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Vacuum energy decay

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Contents

Motivaciones y objetivos

De entre todos los problemas teóricos de la física actual, el más importante desde cierto punto de vista es el problema de la gravedad cuántica (QG).[1] Los principios fundamentales que subyacen en los dos paradigmas considerados como principales en el pasado siglo -o sea: la teoría de la relatividad general y la mecánica cuántica- parecen irreconciliables a primera vista. Desentrañar si lo son o no, y las consecuencias de ello es la definición más exacta de dicho problema. Puede argüirse que una solución satisfactoria para esta cuestión, incluyendo en "satisfactoria" evidencia experimental, está lejos aún dada la formidable escala de energía asociada al regimén cuántico de la gravedad, la masa de Planck M_P, 10^{16} veces superior a la energía que presumiblemente se alcanzará en el acelerador LHC del CERN.

Si resolver el problema de QG pasa por encontrar alguna manifestación del comportamiento cuántico de la gravedad, no puede decirse que, a priori, la teoría cuántica de campos en espacio-tiempo curvo sea un intento en esta dirección. Este marco teórico busca describir la influencia del campo gravitatorio sobre la propagación e interacción de la materia, cuando se toma en plena consideración el comportamiento cuántico de la última y sólo las ecuaciones clásicas de la primera. Así pues, el papel de la gravedad es el de una fuente clásica acoplada -y no la entidad cuántica y dinámica que debería ser en QG- en el espíritu de la descripción del átomo de hidrógeno mediante un potencial central electrostático. Que esta "mezcla" de dos paradigmas distintos describe un cierto régimen de fenómenos físicos es una suposición razonable: se trata de colocarse en un rango de energías en el que la curvatura del espacio-tiempo está lejos de la escala de Planck, pero es lo suficientemente grande como para compararse a las energías de las partículas subatómicas: $\ell_P^2 R \ll 1$ pero $R \cdot \lambda^2 \simeq 1$, con λ la longitud de onda de De Broglie de la materia. Es esta pues una aproximación *semiclásica*.

Sin embargo, de este enfoque se obtienen resultados destacables. En la cosmología inflacionaria la influencia de la gravedad sobre la evolución de la materia es uno de sus elementos clave. Aunque no se aplique directamente el formalismo de la teoría cuántica

[1] Especialmente, atendiendo al primer párrafo de buen número de tesis doctorales en física teórica.

de campos en espacio-tiempo curvo, se corresponde directamente con el régimen que hemos descrito anteriormente. Otro campo en el que han aparecido ideas nuevas y sorprendentes es en la física de agujeros negros. Los resultados sobre su temperatura y entropía, apuntando a su descripción como un colectivo termodinámico, junto con la idea seminal del principio holográfico son las pistas más firmes sobre el comportamiento de una teoría de QG, y una comprobación que ha de verificar cualquier candidato a tal. A la luz de estos resultados, se ha de admitir que el razonamiento que hacíamos anteriormente peca de simplista y no es correcto: la teoría cuántica de campos en espacio-tiempo curvo *sí* abre una primera y pequeña ventana a los fenómenos cuánticos de la gravedad.

Otro enigma crucial, relacionado también con la gravedad, es el llamado problema de la constante cosmológica. Puesto que puede añadirse un término constante a las ecuaciones de Einstein para el campo gravitatorio sin contradicción con ningún principio fundamental, la cuestión de si este término contribuye realmente o no ha de determinarse experimentalmente. Sin embargo, sus efectos son indistinguibles de añadir una energía mínima distinta de cero para la materia que puebla el espacio-tiempo.[2] Esta *energía del vacío* permearía el espacio-tiempo, impulsándolo como una reserva infinita de energía.

El mencionado problema consiste en reconciliar los valores observados para dicha constante con el entendimiento de las teorías cuánticas de campos. Medidas de precisión en el sistema solar descartan rápidamente un valor apreciable a esas escalas. Sólo experimentos más recientes han mostrado que el Universo está dominado por una componente energética con un comportamiento análogo a una constante cosmológica positiva (repulsiva), con un valor aproximado de $(10^{-3}\,\mathrm{eV})^4$. Por otro lado, en el valor "natural" para la energía de vacío de un campo cuántico, contribuyen todas las transiciones de fase por las que atraviesa el Universo. Una estimación cruda de nueva física hasta la escala de Planck arroja valores del orden de M_P^4, con 120 ordenes de magnitud de diferencia.[3] El origen y la naturaleza de la constante cosmológica son verdaderamente misteriosos.

Es por todo esto que el espacio de De Sitter es de especial interés en lo que concierne a estos dos problemas. Siendo la solución más simple y simétrica de las ecuaciones de Einstein con constante cosmológica positiva, es un laboratorio teórico ideal donde testear cualquier teoría de QG, y esclarecer quizá el problema de la constante cos-

[2]Acabando con el criterio generalizado de que sólo las diferencias de energía de un sistema físico son observables: la reacción del campo gravitatorio a dicha materia es la sonda que calibra el valor absoluto de la energía.

[3]Nótese que la constante cosmológica medida es $\sim 10^{-120} M_P^2$, mientras que la *energía de vacío* correspondiente es $\sim 10^{-30} M_P^4 \simeq (10^{-3}\,\mathrm{eV})^4$

mológica. Constituye también una buena descripción del Universo en la etapa inflacionaria así como en la etapa "tardía" del mismo dominada por energía oscura, luego cualquier efecto no trivial asociado a este espacio es susceptible de tener consecuencias físicas directas, si no medibles.

La teoría cuántica de un campo libre en De Sitter es bien conocida. De los múltiples estados de vacío que respetan sus simetrías, el llamado vacío de Bunch-Davies posee propiedades adicionales (analiticidad, la forma de Hadamard, etc.) que indican que es el vacío natural de dicha teoría, y describe un baño térmico alrededor de todo observador inercial.

En cuanto a la definición de una teoría en interacción en De Sitter, existen dos enfoques diferentes. En el enfoque "euclídeo", la teoría se define mediante continuación analítica de los correladores definidos en la esfera. Ninguna divergencia infrarroja puede aparecer mediante este método, puesto que la esfera es compacta. Por otro lado, en el enfoque "lorentziano", el punto de partida es el vacío de la teoría libre (Bunch-Davies), el cual es perturbado mediante un *cutoff* asintótico en la interacción. Ambos procedimientos son equivalentes en una teoría de campos en espacio plano. Sin embargo, arrojan resultados diferentes en De Sitter: el enfoque "lorentziano" da lugar a divergencias infrarrojas que producen una reacción en el campo gravitatorio. Se ha sugerido que la consecuencia última de este fenómeno es un apantallamiento efectivo de la costante cosmológica.

En este trabajo hemos avanzado en este enfoque "lorentziano". En el capítulo 1, hemos revisado algunos detalles del espacio de De Sitter, a nivel cuántico y clásico. En el capítulo 2, estudiamos la propuesta del *eternity test* por A. Polyakov [19] sobre la inestabilidad de un campo cuántico en De Sitter, y mostramos que no hay efectos no triviales a un *loop*. En el capítulo 3, introducimos las reglas de unitariedad para realizar cálculos a órdenes superiores. En el capítulo 4, presentamos un modelo sencillo de decaimiento de vacío en espacio plano, como ejemplo de la técnica del capítulo anterior, y se estudia la dinámica del decaimiento mediante producción de partículas. En el capítulo 5 presentamos nuestras conclusiones. Hemos recogido alguna información útil utilizada en el cuerpo del texto en los apéndices.

El contenido de esta tesis está basado en su mayoría en los resultados obtenidos en [1, 2]. Además de estos, hemos trabajado en otras áreas relacionadas con la gravitación [3, 4], pero cuyos resultados no están recogidos aquí.

Motivation and Objectives

Among all the theoretical problems in physics nowadays, the most important from a certain perspective is the problem of Quantum Gravity (QG)[4]. The fundamental principles in which the two most remarkable paradigms in the last century - i.e, the theory of General Relativity (GR) and Quantum Mechanics (QM) - lie seem to be irreconcilables at first sight. The most accurate statement of the problem is to disentangle whether they are incompatible or not, and the main consequences derived from there. A possible satisfactory solution to this, including in "satisfactory" some experimental evidence, is still far away given the overwhelming energy scale of the quantum regime of gravity, Planck mass M_P, 10^{16} times above the energy which will presumably be reached at the Large Hadron Collider (LHC) at CERN.

If the solution to QG comes through the observation of any indication of the quantum behaviour of gravity, it cannot be said that, *a priori*, quantum field theory in curved spacetime is an attempt in this direction. This theoretical framework aims to describe the influence of the gravitational field on the matter, when the latter's quantum behaviour and the former's classical equations are taken into consideration. Therefore, the role of gravity is that of a coupled classical source - and not the quantum and dynamic entity it should be in QG - following the spirit of the description of the hydrogen atom by means of a central electrostatic potential. It is reasonable to assume that this "mixture" of two different paradigms describes a certain range of physical phenomena. This is achieved focusing on an energy range in which spacetime curvature is far below Planck scale, but sufficiently large in order to be compared to the energy of subatomic particles: $\ell_P^2 R \ll 1$ but $R \cdot \lambda^2 \simeq 1$, λ being the De Broglie matter wavelength. Thus, this constitutes a semi-classical approximation to the problem.

Nevertheless, remarkable results are obtained in this approach. The influence of gravity over matter evolution is one of the key points of Inflationary Cosmology. Even though the formalism of quantum field theory in curved spacetime is not directly applied, there is a direct correspondence with the above described regime. Another field

[4]Specially, paying attention to the first pharagraph of a large fraction of PhD thesis in theoretical physics.

in which new and surprising ideas have arised is that of black hole physics. The results about their temperature and entropy suggest that they should be understood as thermodynamic collectives. These results, together with the idea of the holographic principle, are the strongest hints we have about QG. Any candidate to such a theory should clarify this points. In the light of this considerations, one should accept that the foreseen argument may be too simple and incorrect: quantum field theory in curved spacetime *does* open a first and small window to gravitational quantum phenomena.

Another crucial question, which is also related to gravity, is the so-called problem of the Cosmological Constant (CC). A constant term can be added to Einstein's equations for the gravitational field without contradicting any fundamental principle. Therefore, whether this term really contributes or not is a purely experimental problem. However, its effects are indistinguishable from those of adding a minimum energy different from zero to the matter which is occupying spacetime.[5] This *vacuum energy* would permeate spacetime, driving it as an infinite reservoir of energy.

The problem mentioned above consists in the reconciliation of the observed values for this constant with our understanding of quantum field theory. Precision measurements in the solar system automatically discard a substantial value at such scales. Only more recent experiments have proved that the Universe is dominated by an energetic component with an analogous behaviour to that of a positive cosmological constant (repulsive), with a value $(10^{-3}\,\mathrm{eV})^4$. On the other hand, all phase transitions the Universe goes through should contribute to a "natural" value for the vacuum energy of a quantum field. A rough estimate of new physics below the Planck scale would point to values of $\sim \mathcal{O}(M_P)$, being 120 orders of magnitude above what observations suggest.[6] Therefore, the origin and nature of the CC are completely unknown.

It is precisely due to these reasons that de Sitter space turns out to be of special interest concerning these two problems. This space is the simplest and most symmetric solution to Einstein's equations with a positive CC. Thus, it constitutes an ideal theoretical laboratory where any QG theory can be tested, and even some light over the CC problem could also be shed. Moreover, it also constitutes an approximate description of both the inflationary era and the late dark energy-dominated Universe. Thus, any non-trivial effect associated to this space is susceptible of presenting direct physical consequences, if not measurable.

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[5] Notice that this would end the general statement that only the differences in energy are observables in a physical system: the reaction of the gravitational field to that matter is the probe to the absolute value of the energy.

[6] Notice that the experimentally measured value of the cosmological constant is $\sim 10^{-120} M_P^2$, and the corresponding *vacuum energy* is $\sim 10^{-30} M_P^4 \simeq (10^{-3}\,\mathrm{eV})^4$.

The quantum field theory of a free field in de Sitter is well-known. From the multiple vacuum states that respect its symmetries, the so-called Bunch-Davies vacuum owns certain additional properties (analyticity, Hadamard's form, etc) which indicate that it is the natural vacuum of the theory. It describes a thermal bath of particles, as seen by any inertial observer.

Regarding the definition of an interacting theory in de Sitter, two different approaches exist. In the "euclidean" approach, the theory is defined by means of the analytic continuation of the defined correlators in the sphere. No infrared divergence can appear with this method, given that the sphere is compact. On the other hand, in the "lorentzian" approach the starting point is the own vacuum in the free theory (Bunch-Davies), which is perturbed through an asymptotic *cutoff* in the interaction. Both procedures are equivalent in a field theory in flat space. Nevertheless, the results in de Sitter turn out to be different: the "lorentzian" approach produces infrared divergences which induce a reaction in the gravitational field. It has been suggested that the ultimate consequence from this phenomenon is an effective screening of the CC.

In this work, we have developed some advances this "lorentzian" approach. In chapter 1, we give some details about de Sitter space at the classical and quantum levels. In chapter 2, we study the *eternity* proposal by A. Polyakov [19] about the instability of a quantum field on de Sitter, and we show that there is no non-trivial effect at one loop order. In chapter 3, we introduce the unitariry constraints of such a theory to perform calculations in higher orders. In chapter 4 we present a simple toy model of vacuum decay in flat space as an example of this technique. The dynamical decay through particle production is studied. In chapter 5 we present our conclusions. We have added some useful information used in the main text in the appendices.

