k_2\Pi_1 + \Pi_3 k_5 - \frac{(r^2)'}{r^2}\Pi_2 k_4 + \frac{\lambda}{l(l+1)}\Pi_2 k_3 \right. \\
&\left. - \frac{1}{r^2}(\Pi_1 r^4 k_3')' + \frac{k_4}{a}\frac{\mathcal{H}_\rho}{\mu_g} \right\}. \qquad (8.37)
\end{aligned}$$

We do not display the explicit form of the perturbation of the Hamiltonian constraint $\Delta[\mathcal{H}]$ in terms of these variables because it is a very lengthy expression and does not contribute in any way to the present discussion.

We perform a second canonical transformation which converts the momentum constraints into canonical variables. Because of the two requirements we want to impose in all our transformations, only the momenta π_4 and π_3 can replace the constraints (8.36) and (8.37) respectively,

$$\begin{aligned}
\bar\pi_1 &= \pi_1 - ar^2(\Pi_2 k_4)', \qquad (8.38)\\
\bar\pi_2 &= \pi_2 + \frac{l(l+1)}{2}ar^2\Pi_1 k_3 + ar^2\Pi_1 k_4' - a(r^2\Pi_1)'k_4, \qquad (8.39)\\
\bar\pi_3 &= \frac{1}{2}l(l+1)ar^2\left(\frac{\Delta[\mathcal{H}_a]^{\rm polar}}{\mu_g Z_a}\right) = \pi_3 + ..., \qquad (8.40)\\
\bar\pi_4 &= r^2\left(\frac{\Delta[\mathcal{H}_\rho]}{\mu_g Y}\right) = \pi_4 + ..., \qquad (8.41)\\
\bar\pi_5 &= \pi_5 + \frac{l(l+1)}{2}ar^2\Pi_3 k_3 - a(r^2\Pi_3 k_4)'. \qquad (8.42)
\end{aligned}$$

The division by the tensor harmonics must be understood just as removing them from the above expressions (8.36-8.37). These last transformation for the momenta do not affect the variables,

$$\begin{aligned}
\bar k_1 &= k_1, \qquad (8.43)\\
\bar k_2 &= k_2, \qquad (8.44)\\
\bar k_3 &= k_3, \qquad (8.45)\\
\bar k_4 &= k_4, \qquad (8.46)\\
\bar k_5 &= k_5. \qquad (8.47)
\end{aligned}$$

In terms of these last variables, the constraints take the following simpler form,

$$\Delta[\mathcal{H}] = \mu_g Y \left\{ -\Pi_1\frac{\bar\pi_1}{ar^2} + (\Pi_1 - \Pi_2)\frac{\bar\pi_2}{ar^2} + \Pi_3\frac{\bar\pi_5}{ar^2} + \frac{2}{r^3}(r^3\bar k_1')' - \frac{(r^2)'}{r^2}\bar k_2' + \bar k_5'\Phi' \right.$$

$$+ \left[\Pi_1(\Pi_2 - \Pi_1) - \frac{(l-1)(l+2)}{r^2} - \Pi_3^2\right]\bar{k}_1$$
$$- \left.\left[\Pi_1(\Pi_2 - \Pi_1) + \frac{(l-1)(l+2)+4}{r^2}\right]\bar{k}_2\right\}, \quad (8.48)$$
$$\frac{1}{a}\Delta[\mathcal{H}_\rho] = \mu_g Y \frac{\bar{\pi}_4}{ar^2}, \quad (8.49)$$
$$\Delta[\mathcal{H}_a]^{\rm polar} = \mu_g Z_a \frac{2}{l(l+1)}\frac{\bar{\pi}_3}{ar^2}. \quad (8.50)$$

We have fully isolated the gauge freedom contained in the perturbed momentum constraint. The variables $\bar{k}_3$ and $\bar{k}_4$ are gauge dependent and non-dynamical because their conjugate momenta $\bar{\pi}_3$ and $\bar{\pi}_4$ are constrained to vanish. We are left with a system of three degrees of freedom $(\bar{k}_1, \bar{k}_2, \bar{k}_5)$ and the single constraint (8.48).

Following the same procedure, at this point one should make another canonical transformation and convert the Hamiltonian constraint into one of the variables. Because of the first criterion we want to impose, we can not convert any of the variables $\{\bar{k}_1, \bar{k}_2, \bar{k}_5\}$ that appear in (8.48) into the full constraint. But, because of the second requirement, we can neither do it for any of the momenta $\{\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_5\}$. Therefore, the idea is to first make a transformation that removes second order derivatives of $\bar{k}_1$ from the constraint (8.48), so that all the first derivatives of the perturbed objects can be absorbed in a single term. Later, this object will be promoted to one of the variables of the problem in such a way that the Hamiltonian constraint will have no derivatives of the rest of the variables. We use an arbitrary but constant field x to parameterize the transformation,

$$\tilde{k}_1 = \bar{k}_1, \quad (8.51)$$
$$\tilde{k}_2 = \bar{k}_2 - \frac{r\bar{k}_1'}{r'} - x\bar{k}_1, \quad (8.52)$$
$$\tilde{k}_3 = \bar{k}_3, \quad (8.53)$$
$$\tilde{k}_4 = \bar{k}_4, \quad (8.54)$$
$$\tilde{k}_5 = \bar{k}_5, \quad (8.55)$$

which will introduce a first derivative of $\bar{\pi}_2$ in the Hamiltonian constraint through the transformations of the momenta,

$$\tilde{\pi}_1 = \bar{\pi}_1 + (x-1)\bar{\pi}_2 - ar\left(\frac{a^{-1}\bar{\pi}_2}{r'}\right)', \quad (8.56)$$
$$\tilde{\pi}_2 = \bar{\pi}_2, \quad (8.57)$$
$$\tilde{\pi}_3 = \bar{\pi}_3, \quad (8.58)$$
$$\tilde{\pi}_4 = \bar{\pi}_4, \quad (8.59)$$
$$\tilde{\pi}_5 = \bar{\pi}_5. \quad (8.60)$$

Now we define the $\mathfrak{Z}$ combination that will absorb all derivatives in the expression of the Hamiltonian constraint,

$$r'\mathfrak{Z} \equiv -\Pi_1 \frac{\tilde{\pi}_2}{ar^2} - 2\frac{(r')^2}{r^2}\tilde{k}_2 + \frac{r'}{r}\Phi'\tilde{k}_5 - V_x\tilde{k}_1, \tag{8.61}$$

with the background potentials

$$V \equiv \frac{1+l+l^2}{r^2} + \Pi_1(\Pi_2 - \Pi_1) + \frac{{}^{(3)}R}{2}, \qquad V_x \equiv V + (2x-3)\frac{(r')^2}{r^2}. \tag{8.62}$$

Then the Hamiltonian constraint can be written as a sum of a derivative of $\mathfrak{Z}$ and a linear combination of variables $\tilde{k}_i$ and $\tilde{\pi}_i$ with no derivatives,

$$\begin{aligned}
\frac{\Delta[\mathcal{H}]}{\mu_g Y} &= \mathfrak{Z}' - \Pi_1\frac{\tilde{\pi}_1}{ar^2} + \Pi_3\frac{\tilde{\pi}_3}{ar^2} + \left(\frac{r(\Pi_1)'}{r'} - \Pi_1 - \Pi_2 + x\Pi_1\right)\frac{\tilde{\pi}_2}{ar^2} - \Phi''\tilde{k}_5 \\
&- \left(V + \frac{3(r')^2}{r^2}\right)\tilde{k}_2 + \left\{\frac{r}{r'}V_x' + V_x\frac{r^2}{2(r')^2}\left(\frac{3(r')^2}{r^2} - \frac{1}{r^2} + \frac{{}^{(3)}R}{2}\right) - \Pi_3^2\right. \\
&+ \left.(1-x)\Pi_1(\Pi_2-\Pi_1) - \frac{l(l+1)}{r^2}(1+x) + \frac{2}{r^2}(1-x)\right\}\tilde{k}_1.
\end{aligned} \tag{8.63}$$

As we have anticipated, this clearly motivates a fourth canonical transformation in which $\mathfrak{Z}$ replaces the canonical variable $\tilde{k}_2$ (because using $\tilde{k}_1$ instead would require dividing by V_x, which is a background object that could vanish),

$$\check{k}_1 = \tilde{k}_1, \tag{8.64}$$
$$\check{k}_2 = \mathfrak{Z}, \tag{8.65}$$
$$\check{k}_3 = \tilde{k}_3, \tag{8.66}$$
$$\check{k}_4 = \tilde{k}_4, \tag{8.67}$$
$$\check{k}_5 = \tilde{k}_5, \tag{8.68}$$
$$\check{\pi}_1 = \tilde{\pi}_1 - \frac{V_x}{2(r')^2}\tilde{\pi}_2, \tag{8.69}$$
$$\check{\pi}_2 = -\frac{r\tilde{\pi}_2}{2r'}, \tag{8.70}$$
$$\check{\pi}_3 = \tilde{\pi}_3, \tag{8.71}$$
$$\check{\pi}_4 = \tilde{\pi}_4, \tag{8.72}$$
$$\check{\pi}_5 = \tilde{\pi}_5 + \frac{\Phi'}{2r'}\tilde{\pi}_2. \tag{8.73}$$

Now the Hamiltonian constraint does not contain $\check{\pi}_2$. But, more importantly, we have achieved what we were looking for: it neither contains derivatives of $\check{k}_1$,

$$\frac{\Delta[\mathcal{H}]}{\mu_g Y} = -\Pi_1\frac{\check{\pi}_1}{ar^2} + \Pi_3\frac{\check{\pi}_5}{ar^2} + \check{k}_2' + \frac{\check{k}_2}{2}\left(\frac{V}{r'} + 3r'\right) - \check{k}_5\left[\Phi'' + \frac{\Phi'}{2}\left(\frac{V}{r'} + 3r'\right)\right] + D\check{k}_1, \tag{8.74}$$

where we have defined the background coefficient

$$D = \frac{(r^2V)'}{rr'} + \frac{r^2}{2(r')^2}\left[V + \left(\frac{r'}{r}\right)^2\right]\left[\frac{l(l+1)}{r^2} + \Pi_1(\Pi_2 - \Pi_1) + {}^{(3)}R\right] - \left[2\frac{l(l+1)}{r^2} + \Pi_3^2\right]. \quad (8.75)$$

This fact permits us to perform the final fifth canonical transformation, which converts the Hamiltonian constraint into the first of the variables of the problem,

$$Q_1 = \frac{\Delta[\mathcal{H}]}{\mu_g Y} \quad (8.76)$$

$$Q_2 = \check{k}_2, \quad (8.77)$$

$$Q_3 = \check{k}_3, \quad (8.78)$$

$$Q_4 = \check{k}_4, \quad (8.79)$$

$$Q_5 = \check{k}_5 + \frac{\Pi_3}{ar^2D}\check{\pi}_1, \quad (8.80)$$

$$P_1 = \frac{\check{\pi}_1}{D}, \quad (8.81)$$

$$P_2 = \check{\pi}_2 + a\left(\frac{\check{\pi}_1}{aD}\right)' - \frac{\check{\pi}_1}{2D}\left(\frac{V}{r'} + 3r'\right), \quad (8.82)$$

$$P_3 = \check{\pi}_3, \quad (8.83)$$

$$P_4 = \check{\pi}_4, \quad (8.84)$$

$$P_5 = \check{\pi}_5 + \frac{\check{\pi}_1}{D}\left[\Phi'' + \frac{\Phi'}{2}\left(\frac{V}{r'} + 3r'\right)\right]. \quad (8.85)$$

At this point we have succeeded in separating the physical degrees of freedom (Q_2, P_2) and (Q_5, P_5) from the gauge degrees of freedom (Q_1, P_1), (Q_3, P_3) and (Q_4, P_4). However in this last transformation the background object D appears as denominator and it is not yet clear to us whether this object can vanish or not. In vacuum, for a Schwarzschild solution, defining $\Lambda \equiv (l-1)(l+2)/2$, we have

$$D = \frac{1}{1 - 2M/r}\frac{l(l+1)}{r^3}\left(\Lambda r + 3M\right), \quad (8.86)$$

which is always positive. It is reasonable to assume that for spacetimes close enough to the Schwarzschild exterior (though possibly dynamical), the variable D will also be positive. If this was the case we would have succeeded in implementing to completion the procedure while obeying the two imposed criteria. If not, since the procedure we have followed is more or less unique, one should interpret the result as an indication against the existence of well behaved polar master variables for generic backgrounds. However, one perhaps could still find master functions relaxing the first condition about the algebraic nature of the transformations.

8.4 Equations of motion

The variable $\mathfrak{Z}$ obeys a complicated equation of motion, which we handle using our computer algebra tools. This subsection summarizes its differential structure, and shows that it is indeed a generalization of the Zerilli equation.

In order to simplify the calculations we choose as background coordinates $(t, \rho = r)$. In addition we will take $\beta = 0$ which, because of the background evolution equation (5.72), implies also $\Pi_1 = 0$. The equations of motion for the gauge-invariant variables are obtained by direct variations of the action,

$$\begin{aligned}
P_{2,t} &= A^{(4)}_{11} P_2^{(\mathrm{iv})} + A^{(6)}_{12} Q_2^{(\mathrm{vi})} + A^{(5)}_{13} P_5^{(\mathrm{v})} + A^{(5)}_{14} Q_5^{(\mathrm{v})} + \ldots, && (8.87)\\
Q_{2,t} &= A^{(2)}_{21} P_2'' + A^{(4)}_{22} Q_2^{(\mathrm{iv})} + A^{(3)}_{23} P_5''' + A^{(3)}_{24} Q_5''' + \ldots, && (8.88)\\
P_{5,t} &= A^{(3)}_{31} P_2''' + A^{(5)}_{32} Q_2^{(\mathrm{v})} + A^{(4)}_{33} P_5^{(\mathrm{iv})} + A^{(4)}_{34} Q_5^{(\mathrm{iv})} + \ldots, && (8.89)\\
Q_{5,t} &= A^{(3)}_{41} P_2''' + A^{(5)}_{42} Q_2^{(\mathrm{v})} + A^{(4)}_{43} P_5^{(\mathrm{iv})} + A^{(4)}_{44} Q_5^{(\mathrm{iv})} + \ldots, && (8.90)
\end{aligned}$$

where the dots stand for lower order radial derivatives and the subindices of the background dependent A-coefficients denote positions in a matrix. More precisely, $A^{(k)}_{ij}$ would correspond to the slot (i, j) of the matrix that multiplies the kth order radial derivatives of the vector (P_2, Q_2, P_5, Q_5).

As expected from our experience with the axial case in the previous chapter, the equations are obtained in a form which is more complicated than expected. It turns out that it is possible to replace equations (8.87) and (8.89-8.90) by some linear combinations of them with radial derivatives of equation (8.88) so that their differential order is reduced. The new system takes the following form,

$$\begin{aligned}
B_1 P_{2,t} + B_2 Q_{2,t\rho\rho} + B_3 Q_{2,t\rho} &= \mathcal{A}^{(2)}_{11} P_2'' + \mathcal{A}^{(4)}_{12} Q_2^{(iv)} + \mathcal{A}^{(3)}_{13} P_5''' + \mathcal{A}^{(3)}_{14} Q_5''' + \ldots, && (8.91)\\
Q_{2,t} &= \mathcal{A}^{(2)}_{21} P_2'' + \mathcal{A}^{(4)}_{22} Q_2^{(iv)} + \mathcal{A}^{(3)}_{23} P_5''' + \mathcal{A}^{(3)}_{24} Q_5''' + \ldots, && (8.92)\\
B_4 P_{5,t} + B_5 Q_{2,t\rho} &= \mathcal{A}^{(2)}_{31} P_2'' + \mathcal{A}^{(4)}_{32} Q_2^{(iv)} + \mathcal{A}^{(3)}_{33} P_5''' + \mathcal{A}^{(3)}_{34} Q_5''' + \ldots, && (8.93)\\
B_6 Q_{5,t} + B_7 Q_{2,t\rho} &= \mathcal{A}^{(2)}_{41} P_2'' + \mathcal{A}^{(4)}_{42} Q_2^{(iv)} + \mathcal{A}^{(3)}_{43} P_5''' + \mathcal{A}^{(3)}_{44} Q_5''' + \ldots, && (8.94)
\end{aligned}$$

where the B_i coefficients are given by,

$$B_1 \equiv 8\left[\alpha\frac{(l-2)!}{(l+2)!}\right]^2 \left(\frac{r\Pi_3}{a}\right)^6 \frac{\Pi_2}{aD}, \quad (8.95)$$

$$B_2 \equiv 8\left[\alpha\frac{(l-2)!}{(l+2)!}\right]^2 \left(\frac{r^4\Pi_2\Pi_3^2}{a^4 D}\right)^2, \quad (8.96)$$

$$B_3 \equiv -2\left[\alpha\frac{(l-2)!}{(l+2)!}\right]^2 \left(\frac{\Pi_3 r^3}{a^4 D}\right)^2 \Pi_2 \Big\{ 4r^3\Pi_3^3\Phi_{,\rho\rho} + r\left[r^2\Pi_3^2 + l(l+1)\right]\Pi_2(\Phi_{,\rho})^2$$

$$
\begin{aligned}
&+ \quad a^2 r^3 \Pi_2 \Pi_3^4 + \frac{4r^2}{D}\left(D\Pi_3^2\right)_{,\rho} \Pi_2 + r\left[3a^2 l(l+1) - 8\right]\Pi_2\Pi_3^2 \\
&+ \quad \frac{4}{r} l(l+1)\left[a^2\left(l^2+l+1\right) - 3\right]\Pi_2 \Bigg\}, \qquad (8.97)
\end{aligned}
$$

$$
B_4 \equiv 4\alpha \frac{(l-2)!}{(l+2)!}\left(\frac{r^2\Pi_3}{a^2}\right)^2 \frac{\Pi_2}{D}, \qquad (8.98)
$$

$$
\begin{aligned}
B_5 \equiv\ & 2\alpha r \frac{(l-2)!}{(l+2)!}\left(\frac{r^2}{a^3 D}\right)^2 \Pi_2 \Big\{ \left[\left(l^2+l+2\right)a^2+2\right]\Phi_{,\rho}\Pi_2 \\
&+ 2ra^2 D\Pi_3 + 2r\Phi_{,\rho\rho}\Pi_2 \Big\}, \qquad (8.99)
\end{aligned}
$$

$$
B_6 \equiv -B_4, \qquad (8.100)
$$

$$
B_7 \equiv -4r\alpha \frac{(l-2)!}{(l+2)!}\left(\frac{r}{a}\right)^3 \left(\frac{\Pi_2}{aD}\right)^2 \Pi_3. \qquad (8.101)
$$

For the moment being, this is the simplest way we have found to write this system of equations. Nevertheless, we are currently working in order to give then a more manageable form.

Let us now particularize the above equations of motion (8.87-8.90) to vacuum and recover the results obtained in that case by Moncrief [101]. Then, all background and perturbative fluid variables $\{\Pi_3, \Phi, Q_5, P_5\}$ dissapear from our problem and obviously equations (8.89-8.90) are empty.

At the begining of this section we have chosen coordinates $(t, \rho = r)$ with $\beta = 0$, what imposes $\Pi_1 = 0$ because of equation (5.72). The background Schwarzschild coordinates (t, r) are compatible with this choice and they also imply that the background momentum Π_2 vanishes. Finally, from background equations (5.74,5.75) one obtains the explicit form of the metric components,

$$
\alpha = \frac{1}{a} = \sqrt{1 - \frac{2M}{r}}. \qquad (8.102)
$$

In this way, one can solve equation (8.88) to write down the gauge-invariant momentum P_2 in terms of the time derivative of its conjugate variable Q_2, which is equal to the combination $\mathfrak{Z}$ (8.61) [see transformations (8.65) and (8.77)],

$$
P_2 = \frac{2r^6\Lambda}{l(l+1)(2r\Lambda + 6M)^2}\mathfrak{Z}_{,t}. \qquad (8.103)
$$

For convenience, we also define the rescaled variable,

$$
\mathcal{Z} \equiv -\frac{r^3}{a}\frac{\mathfrak{Z}}{2r\Lambda + 6M}, \qquad (8.104)
$$

which, inverting the canonical transformations that we have performed in the previous section, can be expressed in terms of the initial variables as

$$\mathcal{Z} \equiv \frac{(r-2M)}{3M+\Lambda r}\left\{rH_2 - r^2 K_{,r} - l(l+1)h_1\right\} + rK + \frac{1}{2}l(l+1)rG. \tag{8.105}$$

Introducing relation (8.86) in equation (8.87) we obtain the Zerilli equation,

$$\left(1-\frac{2M}{r}\right)^{-1}\left(-\frac{\partial^2 \mathcal{Z}}{\partial t^2}+\frac{\partial^2 \mathcal{Z}}{\partial r^{*2}}\right) - V_{\mathrm{Z}}\mathcal{Z} = 0, \tag{8.106}$$

where we have made use of the tortoise coordinates (t, r^*), with $r^* = r + 2M\ln(\frac{r}{2M}-1)$, and the potential is given by,

$$V_{\mathrm{Z}} \equiv \frac{l(l+1)}{r^2} - \frac{6M}{r^3}\frac{r^2\Lambda(\Lambda+2)+3M(r-M)}{(r\Lambda+3M)^2}. \tag{8.107}$$

Therefore, the gauge-invariant combination $\mathcal{Z}$ (8.105) reduces to the Zerilli variable when particularized to vacuum. We end this chapter emphasizing the fact that, making use of Hamiltonian gauge techniques, we have obtained a generalization of the Zerilli variable to dynamical scenarios.

Part IV

Second-order Gerlach and Sengupta formalism

Chapter 9

Decomposition of the perturbations

In this chapter we will decompose the perturbations in a spherical harmonic expansion, making use of the Regge-Wheeler-Zerilli harmonics and the Gerlach and Sengupta notation for the background spherical spacetime. We will also identify the part that encodes the gauge freedom in these decompositions. Finally we will present an iterative procedure, applicable up to the desired order, to eliminate this freedom by constructing gauge-invariant quantities.

9.1 Harmonic decomposition

The perturbations of the metric ${}^{\{n\}}h_{\mu\nu}$ are four dimensional symmetric two-tensors. Because of the block decomposition of the background metric (5.3), they can be splitted into two by two boxes in the following way

$$
{}^{\{n\}}h_{\mu\nu} = \left(\begin{array}{c|c} {}^{\{n\}}h_{AB} & {}^{\{n\}}h_{Ab} \\ \hline \text{Sym.} & {}^{\{n\}}h_{ab} \end{array} \right), \tag{9.1}
$$

where the GS notation for the indices has been used. From the point of view of the manifold S^2, each box has a different tensorial rank, i.e., ${}^{\{n\}}h_{AB}$ is a scalar in S^2 and a symmetric two tensor in M^2, ${}^{\{n\}}h_{Ab}$ is a vector on the sphere and in M^2 and, finally, ${}^{\{n\}}h_{ab}$ is a symmetric two tensor in S^2 but a scalar in M^2. We expand all of them in the RWZ basis of tensor spherical harmonics taking into account their tensorial character:

$$
{}^{\{n\}}h_{\mu\nu} = \sum_{l,m} \left(\begin{array}{cc} {}^{\{n\}}H_l^m{}_{AB}\, Z_l^m & {}^{\{n\}}H_l^m{}_A\, Z_l^m{}_b + {}^{\{n\}}h_l^m{}_A\, X_l^m{}_b \\ \text{Sym.} & {}^{\{n\}}K_l^m\, r^2\gamma_{ab} Z_l^m + {}^{\{n\}}G_l^m\, r^2 Z_l^m{}_{ab} + {}^{\{n\}}h_l^m\, X_l^m{}_{ab} \end{array} \right). \tag{9.2}
$$

In principle, the sum over l is infinite. Through this decomposition we have therefore converted ten functions, depending on all of the four coordinates (9.1), into ten infinite collections of functions, which only depend on coordinates of M^2 (9.2). For each label (l,m), the new ten functions are encoded in a symmetric two-tensor ${}^{\{n\}}H_l^m{}_{AB}$, the two vectors $\{{}^{\{n\}}H_l^m{}_A,\ {}^{\{n\}}h_l^m{}_A\}$, and the three scalars $\{{}^{\{n\}}K_l^m,\ {}^{\{n\}}G_l^m,\ {}^{\{n\}}h_l^m\}$. The same decomposition is done for the perturbations of the stress-energy tensor:

$$ {}^{\{n\}}t_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}T_l^m{}_{AB}\, Z_l^m & {}^{\{n\}}T_l^m{}_A\, Z_l^m{}_b + {}^{\{n\}}t_l^m{}_A\, X_l^m{}_b \\ \text{Sym.} & {}^{\{n\}}\tilde{T}_l^m\, r^2\gamma_{ab} Z_l^m + {}^{\{n\}}T_l^m\, Z_l^m{}_{ab} + {}^{\{n\}}t_l^m\, X_l^m{}_{ab} \end{pmatrix}. \tag{9.3} $$

The factors r^2 are introduced so that, for the case $n=1$, equations (9.2) and (9.3) reduce to those of the references [51, 54] up to the mentioned normalization of the axial tensors $X_l^m{}_a$ and $X_l^m{}_{ab}$ and some other small changes in the notation.

For future convenience, let us also define the decomposition in harmonics of the tensors ${}^{\{n\}}\mathcal{K}_{\mu\nu}$ and ${}^{\{n\}}\Psi_{\mu\nu}$ that will encode, respectively, the metric and stress-energy gauge invariants at n-th order:

$$ {}^{\{n\}}\mathcal{K}_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}\mathcal{K}_l^m{}_{AB}\, Z_l^m & {}^{\{n\}}\kappa_l^m{}_A\, X_l^m{}_b \\ \text{Sym.} & {}^{\{n\}}\mathcal{K}_l^m\, r^2\gamma_{ab} Z_l^m \end{pmatrix}, \tag{9.4} $$

$$ {}^{\{n\}}\Psi_{\mu\nu} = \sum_{l,m} \begin{pmatrix} {}^{\{n\}}\Psi_l^m{}_{AB}\, Z_l^m & {}^{\{n\}}\Psi_l^m{}_A\, Z_l^m{}_b + {}^{\{n\}}\psi_l^m{}_A\, X_l^m{}_b \\ \text{Sym.} & {}^{\{n\}}\tilde{\Psi}_l^m\, r^2\gamma_{ab} Z_l^m + {}^{\{n\}}\Psi_l^m\, Z_l^m{}_{ab} + {}^{\{n\}}\psi_l^m\, X_l^m{}_{ab} \end{pmatrix}. \tag{9.5} $$

As can be seen, the tensor $\mathcal{K}_{\mu\nu}$ has only six degrees of freedom instead of ten. That is so because, as it will become clear in the next section, we are going to use the perturbations of the metric to extract the remaining four gauge-dependent degrees of freedom.

Note that in all the decompositions performed in this section the axial harmonic coefficients are denoted with lowercase letters and the polar ones with capital letters. This convention will be very useful to identify the polarity of the functions under consideration.

9.2 Gauge freedom

As explained in Chapter 4, the perturbations (9.2) and (9.3) contain some degrees of freedom corresponding to a change in the definition of the perturbation owing to the possible action of a family of diffeomorphisms. Such freedom is encoded in a gauge vector for each perturbative order. Let us take the gauge vectors and decompose them also in tensor spherical harmonics,

$$ {}^{\{n\}}\xi_\mu = \sum_{l,m} \left({}^{\{n\}}\Xi_l^m{}_A\, Z_l^m,\ r^2\, {}^{\{n\}}\Xi_l^m\, Z_l^m{}_a + r^2\, {}^{\{n\}}\xi_l^m\, X_l^m{}_a \right). \tag{9.6} $$

For every perturbative order n and every pair (l, m), there are three polar gauge freedoms $({}^{\{n\}}\Xi_l^m{}_A$ and ${}^{\{n\}}\Xi_l^m)$ and only one that is axial $({}^{\{n\}}\xi_l^m)$. In order to eliminate those degrees of freedom from the metric and keep only the physically meaningful perturbations there are two options: fixing the gauge or looking for gauge-invariant quantities.

Fixing the gauge is just choosing the gauge vectors in any convenient way or, equivalently, demanding some appropriate constraints on the harmonic coefficients of the decomposition (9.2). A gauge choice might be specified, for $l \geq 2$, by the following algebraic conditions,

$$ {}^{\{n\}}H_l^m{}_A = 0, \qquad {}^{\{n\}}G_l^m = 0 \qquad {}^{\{n\}}h_l^m = 0. \tag{9.7} $$

For the case $n = 1$ this gauge was introduced by Regge and Wheeler [56]. As we will see, this gauge can not be imposed for $l = 0, 1$. Using the property (6.11) of the tensor spherical harmonics, it is easy to see that this gauge condition leads to a full metric $\tilde{g}_{\mu\nu}$ such that

$$ \tilde{g}_{Ab:c}\, g^{bc} = 0, \qquad \tilde{g}_{ab} = \tilde{K} g_{ab}, \tag{9.8} $$

where the four dimensional function $\tilde{K}$ is just the sum

$$ \tilde{K} \equiv \sum_{l=0}^{\infty} {}^{\{n\}}K_l^m Z_l^m. \tag{9.9} $$

As explained in the final part of Chapter 4, the gauge invariants we are going to consider are conceptually quite different from those defined in [100]. They will be linear combinations of the perturbations that functionally do not change under a gauge transformation (acting on those perturbations), rather than the perturbations of some background tensor whose Lie derivative vanishes along all the vector fields.

The procedure to determine them is to fix a gauge and calculate which gauge vectors take a generic perturbation into the mentioned gauge. The arbitrariness of the process resides in the choice of the former gauge. In our case, following GS, it will be the RW gauge, which has a clear geometrical meaning. In this sense, these invariants can be thought of as “what would be measured in the RW gauge", but expressed in a generic gauge. From now on we will use the adjective gauge-invariant in this sense.

The first-order metric gauge invariants were first defined by Moncrief [101] for Schwarzschild background and generalized by GS [51] for any spherically symmetric spacetime. In that paper they also introduced the matter invariants, that is, the gauge invariants corresponding to the stress-energy tensor. Here we are going to give an iterative method to calculate the gauge invariants up to any order and, for the second order, we will give explicit expressions decomposed in harmonics in Chapter 11. We will start explaining what GS did to construct the invariants at first order and demonstrate that the procedure described above is equivalent to their construction. After that discussion we will deduce formulas for the higher-order gauge invariants. We will discuss the two special cases $l = 0, 1$ in the last subsection.

9.2.1 First-order gauge invariants

In the following, and except when the notation might become misleading, we remove the harmonic labels l and m. All equations appearing in this and the following subsections are valid for $l \geq 2$. As has been explained in Chapter 6, this is because, for $l = 0, 1$ there exist several tensor spherical harmonics that are identically zero. Therefore, the equations obtained as the harmonic coefficients of those vanishing harmonics are in fact empty.

At first order the gauge transformation of the perturbation of the metric is given by

$$ {}^{\{1\}}\overline{h}_{\mu\nu} = {}^{\{1\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{1\}}\xi}\, g_{\mu\nu}, \tag{9.10} $$

where ${}^{\{1\}}\xi^{\mu}$ is the gauge generator vector field. If we decompose this equation in harmonics, taking into account the decompositions for the gauge vector (9.6) and for the perturbations (9.2), we get

$$ {}^{\{1\}}\overline{H}_{AB} = {}^{\{1\}}H_{AB} + {}^{\{1\}}\Xi_{A|B} + {}^{\{1\}}\Xi_{B|A}, \tag{9.11} $$
$$ {}^{\{1\}}\overline{H}_{A} = {}^{\{1\}}H_{A} + {}^{\{1\}}\Xi_{A} + r^2\, {}^{\{1\}}\Xi_{|A}, \tag{9.12} $$
$$ {}^{\{1\}}\overline{K} = {}^{\{1\}}K + 2v^A\, {}^{\{1\}}\Xi_{A} - l(l+1)\, {}^{\{1\}}\Xi, \tag{9.13} $$
$$ {}^{\{1\}}\overline{G} = {}^{\{1\}}G + 2\, {}^{\{1\}}\Xi, \tag{9.14} $$
$$ {}^{\{1\}}\overline{h}_{A} = {}^{\{1\}}h_{A} + r^2\, {}^{\{1\}}\xi_{|A}, \tag{9.15} $$
$$ {}^{\{1\}}\overline{h} = {}^{\{1\}}h + 2r^2\, {}^{\{1\}}\xi. \tag{9.16} $$

The overbar quantities are the harmonic coefficients corresponding to the decomposition of ${}^{\{1\}}\overline{h}_{\mu\nu}$. Following GS [51] we can construct the following linear combinations:

$$ {}^{\{1\}}\mathcal{K}_{AB} = {}^{\{1\}}H_{AB} + \left(\frac{r^2}{2}\, {}^{\{1\}}G_{|A} - {}^{\{1\}}H_{A}\right)_{|B} + \left(\frac{r^2}{2}\, {}^{\{1\}}G_{|B} - {}^{\{1\}}H_{B}\right)_{|A}, \tag{9.17} $$
$$ {}^{\{1\}}\mathcal{K} = {}^{\{1\}}K + 2v^A \left(\frac{r^2}{2}\, {}^{\{1\}}G_{|A} - {}^{\{1\}}H_{A}\right) + \frac{l(l+1)}{2}\, {}^{\{1\}}G, \tag{9.18} $$
$$ {}^{\{1\}}\kappa_{A} = {}^{\{1\}}h_{A} - \frac{1}{2}\, {}^{\{1\}}h_{|A}, \tag{9.19} $$

which are invariants under the transformations (9.11–9.16). Recall that we follow the convention of defining polar (axial) objects with capital (lowercase) letters. In this way, the physical degrees of freedom of the perturbations are encoded in these invariants without fixing any gauge: on the one hand the four polar freedoms in the tensor ${}^{\{1\}}\mathcal{K}_{AB}$ and in the scalar ${}^{\{1\}}\mathcal{K}$, and on the other hand the two axial degrees in the vector ${}^{\{1\}}\kappa_A$.

The construction of those invariants, as we have explained and will become clearer below, is tied to the RW gauge. In fact, it is straightforward to see that in the RW gauge

$$
\begin{aligned}
{}^{\{1\}}\mathcal{K}_{AB} &= {}^{\{1\}}H^{(\mathrm{RW})}_{AB}, && (9.20)\\
{}^{\{1\}}\mathcal{K} &= {}^{\{1\}}K^{(\mathrm{RW})}, && (9.21)\\
{}^{\{1\}}\kappa_A &= {}^{\{1\}}h^{(\mathrm{RW})}_A. && (9.22)
\end{aligned}
$$

According to our discussion, the gauge invariants are the representatives of each equivalence class that one constructs taking any general perturbation to the RW form, so we are going to write the gauge invariants ${}^{\{1\}}\mathcal{K}_{\mu\nu}$ as,

$$
{}^{\{1\}}\mathcal{K}_{\mu\nu} = {}^{\{1\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{1\}}p}\, g_{\mu\nu}, \tag{9.23}
$$

where ${}^{\{1\}}h_{\mu\nu}$ is a general perturbation decomposed as (9.2). The invariants are organized in the tensor ${}^{\{1\}}\mathcal{K}_{\mu\nu}$ (9.4) and ${}^{\{1\}}p^\mu$ is the first-order gauge vector that we have to determine and that is decomposed as

$$
{}^{\{n\}}p_\mu \equiv \sum_{l,m} \left({}^{\{n\}}P^m_{l\,A}\; Z^m_l, r^2\, {}^{\{n\}}P^m_l\; Z^m_{l\ a} + r^2\, {}^{\{n\}}q^m_l\; X^m_{l\ a} \right). \tag{9.24}
$$

If we translated equation (9.23) into spherical harmonics, we would obtain a particularization of the general gauge transformation (9.11–9.16). Three of those equations, namely (9.12), (9.14) and (9.16), get a vanishing left-hand side

$$
\begin{aligned}
0 &= {}^{\{1\}}H_A + {}^{\{1\}}P_A + r^2\, {}^{\{1\}}P_{|A}, && (9.25)\\
0 &= {}^{\{1\}}G + 2\, {}^{\{1\}}P, && (9.26)\\
0 &= {}^{\{1\}}h + 2r^2\, {}^{\{1\}}q, && (9.27)
\end{aligned}
$$

and therefore specify the vector ${}^{\{1\}}p^\mu$ in terms of the perturbations:

$$
{}^{\{1\}}P_A = \frac{r^2}{2}\, {}^{\{1\}}G_{|A} - {}^{\{1\}}H_A, \qquad {}^{\{1\}}P = -\frac{1}{2}\, {}^{\{1\}}G, \qquad {}^{\{1\}}q = -\frac{1}{2r^2}\, {}^{\{1\}}h. \tag{9.28}
$$

From the other three equations we can then read the invariants:

$$
\begin{aligned}
{}^{\{1\}}\mathcal{K}_{AB} &= {}^{\{1\}}H_{AB} + {}^{\{1\}}P_{A|B} + {}^{\{1\}}P_{B|A}, && (9.29)\\
{}^{\{1\}}\mathcal{K} &= {}^{\{1\}}K + 2v^A\, {}^{\{1\}}P_A - l(l+1)\, {}^{\{1\}}P, && (9.30)\\
{}^{\{1\}}\kappa_A &= {}^{\{1\}}h_A + r^2\, {}^{\{1\}}q_{|A}. && (9.31)
\end{aligned}
$$

With this analysis we have proven that the RW gauge is well posed in the sense that we can take any generic perturbation $h_{\mu\nu}$ to that gauge. We have also clarified the interpretation of the gauge invariants constructed in [51, 101]. In this respect, a crucial property of the RW gauge is that we only have to solve algebraic equations to impose it and, therefore, the solution to these equations is uniquely determined. It is now easy to understand why it is not possible to impose the RW gauge for the cases $l = 0, 1$. For these values of the harmonic labels, equations (9.26–9.27), as well as (9.25) for $l = 0$, become spurious due to the vanishing of the corresponding tensor harmonics. As a consequence, it is not possible to define the gauge vectors that bring the generic perturbations to the RW form. In fact, it is just in this respect that the RW gauge fails to be a fully rigid gauge. We will discuss this issue in more detail in the last subsection of this chapter.

