Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Matemáticas

Fractional Schrödinger operators, Harnack's inequalities for fractional Laplacians

A dissertation submitted for the degree of

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by

Chao Zhang

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José Luis Torrea Hernández

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Resumen y conclusiones

A finales de los años 60 del siglo pasado, B. Muckenhoupt y E. Stein publican una serie de trabajos en el contexto de los polinomios ortogonales clásicos (Hermite Laguerre y Jacobi). La idea subyacente era analizar el sustituto de serie conjugada del análisis armónico clásico y estudiar el comportamiento en los espacios L^p. En este sentido las ecuaciones de Cauchy-Riemann y los análogos de algunas propiedades de las funciones armónicas también fueron tratados. Por otro lado a comienzos de los años 90, el profesor E. Fabes dirigió dos tesis doctorales (R.Scotto y W. Urbina) en las que se analizaban las "transformadas de Riesz asociadas a la medida gaussiana". En ambos trabajos se definían unos operadores llamados "transformadas de Riesz", y se probaban acotaciones paralelas a las transformadas de Riesz clásicas (tipo fuerte (p, p), $p > 1$ y débil $(1, 1)$). Sin embargo en ningún momento se explicaba la razón del nombre.

Las tesis dirigidas por el profesor Fabes fueron un revulsivo dentro del mundillo de los expertos en Análisis Armónico y comenzó un gran florecimiento de un Análisis de Fourier asociado a laplacianos generales. Además se contó con la ayuda inestimable del libro de E. Stein "Topics in Harmonic Analysis Related to the Littlewood-Paley Theory". Este libro apareció en 1970 y ha sido una guía esencial para un gran número de profesionales. En la monografía de Stein se describe desde varios puntos de vista la importancia de la teoría de semigrupos para entender, mediante una visión muy general, algunos de los conceptos desarrollados en Análisis Armónico. Stein también obtiene acotaciones en espacios de Lebesgue L^p y en espacios de Hardy H^p de algunos operadores clásicos, todo ello apelando a la teoría de semigrupos de difusión. En las dos últimas décadas un gran número de publicaciones se han ocupado de desarrollar un Análisis Armónico asociado a diversos operadores diferenciales.

Por otra parte en Agosto de 2006, L. Caffarelli y L. Silvestre publican el celebrado trabajo "An extension problem related to fractional Laplacian". Este interesante trabajo introduce de pleno derecho en la teoría de ecuaciones en derivadas parciales al operador laplaciano fraccionario $(-\Delta)^{\alpha}$ con $0 < \alpha < 1$. Este operador hasta ese momento había sido estudiado de manera modesta en teoría de potencial y en probabilidad. El trabajo de Caffarelli y Silvestre hizo que pasase a ser uno de los temas candentes de EDP's.

En 2010 Pablo Stinga, presentó su tesis doctoral en la UAM. En dicha tesis se hacía un tratamiento del laplaciano fraccionario utilizando la teora de semigrupos. En concreto la fórmula

$$(-\Delta)^{\sigma} f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} \left(e^{t\Delta} f(x) - f(x)\right) \frac{dt}{t^{1+\sigma}}, \qquad x \in \mathbb{R}^n,\ 0 < \sigma < 1.$$

Permitía un tratamiento del operador fraccionario mucho más versátil que la fórmula $\widehat{(-\Delta)^{\alpha} f}(\xi) = |\xi|^{\alpha/2} \hat{f}(\xi)$, combinada con técnicas de transformada de Fourier.

La tesis de Pablo Stinga sugería que alguno de los temas estudiados por Caffarelli y Silvestre podrían ser abordados con la óptica nueva de la teoría de semigrupos. Esto permitiría obtener resultados nuevos y entender mejor alguno de los resultados ya conocidos.

Uno de los problemas esenciales en ecuaciones en derivadas parciales es la obtención de estimaciones de regularidad (estimaciones de Schauder). Nos propusimos estudiar dichas estimaciones para operadores de Schrödinger desde dos puntos de vista:

1.- A través de las extensiones armónicas. Es decir utilizando el semigrupo subordinado de Poisson.

2.- Utilizando teoremas del análisis armónico relativos a acotaciones en espacios de tipo Hölder pero con descripción como espacios de Campanato. Esto permitiría el aprovechamiento de ideas ya utilizadas en Análisis Armónico.

El trabajo fue realizado con éxito y llegamos a publicar un trabajo en cada una de las lineas de investigación anteriores a saber:

T. Ma, P. R. Stinga, J. L. Torrea, and C. Zhang, *Regularity properties of Schrodinger operators.* J. Math. Anal. Appl. 388, 817837 (2012).

T. Ma, P. R. Stinga, J. L. Torrea and C. Zhang, *Regularity estimates in Holder spaces for Schrodinger operators via a T1 theorem,* Ann. Mat. Pura Appl. (por aparecer).

Sin ninguna duda, otro problema recurrente en la teoría de ecuaciones en derivadas parciales es la obtención de desigualdades de Harnack. Nuestra idea aquí fue la utilización de algunas técnicas de transferencia entre semigrupos que nos permitiesen "transferir" desigualdades de Harnack entre laplacianos fraccionarios. Las técnicas de transferencia fueron introducidas por I.Abu-Falahah y J.L. Torrea en 2006. Nuevamente hubo éxito y esta investigación dió lugar al trabajo

P. R. Stinga and C. Zhang, *Harnack's inequality for fractional operators*, Discrete Contin. Dyn. Syst. (por aparecer).

Además del trabajo descrito hasta el momento también se han estudiado funciones de Littlewood-Paley cuando se toman derivadas fraccionarias. La derivada fraccionaria aparece en distintos tratados, nosotros hemos estudiado la introducida por C. Segovia y R. Wheeden en 1969. Dado $\alpha > 0$, sea m el entero más pequeño estrictamente mayor que α. Llamando P_t al semigrupo de Poisson, se define

$$\partial_t^{\alpha} P_t f(x) = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^{\infty} \partial_t^m P_{t+s}(f)(x) s^{m-\alpha-1} ds, \qquad t > 0,\ x \in \mathbb{R}^n. \qquad (0.1)$$

En la definición puede intervenir cualquier semigrupo subordinado, en particular los semigrupos de Poisson de operadores diferenciales. Esto permite estudiar las funciones de Littlewood-Paley fraccionarias

$$g^{\alpha}(f)(x) = \Big(\int_0^\infty |t^\alpha \partial_t^\alpha P_t f(x)|^2 \frac{dt}{t} \Big)^{1/2}.$$

Estas funciones fueron utilizadas en el caso clásico para obtener caracterizaciones de espacios de Sobolev. Nosotros estudiamos las versiones vectoriales de dichas funciones caracterizando las espacios de Banach para los cuales son acotadas.

CONCLUSIONES

La memoria establece de dos modos, completamente originales, estimaciones de Schauder para potencias fraccionarias del operador de Schrödinger. Un primer método consiste en considerar las extensiones armónicas del operador. Para ello se necesita un estudio muy meticuloso del correspondiente semigrupo de Poisson. El segundo método consiste en la construcción de un criterio general para el estudio de estimaciones de regularidad. Este criterio establece esencialmente que para una amplia familia de operadores, las estimaciones de regularidad dependen del comportamiento del operador cuando actúa sobre la función constante 1.

Por otro lado en la memoria se prueban desigualdades de Harnack para potencias fraccionarias de una amplia familia de operadores. La aportación en este caso, además del propio resultado en sí mismo, la constituye el método de la prueba que utiliza una transferencia muy sencilla entre operadores. La transferencia resulta sencilla, pero su utilización en este tipo de problemas es completamente original.

Finalmente a lo largo de toda la memoria se analizan diversos comportamientos de una derivada fraccionaria unidimensional que extiende a la derivada clásica.

La memoria ha dado lugar a las siguientes publicaciones:

1.- T. Ma, P. R. Stinga, J. L. Torrea, and C. Zhang, *Regularity properties of Schrodinger operators.* Journal of Mathematical Analysis and Applications 388, 817- 837 (2012).

2.- T. Ma, P. R. Stinga, J. L. Torrea and C. Zhang, *Regularity estimates in Holder spaces for Schrodinger operators via a T1 theorem,* Annali di Matematica Pura ed Applicata. (por aparecer).

3.- P. R. Stinga and C. Zhang, *Harnack's inequality for fractional operators*, Discrete and Continuous Dynamical Systems. (por aparecer).

La memoria hace un recorrido por diversos temas de Anlisis Armnico y de Ecuaciones en Derivadas Parciales teniendo como herramienta base la teoría de semigrupos

Contents

Chapter 1

Introduction

In the last decades the theory of diffusion semigroups have been used successfully in the development of a theory of Harmonic Analysis associated to several Laplacians. This theory has meanly dealt with L^p and H^p boundedness of operators like Riesz potentials, Riesz transforms, Litlewood-Paley functions, etc. The idea behind the use of the semigroup theory in Harmonic Analysis has its roots in the book published in 1970 by E. Stein, "Topics in Harmonic Analysis Related to the Littlewood-Paley Theory", [78]. But the big flowering of it was after 1990. It could be said that the starting point of this new wave were the Ph.D. dissertations of R. Scotto and W. Urbina advisered by Professor E. B. Fabes in the University of Minnesota, see [33, 91].

The systematic use of the theory of semigroups to understand Harmonic Analysis started a few years later and papers in this line are [2, 3, 5, 6, 7, 15, 26, 27, 30, 31, 32, 41, 42, 51, 61, 62, 69, 76, 77, 80].

In this very short account of the use of semigroup theory in Harmonic Analysis, we arrive to the use of the theory in PDEs. In general, working in Hamonic Analysis, to get L^p-estimates for one operator it is not necessary to have a pointwise description of the operator acting on functions in L^p. The situation is completely different in PDEs and the pointwise description of the operator is one of the crucial facts that are needed. An example of this situation is the fractional laplacian $(-\Delta)^\sigma$. This operator has become one of the most famous operators in the last five years, after the celebrated work of L. Caffarelli and L. Silvestre [16]. The understanding of the fractional laplacian is completely clear when using Fourier transform, that means $(-\Delta)^\sigma f$ is the function whose Fourier transform is given by $|\xi|^{2\sigma}\hat{f}(\xi)$. However to find a formula for $(-\Delta)^\sigma f(x)$ is rather involve by using Fourier inverse techniques. It turns out that this pointwise description is fundamental when working with C^α-estimates (regularity estimates). An alternative approach to characterize $(-\Delta)^\sigma$, could be the classical formula

$$(-\Delta)^\sigma f(x) = \frac{1}{\Gamma(-\sigma)}\int_0^\infty \left(e^{t\Delta}f(x) - f(x)\right)\,\frac{dt}{t^{1+\sigma}}, \qquad x \in \mathbb{R}^n,\ 0<\sigma<1.$$

This formula can be used in a direct way to get an exact pointwise expression of $(-\Delta)^\sigma f(x)$

for good enough functions, see [82].

The formula above can be considered for any diffusion semgroup, $\{e^{-t\mathcal{L}}\}_{t\geqslant 0}$, generated by a general laplacian $\mathcal{L}$ satisfying certain mild conditions (self-adjointness, positivity, etc) and

$$\mathcal{L}^{\sigma}f(x) = \frac{1}{\Gamma(-\sigma)}\int_0^\infty \left(e^{-t\mathcal{L}}f(x) - f(x)\right)\,\frac{dt}{t^{1+\sigma}}, \qquad x \in \Omega,\ 0 < \sigma < 1. \tag{1.1}$$

Analogously for negative powers we can consider the operators

$$\mathcal{L}^{-\sigma}f(x) = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-t\mathcal{L}}f(x)\,\frac{dt}{t^{1-\sigma}}, \qquad x \in \Omega,\ \sigma > 0, \tag{1.2}$$

where Ω is the domain of the functions f for which $\mathcal{L}$ is acting on, such as the torus, $\mathbb{R}^n$, $(0,\infty)$, etc. In order to make the understanding of these formulas easy we recall the following formulas related to Gamma function

$$\lambda^{\sigma} = \frac{1}{\Gamma(-\sigma)}\int_0^\infty (e^{-t\lambda}-1)\frac{dt}{t^{1+\sigma}}, \quad 0<\sigma<1, \text{ and } \lambda^{-\sigma} = \frac{1}{\Gamma(\sigma)}\int_0^\infty e^{-t\lambda}\frac{dt}{t^{1-\sigma}}, \quad \sigma>0,$$

where Γ denotes the Gamma function and $\Gamma(-\sigma) := \frac{\Gamma(1-\sigma)}{-\sigma} = \int_0^\infty (e^{-s}-1)\frac{ds}{s^{1+\sigma}} < 0.$

Finally, before passing to the description of our work, we want to recall a definition of fractional derivative (one dimensional). The following definition is due to C. Segovia and R. L. Wheeden [70]. Given $\alpha > 0$, let m be the smallest integer which strictly exceeds α. Let f be a reasonable nice function in $L^p_{\mathbb{B}}(\mathbb{R}^n)$. Then

$$\partial_t^{\alpha}\mathcal{P}_t f(x) = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)}\int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f)(x)s^{m-\alpha-1}ds, \qquad t>0,\ x\in\mathbb{R}^n, \tag{1.3}$$

where $\mathcal{P}_t$ is the classical Poisson semigroup. Observe that

$$\partial_t^{\alpha}e^{-t|\xi|} = e^{i\pi\alpha}|\xi|^{\alpha}e^{-t|\xi|}, \quad \alpha > 0.$$

The definition of fractional derivatives appearing in (1.3) can be used in a natural way when dealing with subordinated semigroups. The most interesting for us, will be the case of differential operators that generate diffusion semigroups and subordinated semigroups.

Apart of this, it is a common fact that some results in Probability Theory have some parallels when considering the Poisson semigroup in the torus. In this line of thought, in 1998 Q. Xu [92] found a characterization of a property of Banach spaces (martingale type) that was defined by G. Pisier in probability. The characterization was achieved by the boundedness of some Littlewood-Paley functions defined in the torus. This idea was explored with much more generality in 2006 for subordinated semigroups in [57].

Before entering into a detailed account of our work, we would like to make a naive description of it.

-As we said before formulas (1.1) and (1.2) produce a path to study regularity properties (Schauder estimates) of powers of $\mathcal{L}$. How to exploit this idea in the case of Schrödinger operator will be an important part of our work. In order to do this we shall need to go further and in fact we shall find some new (and useful) definitions for the classes C^α associated to an operator $\mathcal{L}$. These ideas have appeared in [55, 56].

-Sometimes there are some relations (essentially changes of variables) between the semigroups associated to differential operators. For example, due to the fact that an Hermite function is an Hermite polynomial multiplied by a fixed exponential function, the corresponding heat semigroups will have an exact pointwise relation. Because of formulas (1.1) and (1.2), one can think that these relations could be used when analyzing the properties of operators $\mathcal{L}^{\pm\sigma}$. In general this is not true (for example when dealing with L^p boundedness) due to the fact that the relations cannot be controlled globally. However this relations are good enough for some local estimates crucial in PDEs. We shall see how to get some new Harnack's inequalities by using these ideas in Chapter 4, which was produced in [84].

-The introduction of fractional derivative suggests the possibility of defining Littlewood-Paley functions with it. We characterize the geometric properties of Banach spaces for which these new fractional Littlewood-Paley functions are bounded. The condition we found in our paper [88] is the same as the one for the classical square functions previously studied in [57].

Now we shall pass to an explicit description of the manuscript.

1.1 Regularity theory of operators related with Schrödinger operators

Very recently, a great deal of attention was given to nonlinear problems involving fractional integro-differential operators. These problems arise in Physics (fluid dynamics, strange kinetics, anomalous transport) and Mathematical Finance (modeling with Lévy processes), among many other fields, see for instance [17, 18, 19, 73, 74] and the references therein. The main question is the regularity of solutions. In Chapter 2 and 3, we want to get some regularity properties related with the time independent Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 3$, $\mathcal{L} := -\Delta + V$, where the nonnegative potential V satisfies a reverse Hölder inequality. In the last twenty years there exists an increasing interest on the study of these operators. C. Fefferman [35], Z. Shen [72] and J. Zhong [99] obtained some basic results on $\mathcal{L}$, including certain estimates of the fundamental solutions of $\mathcal{L}$ and the boundedness on Lebesgue spaces $L^p(\mathbb{R}^n)$ with some $p \in (1, \infty)$ of Riesz transforms. J. Dziubański and J. Zienkiewicz [30, 31, 32] characterized the Hardy space $H^p_{\mathcal{L}}(\mathbb{R}^n)$ associated with $\mathcal{L}$, see also [76, 77]. In [27], the authors considered the $BMO_{\mathcal{L}}$-spaces. And in [13], the theory related with the $BMO^\alpha_{\mathcal{L}}$-spaces $(0 < \alpha < 1)$ were developed. With a Campanato description, we know that these $BMO^\alpha_{\mathcal{L}}$, $(0 < \alpha < 1)$ are equivalent with the Hölder spaces related with $\mathcal{L}$; see Section 2.4. We shall denote these Hölder spaces by $C^{0,\alpha}_{\mathcal{L}}$. For more results of Schrödinger operators, we refer the readers to [2]

In Chapter 2, our aim is to develop the regularity theory of Hölder spaces adapted to $\mathcal{L}$ and to study estimates of operators like **fractional integrals** $\mathcal{L}^{-\sigma/2}$, and **fractional powers** $\mathcal{L}^{\sigma/2}$ by $\mathcal{L}$-harmonic extensions. The solution of the boundary value problem

$$\begin{cases} \partial_{tt}u - \mathcal{L}u = 0, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & \text{on } \mathbb{R}^n, \end{cases} \tag{1.4}$$

is given by the action of the $\mathcal{L}$-Poisson semigroup on f:

$$u(x,t) = \mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x).$$

By using formulas (1.1) and (1.2), the powers of $\mathcal{L}$ can be described in terms of u as

$$\mathcal{L}^{-\sigma/2} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \mathcal{P}_s f(x) \, \frac{ds}{s^{1-\sigma}}, \quad x \in \mathbb{R}^n,$$

and

$$\mathcal{L}^{\sigma/2} f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \mathcal{P}_s f(x) - f(x) \, \frac{ds}{s^{1+\sigma}}, \quad x \in \mathbb{R}^n.$$

Therefore, to deal with these spaces and operators, we will adopt the point of view based on $\mathcal{L}$-harmonic extensions.

The main result in Chapter 2 is the following theorem.

Theorem 1.1. *Assume that* $q > n$. *Let* σ *be a positive number,* $0 < \alpha < 1$ *and* $f \in C^{0,\alpha}_{\mathcal{L}}$.

(a) If $0 < \alpha + \sigma < 1$ *then* $\mathcal{L}^{-\sigma/2} f \in C^{0,\alpha+\sigma}_{\mathcal{L}}$ *and*

$$\|\mathcal{L}^{-\sigma/2} f\|_{C^{0,\alpha+\sigma}_{\mathcal{L}}} \leqslant C \, \|f\|_{C^{0,\alpha}_{\mathcal{L}}}.$$

(b) If $\sigma < \alpha$ *then* $\mathcal{L}^{\sigma/2} f \in C^{0,\alpha-\sigma}_{\mathcal{L}}$ *and*

$$\|\mathcal{L}^{\sigma/2} f\|_{C^{0,\alpha-\sigma}_{\mathcal{L}}} \leqslant C \, \|f\|_{C^{0,\alpha}_{\mathcal{L}}}.$$

(c) Let a *be a bounded function on* $[0,\infty)$ *and define*

$$m(\lambda) = \lambda^{1/2} \int_0^\infty e^{-s\lambda^{1/2}} a(s) \, ds, \quad \lambda > 0.$$

Then the multiplier operator of Laplace transform type $m(\mathcal{L})$ *is bounded on* $C^{0,\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$.

In general, to study the regularity properties of fractional operators like $(-\Delta)^{1/2}$, or more generally $(-\Delta)^{\sigma/2}$, $0 < \sigma < 2$, and $(-\Delta)^{-\sigma/2}$, there are (essentially) two possible alternatives. Either describe the operators with a pointwise (integro-differential or integral) formula, or characterize the Hölder classes by some norm estimate of harmonic extensions, see (1.4) but replace $\mathcal{L}$ with Δ, that are in fact Poisson integrals, as described above. The first approach was taken by L. Silvestre in [74] to analyze how $(-\Delta)^{\pm\sigma/2}$ acts on classical C^α spaces. Let us point out that he also needed to handle the classical Riesz transforms $\partial_{x_i}(-\Delta)^{-1/2}$ as operators on C^α. The second one, in the spirit of harmonic extensions, is nowadays classical. Indeed, for bounded functions f it is well known that the harmonic extension $u(x,t)$ satisfies $\|tu_t(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha$ for all $t>0$ if, and only if, $f \in C^\alpha$, $0<\alpha<1$.

In order to prove Theorem 1.1, we will use a characterization of functions f in $C^{0,\alpha}_{\mathcal{L}}$ by properties of size and integrability of $\mathcal{L}$-harmonic extensions to the upper half space. The theory of $\mathrm{BMO}_{\mathcal{L}}$ spaces and Carleson measures developed in [27](also [4] in the Bessel seeting) will be a central tool. The characterization is as following.

Theorem 1.2. *Let* $0<\alpha<1$ *and* f *be a function such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+\alpha+\varepsilon}}\, dx < \infty$$

for any $\varepsilon>0$. *Fix any* $\beta>\alpha$ *and assume that* $s>n$. *The following statements are equivalent:*

(i) $f \in C^{0,\alpha}_{\mathcal{L}}$.

(ii) There exists a constant $c_{1,\beta}$ *such that*

$$\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant c_{1,\beta} t^\alpha.$$

(iii) There exists a constant $c_{2,\beta}$ *such that for all balls* $B=B(x_0,r)$ *in* $\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2\, \frac{dx\,dt}{t}\right)^{1/2} \leqslant c_{2,\beta}\, |B|^{\frac{\alpha}{n}},$$

where $\widehat{B}$ *denotes the tent over* B *defined by* $\{(x,t): x\in B, \text{and } 0<t\leqslant r\}$.

Moreover, the constants $c_{1,\beta}$, $c_{2,\beta}$ *and* $\|f\|_{C^{0,\alpha}_{\mathcal{L}}}$ *above are comparable.*

Our choice of the method turns out to be well suited for our purposes. In this Schrödinger context the pointwise description of the operators as in [74] seems to be technically difficult. In fact, even for one of the most simplest cases (the harmonic oscillator, where $V(x)=|x|^2$) it is already rather involved, see [83]. On the other hand, the characterization of $\mathcal{L}$-Hölder spaces via $\mathcal{L}$-harmonic extensions does not appear to be easily obtained as a repetition of the arguments for classical Hölder spaces given in [79]. Once the above characterization is established the following computation shows how to use it for our regularity purpose:

$$\begin{aligned}
|t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x)| &= C t^\beta \int_0^\infty \partial_t^\beta \mathcal{P}_t(\mathcal{P}_s f)(x)\ \frac{ds}{s^{1-\sigma}} \\
&= C t^\beta \int_0^\infty \partial_w^\beta \mathcal{P}_w f(x)\Big|_{w=t+s}\ \frac{ds}{s^{1-\sigma}} \leqslant C\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^\beta \int_0^\infty (t+s)^{\alpha-\beta} \frac{ds}{s^{1-\sigma}} \\
&= C\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^{\alpha+\sigma} \int_0^\infty (1+r)^{\alpha-\beta}\ \frac{dr}{r^{1-\sigma}} \\
&= C\ B(\sigma, \beta-\alpha-\sigma)\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^{\alpha+\sigma}, \quad \text{for all } x \in \mathbb{R}^n.
\end{aligned}$$

Due to the characterization of the spaces $C^{0,\alpha}_{\mathcal{L}}$ via $BMO^\alpha_{\mathcal{L}}$ spaces, it could be thought that some Harmonic Analysis techniques could be adapted to show the boundedness described in Theorem 1.1. The answer is positive. In fact, in order to get the regularity properties related with the Schrödinger operator $\mathcal{L}$, we develop a T1 criterion in Chapter 3. T1-theorem was firstly stated by G. David and J. Journé [22], for the L^2-boundedness of a Calderón-Zygmund operator T, see also in [38]. T. P. Hytönen [45] and, T. P. Hytönen and L. Weis [48], extended it into operator-valued case. In [6], the authors got a T1 criterion for the boundedness in BMO_H-space, where H is the Hermite operator.

The main point of Chapter 3 is to give a similar T1 criterion for boundedness in $BMO^\alpha_{\mathcal{L}}$ of the so called γ-Schrödinger-Calderón-Zygmund operator T, for the definition of T see Definition 3.6. The advantage of this criterion is that everything reduces to check a certain condition on the function T1.

Theorem 1.3. *Let* T *be a* γ*-Schrödinger-Calderón-Zygmund operator,* $\gamma \geqslant 0$*, with smoothness exponent* δ*, such that* $\alpha+\gamma < \min\{1, \delta\}$ *with* $\alpha > 0$*. Then* T *is bounded from* $BMO^\alpha_{\mathcal{L}}$ *into* $BMO^{\alpha+\gamma}_{\mathcal{L}}$ *if and only if there exists a constant* C *such that*

$$\left(\frac{\rho(x)}{s}\right)^\alpha \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y) - (T1)_B|\ dy \leqslant C,$$

for every ball $B = B(x,s)$*,* $x \in \mathbb{R}^n$ *and* $0 < s \leqslant \frac{1}{2}\rho(x)$*. Here* $(T1)_B = \frac{1}{|B|}\int_B T1(y)\ dy$ *and* $\rho(x)$ *is the critical radii function defined in* (2.3)*.*

For the case $\alpha = 0$, we have a similar result.

The criterion is essentially applied to the whole family of operators associated to $\mathcal{L}$, that is maximal operators associated with the semigroups $e^{-t\mathcal{L}}$ and $e^{-t\mathcal{L}^{1/2}}$ (or more general Poisson operators associated to the extension problem for $\mathcal{L}^\sigma$), the $\mathcal{L}$-square functions, the Laplace transform type multipliers $m(\mathcal{L})$, the $\mathcal{L}$-Riesz transforms and the negative powers $\mathcal{L}^{-\gamma/2}$, $\gamma > 0$.

1.2 Harnack's inequality for fractional operators

One of the tools that plays a crucial role in the regularity theory of PDEs is Harnack's inequality. In 1887, C. A. Harnack [44] formulated and proved the classical Harnack's inequality in the case $n = 2$ as in the following, see [50].

Theorem 1.4. *Let* $f : B_R(x_0) \subset \mathbb{R}^n \to \mathbb{R}$ *be a harmonic function* $(\Delta f = 0)$ *which is either nonnegative or nonpositive. Then the valued of* f *at any point in* $B_r(x_0)(0 < r < R)$ *is bounded from above and below by the quantities*

$$f(x_0)\Big(\frac{R}{R+r}\Big)^{n-2}\frac{R-r}{R+r}, \qquad f(x_0)\Big(\frac{R}{R-r}\Big)^{n-2}\frac{R+r}{R-r}.$$

For details of the development of Harnack's inequality, we refer the reader to the paper by M. Kassmann [50]. Some important classical works in the direction of Harnack's inequality are the papers [36, 37, 54, 58, 66, 71, 89, 90]. Particularly, E. B. Fabes, C. Kenig and R. Serapioni [34] proved a scale invariant Harnack's inequality for the degenerate operators and C. E. Gutiérrez [39] proved Harnack's inequality for degenerate Schrödinger operators. Recently, L. Caffarelli and L. Silvestre [16] considered the Harnack's inequality for the fractional Laplacian. A novel proof of Harnack's inequality for the fractional Laplacian was given by L. Caffarelli and L. Silvestre by using the extension problem in [16, 17]. We want to explain it at here because it is crucial in our proof. Consider $f : \mathbb{R}^n \to \mathbb{R}$ as in the hypotheses of Theorem 1.5 below. Let $u(x, y)$ be the extension of f to the upper half space $\mathbb{R}^{n+1}_+$ obtained by solving

$$\begin{cases} \operatorname{div}(y^{1-2\sigma}\nabla u) = 0, & \text{in } \mathbb{R}^n \times (0, \infty); \\ u(x, 0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

Let $\tilde{u}(x, y) = u(x, |y|)$, $y \in \mathbb{R}$, be the reflection of u to $\mathbb{R}^{n+1}$. The hypothesis $(-\Delta)^\sigma f = 0$ in $\mathcal{O}$ implies that $y^{1-2\sigma}u_y(x, y) \to 0$ as $y \to 0^+$, for all $x \in \mathcal{O}$. This is used to show that $\tilde{u}$ is a weak solution of the degenerate elliptic equation with A_2 weight

$$\operatorname{div}(|y|^{1-2\sigma}\nabla\tilde{u}) = 0, \quad \text{in } \mathcal{O} \times (-R, R) \subset \mathbb{R}^{n+1},$$

for some $R > 0$. Recall that a nonnegative function ω on $\mathbb{R}^n$ is an A_2 weight if

$$\sup_{B \text{ ball}} \left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{-1}\right) < \infty.$$

Then the theory of degenerate elliptic equations by E. Fabes, et all in [34] says that $\tilde{u}$ satisfies an interior Harnack's inequality and it is locally Hölder continuous, thus $f(x) = \tilde{u}(x, 0)$ has the same properties. And then, P. R. Stinga and J. L. Torrea [82] extended Harnack's inequality for the fractional harmonic oscillator.

We proved Harnack's inequalities for the following operators:

- Divergence form elliptic operators $\mathcal{L} = -\operatorname{div}(a(x)\nabla) + V(x)$ with bounded measurable coefficients $a(x)$ and locally bounded nonnegative potentials $V(x)$ defined on bounded domains;

- Ornstein-Uhlenbeck operator $\mathbf{O}_B$ and harmonic oscillator $\mathcal{H}_B$ on $\mathbb{R}^n$;
- Laguerre operators L_α, L_α^φ, L_α^ℓ, L_α^ψ and $L_\alpha^\mathcal{L}$ on $(0,\infty)^n$ with $\alpha \in (-1,\infty)^n$;
- Ultraspherical operators L_λ and l_λ on $(0,\pi)$ with $\lambda > 0$;
- Laplacian on domains $\Omega \subseteq \mathbb{R}^n$;
- Bessel operators Δ_λ and S_λ on $(0,\infty)$ with $\lambda > 0$.

For the full description of the operators, see Sections 4.3, 4.5 and 4.6. In general, all these operators L are nonnegative, self-adjoint and have a dense domain $\mathrm{Dom}(L) \subset L^2(\Omega, d\eta)$, where $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 1$, is an open set and $d\eta$ is some positive measure on Ω.

To get Harnack's inequalities for fractional powers of the operators listed above we push further the Caffarelli–Silvestre ideas. We proceed in two steps. First we use two tools: the extension problem of [82] and Harnack's inequality for degenerate Schrödinger operators of C. E. Gutiérrez [39]. These are enough to get Theorem 4.4, from which the result for divergence form elliptic operators with potentials and some Schrödinger operators from orthogonal expansions is deduced. Secondly, we apply systematically a transference method that permits us to derive the results for other operators involving terms of order one and in non-divergence form. The transference method is inspired in ideas from Harmonic Analysis of orthogonal expansions, where it is used to transfer L^p boundedness of operators, see for example [1, 3, 41]. In that case, the dimension, the underlying measure and the parameters that define the operators play a significant role. But we can obtain our estimates without any restrictions on dimensions or parameters in our case. And we have the following theorem.

Theorem 1.5. *Let L be any of the operators listed above and $0 < \sigma < 1$. Let $\mathcal{O}$ be an open and connected subset of Ω and fix a compact subset $K \subset \mathcal{O}$. There exists a positive constant C, depending only on σ, n, K and the coefficients of L such that*

$$\sup_K f \leqslant C \inf_K f,$$

for all functions $f \in \mathrm{Dom}(L)$, $f \geqslant 0$ in Ω, such that $L^\sigma f = 0$ in $L^2(\mathcal{O}, d\eta)$. Moreover, f is a continuous function in $\mathcal{O}$.

Theorem 1.5 is new, except for three cases: the Laplacian on $\mathbb{R}^n$ ([16, Theorem 5.1] and [52, p. 266]), the Laplacian on the one-dimensional torus [67, Theorem 6.1] and the harmonic oscillator [82, Theorem 1.2]. Harnack's inequality is well-known for divergence form Schrödinger operators with locally bounded potentials [39], see also [23, 37, 89]. For the non-divergence form operators listed above the result can be obtained by using our transference method of Section 4.4. We observe that, instead of the theory of [34], Harnack's inequality for degenerate Schrödinger operators of C. E. Gutiérrez [39] had to be applied.

1.3 Fractional vector-valued Littlewood–Paley–Stein theory for semigroups

In Chapter 5, we shall consider a generalized vector-valued Littlewood-Paley-Stein theory for semigroups generated by the Laplacian. We want to get some equivalent connections between the one-side inequalities of generalized Littlewood-Paley g-function and the geometric properties of the Banach space in which the functions taking values. These equivalences are originally from the Probability Theory.

In Probability Theory, the martingale type and cotype properties of a Banach space $\mathbb{B}$ were introduced in the 1970's by G. Pisier [64, 65] in connection with the convexity and smoothness of $\mathbb{B}$. If $M = (M_n)_{n\in\mathbb{N}}$ is a martingale defined on some probability space and with values in $\mathbb{B}$, the q-square function $S_q M$ is defined by $S_q M = \Big(\sum_{n=1}^{\infty} \|M_n - M_{n-1}\|_{\mathbb{B}}^q\Big)^{\frac{1}{q}}$. The Banach space $\mathbb{B}$ is said to be of martingale cotype q, $2 \leqslant q < \infty$, if for every bounded $L^p_{\mathbb{B}}$-martingale $M = (M_n)_{n\in\mathbb{N}}$ we have $\|S_q M\|_{L^p} \leqslant C_p \sup_n \|M_n\|_{L^p_{\mathbb{B}}}$, for some $1 < p < \infty$. The Banach space $\mathbb{B}$ is said to be of martingale type q, $1 < q \leqslant 2$, when the reverse inequality holds for some $1 < p < \infty$. The martingale type and cotype properties do not depend on $1 < p < \infty$ for which the corresponding inequalities are satisfied. $\mathbb{B}$ is of martingale cotype q if and only if its dual, $\mathbb{B}^*$, is of martingale type $q' = q/(q-1)$.

It is a common fact that results in probability theory have parallels in harmonic analysis. In this line of thought, Q. Xu [92] defined the Lusin cotype and Lusin type properties for a Banach space $\mathbb{B}$ as follows. Let f be a function in $L^1_{\mathbb{B}}(\mathbb{T})$, where $\mathbb{T}$ denotes the one dimensional torus and $L^1_{\mathbb{B}}(\mathbb{T})$ stands for the Bochner-Lebesgue space of strong measurable $\mathbb{B}$-valued functions such that the scalar function $\|f\|_{\mathbb{B}}$ is integrable. Consider the generalized Littlewood–Paley g-function

$$g_q(f)(z) = \left(\int_0^1 (1-r)^q \,\|\partial_r P_r * f(z)\|_{\mathbb{B}}^q \,\frac{dr}{1-r}\right)^{\frac{1}{q}},$$

where $P_r(\theta)$ denotes the Poisson kernel. It is said that $\mathbb{B}$ is of Lusin cotype q, $q \geqslant 2$, if for some $1 < p < \infty$ we have $\|g_q(f)\|_{L^p(\mathbb{T})} \leqslant C_p \|f\|_{L^p_{\mathbb{B}}(\mathbb{T})}$; $\mathbb{B}$ is of Lusin type q, $1 \leqslant q \leqslant 2$, if for some $1 < p < \infty$ we have $\|f\|_{L^p_{\mathbb{B}}(\mathbb{T})} \leqslant C_p \left(\|\hat{f}(0)\|_{\mathbb{B}} + \|g_q(f)\|_{L^p(\mathbb{T})}\right)$.

The Lusin cotype and Lusin type properties do not depend on $p \in (1,\infty)$; see [63, 92]. Moreover, a Banach space $\mathbb{B}$ is of Lusin cotype q (Lusin type q) if and only if $\mathbb{B}$ is of martingale cotype q (martingale type q); see [92, Theorem 3.1].

T. Martínez, J. L. Torrea and Q. Xu [57] extended the results in [92] to subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t\geqslant 0}$ of a general symmetric diffusion Markovian semigroup $\{\mathcal{T}_t\}_{t\geqslant 0}$. Being positive operators, $\mathcal{T}_t$ and $\mathcal{P}_t$ have straightforward norm-preserving extensions to $L^p_{\mathbb{B}}(\Omega)$ for every Banach space $\mathbb{B}$, where $L^p_{\mathbb{B}}(\Omega)$ denotes the usual Bochner–Lebesgue L^p-space of $\mathbb{B}$-valued functions defined on a positive measure space $(\Omega, d\mu)$.

As we said before, C. Segovia and R. L. Wheeden [70] motivated by some characterization of potential spaces on $\mathbb{R}^n$, introduced the "fractional derivative" ∂^α. Observe that for reasonable good functions, $\partial_t^\alpha \mathcal{P}_t f(x) = e^{i\pi\alpha}(-\Delta)^{\alpha/2}\mathcal{P}_t f(x)$. In [70], the authors developed a satisfactory theory of euclidean square functions of Littlewood–Paley type in which the usual derivatives are substituted by these fractional derivatives.

It turns out that the notion of partial derivative considered by C. Segovia and R. L. Wheeden can be used in the case of general subordinated Poisson semigroups defined on a measure space $(\Omega, d\mu)$. Of course, without having a pointwise expression but just an identity in $L^p(\Omega)$. This fractional derivative has a nice behavior for iteration and for spectral decomposition. We will consider the following "fractional generalized Littlewood–Paley g-function"

$$g_\alpha^q(f) = \left(\int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f\|_{\mathbb{B}}^q \frac{dt}{t}\right)^{\frac{1}{q}}, \qquad f \in \bigcup_{1\leqslant p\leqslant\infty} L_{\mathbb{B}}^p(\Omega),\ \alpha > 0. \tag{1.5}$$

Then it is natural to ask whether results already known for classical derivatives are still true for the fractional derivative case. In Chapter 5, we shall be concerned with several characterizations of Lusin type and Lusin cotype of Banach spaces by the boundedness of the Littlewood–Paley g-functions defined by using the fractional derivatives. In fact, we get the following results.

Theorem 1.6. *Given a Banach space $\mathbb{B}$ and $2 \leqslant q < \infty$, the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin cotype q.*

(ii) *For every symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t\geqslant 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t\geqslant 0}$, for every (or, equivalently, for some) $p \in (1,\infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant C such that*

$$\|g_\alpha^q(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L_{\mathbb{B}}^p(\Omega)}, \quad \forall f \in L_{\mathbb{B}}^p(\Omega).$$

With the results in [57], some technical results of the g_α^q-functions in Section 5.3 and an extensively using of Calderón-Zygmund theory, we can give the proof of Theorem 1.6 easily.

On the particular Lebesgue measure space $(\mathbb{R}^n, dx)$, we have the following theorem about the Lusin cotype property of the Banach space and the a.e. convergence.

Theorem 1.7. *Given a Banach space $\mathbb{B}$, $2 \leqslant q < \infty$, the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin cotype q.*

(ii) *For every (or, equivalently, for some) $p \in [1,\infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $g_\alpha^q(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L_{\mathbb{B}}^p(\mathbb{R}^n)$.*

(iii) *For every (or, equivalently, for some)* $p \in [1, \infty)$ *and for every (or, equivalently, for some)* $\alpha > 0$, $S_\alpha^q(f)(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$, *for every* $f \in L_{\mathbb{B}}^p(\mathbb{R}^n)$.

(iv) *For every (or, equivalently, for some)* $p \in [q, \infty)$ *and for every (or, equivalently, for some)* $\alpha > 0$, $g_{\lambda,\alpha}^{q,*}(f)(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$, *for every* $f \in L_{\mathbb{B}}^p(\mathbb{R}^n)$.

The proof of Theorem 1.7 contains some new ideas that can be applied to a huge class of operators. Roughly, the method used in the proof is the following. If an operator T with a Calderón–Zygmund kernel is a.e. pointwise finite ($Tf(x) < \infty$) for any function f in $L^{p_0}(\mathbb{R}^n)$ with some $p_0 \in [1, \infty)$, then T is bounded from $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$.

Let us describe the organization of the next chapters. Chapter 2 is devoted to get some regularity properties of the fractional powers of Schrödinger operators by using harmonic extension technical, which contains the results of paper [55]. In Chapter 3, we collect the results of paper [56] which gives some regularity estimates by a T1-type criterion. Chapter 4, aims to get some interior Harnack's inequalities for some fractional operators, which corresponds to the paper [84]. At last, we collect the results in [88] in Chapter 5, related with the Littlewood-Paley-Stein theory on semigroups.

Chapter 2

Regularity properties of Schrödinger operators via $\mathcal{L}$-harmonic extensions

In this chapter, we shall get some regularity estimates of operators related with Schrödinger operators via $\mathcal{L}$-harmonic extensions. In Section 2.1, we give some basic properties of the Schrödinger operators, especially the critical radii function and some estimates of the heat kernel. We list the regular theorem and some characterization theorems in Section 2.2. With the characterization theorems in Section 2.2, we give a simple proof of the regularity theorem in Section 2.3. The proof of the characterization theorems are given in Section 2.4.

2.1 Some properties of the Schrödinger operators

Let

$$\mathcal{L} := -\Delta + V, \tag{2.1}$$

be the time independent Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 3$, where the nonnegative potential V satisfies a reverse Hölder inequality for some $s > n/2$; that is, there exists a constant $C = C(s, V)$ such that

$$\left(\frac{1}{|B|}\int_B V(y)^s \, dy\right)^{1/s} \leqslant \frac{C}{|B|}\int_B V(y) \, dy, \tag{2.2}$$

for all balls $B \subset \mathbb{R}^n$. Associated to this potential, Z. Shen defines in [72] the critical radii function as

$$\rho(x) := \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y) \, dy \leqslant 1\right\}, \qquad x \in \mathbb{R}^n. \tag{2.3}$$

Lemma 2.1 (See [72, Lemma 1.4]). *There exist* $c > 0$ *and* $k_0 \geqslant 1$ *such that for all* $x, y \in \mathbb{R}^n$

$$c^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant c\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \tag{2.4}$$

In particular, there exists a positive constant $C_1 < 1$ *such that*

$$\textit{if} \quad |x-y| \leqslant \rho(x) \quad \textit{then} \quad C_1\rho(x) < \rho(y) < C_1^{-1}\rho(x).$$

Covering by critical balls. According to [30, Lemma 2.3] there exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ such that if $Q_k := B(x_k, \rho(x_k))$, $k \in \mathbb{N}$, then

(a) $\cup_{k=1}^{\infty} Q_k = \mathbb{R}^n$, and

(b) there exists $N \in \mathbb{N}$ such that $\text{card}\{j \in \mathbb{N} : Q_j^{**} \cap Q_k^{**} \neq \emptyset\} \leqslant N$, for every $k \in \mathbb{N}$.

For a ball B, the notation B^* above means the ball with the same center as B and twice radius.

Let $\{\mathcal{T}_t\}_{t>0}$ be the heat–diffusion semigroup associated to $\mathcal{L}$:

$$\mathcal{T}_t f(x) \equiv e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} k_t(x,y) f(y)\, dy, \qquad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n,\ t > 0. \tag{2.5}$$

In the following arguments we need some well known estimates about the kernel $k_t(x,y)$.

Lemma 2.2 (See [31, 51]). *For every* $N > 0$ *there exists a constant* C_N *such that*

$$0 \leqslant k_t(x,y) \leqslant C_N t^{-n/2} e^{-\frac{|x-y|^2}{5t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad x, y \in \mathbb{R}^n,\ t > 0. \tag{2.6}$$

Let

$$h_t(x) := \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}}, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{2.7}$$

be the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ on $\mathbb{R}^n$.

Lemma 2.3 (See [31, 51]). *For every* $N > 0$ *there exists a constant* C_N *such that*

$$0 \leqslant k_t(x,y) \leqslant C_N t^{-n/2} e^{-\frac{|x-y|^2}{5t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad x, y \in \mathbb{R}^n,\ t > 0.$$

Lemma 2.4 (See [31, Proposition 2.16]). *There exists a nonnegative function* $\omega \in \mathcal{S}$ *such that*

$$|k_t(x,y) - h_t(x-y)| \leqslant \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} \omega_t(x-y), \quad x, y \in \mathbb{R}^n,\ t > 0,$$

where $\omega_t(x-y) := t^{-n/2}\omega\left((x-y)/\sqrt{t}\right)$ *and*

$$\delta_0 := 2 - \frac{n}{q} > 0. \tag{2.8}$$

In fact, going through the proof of [31] we see that $\omega(x) = e^{-|x|^2}$.

Lemma 2.5 (See [32, Proposition 4.11]). *For every* $0 < \delta < \delta_0$, *there exists a constant* $c > 0$ *such that for every* $N > 0$ *there exists a constant* $C > 0$ *such that for* $|y - z| < \sqrt{t}$ *we have*

$$|k_t(x, y) - k_t(x, z)| \leqslant C \left(\frac{|y - z|}{\sqrt{t}}\right)^{\delta} t^{-n/2} \, e^{-c|x-y|^2/t} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Lemma 2.6 (See [31, Proposition 2.17]). *For every* $0 < \delta < \min\{1, \delta_0\}$,

$$|(k_t(x, y) - h_t(x - y)) - (k_t(x, z) - h_t(x - z))| \leqslant C \left(\frac{|y - z|}{\rho(x)}\right)^{\delta} \omega_t(x - y),$$

for all $x, y \in \mathbb{R}^n$ *and* $t > 0$, *with* $|y - z| < C\rho(y)$ *and* $|y - z| < \frac{1}{4}|x - y|$.

The Poisson semigroup associated to $\mathcal{L}$ is obtained from the heat semigroup (2.5) through Bochner's subordination formula, see [78]:

$$\mathcal{P}_t f(x) \equiv e^{-t\sqrt{\mathcal{L}}} f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} \, \mathcal{T}_{t^2/(4u)} f(x) \, du = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t^2/(4u)}}{u^{3/2}} \mathcal{T}_u f \, du, \qquad (2.9)$$

for any $x \in \mathbb{R}^n$, $t > 0$. It follows that

$$\mathcal{P}_t f(x) = \int_{\mathbb{R}^n} \mathcal{P}_t(x, y) f(y) \, dy, \quad x \in \mathbb{R}^n, t > 0,$$

where

$$\mathcal{P}_t(x, y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} \, k_{t^2/(4u)}(x, y) \, du = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t^2/(4u)}}{u^{3/2}} \, k_u(x, y) \, du. \qquad (2.10)$$

2.2 Regularity estimates for Schrödinger operator

The concept of Hölder spaces associated with $\mathcal{L}$ is based on the *critical radii function* $\rho(x)$ defined by Z. Shen in [72], see (2.3). In our case, the function $\rho(x)$ is always considered when the potential V satisfies a reverse Hölder inequality for some $q > n/2$.

Definition 2.7 (Hölder spaces for $\mathcal{L}$). A continuous function f defined on $\mathbb{R}^n$ belongs to the space $C^{0,\alpha}_{\mathcal{L}}$, $0 < \alpha \leqslant 1$, if there exists a constant C such that

$$|f(x) - f(y)| \leqslant C\,|x - y|^{\alpha} \quad \text{and} \quad |f(x)| \leqslant C\rho(x)^{\alpha},$$

for all $x, y \in \mathbb{R}^n$. If we define

$$[f]_{C^{\alpha}} = \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \quad \text{and} \quad [f]_{M^{\alpha}_{\mathcal{L}}} = \sup_{x \in \mathbb{R}^n} \left|\rho(x)^{-\alpha} f(x)\right|,$$

then the norm in the spaces $C^{0,\alpha}_{\mathcal{L}}$ is $\|f\|_{C^{0,\alpha}_{\mathcal{L}}} = [f]_{C^{\alpha}} + [f]_{M^{\alpha}_{\mathcal{L}}}$.

Let us present the regularity estimates of fractional integrals, fractional powers of $\mathcal{L}$ and the Laplace transform multiplier operator associated with $\mathcal{L}$.

Theorem 2.8. *Assume that* $q > n$. *Let* σ *be a positive number,* $0 < \alpha < 1$ *and* $f \in C^{0,\alpha}_{\mathcal{L}}$.

(a) If $0 < \alpha + \sigma < 1$ *then* $\mathcal{L}^{-\sigma/2} f \in C^{0,\alpha+\sigma}_{\mathcal{L}}$ *and*

$$\|\mathcal{L}^{-\sigma/2} f\|_{C^{0,\alpha+\sigma}_{\mathcal{L}}} \leqslant C \, \|f\|_{C^{0,\alpha}_{\mathcal{L}}} .$$

(b) If $\sigma < \alpha$ *then* $\mathcal{L}^{\sigma/2} f \in C^{0,\alpha-\sigma}_{\mathcal{L}}$ *and*

$$\|\mathcal{L}^{\sigma/2} f\|_{C^{0,\alpha-\sigma}_{\mathcal{L}}} \leqslant C \, \|f\|_{C^{0,\alpha}_{\mathcal{L}}} .$$

(c) Let a *be a bounded function on* $[0, \infty)$ *and define*

$$m(\lambda) = \lambda^{1/2} \int_0^\infty e^{-s\lambda^{1/2}} a(s) \; ds, \quad \lambda > 0.$$

Then the multiplier operator of Laplace transform type $m(\mathcal{L})$ *is bounded on* $C^{0,\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$.

In order to prove Theorem 2.8, we will use a characterization of functions f in $C^{0,\alpha}_{\mathcal{L}}$ by properties of size and integrability of $\mathcal{L}$-harmonic extensions to the upper half space. The theory of $BMO_{\mathcal{L}}$ spaces and Carleson measures developed in [27] will be a central tool. The characterization is as following.

Theorem 2.9. *Let* $0 < \alpha < 1$ *and* f *be a function such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1 + |x|)^{n+\alpha+\varepsilon}} \; dx < \infty$$

for any $\varepsilon > 0$. *Fix any* $\beta > \alpha$ *and assume that* $q > n$. *The following statements are equivalent:*

(i) $f \in C^{0,\alpha}_{\mathcal{L}}$.

(ii) There exists a constant $c_{1,\beta}$ *such that*

$$\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant c_{1,\beta} t^\alpha.$$

(iii) There exists a constant $c_{2,\beta}$ *such that for all balls* $B = B(x_0, r)$ *in* $\mathbb{R}^n$,

$$\left(\frac{1}{|B|} \int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \; \frac{dx \; dt}{t} \right)^{1/2} \leqslant c_{2,\beta} \, |B|^{\frac{\alpha}{n}} ,$$

where $\widehat{B}$ *denotes the tent over* B *defined by* $\{(x, t) : x \in B, \text{and } 0 < t \leqslant r\}$.

Moreover, the constants $c_{1,\beta}$, $c_{2,\beta}$ *and* $\|f\|_{C^{0,\alpha}_{\mathcal{L}}}$ *above are comparable.*

Note that the conclusion of Theorem 2.9 above is not valid in the cases $\alpha = 1$ or $\alpha = 0$. In fact, we have the following results for $\alpha = 1$:

Theorem 2.10 (Case $\alpha = 1$). *Assume that* $q > n$.

(I) If $f \in C^{0,1}_{\mathcal{L}}$ *then for any* $\beta > 1$ *there exists a constant* c_β *such that*

$$\left(\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \,\frac{dx\,dt}{t}\right)^{1/2} \leqslant c_\beta\,|B|^{\frac{1}{n}},$$

for all balls B. *The converse statement is not true.*

(II) Let $\mathcal{L}_\mu = -\Delta + \mu$, *for* $\mu > 0$. *There exists a function* f *such that for any* $\beta > 1$ *there exists a constant* c_β *that verifies* $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant c_\beta t$, *for all* $t > 0$, *but* f *does not belong to the space* $C^{0,1}_{\mathcal{L}_\mu}$.

It has no sense to take $\alpha = 0$ as a Hölder exponent. By the Campanato-type description of Proposition 2.27 we see that the natural replacement in this situation is the space $BMO_{\mathcal{L}}$.

Theorem 2.11 (Case $\alpha = 0$). *Assume that* $q > n$.

(A) A function f *is in* $BMO_{\mathcal{L}}$ *if and only if for* f *being a function such that*

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+\varepsilon}}\,dx < \infty$$

for any $\varepsilon > 0$, *and for all* $\beta > 0$ *there exists a constant* c_β *such that*

$$\left(\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \,\frac{dx\,dt}{t}\right)^{1/2} \leqslant c_\beta,$$

for all balls B.

(B) Let $\mathcal{L}_\mu = -\Delta + \mu$, *for* $\mu > 0$. *There exists a function* $f \in BMO_{\mathcal{L}_\mu}$ *such that*

$$\sup_{t>0} |t^\beta \partial_t^\beta \mathcal{P}_t f(0)| = \infty,$$

for some $\beta > 0$.