The content in this thesis is mainly based on the results obtained in [1, 2] . In addition, we have worked on other subjects related to gravity [3, 4]. Those results will not be contained here.

Chapter 1

De Sitter space in depth

We collect here a basic summary of the de Sitter physics. One of the basic references is [5].

1.1 Classical de Sitter space

De Sitter space is the lorentzian analogous to an sphere: an embedded hypersurface in a $(1, -1^n)$ Minkowski space given by

$$\eta_{\mu\nu} X^\mu X^\nu \equiv (X^0)^2 - \sum_{I=1}^{n} (X^I)^2 = -l^2 \tag{1.1}$$

As a matter of fact, it is only one of the many possible real sections of the complex sphere. A detailed exposition of these spaces can be found in appendix A. The constant l is the curvature radius of the space, related to the curvature and the cosmological constant.

The metric inherited from the ambient space can be described using the standard "global" embedding (cf. A.1.1) of a generalized spheres:

$$ds^2 = d\tau^2 - l^2 \cosh(\tau/l)^2 d\Omega_{n-1}^2 \tag{1.2}$$

The symmetry group of de Sitter (and any generalized sphere) is easily derived considering its embedding. The subgroup of the ambient Poincare group $ISO(1, n)$ that leaves invariant the whole surface is precisely the Lorentz group, $SO(1, n)$. It can be proved that at the classical level, de Sitter space is stable with respect to linear perturbations [6].

The basic tool to deal with invariant bifunctions in de Sitter space is of course its distance invariant z (cf. A.1.1). Using the defining embedding:

$$z = -\frac{X \cdot Y}{l^2} \tag{1.3}$$

It is obvious that this quantity is de Sitter invariant. In addition, it can be proved that

- $z = 1$ for any pair of points connected by null geodesics (including the trivial geodesic that leaves a point static),
- $z > 1$ for any pair of causally related points. In fact, $z = \cosh\sigma$ where σ is the proper time between the two events.
- $z < 1$ for any pair of causally disconnected points. There is no geodesic connecting two points if $z < -1$, but otherwise, it is still (the cosine of) the geodesic distance.

Another geometric quantity of interest is the Pauli-Jordan function, that corresponds to the field conmmutator. However, this function is intrinsic to the space and can be defined as the difference between the retarded and the advanced propagators:

$$iD(x,y) = G_{>}(x,y) - G_{<}(x,y) \tag{1.4}$$

For de Sitter space, the Pauli-Jordan function is

$$D(x,y) = -\kappa_{n,\mu}\sigma(x,y)\,\mathrm{Im}\,F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}, \frac{n}{2}, \frac{1+z(x,y)}{2}\right) \tag{1.5}$$

where $\sigma(x,y)$ is the sign of the time ordering of x and y, μ is related to the mass, $m^2l^2 = \mu^2 + \frac{(n-1)^2}{4}$ and the constant is

$$\kappa_{n,\mu} = \frac{(-1)^n\left|\Gamma\left(i\mu + \frac{n-1}{2}\right)\right|^2}{(4\pi)^{\frac{n}{2}}\Gamma\left((-1)^n(\frac{n}{2}-1)+1\right)} \tag{1.6}$$

$i\mathrm{Im}\,f = f(\ldots, x + i\epsilon) - f(\ldots, x - i\epsilon)$ is the difference of the hypergeometric function across its branch cut. It is obvious that this function vanishes whenever $z(x,y) < 1$ and that is antisymmetric under time reversal.[1]

[1]Please notice that two different prescriptions can be used for two point functions:

1. The propagator is defined in such a way that: $(\Box + m^2)G(x,y) = \delta(x,y)$
2. The propagator is defined through: $G(x,y) = T\{W(x,y)\} = T\{\langle\mathrm{vac}|\phi(x)\phi(y)|\mathrm{vac}\rangle\}$.

The first prescription is just i times the second one. In this thesis, we will use the first prescription, except in chapter 2, where we will use the second one.

1.2 Free quantum fields in de Sitter space

The quantum theory of free fields can be addressed in several a priori different ways. The standard construction of a Hilbert space over the basic excitations of the fields (cf. appendix B) is possible. In the other hand, the high degree of symmetry allows us to try to repeat the Wigner's construction of particles.[2]

Since we are mostly interested in scalar fields, the basic construction is enough for our purposes. The Klein-Gordon equation in the coordinates (1.2) takes the form:

$$\left\{\frac{\partial^2}{\partial\tau^2}+\frac{n-1}{l}\tanh(\tau/l)\frac{\partial}{\partial\tau}-\frac{1}{l^2\cosh(\tau/l)^2}\Delta_{S^{n-1}}+m^2\right\}\phi=0 \tag{1.7}$$

Since the de Sitter's curvature is constant, we can adopt the convention that any non-minimal coupling can be reabsorbed into the mass of the field, as long as we are careful regarding their different conformal behaviours.[3] The sphere laplacian suggest us to decompose the field in terms of hyperspherical harmonics [7], $\phi\equiv\chi\cdot\Xi_{L\vec{m}}$, so we obtain

$$\left\{\frac{\partial^2}{\partial\tau^2}+\frac{n-1}{l}\tanh(\tau/l)\frac{\partial}{\partial\tau}+\left(\frac{L(L+n-2)}{l^2\cosh(\tau/l)^2}+m^2\right)\right\}\chi=0 \tag{1.8}$$

The two independent solutions for this equation are given in terms of Legendre functions of the second kind [8]:

$$(\operatorname{sech}\tau/l)^{\frac{n-1}{2}}P^{-i\mu}_{L+\frac{n-3}{2}}(\tanh\tau/l)\ ,\ (\operatorname{sech}\tau/l)^{\frac{n-1}{2}}Q^{-i\mu}_{L+\frac{n-3}{2}}(\tanh\tau/l) \tag{1.9}$$

There is no a priori reason to choose one particular combination of these solutions. The fact that the combination:

$$\chi_L(\tau)=\frac{\sqrt{\pi}e^{\pi\mu/2}}{2l^{\frac{n}{2}-1}}(\operatorname{sech}\tau/l)^{\frac{n-1}{2}}\left(P^{-i\mu}_{L+\frac{n-3}{2}}(\tanh\tau/l)+\frac{2i}{\pi}Q^{-i\mu}_{L+\frac{n-3}{2}}(\tanh\tau/l)\right) \tag{1.10}$$

corresponds to the most relevant free vacuum in de Sitter space [9] can be explained only afterwards. This vacuum is the so called Bunch-Davies vacuum, and its two point-function is:

$$\begin{aligned}W_{BD}(x,y)&\equiv {}_{BD}\langle\mathrm{vac}|\phi(x)\phi(y)|\mathrm{vac}\rangle_{BD}=\sum_{L\vec{m}}\phi_{L\vec{m}}(x)\phi_{L\vec{m}}(y)^*=\\&=\kappa_{n,\mu}F\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2},\frac{n}{2},\frac{1+z(x,y)}{2}-i\epsilon\sigma(x,y)\right)\end{aligned} \tag{1.11}$$

[2]This program has been fulfilled [10] constructing the field operators in terms of the "particles" in the principal series of de Sitter's UIRs.

[3]The totally massless case $m^2=0$ (no mass and minimal coupling) is known to be very different from the massive one. There are no de Sitter invariant two-point functions and the construction of the quantum theory should be carried out carefully [11, 12]. We will not study this case.

The Bunch-Davies vacuum is de Sitter invariant as its two-point depends only on the invariant distance.[4] This vacuum as we have mentioned before appears to be the most relevant vacuum for a free field in de Sitter for several reasons [13]:

i) Its two-point function is the boundary value of the analytic continuation of the sphere propagator to the complex sphere. Because of this, sometimes it is called the "Euclidean" vacuum.

ii) The singularities of the two-point function are the same as those of the Minkowski vacuum, i.e., it has the Hadamard form.

iii) The modes in (1.10) reduce to plane waves in the flat space limit $l \to \infty$.

iv) It is the "natural" vacuum when we adopt Gaussian normal coordinates.

In addition to this, the presence of a Killing horizon implies the existence of a temperature associated with it. A geodesic observer in de Sitter perceives a thermal bath of particles [14] -analogous to the Hawking radiation or the Unruh effect- of temperature $T_{dS} = 1/2\pi l$.

However, the constraint of de Sitter invariance does not specify a single vacuum. It can be found a whole 1-complex parameter family of de Sitter invariant vacua, so called the *alpha vacua* [15]:[5]

$$W_\alpha(z) = \frac{\kappa_{n,\mu}}{2}\left\{\cosh 2\alpha \,\mathrm{Re}\, F\left(\frac{1+z}{2}\right) + \sinh 2\alpha \,\mathrm{Re}\, F\left(\frac{1-z}{2}\right) - \right.$$
$$\left. - i\sigma(x,y)\,\mathrm{Im}\, F\left(\frac{1+z}{2}\right)\right\} \tag{1.12}$$

[4]Actually the two-point function is invariant under de Sitter transformations that do not invert the order of time. For the time reversal symmetry T, $W(Tx, Ty) = W(x,y)^*$ because of the antiunitarity of this operator.

[5]The most general expression, de Sitter invariant except for the discrete symmetries, is the α, β vacuum, with $\beta \in (-\pi, \pi]$

$$W_{\alpha,\beta}(x,y) = \frac{\kappa_{n,\mu}}{2}\left\{\cosh 2\alpha \,\mathrm{Re}\, F\left(\frac{1+z}{2}\right) + \sinh 2\alpha \left[\cos\beta \,\mathrm{Re}\, F\left(\frac{1-z}{2}\right) - \right.\right.$$
$$\left.\left. - \sin\beta\, \sigma(x^A, y)\mathrm{Im}\, F\left(\frac{1-z}{2}\right)\right] - i\sigma(x,y)\,\mathrm{Im}\, F\left(\frac{1+z}{2}\right)\right\}$$

The time ordering sign $\sigma(x^A, y)$ is defined only in the case $z < -1$, but for $z > -1$ the imaginary part of $F\left(\frac{1-z}{2}\right)$ vanishes, as in the case of the conmmutator function. This expression for $\beta \neq 0$ is not fully de Sitter invariant, i.e. it does not depend only on z, due precisely to the presence of this sign.

where we are abbreviating the parameters of the hypergeometric functions, and the real part $\mathrm{Re}\, F$ is the average across the branch cut: $\mathrm{Re}\, f(z) = f(z + i\epsilon) + f(z - i\epsilon)$. The free *in* (*out*) vacua, which is defined through a basis of modes with simple asymptotic behaviour in the past (future) timelike infinity, corresponds to the values $\sinh 2\alpha = \mathrm{csch}\, \pi\mu$ and $\beta = \pi/2$ ($\beta = -\pi/2$).

These alpha vacua have an additional singularity for antipodal points $z(x, y) = -1$, that seems unnatural. Although it is hidden beyond the horizon, these alpha vacua are believed to be somehow unphysical [16].

1.3 Interacting quantum fields in de Sitter space

It is well known (cf. appendix B) that in the presence of a generic gravitational field, any kinematically forbidden process for an interacting QFT in flat space, becomes possible. The case of de Sitter has given rise to the idea of an eternal particle production, due to its eternal expansion (in the cosmological expanding patch), and perhaps an associated instability.

The basic idea is simple [13]: to calculate the amplitude for the "forbidden" processes and evaluate the particle production rate. In the S-matrix formalism, this is just a more complicated version of Feynman diagrams, as for example:

$$\Gamma_{0\to4} = \int W(x, y)^4 dxdy \tag{1.13}$$

where the same formula of flat spacetime is applicable, except that the quantities involved (the integration variables, the two-point function) are now referred to de Sitter.

It has been suggested that the ultimate consequence of this particle production is a instability of the space. It is very different in nature from previous proposals [17] (criticized in [18]), since this is not a quantum gravitational effect. In this phenomenon,the backreaction of the quantum fields is neglected, at first stage. Of course, a full understanding of the process involves to take this backreaction into account.

The instability claim has been recently put on a new basis by A. Polyakov in a series of papers [19], where he tried to relate this particle production with the presence of an imaginary part in the effective contribution of the fields to the cosmological constant.

Chapter 2

Eternity up to one loop

In this chapter, we review the so-called "composition principle", and we put it in a new basis through the heat kernel formalism. We use this tool to evaluate the free energy in a constant curvature space to one loop.

2.1 The Composition law

It is well known (cf. for example the discussion in [20]) that in flat space the Klein-Gordon propagator[1] can be recovered from the first quantized path integral

$$G(x,y) \equiv \int \mathcal{D}X(s) e^{-mS(X)}$$

where the integral extends to all paths such that $X(0) = x$ and $X(1) = y$, and the action for each path is

$$S(X) \equiv \int_0^1 d\tau \sqrt{\delta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$$

This representation makes manifest that the propagator enjoys a quantum mechanical composition law, at least in the euclidean case:

$$\int d^n z\, G(x,z) G(z,y) = \int d^n z\, \mathcal{D}X(s)\mathcal{D}Y(s) e^{-m\{S(X)+S(Y)\}} \tag{2.1}$$

where $X(s)$ goes from x to z and $Y(s)$ from z to y. Then

$$\int d^n z G(x,z) G(z,y) = \int \mathcal{D}X(s) e^{-mS(X)} \mathcal{F}\left(m^2, S(X)\right) \tag{2.2}$$

[1]Remember that along this chapter we will use the prescription 1 in p. 16.

where now $X(s)$ goes from x to y, and the extra factor $\mathcal{F}\left(m^2, S(X)\right)$ takes into account the integral over the intermediate point z along the curve and leads to

$$\int d^n z\, G(x,z)G(z,y) = -\frac{\partial}{\partial m^2} G(x,y) \tag{2.3}$$

(This is equivalent to assert that $\mathcal{F}\left(m^2, S(X)\right) = \frac{1}{2m} S(X)$. We are aware of no simple argument for this).