In order to find the gauge invariants for any other tensor field, in particular for the stress-energy tensor, it suffices to transform it to the gauge we have chosen using the vector ${}^{\{1\}}p^{\mu}$,

$$ {}^{\{1\}}\Psi_{\mu\nu} = {}^{\{1\}}t_{\mu\nu} + \mathcal{L}_{\,{}^{\{1\}}p} t_{\mu\nu}, \tag{9.32} $$

where $t_{\mu\nu}$ is the background stress-energy tensor (5.4) and the ten gauge invariants corresponding to the matter content will be encoded in the tensor $\Psi_{\mu\nu}$ (9.5):

$$ \begin{aligned} {}^{\{1\}}\Psi_{AB} &= {}^{\{1\}}T_{AB} + {}^{\{1\}}P^{C}{}_{|A} t_{BC} + {}^{\{1\}}P^{C}{}_{|B} t_{AC} + {}^{\{1\}}P^{C} t_{AB|C}, && (9.33)\\ {}^{\{1\}}\Psi_{A} &= {}^{\{1\}}T_{A} + {}^{\{1\}}P^{B} t_{AB} + \frac{r^2}{2}\,{}^{\{1\}}P_{|A} Q, && (9.34)\\ {}^{\{1\}}\psi_{A} &= {}^{\{1\}}t_{A} + \frac{r^2}{2}\,{}^{\{1\}}q_{|A} Q, && (9.35)\\ {}^{\{1\}}\Psi &= {}^{\{1\}}T + r^2\,{}^{\{1\}}PQ, && (9.36)\\ {}^{\{1\}}\tilde{\Psi} &= {}^{\{1\}}\tilde{T} + \frac{1}{2r^2}\,{}^{\{1\}}P^{A}(Qr^2)_{|A} - \frac{l(l+1)}{2}\,{}^{\{1\}}PQ, && (9.37)\\ {}^{\{1\}}\psi &= {}^{\{1\}}t + r^2\,{}^{\{1\}}qQ. && (9.38) \end{aligned} $$

Here, the objects Q and t_{AB} are the components of the background stress-energy tensor defined in (5.4). Note that we have used the vector ${}^{\{1\}}p^{\mu}$ (9.28) instead of the harmonic coefficients ${}^{\{1\}}H_A$, ${}^{\{1\}}G$ and ${}^{\{1\}}h$, in order to simplify the expression. It is easy to see that these objects are invariant under a gauge transformation parameterized by any vector ${}^{\{1\}}\xi^{\mu}$, that will transform the metric perturbations as (9.11–9.16) and the matter perturbations as

$$ {}^{\{1\}}\overline{t}_{\mu\nu} = {}^{\{1\}}t_{\mu\nu} + \mathcal{L}_{\,{}^{\{1\}}\xi} t_{\mu\nu}. \tag{9.39} $$

Introducing harmonic decompositions, this last equation takes the form

$$
\begin{aligned}
{}^{\{1\}}\overline{T}_{AB} &= {}^{\{1\}}T_{AB} + {}^{\{1\}}\Xi^{C}{}_{|A}t_{BC} + {}^{\{1\}}\Xi^{C}{}_{|B}t_{AC} + {}^{\{1\}}\Xi^{C}t_{AB|C}, &(9.40)\\
{}^{\{1\}}\overline{T}_{A} &= {}^{\{1\}}T_{A} + {}^{\{1\}}\Xi^{B}t_{AB} + \frac{r^2}{2}\,{}^{\{1\}}\Xi_{|A}Q, &(9.41)\\
{}^{\{1\}}\overline{t}_{A} &= {}^{\{1\}}t_{A} + r^2 Q\left(\frac{{}^{\{1\}}\xi}{2}\right)_{|A}, &(9.42)\\
{}^{\{1\}}\overline{T} &= {}^{\{1\}}T + r^2\,{}^{\{1\}}\Xi Q, &(9.43)\\
{}^{\{1\}}\overline{\tilde{T}} &= {}^{\{1\}}\tilde{T} + \frac{1}{2r^2}\,{}^{\{1\}}\Xi^{A}(Qr^2)_{|A} - \frac{l(l+1)}{2}\,{}^{\{1\}}\Xi Q, &(9.44)\\
{}^{\{1\}}\overline{t} &= {}^{\{1\}}t + r^2\,{}^{\{1\}}\xi Q. &(9.45)
\end{aligned}
$$

9.2.2 High-order gauge invariants

The procedure we have used shows how the invariants are associated to a choice of gauge, in this case the RW gauge, and can be generalized to higher orders [114]. From formula (4.9) we see that the gauge transformation for the n-th order perturbation of a background tensor T parameterized by the gauge vectors $\{\,{}^{\{1\}}\xi^{\mu}, ..., {}^{\{n\}}\xi^{\mu}\}$ is schematically given by

$$\overline{\Delta^n[T]} - \Delta^n[T] = \mathcal{L}_{\,{}^{\{n\}}\xi}T + {}^{\{n\}}\mathcal{J}. \qquad (9.46)$$

Here ${}^{\{n\}}\mathcal{J}$ is a source term depending on $\Delta^m[T]$ and ${}^{\{m\}}\xi^{\mu}$ for all $m < n$, but not on ${}^{\{n\}}\xi^{\mu}$, and hence at first order ${}^{\{1\}}\mathcal{J} = 0$. Therefore the dependence of the n-th order gauge transformation on the vector ${}^{\{n\}}\xi^{\mu}$ is the same at every order.

In order to construct the n-th order gauge invariants, we have to obtain all gauge vectors $\{\,{}^{\{1\}}p^{\mu}, ..., {}^{\{n\}}p^{\mu}\}$ that take any generic perturbation ${}^{\{n\}}h_{\mu\nu}$ (9.2) to the RW form ${}^{\{n\}}\mathcal{K}_{\mu\nu}$ (9.4). Hence, the equation we need to solve is the same as at first order (9.23) but with the source term ${}^{\{n\}}\mathcal{J}_{\mu\nu}$ included:

$${}^{\{n\}}\mathcal{K}_{\mu\nu} = {}^{\{n\}}h_{\mu\nu} + \mathcal{L}_{\,{}^{\{n\}}p}g_{\mu\nu} + {}^{\{n\}}\mathcal{J}_{\mu\nu}\,. \qquad (9.47)$$

This hierarchy of equations can be solved iteratively because, as we have said, ${}^{\{n\}}\mathcal{J}_{\mu\nu}$ depends on perturbations of lower order ${}^{\{m\}}h_{\mu\nu}$ and on the gauge vectors ${}^{\{m\}}p^{\mu}$ which are supposed to have been determined in terms of ${}^{\{m\}}h_{\mu\nu}$. As usual, we can decompose ${}^{\{n\}}\mathcal{J}_{\mu\nu}$ in spherical harmonics:

$${}^{\{n\}}\mathcal{J}_{\mu\nu} = \sum_{l,m}\begin{pmatrix} {}^{\{n\}}\mathcal{J}^{m}_{l\,AB}\,Z^{m}_{l} & {}^{\{n\}}\mathcal{J}^{m}_{l\,A}\,Z^{m}_{l\,b} + {}^{\{n\}}\mathfrak{j}^{m}_{l\,A}\,X^{m}_{l\,b} \\ \text{Sym.} & {}^{\{n\}}\tilde{\mathcal{J}}^{m}_{l}\,r^2\gamma_{ab}Z^{m}_{l} + {}^{\{n\}}\mathcal{J}^{m}_{l}\,Z^{m}_{l\,ab} + {}^{\{n\}}\mathfrak{j}^{m}_{l}\,X^{m}_{l\,ab}\end{pmatrix}. \qquad (9.48)$$

As it happens in the first order case, expanding equation (9.47) in harmonics, we obtain six equations. Three of them are

$$
\begin{aligned}
0 &= {}^{\{n\}}H_A + {}^{\{n\}}P_A + r^2\,{}^{\{n\}}P_{|A} + {}^{\{n\}}\mathcal{J}_A, &(9.49)\\
0 &= {}^{\{n\}}G + 2\,{}^{\{n\}}P + \frac{1}{r^2}\,{}^{\{n\}}\mathcal{J}, &(9.50)\\
0 &= {}^{\{n\}}h + 2r^2\,{}^{\{n\}}q + {}^{\{n\}}\mathfrak{j}. &(9.51)
\end{aligned}
$$

From this we can calculate the components of the gauge vector ${}^{\{n\}}p^\mu$ as a function of the perturbations of the metric and the gauge vectors of lower order:

$$
\begin{aligned}
{}^{\{n\}}P_A &= \frac{r^2}{2}\left({}^{\{n\}}G + \frac{1}{r^2}\,{}^{\{n\}}\mathcal{J}\right)_{|A} - {}^{\{n\}}H_A - {}^{\{n\}}\mathcal{J}_A, &(9.52)\\
{}^{\{n\}}P &= -\frac{1}{2}\left({}^{\{n\}}G + \frac{1}{r^2}\,{}^{\{n\}}\mathcal{J}\right), &(9.53)\\
{}^{\{n\}}q &= -\frac{1}{2r^2}\left({}^{\{n\}}h + {}^{\{n\}}\mathfrak{j}\right). &(9.54)
\end{aligned}
$$

The three remaining equations arising from (9.47) provide the gauge invariants:

$$
\begin{aligned}
{}^{\{n\}}\mathcal{K}_{AB} &= {}^{\{n\}}H_{AB} + {}^{\{n\}}P_{A|B} + {}^{\{n\}}P_{B|A} + {}^{\{n\}}\mathcal{J}_{AB}, &(9.55)\\
{}^{\{n\}}\mathcal{K} &= {}^{\{n\}}K + 2v^A\,{}^{\{n\}}P_A - l(l+1)\,{}^{\{n\}}P + {}^{\{n\}}\tilde{\mathcal{J}}, &(9.56)\\
{}^{\{n\}}\kappa_A &= {}^{\{n\}}h_A + r^2\,{}^{\{n\}}q_{|A} + {}^{\{n\}}\mathfrak{j}_A. &(9.57)
\end{aligned}
$$

In order to obtain the n-th order stress-energy invariants ${}^{\{n\}}\Psi_{\mu\nu}$, we simply start with the n-th order perturbation ${}^{\{n\}}t_{\mu\nu}$ and apply a gauge transformation (4.9) parameterized by the vectors $\{{}^{\{1\}}p^\mu, ..., {}^{\{n\}}p^\mu\}$.

9.2.3 The particular cases $l = 0, 1$

It is well-known that, at first order, one cannot construct local gauge invariants for $l = 0, 1$ by the methods explained in the previous subsection. This is because some of the equations (9.25–9.27) are not present in those cases, and therefore it is not possible to attain a local expression for the components of the gauge vector ${}^{\{n\}}p^\mu$ in terms of those of the metric perturbations. Of course, gauge conditions different from the RW ones can be imposed on the metric perturbations and hence one can obtain from them the associated gauge invariants, but these will be nonlocal because the gauge vector will be given by an integral expression (over $\mathcal{M}^2$) of the metric perturbations. Whether this is useful or not will depend on the particular application that one is studying. The following discussion will apply to gauge invariants tied to the RW gauge.

The same obstruction appears as well at every order in perturbation theory, so that one cannot get local gauge invariants for $l = 0, 1$ at any order. However, mode coupling makes the problem worse: the existence of lower-order modes with $l = 0, 1$ may prevent the construction of higher-order local gauge invariants with $l \geq 2$. This will happen when such lower-order modes have a nonzero contribution to the sources $\mathcal{J}_{\mu\nu}$.

At first-order the GS gauge invariants remain unchanged under the restricted group ${}^{\{1\}}\mathcal{G}^{\diamond}$ (see Section 4.1.1), and here we introduce the diamond notation to indicate that no first-order generator with $l = 0, 1$ is included. However, removing all $l = 0, 1$ generators is already inconsistent at second order if we demand invariance under a group of transformations, because the $l = 0, 1$ components of the vectors ${}^{\{2\}}\xi$ will be unavoidably generated by coupling of first-order gauge modes [cf. composition (4.13)]. Fortunately, those offending gauge modes act only on the $l = 0, 1$ second-order perturbations, for which invariants cannot be constructed anyway. All other second-order perturbations admit a gauge-invariant form, as given in the previous subsection, under the gauge group ${}^{\{2\}}\mathcal{G}^{\diamond}$ where again the diamond denotes that no first-order $l = 0, 1$ gauge mode is included, but all other first and second-order (including $l = 0, 1$) modes are allowed. That is one of the main results of this chapter: in spherical symmetry algebraic gauge-invariant combinations of the perturbations can be consistently and simultaneously constructed for all $l \geq 2$ modes at second order, all being invariant under the group ${}^{\{2\}}\mathcal{G}^{\diamond}$.

The situation at third and higher orders is more restrictive. In these cases, one has to restrict to a finite set of lower-order modes both in the gauge generators and in the perturbations in order to define some form of gauge invariance. This is because the presence at first order of any gauge mode $l \geq 2$ will generate, just by self-coupling, the second-order modes with harmonic labels 0 and 1. It will then be imposible to construct the gauge-invariant form of a third-order perturbation whose source $\mathcal{J}$ contains a term coupling any of these second-order $l = 0, 1$ modes with any first-order mode. But those sources generically contain all possible couplings, and so only a problem in which we restrict the number of first-order gauge modes allows some form of gauge-invariance at third order.

Let us then analyze generic mode coupling around spherical symmetry, starting at second order and then proceeding to higher orders. We will later give some bounds on the number of modes that can be present at first order to allow for the construction of a nth-order mode with label l. The second-order l-mode will get a contribution from a pair of first-order modes $\hat{l}$ and $\bar{l}$ if two conditions are obeyed. On the one hand, the harmonic labels must be related by the standard composition formulas

$$|\hat{l} - \bar{l}| \leq l \leq \hat{l} + \bar{l}, \qquad \text{and} \qquad m = \hat{m} + \bar{m}. \tag{9.58}$$

In the following discussion we will not consider the m harmonic labels, but it would be straightforward to include them. On the other hand, mode coupling must conserve parity. To any harmonic coefficient with label l, we associate a polarity sign σ such that, under parity, the harmonic changes by a sign $\sigma(-1)^l$. Polar (axial) harmonics have $\sigma = +1$ ($\sigma = -1$). Then, parity conservation implies the second condition:

$$(-1)^{\bar{l}+\hat{l}-l} \equiv \epsilon = \sigma\bar{\sigma}\hat{\sigma}, \tag{9.59}$$

where the alternating sign ϵ was already defined in Subsection 6.7.3. There is a special case in which the coupling of two modes satisfying equations (9.58) and (9.59) does not contribute to a second-order mode, and the reason comes from the properties of the E-coefficients (6.77) that appear in the product formula for the tensor harmonics (6.83). In axisymmetry ($\bar{m} = \hat{m} = 0$) the Clebsch-Gordan coefficients, and as a consequence the E-coefficients, vanish if $\bar{l} + \hat{l} + l$ is odd.

The above analysis can be extended to higher orders. In particular, the parity condition will be that a collection of k modes with harmonic labels $\{l_1, ..., l_k\}$ and polarities $\{\sigma_1, ..., \sigma_k\}$ will contribute to the mode (l, σ) only if

$$(-1)^l\sigma = \Pi_{i=1}^k (-1)^{l_i}\sigma_i. \tag{9.60}$$

Let us finally consider the case in which we have a first-order finite collection of modes with their harmonic labels taking all the values from $l = 2$ to $l = l_{\rm max}$, with contributions from both the polar and the axial sectors. Coupling of these modes at order n will generate some new modes, following the above rules, so that the highest value of their harmonic label will be $nl_{\rm max}$. The construction of the gauge invariants under the corresponding group of transformations and tied to the RW gauge is only guaranteed for those modes with harmonic label greater than $(n-2)l_{\rm max} + 1$. This number comes from the coupling of the $(n-2)$th-order $(n-2)l_{\rm max}$-mode with the second-order $l = 1$ mode.

As a summary, working at order $n > 2$ only a finite number of gauge generators can be included in the invariance group at orders $1, ..., n-2$. The unavoidable presence of second-order gauge modes with $l = 0, 1$ couples to any metric perturbation at order m and with harmonic label l, preventing the construction of a gauge invariant of order $m + 2$ and label l or $l \pm 1$, with respect to those gauge modes.

Chapter 10

First-order

This chapter summarizes the formalism by Gerlach and Sengupta for first-order perturbations around a spherical spacetime [51,54], and introduces the main notations used in the following chapters. These are geometrical notations which will allow us to show that the master axial function we have obtained in Chapter 7 via Hamiltonian methods coincides with the master scalar introduced by Gerlach and Sengupta.

10.1 Einstein equations

The linearized Einstein equations

$$ {}^{\{1\}}G_{\mu\nu} = 8\pi\, {}^{\{1\}}T_{\mu\nu}, \tag{10.1}$$

expressed in terms of the gauge invariant variables, lead to the six GS equations [51],

$$\begin{aligned} E_{AB}[\mathcal{K}] &\equiv \left[\frac{(l-1)(l+2)}{2r^2} + 3v_C v^C + 2v^C{}_{|C}\right]\mathcal{K}_{AB} + v_C(\mathcal{K}^C{}_{B|A} + \mathcal{K}^C{}_{A|B} - \mathcal{K}_{AB}{}^{|C}) \\ &- (v_B\mathcal{K}_{|A} + v_A\mathcal{K}_{|B} + \mathcal{K}_{|AB}) + g_{AB}\left[r^{-3}(r^3\mathcal{K}^{|C})_{|C} - \frac{(l-1)(l+2)}{2r^2}\mathcal{K}\right. \\ &- \left.\frac{l(l+1)}{2r^2}\mathcal{K}^C{}_C + (\mathcal{K}^C{}_{C|D} - 2\mathcal{K}^C{}_{D|C})v^D - (3v_C v_D + 2v_{C|D})\mathcal{K}^{CD}\right] = 8\pi\Psi_{AB}, \end{aligned} \tag{10.2}$$

$$ E_A[\mathcal{K}] \equiv \frac{1}{2}(\mathcal{K}^B{}_B v_A - \mathcal{K}^B{}_{B|A} + \mathcal{K}_A{}^B{}_{|B} - \mathcal{K}_{|A}) = 8\pi\Psi_A, \tag{10.3}$$

$$\begin{aligned} \tilde{E}[\mathcal{K}] &\equiv \frac{1}{2}\left\{(\mathcal{K}_{AB} - \mathcal{K}g_{AB})\,{}^{(4)}R^{AB} - \frac{l(l+1)}{2r^2}\mathcal{K}^A{}_A + \mathcal{K}^A{}_{A|B}{}^B\right. \\ &- \left.2\mathcal{K}^A{}_{B|A}v^B + \mathcal{K}^A{}_{A|B}v^B - \mathcal{K}^{AB}{}_{|AB} + \mathcal{K}^{|A}{}_A + 2\mathcal{K}_{|A}v^A\right\} = 8\pi\tilde{\Psi}, \end{aligned} \tag{10.4}$$

$$E[\mathcal{K}] \equiv -\frac{1}{2}\mathcal{K}^A{}_A = 8\pi\Psi, \tag{10.5}$$

$$O_A[\kappa] \equiv \frac{(l-1)(l+2)}{2r^2}\kappa_A - \frac{1}{2r^2}\left[r^4\left(\frac{\kappa_A}{r^2}\right)_{|C} - r^4\left(\frac{\kappa_C}{r^2}\right)_{|A}\right]^{|C} + 4\pi Q\kappa_A = 8\pi\psi_A, \tag{10.6}$$

$$O[\kappa] \equiv \kappa^A{}_{|A} = 8\pi\psi, \tag{10.7}$$

where we have defined the four dimensional background Ricci tensor ${}^{(4)}R_{AB}$ and, for future convenience, the linear operators E and O acting respectively in the polar and axial parts of the perturbations. Note that the $n = 1$ label has been dropped, since all the considered harmonic coefficients are of first order.

It is important to remember that the gauge-invariant variables reduce to those obtained using the RW gauge, that is, cancelling the gauge vector p^μ. Hence, the above equations are essentially the ones attained in the RW gauge.

10.2 Energy-momentum conservation equations

A complete set of evolution equations is obtained only after specifying the particular type of matter content of the system. Some simple systems like scalar fields or perfect fluids are completely defined dynamically by energy-momentum conservation, but this is not the case in general. However, we can generally analyze the consequences of perturbing the matter conservation equations, as well as use this analysis as a check of the perturbed Bianchi identities, and hence as a consistency check of the equations given in the previous section.

In the background, the energy-momentum conservation equation is given by (5.16). At each perturbative order it can be decomposed into three geometric parts, given its vectorial character: a vector equation in the polar sector and two scalar (one polar and one axial) equations. We define the following operators acting on the first-order metric and matter perturbations,

$$\begin{aligned} L_A[\Psi,\mathcal{K}] \equiv & -\frac{l(l+1)}{r^2}\Psi_A - 2v_A\tilde{\Psi} + \frac{1}{r^2}(r^2\Psi_{AB})^{|B} \\ & - \frac{1}{2}t^{BC}\mathcal{K}_{BC|A} - \frac{r^2}{2}Q(r^{-2}\mathcal{K})_{|A} + \frac{1}{2}t_{AB}\mathcal{K}^C{}_C{}^{|B} + t_{AB}\mathcal{K}^{|B} - \frac{1}{r^2}(r^2 t_{AB}\mathcal{K}^{BC})_{|C}, \end{aligned} \tag{10.8}$$

$$L[\Psi,\mathcal{K}] \equiv \tilde{\Psi} - \frac{(l-1)(l+2)}{2r^2}\Psi + \frac{1}{r^2}(r^2\Psi^A)_{|A} - (\mathcal{K} - \frac{1}{2}\mathcal{K}^A{}_A)\frac{Q}{2} - \frac{1}{2}\mathcal{K}^{AB}t_{AB}, \tag{10.9}$$

$$\tilde{L}[\psi,\kappa] \equiv \frac{1}{r^2}(r^2\psi^A)_{|A} - \frac{(l-1)(l+2)}{2r^2}\psi - \frac{1}{2r^2}(Qr^2\kappa^A)_{|A}. \tag{10.10}$$

With these at hand, the first order energy-momentum conservation equations can be written in compact notation as

$$L_A[\Psi, \mathcal{K}] = 0, \quad (10.11)$$

$$L[\Psi, \mathcal{K}] = 0, \quad (10.12)$$

$$\tilde{L}[\psi, \kappa] = 0. \quad (10.13)$$

10.3 Axial master equation

10.3.1 Gerlach-Sengupta master scalar

In the axial sector the invariant vector κ_A satisfies equations (10.6-10.7). But, using the energy-momentum conservation equation (10.13), it is easy to prove that equation (10.7) can be obtained by differentiating equation (10.6). Combining the invariants ψ_A (9.35) and κ_A (9.31), let us introduce a new matter invariant

$$\tilde{\psi}_A \equiv \psi_A - \frac{Q}{2}\kappa_A = t_A - \frac{Q}{2}h_A, \quad (10.14)$$

and, as the rotational of the axial vector κ_A, the GS master scalar

$$\Pi \equiv \epsilon^{AB}\left(\frac{\kappa_A}{r^2}\right)_{|B}. \quad (10.15)$$

In terms of them, equation (10.6) takes the form

$$\frac{1}{2r^2}\epsilon_{AB}(r^4\Pi)^{|B} + \frac{(l-1)(l+2)}{2r^2}\kappa_A = 8\pi\tilde{\psi}_A. \quad (10.16)$$

If we take the rotational of this equation we obtain the so-called GS master equation

$$-\left[\frac{1}{2r^2}(r^4\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\Pi = 8\pi\epsilon^{AB}\tilde{\psi}_{A|B}. \quad (10.17)$$

This is a wave equation for the scalar Π. Of course, different matter models will have additional variables and equations coupled to this equation but we stress the fact that both the form of Π (10.15) and its wave equation (10.17) will remain unchanged. This variable is used for historical reasons. But, in fact, it would be better to use the rescaled variable $\tilde{\Pi} \equiv r^3\Pi$, because its evolution equation has no first-order derivatives,

$$\tilde{\Pi}_{|A}{}^{A} - V_{\mathrm{RW}}\tilde{\Pi} = 8\pi r^3\epsilon^{AB}\tilde{\psi}_{A|B}. \quad (10.18)$$

This equation is valid in any spherically symmetric background and the potential is given by

$$V_{\mathrm{RW}} = \frac{l(l+1)}{r^2} + \frac{3}{r^2}(f^2 - 1). \tag{10.19}$$

The RW subscript stands for Regge-Wheeler since, on a Schwarzschild background, equation (10.18) reduces to the RW equation, as we will show in Chapter 12.

The scalar master Π can be considered as containing all information about the axial gravitational wave, since, once Π is found, we could use equation (10.16) to obtain the vector κ_A. This is true once we have solved also for all matter perturbations that define $\tilde{\psi}_A$. But one has to be careful because the vector field κ_A could also appear inside $\tilde{\psi}_A$, what may imply a non-algebraic solution of (10.16). Let us analyze this issue in more detail. It is possible to solve equation (10.16) algebraically for κ_A as long as $\tilde{\psi}_A$ does not contain the symmetrized derivative $\kappa_{(A|B)}$ [since the antysimmetric part defines the master scalar (10.15)] or higher derivatives of κ_A. Second and higher derivatives of κ_A can be ruled out by requiring that the matter stress-energy tensor must not contain second derivatives of the metric, because that would change the principal part of the Einstein equations. Symmetrized first order derivatives of κ_A cannot be ruled out on physical grounds because perturbation of covariant derivatives of tensor fields may introduce the term

$$\Delta^{axial}[\Gamma^a{}_{BC}] = X^a\, \frac{\kappa_{(B|C)}}{r^2}. \tag{10.20}$$

This is the only possible source of symmetrized derivatives of κ_A; all other perturbations of Christoffel symbols give either Π or undifferentiated κ_A terms. Summarizing, it will not be possible to obtain the vector field κ_A in terms of the master scalar Π algebraically for those matter models that have derivatives of tensor fields (no scalars) in their energy-momentum tensor. This is not the case for standard matter models, but we found an example in the Einstein-aether theory [159]. In this theory the aether is described by a vector field that appears differentiated in the energy-momentum tensor. Hence, in order to obtain the vector κ_A in terms of the master scalar Π and the perturbations of the aether, one will have to integrate equation (10.16).

In order to end this subsection, we would like to show the following equation, that relates the master scalar to one of the perturbations of the Riemann tensor (see for instance [136]),

$$\Delta\,[R_{ABcd}] = -\frac{l(l+1)}{2}\, Y\, \epsilon_{AB}\, \epsilon_{cd}\, r^2\Pi, \tag{10.21}$$

which gives another physical interpretation of the master scalar Π.

10.3.2 Comparison with the Hamiltonian approach

Let us now compare the GS master variable with the gauge-invariant variables we have found through the Hamiltonian analysis, for the particular matter content of a scalar field, in Chapter 7. First we note that the background momenta can be rewritten as

$$\Pi_1 = -2u^A v_A, \tag{10.22}$$
$$\Pi_2 = -2u^A v_A - 2u^A{}_{|A}, \tag{10.23}$$
$$\Pi_3 = u^A \Phi_{,A}, \tag{10.24}$$

where u^A is again the vector field orthogonal to the spacelike hypersurfaces Σ_t. We see that Π_1 and Π_3 are essentially time components of vectors in M^2. However Π_2 is a more complicated object. Comparing their definitions (9.2) and (7.1-7.3), it is clear that the vector h_A and the scalar h are related in the following way to the original Hamiltonian variables,

$$h_1 = -h_\rho, \tag{10.25}$$
$$h_2 = 2h, \tag{10.26}$$
$$h_0 = \alpha^2 h^t. \tag{10.27}$$

On the other hand, the components of the gauge-invariant vector field κ_A is given in terms of the gauge-invariant variables Q_1, P_1 and P_2 in the following way,

$$\kappa_\rho = -Q_1, \tag{10.28}$$
$$\kappa^t = \frac{1}{\alpha}\left[\hat{p}_2 + \frac{\Pi_2}{2}h_2\right] = \frac{1}{\lambda a\alpha}\left\{r^2 P_2 - \frac{1}{2}\left[r^2 P_1 + 2l(l+1)\frac{r^2\Pi_1}{a}Q_1\right]_{,\rho}\right\}. \tag{10.29}$$

These last relations can be inverted, obtaining the Hamiltonian variables in terms of GS harmonic coefficients,

$$Q_1 = -\kappa_\rho, \tag{10.30}$$
$$Q_2 = 2h, \tag{10.31}$$
$$\frac{P_1}{l(l+1)} = -\epsilon^{AB}\kappa_{A|B} - 2(n^A u^B + n^B u^A)v_A\kappa_B = -r^2\Pi + \frac{2\Pi_1}{a}\kappa_\rho, \tag{10.32}$$
$$\frac{2}{a\alpha}\frac{r^2 P_2}{l(l+1)} = (l-1)(l+2)\kappa^t + \epsilon^{tC}\left(r^4\Pi\right)_{|C}, \tag{10.33}$$

where Π is the GS master scalar (10.15), not to be confused with the background momenta Π_1 or Π_2. We see that the gauge-invariant Q_1 is the ρ component of the gauge-invariant vector $-\kappa_A$. Then Q_2 is a scalar in M^2, but it is gauge-dependent. The momentum P_1 is the sum of two parts, the first one being a scalar (the curl of the vector κ_A) and the second one being essentially the off-diagonal component of the symmetric tensor $v_{(A}\kappa_{B)}$. Therefore P_1 is not a component of a tensor itself. Finally P_2 is, apart from a factor $a\alpha$, the time component of a contravariant vector. It is important to stress that these properties are very easy to obtain in the GS formalism, but not in the original Hamiltonian formalism, where the variables are well adapted to a three-dimensional point of view.

From the last relations, it is clear that the only independent scalar that can be formed as a linear combination of P_1 and Q_1 is $\mathcal{Q}$ (7.33). Now it is straightforward to show that $\mathcal{Q}$ is related to the GS scalar variable as

$$-\frac{1}{r^2}\frac{\mathcal{Q}}{l(l+1)} = \Pi. \tag{10.34}$$

As a final comment, let us stress again the fact that with the Hamiltonian gauge techniques presented in Chapter 7 and the requirement of finding a scalar variable we have univocally identified the GS master scalar.

Chapter 11

Second-order

Building on the previous chapters, here we introduce the general formalism to handle second-order perturbations of a general spherically-symmetric spacetime. We construct gauge-invariant variables, give their evolution equations (the perturbed Einstein equations) and the perturbation of the equations of energy-momentum conservation. These are the equations on which any application will be based on, for example those in the following chapters. They have been computed in full generality (only under the restriction of spherical symmetry in the background) during this thesis for the first time.

11.1 Gauge invariant variables

Following the general construction explained in Chapter 9, the second order metric invariants will be given by

$$
\begin{aligned}
{}^{\{2\}}\mathcal{K}_{AB} &= {}^{\{2\}}H_{AB} + {}^{\{2\}}P_{A|B} + {}^{\{2\}}P_{B|A} + {}^{\{2\}}\mathcal{J}_{AB}, &(11.1)\\
{}^{\{2\}}\mathcal{K} &= {}^{\{2\}}K + 2v^A\,{}^{\{2\}}P_A - l(l+1)\,{}^{\{2\}}P + {}^{\{2\}}\tilde{\mathcal{J}}, &(11.2)\\
{}^{\{2\}}\kappa_A &= {}^{\{2\}}h_A + r^2\,{}^{\{2\}}q_{|A} + {}^{\{2\}}\mathfrak{j}_A, &(11.3)
\end{aligned}
$$

where the components of the second order gauge vector ${}^{\{2\}}p^{\mu}$ are

$$
\begin{aligned}
{}^{\{2\}}P_A &= \frac{r^2}{2}\left({}^{\{2\}}G + \frac{1}{r^2}\,{}^{\{2\}}\mathcal{J}\right)_{|A} - {}^{\{2\}}H_A - {}^{\{2\}}\mathcal{J}_A, &(11.4)\\
{}^{\{2\}}P &= -\frac{1}{2}\left({}^{\{2\}}G + \frac{1}{r^2}\,{}^{\{2\}}\mathcal{J}\right), &(11.5)\\
{}^{\{2\}}q &= -\frac{1}{2r^2}\left({}^{\{2\}}h + {}^{\{2\}}\mathfrak{j}\right), &(11.6)
\end{aligned}
$$

and the harmonic coefficients $\mathcal{J}$ and $\mathfrak{j}$ come from the decomposition in harmonics of the source

$$ {}^{\{2\}}\mathcal{J}_{\mu\nu} \equiv \mathcal{L}^2_{\{1\}p} g_{\mu\nu} + 2\mathcal{L}_{\{1\}p} {}^{\{1\}}h_{\mu\nu}, \tag{11.7} $$

as given in definition (9.48). As we have explained, the gauge invariants of any other tensor field are constructed just by performing the gauge transformation (4.11) parameterized by the gauge vectors $\{ {}^{\{1\}}p^\mu, {}^{\{2\}}p^\mu \}$. In particular, the matter invariants are given by

$$ {}^{\{2\}}\Psi_{\mu\nu} = {}^{\{2\}}T_{\mu\nu} + \mathcal{L}_{\{2\}p} t_{\mu\nu} + \mathcal{L}^2_{\{1\}p} t_{\mu\nu} + 2\mathcal{L}_{\{1\}p} {}^{\{1\}}T_{\mu\nu}. \tag{11.8} $$

Therefore, in order to make explicit the form of the invariants in terms of the first and second-order perturbations, we only have to obtain the expression of ${}^{\{2\}}\mathcal{J}_{\mu\nu}$ in terms of them. Since this source is quadratic in first-order perturbations, and given the product formula between spherical harmonics (6.83), we see that the source will have the general form (here we obviate the $n = 2$ label for simplicity, but reintroduce the harmonic labels l and m):

$$ \mathcal{J}_l^m{}_{AB} = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}{}_{AB}, \tag{11.9} $$

$$ \mathcal{J}_l^m{}_{A} = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}{}_{A}, \tag{11.10} $$

$$ \tilde{\mathcal{J}}_l^m = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\tilde{\mathcal{J}}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}, \tag{11.11} $$

$$ \mathcal{J}_l^m = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}, \tag{11.12} $$

$$ \mathfrak{j}_l^m{}_{A} = -i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}{}_{A}, \tag{11.13} $$

$$ \mathfrak{j}_l^m = -i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}. \tag{11.14} $$

In these expressios we have used the shortcut

$$ \sum_{\bar{l},\hat{l}} \equiv \sum_{\hat{m}=-\hat{l}}^{\hat{l}} \sum_{\bar{m}=-\bar{l}}^{\bar{l}} \sum_{\hat{l}=0}^{\infty} \sum_{\bar{l}=0}^{\infty}, \tag{11.15} $$

satisfying the standard restrictions $\bar{m}+\hat{m} = m$ and $|\bar{l}-\hat{l}| \leq l \leq |\bar{l}+\hat{l}|$. The structure of the sources is rather peculiar, owing to the mixture of polarities that appears in the product of harmonics. This fact is encoded in the polarity sign σ of the sources ${}^{(\sigma)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}$, which is given in terms of the associated sign $\epsilon \equiv (-1)^{\bar{l}+\hat{l}-l}$ and is thus completely determined for each term of the sum (it cannot be chosen freely). Sources with polarity sign $\sigma = +1$ contain terms *polar* $\times$ *polar* and *axial* $\times$ *axial* with real coefficients. Sources with polarity sign $\sigma = -1$ contain terms of the form *polar* $\times$ *axial* with purely imaginary coefficients. This form ensures an adequate behaviour of the equations under complex conjugation. In particular, using equation (6.78), we have for all sources and for all $\bar{l}, \hat{l}, l$:

$$\left[{}^{(\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}\right]^* = {}^{(\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ \ l}^{-\bar{m}-\hat{m}-m}, \tag{11.16}$$

$$\left[i\ {}^{(-\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}\right]^* = i\ {}^{(-\epsilon)}\mathcal{J}_{\bar{l}\ \hat{l}\ \ l}^{-\bar{m}-\hat{m}-m}. \tag{11.17}$$

This guarantees that the objects $\mathcal{J}_l^m$ and $\mathfrak{j}_l^m$ on the left-hand side of the equations satisfy the reality conditions

$$[\mathcal{J}_l^m]^* = \mathcal{J}_l^{-m} \quad \text{and} \quad [\mathfrak{j}_l^m]^* = \mathfrak{j}_l^{-m}. \tag{11.18}$$

As expected, these conditions imply that the functions determined by these harmonic coefficients are real.

On the other hand, we see that some pairs of equations share the sources: for example, equations (11.10) and (11.13) alternate their sources ${}^{(+)}\mathcal{J}$ and ${}^{(-)}\mathcal{J}$ for particular sets of labels $\hat{l}, \bar{l}, l$. The same thing happens with the pair (11.12) and (11.14). As a result, we need to compute eight sources in total, instead of twelve.

Using equation (9.28) and the decomposition (9.2), without loss of generality, we can expand those sources assuming that there are only two first-order perturbations, but allowing these to be completely arbitrary, in particular assigning arbitrary harmonic labels to them. From now on the coefficients and harmonic labels of those two perturbations will be denoted as $(\hat{h}, \hat{p}, \hat{\mathcal{K}})$ and $(\bar{h}, \bar{p}, \bar{\mathcal{K}})$, with all other perturbation amplitudes vanishing. In this way, we avoid dealing with sums that include (quadratic) couplings between an infinite number of first-order perturbations.