2.3 Proof of regularity theorem

In this section we shall provide the proof of Theorem 2.8. With Theorem 2.9, we can prove the regularity estimates, Theorem 2.8, easily. Let us start with a technical lemma which will be used several times later.

Lemma 2.12. *Let* $0 < \gamma < 1$, *and* g *be a continuous function such that* $|g(x)| \leqslant C\rho(x)^\gamma$, *where* ρ *is the critical radii function defined in* (2.3). *Then*

(i) For any $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \frac{|g(x)|}{(1+|x|)^{n+\gamma+\varepsilon}}\,dx < \infty.$$

(ii) For any $\beta > \gamma$ *and any* $N > 0$ *there exists a constant* $C_{\beta,N,g}$ *such that*

$$|s^\beta \partial_s^\beta \mathcal{P}_s g(x)| \leqslant C_{\beta,N,g}\left(\frac{\rho(x)}{s}\right)^N (\rho(x)^\gamma + s^\gamma), \quad x \in \mathbb{R}^n,\ s > 0.$$

(iii) For any $N > 0$ *there exists a constant* $C_{N,g}$ *such that*

$$|\mathcal{P}_s g(x)| \leqslant C_{N,g}\left(\frac{\rho(x)}{s}\right)^N (\rho(x)^\gamma + s^\gamma), \quad x \in \mathbb{R}^n,\ s > 0.$$

Proof. Let us begin with *(i)*. We write

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \frac{|g(x)|}{(1+|x|)^{n+\gamma+\varepsilon}}\,dx \\ &= \int_{|x|<2\rho(0)} \frac{|g(x)|}{(1+|x|)^{n+\gamma+\varepsilon}}\,dx + \sum_{j=1}^{\infty} \int_{2^j\rho(0)\leqslant|x|<2^{j+1}\rho(0)} \frac{|g(x)|}{(1+|x|)^{n+\gamma+\varepsilon}}\,dx. \end{aligned}$$

To estimate the integrals we apply the hypothesis and some properties of the function ρ contained in Lemma 2.1. The inequality $|x| = |x-0| < 2^{j+1}\rho(0)$, $j \geqslant 0$, and Lemma 2.1 give us

$$\rho(x) \leqslant c\rho(0)\left(1 + \frac{|x-0|}{\rho(0)}\right)^{\frac{k_0}{k_0+1}} \leqslant c\rho(0)\left(1+2^{j+1}\right)^{\frac{k_0}{k_0+1}} \leqslant C\rho(0)2^j.$$

Therefore,

$$I \leqslant C\rho(0)^{\gamma+n} + C\sum_{j=1}^{\infty} \frac{\left(\rho(0)2^j\right)^{\gamma+n}}{\left(1+2^j\rho(0)\right)^{n+\gamma+\varepsilon}} \leqslant C + C\sum_{j=1}^{\infty} 2^{-j\varepsilon} < \infty.$$

For *(ii)*, recall that *(i)* implies that $\mathcal{P}_t g(x)$ is well defined. By Proposition 2.15*(b)* and Lemma 2.1 below, for some constant $C = C_{\beta,N,g}$, we have

$$\begin{aligned} |s^\beta \partial_s^\beta \mathcal{P}_s g(x)| &\leqslant C\int_{\mathbb{R}^n} \frac{s^\beta \rho(x)^N}{(s+|x-y|)^{n+\beta+N}}\,\rho(y)^\gamma\,dy \\ &\leqslant C\int_{\mathbb{R}^n} \frac{s^\beta \rho(x)^N}{(s+|x-y|)^{n+\beta+N}}\,\rho(x)^\gamma\left(1+\frac{|x-y|}{\rho(x)}\right)^\gamma dy \\ &\leqslant C\rho(x)^{\gamma+N}\int_{\mathbb{R}^n} \frac{s^\beta}{(s+|x-y|)^{n+\beta+N}}\,dy \\ &\quad + C\rho(x)^N\int_{\mathbb{R}^n} \frac{s^\beta}{(s+|x-y|)^{n+\beta+N-\gamma}}\,dy \\ &= C\rho(x)^{\gamma+N}s^{-N} + C\rho(x)^N s^{-N+\gamma}. \end{aligned}$$

The third statement *(iii)* can be proved in the same way as *(ii)*. □

Proof of Theorem 2.8. We start with the proof of part *(a)*. For $f \in C^{0,\alpha}_{\mathcal{L}}$, we have

$$\mathcal{L}^{-\sigma/2} f(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \mathcal{P}_s f(x) \, \frac{ds}{s^{1-\sigma}}, \quad x \in \mathbb{R}^n. \tag{2.11}$$

By Lemma 2.12*(iii)*, since $|f(x)| \leqslant C\rho(x)^\alpha$,

$$\begin{aligned} \int_0^\infty |\mathcal{P}_s f(x)| \frac{ds}{s^{1-\sigma}} &\leqslant C \int_0^{\rho(x)} \frac{\rho(x)^{\alpha+N_1}}{s^{N_1}} + \frac{\rho(x)^{N_1}}{s^{N_1-\alpha}} \, \frac{ds}{s^{1-\sigma}} \\ &\quad + C \int_{\rho(x)}^\infty \frac{\rho(x)^{\alpha+N_2}}{s^{N_2}} + \frac{\rho(x)^{N_2}}{s^{N_2-\alpha}} \, \frac{ds}{s^{1-\sigma}} \\ &\leqslant C_{N_1,N_2,\alpha,f} \cdot \rho(x)^{\alpha+\sigma}, \end{aligned} \tag{2.12}$$

by choosing $0 < N_1 < \sigma$ and $N_2 > \alpha + \sigma$. Hence $\mathcal{L}^{-\sigma/2} f(x)$ is well defined. Moreover, it satisfies the required growth $|\mathcal{L}^{-\sigma/2} f(x)| \leqslant C\rho(x)^{\alpha+\sigma}$. So Lemma 2.12 applies to it. Fix any $\beta > \alpha + \sigma$. To obtain the conclusion we apply Theorem 2.9. That is, it is enough to prove that $\|t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)\|_{L^\infty(\mathbb{R}^n)} \leqslant C \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^{\alpha+\sigma}$. By using formula (2.11) and Lemma 2.12 together with Fubini's theorem, we have

$$t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x) = C t^\beta \int_0^\infty \partial_t^\beta \mathcal{P}_t(\mathcal{P}_s f)(x) \, \frac{ds}{s^{1-\sigma}} = C t^\beta \int_0^\infty \partial_w^\beta \mathcal{P}_w f(x) \Big|_{w=t+s} \frac{ds}{s^{1-\sigma}}.$$

Indeed, by (2.12) and Lemma 2.12,

$$|t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x)| \leqslant C_{x,t,f} \left(\frac{\rho(x)^{\alpha+N_0+N_1}}{t^{N_1}} + \frac{\rho(x)^{N_1}}{t^{N_1-\alpha-N_0}} \right) < \infty,$$

for any $x \in \mathbb{R}^n$ and $t > 0$. So Fubini's Theorem can be applied. Since $\beta > \alpha + \sigma$ we can use Theorem 2.9 to get

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t(\mathcal{L}^{-\sigma/2} f)(x)| &\leqslant C \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^\beta \int_0^\infty (t+s)^{\alpha-\beta} \frac{ds}{s^{1-\sigma}} \\ &= C \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^{\alpha+\sigma} \int_0^\infty (1+r)^{\alpha-\beta} \, \frac{dr}{r^{1-\sigma}} \\ &= C \, B(\sigma, \beta-\alpha-\sigma) \|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^{\alpha+\sigma}, \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

This concludes the proof of *(a)*.

To prove part *(b)*, fix any $\beta > \alpha$. Since $0 < \sigma < \alpha < 1$ we can write

$$\begin{aligned} \mathcal{L}^{\sigma/2} f(x) &= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \mathcal{P}_s f(x) - f(x) \, \frac{ds}{s^{1+\sigma}} \\ &= I(x,t) + II(x,t), \end{aligned} \tag{2.13}$$

where

$$I(x,t)=\frac{1}{\Gamma(-\sigma)}\int_0^t \mathcal{P}_sf(x)-f(x)\ \frac{ds}{s^{1+\sigma}}$$

and

$$II(x,t)=\frac{1}{\Gamma(-\sigma)}\int_t^\infty \mathcal{P}_sf(x)-f(x)\ \frac{ds}{s^{1+\sigma}}.$$

To apply Theorem 2.9 we show first that $\left|\mathcal{L}^{\sigma/2}f(x)\right|\leqslant C\rho(x)^{\alpha-\sigma}$. Indeed, since $f\in C^{0,\alpha}_{\mathcal{L}}$,

$$\begin{aligned}|I(x,\rho(x))|&\leqslant\int_0^{\rho(x)}|\mathcal{P}_sf(x)-f(x)|\ \frac{ds}{s^{1+\sigma}}=\int_0^{\rho(x)}\left|\int_0^s\partial_r\mathcal{P}_rf(x)\ dr\right|\ \frac{ds}{s^{1+\sigma}}\\&\leqslant C\int_0^{\rho(x)}\int_0^s r^{\alpha-1}\ dr\ \frac{ds}{s^{1+\sigma}}=C\rho(x)^{\alpha-\sigma}.\end{aligned}$$

Taking $N=\alpha$ in Lemma 2.12*(iii)* and using the growth of f we also have

$$\begin{aligned}|II(x,\rho(x))|&\leqslant\int_{\rho(x)}^\infty|\mathcal{P}_sf(x)-f(x)|\ \frac{ds}{s^{1+\sigma}}\leqslant\int_{\rho(x)}^\infty|\mathcal{P}_sf(x)|+|f(x)|\ \frac{ds}{s^{1+\sigma}}\\&\leqslant C\int_{\rho(x)}^\infty\frac{\rho(x)^{2\alpha}}{s^\alpha}+\rho(x)^\alpha\ \frac{ds}{s^{1+\sigma}}=C\rho(x)^{\alpha-\sigma}.\end{aligned}$$

The computations above also say that (2.13) is well defined. By linearity, it is enough to analyze $t^\beta\partial_t^\beta\mathcal{P}_tI(x,t)$ and $t^\beta\partial_t^\beta\mathcal{P}_tII(x,t)$ separately. Fubini's theorem implies that

$$t^\beta\partial_t^\beta\mathcal{P}_tI(x,t)=\frac{t^\beta}{\Gamma(-\sigma)}\int_0^t\int_0^s\partial_w^{\beta+1}\mathcal{P}_wf(x)\big|_{w=t+r}\ dr\ \frac{ds}{s^{1+\sigma}}.$$

Apply Theorem 2.9 and the fact that $\beta>\alpha$ to obtain

$$\begin{aligned}|t^\beta\partial_t^\beta\mathcal{P}_tI(x,t)|&\leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}t^\beta\int_0^t\int_0^s(t+r)^{\alpha-\beta-1}\ dr\ \frac{ds}{s^{1+\sigma}}\\&=C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}t^\alpha\int_0^t\int_0^{s/t}(1+u)^{\alpha-\beta-1}\ du\ \frac{ds}{s^{1+\sigma}}\\&\leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}t^\alpha\int_0^t\frac{s}{t}\ \frac{ds}{s^{1+\sigma}}=C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}t^{\alpha-\sigma}.\end{aligned}\tag{2.14}$$

Theorem 2.9 and Fubini's theorem give us

$$\begin{aligned}|t^\beta\partial_t^\beta\mathcal{P}_tII(x,t)|&\leqslant C\int_t^\infty\left|t^\beta\partial_w^\beta\mathcal{P}_wf(x)\big|_{w=t+s}\right|+\left|t^\beta\partial_t^\beta\mathcal{P}_tf(x)\right|\ \frac{ds}{s^{1+\sigma}}\\&\leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\int_t^\infty t^\beta(t+s)^{\alpha-\beta}+t^\alpha\ \frac{ds}{s^{1+\sigma}}=C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}t^{\alpha-\sigma}.\end{aligned}\tag{2.15}$$

Collecting estimates (2.14) and (2.15) we get the conclusion of *(b)*.

Let us finally check *(c)*. Fix any $\beta>\alpha$. As

$$m(\lambda)=-\int_0^\infty\partial_s\big(e^{-s\lambda^{1/2}}\big)\ a(s)\ ds,$$

we have

$$m(\mathcal{L})f(x) = -\int_0^\infty \partial_s \mathcal{P}_s f(x)\ a(s)\ ds.$$

As a is a bounded function and $f \in C^{0,\alpha}_{\mathcal{L}}$,

$$\int_0^{\rho(x)} |\partial_s \mathcal{P}_s f(x)\ a(s)|\ ds \leqslant C \int_0^{\rho(x)} s^{\alpha-1}\ ds = C\rho(x)^\alpha.$$

Moreover, by Lemma 2.12*(ii)* with $\beta = 1$ and some $N > \alpha$ at there,

$$\int_{\rho(x)}^\infty |\partial_s \mathcal{P}_s f(x)\ a(s)|\ ds \leqslant C \int_{\rho(x)}^\infty \left(\frac{\rho(x)}{s}\right)^N (\rho(x)^\alpha + s^\alpha)\ \frac{ds}{s} = C\rho(x)^\alpha.$$

Therefore, $|m(\mathcal{L})f(x)| \leqslant C\rho(x)^\alpha$, so by Lemma 2.12*(i)* the hypothesis of Theorem 2.9 holds for $m(\mathcal{L})f$. By Theorem 2.9 and Fubini's theorem we have

$$\begin{aligned}
|t^\beta \partial_t^\beta \mathcal{P}_t\big(m(\mathcal{L})f\big)(x)| &= t^\beta \left| \int_0^\infty \partial_w^{\beta+1} \mathcal{P}_w f(x)\big|_{w=t+s}\ a(s)\ ds \right| \\
&\leqslant C\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\beta \int_0^\infty (t+s)^{\alpha-(\beta+1)}\ ds \\
&= C\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha \int_0^\infty (1+r)^{\alpha-(\beta+1)}\ dr = C\, \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha.
\end{aligned}$$

This completes the proof of *(c)*. □

2.4 Proofs of characterizations of f in Hölder spaces for Schrödinger operator

In this section, we aim to give the proofs of Theorems 2.9–2.11 which characterize the functions in $C^{0,\alpha}_{\mathcal{L}}$. In order to do this, we need do some preparations. We will give some estimates for the kernel that we need first. Secondly, we define the $BMO^\alpha_{\mathcal{L}}$ spaces, give its relation with $C^{0,\alpha}_{\mathcal{L}}$ and prove a duality result $H^p_{\mathcal{L}} - BMO^\alpha_{\mathcal{L}}$. With these results, we give the proof of Theorems 2.9–2.11 at last.

2.4.1 Estimates on the kernel

We define the following kernel that will be useful to obtain estimates for the Poisson kernel in the sequel. Let

$$Q_t(x,y) := t^2\, \frac{\partial k_s(x,y)}{\partial s}\bigg|_{s=t^2}, \quad x, y \in \mathbb{R}^n,\ t > 0, \tag{2.16}$$

where k_s is the heat kernel related with $\mathcal{L}$ as in (2.5).

Lemma 2.13 (See [27, Proposition 4]). *Let* δ_0 *be as in* (2.8). *There exists a constant* c *such that for every* N *there is a constant* C_N *such that*

(a) $|Q_t(x,y)| \leqslant C_N t^{-n} e^{-c\frac{|x-y|^2}{t^2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}$;

(b) $|Q_t(x+h,y) - Q_t(x,y)| \leqslant C_N \left(\frac{|h|}{t}\right)^{\delta_0} t^{-n} e^{-c\frac{|x-y|^2}{t^2}} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N}$, *for all* $|h| \leqslant t$;

(c) $\left|\int_{\mathbb{R}^n} Q_t(x,y)\, dy\right| \leqslant C_N \frac{(t/\rho(x))^{\delta_0}}{(1+t/\rho(x))^N}$.

Remark 2.14. Let $0 < \delta' \leqslant \delta_0$. Then we deduce from Lemma 2.13*(c)* that for any $N > 0$ there exists a constant C_N such that

$$\left|\int_{\mathbb{R}^n} Q_t(x,y)\, dy\right| \leqslant C_N \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}.$$

Indeed, if $t/\rho(x) < 1$,

$$\frac{(t/\rho(x))^{\delta_0}}{(1+t/\rho(x))^N} = \frac{(t/\rho(x))^{\delta'}(t/\rho(x))^{\delta_0-\delta'}}{(1+t/\rho(x))^N} \leqslant \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}.$$

Suppose now that $t/\rho(x) \geqslant 1$. Since N can be arbitrary, we choose it such that $M := N + \delta' - \delta_0 > 0$. Then

$$\frac{(t/\rho(x))^{\delta_0}}{(1+t/\rho(x))^N} \leqslant \frac{(t/\rho(x))^{\delta'}(t/\rho(x))^{\delta_0-\delta'}}{(1+t/\rho(x))^{N+\delta'-\delta_0}(t/\rho(x))^{\delta_0-\delta'}} = \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^M}.$$

To conclude note that $M > 0$ can also be arbitrary.

We will denote the classical Poisson kernel in $\mathbb{R}^{n+1}_+$ by $P_t(x)$,

$$P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}, \tag{2.17}$$

and $P_t f(x) = P_t * f(x)$.

Let us now compute the fractional derivatives of the Poisson kernel. The formula will involve the kernel $Q_t(x,y)$ of (2.16) and the Hermite polynomials $H_m(r)$ defined, for $m \in \mathbb{N}_0$ and $r \in \mathbb{R}$, as

$$H_m(r) = (-1)^m e^{r^2} \frac{d^m}{dr^m}(e^{-r^2}).$$

Observe first that, from the first identity in (3.15) and the definition of Q_t in (2.16),

$$\partial_t \mathcal{P}_t(x,y) = \frac{2}{t\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, Q_{t/(2\sqrt{u})}(x,y)\, du = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2/(4\nu^2)} Q_\nu(x,y)\, \frac{d\nu}{\nu^2}.$$

Hence, for each $m \geqslant 1$,

$$\begin{aligned}\partial_t^m \mathcal{P}_t(x,y) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \partial_t^{m-1}(e^{-\frac{t^2}{4\nu^2}}) Q_\nu(x,y) \, \frac{d\nu}{\nu^2} \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{d^{m-1}}{dr^{m-1}}(e^{-r^2})\Big|_{r=\frac{t}{2\nu}} \frac{1}{(2\nu)^{m-1}} \, Q_\nu(x,y) \, \frac{d\nu}{\nu^2} \\ &= \frac{2(-1)^m}{\sqrt{\pi}} \int_0^\infty H_{m-1}\left(\frac{t}{2\nu}\right) e^{-\frac{t^2}{4\nu^2}} \frac{1}{(2\nu)^{m-1}} \, Q_\nu(x,y) \, \frac{d\nu}{\nu^2}.\end{aligned}$$

With this we can write the derivatives $\partial_t^\beta \mathcal{P}_t(x,y)$, $\beta > 0$, as follows. For $m = [\beta] + 1$,

$$\begin{aligned}\partial_t^\beta \mathcal{P}_t(x,y) &= \frac{e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(x,y) s^{m-\beta} \, \frac{ds}{s} \\ &= \frac{2(-1)^m e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)\sqrt{\pi}} \int_0^\infty \int_0^\infty H_{m-1}\left(\frac{t+s}{2\nu}\right) e^{-\frac{(t+s)^2}{4\nu^2}} \frac{1}{(2\nu)^{m-1}} \, Q_\nu(x,y) \, \frac{d\nu}{\nu^2} s^{m-\beta} \, \frac{ds}{s} \qquad (2.18) \\ &= \frac{2(-1)^m e^{-i\pi(m-\beta)}}{\Gamma(m-\beta)\sqrt{\pi}} \int_0^\infty \left[\int_0^\infty H_{m-1}\left(\frac{t+s}{2\nu}\right) e^{-\frac{(t+s)^2}{4\nu^2}} s^{m-\beta} \, \frac{ds}{s}\right] \frac{1}{(2\nu)^{m-1}} \, Q_\nu(x,y) \, \frac{d\nu}{\nu^2}.\end{aligned}$$

The next proposition collects all the estimates for the Poisson kernel that we need.

Proposition 2.15. *Let $\beta > 0$. For any $0 < \delta' \leqslant \delta_0$ with $0 < \delta' < \beta$, and $N > 0$ there exists a constant $C = C_{N,\beta,\delta'}$ such that*

(a) $|\mathcal{P}_t(x,y)| \leqslant C \dfrac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} \left(1 + \dfrac{(|x-y|^2+t^2)^{1/2}}{\rho(x)} + \dfrac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N}$;

(b) $|t^\beta \partial_t^\beta \mathcal{P}_t(x,y)| \leqslant C \dfrac{t^\beta}{(|x-y|^2+t^2)^{\frac{n+\beta}{2}}} \left(1 + \dfrac{(|x-y|^2+t^2)^{1/2}}{\rho(x)} + \dfrac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N}$;

(c) For all $|h| \leqslant t$,

$$\begin{aligned}&|t^\beta \partial_t^\beta \mathcal{P}_t(x+h,y) - t^\beta \partial_t^\beta \mathcal{P}(x,y)| \\ &\leqslant C \left(\frac{|h|}{t}\right)^{\delta'} \frac{t^\beta}{(|x-y|^2+t^2)^{\frac{n+\beta}{2}}} \left(1 + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)} + \frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N};\end{aligned}$$

(d) $\left|\displaystyle\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,y) \, dy\right| \leqslant C \dfrac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}.$

Proof. Let us prove *(a)* first. Observe that, by the second identity of (3.15) and Lemma 2.3,

$$
\begin{aligned}
|\mathcal{P}_t(x,y)| &\leqslant Ct\int_0^\infty u^{-\frac{n+3}{2}}\, e^{-\frac{|x-y|^2+t^2}{cu}}\left(1+\frac{\sqrt{u}}{\rho(x)}+\frac{\sqrt{u}}{\rho(y)}\right)^{-N}du\\
&= Ct\int_0^{|x-y|^2+t^2} u^{-\frac{n+3}{2}}\, e^{-\frac{|x-y|^2+t^2}{cu}}\left(1+\frac{\sqrt{u}}{\rho(x)}+\frac{\sqrt{u}}{\rho(y)}\right)^{-N}du\\
&\quad+ Ct\int_{|x-y|^2+t^2}^\infty u^{-\frac{n+3}{2}}\, e^{-\frac{|x-y|^2+t^2}{cu}}\left(1+\frac{\sqrt{u}}{\rho(x)}+\frac{\sqrt{u}}{\rho(y)}\right)^{-N}du\\
&=: I+II.
\end{aligned}
$$

For I apply the change of variables $r=(|x-y|^2+t^2)/u$ to get

$$
I\leqslant \frac{Ct}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\left(1+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)}+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N}\int_1^\infty r^{\frac{n+N-1}{2}}e^{-cr}\,dr.
$$

For II,

$$
\begin{aligned}
II &\leqslant Ct\left(1+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)}+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N}\int_{|x-y|^2+t^2}^\infty u^{-\frac{n+3}{2}}\,du\\
&= C\frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}}\left(1+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(x)}+\frac{(|x-y|^2+t^2)^{1/2}}{\rho(y)}\right)^{-N}.
\end{aligned}
$$

Pasting together these last two estimates we conclude the proof of *(a)*.

To prove *(b)*, note that we can estimate the integral in brackets in (2.18) as follows:

$$
\begin{aligned}
&\left|\int_0^\infty H_{m-1}\left(\frac{t+s}{2v}\right)e^{-\frac{(t+s)^2}{4v^2}}s^{m-\beta}\,\frac{ds}{s}\right|\\
&\leqslant C_m\int_0^\infty e^{-c\frac{(t+s)^2}{4v^2}}s^{m-\beta}\,\frac{ds}{s}\leqslant C_m e^{-c\frac{t^2}{v^2}}\int_0^\infty e^{-c\frac{s^2}{v^2}}s^{m-\beta}\,\frac{ds}{s} \qquad (2.19)\\
&= C_m e^{-c\frac{t^2}{v^2}}v^{m-\beta}\int_0^\infty e^{-cr^2}r^{m-\beta}\,\frac{dr}{r}=C_{m,\beta}\,e^{-c\frac{t^2}{v^2}}v^{m-\beta}.
\end{aligned}
$$

Using identity (2.18), this last inequality and Lemma 2.13*(a)*,

$$
\begin{aligned}
|\partial_t^\beta\mathcal{P}_t(x,y)| &\leqslant C\int_0^\infty e^{-c\frac{t^2}{v^2}}v^{-\beta}\,|Q_v(x,y)|\,\frac{dv}{v}\\
&\leqslant C\int_0^\infty v^{-n-\beta}e^{-c\frac{|x-y|^2+t^2}{v^2}}\left(1+\frac{v}{\rho(x)}+\frac{v}{\rho(y)}\right)^{-N}\frac{dv}{v}\\
&= C\int_0^{(|x-y|^2+t^2)^{1/2}} v^{-n-\beta}e^{-c\frac{|x-y|^2+t^2}{v^2}}\left(1+\frac{v}{\rho(x)}+\frac{v}{\rho(y)}\right)^{-N}\frac{dv}{v}\\
&\quad+ C\int_{(|x-y|^2+t^2)^{1/2}}^\infty v^{-n-\beta}e^{-c\frac{|x-y|^2+t^2}{v^2}}\left(1+\frac{v}{\rho(x)}+\frac{v}{\rho(y)}\right)^{-N}\frac{dv}{v}\\
&=: I'+II'.
\end{aligned}
$$

Now I' and II' can be treated as I and II above, respectively. Hence *(b)* is proved.

The proof of part *(c)* follows parallel lines as we have just done for *(b)* by using identity (2.18), estimate (2.19) and Lemma 2.13*(b)*.

For *(d)*, let $0 < \delta' \leqslant \delta_0$ with $0 < \delta' < \beta$. By Remark 2.14 and the change of variables $w = t/v$,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,y)\,dy\right| &\leqslant Ct^\beta \int_0^\infty e^{-c\frac{t^2}{v^2}} v^{-\beta} \left|\int_{\mathbb{R}^n} Q_v(x,y)\,dy\right| \frac{dv}{v} \\ &\leqslant Ct^\beta \int_0^\infty e^{-c\frac{t^2}{v^2}} v^{-\beta} \frac{(v/\rho(x))^{\delta'}}{(1+v/\rho(x))^N} \frac{dv}{v} \\ &= C(t/\rho(x))^{\delta'} \int_0^\infty e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w}.\end{aligned}$$

On one hand,

$$\begin{aligned}\int_{t/\rho(x)}^\infty e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} &\leqslant e^{-c\frac{t^2}{2\rho(x)^2}} \int_0^\infty e^{-c\frac{w^2}{2}} w^{\beta-\delta'} \frac{dw}{w} \\ &\leqslant Ce^{-c\frac{t^2}{\rho(x)^2}} \leqslant \frac{C}{(1+t/\rho(x))^N}.\end{aligned}$$

On the other hand, we consider two cases. If $t/\rho(x) \leqslant 1$ then

$$\int_0^{t/\rho(x)} e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} \leqslant \int_0^1 w^{\beta-\delta'} \frac{dw}{w} \leqslant \frac{C}{(1+t/\rho(x))^N}.$$

If $t/\rho(x) > 1$ then

$$\int_0^{t/\rho(x)} e^{-cw^2} \frac{w^{\beta-\delta'}}{(1+t/(w\rho(x)))^N} \frac{dw}{w} \leqslant \frac{1}{(t/\rho(x))^N} \int_0^\infty e^{-cw^2} w^{\beta-\delta'+N} \frac{dw}{w} \leqslant \frac{C}{(1+t/\rho(x))^N}.$$

This concludes the proof of the proposition. □

To finish this section we show a reproducing formula for the operator $t^\beta \partial_t^\beta \mathcal{P}_t$ on $L^2(\mathbb{R}^n)$.

Lemma 2.16. *The operator* $t^\beta \partial_t^\beta \mathcal{P}_t$ *defines an isometry from* $L^2(\mathbb{R}^n)$ *into* $L^2(\mathbb{R}^{n+1}_+, \frac{dx\,dt}{t})$. *Moreover,*

$$f(x) = \frac{4^\beta}{\Gamma(2\beta)} \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int_\varepsilon^N (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f(x)\,\frac{dt}{t}, \quad \textit{in } L^2(\mathbb{R}^n). \tag{2.20}$$

Proof. The proof is standard by using spectral techniques. Let us denote by $dE(\lambda)$ the spectral resolution of the operator $\mathcal{L}^{1/2}$. Since $\mathcal{P}_t = \int_0^\infty e^{-t\lambda}\,dE(\lambda)$ we have

$$t^\beta \partial_t^\beta \mathcal{P}_t = e^{-i\pi([\beta]+1)} \int_0^\infty (t\lambda)^\beta e^{-t\lambda}\,dE(\lambda).$$

Then, for all $f \in L^2(\mathbb{R}^n)$, we have

$$\|t^\beta \partial_t^\beta \mathcal{P}_t f(x)\|^2_{L^2(\mathbb{R}^{n+1}_+, \frac{dx\,dt}{t})} = \int_0^\infty \int_{\mathbb{R}^n} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \, dx \, \frac{dt}{t} = \int_0^\infty \left\langle (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f, f \right\rangle_{L^2(\mathbb{R}^n)} \frac{dt}{t}$$
$$= \int_0^\infty \int_0^\infty t^{2\beta} \lambda^{2\beta} e^{-2t\lambda} \, \frac{dt}{t} \, dE_{f,f}(\lambda) = \frac{\Gamma(2\beta)}{4^\beta} \|f\|^2_{L^2(\mathbb{R}^n)}. \quad (2.21)$$

In order to prove (2.20), it is enough to show that, for every pair of sequences $n_k \nearrow \infty$, $\varepsilon_k \searrow 0$,

$$\lim_{k\to\infty} \int_{n_k}^{n_{k+m}} (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f(x) \, \frac{dt}{t} = \lim_{k\to\infty} \int_{\varepsilon_k}^{\varepsilon_{k+m}} (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f(x) \, \frac{dt}{t} = 0, \quad \text{for all } m \geqslant 1. \quad (2.22)$$

Indeed, when this is the case, we can find $h \in L^2(\mathbb{R}^n)$ so that $\lim_{k\to\infty} \int_{\varepsilon_k}^{n_k} (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f \, \frac{dt}{t} = h$ and therefore, by using also a polarized version of (2.21),

$$\langle h, g\rangle_{L^2(\mathbb{R}^n)} = \lim_{k\to\infty} \int_{\varepsilon_k}^{n_k} \left\langle t^\beta \partial_t^\beta \mathcal{P}_t f, t^\beta \partial_t^\beta \mathcal{P}_t g \right\rangle_{L^2(\mathbb{R}^n)} \frac{dt}{t} = \int_0^\infty \left\langle t^\beta \partial_t^\beta \mathcal{P}_t f, t^\beta \partial_t^\beta \mathcal{P}_t g \right\rangle_{L^2(\mathbb{R}^n)} \frac{dt}{t}$$
$$= \frac{\Gamma(2\beta)}{4^\beta} \langle f, g\rangle_{L^2(\mathbb{R}^n)}, \quad \text{for all } g \in L^2(\mathbb{R}^n),$$

which implies $h = \frac{\Gamma(2\beta)}{4^\beta} f$. To check (2.22) we use functional calculus again, so that

$$\left\| \int_{n_k}^{n_{k+m}} (t^\beta \partial_t^\beta \mathcal{P}_t)^2 f(x) \, \frac{dt}{t} \right\|^2_{L^2(\mathbb{R}^n)} \leqslant \int_0^\infty \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\beta} e^{-2t\lambda} \, \frac{dt}{t} \, d\left|E_{f,f}\right|(\lambda).$$

Since $\left| \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\beta} e^{-2t\lambda} \, \frac{dt}{t} \right| \to 0$ as $n_k \to \infty$ and $\int_0^\infty d\left|E_{f,f}\right|(\lambda) \leqslant \|f\|^2_{L^2(\mathbb{R}^n)}$, by dominated convergence we have

$$\int_0^\infty \int_{n_k}^{n_{k+m}} t^{2\beta} \lambda^{2\beta} e^{-2t\lambda} \, \frac{dt}{t} \, d\left|E_{f,f}\right|(\lambda) \to 0, \quad \text{as } n_k \to \infty.$$

One proceeds similarly when $\varepsilon_k \to 0$. □

2.4.2 The Campanato-type space $\mathrm{BMO}^\alpha_{\mathcal{L}}$, $0 \leqslant \alpha \leqslant 1$

In this section we give the definition of space $\mathrm{BMO}^\alpha_{\mathcal{L}}$ introduced in [13], also see [25, 93] in a more general setting. For completeness we give the relation with $C^{0,\alpha}_{\mathcal{L}}$ and we prove a duality result $H^p_{\mathcal{L}}$–$\mathrm{BMO}^\alpha_{\mathcal{L}}$.

Definition 2.17 (BMO^α space for $\mathcal{L}$, see [13]). A locally integrable function f is in $\mathrm{BMO}^\alpha_{\mathcal{L}}$, $0 \leqslant \alpha \leqslant 1$, if there exists a constant C such that

(i) $\frac{1}{|B|} \int_B |f(x) - f_B| \, dx \leqslant C\,|B|^{\frac{\alpha}{n}}$, for every ball B in $\mathbb{R}^n$, and

(ii) $\frac{1}{|B|}\int_B |f(x)|\ dx \leqslant C\,|B|^{\frac{\alpha}{n}}$, for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $r_0 \geqslant \rho(x_0)$.

As usual, $f_B := \frac{1}{|B|}\int_B f(x)\ dx$. The norm $\|f\|_{BMO^{\alpha}_{\mathcal{L}}}$ is defined as

$$\|f\|_{BMO^{\alpha}_{\mathcal{L}}} = \inf\{C \geqslant 0 : \text{(i) and (ii) hold}\}. \tag{2.23}$$

Remark 2.18. The space $BMO^0_{\mathcal{L}}$ is the BMO space naturally associated to $\mathcal{L}$ given in [27]. We require $\alpha \leqslant 1$ in the definition above because if $\alpha > 1$ then the space only contains constant functions, see the proof of Proposition 2.27 below. Let us note that if (ii) is true for some ball B then (i) holds true for the same ball, so we might ask for (i) only for balls with radii smaller than $\rho(x_0)$. By using the classical John-Nirenberg inequality it can be seen that if in (i) and (ii) L^1-norms are replaced by L^p-norms, for $1 < p < \infty$, then the space $BMO^{\alpha}_{\mathcal{L}}$ does not change. In this case the conditions read:

(i)$_p$ $\left(\frac{1}{|B|}\int_B |f(x) - f_B|^p\ dx\right)^{1/p} \leqslant C\,|B|^{\frac{\alpha}{n}}$, for every ball B in $\mathbb{R}^n$, and

(ii)$_p$ $\left(\frac{1}{|B|}\int_B |f(x)|^p\ dx\right)^{1/p} \leqslant C\,|B|^{\frac{\alpha}{n}}$, for every $B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $r_0 \geqslant \rho(x_0)$.

Proposition 2.19. *Let $f \in BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha \leqslant 1$, and $B = B(x, r)$ with $r < \rho(x)$. Then there exists a constant $C = C_\alpha$ such that*

$$|f_B| \leqslant C_\alpha\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,\rho(x)^\alpha.$$

Proof. Let j_0 be a positive integer such that $2^{j_0} r \leqslant \rho(x) < 2^{j_0+1} r$. Since $f \in BMO^{\alpha}_{\mathcal{L}}$,

$$\begin{aligned}
|f_B| &\leqslant \frac{1}{|B|}\int_B |f(z) - f_{2B}|\ dz + \sum_{j=1}^{j_0} |f_{2^j B} - f_{2^{j+1}B}| + |f_{2^{j_0+1}B}| \\
&\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|2B|^{\frac{\alpha}{n}} + C\sum_{j=1}^{j_0} \|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|2^{j+1}B|^{\frac{\alpha}{n}} + \|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|2^{j_0+1}B|^{\frac{\alpha}{n}} \\
&\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|B|^{\frac{\alpha}{n}} \sum_{j=1}^{j_0+1} (2^\alpha)^j = C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|B|^{\frac{\alpha}{n}}\,\frac{2^\alpha - 2^{\alpha(j_0+1)}}{1 - 2^\alpha} \\
&\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|B|^{\frac{\alpha}{n}}\,2^{\alpha(j_0+1)} = C2^\alpha\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,\left(2^{j_0} r\right)^\alpha \leqslant C_\alpha\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,\rho(x)^\alpha.
\end{aligned}$$

□

Remark 2.20. From the proof of Proposition 2.19 it can be seen that if f is in $BMO_{\mathcal{L}} = BMO^0_{\mathcal{L}}$ and $B = B(x, r)$ with $r < \rho(x)$ then the conclusion of Lemma 2 in [27] follows:

$$|f_B| \leqslant C\left(1 + \log\frac{\rho(x)}{r}\right)\|f\|_{BMO_{\mathcal{L}}}.$$

Following the works by J. Dziubański and J. Zienkiewicz [30, 32, 31] we introduce the Hardy space naturally associated to $\mathcal{L}$.

Definition 2.21 (Hardy spaces for $\mathcal{L}$). An integrable function f is an element of the $\mathcal{L}$–Hardy space $H^p_{\mathcal{L}}$, $0 < p \leqslant 1$, if the maximal function

$$\mathcal{T}^* f(x) := \sup_{s>0} |\mathcal{T}_s f(x)| = \sup_{s>0} \left| \int_{\mathbb{R}^n} k_s(x,y) f(y)\ dy \right|$$

belongs to $L^p(\mathbb{R}^n)$. The quasi-norm in $H^p_{\mathcal{L}}$ is defined by $\|f\|_{H^p_{\mathcal{L}}} := \|\mathcal{T}^* f\|_{L^p}$.

We also have a description for *atomic* $\mathcal{L}$–Hardy spaces.

Definition 2.22 (Atomic Hardy spaces for $\mathcal{L}$). An atom of the $\mathcal{L}$–Hardy space $H^p_{\mathcal{L}}$, $0 < p \leqslant 1$, associated with a ball $B(x_0, r)$ is a function a such that

- $\operatorname{supp} a \subseteq B(x_0, r)$;
- $\|a\|_{L^\infty} \leqslant \dfrac{1}{|B(x_0,r)|^{1/p}}$;
- $r \leqslant \rho(x_0)$;
- if $r < \rho(x_0)/4$ then $\displaystyle\int a(x)\ dx = 0$.

The atomic $\mathcal{L}$–Hardy space $H^p_{at,\mathcal{L}}$, $0 < p \leqslant 1$, is defined as the set of L^1-functions f with compact support such that f can be written as a sum

$$f = \sum_i \lambda_i a_i,$$

where λ_i are complex numbers with $\sum_i |\lambda_i| < \infty$ and a_i are atoms in $H^p_{\mathcal{L}}$. The quasi-norm is

$$\|f\|_{H^p_{at,\mathcal{L}}} := \inf\Big\{ \sum_i |\lambda_i| : f = \sum_i \lambda_i a_i \Big\}.$$

Theorem 2.23 (See [30, 31]). *Let* $\widetilde{\delta} = \min\{1, \delta_0\}$, *with* δ_0 *as in* (2.8). *Then, for every* $f \in L^1_c(\mathbb{R}^n)$,

$$\|f\|_{H^p_{\mathcal{L}}} \sim \|f\|_{H^p_{at,\mathcal{L}}}, \quad \textit{for all } \tfrac{n}{n+\widetilde{\delta}} < p \leqslant 1.$$

Remark 2.24 (Important remark about atomic decompositions). When $n/2 < q < n$, the conclusion of Theorem 2.23 can be extended to hold for Hardy spaces $H^p_{\mathcal{L}}$ with $\frac{n}{n+1} < p \leqslant \frac{n}{n+\delta_0}$, but atoms must be redefined, see [32].

Lemma 2.25. *Let* $0 < p \leqslant 1$. *Then* $L^2_c(\mathbb{R}^n)$ *is a subset of* $H^p_{\mathcal{L}}$. *More precisely, if* $B = B(x_0, R)$ *with* $R \geqslant \rho(x_0)$ *then*

$$\|g\|_{H^p_{\mathcal{L}}} \leqslant C\,|B|^{\frac{1}{p}-\frac{1}{2}}\,\|g\|_{L^2(B)}, \quad \textit{for all } g \in L^2(B).$$

Proof. Let $g \in L^2(B)$, where $B = B(x_0, R)$ and $R \geqslant \rho(x_0)$. We have to prove that $\mathcal{T}^* g \in L^p(\mathbb{R}^n)$. Let us write

$$\int_{\mathbb{R}^n} (\mathcal{T}^* g(x))^p \, dx = \int_{4B} (\mathcal{T}^* g(x))^p \, dx + \int_{(4B)^c} (\mathcal{T}^* g(x))^p \, dx =: I + II.$$

For I we apply Hölder's inequality with exponents $2/p \geqslant 2$ and $(2/p)'$, and the boundedness of $\mathcal{T}^*$ on $L^2(\mathbb{R}^n)$, so that

$$I \leqslant C\,|B|^{1/(\frac{2}{p})'} \left(\int_{\mathbb{R}^n} (\mathcal{T}^* g(x))^2 \, dx \right)^{p/2} \leqslant C\,|B|^{1-\frac{p}{2}} \, \|g\|^p_{L^2(B)}.$$

Now let $x \in (4B)^c$, that is $|x - x_0| \geqslant 4R$. By Lemma 2.3, Lemma 2.1 and Hölder's inequality,

$$\begin{aligned}
\mathcal{T}_t g(x) &\leqslant C \int_B \left(\frac{\rho(y)}{\sqrt{t}} \right)^N \frac{1}{t^{n/2}} \, e^{-\frac{|x-y|^2}{ct}} \, |g(y)| \, dy \\
&\leqslant C \left(\frac{\rho(x_0)}{\sqrt{t}} \right)^N \frac{e^{-\frac{|x-x_0|^2}{ct}}}{t^{n/2}} \int_B \left(1 + \frac{|x_0 - y|}{\rho(x_0)} \right)^{\frac{k_0}{k_0+1}N} |g(y)| \, dy \\
&\leqslant C \frac{\rho(x_0)^N}{|x - x_0|^{n+N}} \, \|g\|_{L^2(B)} \left(|B|^{1/2} + \rho(x_0)^{-\frac{k_0}{k_0+1}N} \left(\int_B |x_0 - y|^{\frac{2k_0}{k_0+1}N} \, dy \right)^{1/2} \right) \\
&\leqslant C \, \|g\|_{L^2(B)} \, |B|^{1/2} \left(\frac{\rho(x_0)^N}{|x - x_0|^{n+N}} + \frac{\rho(x_0)^{N(1-\frac{k_0}{k_0+1})}}{|x - x_0|^{n+N(1-\frac{k_0}{k_0+1})}} \right), \quad x \in \mathbb{R}^n.
\end{aligned}$$

The estimate above is uniform in t. Therefore for II,

$$\begin{aligned}
II &\leqslant C \, \|g\|^p_{L^2(B)} \, |B|^{\frac{p}{2}} \int_{(4B)^c} \left(\frac{\rho(x_0)^{Np}}{|x - x_0|^{(n+N)p}} + \frac{\rho(x_0)^{N(1-\frac{k_0}{k_0+1})p}}{|x - x_0|^{(n+N(1-\frac{k_0}{k_0+1}))p}} \right) dx \\
&\leqslant C \, \|g\|^p_{L^2(B)} \, |B|^{\frac{p}{2}} \, R^{n(1-p)} \left(\rho(x_0)^{Np} R^{-Np} + \rho(x_0)^{N(1-\frac{k_0}{k_0+1})p} R^{-N(1-\frac{k_0}{k_0+1})p} \right) \\
&\leqslant C \, \|g\|^p_{L^2(B)} \, |B|^{1-\frac{p}{2}},
\end{aligned}$$

by choosing suitable $N > \frac{n(1-p)}{(1-\frac{k_0}{k_0+1})p} > 0$. Pasting together the estimates for I and II above we get $\|g\|_{H^p_{\mathcal{L}}} = \|\mathcal{T}^* g\|_{L^p} \leqslant C\,|B|^{\frac{1}{p}-\frac{1}{2}} \, \|g\|_{L^2(B)}$. □

As mentioned in [14], see also [43, 96, 97], once an atomic decomposition of $H^p_{\mathcal{L}}$ is at hand, the dual space can be described.

Theorem 2.26 (Duality $H^p_{\mathcal{L}}$–$BMO^\alpha_{\mathcal{L}}$). *Let* $q > n$ *and* $0 \leqslant \alpha < 1$. *Then the dual of* $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$ *is the space* $BMO^\alpha_{\mathcal{L}}$. *More precisely, any continuous linear functional* ℓ *over* $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$ *can be represented as*

$$\ell(a) = \int_{\mathbb{R}^n} f(x) a(x) \, dx,$$

for some function $f \in BMO^{\alpha}_{\mathcal{L}}$ *and all atoms* $a \in H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$. *Moreover,* $\|\ell\| \sim \|f\|_{BMO^{\alpha}_{\mathcal{L}}}$.

Proof. The case $\alpha = 0$ is already proved in [27]. Assume then that $0 < \alpha < 1$.

Let us first check that any function f in $BMO^{\alpha}_{\mathcal{L}}$ defines a continuous linear functional on $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$ by

$$\ell_f(a) := \int_{\mathbb{R}^n} f(x)a(x)\ dx, \quad a \text{ an } H^{\frac{n}{n+\alpha}}_{\mathcal{L}}\text{–atom.}$$

Indeed, take an atom a supported in a ball $B = B(x_0, r)$, $r \leqslant \rho(x_0)$, and suppose first that $r \geqslant \rho(x_0)/4$ (so no cancelation happens). Then, by the size condition $\|a\|_{L^\infty(\mathbb{R}^n)} \leqslant |B|^{-(1+\frac{\alpha}{n})}$ and Proposition 2.19,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)a(x)\ dx\right| &\leqslant \int_B |f(x) - f_B|\,|a(x)|\ dx + |f_B|\left|\int_B a(x)\ dx\right| \\ &\leqslant \frac{1}{|B|^{1+\frac{\alpha}{n}}}\int_B |f(x) - f_B|\ dx + C_\alpha \|f\|_{BMO^{\alpha}_{\mathcal{L}}} \frac{\rho(x_0)^\alpha}{|B|^{\frac{\alpha}{n}}} \\ &\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}.\end{aligned}$$

In the remaining case $r \leqslant \rho(x_0)/4$ note that the second term in the first inequality above is zero. Hence ℓ_f is in the dual of $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$.

Now let ℓ be a continuous linear functional on $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$ and $B_N = B(0, N)$ with $N > \rho(0)$. Lemma 2.25 implies that ℓ is a continuous linear functional on $L^2(B_N)$. Hence, by the Riesz Representation Theorem, there exists a function $f_N \in L^2(B_N)$ such that

$$\ell(g) = \int_{B_N} f_N(x)g(x)\ dx, \quad g \in L^2(B_N).$$

Lemma 2.25 gives

$$\left|\int_{B_N} f_N(x)g(x)\ dx\right| \leqslant \|\ell\|\,\|g\|_{H^{\frac{n}{n+\alpha}}_{\mathcal{L}}} \leqslant C\,\|\ell\|\,|B_N|^{\frac{1}{2}+\frac{\alpha}{n}}\,\|g\|_{L^2(B_N)}, \quad g \in L^2(B_N),$$

so $\|f_N\|_{L^2(B_N)} \leqslant C\,\|\ell\|\,|B_N|^{\frac{1}{2}+\frac{\alpha}{n}}$. If we iterate the previous argument in N we get the existence of a function $f \in L^2(\mathbb{R}^n)$ such that $f\big|_{B_N} = f_N$ and

$$\ell(g) = \int_{\mathbb{R}^n} f(x)g(x)\ dx, \quad g \in L^2_c(\mathbb{R}^n). \tag{2.24}$$

Since $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$–atoms belong to $L^2_c(\mathbb{R}^n)$ we have that $\ell \equiv \ell_f$ in $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$. It remains to show that $f \in BMO^{\alpha}_{\mathcal{L}}$ with $\|f\|_{BMO^{\alpha}_{\mathcal{L}}} \leqslant C\,\|\ell\|$. Let $B = B(x_0, r)$. If $r \geqslant \rho(x_0)$ then, by Hölder's inequality,

$$\frac{1}{|B|}\int_B |f(x)|\ dx \leqslant |B|^{-1/2}\,\|f\|_{L^2(B)} \leqslant C\,\|\ell\|\,|B|^{\frac{\alpha}{n}}.$$

Assume that $r \leqslant \rho(x_0)$. Note that the classical Hardy spaces H^p are contained in $H^p_{\mathcal{L}}$, $0 < p \leqslant 1$, since classical H^p–atoms are particular cases of $H^p_{\mathcal{L}}$–atoms supported in small balls. Therefore ℓ is a continuous linear functional on the classical Hardy space $H^{\frac{n}{n+\alpha}}$. Hence there exists a function h in the classical BMO^{α} space such that

$$\ell(a) = \int_{\mathbb{R}^n} h(x)a(x)\ dx, \quad a \text{ an } H_{\mathcal{L}}^{\frac{n}{n+\alpha}}\text{–atom}, \tag{2.25}$$

and $\|h\|_{BMO^{\alpha}} = \|\ell\|$. From (2.24) and (2.25) we get the existence of a constant c_B such that $f(x) - h(x) = c_B$. Therefore,

$$\begin{aligned}\frac{1}{|B|}\int_B |f(x) - f_B|\ dx &\leqslant \frac{1}{|B|}\int_B |h(x) + c_B - h_B - c_B|\ dx \\ &\leqslant C\,\|h\|_{BMO^{\alpha}}\,|B|^{\frac{\alpha}{n}} \leqslant C\,\|\ell\|\,|B|^{\frac{\alpha}{n}},\end{aligned}$$

and the conclusion follows. □

The following result was proved in [13, Proposition 4] for $0 < \alpha < 1$ and in a weighted context, see also in [96]. We collect it here including the case $\alpha = 1$. Just for the sake of completeness we also provide the proof.

Proposition 2.27 (Campanato-type description of $C^{0,\alpha}_{\mathcal{L}}$)**.** *If* $0 < \alpha \leqslant 1$ *then the spaces* $BMO^{\alpha}_{\mathcal{L}}$ *and* $C^{0,\alpha}_{\mathcal{L}}$ *are equal and their norms are equivalent.*

Proof. Let $f \in BMO^{\alpha}_{\mathcal{L}}$. For $x, z \in \mathbb{R}^n$ let $B_x = B(x, |x-z|)$ and $B_z = B(z, |x-z|)$. Then

$$|f(x) - f(z)| \leqslant |f(x) - f_{B_x}| + |f(z) - f_{B_z}| + |f_{B_x} - f_{B_z}|.$$

For the first term in the right hand side above, if x is a Lebesgue point of f,

$$\begin{aligned}|f(x) - f_{B_x}| &\leqslant \lim_{k\to\infty}\left(\left|f(x) - f_{2^{-k}B_x}\right| + \sum_{j=0}^{k-1}\left|f_{2^{-(j+1)}B_x} - f_{2^{-j}B_x}\right|\right) \\ &= \sum_{j=0}^{\infty}\left|f_{2^{-(j+1)}B_x} - f_{2^{-j}B_x}\right| \\ &\leqslant C\sum_{j=0}^{\infty}\frac{1}{|2^{-j}B_x|}\int_{2^{-j}B_x}\left|f(w) - f_{2^{-j}B_x}\right|\ dw \\ &\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\sum_{j=0}^{\infty}|2^{-j}B_x|^{\frac{\alpha}{n}} \leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|x-z|^{\alpha}.\end{aligned}$$

The second term can be handled in the same way. For the third term above,

$$\begin{aligned}|f_{B_x} - f_{B_z}| &\leqslant \frac{|2B_z|}{|B_x|}\frac{1}{|2B_z|}\int_{2B_z}|f(w) - f_{2B_z}|\ dw + |f_{2B_z} - f_{B_z}| \\ &\leqslant C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|B_z|^{\frac{\alpha}{n}} = C\,\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\,|x-z|^{\alpha}.\end{aligned}$$

Let now $B = B(x, \rho(x))$. Then using what we have just proved,

$$\begin{aligned}|f(x)| &\leqslant \frac{1}{|B|}\int_B |f(x) - f(w)|\ dw + \frac{1}{|B|}\int_B |f(w)|\ dw \\ &\leqslant C\,\|f\|_{BMO^\alpha_{\mathcal{L}}}\left(\frac{1}{|B|}\int_B |x-w|^\alpha\ dw + |B|^{\frac{\alpha}{n}}\right) \\ &= C\,\|f\|_{BMO^\alpha_{\mathcal{L}}}\,\rho(x)^\alpha.\end{aligned}$$

Therefore $f \in C^{0,\alpha}_{\mathcal{L}}$.