In a recent paper Polyakov [19] suggests that unitarity in quantum field theory is equivalent to this *path composition*. Asymptotically (for large separation between the points) the propagator should behave as

$$G(x,y) \sim e^{-ims(x,y)} \tag{2.4}$$

where $s(x,y)$ is the geodesic distance between the points x and y.

The flat space Klein-Gordon propagator can be easily recovered [20] through [2]

$$G(x,y) = \int_0^\infty d\tau K(\tau; x, y) \tag{2.7}$$

where $K(\tau; x, y)$ is the Schrödinger functional

$$K(\tau; x, y) \equiv \int \mathcal{D}X e^{-i\int_0^\tau d\sigma\left(\frac{\dot{X}^2}{2\sigma} + \sigma\frac{m^2}{2}\right)} \tag{2.8}$$

and τ is the gauge invariant distance $\tau \equiv \int_0^1 e(\lambda) d\lambda$. Polyakov's path composition is then a simple consequence of Feynman's kernel quantum mechanical composition law

$$\int d^n z K\left(\tau_1; y, z\right) K\left(\tau_2; z, x\right) = K\left(\tau_1 + \tau_2; y, x\right) \tag{2.9}$$

Once these facts are understood, the temptation to choose them as the starting point for the study of quantum fields in a gravitational background is irresistible.

---

[2] In flat space this identity is true in any dimension for true propagators (*id est*, solutions of the inhomogeneous equation) because using the Fourier representation

$$G(x,y) = \int \frac{d^n p}{(2\pi)^n} \frac{e^{ip(x-y)}}{p^2 + m^2} \tag{2.5}$$

and

$$\int d^n z G(x,z)G(z,y) = \int d^n z \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{e^{ip(x-z)}}{p^2+m^2} \frac{e^{ik(z-y)}}{k^2+m^2} = -\frac{\partial}{\partial m^2} G(x,y) \tag{2.6}$$

Direct verification is more laborious.

The preceding results are by no means restricted to flat space. We shall explain in a moment that given the heat kernel, that is, the solution of the heat equation in an arbitrary spacetime $\partial_\tau K = (\Delta - m^2)K$ with the initial conditions $K(0;x) = \delta(x)$ we can obtain a Green's function for the Klein-Gordon equation through

$$G(x) = \int_0^\infty K(\tau;x)\, d\tau = \int \theta(\tau)\, K(\tau;x)\, d\tau \tag{2.10}$$

$$\begin{aligned}(\Delta - m^2)G(x) &= \int_0^\infty (\Delta - m^2)K(\tau;x)\, d\tau = \\ &= \int_0^\infty \partial_\tau K(\tau;x)\, d\tau = K(\tau;x)\Big|_0^\infty = -\delta(x)\end{aligned} \tag{2.11}$$

Whenever the composition principle of Schrödinger (or the heat) equation holds

$$\int K(\tau;x,z)K(\sigma;z,y)\, d^n z = K(\tau+\sigma;x,y) \tag{2.12}$$

this propagator (and others related) enjoys automatically the composition law (2.3)

$$\begin{aligned}\int G(x,z)G(z,y)\, d^n z &= \int_C dt ds\, K(t;x,z)K(s;z,y)\, d^n z = \\ &= \int_C dt ds K(t+s;x,y) = \frac{1}{2}\int_{C'} d\tau d\sigma\, K(\tau;x,y)\ ,\end{aligned} \tag{2.13}$$

where the integration domain in the t,s plane is the upper right quadrant C. We have performed the transformation $\tau = t+s$, $\sigma = t-s$, and the new domain C' can be parametrized as

$$\frac{1}{2}\int d\tau d\sigma\, \theta(\tau+\sigma)\theta(\tau-\sigma)\, K(\tau;x,y) = \int d\tau\, \tau\theta(\tau)K(\tau;x,y) = -\partial_{m^2}G(x,y) \tag{2.14}$$

where we take in account that the heat kernel for mass m is related to the massless one by $K_{m^2} = e^{-m^2\tau}K_{m=0}$. The conclusion of the above is that starting from the heat kernel, the "composition principle" is a simple consequence of the quantum mechanical closure relation

$$\sum_z |z\rangle\langle z| = 1 \tag{2.15}$$

2.2 The heat kernel

What we shall denote by *heat kernel* is what mathematicians call the *fundamental solution of the real heat equation* (FSRHE) made popular by Kac when he asked the question as to whether one could hear the shape of a drum [21] (the short answer is that one cannot in general). The mathematicians call heat equation to

$$\Delta K(x,y;\tau) - \mu^2 \frac{\partial K(x,y;\tau)}{\partial \tau} = 0$$

where $\Delta \equiv \nabla_\mu \nabla^\mu$, and we have introduced a mass scale μ to make τ dimensionless (or, what is equivalent, to consider the operator $\frac{\Delta}{\mu^2}$, whose eigenvalues are also dimensionless). The FSRHE is defined as the solution such that $\lim_{\tau\to 0^+} K(x,y;\tau) = \delta(x,y)$. The importance of the FSRHE is that it is *unique* for compact connected C^∞ riemannian manifolds without boundary [22]. Formally, it can be predicated that

$$K(\tau) \equiv e^{\frac{\tau}{\mu^2}\Delta}$$

(the convention is that the operator in the exponent is negative definite for $\tau \in \mathbb{R}^+$.) so that a Green's function can be defined as

$$G \equiv -\Delta^{-1} \equiv \int_0^\infty K(\tau) d\tau$$

This Green's function is also unique under the same conditions than the FSRHE is.

We will deal with this equation with an additional mass term, as in the previous section. In the particular case of euclidean space $\mathbb{R}^n$ (which is non compact, by the way)

$$K_0\left(x,y;\tau\right) = \frac{\mu^{n-2}}{\left(4\pi\tau\right)^{n/2}} e^{-\frac{\mu^2 (x-y)^2}{4\tau} - \frac{m^2}{\mu^2}\tau}$$

(where μ is an arbitrary mass scale whose physical meaning is the same as the one appearing in dimensional regularization). The famous integral

$$\int_0^\infty dx\, x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} = \left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\beta\gamma}\right) \tag{2.16}$$

leads to the euclidean Green's function

$$G_0\left(x,y\right) \equiv \int_0^\infty d\tau K_0\left(x,y;\tau\right) = \frac{1}{2\pi}\left(\frac{m}{2\pi|x-y|}\right)^{n/2-1} K_{n/2-1}\left(m|x-y|\right)$$

where $|x|^2 \equiv \sum_1^n x_i^2$ and $K_n(x)$ is the Bessel function of imaginary argument. This is the mother of all Green's functions.

This whole procedure can in some sense be reversed. If we consider the heat kernel corresponding to the *massless* Klein-Gordon operator, $K_{m=0}(\tau) \equiv K(\tau) e^{\frac{m^2}{\mu^2}\tau}$, then the relationship between the heat kernel and the (massive) Green's function is just a Laplace transform

$$G_m(x) = \int_0^\infty K_{m=0}(\tau) e^{-\tau\frac{m^2}{\mu^2}} d\tau$$

This means that whenever the Green's function as a function of m^2 is bounded by a polynomial in the half plane $\mathrm{Re}\, m^2 \geq c$, the Laplace transform can be inverted to yield

$$K_{m=0}(\tau) = \frac{1}{\mu^2}\int_{c-i\infty}^{c+i\infty} dm^2 e^{\tau\frac{m^2}{\mu^2}} G_m(x)$$

We shall extend this precise and beautiful mathematical framework in two ways. First of all, physics forces upon us the consideration of operators somewhat more general than the covariant laplacian, for example by allowing a generalized mass term (as well as nonminimal operators for higher spins [23]). Secondly, we are eventually interested in *pseudo*-riemannian, Lorentzian geometries which are moreover non-compact.

One of our main worries will precisely be how to go back and forth from one signature to the other. What we have seen in the previous paragraph is that this particular Green's function also satisfies Polyakov's composition principle.

2.2.1 Heat kernel on the sphere

In their work on the Schrödinger equation, Grosche and Steiner [24] are led towards the following integral, which gives what is essentially the Schrödinger propagator:

$$K\left(\Omega,\Omega';\tau\right) \equiv \int \mathcal{D}\Omega\, e^{i\int_0^\tau d\lambda\left(\frac{ml^2}{2}\dot{\Omega}^2 + \frac{n(n-2)}{8ml^2}\right)} = e^{i\tau\frac{n(n-2)}{8ml^2}} \int \mathcal{D}\Omega\, e^{i\int_0^\tau d\lambda \frac{ml^2}{2}\dot{\Omega}^2} \equiv$$
$$e^{i\tau\frac{n(n-2)}{8ml^2}} Z\left(\Omega,\Omega';\tau\right) \tag{2.17}$$

where $\Omega \equiv \vec{n}$ is a unit vector, defining a point on the unit sphere $\vec{n} \in S_n$, and can be characterized in polar coordinates by a set of angles, $\theta_1 \ldots \theta_n$.

The path integral will be done by means of Feynman's time slicing technique. The action reads

$$S = \frac{ml^2}{2}\sum_{i=1}^n \left(\vec{\Omega}_i - \vec{\Omega}_{i-1}\right)^2 = ml^2 \sum_{i=1}^n \left(1 - \cos\,\psi_{i-1}\right) \tag{2.18}$$

where we have defined

$$\cos\,\psi_{i-1} \equiv \vec{\Omega}_i \cdot \vec{\Omega}_{i-1} \tag{2.19}$$

The expansion discussed in the appendix C conveys the fact that

$$e^{z\cos\psi} = \left(\frac{z}{2}\right)^{-\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right) \sum_{j=0}^{\infty} \left(j + \frac{n-1}{2}\right) I_{j+\frac{n-1}{2}}(z)\, C_j^{\frac{n-1}{2}}(\cos\psi) \tag{2.20}$$

$$Z(\theta, \theta'; \tau) = e^{i\tau\frac{n(n-2)}{8ml^2}} \int \mathcal{D}\Omega\, e^{i\int \frac{ml^2}{2}\dot{\Omega}^2} = e^{i\tau\frac{n(n-2)}{8ml^2}} \int \prod_i d\Omega_i\, e^{iml^2\sum_i(1-\cos\psi_{i-1})} \tag{2.21}$$

the integrations to be done are, schematically,

$$\begin{aligned}
&\int d\Omega_1 \ldots d\Omega_{n-1} \sum_{j_1\vec{m}_1}\sum_{j_2\vec{m}_2} \ldots Y_{j_1\vec{m}_1}(\Omega_1) Y^*_{j_1\vec{m}_1}(\Omega_0) Y_{j_2\vec{m}_2}(\Omega_2) Y^*_{j_2\vec{m}_2}(\Omega_1)\ldots \\
&\ldots \sum Y_{j_n\vec{m}_n}(\Omega_n) Y^*_{j_n\vec{m}_n}(\Omega_{n-1}) = \sum_{j\vec{m}} Y_{j\vec{m}}(\Omega_n) Y^*_{j\vec{m}}(\Omega_0) \sim \sum_j C_j^{\frac{n-1}{2}}(\cos\psi)
\end{aligned}$$

The final result of [24] is

$$K(\Omega, \Omega'; \tau) = \frac{1}{V(S_n)} \sum_{j=0}^{\infty} \frac{2j+n-1}{n-1} C_j^{\frac{n-1}{2}}(\Omega\cdot\Omega') e^{-\frac{i\tau}{2ml^2} j(j+n-1)} \tag{2.22}$$

Our main tool in order to study the effective potential in constant curvature spaces will be the analogous of the preceding computation for our Klein-Gordon equation, as well as the representation of the delta function on the sphere S_{n-1} by means of Gegenbauer polynomials (cf. appendix C) , *id est*,

$$K(\tau; \Omega, \Omega') = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\Omega\cdot\Omega') e^{-\tau(m^2l^2 + j(j+n-1))} \tag{2.23}$$

that is the solution of the heat equation such that

$$\lim_{\tau\to 0^+} K(\tau; \Omega, \Omega') = \delta(\Omega - \Omega') \tag{2.24}$$

where the delta function reads

$$\delta(\Omega - \Omega') = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos\theta_n) \tag{2.25}$$

2.2.2 Free energy

We can see the heat kernel formally as

$$K(\tau) \equiv e^{-\tau \bar{M}^2} \tag{2.26}$$

where $\bar{M}^2$ is the positive definite operator acting on quadratic fluctuations around the background field, *id est*,

$$\bar{M}^2 \equiv -\Delta + \partial^2 V(\bar{\phi}) \tag{2.27}$$

and we include masses in the potential.