The tensorial sources are given by [114]

$$\begin{aligned}
{}^{(+)}\mathcal{J}_{\bar{l}\ \hat{l}\ l\ AB}^{\bar{m}\hat{m}m} &= -4E_{1\ \hat{l}\hat{m}l}^{-1\bar{l}\bar{m}}\left\{\bar{P}\hat{\mathcal{K}}_{AB} + 2\bar{q}_{|(B}\hat{\kappa}_{A)} - (\bar{P}\hat{P}_{(B})_{|A)} - r^2\hat{P}_{|A}\bar{P}_{|B} - r^2\hat{q}_{|A}\bar{q}_{|B}\right\} \\
&+ 2E_{0\hat{l}\hat{m}l}^{0\bar{l}\bar{m}}\left\{2\bar{P}^C{}_{|(A}\hat{\mathcal{K}}_{B)C} + \bar{P}^C\hat{\mathcal{K}}_{AB|C} - \hat{P}^C{}_{|(B}\bar{P}_{A)|C} - \hat{P}^C{}_{|A}\bar{P}_{C|B} - \hat{P}^C\bar{P}_{(A|B)C}\right\}
\end{aligned} \tag{11.19}$$

$${}^{(-)}\mathcal{J}_{\bar{l}\ \hat{l}\ l\ AB}^{\bar{m}\hat{m}m} = -4iE_{1\ \hat{l}\hat{m}l}^{-1\bar{l}\bar{m}}\left\{\hat{q}\bar{\mathcal{K}}_{AB} + 2\bar{P}_{|(A}\hat{\kappa}_{B)} - 2r^2\hat{q}_{|(A}\bar{P}_{|B)} + \bar{q}\hat{P}_{(A|B)} + \bar{q}_{|(A}\hat{P}_{B)}\right\}. \tag{11.20}$$

We have used the fact that the sums in $\bar{l}$ and $\hat{l}$ are symmetric to simplify the form of the sources. Although each individual source ${}^{(\sigma)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m}$ is not symmetric under the interchange $(\bar{l}, \bar{m}) \leftrightarrow (\hat{l}, \hat{m})$, their sum is symmetrized.

The vectorial sources are decomposed as

$$
\begin{aligned}
{}^{(+)}\mathcal{J}_{\bar{l}\ \hat{l}\ l\ A}^{\bar{m}\hat{m}m} &= E^{2\ \bar{l}\bar{m}}_{-1\hat{l}\hat{m}l}\Big\{-2(\bar{q}\hat{\kappa}_A+\hat{q}\bar{\kappa}_A)+(\hat{P}_A\bar{P}+\bar{P}_A\hat{P})+r^2(3\bar{P}\hat{P}_{|A}+\hat{P}\bar{P}_{|A})\\
&+ 2\bar{q}(r^2\hat{q})_{|A}+r^6\left(\frac{\hat{q}\bar{q}}{r^4}\right)_{|A}\Big\}\\
&+ \frac{1}{2}E^{0\bar{l}\bar{m}}_{1\hat{l}\hat{m}l}\Big\{2\bar{l}(\bar{l}+1)(\hat{q}\bar{\kappa}_A-\bar{q}\hat{\kappa}_A)+4\hat{P}^B\bar{\mathcal{K}}_{AB}+4r^2\hat{P}_{|A}\bar{\mathcal{K}} \qquad (11.21)\\
&+ \bar{l}(\bar{l}+1)\left(\bar{P}\hat{P}_A+\hat{P}\bar{P}_A+3r^2\bar{P}\hat{P}_{|A}+r^2\hat{P}\bar{P}_{|A}+r^2\bar{q}\hat{q}_{|A}-r^2\hat{q}\bar{q}_{|A}\right)\\
&- 2r^2\left[\left(\bar{P}^B\hat{P}_{|B}\right)_{|A}+4v^B\bar{P}_B\hat{P}_{|A}\right]-2\left(\bar{P}^B\hat{P}_{A|B}+2\hat{P}^B\bar{P}_{B|A}+\hat{P}^B\bar{P}_{A|B}\right)\Big\},\\
{}^{(-)}\mathcal{J}_{\bar{l}\ \hat{l}\ l\ A}^{\bar{m}\hat{m}m} &= -iE^{2\ \bar{l}\bar{m}}_{-1\hat{l}\hat{m}l}\Big\{2\left(\hat{P}\bar{\kappa}_A-\bar{P}\hat{\kappa}_A\right)+\hat{q}\bar{P}_A-\bar{q}\hat{P}_A+r^2\hat{q}\bar{P}_{|A}+3r^2\bar{P}\hat{q}_{|A}\\
&- 3r^2\bar{q}\hat{P}_{|A}-r^2\hat{P}\bar{q}_{|A}\Big\}\\
&+ iE^{1\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\Big\{2r^2\hat{\mathcal{K}}\bar{q}_{|A}-\hat{l}(\hat{l}+1)\left(\hat{P}\bar{\kappa}_A+\bar{P}\hat{\kappa}_A\right)+2\left(\bar{\kappa}^B\hat{P}_{B|A}+\hat{P}^B\bar{\kappa}_{A|B}\right)\\
&+ \frac{\hat{l}(\hat{l}+1)}{2}\left(r^2\bar{q}\hat{P}_{|A}-r^2\hat{q}\bar{P}_{|A}+3r^2\hat{P}\bar{q}_{|A}+r^2\bar{P}\hat{q}_{|A}-\hat{q}\bar{P}_A+\bar{q}\hat{P}_A\right)\\
&- r^2\left(\hat{P}^B\bar{q}_{|BA}+\hat{P}^B{}_{|A}\bar{q}_{|B}+4v^B\hat{P}_B\bar{q}_{|A}\right)\Big\}, \qquad (11.22)
\end{aligned}
$$

and finally the four scalar sources are given by

$$
\begin{aligned}
{}^{(+)}\tilde{\mathcal{J}}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m} &= -4E^{-2\bar{l}\bar{m}}_{2\ \hat{l}\hat{m}l}\Big\{\hat{q}\bar{q}+\hat{P}\bar{P}\Big\}\\
&+ E^{-1\bar{l}\bar{m}}_{1\ \hat{l}\hat{m}l}\Big\{-4\hat{P}\bar{\mathcal{K}}-\left[\hat{l}(\hat{l}+1)+\bar{l}(\bar{l}+1)\right]\hat{P}\bar{P}+\frac{2}{r^2}\hat{P}^A(r^2\bar{P})_{|A}+\frac{2}{r^2}\hat{P}^A\bar{P}_A\Big\}\\
&+ E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\Big\{-2\bar{l}(\bar{l}+1)\bar{P}\hat{\mathcal{K}}+\frac{2}{r^2}\bar{P}^A(r^2\hat{\mathcal{K}})_{|A}+\frac{\hat{l}(\hat{l}+1)}{r^4}\bar{P}^A(r^4\hat{P})_{|A}\\
&- \hat{l}(\hat{l}+1)\bar{l}(\bar{l}+1)\hat{P}\bar{P}-2\hat{P}^A\left[(\bar{P}^Bv_B)_{|A}+2\bar{P}^Bv_Av_B\right]\Big\}, \qquad (11.23)\\
{}^{(-)}\tilde{\mathcal{J}}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m} &= 8iE^{-2\bar{l}\bar{m}}_{2\ \hat{l}\hat{m}l}\hat{P}\bar{q}\\
&- 2iE^{-1\bar{l}\bar{m}}_{1\ \hat{l}\hat{m}l}\Big\{\frac{2}{r^2}\bar{P}^A\hat{\kappa}_A-2\bar{q}\hat{\mathcal{K}}-\hat{l}(\hat{l}+1)\hat{P}\bar{q}+\frac{1}{r^2}\hat{P}^A(r^2\bar{q})_{|A}\Big\}, \qquad (11.24)\\
{}^{(+)}\mathcal{J}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m} &= 2r^2E^{-1\bar{l}\bar{m}}_{3\ \hat{l}\hat{m}l}\Big\{\hat{q}\bar{q}+\hat{P}\bar{P}\Big\}\\
&+ r^2E^{1\bar{l}\bar{m}}_{1\hat{l}\hat{m}l}\Big\{(\hat{l}+2)(\hat{l}-1)\left[\hat{P}\bar{P}-\hat{q}\bar{q}\right]-2\hat{P}^A\bar{P}_{|A}-\frac{2}{r^2}\hat{P}^A\bar{P}_A\Big\}\\
&+ 2r^2E^{0\bar{l}\bar{m}}_{2\hat{l}\hat{m}l}\Big\{2\hat{P}\bar{\mathcal{K}}+\bar{l}(\bar{l}+1)\left[\hat{q}\bar{q}+2\hat{P}\bar{P}\right]-\frac{1}{r^4}\bar{P}^A\left(r^4\hat{P}\right)_{|A}\Big\}, \qquad (11.25)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(-)}\mathcal{J}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} &= 2ir^2 E_3^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\left\{\hat{q}\bar{P} - \hat{P}\bar{q}\right\} \\
&+ iE^{0\bar{l}\bar{m}}_{2\,\hat{l}\hat{m}l}\left\{4r^2\hat{q}\bar{\mathcal{K}} + 2r^2\bar{l}(\bar{l}+1)\left[2\bar{P}\hat{q} - \hat{P}\bar{q}\right] - \frac{2}{r^2}\bar{P}^A(r^4\hat{q})_{|A}\right\} \\
&+ iE^{1\bar{l}\bar{m}}_{1\,\hat{l}\hat{m}l}\left\{4\bar{P}^A\hat{\kappa}_A + (\hat{l}+2)(\hat{l}-1)r^2\left[\hat{P}\bar{q} + \bar{P}\hat{q}\right] - 2r^2\hat{P}^A\bar{q}_{|A}\right\}.
\end{aligned}
\tag{11.26}
$$

11.2 Einstein equations

The evolution of the second-order perturbations is dictated by the same equations as for first order, except for that now the left-hand side contains extra sources that are quadratic in the first-order perturbations:

$$E_{AB}[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ AB} = 8\pi\, {}^{\{2\}}\Psi^m_{l\ AB}, \tag{11.27}$$

$$E_A[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} = 8\pi\, {}^{\{2\}}\Psi^m_{l\ A}, \tag{11.28}$$

$$\tilde{E}[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\tilde{S}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} = 8\pi\, {}^{\{2\}}\tilde{\Psi}^m_l, \tag{11.29}$$

$$E[{}^{\{2\}}\mathcal{K}^m_l] + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} = 8\pi\, {}^{\{2\}}\Psi^m_l, \tag{11.30}$$

$$O_A[{}^{\{2\}}\kappa^m_l] - i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} = 8\pi\, {}^{\{2\}}\psi^m_{l\ A}, \tag{11.31}$$

$$O[{}^{\{2\}}\kappa^m_l] - i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} = 8\pi\, {}^{\{2\}}\psi^m_l, \tag{11.32}$$

The sign ϵ flips when any of the l labels changes. Therefore, all equations have generically both types of sources.

As it happens with the sources for the gauge invariants, we see that some pairs of equations share the sources: for example, equations (11.30) and (11.32) alternate their sources ${}^{(+)}S$ and ${}^{(-)}S$ for particular sets of labels $\hat{l}, \bar{l}, l$. The same thing happens with the pair (11.28) and (11.31). The operators E_{AB} and $\tilde{E}$, however, have their own pair of sources. As a result, again, we need to compute eight sources in total, instead of twelve.

Using expansion (3.36) and the definition of the gauge-invariant perturbations (11.1–11.3), we expand the Einstein equations and write them for these invariant variables. The expansion contains many terms with products of tensor harmonics: we count 1275, 972 and 1347 source terms in $\Delta[G_{AB}]$, $\Delta[G_{Ab}]$ and $\Delta[G_{ab}]$, respectively (still at the 2+2 abstract level, without any expansion in coordinate ranges). These products of harmonics must then be expanded using formula (6.83). We now analyze the sources separately.

The source of $\Delta[G_{AB}]$ contains products of harmonics of the form $\bar{Z}^{abc}\hat{Z}_{abc}$, $\bar{Z}^{ab}\hat{X}_{ab}$, etc. (harmonics with four indices do not appear in the RW gauge). The final expression can be rearranged to arrive at the sources [113]:

$$
\begin{aligned}
{}^{(+)}S^{\hat{m}\bar{m}m}_{\hat{l}\ \bar{l}\ l\ AB} &= -\frac{2}{r^4}E_2^{-2\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\, g_{AB}\hat{\kappa}_C\bar{\kappa}^C + \frac{\hat{l}(\hat{l}+1)}{r^2}\frac{\bar{l}(\bar{l}+1)}{r^2}E_{0\,\hat{l}\hat{m}l}^{0\bar{l}\bar{m}}\left(\hat{\kappa}_A\bar{\kappa}_B - g_{AB}\hat{\kappa}_C\bar{\kappa}^C\right)\\
&+ \frac{1}{r^2}E_1^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{4\hat{\kappa}^C(\bar{\kappa}_{(A|B)C} - \bar{\kappa}_{C|AB} + \bar{\kappa}_C v_{A|B} + 2\bar{\kappa}_{C|(A}v_{B)}) + 4\hat{\kappa}^C{}_{|C}\bar{\kappa}_{(A|B)}\\
&- 2\hat{\kappa}^C{}_{|A}\bar{\kappa}_{C|B} - 2\bar{\kappa}_A{}^{|C}\hat{\kappa}_{B|C} + g_{AB}\Big[2\hat{\kappa}^{(C|D)}\bar{\kappa}_{(C|D)} - \hat{\kappa}^C\bar{\kappa}^D\left(12\frac{r_{|CD}}{r} - 4v_Cv_D\right)\\
&- 2r^{-4}(r^2\hat{\kappa}^C)_{|C}(r^2\bar{\kappa}^D)_{|D} + \left(2R - 4\frac{\hat{l}^2+\hat{l}-1}{r^2} + \frac{4}{3}\frac{(r^3)^{|D}{}_D}{r^3}\right)\hat{\kappa}^C\bar{\kappa}_C\\
&+ \frac{4}{r^2}\epsilon^{CD}\hat{\kappa}_C(r^4\bar{\Pi})_{|D} - 2r^4\hat{\Pi}\bar{\Pi} + 4r^2\epsilon^{CD}\bar{\kappa}_Cv_D\hat{\Pi}\Big]\Big\}\\
&+ \frac{1}{2r^2}E_1^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big[2\hat{\mathcal{K}}^C{}_C\bar{\mathcal{K}}_{AB} - 4\bar{\mathcal{K}}_{AC}\hat{\mathcal{K}}^C{}_B \qquad (11.33)\\
&+ g_{AB}\left(4\hat{\mathcal{K}}\bar{\mathcal{K}} + 3\hat{\mathcal{K}}^{CD}\bar{\mathcal{K}}_{CD} - \hat{\mathcal{K}}^C{}_C\bar{\mathcal{K}}^D{}_D\right)\Big]\\
&+ E_{0\,\hat{l}\hat{m}l}^{0\bar{l}\bar{m}}\Big\{-\frac{\hat{l}^2+\hat{l}+\bar{l}^2+\bar{l}-2}{r^2}\hat{\mathcal{K}}_{AB}\bar{\mathcal{K}} + \bar{\mathcal{K}}^C{}_{AB}\hat{\mathcal{K}}_{|C} + \frac{2}{r^3}\hat{\mathcal{K}}_{AB}(r^3\bar{\mathcal{K}}_{|C})^{|C}\\
&+ (\hat{\mathcal{K}}\bar{\mathcal{K}})_{|AB} - \hat{\mathcal{K}}_{|A}\bar{\mathcal{K}}_{|B} + 4v_{(A}\hat{\mathcal{K}}_{|B)}\bar{\mathcal{K}} + \frac{1}{2}\hat{\mathcal{K}}_{CD(A}\bar{\mathcal{K}}^C{}_{B)}{}^D\\
&- \hat{\mathcal{K}}^{CD}\left[r^{-2}(r^2\bar{\mathcal{K}}_{CAB})_{|D} - \bar{\mathcal{K}}_{CD|BA}\right] - \frac{1}{2}\hat{\mathcal{K}}_{DAB}\bar{\mathcal{K}}^{DC}{}_C\\
&+ (\hat{\mathcal{K}}_{AB} - g_{AB}\hat{\mathcal{K}}^F{}_F)\Big[\bar{\mathcal{K}}^C{}_C{}^{|D}{}_D - \bar{\mathcal{K}}^{CD}{}_{|CD} + 2\bar{\mathcal{K}}^C{}_{C|D}v^D - 4\bar{\mathcal{K}}^{CD}{}_{|C}v_D\\
&- 2\bar{\mathcal{K}}^{CD}(2v_{C|D} + 3v_Cv_D) + \bar{\mathcal{K}}^C{}_C\left(\frac{R}{2} - \frac{\bar{l}^2+\bar{l}}{r^2}\right)\Big] + g_{AB}\Big[\frac{\hat{l}^2+\hat{l}}{r^2}(\hat{\mathcal{K}}^C{}_C + 2\hat{\mathcal{K}})\bar{\mathcal{K}}\\
&- \bar{\mathcal{K}}^{DC}{}_C\hat{\mathcal{K}}_{|D} - \frac{2}{r^3}\hat{\mathcal{K}}^{CD}(r^3\bar{\mathcal{K}}_{|C})_{|D} - \frac{2}{r^2}\hat{\mathcal{K}}\bar{\mathcal{K}} - \frac{1}{r^3}[r^3(\hat{\mathcal{K}}\bar{\mathcal{K}})_{|C}]^{|C} + \frac{3}{2}\hat{\mathcal{K}}_{|C}\bar{\mathcal{K}}^{|C}\\
&+ 2\bar{\mathcal{K}}^{FDE}[(\hat{\mathcal{K}}_{FC}v^C - \hat{\mathcal{K}}^C{}_Cv_F)g_{DE} + v_F\hat{\mathcal{K}}_{DE}] + \frac{1}{4}\hat{\mathcal{K}}_{FC}{}^C\bar{\mathcal{K}}^{FD}{}_D - \frac{1}{4}\hat{\mathcal{K}}_{CDF}\bar{\mathcal{K}}^{CDF}\\
&+ (\hat{\mathcal{K}}^{CD}\bar{\mathcal{K}}_C{}^F - \hat{\mathcal{K}}^C{}_C\bar{\mathcal{K}}^{DF})\Big[g_{DF}\left(\frac{\bar{l}^2+\bar{l}}{r^2} - \frac{R}{2}\right) + 2(2v_{D|F} + 3v_Dv_F)\Big]\Big]\Big\},\\
{}^{(-)}S^{\hat{m}\bar{m}m}_{\hat{l}\ \bar{l}\ l\ AB} &= \frac{2i}{r^2}E_1^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{\hat{\mathcal{K}}_{AB}\bar{\kappa}^C{}_{|C} + \hat{\mathcal{K}}^C{}_C\bar{\kappa}_{(A|B)} - 2\hat{\mathcal{K}}^C{}_{(A}\bar{\kappa}_{B)|C}\\
&+ 2(\hat{\mathcal{K}}_{AB|C} - \hat{\mathcal{K}}_{C(A|B)})\bar{\kappa}^C \qquad (11.34)\\
&+ g_{AB}\Big[\hat{\mathcal{K}}^{CD}\frac{1}{r^2}(r^2\bar{\kappa}_C)_{|D} - \hat{\mathcal{K}}^C{}_C\bar{\kappa}^D{}_{|D} + 2(\hat{\mathcal{K}}^C{}_{D|C} - \hat{\mathcal{K}}^C{}_{C|D} - \hat{\mathcal{K}}_{|D})\bar{\kappa}^D\Big]\Big\}.
\end{aligned}
$$

We have tried to simplify the expressions as much as possible by using the GS master scalar

Π introduced in Chapter 10, and we have defined $\mathcal{K}_{ABC} \equiv \mathcal{K}_{AB|C} + \mathcal{K}_{AC|B} - \mathcal{K}_{BC|A}$.

On the other hand, the source of $\Delta[G_{Ab}]$ can be decomposed as

$$\begin{aligned}
{}^{(+)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A} &= \frac{2}{r^2}E^{2\;\;\bar{l}\bar{m}}_{-1\hat{l}\hat{m}l}\left\{(\hat{\kappa}_{[A|B]} + \hat{\kappa}_B v_A)\bar{\kappa}^B - \hat{\kappa}^B\bar{\kappa}_{(A|B)}\right\} \\
&+ E^{1\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\left\{\frac{1}{2}\hat{\mathcal{K}}_{BC|A}\bar{\mathcal{K}}^{BC} + \hat{\mathcal{K}}^{BC}(\bar{\mathcal{K}}_{BC|A} - \bar{\mathcal{K}}_{AB|C} - \bar{\mathcal{K}}_{BC}v_A)\right. \\
&+ \frac{1}{2}(\hat{\mathcal{K}}^B{}_{B|C} - 2\hat{\mathcal{K}}^B{}_{C|B})\bar{\mathcal{K}}_A{}^C + (\hat{\mathcal{K}}\bar{\mathcal{K}})_{|A} + \frac{1}{2}\hat{\mathcal{K}}_{|A}\bar{\mathcal{K}}^B{}_B \qquad (11.35) \\
&+ \left.\frac{\hat{l}^2+\hat{l}}{r^2}\left[3(\hat{\kappa}_{[A|B]} + \hat{\kappa}_B v_A)\bar{\kappa}^B - \hat{\kappa}^B\bar{\kappa}_{(A|B)} + r^2\hat{\kappa}_A(r^{-2}\bar{\kappa}^B)_{|B}\right]\right\}, \\
{}^{(-)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A} &= \frac{-i}{r^2}E^{-1\bar{l}\bar{m}}_{2\;\;\hat{l}\hat{m}l}\left\{\hat{\mathcal{K}}_{AB}\bar{\kappa}^B + \hat{\kappa}^B\bar{\mathcal{K}}_{AB}\right\} \\
&+ \frac{i}{2}E^{0\bar{l}\bar{m}}_{1\hat{l}\hat{m}l}\left\{\frac{\bar{l}^2+\bar{l}}{r^2}\left[-\hat{\mathcal{K}}_{AB}\bar{\kappa}^B + \hat{\kappa}^B\bar{\mathcal{K}}_{AB} + (\hat{\mathcal{K}}^B{}_B - 2\hat{\mathcal{K}})\bar{\kappa}_A - \hat{\kappa}_A(\bar{\mathcal{K}}^B{}_B + 2\bar{\mathcal{K}})\right]\right. \\
&+ 2\hat{\kappa}^B{}_{|A}\bar{\mathcal{K}}_{|B} - 2r^{-2}(r^2\hat{\kappa}^B)_{|B}\bar{\mathcal{K}}_{|A} - 2r^{-2}\hat{\kappa}^B(r^2\bar{\mathcal{K}}_{|B})_{|A} \\
&+ \frac{2}{r^2}\hat{\kappa}_A\left(2\tilde{E}\left[{}^{\{1\}}h^{\bar{m}}_{\bar{l}}\right] + \bar{\mathcal{K}}(r^2\,R - \hat{l}^2 - \hat{l}) + (r^2\bar{\mathcal{K}})^{|B}{}_B\right) \qquad (11.36) \\
&+ \left.2r^2\epsilon^{BC}\hat{\Pi}\bar{\mathcal{K}}_{AB|C} - r^2\epsilon_{AB}\hat{\Pi}(\bar{\mathcal{K}}_C{}^{C|B} - 2\bar{\mathcal{K}}^{BC}{}_{|C}) + 2r^{-2}\epsilon_{AB}(r^4\hat{\Pi})_{|C}\bar{\mathcal{K}}^{BC}\right\}.
\end{aligned}$$

Finally, the manipulations for $\Delta[G_{ab}]$ are more complicated, involving up to 3091 terms in some intermediate steps. The resulting expression can be organized in the following four sources:

$$\begin{aligned}
{}^{(+)}\tilde{S}^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l} &= \frac{1}{r^2}E^{1\;\;\bar{l}\bar{m}}_{-1\hat{l}\hat{m}l}\left\{\hat{\mathcal{K}}^{AB}\bar{\mathcal{K}}_{AB} - \frac{1}{2}\hat{\mathcal{K}}^A{}_A\bar{\mathcal{K}}^B{}_B + 2\hat{\kappa}^{A|B}\bar{\kappa}_{A|B} - \frac{2}{r^2}(r\hat{\kappa}^A)_{|A}(r\bar{\kappa}^B)_{|B}\right. \\
&+ \left.2\hat{\kappa}^A\bar{\kappa}^B\left[2\;{}^{(4)}R_{AB} + 3v_Av_B - g_{AB}\frac{\bar{l}^2+\bar{l}-1-2rr^{|C}{}_{|C}}{r^2}\right] + \frac{4}{r}\epsilon^{AB}\hat{\kappa}_A(r^3\bar{\Pi})_{|B}\right\} \\
&+ \frac{1}{2}E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\left\{-\frac{\hat{l}^2+\hat{l}}{r^2}\frac{\bar{l}^2+\bar{l}}{r^2}\hat{\kappa}^A\bar{\kappa}_A + 2(\hat{\mathcal{K}}^A{}_A - \hat{\mathcal{K}})(\bar{\mathcal{K}}^{BC}{}_{|BC} - \bar{\mathcal{K}}^B{}_B{}^{|C}{}_C)\right. \\
&+ \frac{1}{2}\hat{\mathcal{K}}^{AB|C}(2\bar{\mathcal{K}}_{AC|B} - 3\bar{\mathcal{K}}_{AB|C}) + \frac{1}{2}(2\hat{\mathcal{K}}^A{}_{B|A} - \hat{\mathcal{K}}^A{}_{A|B})(2\bar{\mathcal{K}}^{CB}{}_{|C} - \bar{\mathcal{K}}^C{}_C{}^{|B}) \\
&+ \hat{\mathcal{K}}^{AB}\left[\bar{\mathcal{K}}_{AB}\left(\frac{\bar{l}^2+\bar{l}}{r^2} - R\right) - 2v_B\bar{\mathcal{K}}^C{}_{C|A} + 4(\bar{\mathcal{K}}_{AC}v^C)_{|B} + \frac{4}{r}(r\bar{\mathcal{K}}_{AC})^{|C}v_B\right. \\
&- \left.2\bar{\mathcal{K}}_{AB|C}v^C\right] - 2[\hat{\mathcal{K}}^{AB}r^{-2}(r^2\bar{\mathcal{K}})_{|B}]_{|A} + \hat{\mathcal{K}}^A{}_{A|B}r^{-2}(r^2\bar{\mathcal{K}})^{|B} + R\,\bar{\mathcal{K}}^A{}_A\hat{\mathcal{K}} \\
&- \left.4\hat{\mathcal{K}}^{AB}\bar{\mathcal{K}}v_Av_B - \hat{\mathcal{K}}^{|A}\bar{\mathcal{K}}_{|A}\right\}, \qquad (11.37) \\
{}^{(-)}\tilde{S}^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l} &= \frac{2i}{r^2}E^{1\;\;\bar{l}\bar{m}}_{-1\hat{l}\hat{m}l}\left\{\left(g^{AB}\hat{\mathcal{K}}^C{}_C - \hat{\mathcal{K}}^{AB}\right)\bar{\kappa}_{A|B} + \left(\hat{\mathcal{K}}_{|B} + 4\hat{\mathcal{K}}^A{}_{[A|B]} - \hat{\mathcal{K}}^A{}_Av_B\right)\bar{\kappa}^B\right\}, \qquad (11.38)
\end{aligned}$$

$$ {}^{(+)}S_{\hat{l}\ \bar{l}\ l}^{\hat{m}\bar{m}m} = E_{1\hat{l}\hat{m}l}^{1\bar{l}\bar{m}} \left\{ \frac{1}{2}\hat{\mathcal{K}}^{AB}\bar{\mathcal{K}}_{AB} + \hat{\mathcal{K}}^{A}{}_{A}\bar{\mathcal{K}} + r^4\hat{\Pi}\bar{\Pi} - 2\frac{\hat{l}^2+\hat{l}-1}{r^2}\hat{\kappa}^A\bar{\kappa}_A \right\} + E_{0\hat{l}\hat{m}l}^{2\bar{l}\bar{m}}\hat{\mathcal{K}}^{AB}\bar{\mathcal{K}}_{AB}, \tag{11.39} $$

$$ {}^{(-)}S_{\hat{l}\ \bar{l}\ l}^{\hat{m}\bar{m}m} = -2iE_{1\hat{l}\hat{m}l}^{1\bar{l}\bar{m}}(\hat{\mathcal{K}}\bar{\kappa}^A)_{|A} - iE_{0\hat{l}\hat{m}l}^{2\bar{l}\bar{m}} \left\{ 2(\hat{\mathcal{K}}^{AB}\bar{\kappa}_A)_{|B} - \hat{\mathcal{K}}^{A}{}_{A|B}\bar{\kappa}^B \right\}, \tag{11.40} $$

where we have denoted the four dimensional Ricci tensor as ${}^{(4)}R_{AB}$.

In spite of the obvious increase of complexity from the first to the second-order equations, we want to stress that the final expressions given above are still manageable and fully general. They are valid for any gauge since only gauge-invariant objects appear. They can be particularized to the case of any spherical background, dynamical or not, containing any type of matter, and expressed in any kind of background coordinates (polar-radial, null, comoving, etc.)

In situations with just a single first-order perturbation we have $\hat{l} = \bar{l}$ and $\hat{m} = \bar{m}$. In these circumstances, the E-coefficients vanish for odd l. This implies that $\hat{l}+\bar{l}-l$ is always even and therefore the sources ${}^{(-)}S$ are never excited in the polar equations (11.27–11.30), neither are the sources ${}^{(+)}S$ in the axial equations (11.31, 11.32). In particular, ${}^{(-)}S_{AB}$ and ${}^{(-)}\tilde{S}$ are never excited. This has been the case encountered in many previous investigations in second-order perturbation theory. For instance, for the perturbations of a slowly rotating star a single axial $l = 1$ mode has been assumed [44], and the studies of the close-limit black hole collisions have considered a single polar $l = 2$ first-order perturbative mode [82].

11.3 Energy-momentum conservation equations

The second-order perturbation of the energy-momentum conservation equations adopts the same form as the first-order one (10.11-10.13), with additional quadratic sources,

$$ L_A[{}^{\{2\}}\psi_l^m, {}^{\{2\}}h_l^m] + \sum_{\bar{l},\hat{l}} {}^{(\varepsilon)}\mathcal{I}_{\bar{l}\ \hat{l}\ l\ A}^{\bar{m}\hat{m}m} = 0, \tag{11.41} $$

$$ L[{}^{\{2\}}\psi_l^m, {}^{\{2\}}h_l^m] + \sum_{\bar{l},\hat{l}} {}^{(\varepsilon)}\mathcal{I}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m} = 0, \tag{11.42} $$

$$ \tilde{L}[{}^{\{2\}}\psi_l^m, {}^{\{2\}}h_l^m] - i\sum_{\bar{l},\hat{l}} {}^{(-\varepsilon)}\mathcal{I}_{\bar{l}\ \hat{l}\ l}^{\bar{m}\hat{m}m} = 0. \tag{11.43} $$

Such sources can be computed starting from the equation

$$ \Delta^2[T_{\mu\nu}{}^{;\nu}] = 0, \tag{11.44} $$

by decomposing it using formulas (9.2) and (9.3), changing variables to the gauge invariants (11.1–11.3), and finally applying the tools that we have developed to deal with products of harmonics. The result, with the same notation used for the sources of the main equations, is

$$
\begin{aligned}
{}^{(+)}\mathcal{I}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} &= \frac{1}{r^2}E_1^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{2r^2\hat{\psi}^B(r^{-2}\bar{\kappa}_B)_{|A} - \bar{\mathcal{K}}^B{}_B\hat{\Psi}_A - \frac{r^4}{2}Q(r^{-4}\hat{\kappa}_B\bar{\kappa}^B)_{|A} \\
&+ 2(\hat{\psi}_A\bar{\kappa}^B)_{|B} + r^2 t_A{}^B(r^{-2}\hat{\kappa}^C\bar{\kappa}_C)_{|B} - 2(\hat{\kappa}^C\bar{\kappa}^B t_{AB})_{|C} - r^2 t_{BC}(r^{-2}\hat{\kappa}^B\bar{\kappa}^C)_{|A}\Big\} \\
&+ \frac{1}{2}E_0^{0\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{\frac{2\hat{l}(\hat{l}+1)}{r^2}\bar{\mathcal{K}}\hat{\Psi}_A + 2\bar{\Psi}_A{}^B\hat{\mathcal{K}}_{|B} + 2Qr\bar{\mathcal{K}}(r^{-1}\hat{\mathcal{K}})_{|A} - \frac{2}{r^2}(r^2\hat{\Psi}_{AB}\bar{\mathcal{K}}^{BC})_{|C} \\
&- 2r^2\bar{\bar{\Psi}}(r^{-2}\hat{\mathcal{K}})_{|A} - 2\hat{\mathcal{K}}^{BC}t_{AB}\bar{\mathcal{K}}_{|C} - \hat{\Psi}^{BC}\bar{\mathcal{K}}_{BC|A} + \hat{\Psi}_A{}^B\bar{\mathcal{K}}^C{}_{C|B} - 2\hat{\mathcal{K}}\bar{\mathcal{K}}^{|B}t_{AB} \\
&+ \hat{\mathcal{K}}^{BC}\Big[4\bar{\mathcal{K}}_C{}^D{}_{|(A}t_{D)B} - \bar{\mathcal{K}}^D{}_{D|C}t_{AB} - \bar{\mathcal{K}}_{BC|D}t_A{}^D + \frac{2}{r^2}(r^2\bar{\mathcal{K}}_{BD}t_A{}^D)_{|C}\Big]\Big\}, \qquad (11.45)\\
{}^{(-)}\mathcal{I}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} &= -\frac{i}{r^2}E_1^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{\bar{\mathcal{K}}^B{}_B\hat{\psi}_A + \big(2\bar{\mathcal{K}}^{BC}\hat{\kappa}_C - \bar{\mathcal{K}}^C{}_C\hat{\kappa}^B\big)t_{AB} - 2\hat{\kappa}^B\bar{\Psi}_{AB} \\
&- 2r^2\bar{\Psi}^B(r^{-2}\hat{\kappa}_B)_{|A} - 2(\bar{\Psi}_A\hat{\kappa}^B)_{|B}\Big\}, \qquad (11.46)\\
{}^{(+)}\mathcal{I}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} &= \frac{1}{r^2}E_2^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{2\bar{\kappa}_A\hat{\psi}^A + 2\bar{\psi}_A\hat{\kappa}^A + 2(\hat{\psi}\bar{\kappa}^A)_{|A} - \hat{\Psi}\bar{\mathcal{K}}^A{}_A - Q\hat{\kappa}^A\bar{\kappa}_A - 2\hat{\kappa}^A\bar{\kappa}^B t_{AB}\Big\} \\
&+ \frac{1}{r^2}E_0^{1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{\hat{l}(\hat{l}+1)\Big[\bar{\psi}_A\hat{\kappa}^A - \hat{\psi}_A\bar{\kappa}^A - \bar{\kappa}^A\hat{\kappa}^B t_{AB} + \frac{Q}{2}\bar{\kappa}^A\hat{\kappa}_A\Big] + (\bar{l}-1)(\bar{l}+2)\hat{\mathcal{K}}\bar{\Psi} \\
&- 2(r^2\bar{\Psi}_A\hat{\mathcal{K}}^{AB})_{|B} + 2r^2\Big[\bar{\Psi}^A\hat{\mathcal{K}}_{|A} - \hat{\bar{\Psi}}\bar{\mathcal{K}} - \bar{\bar{\Psi}}\hat{\mathcal{K}} + Q\hat{\mathcal{K}}\bar{\mathcal{K}} + \frac{1}{2}\bar{\Psi}^B\hat{\mathcal{K}}^A{}_{A|B}\Big] \\
&+ r^2\bar{\mathcal{K}}^{AB}\Big[2\hat{\mathcal{K}}_{BC}t^C{}_A - \hat{\Psi}_{AB} - \frac{Q}{2}\hat{\mathcal{K}}_{AB} + g_{AB}\Big(\hat{\bar{\Psi}} - \frac{Q}{2}\hat{\mathcal{K}}\Big)\Big]\Big\}, \qquad (11.47)\\
{}^{(-)}\mathcal{I}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l} &= \frac{i}{r^2}E_2^{-1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{2\hat{\kappa}_A\bar{\Psi}^A - 2\bar{\kappa}_A\hat{\Psi}^A - \hat{\psi}\bar{\mathcal{K}}^A{}_A - 2(\hat{\Psi}\bar{\kappa}^A)_{|A}\Big\} \\
&+ \frac{i}{r^2}E_0^{1\bar{l}\bar{m}}{}_{\hat{l}\hat{m}l}\Big\{\hat{l}(\hat{l}+1)(\bar{\kappa}_A\hat{\Psi}^A - \hat{\kappa}_A\bar{\Psi}^A) + (\bar{l}-1)(\bar{l}+2)\bar{\psi}\hat{\mathcal{K}} - 2(r^2\bar{\psi}_A\hat{\mathcal{K}}^{AB})_{|B} \\
&- 2(r^2\hat{\bar{\Psi}}\bar{\kappa}^A)_{|A} + 2r^2\bar{\psi}^A\hat{\mathcal{K}}_{|A} + \hat{\mathcal{K}}(Qr^2\bar{\kappa}^A)_{|A} + (Qr^2\hat{\mathcal{K}}^{AB}\bar{\kappa}_B)_{|A} \\
&+ r^2\hat{\mathcal{K}}^A{}_{A|B}\Big(\bar{\psi}^B - \frac{Q}{2}\bar{\kappa}^B\Big)\Big\}. \qquad (11.48)
\end{aligned}
$$

11.4 Gerlach and Sengupta master equation

The GS master scalar is defined in the same way as in first-order perturbation theory:

$$
{}^{\{2\}}\Pi \equiv \epsilon^{AB}\left(\frac{{}^{\{2\}}\kappa_A}{r^2}\right)_{|B}. \qquad (11.49)
$$

Its evolution equation, which can be easily obtained from equation (11.31), is similar to that found at first-order (10.16), but with a source term,

$$-\left[\frac{1}{2r^2}(r^4\,{}^{\{2\}}\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\,{}^{\{2\}}\Pi = 8\pi\epsilon^{AB}\,{}^{\{2\}}\tilde{\psi}_{A|B} + {}^{\{2\}}\mathcal{S}_\Pi, \tag{11.50}$$

where we have made use of the matter invariant ${}^{\{2\}}\tilde{\psi}$ [see equation (10.14)] and the source can be given explicitly as,

$${}^{\{2\}}\mathcal{S}_\Pi = i\epsilon^{AB}\sum_{\bar{l},\hat{l}}\sum_{\bar{m},\hat{m}}{}^{(-\epsilon)}S^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l\;A|B}. \tag{11.51}$$

As was the case for first-order perturbations, the master equation (11.50) fully describes the evolution of the second-order part of the gravitational wave with axial polarity.