Assume that $f \in C^{0,\alpha}_{\mathcal{L}}$ and let $B' = B(x_0, r)$, $r > 0$. Then

$$\begin{aligned}\frac{1}{|B'|}\int_{B'} |f(x) - f_{B'}|\ dx &\leqslant \frac{1}{|B'|^2}\int_{B'}\int_{B'} |f(x) - f(y)|\ dx\ dy \\ &\leqslant \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\frac{1}{|B'|^2}\int_{B'}\int_{B'} |x-y|^\alpha\ dx\ dy \\ &\leqslant \|f\|_{C^{0,\alpha}_{\mathcal{L}}}(2r)^\alpha = C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\left|B'\right|^{\frac{\alpha}{n}}.\end{aligned}$$

Suppose that $r \geqslant \rho(x_0)$. Then, when $|x_0 - w| \leqslant r$, we have

$$\rho(w) \leqslant c\rho(x_0)\left(1 + \frac{|x_0 - w|}{\rho(x_0)}\right)^{\frac{k_0}{k_0+1}} \leqslant c\rho(x_0)^{1-\frac{k_0}{k_0+1}}(2r)^{\frac{k_0}{k_0+1}} \leqslant cr,$$

see Lemma 2.1. Thus

$$\begin{aligned}\frac{1}{|B'|}\int_{B'} |f(w)|\ dw &\leqslant \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\frac{1}{|B'|}\int_{B'} \rho(w)^\alpha\ dw \\ &\leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\frac{1}{|B'|}\int_{B'} r^\alpha\ dw = C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\left|B'\right|^{\frac{\alpha}{n}}.\end{aligned}$$

Thus $f \in BMO^\alpha_{\mathcal{L}}$ and the proof of Proposition 2.27 is completed. □

Remark 2.28. Proposition 2.27 implies, in particular, that functions in $BMO^\alpha_{\mathcal{L}}$ can be modified in a set of measure zero so they become α-Hölder continuous, $0 < \alpha \leqslant 1$.

2.4.3 Proofs of Theorems 2.9–2.11

The proof of Theorem 2.9 will follow the scheme *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)* $\Longrightarrow$ *(i)*. The statement *(iii)* $\Longrightarrow$ *(i)* relies heavily on the duality $H^{\frac{n}{n+\alpha}}_{\mathcal{L}} - BMO^\alpha_{\mathcal{L}}$ developed in the last section, so the method, rather technical, will work only for $0 < \alpha < 1$.

To prove Theorem 2.10*(I)* we just note that the proofs of *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)* in Theorem 2.9 also hold for $\alpha = 1$. A simple contradiction argument shows that the converse is false: if it was true then, by the comment just made, $f \in C^{0,1}_{\mathcal{L}}$ would be equivalent to *(ii)* in Theorem 2.9 with $\alpha = 1$. But that contradicts the statement of Theorem 2.10*(II)* (which is proved by a counterexample).

For Theorem 2.11*(A)* we only have to prove the necessity part since the sufficiency for $\beta = 1$ follows the same lines as in [27]. For part *(B)* we give a counterexample.

Proof of Theorem 2.9: *(i)$\Longrightarrow$(ii)*
Let $f \in C^{0,\alpha}_{\mathcal{L}}$. Then

$$\begin{aligned} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)| &= \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,z)\,(f(z)-f(x))\ dz + f(x)\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,z)\ dz \right| \\ &\leqslant \|f\|_{C^{0,\alpha}_{\mathcal{L}}} \int_{\mathbb{R}^n} |t^\beta \partial_t^\beta \mathcal{P}_t(x,z)|\,|x-z|^\alpha\, dz + \|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, \rho(x)^\alpha \left| \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,z)\ dz \right| \\ &=: I + II. \end{aligned}$$

Applying Proposition 2.15*(b)*,

$$I \leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}} \int_{\mathbb{R}^n} \frac{t^\beta\,|x-z|^\alpha}{(t+|x-z|)^{n+\beta}}\ dz = C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha.$$

For II we consider two cases. Assume first that $\rho(x) \leqslant t$. Then Proposition 2.15*(b)* gives

$$II \leqslant C\|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^\alpha \int_{\mathbb{R}^n} \frac{t^\beta}{(t+|x-z|)^{n+\beta}}\ dz = C\|f\|_{C^{0,\alpha}_{\mathcal{L}}} t^\alpha.$$

Suppose now that $\rho(x) > t$. Since $q > n$, we have $\delta_0 > 1$ in (2.8). Therefore we can choose δ' such that $\alpha < \delta' \leqslant \delta_0$ with $\delta' < \beta$. By Proposition 2.15*(d)*,

$$II \leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha (t/\rho(x))^{\delta'-\alpha} \leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha.$$

Hence $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant C\,\|f\|_{C^{0,\alpha}_{\mathcal{L}}}\, t^\alpha$.

Proof of Theorem 2.9: *(ii)$\Longrightarrow$(iii)*
For any ball $B = B(x_0, r)$,

$$\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2 \frac{dx\ dt}{t} \leqslant \|f\|^2_{C^{0,\alpha}_{\mathcal{L}}} \frac{1}{|B|}\int_B \int_0^r t^{2\alpha}\ \frac{dt\ dx}{t} = C\,\|f\|^2_{C^{0,\alpha}_{\mathcal{L}}}\, r^{2\alpha}.$$

Proof of Theorem 2.9: *(iii)$\Longrightarrow$(i)*
Assume that $f \in L^1(\mathbb{R}^n, (1+|x|)^{-(n+\alpha+\varepsilon)}\ dx)$ for any $0 < \varepsilon < \min\{\beta-\alpha, 1-\alpha\}$, and that the Carleson condition in *(iii)* holds. Let

$$[d\mu_f]_{\alpha,\beta} := \sup_B \frac{1}{|B|^{\frac{\alpha}{n}}} \left(\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f(x)|^2\ \frac{dx\ dt}{t} \right)^{1/2}.$$

To show that $f \in BMO^\alpha_{\mathcal{L}}$, by Theorem 2.26, it is enough to prove that the linear functional

$$H^{\frac{n}{n+\alpha}}_{\mathcal{L}} \ni g \longmapsto \Phi_f(g) := \int_{\mathbb{R}^n} f(x)g(x)\ dx,$$

is continuous on $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$. In fact, we are going to prove that there exists a constant C such that

$$|\Phi_f(g)| \leqslant C[d\mu_f]_{\alpha,\beta} \, \|g\|_{H_{\mathcal{L}}^{\frac{n}{n+\alpha}}},$$

which implies that $f \in BMO_{\mathcal{L}}^{\alpha}$ with $\|f\|_{BMO_{\mathcal{L}}^{\alpha}} \leqslant C[d\mu_f]_{\alpha,\beta}$.

To that end we shall proceed in three steps.

Step 1. It consists in writing the functional Φ by using extensions of f and g to the upper half-space. Define the extended functions

$$F(x,t) := t^{\beta}\partial_t^{\beta}\mathcal{P}_t f(x), \quad G(x,t) := t^{\beta}\partial_t^{\beta}\mathcal{P}_t g(x),$$

for $x \in \mathbb{R}^n$, $t > 0$. The following reproducing formula holds:

Lemma 2.29. *Let* $f \in L^1(\mathbb{R}^n, (1+|x|)^{-(n+\alpha+\varepsilon)}dx)$ *for any* $\varepsilon > 0$ *and* g *be an* $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$*–atom. Then*

$$\frac{4^{\beta}}{\Gamma(2\beta)} \int_{\mathbb{R}^n} f(x)\overline{g(x)} \, dx = \int_{\mathbb{R}^{n+1}_+} F(x,t)\overline{G(x,t)} \, \frac{dx \, dt}{t}.$$

The rather technical proof of the lemma above will be given at the end of this subsection. To continue we assume its validity. Therefore we are reduced to study the integral in the right-hand side appearing in the lemma.

Step 2. To handle the integral in Lemma 2.29 we take a result of E. Harboure, O. Salinas and B. Viviani about tent spaces into our particular case.

Lemma 2.30 (See [43, p. 279]). *For any pair of measurable functions* F *and* G *on* $\mathbb{R}^{n+1}_+$ *we have*

$$\int_{\mathbb{R}^{n+1}_+} |F(x,t)|\,|G(x,t)| \, \frac{dx \, dt}{t}$$
$$\leqslant C \sup_B \left(\frac{1}{|B|^{1+\frac{2\alpha}{n}}} \int_{\widehat{B}} |F(x,t)|^2 \, \frac{dx \, dt}{t} \right)^{1/2} \times \left(\int_{\mathbb{R}^n} \left(\int_{\Gamma(x)} |G(y,t)|^2 \, \frac{dy \, dt}{t^{n+1}} \right)^{\frac{n}{2(n+\alpha)}} dx \right)^{\frac{n+\alpha}{n}},$$

where $\Gamma(x)$ *denotes the cone with vertex at* x *and aperture* 1*:* $\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

If we take $F(x,t) = t^{\beta}\partial_t^{\beta}\mathcal{P}_t f(x)$ in Lemma 2.30 then the supremum that appears in the inequality is exactly $[d\mu_f]_{\alpha,\beta}$. Hence it remains to handle the term with $G(x,t)$, which is done in the last step.

Step 3. The area function S_{β} defined by

$$S_{\beta}(h)(z) = \left(\int_{\Gamma(z)} |t^{\beta}\partial_t^{\beta}\mathcal{P}_t h(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}, \quad z \in \mathbb{R}^n, \tag{2.26}$$

is a bounded operator on $L^2(\mathbb{R}^n)$. Indeed, by the Spectral Theorem, the square function

$$g_\beta(h)(x) = \left(\int_0^\infty |t^\beta \partial_t^\beta \mathcal{P}_t h(x)|^2 \,\frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n, \tag{2.27}$$

satisfies $\left\|g_\beta(h)\right\|_{L^2(\mathbb{R}^n)} = \Gamma(\beta)\,\|h\|_{L^2(\mathbb{R}^n)}$ and it is easy to check that $\left\|S_\beta(h)\right\|_{L^2(\mathbb{R}^n)} = \left\|g_\beta(h)\right\|_{L^2(\mathbb{R}^n)}$. Now, in view of Steps 1 and 2, we will finish the proof of *(iii)* $\Longrightarrow$ *(i)* in Theorem 2.9 as soon as we have proved the following

Lemma 2.31. *There exists a constant* C *such that for any function* g *which is a linear combination of* $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$*–atoms we have*

$$\left\|S_\beta(g)\right\|_{L^{\frac{n}{n+\alpha}}} \leqslant C\,\|g\|_{H_{\mathcal{L}}^{\frac{n}{n+\alpha}}}.$$

Proof. Let g be an $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$–atom associated to a ball $B = B(x_0, r)$. We apply Hölder's inequality and the L^2-boundedness of the area function (2.26) to get

$$\begin{aligned}\int_{8B} \left|S_\beta(g)(x)\right|^{\frac{n}{n+\alpha}}\,dx &\leqslant C\,|B|^{\frac{n+2\alpha}{2(n+\alpha)}} \left(\int_{8B} \left|S_\beta(g)(x)\right|^2\,dx\right)^{\frac{n}{2(n+\alpha)}} \leqslant C\,|B|^{\frac{n+2\alpha}{2(n+\alpha)}}\,\|g\|_{L^2(8B)}^{\frac{n}{n+\alpha}} \\ &\leqslant C\,|B|^{\frac{n+2\alpha}{2(n+\alpha)}}\,|B|^{\frac{n}{2(n+\alpha)}}\,\|g\|_{L^\infty}^{\frac{n}{n+\alpha}} \leqslant C.\end{aligned}$$

In order to complete the proof of Lemma 2.31, we must find a uniform bound for

$$\int_{(8B)^c} |S_\beta(g)(x)|^{\frac{n}{n+\alpha}}\,dx. \tag{2.28}$$

Let us consider first the case when $r < \frac{\rho(x_0)}{4}$. Then, by the moment condition on g,

$$\begin{aligned}\left(S_\beta(g)(x)\right)^2 &= \int_0^\infty \int_{|x-y|<t} \left(\int_{\mathbb{R}^n} \left(t^\beta \partial_t^\beta \mathcal{P}_t(y,x') - t^\beta \partial_t^\beta \mathcal{P}_t(y,x_0)\right) g(x')\,dx'\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &\leqslant \int_0^{\frac{|x-x_0|}{2}} \int_{|x-y|<t} \left(\int_B |t^\beta \partial_t^\beta \mathcal{P}_t(y,x') - t^\beta \partial_t^\beta \mathcal{P}_t(y,x_0)|\,\frac{dx'}{|B|^{\frac{n+\alpha}{n}}}\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &\quad + \int_{\frac{|x-x_0|}{2}}^\infty \int_{|x-y|<t} \left(\int_B |t^\beta \partial_t^\beta \mathcal{P}_t(y,x') - t^\beta \partial_t^\beta \mathcal{P}_t(y,x_0)|\,\frac{dx'}{|B|^{\frac{n+\alpha}{n}}}\right)^2 \frac{dy\,dt}{t^{n+1}} \\ &=: I_1(x) + I_2(x).\end{aligned}$$

We now use the smoothness of $t^\beta \partial_t^\beta \mathcal{P}_t(y,x) = t^\beta \partial_t^\beta \mathcal{P}_t(x,y)$ established in Proposition 2.15*(c)* with $\alpha < \delta' < \beta$ and $N > 0$. In the domain of integration of $I_1(x)$ we have

$|x-x_0| \leqslant 2|y-x_0|$. So

$$I_1(x) \leqslant C\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\left(\int_B\left(\frac{|x'-x_0|}{t}\right)^{\delta'}\frac{t^\beta}{(|x_0-y|^2+t^2)^{\frac{n+\beta}{2}}}\frac{dx'}{|B|^{\frac{n+\alpha}{n}}}\right)^2\frac{dy\,dt}{t^{n+1}}$$
$$\leqslant C\int_0^{\frac{|x-x_0|}{2}}\int_{|x-y|<t}\left(\frac{r}{t}\right)^{2\delta'}\frac{1}{t^{2n}\left(\frac{|x_0-y|}{t}+1\right)^{2(n+\beta)}}\frac{1}{|B|^{\frac{2\alpha}{n}}}\frac{dy\,dt}{t^{n+1}}$$
$$\leqslant C\int_0^{\frac{|x-x_0|}{2}}\left(\frac{r}{t}\right)^{2\delta'}\frac{1}{t^{2n}\left(\frac{|x_0-x|}{t}\right)^{2(n+\beta)}}\frac{1}{|B|^{\frac{2\alpha}{n}}}\frac{dt}{t}$$
$$\leqslant C\frac{r^{2(\delta'-\alpha)}}{|x-x_0|^{2(n+\beta)}}\int_0^{\frac{|x-x_0|}{2}}t^{2(\beta-\delta')}\frac{dt}{t}=C\frac{r^{2(\delta'-\alpha)}}{|x-x_0|^{2(n+\delta')}}.$$

Thus, integrating over $(8B)^c$, we have

$$\int_{(8B)^c}|I_1(x)^{1/2}|^{\frac{n}{n+\alpha}}\,dx \leqslant C\int_{(8B)^c}\left(\frac{r^{\delta'-\alpha}}{|x-x_0|^{n+\delta'}}\right)^{\frac{n}{n+\alpha}}dx = C.$$

Let us continue with $I_2(x)$. If $x\in(8B)^c$ then we have $|x'-x_0|\leqslant r<\dfrac{|x-x_0|}{2}\leqslant t$. Then, by Proposition 2.15*(c)* and $x\in(8B)^c$,

$$I_2(x)\leqslant C\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\left(\int_B\left(\frac{|x'-x_0|}{t}\right)^{\delta'}\frac{1}{t^n}\frac{dx'}{|B|^{\frac{n+\alpha}{n}}}\right)^2\frac{dy\,dt}{t^{n+1}}$$
$$\leqslant C\int_{\frac{|x-x_0|}{2}}^{\infty}\int_{|x-y|<t}\left(\frac{r}{t}\right)^{2\delta'}\frac{1}{t^{2n}}\frac{1}{|B|^{\frac{2\alpha}{n}}}\frac{dy\,dt}{t^{n+1}}$$
$$=Cr^{2(\delta'-\alpha)}\int_{\frac{|x-x_0|}{2}}^{\infty}t^{-2(n+\delta')}\frac{dt}{t}=C\frac{r^{2(\delta'-\alpha)}}{|x-x_0|^{2(n+\delta')}}.$$

Therefore,

$$\int_{(8B)^c}|(I_2(x))^{1/2}|^{\frac{n}{n+\alpha}}\,dx\leqslant C\int_{(8B)^c}\left(\frac{r^{\delta'-\alpha}}{|x-x_0|^{n+\delta'}}\right)^{\frac{n}{n+\alpha}}dx\leqslant C.$$

Collecting terms we see that if $r<\frac{\rho(x_0)}{4}$ then a uniform bound for (2.28) is obtained.

We now turn to the estimate of (2.28) when r is comparable to $\rho(x_0)$, namely, $\frac{\rho(x_0)}{4}<r\leqslant\rho(x_0)$. For $x\in(8B)^c$ we can split the integral in $t>0$ in the definition of $S_\beta g(x)$ into three parts:

$$(S_\beta(g)(x))^2=\left(\int_0^{\frac{r}{2}}+\int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}}+\int_{\frac{|x-x_0|}{4}}^{\infty}\right)\int_{|x-y|<t}\left|\int_{\mathbb{R}^n}t^\beta\partial_t^\beta\mathcal{P}_t(y,x')g(x')\,dx'\right|^2\frac{dy\,dt}{t^{n+1}}$$
$$=: I_1'(x)+I_2'(x)+I_3'(x).$$

In the integrand of $I_1'(x)$ we have $|x'-y| \sim |x-x_0|$, so by Proposition 2.15*(b)*,

$$\begin{aligned} I_1'(x) &\leqslant C \int_0^{\frac{r}{2}} \int_{|x-y|<t} \left(\int_B \frac{t^\beta}{(|y-x'|+t)^{n+\beta}} \frac{1}{|B|^{\frac{n+\alpha}{n}}} \, dx' \right)^2 \frac{dy\, dt}{t^{n+1}} \\ &\leqslant C r^{-2\alpha} \int_0^{\frac{r}{2}} \int_{|x-y|<t} \frac{t^{2\beta}}{(|x-x_0|+t)^{2(n+\beta)}} \frac{dy\, dt}{t^{n+1}} \\ &= C r^{-2\alpha} \int_0^{\frac{r}{2}} \frac{t^{2\beta}}{(|x-x_0|+t)^{2(n+\beta)}} \frac{dt}{t} \leqslant C \frac{r^{2(\beta-\alpha)}}{|x-x_0|^{2(n+\beta)}}. \end{aligned}$$

For $I_2'(x)$, by applying Proposition 2.15*(b)* for any $M > \alpha$, together with $|x'-y| \sim |x-x_0|$ and $\rho(x') \sim \rho(x_0) \sim r$, we get

$$\begin{aligned} I_2'(x) &\leqslant C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left(\int_B \frac{t^\beta}{(|y-x'|+t)^{n+\beta}} \left(\frac{\rho(x')}{t}\right)^M \frac{1}{|B|^{\frac{n+\alpha}{n}}} \, dx' \right)^2 \frac{dy\, dt}{t^{n+1}} \\ &\leqslant C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left(\int_B \frac{1}{t^n \left(\frac{|x-x_0|}{t}+1\right)^{n+\beta}} \left(\frac{\rho(x_0)}{t}\right)^M \frac{1}{|B|^{\frac{n+\alpha}{n}}} \, dx' \right)^2 \frac{dy\, dt}{t^{n+1}} \\ &\leqslant C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \int_{|x-y|<t} \left(\frac{t^{\beta-M}\rho(x_0)^M}{|x-x_0|^{n+\beta} r^\alpha} \right)^2 \frac{dy\, dt}{t^{n+1}} \leqslant C \int_{\frac{r}{2}}^{\frac{|x-x_0|}{4}} \left(\frac{t^{\beta-M} r^{M-\alpha}}{|x-x_0|^{n+\beta}} \right)^2 \frac{dt}{t} \\ &\leqslant C \frac{r^{2(\beta-\alpha)}}{|x-x_0|^{2(n+\beta)}} \int_1^{\frac{|x-x_0|}{2r}} u^{2(\beta-M)} \frac{du}{u} \leqslant C \frac{r^{2(M-\alpha)}}{|x-x_0|^{2(n+M)}}. \end{aligned}$$

Finally, for the last term above $I_3'(x)$, with the same method that was used to estimate $I_2'(x)$, we obtain

$$I_3'(x) \leqslant C \frac{r^{2(M-\alpha)}}{|x-x_0|^{2(n+M)}}.$$

Hence,

$$\int_{(8B)^c} |I_j'(x)^{1/2}|^{\frac{n}{n+\alpha}} \, dx \leqslant C,$$

for $j = 1, 2, 3$ and the uniform bound for (2.28) is established also when $r \sim \rho(x_0)$. This completes the proof of Lemma 2.31. □

Now the three steps of the proof of *(iii)* $\Longrightarrow$ *(i)* in Theorem 2.9 are completed. It only remains to prove Lemma 2.29, that we took for granted before. To that end, we need the following result.

Lemma 2.32. *Let* $q_t(x,y)$ *be a function of* $x, y \in \mathbb{R}^n$, $t > 0$. *Assume that for each* $N > 0$ *there exists a constant* C_N *such that*

$$|q_t(x,y)| \leqslant C_N \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^{-N} t^{-n} \left(1 + \frac{|x-y|}{t}\right)^{-(n+\gamma)}, \tag{2.29}$$

for some $\gamma \geqslant \alpha$. *Then, for every* $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$*–atom* g *supported on* $B(x_0, r)$*, there exists* $C_{N,x_0,r} > 0$ *such that*

$$\sup_{t>0} \left| \int_{\mathbb{R}^n} q_t(x,y) g(y)\, dy \right| \leqslant C_{N,x_0,r} \left(1 + |x|\right)^{-(n+\gamma)}, \quad x \in \mathbb{R}^n.$$

Proof. If $x \in B(x_0, 2r)$ then, since $\|g\|_{L^\infty(\mathbb{R}^n)} \leqslant |B(x_0,r)|^{-(1+\frac{\alpha}{n})}$,

$$\begin{aligned}\left| \int_{\mathbb{R}^n} q_t(x,y) g(y)\, dy \right| &\leqslant C_N \frac{1}{r^{n+\alpha}} \int_{\mathbb{R}^n} t^{-n} \left(1 + \frac{|x-y|}{t}\right)^{-(n+\gamma)} dy \\ &\leqslant C_N \frac{1}{r^{n+\alpha}} \int_{\mathbb{R}^n} \frac{1}{(1+|u|)^{n+\gamma}}\, du \leqslant C_{N,r}.\end{aligned}$$

Since $|x - x_0| \leqslant 2r$, we have $1 + |x| \leqslant 1 + |x - x_0| + |x_0| \leqslant 1 + 2r + |x_0|$. Hence

$$\left| \int_{\mathbb{R}^n} q_t(x,y) g(y)\, dy \right| \leqslant C_{N,r} \frac{(1 + 2r + |x_0|)^{n+\gamma}}{(1 + 2r + |x_0|)^{n+\gamma}} \leqslant C_{N,x_0,r} (1 + |x|)^{-(n+\gamma)}.$$

If $x \notin B(x_0, 2r)$ then for $y \in B(x_0, r)$ we have $|x - y| \sim |x - x_0|$ and, since $r < \rho(x_0)$, we get that $\rho(x_0) \sim \rho(y)$, see Lemma 2.1. Hence, choosing $N = \gamma$ in (2.29),

$$\begin{aligned}\left| \int_{\mathbb{R}^n} q_t(x,y) g(y)\, dy \right| &\leqslant C_\gamma \left(\frac{t}{\rho(x_0)}\right)^{-\gamma} t^{-n} \left(\frac{|x - x_0|}{t}\right)^{-(n+\gamma)} \|g\|_{L^1(\mathbb{R}^n)} \\ &\leqslant C_{\gamma,x_0,r} \rho(x_0)^\gamma |x - x_0|^{-(n+\gamma)} r^{-\gamma} \leqslant C_{\gamma,x_0,r} |x - x_0|^{-(n+\gamma)}.\end{aligned}$$

Since $x \notin B(x_0, 2r)$, we can set $x = x_0 + 2rz$, $|z| \geqslant 1$. Then $1 + |x| \leqslant 1 + |x_0| + 2r|z|$, and $\frac{1+|x_0|+2r}{2r} \left| x - x_0 \right| = (1 + |x_0| + 2r)|z| \geqslant 1 + |x_0| + 2r|z|$. It means that $c_{x_0,r}|x - x_0| \geqslant 1 + |x|$. Therefore

$$\left| \int_{\mathbb{R}^n} q_t(x,y) g(y)\, dy \right| \leqslant C_{\gamma,x_0,r} |x - x_0|^{-(n+\gamma)} \leqslant C_{\gamma,x_0,r} (1 + |x|)^{-(n+\gamma)}.$$

We complete the proof of Lemma 2.32. □

Proof of Lemma 2.29. Assume that g is an $H_{\mathcal{L}}^{\frac{n}{n+\alpha}}$–atom associated to a ball $B = B(x_0, r)$. By Lemma 2.30 and Lemma 2.31, the following integral is absolutely convergent and therefore it can be described as

$$I = \int_{\mathbb{R}^{n+1}_+} F(x,t) \overline{G(x,t)}\, \frac{dx\, dt}{t} = \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x) \overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)}\, \frac{dx\, dt}{t}.$$

Proposition 2.15*(b)* and $\beta > \alpha + \varepsilon$ imply that $q_t(x,y) := t^\beta \partial_t^\beta \mathcal{P}_t(x,y)$ satisfies (2.29) in Lemma 2.32. Therefore, since $f \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+\alpha+\varepsilon)} dx)$, Fubini's theorem can be applied in the following:

$$\begin{aligned}\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x) \overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)}\, dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t(x,y) f(y) \overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)}\, dy\, dx \\ &= \int_{\mathbb{R}^n} f(y) \overline{(t^\beta \partial_t^\beta \mathcal{P}_t)^2 g(y)}\, dy.\end{aligned}$$

In fact, by Proposition 2.15*(b)* and Theorem 2.32, we have

$$\begin{aligned}\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x)\overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)}\, dx &\leqslant C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{t^\beta}{(|x-y|+t)^{n+\beta}} f(y)\, dy \frac{1}{(1+|x|)^{n+\beta}}\, dx\\ &\leqslant C_t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{1}{(|x-y|+1)^{n+\beta}} f(y)\, dy \frac{1}{(1+|x|)^{n+\beta}}\, dx\\ &= C_t \iint_{|x-y|>\frac{|y|}{2}} \frac{1}{(|x-y|+1)^{n+\beta}} f(y)\, dy \frac{1}{(1+|x|)^{n+\beta}}\, dx\\ &\quad + C_t \iint_{|x-y|\leqslant\frac{|y|}{2}} \frac{1}{(|x-y|+1)^{n+\beta}} f(y)\, dy \frac{1}{(1+|x|)^{n+\beta}}\, dx.\end{aligned}$$

If $|x-y|\leqslant \frac{|y|}{2}$, then $|y|\leqslant |y-x|+|x|\leqslant \frac{|y|}{2}+|x|$, and so $\frac{|y|}{2}\leqslant |x|$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} t^\beta \partial_t^\beta \mathcal{P}_t f(x)\overline{t^\beta \partial_t^\beta \mathcal{P}_t g(x)}\, dx &\leqslant C_t \iint_{|x-y|>\frac{|y|}{2}} \frac{1}{(|y|+1)^{n+\beta}} f(y)\, dy \frac{1}{(1+|x|)^{n+\beta}}\, dx\\ &\quad + C_t \iint_{|x-y|\leqslant\frac{|y|}{2}} \frac{1}{(|x-y|+1)^{n+\beta}} f(y)\, dy \frac{1}{(1+|y|)^{n+\beta}}\, dx\\ &\leqslant C.\end{aligned}$$

Hence, we can apply Fubini's theorem.

So that,

$$\begin{aligned} I &= \lim_{\epsilon\to 0}\int_\epsilon^{1/\epsilon}\left[\iint_{\mathbb{R}^n} f(y)\overline{(t^\beta\partial_t^\beta\mathcal{P}_t)^2 g(y)}\, dy\right]\frac{dt}{t}\\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} f(y)\left[\int_\epsilon^{1/\epsilon}\overline{t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t} g(y)}\,\frac{dt}{t}\right] dy. \end{aligned} \tag{2.30}$$

We claim that

$$\sup_{\epsilon>0}\left|\int_\epsilon^{1/\epsilon} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t} g(y)\,\frac{dt}{t}\right| \leqslant C(1+|y|)^{-(n+\alpha+\varepsilon)}, \tag{2.31}$$

for any $y\in\mathbb{R}^n$. To prove (2.31) we first note that

$$\begin{aligned}\left|\int_\epsilon^{1/\epsilon} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t} g(y)\,\frac{dt}{t}\right| &\leqslant \left|\int_\epsilon^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t} g(y)\,\frac{dt}{t}\right| + \left|\int_{1/\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t} g(y)\,\frac{dt}{t}\right|\\ &= \left|\int_{\mathbb{R}^n}\int_\epsilon^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t}(x,y)\,\frac{dt}{t}\, g(x)\, dx\right| + \left|\int_{\mathbb{R}^n}\int_{1/\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t}(x,y)\,\frac{dt}{t}\, g(x)\, dx\right|.\end{aligned}$$

Hence, to prove (2.31) it is enough to check that the kernel

$$\int_\epsilon^\infty t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{2t}(x,y)\,\frac{dt}{t} = 2^{[2\beta]-2\beta+1}\int_{2\epsilon}^\infty t^{2\beta}\partial_t^{2\beta}\mathcal{P}_{t}(x,y)\,\frac{dt}{t}, \tag{2.32}$$

satisfies estimate (2.29) of Lemma 2.32, for any $\epsilon>0$. To verify this we consider three cases.

Case I: $2\beta < 1$. Making a change of variables in the definition of the fractional derivative (5.5), applying Fubini's theorem and integrating by parts,

$$\begin{aligned}
\int_{2\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_t(x,y)\,\frac{dt}{t} &= C\int_{2\epsilon}^{\infty} t^{2\beta}\int_t^{\infty}\partial_u\mathcal{P}_u(x,y)(u-t)^{-2\beta}\,du\,\frac{dt}{t}\\
&= C\int_{2\epsilon}^{\infty}\partial_u\mathcal{P}_u(x,y)\int_{2\epsilon}^{u}\left(\frac{t}{u-t}\right)^{2\beta}\frac{dt}{t}\,du\\
&= C\int_{2\epsilon}^{\infty}\partial_u\mathcal{P}_u(x,y)\int_{\frac{2\epsilon}{u}}^{1}\left(\frac{w}{1-w}\right)^{2\beta}\frac{dw}{w}\,du = C\int_{2\epsilon}^{\infty}\mathcal{P}_u(x,y)\left(\frac{2\epsilon}{u-2\epsilon}\right)^{2\beta}\frac{du}{u}\\
&= C\int_{2\epsilon}^{\infty}\mathcal{P}_u(x,y)\left(\frac{2\epsilon}{u-2\epsilon}\right)^{2\beta}\chi_A(u)\,\frac{du}{u} + C\int_{2\epsilon}^{\infty}\mathcal{P}_u(x,y)\left(\frac{2\epsilon}{u-2\epsilon}\right)^{2\beta}\chi_{A^c}(u)\,\frac{du}{u}\\
&=: I' + II',
\end{aligned}$$

where $A = \{u - 2\epsilon \leqslant \epsilon + |x-y|\}$. Observe that in the equalities above we applied the assumption $2\beta < 1$ to have convergent integrals. Let us first estimate I'. By Proposition 2.15*(a)* and since $\alpha + \varepsilon < 2\beta$ we get that for any $N > 0$,

$$\begin{aligned}
|I'| &\leqslant C\int_{2\epsilon}^{\infty}\frac{u}{(|x-y|+u)^{n+1}}\left(1+\frac{u}{\rho(x)}+\frac{u}{\rho(y)}\right)^{-N}\left(\frac{2\epsilon}{u-2\epsilon}\right)^{2\beta}\chi_A(u)\,\frac{du}{u}\\
&\leqslant C\frac{\epsilon^{2\beta}}{(|x-y|+\epsilon)^{n+1}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{2\epsilon}^{3\epsilon+|x-y|}(u-2\epsilon)^{-2\beta}\,du\\
&\leqslant C\epsilon^{2\beta}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}(|x-y|+\epsilon)^{-n-2\beta}\\
&\leqslant C\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\epsilon^{-n}\left(1+\frac{|x-y|}{\epsilon}\right)^{-(n+\alpha+\varepsilon)}.
\end{aligned}$$

We continue now with II'. Note that in II' we have $u - 2\epsilon > |x-y| + \epsilon$ so, again by Proposition 2.15*(a)*,

$$\begin{aligned}
|II'| &\leqslant C\left(\frac{\epsilon}{\epsilon+|x-y|}\right)^{2\beta}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{2\epsilon}^{\infty}(|x-y|+u)^{-n-1}\,du\\
&= C\left(\frac{\epsilon}{\epsilon+|x-y|}\right)^{2\beta}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}(\epsilon+|x-y|)^{-n}\\
&\leqslant C\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\epsilon^{-n}\left(1+\frac{|x-y|}{\epsilon}\right)^{-(n+\alpha+\varepsilon)}.
\end{aligned}$$

Case II: $2\beta = 1$. By Proposition 2.15*(b)* and integrating by parts it is easy to verify that $\int_{\epsilon}^{\infty}\partial_t\mathcal{P}_{2t}(x,y)\,dt$ satisfies condition (2.29) for any $\epsilon > 0$.

Case III: $2\beta > 1$. Let $k \geqslant 2$ be the integer such that $k - 1 < 2\beta \leqslant k$. Note that the estimate is easy when $2\beta = k$, just integrating by parts. When $k - 1 < 2\beta < k$ we make a computation similar to the case $2\beta < 1$. In fact,

$$\begin{aligned}\int_{2\epsilon}^{\infty} t^{2\beta}\partial_t^{2\beta}\mathcal{P}_t(x,y)\ \frac{dt}{t} &= C\int_{2\epsilon}^{\infty} t^{2\beta}\int_t^{\infty}\partial_u^k\mathcal{P}_u(x,y)(u-t)^{k-2\beta-1}\ du\ \frac{dt}{t}\\ &= C\int_{2\epsilon}^{\infty}\partial_u^k\mathcal{P}_u(x,y)\int_{2\epsilon}^{u} t^{2\beta}(u-t)^{k-2\beta-1}\ \frac{dt}{t}\ du\\ &= C\int_{2\epsilon}^{\infty} u^{k-1}\partial_u^k\mathcal{P}_u(x,y)\int_{\frac{2\epsilon}{u}}^{1} w^{2\beta}(1-w)^{k-2\beta-1}\ \frac{dw}{w}\ du\\ &= C\int_{2\epsilon}^{\infty} u^{k-1}\partial_u^{k-1}\mathcal{P}_u(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u}\\ &\quad + C\int_{2\epsilon}^{\infty} u^{k-2}\partial_u^{k-2}\mathcal{P}_u(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u}\\ &\quad + \cdots + C\int_{2\epsilon}^{\infty} u\partial_u\mathcal{P}_u(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u}\\ &\quad + C\int_{2\epsilon}^{\infty}\mathcal{P}_u(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u}.\end{aligned} \tag{2.33}$$

In the above equalities, we have used equality

$$\begin{aligned}&u^{k-1}\partial_u^k\mathcal{P}_u(x,y)\\ &= \partial_u(u^{k-1}\partial_u^{k-1}\mathcal{P}_u(x,y)) - (k-1)\partial_u(u^{k-2}\partial_u^{k-2}\mathcal{P}_u(x,y)) + \cdots + (-1)^{(k-1)}(k-1)!\mathcal{P}_u(x,y).\end{aligned}$$

For any $1 \leqslant m \leqslant k-1$ apply Proposition 2.15*(b)* to get that for any $N > 0$

$$\begin{aligned}&\left|\int_{2\epsilon}^{\infty} u^m\partial_u^m\mathcal{P}_u(x,y)\frac{(2\epsilon)^{2\beta}u^{1-k}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u}\right|\\ &\leqslant C\int_{2\epsilon}^{\infty}\frac{u^m}{(u+|x-y|)^{n+m}}\left(1+\frac{u}{\rho(x)}+\frac{u}{\rho(y)}\right)^{-N}\frac{(2\epsilon)^{2\beta}}{(u-2\epsilon)^{1+2\beta-k}}\ \frac{du}{u^k}\\ &\leqslant C\frac{\epsilon^{2\beta}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{2\epsilon}^{\infty}(u-2\epsilon)^{k-2\beta-1}\ \frac{du}{u^{k-m}}\\ &= C\frac{\epsilon^{2\beta}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{2\epsilon}^{3\epsilon}(u-2\epsilon)^{k-2\beta-1}\ \frac{du}{u^{k-m}}\\ &\quad + C\frac{\epsilon^{2\beta}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{3\epsilon}^{\infty}(u-2\epsilon)^{k-2\beta-1}\ \frac{du}{u^{k-m}}\\ &=: I'' + II''.\end{aligned}$$

For I'', since $2\beta < k$ and $m \geqslant 1 > \alpha + \varepsilon$, we obtain

$$\begin{aligned}
I'' &\leqslant C\frac{\epsilon^{2\beta}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\frac{1}{\epsilon^{k-m}}\int_{2\epsilon}^{3\epsilon}(u-2\epsilon)^{k-2\beta-1}\,du \\
&= C\frac{\epsilon^{m}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N} \\
&\leqslant C\frac{1}{(\epsilon+|x-y|)^{n}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\left(\frac{\epsilon}{\epsilon+|x-y|}\right)^{\alpha+\varepsilon} \\
&= C\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\epsilon^{-n}\left(1+\frac{|x-y|}{\epsilon}\right)^{-(n+\alpha+\varepsilon)}.
\end{aligned}$$

For II'', since $\frac{1}{u} < \frac{1}{u-2\epsilon}$ and $m < 2\beta$, we also have

$$\begin{aligned}
II'' &\leqslant C\frac{\epsilon^{2\beta}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\int_{3\epsilon}^{\infty}(u-2\epsilon)^{m-2\beta-1}\,du \\
&\leqslant C\frac{\epsilon^{m}}{(\epsilon+|x-y|)^{n+m}}\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N} \\
&\leqslant C\left(1+\frac{\epsilon}{\rho(x)}+\frac{\epsilon}{\rho(y)}\right)^{-N}\epsilon^{-n}\left(1+\frac{|x-y|}{\epsilon}\right)^{-(n+\alpha+\varepsilon)}.
\end{aligned}$$

For the last term of (2.33) we have the same estimate as above by Proposition 2.15*(b)*.

Hence, from the three cases above we see that the kernel (2.32) satisfies condition (2.29) in Lemma 2.32, for any $\epsilon > 0$. Therefore can pass the limit inside the integral in (2.30). Then, by Lemma 2.16,

$$I = \frac{4^{\beta}}{\Gamma(2\beta)}\int_{\mathbb{R}^n} f(y)\overline{g(y)}\,dy.$$

This establishes Lemma 2.29 and it finally completes the proof of *(iii)* $\Longrightarrow$ *(i)*. □

Proof of Theorem 2.10*(II)*

Recall that the proof of Theorem 2.10*(I)* is contained in the proof of Theorem 2.9, since it works also when $\alpha = 1$. The argument for the converse statement was given at the beginning of this section. Let us continue with the proof of Theorem 2.10*(II)*. To that end we need the following proposition.

Proposition 2.33. *Let* $0 < \alpha \leqslant 1$ *and* f *be a function in* $L^\infty(\mathbb{R}^n)$ *such that* $|f(x)| \leqslant C\rho(x)^{\alpha}$, *for some constant* C *and all* $x \in \mathbb{R}^n$. *Then*

$$\|t^{\beta}\partial_t^{\beta}\mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^{\alpha}, \quad \textit{for any } \beta > \alpha,$$

if and only if

$$|f(x+y) + f(x-y) - 2f(x)| \leqslant C\,|y|^{\alpha}, \quad \textit{for all } x, y \in \mathbb{R}^n.$$

Just for a moment we take the proposition for granted. Let us show how it can be applied to prove Theorem 2.10*(II)*.

Proof of Theorem 2.10(II). In a first step we take $n = 1$. The idea is to consider the Weierstrass-Hardy non-differentiable function as in [79]:

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} e^{2\pi i 2^k x}, \quad x \in \mathbb{R}.$$

Observe that for $\mathcal{L}_\mu$ we have $\rho(x) \equiv \frac{1}{\sqrt{\mu}}$. Therefore there exists a constant $C = 2\sqrt{\mu}$ such that $|f(x)| \leqslant \sum_{k=1}^{\infty} 2^{-k} = 1 \leqslant \frac{C}{\sqrt{\mu}} = C\rho(x)$, for all $x \in \mathbb{R}$. Now, for any $y \in \mathbb{R}$,

$$f(x+y) + f(x-y) - 2f(x) = 2\sum_{k=1}^{\infty} 2^{-k}\big(\cos(2\pi 2^k y) - 1\big)e^{2\pi i 2^k x}.$$

Since $\left|\cos(2\pi 2^k y) - 1\right| \leqslant C(2^k y)^2$ and $\left|\cos(2\pi 2^k y) - 1\right| \leqslant 2$, we have

$$|f(x+y) + f(x-y) - 2f(x)| \leqslant C \sum_{2^k|y| \leqslant 1} 2^{-k}(2^k y)^2 + C \sum_{2^k|y|>1} 2^{-k} \leqslant C\,|y|\,.$$

So, by Proposition 2.33, $\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct$. Let us see that f can not be a function in $C^{0,1}_{\mathcal{L}_\mu}$. To arrive to a contradiction suppose that $|f(x+y) - f(x)| \leqslant C_f\,|y|$, for any $x, y \in \mathbb{R}$. Then by Bessel's inequality for L^2 periodic functions we would have

$$(C_f\,|y|)^2 \geqslant \int_0^1 |f(x+y) - f(x)|^2 \, dx = \sum_{k=1}^{\infty} 2^{-2k}|e^{2\pi i 2^k y} - 1|^2 \geqslant |y|^2 \sum_{2^k|y| \leqslant 1} |e^{2\pi i 2^k y} - 1|^2.$$

Note that in the range $2^k\,|y| \leqslant 1$ we have $|e^{2\pi i 2^k y} - 1|^2 \geqslant c(2^k y)^2$. Hence we arrive to the contradiction

$$C_f^2 \geqslant c\,|y|^2 \sum_{2^k|y| \leqslant 1} 2^{2k}.$$

For the case $n > 1$, note that we can write

$$\mathcal{L}_\mu = \mathcal{L}_\mu^1 - \frac{\partial^2}{\partial {x_2}^2} - \cdots - \frac{\partial^2}{\partial {x_n}^2},$$

where

$$\mathcal{L}_\mu^1 = -\frac{\partial^2}{\partial {x_1}^2} + \mu.$$

The operator $\mathcal{L}_\mu^1$ acts only in the one dimensional variable x_1. Let us define $g(x_1, \dots, x_n) = f(x_1)$, with f as above. Then, with an easy computation using the subordination formula (2.9), we have

$$\|t^\beta \partial_t^\beta \mathcal{P}_t g\|_{L^\infty(\mathbb{R}^n)} = \|t^\beta \partial_t^\beta e^{-t\sqrt{\mathcal{L}_\mu^1}} f\|_{L^\infty(\mathbb{R})} \leqslant Ct,$$

and, for any $x, x' \in \mathbb{R}^n$, the inequality

$$\left|g(x) - g(x')\right| = \left|f(x_1) - f(x_1')\right| \leqslant C\left|x_1 - x_1'\right| \leqslant C\left|x - x'\right|,$$

fails for any $C > 0$. Hence, we complete the proof. □

To prove Proposition 2.33 we need the following two lemmas.

Lemma 2.34. *Let f be a locally integrable function on $\mathbb{R}^n$, $n \geqslant 3$, and $\alpha > 0$. If there exists $\beta > \alpha$ such that*

$$\|t^\beta \partial_t^\beta \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant C_\beta t^\alpha, \quad \textit{for all } t > 0,$$

then for any $\sigma > \alpha$ we also have

$$\|t^\sigma \partial_t^\sigma \mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant C_\sigma t^\alpha, \quad \textit{for all } t > 0.$$

Moreover, the constants C_β and C_σ are comparable.

Proof. Assume first that $\sigma > \beta > \alpha$. Then, by hypothesis and Proposition 2.15*(b)*,

$$|t^\sigma \partial_t^\sigma \mathcal{P}_t f(x)| = |t^\sigma \partial_t^{\sigma-\beta} \mathcal{P}_{t/2}(\partial_t^\beta \mathcal{P}_{t/2} f)(x)| = t^\sigma \left|\int_{\mathbb{R}^n} \partial_t^{\sigma-\beta} \mathcal{P}_{t/2}(x,y) \partial_t^\beta \mathcal{P}_{t/2} f(y)\, dy\right|$$
$$\leqslant C t^{\sigma+\alpha-\beta} \int_{\mathbb{R}^n} \frac{1}{(|y|+t)^{n+\sigma-\beta}}\, dy = C t^\alpha.$$

Suppose now that $\alpha < \sigma < \beta$. Let k be the least positive integer for which $\sigma < \beta \leqslant \sigma + k$. Applying the case just proved above,

$$|t^\sigma \partial_t^\sigma \mathcal{P}_t f(x)| \leqslant t^\sigma \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty \left|\partial_{s_k}^{k+\sigma} \mathcal{P}_{s_k} f(x)\right|\, ds_k \cdots ds_2\, ds_1$$
$$\leqslant C t^\sigma \int_t^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty s_k^{\alpha-(k+\sigma)} ds_k \cdots ds_2\, ds_1 = C t^\alpha.$$

□

Lemma 2.35. *Let $0 < \alpha \leqslant 1$. If a function f satisfies $|f(x)| \leqslant C\rho(x)^\alpha$ for all $x \in \mathbb{R}^n$ then for any $\beta > \alpha$,*

$$\|t^\beta \partial_t^\beta (\mathcal{P}_t - P_t) f\|_{L^\infty(\mathbb{R}^n)} \leqslant C t^\alpha, \quad \textit{for all } t > 0,$$

where P_t is the classical Poisson semigroup with kernel (2.17).

Proof. Let $\beta > \alpha$ and $m = [\beta] + 1$. In a parallel way as in (2.18), we can derive a formula for the kernel $D_\beta(x,y,t)$ of the operator $t^\beta \partial_t^\beta (\mathcal{P}_t - P_t)$ in terms of the heat kernels for $\mathcal{L}$ and $-\Delta$ given in (2.5) and (2.7):

$$D_\beta(x,y,t) = t^\beta \partial_t^\beta \int_0^\infty \frac{t e^{-\frac{t^2}{4u}}}{2\sqrt{\pi}} \left(k_u(x,y) - h_u(x-y)\right) \frac{du}{u^{3/2}}$$
$$= C t^\beta \int_0^\infty \int_0^\infty H_{m+1}\left(\frac{t+s}{2\sqrt{u}}\right) e^{-\frac{(t+s)^2}{4u}} \left(\frac{1}{\sqrt{u}}\right)^{m+1} s^{m-\beta} \frac{ds}{s} \left(k_u(x,y) - h_u(x-y)\right) \frac{du}{u^{1/2}}.$$

Then, by Lemma 2.4, we have

$$\begin{aligned}\left|D_\beta(x,y,t)\right| &\leqslant Ct^\beta \int_0^\infty \int_0^\infty e^{-c\frac{(t+s)^2}{4u}} \left(\frac{1}{\sqrt{u}}\right)^{m+1} s^{m-\beta}\,\frac{ds}{s}\,|k_u(x,y)-h_u(x-y)|\,\frac{du}{u^{1/2}}\\ &\leqslant C\int_0^\infty e^{-c\frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta \left(\frac{\sqrt{u}}{\rho(y)}\right)^\alpha w_u(x-y)\,\frac{du}{u},\end{aligned}$$

where w is a nonnegative Schwartz class function on $\mathbb{R}^n$. Hence, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}|t^\beta \partial_t^\beta(\mathcal{P}_t - P_t)f(x)| &\leqslant C\int_{\mathbb{R}^n} \left|D_\beta(x,y,t)\right| |f(y)|\,dy\\ &\leqslant C\int_{\mathbb{R}^n}\int_0^\infty e^{-c\frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta \left(\frac{\sqrt{u}}{\rho(y)}\right)^\alpha w_u(x-y)\,\frac{du}{u}\,\rho(y)^\alpha\,dy\\ &\leqslant C\int_0^\infty e^{-c\frac{t^2}{4u}} \left(\frac{t}{\sqrt{u}}\right)^\beta (\sqrt{u})^\alpha\,\frac{du}{u}\\ &= Ct^\alpha \int_0^\infty e^{-v} v^{\frac{\beta-\alpha}{2}}\,\frac{dv}{v} = Ct^\alpha.\end{aligned}$$

□

Proof of Proposition 2.33. Assume that $\|t^\beta\partial_t^\beta\mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha$, for any $\beta > \alpha$. Then, by Lemma 2.35,

$$\|t^\beta\partial_t^\beta P_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant \|t^\beta\partial_t^\beta(P_t-\mathcal{P}_t)f\|_{L^\infty(\mathbb{R}^n)} + \|t^\beta\partial_t^\beta\mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha.$$

Therefore, as f is bounded, $f \in \Lambda^\alpha$, where Λ^α denotes the classical α-Lipschitz space, see [79, Ch. V]. Hence

$$|f(x+y)+f(x-y)-2f(x)| \leqslant C\,|y|^\alpha, \quad \text{for all } x,y \in \mathbb{R}^n.$$

For the converse, if $f \in L^\infty(\mathbb{R}^n)$ and $|f(x+y)+f(x-y)-2f(x)| \leqslant C\,|y|^\alpha$, $0<\alpha\leqslant 1$, then, by [79, Ch. V], we have $\|t^2\partial_t^2 P_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha$. So Lemma 2.35 gives

$$\|t^2\partial_t^2\mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant \|t^2\partial_t^2(\mathcal{P}_t-P_t)f\|_{L^\infty(\mathbb{R}^n)} + \|t^2\partial_t^2 P_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha.$$

Thus, by Lemma 2.34, $\|t^\beta\partial_t^\beta\mathcal{P}_t f\|_{L^\infty(\mathbb{R}^n)} \leqslant Ct^\alpha$ for any $\beta > \alpha$. □

Proof of Theorem 2.11*(A)*

As explained at the beginning of this section, we only need to prove the necessity part. Let $f \in BMO_\mathcal{L}$. Let us fix a ball $B = B(x_0, r)$ and write

$$f = (f-f_B)\chi_{2B} + (f-f_B)\chi_{(2B)^c} + f_B = f_1 + f_2 + f_3.$$

For f_1, by the boundedness of the area function (2.26) on $L^2(\mathbb{R}^n)$ and Remark 2.18 with $p=2$,

$$\begin{aligned}\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \frac{dx\,dt}{t} &= \frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \int_{\mathbb{R}^n} \chi_{|x-z|<t}(z)\,dz\,\frac{dx\,dt}{t^{n+1}} \\ &\leqslant \frac{1}{|B|}\int_{|x_0-z|<2r}\int_0^\infty\int_{\mathbb{R}^n} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \chi_{|x-z|<t}(z)\,\frac{dx\,dt}{t^{n+1}}\,dz \\ &= \frac{1}{|B|}\int_{|x_0-z|<2r}\iint_{\Gamma(z)} |t^\beta \partial_t^\beta \mathcal{P}_t f_1(x)|^2 \frac{dx\,dt}{t^{n+1}}\,dz \\ &\leqslant \frac{C}{|B|}\int_{2B}|f(z)-f_B|^2\,dz \leqslant C\,\|f\|^2_{BMO_{\mathcal{L}}}.\end{aligned}$$

For f_2 and $x\in B$, apply Proposition 2.15*(b)* to get

$$\begin{aligned}|t^\beta \partial_t^\beta \mathcal{P}_t f_2(x)| &\leqslant \sum_{k=2}^\infty \int_{2^kB\setminus 2^{k-1}B} |f(z)-f_B|\,|t^\beta \partial_t^\beta \mathcal{P}_t(x,z)|\,dz \\ &\leqslant \sum_{k=2}^\infty \int_{2^kB\setminus 2^{k-1}B} |f(z)-f_{2^kB}|\,|t^\beta \partial_t^\beta \mathcal{P}_t(x,z)|\,dz \\ &\quad + \sum_{k=2}^\infty\sum_{j=1}^k |f_{2^jB}-f_{2^{j-1}B}| \int_{2^kB\setminus 2^{k-1}B} |t^\beta \partial_t^\beta \mathcal{P}_t(x,z)|\,dz \\ &\leqslant C\sum_{k=2}^\infty \int_{2^kB\setminus 2^{k-1}B} |f(z)-f_{2^kB}|\,\frac{t^\beta}{(t+|x-z|)^{n+\beta}}\,dz \\ &\quad + C\,\|f\|_{BMO_{\mathcal{L}}}\sum_{k=2}^\infty\sum_{j=1}^k \int_{2^kB\setminus 2^{k-1}B} \frac{t^\beta}{(t+|x-z|)^{n+\beta}}\,dz \\ &\leqslant C\left(\frac{t}{r}\right)^\beta\left(\sum_{k=2}^\infty \frac{1}{2^{k\beta}}\,\frac{1}{(2^kr)^n}\int_{2^kB}|f(z)-f_{2^kB}|\,dz + \|f\|_{BMO_{\mathcal{L}}}\sum_{k=2}^\infty \frac{k}{2^{k\beta}}\right) \\ &\leqslant C\left(\frac{t}{r}\right)^\beta \|f\|_{BMO_{\mathcal{L}}}\sum_{k=2}^\infty \frac{1+k}{2^{k\beta}} = C\left(\frac{t}{r}\right)^\beta \|f\|_{BMO_{\mathcal{L}}}.\end{aligned}$$

Therefore

$$\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_2(x)|^2 \frac{dx\,dt}{t} \leqslant C\,\|f\|^2_{BMO_{\mathcal{L}}}\int_0^r \left(\frac{t}{r}\right)^{2\beta}\frac{dt}{t} = C\,\|f\|^2_{BMO_{\mathcal{L}}}.$$

Let us finally consider f_3. Assume that $r\geqslant \rho(x_0)$. By Proposition 2.15*(d)*, for some $0<\delta'\leqslant\delta_0$ with $\delta'<\beta$,

$$|t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)| \leqslant C\,|f_B|\,\frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N} \leqslant C\,\|f\|_{BMO_{\mathcal{L}}}\frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}.$$

Hence

$$\begin{aligned}\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \,\frac{dx\,dt}{t} &\leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2 \frac{1}{|B|}\int_{\widehat{B}} \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}}\,\frac{dx\,dt}{t}\\ &\leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2 \frac{1}{|B|}\int_B \left(\int_0^{\rho(x)} + \int_{\rho(x)}^\infty\right) \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}}\,\frac{dt}{t}\,dx.\end{aligned} \tag{2.34}$$

On one hand,

$$\int_0^{\rho(x)} \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}}\,\frac{dt}{t} \leqslant \int_0^{\rho(x)} (t/\rho(x))^{2\delta'}\,\frac{dt}{t} = C.$$

On the other hand,

$$\int_{\rho(x)}^\infty \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}}\,\frac{dt}{t} \leqslant \int_{\rho(x)}^\infty (t/\rho(x))^{2\delta'-2N}\,\frac{dt}{t} = C.$$

Therefore from (2.34) we obtain that if $r \geqslant \rho(x_0)$ then

$$\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \,\frac{dx\,dt}{t} \leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2.$$

Suppose that $r < \rho(x_0)$. Apply Remark 2.20, Proposition 2.15*(d)* with some $\delta' > 1/2$ and Lemma 2.1 to get

$$\begin{aligned}\frac{1}{|B|}\int_{\widehat{B}} |t^\beta \partial_t^\beta \mathcal{P}_t f_3(x)|^2 \,\frac{dx\,dt}{t} &\leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2 \left(1+\log\frac{\rho(x_0)}{r}\right)^2 \frac{1}{|B|}\int_{\widehat{B}} \frac{(t/\rho(x))^{2\delta'}}{(1+t/\rho(x))^{2N}}\,\frac{dx\,dt}{t}\\ &\leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2 \left(1+\log\frac{\rho(x_0)}{r}\right)^2 \frac{1}{|B|}\int_B\int_0^r (t/\rho(x_0))^{2\delta'}\,\frac{dt}{t}\,dx\\ &= C\,\|f\|_{BMO_{\mathcal{L}}}^2 \left(1+\log\frac{\rho(x_0)}{r}\right)^2 \left(\frac{r}{\rho(x_0)}\right)^{2\delta'}\\ &\leqslant C\,\|f\|_{BMO_{\mathcal{L}}}^2, \qquad \text{for all } r < \rho(x_0).\end{aligned}$$

This finishes the proof.