Let us mention that whenever the full eigenvalue problem for the operator $\bar{M}^2$ is known, there is a formal FSRHE. Using the discrete notation,

$$\bar{M}^2 u_n(x) = \lambda_n u_n(x) \tag{2.28}$$

with eigenfunctions which can be chosen to obey

$$(u_n, u_m) \equiv \int d\mu(x)\, u_n^*(x) u_m(x) = \delta_{nm} \tag{2.29}$$

(where the measure $d\mu(x)$ is usually $\sqrt{|g|}d^n x$) as well as a completeness relationship of the type

$$\sum_n u_n^*(x) u_n(y) = \delta(x-y) \tag{2.30}$$

then the following is the sought for FSRHE

$$K(x,y|\tau) = \sum_n e^{-\lambda_n \tau} u_n^*(x) u_n(y) \tag{2.31}$$

whose imaginary part is determined by the one of the eigenvalues themselves.

As we have already advertised, in order to study the free energy up to one loop order, it is much more convenient to study the heat kernel, than the Green's function, because it gives the desired result directly

$$W = \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau} \mathrm{tr} \int d^n x \sqrt{|g|} K(\tau; x, x) \tag{2.32}$$

This definition *includes* the definition based to the zeta-function (which is the finite part) as well as the divergent counterterms.

2.3 Green's functions in constant curvature spaces.

Before computing the free energy, let us clarify a few points on the relationship between Green's functions in constant curvature spaces. Although the defining equations of the different spaces themselves in Weierstrass coordinates are analytic continuations of the equation of the sphere, some subtleties appear with the analytic continuation of Green's functions.

We shall mainly be concerned in this section with fundamental solutions of the Klein-Gordon equation in the real sections of the sphere, invariant under the full group of isometries. Related analysis have been performed in [25, 26]. The homogeneous version of this equation takes always the same form in these spaces:

$$(z^2-1)G'' + nzG' \pm m^2 l^2 = 0 \tag{2.33}$$

where z is the corresponding geodesic distance for each space (cf. A.1.1).

The problem of finding the invariant Green's functions of this equation can be solved in a simple and general way. The full space of solutions is two-dimensional. All we have to do is extending the domain of definition of these functions to the appropiate region of the real axis for each surface.

We have to take care also of the singularities we obtain. We are interested in a single source (tipically in the "north pole" $z=1$), or perhaps in symmetric solutions under $\mathbb{Z}_2$ in order to obtain Green's functions for the projective case.

In the Fig. 2.1 we have summarized the results. Combining solutions of the generic Klein-Gordon equation (hypergeometric functions) with the appropriate singularity $(F\left(\frac{1+z}{2}\right),\ R)$, we can build several different propagators for each space. Here R is proportional to a Legendre Q function, finite at $z=\infty$. G_∞ means a Green's function that diverges at infinity. G_α stands for the Green's functions of the α-vacua.

2.3.1 Flat spacetime

The flat spacetime case is interesting in order to know the appropriate short distance behaviour. We saw in the previous that the calculation of the n-dimensional Green's function in an euclidean flat spacetime gives

$$G(x) = \int \frac{e^{ipx}}{p^2+m^2}\frac{d^n p}{(2\pi)^n} = \frac{1}{2\pi}\left(\frac{m}{2\pi r}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(mr) \tag{2.34}$$

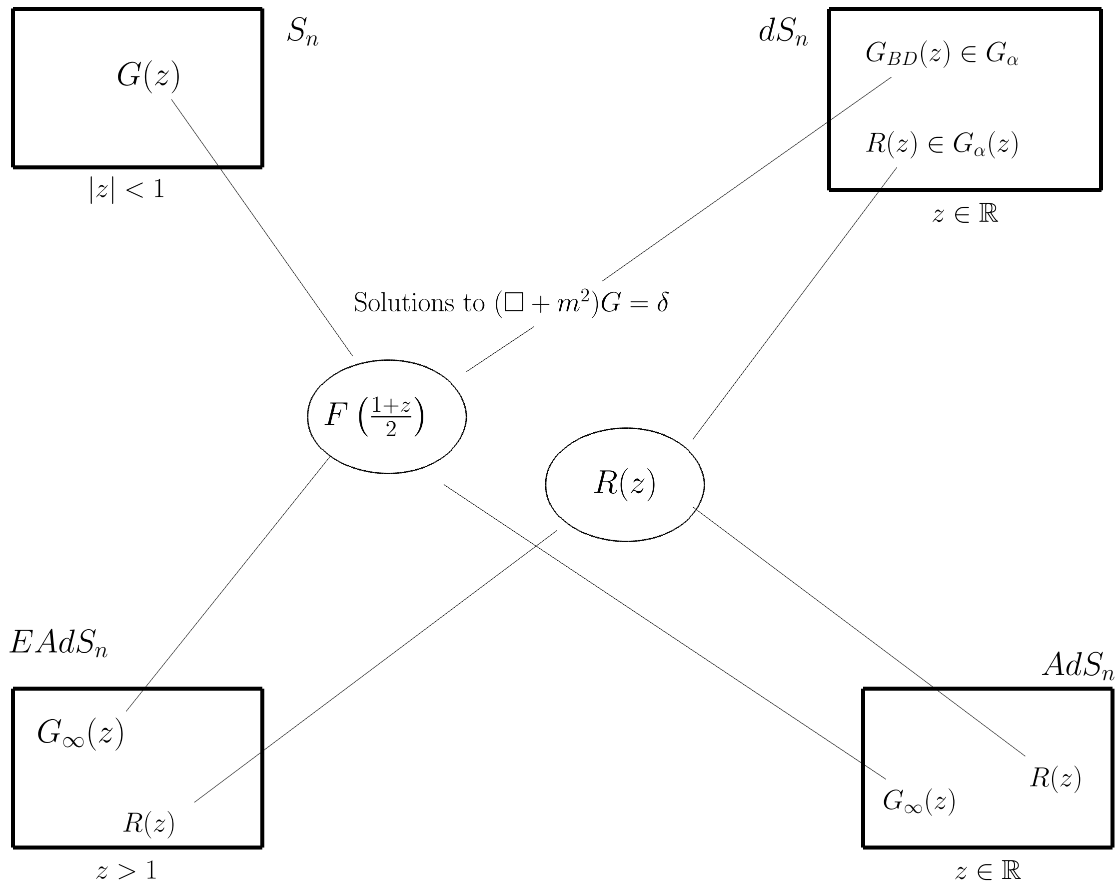

Figure 2.1: Route sheet of analytic continuations.

When we perform the analytic continuation to the Feynman propagator in lorentzian signature, we implicitly chose the prescription such that the result is still a propagator, i.e. that keeps the appropriate singularity:

$$G_F(x) = \frac{i}{2\pi}\left(\frac{m}{2\pi\sqrt{-x^2+i\epsilon}}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(m\sqrt{-x^2+i\epsilon})$$

That this is correct, can be checked performing the integral $\displaystyle\int \frac{d^n k}{(2\pi)^n}\frac{e^{ikx}}{-k^2+m^2-i\epsilon}$ explicitly. The branch cut of $\sqrt{-x^2}$ does not depend on the sign on time, but just on $|t|$, as was expected from a time ordering.

The singularity of this propagator is:

$$G(x) \xrightarrow{x^2 \to 0} \frac{i}{(2\pi)^{\frac{n}{2}}} 2^{\frac{n}{2}-2}\Gamma\left(\frac{n}{2}-1\right)(-x^2+i\epsilon)^{1-\frac{n}{2}} + [\log(-x^2+i\epsilon)] \tag{2.35}$$

where the term in brackets appears when n is even.

This prescription precisely gives us the correct singularity to recover a delta function. Other possibilities lead to homogeneous solutions which correspond to important functions:

- Wightman function $-iW$: $x^2 \to -x^2 + i\epsilon t$
- Symmetric function $G^{(1)}$: Re W
- Pauli-Jordan function (conmmutator) D: Im W

2.3.2 Sphere

In the appendix A we give some details on different metrics for constant curvature spaces with different signatures. The Klein-Gordon equation in the n-dimensional sphere reads:

$$\frac{1}{\sin\theta^{n-1}}\partial_\theta(\sin\theta^{n-1}\partial_\theta G) - m^2l^2G = 0 = \frac{1}{(1-z^2)^{\frac{n-2}{2}}}\partial_z((1-z^2)^{\frac{n}{2}}\partial_z G) - m^2l^2G \tag{2.36}$$

where $z = \cos\theta$. This is almost an hypergeometric equation:

$$(z^2-1)G'' + nzG' + m^2l^2G = 0 \tag{2.37}$$

with the solutions[3]:

$$G(z) = F_\pm(z) = F\left(\frac{1\pm z}{2}\right) \equiv F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1\pm z}{2}\right) \tag{2.38}$$

where $m^2l^2 = \mu^2 + \frac{(n-1)^2}{4}$. Each one is singular respectively in $z = \pm 1$, and this singularity corresponds precisely to delta function in opposite points. in this way we recover the well known fact that there is a single Green's function in the sphere.

The composition law holds for this Green's function, given that is unique and therefore, proportional to the alternate expression:

$$G(\Omega\cdot\Omega') = \sum_{j\vec{k}} \frac{Y_{j\vec{k}}(\Omega)Y_{j\vec{k}}(\Omega')^*}{j(j+n-1)+m^2} \tag{2.39}$$

given in terms of eigenfunctions of Δ, i.e. spherical harmonics, and their eigenvalues. It is straightforward to check the composition law with this formula.

[3]The possible values of μ are real and positive, or imaginary, with $\frac{n-1}{2} > -i\mu > 0$

2.3.3 de Sitter space

The Klein-Gordon equation in this case reads

$$\frac{1}{\cosh\tau^{n-1}}\partial_\tau\left(\cosh\tau^{n-1}\partial_\tau G\right)-\frac{1}{\cosh\tau^2\sin\theta^{n-2}}\partial_\theta\left(\sin\theta^{n-2}\partial_\theta G\right)+m^2l^2G=0$$

$$(z^2-1)G''+nzG'+m^2l^2G=0\ ,\ z=\cosh\tau\cos\theta \tag{2.40}$$

The solution is given by the same expression as before. In order to provide a function defined over the full de Sitter space (for all $z\in\mathbb{R}$), we must specify the values in the branch cuts. In addition, since the signature of spacetime has changed, this prescription will determine the character of the singularity, i.e. homogeneous or not.

Looking to the flat spacetime case, the solution is simple, since the short distance behaviour should match. The correct analytic continuation is:

$$G_{BD}(z)=F\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2};\frac{n}{2};\frac{1+z}{2}-i\epsilon\right) \tag{2.41}$$

and this is (proportional to) the euclidean or Bunch-Davies propagator. In addition we can continue the both solutions in such a way that they remain homogeneous, for example:

$$\mathrm{Re}F_\pm(z)=\mathrm{Re}\,F\left(i\mu+\frac{n-1}{2},-i\mu+\frac{n-1}{2};\frac{n}{2};\frac{1\pm z}{2}\right) \tag{2.42}$$

where we denote by $\mathrm{Re}, i\mathrm{Im}f(z)=f(z+i\epsilon)\pm f(z-i\epsilon)$. This combination cancels the delta divergence.

The above expression spans the space of homogeneous invariant solutions that originates the ambiguity in the propagator:

$$G(z)=G_{BD}(z)+\alpha\,\mathrm{Re}\,F_+(z)+\beta\,\mathrm{Re}\,F_-(z) \tag{2.43}$$

However, if the propagator comes from a vacuum expectation value, we know [15] that just a 1-parameter family survives, the α ($\alpha>0$) vacuum:

$$G_\alpha(z)=\frac{i|\Gamma\left(i\mu+\frac{n-1}{2}\right)|^2}{2(4\pi)^{\frac{n}{2}}\{-\Gamma(2-\frac{n}{2})|\Gamma(\frac{n}{2})\}}\left\{\cosh 2\alpha\,\mathrm{Re}F\left(\frac{1+z}{2}\right)+\right.$$

$$\left.+\sinh 2\alpha\,\mathrm{Re}F\left(\frac{1-z}{2}\right)-i\,\mathrm{Im}F\left(\frac{1+z}{2}\right)\right\} \tag{2.44}$$

The term in the $\{|\}$ corresponds to the $\{$odd$|$even$\}$ case.

2.3.4 Euclidean Anti de Sitter space

Now the Klein-Gordon equation reads

$$(z^2-1)G'' + nzG' - m^2l^2G = 0 \tag{2.45}$$

The solutions are pretty similar to the sphere case:

$$G(z) = F\left(\mu + \frac{n-1}{2}, -\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 \pm z}{2}\right) \tag{2.46}$$

where $\mu^2 = m^2l^2 + \left(\frac{n-1}{2}\right)^2$. This time $\mu > \frac{n-1}{2}$.

The negative sign solution is regular in $z = 1$ so it is purely homogeneous. Given that now $z \geq 1$, the positive sign solution needs a prescription in the branch cut to be meaningful. The exact behaviour near $z = 1$ depends on the parity of n, but in both cases the expressions are like:

$$F\left(\frac{1+z}{2}\right) = \ldots + \ldots \cdot \left(\frac{1-z}{2}\right)^{1-\frac{n}{2}} \tag{2.47}$$

where $\ldots$ something regular in $z = 1$ (or a logarithm). We can see from this equation that taking the upper or lower limit in the real axis, $z \pm i\epsilon$ gives us a Green's function G_∞.