Part V

Applications

Chapter 12

Vacuum

In the previous chapter, a gauge-invariant formalism to deal with second-order perturbations on an arbitrary spherical spacetime has been introduced. The aim of the present chapter is to apply that formalism to the particular background of a Schwarzschild black hole. In such a background, the perturbed Einstein equations can be reduced to two non-constrained wave equations: one per degree of freedom of the gravitational wave. These two equations are the so-called the Regge-Wheeler [56] equation, namely, the particularization of the GS master equation to vacuum, and the Zerilli [57] equation.

We will derive the formula of the power radiated to null infinity in terms of the harmonic coefficients of the perturbed metric up to any order. We will obtain the sources for the second-order Regge-Wheeler and Zerilli equations. As we will show, these sources diverge at infinity and we will regularize them. The formula of the emitted power will be given in terms of the regularized second-order master variables.

Throughout this chapter we will mainly use Schwarzschild coordinates (t, r). We define the components of the metric perturbations in Schwarzschild coordinates as,

$$ {}^{\{n\}}H_{AB} \equiv \begin{pmatrix} {}^{\{n\}}H_{tt} & {}^{\{n\}}H_{tr} \\ {}^{\{n\}}H_{tr} & {}^{\{n\}}H_{rr} \end{pmatrix}, \quad {}^{\{n\}}H_{A} \equiv \begin{pmatrix} {}^{\{n\}}H_{t} \\ {}^{\{n\}}H_{r} \end{pmatrix}, \quad {}^{\{n\}}h_{A} \equiv \begin{pmatrix} {}^{\{n\}}h_{t} \\ {}^{\{n\}}h_{r} \end{pmatrix}. \tag{12.1} $$

We also introduce the following shorthand for coordinate derivatives acting on any scalar function ζ,

$$ \dot{\zeta} \equiv \frac{\partial \zeta}{\partial t}, \qquad \zeta' \equiv \frac{\partial \zeta}{\partial r}. \tag{12.2} $$

12.1 The radiated power

In order to compute the power that is radiated to infinity by gravitational waves, we will use the Landau and Lifshitz formula [14],

$$\frac{d\text{Power}}{d\Omega} = \frac{1}{16\pi r^2}\left\{\frac{1}{\sin^2\theta}\left|\frac{\partial\tilde{g}_{\theta\phi}}{\partial t}\right|^2 + \frac{1}{4}\left|\frac{\partial\tilde{g}_{\theta\theta}}{\partial t} - \frac{1}{\sin^2\theta}\frac{\partial\tilde{g}_{\phi\phi}}{\partial t}\right|^2\right\}, \tag{12.3}$$

where the vertical bars denote absolute value, and which is valid for any asymptotically flat (AF) gauge. By this we mean any gauge in which the components of the metric perturbations fall off as [46],

$$\begin{aligned} {}^{\{n\}}h_{tt},\ {}^{\{n\}}h_{rr},\ {}^{\{n\}}h_{tr} &= O(r^{-2}), &(12.4)\\ {}^{\{n\}}h_{t\theta},\ {}^{\{n\}}h_{t\phi},\ {}^{\{n\}}h_{r\theta},\ {}^{\{n\}}h_{r\phi} &= O(r^{-1}), &(12.5)\\ {}^{\{n\}}h_{\theta\theta},\ {}^{\{n\}}h_{\phi\phi},\ {}^{\{n\}}h_{\theta\phi} &= O(r), &(12.6)\\ \gamma^{ab}\,{}^{\{n\}}h_{ab} &= O(r^0), &(12.7)\end{aligned}$$

or more rapidly up to the desired order n. These conditions imply the following decay rates for the harmonic coefficients,

$$\begin{aligned} {}^{\{n\}}H^{AF}_{tt} = {}^{\{n\}}H^{AF}_{tr} = {}^{\{n\}}H^{AF}_{rr} = {}^{\{n\}}K^{AF} = O(r^{-2}),\\ {}^{\{n\}}H^{AF}_{t} = {}^{\{n\}}H^{AF}_{r} = {}^{\{n\}}h^{AF}_{t} = {}^{\{n\}}h^{AF}_{r} = {}^{\{n\}}G^{AF} = {}^{\{n\}}h^{AF} = O(r^{-1}), \end{aligned} \tag{12.8}$$

where the superscript AF stands for asymptotically flat gauge. In particular, as we will see, the RW gauge is not asymptotically flat. Nonetheless, we will exploit the fact that we are working with gauge-invariant objects and, hence, we can recover straightforwardly any perturbative expression in any gauge, in particular an AF gauge.

The Landau and Lifshitz formula is expressed in a spherical coordinate system (θ, ϕ), but it can be given in covariant form. Making use of the round metric defined by the background spherical spacetime, let us define on the sphere the projector

$$P_{ab}{}^{cd} \equiv \gamma_a{}^c\gamma_b{}^d - \frac{1}{2}\gamma_{ab}\gamma^{cd}, \tag{12.9}$$

which maps any rank-two tensor A_{ab} into a trace-free tensor on the sphere $P_{ab}{}^{cd}A_{cd}$. With this projector at hand, we define the projected trace-free metric on the sphere,

$$\iota_{ab} \equiv P_{ab}{}^{cd}\tilde{g}_{cd}, \tag{12.10}$$

which allows us to rewrite formula (12.3) in the following way,

$$\frac{d\text{Power}}{d\Omega} = \frac{1}{32\pi r^2}\left(\frac{\partial\iota_{ab}}{\partial t}\right)\gamma^{ac}\gamma^{bd}\left(\frac{\partial\iota_{cd}}{\partial t}\right)^*. \tag{12.11}$$

Note that for metrics of the form $\tilde{g}_{ab} = r^2 K \gamma_{ab}$, K being any generic function of the four coordinates, the induced metric ι_{ab} will be zero and no power will arrive at null infinity. The spherically symmetric spacetime, with $\tilde{g}_{ab} = r^2\gamma_{ab}$, is one such particular case.

Now we replace the perturbative expansion for the metric decomposed in spherical harmonics (9.2) and find that the projected trace-free metric in an AF gauge is

$$\iota_{ab} = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \sum_{l,m} \{ r^2\, {}^{\{n\}}G_l^{m\mathrm{AF}} Z_l^m{}_{ab} + {}^{\{n\}}h_l^{m\mathrm{AF}} X_l^m{}_{ab} \}. \tag{12.12}$$

Making use of the fact that the tensor spherical harmonics are trace-free and normalized as shown in (6.21), it is easy to integrate the emitted power over the solid angle and obtain the total radiated power,

$$\begin{aligned} \text{Power} &= \frac{1}{64\pi r^2} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\varepsilon^{j+k}}{j!k!} \sum_{l,m} \frac{(l+2)!}{(l-2)!} \left\{ r^4 \left(\frac{\partial\, {}^{\{j\}}G_l^{m\mathrm{AF}}}{\partial t} \right) \left(\frac{\partial\, {}^{\{k\}}G_l^{m\mathrm{AF}}}{\partial t} \right)^* \right. \\ &+ \left. \left(\frac{\partial\, {}^{\{j\}}h_l^{m\mathrm{AF}}}{\partial t} \right) \left(\frac{\partial\, {}^{\{k\}}h_l^{m\mathrm{AF}}}{\partial t} \right)^* \right\}. \end{aligned} \tag{12.13}$$

Therefore, the problem of extracting the radiated power at order ε^n reduces to find the value of the time derivative of the harmonic coefficients ${}^{\{k\}}G_l^m$ and ${}^{\{k\}}h_l^m$, for all $k < n$, in an AF gauge. Note that in the last formula there is no coupling between modes with different harmonic labels, a characteristic phenomenon of high-order perturbation theory. This is because of the integrated character of the total emitted power. In fact, the orthogonality between different spherical harmonics prevents it to happen. This issue has important consequences when one wants to obtain the radiated power up to a given order ε^n in a consistent way, as we will analyze in the last section of the present chapter.

12.2 Master equations

When perturbing vacuum, it is possible to reduce the perturbed Einstein equations to two wave equations for two scalars, one axial and one polar. In such a way that if these equations are satisfied, the Einstein equations will also be trivially fulfilled. These scalars are called the master scalars because they contain all the physical information of the system since the perturbed metric can be fully reconstructed from them.

12.2.1 Polar sector

We define the nth order Zerilli scalar function as the following combination of gauge-invariant polar harmonic coefficients,

$$ {}^{\{n\}}\mathcal{Z} \equiv \frac{r^4}{3M+\Lambda}(v^B\, {}^{\{n\}}\mathcal{K}_{AB} - {}^{\{n\}}\mathcal{K}_{|A})v^A + r\, {}^{\{n\}}\mathcal{K}, \tag{12.14} $$

The symbol Λ was defined in (8.86). Note that the Zerilli scalar (12.14) is given in terms of the nth order gauge invariants tied to the Regge-Wheeler gauge. Then, in Schwarzschild coordinates,the Zerilli function takes the following form,

$$ \begin{aligned} {}^{\{n\}}\mathcal{Z} &\equiv \frac{(2M-r)}{3M+\Lambda r}\left\{(2M-r)\, {}^{\{n\}}H_{rr} + r^2\, {}^{\{n\}}K'\right\} + r\, {}^{\{n\}}K \\ &+ \frac{l(l+1)}{3M+\Lambda r}(2M-r)\, {}^{\{n\}}H_r + \frac{1}{2}l(l+1)r\, {}^{\{n\}}G + {}^{\{n\}}Q_{\mathcal{Z}}, \end{aligned} \tag{12.15} $$

where ${}^{\{n\}}Q_{\mathcal{Z}}$ depends on lower order perturbations. Note that all terms in the second line, including ${}^{\{n\}}Q_{\mathcal{Z}}$, are zero when imposing the RW gauge. From this definition, it is easy to see that, at first order, this variable coincides with the Zerilli variable that we have found with Hamiltonian techniques particularized to the vacuum (8.105). In order to compare them, one has to employ that $h_1 = H_r$ and $H_2 = (1-2M/r)H_{rr}$, which can be obtained relating the different harmonic expansions.

As we have anticipated, the Zerilli scalar satisfies the following wave equation

$$ {}^{\{n\}}\mathcal{Z}^{|A}{}_{A} - V_{\mathrm{Z}}\, {}^{\{n\}}\mathcal{Z} = {}^{\{n\}}\mathcal{S}_{\mathcal{Z}}, \tag{12.16} $$

${}^{\{n\}}\mathcal{S}_{\mathcal{Z}}$ being a source that depends on the perturbations of lower order and the potential V_{Z} defined in (8.107). As we have shown in (8.106), when it is expressed in the tortoise coordinates (t, r^*), the differential operator takes the following simple form,

$$ {}^{\{n\}}\mathcal{Z}^{|A}{}_{A} \equiv \left(1-\frac{2M}{r}\right)^{-1}\left(-\frac{\partial^2\, {}^{\{n\}}\mathcal{Z}}{\partial t^2} + \frac{\partial^2\, {}^{\{n\}}\mathcal{Z}}{\partial r^{*2}}\right). \tag{12.17} $$

In addition to having vacuum in the background spacetime, we will also assume vacuum as perturbed system. This is not mandatory, and some authors have indeed considered non zero perturbations of a vanishing background stress-energy tensor [109]. In this case, already at linear order, one finds matter sources in the equation (12.16).

The sources for the second-order Zerilli scalar (12.15) and its wave equation (12.16) were recently obtained for the case $\bar{l} = \hat{l} = 2$, considering only first-order polar perturbations [139]. As we have commented, this case is simpler than the generic one analyzed here, because the necessary product formula for the tensor spherical harmonics can be obtained by using just the product formula for the scalar harmonics and the Leibnitz rule. For completeness, we now give the second-order source in terms of the sources for the polar

Einstein equations (11.27–11.30),

$$
\begin{aligned}
{}^{\{2\}}\mathcal{S}_{\mathcal{Z}} \equiv & \sum_{\bar{l},\hat{l}} \frac{r^4 v^A}{3M+\Lambda r}\left[{}^{(\epsilon)}S^{\hat{m}\bar{m}mB}_{\hat{l}\;\bar{l}\;l\;\;B|A} - {}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;AB}{}^{|B} - \frac{l(l+1)}{r^2}{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A}\right] \\
- & \frac{r}{4(3M+\Lambda r)^2}\left[\left(84M^2+12(l^2+l-5)Mr+2\Lambda(l^2+l-4)r^2\right){}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A}{}^{A}\right. \\
+ & \left. 4r^3(12M+2\Lambda r)v^A v^B\,{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;AB}\right] + \frac{l(l+1)}{r}{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l}. \qquad (12.18)
\end{aligned}
$$

It is interesting to note that the source ${}^{(\epsilon)}\tilde{S}^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l}$ does not appear in this expression. In fact, it can be obtained from the scalar and the vectorial sources in the following way:

$$
{}^{(\epsilon)}\tilde{S}^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l} = \Lambda\,{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l} - r^2\,{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A}{}^{|A} - 2r^2 v^A\,{}^{(\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A}. \qquad (12.19)
$$

This relation comes from the fact that their first-order equation counterparts are not independent, actually, equation (10.4) is a consequence of equations (10.3–10.5).

Note finally that the definition of the high-order Zerilli function (12.14) is essentially determined up to addition of low-order gauge-invariant terms. That is, the addition of such low-order terms would keep the same form of the Zerilli equation (12.16), in particular with the same potential V_Z, but would change the source ${}^{\{n\}}S_{\mathcal{Z}}$. The definition given in (12.14) is just the simplest possibility, and follows [139]. We will later make use of this freedom.

12.2.2 Axial sector

The Gerlach and Sengupta master scalar is defined as the rotational of the axial invariant vector κ_A (10.15). Adopting Schwarzschild coordinates, it takes the following form in any generic gauge,

$$
{}^{\{n\}}\Pi = \frac{1}{r^3}\left[2\,{}^{\{n\}}h_t + r\left({}^{\{n\}}\dot{h}_r - {}^{\{n\}}h_t'\right)\right] + {}^{\{n\}}Q_\Pi, \qquad (12.20)
$$

where ${}^{\{n\}}Q_\Pi$ would be a source term that depends on lower-order perturbations and that is zero in the RW gauge. It obeys the GS master equation, which at second order is straightforwardly obtained from (11.50) after letting the matter perturbation $\tilde{\psi}_A$ vanish,

$$
-\left[\frac{1}{2r^2}(r^4\,{}^{\{2\}}\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\,{}^{\{2\}}\Pi = {}^{\{2\}}\mathcal{S}_\Pi. \qquad (12.21)
$$

The source term is related to the sources of the Einstein equations by (11.51). This equation is equivalent to the RW equation, which is satisfied by the rescaled variable ${}^{\{n\}}\tilde{\Pi} \equiv r^3\,{}^{\{n\}}\Pi$,

$$
{}^{\{n\}}\tilde{\Pi}_{|A}{}^{A} - V_{\rm RW}\,{}^{\{n\}}\tilde{\Pi} = 2r\,{}^{\{n\}}\mathcal{S}_\Pi, \qquad (12.22)
$$

where the potential is given by,

$$V_{\rm RW} = \frac{l(l+1)}{r^2} - \frac{6M}{r^3}. \tag{12.23}$$

There are some other variables that obey the same master equation. In particular, one that will be very useful for our purposes is that introduced by RW themselves in their original paper [56]. Its gauge-invariant generalization to higher orders takes the following form,

$${}^{\{n\}}\mathcal{X} \equiv v^A\, {}^{\{n\}}\kappa_A, \tag{12.24}$$

which in a general gauge and Schwarzschild coordinates is

$$\mathcal{X} = \frac{2M-r}{2r^2}\left[h' - 2h_r - \frac{2}{r}h\right] + {}^{\{n\}}Q_{\mathcal{X}}, \tag{12.25}$$

where ${}^{\{n\}}Q_{\mathcal{X}}$ is the usual term that depends on lower order perturbations and vanishes when particularized to the RW gauge. At linear order, ${}^{\{1\}}\mathcal{X}$ satisfies the same equation as $\tilde{\Pi}$. But already at second order the source term changes,

$${}^{\{2\}}\mathcal{X}_{|A}{}^{A} - V_{\rm RW}\, {}^{\{2\}}\mathcal{X} = {}^{\{2\}}\mathcal{S}_{\mathcal{X}}, \tag{12.26}$$

with

$${}^{\{2\}}\mathcal{S}_{\mathcal{X}} \;\equiv\; \sum_{\bar{l},\hat{l}} \frac{2i}{r^3}(3M-r)\,{}^{(-\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l} + iv^A\left[{}^{(-\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;|A} - 2\,{}^{(-\epsilon)}S^{\hat{m}\bar{m}m}_{\hat{l}\;\bar{l}\;l\;A}\right]. \tag{12.27}$$

As we have shown in Chapter 10 for standard matter models, which also includes vacuum, all perturbations of the metric can be algebraically reconstructed in terms of the master scalar Π. This is a great advantage, since for other variables, like $\mathcal{X}$, the reconstruction is non-algebraic.

12.3 First order

In this section we present the reconstruction of the metric components in the RW gauge (or the gauge invariants) in terms of first-order master scalars. In order to use the Landau and Lifshitz formula (12.3) we must use an asymptotically flat gauge. As we will see RW is not asymptotically flat. We will use two different methods to express the emitted power in terms of the master scalar. Firstly, we will perform an explicit gauge transformation from the RW to an AF gauge following [140]. Secondly, we will exploit the gauge invariant form of the master scalars presented in the previous section. This last method was already used in [139]. We want to show that the second method is much more straightforward and easier to apply. In this section we will remove all harmonic labels, as well as the $n=1$ label since all the objects will be of first-order and correspond to a generic harmonic pair (l,m).

12.3.1 Polar sector

At first order, the Zerilli equation is a wave equation without sources,

$$\mathcal{Z}^{|A}{}_{A} - V_{\rm Z}\mathcal{Z} = 0. \tag{12.28}$$

This master scalar allows full reconstruction of all components of the perturbation of the metric, in the RW gauge. Here we provide the explicit relations,

$$\begin{aligned}
H_{tt} &= \frac{2M-r}{4l(l+1)r^3\left(3M+\Lambda r\right)^2}\left\{2(2M-r)r^2\left(6M+2\Lambda r\right)^2\mathcal{Z}''\right. &(12.29)\\
&+ 4r\left[\Lambda\left(l^2+l-8\right)r^2M-2\Lambda^2r^3-18M^3\right]\mathcal{Z}'\\
&+ \left.4\left[18M^3+18\Lambda rM^2+6\Lambda^2r^2M+l(l+1)\Lambda^2r^3\right]\mathcal{Z}\right\},\\
H_{rr} &= \frac{r^2}{(2M-r)^2}H_{tt}, &(12.30)\\
H_{tr} &= \frac{2\left(3M^2+3\Lambda rM-\Lambda r^2\right)}{l(l+1)\left[6M^2+\left(l^2+l-5\right)rM-\Lambda r^2\right]}\dot{\mathcal{Z}}+\frac{2r}{l(l+1)}\dot{\mathcal{Z}}', &(12.31)\\
K &= \frac{1}{2l(l+1)r^2\left(3M+\Lambda r\right)}\left\{2r\left[-12M^2-2\left(l^2+l-5\right)rM+2\Lambda r^2\right]\mathcal{Z}'\right.\\
&+ \left.\left[24M^2+12\Lambda rM+(l-1)l(l+1)(l+2)r^2\right]\mathcal{Z}\right\}. &(12.32)
\end{aligned}$$

Introducing the above relations into the linearized Einstein equations, one can show that all of them are trivially satisfied if the Zerilli equation (12.28) holds.

In order to extract the radiated power (12.13), let us now perform a gauge transformation from the RW to an asymptotically flat gauge near infinity. Since the Zerilli variable $\mathcal{Z}$ obeys a wave equation, near null infinity, it can be expanded in inverse powers of r, with coefficients that depend on the retarded time $u \equiv t - r^*$ (note that both coordinate derivatives $(\partial/\partial u)_r$ and $(\partial/\partial t)_r$ coincide),

$$\mathcal{Z} \equiv \mathcal{Z}_0(u) + \frac{\mathcal{Z}_1(u)}{r} + \frac{\mathcal{Z}_2(u)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{12.33}$$

But these coefficients are not independent. Introducing this expansion into the Zerilli equation and solving it for each power of r independently we obtain the following relations,

$$\mathcal{Z}_0(u) = \frac{2}{l(l+1)}\ddot{F}(u), \qquad \mathcal{Z}_1(u) = \dot{F}(u), \qquad \mathcal{Z}_2(u) = \frac{\Lambda}{2}F(u) - \frac{3M(\Lambda+2)}{2\Lambda(\Lambda+1)}\dot{F}(u), \tag{12.34}$$

where the function $F(u)$ can be understood as the free data at null infinity.

In order to see the divergent behaviour of the harmonic coefficients in the RW gauge at null infinity, we replace the expansion (12.33) in (12.29-12.32),

$$
\begin{aligned}
H_{tt} &= \frac{4\ddddot{F}}{l^2(l+1)^2}r + \frac{4\Lambda\dddot{F}}{l^2(l+1)^2} + \mathcal{O}\left(\frac{1}{r}\right), && (12.35)\\
H_{rr} &= \frac{4\ddddot{F}}{l^2(l+1)^2}r + \frac{16M\ddddot{F} + 4\Lambda\dddot{F}}{l^2(l+1)^2} + \mathcal{O}\left(\frac{1}{r}\right), && (12.36)\\
H_{tr} &= -\frac{4\ddddot{F}}{l^2(l+1)^2}r - \frac{8M\ddddot{F} + 4\Lambda\dddot{F}}{l^2(l+1)^2} + \mathcal{O}\left(\frac{1}{r}\right), && (12.37)\\
K &= -\frac{4\dddot{F}}{l^2(l+1)^2}r + \mathcal{O}\left(\frac{1}{r^2}\right), && (12.38)
\end{aligned}
$$

where the orders r^0 and r^{-1} vanish for the harmonic coefficient K. Now, we can make an explicit gauge transformation, from the RW gauge to an asymptotically flat gauge (12.8) and obtain,

$$
\begin{aligned}
H_{tt}^{\mathrm{AF}} &= \mathcal{O}\left(\frac{1}{r^3}\right), && (12.39)\\
H_{rr}^{\mathrm{AF}} &= \mathcal{O}\left(\frac{1}{r^3}\right), && (12.40)\\
H_{tr}^{\mathrm{AF}} &= \mathcal{O}\left(\frac{1}{r^3}\right), && (12.41)\\
H_{t}^{\mathrm{AF}} &= \left\{\dot{\Xi}_1 - \frac{1}{4l^2(l+1)^2}\left[4M\ddot{F} + \frac{(l+2)!}{(l-2)!}\dot{F}\right]\right\}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), && (12.42)\\
H_{r}^{\mathrm{AF}} &= -\left\{\dot{\Xi}_1 - \frac{1}{2l^2(l+1)^2}\left[2M\ddot{F} + \Lambda(l^2+l-8)\dot{F}\right]\right\}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), && (12.43)\\
G^{\mathrm{AF}} &= \frac{4\ddot{F}}{l^2(l+1)^2}\frac{1}{r} + \frac{4\Lambda\dot{F}}{l^2(l+1)^2}\frac{1}{r^2} + 2\Xi_1\frac{1}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right), && (12.44)\\
K^{\mathrm{AF}} &= \mathcal{O}\left(\frac{1}{r^3}\right). && (12.45)
\end{aligned}
$$

Note that (12.39-12.41) and (12.45) show that one could actually ask for faster decay rates than those defined in (12.8). On the other hand, $\Xi_1 = \Xi_1(u)$ is a gauge freedom that is not fixed by the requirement of asymptotic flatness. From the behaviour of the harmonic coefficient G in an asymptotically flat gauge (12.44) and the asymptotic expansion of the Zerilli function (12.33), it is easy to obtain that

$$
G^{\mathrm{AF}} = \frac{2\mathcal{Z}}{l(l+1)r} + \mathcal{O}\left(\frac{1}{r^2}\right). \qquad (12.46)
$$

This last relation can be directly obtained from the gauge invariant definition of the Zerilli variable (12.14-12.15). Since that definition is valid for any gauge we can suppose that we are in an AF gauge. Imposing the decay rates (12.8) it is straightforward to obtain

(12.46). The great advantage of this last method is that we do not have to do an explicit gauge transformation. However, we need to assume that (12.8) is indeed possible.

Using relation (12.46) we obtain the radiated power in terms of the Zerilli function,

$$\text{Power} = \frac{\varepsilon^2}{16\pi}\sum_{l,m}\frac{(l-1)(l+2)}{l(l+1)}\left|\frac{\partial \mathcal{Z}_l^m}{\partial t}\right|^2. \tag{12.47}$$

12.3.2 Axial sector

In the first-order axial sector we have to solve the master equation,

$$-\left[\frac{1}{2r^2}(r^4\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\Pi = 0. \tag{12.48}$$

Once we have obtained the master scalar Π, we can reconstruct the metric in the RW gauge,

$$h_t = \frac{r^2}{2\Lambda}(2M-r)(4\Pi + r\Pi'), \tag{12.49}$$

$$h_r = \frac{r^5}{2\Lambda(2M-r)}\dot{\Pi}. \tag{12.50}$$

In order to obtain the radiated power in the axial sector, we will use again formula (12.13), valid for an asymptotically flat gauge. The rest of this subsection will be devoted to obtaining the harmonic coefficient h_l^{mAF} in an asymptotically flat gauge in terms of the master scalar Π.

We start by expanding the master scalar in inverse powers of r near the asymptotic null infinity ($r\to\infty, u = const.$). Since the function $\tilde{\Pi} = r^3\Pi$ satisfies a standard wave equation, our master scalar will have the following behaviour,

$$\Pi \equiv \frac{\Pi_0(u)}{r^3} + \frac{\Pi_1(u)}{r^4} + \frac{\Pi_2(u)}{r^5} + \mathcal{O}\left(\frac{1}{r^6}\right), \tag{12.51}$$

We can define a function $J(u)$ such that,

$$\Pi_0(u) = \frac{2}{l(l+1)}\ddot{J}(u), \qquad \Pi_1(u) = \dot{J}(u), \qquad \Pi_2(u) = \frac{\Lambda}{2}J(u) - \frac{3M}{l(l+1)}\dot{J}(u). \tag{12.52}$$

With these expansions at hand, we can obtain the precise divergent behaviour of the metric perturbation in RW gauge in terms of the function $J(u)$,

$$h_t = \frac{r}{\Lambda l(l+1)}\dddot{J} + \frac{1}{l(l+1)}\ddot{J} + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.53}$$

$$h_r = -\frac{r}{\Lambda l(l+1)}\dddot{J} - \frac{2M}{\Lambda l(l+1)}\dddot{J} - \frac{1}{2\Lambda}\ddot{J} + \mathcal{O}\left(\frac{1}{r}\right). \tag{12.54}$$

Now we perform a gauge transformation to an asymptotically flat gauge,

$$h_t^{AF} = \left\{\dot{\xi}_0 + \frac{1}{4}\dot{J} + \frac{(l-2)!}{(l+2)!}M\ddot{J}\right\}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{12.55}$$

$$h_r^{AF} = -\left\{\dot{\xi}_0 + \frac{(l-2)!}{(l+2)!}\left[M\ddot{J} + \frac{\Lambda}{2}(l^2+l-8)\dot{J}\right]\right\}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{12.56}$$

$$h^{AF} = -\frac{2r}{\Lambda l(l+1)}\ddot{J} - \frac{2}{l(l+1)}\dot{J} + \frac{2}{r}\xi_0 + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{12.57}$$

where $\xi_0 = \xi_0(u)$ is a residual gauge freedom. From the last relation it is easy to obtain that asymptotically,

$$h^{AF} = -\frac{r^4}{\Lambda}\Pi + \mathcal{O}(r^0). \tag{12.58}$$

Replacing this result in the formula for the emitted power,

$$\text{Power} = \frac{\varepsilon^2 r^6}{16\pi}\sum_{l,m}\frac{l(l+1)}{(l-1)(l+2)}\left|\frac{\partial \Pi_l^m}{\partial t}\right|^2 + \mathcal{O}\left(\frac{1}{r}\right). \tag{12.59}$$

One can try to obtain relation (12.58) with a gauge invariant approach, as we did in the polar case. But it can not be done since the gauge invariant form of the master variable Π (12.20) does not contain the harmonic coefficient h. Hence, one has to face the explicit gauge transformation. But at this point we note that there is another master variable $\mathcal{X}$ whose gauge-invariant form (12.25) does contain the harmonic coefficient h. Making a transformation to outgoing coordinates and supossing that the last definition is in an AF gauge, the relation between $\mathcal{X}$ and the time derivative of h at null infinity is easily obtained,

$$\dot{h}^{\rm AF} = 2r\mathcal{X} + \mathcal{O}(r^0). \tag{12.60}$$

Therefore, the emitted power can also be given in terms of this last variable,

$$\text{Power} = \frac{\varepsilon^2}{16\pi}\sum_{l,m}\frac{(l+2)!}{(l-2)!}|\mathcal{X}_l^m|^2 + \mathcal{O}\left(\frac{1}{r}\right). \tag{12.61}$$

Because we can apply this gauge-invariant approach to relate the master variable with the harmonic coefficient h at null infinity, at second-order we will use the variable ${}^{\{2\}}\mathcal{X}$. But there is one disadvantage in using $\mathcal{X}$ instead of Π. We have shown above how to reconstruct the perturbations of the metric in the RW gauge in terms of Π (12.49). These relations are algebraic. If we try to do the same with the variable $\mathcal{X}$, we find that the reconstruction of the metric is not algebraic, but differential,

$$h_r = \frac{r^2}{(r-2M)}\mathcal{X}, \tag{12.62}$$

$$\dot{h}_t = \left(1-\frac{2M}{r}\right)^2\left(h_r' + \frac{2M}{r-2M}\frac{h_r}{r}\right). \tag{12.63}$$

This is why we will use Π at linear order, so that we can give the sources explicitly in terms of it, and $\mathcal{X}$ at second order.

12.4 Second order

12.4.1 Regularization of the sources

At first sight one would say that in order to solve the system at second order it is enough to solve the Zerilli (12.16) and RW (12.26) equations with their corresponding sources (12.18) and (12.27), respectively. But we encounter another difficulty: as we have defined it, the second-order Zerilli function ${}^{\{2\}}\mathcal{Z}$ diverges at large distances, and as a consequence also the source ${}^{\{2\}}S_{\mathcal{Z}}$ of the equation it satisfies. In order to see this, it is enough to take its gauge-invariant definition (12.14-12.15), suppose that we are in an asymptotically flat gauge, and impose the conditions (12.39-12.45) and (12.55-12.57). In this way, we find that the quadratic source ${}^{\{2\}}Q_{\mathcal{Z}}$ diverges as,

$$ {}^{\{2\}}Q_{\mathcal{Z}} = Q_2 r^2 + Q_1 r + Q_0 + \mathcal{O}\left(\frac{1}{r}\right), \tag{12.64} $$

where Q_0, Q_1 and Q_2 are quadratic functions of $\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}$. The hat and bar on F and J functions denote, again, the generic harmonic labels $(\hat{l}, \hat{m})$ and $(\bar{l}, \bar{m})$, respectively. For instance, the dominant term is given by

$$ Q_2 = \sum_{\bar{l},\hat{l}} \frac{32}{\Lambda \bar{l}^2(\bar{l}+1)^2 \hat{l}^2(\hat{l}+1)^2} E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l} \; \ddot{\ddot{\hat{F}}} \; \dddot{\bar{F}}. \tag{12.65} $$

In this formula, because of the properties of Clebsch-Gordan coefficients, the E-coefficient (6.74) restricts the sums to those harmonic labels $\{\hat{l}, \bar{l}, l\}$ for which $\hat{l} + \bar{l} + l$ is an even number. That is, in the cases for which $\hat{l} + \bar{l} + l$ is an odd number, the term Q_2 cancels out. However, the term Q_1 contributes in all the cases and the divergence does not dissappear.

These divergences are non-physical and can be regularized by making use of the fact that we have absolute freedom to add any first-order gauge-invariant quadratic terms to the definition of the second-order master scalars and they will still satisfy the same equation with a new source term. Therefore, our aim is to obtain some quadratic terms on the first-order Zerilli and GS master scalars $Q_{reg} = Q_{reg}[\{\hat{\mathcal{Z}}, \overline{\mathcal{Z}}, \hat{\Pi}, \overline{\Pi}\}]$ so that, near null infinity, they reproduce the asymptotic divergent behaviour of the source $Q_{\mathcal{Z}}$. That is, for large r

while keeping $u = const$,

$$Q_{reg}[\{\hat{\mathcal{Z}}, \overline{\mathcal{Z}}, \hat{\Pi}, \overline{\Pi}\}] = Q_2[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}]r^2 + Q_1[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}]r + Q_0[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}] + \mathcal{O}\left(\frac{1}{r}\right). \quad (12.66)$$

In order to construct the function Q_{reg} we will make the following replacements in Q_2, Q_1 and Q_0,

$$\ddot{F} \to \frac{1}{2}l(l+1)\mathcal{Z}, \qquad \ddot{J} \to \frac{r^3}{2}l(l+1)\Pi. \quad (12.67)$$

These rules can be applied to replace second and higher derivatives of F and J. There are no F or J terms without derivatives in the divergent terms, but there are some first-order derivatives. Hence, the straightforward definition for the first derivative of F would be

$$\dot{F} \to -r^2 \left(\frac{\partial \mathcal{Z}}{\partial r}\right)_u. \quad (12.68)$$

But this replacement introduces divergences at the horizon $r = 2M$. In order to see this, we choose ingoing Eddington-Finkelstein coordinates, which are smooth at the horizon. They are obtained from the Schwarzschild coordinates (t, r) by making the following transformation of the time coordinate,

$$t \to w \equiv t + 2M \ln\left|\frac{r}{2M} - 1\right|. \quad (12.69)$$

In these coordinates the two-dimensional metric takes the form,

$$g_{AB}dx^A dx^B = -\left(1 - \frac{2M}{r}\right)dw^2 + \frac{4M}{r}dwdr + \left(1 + \frac{2M}{r}\right)dr^2. \quad (12.70)$$

Therefore, we have the following relation between derivatives,

$$\left(\frac{\partial \mathcal{Z}}{\partial r}\right)_u = \left(\frac{\partial \mathcal{Z}}{\partial r}\right)_\omega + \frac{r+2M}{r-2M}\left(\frac{\partial \mathcal{Z}}{\partial \omega}\right)_r, \quad (12.71)$$

which makes explicit the divergence of the radial derivative in outgoing coordinates at the horizon $r = 2M$.

Hence, instead of using the replacement rule (12.68), we can make a Taylor expansion in inverse powers of r of the right-hand side of the last relation and define a derivative that, for large r, will be equal to $(\partial/\partial r)_u$, but without being divergent at the horizon. Following this method we arrive at the relation

$$\dot{F} = -r^2 \left(\frac{\partial \mathcal{Z}}{\partial r}\right)_\omega - (r^2 + 4Mr + 8M^2)\left(\frac{\partial \mathcal{Z}}{\partial \omega}\right)_r + \mathcal{O}\left(\frac{1}{r}\right), \quad (12.72)$$

that converges at the horizon. Converting it back into Schwarzschild coordinates, gives the following rules to reconstruct the divergent terms,

$$\begin{aligned}\dot{F} &\to -r^2\mathcal{Z}' + \frac{r^3-16M^3}{2M-r}\dot{\mathcal{Z}},\\ \dot{J} &\to -r^2\left(r^3\Pi\right)' + \frac{r^3-16M^3}{2M-r}r^3\dot{\Pi}.\end{aligned} \tag{12.73}$$

The replacements (12.67) and (12.73) must be done systematically. That is, first take $Q_2 r^2$ and reconstruct the term that will reproduce it,

$$\sum_{\bar{l},\hat{l}} \frac{8r^2}{\Lambda\bar{l}(\bar{l}+1)\hat{l}(\hat{l}+1)} E_{0\hat{l}\hat{m}l}^{0\bar{l}\bar{m}}\ddot{\hat{\mathcal{Z}}}\dot{\overline{\mathcal{Z}}}. \tag{12.74}$$

When expanding near null infinity, this term will go as $Q_2 r^2 + R_1 r + R_0 + \mathcal{O}(r^{-1})$. In order to remove the divergent terms of order $\mathcal{O}(r)$, it is not enough to find a term that will reproduce $Q_1 r$, it must reproduce $(Q_1 - R_1)r$ to compensate the new term that we have just introduced. Therefore, we take $(Q_1 - R_1)r$ and make the above replacements (12.67) and (12.73) again. And so on, until we achieve the desired quadratic function $Q_{reg}[\{\hat{\mathcal{Z}},\overline{\mathcal{Z}},\hat{\Pi},\overline{\Pi}\}]$ that behaves as (12.66).