Proof of Theorem 2.11*(B)*

As in the argument of the proof of Theorem 2.10*(II)*, we only need to consider the case $n = 1$. Let

$$f(x) = \max\left\{\log\frac{1}{|x|},\ 0\right\}, \quad x \in \mathbb{R}.$$

It is well known that f belongs to the classical $BMO(\mathbb{R})$. Observe that the function f is nonnegative and it is supported in $[-1,1]$. For every x we have $\rho(x) = \frac{1}{\sqrt{\mu}}$. Hence, for $r \geqslant \rho(x)$ and $B(x_0,r) = [x_0 - r, x_0 + r]$,

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(x)|\,dx \leqslant \frac{1}{2r}\int_{B(0,1)} |f(x)|\,dx \leqslant C\sqrt{\mu}.$$

So $f \in BMO_{\mathcal{L}_\mu}$.

Now we want to prove that $\sup_{t>0} |t\partial_t \mathcal{P}_t f(0)| = \infty$. In fact, we have

$$\begin{aligned}
t\partial_t \mathcal{P}_t f(0) &= C\int_0^\infty t\left(1-\frac{t^2}{2s}\right)\frac{e^{-t^2/(4s)}}{s^{3/2}}\int_{|y|<1}\frac{e^{-y^2/(4s)}}{s^{1/2}}(-\log|y|)\,dy\;e^{-s\mu}\,ds\\
&= C\int_0^\infty t\left(1-\frac{t^2}{2s}\right)\frac{e^{-t^2/(4s)}}{s^{3/2}}\int_{|zt|<1}\frac{e^{-(zt)^2/(4s)}}{s^{1/2}}\;t(-\log|zt|)\,dz\;e^{-s\mu}\,ds\\
&= C\int_0^\infty w^2\left(1-w^2\right)e^{-w^2/2}\int_{|zt|<1}\frac{e^{-(zw)^2/2}}{s^{1/2}}\;(-\log|zt|)\,dz\;e^{-\frac{t^2}{2w^2}\mu}\,\frac{dw}{w}\\
&= C\int_0^\infty w\left(1-w^2\right)e^{-w^2/2}\int_{|zt|<1}e^{-(zw)^2/2}(-\log|z|)\,dz\;e^{-\frac{t^2}{2w^2}\mu}\,dw\\
&\quad + C\int_0^\infty w\left(1-w^2\right)e^{-w^2/2}\int_{|zt|<1}e^{-(zw)^2/2}(-\log|t|)\,dz\;e^{-\frac{t^2}{2w^2}\mu}\,dw\\
&=: I + II.
\end{aligned}$$

Observe that

$$\begin{aligned}
|I| &\leqslant C\int_0^\infty we^{-w^2/c}\int_{\mathbb{R}}e^{-(zw)^2/2}\left|\log|z|\right|\,dz\,dw\\
&\leqslant C\int_0^\infty we^{-w^2/c}\left(\int_{|z|<1}e^{-(zw)^2/2}(-\log|z|)\,dz+\int_{|z|>1}e^{-(zw)^2/2}\log|z|\,dz\right)\,dw\\
&\leqslant C\int_0^\infty we^{-w^2/c}\left(\int_{|z|<1}(-\log|z|)\,dz+\int_{|z|>1}e^{-(zw)^2/2}|z|^{\delta_0}\,dz\right)\,dw\\
&\leqslant C\int_0^\infty we^{-w^2/c}\left(1+\frac{1}{w^{\delta_0}}\right)\,dw \leqslant C,
\end{aligned}$$

where $\delta_0 < 1$. For the second integral,

$$|II| \leqslant C\left|\log|t|\right|\int_0^\infty we^{-w^2/c}\int_{\mathbb{R}}e^{-(zw)^2/2}\,dz\,dw = C\left|\log|t|\right|\int_0^\infty e^{-w^2/c}\,dw = C\left|\log|t|\right|.$$

Therefore the two integrals that define $t\partial_t\mathcal{P}_t f(0)$ are (absolutely) convergent. The limit when $t \to 0$ of the second term II above is infinity. Thus $t\partial_t\mathcal{P}_t f(0) \to \infty$ as $t \to 0$. We complete the proof with $\beta = 1$.

Chapter 3

Regularity estimates in Hölder spaces for Schrödinger operators via a T1 theorem

In this chapter we shall study the regularity estimates in the Hölder classes $C^{0,\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$, of operators associated with the time independent Schrödinger operator in $\mathbb{R}^n$, $n \geqslant 3$, $\mathcal{L} = -\Delta + V$.

It is well-known that the classical Hölder space $C^{\alpha}(\mathbb{R}^n)$ can be identified with the Campanato space BMO^{α}, see [20]. In the Schrödinger case the analogous result was proved by B. Bongioanni, E. Harboure and O. Salinas in [13]. They identified the Hölder space associated to $\mathcal{L}$ with a Campanato type $BMO^{\alpha}_{\mathcal{L}}$ space, see Proposition 2.27. Therefore, in order to study regularity estimates we can take advantage of this characterization. In fact we shall present our results as boundedness of operators between $BMO^{\alpha}_{\mathcal{L}}$ spaces.

We will give a T1-type criterion for the boundedness of some operators in $BMO^{\alpha}_{\mathcal{L}}$ spaces in Section 3.1 first. With this T1-type criterion, we get the regularity estimates for some operators related to $\mathcal{L}$ as applications in Section 3.2.

3.1 T1-type criterions on $BMO^{\alpha}_{\mathcal{L}}$-spaces

The main point of this section is to give a simple T1 criterion for boundedness in $BMO^{\alpha}_{\mathcal{L}}$ of the so-called γ-Schrödinger-Calderón-Zygmund operators T, see Definition 3.6. The advantage of this criterion is that everything reduces to check a certain condition on the function T1.

We use the notation $f_B = \frac{1}{|B|}\int_B f$. The first result reads as follows.

Theorem 3.1 (T1-type criterion for $BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$). *Let* T *be a* γ*-Schrödinger-Calderón-Zygmund operator,* $\gamma \geqslant 0$*, with smoothness exponent* δ*, such that* $\alpha + \gamma < \min\{1, \delta\}$*. Then* T *is bounded from* $BMO^{\alpha}_{\mathcal{L}}$ *into* $BMO^{\alpha+\gamma}_{\mathcal{L}}$ *if and only if there exists a* *constant* C *such that*

$$\left(\frac{\rho(x)}{s}\right)^{\alpha}\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B |T1(y)-(T1)_B|\,dy \leqslant C,$$

for every ball $B = B(x,s)$, $x \in \mathbb{R}^n$ *and* $0 < s \leqslant \frac{1}{2}\rho(x)$. *Here* $\rho(x)$ *is the critical radii function defined in* (2.3).

We can also consider the endpoint case $\alpha = 0$.

Theorem 3.2 (T1-type criterion for $BMO_{\mathcal{L}}$). *Let* T *be a* γ*-Schrödinger-Calderón-Zygmund operator,* $0 \leqslant \gamma < \min\{1,\delta\}$, *with smoothness exponent* δ. *Then* T *is a bounded operator from* $BMO_{\mathcal{L}}$ *into* $BMO^{\gamma}_{\mathcal{L}}$ *if and only if there exists a constant* C *such that*

$$\log\left(\frac{\rho(x)}{s}\right)\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B |T1(y)-(T1)_B|\,dy \leqslant C,$$

for every ball $B = B(x,s)$, $x \in \mathbb{R}^n$ *and* $0 < s \leqslant \frac{1}{2}\rho(x)$.

Observe that for any $x \in \mathbb{R}^n$ and $0 < \alpha \leqslant 1$, if $0 < s \leqslant \frac{1}{2}\rho(x)$ then $1 + \log\frac{\rho(x)}{s} \sim \log\frac{\rho(x)}{s}$ and $1 + \frac{2^{\alpha}\left(\left(\frac{\rho(x)}{s}\right)^{\alpha}-1\right)}{2^{\alpha}-1} \sim \left(\frac{\rho(x)}{s}\right)^{\alpha}$. Therefore, tracking down the exact constants in the proof we can see that Theorem 3.2 is indeed the limit case of Theorem 3.1.

Theorem 3.2 is a generalization of the T1-type criterion given in [6] for the case of the harmonic oscillator $H = -\Delta + |x|^2$. Here we require the dimension to be $n \geqslant 3$, while in [6] the dimension can be any $n \geqslant 1$.

For $BMO^{\alpha}_{\mathcal{L}}$-spaces, we have the following propositions and lemmas.

Proposition 3.3. *Let* $B = B(x,r)$ *with* $r < \rho(x)$.

(1) (See [27, Lemma 2]) If $f \in BMO_{\mathcal{L}}$ *then* $|f_B| \leqslant C\left(1+\log\frac{\rho(x)}{r}\right)\|f\|_{BMO_{\mathcal{L}}}$.

(2) (See [55, Proposition 4.3]) If $f \in BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha \leqslant 1$, *then* $|f_B| \leqslant C_{\alpha}\|f\|_{BMO^{\alpha}_{\mathcal{L}}}\rho(x)^{\alpha}$.

(3) (See [13, Proposition 3]) A function f *belongs to* $BMO^{\alpha}_{\mathcal{L}}$, $0 \leqslant \alpha \leqslant 1$, *if and only if* f *satisfies* (i) *for every ball* $B = B(x_0,r_0)$ *with* $r_0 < \rho(x_0)$ *and* $|f|_{Q_k} \leqslant C|Q_k|^{1+\frac{\alpha}{n}}$, *for all balls* Q_k *given in the covering by critical balls above.*

Lemma 3.4 (Boundedness criterion). *Let* S *be a linear operator defined on* $BMO^{\alpha}_{\mathcal{L}}$, $0 \leqslant \alpha \leqslant 1$. *Then* S *is bounded from* $BMO^{\alpha}_{\mathcal{L}}$ *into* $BMO^{\alpha+\gamma}_{\mathcal{L}}$, $\alpha+\gamma \leqslant 1$, $\gamma \geqslant 0$, *if there exists* $C > 0$ *such that, for every* $f \in BMO^{\alpha}_{\mathcal{L}}$ *and* $k \in \mathbb{N}$,

(A_k) $\displaystyle\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}}\int_{Q_k}|Sf(x)|\,dx \leqslant C\|f\|_{BMO^{\alpha}_{\mathcal{L}}}$, *and*

(B_k) $\|Sf\|_{BMO^{\alpha+\gamma}(Q^*_k)} \leqslant C\|f\|_{BMO^{\alpha}_{\mathcal{L}}}$, *where* $BMO^{\alpha}(Q^*_k)$ *denotes the usual* BMO^{α} *space on the ball* Q^*_k.

Proof. For $\alpha = 0$ the result is already contained in [27, p. 346]. The general statement follows immediately from the definition of $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ and Lemma 2.1 (see Proposition 3.3). □

The duality of the $\mathcal{L}$-Hardy space $H^1_{\mathcal{L}}$ with $\mathrm{BMO}_{\mathcal{L}}$ was proved in [27]. As mentioned in [13], the $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ spaces are the duals of the $H^p_{\mathcal{L}}$ spaces defined in [30, 32, 31]. In fact, if $q > n$ and $0 \leqslant \alpha < 1$ then the dual of $H^{\frac{n}{n+\alpha}}_{\mathcal{L}}$ is $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$, see also [43].

In the following lemma we present examples of families of functions indexed by $x_0 \in \mathbb{R}^n$ and $0 < s \leqslant \rho(x_0)$ that are uniformly bounded in $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$. They will be very useful in the sequel.

Lemma 3.5. *There exists constants* $C, C_\alpha > 0$ *such that for every* $x_0 \in \mathbb{R}^n$ *and* $0 < s \leqslant \rho(x_0)$,

(a) the function $g_{x_0,s}(x) := \chi_{[0,s]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{s}\right) + \chi_{(s,\rho(x_0))]}(|x-x_0|)\log\left(\frac{\rho(x_0)}{|x-x_0|}\right)$, $x \in \mathbb{R}^n$, *belongs to* $\mathrm{BMO}_{\mathcal{L}}$ *and* $\|g_{x_0,s}\|_{\mathrm{BMO}_{\mathcal{L}}} \leqslant C$;

(b) the function $f_{x_0,s}(x) = \chi_{[0,s]}(|x-x_0|)\,(\rho(x_0)^\alpha - s^\alpha) + \chi_{(s,\rho(x_0))]}(|x-x_0|)\,(\rho(x_0)^\alpha - |x-x_0|^\alpha)$, $x \in \mathbb{R}^n$, *belongs to* $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$, $0 < \alpha \leqslant 1$, *and* $\|f_{x_0,s}\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}} \leqslant C_\alpha$.

Proof. The proof of part *(a)* follows the same lines as the proof of Lemma 2.1 in [6]. We omit the details.

Let us continue with *(b)*. Recall that the function $h(x) = (1 - |x|^\alpha)\chi_{[0,1]}(|x|)$ is in $\mathrm{BMO}^\alpha(\mathbb{R}^n)$. Hence, for every $R > 0$, the function $h_R(x) := R^\alpha h(x/R)$ is in $\mathrm{BMO}^\alpha(\mathbb{R}^n)$ and $\|h_R\|_{\mathrm{BMO}^\alpha(\mathbb{R}^n)} \leqslant C$, where $C > 0$ is independent of R. Moreover, for every $R > 0$ and $S \geqslant 1$, the function $h_{R,S}(x) = \min\{R^\alpha(1 - S^{-\alpha}), R^\alpha h(x/R)\}$ belongs to $\mathrm{BMO}^\alpha(\mathbb{R}^n)$ and $\|h_{R,S}\|_{\mathrm{BMO}^\alpha(\mathbb{R}^n)} \leqslant C$, where $C > 0$ does not depend on R and S. Then, since for every $x_0 \in \mathbb{R}^n$ and $0 < s \leqslant \rho(x_0)$,

$$f_{x_0,s}(x) = h_{\rho(x_0),\frac{\rho(x_0)}{s}}(x - x_0), \quad x \in \mathbb{R}^n,$$

we get $f_{x_0,s} \in \mathrm{BMO}^\alpha(\mathbb{R}^n) = C^\alpha(\mathbb{R}^n)$ and $\|f_{x_0,s}\|_{\mathrm{BMO}^\alpha(\mathbb{R}^n)} \leqslant C$. This, the obvious inequality $|f_{x_0,s}(x)| \leqslant C\rho(x)^\alpha$, for all x, uniformly in x_0 and $s \leqslant \rho(x_0)$, and Proposition 2.27 imply the conclusion. □

We denote by $L^p_c(\mathbb{R}^n)$ the set of functions $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, whose support $\mathrm{supp}(f)$ is a compact subset of $\mathbb{R}^n$.

Definition 3.6. Let $0 \leqslant \gamma < n$, $1 < p \leqslant q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$. Let T be a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\, dy, \quad f \in L^p_c(\mathbb{R}^n) \text{ and a.e. } x \notin \mathrm{supp}(f).$$

We shall say that T is a γ-Schrödinger-Calderón-Zygmund operator with regularity exponent $\delta > 0$ if for some constant C

(1) $|K(x,y)| \leqslant \frac{C}{|x-y|^{n-\gamma}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}$, for all $N>0$ and $x\neq y$,

(2) $|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leqslant C\frac{|y-z|^{\delta}}{|x-y|^{n-\gamma+\delta}}$, when $|x-y|>2|y-z|$.

Definition of Tf **for** $f\in BMO^{\alpha}_{\mathcal{L}}$, $0\leqslant\alpha\leqslant 1$**.** Suppose that $f\in BMO^{\alpha}_{\mathcal{L}}$ and $R\geqslant\rho(x_0)$, $x_0\in\mathbb{R}^n$. We define

$$Tf(x) = T\left(f\chi_{B(x_0,R)}\right)(x) + \int_{B(x_0,R)^c} K(x,y)f(y)\,dy, \quad \text{a.e. } x\in B(x_0,R).$$

Note that the first term in the right hand side makes sense since $f\chi_{B(x_0,R)}\in L^p_c(\mathbb{R}^n)$. The integral in the second term is absolutely convergent. Indeed, by Lemma 2.1, there exists a constant C such that for any $x\in B(x_0,R)$,

$$\begin{aligned}\rho(x) &\leqslant c\rho(x_0)\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{\frac{k_0}{k_0+1}} \leqslant C\left(\rho(x_0)+\rho(x_0)^{1-\frac{k_0}{k_0+1}}|x-x_0|^{\frac{k_0}{k_0+1}}\right)\\ &\leqslant C\left(R+R^{1-\frac{k_0}{k_0+1}}|x-x_0|^{\frac{k_0}{k_0+1}}\right)\leqslant C2R.\end{aligned}$$

Hence, using the γ-Schrödinger-Calderón-Zygmund condition (1) for K with $N-\gamma>\alpha$,

$$\begin{aligned}\int_{B(x_0,2R)^c}|K(x,y)||f(y)|\,dy &\leqslant C\sum_{j=1}^{\infty}\int_{2^jR<|y-x_0|\leqslant 2^{j+1}R}\frac{\rho(x)^N}{|x-y|^{n+N-\gamma}}\,|f(y)|\,dy\\ &\leqslant C\sum_{j=1}^{\infty}\frac{\rho(x)^N}{(2^jR-R)^{n+N-\gamma}}\int_{|y-x_0|\leqslant 2^{j+1}R}|f(y)|\,dy \qquad (3.1)\\ &\leqslant CR^{\alpha+\gamma}\|f\|_{BMO^{\alpha}_{\mathcal{L}}}, \quad \text{a.e. } x\in B(x_0,R).\end{aligned}$$

The definition of $Tf(x)$ is also independent of R in the sense that if $B(x_0,R)\subset B(x_0',R')$, with $R'\geqslant\rho(x_0)$, then the definition using $B(x_0',R')$ coincides almost everywhere in $B(x_0,R)$ with the one just given, because in that situation,

$$\begin{aligned}&T\left(f\chi_{B(x_0',R')}\right)(x) - T\left(f\chi_{B(x_0,R)}\right)(x)\\ &= T\left(f\chi_{B(x_0',R')\setminus B(x_0,R)}\right)(x) = \int_{B(x_0',R')\setminus B(x_0,R)} K(x,y)f(y)\,dy\\ &= \int_{B(x_0,R)^c} K(x,y)f(y)\,dy - \int_{B(x_0',R')^c} K(x,y)f(y)\,dy, \quad \text{a.e. } x\in B(x_0,R).\end{aligned}$$

The definition just given above is equally valid for $f\equiv 1\in BMO_{\mathcal{L}}$.

Next we derive an expression for Tf where T1 appears that will be useful in the proof of our main results. Let $x_0\in\mathbb{R}^n$ and $r_0>0$. For $B=B(x_0,r_0)$ we clearly have

$$f = (f-f_B)\chi_{B^{***}} + (f-f_B)\chi_{(B^{***})^c} + f_B =: f_1+f_2+f_3. \qquad (3.2)$$

Let us choose $R \geqslant \rho(x_0)$ such that $B^{***} \subset B(x_0, R)$. Using (3.2) we get

$$
\begin{aligned}
Tf(x) &= T\left(f\chi_{B(x_0,R)}\right)(x) + \int_{B(x_0,R)^c} K(x,y)f(y)\, dy \\
&= T\left((f-f_B)\chi_{B^{***}}\right)(x) + T\left((f-f_B)\chi_{B(x_0,R)\setminus B^{***}}\right)(x) + f_B T\left(\chi_{B(x_0,R)}\right)(x) \\
&\quad + \int_{B(x_0,R)^c} K(x,y)(f(y)-f_B)\, dy + f_B \int_{B(x_0,R)^c} K(x,y)\, dy \qquad (3.3)\\
&= T\left((f-f_B)\chi_{B^{***}}\right)(x) + \int_{(B^{***})^c} K(x,y)(f(y)-f_B)\, dy + f_B T1(x), \quad \text{a.e. } x \in B^{***}.
\end{aligned}
$$

We observe that there exists a constant C such that

$$
\frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T1(y)|\, dy \leqslant C, \quad \text{for all } B = B(x,\rho(x)),\ x \in \mathbb{R}^n. \qquad (3.4)
$$

Indeed, by Hölder's inequality and the $L^p - L^q$ boundedness of T,

$$
\frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |T(\chi_{B^*})(y)|\, dy \leqslant \frac{1}{|B|^{\frac{1}{q}+\frac{\gamma}{n}}} \left(\int_B |T(\chi_{B^*})(y)|^q\, dy \right)^{1/q} \leqslant C \frac{|B|^{1/p}}{|B|^{\frac{1}{q}+\frac{\gamma}{n}}} = C.
$$

By the integral representation of T and the size condition (1) on K with $N = n+\gamma$, for $y \in B(x,\rho(x))$ we have

$$
\begin{aligned}
\left|T\left(\chi_{(B^*)^c}\right)(y)\right| &\leqslant C \sum_{k=1}^{\infty} \int_{2^j\rho(x)\leqslant|x-z|<2^{j+1}\rho(x)} \frac{\rho(y)^{n+\gamma}}{|y-z|^{2n}}\, dz \\
&\leqslant C\rho(y)^{n+\gamma} \sum_{k=1}^{\infty} \frac{(2^{j+1}\rho(x))^n}{(2^j\rho(x)-\rho(x))^{2n}} \leqslant C\rho(x)^{\gamma},
\end{aligned}
$$

because $\rho(x) \sim \rho(y)$. Thus (3.4) follows by linearity.

Proof of Theorem 3.1. First we shall see that the condition on T1 implies that T is bounded from $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ into $\mathrm{BMO}^{\alpha+\gamma}_{\mathcal{L}}$. In order to do this, we will show that there exists $C > 0$ such that the properties (A_k) and (B_k) stated in Lemma 3.4 hold for every $k \in \mathbb{N}$ and $f \in \mathrm{BMO}^{\alpha}_{\mathcal{L}}$.

We begin with (A_k). According to (3.3) with $B = Q_k$,

$$
Tf(x) = T\left((f-f_{Q_k})\chi_{Q_k^{***}}\right)(x) + \int_{(Q_k^{***})^c} K(x,y)(f(y)-f_{Q_k})\, dy + f_{Q_k} T1(x), \quad \text{a.e. } x \in Q_k.
$$

As T maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$, by Hölder's inequality,

$$
\begin{aligned}
\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}} \int_{Q_k} \left|T\left((f-f_{Q_k})\chi_{Q_k^{***}}\right)(x)\right| dx &\leqslant \frac{1}{|Q_k|^{\frac{1}{q}+\frac{\alpha+\gamma}{n}}} \left(\int_{Q_k} \left|T\left((f-f_{Q_k})\chi_{Q_k^{***}}\right)(x)\right|^q dx \right)^{1/q} \\
&\leqslant \frac{C}{|Q_k|^{\frac{\alpha}{n}}} \left(\frac{1}{|Q_k|} \int_{Q_k^{***}} |f(x)-f_{Q_k}|^p\, dx \right)^{1/p} \leqslant C\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}.
\end{aligned}
$$

On the other hand, given $x \in Q_k$, we have $\rho(x) \sim \rho(x_k)$ and if $|x_k - y| > 2^j\rho(x_k)$, $j \in \mathbb{N}$, then $|x-y| \geqslant 2^{j-1}\rho(x_k)$. By the size condition (1) of the kernel K, for any $N > \alpha$ we have

$$\begin{aligned}
\frac{1}{|Q_k|^{\frac{\alpha+\gamma}{n}}}\left|\int_{(Q_k^{***})^c} K(x,y)\big(f(y)-f_{Q_k}\big)\,dy\right| &\leqslant \frac{1}{|Q_k|^{\frac{\alpha+\gamma}{n}}}\int_{(Q_k^{***})^c}|K(x,y)|\left|f(y)-f_{Q_k}\right|\,dy \\
&\leqslant \frac{C}{|Q_k|^{\frac{\alpha+\gamma}{n}}}\int_{(Q_k^{***})^c}\frac{1}{|x-y|^{n-\gamma}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}\left|f(y)-f_{Q_k}\right|\,dy \\
&\leqslant \frac{C}{|Q_k|^{\frac{\alpha+\gamma}{n}}}\sum_{j=3}^{\infty}\int_{2^j\rho(x_k)<|x_k-y|\leqslant 2^{j+1}\rho(x_k)}\frac{\rho(x)^N}{|x-y|^{n-\gamma+N}}\left|f(y)-f_{Q_k}\right|\,dy \\
&\leqslant \frac{C}{\rho(x_k)^{\alpha}}\sum_{j=3}^{\infty}\frac{\rho(x_k)^N}{\big(2^j\rho(x_k)\big)^{n+N}}\int_{|x_k-y|\leqslant 2^{j+1}\rho(x_k)}\left|f(y)-f_{Q_k}\right|\,dy \\
&\leqslant C\sum_{j=3}^{\infty}2^{-j(N-\alpha)}(j+1)\,\|f\|_{BMO_{\mathcal{L}}^{\alpha}} \leqslant C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.
\end{aligned}$$

Finally, by (3.4),

$$\frac{1}{|Q_k|^{1+\frac{\alpha+\gamma}{n}}}\int_{Q_k}\left|f_{Q_k}T1(x)\right|\,dx = \frac{|f_{Q_k}|}{|Q_k|^{\frac{\alpha}{n}}}\frac{1}{|Q_k|^{1+\frac{\gamma}{n}}}\int_{Q_k}|T1(x)|\,dx \leqslant C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

Hence, we conclude that (A_k) holds for T with a constant C that does not depend on k.

Let us continue with (B_k). Let $B = B(x_0, r_0) \subseteq Q_k^*$, where $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. Note that if $r_0 \geqslant \frac{1}{2}\rho(x_0)$ then $\rho(x_0) \sim \rho(x_k) \sim r_0$, so proceeding as above we have

$$\frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B |Tf(x)-(Tf)_B|\,dx \leqslant \frac{2}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B |Tf(x)|\,dx \leqslant C\|f\|_{BMO_{\mathcal{L}}^{\alpha}}.$$

Assume next that $0 < r_0 < \frac{1}{2}\rho(x_0)$. Using (3.3) we have

$$\begin{aligned}
\frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B |Tf(x)-(Tf)_B|\,dx &\leqslant \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B\frac{1}{|B|}\int_B |Tf_1(x)-Tf_1(z)|\,dz\,dx \\
&\quad + \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B\frac{1}{|B|}\int_B |F_2(x)-F_2(z)|\,dz\,dx \\
&\quad + \frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B |Tf_3(x)-(Tf_3)_B|\,dx =: L_1 + L_2 + L_3,
\end{aligned}$$

where $f = f_1 + f_2 + f_3$ as in (3.3) and we defined

$$F_2(x) = \int_{(B^{***})^c} K(x,y)f_2(y)\,dy, \quad x \in B.$$

Again Hölder's inequality and $L^p - L^q$ boundedness of T give $L_1 \leqslant C\,\|f\|_{BMO_{\mathcal{L}}^{\alpha}}$. Let us estimate L_2. Take $x, z \in B$ and $y \in (B^{***})^c$. Then $8r_0 < |y-x_0| \leqslant |y-x| + r_0$ and therefore

$2|x-x_0| < 4r_0 < |y-x|$. Under these conditions we can apply the smoothness of the kernel (2) and the restriction $\alpha+\gamma < \min\{1,\delta\}$ to get

$$\begin{aligned}
\frac{1}{|B|^{\frac{\alpha+\gamma}{n}}}|F_2(x)-F_2(z)| &\leqslant \frac{C}{r_0^{\alpha+\gamma}}\int_{(B^{***})^c}|K(x,y)-K(z,y)|\,|f(y)-f_B|\,dy \\
&\leqslant \frac{C}{r_0^{\alpha+\gamma}}\sum_{j=3}^{\infty}\int_{2^j r_0\leqslant|x_0-y|<2^{j+1}r_0}\frac{|x-z|^{\delta}}{|x-y|^{n-\gamma+\delta}}|f(y)-f_B|\,dy \\
&\leqslant \frac{C}{r_0^{\alpha+\gamma}}\sum_{j=3}^{\infty}\frac{r_0^{\delta}}{((2^j-1)r_0)^{n-\gamma+\delta}}\int_{2^j r_0\leqslant|x_0-y|<2^{j+1}r_0}|f(y)-f_B|\,dy \\
&\leqslant C\sum_{j=3}^{\infty}\frac{2^{-j(\delta-(\alpha+\gamma))}}{(2^{j+1}r_0)^{n+\alpha}}\int_{|x_0-y|<2^{j+1}r_0}|f(y)-f_B|\,dy \\
&\leqslant C\,\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}\sum_{j=3}^{\infty}2^{-j(\delta-(\alpha+\gamma))}(j+1) = C\,\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}.
\end{aligned}$$

Therefore, $L_2 \leqslant C\,\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}$. We finally consider L_3. Using Proposition 3.3*(2)* and the assumption on T1 it follows that

$$\begin{aligned}
L_3 &= \frac{|f_B|}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B |T1(x)-(T1)_B|\,dx \qquad (3.5)\\
&\leqslant C\,\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}\left(\frac{\rho(x_0)}{r_0}\right)^{\alpha}\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B|T1(x)-(T1)_B|\,dx \leqslant C\,\|f\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}}.
\end{aligned}$$

This concludes the proof of (B_k). Hence T is bounded from $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ into $\mathrm{BMO}^{\alpha+\gamma}_{\mathcal{L}}$.

Let us now prove the converse statement. Suppose that T is bounded from $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ into $\mathrm{BMO}^{\alpha+\gamma}_{\mathcal{L}}$. Let $x_0 \in \mathbb{R}^n$ and $0 < s \leqslant \frac{1}{2}\rho(x_0)$ and $B = B(x_0,s)$. For such x_0 and s consider the nonnegative function $f_0(x) \equiv f_{x_0,s}(x)$ defined in Lemma 3.5. Using the decomposition $f_0 = (f_0-(f_0)_B)\chi_{B^{***}}+(f_0-(f_0)_B)\chi_{(B^{***})^c}+(f_0)_B =: f_1+f_2+(f_0)_B$ we can write $(f_0)_B T1(y) = Tf_0(y)-Tf_1(y)-Tf_2(y)$, so

$$(f_0)_B\frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B|T1(y)-T1_B|\,dy \leqslant \sum_{i=0}^{2}\frac{1}{|B|^{1+\frac{\alpha+\gamma}{n}}}\int_B|Tf_i(y)-(Tf_i)_B|\,dy.$$

We can check that each of the three terms above is controlled by $C\,\|f_0\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}} \leqslant C$, where C is independent of x_0 and s. Indeed, the case $i=0$ follows by the hypothesis about the boundedness of T. For $i=1$ the estimate follows, as usual, by Hölder's inequality and L^p-L^q boundedness of T. The term for $i=2$ is done as L_2 above. Thus, since $(f_0)_B = C(\rho(x_0)^{\alpha}-s^{\alpha})$ we obtain

$$\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B|T1(y)-(T1)_B|\,dy \leqslant C.$$

□

Proof of Theorem 3.2. The proof is the same as the proof of Theorem 3.1 putting $\alpha = 0$ everywhere, except for just two differences. The first one is the estimate of the term L_3, where we must apply Proposition 3.3*(1)* instead of *(2)*. The second difference is the proof of the converse, where instead of $f_{x_0,s}(x)$ we have to consider the function $g_{x_0,s}(x)$ of Lemma 3.5. □

At the end of this section, we give an easy application of Theorem 3.1 and Theorem 3.2 about the pointwise multipliers in $BMO^{\alpha}_{\mathcal{L}}$, $0 \leqslant \alpha < 1$. For pointwise multipliers of the classical BMO^{α} spaces see the papers by S. Bloom [11], S. Janson [49] and E. Nakai and K. Yabuta [60].

Proposition 3.7. *Let* ψ *be a measurable function on* $\mathbb{R}^n$. *We denote by* T_ψ *the multiplier operator defined by* $T_\psi(f) = f\psi$. *Then*

(A) T_ψ *is a bounded operator in* $BMO_{\mathcal{L}}$ *if and only if* $\psi \in L^\infty(\mathbb{R}^n)$ *and there exists* $C > 0$ *such that, for all balls* $B = B(x_0, s)$ *with* $0 < s < \frac{1}{2}\rho(x_0)$,

$$\log\left(\frac{\rho(x_0)}{s}\right)\frac{1}{|B|}\int_B |\psi(y) - \psi_B|\, dy \leqslant C.$$

(B) T_ψ *is a bounded operator in* $BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$, *if and only if* $\psi \in L^\infty(\mathbb{R}^n)$ *and there exists* $C > 0$ *such that, for all balls* $B = B(x_0, s)$ *with* $0 < s < \frac{1}{2}\rho(x_0)$,

$$\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\frac{1}{|B|}\int_B |\psi(y) - \psi_B|\, dy \leqslant C.$$

Remark 3.8. If $\psi \in C^{0,\beta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $0 < \beta \leqslant 1$, then T_ψ is bounded on $BMO_{\mathcal{L}}$. Moreover, if for some γ-Schrödinger-Calderón-Zygmund operator T and T1 defines a pointwise multiplier in $BMO^{\alpha}_{\mathcal{L}}$ then the proposition above and Theorems 3.2 and 3.1 imply that T is a bounded operator on $BMO^{\alpha}_{\mathcal{L}}$.

Proof of Proposition 3.7. Let us first prove *(B)*. Suppose that T_ψ is a bounded operator on $BMO^{\alpha}_{\mathcal{L}}$, $0 < \alpha < 1$. For the function $f_{x_0,s}(x)$ defined in Lemma 3.5 and any ball $B = B(x_0, s)$ with $0 < s \leqslant \frac{1}{2}\rho(x_0)$, by Proposition 3.3*(2)* applied to $f\psi$ and the hypothesis, we get

$$\begin{aligned}\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\frac{1}{|B|}\int_B |\psi(x)|\, dx &\leqslant C_\alpha \frac{(\rho(x_0)^\alpha - s^\alpha)}{|B|^{1+\frac{\alpha}{n}}}\int_B |\psi(x)|\, dx = \frac{C_\alpha}{|B|^{1+\frac{\alpha}{n}}}\int_B |\psi(x) f_{x_0,s}(x)|\, dx \\ &\leqslant \frac{C_\alpha}{|B|^{1+\frac{\alpha}{n}}}\int_B |(\psi f_{x_0,s})(x) - (\psi f_{x_0,s})_B|\, dx + \frac{C_\alpha}{|B|^{\frac{\alpha}{n}}}(\psi f_{x_0,s})_B \\ &\leqslant C_\alpha \|f_{x_0,s}\|_{BMO^{\alpha}_{\mathcal{L}}} + C_\alpha\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\|\psi f_{x_0,s}\|_{BMO^{\alpha}_{\mathcal{L}}} \\ &\leqslant C_\alpha\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\|f_{x_0,s}\|_{BMO^{\alpha}_{\mathcal{L}}} \leqslant C\left(\frac{\rho(x_0)}{s}\right)^{\alpha}.\end{aligned}$$

Hence $|\psi|_B \leqslant C$ with C independent of B, so that ψ is bounded. Next we check the condition on ψ. We have

$$\begin{aligned}\left(\frac{\rho(x_0)}{s}\right)^{\alpha}\frac{1}{|B|}\int_B|\psi(x)-\psi_B|\,dx &\leqslant C_\alpha\frac{(\rho(x_0)^\alpha-s^\alpha)}{|B|^{1+\frac{\alpha}{n}}}\int_B|\psi(x)-\psi_B|\,dx\\ &\leqslant \frac{C_\alpha}{|B|^{1+\frac{\alpha}{n}}}\int_B|\psi(x)f_{x_0,s}(x)-(\psi f_{x_0,s})_B|\,dx\\ &\leqslant C_\alpha\|\psi f_{x_0,s}\|_{BMO^\alpha_{\mathcal{L}}}\leqslant C_\alpha\|f_{x_0,s}\|_{BMO^\alpha_{\mathcal{L}}}\leqslant C.\end{aligned}$$

The constants C and C_α appearing in this proof do not depend on $x_0 \in \mathbb{R}^n$ and $0 < s \leqslant \frac{1}{2}\rho(x_0)$.

For the converse statement, assume ψ satisfies the properties required in the hypothesis. The kernel of the operator $T = T_\psi$ is zero and $T_\psi 1(x) = \psi(x)$, so the conclusion follows by Theorem 3.1.

The proof of *(A)* is completely analogous by using the function $g_{x_0,s}(x)$ of Lemma 3.5 instead of $f_{x_0,s}(x)$ and by applying Theorem 3.2. □

3.2 Regularity estimates by T1-type criterions

In this section, we will use our T1-criterion to get some regularity estimates in $C^{0,\alpha}_{\mathcal{L}}$.

First, we need the following remark to extend our T1-criterion to vector-valued case.

Remark 3.9 (Vector-valued setting). Theorems 3.2 and 3.1 can also be stated in a vector valued setting. If Tf takes values in a Banach space $\mathbb{B}$ and the absolute values in the conditions are replaced by the norm in $\mathbb{B}$ then both results hold.

By the T1-type criterions we can get the following regularity estimates.

Theorem 3.10. *Let $0 \leqslant \alpha < \min\{1, 2-\frac{n}{q}\}$. The maximal operators associated with the heat semigroup $\{\mathcal{T}_t\}_{t>0}$ and with the generalized Poisson operators $\{\mathcal{P}^\sigma_t\}_{t>0}$, the Littlewood-Paley g-functions given in terms of the heat and the Poisson semigroups, and the Laplace transform type multipliers $m(\mathcal{L})$, are bounded from $BMO^\alpha_{\mathcal{L}}$ into itself.*

3.2.1 Maximal operators for the heat–diffusion semigroup $e^{-t\mathcal{L}}$

Let $\{\mathcal{T}_t\}_{t>0}$ be the heat–diffusion semigroup associated to $\mathcal{L}$. To prove that the maximal operator $\mathcal{T}^*$ defined by $\mathcal{T}^*f(x) = \sup_{t>0}|\mathcal{T}_tf(x)|$ is bounded from $BMO^\alpha_{\mathcal{L}}$ into itself we give a vector-valued interpretation of the operator and apply Remark 3.9. Indeed, it is clear that $\mathcal{T}^*f = \|\mathcal{T}_tf\|_E$, with $E = L^\infty((0,\infty), dt)$. Hence, it is enough to show that the operator $V(f) := (\mathcal{T}_tf)_{t>0}$ is bounded from $BMO^\alpha_{\mathcal{L}}$ into $BMO^\alpha_{\mathcal{L},E}$, where the space $BMO^\alpha_{\mathcal{L},E}$ is defined in the obvious way by replacing the absolute values $|\cdot|$ by norms $\|\cdot\|_E$.

By the Spectral Theorem, V is bounded from $L^2(\mathbb{R}^n)$ into $L^2_E(\mathbb{R}^n)$. The desired result is then deduced from the following proposition.

Proposition 3.11. *Let* $x, y, z \in \mathbb{R}^n$ *and* $N > 0$. *Then*

(i) $\|k_t(x,y)\|_E \leqslant \frac{C}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}$;

(ii) $\|k_t(x,y)-k_t(x,z)\|_E + \|k_t(y,x)-k_t(z,x)\|_E \leqslant C_\delta \frac{|y-z|^\delta}{|x-y|^{n+\delta}}$, *when* $|x-y| > 2|y-z|$, *for any* $0<\delta<2-\frac{n}{q}$;

(iii) there exists a constant C *such that for every ball* $B = B(x,s)$ *with* $0 < s \leqslant \frac{1}{2}\rho(x)$,

$$\log\left(\frac{\rho(x)}{s}\right)\frac{1}{|B|}\int_B \|\mathcal{T}_t 1(y) - (\mathcal{T}_t 1)_B\|_E \, dy \leqslant C,$$

and, if $\alpha < \min\{1, 2-\frac{n}{q}\}$ *then*

$$\left(\frac{\rho(x)}{s}\right)^\alpha \frac{1}{|B|}\int_B \|\mathcal{T}_t 1(y) - (\mathcal{T}_t 1)_B\|_E \, dy \leqslant C.$$

Proof. Let us begin with *(i)*. If $t > |x-y|^2$ then the conclusion is immediate from the estimate of Lemma 2.3. Assume that $t \leqslant |x-y|^2$. Then

$$\begin{aligned}
0 \leqslant k_t(x,y) &\leqslant \frac{C}{|x-y|^n}\, e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\\
&= \frac{C}{|x-y|^n}\, e^{-c\frac{|x-y|^2}{t}}\left(\frac{\sqrt{t}}{|x-y|}\right)^{-N}\left(\frac{|x-y|}{\sqrt{t}}+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\\
&\leqslant \frac{C}{|x-y|^n}\, e^{-c\frac{|x-y|^2}{t}}\left(\frac{\sqrt{t}}{|x-y|}\right)^{-N}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\\
&\leqslant \frac{C}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}.
\end{aligned}$$

We prove *(ii)*. Observe that if $|x-y| > 2|y-z|$ then $|x-y| \sim |x-z|$. For any $0 < \delta < \delta_0$, if $|y-z| \leqslant \sqrt{t}$, by Lemma 2.5,

$$|k_t(x,y)-k_t(x,z)| \leqslant C\left(\frac{|y-z|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c\frac{|x-y|^2}{t}} \leqslant C\frac{|y-z|^\delta}{|x-y|^{n+\delta}}. \tag{3.6}$$

Consider the situation $|y-z| > \sqrt{t}$. Then Lemma 2.3 gives

$$|k_t(x,y)| \leqslant C\left(\frac{|y-z|}{\sqrt{t}}\right)^\delta t^{-n/2} e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N} \leqslant C\frac{|y-z|^\delta}{|x-y|^{n+\delta}}.$$

The same bound is valid for $\mathcal{T}_t(x,z)$ because $|x-z| \sim |x-y|$. Then the estimate follows directly since $|k_t(x,y)-k_t(x,z)| \leqslant |k_t(x,y)| + |k_t(x,z)|$. The symmetry of the kernel $k_t(x,y) = k_t(y,x)$ gives the conclusion of *(ii)*.

Let us prove the first statement of *(iii)*. Let $B = B(x,s)$ with $0 < s \leqslant \frac{1}{2}\rho(x)$. The triangle inequality gives

$$\|\mathcal{T}_t 1(y) - (\mathcal{T}_t 1)_B\|_E \leqslant \frac{1}{|B|}\int_B \|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)\|_E \, dz \tag{3.7}$$

We estimate the integrand $\|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)\|_E$. Because $y, z \in B$, we have $\rho(y) \sim \rho(z) \sim \rho(x)$ (see Lemma 2.1). The fact that $T_t 1(x) \equiv 1$ and Lemma 2.4 entail

$$\begin{aligned} |\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| &\leqslant |\mathcal{T}_t 1(y) - T_t 1(y)| + |\mathcal{T}_t 1(z) - T_t 1(z)| \\ &\leqslant \int_{\mathbb{R}^n} \left[\left(\frac{\sqrt{t}}{\rho(y)}\right)^{\delta_0} \omega_t(y-w) + \left(\frac{\sqrt{t}}{\rho(z)}\right)^{\delta_0} \omega_t(z-w)\right] dw \\ &\leqslant \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0} \int_{\mathbb{R}^n} [\omega_t(y-w) + \omega_t(z-w)] \, dw = C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}. \end{aligned} \tag{3.8}$$

So (3.8) gives

$$|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta_0}, \quad \text{when } \sqrt{t} \leqslant 2s. \tag{3.9}$$

If $\sqrt{t} > 2s$ then $|y-z| \leqslant 2s < \sqrt{t}$. Hence Lemma 2.5 implies that

$$|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| \leqslant \int_{\mathbb{R}^n} |k_t(y,w) - k_t(z,w)| \, dw \leqslant C\left(\frac{|y-z|}{\sqrt{t}}\right)^{\delta} \leqslant C\left(\frac{s}{\sqrt{t}}\right)^{\delta}, \tag{3.10}$$

where $0 < \delta < \delta_0$. Therefore estimate (3.10) gives

$$|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}, \quad \text{when } \sqrt{t} > \rho(x). \tag{3.11}$$

When $2s < \sqrt{t} < \rho(x)$ we write

$$\begin{aligned} |\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| &= |(\mathcal{T}_t 1(y) - T_t 1(y)) - (\mathcal{T}_t 1(z) - T_t 1(z))| \\ &= \Big|\Big(\int_{|w-y|>C\rho(y)} + \int_{4|y-z|<|w-y|<C\rho(y)} + \int_{|w-y|<4|y-z|}\Big) \\ &\qquad (k_t(y,w) - h_t(y,w)) - (k_t(z,w) - h_t(z,w)) \, dw\Big| \\ &= |I + II + III|. \end{aligned}$$

For I we use the smoothness proved in part *(ii)* of this proposition. Note that the same smoothness estimate is valid for the classical heat kernel. So we get

$$|I| \leqslant C\int_{|w-y|>C\rho(y)} \frac{|y-z|^{\delta}}{|w-y|^{n+\delta}} \, dw \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}.$$

In II we apply Lemma 2.6 and the fact that $\rho(w) \sim \rho(y)$ in the region of integration:

$$|II| \leqslant C\,|y-z|^{\delta} \int_{C\rho(y)>|w-y|>4|y-z|} \frac{\omega_t(w-y)}{\rho(w)^{\delta}}\, dw \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}.$$

The estimate of III is obtained by applying Lemma 2.4:

$$\begin{aligned}|III| &\leqslant C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}\left(\int_{|w-y|<4|y-z|}\omega_t(y-w)\, dw + \int_{|w-z|\leqslant 5|y-z|}\omega_t(z-w)\, dw\right)\\ &\leqslant C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}\int_{|\xi|\leqslant 5\frac{|y-z|}{\sqrt{t}}}\omega(\xi)\, d\xi \leqslant C\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta_0}\left(\frac{|y-z|}{\sqrt{t}}\right)^{n} \leqslant C\frac{s^n}{\rho(x)^{\delta_0}(\sqrt{t})^{n-\delta_0}}\\ &\leqslant C\frac{s^n}{\rho(x)^{\delta_0}s^{n-\delta_0}} = C\left(\frac{s}{\rho(x)}\right)^{\delta_0},\end{aligned}$$

since $2s < \sqrt{t}$ and $n - \delta_0 > 0$. Thus

$$|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)| \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}, \quad \text{when } 2s < \sqrt{t} < \rho(x). \tag{3.12}$$

Combining (3.9), (3.11) and (3.12), we get

$$\|\mathcal{T}_t 1(y) - \mathcal{T}_t 1(z)\|_E \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}. \tag{3.13}$$

Therefore, from (3.7) and (3.13) we get

$$\log\left(\frac{\rho(x)}{s}\right)\frac{1}{|B|}\int_B \|\mathcal{T}_t 1(y) - (\mathcal{T}_t 1)_B\|_E\, dy \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}\log\left(\frac{\rho(x)}{s}\right) \leqslant C,$$

which is the first conclusion of *(iii)*.

For the second estimate of *(iii)*, by (3.13), we have

$$\left(\frac{\rho(x)}{s}\right)^{\alpha}\frac{1}{|B|}\int_B \|\mathcal{T}_t 1(y) - (\mathcal{T}_t 1)_B\|_E\, dy \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta-\alpha} \leqslant C,$$

as soon as $\delta - \alpha \geqslant 0$, which can be guaranteed if $\alpha < \min\{1, 2 - \frac{n}{q}\}$ and we choose $\delta \geqslant \alpha$. □

3.2.2 Maximal operators for the generalized Poisson operators $\mathcal{P}_t^{\sigma}$

For $0 < \sigma < 1$ we define the generalized Poisson operators $\mathcal{P}_t^{\sigma}$ as

$$u(x,t) \equiv \mathcal{P}_t^{\sigma} f(x) = \frac{t^{2\sigma}}{4^{\sigma}\Gamma(\sigma)}\int_0^{\infty} e^{-\frac{t^2}{4r}}\mathcal{T}_r f(x)\, \frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)}\int_0^{\infty} e^{-r}\mathcal{T}_{\frac{t^2}{4r}} f(x)\, \frac{dr}{r^{1-\sigma}}, \tag{3.14}$$

for $x \in \mathbb{R}^n$ and $t > 0$. The function u satisfies the following boundary value (extension) problem:

$$\begin{cases} -\mathcal{L}_x u + \frac{1-2\sigma}{t} u_t + u_{tt} = 0, & \text{in } \mathbb{R}^n \times (0,\infty); \\ u(x,0) = f(x), & \text{on } \mathbb{R}^n. \end{cases}$$

Moreover, u is useful to characterize the fractional powers of $\mathcal{L}$ since $-t^{1-2\sigma}u_t(x,t)\big|_{t=0} = c_\sigma \mathcal{L}^\sigma f(x)$ for some constant $c_\sigma > 0$, see [82]. The fractional powers $\mathcal{L}^\sigma$ can be defined in a spectral way. When $\sigma = 1/2$ we get that $\mathcal{P}_t^{1/2} = e^{-t\mathcal{L}^{1/2}}$ is the classical Poisson semigroup generated by $\mathcal{L}$ given by Bochner's subordination formula, see [78]. It follows that

$$\mathcal{P}_t^\sigma f(x) = \int_{\mathbb{R}^n} \mathcal{P}_t^\sigma(x,y) f(y)\, dy,$$

where

$$\mathcal{P}_t^\sigma(x,y) = \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{t^2}{4r}} k_r(x,y)\, \frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-r} k_{\frac{t^2}{4r}}(x,y)\, \frac{dr}{r^{1-\sigma}}. \tag{3.15}$$

To get the boundedness of the maximal operator $\mathcal{P}^{\sigma,*}f(x) := \sup_{t>0} |\mathcal{P}_t^\sigma f(x)| = \|\mathcal{P}_t^\sigma f(x)\|_E$ in $BMO_{\mathcal{L}}^\alpha$, we proceed using the vector-valued approach and the boundedness of the maximal heat semigroup $\mathcal{T}^* f$. The following proposition completely analogous to Proposition 3.11 holds.