However, this propagator G_∞ diverges in the infinity, as we can see from the expansion of the hypergeometric function near the infinity:

$$F(\alpha, \beta; \gamma; z) \xrightarrow{z\to\infty} \text{const}\,(-z)^{-\alpha} + \text{const}\,(-z)^{-\beta} \tag{2.48}$$

from wich we get:

$$G_\infty(z) \xrightarrow{z\to\infty} \text{const}\left(-\frac{1+z}{2}\right)^{-\mu-\frac{n-1}{2}} + \text{const}\left(-\frac{1+z}{2}\right)^{\mu-\frac{n-1}{2}} \tag{2.49}$$

Both the imaginary and the real part of this expression diverge (this is due to the second term), so in general no prescription gives us a propagator that vanishes at infinity[4].

An appropiate solution can be obtained combining the G_∞ with the homogeneous solutions. The exact expression can be given in terms of Legendre associated functions:

$$G(z) = (z^2-1)^{\frac{1-n}{4}} Q^{\frac{n-1}{2}}_{\mu-\frac{1}{2}}(z) \sim z^{-\mu-\frac{n-1}{2}} F\left(\frac{\mu}{2} + \frac{n+1}{4}, \frac{\mu}{2} + \frac{n-1}{4}; \mu+1; \frac{1}{z^2}\right) \tag{2.50}$$

[4] In fact, some specific values of m are such that taking only the imaginary [real] part of the function, for n odd [even], this term is cancelled.

This special combination, that we will abbreviate $R^{\frac{n-1}{2}}_{\mu-\frac{1}{2}}$, is a solution of (2.45). The composition principle holds for this propagator, given that this solution is the Laplace transform of the Schrödinger propagator of $EAdS$ [24].

2.3.5 Anti de Sitter space

The Klein-Gordon equation in AdS is identical to the $EAdS$ case. The variable z can take any real value again, as in de Sitter, so the the solutions to (2.45) can be continued in the same way as in (2.41), (2.42). We have just to take in account that now $i\mu \to \mu$, where μ means the same as in the $EAdS$ case.

Since the Anti de Sitter space has a well defined spatial infinity at $z = \infty$, if we require the propagator to vanish there, we will obtain the same R expression as in the $EAdS$ case (2.50). However, in this case we have to extend the domain to the full real axis. In order to get the correct prescription, we need the relationship between the R and the hypergeometric solutions:

$$R_\nu^{\frac{n-2}{2}}(z) = \rho_{n,\nu}\left\{e^{\mp i\pi\nu}F\left(\frac{1-z}{2}\right) + \varphi_\pm\, F\left(\frac{1+z}{2}\right)\right\} \tag{2.51}$$

$$\rho_{n,\nu} = \frac{2^{-\frac{n}{2}}\pi\Gamma(\frac{n}{2}+\nu)}{\Gamma(\frac{n}{2})\Gamma(2-\frac{n}{2}+\nu)\{i\cos\pi\nu|\sin\pi\nu\}}\ ;\ \varphi_\pm = \{i(-1)^{\frac{n\pm1}{2}}|(-1)^{\frac{n}{2}}\}$$

where again we write togheter the {odd|even} case, and the upper (lower) sign is for positive (negative) imaginary part of z.

An expression like (2.43) is the most general Green's function. Since the delta singularities are in the imaginary part of the F solutions, and the homogeneous pieces are the real parts, we have to eliminate the imaginary part of $F_- \equiv F\left(\frac{1-z}{2}\right)$, and it is easy to see that the appropriate combination to achieve it is

$$\tilde{R}_\nu^{\frac{n-2}{2}}(z) = e^{i\pi\nu}R_\nu^{\frac{n-2}{2}}(z+i\epsilon) + e^{-i\pi\nu}R_\nu^{\frac{n-2}{2}}(z-i\epsilon) \tag{2.52}$$

The detailed expressions in the even and odd cases are respectively:

$$\tilde{R}_\nu^{\frac{n-2}{2}}(z) \sim \mathrm{Re}F_-(z) + (-1)^{\frac{n}{2}}\cos\pi\nu\,\mathrm{Re}F_+(z) - (-1)^{\frac{n}{2}}\sin\pi\nu\,\mathrm{Im}F_+(z) =$$

$$= \mathrm{Re}F_-(z) + (-1)^{\frac{n}{2}}i\sinh\pi\mu\,\mathrm{Re}F_+(z) + (-1)^{\frac{n}{2}}\cosh\pi\mu\,\mathrm{Im}F_+(z) \tag{2.53}$$

$$\tilde{R}_\nu^{\frac{n-2}{2}}(z) \sim \mathrm{Re}F_-(z) + (-1)^{\frac{n-1}{2}}\sin\pi\nu\,\mathrm{Re}F_+(z) + (-1)^{\frac{n-1}{2}}\cos\pi\nu\,\mathrm{Im}F_+(z) =$$

$$= \text{Re}F_-(z) - (-1)^{\frac{n-1}{2}} \cosh \pi\mu \, \text{Re}F_+(z) + (-1)^{\frac{n-1}{2}} i \sinh \pi\mu \, \text{Im}F_+(z) \tag{2.54}$$

The second line in each case come from $\nu = i\mu - \frac{1}{2}$, i.e. the de Sitter case. As we can see, if and only if the dimension n is odd the R solution can be analitically continued into an alpha-beta vacuum, because of the inappropiate i factors in the even case. The parameters of that vacuum are $\sinh 2\alpha = \text{csch}\pi\mu$, and $\beta = 0$ ($\beta = \pi$) for $(-1)^{\frac{n+1}{2}}$ positive (negative).[5]

2.3.6 Projective spaces

A function defined over the projective version of these spaces can always be lifted to an symmetric function defined over the original space. It is very easy to obtain the most general Green's function of such an space, given the previous classification.

For the projective plane $\mathbb{R}P_n = S_n/\mathbb{Z}_2$, there is a single Green function corresponding to the projection of $G(z) + G(-z)$, where $G(z)$ is the propagator in 2.38 with the positive sign.

In the projective versions of de Sitter or Anti de Sitter, $dS_n/\mathbb{Z}_2$ and $AdS_n/\mathbb{Z}_2$, we found that the most general Green's function is:

$$G(z) = G_{BD}(z) + \alpha \, \text{Re} \, F_+(z) + \beta \, \text{Re} \, F_-(z) \tag{2.55}$$

where α and β are arbitrary constants. If we symmetrize this expression, we get the general propagator for these spacetimes:

$$G_P(z) = G_{BD}(z) + G_{BD}(-z) + \alpha \left(\text{Re} \, F_+(z) + \text{Re} \, F_-(z) \right) \tag{2.56}$$

In particular, we can symmetrize the $\tilde{R}$ solution finite at $z = \pm\infty$.

2.4 The imaginary part of the effective potential.

In flat space there is a systematic way of determining the ground state of a physical system, namely, to minimize the effective potential (the effective action for constant backgrounds). This is the physical principle that generalizes minimization of energy for classical systems. Things get more complicated when gravitational fields are present.

[5] This is valid only in the case of $m > \frac{n-1}{2}$ in de Sitter. For lower masses there is no possibility of analytic continuation, because of the i factors again.

First of all there is no fully satisfactory concept of energy in general gravitational backgrounds. In de Sitter space a Killing energy with support on the space orthogonal to a given observer, u, is well-defined through

$$E(u) \equiv \int d^{n-1}x\, u_\mu T^{\mu\nu} k_\nu \tag{2.57}$$

where the energy-momentum tensor is defined by expanding à la Abbott-Deser around a background. The lack of global existence of the Killings means that precise statements are only possible outside the corresponding horizons. In the general situation the situation is even worse, and several definitions (such as the Hawking-Geroch, Penrose, Nester-Witten or Brown-York, [27]) of quasilocal energy exist, none of which is fully satisfactory, and besides all of them seem difficult to compute in quantum field theory. What we have done instead is to compute the simplest and most naive expression for the energy, namely the effective potential.

As a matter of fact, the formula (2.23) for the sphere S_n could be directly continued to de Sitter space, given that the Gegenbauer polynomials $C_j^{\frac{n-1}{2}}$ are defined for all real z. Then, the expression:

$$K(\tau; z) = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(z) e^{-\tau(m^2 l^2 + V''(\bar{\phi}) + j(j+n-1))} \tag{2.58}$$

is a natural candidate for the heat kernel in de Sitter as well.

Then we can evaluate the free energy given by formula (2.32):

$$W = \frac{1}{2}\int_0^\infty \frac{d\tau}{\tau} \int d^n x \sqrt{|g|} K(\tau; x, x) = \frac{\mathrm{Vol}_{dS}}{2} \int_0^\infty \frac{d\tau}{\tau} K(\tau; 1) =$$

$$= \frac{\mathrm{Vol}_{dS}}{2V(S_n)} \int_0^\infty \frac{d\tau}{\tau} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(1) e^{-\frac{\tau}{\mu^2}(m^2 + V''(\bar{\phi}) + j(j+n-1)/l^2)} \tag{2.59}$$

where we have redefined the heat kernel in order to get a mass dimension 2 equation. Here $C_j^{\frac{n-1}{2}}(1) = \binom{j+n-2}{j}$. This expression, which is divergent,[6] is purely real (the $C_l^{\frac{n-1}{2}}(1)$ are integers), so no imaginary parts appear.

It seems plain that the analytic continuation, should it work at all, it not will do it term by term. The eigenvalues are not the same in the sphere as in de Sitter space, not

[6] General theorems imply that the trace of the heat kernel must diverge when $\tau \to 0$ as $K \sim \mu^n \tau^{-n/2}$. This just means that the sum and the integral do not commute.

to mention the fact that the sphere is a compact space whereas de Sitter is not. Nevertheless, there is a well-known *duality* between compact and non-compact symmetric spaces [28]. Some further caveats on the analytic continuation of the heat kernel have been made in [29]. It is true that until the whole sum is performed and then the explicit continuation is made, surprises may appear, so perhaps some wise restrain is called for.

Laplacian spectrum In the reference [30] the spectrum of the laplacian for de Sitter space, dS_n, anti de Sitter space AdS_n and euclidean (anti) de Sitter space $EAdS_n$ is computed and the eingenfunctions are constructed as well. The spectrum is identical[7] for both dS_n and AdS_n and has got a discrete part (similar to the one corresponding to the sphere)

$$-L\left(L+n-1\right)/l^2$$

where

$$L=-\left[\frac{n}{2}\right]+1,-\left[\frac{n}{2}\right]+2,\ldots-\left[\frac{n}{2}\right]+j\ldots$$

and we represent by $[z]$ the integer part of z. The starting point of the spectrum is actually the only difference between the sphere and both de Sitter and anti de Sitter spaces, as long as the discrete part of the said spectrum is concerned. In terms of $j\in\mathbb{N}$, for even dimension, $n=2m$, or else for odd dimension $n=2m+1$

$$L=-\frac{-j\left(j-1\right)+m\left(m-1\right)}{4l^2}$$

There is also a continuous piece of the spectrum, which can be written in the form

$$\frac{1}{l^2}\left(\Lambda^2+\frac{\left(n-1\right)^2}{4}\right)\text{ where }\Lambda\in[0,\infty)$$

In the case of $EAdS_n$ only the continuous spectrum appears. So the situation is as follows: the two euclidean spaces enjoy only one type of spectrum; discrete in the case of the sphere S_n and continuum in the case of $EAdS_n$; whereas the two manifolds with lorentzian signature (AdS_n and dS_n) carry both discrete and continuous spectra. In all cases the eigenvalues are of course real.

The eigenfunctions are explicitly known and can be find in the references just quoted. It is enough for our purposes though to point out that they obey a completeness relationship,

$$\sum_L Y_L(x)^*Y_L(y)+\int d\Lambda Z_\Lambda(x)^*Z_\Lambda(y)=\delta\left(x,y\right)\tag{2.60}$$

[7]Except for a sign perhaps, depending on the sign chosen for the metric for each space.

Let us nevertheless perform a simple approximation (in the case of the sphere; the other cases are very similar), just to get an idea of the result. We shall explore the high angular momentum region,

$$\sum_j j^{n-1} e^{-\frac{\tau}{\mu^2}(j+n-1)j/l^2} \sim \int_0^\infty dj j^{n-1} e^{-\frac{\tau}{\mu^2 l^2} j^2} = \frac{(\mu l)^n}{2\tau^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right)$$

We then get in this approximation

$$W \sim \mu^n l^n \int_{\frac{\mu^2}{\Lambda^2}}^\infty \frac{d\tau}{\tau^{1+\frac{n}{2}}} e^{-\frac{m^2+V''(\bar{\phi})}{\mu^2}\tau} = (m^2 l^2 + V''(\bar{\phi}) l^2)^{\frac{n}{2}} \, \Gamma\left(-\frac{n}{2}, \frac{m^2+V''(\bar{\phi})}{\Lambda^2}\right) =$$

$$= \begin{cases} \text{odd } n: \ 0 \\ \text{even } n: \ -\frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!} (m^2 l^2 + V''(\bar{\phi}) l^2)^{\frac{n}{2}} \log \frac{\Lambda^2}{m^2+V''(\bar{\phi})} + \frac{2\Lambda^n l^n}{n} + \ldots \end{cases} \tag{2.61}$$

Here, as in flat space, the only possible imaginary part comes from the logarithm, that is, when

$$\frac{m^2 + V''\left(\bar{\phi}\right)}{\mu^2} \leq 0$$

This is in agreement with general theorems [31] asserting that the only way a non vanishing imaginary part can appear in a manifestly real integral is from the region in which the integral diverges.