We have made the calculation for generic mode coupling, but the results are quite lengthy. Hence here we just want to present a particular example to make an idea of what kind of terms appear. In the particular case $(\hat{l},\hat{m}) = (\bar{l},\bar{m}) = (l,m) = (2,0)$ the previous construction gives rise to the following regularization term,

$$\begin{aligned} Q_{reg}^{\text{particular}} &= -\frac{1}{252\,(2M-r)}\sqrt{\frac{5}{\pi}}\Big\{2(2M-r)\Big((9M+r)\,{}^{\{1\}}\dot{\mathcal{Z}} + 6\,{}^{\{1\}}\mathcal{Z}\Big)\,{}^{\{1\}}\dot{\mathcal{Z}}\\ &+ \left(110M^3 - 21rM^2 + 14r^2M + 4r^3\right)\,{}^{\{1\}}\dot{\mathcal{Z}}\,{}^{\{1\}}\ddot{\mathcal{Z}}\\ &- 2(2M-r)\left(4r^2\,{}^{\{1\}}\mathcal{Z}' - (15M-6r)\,{}^{\{1\}}\mathcal{Z}\right)\,{}^{\{1\}}\ddot{\mathcal{Z}}\Big\}\\ &- \frac{3r^6}{224}\sqrt{\frac{5}{\pi}}\left\{16\,{}^{\{1\}}\dot{\Pi}\,{}^{\{1\}}\Pi + (2r-3M)\,{}^{\{1\}}\dot{\Pi}\,{}^{\{1\}}\dot{\Pi}\right\}. \end{aligned} \tag{12.75}$$

In this way, we define the regularized second-order Zerilli function as

$${}^{\{2\}}\mathcal{Z}_{reg} \equiv {}^{\{2\}}\mathcal{Z} + Q_{reg}. \tag{12.76}$$

It satisfies the following wave equation,

$${}^{\{2\}}\mathcal{Z}_{reg}{}_{l}^{m}{}_{|A}{}^{|A} - V_{\rm Z}\,{}^{\{2\}}\mathcal{Z}_{reg}{}_{l}^{m} = {}^{\{2\}}\mathcal{S}_{\mathcal{Z}}^{reg}, \tag{12.77}$$

where the regularized source is given by

$${}^{\{2\}}\mathcal{S}_{\mathcal{Z}}^{reg} \equiv {}^{\{2\}}\mathcal{S}_{\mathcal{Z}} + Q_{reg|A}{}^{|A} - V_{\rm Z} Q_{reg}. \tag{12.78}$$

On the other hand, in the axial case, there is no such a divergence. Following the same steps as above, one finds that near null infinity the quadratic part of the RW function ${}^{\{2\}}\mathcal{X}$ tends to

$$ {}^{\{2\}}Q_{\mathcal{X}} = Q^{(0)}_{\mathcal{X}}[\{\hat{F}, \overline{F}, \hat{J}, \overline{J}\}] + \mathcal{O}\left(\frac{1}{r}\right). \tag{12.79}$$

Therefore, in principle there is no need to regularize the second-order RW variable. But, as it will be clear in the next section, we are interested in removing the term of order $\mathcal{O}(1)$. Then, we apply the same procedure as in the polar case and obtain a term $Q^{reg}_{\mathcal{X}}[\hat{\mathcal{Z}}, \overline{\mathcal{Z}}, \hat{\Pi}, \overline{\Pi}]$ that, at null infinity, will reproduce $Q^{(0)}_{\mathcal{X}}[\hat{F}, \overline{F}, \hat{J}, \overline{J}]$. Following the above example, for the case $(\hat{l}, \hat{m}) = (\bar{l}, \bar{m}) = (l, m) = (2, 0)$ this regularizing term is given by

$$ Q^{reg}_{\mathcal{X}} = -\frac{r^3}{84}\sqrt{\frac{5}{\pi}}\{3\dot{\Pi}\dot{\mathcal{Z}} + \ddot{\Pi}\mathcal{Z} + \ddot{\mathcal{Z}}\Pi\}. \tag{12.80}$$

We define the regularized second-order RW variable as

$$ {}^{\{2\}}\mathcal{X}_{reg} \equiv {}^{\{2\}}\mathcal{X} + Q^{reg}_{\mathcal{X}}, \tag{12.81}$$

and its evolution equation

$$ {}^{\{2\}}\mathcal{X}_{regl}{}^{m}{}_{|A}{}^{|A} - V_{\rm RW}\, {}^{\{2\}}\mathcal{X}_{regl}{}^{m} = {}^{\{2\}}\mathcal{S}^{reg}_{\mathcal{X}}, \tag{12.82}$$

where the regularized source is again given by

$$ {}^{\{2\}}\mathcal{S}^{reg}_{\mathcal{X}} \equiv {}^{\{2\}}\mathcal{S}_{\mathcal{X}} + Q^{reg}_{\mathcal{X}}{}_{|A}{}^{|A} - V_{\rm RW} Q^{reg}_{\mathcal{X}}. \tag{12.83}$$

The regularized sources for the equations of motion (12.77) and (12.82) are the main result of this chapter. We have calculated them for the presence of any first- and second-order axial or polar modes. We do not include them here because they are quite lengthy and do not contribute to the discussion. Nevertheless, in Appendix C we have written down their explicit form for two particular cases.

12.4.2 Radiated power

After solving for the second-order perturbations of the metric, we can obtain the corresponding radiated power by making use of the Landau and Lifshitz formula (12.13). Writing it explicitly up to order ε^3, it takes the following form,

$$\begin{aligned} \text{Power} &= \frac{\varepsilon^2}{64\pi r^2}\sum_{l,m}\frac{(l+2)!}{(l-2)!}\left\{ r^4 \left|\frac{\partial\, {}^{\{1\}}G_l^{mAF}}{\partial t}\right|^2 + \left|\frac{\partial\, {}^{\{1\}}h_l^{mAF}}{\partial t}\right|^2 \right. \\ &+ \left. \varepsilon \mathrm{Re}\left[r^4 \frac{\partial\, {}^{\{1\}}G_l^{mAF}}{\partial t}\left(\frac{\partial\, {}^{\{2\}}G_l^{mAF}}{\partial t}\right)^* + \frac{\partial\, {}^{\{1\}}h_l^{mAF}}{\partial t}\left(\frac{\partial\, {}^{\{2\}}h_l^{mAF}}{\partial t}\right)^*\right]\right\} + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{12.84}$$

where Re means the real part. Again, the problem of finding the radiated power, reduces to calculate the harmonic coefficients G_l^m and h_l^m, near null infinity, in an asymptotically flat gauge. More precisely, we want to relate them with the regularized master scalars that would be used to perform a numerical implementation of this problem. In the previous section, we have regularized the second-order master variables so that the quadratic contributions from first-order modes decay as $\mathcal{O}(1/r)$ near null infinity. Hence, we can use their gauge-invariant definitions, (12.15) and (12.20), and suppose that we are in an AF gauge (12.8), which is also valid at second order (this has been explicitly shown in [140]). This leads to the very same relations as at first-order, namely,

$$
\begin{aligned}
{}^{\{2\}}G_l^{m\mathrm{AF}} &= \frac{2\,{}^{\{2\}}\mathcal{Z}_l^m{}_{reg}}{l(l+1)r} + \mathcal{O}\left(\frac{1}{r^2}\right), && (12.85)\\
{}^{\{2\}}\dot{h}_l^{m\mathrm{AF}} &= 2r\,{}^{\{2\}}\mathcal{X}_l^m{}_{reg} + \mathcal{O}(r^0). && (12.86)
\end{aligned}
$$

Replacing these relations in the above formula, the radiated power, up to order ε^3, is given in terms of the master scalars by

$$
\begin{aligned}
\text{Power} &= \frac{\varepsilon^2}{64\pi}\sum_{l,m}\frac{(l+2)!}{(l-2)!}\left\{\frac{4}{l^2(l+1)^2}\left|\frac{\partial\,{}^{\{1\}}\mathcal{Z}_l^m}{\partial t}\right|^2 + \frac{r^6}{\Lambda^2}\left|\frac{\partial\,{}^{\{1\}}\Pi_l^m}{\partial t}\right|^2\right. && (12.87)\\
&+ \left.\varepsilon Re\left[\frac{4}{l^2(l+1)^2}\frac{\partial\,{}^{\{1\}}\mathcal{Z}_l^m}{\partial t}\left(\frac{\partial\,{}^{\{2\}}\mathcal{Z}_l^m{}_{reg}}{\partial t}\right)^* - \frac{2r^3}{\Lambda}\frac{\partial\,{}^{\{1\}}\Pi_l^m}{\partial t}\,{}^{\{2\}}\mathcal{X}_l^{m*}{}_{reg}\right]\right\} + \mathcal{O}(\varepsilon^4).
\end{aligned}
$$

This last formula, complemented with the evolution equations for the regularized master scalars (12.77) and (12.82), provides a closed set of formulas that permit us to obtain the complete radiated power up to order ε^3.

Note that even solving the problem up to second-order in perturbations, we only can obtain the complete radiated power up to third-order in ε. In order to obtain the following order ε^4, one should also consider third-order perturbations.

Let us suppose the simplest scenario: a unique first-order mode with harmonic labels (l, m) and polarity σ, as has been defined in Subsection 9.2.3. Because of the reality conditions, if we have the mode (l, m, σ) we must also have its conjugate $(l, -m, \sigma)$. Following the selection rules (9.58) and (9.59), the self-coupling of this mode will generate several second-order modes but not necessarily the one with labels $(l, \pm m, \sigma)$. In contrast, at third order the mentioned mode $(l, \pm m, \sigma)$ will always be generated by the mere coupling of the first-order mode under consideration with the second-order $(l, 0, (-1)^l)$. This means that the third-order modes will always contribute to the emitted power at order ε^4, coupled to the first-order mode with the same harmonic labels. Therefore, without considering third-order modes, one can only obtain the radiated power consistently up to order ε^3. In order that the emitted power (12.13) has a contribution of that order (ε^3), the self-coupling of the first-order mode must give a second-order mode with the same labels (l, m, σ). It is easy to see from the selection rules (9.58) and (9.59) that when considering a unique first-order mode $(l, \pm m, \sigma)$, this will happen if and only if $m = 0$ and if, for $\sigma = 1$ ($\sigma = -1$), l is an even (odd) number.

In order to clarify the above discussion and analyze which particular problems can be addressed consistently, let us consider the particular case of a first-order polar mode with harmonic labels $l = m = 2$ and $l = m = -2$. These modes will generate the second-order $\{l = 4, m = \pm 4, 0\}$, $\{l = 2, m = 0\}$ and $\{l = 0, m = 0\}$ polar modes as well as the $\{l = 3, m = 0\}$ axial mode. Particularizing the power formula (12.13) in terms of the master scalars to this case, we obtain the following contributions from the considered modes,

$$\begin{aligned} \text{Power} &= \frac{\varepsilon^2}{12\pi} \left|\partial_t\, {}^{\{1\}}\mathcal{Z}^2_{2\,reg}\right|^2 + \frac{9\varepsilon^4}{640\pi} \left\{\left|\partial_t\, {}^{\{2\}}\mathcal{Z}^0_{4\,reg}\right|^2 + 2\left|\partial_t\, {}^{\{2\}}\mathcal{Z}^4_{4\,reg}\right|^2\right\} \\ &+ +\frac{15\varepsilon^4}{8\pi} \left|{}^{\{2\}}\mathcal{X}^0_{3\,reg}\right|^2 + \frac{\varepsilon^4}{96\pi} \left|\partial_t\, {}^{\{2\}}\mathcal{Z}^0_{2\,reg}\right|^2, \end{aligned} \tag{12.88}$$

where the order ε^3 is not present. The problem with this formula is that it is not complete since the third-order $\{l = 2, m = \pm 2\}$ polar mode would contribute to the power at order ε^4.

On the other hand, let us consider the first-order mode $l = 2$ with all its possible harmonic labels $m = 0, \pm 1, \pm 2$. By coupling, they will generate the second-order polar modes $l = 0$, $l = 2$ and $l = 4$ with all their possible m. That is, we will have the second-order $\{l = 0, m = 0\}$, $\{l = 2, m = 0, \pm 1, \pm 2\}$ and $\{l = 4, m = 0, \pm 1, \pm 2, \pm 3, \pm 4\}$ polar modes. This particular case will provide a non-vanishing ε^3-order term to the power,

$$\begin{aligned} \text{Power} &= \frac{\varepsilon^2}{24\pi} \left\{2\left|\partial_t\, {}^{\{1\}}\mathcal{Z}^2_{2\,reg}\right|^2 + 2\left|\partial_t\, {}^{\{1\}}\mathcal{Z}^1_{2\,reg}\right|^2 + \left|\partial_t\, {}^{\{1\}}\mathcal{Z}^0_{2\,reg}\right|^2\right\} \\ &+ \frac{\varepsilon^3}{24\pi} Re\left[2\partial_t({}^{\{1\}}\mathcal{Z}^2_{2\,reg})\partial_t({}^{\{2\}}\mathcal{Z}^2_{2\,reg})^* + 2\partial_t({}^{\{1\}}\mathcal{Z}^1_{2\,reg})\partial_t({}^{\{2\}}\mathcal{Z}^1_{2\,reg})^*\right. \\ &+ \left.\partial_t({}^{\{1\}}\mathcal{Z}^0_{2\,reg})\partial_t({}^{\{2\}}\mathcal{Z}^0_{2\,reg})^*\right] + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{12.89}$$

In this last case the formula is exact up to the displayed order because at order ε^3 the generated second-order axial modes and third-order polar modes do not contribute.

12.5 Numerical implementation

In this section we present some preliminary results about the numerical investigation on second-order perturbations of Schwarzschild, that we are performing at this moment [125]. We will just consider a first-order $(l = 2, m = \pm 2)$ polar mode described by a real Zerilli function ${}^{\{1\}}\mathcal{Z}$. As we have seen in the previous section, and can be easily verified with the selection rules (9.58) and (9.59), by selfcoupling this mode generates several second-order axial as well as polar modes. In particular, here we will only focus on the $(l = 2, m = 0)$ polar mode, which will be encoded in the (also real) Zerilli function ${}^{\{2\}}\mathcal{Z}$.

Making use of a pseudo-spectral code, we have solved the equations (12.28) and (12.77) for the evolution of the two (first-order and second-order) variables. The numerical scheme is a fourth-order Runge-Kutta integrator in time, whereas the spatial domain is decomposed in blocks of length $10M$, so that, at each block, the function is proyected into Tchebychev polynomials. The grid points are the so-called Gauss-Lobatto points [160], which are not equidistant. In fact, they accumulate at the boundaries of each block. The spatial domain in decomposed in blocks to avoid too too coars spatial discretization. Different blocks are matched usign a penalty technique [161].

The mass of the black hole has been fixed to $M = 1$, which locates the event horizon at $r = 2$. The initial data have been taken as a Gaussian centered in the position $r = 20M$ for the velocity of the first-order variable and vanishing for the rest of the components,

$$
\begin{aligned}
{}^{\{1\}}\mathcal{Z}(t=0,r) &= 0, && (12.90)\\
\partial_t\, {}^{\{1\}}\mathcal{Z}(t=0,r) &= \exp(-(r-20M)^2/4^2), && (12.91)\\
{}^{\{2\}}\mathcal{Z}(t=0,r) &= 0, && (12.92)\\
\partial_t\, {}^{\{2\}}\mathcal{Z}(t=0,r) &= 0. && (12.93)
\end{aligned}
$$

Therefore all the information for the evolution of the second-order Zerilli function will come from the quadratic source $S_{\mathcal{Z}}$.

In the following plot we show the time dependence of both first and second Zerilli functions for an observer located at position $r = 41M$. The exterior boundary for the integration domain has been chosen large enough $(r = 801.8M)$ to prevent spureous reflections from the outer boundary. The convergency test done for this case estimates an accuracy of the order 10^{-10} for the plotted solution.

In this plot we can observe the expected quasinormal mode ringing (of both first and second-order Zerilli functions) dominating the evolution for times greater than $50M$. The quasinormal frecuency for both Zerilli functions is similar. For late times $(> 250M)$ one can observe the decaying power-law tails, also with similar slopes. These results are very preliminary and need further study. In particular, there is a remarkable effect regarding the tails in the previous plot. It happens that, for large times, ${}^{\{2\}}\mathcal{Z}$ is greater than ${}^{\{1\}}\mathcal{Z}$. We still do not understand this behavior and it could indicate that there exists a limit of validity in the second order approximation.

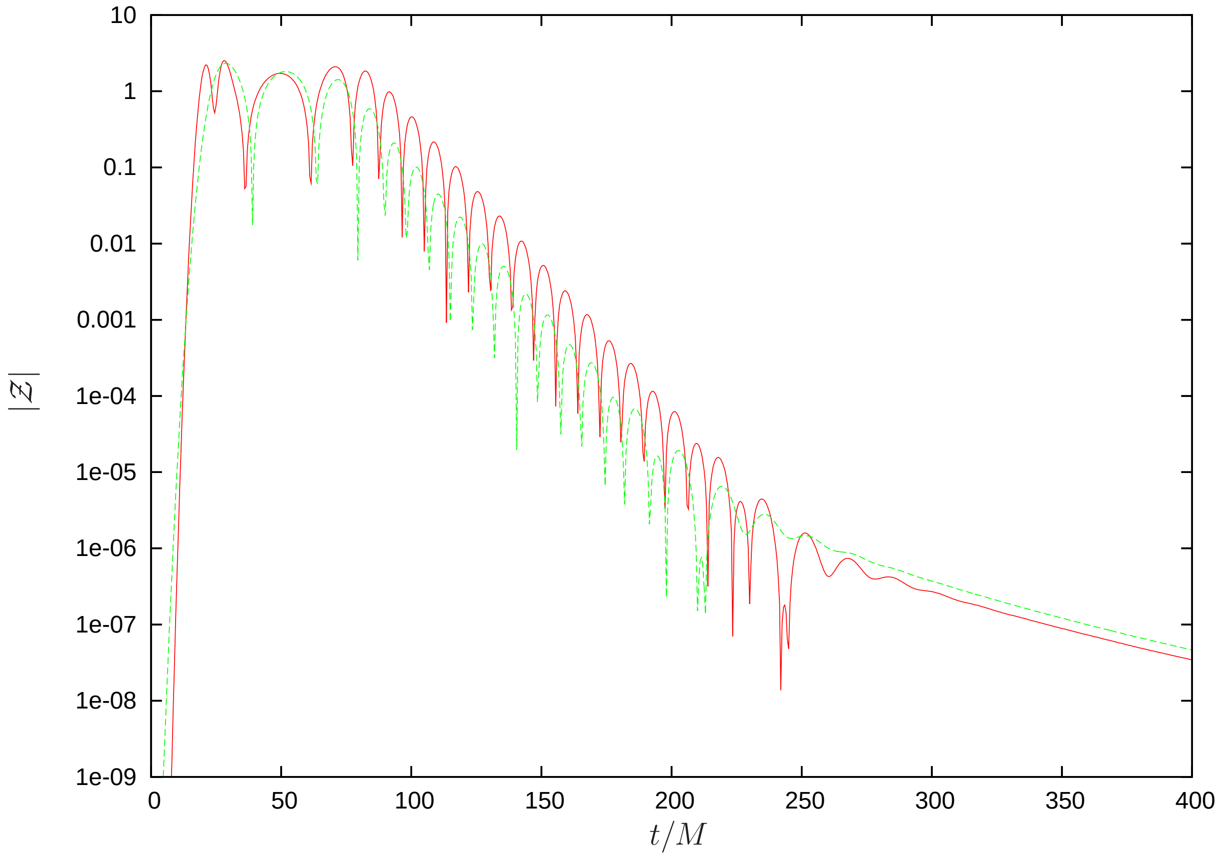

Figure 12.1: The evolution of the first-order Zerilli variable with harmonic labels ($l = 2, m \pm 2$) (in red) generates, by self-coupling, the second-order Zerilli function ($l = 2, m = 0$) (in green). The plot is for an observer stationary at position $r = 41M$.

Chapter 13

Perfect fluid

In this chapter we apply the second-order GS formalism to a spacetime containing perfect fluid matter. This generalizes the work by Gundlach and Martín-García [55] beyond linear perturbation theory, and closely follows their notations. It provides a general framework to study the evolution of perturbations of both first and second orders on a dynamical fluid background, and together with the results in the previous and the next chapter, gives a complete set of tools to study the emission of gravitational radiation during the collapse of a slowly rotating star, a project now under development [124].

The chapter is organized in two sections. In the first one we will introduce the notation for the perturbations of the basic fluid variables and calculate the perturbed energy-momentum tensor in terms of them. At second order this will introduce matter sources in the Einstein equations. The second section analyzes how to solve the evolution equations, separating the cases $l \geq 2$, $l = 1$ and $l = 0$. Several long expressions have been moved to appendices. The equations of motion and notations for the background have been already displayed in Section 5.1.3. In particular the vector field u^μ will be the four-velocity of the perfect fluid. Together with its unit normal n^μ they define the following frame derivatives that act on any scalar function ζ,

$$\dot{\zeta} \equiv u^A \zeta_{|A}, \qquad \zeta' \equiv n^A \zeta_{|A}. \tag{13.1}$$

13.1 High-order perfect fluid perturbations

Using the nth order Leibnitz rule (3.6), it is easy to find the explicit form for the nth perturbation of the energy-momentum tensor of a perfect fluid (5.44),

$$\Delta^n[t_{\mu\nu}] = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \left\{ {}^{\{k\}}p\,{}^{\{n-k\}}h_{\mu\nu} + \sum_{j=k}^{n} \left(\begin{array}{c} n-k \\ j-k \end{array} \right) ({}^{\{k\}}\rho + {}^{\{k\}}p)\,{}^{\{j-k\}}u_\mu\,{}^{\{n-j\}}u_\nu \right\}, \quad (13.2)$$

where we define the following notations for the perturbations: ${}^{\{n\}}u_\mu \equiv \Delta^n[u_\mu]$, ${}^{\{n\}}p \equiv \Delta^n[p]$ and ${}^{\{n\}}\rho \equiv \Delta^n[\rho]$. This can be further expanded, using the high-order generalization of the chain rule (Faá di Bruno formula) on the general equation of state $p = p(\rho, s)$,

$${}^{\{n\}}p \;\; = \;\; \sum \frac{n!}{2!^{k_2}...i!^{k_i}2!^{r_2}...j!^{r_j}k_1!...k_i!r_1!...r_j!} \frac{\partial^{K+R}p}{\partial\rho^K \partial s^R}\;{}^{\{1\}}\rho^{k_1} \ldots {}^{\{i\}}\rho^{k_i}\,{}^{\{1\}}s^{r_1} \ldots {}^{\{j\}}s^{r_j}, \quad (13.3)$$

where the sum is restricted to the constraint

$$\sum_{m=1}^{i} m k_m + \sum_{m=1}^{j} m r_m = n$$

and we have defined $K \equiv \sum_{m=1}^{i} k_m$ and $R \equiv \sum_{m=1}^{j} r_m$. Derivatives of the pressure must be replaced by the sound speed c_s and thermodynamic factor C, defined in equations (5.45), and their derivatives. Note that these combinatorial formulas are valid for any background spacetime, not necessarily spherical.

The fluid four-velocity vector is normalized to -1, and this must be satisfied at all perturbative orders,

$$\Delta^n[u_\mu g^{\mu\nu} u_\nu] = 0. \quad (13.4)$$

Applying again the Leibniz rule n times and separating the terms linear in perturbations of order n, we can rewrite this equation as follows,

$$\begin{aligned} &-2\,{}^{\{n\}}u^\mu u_\mu - u_\mu u_\nu \Delta^n[g^{\mu\nu}] = \quad (13.5) \\ &\sum_{k=1}^{n-1}\sum_{i=k}^{n} \frac{n!}{k!(i-k)!(n-i)!}\Delta^k[g^{\mu\nu}]\,{}^{\{i-k\}}u_\mu\,{}^{\{n-i\}}u_\nu + \sum_{i=1}^{n-1} \left(\begin{array}{c} n \\ i \end{array} \right) {}^{\{i\}}u_\mu\,{}^{\{n-i\}}u^\mu, \end{aligned}$$

where the nth order perturbation of the inverse metric is given by formula (3.17). Because of spherical symmetry, the background four-velocity has no angular components ($u_a = 0$). Therefore, the constraint (13.5) contains only non-angular components of the nth perturbation of the four-velocity. Calling ${}^{\{n\}}\Upsilon$ the right-hand side of equation (13.5), we choose the following ansatz,

$${}^{\{n\}}u^A \;\; = \;\; \frac{1}{2}\,{}^{\{n\}}\Upsilon u^A - \frac{1}{2}\Delta^n[g^{AB}]u_B + {}^{\{n\}}\tilde{\gamma} n^A, \quad (13.6)$$

which reproduces the ansatz used in [55] and trivially satisfies the constraint (13.5). In this way, the three independent components of the perturbations of the four-velocity are encoded in the generic function ${}^{\{n\}}\tilde{\gamma}$ and in the angular components ${}^{\{n\}}u^a$. The scalar function ${}^{\{n\}}\tilde{\gamma}$ will be expanded in harmonic series and its harmonic coefficients will be denoted as ${}^{\{n\}}\gamma_l^m$.

Following this ansatz, at first order we decompose the four-velocity as [55]

$$ {}^{\{1\}}u_\mu dx^\mu \equiv \sum_{l,m} \left({}^{\{1\}}\gamma_l^m n_A + \frac{1}{2}\, {}^{\{1\}}\mathcal{K}_l^m{}_{AB} u^B \right) Z_l^m dx^A + \left({}^{\{1\}}\alpha_l^m Z_l^m{}_a + {}^{\{1\}}\beta_l^m X_l^m{}_a \right) dx^a, \quad (13.7) $$

Note that we are supposing that everything in this equation is gauge invariant, and hence so must be the harmonic component ${}^{\{1\}}\gamma_l^m$. At second order, owing to the two first terms in (13.6), first-order quadratic terms appear in the harmonic decomposition of the four-velocity,

$$ \begin{aligned} {}^{\{2\}}u_\mu dx^\mu \;&\equiv\; \sum_{l,m} \left({}^{\{2\}}\gamma_l^m n_A + \frac{1}{2}\, {}^{\{2\}}\mathcal{K}_l^m{}_{AB} u^B + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{U}_{\bar{l}}^{\bar{m}}{}_{\hat{l}}^{\hat{m}}{}_l^m{}_A \right) Z_l^m dx^A \\ &+\; \left({}^{\{2\}}\alpha_l^m Z_l^m{}_a + {}^{\{2\}}\beta_l^m X_l^m{}_a \right) dx^a. \end{aligned} \quad (13.8) $$

We have decomposed the sources of the four-velocity in spherical harmonics,

$$ \begin{aligned} {}^{(+)}\mathcal{U}_{\bar{l}}^{\bar{m}}{}_{\hat{l}}^{\hat{m}}{}_l^m{}_A \;&=\; E_{0}^{0}{}_{\hat{l}\hat{m}l}^{\bar{l}\bar{m}} \{ u_A [\hat{\gamma}\bar{\gamma} - \hat{\gamma}\bar{\mathcal{K}}_{BC} n^B u^C + \frac{1}{4}\hat{\mathcal{K}}^{BD}\bar{\mathcal{K}}_{CD} u_B u^C - \hat{\mathcal{K}}^{BC}\bar{\mathcal{K}}_{CD} u_B u^D] \quad (13.9) \\ &\qquad - u_B \hat{\mathcal{K}}^{BC} \bar{\mathcal{K}}_{AC} \} \\ &-\; \frac{2}{r^2} E_{1}^{-1}{}_{\hat{l}\hat{m}l}^{\bar{l}\bar{m}} \{ u_A [\hat{\alpha}\bar{\alpha} + \hat{\beta}\bar{\beta} + \hat{\kappa}_B u^B \bar{\kappa}_C u^C - 2\hat{\kappa}_B u^B \bar{\beta} - \hat{\kappa}_B \bar{\kappa}_C u^B u^C] - \hat{\kappa}_B u^B \bar{\kappa}_A \}, \\ {}^{(-)}\mathcal{U}_{\bar{l}}^{\bar{m}}{}_{\hat{l}}^{\hat{m}}{}_l^m{}_A \;&=\; \frac{4i}{r^2} E_{1}^{-1}{}_{\hat{l}\hat{m}l}^{\bar{l}\bar{m}} u_A [\hat{\alpha}\bar{\beta} + \bar{\alpha}\hat{\kappa}^B u_B]. \quad (13.10) \end{aligned} $$

Again, objects with hats and overbars denote first-order harmonic components with harmonic labels $(\hat{l}, \hat{m})$ and $(\bar{l}, \bar{m})$, respectively.

We define the harmonic decomposition of the perturbations of the density and entropy,

$$ {}^{\{n\}}\rho \equiv \sum_{l,m} \rho\, {}^{\{n\}}\omega_l^m Z_l^m, \quad (13.11) $$

$$ {}^{\{n\}}s \equiv \sum_{l,m} {}^{\{n\}}\sigma_l^m Z_l^m. \quad (13.12) $$

From now on we particularize our formulas to second order and will not write the $n = 2$ and $n = 1$ labels anymore, since it will be clear from the context. On the other hand, the pure first order can be considered a particular case of the second-order case just by removing all quadratic sources. We perform the harmonic decomposition of the second-order perturbation of the energy-momentum tensor (13.2). In terms of metric and matter perturbations, the axial components are given by,

$$\psi^m_{l\ A} = p\kappa^m_l + \beta^m_l(p+\rho)u_A - i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A}, \tag{13.13}$$

$$\psi^m_l = -i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l}, \tag{13.14}$$

where $\mathcal{E}$ are first-order quadratic sources that will be given below. The polar sector takes the following form,

$$\begin{aligned}\Psi^m_{l\ AB} &= (\rho+p)\left[\gamma^m_l(u_A n_B + n_A u_B) + \frac{1}{2}(\mathcal{K}^m_{l\ AC}u_B + \mathcal{K}^m_{l\ BC}u_A)\right]u^C \\ &+ p\mathcal{K}^m_{l\ AB} + \rho\omega^m_l(u_A u_B + c_s^2 n_A n_B) + C\rho\sigma^m_l n_A n_B + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ AB},\end{aligned} \tag{13.15}$$

$$\Psi^m_{l\ A} = (p+\rho)\alpha^m_l u_A + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A}, \tag{13.16}$$

$$\tilde{\Psi}^m_l = p\kappa^m_l + c_s^2\rho\omega^m_l + C\rho\sigma^m_l + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\tilde{\mathcal{E}}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l}, \tag{13.17}$$

$$\Psi^m_l = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l}. \tag{13.18}$$

Finally, we provide the explicit sources in terms of first-order perturbations and E-coefficients,

$$\begin{aligned}{}^{(+)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ AB} &= \frac{4}{r^2}(p+\rho)E^{-1\bar{l}\bar{m}}_{1\ \hat{l}\hat{m}l}\Big\{\hat{\kappa}^C\bar{\kappa}_{(A}u_{B)}u_C - u_A u_B[\hat{\alpha}\bar{\alpha} + \hat{\beta}\bar{\beta} - 2\hat{\beta}\bar{\kappa}_C u^C]\Big\} \\ &+ E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\Big\{2\rho c_s^2\hat{\omega}\bar{\mathcal{K}}_{AB} + 2\rho\bar{\gamma}\left[C\hat{\sigma} + \hat{\omega}\left(1+c_s^2\right)\right](n_B u_A + n_A u_B) \\ &+ 2C\rho\hat{\sigma}(\bar{\mathcal{K}}_A{}^C n_B n_C + \bar{\mathcal{K}}_{C[B}u_{A]}u^C) + 2\rho\hat{\omega}\left(1+c_s^2\right)\bar{\mathcal{K}}_{C(B}u_{A)}u^C \\ &+ \rho n_A n_B\left(\bar{\sigma}\hat{\sigma}\frac{\partial C}{\partial s} + 2\bar{\omega}\hat{\sigma}\frac{\partial c_s^2}{\partial s} + \rho\bar{\omega}\hat{\omega}\frac{\partial c_s^2}{\partial\rho}\right) \\ &+ (P+\rho)\Big[2\bar{\gamma}\hat{\gamma}(n_A n_B + u_A u_B) + 2\bar{\gamma}\hat{\mathcal{K}}_{C(A}n_{B)}u^C - 2\bar{\gamma}\hat{\mathcal{K}}_{CD}n^C u^D u_A u_B \\ &- \frac{3}{2}\hat{\mathcal{K}}_C{}^F\bar{\mathcal{K}}_{DF}u^C u^D u_A u_B - 2\hat{\mathcal{K}}^{CD}\bar{\mathcal{K}}_{D(A}u_{B)}u_C + \frac{1}{2}\hat{\mathcal{K}}_{AC}u^C\bar{\mathcal{K}}_{BD}u^D\Big]\Big\},\end{aligned} \tag{13.19}$$

$${}^{(-)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ AB} = \frac{8i}{r^2}E^{-1\bar{l}\bar{m}}_{1\ \hat{l}\hat{m}l}\bar{\alpha}(\hat{\kappa}^C u_C - \hat{\beta})(p+\rho)u_A u_B, \tag{13.20}$$

$${}^{(+)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} = 2E^{1\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\bar{\alpha}\Big\{(p+\rho)\left[\hat{\gamma}n_A + \frac{1}{2}\hat{\mathcal{K}}_{AB}u^B\right] + \rho\hat{\omega}(1+c_s^2)u_A + C\rho\hat{\sigma}u_A\Big\}, \tag{13.21}$$

$$\begin{aligned}{}^{(-)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\ \hat{l}\ l\ A} &= 2iE^{1\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\Big\{(p+\rho)[\hat{\gamma}n_A + \frac{1}{2}\hat{\mathcal{K}}_{AB}u^B]\bar{\beta} + \rho\bar{\beta}[(c_s^2+1)\hat{\omega} + C\hat{\sigma}]u_A \\ &+ \rho(C\hat{\sigma} + c_s^2\hat{\omega})\bar{\kappa}_A\Big\},\end{aligned} \tag{13.22}$$

$$
\begin{aligned}
{}^{(+)}\tilde{\mathcal{E}}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} &= -\frac{2}{r^2}(p+\rho)E^{-1\bar{l}\bar{m}}_{1\;\hat{l}\hat{m}l}\Big\{\hat{\alpha}\bar{\alpha}+\hat{\beta}\bar{\beta}\Big\}+2\rho E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\Big\{\hat{\mathcal{K}}(\bar{\omega}c_s^2+\bar{\sigma}C) \\
&+\ \hat{\omega}\bar{\sigma}\frac{\partial c_s^2}{\partial s}+\frac{1}{2}\hat{\sigma}\bar{\sigma}\frac{\partial C}{\partial s}+\frac{\rho}{2}\hat{\omega}\bar{\omega}\frac{\partial c_s^2}{\partial\rho}\Big\}, \qquad (13.23) \\
{}^{(-)}\tilde{\mathcal{E}}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} &= \frac{4i}{r^2}E^{-1\bar{l}\bar{m}}_{1\;\hat{l}\hat{m}l}(p+\rho)\hat{\alpha}\bar{\beta}, \qquad (13.24) \\
{}^{(+)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} &= 2E^{1\bar{l}\bar{m}}_{1\hat{l}\hat{m}l}(\hat{\alpha}\bar{\alpha}-\hat{\beta}\bar{\beta})(p+\rho), \qquad (13.25) \\
{}^{(-)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} &= 4iE^{1\bar{l}\bar{m}}_{1\hat{l}\hat{m}l}\hat{\alpha}\bar{\beta}(p+\rho). \qquad (13.26)
\end{aligned}
$$

13.2 Second-order evolution equations

We now focus on the second-order perturbations of the evolution equations, both for the fluid and the metric variables, separating the polar and axial problems for arbitrary harmonic label l. Cases with $l = 0, 1$ are special because the corresponding harmonic bases are degenerated. In particular, because of the impossibility of constructing gauge-invariants for $l = 0, 1$, we will need to resort to an adequate gauge fixing.

This section generalizes the results [55] for first-order perturbations to second order. The equations at second order share the linear part with those at first order, but contain complicated additional sources which are quadratic in first-order perturbations. Our main task here will be, hence, to compute those quadratic sources.

In perturbing the Einstein equations, we find two types of quadratic sources. First, those coming from second-order perturbation of the Einstein tensor, which have been denoted as ${}^{(\epsilon)}\mathcal{S}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l}$ and displayed in Chapter 11. Second, sources ${}^{(\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l}$, arising from the perturbation of the energy-momentum tensor (13.13)-(13.18), which have been given explicitly in (13.19)-(13.26) for perfect fluid matter. It is convenient to combine those two types as

$$
{}^{(\epsilon)}\mathcal{B}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} \equiv 8\pi\,{}^{(\epsilon)}\mathcal{E}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l} - {}^{(\epsilon)}\mathcal{S}^{\bar{m}\hat{m}m}_{\bar{l}\;\hat{l}\;l}, \qquad (13.27)
$$

for all tensorial, vectorial and scalar sources. From now on we will remove the harmonic labels $\{l, m, \hat{l}, \hat{m}, \bar{l}, \bar{m}\}$ appearing as super and sub indices of the harmonic coefficients, as well as in the sources.

13.2.1 Axial perturbations ($l \geq 2$)

The energy-momentum conservation equation $t^{\mu\nu}{}_{;\mu} = 0$ contains the evolution equations for a perfect fluid, except the conservation of the entropy per particle (5.60). The perturbation of the former of these equations is composed by three polar and one axial equations for each perturbative order n, whereas perturbing the latter gives rise to a polar equation only. In the axial sector there is only a transport equation for β (11.43),

$$\begin{aligned}(p+\rho)[\dot\beta - c_s^2(\mu+2U)\beta] = \\ = i\sum_{\bar l,\hat l}\left\{{}^{(-\epsilon)}\mathcal{I} - \frac{(l-1)(l+2)}{2r^2}{}^{(-\epsilon)}\mathcal{E} + \frac{1}{r^2}\left(r^{2}{}^{(-\epsilon)}\mathcal{E}_A\right)^{|A}\right\}.\end{aligned} \tag{13.28}$$

After expanding the energy-momentum perturbation in terms of fluid variables (13.13), we arrive at the following form for the second-order GS master equation (11.50) on the background that we are considering,

$$\begin{aligned}-\left[\frac{1}{2r^2}(r^4\,{}^{\{2\}}\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\,{}^{\{2\}}\Pi = \\ 8\pi(p+\rho)\,{}^{\{2\}}\beta' - \frac{8\pi}{c_s^2}(\rho C s' + \nu(p+\rho))\,{}^{\{2\}}\beta - i\epsilon^{AB}\sum_{\bar l,\hat l}{}^{(-\epsilon)}\mathcal{B}_{A|B}.\end{aligned} \tag{13.29}$$

Since there are no covariant derivatives in the energy-momentum tensor of the perfect fluid, the vector κ_A can be reconstructed in terms of Π and matter perturbative variables,

$$\kappa_A = \frac{1}{(l-1)(l+2)}\left[16\pi r^2\tilde\psi_A - \epsilon_{AB}(r^4\Pi)^{|B} + 2ir^2\sum_{\bar l,\hat l}{}^{(-\epsilon)}S_A\right]. \tag{13.30}$$

We expand this vector field in the frame provided by the perfect fluid,

$$\kappa_A \equiv \delta u_A + \lambda n_A, \tag{13.31}$$

and obtain the following components in terms of fluid variables and $\mathcal{B}$ sources,

$$\delta = \frac{r^2}{(l-1)(l+2)}(r^2\Pi' + 4r^2W\Pi + 16\pi\beta(P+\rho) + 2i\sum_{\bar l,\hat l}{}^{(-\epsilon)}\mathcal{B}_A u^A), \tag{13.32}$$

$$\lambda = -\frac{r^2}{(l-1)(l+2)}(r^2\dot\Pi + 4r^2U\Pi + 2i\sum_{\bar l,\hat l}{}^{(-\epsilon)}\mathcal{B}_A n^A). \tag{13.33}$$

Therefore, for the case that the second-order harmonic label is $l \geq 2$, the evolution can be performed with two equations. On the one hand, the matter equation (13.28) evolves β, and on the other hand, the GS master equation (13.29) evolves the GS master scalar Π. Once these two variables are obtained, we can reconstruct the perturbed metric by using equations (13.31-13.33).