Proposition 3.12. *The estimates of Proposition 3.11 are valid when* $\mathcal{T}_t$ *is replaced by* $\mathcal{P}_t^\sigma$.

Proof. The proof follows by transferring the estimates for $k_t(x,y)$ to $\mathcal{P}_t^\sigma(x,y)$ through formula (3.15). We just sketch the proof of *(iii)*. For any $y, z \in B = B(x,s)$, $x \in \mathbb{R}^n$, $0 < s \leqslant \frac{1}{2}\rho(x)$, by (3.15), Minkowski's integral inequality and (3.13) we have

$$\begin{aligned} \|\mathcal{P}_t^\sigma 1(y) - \mathcal{P}_t^\sigma 1(z)\|_E &=\leqslant C_\sigma \int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \|\mathcal{T}_r 1(y) - \mathcal{T}_r 1(z)\|_E \frac{dr}{r^{1+\sigma}} \\ &\leqslant C \left(\frac{s}{\rho(x)}\right)^\delta \int_0^\infty t^{2\sigma} e^{-\frac{t^2}{4r}} \frac{dr}{r^{1+\sigma}} = C \left(\frac{s}{\rho(x)}\right)^\delta. \end{aligned}$$

Then the same computations for the heat semigroup apply in this case and give *(iii)*. □

3.2.3 Littlewood–Paley g-function for the heat–diffusion semigroup

The Littlewood–Paley g-function associated with $\{\mathcal{T}_t\}_{t>0}$ is defined by

$$g_{\mathcal{T}}(f)(x) = \left(\int_0^\infty |t\partial_t \mathcal{T}_t f(x)|^2 \frac{dt}{t}\right)^{1/2} = \|t\partial_t \mathcal{T}_t f(x)\|_F,$$

where $F := L^2\big((0,\infty), \frac{dt}{t}\big)$. The Spectral Theorem implies that $g_{\mathcal{T}}$ is an isometry on $L^2(\mathbb{R}^n)$, see [27, Lemma 3]. As before, to get the boundedness of $g_{\mathcal{T}}$ from $BMO_{\mathcal{L}}^\alpha$ into itself it is sufficient to prove the following result.

Proposition 3.13. *The estimates of Proposition 3.11 are valid when $\mathcal{T}_t$ is replaced by $t\partial_t\mathcal{T}_t$ and the Banach space E is replaced by F.*

Proof. Part *(i)* is proved using Lemma 2.13*(a)* and the same argument of the proof of Proposition 3.11*(i)*.

Similarly *(ii)* follows by Lemma 2.13*(b)* and the symmetry $k_t(x,y) = k_t(y,x)$.

To prove *(iii)* let us fix $y, z \in B = B(x_0, s)$, $0 < s \leqslant \frac{1}{2}\rho(x_0)$. In view of an estimate like (3.7), we must handle $\|t\partial_t\mathcal{T}_t 1(y) - t\partial_t\mathcal{T}_t 1(z)\|_F$ first. We can write

$$\begin{aligned}\|t\partial_t\mathcal{T}_t 1(y) - t\partial_t\mathcal{T}_t 1(z)\|_F^2 &= \int_0^\infty \left|\int_{\mathbb{R}^n} (t\partial_t k_t(x,y) - t\partial_t k_t(x,z))\, dx\right|^2 \frac{dt}{t} \\ &= \left(\int_0^{4s^2} + \int_{4s^2}^{\rho(x_0)^2} + \int_{\rho(x_0)^2}^\infty\right)\left|\int_{\mathbb{R}^n} (t\partial_t k_t(x,y) - t\partial_t k_t(x,z))\, dx\right|^2 \frac{dt}{t} =: A_1 + A_2 + A_3.\end{aligned} \tag{3.16}$$

Since $y, z \in B \subset B(x_0, \rho(x_0))$, it follows that $\rho(y) \sim \rho(x_0) \sim \rho(z)$. By Lemma 2.13*(c)*,

$$A_1 \leqslant C\int_0^{4s^2} \frac{(\sqrt{t}/\rho(x_0))^{2\delta}}{(1+\sqrt{t}/\rho(x_0))^{2N}}\frac{dt}{t} \leqslant C\int_0^{4s^2}\left(\frac{\sqrt{t}}{\rho(x_0)}\right)^{2\delta}\frac{dt}{t} = C\left(\frac{s}{\rho(x_0)}\right)^{2\delta}. \tag{3.17}$$

Also, by Lemma 2.13*(b)*,

$$\begin{aligned}A_3 &\leqslant C\int_{\rho(x_0)^2}^\infty \left(\frac{|y-z|}{\sqrt{t}}\right)^{2\delta}\left|\int_{\mathbb{R}^n} t^{-n/2} e^{-c\frac{|x-y|^2}{t}}\, dx\right|^2 \frac{dt}{t} \\ &= C\int_{\rho(x_0)^2}^\infty \left(\frac{|y-z|}{\sqrt{t}}\right)^{2\delta}\frac{dt}{t} \leqslant C\left(\frac{s}{\rho(x_0)}\right)^{2\delta}.\end{aligned} \tag{3.18}$$

It remains to estimate the term A_2. Recall from [27, Eq. (2.8)] that, because the potential V is in the reverse Hölder class,

$$\int_{\mathbb{R}^n} \omega_t(x-y)V(y)\, dy \leqslant \frac{C}{t}\left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta, \quad \text{for } t \leqslant \rho(x)^2. \tag{3.19}$$

Clearly $\partial_t\mathcal{T}_t 1(x) = \mathcal{L}\mathcal{T}_t 1(x) = \mathcal{T}_t V(x)$, that is

$$\int_{\mathbb{R}^n} \partial_t k_t(x,y)\, dy = \int_{\mathbb{R}^n} k_t(x,y)V(y)\, dy. \tag{3.20}$$

We then have, by Lemma 2.5 (remember that $|y-z|\leqslant 2s\leqslant\sqrt{t}$),

$$\begin{aligned}
A_2 &= \int_{4s^2}^{\rho(x_0)^2}\left|\int_{\mathbb{R}^n}\left(t\partial_t k_t(x,y)-t\partial_t k_t(x,z)\right)\,dx\right|^2\frac{dt}{t}\\
&= \int_{4s^2}^{\rho(x_0)^2} t\left|\int_{\mathbb{R}^n}\left(k_t(y,x)-k_t(z,x)\right)V(x)\,dx\right|^2 dt\\
&\leqslant C\,|y-z|^{2\delta}\int_{4s^2}^{\rho(x_0)^2} t^{1-\delta}\left|\int_{\mathbb{R}^n} t^{-n/2}e^{-c\frac{|y-x|}{t}}V(x)\,dx\right|^2 dt \qquad (3.21)\\
&\leqslant Cs^{2\delta}\int_{4s^2}^{\rho(x_0)^2} t^{1-\delta}t^{-2}\left(\frac{\sqrt{t}}{\rho(y)}\right)^{2\delta} dt\\
&\leqslant C\left(\frac{s}{\rho(x_0)}\right)^{2\delta}\int_{s^2}^{\rho(x_0)^2}\frac{dt}{t} = C\left(\frac{s}{\rho(x_0)}\right)^{2\delta}\log\left(\frac{\rho(x_0)}{s}\right).
\end{aligned}$$

Combining (3.16), (3.17), (3.18) and (3.21) we get

$$\|t\partial_t\mathcal{T}_t1(y)-t\partial_t\mathcal{T}_t1(z)\|_F\leqslant C\left(\frac{s}{\rho(x_0)}\right)^{\delta}\left(\log\left(\frac{\rho(x_0)}{s}\right)\right)^{1/2}. \qquad (3.22)$$

Thus *(iii)* readily follows. □

3.2.4 Littlewood–Paley g-function for the Poisson semigroup

The Littlewood–Paley g-function associated with the Poisson semigroup $\{\mathcal{P}_t\}_{t>0}\equiv\{\mathcal{P}_t^{1/2}\}_{t>0}$ (see (3.14) and (3.15)) is defined analogously as $g_{\mathcal{T}}$ by replacing the heat semigroup by the Poisson semigroup:

$$g_{\mathcal{P}}(f)(x)=\left(\int_0^\infty|t\partial_t\mathcal{P}_tf(x)|^2\frac{dt}{t}\right)^{1/2}=\|t\partial_t\mathcal{P}_tf(x)\|_F.$$

The Spectral Theorem shows that $g_{\mathcal{P}}$ is an isometry on $L^2(\mathbb{R}^n)$, see [55, Lemma 3.7]. We also have

Proposition 3.14. *The estimates of Proposition 3.11 are valid when $\mathcal{T}_t$ is replaced by $t\partial_t\mathcal{P}_t$ and the Banach space E is replaced by F.*

Proof. First we derive a convenient formula to treat the operator $t\partial_t\mathcal{P}_t$. By the second identity of (3.15) with $\sigma=1/2$ (Bochner's subordination formula) and a change of variables,

$$\begin{aligned}
t\partial_t\mathcal{P}_t(x,y) &= \frac{t}{\sqrt{\pi}}\int_0^\infty\frac{e^{-r}}{r^{1/2}}\,\partial_t\left(k_{\frac{t^2}{4r}}(x,y)\right)\,dr=\frac{t^2}{2\sqrt{\pi}}\int_0^\infty\frac{e^{-r}}{r^{1/2}}\,\partial_\nu\left(k_\nu(x,y)\right)\Big|_{\nu=\frac{t^2}{4r}}\frac{dr}{r}\\
&= \frac{t}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4\nu}}\,\nu\partial_\nu k_\nu(x,y)\,\frac{d\nu}{\nu^{3/2}}. \qquad (3.23)
\end{aligned}$$

Formula (3.23) should be compared with the first identity of (3.15) for $\sigma=1/2$. It will allow us to transfer the estimates for $\nu\partial_\nu\mathcal{T}_\nu$ to $t\partial_t\mathcal{P}_t$.

For *(i)* we use (3.23), Minkowski's integral inequality and the estimate for $\nu\partial_\nu\mathcal{T}_\nu$:

$$\begin{aligned}\|t\partial_t\mathcal{P}_t(x,y)\|_F^2 &\leqslant C\int_0^\infty |\nu\partial_\nu k_\nu(x,y)|^2\int_0^\infty te^{-\frac{t^2}{4\nu}}\,\frac{dt}{t}\,\frac{d\nu}{\nu^{3/2}}\\ &= C\int_0^\infty |\nu\partial_\nu k_\nu(x,y)|^2\,\frac{d\nu}{\nu}\leqslant \frac{C}{|x-y|^{2n}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-2N}.\end{aligned}$$

The estimate for *(ii)* follows in the same way.

By (3.23), Fubini's Theorem and (3.22),

$$\|t\partial_t\mathcal{P}_t1(y)-t\partial_t\mathcal{P}_t1(z)\|_F\leqslant C\left(\frac{s}{\rho(x_0)}\right)^{\delta}\log\left(\frac{\rho(x_0)}{s}\right)^{1/2},$$

which is sufficient for *(iii)*. □

3.2.5 Laplace transform type multipliers

Given a bounded function a on $[0,\infty)$ we let

$$m(\lambda)=\lambda\int_0^\infty a(t)e^{-t\lambda}\,dt.$$

The Spectral Theorem allows us to define the Laplace transform type multiplier operator $m(\mathcal{L})$ associated to a that is bounded on $L^2(\mathbb{R}^n)$. Observe that

$$m(\mathcal{L})f(x)=\int_0^\infty a(t)\mathcal{L}e^{-t\mathcal{L}}f(x)\,dt=\int_0^\infty a(t)\partial_t\mathcal{T}_tf(x)\,dt,\quad x\in\mathbb{R}^n.$$

Then the kernel $\mathcal{M}(x,y)$ of $m(\mathcal{L})$ can be written as

$$\mathcal{M}(x,y)=\int_0^\infty a(t)\partial_tk_t(x,y)\,dt.$$

Proposition 3.15. *Let* $x,y,z\in\mathbb{R}^n$, $N>0$, $0\leqslant\alpha<1$ *and* $B=B(x,s)$ *for* $0<s\leqslant\rho(x)$. *Then*

(a) $|\mathcal{M}(x,y)|\leqslant\dfrac{C}{|x-y|^n}\left(1+\dfrac{|x-y|}{\rho(x)}+\dfrac{|x-y|}{\rho(y)}\right)^{-N}$*;*

(b) $|\mathcal{M}(x,y)-\mathcal{M}(x,z)|+|\mathcal{M}(y,x)-\mathcal{M}(z,x)|\leqslant C_\delta\dfrac{|y-z|^\delta}{|x-y|^{n+\delta}}$, *for all* $|x-y|>2|y-z|$ *and any* $0<\delta<\delta_0$*;*

(c) $\log\left(\dfrac{\rho(x)}{s}\right)\dfrac{1}{|B|}\displaystyle\int_B|m(\mathcal{L})1(y)-(m(\mathcal{L})1)_B|\,dy\leqslant C$*;*

(d) $\left(\dfrac{\rho(x)}{s}\right)^\alpha\dfrac{1}{|B|}\displaystyle\int_B|m(\mathcal{L})1(y)-(m(\mathcal{L})1)_B|\,dy\leqslant C$, *for any* $0\leqslant\alpha<\min\{1,2-\frac{n}{q}\}$.

Proof. The reader should recall the estimates for $\partial_t \mathcal{T}_t(x,y)$ stated in Lemma 2.13.

For *(a)*, by Lemma 2.13*(a)*,

$$\begin{aligned}\int_0^{|x-y|^2} |a(t)\partial_t k_t(x,y)|\,dt &\leqslant C\int_0^{|x-y|^2} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\frac{dt}{t}\\ &= C\int_0^{|x-y|^2} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\left(\frac{\sqrt{t}}{|x-y|}\right)^{-N}\left(\frac{|x-y|}{\sqrt{t}}+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\frac{dt}{t}\\ &\leqslant C\int_0^{|x-y|^2} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\frac{dt}{t}\\ &\leqslant \frac{C}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N},\end{aligned}$$

and

$$\begin{aligned}\int_{|x-y|^2}^{\infty} |a(t)\partial_t k_t(x,y)|\,dt &\leqslant C\int_{|x-y|^2}^{\infty} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\frac{dt}{t}\\ &\leqslant C\int_{|x-y|^2}^{\infty} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}\frac{dt}{t}\\ &\leqslant \frac{C}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-N}.\end{aligned}$$

To check *(b)* we apply Lemma 2.13*(b)* to see that

$$\begin{aligned}\int_{|x-y|^2}^{\infty} |a(t)||\partial_t k_t(x,y)-\partial_t k_t(x,z)|\,dt &\leqslant C\int_{|x-y|^2}^{\infty}\left(\frac{|y-z|}{\sqrt{t}}\right)^{\delta} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\frac{dt}{t}\\ &\leqslant C\frac{|y-z|^{\delta}}{|x-y|^{n+\delta}}.\end{aligned}$$

Moreover, by Lemma 2.13*(a)*,

$$\int_0^{|x-y|^2} |a(t)\partial_t k_t(x,y)|\,dt \leqslant C\int_0^{|x-y|^2}\left(\frac{|y-z|}{\sqrt{t}}\right)^{\delta} t^{-n/2}e^{-c\frac{|x-y|^2}{t}}\,\frac{dt}{t} \leqslant C\frac{|y-z|^{\delta}}{|x-y|^{n+\delta}}.$$

The same bound is valid for $\int_0^{|x-y|^2}|a(t)||\partial_t k_t(x,z)|\,\frac{dt}{t}$ because $|x-z|\sim|x-y|$. The symmetry of the kernel $\mathcal{M}(x,y)=\mathcal{M}(y,x)$ gives the conclusion of *(b)*.

Fix $y,z\in B$. For *(c)* and *(d)*, let us estimate the difference

$$|m(\mathcal{L})1(y)-m(\mathcal{L})1(z)| \leqslant \|a\|_{L^\infty([0,\infty))}\int_0^\infty\left|\int_{\mathbb{R}^n}(\partial_t k_t(y,w)-\partial_t k_t(z,w))\,dw\right|\,dt.$$

To that end we split the integral in t into three parts. We start with the part from 0 to $4s^2$. From Lemma 2.13*(c)*,

$$\left|\int_0^{4s^2}\int_{\mathbb{R}^n}(\partial_t k_t(y,w)-\partial_t k_t(z,w))\,dw\,dt\right| \leqslant C\int_0^{4s^2}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta}\frac{dt}{t} = C\left(\frac{s}{\rho(x)}\right)^{\delta}.$$

Let us continue with the integral from $\rho(x)^2$ to ∞. We apply Lemma 2.13*(b)*:

$$\left|\int_{\rho(x)^2}^{\infty}\int_{\mathbb{R}^n}(\partial_t k_t(y,w)-\partial_t k_t(z,w))\ dw\ dt\right| \leqslant C\int_{\rho(x)^2}^{\infty}\left(\frac{|y-z|}{\sqrt{t}}\right)^{\delta}\ \frac{dt}{t}$$
$$\leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}.$$

Finally we consider the part from $4s^2$ to $\rho(x)^2$. Applying (3.20), Lemma 2.5 and (3.19),

$$\left|\int_{4s^2}^{\rho(x)^2}\int_{\mathbb{R}^n}(\partial_t k_t(y,w)-\partial_t k_t(z,w))\ dw\ dt\right| = \int_{4s^2}^{\rho(x)^2}\left|\int_{\mathbb{R}^n}(k_t(y,w)-k_t(z,w))\,V(w)\ dw\ \right| dt$$
$$\leqslant C\,|y-z|^{\delta}\int_{4s^2}^{\rho(x)^2}\int_{\mathbb{R}^n} t^{-n/2}e^{-c\frac{|y-w|^2}{t}}V(w)\ dw\ \frac{dt}{t^{\delta/2}}$$
$$\leqslant C\left(\frac{s}{\rho(y)}\right)^{\delta}\int_{s^2}^{\rho(x)^2}\frac{dt}{t}\leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}\log\left(\frac{\rho(x)}{s}\right).$$

Hence

$$\frac{1}{|B|}\int_B|m(\mathcal{L})1(y)-(m(\mathcal{L})1)_B|\ dy \leqslant \frac{C}{s^{2n}}\int_B\int_B|m(\mathcal{L})1(y)-m(\mathcal{L})(z)|\,dy\,dz$$
$$\leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}\log\left(\frac{\rho(x)}{s}\right).$$

Thus *(c)* is valid and also *(d)* holds when $\alpha<\delta$. □

3.2.6 $\mathcal{L}$-Riesz transforms and negative powers

Following the pattern of the proof of Theorem 3.10 we can recover the results from [13] and [14]. We state them as a theorem for further reference.

Theorem 3.16. *Let* $\alpha \geqslant 0$ *and* $0<\gamma<n$*. Then:*

- *The* $\mathcal{L}$*-Riesz transforms are bounded from* $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ *into itself, for any* $0\leqslant\alpha<1-\frac{n}{q}$*, with* $q>n$*.*
- *The negative powers* $\mathcal{L}^{-\gamma/2}$ *are bounded from* $\mathrm{BMO}^{\alpha}_{\mathcal{L}}$ *into* $\mathrm{BMO}^{\alpha+\gamma}_{\mathcal{L}}$ *for* $\alpha+\gamma<\min\{1,2-\frac{n}{q}\}$*.*

Let us prove Theorem 3.16 for the $\mathcal{L}$-Riesz transforms and the negative powers $\mathcal{L}^{-\gamma/2}$ separately.

$\mathcal{L}$-Riesz transforms. For every $i=1,2,\dots,n$, the i-th Riesz transform $\mathcal{R}_i$ associated to $\mathcal{L}$ is defined by

$$\mathcal{R}_i=\partial_{x_i}\mathcal{L}^{-1/2}=\partial_{x_i}\frac{1}{\sqrt{\pi}}\int_0^{\infty}e^{-t\mathcal{L}}\ \frac{dt}{t^{1/2}}.$$

We denote by $\mathcal{R}$ the vector $\nabla\mathcal{L}^{-1/2} = (\mathcal{R}_1, \dots, \mathcal{R}_n)$. The Riesz transforms associated to $\mathcal{L}$ were first studied by Z. Shen in [72]. He showed (Theorem 0.8 of [72]) that if the potential $V \in RH_q$ with $q > n$ then $\mathcal{R}$ is a Calderón–Zygmund operator. In particular, the $\mathbb{R}^n$–valued operator $\mathcal{R}$ is bounded from $L^2(\mathbb{R}^n)$ into $L^2_{\mathbb{R}^n}(\mathbb{R}^n)$ and its kernel $\mathcal{K}$ satisfies, for any $0 < \delta < 1 - \frac{n}{q}$,

$$|\mathcal{K}(x,y) - \mathcal{K}(x,z)| + |\mathcal{K}(y,x) - \mathcal{K}(z,x)| \leqslant C\frac{|y-z|^\delta}{|x-y|^{n+\delta}}, \tag{3.24}$$

whenever $|x-y| > 2|y-z|$. Moreover, when $q > n$ we have for any $x, y \in \mathbb{R}^n$, $x \neq y$, and $N > 0$ there exists a constant C_N such that

$$|\mathcal{K}(x,y)| \leqslant \frac{C_N}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}, \tag{3.25}$$

see [72, Eq. (6.5)] and also [14, Lemma 3]. Hence $\mathcal{R}$ is a γ-Schrödinger-Calderón-Zygmuund operator with $\gamma = 0$. For more information about $\mathcal{R}$, we refer the reader to [14, 15, 28, 29, 61, 77, 94].

The boundedness results of $\mathcal{R}$ in $BMO^\alpha_\mathcal{L}$ follow by checking the properties of $\mathcal{R}1$.

Proposition 3.17. *Let* $V \in RH_q$ *with* $q > n$, $B = B(x_0, s)$ *for* $x_0 \in \mathbb{R}^n$ *and* $0 < s \leqslant \frac{1}{2}\rho(x_0)$. *Then*

(i) $\log\left(\frac{\rho(x_0)}{s}\right)\frac{1}{|B|}\int_B |\mathcal{R}1(y) - (\mathcal{R}1)_B|\ dy \leqslant C$;

(ii) $\left(\frac{\rho(x_0)}{s}\right)^\alpha \frac{1}{|B|}\int_B |\mathcal{R}1(y) - (\mathcal{R}1)_B|\ dy \leqslant C$, *for* $\alpha < 1 - \frac{n}{q}$.

To prove Proposition 3.17, we collect some well-known estimates on $\mathcal{K}(x,y)$. Let us denote by $\mathcal{K}_0$ the kernel of the ($\mathbb{R}^n$–valued) classical Riesz transform $\mathcal{R}_0 = \nabla(-\Delta)^{-1/2}$.

Lemma 3.18 ([14, Lemmas 3 and 4]). *Suppose that* $V \in RH_q$ *with* $q > n$.

(a) For any $x, y \in \mathbb{R}^n$, $x \neq y$,

$$|\mathcal{K}(x,y) - \mathcal{K}_0(x,y)| \leqslant \frac{C}{|x-y|^n}\left(\frac{|x-y|}{\rho(x)}\right)^{2-n/q}.$$

(b) For any $0 < \delta < 1 - \frac{n}{q}$ *there exists a constant* C *such that if* $|z-y| \geqslant 2|x-y|$ *then*

$$|(\mathcal{K}(x,z) - \mathcal{K}_0(x,z)) - (\mathcal{K}(y,z) - \mathcal{K}_0(y,z))| \leqslant C\frac{|x-y|^\delta}{|z-y|^{n+\delta}}\left(\frac{|z-y|}{\rho(z)}\right)^{2-n/q}.$$

Proof of Proposition 3.17. Let $y, z \in B$. Then $\rho(y) \sim \rho(x_0) \sim \rho(z)$. Since

$$\mathcal{R}1(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} \mathcal{K}(x,y)\ dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

we have

$$\begin{aligned}|\mathcal{R}1(y) - \mathcal{R}1(z)| \leqslant \lim_{\varepsilon\to 0^+} &\left| \int_{\varepsilon<|x-y|\leqslant 4\rho(x_0)} \mathcal{K}(y,x)\ dx - \int_{\varepsilon<|x-z|\leqslant 4\rho(x_0)} \mathcal{K}(z,x)\ dx \right| \\ &+ \left| \int_{|x-y|>4\rho(x_0)} \mathcal{K}(y,x)\ dx - \int_{|x-z|>4\rho(x_0)} \mathcal{K}(z,x)\ dx \right| =: \lim_{\varepsilon\to 0^+} A_\varepsilon + B,\end{aligned}$$

First, let us consider A_ε. Since we will consider the limit as ε tends to zero, we can assume that $0 < \varepsilon < 4\rho(x_0) - 2s$. For every annulus E we have $\int_E \mathcal{K}_0(x,y)\ dy = 0$. Therefore,

$$\begin{aligned}A_\varepsilon &= \left| \int_{\varepsilon<|x-y|\leqslant 4\rho(x_0)} (\mathcal{K}(y,x) - \mathcal{K}_0(y,x))\ dx - \int_{\varepsilon<|x-z|\leqslant 4\rho(x_0)} (\mathcal{K}(z,x) - \mathcal{K}_0(z,x))\ dx \right| \\ &\leqslant \left| \int_{\mathbb{R}^n} (\mathcal{K}(y,x) - \mathcal{K}_0(y,x)) \left(\chi_{\varepsilon<|x-y|\leqslant 4\rho(x_0)}(x) - \chi_{\varepsilon<|x-z|\leqslant 4\rho(x_0)}(x)\right)\ dx \right| \\ &\quad + \left| \int_{\mathbb{R}^n} [(\mathcal{K}(y,x) - \mathcal{K}_0(y,x)) - (\mathcal{K}(z,x) - \mathcal{K}_0(z,x))]\, \chi_{\varepsilon<|x-z|\leqslant 4\rho(x_0)}(x)\ dx \right| =: A_\varepsilon^1 + A_\varepsilon^2.\end{aligned} \tag{3.26}$$

The term A_ε^1 is not zero when $\left|\chi_{\varepsilon<|x-y|\leqslant 4\rho(x_0)}(x) - \chi_{\varepsilon<|x-z|\leqslant 4\rho(x_0)}(x)\right| = 1$, namely, when

- $\varepsilon < |x-y| \leqslant 4\rho(x_0)$ and $|x-z| \leqslant \varepsilon$; or
- $\varepsilon < |x-y| \leqslant 4\rho(x_0)$ and $|x-z| > 4\rho(x_0)$; or
- $\varepsilon < |x-z| \leqslant 4\rho(x_0)$ and $|x-y| \leqslant \varepsilon$; or
- $\varepsilon < |x-z| \leqslant 4\rho(x_0)$ and $|x-y| > 4\rho(x_0)$.

In the first case we have $\varepsilon < |x-y| \leqslant |x-z| + |z-y| < \varepsilon + 2s$. Then, by Lemma 3.18*(a)*,

$$A_\varepsilon^1 \leqslant \int_{\varepsilon<|x-y|\leqslant 2s+\varepsilon} \frac{C}{|x-y|^n} \left(\frac{|x-y|}{\rho(y)}\right)^{2-n/q} dx \leqslant C \left(\frac{s}{\rho(x_0)}\right)^{2-n/q}. \tag{3.27}$$

In the second case, by the assumption on ε, we get $\max\{\varepsilon, 4\rho(x_0) - 2s\} = 4\rho(x_0) - 2s < |x-y| \leqslant 4\rho(x_0)$. Then Lemma 3.18*(a)* and the Mean Value Theorem give

$$A_\varepsilon^1 \leqslant \frac{C}{\rho(x_0)^{2-n/q}} \int_{4\rho(x_0)-2s<|x-y|\leqslant 4\rho(x_0)} |x-y|^{2-n/q-n}\ dx \leqslant C\frac{s}{\rho(x_0)}. \tag{3.28}$$

In the third and fourth cases we obtain the same bounds as in (3.27) and (3.28) by replacing y by z. Thus, when $0 < \delta < 1 - n/q$,

$$A_\varepsilon^1 \leqslant C \left(\frac{s}{\rho(x_0)}\right)^\delta. \tag{3.29}$$

For A_ε^2,

$$\begin{aligned} A_\varepsilon^2 &\leqslant \left| \iint_{|x-z|>2|y-z|} [(\mathcal{K}(y,x)-\mathcal{K}_0(y,x))-(\mathcal{K}(z,x)-\mathcal{K}_0(z,x))]\, \chi_{\varepsilon<|x-z|\leqslant 4\rho(x_0)}(x)\, dx \right| \\ &\quad + \left| \iint_{|x-z|\leqslant 2|y-z|} [(\mathcal{K}(y,x)-\mathcal{K}_0(y,x))-(\mathcal{K}(z,x)-\mathcal{K}_0(z,x))]\, \chi_{\varepsilon<|x-z|\leqslant 4\rho(x_0)}(x)\, dx \right| \\ &=: A_\varepsilon^{2,1} + A_\varepsilon^{2,2}. \end{aligned} \tag{3.30}$$

By Lemma 3.18*(b)*,

$$A_\varepsilon^{2,1} \leqslant C \frac{|y-z|^\delta}{\rho(z)^{2-n/q}} \int_{|x-z|\leqslant 4\rho(x_0)} |x-z|^{2-n/q-n-\delta}\, dx \leqslant C \left(\frac{s}{\rho(x_0)}\right)^\delta. \tag{3.31}$$

On the other hand, Lemma 3.18*(a)* gives

$$\begin{aligned} A_\varepsilon^{2,2} &\leqslant \int_{|x-z|\leqslant 2|y-z|} \frac{C}{|x-y|^n}\left(\frac{|x-y|}{\rho(y)}\right)^{2-n/q} dx + \int_{|x-z|\leqslant 2|y-z|} \frac{C}{|x-z|^n}\left(\frac{|x-z|}{\rho(z)}\right)^{2-n/q} dx \\ &\leqslant \frac{C}{\rho(x_0)^{2-n/q}} \left(\int_{|x-y|\leqslant 3|y-z|} |x-y|^{2-n/q-n}\, dx + \int_{|x-z|\leqslant 2|y-z|} |x-z|^{2-n/q-n}\, dx \right) \\ &\leqslant C\left(\frac{s}{\rho(x_0)}\right)^{2-n/q} \leqslant C\left(\frac{s}{\rho(x_0)}\right)^\delta, \end{aligned} \tag{3.32}$$

for any $0<\delta<1-n/q$. Hence, from (3.26), (3.29), (3.30), (3.31) and (3.32) we obtain that for all $\varepsilon>0$ sufficiently small,

$$A_\varepsilon \leqslant C\left(\frac{s}{\rho(x_0)}\right)^\delta. \tag{3.33}$$

Let us now estimate B. In a similar way,

$$\begin{aligned} B &\leqslant \int_{|x-y|>4\rho(x_0)} |\mathcal{K}(y,x)-\mathcal{K}(z,x)|\, dx \\ &\quad + \int_{\mathbb{R}^n} |\mathcal{K}(z,x)| \left|\chi_{|x-z|>4\rho(x_0)}(x) - \chi_{|x-z|>4\rho(x_0)}(x)\right|\, dx =: B_1 + B_2. \end{aligned}$$

In the integrand of B_1 we have $|x-y|>4\rho(x_0)\geqslant 8s>2\,|y-z|$. Therefore the smoothness of the Riesz kernel (3.24) can be applied to get

$$B_1 \leqslant C\int_{|x-y|>4\rho(x_0)} \frac{|y-z|^\delta}{|x-y|^{n+\delta}}\, dx \leqslant C\left(\frac{s}{\rho(x_0)}\right)^\delta.$$

It is possible to deal with B_2 as with A_ε^1 above to derive the same bound. Hence,

$$B \leqslant C\left(\frac{s}{\rho(x_0)}\right)^\delta.$$

This last estimate together with (3.33) imply

$$|\mathcal{R}1(y) - \mathcal{R}1(z)| \leqslant C \left(\frac{s}{\rho(x_0)} \right)^{\delta},$$

where $0 < \delta < 1 - n/q$. From here *(i)* and *(ii)* readily follow. □

Negative powers. For any $\gamma > 0$ the negative powers of $\mathcal{L}$ are defined as

$$\mathcal{L}^{-\gamma/2} f(x) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t\mathcal{L}} f(x) \; \frac{dt}{t^{1-\gamma/2}} = \int_{\mathbb{R}^n} \mathcal{K}_\gamma(x,y) f(y) \; dy,$$

where

$$\mathcal{K}_\gamma(x,y) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty k_t(x,y) \; \frac{dt}{t^{1-\gamma/2}}, \quad x \in \mathbb{R}^n.$$

Therefore, by Lemma 2.3 and a similar argument as in the proof of Proposition 3.11*(i)*, for every $N > 0$,

$$|\mathcal{K}_\gamma(x,y)| \leqslant \frac{C}{|x-y|^{n-\gamma}} \left(1 + \frac{|x-y|}{\rho(x)} + \frac{|x-y|}{\rho(y)} \right)^{-N}.$$

In particular, $\mathcal{L}^{-\gamma/2}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, for $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ with $1 < p < q < \infty$ and $0 < \gamma < n$. Using similar arguments to those in the proof of Proposition 3.11*(ii)* it can be checked that

$$|\mathcal{K}_\gamma(x,y) - \mathcal{K}_\gamma(x,z)| + |\mathcal{K}_\gamma(y,x) - \mathcal{K}_\gamma(z,x)| \leqslant C \frac{|y-z|^\delta}{|x-y|^{n-\gamma+\delta}},$$

when $|x-y| > 2|y-z|$, for any $0 < \delta < 2 - \frac{n}{q}$. Thus $\mathcal{L}^{-\gamma}$ is a γ-Schrödinger-Calderón-Zygmund operator according to Definition 3.6.

The second item of Theorem 3.16 is a consequence of the following proposition and our two main theorems.

Proposition 3.19. *Let* $B = B(x,s)$ *with* $0 < s \leqslant \frac{1}{2}\rho(x)$. *Then*

(i) $\log \left(\frac{\rho(x)}{s} \right) \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |\mathcal{L}^{-\gamma/2} 1(y) - (\mathcal{L}^{-\gamma/2} 1)_B| \; dy \leqslant C$ *if* $\gamma \leqslant 2 - \frac{n}{q}$;

(ii) $\left(\frac{\rho(x)}{s} \right)^{\alpha} \frac{1}{|B|^{1+\frac{\gamma}{n}}} \int_B |\mathcal{L}^{-\gamma/2} 1(y) - (\mathcal{L}^{-\gamma/2} 1)_B| \; dy \leqslant C$ *if* $\alpha + \gamma < \min\{1, 2 - \frac{n}{q}\}$.

Proof. Fix $y, z \in B$, so that $\rho(x) \sim \rho(y) \sim \rho(z)$. We can write

$$\mathcal{L}^{-\gamma/2} 1(y) - \mathcal{L}^{-\gamma/2} 1(z) = \int_0^\infty \int_{\mathbb{R}^n} (k_t(y,w) - k_t(z,w)) \; dw \; t^{\gamma/2} \, \frac{dt}{t}. \qquad (3.34)$$

We split the integral in t of the difference (3.34) into two parts. From (3.13) we have

$$\left|\int_0^{\rho(x)^2}\int_{\mathbb{R}^n}\left(k_t(y,w)-k_t(z,w)\right)\,dw\;t^{\gamma/2}\,\frac{dt}{t}\right| \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}\int_0^{\rho(x)^2} t^{\gamma/2}\,\frac{dt}{t} = C\left(\frac{s}{\rho(x)}\right)^{\delta}\rho(x)^{\gamma}.$$

On the other hand we can use (3.10) to get

$$\left|\int_{\rho(x)^2}^{\infty}\int_{\mathbb{R}^n}\left(k_t(y,w)-k_t(z,w)\right)\,dw\;t^{\gamma/2}\,\frac{dt}{t}\right| \leqslant C\int_{\rho(x)^2}^{\infty}\left(\frac{s}{\sqrt{t}}\right)^{\delta} t^{\gamma/2}\,\frac{dt}{t} \leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta}\rho(x)^{\gamma},$$

since $\gamma < \delta$. An application of these last two estimates to (3.34) finally gives

$$\begin{aligned}\frac{1}{|B|^{1+\frac{\gamma}{n}}}\int_B |\mathcal{L}^{-\gamma/2}1(y)-(\mathcal{L}^{-\gamma/2}1)_B|dy &\leqslant \frac{C}{s^{2n+\gamma}}\int_B\int_B |\mathcal{L}^{-\gamma/2}1(y)-\mathcal{L}^{-\gamma/2}1(z)|\,dy\,dz \\ &\leqslant C\left(\frac{s}{\rho(x)}\right)^{\delta-\gamma}.\end{aligned}$$

Thus *(i)* is valid if $\gamma < 2-\frac{n}{q}$ and $\delta < 2-\frac{n}{q}$ is chosen such that $\gamma \leqslant \delta$. Also *(ii)* holds when $\alpha+\gamma < \min\{1, 2-\frac{n}{q}\}$. □

Chapter 4

Harnack's inequality for fractional operators

In this chapter, we will prove interior Harnack's inequalities for fractional powers of second order partial differential operators. In Section 4.1, we give the theorem of Harnack's inequality for fractional powers of second order partial differential operators. The Caffarelli-Silvestre extension problem and the Harnack's inequality for degenerate Schrödinger operators proved by C. E. Gutiérrez for fractional second order partial differential operators are developed in Section 4.2 and Section 4.3 separately. We give a transference method in Section 4.4 to obtain the non-divergence form operators from the Harnack's inequality for the related divergence form operators. With Harnack's inequality developed in Section 4.3 and the transfer method in Section 4.4, we give the proof of the Harnack's inequality for fractional operators in Section 4.5 and Section 4.6.

4.1 Harnack's inequality for fractional operators

In this section we shall give the interior Harnack's inequalities for fractional powers of second order partial differential operators. The operators we consider are:

- Divergence form elliptic operators $\mathcal{L} = -\operatorname{div}(a(x)\nabla) + V(x)$ with bounded measurable coefficients $a(x)$ and locally bounded nonnegative potentials $V(x)$ defined on bounded domains;
- Ornstein-Uhlenbeck operator $\mathbf{O}_B$ and harmonic oscillator $\mathcal{H}_B$ on $\mathbb{R}^n$;
- Laguerre operators L_α, L_α^φ, L_α^ℓ, L_α^ψ and $L_\alpha^\mathcal{L}$ on $(0,\infty)^n$ with $\alpha \in (-1,\infty)^n$;
- Ultraspherical operators L_λ and l_λ on $(0,\pi)$ with $\lambda > 0$;
- Laplacian on domains $\Omega \subseteq \mathbb{R}^n$;
- Bessel operators Δ_λ and S_λ on $(0,\infty)$ with $\lambda > 0$.

For the full description of the operators see Sections 4.3, 4.5 and 4.6. In general, all these operators L are nonnegative, self-adjoint and have a dense domain $\mathrm{Dom}(L) \subset L^2(\Omega, d\eta)$, where $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 1$, is an open set and $d\eta$ is some positive measure on Ω. In Section 4.2 we show how the fractional powers L^σ, $0 < \sigma < 1$, can be defined by using the spectral theorem.

Theorem A (Harnack's inequality for L^σ). *Let L be any of the operators listed above and $0 < \sigma < 1$. Let $\mathcal{O}$ be an open and connected subset of Ω and fix a compact subset $K \subset \mathcal{O}$. There exists a positive constant C, depending only on σ, n, K and the coefficients of L such that*

$$\sup_K f \leqslant C \inf_K f,$$

for all functions $f \in \mathrm{Dom}(L)$, $f \geqslant 0$ in Ω, such that $L^\sigma f = 0$ in $L^2(\mathcal{O}, d\eta)$. Moreover, f is a continuous function in $\mathcal{O}$.

We will prove Theorem A in each case in Section 4.5 and Section 4.6.

4.2 Fractional operators and extension problem

Along this chapter all the operators will verify the following

General assumption. *By $L = L_x$ we denote a nonnegative self-adjoint second order partial differential operator with dense domain $\mathrm{Dom}(L) \subset L^2(\Omega, d\eta) \equiv L^2(\Omega)$. Here Ω is an open subset of $\mathbb{R}^n$, $n \geqslant 1$, and $d\eta$ is a positive measure on Ω. The operator L acts in the variables $x \in \mathbb{R}^n$.*

The Spectral Theorem can be applied to an operator L as in the general assumption, see [68, Chapter 13]. We recall it at here. Given a real measurable function ϕ on $[0, \infty)$, the operator $\phi(L)$ is defined as $\phi(L) = \int_0^\infty \phi(\lambda)\, dE(\lambda)$, where E is the unique resolution of the identity of L. The domain $\mathrm{Dom}(\phi(L))$ of $\phi(L)$ is the set of functions $f \in L^2(\Omega)$ such that $\int_0^\infty |\phi(\lambda)|^2 \, dE_{f,f}(\lambda) < \infty$.

In this chapter we are going to use:

- The heat-diffusion semigroup generated by L, defined as $\phi(L) = e^{-tL}$, $t \geqslant 0$. For $f \in L^2(\Omega)$, we have that $v = e^{-tL} f$ solves the evolution equation $v_t = -Lv$, for $t > 0$. Moreover, $\|e^{-tL} f\|_{L^2(\Omega)} \leqslant \|f\|_{L^2(\Omega)}$, for all $t \geqslant 0$, and $e^{-tL} f \to f$ in $L^2(\Omega)$ as $t \to 0^+$.
- The fractional powers of L, given by $\phi(L) = L^\sigma$, with domain $\mathrm{Dom}(L^\sigma) \supset \mathrm{Dom}(L)$. When $f \in \mathrm{Dom}(L^\sigma)$ we have $L^\sigma e^{-tL} f = e^{-tL} L^\sigma$. If $f \in \mathrm{Dom}(L)$ then $\langle Lf, f\rangle = \|L^{1/2} f\|^2_{L^2(\Omega)}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$. Also, for $f \in \mathrm{Dom}(L)$,

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-tL} f(x) - f(x)) \frac{dt}{t^{1+\sigma}}, \quad \text{in } L^2(\Omega), \tag{4.1}$$

where Γ is the Gamma function, see for example [98, p. 260].

We will usually assume that the heat-diffusion semigroup e^{-tL} is positivity-preserving, that is,

$$f \geqslant 0 \text{ on } \Omega \text{ implies } e^{-tL}f \geqslant 0 \text{ on } \Omega, \text{ for all } t > 0. \tag{4.2}$$

Remark 4.1 (Maximum and comparison principle for L^σ)**.** Let L be as in the general assumption. Under the additional hypothesis (4.2), the following comparison principle holds. If $f, g \in \mathrm{Dom}(L)$, $f \geqslant g$ in Ω and $f(x_0) = g(x_0)$ at a point $x_0 \in \Omega$, then $L^\sigma f(x_0) \leqslant L^\sigma g(x_0)$. This comparison principle is a direct consequence of the maximum principle: if $f \in \mathrm{Dom}(L)$, $f \geqslant 0$, $f(x_0) = 0$, then $L^\sigma f(x_0) \leqslant 0$ (for the proof just observe in (4.1) that $\Gamma(-\sigma) < 0$ and $e^{-tL}f(x_0) \geqslant 0$).

Theorem 4.2 (Extension problem [82, Theorem 1.1])**.** *Let* L *be as in the general assumptions and* $f \in \mathrm{Dom}(L^\sigma)$*. Let* u *be defined as*

$$\begin{aligned} u(x,y) &:= \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1-\sigma}}, \end{aligned} \tag{4.3}$$

for $x \in \Omega$, $y > 0$*. Then* $u \in C^\infty((0,\infty) : \mathrm{Dom}(L)) \cap C([0,\infty) : L^2(\Omega))$ *and it satisfies the extension problem*

$$\begin{cases} -L_x u + \frac{1-2\sigma}{y} u_y + u_{yy} = 0, & x \in \Omega,\ y > 0, \\ u(x,0) = f(x), & x \in \Omega. \end{cases} \tag{4.4}$$

In addition, for $c_\sigma = \frac{4^{\sigma-1/2}\Gamma(\sigma)}{\Gamma(1-\sigma)} > 0$*,*

$$-c_\sigma \lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = L^\sigma f(x). \tag{4.5}$$

We must clarify in which sense the identities in Theorem 4.2 are taken. The first equality in (4.3) means that for any $g \in L^2(\Omega)$,

$$\langle u(\cdot,y), g(\cdot)\rangle = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \langle e^{-tL} f, g\rangle e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}}, \quad y > 0,$$

and similarly for the second one. Also (4.4) in general means that $\langle \frac{1-2\sigma}{y} u_y(\cdot,y) + u_{yy}(\cdot,y), g(\cdot)\rangle = \langle Lu(\cdot,y), g(\cdot)\rangle$, for all $y > 0$, with $\langle u(\cdot,y), g(\cdot)\rangle \to \langle f, g\rangle$, as $y \to 0^+$, and analogously for (4.5). By the second identity of (4.3), a change of variables and dominated convergence, we have

$$\begin{aligned} \limsup_{y\to 0^+} \|y^{1-2\sigma} u_y(x,y)\|^2_{L^2(\Omega')} &\leqslant \frac{4^{1/2-\sigma}}{\Gamma(\sigma)} \limsup_{y\to 0^+} \int_0^\infty \|e^{-\frac{y^2}{4s}L}(L^\sigma f)\|^2_{L^2(\Omega')} e^{-s} \frac{ds}{s^\sigma} \\ &= c_\sigma^{-1} \|L^\sigma f\|_{L^2(\Omega')}, \quad \text{for any measurable set } \Omega' \subseteq \Omega. \end{aligned} \tag{4.6}$$

4.3 Harnack's inequality for fractional Schrödinger operators

In this section we consider a uniformly elliptic Schrödinger operator of the form

$$\mathcal{L} = -\operatorname{div}(a(x)\nabla) + V, \quad \text{on } \Omega \subseteq \mathbb{R}^n.$$

Here $a = (a^{ij})$ is a symmetric matrix of real-valued measurable coefficients such that $\mu^{-1}|\xi|^2 \leqslant a(x)\xi \cdot \xi \leqslant \mu|\xi|^2$, for some constant $\mu > 0$, for almost every $x \in \Omega$ and for all $\xi \in \mathcal{R}^n$. The potential V is a locally bounded function on Ω. Here Ω can be an unbounded set. We assume that $\mathcal{L}$ satisfies the general assumption at the beginning of Section 4.2, with $d\eta(x) = dx$, the Lebesgue measure. The domain of $\mathcal{L}$ is $\operatorname{Dom}(\mathcal{L}) = W_0^{1,2}(\Omega) \cap L^2(\Omega, V(x)\,dx)$. The Sobolev space $W_0^{1,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ under the norm $\|f\|^2_{W^{1,2}(\Omega)} = \|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)}$. Note that $\operatorname{Dom}(\mathcal{L})$ is dense in $L^2(\Omega)$. For $f \in \operatorname{Dom}(\mathcal{L})$,

$$\langle \mathcal{L}f, g\rangle = \int_\Omega (a(x)\nabla f \cdot \nabla g + V(x) f g)\,dx, \quad g \in \operatorname{Dom}(\mathcal{L}).$$

Theorem 4.3 (Reflection extension). *Fix a ball $B_R(x_0) \subset \Omega$, $x_0 \in \Omega$, $R > 0$. Let $u : \Omega \times [0, R) \to \mathbb{R}$ be a solution of the extension equation in (4.4) with $L = \mathcal{L}$ in $B_R(x_0) \times (0, R)$. Define the reflection of u to $\Omega \times (-R, R)$ by $\tilde{u}(x, y) = u(x, |y|)$, $x \in \Omega$, $y \in (-R, R)$. Suppose that*

(I) $\lim_{y\to 0^+} \|y^{1-2\sigma} u_y(x, y)\|_{L^2(B_R(x_0), dx)} = 0$; *and*

(II) $\|\nabla_x u(x, y)\|_{L^2(B_R(x_0), dx)}$ *remains bounded as* $y \to 0^+$.

Then $\tilde{u}$ verifies the degenerate Schrödinger equation

$$\operatorname{div}(|y|^{1-2\sigma} b(x)\nabla\tilde{u}) - |y|^{1-2\sigma} V(x)\tilde{u} = 0, \tag{4.7}$$

in the weak sense in $\tilde{B} := \{(x, y) \in \mathcal{R}^{n+1} : |x - x_0|^2 + y^2 < R^2\}$, where the matrix of coefficients $b = (b^{ij})$ is given by $b^{ij} = a^{ij}$, $b^{n+1,j} = b^{i,n+1} = 0$, $1 \leqslant i, j \leqslant n$, and $b^{n+1,n+1} = 1$.

Proof. Let $\varphi \in C_c^\infty(\tilde{B})$. Take any $0 < \delta < R$. Since u is a solution of the extension equation in (4.4) for $\mathcal{L}$, for any fixed $y \in (\delta, R)$, we have

$$\int_{B_R(x_0)} (a(x)\nabla_x u \cdot \nabla_x \varphi + V(x) u\varphi)\,dx = \int_{B_R(x_0)} |y|^{2\sigma-1} \partial_y(|y|^{1-2\sigma} u_y)\varphi\,dx.$$

Recall that we are assuming that $u \in C^\infty((0, R) : \operatorname{Dom}(\mathcal{L}))$. By integrating the last identity in y, applying Fubini's theorem and integration by parts,

$$\begin{aligned}\int_\delta^R |y|^{1-2\sigma} \int_{B_R(x_0)} &(a(x)\nabla_x u \cdot \nabla_x \varphi + V(x) u\varphi)\,dx\,dy \\ &= -\int_{B_R(x_0)} \delta^{1-2\sigma} u_y(x, \delta)\varphi(x, \delta)\,dx - \int_{B_R(x_0)} \int_\delta^R |y|^{1-2\sigma} u_y(x, y)\varphi_y(x, y)\,dy\,dx.\end{aligned}$$

From here we get

$$\int_{B_R(x_0)\times\{|y|\geqslant\delta\}} (b(x)\nabla\tilde{u}\cdot\nabla\varphi + V(x)\tilde{u}\varphi)|y|^{1-2\sigma}\,dx\,dy$$
$$= \int_{B_R(x_0)} \delta^{1-2\sigma}u_y(x,\delta)\varphi(x,-\delta)\,dx - \int_{B_R(x_0)} \delta^{1-2\sigma}u_y(x,\delta)\varphi(x,\delta)\,dx. \quad (4.8)$$

We are ready to prove that $\tilde{u}$ is a weak solution of (4.7) in $\tilde{B}$. We have to check that

$$I := \int_{\tilde{B}} (b(x)\nabla\tilde{u}\cdot\nabla\varphi + V(x)\tilde{u}\varphi)|y|^{1-2\sigma}\,dx\,dy = 0.$$

By using (4.8),

$$I = \left(\int_{\tilde{B}\cap\{|y|\geqslant\delta\}} + \int_{\tilde{B}\cap\{|y|<\delta\}}\right)dx\,dy$$
$$= \int_{B_R(x_0)} \delta^{1-2\sigma}u_y(x,\delta)\varphi(x,-\delta)\,dx - \int_{B_R(x_0)} \delta^{1-2\sigma}u_y(x,\delta)\varphi(x,\delta)\,dx$$
$$+ \int_{\tilde{B}\cap\{|y|<\delta\}} b^{ij}\nabla\tilde{u}\cdot\nabla\varphi|y|^{1-2\sigma}\,dx\,dy + \int_{\tilde{B}\cap\{|y|<\delta\}} V(x)\tilde{u}\varphi|y|^{1-2\sigma}\,dx\,dy.$$

As $\delta\to0^+$, the first and second terms above tend to zero because of (I). Also the fourth term goes to zero because $V(x)\tilde{u}|y|^{1-2\sigma}\in L^1_{loc}$. Since $\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),dx)}$ remains bounded as $y\to0^+$, for any small $\delta>0$ there exists a constant $c>0$ such that if $|y|<\delta$ then $\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),dx)}\leqslant c$. This property and (I) imply that the third term above tends to zero as $\delta\to0^+$. □

Theorem 4.4 (Harnack's inequality for $\mathcal{L}^\sigma$). *Let $\mathcal{L}$ be as above. Assume that the heat-diffusion semigroup $e^{-t\mathcal{L}}$ is positivity-preserving, see (4.2). Let $f\in\mathrm{Dom}(\mathcal{L})$ be a non-negative function such that $\mathcal{L}^\sigma f=0$ in $L^2(B_R(x_0),dx)$ for some ball $B_R(x_0)\subset\Omega$. Suppose that $\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),dx)}$ remains bounded as $y\to0^+$, where u is a solution to the extension problem (4.4) for $\mathcal{L}$ and f. There exist constants $R_0<R$ and C depending only on n, σ, μ, and V, but not on f, such that,*

$$\sup_{B_r} f\leqslant C\inf_{B_r} f,$$

for any ball B_r with $B_{8r}\subset B_R(x_0)$ and $0<r\leqslant R_0$. Moreover, f is continuous in $B_R(x_0)$.