On the other hand, this is exactly the situation when spontaneous symmetry breaking occurs in flat space, that it is believed to be well understood.

Chapter 3

Unitarity and vacuum decay

In this chapter, we analyze the presence of an imaginary part in the free energy to two loops. Assuming some basic form of unitarity, it is showed how it can be established a direct relation between this imaginary part and the production of particles in "forbidden processes".

3.1 Unitarity relations

In an interesting series of papers, Bros, Epstein and Moschella [32] following early work [1] in [33] and [13], have shown that one particle decays in ϕ^3 or ϕ^4 theories are not forbidden kinematically in de Sitter space. Representing by $h(x)$ the scalar field, such decays imply a nonvanishing width

$$\Gamma\left(h \to hh\right) \text{ or else } \Gamma\left(h \to hhh\right)$$

This is in sharp contrast with the situation in flat space, where momentum conservation forbids them. The reason for that is the lack of translational invariance (there is no abelian translation subgroup of the de Sitter group; pseudotranslations do not commute), so that two-point functions are not necessarily functions of the difference between spacetime coordinates of the two points, which is the root of global momentum conservation in any physical process. In fact this effect is common to any quantum field theory in a nontrivial gravitational background.

Once momentum conservation is not working, nothing forbids the vacuum decay in to physical particles, which essentially related to the effect pointed out by A. Polyakov [19]. Assuming, as we do, *crossing symmetry*, the preceding channels are related to the

[1] In the book by Birrell and Davies [34] some earlier references can be found.

vacuum decay in the tree approximation

$$\Gamma\left(0 \to hhh\right) \text{ or else } \Gamma\left(0 \to hhhh\right)$$

Once there is a nonvanishing amplitude for this sort of decay into several particles, it seems plain that the inverse reaction is much less likely, so that there is an enhanced production until the particle density n is so high that

$$n^{1/3} \sim \Gamma$$

at which point detailed balance should establish itself and the particle production growing stops.

We will use the formalism for interacting quantum fields detailed in B.2. Even when S-matrix elements are not defined *sensu stricto* (such as in de Sitter space) transition amplitudes for finite time intervals can still be computed using Feynman's rules. Our point of view is similar to the one in [35] in that we assume that enough of the analytical scheme of flat space quantum field theory survives to justify the formal use of the interaction representation and related path integral techniques.

In any S-matrix perturbative framework, unitarity precisely relates the imaginary part of the vacuum diagrams to creation and absorption of physical particles from the vacuum,

$$S \equiv 1 + i\mathcal{T}$$

Unitarity means that for any couple of states $|a\rangle$ and $|b\rangle$, and any closure relation

$$\sum |n\rangle\langle n| = 1$$

the following is true

$$\langle a|b\rangle = \langle a|SS^{\dagger}|b\rangle = \langle a|\left(1 + i\mathcal{T}\right)\left(1 - i\mathcal{T}^{\dagger}\right)|b\rangle$$

so that

$$\langle a|i\left(\mathcal{T} - \mathcal{T}^{\dagger}\right)|b\rangle = -\langle a|\mathcal{T}\mathcal{T}^{\dagger}|b\rangle = -\sum_n \langle a|\mathcal{T}|n\rangle\langle n|\mathcal{T}^{\dagger}|b\rangle$$

In a $\lambda\phi^4$ theory, to second order we have for the vacuum-to-vacuum amplitude:

$$\langle \text{vac}|i\left(\mathcal{T}^{(2)} - \mathcal{T}^{(2)\dagger}\right)|\text{vac}\rangle = -\sum_n \langle \text{vac}|\mathcal{T}^{(1)}|n\rangle \cdot \langle n|\mathcal{T}^{(1)\dagger}|\text{vac}\rangle$$

where we do not specify the appropriate asymptotic limit (i.e. *in* or *out*), since the obtained result is identical

$$2\text{Im}\,\langle \text{vac}|\mathcal{T}^{(2)}|\text{vac}\rangle = \sum_n |\langle \text{vac}|\mathcal{T}^{(1)}|n\rangle|^2 \tag{3.1}$$

Up to second order, the $\mathcal{T}$ matrix for a $\lambda\phi^4$ theory is

$$S = T \exp\left\{-i\frac{\lambda}{4!}\int dx\ :\phi(x)^4:\right\}$$
$$\mathcal{T} = -\frac{\lambda}{4!}\int dx : \phi(x)^4 : +i\frac{\lambda^2}{2\cdot 4!^2}\int dxdy\, T(:\phi(x)^4 : \times : \phi(y)^4 :) + \dots \tag{3.2}$$

so the previous relation gives[2]

$$\mathrm{Re}\int dxdy\, G(x,y)^4 = \int dxdy\, W(x,y)^4 \tag{3.3}$$

where $G(x,y) \equiv \langle \mathrm{vac}|T\phi(x)\phi(y)|\mathrm{vac}\rangle$ is the "Feynman" propagator and $W(x,y) \equiv \langle \mathrm{vac}|\phi(x)\phi(y)|\mathrm{vac}\rangle$ the Wightman function. This algebraic relation, that is equally valid in the non-flat case, is the basis of our study.

3.2 Vacuum decay

In Poincaré coordinates the metric of de Sitter space reads

$$ds^2 = \left(\frac{l}{u}\right)^2 (du^2 - d\mathbf{x}^2)$$

A conformally coupled scalar field is massless and the value of the curvature coupling is $\xi = \frac{1}{4}\frac{n-2}{n-1}$, where n is the dimension of spacetime. In de Sitter space, where the curvature is constant, $R = \frac{n(n-1)}{l^2}$, this is equivalent to a minimally coupled ($\xi = 0$) scalar field with mass $m^2 = \frac{1}{4}\frac{n-2}{n-1}R = \frac{n(n-2)}{4l^2}$.

This mass is in the complementary series of the de Sitter group $SO(n,1)$, with a $i\mu = \frac{1}{2}$ parameter $(m^2 l^2 = \mu^2 + \frac{(n-1)^2}{4})$, and the functional form of their two-point functions (without entering into the $i\epsilon$ prescriptions for the time being) is particularly simple

$$\frac{\Gamma\left(\frac{n}{2}-1\right)}{l^{n-2}(4\pi)^{\frac{n}{2}}} F\left(\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}; \frac{1+z}{2}\right) = \frac{\Gamma\left(\frac{n}{2}-1\right)}{2(2\pi)^{\frac{n}{2}} l^{n-2}} (1-z)^{1-\frac{n}{2}}$$

Let us concentrate in this simplest example for the time being. In the coordinates we are using

$$1 - z = \frac{-\left(u-u'\right)^2 + \left(\vec{x}-\vec{x}'\right)^2}{2uu'} \tag{3.4}$$

[2] We ignore the tadpole contributions that disappear by considering normal order in the interaction term.

Exploiting the conformal invariance of the setup [3], we can expand the free field expansion in term of the n-dimensional modes

$$f_{\mathbf{k}}(u,\mathbf{x}) = \frac{u^{n/2-1}}{(2\pi)^{\frac{n-1}{2}} l^{n/2-1}\sqrt{2k}} e^{-iku} e^{i\mathbf{k}\mathbf{x}}$$

which are equivalent to choosing the Euclidean or Bunch-Davies vacuum.

It is a fact that the closure relation for the complete set of solutions gives a particular solution of the homogeneous equation (*id est*, without the delta function source) namely[4]

$$\begin{aligned}\int d^{n-1}\mathbf{k}\, f_{\mathbf{k}}(u,\mathbf{x})\, f^*_{\mathbf{k}}(u',\mathbf{x}') &= \frac{(uu')^{n-2}}{l^{n-2}} \int \frac{d^{n-1}\mathbf{k}}{(2\pi)^{n-1}2k} e^{-ik(u-u')} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} = \\ &= \frac{\Gamma\left(\frac{n}{2}-1\right)}{2(2\pi)^{\frac{n}{2}} l^{n-2}} \left(\frac{2uu'}{(i(u-u'-i\epsilon))^2 + r^2}\right)^{\frac{n}{2}-1} \qquad (3.7)\end{aligned}$$

where the integrations needs an small negative imaginary part in $u-u'$. This closure relation appears in the derivation of (3.3), and this particular homogeneous solution is precisely the Wightman function. It is plain in (3.7) that the $i\epsilon$ prescription depends on the time ordering of the arguments. The unitarity relations rely on this subtle difference with the Feynman propagator.

In the $\lambda\phi^4$ theory, the bubble diagram $G(x,y)^4$ can be computed for the conformally coupled case in 4 dimensions. The propagator is $\frac{1}{l^2}\frac{1}{z-1-i\epsilon}$ where z is the geodesic distance, times a constant we will ignore. Given that de Sitter space is homogeneous, the diagram is proportional to the infinite spacetime volume $V_4 \equiv \int d^4y\sqrt{\|g\|}$:

$$M_{0\to 0} = i\frac{\lambda^2}{2\cdot 4! l^8} V_4 \int d^4x \sqrt{\|g\|} \frac{1}{(z(x,x_0)-1-i\epsilon)^4}$$

[3] Broken by the interactions, however.

[4] The integrals can be done with

$$\int_{-1}^{1} dy (1-y^2)^{\frac{n-4}{2}} e^{iay} = \sqrt{\pi}\left(\frac{2}{a}\right)^{\frac{n-3}{2}} \Gamma\left(\frac{n}{2}-1\right) J_{\frac{n-3}{2}}(a) \qquad (3.5)$$

$$\int_0^\infty dx J_{\frac{n-3}{2}}(\beta x) e^{-\alpha x} x^{\frac{n-3}{2}} = \frac{(2\beta)^{\frac{n-3}{2}}\Gamma\left(\frac{n}{2}-1\right)}{\sqrt{\pi}(\alpha^2+\beta^2)^{\frac{n}{2}-1}} \qquad (3.6)$$

and we can choose x_0 as an arbitrary point. Choosing x_0 as the north pole $N = (0, l, 0, \ldots, 0)$, we get

$$M_{0\to 0} = i\frac{\lambda^2}{2\cdot 4!l^8}V_4\int d^4x\sqrt{\|g\|}\frac{1}{(z(x,x_0)-1-i\epsilon)^4} = i\frac{\lambda^2(4\pi)}{2\cdot 4!l^4}V_4\int dt\,d\theta\frac{\cosh^3 t\sin^2\theta}{(\cosh t\cos\theta - 1 - i\epsilon)^4}$$

To compute the imaginary part, we can use the well-known formula [36]

$$\frac{1}{(x-i0)^4} = \frac{1}{x^4} - i\frac{\pi}{6}\delta^{(\mathrm{iii})}(x)$$

It follows that

$$\begin{aligned}\mathrm{Im}M_{0\to 0} \quad &= \frac{\lambda^2(4\pi)}{2\cdot 4!l^4}V_4\int dt\,d\theta\,\frac{\cosh^3 t\sin^2\theta}{(\cosh t\cos\theta-1)^4} = \\ &= \frac{\lambda^2(4\pi)}{2\cdot 4!l^4}V_4 P\int\frac{dx}{x^4}d\theta\frac{\sin^2\,\theta}{\cos\,^3\theta}\frac{(1+x)^3\mathrm{sgn}(x,\theta)}{\sqrt{(1+x)^2-\cos^2\theta}}\end{aligned} \qquad (3.8)$$

where the principal part P regularizes the divergence at $x = 0$ and $\mathrm{sgn}(x,\theta)$ is zero except in the case $(1+x)\cos\theta > 0$. It seems clear that the angular integral diverges owing to the behaviour of the integrand in a neighborhood of $\theta = \frac{\pi}{2}$.

This is a very strange result indeed, because as we shall see in a moment, this is equivalent to a corresponding divergence in the vacuum decay amplitude *in the tree approximation*, which has been reported [37] to be finite in the literature. This interesting calculation will be reviewed later on.

Keeping ϵ finite, we get an exact result

$$M_{0\to 0} = i\frac{\lambda^2(4\pi)}{2\cdot 4!l^4}V_4\lim_{\epsilon\to 0^+}\mathrm{sgn}(\epsilon)\left(\frac{i\pi}{6\epsilon^2}+\frac{\pi}{6\epsilon}+\frac{i\pi}{8}+\ldots\right) \equiv i\frac{\lambda^2(4\pi)}{2\cdot 4!l^4}V_4\lim_{\epsilon\to 0^+}\cdot I(\epsilon) \qquad (3.9)$$

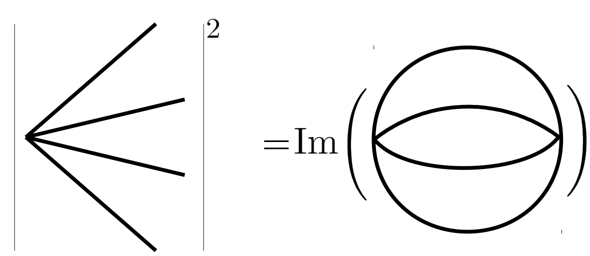

Figure 3.1: Unitarity relation in the vacuum-to-vacuum amplitude to second order.