13.2.2 Axial perturbations ($l = 1$)

In this case, the fluid equation is the same as in the general case (13.28) particularized to $l = 1$. This makes one of the right-hand side terms vanish, but the equation continues being a transport equation for β,

$$(p+\rho)(\dot{\beta} - c_s^2(\mu + 2U)\beta) = i \sum_{\bar{l},\hat{l}} \left\{ {}^{(-\epsilon)}\mathcal{I} + \frac{1}{r^2}\left(r^{2\,(-\epsilon)}\mathcal{E}_A\right)^{|A} \right\}. \tag{13.34}$$

For $l = 1$ there cannot be gravitational waves, and so the metric perturbations cannot obey a wave equation. Instead, the metric perturbation equation is given by,

$$\frac{1}{2r^2}\epsilon_{AB}(r^4\Pi)^{|B} = 8\pi\tilde{\psi}_A + i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}S_A. \tag{13.35}$$

Projecting this equation into the frame vectors u^A and n^A respectively, and introducing the dependence in the perturbative fluid variables, it gives rise to

$$\frac{r^2}{2}(\Pi' + 4W\Pi) = -8\pi(p+\rho)\beta - i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{B}_A u^A, \tag{13.36}$$

$$\frac{r^2}{2}(\dot{\Pi} + 4U\Pi) = -i\sum_{\bar{l},\hat{l}} {}^{(-\epsilon)}\mathcal{B}_A n^A. \tag{13.37}$$

This last equation will be used to evolve Π. One could employ equation (13.36) to obtain β algebraically from Π, but we know that for certain equation of states the term $(p+\rho)$ vanishes at the surface of the star, what makes the equation (13.36) inappropriate for numerical resolution. Instead, we can always determine β from the matter equation (13.34).

We cannot employ formula (13.30) to reconstruct the vector κ_A from Π and β, and actually it cannot be uniquely reconstructed. Any two-dimensional vector can be written in terms of two scalar functions $\zeta_1(x^A)$ and $\zeta_2(x^A)$ in the following way,

$$\frac{1}{r^2}\kappa_A \equiv \zeta_{1|A} + \epsilon_A{}^B\zeta_{2|B} = -(\dot{\zeta}_1 + \zeta_2')u_A + (\zeta_1' + \dot{\zeta}_2)n_A. \tag{13.38}$$

From the definition of the master scalar Π (10.15), we obtain an equation to solve for ζ_2,

$$\Pi = -\zeta_2{}^{|A}{}_A = \ddot{\zeta}_2 + \mu\dot{\zeta}_2 - \nu\zeta_2' - \zeta_2''. \tag{13.39}$$

There is still one axial gauge degree of freedom because the RW gauge (9.7) has not imposed anything in this case, since the harmonic coefficient h does not exist for $l = 1$. Thus, the other scalar ζ_1 is this gauge freedom. Under a gauge transformation the vector κ_A changes in the following way,

$$\kappa_A \longrightarrow \kappa_A + r^2\xi_{|A} + \ldots, \tag{13.40}$$

where the dots indicate a source term quadratic in first-order perturbations and is explicitly given in [114]. In the notation of the previous section, the gauge transformation is

$$\delta \longrightarrow \delta - r^2\dot{\xi} + \ldots, \qquad (13.41)$$

$$\lambda \longrightarrow \lambda + r^2\xi' + \ldots. \qquad (13.42)$$

Therefore, to fix the gauge, we can make δ or λ vanish, or any combination of them. The best option is to make the gauge choice $\lambda = 0$, which leaves a residual gauge under all those vectors such that $\xi' = 0$. This can be interpreted as a free function of time at the center. In this gauge, the vector κ_A is given by,

$$\kappa_A = -r^2(\dot{\zeta}_1 + \zeta_2')u_A, \qquad (13.43)$$

where ζ_2 is obtained from equation (13.39) and $\zeta_1' = -\dot{\zeta}_2$.

13.2.3 Polar perturbations ($l \geq 2$)

In order to deal only with scalar quantities, we decompose the tensorial metric gauge-invariant into three scalars making use of the frame defined by the fluid,

$$\mathcal{K}_{AB} \equiv \eta(n_A n_B - u_A u_B) + \phi(u_A u_B + n_A n_B) + \psi(u_A n_B + n_A u_B). \qquad (13.44)$$

The second-order Einstein equations can be schematically given in the following way. For $l \geq 0$,

$$u^A n^B E_{AB}[\mathcal{K}] = -8\pi(\rho+p)\gamma + 4\pi(\rho-p)\psi + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}_{AB} u^A n^B, \qquad (13.45)$$

$$u^A u^B E_{AB}[\mathcal{K}] = 8\pi\rho\omega + 8\pi\rho(\eta-\phi) + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}_{AB} u^A u^B, \qquad (13.46)$$

$$n^A n^B E_{AB}[\mathcal{K}] = 8\pi\rho(c_s^2\omega + C\sigma) + 8\pi p(\eta+\phi) + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}_{AB} n^A n^B, \qquad (13.47)$$

$$\tilde{E}[\mathcal{K}] = 8\pi\rho(c_s^2\omega + C\sigma) + 8\pi p\mathcal{K} + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\tilde{\mathcal{B}}. \qquad (13.48)$$

For $l \geq 1$,

$$u^A E_A[\mathcal{K}] = -(\rho+p)\alpha + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}_A u^A, \qquad (13.49)$$

$$n^A E_A[\mathcal{K}] = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}_A n^A. \qquad (13.50)$$

And finally, for $l \geq 2$,

$$E[\mathcal{K}] = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}. \tag{13.51}$$

The E's are the GS differential operators (10.2-10.5) acting on the metric polar perturbations $\{\mathcal{K}_{AB}, \mathcal{K}\}$. Writing explicitly the linear dependences of equation (13.51) in second-order objects, we obtain the following relation,

$$\eta = -\sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\mathcal{B}. \tag{13.52}$$

This equation determines the value of η in terms of first-order perturbations. Note that in the case of first-order perturbation theory η vanishes. Therefore, this variable is fixed and we do not have to worry about it. This equation makes the treatment for the case $l \geq 2$ differ from the case $l = 1$ analyzed in the next section, since in the latter case this equation does not exist.

We define the following linear combinations of first-order sources,

$$\begin{aligned}
{}^{(\epsilon)}C_\gamma &\equiv {}^{(\epsilon)}\mathcal{B}_{AB}u^A n^B, && (13.53)\\
{}^{(\epsilon)}C_\omega &\equiv -{}^{(\epsilon)}\mathcal{B}_{AB}u^A u^B, && (13.54)\\
{}^{(\epsilon)}C_\alpha &\equiv 2{}^{(\epsilon)}\mathcal{B}_A u^A, && (13.55)\\
{}^{(\epsilon)}S_\chi &\equiv 2{}^{(\epsilon)}\tilde{\mathcal{B}} + 4({}^{(\epsilon)}\mathcal{B}_A n^A)' - 2{}^{(\epsilon)}\mathcal{B}_{AB}n^A n^B + 4(2\nu - W){}^{(\epsilon)}\mathcal{B}_A n^A, && (13.56)\\
{}^{(\epsilon)}S_\mathcal{K} &\equiv (-c_s^2 u^A u^B + n^A n^B){}^{(\epsilon)}\mathcal{B}_{AB} + 4W{}^{(\epsilon)}\mathcal{B}_A n^A, && (13.57)\\
{}^{(\epsilon)}S_\psi &\equiv -2{}^{(\epsilon)}\mathcal{B}_A n^A. && (13.58)
\end{aligned}$$

And we also introduce the following change to a new variable χ,

$$\phi \longrightarrow \chi + \mathcal{K} - \eta. \tag{13.59}$$

With this new notation at hand, the system of equations (13.45)-(13.50) can be rewritten,

$l \geq 0:$

$$\begin{aligned}
8\pi(p+\rho)\gamma &= (\dot{\mathcal{K}})' + C_\gamma + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}C_\gamma, && (13.60)\\
8\pi\rho\omega &= -\mathcal{K}'' + 2U\psi' + C_\omega + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}C_\omega, && (13.61)
\end{aligned}$$

$l \geq 1$:

$$16\pi(p+\rho)\alpha = \psi' + C_\alpha + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}C_\alpha, \tag{13.62}$$

$$-\ddot{\chi} + \chi'' + 2(\mu - U)\psi' = S_\chi + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\chi, \tag{13.63}$$

$$-\ddot{\mathcal{K}} + c_s^2\mathcal{K}'' - 2c_s^2 U\psi' = S_\mathcal{K} + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\mathcal{K}, \tag{13.64}$$

$$-\dot{\psi} = S_\psi + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\psi, \tag{13.65}$$

where only higher-order derivatives of the variables have been given explicitly. The linear sources C_γ, C_ω, C_α, S_χ, $S_\mathcal{K}$, and S_ψ are linear in the variables χ, $\mathcal{K}$ and η and their first-order derivatives, as well as in the undifferentiated variables ψ and σ. These sources coincide with those given in [55] and are included in Appendix D.1 for completeness. Note that their only dependence in matter perturbations is through the entropy σ, which, as we will see, obeys an equation decoupled from the rest of matter perturbations. This fact allows us to interpret equations (13.63)-(13.65) as coupled evolution equations for the variables χ, $\mathcal{K}$ and ψ. The other three equations (13.60)-(13.62) provide the matter perturbations γ, ω and α algebraically in terms of the metric perturbations.

In order to close the system of equations, we give here the second-order perturbation of the entropy conservation equation $\dot{s} = 0$, for $l \geq 0$,

$$\dot{\sigma} + (\gamma + \frac{\psi}{2})s' = \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\sigma, \tag{13.66}$$

where the source terms are given by,

$$\begin{aligned} {}^{(+)}S_\sigma &= -\frac{2}{r^2} E_1^{-1}{}^{\bar{l}\bar{m}}_{\hat{l}\hat{m}l} \Big\{ (2\hat{\beta}\bar{\lambda} + \hat{\delta}\bar{\lambda})s' - 2\hat{\alpha}\bar{\sigma} \Big\} \\ &+ E_0^{0}{}^{\bar{l}\bar{m}}_{\hat{l}\hat{m}l} \Big\{ 2\hat{\gamma}(\bar{\mathcal{K}} + \bar{\chi})s' - (2\bar{\gamma} + \bar{\psi})\hat{\sigma}' + (2\bar{\eta} - \bar{\mathcal{K}} - \bar{\chi})\dot{\hat{\sigma}} \Big\}, \end{aligned} \tag{13.67}$$

$${}^{(-)}S_\sigma = -\frac{4i}{r^2} E_1^{-1}{}^{\bar{l}\bar{m}}_{\hat{l}\hat{m}l} \Big\{ \bar{\alpha}\hat{\lambda}s' - (\hat{\beta} + \hat{\delta})\bar{\sigma} \Big\}. \tag{13.68}$$

At this point we have succeeded in giving a complete set of equations [namely (13.60-13.66)] to evolve the second-order perturbations of both the metric and the perfect fluid, for $l \geq 2$. The matter perturbations are obtained algebraically from the metric perturbations, though they could also be evolved using conservation of energy-stress,

$l \geq 0$:

$$-\dot{\omega} - \left(1 + \frac{p}{\rho}\right)\left(\gamma + \frac{\psi}{2}\right)' = S_\omega + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\omega, \quad (13.69)$$

$$\left(1 + \frac{p}{\rho}\right)\left(\gamma - \frac{\psi}{2}\right)^{\cdot} + c_s^2 \omega' = S_\gamma + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\gamma, \quad (13.70)$$

$l \geq 1$:

$$-\dot{\alpha} = S_\alpha + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}S_\alpha, \quad (13.71)$$

where, again, we have only made explicit the highest derivatives. These equations are redundant for $l \geq 2$, but will play a fundamental role for $l = 0, 1$. The linear sources S_ω, S_γ, and S_α are also provided in Appendix D.1. They depend just on the undifferentiated matter variables α, γ, and ω and on the entropy perturbation σ and its spatial derivative, as well as on the metric perturbations χ, $\mathcal{K}$, ψ, η, and their derivatives. Besides, we have defined the following quadratic sources,

$$\rho\, {}^{(\epsilon)}S_\omega \equiv \frac{l(l+1)}{r^2} {}^{(\epsilon)}\mathcal{E}_A u^A + 2U\, {}^{(\epsilon)}\tilde{\mathcal{E}} - \frac{1}{r^2}(r^2\, {}^{(\epsilon)}\mathcal{E}_{AB})^{|B} u^A - {}^{(\epsilon)}\mathcal{I}_A u^A, \quad (13.72)$$

$$\rho\, {}^{(\epsilon)}S_\gamma \equiv \frac{l(l+1)}{r^2} {}^{(\epsilon)}\mathcal{E}_A n^A + 2W\, {}^{(\epsilon)}\tilde{\mathcal{E}} - \frac{1}{r^2}(r^2\, {}^{(\epsilon)}\mathcal{E}_{AB})^{|B} n^A - {}^{(\epsilon)}\mathcal{I}_A n^A, \quad (13.73)$$

$$(p+\rho)\, {}^{(\epsilon)}S_\alpha \equiv {}^{(\epsilon)}\tilde{\mathcal{E}} - \frac{(l-1)(l+2)}{2r^2} {}^{(\epsilon)}\mathcal{E} + \frac{1}{r^2}(r^2\, {}^{(\epsilon)}\mathcal{E}^A)_{|A} + {}^{(\epsilon)}\mathcal{I}. \quad (13.74)$$

It is possible to use equations (13.62) and (13.65) to remove the dependence on the derivative of ψ from equations (13.69) and (13.70). In this way, with the exception of $\dot{\mathcal{K}}$ and $\mathcal{K}'$, no derivatives of the metric perturbations are present in the matter equations,

$l \geq 1$:

$$-\dot{\omega} - \left(1 + \frac{p}{\rho}\right)\gamma' = \overline{S}_\omega + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\overline{S}_\omega, \quad (13.75)$$

$$\left(1 + \frac{p}{\rho}\right)\dot{\gamma} + c_s^2 \omega' = \overline{S}_\gamma + \sum_{\bar{l},\hat{l}} {}^{(\epsilon)}\overline{S}_\gamma. \quad (13.76)$$

The linear sources $\overline{S}_\omega$ and $\overline{S}_\gamma$ can be found again in Appendix D.1.

13.2.4 Polar perturbations ($l = 1$)

As explained in Chapter 9, there are also three polar degrees of freedom in the case $l = 1$, but the RW gauge (9.7) only imposes two conditions ($H_A = 0$). Therefore, the RW gauge is not rigid for $l = 1$ and we do not have a complete set of invariants. Under a gauge transformation, these "invariants" change as

$$\mathcal{K}_{AB} \to \mathcal{K}_{AB} - (r^2\xi_{|A})_{|B} - (r^2\xi_{|B})_{|A} + \dots, \tag{13.77}$$

$$\mathcal{K} \to \mathcal{K} - 2\xi - 2r^2v^A\xi_{|A} + \dots, \tag{13.78}$$

for a generic function $\xi(x^A)$ and where the dots indicate first-order quadratic terms. These quadratic terms will arise only if there is some contribution from a first-order $l = 0, 1$ mode, otherwise the quadratic terms in the definition of the invariants will absorb them.

At this point, we follow Campolattaro and Thorne [42] and impose that $\mathcal{K}$ vanish as the leading gauge. This still leaves a residual gauge freedom corresponding to functions ξ such that $\xi + r^2v^A\xi_{|A} = 0$.

From the point of view of the evolution equations, as we have commented, the main difference between this and the general $l \geq 2$ case is that equation (13.52) disappears now. Therefore, we need to use equations (13.66), (13.71), (13.75), and (13.76) in order to evolve the matter perturbations $\{\sigma, \alpha, \omega, \gamma\}$. This is possible because those equations contain only the metric perturbations $\{\eta, \psi, \chi\}$, but not their derivatives. Once all matter perturbations are obtained, one can determine the metric perturbations by making use of the Einstein equations (13.60-13.62) and (13.64-13.65). These five equations do not depend on second and higher-order (time and space) derivatives of $\{\eta, \psi, \chi\}$. Hence, we have five equations for six unknowns. We can obtain $\{\dot{\psi}, \psi', \dot{\chi}, \chi'\}$ but only the following combination between $\dot{\eta}$ and η',

$$D\eta \equiv \frac{r^{|A}\eta_{|A}}{r^{|B}r_{|B}} = \frac{1}{r|v|^2}(W\eta' - U\dot{\eta}), \tag{13.79}$$

where we have defined $|v|^2 \equiv v^Av_A$. This means that we can only integrate η in a spatial surface that is everywhere normal to the $r = const.$ surfaces. Therefore, in Appendix D.2 we provide the equations exclusively for $D\eta$, $D\psi$ and $D\chi$.

13.2.5 Polar perturbations ($l = 0$)

The spherical perturbations are treated in a very similar way as in the case $l = 1$. We first evolve the matter perturbations and then obtain the metric perturbations as constraint equations. This was expected, since in spherical symmetry the metric has no radiative freedom and it is totally constrained.

First of all, since the RW gauge does not impose any restriction in the case $l = 0$, there are still two gauge degrees of freedom to fix. We extend the gauge used in the previous section by choosing

$$\mathcal{K} = 0, \qquad \psi = \frac{2UW}{U^2 + W^2}(\eta - \chi). \tag{13.80}$$

The second condition implies the vanishing of the variable ψ when working in polar-radial coordinates. Another new feature of this case is that the velocity perturbation α is also equal to zero.

The matter perturbations σ, ω, and γ are evolved by making use of equations (13.66), (13.69), and (13.70) respectively. The last two of these equations contain derivatives of the metric perturbations that can be removed by making use of the perturbed Einstein equations (13.45-13.47). In Appendix D.3, we present the resulting equations, as well as the constraint equations for the only two non-vanishing metric perturbations η and χ.

Chapter 14

Perturbative matching

In the previous two chapters we have studied the perturbative problem for a perfect fluid as well as a vacuum background. We now assume that our background system is a spherical fluid star surrounded by vacuum, with both regions, interior and exterior, separated by a timelike surface Σ where the pressure vanishes. Therefore, in order to complete the analysis of the given physical situation, we need to find out the matching conditions between those two parts of the spacetime (both at the background and perturbative levels), ensuring continuity and the correct exchange of radiative information through the stellar surface Σ.

The first section of this chapter describes high-order perturbative matching on any timelike surface in a general background spacetime. The second section is particularized to the case of a fluid interior matched to a vacuum exterior. This generalizes the first-order results of [112] for the same scenario, and we closely follow the notation of that reference.

14.1 High-order matching conditions

We describe the matching surface Σ as the zero level set of a smooth scalar field $\mathcal{P}(x^\mu)$, whose continuation off the surface is irrelevant. The unit vector normal to the surface is defined as

$$n_\mu \equiv \mathfrak{p}\mathcal{P}_{,\mu}, \quad \text{where} \quad \mathfrak{p} \equiv (\mathcal{P}_{,\nu}\mathcal{P}^{,\nu})^{-1/2}\,. \tag{14.1}$$

From this vector we construct the induced metric $i_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu$ and the extrinsic curvature $e_{\mu\nu} \equiv n_{\mu;\alpha}i^\alpha{}_\nu$ of the surface. In order to ensure a smooth matching at Σ, the Israel junction conditions [162] require that the induced metric $i_{\mu\nu}$ must be continuous through the surface, whereas the extrinsic curvature $e_{\mu\nu}$ may have a discontinuity as given by the Lanczos tensor $S_{\mu\nu}$ [163, 164]. This tensor can be understood as the surface energy-momentum tensor and is defined with respect to the proper distance δ to the surface in the direction of its normal n_μ,

$$S^\mu{}_\nu \equiv \lim_{\delta\to 0}\int_{-\delta}^{\delta} t^\mu{}_\nu ds. \tag{14.2}$$

The Lanczos tensor is non-zero only in the presence of surface layers, that is, when there is some delta-like contribution to the energy-momentum tensor. From now on we will consider smooth energy-momentum tensors, so that, the junction conditions reduce to continuity of both $i_{\mu\nu}$ and $e_{\mu\nu}$. Before discussing high-order perturbations of these objects we need to address two important new problems in the perturbative version of the matching problem: index positioning and gauge dependence.

As has been explained in Chapter 3, for a generic tensor field T_μ we have

$$\Delta[T_\mu] \neq g_{\mu\nu}\Delta[T^\nu], \tag{14.3}$$

because in general the perturbation of the metric field does not vanish. Imposing continuity on the perturbations of the covariant or contravariant forms of the tensors i and e can lead to different results, and so we must decide which are the adequate conditions. The discussion of the first-order problem in [112] shows that we must use perturbations of the contravariant fundamental forms. Essentially, this is because a contravariant tensor field $T^{\mu\nu\cdots}$ is intrinsic to the surface Σ if and only if $T^{\mu\nu\cdots}\mathcal{P}_{,\mu} = 0$ on any of its indices. However, the equivalent condition for a covariant tensor field $T_{\mu\nu\ldots}$ would be $T_{\mu\nu\ldots}g^{\mu\alpha}\mathcal{P}_{,\alpha} = 0$, which involves the metric, and therefore introduces additional information, not intrinsic to the surface. This argument is valid also for the perturbed objects, hence we will impose continuity of $\Delta^n[i^{\mu\nu}]$ and $\Delta^n[e^{\mu\nu}]$ for all n.

On the other hand, we need to deal with the gauge freedom arising from the arbitrariness of the choice of mapping Ψ between the perturbed and the background spacetimes. Under a general perturbation, the scalar $\mathcal{P}$ will also change, so that the perturbed surface will be described by the level surfaces of $\mathcal{P} + \Delta_\Psi[\mathcal{P}] + \ldots$ (a field on the background manifold), where we have made explicit the gauge Ψ relating the background and perturbed spacetimes. This allows having a perturbed interior point at a coordinate position corresponding to the background exterior, a situation that can be handled consistently using one-sided derivatives [91]. However, the question remains of what are the correct continuity conditions, because these conditions could be inequivalent if expressed in different gauges. There is a privileged class of gauges Θ, characterized by the condition $\Delta^n_\Theta[\mathcal{P}] = 0$, generalizing the first-order surface gauge of [112]. This does not mean that the shape of the surface will not change. It could be highly distorted in the perturbed manifold, but when mapping it to the background manifold a point p of the perturbed surface will be mapped onto another point (not necessarily p) of the background surface. Hence, we will have as matching conditions continuity of the perturbations of the induced metric $\Delta^n_\Theta[i^{\mu\nu}]$ and extrinsic curvature $\Delta^n_\Theta[e^{\mu\nu}]$ in surface gauge at any perturbative order.

Surface gauge is only used to define the continuity conditions. We can still work using any other gauge, which is convenient for the interior or exterior problems. Then, the best way of treating gauge freedom is by constructing gauge-invariants associated to surface gauge, that is combinations of the perturbations in an arbitrary gauge whose value coincide with the result in surface gauge. As has been explained in Chapter 4, this can be achieved by finding the general form of a gauge transformation from a general gauge to a surface gauge. Such transformation will be parameterized by the gauge vectors $\{{}^{\{1\}}\xi^\mu, ..., {}^{\{n\}}\xi^\mu\}$ defined by solving the expression for the gauge transformation (4.9) order by order,

$$0 = \Delta^n[\mathcal{P}] + \sum_{m=1}^{n} \frac{n!}{(n-m)!} \sum_{[k_m]} \frac{1}{2!^{k_2}...(m!)^{k_m} k_1!...k_m!} \quad \mathcal{L}^{k_1}_{\{1\}\xi}...\mathcal{L}^{k_m}_{\{m\}\xi}\Delta^{n-m}[\mathcal{P}], \tag{14.4}$$

where now Δ represents again perturbations in an arbitrary gauge. Particular solutions at first and second orders are given by

$$ {}^{\{1\}}\xi_\mu = -\Delta[\mathcal{P}]n_\mu, \tag{14.5}$$
$$ {}^{\{2\}}\xi_\mu = -\Delta^2[\mathcal{P}]n_\mu + \mathfrak{p}^2\Delta[\mathcal{P}]\Delta[\mathcal{P}]_{,\mu}. \tag{14.6}$$

These solutions fix only one of the four degrees of freedom in each generator. The general solutions would have three additional degrees of freedom, which represent gauge changes among different surface gauges, corresponding to the same surface Σ. Owing to its non-rigidity, the gauge-invariant objects we will construct using the surface gauge will be invariant only under very restricted transformations. These gauge-invariant combinations of perturbations will be denoted using a $\overline{\Delta}$ operator. On any background tensor field T we define

$$\overline{\Delta}^n[T] \equiv \Delta^n[T^\mu] + \sum_{m=1}^{n} \frac{n!}{(n-m)!} \sum_{(K_m)} \frac{1}{k_1!...k_m!} \frac{1}{2!^{k_2}...(m!)^{k_m}} \mathcal{L}^{k_1}_{\{1\}\xi}...\mathcal{L}^{k_m}_{\{m\}\xi}\Delta^{n-m}[T], \tag{14.7}$$

where the gauge vectors $\{{}^{\{1\}}\xi, ..., {}^{\{n\}}\xi\}$ are those obtained by solving equation (14.4). The fact that the gauge generators depend implicitly on the metric perturbations makes this formula highly nontrivial and nonlinear.

The $\bar{\Delta}$ operator has been constructed explicitly so that it satisfies $\overline{\Delta}^n[\mathcal{P}] = 0$. This leads to the important result that the barred perturbation of the one-form n_μ is proportional to itself,

$$\overline{\Delta}^n[n_\mu] = \frac{\overline{\Delta}^n[\mathfrak{p}]}{\mathfrak{p}} n_\mu. \tag{14.8}$$

Again, this formula contains nontrivial information in the barred perturbation of the scalar $\mathfrak{p}$. For example, at first order it is given by,

$$\overline{\Delta}[\mathfrak{p}] = \frac{\mathfrak{p}}{2} n^\alpha \left[n^\beta h_{\alpha\beta} - 2(\mathfrak{p}\Delta[\mathcal{P}])_{,\alpha} \right]. \qquad (14.9)$$

We do not display explicit second and higher-order formulas here, as they are rather complicated and do not contribute to enlighten the discussion.

Result (14.8) implies that perturbations of any contravariant tensor intrinsic to the surface Σ will also be intrinsic to the perturbed surface. That is, if the background tensor T^μ is orthogonal to n_μ, then its barred perturbations will not have orthogonal components either,

$$\overline{\Delta}^n [T^\mu] n_\mu = 0. \qquad (14.10)$$

Note that this property is not shared by covariant tensor fields.

Summarizing, we require the barred perturbations of the contravariant fundamental forms, $\overline{\Delta}^n[i^{\mu\nu}]$ and $\overline{\Delta}^n[e^{\mu\nu}]$, to be continuous through the surface Σ at any perturbative order n. Note that we have not imposed surface gauge, as was done in reference [112], since these conditions are given for any general gauge.

We end this subsection by mentioning the alternative approach to perturbative matching introduced by Mukohyama [165] and further developed in [166]. These are results for first-order perturbations, and coincide with those obtained here and in [112], though not imposing surface gauge and also including gauge freedom within the matching surface Σ. The main difference is that Mukohyama introduces an abstract copy of the surface Σ and the matching is performed separately between the boundaries of the interior and exterior spacetimes and that new surface. This introduces an additional geometric structure and gives rise to a new type of gauge invariance, and so the concept of double gauge-invariants appears. This is a nice feature in problems like reduction from a N-dimensional spacetime to a $(N-1)$-dimensional brane, but it would be a complication in the problem of spherical backgrounds, where the geometry of the matching surface is trivial.

14.2 Matching to vacuum

Around spherical symmetry, the matching conditions can be decomposed in tensor spherical harmonics. We have obtained the decomposed form of the continuity conditions for second-order perturbations in the most general case, but we will analyze these conditions only for a very particular example of interest, since the general expressions are quite large. In this example we will consider the presence of a first-order $\{l = 1, m = 0\}$ axial mode that, by self coupling, generates the second-order $\{l = 2, m = 0\}$ polar mode. This is a particularly interesting situation since the non-radiative first-order mode can be understood as a slow rotation of the star that produces gravitational radiation through self coupling. The shorthands dot and prime that will be used in this section are those related to the fluid frame (13.1). In order to facilitate the identification of the perturbative order of each object that will appear throughout this section, we will use the left superindex $\{1\}$ on first-order perturbations.

The background junction conditions are straightforwardly deduced from continuity of the two first fundamental forms of the surface. The continuous quantities include the scalars r, ν, and W defined in (5.50) and (5.52) respectively. Derivatives of continuous quantities in the direction of the fluid velocity u^A must also be continuous. This leads to continuity of U (5.52) and, since both derivatives of r are continuous, to continuity of the Hawking mass (5.23). In our particular case, the negative pressure of the fluid $-p$ will be interpreted as the scalar function $\mathcal{P}$ of the previous section, since the stellar surface is characterized by $p = 0$. The negative sign has been chosen so that $\mathcal{P}$ increases with the radius r. The pressure must be continuous trough the surface, whereas the energy density ρ may jump there.

The first-order axial matching is simplified by the existence of the Gerlach and Sengupta master scalar, which can be defined both in the interior and the exterior without using fluid information. In fact, a first-order junction condition for our particular example is simply continuity of the master scalar ${}^{\{1\}}\Pi$ through the surface. Regarding the axial vector ${}^{\{1\}}\kappa_A$, we have just continuity of its timelike component ${}^{\{1\}}\kappa_A u^A$, that has been defined as ${}^{\{1\}}\delta$ in (13.31). In particular, in the gauge we have suggested in Section 13.2.2, ${}^{\{1\}}\lambda \equiv {}^{\{1\}}\kappa_A n^A = 0$, the full axial vector ${}^{\{1\}}\kappa_A$ would be continuous. For simplicity and consistency with that section, this is the gauge we will use here. Another quantity that must be continuous is the following combination,

$$ {}^{\{1\}}\Pi' + \frac{16\pi}{r^2}\rho\, {}^{\{1\}}\beta, \tag{14.11} $$

which depends on the axial fluid perturbation ${}^{\{1\}}\beta$.

The second-order polar problem is more complicated because none of the internal perturbations matches easily with the natural variable describing vacuum perturbations, the Zerilli scalar. Decomposing into harmonics the objects $\overline{\Delta}^2[i^{\mu\nu}]$ and $\overline{\Delta}^2[e^{\mu\nu}]$, we find that the following second-order polar quantities must be continuous,

$$ A_1 = N + S \tag{14.12} $$

$$ A_2 = \chi + 2(\nu + W)N - 2\eta \tag{14.13} $$

$$ A_3 = \mathcal{K} - 2NW \tag{14.14} $$

$$(14.15)$$

$$A_4 = \psi + 2\dot{N} - 2UN \tag{14.16}$$

$$A_5 = \mathcal{K}' - 2\eta W + 2U\dot{N} + 2\left(4\pi\rho - 2U^2 + W^2 + \nu W - \frac{M+4r}{r^3}\right)N, \tag{14.17}$$

$$\begin{aligned} A_6 &= \chi' + 2\mu\psi - 2\eta' + 2(W+\nu)\eta - 2\ddot{N} + 2U\dot{N} \\ &- 2\left(4\pi p + 2\nu^2 + U^2 - \nu W + \frac{M-5r}{r^3}\right)N, \end{aligned} \tag{14.18}$$

where the quadratic source

$$S \equiv -\frac{1}{\sqrt{5\pi}}\,{}^{\{1\}}\delta\,{}^{\{1\}}\delta', \tag{14.19}$$

appears only in the first continuous combination A_1, and η is given algebraically in terms of first-order perturbations (13.52),

$$\eta = \frac{1}{\sqrt{5\pi}}\left\{8\pi(p+\rho)\,{}^{\{1\}}\beta^2 + \frac{1}{2}\left[(\nu - 2W)\,{}^{\{1\}}\delta + {}^{\{1\}}\delta'\right]^2 + \frac{{}^{\{1\}}\delta^2}{r^2}\right\}. \tag{14.20}$$

Note that some terms of the right-hand side of this expression are continuous, e. g. ${}^{\{1\}}\delta^2/r^2$. Hence, when introducing this expression in the definition of the continuous objects (14.12-14.18), those terms can be removed. But one has to be careful with the term η' that appears in (14.18), since prime derivatives of continuous objects do not have to be continuous. All perturbative objects that form part of the expressions for the continuous objects, except N, have been defined in Chapter 13. N is proportional to the gauge invariant (tied to the RW gauge) associated to the pressure perturbations,

$$-\frac{N}{\mathfrak{p}} \equiv \Delta^2[p] + 2\mathcal{L}_{\{1\}p}\Delta[p] + \left(\mathcal{L}_{\{2\}p} + \mathcal{L}^2_{\{1\}p}\right)p. \tag{14.21}$$

The pressure p must not be confused with the vector ${}^{\{n\}}p^\mu$, whose harmonic components are given in (9.52). This can be written in terms of the second-order gauge-invariant perturbations of the energy-density (13.11) and entropy (13.12),

$$N \equiv \Delta^2_{\rm GI}[p] = -\mathfrak{p}(c_s^2\omega + C\sigma). \tag{14.22}$$

The subindex GI stands to denote the perturbation expressed in terms of the gauge-invariant objects (tied to the RW gauge) that, again, has a form equivalent to the perturbation in the RW gauge ${}^{\{n\}}p^\mu = 0$. This last equation has no quadratic terms in first-order energy-density and entropy perturbations, because both of them are polar and we are assuming that there are only first-order axial perturbations. The norm of the normal vector is defined by (14.1),

$$\mathfrak{p} = \left(p_{,A}p^{,A}\right)^{-1/2} = -\frac{1}{p'}. \tag{14.23}$$

The second equality holds because the pressure vanishes on the surface at all times; therefore $\dot{p} = 0$. The minus sign comes from the fact that $p' < 0$. On the other hand, making use of the background Euler equation for the fluid (5.59), we obtain

$$p' = -\nu(p + \rho). \tag{14.24}$$

Combining the last three equations, we finally obtain N in terms of the fluid variables,

$$N = -\frac{c_s^2\omega + C\sigma}{\nu}. \tag{14.25}$$

The continuity conditions (14.12-14.18) coincide with the expressions given in [112] if we remove the source S and the component η, which are vanishing at first order.

14.2.1 Extraction

The natural exterior frame is given by the unitary radial vector r^A and its orthogonal, the unit vector, t^A, that were defined in (5.21). In the interior of the star, that frame as well as the frame defined by the fluid velocity (u^A, n^A) are well defined. The relation between them is given by a hyperbolic rotation,

$$\begin{aligned} r^A &= -f^{-1}\, r\, U u^A + f^{-1}\, r\, W n^A, & (14.26)\\ t^A &= -f^{-1}\, r\, W u^A + f^{-1}\, r\, U n^A, & (14.27) \end{aligned}$$

where again $f \equiv \sqrt{g^{AB} r_{|A} r_{|B}}$. Replacing this form of the radial vector r^A in the definition of the Zerilli function (12.14) and rewriting it in terms of the fluid variables, one obtains

$$\begin{aligned} \mathcal{Z} &= \frac{2r^4}{6M + (l+2)\,(l-1)\,r}\Big\{U\dot{\mathcal{K}}_{\rm out} + 2UW\psi_{\rm out} - W\mathcal{K}'_{\rm out} - 2U^2\eta_{\rm out} \\ &+ \left(U^2 + W^2\right)(\chi_{\rm out} + \mathcal{K}_{\rm out})\Big\} + r\mathcal{K}_{\rm out}, \qquad (14.28) \end{aligned}$$

where primes and dots are always expressed in the fluid frame, and the expression is evaluated just outside the surface. Making use of the continuous quantities (14.12-14.18), we arrive at the formula that gives the outside Zerilli function in terms of the fluid inner variables,

$$\begin{aligned} \mathcal{Z} &= \frac{2r^4}{6M + (l+2)\,(l-1)\,r}\big\{U\dot{\mathcal{K}}_{\rm in} + 2UW\psi_{\rm in} - W\mathcal{K}'_{\rm in} - 2U^2\eta_{\rm in} \qquad (14.29)\\ &+ \left(U^2 + W^2\right)(\chi_{\rm in} + \mathcal{K}_{\rm in}) - 8\pi r^2\rho W N_{\rm in} - \frac{W}{r^2}(-8 + l + l^2)(S_{\rm out} - S_{\rm in})\Big\} + r\mathcal{K}_{\rm in}. \end{aligned}$$

At first order, because of the vanishing of the source S, the last term in curly brackets would dissappear.

14.2.2 Injection

Inside the star, the polar variables χ and $\mathcal{K}$ satisfy a wave equation. Therefore, boundary conditions on the surface of the star must be given for these variables.

Outside the star we know how the polar metric perturbations $\{H_{rr}, H_{rt}, H_{tt}, \mathcal{K}\}$ can be recovered once the Zerilli function has been determined. So, again, we only have to make a hyperbolic rotation and then use the continuous quantities (14.12-14.18) in order to transfer information from the exterior to the interior of the star. Making use of the continuous quantities (14.13) and (14.14), we get

$$\chi_{\rm in} = \chi_{\rm out} - 2(\nu + W)(S_{\rm out} - S_{\rm in}) - 2(\eta_{\rm out} - \eta_{\rm in}), \tag{14.30}$$

$$\mathcal{K}_{\rm in} = \mathcal{K}_{\rm out} + 2W(S_{\rm out} - S_{\rm in}), \tag{14.31}$$

where, making the hyperbolic rotation, we find that

$$\chi_{\rm out} = \frac{1}{(r-2M)^2}\left\{r^4U^2H_{tt} + 2r^3(r-2M)UWH_{rt} + r^2(r-2M)^2W^2H_{rr}\right\} - \mathcal{K}_{\rm out}. \tag{14.32}$$

So, we have succeeded in writing the interior variables $\{\mathcal{K}, \chi\}$ in terms of the exterior variables $\{\mathcal{K}, H_{tt}, H_{tr}, H_{rr}\}$, which, in turn, can be obtained in terms of the Zerilli function.