In order to prove Theorem 4.4 we use Theorem 4.3 and the following version of

Gutiérrez's Harnack inequality for degenerate Schrödinger equations. Consider a degenerate Schrödinger equation of the form

$$-\operatorname{div}(\tilde{a}(X)\nabla v)+\tilde{V}(X)v=0,\quad X\in\mathbb{R}^N, \quad (4.9)$$

where $\tilde{a} = (\tilde{a}^{ij})$ is an $N \times N$ symmetric matrix of real-valued measurable coefficients such that $\lambda^{-1}\omega(X)|\xi|^2 \leqslant \tilde{a}(X)\xi \cdot \xi \leqslant \lambda\omega(X)|\xi|^2$, for some $\lambda > 0$, for almost every $X \in \mathbb{R}^N$ and for all $\xi \in \mathcal{R}^N$. The function ω is an A_2 weight. The potential $\tilde{V}$ satisfies $\tilde{V}/\omega \in L^p_\omega$ locally, for some large $p = p_{N,\omega}$. Let $\mathcal{O}$ be any open bounded subset of $\mathcal{R}^N$. Then there exist positive constants r_0, γ depending only on λ, N, ω, $\mathcal{O}$ and $\tilde{V}$ such that if ν is any nonnegative weak solution of (4.9) in $\mathcal{O}$ then for every ball B_r with $B_{8r} \subset \mathcal{O}$ and $0 < r \leqslant r_0$ we have

$$\sup_{B_{r/2}} \nu \leqslant \gamma \inf_{B_{r/2}} \nu.$$

As a consequence, ν is continuous in $\mathcal{O}$; see [39].

Proof of Theorem 4.4. Since $\mathcal{L}^\sigma f = 0$ in $L^2(B_R(x_0), dx)$, by (4.6) and the hypothesis on $\nabla_x u$, we see that u satisfies the conditions of Theorem 4.3. Now, equation (4.7) is a degenerate Schrödinger equation with A_2 weight $\omega(x,y) = |y|^{1-2\sigma}$ and potential $\tilde{V} = |y|^{1-2\sigma}V(x)$ such that $\tilde{V}/\omega \in L^p_\omega$ locally for all p sufficiently large. By C. E. Gutiérrez's result just explained above, Harnack's inequality for $\tilde{u}$ holds. By restricting $\tilde{u}$ to $y = 0$ we get Harnack's inequality for f. Moreover, $\tilde{u}$ is continuous in $B_R(x_0)$ and thus f. □

The case of nonnegative potentials. Under the additional assumptions that Ω is a bounded set and that the potential V is a nonnegative function in Ω, we can prove Theorem A for $\mathcal{L}^\sigma$. In this case the domain of $\mathcal{L}$ is $\mathrm{Dom}(\mathcal{L}) = W^{1,2}_0(\Omega)$ and it is known that $e^{-t\mathcal{L}}$ is positivity-preserving, see [23, Chapter 1]. Let $f \in W^{1,2}_0(\Omega)$, $f \geqslant 0$, such that $\mathcal{L}^\sigma f = 0$ in $L^2(B_R(x_0), dx)$ for some ball $B_R(x_0) \subset \Omega$, $R > 0$. Denote by u the solution of the extension problem for f as in Theorem 4.2. By virtue of Theorem 4.4, to prove Harnack's inequality for $\mathcal{L}^\sigma$ we just have to verify that u satisfies condition (II) of Theorem 4.3. As $f \in W^{1,2}_0(\Omega)$, by the ellipticity condition,

$$\mu^{-1}\|\nabla f\|^2_{L^2(\Omega,dx)} \leqslant \int_\Omega a(x)\nabla f \cdot \nabla f \, dx \leqslant \langle \mathcal{L}f, f\rangle = \|\mathcal{L}^{1/2}f\|^2_{L^2(\Omega,dx)}, \tag{4.10}$$

(for the last equality see Section 4.2). Now, since $u \in C^2((0,\infty) : W^{1,2}_0(\Omega))$, $\nabla_x u(x,y)$ is well defined and belongs to $L^2(\Omega, dx)$ for each $y > 0$. We can apply (4.3), (4.10) and the properties of the heat-diffusion semigroup $e^{-t\mathcal{L}}$ stated at the beginning of Section 4.2 to get

$$\begin{aligned}
\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),dx)} &\leqslant \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \|\nabla e^{-t\mathcal{L}} f\|_{L^2(\Omega,dx)} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} \\
&\leqslant \mu^{1/2} \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)} \int_0^\infty \|e^{-t\mathcal{L}}\mathcal{L}^{1/2} f\|_{L^2(\Omega,dx)} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} \\
&\leqslant \mu^{1/2} \frac{\|\mathcal{L}^{1/2}f\|_{L^2(\Omega,dx)}}{\Gamma(\sigma)} \int_0^\infty \left(\frac{y^2}{4t}\right)^\sigma e^{-\frac{y^2}{4t}} \frac{dt}{t} = \mu^{1/2}\|\mathcal{L}^{1/2}f\|_{L^2(\Omega,dx)}.
\end{aligned}$$

Thus $\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),dx)}$ remains bounded as $y \to 0^+$ and (II) in Theorem 4.3 is valid. Hence Theorem A is proved for this case. Observe that, in particular, Theorem A is valid for the Laplacian in bounded domains with Dirichlet boundary conditions.

Remark 4.5 (Liouville theorem for fractional divergence form elliptic operators)**.** Let $\Omega = \mathbb{R}^n$ and $V \equiv 0$, that is, $\mathcal{L} = -\operatorname{div}(a(x)\nabla)$. Take $f \in \operatorname{Dom}(\mathcal{L}) = W^{1,2}(\mathbb{R}^n)$. The following Liouville theorem is true: If $f \geqslant 0$ on $\mathbb{R}^n$ and $\mathcal{L}^\sigma f = 0$ in $L^2(\mathbb{R}^n)$ then f must be a constant function. Indeed, for this f, the reflection $\tilde{u}$ of u is a nonnegative weak solution of (4.7) with $V \equiv 0$ in $\mathbb{R}^{n+1}$, so $\tilde{u}$ is constant and therefore f is a constant function. Here we have applied the Liouville theorem for degenerate elliptic equations in divergence form with A_2 weights, which is a simple consequence of Harnack's inequality of [34].

Remark 4.6. Since our method is based on C. E. Gutiérrez's result [39], we are not able to get the exact dependence on σ of the constant C in Harnack's inequality of Theorem 4.4.

4.4 Transference method for Harnack's inequality

In this section, we assume that L satisfies the general assumptions of Section 4.2. We will develop a transference method to get Harnack's inequality from L^σ to another operator $\bar{L}^\sigma$ related to L. This method will be useful when considering differential operators arising in classical orthogonal expansions and also for the Bessel operator.

Firstly, by a change of measure, we have the following trivial result.

Lemma 4.7. *Let $M(x) \in C^\infty(\Omega)$ be a positive function. Define the isometry operator U from $L^2(\Omega, M(x)^2 d\eta(x))$ into $L^2(\Omega, d\eta(x))$ as $(Uf)(x) = M(x)f(x)$. Then if $\{\varphi_k\}_{k\in\mathbb{N}_0^n}$ is an orthonormal system in $L^2(\Omega, M(x)^2 d\eta(x))$ then $\{U\varphi_k\}_{k\in\mathbb{N}_0^n}$ is also an orthonormal system in $L^2(\Omega, d\eta(x))$.*

Next we set up the notation for the change of variables.

Definition 4.8 (Change of variables)**.** Let $h : \Omega \to \bar{\Omega} \subseteq \mathbb{R}^n$ be a one-to-one C^∞ transformation on Ω. Denote the Jacobian of the inverse map $h^{-1} : \bar{\Omega} \to \Omega$ by $|J_{h^{-1}}|$. We define the change of variables operator W from $L^2(\bar{\Omega}, M(h^{-1}(\bar{x}))^2 \, |J_{h^{-1}}| \, d\eta(\bar{x}))$ into $L^2(\Omega, M(x)^2 d\eta(x))$ as

$$(Wf)(x) = f(h(x)), \quad x \in \Omega.$$

Now we are in position to describe the transference method. By using the definition above and Lemma 4.7 we construct a new differential operator. This new operator will be nonnegative and self-adjoint in $L^2(\bar{\Omega}, d\bar{\eta}(\bar{x}))$, where $\bar{\Omega} = h(\Omega)$ and $d\bar{\eta}(\bar{x}) := M(h^{-1}(\bar{x}))^2 \, |J_{h^{-1}}| \, d\eta(\bar{x})$. Let

$$\bar{L} := (U \circ W)^{-1} \circ L \circ (U \circ W).$$

If E is the resolution of the identity of L then the resolution of the identity $\bar{E}$ of $(U \circ W) \circ \bar{L}$ verifies

$$d\bar{E}_{f,g}(\lambda) = dE_{(U\circ W)f,(U\circ W)g}(\lambda), \quad f, g \in L^2(\bar{\Omega}, d\bar{\eta}).$$

Therefore if $f \in \operatorname{Dom}(\bar{L}^\sigma)$ then we see that the fractional powers of $\bar{L}$ satisfy

$$\bar{L}^\sigma f = (U \circ W)^{-1} \circ L^\sigma \circ (U \circ W) f.$$

Lemma 4.9 (Transference method). *If Theorem A for L^σ is true, then the analogous statement for $\bar{L}^\sigma$ is also true.*

Proof. Let $f \in \mathrm{Dom}(\bar{L}^\sigma)$, $f \geqslant 0$, such that $\bar{L}^\sigma f = 0$ in $L^2(\bar{\mathcal{O}}, d\bar{\eta})$, for some open set $\bar{\mathcal{O}} \subset \bar{\Omega}$. Take a compact set $\bar{K} \subset \bar{\mathcal{O}}$. We want to see that there is a constant C depending on $\bar{K}$ and $\bar{L}^\sigma$ such that

$$\sup_{\bar{K}} f \leqslant C \inf_{\bar{K}} f. \tag{4.11}$$

Observe that

$$\int_{h^{-1}(\bar{\mathcal{O}})} |L^\sigma \circ (U \circ W) f(x)|^2 \, d\eta(x) = \int_{\bar{\mathcal{O}}} |\bar{L}^\sigma f(\bar{x})|^2 \, d\bar{\eta}(\bar{x}) = 0,$$

and $(U \circ W)f \in \mathrm{Dom}(L)$ is nonnegative. By the assumption on L^σ, there exists C depending on $h^{-1}(\bar{K})$ and L^σ such that

$$\sup_{h^{-1}(\bar{K})} (U \circ W) f \leqslant C \inf_{h^{-1}(\bar{K})} (U \circ W) f,$$

and $(U \circ W)f$ is continuous. In particular, f is continuous. Since $M(x)$ is positive, continuous and bounded in $h^{-1}(\bar{K})$,

$$\sup_{h^{-1}(\bar{K})} W f \leqslant C' \inf_{h^{-1}(\bar{K})} W f.$$

This in turn implies (4.11) as desired. □

4.5 Classical orthogonal expansions

In this section we consider operators L (as in the general assumptions of Section 4.2) for which there exists a family $\{\varphi_k\}_{k \in \mathbb{N}_0^n}$ of eigenfunctions of L, with associated nonnegative eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}_0^n}$, namely, $L\varphi_k(x) = \lambda_k \varphi_k(x)$, such that $\{\varphi_k\}$ is an orthonormal basis of $L^2(\Omega, d\eta)$. In all our examples, the eigenvalues will satisfy the following: there exists a constant $c \geqslant 1$ such that $\lambda_k \sim |k|^c$, for any $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, $|k| = k_1 + \cdots + k_n$. We also suppose that the eigenfunctions φ_k are in $C^2(\Omega)$ and that their derivatives satisfy the following local estimate. For any compact subset $K \subset \Omega$ and any multi-index $\beta \in \mathbb{N}_0^n$, $|\beta| \leqslant 2$, there exist $\varepsilon = \varepsilon_{K,\beta} \geqslant 0$ and a constant $C = C_{K,\beta}$ such that

$$\|D^\beta \varphi_k\|_{L^\infty(K, d\eta)} \leqslant C \, |k|^\varepsilon, \tag{4.12}$$

for any $k \in \mathbb{N}_0^n$. For $f \in L^2(\Omega, d\eta)$ the heat-diffusion semigroup can be written as $e^{-tL} f(x) = \sum_{|k|=0}^\infty e^{-t\lambda_k} c_k \varphi_k(x)$. For $0 < \sigma < 1$, the domain of L^σ is given as $\mathrm{Dom}(L^\sigma) = \{f \in L^2(\Omega, d\eta) : \sum_{|k|=0}^\infty \lambda_k^{2\sigma} |c_k|^2 < \infty\}$, where c_k denotes the Fourier coefficient of f in the basis φ_k: $c_k = \langle f, \varphi_k \rangle = \int_\Omega f \varphi_k \, d\eta$. Given $f \in \mathrm{Dom}(L^\sigma)$ we have $L^\sigma f(x) = \sum_{|k|=0}^\infty \lambda_k^\sigma c_k \varphi_k(x)$.

Under these assumptions we can show that the solution u of the extension problem is classical. To this end, let K be any compact subset of Ω. First we show that the series that defines $e^{-tL}f(x)$ is uniformly convergent in $K \times (0,T)$, for every $T > 0$. Indeed, by Cauchy-Schwartz's inequality,

$$\begin{aligned}
|e^{-tL}f(x)| &\leqslant \sum_{|k|\geqslant 0} |e^{-t\lambda_k} c_k \varphi_k(x)| \leqslant C \sum_{|k|\geqslant 0} e^{-Ct|k|^c} |c_k|\,|k|^\varepsilon \\
&\leqslant \frac{C}{t^{\varepsilon/c}} \left(\sum_{|k|\geqslant 0} e^{-2Ct|k|^c}\right)^{1/2} \left(\sum_{|k|\geqslant 0} c_k^2\right)^{1/2} \leqslant \frac{C}{t^{\varepsilon/c}} \left(\sum_{j\geqslant 0} j^n e^{-2Ctj^c}\right)^{1/2} \|f\|_{L^2(\Omega,d\eta)} \\
&\leqslant \frac{C}{t^{\frac{\varepsilon+n}{c}}} \left(\sum_{j\geqslant 0} e^{-C' t j^c}\right)^{1/2} \|f\|_{L^2(\Omega,d\eta)} \leqslant \frac{C}{t^{\frac{\varepsilon+n+1/2}{c}}} \|f\|_{L^2(\Omega,d\eta)}, \quad x \in K,
\end{aligned}$$

and the uniform convergence follows. As a consequence, u in (4.3) is well defined, for by the estimate above, for any $x \in K$ and $y > 0$,

$$\int_0^\infty |e^{-tL}f(x)e^{-\frac{y^2}{4t}}| \frac{dt}{t^{1+\sigma}} \leqslant C \|f\|_{L^2(\Omega,d\eta)} \int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{\frac{\varepsilon+n+1/2}{c}}} \frac{dt}{t^{1+\sigma}} \leqslant F(y),$$

for some function $F = F(y)$. This estimate also implies that in the first identity of (4.3) we can interchange the integration in t with the summation that defines $e^{-tL}f(x)$ to get

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \sum_{|k|\geqslant 0} c_k \varphi_k(x) \int_0^\infty e^{-t\lambda_k} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}}. \tag{4.13}$$

By using (4.12) and the same arguments as above, it is easy to see that this series defines a function in $C^2(\Omega) \cap C^1(0,\infty)$. Moreover, since each term of the series in (4.13) satisfies equation (4.4) in the classical sense, we readily see that u is a classical solution to (4.4).

Next we will present the concrete applications.

We will take advantage of well-known formulas, see for instance [1, 3], to apply our transference method to get Harnack's inequality for operators of classical orthogonal expansions which are not of the form considered in Section 4.3. A remarkable advantage of the transference method is that we do not need to check that the semigroup e^{-tL} is positivity-preserving.

4.5.1 Ornstein-Uhlenbeck operator and harmonic oscillator

In [40], C. E. Gutiérrez dealt with the Ornstein-Uhlenbeck operator

$$\mathbf{O}_B = -\Delta + 2Bx \cdot \nabla,$$

where B is an $n \times n$ positive definite symmetric matrix. The operator $\mathbf{O}_B$ is positive and symmetric in $L^2(\mathcal{R}^n, d\gamma_B(x))$, where $d\gamma_B(x) = (\det B)^{n/2}\pi^{-n/2}e^{-Bx\cdot x}dx$ is the B-Gaussian measure. Let us consider the eigenvalue problem $\mathbf{O}_B w = \lambda w$, with boundary conditions

$w(x) = O(|x|^k)$, for some $k \geqslant 0$ as $|x| \to \infty$. Firstly, let us assume that the matrix B is diagonal, which means that

$$B = D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

with $d_i > 0$ for $1 \leqslant i \leqslant n$. It is not difficult to see that in this case the eigenfunctions w are the multidimensional Hermite polynomials defined by $H_k^D(x) = H_{k_1}(\sqrt{d_1}x_1)\cdots H_{k_n}(\sqrt{d_n}x_n)$, $k \in \mathbb{N}_0^n$, with eigenvalues $2(k \cdot d)$, $d = (d_1, \dots, d_n)$, where H_{k_i} is the one-dimensional Hermite polynomial of degree k_i, see [40]. For the general case, since B is a positive definite symmetric matrix, there exists an orthogonal matrix A such that $ABA^t = D$, where A^t is the transpose of A. Then the eigenfunctions become $H_k^B(x) = H_k^D(Ax)$.

Let us also consider the harmonic oscillator

$$\mathcal{H}_D = -\Delta + |Dx|^2,$$

where D is a matrix as above, with zero boundary condition at infinity. Under these assumptions $\mathcal{H}_D$ is positive and symmetric in $L^2(\mathbb{R}^n, dx)$. It is well known that the multidimensional Hermite functions $h_k^D(x) = (\det D)^{n/4}\pi^{-n/4}e^{-\frac{Dx\cdot x}{2}}H_k^D(x)$, are the eigenfunctions of $\mathcal{H}_D$ and $\mathcal{H}_D h_k^D = (2(k \cdot d) + \sum_{i=1}^n d_i)\, h_k^D$. The Hermite functions form an orthonormal basis of $L^2(\mathbb{R}^n, dx)$.

Observe that we may also consider

$$\mathcal{H}_D - \sum_{i=1}^n d_i,$$

since it has the same eigenfunctions as $\mathcal{H}_D$ with eigenvalues $2(k \cdot d) \geqslant 0$. We can also put a more general matrix B in the place of D; we will prove Harnack's inequality for it by using the transference method.

A. Proof of Harnack's inequality for $(\mathcal{H}_D)^\sigma$: To show Harnack's inequality for $(\mathcal{H}_D)^\sigma$ we have to check that all the conditions of Theorem 4.4 hold.

The potential here is $V(x) = |Dx|^2$, which is a locally bounded function on $\mathbb{R}^n$.

By Mehler's formula [40, 85, 86], $e^{-t\mathcal{H}_D}$ is positivity-preserving.

In [86], it is shown that there exists C such that $\|h_k^D\|_{L^\infty(\mathbb{R}^n,dx)} \leqslant C$ for all k. Using the relation

$$2\partial_{x_i}h_k^D(x) = \sqrt{d_i}\left((2k_i)^{1/2}h_{k-e_i}^D(x) - (2k_i+2)^{1/2}h_{k+e_i}^D(x)\right),$$

where e_i is the i-th coordinate vector in $\mathbb{N}_0^n$, we see that (4.12) is valid for $h_k^D(x)$. Therefore the solution u to the extension problem given in (4.3) for $\mathcal{H}_D$ is a classical solution.

Let $f \in \mathrm{Dom}(\mathcal{H}_D)$, $f \geqslant 0$, such that $(\mathcal{H}_D)^\sigma f = 0$ in $L^2(B_R(x_0), dx)$. We have to verify that $\|\nabla_x u(x, y)\|_{L^2(B_R(x_0),dx)}$ remains bounded as $y \to 0^+$. In fact, we will have

$\|\nabla_x u(x,0)\|_{L^2(B_R(x_0),dx)} = \|\nabla_x f(x)\|_{L^2(B_R(x_0),dx)}$. Indeed, as we can write $f = \sum_{|k|=0}^{\infty} c_k h_k^D$, by (4.13) and the identity for the derivatives of the Hermite functions h_k^D given above,

$$\begin{aligned}&\left(\partial_{x_i} + \sqrt{d_i}x_i\right)(u(x,y) - f(x)) \\ &= \sum_k c_k \sqrt{d_i}(2k_i)^{1/2} h_{k-e_i}^D(x) \left(\frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-t(k\cdot d + \sum_{l=1}^n d_l)} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} - 1 \right).\end{aligned} \tag{4.14}$$

Observe that the term in parenthesis above is uniformly bounded in y and, since

$$\frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-t(k\cdot d + \sum_{l=1}^n d_l)} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4w}(k\cdot d + \sum_{l=1}^n d_l)} e^{-w} \frac{dw}{w^{1-\sigma}},$$

we readily see that it converges to 1 when $y \to 0^+$. Moreover, as $f \in \mathrm{Dom}(\mathcal{H}_D)$,

$$\left\| \sum_{|k|=0}^{\infty} c_k \sqrt{d_i}(2k_i)^{1/2} h_{k-e_i}^D(x) \right\|_{L^2(\mathbb{R}^n,dx)} = \left(2 \sum_{|k|=0}^{\infty} c_k^2 k_i d_i \right)^{1/2} < \infty.$$

Hence, by dominated convergence in (4.14), we get that $\left(\partial_{x_i} + \sqrt{d_i}x_i\right) u(x,y) \to \left(\partial_{x_i} + \sqrt{d_i}x_i\right) f(x)$ in $L^2(\mathbb{R}^n, dx)$ as $y \to 0^+$. Since $u(x,y) \to f(x)$ in $L^2(\mathbb{R}^n, dx)$ and $\sqrt{d_i}x_i$ is a bounded function in $B_R(x_0)$, we have that $\sqrt{d_i}x_i u(x,y)$ converges to $\sqrt{d_i}x_i f(x)$ in $L^2(B_R(x_0), dx)$ as $y \to 0^+$. Hence $\nabla_x u(x,y) \to \nabla_x f(x)$, as $y \to 0^+$, in $L^2(B_R(x_0), dx)$.

B. Proof of Harnack's inequality for $(\mathbf{O}_D)^\sigma$: We apply the transference method explained in Section 4.4. For this case we take $M(x) = (\det D)^{n/4}\pi^{-n/4}e^{-\frac{Dx\cdot x}{2}}$ and $h(x) = x$. Clearly $h_k^D(x) = (U \circ W)H_k^D(x)$ and we have the relation

$$\mathbf{O}_D H_k^D = (U \circ W)^{-1} \circ \left(\mathcal{H}_D - \sum_{i=1}^n d_i \right) \circ (U \circ W) H_k^D. \tag{4.15}$$

See also [3]. It can be easily checked, as done for $(\mathcal{H}_D)^\sigma$ above, that the operator $(\mathcal{H}_D - \sum_{i=1}^n d_i)^\sigma$ satisfies Harnack's inequality. Hence the conclusion for $(\mathbf{O}_D)^\sigma$ follows from Lemma 4.9.

C. Proof of Harnack's inequality for $(\mathbf{O}_B)^\sigma$: Consider the change of variables $h(x) = A^t x$ and call W the corresponding operator as in Definition 4.8. Then it is easy to check that

$$\mathbf{O}_B(H_k^D \circ h^{-1})(h(x)) = \mathbf{O}_D H_k^D(x).$$

Then we have $\mathbf{O}_B = W^{-1} \circ \mathbf{O}_D \circ W$ and the result follows by the transference method.

D. Proof of Harnack's inequality for $(\mathcal{H}_B)^\sigma$:

We observe that parallel to the case of the operator $\mathbf{O}_B$ we can get $\mathcal{H}_B = W^{-1} \circ \mathcal{H}_D \circ W$ with W as in Subsection 4.5.1 above and then we get Harnack's inequality for the operator $(\mathcal{H}_B)^\sigma$.

4.5.2 Laguerre operators

We suggest the reader to check [1, 41, 53, 85, 86] for the proof of the basics about Laguerre expansions we use here. Let us consider the system of multidimensional Laguerre polynomials $L_k^\alpha(x)$, where $k \in \mathbb{N}_0^n$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in (-1,\infty)^n$ and $x \in (0,\infty)^n$. It is well known that the Laguerre polynomials form a complete orthogonal system in $L^2((0,\infty)^n, d\gamma_\alpha(x))$, where $d\gamma_\alpha(x) = x_1^{\alpha_1}e^{-x_1}\,dx_1 \cdots x_n^{\alpha_n}e^{-x_n}\,dx_n$. We denote by $\tilde{L}_k^\alpha$ the orthonormalized Laguerre polynomials. The polynomials $\tilde{L}_k^\alpha$ are eigenfunctions of the Laguerre differential operator

$$L_\alpha = \sum_{i=1}^n \left(-x_i\frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}\right),$$

namely, $L_\alpha(\tilde{L}_k^\alpha) = |k|\tilde{L}_k^\alpha$. There are several systems of Laguerre functions. We first prove Harnack's inequality for the operator L_α^φ (related to the system φ_k^α below) and then we apply the transference method of Section 4.4 to get the result for the remaining systems.

A. Laguerre functions φ_k^α

This multidimensional system in $L^2((0,\infty)^n, d\mu_0(x))$, where $d\mu_0(x) = dx_1 \cdots dx_n$, is given as a tensor product $\varphi_k^\alpha(x) = \varphi_{k_1}^{\alpha_1}(x_1)\cdots\varphi_{k_n}^{\alpha_n}(x_n)$, where each factor $\varphi_{k_i}^{\alpha_i}(x_i) = x_i^{\alpha_i}(2x_i)^{1/2}e^{-x_i^2/2}\tilde{L}_{k_i}^{\alpha_i}(x_i^2)$. The functions φ_k^α are eigenfunctions of the differential operator

$$L_\alpha^\varphi = \frac{1}{4}\left(-\Delta + |x|^2\right) + \sum_{i=1}^n \frac{1}{4x_i^2}\left(\alpha_i^2 - \frac{1}{4}\right), \tag{4.16}$$

namely,

$$L_\alpha^\varphi \varphi_k^\alpha(x) = \sum_{i=1}^n \left(k_i + \frac{\alpha_i+1}{2}\right)\varphi_{k_i}^{\alpha_i}(x_i). \tag{4.17}$$

Clearly, the functions φ_k^α are locally bounded in $(0,\infty)^n$. Observe that

$$\partial_{x_i}\varphi_k^\alpha(x) = -|k|^{1/2}\varphi_{k-e_i}^{\alpha_i+e_i}(x) - \left(x_i - \frac{1}{x_i}\left(\alpha_i + \frac{1}{2}\right)\right)\varphi_k^\alpha(x). \tag{4.18}$$

Therefore, (4.12) holds for this system and we get that the solution u in (4.3) of the extension problem for L_α^φ is classical. Moreover, it can be easily seen from [85, p. 102] that $e^{-tL_\alpha^\varphi}$ is positivity-preserving.

Let us prove Theorem A for $(L_\alpha^\varphi)^\sigma$. We can do this as we did for $(\mathcal{H}_D)^\sigma$ above by following the reasoning line by line, but with some modifications as follows. Let $f \in \mathrm{Dom}(L_\alpha^\varphi)$, $f \geqslant 0$, such that $(L_\alpha^\varphi)^\sigma f = 0$ in $L^2(B_R(x_0), d\mu_0(x))$, and let u be the corresponding solution to the extension problem. By (4.18) and a similar argument for that of $\mathcal{H}_D^\sigma$ we can check that $\|\nabla_x u(x,y)\|_{L^2(B_R(x_0),d\mu_0(x))}$ converges to $\|\nabla_x f\|_{L^2(B_R(x_0),d\mu_0(x))}$, as $y \to 0^+$. Moreover, the potential in (4.16) is locally bounded. Hence, by Theorem 4.4, f satisfies Harnack's inequality and it is continuous.

Note that the same arguments above can be used for $(L^{\varphi}_{\alpha} - \frac{\alpha+1}{2})^{\sigma}$ instead of $(L^{\varphi}_{\alpha})^{\sigma}$, so it also satisfies Theorem A.

B. Laguerre functions ℓ^{α}_{k}

The Laguerre functions ℓ^{α}_{k} are defined as $\ell^{\alpha}_{k}(x) = \ell^{\alpha_1}_{k_1}(x_1)\cdots\ell^{\alpha_n}_{k_n}(x_n)$, where $\ell^{\alpha_i}_{k_i}$ are the one-dimensional Laguerre functions $\ell^{\alpha_i}_{k_i}(x_i) = e^{-x_i/2}\tilde{L}^{\alpha_i}_{k_i}(x_i)$. Each ℓ^{α}_{k} is an eigenfunction of the differential operator

$$L^{\ell}_{\alpha} = \sum_{i=1}^{n}\left(-x_i\frac{\partial^2}{\partial x_i^2} - (\alpha_i+1)\frac{\partial}{\partial x_i} + \frac{x_i}{4}\right).$$

More explicitly, $L^{\ell}_{\alpha}\ell^{\alpha}_{k} = \sum_{i=1}^{n}\left(k_i + \frac{\alpha_i+1}{2}\right)\ell^{\alpha_i}_{k_i}$. For $d\mu_{\alpha}(x) = x_1^{\alpha_1}\cdots x_n^{\alpha_n}dx$, the operator L^{ℓ}_{α} is positive and symmetric in $L^2((0,\infty)^n, d\mu_{\alpha}(x))$. The system $\{\ell^{\alpha}_{k} : k \in \mathbb{N}_0^n\}$ is an orthonormal basis of $L^2((0,\infty)^n, d\mu_{\alpha}(x))$.

To apply the transference method we set $M(x) = 2^{n/2}x_1^{\alpha_1+1/2}\cdots x_n^{\alpha_n+1/2}$ and $h(x) = (x_1^2,\ldots,x_n^2)$. Then $U \circ W$ is an isometry from $L^2((0,\infty)^n, d\mu_{\alpha}(x))$ into $L^2((0,\infty)^n, d\mu_0(x))$ and $L^{\ell}_{\alpha} = (U\circ W)^{-1}\circ L^{\varphi}_{\alpha}\circ(U\circ W)$, see [1].

C. Laguerre functions ψ^{α}_{k}

Consider the Laguerre system $\psi^{\alpha}_{k}(x) = \psi^{\alpha_1}_{k_1}(x_1)\cdots\psi^{\alpha_n}_{k_n}(x_n)$, which is orthonormal in $L^2((0,\infty)^n, d\mu_{2\alpha+1}(x))$, where $d\mu_{2\alpha+1}(x) = x_1^{2\alpha_1+1}dx_1\cdots x_n^{2\alpha_n+1}dx_n$ and $\psi^{\alpha_i}_{k_i}$ is the one-dimensional Laguerre function $\psi^{\alpha_i}_{k_i}(x_i) = \sqrt{2}\,\ell^{\alpha_i}_{k_i}(x_i^2)$. The functions ψ^{α}_{k} are eigenfunctions of the operator

$$L^{\psi}_{\alpha} = \frac{1}{4}\left(-\Delta + |x|^2\right) - \sum_{i=1}^{n}\frac{2\alpha_i+1}{4x_i}\frac{\partial}{\partial x_i}.$$

In fact, $L^{\psi}_{\alpha}(\psi^{\alpha}_{k}) = \sum_{i=0}^{n}\left(k_i + \frac{\alpha_i+1}{2}\right)\psi^{\alpha_i}_{k_i}$.

For the transference method we have to take $M(x) = x_1^{\alpha_1+1/2}\cdots x_n^{\alpha_n+1/2}$ and $h(x) = x$. Then $U\circ W$ is an isometry from $L^2((0,\infty)^n, d\mu_{2\alpha+1}(x))$ into $L^2((0,\infty)^n, d\mu_0(x))$ and $L^{\psi}_{\alpha} = (U\circ W)^{-1}\circ L^{\varphi}_{\alpha}\circ(U\circ W)$, see [1].

D. Laguerre functions $\mathcal{L}^{\alpha}_{k}$

The functions $\mathcal{L}^{\alpha}_{k}(x) = \mathcal{L}^{\alpha_1}_{k_1}(x_1)\cdots\mathcal{L}^{\alpha_n}_{k_n}(x_n)$ form an orthonormal system in $L^2((0,\infty)^n, d\mu_0(x))$, where $\mathcal{L}^{\alpha_i}_{k_i}$ is the one-dimensional Laguerre function given by $\mathcal{L}^{\alpha_i}_{k_i}(x_i) = x_i^{\alpha_i/2}\ell^{\alpha_i}_{k_i}(x_i)$. The functions $\mathcal{L}^{\alpha}_{k}$ are eigenfunctions of the operator

$$L^{\mathcal{L}}_{\alpha} = \sum_{i=1}^{n}\left(-x_i\frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i} + \frac{x_i}{4} + \frac{\alpha_i^2}{4x_i}\right).$$

In fact, $L^{\mathcal{L}}_{\alpha}(\mathcal{L}^{\alpha}_{k}) = \sum_{i=0}^{n}\left(k_i + \frac{\alpha_i+1}{2}\right)\mathcal{L}^{\alpha_i}_{k_i}$.

Apply the transference method with $M(x) = 2^{n/2}x_1^{1/2}\cdots x_n^{1/2}$ and $h(x) = (x_1^2,\ldots,x_n^2)$. Then $U\circ W$ is an isometry from $L^2((0,\infty)^n, d\mu_0(x))$ into itself and $L^{\mathcal{L}}_{\alpha} = (U\circ W)^{-1}\circ L^{\varphi}_{\alpha}\circ(U\circ W)$, see [1].

E. Laguerre polynomials $\tilde{L}_k^\alpha$

Finally consider the Laguerre polynomials operator L_α. Let $h(x) = (x_1^2, \dots, x_n^2)$ and $M(x) = 2^{n/2}e^{-|x|^2/2}x_1^{\alpha_1+1/2}\cdots x_n^{\alpha_n+1/2}$. We have that the operator $U \circ W$ is an isometry from $L^2((0,\infty)^n, d\gamma_\alpha(x))$ into $L^2((0,\infty)^n, d\mu_0(x))$ and $L_\alpha = (U\circ W)^{-1}\circ(L_\alpha^\varphi - \frac{\alpha+1}{2})\circ(U\circ W)$, see [1], so the transference method applies.

4.5.3 Ultraspherical operators

Here we restrict ourselves to one-dimensional expansions. We denote the ultraspherical polynomials of type $\lambda > 0$ and degree $k \in \mathbb{N}_0$ by $P_k^\lambda(x)$, $x \in (-1,1)$, see [53, 59, 85]. It is well-known that the set of trigonometric polynomials $\{P_k^\lambda(\cos\theta) : \theta \in (0,\pi)\}$ forms an orthogonal basis of $L^2((0,\pi), dm_\lambda(\theta))$, where $dm_\lambda(\theta) = \sin^{2\lambda}\theta\, d\theta$. The polynomials $P_k^\lambda(\cos\theta)$ are eigenfunctions of the ultraspherical operator

$$L_\lambda = -\frac{d^2}{d\theta^2} - 2\lambda\cot\theta\frac{d}{d\theta} + \lambda^2,$$

that is, $L_\lambda P_k^\lambda(\cos\theta) = (k+\lambda)^2P_k^\lambda(\cos\theta)$. We denote by $\tilde{P}_k^\lambda(\cos\theta)$ the orthonormalized polynomials given by $\frac{\Gamma(\lambda)^2(n+\lambda)n!}{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}P_k^\lambda(\cos\theta)$. There exists a constant A such that $|P_k^\lambda(\cos\theta)| \leqslant Ak^{2\lambda-1}$, see [59]. This and Stirling's formula for the Gamma function [53] imply that there exists C such that $|\tilde{P}_k^\lambda(\theta)| \leqslant Ck$ for all k. A similar estimate holds for the derivatives of $\tilde{P}_k^\lambda$ since $\frac{d}{dx}P_k^\lambda(x) = 2\lambda P_{k-1}^{\lambda+1}(x)$, see [85].

The set of orthonormal ultraspherical functions $p_k^\lambda(\theta) = \sin^\lambda\theta\tilde{P}_k^\lambda(\cos\theta)$ is a basis of $L^2((0,\pi), dx)$. The ultraspherical functions are eigenfunctions of the differential operator

$$l_\lambda = -\frac{d^2}{d\theta^2} + \frac{\lambda(\lambda-1)}{\sin^2\theta},$$

namely, $l_\lambda p_k^\lambda(\theta) = (k+\lambda)^2p_k^\lambda(\theta)$. By using the estimates for $\tilde{P}_k^\lambda$ given above, we can easily check that this system satisfies (4.12). Moreover, the heat-diffusion semigroup e^{-tl_λ} is positivity-preserving. This last assertion can be deduced directly from the facts that the heat-diffusion semigroup for the ultraspherical polynomials e^{-tL_λ} is positivity preserving, see [12], and $e^{-tl_\lambda} = (U\circ W)\circ(e^{-tL_\lambda})\circ(U\circ W)^{-1}$, see Subsection 4.5.3 below.

A. Proof of Harnack's inequality for $(l_\lambda)^\sigma$

We do this as we did for $(\mathcal{H}_D)^\sigma$ above by following parallel arguments. Let $f \in \mathrm{Dom}(l_\lambda)$, $f \geqslant 0$, such that $(l_\lambda)^\sigma f = 0$ in $L^2(I, d\theta)$, for some interval $I \subset (0,\pi)$. Let u be the solution to the extension problem for l_λ and this f. By the estimates mentioned above, u is classical. The potential here is $V(\theta) = \frac{\lambda(\lambda-1)}{\sin^2\theta}$, which is a locally bounded function. Observe that $\frac{d}{d\theta}p_k^\lambda(\theta) = -2\lambda p_{k-1}^{\lambda+1}(\theta) + \lambda\cot\theta p_k^\lambda(\theta)$. Since $\cot\theta$ is bounded in I, by following the same arguments as those for $(\mathcal{H}_D)^\sigma$, we can get $\|\frac{\partial}{\partial\theta}u(\theta,y)\|_{L^2(I,d\theta)} \to \|f'(\theta)\|_{L^2(I,d\theta)}$, as $y \to 0^+$. The conclusion follows by Theorem 4.4.

B. Proof of Harnack's inequality for $(L_\lambda)^\sigma$

This is achieved by applying the transference method with $M(\theta) = \sin^\lambda \theta$ and $h(\theta) = \theta$. It readily follows that $(L_\lambda)^\sigma = (U \circ W)^{-1} \circ (l_\lambda)^\sigma \circ (U \circ W)$.

4.6 Laplacian and Bessel operators

In this section we will prove Theorem A for the fractional powers of the Bessel operator. This operator is a generalization of the radial Laplacian. For the sake of completeness and to show how the proof works, we present first the case of the fractional Laplacian on $\mathbb{R}^n$, for which the more familiar Fourier transform applies.

The main difference with respect to the examples given before is that these operators have a continuous spectrum and the Fourier and Hankel transforms come into play.

4.6.1 The Laplacian on $\mathbb{R}^n$

Consider the fractional Laplacian defined by $\widehat{(-\Delta)^\sigma f}(\xi) = |\xi|^{2\sigma}\widehat{f}(\xi)$, where $\widehat{f}$ denotes the Fourier transform: $\widehat{f}(\xi) \equiv c_\xi(f) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx$, $\xi \in \mathbb{R}^n$. The eigenfunctions of $-\Delta$, indexed by the continuous parameter ξ, are $\varphi_\xi(x) = e^{-ix\cdot\xi}$, $x \in \mathcal{R}^n$, and $(-\Delta)\varphi_\xi(x) = |\xi|^2\varphi_\xi(x)$. Note that for any compact subset $K \subset \mathcal{R}^n$ and any multi-index $\beta \in \mathbb{N}_0^n$, $|\beta| \leqslant 2$, we have

$$\left\|D^\beta \varphi_\xi\right\|_{L^\infty(K)} \leqslant |\xi|^{|\beta|}. \tag{4.19}$$

For any $f \in L^2(K, dx)$, the heat semigroup is defined by $e^{t\Delta}f(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathcal{R}^n} e^{-t|\xi|^2}c_\xi(f)\varphi_{-\xi}(x)\,d\xi$. As

$$\left|e^{t\Delta}f(x)\right| \leqslant C\int_{\mathcal{R}^n}\left|e^{-t|\xi|^2}c_\xi(f)\varphi_{-\xi}(x)\right|\,d\xi \leqslant Ct^{-n/4}\|f\|_{L^2(K,dx)}, \quad x \in K, \tag{4.20}$$

the integral that defines $e^{t\Delta}f(x)$ is absolutely convergent in $K \times (0, T)$ with $T > 0$. Moreover, $e^{t\Delta}$ is positivity-preserving in the sense of (4.2) because it is given by convolution with the Gauss-Weierstrass kernel. Note that, in this spectral language, $\mathrm{Dom}(-\Delta) = \left\{f \in L^2(\mathcal{R}^n, dx) : |\xi|^2\hat{f}(\xi) \in L^2(\mathcal{R}^n, dx)\right\} = \left\{f \in L^2(\mathcal{R}^n, dx) : D^2 f \in L^2(\mathcal{R}^n, dx)\right\} = W^{2,2}(\mathbb{R}^n)$, the Sobolev space of functions in $L^2(\mathbb{R}^n)$ with Hessian $D^2 f$ in $L^2(\mathbb{R}^n)$.

Let us show Theorem A for $(-\Delta)^\sigma$. Assume that $f \in W^{2,2}(\mathbb{R}^n)$, $f \geqslant 0$ and $(-\Delta)^\sigma f = 0$ in $L^2(B_R, dx)$, for some ball $B_R \subset \mathcal{R}^n$. By Theorem 4.4, we just must check that $\|\nabla_x u(x, y)\|_{L^2(B_R,dx)}$ remains bounded as $y \to 0^+$. To that end, observe that for any $x \in B_R$ and $y > 0$, by (4.20),

$$\int_0^\infty |e^{t\Delta}f(x)e^{-\frac{y^2}{4t}}|\,\frac{dt}{t^{1+\sigma}} \leqslant C\|f\|_{L^2(B_R,dx)}\int_0^\infty t^{-n/4}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}} \leqslant F(y),$$

for some function $F(y)$. This means that we can interchange integrals in u to get

$$u(x, y) = \frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)(2\pi)^{n/2}}\int_{\mathcal{R}^n} c_\xi(f)\varphi_{-\xi}(x)\int_0^\infty e^{-t|\xi|^2}e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}\,d\xi. \tag{4.21}$$

By (4.19) and using the same arguments as above, it is easy to see that this double integral defines a function in $C^2(B_R \times (0,\infty))$. So in this case u is a classical solution of (4.4). By using Plancherel's Theorem and (4.21) we have

$$\begin{aligned}
\|\partial_{x_j}(u(x,y)-f(x))\|^2_{L^2(\mathcal{R}^n,dx)} &= \left\|\left(\partial_{x_j}(u(x,y)-f(x))\right)^{\widehat{}}(\xi)\right\|^2_{L^2(\mathcal{R}^n,d\xi)} \\
&= \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n}\left|(-i\xi_j)c_\xi(f)\left[\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-t|\xi|^2}e^{-\frac{y^2}{4t}}\frac{dt}{t^{1+\sigma}}-1\right]\right|^2 d\xi \qquad (4.22)\\
&= \frac{1}{(2\pi)^n}\int_{\mathcal{R}^n}\left|(-i\xi_j)c_\xi(f)\varphi_{-\xi}(x)\right|^2\left[\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty\left(e^{-t|\xi|^2}-1\right)e^{-\frac{y^2}{4t}}\frac{dt}{t^{1+\sigma}}\right]^2 d\xi.
\end{aligned}$$

Observe that the expression in square brackets above is uniformly bounded in y and it converges to 0 when $y \to 0^+$. Moreover, as $f \in W^{2,2}(\mathbb{R}^n)$, $\left\|(-i\xi_j)c_\xi(f)\right\|_{L^2(\mathcal{R}^n,d\xi)} = \|\partial_{x_j}f\|_{L^2(\mathbb{R}^n,dx)} < \infty$. Hence, by dominated convergence in (4.22), $\partial_{x_j}u(x,y)$ converges to $\partial_{x_j}f$ in $L^2(\mathcal{R}^n,dx)$ as $y \to 0^+$. Whence $\nabla_x u(x,y) \to \nabla_x f(x)$ as $y \to 0^+$, in $L^2(B_R,dx)$.

4.6.2 The Bessel operators on $(0,\infty)$

Let $\lambda > 0$. Let us denote by Δ_λ the Bessel operator

$$\Delta_\lambda = -\frac{d^2}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}, \quad x > 0,$$

which is positive and symmetric in $L^2((0,\infty), dm_\lambda(x))$, where $dm_\lambda(x) = x^{2\lambda}dx$, see [7, 59]. If $2\lambda = n-1$, $n \in \mathbb{N}$, then we recover the radial Laplacian on $\mathbb{R}^n$. Let J_ν denote the Bessel function of the first kind with order ν and let us define $\varphi_\xi^\lambda(x) = x^{-\lambda}(\xi x)^{1/2}J_{\lambda-1/2}(\xi x)$, $x, \xi \in (0,\infty)$. Then, $\Delta_\lambda\varphi_\xi^\lambda(x) = \xi^2\varphi_\xi^\lambda(x)$, see [7]. These functions will play the role of the exponentials $e^{-ix\xi}$ in the case of the Laplacian.

We also consider the Bessel operator

$$S_\lambda = -\frac{d^2}{dx^2} + \frac{\lambda^2-\lambda}{x^2},$$

which is positive and symmetric in $L^2((0,\infty),dx)$. Observe that the potential $V(x) = \frac{\lambda^2-\lambda}{x^2}$ is a locally bounded function. If we let $\psi_\xi^\lambda(x) = x^\lambda\varphi_\xi^\lambda(x)$ then $S_\lambda\psi_\xi^\lambda(x) = \xi^2\psi_\xi^\lambda(x)$, see [7]. The Hankel transform

$$f \longmapsto \int_0^\infty \psi^\lambda(\xi x)f(x)\,dx$$

is a unitary transformation in $L^2((0,\infty),dx)$, see [87, Chapter 8]. On the other hand, it is known that for any compact subset $K \subset (0,\infty)$ and $k \in \mathbb{N}_0$, there exist a nonnegative number $\varepsilon = \varepsilon_{K,k}$ and a constant $C = C_{K,k}$ such that $\|\psi_\xi^\lambda(x)\|_{L^\infty(K,dx)} \leqslant C$, and $\|\frac{d^k}{dx^k}\psi_\xi^\lambda(x)\|_{L^\infty(K,dx)} \leqslant C|\xi|^\varepsilon$, see [53]. Therefore parallel to the case of the Laplacian we can define the heat semigroup as

$$e^{-tS_\lambda}f(x) = \int_0^\infty e^{-t\xi^2}c_\xi(f)\psi_\xi^\lambda(x)\,d\xi,$$

where $c_\xi(f) = \int_0^\infty f(x)\psi_\xi^\lambda(x)\,dx$. Moreover,

$$\left|e^{-tS_\lambda}f(x)\right| \leqslant \int_0^\infty \left|e^{-t\xi^2}c_\xi(f)\psi_\xi^\lambda(x)\right| d\xi \leqslant Ct^{-1/4}\|f\|_{L^2(K,dx)}, \quad x \in K,$$

so the integral that defines $e^{-tS_\lambda}f(x)$ is absolutely convergent in $K \times (0, T)$ with $T > 0$. Since e^{-tS_λ} is positivity-preserving (see [7]), we can follow step by step the arguments we gave for the case of the classical Laplacian to derive Theorem A for the operator $(S_\lambda)^\sigma$.

In order to get Theorem A for $(\Delta_\lambda)^\sigma$ we apply the transference method. Indeed, an obvious modification of Lemma 4.7 is applied with $M(x) = x^\lambda$ to get $(\Delta_\lambda)^\sigma = U^{-1} \circ (S_\lambda)^\sigma \circ U$.

Chapter 5

Fractional vector-valued Littlewood-Paley-Stein theory for semigroups

In this chapter, we consider the fractional derivative of a general Poisson semigroup. With this fractional derivative, we define the generalized fractional Littlewood–Paley g-function for semigroups acting on L^p-spaces of functions with values in Banach spaces. In Section 5.1, we give a characterization of the classes of Banach spaces for which the fractional Litlewood–Paley g-function is bounded on L^p-spaces. It is also shown that the same kind of results exist for the case of the fractional area function and the fractional g_λ^*-function on $\mathbb{R}^n$, see in Section 5.5. In Section 5.6, we consider the relationship of the almost sure finiteness of the fractional Littlewood–Paley g-function, area function, and g_λ^*-function with the Lusin cotype property of the underlying Banach space.

5.1 Main theorems about the fractional Littlewood-Paley-Stein theory

Let $\{\mathcal{T}_t\}_{t\geqslant 0}$ be a collection of linear operators defined on $L^p(\Omega, d\mu)$ over a positive measure space $(\Omega, d\mu)$ satisfying the following properties:

$$\mathcal{T}_0 = \mathrm{Id}, \quad \mathcal{T}_t\mathcal{T}_s = \mathcal{T}_{t+s}, \quad \|\mathcal{T}_t\|_{L^p\to L^p} \leqslant 1 \quad \forall p \in [1,\infty], \tag{5.1}$$

$$\lim_{t\to 0} \mathcal{T}_t f = f \quad \text{in} \quad L^2 \quad \forall f \in L^2, \tag{5.2}$$

$$\mathcal{T}_t^* = \mathcal{T}_t \quad \text{on} \quad L^2, \quad \mathcal{T}_t f \geqslant 0 \quad \text{if } f \geqslant 0, \quad \mathcal{T}_t 1 = 1. \tag{5.3}$$

The subordinated Poisson semigroup $\{\mathcal{P}_t\}_{t\geqslant 0}$ is defined as

$$\mathcal{P}_t f = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\mathcal{T}_{\frac{t^2}{4u}}\,du = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4u}}}{u^{\frac{3}{2}}}\mathcal{T}_u f du. \tag{5.4}$$

$\{\mathcal{P}_t\}_{t\geqslant 0}$ is again a symmetric diffusion semigroup, see [78]. Recall that if A denotes the infinitesimal generator of $\{\mathcal{T}_t\}_{t\geqslant 0}$, then that of $\{\mathcal{P}_t\}_{t\geqslant 0}$ is $-(-A)^{1/2}$. $\mathcal{T}_t$ and $\mathcal{P}_t$ have straightforward extensions to $L^p_{\mathbb{B}}(\Omega)$ for every Banach space $\mathbb{B}$, where $L^p_{\mathbb{B}}(\Omega)$ denotes the usual Bochner–Lebesgue L^p-space of $\mathbb{B}$-valued functions defined on Ω and the extensions are also contractive. So we shall consider $\mathcal{T}_t$ and $\mathcal{P}_t$ as semigroups on $L^p_{\mathbb{B}}(\Omega)$ too. Recall that the generalized Littlewood–Paley g-function associated to the semigroup is defined as

$$g_1^q(f)(x) = \left(\int_0^\infty \|t\partial_t\mathcal{P}_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

C. Segovia and R. L. Wheeden, see [70], motivated by some characterization of potential spaces on $\mathbb{R}^n$, introduced the fractional derivative ∂^α. Parallel to C. Segovia and R. L. Wheeden, we define

$$\partial_t^\alpha \mathcal{P}_t f = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)}\int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f) s^{m-\alpha-1} ds, \qquad t>0, \tag{5.5}$$

where m is the smallest integer which strictly exceeds α. In [70], the authors developed a satisfactory theory of euclidean square functions of Littlewood–Paley type in which the usual derivatives are substituted by these fractional derivatives. In Section 5.2, we shall see that for any $f\in L^p(\Omega)$, this partial derivative is well defined and then we are allowed to consider the following "fractional generalized Littlewood–Paley g-function" associated to the semigroup as

$$g_\alpha^q(f)(x) = \left(\int_0^\infty \|t^\alpha\partial_t^\alpha\mathcal{P}_t f(x)\|_{\mathbb{B}}^q \frac{dt}{t}\right)^{\frac{1}{q}}, \qquad \forall f\in\bigcup_{1\leqslant p\leqslant\infty} L^p_{\mathbb{B}}(\Omega). \tag{5.6}$$

Let $\mathbb{E}\subset L^2(\Omega)$ be the subspace of the fixed points of $\{\mathcal{P}_t\}_{t\geqslant 0}$, that is, the subspace of all f such that $\mathcal{P}_t(f)=f$ for all $t\geqslant 0$. Let $E: L^2(\Omega)\longrightarrow\mathbb{E}$ be the orthogonal projection. It is clear that E extends to be a contractive projection (still denoted by E) on $L^p(\Omega)$ for every $1\leqslant p\leqslant\infty$ and that $E(L^p(\Omega))$ is exactly the fix point space of $\{\mathcal{P}_t\}_{t\geqslant 0}$ on $L^p(\Omega)$. Moreover, for any Banach space $\mathbb{B}$, E extends to be a contractive projection on $L^p_{\mathbb{B}}(\Omega)$ for every $1\leqslant p\leqslant\infty$ and that $E(L^p_{\mathbb{B}}(\Omega))$ is again the fix point space of $\{\mathcal{P}_t\}_{t\geqslant 0}$ considered as a semigroup on $L^p_{\mathbb{B}}(\Omega)$. In the particular case on $\mathbb{R}^n$, $\mathbb{E}=0$ and so $E(L^p_{\mathbb{B}}(\mathbb{R}^n))=0$. According to our convention, in the sequel, we shall use the same symbol E to denote any of these contractive projections. Our main goal in this chapter is to extend the results in [57] to the fractional derivative case. Now we list our main theorems.