This expression is divergent in the limit when $\epsilon \to 0^+$, and it has got a finite part in this regularization of sorts we are using. Of course we are aware that only the limit when $\epsilon \to 0^+$ has got any physical sense, so that the divergent parts are regularization dependent and just have to be renormalized away.

Let us now turn our attention to the computation of the vacuum decay rate. We use again the de Sitter homogeneity as we did for the vacuum bubble[5]. The corresponding integral contains Wightman functions instead of propagators

$$\begin{aligned}
&|M_{0\to 4}|^2 = \frac{\lambda^2 V_4}{4!} \int d^4x \sqrt{\|g\|} \prod_{i=1}^{4} d\mathbf{k}_i f_{\mathbf{k}_i}(x) f^*_{\mathbf{k}_i}(x_0) = \\
&= \frac{\lambda^2}{4! l^8} V_4 \int d^4x \sqrt{\|g\|}\, W(x,x_0)^4 = \frac{\lambda^2}{4! l^8} V_4 \Big(\int_{X^0>0} d^4x \sqrt{\|g\|}\, G(x,x_0)^4 + \\
&+ \int_{X^0<0} d^4x \sqrt{\|g\|}\, (G(x,x_0)^*)^4 \Big) = \frac{\lambda^2 (4\pi)}{4! l^4} V_4 \left(\frac{I(\epsilon)}{2} + \frac{I(-\epsilon)}{2} \right) = \frac{\lambda^2 (4\pi)}{4! l^4} V_4 \cdot \frac{\pi}{6|\epsilon|}
\end{aligned} \tag{3.10}$$

where we are again ignoring the concrete normalization factors of the two-point functions; and we have used that the integral of the propagator over *half* of the full hyperboloid (over $t \in [0,\infty)$) is precisely the integral in (3.9), $I(\epsilon)$, divided by two.

The vacuum decay matches exactly the imaginary part of the vacuum energy in the regularized theory (as in (3.1)), which is more than is necessary for unitarity, which does not necessarily hold for the regularized theory (*id est*, before taking the limit $\epsilon \to 0^+$ in our case).

It is worth remarking that the total *vacuum decay rate* per unit volume and unit time interval at tree level appears to be divergent in the $\epsilon \to 0$ limit. Given the fact that the corresponding tree level amplitude is indeed finite, the divergence is entirely due to the integration over the *phase space*. We are not aware of any such divergences, except the ones associated to bremsstrahlung corrections in the external legs [38]. It would be interesting to study what happens in the non-conformal case as well as to investigate whether this divergences are related to the ones found in [39] in a classical context.

In the next section we discuss a different way of computing the vacuum decay rate, in a form that results proportional to the covariant four-volume in a way consistent with Fermi's Golden Rule; there seems to be an ambiguity as to whether the Rule can

[5]The Wightman functions are not fully de Sitter invariant (not under time reversal). However this is enough to repeat the procedure.

be applied in arbitrary coordinates.

Indeed, in [19] some arguments are given for finiteness (after substraction) of the massive contribution to the imaginary part of the effective action, and in [13, 32] it is similarly argued for the finiteness of the *forbidden decay width*, which is, as is has been already argued for, a closely related quantity through crossing symmetry.

General case The result above is quite general, as these unitarity relations are "built-in" in the S matrix formalism. In an $\lambda\phi^n$ theory with an arbitrary mass, the relevant identity to second order is

$$\mathrm{Re}\int dxdy\, G(x,y)^n = \int dxdy\, W(x,y)^n \tag{3.11}$$

Using again the homogeneity of de Sitter, we obtain

$$\mathrm{Re}\int dx\, G(x,x_0)^n = \int dx\, W(x,x_0)^n \tag{3.12}$$

and the relationship between the propagator and the Wightman function $G(x,y) = T(W(x,y))$ allows us to decompose these integrals

$$\int dx\, G(x,x_0)^n = I^+ + I^- \ , \ \int dx\, W(x,x_0)^n = I^+ + (I^-)^* \tag{3.13}$$

where we separate de Sitter space in two regions, future and past of x_0, and those are their respective contributions. More in detail, the argument x can be in the future, the past or be causally disconnected from x_0. We are splitting artificially the spatial region in two pieces (respect to some time parameter) and including them in the two causal contributions. This procedure is correct, because both the propagator and the Wightman function have the same real value in that region. So now our identity looks

$$\mathrm{Im}I^+ = \mathrm{Im}I^- \tag{3.14}$$

The decomposition of the Feynman propagator is

$$G(x,y) = \frac{1}{2}(G^{(1)}(x,y) + i\sigma(x,y)D(x,y)),$$

where the Hadamard symmetric function $G^{(1)}$ and the Pauli-Jordan commutator function D are real, $\sigma(x,y)$ is the sign of the time ordering of x and y, and D is antisymmetric and zero for causally disconnected points, i.e. has the form $D(x,y) = \sigma(x,y)\Delta(x,y)$

with $\Delta(x,y)$ symmetric.

The imaginary part of the nth power of the propagator is then proportional to $i\Delta(x,y)$ times asymmetric function (built with odd powers of $G^{(1)}$ and even powers of D), in such a way that the integrand depends only on $z(x,x_0)$, and not on the sign of the time ordering, so the transformation $X^0 \to -X^0$ leaves unaltered the value of the integral. This proves the unitarity relation.

Largest time equation There is a formulation of unitarity called the *largest time equation*[6] which acts of position space Feynman diagrams themselves and, as such is most suitable for application in curved spacetimes. This set up is due to Veltman [40], and asserts that for any diagram $F\left(x_1,\dots,x_n\right)$,

$$2\mathrm{Re}\,F\left(x_1,\dots,x_n\right) = -\sum_{cuttings} F\left(x_1,\dots,x_n\right)$$

This formula stems from a representation of the Feynman propagator as

$$G_F\left(x,y\right) = \theta(t)G_F^+(x,y) + \theta(-t)G_F^-(x,y)$$

such that

$$\left(G_F^+(x,y)\right)^* = G_F^-(x,y)$$

Those conditions are fulfilled in our case. It is always possible to argue that all we are doing is to check the largest time equation, without necessarily committing ourselves to the thorny [16] issue of unitarity in de Sitter space.

3.2.1 Alternative computation of the vacuum decay rate.

The particle production in the conformally coupled case has been calculated in the expanding patch of de Sitter by A. Higuchi [37]. Let us review this calculation. In the conformally flat patch

$$ds^2 = \frac{l}{u}^2\left(du^2 - d\mathbf{x}^2\right)$$

For a conformally coupled scalar field, we can expand the free field expansion in term of the modes (we represent by $k \equiv |\mathbf{k}|$).

$$f_{\mathbf{k}}(u,\mathbf{x}) = \frac{u}{l(2\pi)^3\sqrt{2k}}e^{iku}e^{i\mathbf{k}\mathbf{x}}$$

[6] To be precise, the statement is that in flat space the largest time equation can be shown to imply unitarity of the S-matrix.

which are equivalent to choosing the Euclidean or Bunch-Davies vacuum. The normalization is such that $[a_1, a_2^\dagger] = (2\pi)^3\delta_{12}$. This expansion is the same for the interacting field in the interaction picture, so the amplitude $0 \to \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4$ is simply

$$
\begin{aligned}
M(\mathbf{k}_i) = \int_0^\infty du\, d\mathbf{x} \langle \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4| \frac{\lambda}{4!}\phi(u,\mathbf{x})|\text{vac}\rangle = \lambda(2\pi)^{12}\int du\, d\mathbf{x} \prod_{i=1}^4 f^*_{\mathbf{k}_i}(u,\mathbf{x}) = \\
= \lambda \int \frac{dud\mathbf{x}}{4\sqrt{k_1k_2k_3k_4}} e^{-iu\sum_i k_i} e^{-i\sum_i \mathbf{x}.\mathbf{k}_i} = \lambda(2\pi)^3 \int \frac{du}{4\sqrt{k_1k_2k_3k_4}} e^{-iu\sum_i k_i}\delta(\sum_i \mathbf{k}_i)
\end{aligned}
\tag{3.15}
$$

The total decay rate is

$$
P = \frac{1}{4!}\int \prod_{i=1}^4 \frac{d\mathbf{k}_i}{(2\pi)^3}|M(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)|^2 = \frac{\lambda^2(2\pi)^3 V_c}{4!(2\pi)^{3\cdot 4}16}\int dudv \prod_{i=1}^4 \frac{d\mathbf{k}_i}{k_i} e^{-i(u-v)\sum_i k_i}\delta(\sum\nolimits_i \vec{k}_i)
\tag{3.16}
$$

We consider that the infinite quantity $V_c = (2\pi)^3\delta(\mathbf{0})$ represents the volume of 3-space. The following changes of variables are considered:

$$
\begin{aligned}
u \equiv l\, e^{-\frac{t_1}{l}}, \quad 2T = t_1 + t_2 \\
v \equiv l\, e^{-\frac{t_2}{l}}, \quad \tau = t_1 - t_2;
\end{aligned}
\tag{3.17}
$$

yielding

$$
P = \frac{\lambda^2 V_c}{4!(2\pi)^9 16}\int_{-\infty}^{\infty} dT\, d\tau \prod_{i=1}^4 \frac{d\mathbf{k}_i}{k_i}\delta(\sum_{i=1}^4 \mathbf{k}_i) e^{-2\frac{T}{l}} \exp\left\{2il\sum_i k_i\, e^{-HT}\sinh\frac{\tau}{2l}\right\}
$$

In the reference we are annotating unity is introduced in the form $\int_0^\infty dK\delta(\sum_{i=1}^4 k_i - K) = 1$ and the momenta are normalized by $\mathbf{k}_i = K\mathbf{y}_i$, so that using

$$
\int \prod_{i=1}^4 \frac{d\mathbf{y}_i}{y_i}\delta(\sum_i y_i - 1)\delta(\sum_i \mathbf{y}_i) = \frac{\pi^3}{4}
$$

yields

$$
\begin{aligned}
P = \frac{\lambda^2 V_c}{3(8\pi)^6}\int dTd\tau dK K^4 e^{-2\frac{T}{l}} \exp\left\{\frac{2iK}{H}e^{-\frac{T}{l}}\sinh\frac{\tau}{2}l\right\} = \\
= \int dT e^{3\frac{T}{l}} V_c \left(\frac{\lambda^2}{48l^4(8\pi)^6}\int_0^\infty d\kappa\kappa^4 \int d\eta \exp(i\kappa\sinh\eta)\right)
\end{aligned}
$$

where $\kappa = 2Kle^{-\frac{T}{l}}$ and $\eta = \frac{\tau}{2l}$ the particle production rate per unit volume is

$$
\Gamma = \frac{\lambda^2}{48l^4(8\pi)^6}\int_0^\infty d\kappa\kappa^4 \int d\eta \exp(i\kappa\sinh\eta) = \frac{\lambda^2}{48l^4(8\pi)^6}\int_0^\infty d\kappa\kappa^4 2K_0(\kappa) = \frac{3\lambda^2}{4l^4(16\pi)^5}
$$

This formula is very appealing physically; it gives a finite result for a tree-level cross section, which is a good thing, because the only known place in which divergent cross sections appear at tree level is in bremsstrahlung effect (correction to external legs by emission of massless particles), and this is not our case. [7]

This means that whereas the calculation in section 2 was proportional to the covariant spacetime volume, V_4 (times a divergent expression), Higuchi's is proportional to V_c times the integral over $dTe^{3\frac{T}{l}}$ times a *finite* expression. If we identify

$$V_4 \sim V_3 \int dTe^{3\frac{T}{l}}$$

both calculations are inconsistent. But this identification is not compulsory. As has been already pointed out in [32] it is not clear in which coordinates the particle production per unit volume per unit time is to be defined.

Ay any rate, by consistency, in case the computation of [37] is preferred, the imaginary part of the free energy should also be finite.

3.3 Imaginary part in the vacuum energy

The standard lore on renormalization in Quantum Field Theory [41] includes the fact that the cosmological constant is additively renormalized away, the finite part being fully undetermined. There is an exception to this, however. In some cases, when nontrivial boundary conditions are imposed on the fields (like vanishing electromagnetic field in two fixed parallel plates in the classical example) the *difference* between the vacuum energy corresponding to nontrivial boundary conditions and the one with "trivial" boundary conditions is computable and in many cases, finite. Confer [42] for a clarifying review.

This effect is usually known as the Casimir effect and has nothing to do with the present work; boundary conditions are kept fixed ("trivial") in our setting.

Let us review this in the presence of an external gravitational field. In order to do that, we shall consider a simple model of a scalar quantum field

$$S = \int d^n x\sqrt{|g|}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{\lambda}{4!}\phi^4 - \frac{1}{2}\xi R\phi^2 + \alpha_3 R^2 - \alpha_1 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \alpha_2 R^{\mu\nu}R_{\mu\nu}\right)$$

[7] In the last formula the integral representation for the Bessel function $K_0(z)$ has been used. This is for $-\pi/2 \leq \arg z \leq \pi/2$

$$K_0(z) = \int_0^\infty e^{-z\,cosh\,t}dt = \int_0^\infty e^{iz\,sinh\,(t+i\pi/2)}dt = \int_{i\pi/2}^{\infty+i\pi/2} e^{iz\,sinh\,t}dt$$

which seems to require that $Im\,\eta = \pi/2$, which is not the case, but again the integral can be interpreted as the analytical continuation of a convergent one.