Part VI

Algebraic implementation

Chapter 15

Mathematica packages

The calculations involved in high-order perturbation theory can hardly be done by hand, as can be realized from the length of the sources for the gauge invariants (11.19-11.26) or for the Einstein equations (11.33-11.40). That is why computer algebra turns out to be necessary. Most calculations in this thesis have been performed using the free system *xAct* [111] for efficient tensor computer algebra in *Mathematica*. A number of packages and notebooks have been created for that purpose, two of which, called *xPert* and *Harmonics*, are also freely distributed from `http://metric.iem.csic.es/Martin-Garcia/xAct/xPert/`, under the GNU Public License. Other codes, specialized for vacuum or fluid perturbation theory, will be kept private while we are still applying them to particular scenarios, though their results will be provided to other authors upon request.

This chapter follows [121] and describes in detail *xPert*, a package for high-order metric perturbation theory around arbitrary backgrounds. We will also briefly describe *Harmonics*, specialized in the computations and products of tensor spherical harmonics.

The notation that will be used is that of *Mathematica* and *xAct*. For instance, a tensor ${T_{ab}}^c$ will be written as `T[-a, -b, c]` and the partial derivative operator ∂_a is represented as `PD[-a]`, such that the divergence $\partial_a {T_{bc}}^a$ is `PD[-a][ T[-b,-c,a] ]`.

15.1 *xPert*

xPert has been specifically designed to manipulate the expressions appearing in problems of high-order perturbation theory in General Relativity around arbitrary backgrounds. Essentially it implements the perturbative formulas presented in Part I of this thesis.

We have used *xPert* in the investigation developed here, but it has also been employed by other researchers in several projects. In a cosmological setting, high-order perturbations of non-linear radiation transfer have been studied using *xPert*, both in kinetic theory [115] and in a fluid approximation [116], also including the effects of polarization [117] in the Boltzmann equation. *xPert* has also been essential in constructing the equations for first-order perturbations of scalar field inflation for anisotropic spacetimes [118], or in the recent study of the interaction of cosmological gravitational waves and magnetic fields [119]. *xPert* is also now being used by several authors to perform complicated variational derivatives of general diffeomorphism-invariant Lagrangians.

15.1.1 The code

This subsection describes the main commands and features of *xPert*, simultaneously constructing a very simple example session. (The *In[*]:=* and *Out[*]:=* prompts represent respectively input and output lines in *Mathematica*. Code lines without prompt indicate internal definitions in the package.) Next subsection will provide timings with more complicated examples. Let us start by loading *xTensor*, the *xAct* package for abstract tensor calculus,

In[1] := `«xAct‘xTensor‘`

(Version and copyright messages)

We first define our background structure: a four-dimensional manifold `M` whose tangent vector space will have (abstract) indices `{a,b,c,d,e}`,

In[2] := `DefManifold[ M, 4, {a,b,c,d,e} ]`

Then we define a metric tensor field `g` with negative determinant and associated Levi-Civita covariant derivative `CD`,

In[3] := `DefMetric[ -1, g[-a,-b], CD, {";","∇"} ]`

(Info messages on construction of associated tensors)

We have provided the symbols `{";","∇"}` to format the derivative in postfix or prefix output notation, respectively. `DefMetric` automatically defines all tensors normally associated to the metric or its connection, like `ChristoffelCD[a,-b,-c]`, `RiemannCD[-a,-b,-c,-d]`, `EinsteinCD[-a,-b]`, `Detg[]`, and so on, with obvious meanings. We can define other tensors with the syntax

In[4] := `DefTensor[MaxwellF[a,b],M,Antisymmetric[{a,b}],PrintAs->"F"]`

The arrow `->` is the *Mathematica* representation for an optional named argument.

Now we load *xPert* (this would also load automatically *xTensor* if it was not already in memory):

In[5] := `<<xAct`xPert``

```
------------------------------------------
```

Package xAct‘xPert‘ version 1.0.0, {2008, 6, 30}

Copyright (C) 2005–2008 David Brizuela, Jose M. Martin-Garcia

and Guillermo A. Mena Marugan, under GPL

```
------------------------------------------
```

This adds several new commands and reserved words to the system, of which we will here describe the four most important ones, namely `DefMetricPerturbation`, `Perturbation`, `ExpandPerturbation` and `GaugeChange`.

A perturbative structure having metric `g` as background and the tensor `h` as its perturbation is defined using

In[6] := `DefMetricPerturbation[ g, h, ε ]`

which also identifies ε as the perturbative parameter of the expansions. From now on, the n-th perturbation of the metric `g[-a,-b]` will be denoted as `h[LI[n],-a,-b]`, where `LI` is the *xTensor* head to denote the so-called ‘label indices’, that is, indices with no vector space association. Labels can be considered as general non-geometric purpose indices.

The perturbative operator Δ is represented by the head `Perturbation`. It has two arguments: the background expression being perturbed and the perturbative order (with default value 1):

In[7] := `Perturbation[ MaxwellF[a,b], 3 ]`

Out[7] := $\Delta^3[F^{ab}]$

Note that the tensor is represented with its symbol F and that the perturbation order is an exponent of Δ. Following normal *Mathematica*, the output is formatted for the sake of clarity, but the internal notation is still the same. `Perturbation` acts mainly as a wrapper for tensor expressions, but has been instructed to evaluate them under certain conditions. First, it automatically combines perturbative orders of composed heads (symbols with an underscore are named patterns in *Mathematica*):

```
Perturbation[ expr_, 0 ] := expr

Perturbation[Perturbation[expr_,n_ ],m_ ] := Perturbation[expr,n+m]
```

Being a derivative, `Perturbation` is linear and gives zero on the `delta` identity tensor and constants:

```
Perturbation[ x_+y_,n_ ] := Perturbation[x,n] + Perturbation[y,n]

Perturbation[ delta[a_, b_ ], n_ ] := 0

Perturbation[ expr_?ConstantQ, n_ ] := 0
```

The question mark in a pattern is the *Mathematica* notation to restrict the pattern to those expressions obeying a condition. For instance, the last definition will only be used if `expr` is a constant quantity. The Leibnitz rule is also automatic, and has been implemented following equation (3.6) for any number of factors and any perturbative order, using fast algorithms to compute partitions implemented in *xPert*. `Perturbation` commutes with partial derivatives of general expressions and with any covariant derivative of a scalar expression:

```
Perturbation[ PD[-a_][ expr_ ], n_ ] := PD[-a][Perturbation[expr,n]]

Perturbation[ CD_?CovDQ[-a_][ expr_?ScalarQ ], n_ ] :=

    CD[-a][ Perturbation[ expr, n ] ]
```

The index of the derivatives is required to be always covariant, to avoid a metric mismatch, and that is implemented through a pattern index `-a_` . Finally, `Perturbation` does not change the density weight of the perturbed expression,

```
WeightOf[ Perturbation[ expr_, n_ ] ] := WeightOf[ expr ]
```

The `DefMetricPerturbation` in *In[6]* defines special rules for the metric `g` and its perturbations `h` with covariant indices:

```
Perturbation[ g[-a_,-b_], n_ ] := h[LI[n],-a,-b]

Perturbation[ h[LI[n_],-a_,-b_], m_ ] := h[LI[n+m],-a,-b]
```

With the setup and internal definitions so far we can now perform computations like this second order perturbation

In[8] := `Perturbation[g[-a,-b]RicciCD[c,d]+RiemannCD[-a,-b,c,d],2]`

Out[8] := $2\, h^1{}_{ab}\, \Delta[R_{cd}] + g_{ab}\, \Delta^2[R_{cd}] + \Delta^2[R_{ab}{}^{cd}] + h^2{}_{ab}\, R_{cd}$

(Note that in the output notation, as opossed to the notation used in the rest of the thesis, the perturbative order appears after the symbol `h`.) Actually, we could now proceed to perform any computation in metric perturbation theory by decomposing the curvature tensors in partial derivatives of the metric and using the code already given recursively. Only the definition $\Delta[g^{ab}] = -\,{}^{\{1\}}h^{ab}$ would be missing. However, that would be highly inefficient already for moderate perturbative order n. Instead, we will use the expansion formulas of Chapter 3, which allow the nonrecursive construction of perturbations at any order n.

Formulas (3.8, 3.10) for derivative expansions and formulas (3.17, 3.22, 3.26, 3.28, 3.31, 3.33, 3.36) for the relevant curvature tensors have all been encoded in a single command called `ExpandPerturbation`, the most powerful part of *xPert*. `ExpandPerturbation` takes any expression and replaces the arbitrary-order perturbations of known background objects by their expansions in terms of metric perturbations, but only if those objects have their indices in the appropriate positions. For example the perturbation of the Einstein tensor has only been stored for covariant indices. In all other cases there is an internal call to the *xTensor* function `SeparateMetric`, which introduces metric factors to bring all indices to their *natural* positions, which are those given at definition time. To show how this works, we perform an explicit metric separation by hand (symbol `%` represents the previous output):

In[9] := `Perturbation[ EinsteinCD[a,b] ]`

Out[9] := $\Delta[G^{ab}]$

In[10] := `SeparateMetric[ ][ % ]`

$$Out[10] := \; G_{cd}\, g^{bd}\, \Delta[g^{ac}] + g^{ac}\left(g^{bd}\,\Delta[G_{cd}] + G_{cd}\,\Delta[g^{bd}]\right)$$

Now `ExpandPerturbation` can expand the perturbation of the Einstein tensor with covariant indices, and the perturbation of the inverse metric.

In[11] := `ContractMetric[ ExpandPerturbation[ % ] ]`

$$Out[11] := \; -G_c{}^b h^{1ac} - G^a{}_c h^{1bc} + \tfrac{1}{2} g^{ab} h^{1cd} R_{cd} - \tfrac{1}{2} h^{1ab} R - \tfrac{1}{2} h^{1c}{}_c{}^{;b;a}$$
$$-\tfrac{1}{2} h^{1cb}{}_{;c}{}^{;a} + \tfrac{1}{2} h^{1b}{}_c{}^{;c;a} + \tfrac{1}{2} h^{1cb;a}{}_{;c} + \tfrac{1}{2} h^{1ca;b}{}_{;c} - \tfrac{1}{2} h^{1ba;c}{}_{;c}$$
$$+\tfrac{1}{4} g^{ab} h^{1d}{}_d{}^{;c}{}_{;c} + \tfrac{1}{4} g^{ab} h^{1dc}{}_{;d;c} - \tfrac{1}{4} g^{ab} h^{1c}{}_d{}^{;d}{}_{;c} - \tfrac{1}{2} g^{ab} h^{1dc}{}_{;c;d} + \tfrac{1}{4} g^{ab} h^{1c}{}_c{}^{;d}{}_{;d}$$

The *xTensor* command `ContractMetric` has been used to absorb all possible metric factors. Finally, `ToCanonical` moves indices around, bringing equal terms together,

In[12] := `ToCanonical[ % ]`

$$Out[12] := \; -G^{bc} h^{1a}{}_c - G^{ac} h^{1b}{}_c + \tfrac{1}{2} g^{ab} h^{1cd} R_{cd} - \tfrac{1}{2} h^{1ab} R - \tfrac{1}{2} h^{1c}{}_c{}^{;a;b}$$
$$+\tfrac{1}{2} h^{1bc;a}{}_{;c} + \tfrac{1}{2} h^{1ac;b}{}_{;c} - \tfrac{1}{2} h^{1ab;c}{}_{;c} - \tfrac{1}{2} g^{ab} h^{1cd}{}_{;c;d} + \tfrac{1}{2} g^{ab} h^{1c}{}_c{}^{;d}{}_{;d}$$

Figure 15.1 shows the output of the second perturbation of the covariant Einstein tensor, computed with the same combination of commands.

Another useful command in *xPert* is `GaugeChange`, that implements the general gauge transformation at any order (4.9). In order to use it, first we must define the family of generator vector fields on the manifold `M` transforming from the current to a new gauge,

In[13] := `DefTensor[ ξ[LI[n],a], M ]`

The third-order perturbation of F^{ab} can be changed to the new gauge using

In[14] := `GaugeChange[ Perturbation[ MaxwellF[a,b], 3 ], ξ ]`

$$Out[14] := \; \Delta^3[F^{ab}] + 3\,\mathcal{L}_{\xi^1}\Delta^2[F^{ab}] + 3\,\mathcal{L}_{\xi^1}\mathcal{L}_{\xi^1}\Delta[F^{ab}] + \mathcal{L}_{\xi^1}\mathcal{L}_{\xi^1}\mathcal{L}_{\xi^1}F^{ab}$$
$$+3\,\mathcal{L}_{\xi^1}\mathcal{L}_{\xi^2}F^{ab} + 3\,\mathcal{L}_{\xi^2}\Delta[F^{ab}] + \mathcal{L}_{\xi^3}F^{ab}$$

We finish this subsection by turning back to the problem of perturbation theory using the background field method, in which all but the first metric perturbations vanish, as stated in Section 3.2. This can be easily implemented setting

In[15] := `h[LI[n_],a_,b_]:=0 /; n > 1`

```
AbsoluteTiming[
 Perturbation[EinsteinCD[-a, -b], 2] // ExpandPerturbation // ContractMetric //
  ToCanonical]
```

$$\{0.783872\,\text{Second},\ h^{1}{}_{ab}\, h^{1\,cd} R_{cd} + \frac{1}{2} g_{ab}\, h^{2\,cd} R_{cd} - g_{ab}\, h^{1}{}_{c}{}^{e}\, h^{1\,cd} R_{de} - \frac{1}{2} h^{2}{}_{ab} R + h^{1\,cd}\, h^{1}{}_{cd;b;a} +$$

$$\frac{1}{2} h^{1\,cd}{}_{;a}\, h^{1}{}_{cd;b} - \frac{h^{2\,c}{}_{c;a;b}}{2} + \frac{1}{2} h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1\,d}{}_{d;c} + \frac{1}{2} h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1\,d}{}_{d;c} + \frac{h^{2}{}_{b}{}^{c}{}_{;a;c}}{2} + \frac{h^{2}{}_{a}{}^{c}{}_{;b;c}}{2} - \frac{h^{2}{}_{ab}{}^{;c}{}_{;c}}{2} -$$

$$\frac{1}{2} h^{1\,d}{}_{d;c}\, h^{1}{}_{ab}{}^{;c} - h^{1}{}_{b}{}^{c}{}_{;a}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1}{}_{a}{}^{c}{}_{;b}\, h^{1}{}_{c}{}^{d}{}_{;d} + h^{1}{}_{ab}{}^{;c}\, h^{1}{}_{c}{}^{d}{}_{;d} - h^{1\,cd}\, h^{1}{}_{bc;a;d} - h^{1\,cd}\, h^{1}{}_{ac;b;d} +$$

$$h^{1\,cd}\, h^{1}{}_{ab;c;d} - h^{1}{}_{ab}\, h^{1\,cd}{}_{;c;d} - g_{ab}\, h^{1\,cd}\, h^{1\,e}{}_{e;c;d} - \frac{1}{2} g_{ab}\, h^{2\,cd}{}_{;c;d} + h^{1}{}_{ab}\, h^{1\,c}{}_{c}{}^{;d}{}_{;d} + \frac{1}{2} g_{ab}\, h^{2\,c}{}_{c}{}^{;d}{}_{;d} +$$

$$g_{ab}\, h^{1\,cd}\, h^{1}{}_{c}{}^{e}{}_{;e;d} - h^{1}{}_{bd;c}\, h^{1}{}_{a}{}^{c;d} + h^{1}{}_{bc;d}\, h^{1}{}_{a}{}^{c;d} + \frac{1}{4} g_{ab}\, h^{1\,e}{}_{e;d}\, h^{1\,c}{}_{c}{}^{;d} + g_{ab}\, h^{1\,cd}{}_{;c}\, h^{1}{}_{d}{}^{e}{}_{;e} -$$

$$g_{ab}\, h^{1\,c}{}_{c}{}^{;d}\, h^{1}{}_{d}{}^{e}{}_{;e} + g_{ab}\, h^{1\,cd}\, h^{1}{}_{c}{}^{e}{}_{;d;e} - g_{ab}\, h^{1\,cd}\, h^{1}{}_{cd}{}^{;e}{}_{;e} + \frac{1}{2} g_{ab}\, h^{1}{}_{ce;d}\, h^{1\,cd;e} - \frac{3}{4} g_{ab}\, h^{1}{}_{cd;e}\, h^{1\,cd;e}\}$$

Figure 15.1: The second-order perturbation of the Einstein tensor is constructed and canonicalized in less than one second. The first (blue) label of each `h` tensor denotes the perturbative order. In *Mathematica* the action of a command `command` on an expression `expr` is denoted by either `expr//command` or `command[expr]`.

15.1.2 Timings

We now focus on the dependence of the timings of standard computations on the perturbative order and the number of objects being perturbed. The intrinsic combinatorial nature of the problem will always imply exponential dependence, but we will see that the timings in *xPert* are short enough to handle all useful cases. These examples have been performed using a Linux box with a 3 GHz Pentium IV processor and 2 Gb of RAM.

In perturbation theory the overall level of complexity is mainly determined by the perturbative order n. It affects the computation in two different ways: on the one hand the expressions to manipulate are sums with a number of terms which grows exponentially with n; on the other hand each term is a product of objects and the number of factors also grows (typically linearly) with n. Canonicalizing a sum of terms is obviously a linear process because each term can be dealt with independently, but canonicalization of a product of objects is naturally factorial in the number of indices and this could prevent any practical computation. The algorithms in *xTensor* are fast enough to render the problem effectively polynomial in the number of indices, allowing us to deal with expressions of a few dozen indices in hundredths of a second. Figure 15.2 shows the number of terms and the timing of canonicalization of the perturbation of the Riemann tensor at different perturbative orders. The 10-th order perturbation contains 44544 terms and is canonicalized in slightly less than 20 minutes, whereas third-order expressions can be manipulated in 1 second. We see clearly the exponential growth of both curves, but with manageable timings.

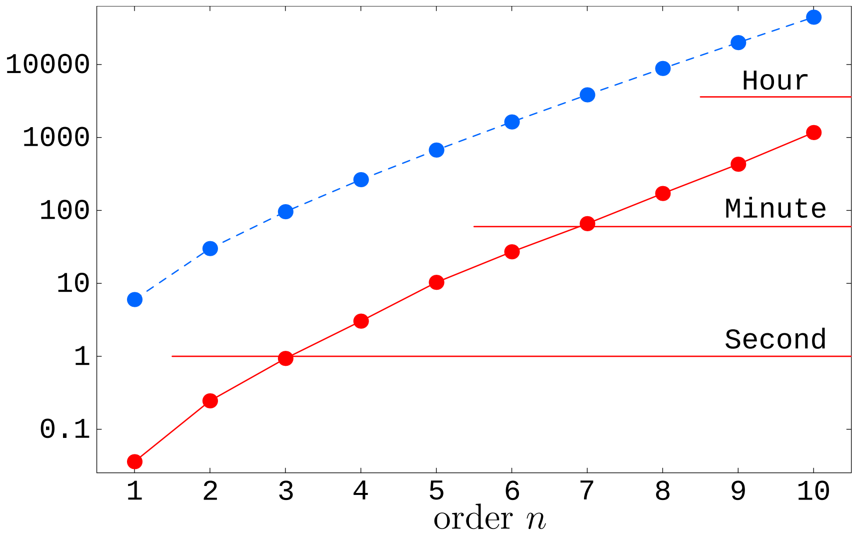

Figure 15.2: Canonicalization timings (in seconds) for the perturbation of the Riemann tensor at perturbative orders $n = 1...10$ (lower, red line). Also shown number of terms in the expression (upper, blue dashed line). Both lines are clear exponentials in n, with the timings growing slightly faster because terms with larger n are harder to canonicalize owing to their larger average number of indices.

Other possible sources of complexity in perturbative computations are the expansion of perturbations of a product of tensors and the expansion of perturbations of a function of a number of scalar arguments. Our implementations of the nth-order Leibnitz rule and Faà di Bruno formula are fast enough to neglect their timings in comparison with those of canonicalization. Figures 15.3 and 15.4 display example timings for those problems respectively. The Leibnitz rule is simpler than the Faà di Bruno formula and produces faster results, also taking less memory.

Overall, we see that we are limited in size by RAM memory, which allows us to work with up to roughly 10^5 terms with a few Gbytes, corresponding to $n = 10$ approximately. Time limitations come mainly from the canonicalization process (other expansions are faster). Within seconds we can manipulate all equations up to orders $n = 4$ or $n = 5$. The $n = 10$ equations require canonicalization times of the order of 1 hour. This gives an idea of the power and efficiency of *xPert*, and what can be achieved with it.

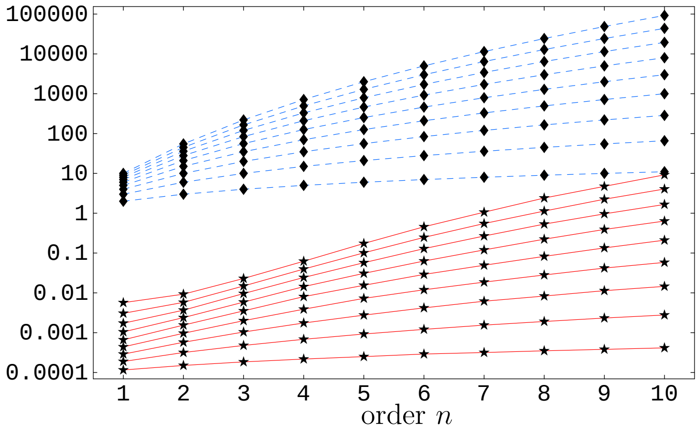

Figure 15.3: Timings of expansion (in seconds; red lines, stars) and number of terms (blue dashed lines, diamonds) of the perturbation of the product of m factors, for different perturbative orders $n = 1...10$ [the generalized Leibnitz rule (3.6)]. Different lines correspond to increasing values of m, from $m = 2$ to $m = 10$ starting from below. All practical cases stay below one second.

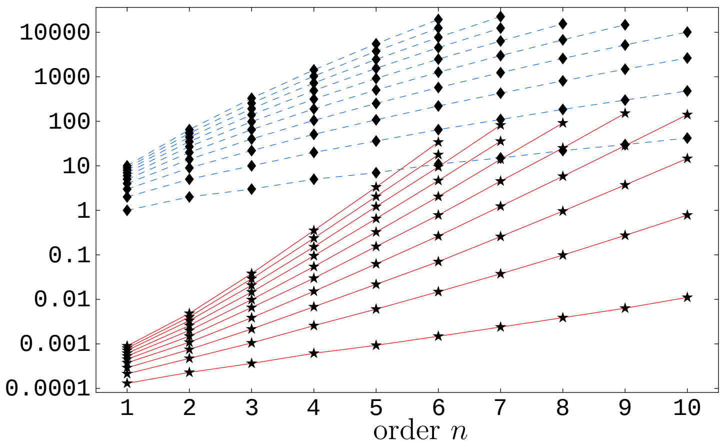

Figure 15.4: Timings of expansion (in seconds; red lines, stars) and number of terms (blue dashed lines, diamonds) of the perturbation of a scalar function of m scalar arguments, for different perturbative orders $n = 1...10$ [the generalization to many arguments of Faà di Bruno formula]. Different lines correspond to increasing values of m, from $m = 1$ to $m = 10$ starting from below. We include only those cases which can be handled with 2Gb of RAM memory, corresponding to a few tens of thousands of terms. For example, for $n = 10$ we can only handle up to $m = 4$.

15.2 *Harmonics*

The package *Harmonics* implements all geometric structures on S^2 defined in Part II and Appendices A and B of this work. The commands `PureSpin[j, ±1, m][a, b, ...]` and `PureOrbital[j, l, m][a, b, ...]` provide any pure-spin or pure-orbital harmonics (analyzed in Sections 6.3 and 6.4 respectively) , both in abstract form (in terms of the bases m or t) or giving their components in any coordinate or non-coordinate basis. The generalized RWZ harmonics `Z[LI[l], LI[m],a, b, ...]` and `X[LI[l], LI[m],a, b, ...]` have also been defined, incorporating all their symmetries and properties (6.59–6.65). The product formula (6.83) has also been included defining a rule named `productRule`.

As an example of the use of this package, we present in figure 15.5 the expansion of the product between two tensor spherical harmonics into a linear combination of harmonics, in spherical coordinates (θ, ϕ).

```
X[LI[7], LI[2], {-2, spherical}, {3, spherical}]
 Z[LI[4], LI[-1], {-2, spherical}, {-2, spherical}, {-3, spherical}]
```

$$X^{72}{}_{2}{}^{3}\ Z^{4-1}{}_{223}$$

```
% /. productRule // Simplify
```

0.736047 Second

$$\frac{1}{4}\mathrm{Csc}[\theta]\Big(i\,E^{-2,7,2}_{3,4,-1;4}\,X^{41}{}_{3} + i\,E^{-2,7,2}_{3,4,-1;6}\,X^{61}{}_{3} + i\,E^{-2,7,2}_{3,4,-1;8}\,X^{81}{}_{3} + i\,E^{-2,7,2}_{3,4,-1;10}\,X^{101}{}_{3} + 4\,E^{2,7,2}_{3,4,-1;5}\,\mathrm{Sin}[\theta]\,X^{51}{}_{2232}{}^{3} + 4\,E^{2,7,2}_{3,4,-1;7}\,\mathrm{Sin}[\theta]\,X^{71}{}_{2232}{}^{3} + 4\,E^{2,7,2}_{3,4,-1;9}\,\mathrm{Sin}[\theta]\,X^{91}{}_{2232}{}^{3} + 4\,E^{2,7,2}_{3,4,-1;11}\,\mathrm{Sin}[\theta]\,X^{111}{}_{2232}{}^{3} - E^{-2,7,2}_{3,4,-1;3}\,Z^{31}{}_{3} - E^{-2,7,2}_{3,4,-1;5}\,Z^{51}{}_{3} - E^{-2,7,2}_{3,4,-1;7}\,Z^{71}{}_{3} - E^{-2,7,2}_{3,4,-1;9}\,Z^{91}{}_{3} - E^{-2,7,2}_{3,4,-1;11}\,Z^{111}{}_{3} + 4\,i\,E^{2,7,2}_{3,4,-1;6}\,\mathrm{Sin}[\theta]\,Z^{61}{}_{2232}{}^{3} + 4\,i\,E^{2,7,2}_{3,4,-1;8}\,\mathrm{Sin}[\theta]\,Z^{81}{}_{2232}{}^{3} + 4\,i\,E^{2,7,2}_{3,4,-1;10}\,\mathrm{Sin}[\theta]\,Z^{101}{}_{2232}{}^{3}\Big)$$

```
% /. coeffERule // HarmonicComponent
```

0.64404 Second

$$\frac{1}{131072\,\pi}\left(8505\,i\,\sqrt{7}\,e^{i\phi}\,(-338 + 1351\,\mathrm{Cos}[2\theta] - 990\,\mathrm{Cos}[4\theta] + 1001\,\mathrm{Cos}[6\theta])\,\mathrm{Sin}[2\theta]^2\right)$$

Figure 15.5: The product between two tensor spherical harmonics is expanded as a linear combination of other tensor harmonics with E-coefficients via the command `productRule` that is encoded in the package *Harmonics*. After that, the E-coefficients are replaced by their numerical values and the component of the resultant harmonic in the spherical basis (θ, ϕ) is obtained.

The output of the tensorial harmonics is different from the input since, as it is usual in *Mathematica*, the output is formatted but the internal structure is maintained. In the input, the harmonic labels l and m are considered as label indices (`LI`), whereas the basis is named `spherical` and the components are denoted by $(\mathtt{2}, \mathtt{3}) \equiv (\theta, \phi)$. In the output, the harmonic labels are the first set of indices (in black), while the second set (in red) stands for the coordinate indices in the mentioned basis.

The command `coeffERule` replace the E-coefficients by their numerical values, and `HarmonicComponent` returns the explicit component of the expression in the given spherical basis. The time spent by the computer to make the computation between each input and output is also shown above.

Conclusions

The main part of the present work provides a systematic approach to high-order perturbation theory in General Relativity. It is based on the combination of an appropriate theoretical formalism for the description of the problem (implementing symmetry reductions, covariant notation, gauge invariance, and other nice features) and the intensive use of abstract computer algebra to manipulate the enormous expressions that unavoidably appear in this field. In addition, this theoretical formalism has been proven in situations of astrophysical relevance through its application to a background of a spherical, but still dynamical, perfect-fluid star.

As another parallel and complementary line of research, we have also considered the linear perturbative problem in a canonical framework. It offers an alternative point of view to analyze the gauge freedom in perturbation theory and a systematic way of constructing gauge invariant objects, whose evolution is given by *master* unconstrained equations.

The main results presented in this thesis are the following:

- We have given a number of closed formulas which allow us to compute at any order the perturbation of all relevant curvature tensors in General Relativity. These formulas can be used in very different areas of gravitational physics, including theories that depart from standard General Relativity (like in the case of models with extra dimensions, curvature corrections, or in braneworld scenarios).

- These formulas are combinatorial, what makes them very effective from the point of view of an algebraic implementation. So, they have been implemented in the *Mathematica* package *xPert*, which enables us to employ them up to very high orders. Apart from its obvious application in field theory, *xPert* can also be easily adapted to compute variational derivatives with respect to a metric field, because the computation is equivalent to performing first-order perturbations. This is of great help in deriving the evolution equations from the most general diffeomorphism-invariant Lagrangian, which includes as a special case the $f(R)$ theories, under intensive study currently.

- We have analyzed the general issue of gauge invariance in high-order perturbation theory. The two classical approaches by Sachs and Moncrief have been related and generalized. We have shown how invariants can be constructed, for any background spacetime and up to any order in perturbation theory, provided that a rigid gauge is chosen. For highly symmetric backgrounds, the invariants can be explicitly expressed.

- We have generalized to higher orders the well known Gerlach and Sengupta formalism for nonspherical first-order perturbations of a spherical spacetime. This formalism is considered to be optimal for the perturbative study of a number of astrophysical scenarios of interest. The generalization put forward here makes it even more powerful, leading to more precise results and allowing to describe interactions between different modes, and other nonlinear effects.

- With this purpose, we have constructed a generalization of the Regge-Wheeler-Zerilli harmonics to any number of indices, closely related to the Wigner rotation matrices (the so-called spin-weighted harmonics in the General Relativity community).

- We have obtained a general formula to expand the product between any pair of generalized tensorial harmonics into a linear combination of harmonics. This formula is essential in the generalization of the GS formalism to higher orders.

- We have written the *Mathematica* package *Harmonics*, able to work with the different kinds of tensorial harmonics presented in this thesis. It stores all the symmetry properties of the harmonics, as well as the product formula between any two of them.

- We have proven that the Regge-Wheeler gauge can be imposed to any order in perturbation theory and given an iterative procedure to construct the gauge invariant quantities tied to this gauge.

- Making use of this procedure, we have explicitly calculated and simplified the second-order gauge invariants for spherical backgrounds. These invariants have a similar form to that of the GS first-order invariants, corrected with terms that are quadratic in first-order perturbations.

- We have explicitly computed all equations of the generalized GS formalism at second order, including those of energy-momentum conservation, and simplified them to a manageable form. These equations are completely general except for the restriction to a spherical background: they can be used with any background, dynamical or not, they can be coupled to any matter model, and they have been given in covariant form, so that any coordinate system can be used on the background manifold.

- The second-order equations are essentially the same as the first-order equations, but they also include complicated quadratic sources. We have disentangled the structure of these sources and shown that, in previous investigations considering just a single first-order perturbative mode, many of such sources were not excited.

- We have applied the second-order GS formalism to vacuum. A formula to obtain the power that the gravitational waves carry to null infinity at any perturbative order has been given in terms of the trace-free part of the projected perturbation on the sphere. There are two important features of this formula that we would like to mention. On the one hand, owing to its integrated character, only the coupling between harmonic coefficients with the same harmonic labels contribute to the total power. On the other hand, one can see that the emitted power at order $\mathcal{O}(\varepsilon^n)$ depends on all the lower $k < n$ orders.

- The simplest generalizations of the Zerilli and RW variables at second order do not decay with radius when approaching null infinity. This unphysical behavior has been regularized by adding appropriate first-order quadratic sources to the definitions of the master variables. The evolution equations for the new master scalars have the same differential part as the non-regularized ones; only their corresponding sources change. These sources have been explicitly obtained in full generality.

- The trace-free part of the projected perturbation on the sphere has been reconstructed in terms of the regularized master scalars at null infinity. This permits us to solve the problem of the radiated power up to order $\mathcal{O}(\varepsilon^3)$ just by solving the two (first- and second-order) master equations.

- We have also applied the second-order GS formalism to a perfect fluid. There are first-order quadratic sources that appear when writing down the perturbations of the energy-momentum tensor in terms of the fluid variables. We have obtained them at second order.

- The evolution equations have been converted into scalar equations by projecting them into the frame provided by the background fluid four-velocity u^μ. They have been simplified for different harmonic labels. In particular, for $l > 2$ the second-order axial problem reduces to a wave equation for the perturbations of the metric and a transport equation for the matter perturbations. Whereas in the polar case there are two wave equations (one for the gravitational and another for the sound waves) and a transport equation. The rest of the perturbations are recovered in terms of the variables that are obtained by solving the mentioned equations.

- We have provided the matching or junction conditions at any perturbative order through a time-like surface defined by the zero level surface of a scalar function $\mathcal{P}$. Summarizing, the junction conditions are given by the continuity of the gauge-invariant (tied to the surface gauge, which is defined by the requirement $\Delta^n[\mathcal{P}] = 0$) perturbations of the contravariant induced metric and extrinsic curvature. As far as we know, this is the first time that arbitrary-order perturbative matching conditions are analyzed.

- In order to analyze the second-order perturbations of a spherical star, we have decomposed the junction conditions into spherical harmonics for any harmonic label. For simplicity and explicitness, we have restricted the presentation to a particular example of a spherical background with a first-order $l = 1$ axial mode. Second order perturbations represent gravitational radiation generated by a slowly rotating star by the mere self-coupling of the rotation on a dynamical background. For this case, we have also solved the problem of the injection and extraction of information through the matching surface.

- On a different matter, we have studied the linear perturbations of a spherical spacetime in a canonical setting. The background matter content is taken to be a scalar field, which makes the spacetime non static. We have found gauge-invariant objects that contain all physical information of the system. In the axial sector we have recovered the GS master scalar, whereas in the polar sector we have found a master variable with a complicated equation of motion, still under analysis. This study paves the way to obtain systematically master variables for different background spacetimes.

Conclusiones

El trabajo presentado proporciona un formalismo sistemático para la teoría de perturbaciones a altos órdenes en Relatividad General. Está basado en la combinación de una buena elección del formalismo teórico empleado para la descripción del problema (implementando reducción de simetrías, notación covariante y otras buenas características) y el uso intensivo de álgebra computacional abstracta para manipular las enormes expresiones que irremediablemente aparecen en este campo. Además, este formalismo teórico se ha utilizado en situaciones de relevancia astrofísica mediante su aplicación a un fondo correspondiente a una estrella de fluido perfecto esférica, pero aún dinámica.

Como otra línea de investigación paralela y complementaria a la anterior, se ha considerado el problema perturbativo lineal en un marco canónico. La libertad *gauge* perturbativa aparece de forma más explícita en este formalismo. Además, este tratamiento proporciona una manera sistemática de construir objetos invariantes *gauge*, cuya evolución está dada por una ecuación *master* libre (sin ligaduras).

Los resultados principales que se han alcanzado con este trabajo son los siguientes:

- Se han dado fórmulas cerradas que permiten calcular a cualquier orden la perturbación de todos los tensores de curvatura relevantes en Relatividad General. Estas fórmulas pueden utilizarse en diversas áreas de la física gravitacional, incluyendo teorías que difieren de la Relatividad General estándar (como pueden ser los modelos con dimensiones extra, con correciones de curvatura o bien en escenarios de mundos brana [*braneworlds*]).

- Todas estas fórmulas son combinatorias, lo que las convierte en altamente efectivas desde el punto de vista de la implementación algebraica. Se han implementado en el módulo *xPert* para *Mathematica*, lo que permite utilizarlas a órdenes muy elevados. Aparte de su aplicación obvia en teorías de campo, *xPert* puede ser adaptado fácilmente para calcular derivadas variacionales con respecto de una métrica, ya que el cálculo es equivalente a realizar perturbaciones a primer orden. Esto es de gran ayuda al derivar las ecuaciones de evolución para el lagrangiano más general invariante bajo difeomorfismos, lo que incluye como una caso especial las teorías $f(R)$, bajo amplio estudio hoy en día.

- Se ha aclarado la confusión existente en la literatura en lo que respecta a los dos puntos de vista diferentes sobre las cantidades invariantes *gauge*. Hemos comprobado que los invariantes *gauge* pueden construirse, para cualquier espaciotiempo de fondo y hasta cualquier orden en teoría de perturbaciones, dado que se pueda hallar un *gauge* rígido. De esta manera, hemos resuelto la cuestión sobre la existencia de invariantes *gauge* a cualquier orden superior.

- Hemos generalizado a órdenes superiores el conocido formalismo de GS para primeras perturbaciones no esféricas de un espaciotiempo esférico. Este formalismo se considera óptimo para el estudio perturbativo de varios escenarios astrofísicos de interés. La generalización realizada aquí lo hace aún más poderoso, llevando a resultados más precisos y permitiendo describir las interacciones entre diferentes modos.

- Con este objetivo, hemos construido la generalización de los harmónicos de RWZ y de los *pure-spin* a cualquier número de índices. Dicha generalización resulta estar muy relacionada con las matrices de rotación de Wigner (que son proporcionales a los harmónicos llamados *spin-weighted* por la comunidad de Relatividad General).

- Hemos obtenido una fórmula general para expandir el producto entre cualquier par de harmónicos tensoriales generalizados como una combinación lineal de harmónicos. Esta fórmula es esencial para llevar a cabo la generalización del formalismo de GS a altos órdenes.