Theorem 5.1. *Given a Banach space $\mathbb{B}$ and $2\leqslant q<\infty$, the following statements are equivalent:*

(i) $\mathbb{B}$ *is of Lusin cotype* q.

(ii) *For every symmetric diffusion semigroup* $\{\mathcal{T}_t\}_{t\geqslant 0}$ *with subordinated semigroup* $\{\mathcal{P}_t\}_{t\geqslant 0}$, *for every (or, equivalently, for some)* $p \in (1,\infty)$, *and for every (or, equivalently, for some)* $\alpha > 0$, *there is a constant* C *such that*

$$\|g_\alpha^q(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

Theorem 5.2. *Given a Banach space* $\mathbb{B}$ *and* $1 < q \leqslant 2$, *the following statements are equivalent:*

(i) $\mathbb{B}$ *is of Lusin type* q.

(ii) *For every symmetric diffusion semigroup* $\{\mathcal{T}_t\}_{t\geqslant 0}$ *with subordinated semigroup* $\{\mathcal{P}_t\}_{t\geqslant 0}$, *for every (or, equivalently, for some)* $p \in (1,\infty)$, *and for every (or, equivalently, for some)* $\alpha > 0$, *there is a constant* C *such that*

$$\|f\|_{L^p_{\mathbb{B}}(\Omega)} \leqslant C\left(\|E_0(f)\|_{L^p_{\mathbb{B}}(\Omega)} + \|g_\alpha^q(f)\|_{L^p(\Omega)}\right), \qquad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

5.2 Fractional derivatives

In this section, we shall give some properties of the fractional derivatives.

Theorem 5.3. *Given a Banach space* $\mathbb{B}$, $1 \leqslant p \leqslant \infty$, $\alpha > 0$, *and* $t > 0$, $\partial_t^\alpha \mathcal{P}_t f$ *is well defined as a function in* $L^p_{\mathbb{B}}(\Omega)$ *for any* $f \in L^p_{\mathbb{B}}(\Omega)$. *Moreover, there exists a constant* C_α *such that*

$$\|\partial_t^\alpha \mathcal{P}_t f\|_{L^p_{\mathbb{B}}(\Omega)} \leqslant \frac{C_\alpha}{t^\alpha}\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f \in L^p_{\mathbb{B}}(\Omega). \tag{5.7}$$

Proof. Firstly, let us consider the case $\alpha = m,\ m = 1, 2, \ldots$. We know that, for any $m = 1, 2, \ldots$, there exist constants C_m such that

$$\partial_t^m\left(\frac{t}{\sqrt{u}}\, e^{-\frac{t^2}{4u}}\right) \leqslant C_m \frac{1}{\left(\sqrt{u}\right)^m}\, e^{-\frac{t^2}{4u}}.$$

Then, by using formula (5.4), we have

$$\begin{aligned}\|\partial_t^m \mathcal{P}_t f\|_{L^p_{\mathbb{B}}(\Omega)} &\leqslant C\int_0^\infty \left|\partial_t^m\left(\frac{t}{\sqrt{u}}\, e^{-\frac{t^2}{4u}}\right)\right| \|\mathcal{T}_u f\|_{L^p_{\mathbb{B}}(\Omega)}\frac{du}{u} \\ &\leqslant C_m \int_0^\infty \frac{1}{\left(\sqrt{u}\right)^m}\, e^{-\frac{t^2}{4u}}\frac{du}{u}\,\|f\|_{L^p_{\mathbb{B}}(\Omega)} = \frac{C_m}{t^m}\|f\|_{L^p_{\mathbb{B}}(\Omega)}. \end{aligned} \tag{5.8}$$

So we have proved (5.7) when α is integer. Therefore, given $\alpha > 0$, we have

$$
\begin{aligned}
\|\partial_t^\alpha \mathcal{P}_t f\|_{L^p_{\mathbb{B}}(\Omega)} &= \left\| \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \mathcal{P}_{t+s}(f) s^{m-\alpha-1} ds \right\|_{L^p_{\mathbb{B}}(\Omega)} \\
&\leqslant \frac{C_m}{\Gamma(m-\alpha)} \|f\|_{L^p_{\mathbb{B}}(\Omega)} \int_0^\infty \frac{1}{(t+s)^m} s^{m-\alpha-1} ds \\
&= \frac{C_m}{\Gamma(m-\alpha)} B(m-\alpha, \alpha) \frac{\|f\|_{L^p_{\mathbb{B}}(\Omega)}}{t^\alpha} = C_\alpha \frac{\|f\|_{L^p_{\mathbb{B}}(\Omega)}}{t^\alpha},
\end{aligned} \tag{5.9}
$$

where B denotes the Beta function, see [53]. □

Observe that by estimate (5.8), we can perform integration by parts in the formula (5.5). In particular, the formula (5.5) is valid for α being integer.

Theorem 5.4. *Given a Banach space $\mathbb{B}$ and $0 < \beta < \gamma$, we have*

$$
\partial_t^\beta \mathcal{P}_t f = \frac{e^{-i\pi(\gamma-\beta)}}{\Gamma(\gamma-\beta)} \int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds, \quad \forall f \in \bigcup_{1 \leqslant p \leqslant \infty} L^p_{\mathbb{B}}(\Omega). \tag{5.10}
$$

Proof. Assume that $f \in L^p_{\mathbb{B}}(\Omega)$ for some $1 \leqslant p \leqslant \infty$, by changing variables and Fubini's theorem, we have the following computation as in (5.9)

$$
\begin{aligned}
\int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds &= \int_0^\infty \frac{e^{-i\pi(k-\gamma)}}{\Gamma(k-\gamma)} \int_0^\infty \partial_t^k \mathcal{P}_{t+s+u}(f) u^{k-\gamma-1} du s^{\gamma-\beta-1} ds \\
&= \frac{e^{-i\pi(k-\gamma)}}{\Gamma(k-\gamma)} \int_0^\infty \int_s^\infty \partial_t^k \mathcal{P}_{t+\bar{u}}(f) (\bar{u}-s)^{k-\gamma-1} s^{\gamma-\beta-1} d\bar{u} ds \\
&= \frac{e^{-i\pi(k-\gamma)} B(k-\gamma, \gamma-\beta)}{\Gamma(k-\gamma)} \int_0^\infty \partial_t^k \mathcal{P}_{t+\bar{u}}(f) \bar{u}^{k-\beta-1} d\bar{u},
\end{aligned} \tag{5.11}
$$

where k is the smallest integer which is bigger than γ. By (5.8), we know that we can integrate by parts in the last integral of (5.11). Let m be the smallest integer which is bigger than β. Then by integrating by parts $k - m$ times, we obtain

$$
\begin{aligned}
&\int_0^\infty \partial_t^\gamma \mathcal{P}_{t+s}(f) s^{\gamma-\beta-1} ds \\
&= \frac{B(k-\gamma, \gamma-\beta) e^{-i\pi(m-\gamma)}}{\Gamma(k-\gamma)} (k-\beta-1) \cdots (m-\beta) \int_0^\infty \partial_t^m \mathcal{P}_{t+\bar{u}}(f) \bar{u}^{m-\beta-1} d\bar{u} \\
&= e^{-i\pi(\gamma-\beta)} \Gamma(\gamma-\beta) \partial_t^\beta \mathcal{P}_t f.
\end{aligned}
$$

Hence we get (5.10). □

Theorem 5.5. *Given a Banach space $\mathbb{B}$ and $\alpha,\ \beta > 0$, $\partial_t^\alpha \left(\partial_t^\beta \mathcal{P}_t f\right)$ can be defined as*

$$
\partial_t^\alpha \left(\partial_t^\beta \mathcal{P}_t f\right) = \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m \left(\partial_{t+s}^\beta \mathcal{P}_{t+s} f\right) s^{m-\alpha-1} ds, \quad \forall f \in \bigcup_{1 \leqslant p \leqslant \infty} L^p_{\mathbb{B}}(\Omega), \tag{5.12}
$$

where m is the smallest integer which is bigger than α. Then

$$\partial_t^{\alpha}\left(\partial_t^{\beta}\mathcal{P}_t f\right) = \partial_t^{\alpha+\beta}\mathcal{P}_t f, \quad \forall f \in \bigcup_{1\leqslant p\leqslant\infty} L^p_{\mathbb{B}}(\Omega). \tag{5.13}$$

Proof. For any $f \in L^p_{\mathbb{B}}(\Omega)$ for some $1 \leqslant p \leqslant \infty$, by (5.5) and Theorem 5.3 we have the following computation for the latter of (5.12):

$$\begin{aligned}&\frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)}\int_0^{\infty}\partial_t^m\left(\partial_{t+s}^{\beta}\mathcal{P}_{t+s}f\right)s^{m-\alpha-1}ds\\ &\qquad= \frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^{\infty}\partial_t^m\left(\int_0^{\infty}\partial_{t+s}^{k}\mathcal{P}_{t+s+u}(f)u^{k-\beta-1}du\right)s^{m-\alpha-1}ds,\end{aligned} \tag{5.14}$$

where k is the smallest integer which is bigger than β. For any fixed $s \in (0,\infty)$, $t \in (t_0-\varepsilon, t_0+\varepsilon) \subset (0,\infty)$ for some $t_0 \in (0,\infty)$, and $\varepsilon > 0$, by (5.7) we have

$$\begin{aligned}&\left\|\partial_t^m\left(\partial_{t+s}^k\mathcal{P}_{t+s+u}(f)u^{k-\beta-1}\right)\right\|_{L^p_{\mathbb{B}}(\Omega)} = \left\|\partial_t^{m+k}\mathcal{P}_{t+s+u}(f)\right\|_{L^p_{\mathbb{B}}(\Omega)}u^{k-\beta-1}\\ &\qquad\leqslant \frac{C}{(t+s+u)^{m+k}}u^{k-\beta-1}\|f\|_{L^p_{\mathbb{B}}(\Omega)} \leqslant \frac{C}{(t_0-\varepsilon+s+u)^{m+k}}u^{k-\beta-1}\|f\|_{L^p_{\mathbb{B}}(\Omega)},\end{aligned} \tag{5.15}$$

for any $1 \leqslant p \leqslant \infty$. And

$$\begin{aligned}&\int_0^{\infty}\left|\frac{u^{k-\beta-1}}{(t_0-\varepsilon+s+u)^{m+k}}\right|du\,\|f\|_{L^p_{\mathbb{B}}(\Omega)}\\ &= \left(\int_0^{t_0-\varepsilon+s}\left|\frac{u^{k-\beta-1}}{(t_0-\varepsilon+s+u)^{m+k}}\right|du + \int_{t_0-\varepsilon+s}^{\infty}\left|\frac{u^{k-\beta-1}}{(t_0-\varepsilon+s+u)^{m+k}}\right|du\right)\|f\|_{L^p_{\mathbb{B}}(\Omega)}\\ &\leqslant C\frac{1}{(t_0-\varepsilon+s)^{\beta+m}}\|f\|_{L^p_{\mathbb{B}}(\Omega)} < \infty.\end{aligned} \tag{5.16}$$

Combining (5.15) and (5.16), we know that $\partial_t^m\left(\partial_{t+s}^k\mathcal{P}_{t+s+u}(f)u^{k-\beta-1}\right)$ is controlled by an integrable function. Hence we can interchange the order of the inner integration and the partial derivative ∂_t^m in (5.14) to obtain

$$\begin{aligned}
&\frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)}\int_0^\infty \partial_t^m\left(\partial_{t+s}^\beta \mathcal{P}_{t+s}f\right)s^{m-\alpha-1}ds\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^\infty\int_0^\infty \partial_t^m\partial_{t+s}^k\mathcal{P}_{t+s+u}(f)u^{k-\beta-1}du s^{m-\alpha-1}ds\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^\infty\int_0^\infty \partial_t^{m+k}\mathcal{P}_{t+s+u}(f)u^{k-\beta-1}du s^{m-\alpha-1}ds\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^\infty\int_s^\infty \partial_t^{m+k}\mathcal{P}_{t+w}(f)(w-s)^{k-\beta-1}dw s^{m-\alpha-1}ds \qquad (5.17)\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^\infty\int_0^w \partial_t^{m+k}\mathcal{P}_{t+w}(f)(w-s)^{k-\beta-1}s^{m-\alpha-1}ds dw\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}B(m-\alpha,k-\beta)}{\Gamma(m-\alpha)\Gamma(k-\beta)}\int_0^\infty \partial_t^{m+k}\mathcal{P}_{t+w}(f)w^{k+m-\alpha-\beta-1}dw\\
&=\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)}\int_0^\infty \partial_t^{m+k}\mathcal{P}_{t+w}(f)w^{k+m-\alpha-\beta-1}dw.
\end{aligned}$$

Since $m-1\leqslant\alpha<m$ and $k-1\leqslant\beta<k$, $m+k-2\leqslant\alpha+\beta<m+k$. If $m+k-1\leqslant\alpha+\beta<m+k$, we have

$$\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)}\int_0^\infty \partial_t^{m+k}\mathcal{P}_{t+w}(f)w^{k+m-\alpha-\beta-1}dw=\partial_t^{\alpha+\beta}\mathcal{P}_tf. \qquad (5.18)$$

If $m+k-2\leqslant\alpha+\beta<m+k-1$, then integrating by parts, we get

$$\begin{aligned}
&\frac{e^{-i\pi(m+k-\alpha-\beta)}}{\Gamma(m+k-\alpha-\beta)}\int_0^\infty \partial_t^{m+k}\mathcal{P}_{t+w}(f)w^{k+m-\alpha-\beta-1}dw\\
&\quad=\frac{e^{-i\pi(m+k-1-\alpha-\beta)}}{\Gamma(m+k-1-\alpha-\beta)}\int_0^\infty \partial_t^{m+k-1}\mathcal{P}_{t+w}(f)w^{k+m-\alpha-\beta-2}dw=\partial_t^{\alpha+\beta}\mathcal{P}_tf. \qquad (5.19)
\end{aligned}$$

So, combining (5.14) and (5.17)–(5.19), we get

$$\frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)}\int_0^\infty \partial_t^m\left(\partial_{t+s}^\beta\mathcal{P}_{t+s}f\right)s^{m-\beta-1}ds=\partial_t^{\alpha+\beta}\mathcal{P}_tf,$$

for any $f\in\bigcup_{1\leqslant p\leqslant\infty}L^p_{\mathbb{B}}(\Omega)$. □

Write the spectral decomposition of the semigroup $\{\mathcal{P}_t\}_{t\geqslant0}$: for any $f\in L^2(\Omega)$

$$\mathcal{P}_tf=\int_0^\infty e^{-\lambda t}dE_f(\lambda),$$

where $E(\lambda)$ is a resolution of the identity. Thus

$$\partial_t^k\mathcal{P}_tf=e^{-i\pi k}\int_{0^+}^\infty \lambda^k e^{-\lambda t}dE_f(\lambda),\quad k=1,2,\dots. \qquad (5.20)$$

We have the following proposition.

Proposition 5.6. *Let* $f \in L^2(\Omega)$ *and* $0 < \alpha < \infty$. *We have*

$$\partial_t^\alpha \mathcal{P}_t f = e^{-i\pi\alpha} \int_{0^+}^\infty \lambda^\alpha e^{-\lambda t} dE_f(\lambda). \tag{5.21}$$

Proof. Assume that $k-1 \leqslant \alpha < k$, $0 < k \in \mathbb{Z}$. By (5.5) and (5.20), we have

$$\begin{aligned} \partial_t^\alpha \mathcal{P}_t f &= \frac{1}{\Gamma(k-\alpha)} \int_0^\infty \partial_t^k \mathcal{P}_{t+s} f s^{k-\alpha-1} ds \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)} \int_0^\infty \int_{0^+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda) s^{k-\alpha-1} ds. \end{aligned} \tag{5.22}$$

Then $\int_0^\infty \int_{0^+}^\infty \lambda^k \; e^{-(t+s)\lambda} |dE(\lambda)| \, s^{k-\alpha-1} ds$ is absolutely convergent. Indeed, for any $t \in (0, \infty)$, we have

$$\begin{aligned} &\int_0^\infty \int_{0^+}^\infty \lambda^k e^{-(t+s)\lambda} |dE(\lambda)| \, s^{k-\alpha-1} ds = \int_{0^+}^\infty \int_0^\infty \lambda^k e^{-(t+s)\lambda} s^{k-\alpha-1} ds \, |dE(\lambda)| \\ &= \int_{0^+}^\infty \left(\int_0^t \lambda^k e^{-(t+s)\lambda} s^{k-\alpha-1} ds + \int_t^\infty (\lambda s)^k e^{-(t+s)\lambda} s^{-\alpha} \frac{ds}{s} \right) |dE(\lambda)| \\ &\leqslant C \int_{0^+}^\infty t^{-\alpha} |dE(\lambda)| \leqslant \frac{C}{t^\alpha} < \infty. \end{aligned}$$

By Theorem 5.3, we know that the integral in (5.5) is absolutely convergent in $L^2(\Omega)$. So by (5.22), we get

$$\begin{aligned} \langle \partial_t^\alpha \mathcal{P}_t f, \; g \rangle &= \left\langle \frac{(-1)^k}{\Gamma(k-\alpha)} \int_0^\infty \int_{0^+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda) s^{k-\alpha-1} ds, \; g \right\rangle \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)} \int_0^\infty \left\langle \int_{0^+}^\infty \lambda^k e^{-(t+s)\lambda} dE_f(\lambda), \; g \right\rangle s^{k-\alpha-1} ds \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)} \int_0^\infty \int_{0^+}^\infty \lambda^k \; e^{-(t+s)\lambda} dE_{\langle f,g \rangle}(\lambda) s^{k-\alpha-1} ds \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)} \int_{0^+}^\infty \int_0^\infty \lambda^k \; e^{-(t+s)\lambda} s^{k-\alpha-1} ds \; dE_{\langle f,g \rangle}(\lambda) \\ &= (-1)^k \int_{0^+}^\infty \lambda^\alpha \; e^{-t\lambda} dE_{\langle f,g \rangle}(\lambda) \\ &= \left\langle (-1)^k \int_{0^+}^\infty \lambda^\alpha \; e^{-t\lambda} dE_f(\lambda), \; g \right\rangle, \quad \forall g \in L^2(\Omega). \end{aligned}$$

Hence we get (5.21). □

5.3 Some technical results for Littlewood-Paley g-function

In this section, we will give some properties of the fractional Littlewood–Paley g-function.

Proposition 5.7. *Given a Banach space* $\mathbb{B}$, $1 < q < \infty$, *and* $0 < \beta < \gamma$, *there exists a constant* C *such that*

$$g_\beta^q(f) \leqslant C g_\gamma^q(f), \quad \forall f \in \bigcup_{1\leqslant p\leqslant\infty} L_{\mathbb{B}}^p(\Omega). \tag{5.23}$$

Proof. Assume that $f \in L_{\mathbb{B}}^p(\Omega)$ for some $1 \leqslant p \leqslant \infty$. By Theorem 5.4 and Hölder's inequality, we have

$$\begin{aligned}\left\|\partial_t^\beta \mathcal{P}_t f\right\|_{\mathbb{B}} &\leqslant \frac{1}{\Gamma(\gamma-\beta)}\int_t^\infty \left\|\partial_s^\gamma \mathcal{P}_s f\right\|_{\mathbb{B}}(s-t)^{\gamma-\beta-1}ds \\ &\leqslant \frac{1}{\Gamma(\gamma-\beta)}\left(\int_t^\infty \left\|\partial_s^\gamma \mathcal{P}_s f\right\|_{\mathbb{B}}^q (s-t)^{\gamma-\beta-1}s^{\gamma(q-1)}ds\right)^{\frac{1}{q}}\left(\int_t^\infty (s-t)^{\gamma-\beta-1}s^{-\gamma}ds\right)^{\frac{1}{q'}}.\end{aligned}$$

By changing variables, we have

$$\begin{aligned}\int_t^\infty (s-t)^{\gamma-\beta-1}s^{-\gamma}ds &= \int_t^\infty \left(1-\frac{t}{s}\right)^{\gamma-\beta-1}\left(\frac{t}{s}\right)^{\beta+1}t^{-\beta-1}ds \\ &= t^{-\beta}\int_0^1 (1-u)^{\gamma-\beta-1}u^{\beta-1}du = t^{-\beta}B(\gamma-\beta,\beta).\end{aligned}$$

So we have

$$\begin{aligned}\left\|\partial_t^\beta \mathcal{P}_t f\right\|_{\mathbb{B}} &\leqslant \frac{1}{\Gamma(\gamma-\beta)}\left(t^{-\beta}B(\gamma-\beta,\beta)\right)^{\frac{1}{q'}}\left(\int_t^\infty \left\|\partial_s^\gamma \mathcal{P}_s f\right\|_{\mathbb{B}}^q (s-t)^{\gamma-\beta-1}s^{\gamma(q-1)}ds\right)^{\frac{1}{q}} \\ &= \frac{\left(B(\gamma-\beta,\beta)\right)^{\frac{1}{q'}}}{\Gamma(\gamma-\beta)}\left(t^{-\beta}\right)^{\frac{1}{q'}}\left(\int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_{\mathbb{B}}^q (s-t)^{\gamma-\beta-1}s^{\gamma(q-1)}ds\right)^{\frac{1}{q}}.\end{aligned} \tag{5.24}$$

Using Fubini's theorem, by (5.24) we get

$$\begin{aligned}\int_0^\infty \left\|t^\beta \partial_t^\beta \mathcal{P}_t f\right\|_{\mathbb{B}}^q \frac{dt}{t} &\leqslant \frac{\left(B(\gamma-\beta,\beta)\right)^{\frac{q}{q'}}}{\Gamma(\gamma-\beta)^q}\int_0^\infty t^{\beta q}\left(t^{-\beta}\right)^{\frac{q}{q'}}\int_t^\infty \|\partial_s^\gamma \mathcal{P}_s f\|_{\mathbb{B}}^q (s-t)^{\gamma-\beta-1}s^{\gamma(q-1)}ds\frac{dt}{t} \\ &= \frac{\left(B(\gamma-\beta,\beta)\right)^{q-1}}{\Gamma(\gamma-\beta)^q}\int_0^\infty s^{\gamma(q-1)}\|\partial_s^\gamma \mathcal{P}_s f\|_{\mathbb{B}}^q \int_0^s t^{\beta-1}(s-t)^{\gamma-\beta-1}dtds \\ &= \left(\frac{\Gamma(\beta)}{\Gamma(\gamma)}\right)^q \int_0^\infty \left\|s^\gamma \partial_s^\gamma \mathcal{P}_s f\right\|_{\mathbb{B}}^q \frac{ds}{s}.\end{aligned}$$

Hence we get the inequality (5.23) with the constant $C = \dfrac{\Gamma(\beta)}{\Gamma(\gamma)}$. □

In the following, we shall need the theory of Calderón–Zygmund on $\mathbb{R}^n$. So we should recall briefly the definition of the Calderón–Zygmund operator. Given two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, let T be a linear operator. Then we call that T is a Calderón–Zygmund operator on $\mathbb{R}^n$, with associated Calderón–Zygmund kernel K if T maps $L_{c,\mathbb{B}_1}^\infty$, the space of the essentially bounded $\mathbb{B}_1$-valued functions on $\mathbb{R}^n$ with compact support, into the space of $\mathbb{B}_2$-valued and strongly measurable functions on $\mathbb{R}^n$, and for any function $f \in L^\infty_{c,\mathbb{B}_1}$ we have

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad \text{a.e. } x \in \mathbb{R}^n \text{ outside the support of } f,$$

where the kernel $K(x,y)$ is a regular kernel, that is, $K(x,y) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ satisfies $\|K(x,y)\| \leqslant C\frac{1}{|x-y|^n}$ and $\|\nabla_x K(x,y)\| + \|\nabla_y K(x,y)\| \leqslant C\frac{1}{|x-y|^{n+1}}$, for any $x, y \in \mathbb{R}^n$ and $x \neq y$, where as usual $\nabla_x = (\partial_{x_1}, \cdots, \partial_{x_n})$.

Let us recall the $\mathbb{B}$-valued BMO and H^1 spaces on $\mathbb{R}^n$. It is well known that

$$BMO_{\mathbb{B}}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathbb{B},loc}(\mathbb{R}^n) : \sup_{\text{cubes } Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \left\| f(x) - \frac{1}{|Q|}\int_Q f(y)dy \right\|_{\mathbb{B}} dx < \infty \right\}.$$

The $\mathbb{B}$-valued H^1 space is defined in the atomic sense. We say that a function $a \in L^\infty_{\mathbb{B}}(\mathbb{R}^n)$ is a $\mathbb{B}$-valued atom if there exists a cube $Q \subset \mathbb{R}^n$ containing the support of a, and such that $\|a\|_{L^\infty_{\mathbb{B}}(\mathbb{R}^n)} \leqslant |Q|^{-1}$ and $\int_Q a(x)dx = 0$. Then, we can define $H^1_{\mathbb{B}}(\mathbb{R}^n)$ as

$$H^1_{\mathbb{B}}(\mathbb{R}^n) = \left\{ f : f = \sum_i \lambda_i a_i, \ a_i \text{ are } \mathbb{B}\text{-valued atoms and } \sum_i |\lambda_i| < \infty \right\}.$$

We define $\|f\|_{H^1_{\mathbb{B}}(\mathbb{R}^n)} = \inf\left\{ \sum_i |\lambda_i| \right\}$, where the infimum runs over all those such decompositions.

Remark 5.8. [57, Theorem 4.1] Given a pair of Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, let T be a Calderón–Zygmund operator on $\mathbb{R}^n$ with regular vector-valued kernel. Then the following statements are equivalent:

(i) T maps $L^\infty_{c,\mathbb{B}_1}(\mathbb{R}^n)$ into $BMO_{\mathbb{B}_2}(\mathbb{R}^n)$.

(ii) T maps $H^1_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^1_{\mathbb{B}_2}(\mathbb{R}^n)$.

(iii) T maps $L^p_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^p_{\mathbb{B}_2}(\mathbb{R}^n)$ for any (or, equivalently, for some) $p \in (1, \infty)$.

(iv) T maps $BMO_{c,\mathbb{B}_1}(\mathbb{R}^n)$ into $BMO_{\mathbb{B}_2}(\mathbb{R}^n)$.

(v) T maps $L^1_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^{1,\infty}_{\mathbb{B}_2}(\mathbb{R}^n)$.

Proposition 5.9. *Given a Banach space $\mathbb{B}$, $1 < q < \infty$, and $0 < \alpha < \infty$, $g^q_\alpha(f)$ can be expressed as an $L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$-norm of a Calderón–Zygmund operator on $\mathbb{R}^n$ with regular vector-valued kernel.*

Proof. Assume that $m-1 \leqslant \alpha < m$ for some positive integer m. For any $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$, we have

$$g_\alpha^q(f)(x) = \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(x)\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})} = \left\| \int_{\mathbb{R}^n} K_t(x-y) f(y) dy \right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})},$$

with

$$K_t(x-y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}\Gamma(m-\alpha)} t^\alpha \int_0^\infty \partial_t^m \left(\frac{t+s}{((t+s)^2+|x-y|^2)^{\frac{n+1}{2}}} \right) s^{m-\alpha-1} ds, \quad x,y \in \mathbb{R}^n, x \neq y, t>0.$$

Then for every $x, y \in \mathbb{R}^n, x \neq y, t > 0$ and $f \in S(\mathbb{R}^n) \otimes \mathbb{B}$, the integral $\int_{\mathbb{R}^n} K_t(x-y) f(y) dy$ is absolutely convergent because

$$\begin{aligned} |K_t(x-y)| &\leqslant Ct^\alpha \int_0^\infty \frac{1}{((t+s)^2+|x-y|^2)^{\frac{n+m}{2}}} s^{m-\alpha-1} ds \\ &\leqslant Ct^\alpha \int_0^{t+|x-y|} \frac{1}{(t+s+|x-y|)^{n+m}} s^{m-\alpha-1} ds \\ &\quad + Ct^\alpha \int_{t+|x-y|}^\infty \frac{1}{(t+s+|x-y|)^{n+m}} s^{m-\alpha-1} ds \\ &\leqslant C \frac{t^\alpha}{(t+|x-y|)^{n+\alpha}}, \qquad x,y \in \mathbb{R}^n, x \neq y, t>0. \end{aligned} \tag{5.25}$$

By the comment about the Calderón–Zygmund operator above, we need only prove that $K_t(x,y) = K_t(x-y)$ is a regular kernel. For any $b \in \mathbb{B}$,

$$\|K_t(x-y)b\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})} = \left(\int_0^\infty \|K_t(x-y)b\|_{\mathbb{B}}^q \frac{dt}{t} \right)^{\frac{1}{q}} \leqslant \|b\|_{\mathbb{B}} \left(\int_0^\infty |K_t(x-y)|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

So, by (5.25), we have

$$\begin{aligned} \|K_t(x-y)\|_{\mathcal{L}\left(\mathbb{B},\ L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})\right)} &\leqslant \left(\int_0^\infty |K_t(x-y)|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leqslant C \left(\int_0^\infty \left(\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ = C \left(\int_0^{|x-y|} \left(\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \right)^q \frac{dt}{t} + \int_{|x-y|}^\infty \left(\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ \leqslant C \frac{1}{|x-y|^n}, \qquad x,y \in \mathbb{R}^n,\ x \neq y. \end{aligned} \tag{5.26}$$

Next, we observe that, for each $i = 1, \ldots, n$,

$$\begin{aligned} |\partial_{x_i}(K_t(x-y))| &= Ct^\alpha \left| \int_0^\infty \partial_{x_i} \left[\partial_t^m \left(\frac{t+s}{((t+s)^2+|x-y|^2)^{\frac{n+1}{2}}} \right) \right] s^{m-\alpha-1} ds \right| \\ &\leqslant C \left| t^\alpha \int_0^\infty \frac{1}{(t+s+|x-y|)^{n+m+1}} s^{m-\alpha-1} ds \right| \\ &\leqslant C \left| \frac{t^\alpha}{(t+|x-y|)^{n+\alpha+1}} \right|, \qquad x,y \in \mathbb{R},\ x \neq y,\ t>0. \end{aligned}$$

Then for any $b \in \mathbb{B}$

$$
\begin{aligned}
&\|\partial_{x_i}(K_t(x-y))\,b\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} = \left(\int_0^\infty \|\partial_{x_i}(K_t(x-y))\,b\|^q_{\mathbb{B}}\frac{dt}{t}\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_0^\infty |\partial_{x_i}K_t(x-y)|^q \frac{dt}{t}\right)^{\frac{1}{q}}\|b\|_{\mathbb{B}} \leqslant C\left(\int_0^\infty \left|\frac{t^\alpha}{(t+|x-y|)^{n+\alpha+1}}\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}\|b\|_{\mathbb{B}} \\
&\leqslant C\frac{1}{|x-y|^{n+1}}\|b\|_{\mathbb{B}}, \qquad x,y\in\mathbb{R}^n, x\neq y.
\end{aligned}
$$

As the argument in (5.26), we obtain

$$\|\nabla_x K_t(x-y)\|_{\mathcal{L}(\mathbb{B},L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t}))} \leqslant C\frac{1}{|x-y|^{n+1}}, \qquad x,y\in\mathbb{R}^n, x\neq y. \tag{5.27}$$

Also, we can prove that

$$\|\nabla_y K_t(x-y)\|_{\mathcal{L}(\mathbb{B},L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t}))} \leqslant C\frac{1}{|x-z|^{n+1}}, \qquad x,y\in\mathbb{R}^n, x\neq y \tag{5.28}$$

in the same way as the proof of (5.27). Combining (5.26)–(5.28), we know that $K_t(x,y)$ is a regular kernel. The proposition is established. The proof of this proposition can be found in [8] also. □

Proposition 5.10. *Let $\mathbb{B}$ be a Banach space which is of Lusin cotype q, $2\leqslant q<\infty$. Then for every symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t\geqslant 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t\geqslant 0}$ and for every (or, equivalently, for some) $p\in(1,\infty)$, there is a constant C such that*

$$\|g^q_k(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad k=1,2,\ldots, \quad \forall f\in L^p_{\mathbb{B}}(\Omega). \tag{5.29}$$

Moreover, for any $0<\alpha<\infty$, if

$$\|g^q_\alpha(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f\in L^p_{\mathbb{B}}(\Omega), \tag{5.30}$$

then we have

$$\|g^q_{k\alpha}(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad k=1,2,\ldots, \quad \forall f\in L^p_{\mathbb{B}}(\Omega). \tag{5.31}$$

Proof. For the case $k=1$, the inequality (5.29) have been proved in [57]. We only need prove the cases $k=2,3,\ldots$. We can prove it by induction. Assume that the inequality (5.29) is true for some $1\leqslant k\in\mathbb{Z}$. Let us prove that it is true for $k+1$ also. Since the inequality (5.29) is true for k, we know that the following operator

$$T: L^q_{\mathbb{B}}(\mathbb{R}^n) \longrightarrow L^q_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(\mathbb{R}^n),$$

$$Tf(x,t) = t^k\partial^k_t\mathcal{P}_t f(x), \qquad \forall f\in L^q_{\mathbb{B}}(\mathbb{R}^n)$$

is bounded. By Fubini's theorem we know that the operator

$$\tilde{T} : L^q_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\mathbb{R}^n) \longrightarrow L^q_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\mathbb{R}^n),$$

$$\tilde{T}F(x,s,t) = s\partial_s \mathcal{P}_s(F)(x,t), \qquad \forall F(x,t) \in L^q_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\mathbb{R}^n)$$

is also bounded. Since $\tilde{T}$ can be expressed as a Calderón–Zygmund operator with regular vector-valued kernel, by Remark 5.8 we get that $\tilde{T} : L^p_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\mathbb{R}^n) \longrightarrow L^p_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\mathbb{R}^n)$ is bounded for any $1 < p < \infty$. Hence, by Theorem 5.2 of [57], we know that $L^q_{\mathbb{B}}\left(\mathbb{R}_+, \frac{ds}{s}\right)$ is of Lusin cotype q.

Now given a symmetric diffusion semigroup $\{\mathcal{T}_t\}_{t\geqslant 0}$ with subordinated semigroup $\{\mathcal{P}_t\}_{t\geqslant 0}$. As $\mathbb{B}$ is of Lusin cotype q and $L^q_{\mathbb{B}}\left(\mathbb{R}_+, \frac{ds}{s}\right)$ also is of Lusin cotype q, we get that T is bounded from $L^p_{\mathbb{B}}(\Omega)$ to $L^p_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\Omega)$ and $\tilde{T}$ is bounded from $L^p_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\Omega)$ to $L^p_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\Omega)$, for any $1 < p < \infty$. So the operator $\tilde{T} \circ T$ is bounded from $L^p_{\mathbb{B}}(\Omega)$ to $L^p_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\Omega)$, for any $1 < p < \infty$, and by (5.13) we have

$$\begin{aligned}\tilde{T} \circ Tf(x,t,s) &= \tilde{T}(Tf(x,t))(s) = \tilde{T}(t^k\partial_t^k\mathcal{P}_t f(x))(s)\\ &= s\partial_s\mathcal{P}_s(t^k\partial_t^k\mathcal{P}_t f)(x) = st^k\partial_s\partial_t^k\mathcal{P}_s\mathcal{P}_t f(x)\\ &= st^k\partial_s\partial_t^k\mathcal{P}_{s+t}f(x) = st^k\partial_u^{k+1}\mathcal{P}_u f\big|_{u=t+s}(x).\end{aligned} \tag{5.32}$$

So there exists a constant C such that

$$\begin{aligned}\|f\|^p_{L^p_{\mathbb{B}}(\Omega)} &\geqslant C\left\|\tilde{T}\circ Tf\right\|^p_{L^p_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\Omega)}\\ &= C\left\|st^k\partial_u^{k+1}\mathcal{P}_u f|_{u=t+s}(x)\right\|^p_{L^p_{L^q_{\mathbb{B}}(\frac{ds}{s}\frac{dt}{t})}(\Omega)}\\ &= C\left\|\left(\int_0^\infty\int_0^\infty \left\|st^k\partial_u^{k+1}\mathcal{P}_u f\big|_{u=t+s}(x)\right\|^q_{\mathbb{B}}\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{q}}\right\|^p_{L^p(\Omega)}\\ &= C\left\|\left(\int_0^\infty\int_t^\infty t^{kq}(s-t)^q\left\|\partial_s^{k+1}\mathcal{P}_s f\right\|^q_{\mathbb{B}}\frac{ds}{s-t}\frac{dt}{t}\right)^{\frac{1}{q}}\right\|^p_{L^p(\Omega)}\\ &= C\left\|\left(\int_0^\infty\left\|\partial_s^{k+1}\mathcal{P}_s f\right\|^q_{\mathbb{B}}\int_0^s t^{kq-1}(s-t)^{q-1}dtds\right)^{\frac{1}{q}}\right\|^p_{L^p(\Omega)}\\ &= C(B(kq,q))^{\frac{p}{q}}\left\|\left(\int_0^\infty s^{(k+1)q}\left\|\partial_s^{k+1}\mathcal{P}_s f\right\|^q_{\mathbb{B}}\frac{ds}{s}\right)^{\frac{1}{q}}\right\|^p_{L^p(\Omega)}\\ &= C(B(kq,q))^{\frac{p}{q}}\left\|g^q_{k+1}(f)\right\|^p_{L^p(\Omega)}.\end{aligned} \tag{5.33}$$

Whence

$$\|g^q_{k+1}(f)\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

Then we get the inequality (5.29) for any $k \in \mathbb{Z}_+$.

We can prove inequality (5.31) under the assumption (5.30) with the similar argument as above. The only difference is that we should define T by

$$Tf(x,t) = t^{k\alpha}\partial_t^{k\alpha}\mathcal{P}_t f(x), \qquad \forall f \in L^q_{\mathbb{B}}(\mathbb{R}^n),$$

and define $\tilde{T}$ by

$$\tilde{T}F(x,s,t) = s^{\alpha}\partial_s^{\alpha}\mathcal{P}_s F(x,t), \qquad \forall F(x,t) \in L^q_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(\mathbb{R}^n).$$

And by Proposition 5.9 we know that in this case $\tilde{T}$ can be expressed as a Calderón–Zygmund operator also. □

The following theorem is proved in [57].

Theorem 5.11 (See [57, Theorem 3.2]). *Let $\mathbb{B}$ be a Banach space and $1 < p, q < \infty$. Let $h(x,t)$ be a function in $L^p_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(\Omega)$. Consider the operator Q defined by $Qh(x) = \int_0^\infty \partial_t\mathcal{P}_t h(x,t)dt$, $x \in \Omega$. Then for nice function h we have*

$$\left\|g_1^q(Qh)\right\|_{L^p(\Omega)} \leqslant C_{p,q}\|h\|_{L^p_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(\Omega)},$$

where the constant $C_{p,q}$ depends only on p and q.

5.4 Proofs of the main results

In this section, we will give the proofs of Theorem 5.1 and Theorem 5.2.

Proof of Theorem 5.1. (i) ⇒ (ii). Since $\mathbb{B}$ is of Lusin cotype q, by Proposition 5.10 we have

$$\left\|g_k^q(f)\right\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad k = 1, 2, \dots, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

Then, for any $\alpha > 0$, there exists $k \in \mathbb{N}$ such that $\alpha < k$. By Proposition 5.7, we have

$$\|g_\alpha^q(f)\|_{L^p(\Omega)} \leqslant C\left\|g_k^q(f)\right\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}, \quad \forall f \in L^p_{\mathbb{B}}(\Omega).$$

(ii) ⇒ (i). Since $\left\|g_\alpha^q(f)\right\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}$ for any $f \in L^p_{\mathbb{B}}(\Omega)$, by Proposition 5.10 there exists an integer k such that $k\alpha > 1$ and

$$\left\|g_{k\alpha}^q(f)\right\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}$$

for any $f \in L^p_{\mathbb{B}}(\Omega)$. By Proposition 5.7, we have

$$\left\|g_1^q(f)\right\|_{L^p(\Omega)} \leqslant C\left\|g_{k\alpha}^q(f)\right\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\Omega)}$$

for any $f \in L^p_{\mathbb{B}}(\Omega)$. Hence, by Theorem 2.1 in [57], $\mathbb{B}$ is of Lusin cotype q. □

Proof of Theorem 5.2. (i) $\Rightarrow$ (ii). It is easy to deduce from (5.21) that for any $f, g \in L^2(\Omega)$

$$\int_\Omega (f - E_0(f))(g - E_0(g))d\mu = \frac{4^\alpha}{\Gamma(2\alpha)} \int_\Omega \int_0^\infty (t^\alpha \partial_t^\alpha \mathcal{P}_t f)(t^\alpha \partial_t^\alpha \mathcal{P}_t g) \frac{dt}{t} d\mu. \tag{5.34}$$

Now we use duality. Fix two functions $f \in L^p_{\mathbb{B}}(\Omega)$ and $g \in L^{p'}_{\mathbb{B}^*}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Without loss of generality, we may assume that f and g are in the algebraic tensor products $\big(L^p(\Omega) \cap L^2(\Omega)\big) \otimes \mathbb{B}$ and $\big(L^{p'}(\Omega) \cap L^2(\Omega)\big) \otimes \mathbb{B}^*$, respectively. With $\langle\ ,\ \rangle$ denoting the duality between $\mathbb{B}$ and $\mathbb{B}^*$, we have

$$\int_\Omega \langle f, g\rangle \, d\mu = \int_\Omega \langle E_0(f), E_0(g)\rangle \, d\mu + \int_\Omega \langle f - E_0(f), g - E_0(g)\rangle \, d\mu. \tag{5.35}$$

The first term on the right is easy to be estimated:

$$\left|\int_\Omega \langle E_0(f), E_0(g)\rangle d\mu\right| \leqslant \|E_0(f)\|_{L^p_{\mathbb{B}}(\Omega)} \|E_0(g)\|_{L^{p'}_{\mathbb{B}^*}(\Omega)} \leqslant \|E_0(f)\|_{L^p_{\mathbb{B}}(\Omega)} \|g\|_{L^{p'}_{\mathbb{B}^*}(\Omega)}. \tag{5.36}$$

For the second one, by (5.34) and Hölder's inequality

$$\begin{aligned} \left|\int_\Omega \langle f - E_0(f), g - E_0(g)\rangle d\mu\right| &= \frac{4^\alpha}{\Gamma(2\alpha)} \left|\int_\Omega \int_0^\infty \langle t^\alpha \partial_t^\alpha \mathcal{P}_t f, t^\alpha \partial_t^\alpha \mathcal{P}_t g\rangle \frac{dt}{t} d\mu\right| \\ &\leqslant \frac{4^\alpha}{\Gamma(2\alpha)} \int_\Omega \int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f\|_{\mathbb{B}} \|t^\alpha \partial_t^\alpha \mathcal{P}_t g\|_{\mathbb{B}^*} \frac{dt}{t} d\mu \\ &\leqslant \frac{4^\alpha}{\Gamma(2\alpha)} \left\|g^q_\alpha(f)\right\|_{L^p(\Omega)} \left\|g^{q'}_\alpha(g)\right\|_{L^{p'}(\Omega)}. \end{aligned} \tag{5.37}$$

Now since $\mathbb{B}$ is of Lusin type q, $\mathbb{B}^*$ is of Lusin cotype q'. Thus by Theorem A,

$$\left\|g^{q'}_\alpha(g)\right\|_{L^{p'}(\Omega)} \leqslant C \|g\|_{L^{p'}_{\mathbb{B}^*}(\Omega)}. \tag{5.38}$$

Combining (5.35)–(5.38), we get

$$\left|\int_\Omega \langle f, g\rangle d\mu\right| \leqslant \Big(\|E_0(f)\|_{L^p_{\mathbb{B}}(\Omega)} + C \|g^q_\alpha(f)\|_{L^p(\Omega)} \Big) \|g\|_{L^{p'}_{\mathbb{B}^*}(\Omega)},$$

which gives (ii) by taking the supremum over all g as above such that $\|g\|_{L^{p'}_{\mathbb{B}^*}(\Omega)} \leqslant 1$.

(ii) $\Rightarrow$ (i). We only need consider the particular case on $\mathbb{R}^n$. In this case, $E_0(f) = 0$ for any $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$. Assuming $p = q$ and $k - 1 \leqslant \alpha < k$ for some $k \in \mathbb{Z}_+$, by Proposition 5.7 we have

$$\|f\|_{L^q_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C \|g^q_\alpha(f)\|_{L^q(\mathbb{R}^n)} \leqslant C \left\|g^q_k(f)\right\|_{L^q(\mathbb{R}^n)}, \tag{5.39}$$

for any $f \in L^q_{\mathbb{B}}(\mathbb{R}^n)$. By using (5.33) and (5.32), we have

$$\left(\int_0^\infty s^{kq} \left\|\partial_s^k \mathcal{P}_s f\right\|^q_{\mathbb{B}} \frac{ds}{s}\right)^{\frac{1}{q}} = C \left(\int_0^\infty \int_0^\infty s_1^q s_2^{(k-1)q} \left\|\partial_{s_2}^{k-1} \mathcal{P}_{s_2} (\partial_{s_1} \mathcal{P}_{s_1}) f\right\|^q_{\mathbb{B}} \frac{ds_2}{s_2} \frac{ds_1}{s_1}\right)^{\frac{1}{q}}.$$

By iterating the argument, we can get

$$\left(\int_0^\infty s^{kq}\left\|\partial_s^k\mathcal{P}_s f\right\|_{\mathbb{B}}^q \frac{ds}{s}\right)^{\frac{1}{q}} = C\left(\int_0^\infty\cdots\int_0^\infty s_1^q\cdots s_k^q\left\|\partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}f\right\|_{\mathbb{B}}^q\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}\right)^{\frac{1}{q}}.$$

Therefore we can choose a function $b(x, s_1, \ldots, s_k) \in L^q_{L^q_{\mathbb{B}}\left(\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}\right)}(\mathbb{R}^n)$ of unit norm such that

$$\begin{aligned}&\left\|g_k^{q'}(f)\right\|_{L^{q'}(\mathbb{R}^n)}\\ &= C\int_{\mathbb{R}^n}\int_0^\infty\cdots\int_0^\infty\langle s_1\cdots s_k\ \partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}f(x),\ b(x,s_1,\ldots,s_k)\rangle\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}dx.\end{aligned}$$

We may assume that f and b are nice enough to legitimate the calculations below. By Fubini's theorem, Hölder's inequality and (5.39), we have

$$\begin{aligned}&\left\|g_k^{q'}(f)\right\|_{L^{q'}(\mathbb{R}^n)}\\ &= C\int_{\mathbb{R}^n}\int_0^\infty\cdots\int_0^\infty\left\langle s_1\cdots s_k\ \partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}f(x),\ b(x,s_1,\ldots,s_k)\right\rangle\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}dx\\ &= C\int_{\mathbb{R}^n}\left\langle f(x),\int_0^\infty\cdots\int_0^\infty s_1\cdots s_k\ \partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}b(x,s_1,\ldots,s_k)\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}\right\rangle dx\\ &\leqslant C\|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)}\left\|\int_0^\infty\cdots\int_0^\infty s_1\cdots s_k\partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}b(x,s_1,\ldots,s_k)\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}\right\|_{L^q_{\mathbb{B}}(\mathbb{R}^n)}\\ &\leqslant C\|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)}\left\|g_k^q\left(\int_0^\infty\cdots\int_0^\infty s_1\cdots s_k\partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}b(x,s_1,\ldots,s_k)\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}\right)\right\|_{L^q(\mathbb{R}^n)}\\ &=: C\|f\|_{L^{q'}_{\mathbb{B}^*}(\mathbb{R}^n)}\left\|g_k^q\left(G_k(b)\right)\right\|_{L^q(\mathbb{R}^n)},\end{aligned} \tag{5.40}$$

where

$$G_k(b) = \int_0^\infty\cdots\int_0^\infty s_1\cdots s_k\partial_{s_1}\mathcal{P}_{s_1}\cdots\partial_{s_k}\mathcal{P}_{s_k}b(x,s_1,\ldots,s_k)\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k},\quad k\in\mathbb{Z}_+.$$

Using (5.33), Fubini's theorem and Theorem 5.11 repeatedly, we have

$$
\begin{aligned}
&\left\|g_k^q\left(G_k(b)\right)\right\|_{L^q(\mathbb{R}^n)}^q \qquad (5.41)\\
&\leqslant C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \left\| t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_k \partial_{t_k} \mathcal{P}_{t_k}\left(G_k(b)\right)\right\|_{\mathbb{B}}^q \frac{dt_1}{t_1} \cdots \frac{dt_k}{t_k} dx \\
&= C \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \left\| t_k \partial_{t_k} \mathcal{P}_{t_k}\left[\int_0^\infty s_k \partial_{s_k} \mathcal{P}_{s_k}\left(t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_{k-1} \partial_{t_{k-1}} \mathcal{P}_{t_{k-1}} G_{k-1}(b)\right) \frac{ds_k}{s_k}\right]\right\|_{\mathbb{B}}^q \\
&\quad \frac{dt_k}{t_k} dx \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \\
&\leqslant C \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \left\| t_1 \cdots t_{k-1} \partial_{t_1} \mathcal{P}_{t_1} \cdots \partial_{t_{k-1}} \mathcal{P}_{t_{k-1}} G_{k-1}(b)\right\|_{\mathbb{B}}^q \frac{ds_k}{s_k} dx \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \\
&\vdots \\
&\leqslant C \int_{\mathbb{R}^n} \int_0^\infty \cdots \int_0^\infty \left\| b(x, s_1, \ldots, s_k)\right\|_{\mathbb{B}}^q \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k} dx = C.
\end{aligned}
$$

Combining (5.40) and (5.41), we get

$$
\left\|g_k^{q'}(f)\right\|_{L^{q'}(\mathbb{R}^n)} \leqslant C\|f\|_{L_{\mathbb{B}^*}^{q'}(\mathbb{R}^n)}.
$$

By Theorem A, $\mathbb{B}^*$ is of Lusin cotype q'. Hence $\mathbb{B}$ is of Lusin type q.