- Hemos escrito el módulo *Harmonics* para *Mathematica* capaz de trabajar con los diferentes tipos de harmónicos tensoriales que se han considerado en esta tesis. Contiene todas las propiedades de simetría de los harmónicos, así como la fórmula del producto entre cualquier par de ellos.

- Hemos probado que el *gauge* de RW se puede imponer a cualquier orden en teoría de perturbaciones y hemos dado un procedimiento iterativo para construir las cantidades invariantes *gauge* ancladas a este *gauge*.

- Haciendo uso de este procedimiento, hemos calculado y simplificado explícitamente los invariantes *gauge* de segundo orden para fondos esféricos. Estos invariantes han resultado tener una forma similar a los invariantes GS de primer orden pero corregidos con términos cuadráticos en primeras perturbaciones.

- Hemos calculado explícitamente todas las ecuaciones del formalismo de GS generalizado a segundo orden, incluyendo las de conservación de energía-momento, y las hemos simplificado hasta una forma manejable. Estas ecuaciones son completamente generales excepto por la restricción a un fondo esférico: son válidas para cualquier fondo, sea dinámico o no, se pueden acoplar a cualquier modelo de materia y están dadas de manera covariante, lo que permite elegir cualquier sistema de coordenadas en la variedad de fondo.

- Las ecuaciones de segundo orden son esencialmente las mismas que a primer orden, pero también contienen complejas fuentes cuadráticas. Hemos desenmarañado la estructura de estas fuentes y mostrado que en las investigaciones que se habían realizado previamente, considerando un único modo perturbativo de primer orden, muchas de esas fuentes no estaban excitadas.

- Hemos aplicado el formalismo GS de segundo orden a vacío. Se ha dado una fórmula para obtener la potencia que las ondas gravitatorias transportan hasta el infinito nulo a cualquier orden perturbativo en función de la parte sin traza de la perturbación proyectada sobre la esfera. Hay dos características importantes de esta fórmula que nos gustaría mencionar. Por un lado, debido a su carácter integrado, sólo el acoplo entre los coeficientes con los mismos números harmónicos contribuye a la potencia total. Por otro lado, se puede ver que la potencia emitida a orden $\mathcal{O}(\varepsilon^n)$ depende de todos los órdenes inferiores $k < n$.

- La generalización más sencilla de las variables de Zerilli y de RW a segundo orden no decae con el radio al aproximarse al infinito nulo. No obstante, este comportamiento no es físico, y se ha corregido simplemente sumando a sus definiciones fuentes cuadráticas adecuadas de los términos perturbativos de primer orden. Las ecuaciones de evolución para los escalares *master* regularizados resultantes tienen la misma parte diferencial que los no regularizados; solamente cambian las fuentes correspondientes. Estas fuentes se han obtenido explícitamente.

- La parte sin traza de la perturbación proyectada sobre la esfera se ha reconstruido en términos de los escalares *master* regularizados en el infinito nulo. Esto permite solucionar el problema de la potencia radiada hasta orden $\mathcal{O}(\varepsilon^3)$ simplemente mediante la resolución de las dos ecuaciones *master* (de primer y segundo orden).

- También hemos aplicado el formalismo de GS de segundo orden a un fluido perfecto. Se han obtenido las fuentes cuadráticas de primer orden que aparecen al determinar las perturbaciones del tensor energía-momento en función de las variables del fluido.

- Las ecuaciones de evolución se han convertido en ecuaciones escalares proyectándolas en el *frame* dado por la cuadrivelocidad u^μ del fluido de fondo. Han sido simplificadas para los diferentes números harmónicos. En particular, para $l > 2$ el problema axial de segundo orden se reduce a una ecuación de onda para las perturbaciones métricas y, a una ecuación de transporte para las perturbaciones de la materia. Por su parte, en el caso polar hay dos ecuaciones de onda (una para la ondas gravitatorias y otra para las sónicas) además de una ecuación de transporte. El resto de las perturbaciones se recuperan en términos de las variables que se obtienen al resolver las ecuaciones mencionadas.

- Hemos dado las condiciones de *matching*, para cualquier orden perturbativo, a través de una superficie temporal definida como la superficie de nivel cero de una función escalar $\mathcal{P}$. En resumen, estas condiciones están dadas por la continuidad de las pertubaciones invariantes *gauge* (ancladas al *gauge* de superficie, que está definido por el requisito $\Delta^n[\mathcal{P}] = 0$) de la métrica inducida y la curvatura extrínseca contra-variantes. Por lo que sabemos, esta es la primera vez que se analizan las condiciones de *matching* para órdenes perturbativos arbitrarios.

- Para analizar las perturbaciones a segundo orden de una estrella esférica, hemos descompuesto las condiciones de *matching* en harmónicos esféricos para cualquier número harmónico. Por simplicidad hemos restringido la presentaciń al caso particular de una perturbaciíon axial $l = 1$. Las segundas perturbaciones representan las ondas gravitatorias generadas por una estrella en rotación lenta debido al autoacoplo de la rotación. Para este caso, también hemos resuelto el problema de la inyección y la extracción. Esencialmente, esto quiere decir que hemos fijado las variables de vacío externas en términos de las variables del fluido internas y viceversa.

- En otro orden de cosas, hemos estudiado las perturbaciones lineales de un espacio-tiempo esférico en un tratamiento canónico. Se ha supuesto que el contenido material de fondo es un campo escalar, que hace que el espaciotiempo no sea estático. Hemos hallado los objetos invariantes *gauge* que contienen toda la información físicamente relevante del sistema. En el sector axial hemos recuperado el escalar *master* de GS, mientras que en el sector polar hemos encontrado una variable *master* con una complicada ecuación de movimiento. A pesar de esta complicación, el análisis realizado abre el camino para obtener sistemáticamente variables *master* para diferentes espaciotiempos de fondo.

Appendix A

Spherical functions

Several conventions are employed in the literature for the special functions used in the theory of representations of the 3-dimensional rotation group. Here we follow the conventions of Edmonds (E) [158], and briefly compare them with those of Galindo and Pascual (GP) [156], Goldberg et al. (G) [152], and *Mathematica* [167].

The spherical harmonics $Y_l^m(\theta, \phi)$ are

$$Y_l^m(\theta, \phi) \equiv \sqrt{\frac{(2l+1)(l-m)!}{4\pi\,(l+m)!}}\, P_l^m(\cos\theta) e^{im\phi}, \tag{A.1}$$

where P_l^m is the associated Legendre function

$$P_l^m(x) \equiv \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2} \frac{d^{\,l+m}}{dx^{l+m}}(x^2-1)^l. \tag{A.2}$$

The so-called Condon-Shortley phase $(-1)^m$ [156] is already included in these polynomials, as it is nowadays standard, rather than in the definition of Y_l^m, as done by Edmonds [cf. his equation (2.5.29)]. The *Mathematica* functions `SphericalHarmonicY` and `LegendreP` are indeed those defined in equations (A.1) and (A.2), respectively.

All the references provided above agree in the definition of the Clebsch-Gordan coefficients and we refer to any of them for explicit expressions. However, there is no universally accepted convention for the rotation matrices in a representation $\mathcal{D}^{(l)}$ of $SU(2)$. For a given rotation of the reference frame described by the Euler angles (α, β, γ), Edmonds defines the unitary matrices

$$\mathcal{D}^{(l)}_{m'm}(\alpha, \beta, \gamma) = e^{im'\alpha} d^{(l)}_{m'm}(\beta) e^{im\gamma}. \tag{A.3}$$

In fact, we have corrected here a mistake in Edmonds' equation (4.1.12): the angles α and γ have been exchanged (see e.g. [168] for an independent mention of this mistake). The

β-transformation is given by

$$d^{(l)}_{m'm}(\beta) = \sum_{\sigma} \frac{(-1)^{l-m'-\sigma}\sqrt{(l+m)!(l-m)!(l+m')!(l-m')!}}{(l-m'-\sigma)!\,(l-m-\sigma)!\,(m'+m+\sigma)!\,\sigma!} \times \left(\sin\frac{\beta}{2}\right)^{2l-m'-m-2\sigma}\left(\cos\frac{\beta}{2}\right)^{m+m'+2\sigma}, \tag{A.4}$$

where the sum ranges over those integers σ for which the arguments of the factorials are all nonnegative. The order and sign of the Euler angles differ among authors, the relation being

$$\mathcal{D}^{(l)}_{m'm}{}^{[\mathrm{G}]}(\gamma,\beta,\alpha) = \mathcal{D}^{(l)}_{m'm}{}^{[\mathrm{GP}]}(-\alpha,-\beta,-\gamma) \tag{A.5}$$

$$= \mathcal{D}^{(l)}_{m'm}{}^{[\mathrm{E}]}(\alpha,\beta,\gamma), \tag{A.6}$$

$$d^{(l)}_{m'm}{}^{[\mathrm{G}]}(\beta) = d^{(l)}_{m'm}{}^{[\mathrm{GP}]}(\beta) = d^{(l)}_{m'm}{}^{[\mathrm{E}]}(-\beta). \tag{A.7}$$

Throughout this article and in our computational implementation, definitions (A.3) and (A.4) have been adopted.

Appendix B

Symmetric trace-free tensors

Given any tensor $T_{i_1\dots i_l}$ over a vector space of dimension d with a metric g_{ij}, we construct its symmetric trace-free part as

$$[T_{i_1\dots i_l}]^{STF} = \sum_{m=0}^{[l/2]} a_{l,d}^{(m)} g_{(i_1 i_2}\dots g_{i_{2m-1}i_{2m}} S_{i_{2m+1}\dots i_l)}{}^{j_1}{}_{j_1\dots}{}^{j_m}{}_{j_m} \tag{B.1}$$

with $S_{i_1\dots i_l} = T_{(i_1\dots i_l)}$ and $[l/2]$ the integer part of $l/2$. The coefficients of the expansion are determined by the trace-free condition, and are given by

$$a_{l,d}^{(m)} = \frac{l!}{(-4)^m m!(l-2m)!}\frac{\Gamma[l+d/2-1-m]}{\Gamma[l+d/2-1]}. \tag{B.2}$$

In our case, we have $d=2$ for the unit sphere. These formulas allow us to compute any of the $Z_l^m{}_{a_1\dots a_s}$ in terms of the derivatives $Y_l^m{}_{:a_1\dots a_s}$ or viceversa. Note that derivatives of Y_l^m with indices sorted differently are not equal, but can always be transformed into a term with the desired order of indices plus terms with a lower number of derivatives. For example (eliminating the harmonic labels l and m)

$$Y_{:abc} = Z_{abc} - \frac{l(l+1)+2}{4}\gamma_{ab}Y_{:c} - \frac{(l+2)(l-1)}{2}\gamma_{c(a}Y_{:b)}, \tag{B.3}$$

and

$$\begin{aligned} Y_{:abcd} &= Z_{abcd} - 2\gamma_{ab}Y_{:cd} - \frac{(l+2)(l-1)}{2}\gamma_{c(a}Y_{:b)d} \\ &- \frac{(l+3)(l-2)}{2}\gamma_{d(a}Y_{:b)c} - \frac{(l+3)(l+1)l(l-2)}{4}\left[\gamma_{c(a}\gamma_{b)d} - \frac{1}{2}\gamma_{ab}\gamma_{cd}\right]. \end{aligned} \tag{B.4}$$

Appendix C

Vacuum sources

In this appendix we show two particular examples for the regularized sources of the RW (12.82) and Zerilli equations (12.77). We assume that we have the first-order $\{l=2, m=1\}$-polar and $\{l=8, m=-4\}$-axial modes. The regularized source generated by them for the equation of motion of the second-order $\{l=7, m=3\}$-polar mode is given by,

$$
\begin{aligned}
S_{\mathcal{Z}}^{reg} &= -\frac{i\sqrt{\frac{22}{51\pi}}r}{945(\mathfrak{u}+9)^2(2\mathfrak{u}-1)(3\mathfrak{u}+2)^2}\Big\{-60r^3\left(6\mathfrak{u}^3+55\mathfrak{u}^2+7\mathfrak{u}-18\right)^2\mathcal{Z}_{,rr}\Pi_{,tr}\\
&+ 60r^3\left(6\mathfrak{u}^3+55\mathfrak{u}^2+7\mathfrak{u}-18\right)^2\mathcal{Z}_{,tr}\Pi_{,rr}\\
&- 20r^2(\mathfrak{u}+9)^2\left(18\mathfrak{u}^4+51\mathfrak{u}^3-58\mathfrak{u}^2-26\mathfrak{u}+20\right)\mathcal{Z}_{,r}\Pi_{,tr}\\
&- 5r^2(3\mathfrak{u}+2)^2\left(212\mathfrak{u}^4+3364\mathfrak{u}^3+11603\mathfrak{u}^2-13149\mathfrak{u}+3240\right)\mathcal{Z}_{,rr}\Pi_{,t}\\
&+ 20r^2(\mathfrak{u}+9)^2\left(126\mathfrak{u}^4+141\mathfrak{u}^3-82\mathfrak{u}^2-50\mathfrak{u}+20\right)\mathcal{Z}_{,t}\Pi_{,rr}\\
&+ 5r^2(3\mathfrak{u}+2)^2\left(308\mathfrak{u}^4+4972\mathfrak{u}^3+17255\mathfrak{u}^2-22221\mathfrak{u}+6156\right)\mathcal{Z}_{,tr}\Pi_{,r}\\
&- 180r(\mathfrak{u}+9)^2\left(6\mathfrak{u}^4+9\mathfrak{u}^3+2\mathfrak{u}^2+4\mathfrak{u}-4\right)\mathcal{Z}\Pi_{,tr}\\
&+ r\left(360\mathfrak{u}^6+2568\mathfrak{u}^5+57529\mathfrak{u}^4-14036\mathfrak{u}^3+375894\mathfrak{u}^2+88254\mathfrak{u}\right.\\
&- \left.123444\right)\mathcal{Z}_{,r}\Pi_{,t}+r\left(14220\mathfrak{u}^6+253302\mathfrak{u}^5+1234181\mathfrak{u}^4+854111\mathfrak{u}^3\right.\\
&- \left.966354\mathfrak{u}^2-375354\mathfrak{u}+220644\right)\mathcal{Z}_{,t}\Pi_{,r}+15r(3\mathfrak{u}+2)^2\left(92\mathfrak{u}^4+1492\mathfrak{u}^3\right.\\
&+ \left.8507\mathfrak{u}^2+14154\mathfrak{u}-9396\right)\mathcal{Z}_{,tr}\Pi-\left(6840\mathfrak{u}^6+112128\mathfrak{u}^5+422069\mathfrak{u}^4\right.\\
&+ \left.1424\mathfrak{u}^3-213006\mathfrak{u}^2+271944\mathfrak{u}-48924\right)\mathcal{Z}\Pi_{,t}\\
&+ \left(6210\mathfrak{u}^6+116313\mathfrak{u}^5+1283789\mathfrak{u}^4+6929894\mathfrak{u}^3+6649074\mathfrak{u}^2\right.\\
&- \left.316926\mathfrak{u}-1359504\right)\mathcal{Z}_{,t}\Pi\Big\},
\end{aligned}
\tag{C.1}
$$

where the symbol $\mathfrak{u}$ stands for the dimensionless mass $\mathfrak{u}\equiv M/r$. In order to show the regularized source for the RW equation (12.82) we take the particular case in which the first-order polar modes ($\bar{l} = 3, \bar{m} = 0$) and ($\hat{l} = 4, \hat{m} = -1$) generate a second-order axial mode with labels ($l = 4, m = -1$):

$$
\begin{aligned}
S_{\mathcal{X}}^{reg} &= \frac{3i}{8800\sqrt{7\pi}(\mathrm{u}+3)^4(2\mathrm{u}-1)^2(3\mathrm{u}+5)^4}\Big\{-10r\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^4(2\mathrm{u}-1)^5\hat{\mathcal{Z}}_{,rrr}\bar{\mathcal{Z}}_{,rr} \\
&+ 26r\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^4(2\mathrm{u}-1)^5\hat{\mathcal{Z}}_{,rr}\bar{\mathcal{Z}}_{,rrr} \\
&- \frac{(3\mathrm{u}+5)^2}{r^2}\big(3060\mathrm{u}^8+49401\mathrm{u}^7+332356\mathrm{u}^6+1197973\mathrm{u}^5+2636572\mathrm{u}^4 \\
&+ 3760905\mathrm{u}^3+2764530\mathrm{u}^2-467775\mathrm{u}-1518750\big)(2\mathrm{u}-1)^3\hat{\mathcal{Z}}_{,rr}\bar{\mathcal{Z}} \\
&+ \frac{(\mathrm{u}+3)^2}{r^2}\big(1620\mathrm{u}^8+28161\mathrm{u}^7+173844\mathrm{u}^6+197637\mathrm{u}^5-1511900\mathrm{u}^4 \\
&- 5534775\mathrm{u}^3-7023550\mathrm{u}^2-3510375\mathrm{u}-573750\big)(2\mathrm{u}-1)^3\hat{\mathcal{Z}}\bar{\mathcal{Z}}_{,rr} \\
&+ 10r\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^4(2\mathrm{u}-1)^3\hat{\mathcal{Z}}_{,trr}\bar{\mathcal{Z}}_{,tr}-26r\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^4(2\mathrm{u}-1)^3\hat{\mathcal{Z}}_{,tr}\bar{\mathcal{Z}}_{,trr} \\
&- \frac{16(1-2\mathrm{u})^2}{r^4}\big(1701\mathrm{u}^{11}+35262\mathrm{u}^{10}+320166\mathrm{u}^9+1720086\mathrm{u}^8+6285736\mathrm{u}^7 \\
&+ 16821825\mathrm{u}^6+34748135\mathrm{u}^5+56990175\mathrm{u}^4+68601150\mathrm{u}^3+42931125\mathrm{u}^2 \\
&- 5703750\mathrm{u}-15946875)\,\hat{\mathcal{Z}}\bar{\mathcal{Z}} \\
&- \frac{2}{r^3}\left(6\mathrm{u}^2+7\mathrm{u}-5\right)^2\big(1530\mathrm{u}^9+34221\mathrm{u}^8+303099\mathrm{u}^7+1485635\mathrm{u}^6+4592169\mathrm{u}^5 \\
&+ 9179205\mathrm{u}^4+10353033\mathrm{u}^3+3316365\mathrm{u}^2-2994975\mathrm{u}-1478250\big)\hat{\mathcal{Z}}_{,r}\bar{\mathcal{Z}} \\
&- \frac{10}{r}(\mathrm{u}+3)^2\left(6\mathrm{u}^2+7\mathrm{u}-5\right)^4\left(\mathrm{u}^3+9\mathrm{u}^2+27\mathrm{u}+90\right)\hat{\mathcal{Z}}_{,rrr}\bar{\mathcal{Z}} \\
&+ \frac{2}{r^3}\left(2\mathrm{u}^2+5\mathrm{u}-3\right)^2\big(24138\mathrm{u}^9+399357\mathrm{u}^8+2535795\mathrm{u}^7+8866263\mathrm{u}^6 \\
&+ 20189321\mathrm{u}^5+31979265\mathrm{u}^4+34936825\mathrm{u}^3+20024625\mathrm{u}^2-4674375\mathrm{u} \\
&- 9618750)\,\hat{\mathcal{Z}}\bar{\mathcal{Z}}_{,r}+\frac{8}{r^2}\left(6\mathrm{u}^3+25\mathrm{u}^2+16\mathrm{u}-15\right)^2\big(90\mathrm{u}^7+468\mathrm{u}^6+1763\mathrm{u}^5 \\
&+ 5632\mathrm{u}^4+22704\mathrm{u}^3-6480\mathrm{u}^2-35025\mathrm{u}+15750\big)\hat{\mathcal{Z}}_{,r}\bar{\mathcal{Z}}_{,r} \\
&+ \frac{2}{r}(\mathrm{u}+3)^2\left(6\mathrm{u}^2+7\mathrm{u}-5\right)^3\big(675\mathrm{u}^5+5741\mathrm{u}^4+15946\mathrm{u}^3+24570\mathrm{u}^2 \\
&+ 10590\mathrm{u}-13950)\,\hat{\mathcal{Z}}_{,rr}\bar{\mathcal{Z}}_{,r}-5(\mathrm{u}+3)^3(11\mathrm{u}-9)\left(6\mathrm{u}^2+7\mathrm{u}-5\right)^4\hat{\mathcal{Z}}_{,rrr}\bar{\mathcal{Z}}_{,r} \\
&- \frac{2}{r}(3\mathrm{u}+5)^2\left(2\mathrm{u}^2+5\mathrm{u}-3\right)^3\big(1377\mathrm{u}^5+11403\mathrm{u}^4+25994\mathrm{u}^3+19250\mathrm{u}^2 \\
&- 410\mathrm{u}-10350)\,\hat{\mathcal{Z}}_{,r}\bar{\mathcal{Z}}_{,rr}-8(1-2\mathrm{u})^4\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^3\big(30\mathrm{u}^3+91\mathrm{u}^2 \\
&+ 99\mathrm{u}+15)\,\hat{\mathcal{Z}}_{,rr}\bar{\mathcal{Z}}_{,rr}+13(3\mathrm{u}+5)^3(17\mathrm{u}-15)\left(2\mathrm{u}^2+5\mathrm{u}-3\right)^4\hat{\mathcal{Z}}_{,r}\bar{\mathcal{Z}}_{,rrr} \\
&+ \frac{78}{r}(3\mathrm{u}+5)^2\left(2\mathrm{u}^2+5\mathrm{u}-3\right)^4\left(3\mathrm{u}^3+15\mathrm{u}^2+25\mathrm{u}+50\right)\hat{\mathcal{Z}}\bar{\mathcal{Z}}_{,rrr} \\
&- \frac{8}{r^2}\left(3\mathrm{u}^2+14\mathrm{u}+15\right)^2\big(198\mathrm{u}^7+1164\mathrm{u}^6+2627\mathrm{u}^5+17137\mathrm{u}^4+45972\mathrm{u}^3 \\
&+ 3140\mathrm{u}^2-38100\mathrm{u}+11475\big)\hat{\mathcal{Z}}_{,t}\bar{\mathcal{Z}}_{,t}
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{2}{r}(\mathfrak{u}+3)^2(3\mathfrak{u}+5)^3\left(474\mathfrak{u}^6+4575\mathfrak{u}^5+13106\mathfrak{u}^4+20156\mathfrak{u}^3+1014\mathfrak{u}^2\right.\\
- & \left.23685\mathfrak{u}+8100\right)\hat{\mathcal{Z}}_{,tr}\bar{\mathcal{Z}}_{,t}+\frac{2}{r}(\mathfrak{u}+3)^3(3\mathfrak{u}+5)^2\left(2142\mathfrak{u}^6+13005\mathfrak{u}^5\right.\\
+ & \left.21826\mathfrak{u}^4+13968\mathfrak{u}^3-4370\mathfrak{u}^2-21115\mathfrak{u}+8100\right)\hat{\mathcal{Z}}_{,t}\bar{\mathcal{Z}}_{,tr}\\
+ & 5(1-2\mathfrak{u})^2(\mathfrak{u}+3)^3(3\mathfrak{u}+5)^4\left(4\mathfrak{u}^2+23\mathfrak{u}-9\right)\hat{\mathcal{Z}}_{,trr}\bar{\mathcal{Z}}_{,t}\\
+ & 24\left(\mathfrak{u}^2-3\mathfrak{u}-5\right)\left(6\mathfrak{u}^3+25\mathfrak{u}^2+16\mathfrak{u}-15\right)^3\hat{\mathcal{Z}}_{,tr}\bar{\mathcal{Z}}_{,tr}\\
- & 13(1-2\mathfrak{u})^2(\mathfrak{u}+3)^4(3\mathfrak{u}+5)^3\left(12\mathfrak{u}^2+37\mathfrak{u}-15\right)\hat{\mathcal{Z}}_{,t}\bar{\mathcal{Z}}_{,trr}\Big\}. \qquad \text{(C.2)}
\end{aligned}
$$

Appendix D

Fluid sources

D.1 Polar linear sources

In this appendix we give the linear sources that appears in the polar evolution equations of the fluid. These are the same sources that were given in [55].

$$
\begin{aligned}
S_\chi &= -2\left[2\nu^2 + 8\pi\rho - \frac{6m}{r^3} - 2U(\mu - U)\right](\chi + \mathcal{K}) + \frac{(l-1)(l+2)}{r^2}\chi \\
&+ 3\mu\dot{\chi} + 4(\mu - U)\dot{\mathcal{K}} - (5\nu - 2W)\chi' - 2[2\mu\nu - 2(\mu - U)W + \mu' - \dot{\nu}]\psi \\
&+ 2\eta'' - 2(\mu - U)\dot{\eta} + (8\nu - 6W)\eta' - \Bigg[-4\nu^2 + \frac{l(l+1)+8}{r^2} + 8\nu W \\
&+ 4(2\mu U + U^2 - 4W^2 - 8\pi\rho)\Bigg]\eta, \qquad \text{(D.1)} \\
S_{\mathcal{K}} &= (1 + c_s^2)U\dot{\chi} + [4U + c_s^2(\mu + 2U)]\dot{\mathcal{K}} - W(1 - c_s^2)\chi' - (\nu + 2Wc_s^2)\mathcal{K}' \\
&- \left[2\left(\frac{1}{r^2} - W^2\right) + 8\pi p - c_s^2\left(\frac{l(l+1)}{r^2} + 2U(2\mu + U) - 8\pi\rho\right)\right](\chi + \mathcal{K}) \\
&- \frac{(l-1)(l+2)}{2r^2}(1 + c_s^2)\chi + 2[-\mu W(1 - c_s^2) + (\nu + W)U(1 + c_s^2)]\psi \\
&+ 8\pi C\rho\sigma - 2U\dot{\eta} + 2W\eta' + \left[\frac{l(l+1)+2}{r^2} - 6W^2 + 16\pi p - 2U(2\mu + U)c_s^2\right]\eta, \qquad \text{(D.2)} \\
S_\psi &= 2\nu(\chi + \mathcal{K}) + 2\mu\psi + \chi' - 2\eta(\nu - W) - 2\eta', \qquad \text{(D.3)} \\
C_\gamma &= -W\dot{\chi} + U\chi' - (\mu - 2U)\mathcal{K}' + \frac{1}{2}\Bigg[\frac{l(l+1)+2}{r^2} + 2U(2\mu + U) \\
&- 2W(2\nu + W) + 8\pi(p - \rho)\Bigg]\psi - 2U\eta', \qquad \text{(D.4)}
\end{aligned}
$$

$$C_\omega = \left[\frac{l(l+1)}{r^2} + 2U(2\mu+U) - 8\pi\rho\right](\chi+\mathcal{K}) + 2[\nu U + (\mu+U)W]\psi$$
$$+ U\dot{\chi} + (\mu+2U)\dot{\mathcal{K}} + W\chi' - 2W\mathcal{K}' - 2\eta U(2\mu+U) - \frac{(l-1)(l+2)}{2r^2}\chi, \quad \text{(D.5)}$$

$$C_\alpha = 2\mu(\chi+\mathcal{K}) + 2\nu\psi + \dot{\chi} + 2\dot{\mathcal{K}} - 2\eta(\mu+U), \quad \text{(D.6)}$$

$$S_\omega = \left(1+\frac{p}{\rho}\right)\left[-\frac{l(l+1)}{r^2}\alpha + \frac{\dot{\chi}+3\dot{\mathcal{K}}}{2} + \left(\nu + 2W - \frac{\nu}{c_s^2}\right)\left(\gamma+\frac{\psi}{2}\right)\right]$$
$$+(\mu+2U)\left(c_s^2 - \frac{p}{\rho}\right)\omega - C\left[\left(\gamma+\frac{\psi}{2}\right)\frac{s'}{c_s^2} - \sigma(\mu+2U)\right], \quad \text{(D.7)}$$

$$S_\gamma = \left(1+\frac{p}{\rho}\right)\left[\frac{\chi'+\mathcal{K}'-2\eta'}{2} + \left[c_s^2(\mu+2U)-\mu\right]\left(\gamma-\frac{\psi}{2}\right)\right]$$
$$-C\sigma' - \sigma\left[C\left(\nu - \frac{s'}{c_s^2}\frac{\partial c_s^2}{\partial s}\right) + s'\frac{\partial C}{\partial s} - \nu\left(1+\frac{p}{\rho}\right)\frac{1}{c_s^2}\frac{\partial c_s^2}{\partial s}\right]$$
$$-\nu\left(c_s^2 - \frac{p}{\rho} - \frac{\rho+p}{c_s^2}\frac{\partial c_s^2}{\partial \rho}\right)\omega - \omega s'\left[\frac{\partial c_s^2}{\partial s} - C\left(1+\frac{\rho}{c_s^2}\frac{\partial c_s^2}{\partial\rho}\right)\right], \quad \text{(D.8)}$$

$$S_\alpha = -\frac{\mathcal{K}+\chi}{2} + \eta - c_s^2(\mu+2U)\alpha + \frac{c_s^2\omega + C\sigma}{1+\frac{p}{\rho}}, \quad \text{(D.9)}$$

$$\bar{S}_\omega = \left(1+\frac{p}{\rho}\right)\left[\left(-\frac{l(l+1)}{r^2} + 8\pi(\rho+p)\right)\alpha + \frac{\dot{\mathcal{K}}}{2} + (\mu+U)\eta - \mu(\chi+\mathcal{K})\right]$$
$$+C(\mu+2U)\sigma - \frac{1}{c_s^2}\left[s'C + \left(1+\frac{p}{\rho}\right)(\nu - 2Wc_s^2)\right]\left(\gamma+\frac{\psi}{2}\right)$$
$$+(\mu+2U)\left(c_s^2-\frac{p}{\rho}\right)\omega + \nu\left(1+\frac{p}{\rho}\right)\left(\gamma-\frac{\psi}{2}\right), \quad \text{(D.10)}$$

$$\bar{S}_\gamma = \left(1+\frac{p}{\rho}\right)\left[\frac{\mathcal{K}'}{2} + \left(c_s^2(\mu+2U)-\mu\right)\left(\gamma-\frac{\psi}{2}\right) - \mu\psi - \nu(\chi+\mathcal{K}) + (\nu-W)\eta\right]$$
$$-C\sigma' - \sigma C\left[\nu + \frac{s'}{C}\frac{\partial C}{\partial s} - \left(\frac{\nu}{C}\left(1+\frac{p}{\rho}\right) + s'\right)\frac{1}{c_s^2}\frac{\partial c_s^2}{\partial s}\right]$$
$$+\omega\left[\nu\left(\frac{p}{\rho} - c_s^2\right) + s'\left(C - \frac{\partial c_s^2}{\partial s}\right) + \left[\nu(\rho+p) + \rho C s'\right]\frac{1}{c_s^2}\frac{\partial c_s^2}{\partial\rho}\right]. \quad \text{(D.11)}$$

D.2 Polar constraint equations for the case $l = 1$

Here we give the second-order polar constraint equations for the case $l = 1$. Once one knows the matter perturbations $\{\alpha, \gamma, \omega, \sigma\}$, these must be solved in a surface everywhere orthogonal to the $r = const.$ surfaces to obtain the metric perturbations $\{\eta, \chi, \psi\}$.

$$
\begin{aligned}
r|v|^2 D\eta &= 4\pi\rho(1-c_s^2)\omega - 16\pi(\rho+p)U\alpha - \Big(\frac{2}{r^2} - 3W^2 + 8\pi p + U^2\Big)\eta \\
&- \big(W^2 + U^2 - 4\pi(\rho+p)\big)\chi - 2UW\psi - 4\pi\rho C\sigma \\
&- \frac{1}{2}\sum_{\bar{l},\hat{l}} \{{}^{(\epsilon)}S_{\mathcal{K}} + (1-c_s^2){}^{(\epsilon)}C_\omega - 2U{}^{(\epsilon)}C_\alpha\}, \qquad \text{(D.12)}\\
r|v|^2 D\chi &= 8\pi\rho\omega - \Big(\frac{2}{r^2} + 2U^2 - 8\pi\rho\Big)\chi + 2\big(\nu U - (\mu+U)W\big)\psi - 2U^2\eta \\
&- 32\pi(p+\rho)U\alpha + \sum_{\bar{l},\hat{l}} \{2U{}^{(\epsilon)}C_\alpha - {}^{(\epsilon)}C_\omega\}, \qquad \text{(D.13)}\\
r|v|^2 D\psi &= 8\pi(\rho+p)(\gamma + 2W\alpha) + 4UW\eta - \Big(U^2 - W^2 + \frac{2}{r^2} + 4\pi(p-\rho)\Big)\psi \\
&+ 2(\mu W - \nu U)(\eta - \chi) - \sum_{\bar{l},\hat{l}} \{{}^{(\epsilon)}C_\gamma + W{}^{(\epsilon)}C_\alpha - U{}^{(\epsilon)}S_\psi\}. \qquad \text{(D.14)}
\end{aligned}
$$

D.3 Polar equations for the case $l = 0$

We give now all the equations needed to solve the second-order polar sector for the particular case $l = 0$. On the one hand, there are two evolution equations for $\{\omega, \gamma\}$ and, on the other hand, two constraint equations for $\{\eta, \chi\}$ that must be solved in a surface orthogonal to $r = const.$ surfaces,

$$
\begin{aligned}
-\dot{\omega} - \Big(1 + \frac{p}{\rho}\Big)\gamma' &= (\mu + 2U)C\sigma - \omega\Big[4\pi\frac{U}{|v|^2}(p+\rho) + (\mu+2U)\Big(\frac{p}{\rho} - c_s^2\Big)\Big] \\
&- \gamma\Big[\frac{Cs'}{c_s^2} + \Big(1+\frac{p}{\rho}\Big)\Big(\frac{4\pi W}{|v|^2}(p+\rho) - \nu - 2W + \frac{\nu}{c_s^2}\Big)\Big] \\
&+ (\chi - \eta)\Big[\frac{UW}{U^2+W^2}\frac{Cs'}{c_s^2} - \Big(1+\frac{p}{\rho}\Big)\Big(-\nu\frac{UW}{U^2+W^2}\frac{1+c_s^2}{c_s^2} \\
&+ \frac{U}{2}\frac{|v|^2}{U^2+W^2} + 4\pi\frac{UW^2}{W^4-U^4}(p+\rho) + U + \mu\Big)\Big] \\
&- \eta\frac{U}{|v|^2}\Big(1+\frac{p}{\rho}\Big)\Big(4\pi\rho - \frac{1}{2r^2}\Big) \\
&+ \sum_{\bar{l},\hat{l}} \Big\{{}^{(\epsilon)}S_\omega - \frac{(p+\rho)}{2\rho|v|^2}(Uu^Au^B - Wu^An^B){}^{(\epsilon)}\mathcal{P}_{AB}\Big\}, \qquad \text{(D.15)}
\end{aligned}
$$

$$
\begin{aligned}
\Big(1+\frac{p}{\rho}\Big)\dot{\gamma}+c_s^2\omega' &= \gamma\Big(1+\frac{p}{\rho}\Big)\Big(-4\pi\frac{U}{|v|^2}(p+\rho)-\mu+(\mu+2U)c_s^2\Big)\\
&- \sigma\Big\{\nu\Big[C-\Big(1+\frac{p}{\rho}\Big)\frac{1}{c_s^2}\frac{\partial c_s^2}{\partial s}\Big]+4\pi\frac{WC}{|v|^2}(p+\rho)+s'\Big(\frac{\partial C}{\partial s}-\frac{C}{c_s^2}\frac{\partial c_s^2}{\partial s}\Big)\Big\}\\
&- \omega\Big\{\nu\Big[c_s^2-\frac{p}{\rho}-(p+\rho)\frac{1}{c_s^2}\frac{\partial c_s^2}{\partial \rho}\Big]+4\pi\frac{Wc_s^2}{|v|^2}(p+\rho)\\
&- s'\Big[C-\frac{\partial c_s^2}{\partial s}+\frac{\rho C}{c_s^2}\frac{\partial c_s^2}{\partial \rho}\Big]\Big\}-\eta\frac{W}{|v|^2}\Big(1+\frac{p}{\rho}\Big)\Big(\frac{1}{2r^2}+4\pi p\Big)-C\sigma'\\
&- (\chi-\eta)\Big(1+\frac{p}{\rho}\Big)\Big(\nu-\mu\frac{UW}{U^2+W^2}(1+c_s^2)-\frac{2U^2W}{U^2+W^2}c_s^2\\
&+ \frac{W}{2}\frac{|v|^2}{U^2+W^2}+4\pi\frac{U^2W}{U^4-W^4}(p+\rho)\Big) \qquad \text{(D.16)}\\
&+ \sum_{\bar{l},\hat{l}}\Big\{{}^{(\epsilon)}S_\gamma+\frac{(p+\rho)}{2\rho|v|^2}(Uu^An^B-Wn^An^B)^{(\epsilon)}\mathcal{P}_{AB}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
r|v|^2D\eta &= 4\pi(p+\rho)\Big(\chi+\frac{2U^2}{|v|^2}\eta\Big)+8\pi(p+\rho)\frac{2UW}{|v|^2}\gamma\\
&+ 4\pi\rho\frac{U^2+W^2}{|v|^2}\big(C\sigma+(1+c_s^2)\omega\big) \qquad \text{(D.17)}\\
&+ \frac{1}{2|v|^2}\sum_{\bar{l},\hat{l}}\big\{(U^2+W^2)(u^Au^B+n^An^B)-4UWu^An^B\big\}^{(\epsilon)}\mathcal{P}_{AB},
\end{aligned}
$$

$$
\begin{aligned}
r|v|^2D\chi &= \frac{4UW}{U^2+W^2}\Big(\mu W-\nu U+4\pi\frac{UW}{|v|^2}(p+\rho)\Big)(\chi-\eta)\\
&- \Big(8\pi\rho-\frac{1}{r^2}\Big)\Big(\chi+\frac{2U^2}{|v|^2}\eta\Big)+8\pi(p+\rho)\frac{2UW}{|v|^2}\gamma+8\pi\rho\frac{U^2+W^2}{|v|^2}\omega\\
&+ \frac{1}{|v|^2}\sum_{\bar{l},\hat{l}}\big\{(U^2+W^2)u^Au^B-2UWu^An^B\big\}^{(\epsilon)}\mathcal{P}_{AB}. \qquad \text{(D.18)}
\end{aligned}
$$

Last equation corrects the sign of the second term of the sum from reference [55].

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