If $p \neq q$, it suffices to prove that the operator $b \to g_k^q(G_k(b))$ maps $L^p_{L^q_{\mathbb{B}}\left(\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}\right)}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. To that end we shall use the theory of vector-valued Calderón–Zygmund operators. We borrow this idea from [63]. Let us consider the operator

$$
\begin{aligned}
&T(b)(x, t_1, \ldots, t_k) \\
&\quad = t_1 \partial_{t_1} \mathcal{P}_{t_1} \cdots t_k \partial_{t_k} \mathcal{P}_{t_k} \int_0^\infty \cdots \int_0^\infty s_1 \partial_{s_1} \mathcal{P}_{s_1} \cdots s_k \partial_{s_k} \mathcal{P}_{s_k} b(x, s_1, \ldots, s_k) \frac{ds_1}{s_1} \cdots \frac{ds_k}{s_k}.
\end{aligned}
$$

Clearly,

$$
\|T(b)(x, t_1, \ldots, t_k)\|_{L^p_{L^q_{\mathbb{B}}\left(\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}\right)}(\mathbb{R}^n)} = \left\|g_k^q\left(G_k(b)\right)\right\|_{L^p(\mathbb{R}^n)}.
$$

Therefore it is enough to prove

$$
T : L^p_{L^q_{\mathbb{B}}\left(\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}\right)}(\mathbb{R}^n) \longrightarrow L^p_{L^q_{\mathbb{B}}\left(\frac{ds_1}{s_1}\cdots\frac{ds_k}{s_k}\right)}(\mathbb{R}^n).
$$

Hence as we already know that T is bounded in the case $p = q$, in order to get the case $p \neq q$ it suffices to show that the kernel of T satisfies the standard estimates, see Remark 5.8. For simply and essentially, we only need consider the case when $k = 2$. So

$$
\begin{aligned}
T(b)(x, t_1, t_2) &= t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} \int_0^\infty \int_0^\infty s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(b)(x, s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} \\
&= \int_{\mathbb{R}^n} \int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(x - y) b(y, s_1, s_2) \frac{ds_1}{s_1} \frac{ds_2}{s_2} dy.
\end{aligned}
$$

Then the operator-valued kernel $K(x)$ is $\int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(x) \frac{ds_1}{s_1}\frac{ds_2}{s_2}$.
For any $b(s_1, s_2) \in L^q_{\mathbb{B}}\left(\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)$ with unit norm, we have

$$\begin{aligned}
\|K(x)b\|_{\mathbb{B}} &= \left\| \int_0^\infty \int_0^\infty t_1 t_2 \partial_{t_1} \mathcal{P}_{t_1} \partial_{t_2} \mathcal{P}_{t_2} s_1 s_2 \partial_{s_1} \mathcal{P}_{s_1} \partial_{s_2} \mathcal{P}_{s_2}(x) b(s_1, s_2) \frac{ds_1}{s_1}\frac{ds_2}{s_2} \right\|_{\mathbb{B}} \\
&= \left\| \int_0^\infty \int_0^\infty t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} b(s_1, s_2) \frac{ds_1}{s_1}\frac{ds_2}{s_2} \right\|_{\mathbb{B}} \\
&\leqslant \int_0^\infty \int_0^\infty t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} \|b(s_1, s_2)\|_{\mathbb{B}} \frac{ds_1}{s_1}\frac{ds_2}{s_2} \\
&\leqslant \|b\|_{L^q_{\mathbb{B}}\left(\frac{ds_1}{s_1}\frac{ds_2}{s_2}\right)} \left\{ \int_0^\infty \int_0^\infty \left(t_1 t_2 s_1 s_2 \partial_u^4 \mathcal{P}_u(x) \Big|_{u=t_1+t_2+s_1+s_2} \right)^{q'} \frac{ds_1}{s_1}\frac{ds_2}{s_2} \right\}^{\frac{1}{q'}} \\
&\leqslant C \left\{ \int_0^\infty \int_0^\infty \left(\frac{t_1 t_2 s_1 s_2}{(t_1 + t_2 + s_1 + s_2 + |x|)^{n+4}} \right)^{q'} \frac{ds_1}{s_1}\frac{ds_2}{s_2} \right\}^{\frac{1}{q'}} \\
&\leqslant C \frac{t_1 t_2}{(t_1 + t_2 + |x|)^{n+2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|K(x)b\|_{L^q_{\mathbb{B}}\left(\frac{dt_1}{t_1}\frac{dt_2}{t_2}\right)} &= \left(\int_0^\infty \int_0^\infty \|K(x)b\|^q_{\mathbb{B}} \frac{dt_1}{t_1}\frac{dt_2}{t_2} \right)^{\frac{1}{q}} \\
&\leqslant C \left(\int_0^\infty \int_0^\infty \left(\frac{t_1 t_2}{(t_1 + t_2 + |x|)^{n+2}} \right)^q \frac{dt_1}{t_1}\frac{dt_2}{t_2} \right)^{\frac{1}{q}} \leqslant \frac{C}{|x|^n}.
\end{aligned}$$

Similarly, we can show that

$$\|\nabla K(x)\| \leqslant \frac{C}{|x|^{n+1}}.$$

Therefore, K is a regular vector-valued kernel and the proof is finished. □

5.5 Some results with Poisson semigroup on $\mathbb{R}^n$

In this section, we devote to study the fractional area function and the fractional g^*_λ-function on $\mathbb{R}^n$ in the vector-valued case. Our main goal is to prove the analogous results with Theorem A and Theorem B related to these two functions on $\mathbb{R}^n$.

Let $\mathbb{B}$ be a Banach space, $0 < \alpha < \infty$, $\lambda > 1$, and $1 < q < \infty$. We define the fractional area function on $\mathbb{R}^n$ as

$$S^q_\alpha(f)(x) = \left(\iint_{\Gamma(x)} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{q}}, \quad \forall f \in \bigcup_{1 \leqslant p \leqslant \infty} L^p_{\mathbb{B}}(\mathbb{R}^n),$$

where $\Gamma(x) = \left\{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\right\}$, and define the fractional g_λ^*-function on $\mathbb{R}^n$ as

$$g_{\lambda,\alpha}^{q,*}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}}, \quad \forall f \in \bigcup_{1\leqslant p\leqslant\infty} L_{\mathbb{B}}^p(\mathbb{R}^n).$$

In [9, 10, 21], the authors considered some area square functions in some general setting.

The following proposition demonstrate that the vector-valued fractional area function S_α^q can be treated as an $L_{\mathbb{B}}^q(\Gamma(0), \frac{dydt}{t^{n+1}})$-norm of a Calderón–Zygmund operator.

Proposition 5.12. *Given a Banach space $\mathbb{B}$, $1<q<\infty$ and $0<\alpha<\infty$, then $S_\alpha^q(f)$ can be expressed as an $L_{\mathbb{B}}^q(\Gamma(0), \frac{dydt}{t^{n+1}})$-norm of a Calderón–Zygmund operator on $\mathbb{R}^n$ with regular vector-valued kernel.*

Proof. Assume that $m-1 \leqslant \alpha < m$ for some positive integer m. For any $f \in S(\mathbb{R}^n)\otimes\mathbb{B}$, by changing of variables, we have

$$S_\alpha^q(f)(x) = \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(x+y)\|_{L_{\mathbb{B}}^q\left(\Gamma(0), \frac{dydt}{t^{n+1}}\right)} = \left\|\int_{\mathbb{R}^n} K_{y,t}(x,z) f(z)dz\right\|_{L_{\mathbb{B}}^q\left(\Gamma(0), \frac{dydt}{t^{n+1}}\right)},$$

where

$$K_{y,t}(x,z) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}\Gamma(m-\alpha)} t^\alpha \int_0^\infty \partial_t^m \left(\frac{t+s}{((t+s)^2+|x+y-z|^2)^{\frac{n+1}{2}}}\right) s^{m-\alpha-1} ds,$$

for any $x,y,z \in \mathbb{R}^n, t>0$. Then for every $x,y,z\in\mathbb{R}^n, t>0$ and $f\in S(\mathbb{R}^n)\otimes\mathbb{B}$, the integral $\int_{\mathbb{R}^n} K_{y,t}(x,z)f(z)dz$ is absolutely convergent because

$$\begin{aligned} |K_{y,t}(x,z)| &\leqslant Ct^\alpha \int_0^\infty \frac{1}{((t+s)^2+|x+y-z|^2)^{\frac{n+m}{2}}} s^{m-\alpha-1} ds \\ &\leqslant Ct^\alpha \int_0^\infty \frac{1}{(t+s+|x+y-z|)^{n+m}} s^{m-\alpha-1} ds \qquad (5.42) \\ &\leqslant C\frac{1}{(t+|x+y-z|)^n}, \qquad x,y,z\in\mathbb{R}^n, t>0. \end{aligned}$$

And for any $b\in\mathbb{B}$

$$\begin{aligned} \|K_{y,t}(x,z)b\|_{L_{\mathbb{B}}^q(\Gamma(0),\frac{dydt}{t^{n+1}})} &= \left(\iint_{\Gamma(0)} \|K_{y,t}(x,z)b\|_{\mathbb{B}}^q \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant \|b\|_{\mathbb{B}} \left(\iint_{\Gamma(0)} |K_{y,t}(x,z)|^q \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}}, \quad x,z\in\mathbb{R}^n. \quad (5.43) \end{aligned}$$

So, by (5.42) and (5.43), we have

$$\begin{aligned} \|K_{y,t}(x,z)\|_{\mathcal{L}\left(\mathbb{B}, L_{\mathbb{B}}^q(\Gamma(0),\frac{dydt}{t^{n+1}})\right)} &\leqslant \left(\iint_{\Gamma(0)} |K_{y,t}(x,z)|^q \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant C\left(\iint_{\Gamma(0)} \left|\frac{1}{(t+|x+y-z|)^n}\right|^q \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}}, \quad x,z\in\mathbb{R}^n. \end{aligned}$$

We split the integral in two parts. If $x, y, z \in \mathbb{R}^n$ and $2|y| < |x-z|$, then$|x+y-z| > \frac{|x-z|}{2}$. Hence

$$\left(\int_0^\infty \int_{\left\{|y|<t, |y|<\frac{|x-z|}{2}\right\}} \left|\frac{1}{(t+|x+y-z|)^n}\right|^q \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}}$$

$$\leqslant C \left(\int_0^\infty \frac{1}{\left(t+\frac{|x-z|}{2}\right)^{nq}} \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant C \frac{1}{|x-z|^n}, \quad x, z \in \mathbb{R}^n, x \neq z.$$

On the other hand, if $2|y| > |x-z|$, then $|x-z| < 2t$. So

$$\left(\int_0^\infty \int_{\left\{|y|<t, |y|>\frac{|x-z|}{2}\right\}} \frac{1}{(t+|x+y-z|)^{nq}} \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}}$$

$$\leqslant C \left(\int_0^\infty \frac{1}{(t+|x-z|)^{nq}} \frac{dt}{t}\right)^{\frac{1}{q}} \leqslant C \frac{1}{|x-z|^n}, \quad x, z \in \mathbb{R}^n, x \neq z.$$

Then we get

$$\|K_{y,t}(x,z)\|_{\mathcal{L}\left(\mathbb{B}, L^q_{\mathbb{B}}\left(\Gamma(0), \frac{dy\,dt}{t^{n+1}}\right)\right)} \leqslant C \frac{1}{|x-z|^n}, \quad x, z \in \mathbb{R}^n, x \neq z. \tag{5.44}$$

Next, we observe that, for each $i = 1, \dots, n$,

$$\begin{aligned} |\partial_{x_i} K_{y,t}(x,z)| &= C \left| t^\alpha \int_0^\infty \partial_{x_i} \left[\partial_t^m \left(\frac{t+s}{\left((t+s)^2 + |x+y-z|^2\right)^{\frac{n+1}{2}}}\right)\right] s^{m-\alpha-1} ds \right| \\ &\leqslant C \left| t^\alpha \int_0^\infty \frac{1}{(t+s+|x+y-z|)^{n+m+1}} s^{m-\alpha-1} ds \right| \qquad (5.45) \\ &\leqslant C \frac{t^\alpha}{(t+|x+y-z|)^{n+\alpha+1}}, \quad x, y, z \in \mathbb{R}^n, t > 0. \end{aligned}$$

Then for any $b \in \mathbb{B}$

$$\begin{aligned} \|\partial_{x_i} K_{y,t}(x,z) b\|_{L^q_{\mathbb{B}}\left(\Gamma(0), \frac{dy\,dt}{t^{n+1}}\right)} &= \left(\iint_{\Gamma(0)} \|\partial_{x_i} K_{y,t}(x,z) b\|^q_{\mathbb{B}} \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant \|b\|_{\mathbb{B}} \left(\iint_{\Gamma(0)} |\partial_{x_i} K_{y,t}(x,z)|^q \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}}, \quad x, z \in \mathbb{R}^n. \qquad (5.46) \end{aligned}$$

So, by (5.45) and (5.46), we have

$$\begin{aligned} \|\partial_{x_i} K_{y,t}(x,z)\|_{\mathcal{L}\left(\mathbb{B}, L^q_{\mathbb{B}}\left(\Gamma(0), \frac{dy\,dt}{t^{n+1}}\right)\right)} &\leqslant \left(\iint_{\Gamma(0)} |\partial_{x_i} K_{y,t}(x,z)|^q \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant C \left(\iint_{\Gamma(0)} \left|\frac{t^\alpha}{(t+|x+y-z|)^{n+\alpha+1}}\right|^q \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &= C \left(\int_0^\infty \int_{\{|y|<t\}} \frac{t^{\alpha q-n-1}}{(t+|x+y-z|)^{(n+\alpha+1)q}} dy\,dt\right)^{\frac{1}{q}} \quad x, z \in \mathbb{R}^n. \end{aligned}$$

Then, proceeding as above, we have

$$\|\nabla_x K_{y,t}(x,z)\|_{\mathcal{L}\left(\mathbb{B}, L^q_{\mathbb{B}}(\Gamma(0), \frac{dydt}{t^{n+1}})\right)} \leqslant C\frac{1}{|x-z|^{n+1}}, \tag{5.47}$$

for $x, z \in \mathbb{R}^n, x \neq z$. Also, we can prove that

$$\|\nabla_z K_{y,t}(x,z)\|_{\mathcal{L}\left(\mathbb{B}, L^q_{\mathbb{B}}(\Gamma(0), \frac{dydt}{t^{n+1}})\right)} \leqslant C\frac{1}{|x-z|^{n+1}}, \tag{5.48}$$

for $x, z \in \mathbb{R}^n, x \neq z$, in the same way as the proof of (5.47). Combining (5.44) and (5.47)–(5.48), we know that $K_{y,t}(x,z)$ is a regular kernel. So we get the proof. □

Together with Proposition 5.9, Proposition 5.12 and Remark 5.8, we can immediately get the following theorem for g^q_α and S^q_α with $1 < q < \infty$ and $0 < \alpha < \infty$.

Theorem 5.13. *Given a Banach space $\mathbb{B}$, $1 < q < \infty$ and $0 < \alpha < \infty$, let U be either the fractional Littlewood–Paley g-function g^q_α or the fractional area function S^q_α, then the following statements are equivalent:*

(i) U *maps* $L^\infty_{c,\mathbb{B}}(\mathbb{R}^n)$ *into* $BMO(\mathbb{R}^n)$.

(ii) U *maps* $H^1_{\mathbb{B}}(\mathbb{R}^n)$ *into* $L^1(\mathbb{R}^n)$.

(iii) U *maps* $L^p_{\mathbb{B}}(\mathbb{R}^n)$ *into* $L^p(\mathbb{R}^n)$ *for any (or, equivalently, for some)* $p \in (1,\infty)$.

(iv) U *maps* $BMO_{c,\mathbb{B}}(\mathbb{R}^n)$ *into* $BMO(\mathbb{R}^n)$.

(v) U *maps* $L^1_{\mathbb{B}}(\mathbb{R}^n)$ *into* $L^{1,\infty}(\mathbb{R}^n)$.

Theorem 5.14. *Given a Banach space $\mathbb{B}$ and $2 \leqslant q < \infty$, the following statements are equivalent:*

(i) $\mathbb{B}$ *is of Lusin cotype q.*

(ii) *For every (or, equivalently, for some) positive integer n, for every (or, equivalently, for some) $p \in (1,\infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that*

$$\|S^q_\alpha(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

Proof. (i) ⇒ (ii). By Fubini's theorem, we have

$$\begin{aligned}\|S^q_\alpha(f)\|^q_{L^q(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_0^\infty \|t^\alpha \partial^\alpha_t \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \left(\int_{\mathbb{R}^n} \chi_{|x-y|<t}dx\right)\frac{dydt}{t^{n+1}} \\ &= \int_{\mathbb{R}^n}\int_0^\infty \|t^\alpha \partial^\alpha_t \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \frac{dydt}{t} = \|g^q_\alpha(f)\|^q_{L^q(\mathbb{R}^n)}. \end{aligned} \tag{5.49}$$

Since $\mathbb{B}$ is of Lusin cotype q, by (5.49) and Theorem A we get

$$\|S_\alpha^q(f)\|_{L^q(\mathbb{R}^n)} = \|g_\alpha^q(f)\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^q_{\mathbb{B}}(\mathbb{R}^n)}, \quad \forall f \in L^q_{\mathbb{B}}(\mathbb{R}^n).$$

Hence, by Theorem 5.13

$$\|S_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n), 1 < p < \infty.$$

(ii) $\Rightarrow$ (i). We only need prove that there exists a constant C such that

$$g_\alpha^q(f)(x) \leqslant CS_\alpha^q(f)(x), \quad \forall x \in \mathbb{R}^n, \tag{5.50}$$

for a big enough class of nice functions in $L^p_{\mathbb{B}}(\mathbb{R}^n)$. Then we have $\|g_\alpha^q(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}$. By Theorem A, $\mathbb{B}$ is of Lusin cotype q.

Now, let us prove (5.50). We shall follow those ideas in [79]. It suffices to prove it for $x = 0$. Let us denote by $B(0,t)$ the ball in $\mathbb{R}^{n+1}$ centered at $(0,t)$ and tangent to the boundary of the cone $\Gamma(0)$. Then the radius of $B(0,t)$ is $\frac{\sqrt{2}}{2}t$. Now the partial derivative $\partial_t^\alpha \mathcal{P}_t f(x)$ is, like $\mathcal{P}_t f(x)$, harmonic function. Thus by the mean-value theorem, we have

$$\partial_t^\alpha \mathcal{P}_t f(0) = \frac{1}{|B(0,t)|}\iint_{B(0,t)} \partial_s^\alpha \mathcal{P}_s f(x) dx ds.$$

By Hölder's inequality,

$$\begin{aligned}
\|\partial_t^\alpha \mathcal{P}_t f(0)\|_{\mathbb{B}} &\leqslant \frac{1}{|B(0,t)|}\iint_{B(0,t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}} dx ds \\
&\leqslant \frac{1}{|B(0,t)|^{\frac{1}{q}}}\left(\iint_{B(0,t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds\right)^{\frac{1}{q}}.
\end{aligned}$$

Integrating this inequality, we obtain

$$\begin{aligned}
\int_0^\infty t^{\alpha q}\|\partial_t^\alpha \mathcal{P}_t f(0)\|_{\mathbb{B}}^q \frac{dt}{t} &\leqslant C\int_0^\infty t^{\alpha q-n-2}\iint_{B(0,t)} \|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds dt \\
&\leqslant C\iint_{\Gamma(0)}\left(\int_{c_1 s}^{c_2 s} t^{\alpha q-n-2} dt\right)\|\partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q dx ds \leqslant C\iint_{\Gamma(0)} \|s^\alpha \partial_s^\alpha \mathcal{P}_s f(x)\|_{\mathbb{B}}^q \frac{dx ds}{s^{n+1}}
\end{aligned}$$

by using Fubini's theorem and $(x,s) \in B(0,t)$ implying $c_1 s \leqslant t \leqslant c_2 s$, for two positive constants c_1 and c_2. Hence, we get inequality (5.50). □

Theorem 5.15. *Given a Banach space $\mathbb{B}$ and $1 < q \leqslant 2$, the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin type q.*

(ii) *For every (or, equivalently, for some) positive integer* n, *for every (or, equivalently, for some)* $p \in (1, \infty)$, *and for every (or, equivalently, for some)* $\alpha > 0$, *there is a constant* $C > 0$ *such that*

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C \, \|S^q_\alpha(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

Proof. (i) ⇒ (ii). Since $\mathbb{B}$ is of Lusin type q, by Theorem B and (5.50) we have

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C \, \|g^q_\alpha(f)\|_{L^p(\mathbb{R}^n)} \leqslant C \, \|S^q_\alpha(f)\|_{L^p(\mathbb{R}^n)}.$$

(ii) ⇒ (i). We shall prove $\|S^{q'}_\alpha(g)\|_{L^{p'}(\mathbb{R}^n)} \leqslant C\|g\|_{L^{p'}_{\mathbb{B}^*}(\mathbb{R}^n)}$. We can choose $b \in L^p_{L^q_{\mathbb{B}}\left(\Gamma(0), \frac{dzdt}{t^{n+1}}\right)}(\mathbb{R}^n)$ of unit norm such that

$$\begin{aligned}
\|S^{q'}_\alpha(g)\|_{L^{p'}(\mathbb{R}^n)} &= \|t^\alpha \partial^\alpha_t \mathcal{P}_t g(y-z)\|_{L^{p'}_{L^{q'}_{\mathbb{B}}\left(\Gamma(0), \frac{dzdt}{t^{n+1}}\right)}(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \int_{\Gamma(0)} \langle t^\alpha \partial^\alpha_t \mathcal{P}_t g(y-z), b(y,z,t) \rangle \frac{dzdt}{t^{n+1}} dy \\
&= \int_{\mathbb{R}^n} \int_{\Gamma(0)} \left\langle \int_{\mathbb{R}^n} t^\alpha \partial^\alpha_t \mathcal{P}_t (y-z-\tilde{z}) \, g(\tilde{z}) \, d\tilde{z}, b(y,z,t) \right\rangle \frac{dzdt}{t^{n+1}} dy \\
&= \int_{\mathbb{R}^n} \left\langle g(\tilde{z}), \int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial^\alpha_t \mathcal{P}_t (y-z-\tilde{z}) b(y,z,t) dy \frac{dzdt}{t^{n+1}} \right\rangle d\tilde{z} \\
&\leqslant \|g\|_{L^{p'}_{\mathbb{B}^*}(\mathbb{R}^n)} \|G(b)\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant \|g\|_{L^{p'}_{\mathbb{B}^*}(\mathbb{R}^n)} \|S^q_\alpha(G(b))\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)},
\end{aligned}$$

where $G(b)(\tilde{z}) = \displaystyle\int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial^\alpha_t \mathcal{P}_t (y-z-\tilde{z}) b(y,z,t) dy \frac{dzdt}{t^{n+1}}$ and in the last inequality we used the hypothesis. Let us observe that we will have proved the result as soon as we prove $\|S^q_\alpha(G(b))\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C\|b\|_{L^p_{L^q_{\mathbb{B}}\left(\Gamma(0), \frac{dzdt}{t^{n+1}}\right)}(\mathbb{R}^n)}$. We shall prove this by following a parallel argument to the proof of (ii) $\Longrightarrow$ (i) in Theorem B and we also borrow the ideal from [63]. Observe that

$$\begin{aligned}
&S^q_\alpha(G(b))(x) \\
&= \left(\int_{\Gamma(0)} \left\| s^\alpha \partial^\alpha_s \mathcal{P}_s \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial^\alpha_t \mathcal{P}_t (y-z-\cdot) b(y,z,t) dy \frac{dzdt}{t^{n+1}} \right)(x+u) \right\|^q_{\mathbb{B}} \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| s^\alpha \partial^\alpha_s \mathcal{P}_s \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} t^\alpha \partial^\alpha_t \mathcal{P}_t (-y+z+\cdot) b(y,z,t) dy \frac{dzdt}{t^{n+1}} \right)(x+u) \right\|^q_{\mathbb{B}} \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} s^\alpha \partial^\alpha_s \mathcal{P}_s t^\alpha \partial^\alpha_t \mathcal{P}_t (-y+z+x+u) b(y,z,t) dy \frac{dzdt}{t^{n+1}} \right) \right\|^q_{\mathbb{B}} \frac{duds}{s^{n+1}} \right)^{1/q} \\
&= \left(\int_{\Gamma(0)} \left\| \left(\int_{\Gamma(0)} \int_{\mathbb{R}^n} s^\alpha t^\alpha \partial^{2\alpha}_u \mathcal{P}_u \big|_{u=s+t} (-y+z+x+u) b(y,z,t) dy \frac{dzdt}{t^{n+1}} \right) \right\|^q_{\mathbb{B}} \frac{duds}{s^{n+1}} \right)^{1/q}.
\end{aligned}$$

It is an easy exercise to prove that

$$|s^\alpha t^\alpha \partial^{2\alpha}_u \mathcal{P}_u|_{u=s+t}| \leqslant C \frac{s^\alpha t^\alpha}{(s+t+|x|)^{n+2\alpha}}.$$

In this circumstances, it can be proved that the operator

$$b \longrightarrow \mathcal{U}(b)(x,u,s) = \int_{\mathbb{R}^n}\int_{\Gamma(0)} s^\alpha t^\alpha \partial_u^{2\alpha}\mathcal{P}_u|_{u=s+t}(-y+z+x+u)b(y,z,t)\frac{dzdt}{t^{n+1}}dy$$

can be handled by using Calderón–Zygmund techniques and $\mathcal{U}$ is bounded on $L^p_{L^q_{\mathbb{B}}(\Gamma(0),\frac{duds}{s^{n+1}})}(\mathbb{R}^n)$ for every $1 < p, q < \infty$ and every Banach space $\mathbb{B}$, see the details in [63, Section 2]. The proof of the theorem ends by observing that $S^q_\alpha(G(b)) = \|\mathcal{U}(b)\|_{L^q_{\mathbb{B}}(\Gamma(0),\frac{duds}{s^{n+1}})}$. □

Now, let us consider the relationship between the geometry properties of the Banach space $\mathbb{B}$ and the fractional g^*_λ-function $g^{q,*}_{\lambda,\alpha}$.

Theorem 5.16. *Given a Banach space $\mathbb{B}$, $2 \leqslant q < \infty$ and $\lambda > 1$, the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin cotype q.*

(ii) *For every (or, equivalently, for some) positive integer n, for every (or, equivalently, for some) $p \in [q,\infty)$, and for every (or, equivalently, for some) $\alpha > 0$, there is a constant $C > 0$ such that*

$$\left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

Proof. (i) $\Rightarrow$ (ii). Since $\lambda > 1$, the function $(1+|x|)^{-\lambda n}$ is integrable and hence for good enough function $h(x) \geqslant 0$, we have

$$\sup_{t>0}\int_{\mathbb{R}^n}\frac{1}{t^n}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}h(y)dy \leqslant CMh(x), \tag{5.51}$$

where Mh is the Hardy–Littlewood maximal function of h. By (5.51) and Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left(g^{q,*}_{\lambda,\alpha}(f)(x)\right)^q h(x)dx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^\infty \|t^\alpha\partial_t^\alpha\mathcal{P}_t f(y)\|^q_{\mathbb{B}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{dt}{t^{n+1}}dyh(x)dx\\ &\leqslant C\int_{\mathbb{R}^n}\left(g^q_\alpha(f)(y)\right)^q Mh(y)dy \leqslant C\left\|g^q_\alpha(f)\right\|^q_{L^p(\mathbb{R}^n)}\|Mh\|_{L^{\frac{p}{p-q}}(\mathbb{R}^n)}.\end{aligned}$$

Here, when $p = q$, let $L^{\frac{p}{p-q}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Since M is bounded on $L^r(\mathbb{R}^n)$ $(1 < r \leqslant \infty)$, we get

$$\int_{\mathbb{R}^n}\left(g^{q,*}_{\lambda,\alpha}(f)(x)\right)^q h(x)dx \leqslant C\left\|g^q_\alpha(f)\right\|^q_{L^p(\mathbb{R}^n)}\|h\|_{L^{\frac{p}{p-q}}(\mathbb{R}^n)}.$$

Taking supremum over all h in $L^{\frac{p}{p-q}}(\mathbb{R}^n)$, we get

$$\left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\left\|g^q_\alpha(f)\right\|_{L^p(\mathbb{R}^n)}, \quad q \leqslant p. \tag{5.52}$$

Since $\mathbb{B}$ is of Lusin cotype q, by Theorem A and (5.52) we get $\|g^{q,*}_{\lambda,\alpha}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}$.

(ii) $\Rightarrow$ (i). On the domain $\Gamma(x) = \{(y,t) \in \mathbb{R}^n_+ : |y-x| < t\}$, we have

$$\left(\frac{t}{|x-y|+t}\right)^{\lambda n} > \left(\frac{1}{2}\right)^{\lambda n}.$$

Hence

$$\begin{aligned} S^q_\alpha(f)(x) &= \left(\iint_{\Gamma(x)} \|t^\alpha \partial^\alpha_t \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant \left(\iint_{\Gamma(x)} 2^{\lambda n}\left(\frac{t}{|x-y|+t}\right)^{\lambda n} \|t^\alpha \partial^\alpha_t \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}} \\ &\leqslant 2^{\frac{\lambda n}{q}} \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-y|+t}\right)^{\lambda n} \|t^\alpha \partial^\alpha_t \mathcal{P}_t f(y)\|^q_{\mathbb{B}} \frac{dydt}{t^{n+1}}\right)^{\frac{1}{q}} \qquad (5.53) \\ &= 2^{\frac{\lambda n}{q}} g^{q,*}_\lambda(f)(x), \quad \forall x \in \mathbb{R}^n. \end{aligned}$$

Hence $\|S^q_\alpha(f)\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant 2^{\frac{\lambda n}{q}} \left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}$, for any $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$. Then, by Theorem 5.14, $\mathbb{B}$ is of Lusin cotype q. □

Theorem 5.17. *Given a Banach space $\mathbb{B}$, $1 < q \leqslant 2$ and $\lambda > 1$, the following statements are equivalent:*

(i) $\mathbb{B}$ *is of Lusin type* q.

(ii) *For every (or, equivalently, for some) positive integer* n, *for every (or, equivalently, for some)* $p \in [q, \infty)$, *and for every (or, equivalently, for some)* $\alpha > 0$, *there is a constant* $C > 0$ *such that*

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C\left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

Proof. (i) $\Rightarrow$ (ii). Since $\mathbb{B}$ is of Lusin type q, by Theorem 5.15 and (5.53) we get

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C\,\|S^q_\alpha(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

(ii) $\Rightarrow$ (i). By (5.52), we get

$$\|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant C\left\|g^{q,*}_{\lambda,\alpha}(f)\right\|_{L^p(\mathbb{R}^n)} \leqslant C\,\|g^q_\alpha(f)\|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p_{\mathbb{B}}(\mathbb{R}^n).$$

Then by Theorem B, $\mathbb{B}$ is of Lusin type q. □

5.6 Another characterization of Lusin cotype

In this section, we will give another characterization of Lusin cotype q property by almost everywhere finiteness, see Theorem 5.18. It is worth to mention that the proof of Theorem 5.18 contains some new ideas that can be applied to a huge class of operators. Roughly, the method used in the proof is the following. If an operator T with a Calderón–Zygmund kernel is a.e. pointwise finite ($Tf(x) < \infty$) for any function f in $L^{p_0}(\mathbb{R}^n)$ and some $p_0 \in [1, \infty)$, then T is bounded from $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$. This philosophy can be translated to the vector-valued case and with this we can get a characterization of the UMD property of a Banach space.

5.6.1 Characterization of Lusin cotype by almost everywhere finiteness

On the particular Lebesgue measure space $(\mathbb{R}^n, dx)$, we have the following theorem.

Theorem 5.18. *Given a Banach space $\mathbb{B}$, $2 \leqslant q < \infty$, the following statements are equivalent:*

(i) *$\mathbb{B}$ is of Lusin cotype* q.

(ii) *For every (or, equivalently, for some) $p \in [1, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $g^q_\alpha(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$.*

(iii) *For every (or, equivalently, for some) $p \in [1, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $S^q_\alpha(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$.*

(iv) *For every (or, equivalently, for some) $p \in [q, \infty)$ and for every (or, equivalently, for some) $\alpha > 0$, $g^{q,*}_{\lambda,\alpha}(f)(x) < \infty$ for a.e. $x \in \mathbb{R}^n$, for every $f \in L^p_{\mathbb{B}}(\mathbb{R}^n)$.*

Proof. By Theorem A, Theorem 5.13, Theorem 5.14 and Theorem 5.16, we have (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv).
Let us prove the converse. (ii) $\Rightarrow$ (i). Let $p_0 \in (1, \infty)$. Observe that

$$g^q_\alpha(f)(x) = \left(\int_0^\infty \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(x)\|^q_{\mathbb{B}} \frac{dt}{t}\right)^{\frac{1}{q}}$$

$$= \sup_{j \in \mathbb{Z}^+} \left(\int_{\frac{1}{j}}^j \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(x)\|^q_{\mathbb{B}} \frac{dt}{t}\right)^{\frac{1}{q}} = \sup_{j \in \mathbb{Z}^+} \left\|T^j(f)(x,t)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})},$$

where $T^j(f)(x,t) = t^\alpha \partial_t^\alpha \mathcal{P}_t f(x) \chi_{\{\frac{1}{j} < t < j\}}$ is the operator which sends $\mathbb{B}$-valued functions to $L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$-valued functions. It is clear that T^j is bounded from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^{p_0}_{L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})}(\mathbb{R}^n)$. Indeed, since for any $m \in \mathbb{Z}_+$, we have

$$|t^m \partial_t^m \mathcal{P}_t(x,y)| \leqslant C_m \frac{t}{(t + |x-y|)^{n+1}}.$$

Therefore, for $\frac{1}{j} < t < j, j \in \mathbb{Z}_+$, we have

$$|t^m \partial_t^m \mathcal{P}_t(x,y)| \leqslant C_m \frac{j}{(\frac{1}{j} + |x-y|)^{n+1}}.$$

Hence

$$\|t^m \partial_t^m \mathcal{P}_t f(x)\|_{\mathbb{B}} \leqslant C_m \int_{\mathbb{R}^n} \frac{j}{(\frac{1}{j} + |y|)^{n+1}} \|f(x-y)\|_{\mathbb{B}}\, dy =: L_j(\|f\|_{\mathbb{B}})(x).$$

Since $\left\|L_j(\|f\|_{\mathbb{B}})\right\|_{L^p(\mathbb{R}^n)} = \left\|\frac{j}{(\frac{1}{j} + |y|)^{n+1}}\right\|_{L^p(\mathbb{R}^n)} \|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)} \leqslant j^{n+2-\frac{n}{p}} \|f\|_{L^p_{\mathbb{B}}(\mathbb{R}^n)}$. Therefore,

$$\begin{aligned}\left\|T^j(f)\right\|_{L^{p_0}_{L^q_{\mathbb{B}}(\mathbb{R}_+,\ \frac{dt}{t})}(\mathbb{R}^n)} &\leqslant \left(\int_{\mathbb{R}^n} \left(\int_{\frac{1}{j}}^{j} \left|L_j(\|f\|_{\mathbb{B}})(x)\right|^q \frac{dt}{t}\right)^{p_0/q} dx\right)^{1/p_0} \\ &= C_j \left\|L_j(\|f\|_{\mathbb{B}})\right\|_{L^{p_0}(\mathbb{R}^n)} \leqslant C_j \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}.\end{aligned}$$

Let $T^j_N(f)(x) = T^j(f)(x)\chi_{B_N}(x)$, where $B_N = B(0,N)$ is the ball in $\mathbb{R}^n$, for any $N > 0$. So T^j_N is bounded from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^{p_0}_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(B_N)$. Then we have

$$\begin{aligned}&\left|\left\{x \in B_N : \left\|T^j_N(f)(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \lambda \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}\right\}\right| \\ &\qquad \leqslant \frac{1}{\lambda^{p_0}\|f\|^{p_0}_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}} \int_{B_N} \left\|T^j_N(f)(x)\right\|^{p_0}_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} dx \leqslant \frac{C}{\lambda^{p_0}}. \qquad (5.54)\end{aligned}$$

Let $\mathcal{M} = \left\{f : f \text{ is } L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})\text{-valued and strong measurable on } B_N\right\}$. In the finite measurable space, $(B_N, \mathcal{M})$, we introduce the following topology basis. For any $\varepsilon > 0$, let

$$V_{B_N,\varepsilon} = \left\{f \in \mathcal{M} : \left|\left\{x \in B_N : \left\|f(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \varepsilon\right\}\right| < \varepsilon\right\}.$$

We denote the topology space on B_N by $L^0_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(B_N)$. By (5.54), we have

$$\lim_{\lambda\to\infty} \left|\left\{x \in B_N : \left\|T^j_N(f)(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \lambda \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}\right\}\right| = 0.$$

So for any $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that

$$\left|\left\{x \in B_N : \left\|T^j_N(f)(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \lambda \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}\right\}\right| < \varepsilon, \quad \lambda \geqslant \lambda_\varepsilon.$$

Then for ε given above, there exists a constant $\delta_\varepsilon = \dfrac{\varepsilon}{\lambda_\varepsilon}$, such that for any $\|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} < \delta_\varepsilon$ we have

$$\begin{aligned}&\left|\left\{x \in B_N : \left\|T^j_N(f)(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \varepsilon\right\}\right| \\ &\leqslant \left|\left\{x \in B_N : \left\|T^j_N(f)(x)\right\|_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})} > \lambda_\varepsilon \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}\right\}\right| < \varepsilon.\end{aligned}$$

This means that $T_N^j(f) \in V_{B_N,\varepsilon}$ for any $f \in L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ with $\|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} < \delta_\varepsilon$. Hence T_N^j is continuous from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^0_{L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t})}(B_N)$. Let $U_N = \left\{T_N^j(f)\right\}_{j=1}^{\infty}$. Since $g^q_\alpha(f)(x) < \infty$ a.e., U_N is a well defined linear operator from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^0_{\ell^\infty(L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t}))}(B_N)$. As B_N has finite measure, the space $L^0_{\ell^\infty(L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t}))}(B_N)$ is metrizable and complete. Then by the closed graph theorem, the operator U_N is continuous. As $g^q_{\alpha,N}(f)(x) = \left\|T_N^j(f)(x)\right\|_{\ell^\infty(L^q_{\mathbb{B}}(\mathbb{R}_+,\frac{dt}{t}))}$, we get that $g^q_{\alpha,N}$ is continuous from $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ to $L^0(B_N)$. Therefore for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|\{x \in B_N : |g^q_\alpha(h)(x)| > \varepsilon\}| < \varepsilon, \quad \text{for} \quad \|h\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} < \delta_\varepsilon.$$

In particular, for any $0 < r < \varepsilon$, there exists $\delta_r > 0$ such that

$$|\{x \in B_N : |g^q_\alpha(h)| > r\}| < \varepsilon, \quad \text{for} \quad \|h\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} < \delta_r.$$

Now let g be an element of $L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)$ with $\|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \neq 0$ and $h = \frac{g}{\|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}}\frac{\delta_r}{2}$. Then we have $\|h\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} < \frac{\delta_r}{2}$ and

$$\begin{aligned}\varepsilon &> \left|\left\{x \in B_N : |g^q_\alpha(h)| > r\right\}\right| > \left|\left\{x \in B_N : |g^q_\alpha(h)| > \varepsilon\right\}\right| \\ &= \left|\left\{x \in B_N : |g^q_\alpha(g)| > \frac{2\varepsilon \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}}{\delta_r}\right\}\right|.\end{aligned}$$

Let $\mu_\varepsilon = \frac{2\varepsilon}{\delta_r}$. Then when $\mu \geqslant \mu_\varepsilon$, we have

$$\left|\left\{x \in B_N : |g^q_\alpha(g)| > \mu \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}\right\}\right| \leqslant \left|\left\{x \in B_N : |g^q_\alpha(g)| > \frac{2\varepsilon \|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)}}{\delta_r}\right\}\right| < \varepsilon. \tag{5.55}$$

Let $f \in L^1_{\mathbb{B}}(\mathbb{R}^n)$ and $\lambda > 0$, we perform the Calderón–Zygmund decomposition as the sum $f = g + b$ such that $\|g\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \leqslant \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}$ and $\|g\|_{L^\infty_{\mathbb{B}}(\mathbb{R}^n)} \leqslant 2\lambda$. Then we have

$$\|g\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R}^n)} \leqslant (2\lambda)^{\frac{p_0-1}{p_0}} \|f\|^{\frac{1}{p_0}}_{L^1_{\mathbb{B}}(\mathbb{R}^n)} \tag{5.56}$$

and

$$\left|\left\{x \in \mathbb{R}^n : \left|g^q_\alpha(b)(x)\right| > \frac{\lambda}{2}\right\}\right| \leqslant \frac{C}{\lambda} \|f\|_{L^1_{\mathbb{B}}(\mathbb{R}^n)}. \tag{5.57}$$

Indeed, (5.56) is trivial from the estimates of g. For (5.57), we observe that by Proposition 5.9, $g^q_\alpha(f)$ can be expressed as an $L^q_{\mathbb{B}}(\mathbb{R}_+, \frac{dt}{t})$-norm of a Calderón–Zygmund operator with a regular kernel. In these circumstances, it can be observed that the boundedness of the measure of the set appearing in (5.57) depends only on the kernel of the operator and not on the boundedness of the operator, see [24]. Therefore, by (5.56) and (5.57), we have

$$\begin{aligned}
&\left|\left\{x \in B_N : \left|g_\alpha^q(f)(x)\right| > \lambda\right\}\right| \leqslant \left|\left\{x \in B_N : \left|g_\alpha^q(g)(x)\right| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : \left|g_\alpha^q(b)(x)\right| > \frac{\lambda}{2}\right\}\right| \\
&= \left|\left\{x \in B_N : \left|g_\alpha^q(g)(x)\right| > \frac{\lambda}{2\,\|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}} \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\}\right| \\
&\quad + \left|\left\{x \in \mathbb{R}^n : |g_\alpha^q(b)(x)| > \frac{\lambda}{2}\right\}\right| \\
&\leqslant \left|\left\{x \in B_N : |g_\alpha^q(g)(x)| > \frac{\lambda^{\frac{1}{p_0}}}{2^{2-\frac{1}{p_0}}\,\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)}^{\frac{1}{p_0}}} \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\}\right| + \frac{C}{\lambda}\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)} \\
&= \left|\left\{x \in B_N : \left|g_{\alpha,N}^q(g)(x)\right| > \frac{\lambda^{\frac{1}{p_0}}}{2^{2-\frac{1}{p_0}}\,\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)}^{\frac{1}{p_0}}} \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)} \right\}\right| + \frac{C}{\lambda}\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)}.
\end{aligned}$$

Now, given $\varepsilon > 0$ we perform the Calderón-Zygmund decomposition with λ such that $\lambda^{\frac{1}{p_0}} > 2^{2-\frac{1}{p_0}}\,\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)}^{\frac{1}{p_0}}\,\mu_\varepsilon$. Then, by (5.55), we have

$$\begin{aligned}
|\{x \in B_N : |g_\alpha^q(f)(x)| > \lambda\}| &\leqslant \left|\left\{x \in B_N : |g_\alpha^q(g)(x)| > \mu_\varepsilon \|g\|_{L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)}\right\}\right| + \frac{C}{\lambda}\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)} \\
&\leqslant \varepsilon + \frac{C}{\lambda}\|f\|_{L_{\mathbb{B}}^1(\mathbb{R}^n)}.
\end{aligned}$$

This clearly implies $g_\alpha^q(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$, for any $f \in L_{\mathbb{B}}^1(\mathbb{R}^n)$. We apply Theorem 7.1 in [57] and get the result.

To prove that (iii) $\Rightarrow$ (i), we can use the same argument as above but with a very small modification. We only need note that

$$\begin{aligned}
S_\alpha^q(f)(x) &= \left(\iint_{\Gamma(x)} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{q}} = \left(\int_0^\infty \int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{q}} \\
&= \sup_{j\in\mathbb{Z}^+}\left(\int_{\frac{1}{j}}^{j} \int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy}{t^n}\frac{dt}{t}\right)^{\frac{1}{q}} = \sup_{j\in\mathbb{Z}^+}\left\|T^j(f)(x,t)\right\|_{L_{\mathbb{B}}^q(\mathbb{R}_+,\frac{dt}{t})},
\end{aligned}$$

where $T^j(f)(x,t) = \displaystyle\int_{|y-x|<t} \|t^\alpha \partial_t^\alpha \mathcal{P}_t f(y)\|_{\mathbb{B}}^q \frac{dy}{t^n}\chi_{\{\frac{1}{j}<t<j\}}$ is the operator which sends $\mathbb{B}$-valued functions to $L^q(\mathbb{R}_+, \frac{dt}{t})$-valued functions. And T^j is bounded from $L_{\mathbb{B}}^{p_0}(\mathbb{R}^n)$ to $L_{L^q(\mathbb{R}_+,\frac{dt}{t})}^{p_0}(\mathbb{R}^n)$, $1 < p_0 < \infty$ also. Now we can continue the proof as in the case of g_α^q.

(iv) $\Rightarrow$ (i). Assuming that $g_{\lambda,\alpha}^{q,*}(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$, by (5.53) we know that $S_\alpha^q(f)(x) \leqslant C g_{\lambda,\alpha}^{q,*}(f)(x) < \infty$ a.e. $x \in \mathbb{R}^n$. Then by (iii) $\Rightarrow$ (i), $\mathbb{B}$ is of Lusin cotype q. □

5.6.2 UMD spaces

By using the method used in the proof of Theorem 5.18, we can get a characterization of UMD spaces as follows. For some information about the Littlewood-Paley-Stein theory for semigroups in UMD spaces, see [46, 47].

Theorem 5.19. *Given a Banach space $\mathbb{B}$, the following statements are equivalent:*

(i) $\mathbb{B}$ *is UMD.*

(ii) For every (or, equivalently, for some) $p \in [1, \infty)$,

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \text{ exists a.e. } x \in \mathbb{R}, \quad \text{for every } f \in L^p_{\mathbb{B}}(\mathbb{R}).$$

Proof. Clearly it is enough to prove (ii) $\Rightarrow$ (i). Let $1 < p_0 < \infty$ and assume that $\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$ exists a.e. $x \in \mathbb{R}$ for any $f \in L^{p_0}_{\mathbb{B}}(\mathbb{R})$. Then the maximal operator $H^*f(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \right\|_{\mathbb{B}}$ is finite a.e. $x \in \mathbb{R}$. Indeed, given $\varepsilon_0 = 1$, there exists $\delta_1 > 0$ such that

$$\|H_\varepsilon f(x) - F(x)\|_{\mathbb{B}} < 1, \quad \text{for all } \varepsilon < \varepsilon_0,$$

where $F(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$. So

$$\|H_\varepsilon f(x)\|_{\mathbb{B}} \leqslant \|H_\varepsilon f(x) - F(x)\|_{\mathbb{B}} + \|F(x)\|_{\mathbb{B}} < 1 + \|F(x)\|_{\mathbb{B}}.$$

If $\varepsilon > \varepsilon_0$, we have

$$\begin{aligned} \|H_\varepsilon f(x)\|_{\mathbb{B}} &= \left\| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \right\|_{\mathbb{B}} \leqslant \left(\int_{|x-y|>\varepsilon} \|f(y)\|^{p_0}_{\mathbb{B}} \, dy \right)^{\frac{1}{p_0}} \left(\int_{|x-y|>\varepsilon} \frac{1}{|x-y|^{p_0'} \, dy} \right)^{\frac{1}{p_0'}} \\ &\leqslant \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R})} \left(\int_{|x-y|>\varepsilon_0} \frac{1}{|x-y|^{p_0'} \, dy} \right)^{\frac{1}{p_0'}} = \|f\|_{L^{p_0}_{\mathbb{B}}(\mathbb{R})} \frac{\varepsilon_0^{1-p_0'}}{p_0' - 1}. \end{aligned}$$

So $H^*f(x) = \sup_\varepsilon H_\varepsilon f(x)$ is finite a.e. $x \in \mathbb{R}$. Our idea is to apply the method developed in the proof of (ii) $\Rightarrow$ (i) of Theorem C. However, we cannot apply it directly since H^* can't be expressed as a norm of a Calderón–Zygmund operator with a regular kernel. Let φ be a smooth function such that $\chi_{[\frac{3}{2},\infty)} \leqslant \varphi \leqslant \chi_{[\frac{1}{2},\infty)}$. Consider the operator $H^*_\varphi f(x) = \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}} \varphi\Big(\frac{|x-y|}{\varepsilon}\Big) f(y) dy \right\|_{\mathbb{B}}$. It can be easily checked that

$$|H^*_\varphi f(x) - H^*f(x)| \leqslant CM(\|f\|_{\mathbb{B}})(x), \quad \text{a.e. } x \in \mathbb{R}, \tag{5.58}$$

where M denotes the Hardy–Littlewood maximal function. Then

$$\begin{aligned} H_\varphi^* f(x) &= \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \varphi\left(\frac{|x-y|}{\varepsilon}\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} \\ &\leqslant \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \left(\varphi\left(\frac{|x-y|}{\varepsilon}\right) - \chi_{[\frac{3}{2},\infty)}\left(\frac{|x-y|}{\varepsilon}\right)\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} \\ &\quad + \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \chi_{[\frac{3}{2},\infty)}\left(\frac{|x-y|}{\varepsilon}\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} \\ &\leqslant \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \chi_{(\frac{1}{2},\frac{3}{2})}\left(\frac{|x-y|}{\varepsilon}\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} + \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \chi_{[\frac{3}{2},\infty)}\left(\frac{|x-y|}{\varepsilon}\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} \\ &\leqslant \sup_{\varepsilon>0} \frac{C}{\varepsilon} \int_{\{y\in\mathbb{R}^n\}:|x-y|<\frac{3}{2}\varepsilon} \|f\|_{\mathbb{B}}\, dy + \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^n} \chi_{[\frac{3}{2},\infty)}\left(\frac{|x-y|}{\varepsilon}\right) \frac{f(y)}{|x-y|} dy \right\|_{\mathbb{B}} \\ &= CM\left(\|f\|_{\mathbb{B}}\right)(x) + H^* f(x). \end{aligned}$$

Therefore, the operator $H_\varphi^* f(x) < \infty$, a.e. $x \in \mathbb{R}$. Observe that this operator can be expressed as

$$H_\varphi^* f(x) = \left\| \left\{ \int_{\mathbb{R}} \varphi\Big(\frac{|x-y|}{\varepsilon}\Big) f(y) dy \right\}_\varepsilon \right\|_{L_{\mathbb{B}}^\infty}.$$

It is well known that the last operator can be viewed as the $L_{\mathbb{B}}^\infty$-norm of a Calderón–Zygmund operator with regular kernel. Now we are in the situation of the proof of part (ii) $\Rightarrow$ (i) of Theorem C and with some obvious changes we get

$$\lim_{\lambda\to\infty} |\{x \in B_N : |H_\varphi^*(f)(x)| > \lambda\}| = 0, \quad \forall f \in L_{\mathbb{B}}^1(\mathbb{R}),\ N > 0.$$

In particular, this implies that the operator H_φ^* maps $L_{\mathbb{B}}^1(\mathbb{R})$ into $L^0(\mathbb{R})$. By (5.58) and the fact that M maps $L_{\mathbb{B}}^1(\mathbb{R})$ into weak-$L^1(\mathbb{R})$ for every Banach space $\mathbb{B}$, H^* maps $L_{\mathbb{B}}^1(\mathbb{R})$ into $L^0(\mathbb{R})$. Now we can apply the following lemma.

Lemma 5.20. *[57, Lemma 7.3] Let $\mathbb{B}$ be a Banach space. Then every translation and dilation invariant continuous sublinear operator $T : L_{\mathbb{B}}^1(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ is of weak type* $(1,1)$.

Then we get $H^* : L_{\mathbb{B}}^1(\mathbb{R}) \to$ weak-$L^1(\mathbb{R})$ which implies that the Banach space $\mathbb{B}$ is UMD. □

Remark 5.21. The above thoughts can be apply to the following general situation.
Given two Banach spaces $\mathbb{B}_1$, $\mathbb{B}_2$ and $1 \leqslant p < \infty$, let $K(x,y) \in \mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$ be a regular Calderón–Zygmund kernel. Define $T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) dy$ and

$$Sf(x) = \lim_{\varepsilon\to 0^+} T_\varepsilon f(x), \quad x \in \mathbb{R}^n.$$

Then the following statements are equivalent:

- For any $p \in (1, \infty)$, the operator S maps $L^p_{\mathbb{B}_1}(\mathbb{R}^n)$ into $L^p_{\mathbb{B}_2}(\mathbb{R}^n)$.
- For any (or, equivalently, for some) $p \in (1, \infty)$, the maximal operator $S^*f(x) = \sup_{\varepsilon>0} \|T_\varepsilon f(x)\|_{\mathbb{B}_2} < \infty$, a.e. $x \in \mathbb{R}^n$ for every $f \in L^p_{\mathbb{B}_1}(\mathbb{R}^n)$.

In other words, the following statement

"There exists a number $p_0 \in [1, \infty)$ such that $\|Tf(x)\|_{\mathbb{B}_2} < \infty$ a.e $x \in \mathbb{R}^n$, for every $f \in L^{p_0}_{\mathbb{B}_1}(\mathbb{R}^n)$."

could be added to the list of those statements in Remark 5.8, after an appropriated description of T.

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