(the terms quadratic in the curvature are necessary for renormalization). We do not believe that our physical conclusions depend upon the details of the model. Standard computation of the one-loop effective potential [43] yields the result

$$V_{eff} = \Lambda_0 + \frac{1}{2}\xi R\bar{\phi}^2 - \alpha_3 R^2 - \alpha_1 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - \alpha_2 R^{\mu\nu}R_{\mu\nu} + \frac{1}{2}m_0^2\bar{\phi}_0^2 + \frac{\lambda_0}{4!}\bar{\phi}_0^4 +$$

$$\frac{\hbar}{32\pi^2}\left(\frac{1}{n-4} + \frac{\gamma}{2} - \frac{3}{4}\right)\left(m_0^2 + \frac{\lambda_0}{2}\bar{\phi}_0^2\right)^2 + \frac{\hbar}{64\pi^2}\left(m_0^2 + \frac{\lambda_0}{2}\bar{\phi}_0^2\right)^2 \log\frac{m_0^2 + \frac{\lambda_0}{2}\bar{\phi}_0^2}{4\pi\mu^2}$$

Defining renormalized quantities in the MS scheme

$$\Lambda_0 = \mu^{4+\epsilon}\left(\Lambda + \frac{a_\Lambda(\lambda)}{n-4}\right)$$

(where Λ is dimensionless) leads to

$$\beta_\Lambda = -4\Lambda + \frac{\hbar}{32\pi^2}\frac{m^4}{\mu^4}$$

The physical Cosmological Constant (CC) [8] is given by

$$\Lambda_{phys} \equiv \mu^4\Lambda$$

so that

$$\beta_{\Lambda_{phys}} = \frac{\hbar}{32\pi^2}m^4$$

Let us define the *cosmological constant*[9] in the MS scheme as

$$\Lambda(\lambda) \equiv V_{eff}(\bar{\phi} = 0)$$

[8] This is entirely analogous to 't Hooft's [44] beta function for the dimensionless mass

$$\beta_m = -m + \frac{\lambda m}{32\pi^2}$$

in such a way that the physical mass

$$m_{phys} \equiv m\mu$$

and

$$\beta_{m_{phys}} = \frac{\lambda m_{phys}}{32\pi^2}$$

[9] Of course the CC is put by hand equal to zero if we were to use Coleman-Weinberg's [45] renormalization condition

$$V_{eff}(\bar{\phi} = 0) = 0$$

Assuming $\frac{\partial V_{eff}}{\partial \bar{\phi}}\Big|_{\bar{\phi}=0} = 0$, that is, absence of spontaneous symmetry breaking, the CC obeys the renormalization group (RG) equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \beta_\xi\frac{\partial}{\partial\xi} + \sum_i \beta_{\alpha_i}\frac{\partial}{\partial\alpha_i}\right)\Lambda(\lambda) = 0$$

A successfully employed technique to compute the effective potential in many cases [35] stems precisely from using this equation in a recursive way. Feeding the RG equation with the MS one loop result

$$\Lambda^1 = \Lambda_0 + \frac{\hbar}{32\pi^2}\left(\frac{\gamma}{2} - \frac{3}{4}\right)m^4 + \frac{\hbar}{64\pi^2}m^4 \log \frac{m^2}{\mu^2}$$

and using the well-known one-loop beta functions and anomalous dimensions [10] yields the μ dependence of the two loop piece:

$$\Lambda^2 = \left(\frac{\lambda\hbar}{256\pi^4}\left(\frac{\gamma}{2} - \frac{3}{4}\right)m^4 + \frac{\hbar}{512\pi^4}\lambda m^4 \log \frac{m^2}{4\pi} + \frac{\hbar\lambda m^4}{1024\pi^4}\right)\log \mu -$$
$$\frac{\hbar}{512\pi^4}\lambda m^4 \left(\log \mu\right)^2 + f(g_i)$$

where g_i stands for *all* coupling constants not including the CC itself, but including the α_i; this piece is not determined by the RG equations.

No imaginary part appears ever by this procedure. Some ambiguity remains in the finite part independent of the coupling constants, which is usually fixed by the renormalization conditions, that is, renormalized to zero ([35]).

Following the logic of [19] and the previous section *generically*, owing to lacking of translation invariance, the matter vacuum is unstable (that is, it can decay to physical particles with a certain computable width) so that by unitarity the vacuum energy must have a unambiguous finite imaginary part. Assuming analyticity in the coupling constants, this in turn puts restrictions on the possible real part of the cosmological constant through the dispersion relations as a consequence of Cauchy's theorem. This explicitly contradicts previous claims in the literature [46].

[10] To wit:

$$\beta_{\Lambda_{phys}} = \frac{\hbar}{32\pi^2}m^4\ , \quad \beta_{m_{phys}} = \frac{\lambda m_{phys}}{32\pi^2}\ , \quad \beta_\lambda = \frac{3}{16\pi^2}\lambda^2$$
$$\beta_\xi = \frac{\lambda}{16\pi^2}\left(\xi - \frac{1}{6}\right)\ , \quad \beta_{\alpha_1} = -\frac{1}{180}\frac{1}{16\pi^2}\ , \quad \beta_{\alpha_2} = \frac{1}{180}\frac{1}{16\pi^2}$$
$$\beta_{\alpha_3} = -\frac{1}{72}\frac{1}{16\pi^2}\frac{\xi}{2}\left(\xi - \frac{1}{3}\right) \tag{3.18}$$

Another consequence of all this is that the usual analysis of quantum field theory in curved space-times is incomplete without a careful self-consistent consideration of the back-reaction problem, owing precisely to this imaginary part. We shall comment on this point in the conclusions.

Chapter 4

Unstable vacuum in flat space

We present in this chapter a toy model in flat space coupled to an external source. In this way we can examine the phenomenon of the vacuum decay in a simple framework. The particle production is studied through the kinematical equations of the model.

4.1 Toy model for vacuum decay

We have just witnessed a few paragraphs ago that explicit computations even in the simplest of background spacetimes, such as constant curvature de Sitter or anti de Sitter spaces, are quite involved [32], and besides, the physical interpretation of the divergences found is not clear.

This is the rationale for first performing the complete analysis in a much simpler context in flat space, a *toy model* of sorts. It is clear that the main new ingredient of a curved space in this context is the non-conservation of four momentum. This can be achieved through an explicit time-dependent interaction term in flat space which violates energy-conservation. The mapping between a subclass of scalar models in Friedmann spacetimes and flat models with spacetime dependent coupling constants is spelled out at the end of the section.

Let be the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda(t)}{3!}\phi^3 \tag{4.1}$$

with a non-homogeneous coupling $\lambda(t) = \frac{\lambda}{2}(1 + \cos\eta t)$. With this choice, energy is no longer conserved and a fixed amount (η) can be created or destroyed at any vertex. In the limit $\eta \to 0$ we recover an standard $\lambda\phi^3$ theory.

With this interaction, the second order vacuum-to-vacuum amplitude reads:

$$S = T \exp\left\{-\frac{i}{3!}\int \lambda(x^0) : \phi(x)^3 : dx\right\} \tag{4.2}$$

$$\begin{aligned}\langle \mathrm{vac}|S^{(2)}|\mathrm{vac}\rangle &= -\frac{1}{2\cdot 3!}\int \lambda(x^0)\lambda(y^0)G(x,y)^3\, dxdy = \\ &= i\frac{(2\pi)^3 V\lambda^2}{8\cdot 3!}\int dx^0 dy^0 \left\{\frac{dp_i}{(2\pi)^4}\right\}(1+\cos\eta x^0)(1+\cos\eta y^0)\frac{\delta^3(\mathbf{p}_T)\, e^{ip_T^0(x^0-y^0)}}{D_{p_1}D_{p_2}D_{p_3}}\end{aligned} \tag{4.3}$$

where we abbreviate by D_p the $p^2 - m^2 + i\epsilon$ denominators. $\{dp_i\}$ and p_T mean $\prod_i dp_i$ and $\sum_i p_i$; V is an infinite volume factor.

The time dependent coupling can be expressed as proportional to $1+\frac{1}{2}e^{i\eta x^0}+\frac{1}{2}e^{-i\eta x^0}$, so the "energy conservation" factors are now different

$$\begin{aligned}&\langle \mathrm{vac}|S^{(2)}|\mathrm{vac}\rangle = \\ &\quad = i\frac{(2\pi)^4 VT\lambda^2}{8\cdot 3!}\int\left\{\frac{dp_i}{(2\pi)^4}\right\}\frac{\delta^3(\mathbf{p}_T)}{D_{p_1}D_{p_2}D_{p_3}}\left(\delta(p_T^0)+\frac{1}{4}(\delta(p_T^0-\eta)+\delta(p_T^0+\eta))\right) = \\ &\quad = i\frac{(2\pi)^4 VT\lambda^2}{8\cdot 3!}\left(T_{234}(0,m,m,m)+\frac{1}{2}T_{234}(\eta^2,m,m,m)\right)\end{aligned} \tag{4.4}$$

so the creation of energy at one vertex should be compensated in the other. T is an infinite time factor. The standard integral T_{234} is well known [47] in general dimension:

$$T_{234}(p^2,m_2,m_3,m_4) = \int \frac{d^n k_2 d^n k_3 d^n k_4}{(2\pi)^{3n}}\frac{\delta^n(k_1+k_2+k_3-p)}{(k_2^2-m_2^2+i\epsilon)(k_3^2-m_3^2+i\epsilon)(k_4^2-m_4^2+i\epsilon)} \tag{4.5}$$

and corresponds to the self energy "setting sun" diagram $\Sigma(\eta^2)$ in a $\lambda\phi^4$ theory.

Notice that we have assumed that $\eta \neq 0$, so the regular $\lambda\phi^3$ theory result is not recovered when $\eta \to 0$ in the last formula, i.e. the limit is discontinuous.

The corresponding vacuum decay rate can be calculated as the square of the T-matrix up to first order, $\mathcal{T}^{(1)} = -\frac{1}{3!}\int dx\lambda(x^0) : \phi(x)^3 :$

$$\begin{aligned}\Gamma_{0\to 3} &= \langle \mathrm{vac}\rangle|\mathcal{T}^{(1)}\mathcal{T}^{(1)\dagger}|\mathrm{vac}\rangle = \\ &= \frac{1}{3!^2}\int dxdy\langle \mathrm{vac}| : \phi(x)^3 :: \phi(y)^3 : |\mathrm{vac}\rangle = \frac{1}{3!}\int \lambda(x^0)\lambda(y^0)W(x,y)^3\, dxdy\end{aligned} \tag{4.6}$$

We ca use now the expression for the Wightman function of the scalar field

$$W(x,y) = \int \frac{dp}{(2\pi)^4}(2\pi)\delta_+(p^2-m^2)e^{-ip(x-y)} = \int \frac{d\mathbf{p}}{(2\pi)^3 2E_p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} e^{-iE_p(x^0-y^0)} \quad (4.7)$$

and then

$$\begin{aligned}
\Gamma_{0\to 3} &= \frac{(2\pi)^3 V\lambda^2}{3!4}\int dx^0 dy^0 \left\{\frac{(2\pi)\delta_+(p_i^2-m^2)dp_i}{(2\pi)^4}\right\}(1+\cos\eta x^0)(1+\cos\eta y^0)e^{-ip_T^0(x^0-y^0)}\delta^3(\mathbf{p}_T) = \\
&= \frac{(2\pi)^4 VT\lambda^2}{3!4}\int \frac{(2\pi)\delta_+(p_i^2-m^2)dp_i}{(2\pi)^4}\left(\delta(p_T^0)+\frac{1}{4}(\delta(p_T^0-\eta)+\delta(p_T^0+\eta))\right)\delta^3(\mathbf{p}_T) = \\
&= \frac{(2\pi)^4 VT\lambda^2}{3!16}\int \frac{(2\pi)\delta_+(p_i^2-m^2)dp_i}{(2\pi)^4}\delta(p_T^0-\eta)\delta^3(\mathbf{p}_T) = \\
&= \frac{(2\pi)^4 VT\lambda^2}{3!16}\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^9 8E_1E_2E_{12}}\delta(E_1+E_2+E_{12}-\eta) \qquad (4.8)
\end{aligned}$$

where $E_{12} = \sqrt{(\mathbf{p}_1+\mathbf{p}_2)^2+m^2}$. From the three delta function factors for te energy, only the one with positive increment of energy contributes, for kinematical reasons.

This expression is just the standard three-body phase space factor

$$\Gamma_{0\to 3} = \frac{(2\pi)^4 VT\lambda^2}{3!16}\int \frac{dE_1 dE_2}{4(2\pi)^7}\theta(E_T^{\max}-\eta)\theta(\eta-E_T^{\min}) \qquad (4.9)$$

where $E^{\max}_{\min} = E_1+E_2+\sqrt{(p_1\pm p_2)^2+m^2}$ correspond to parallel and anti-parallel configurations of momenta. There is also an implicit kinematic factor $\theta(\eta-3m)$ indicating the minimun energy necessary to create 3 particles.

The final expression for this rate is

$$\Gamma_{0\to 3} = \frac{(2\pi)^4 VT\lambda^2}{3!16}\int_m^{\frac{\eta^2-3m^2}{2\eta}} \frac{dE}{4(2\pi)^7} I(E,\eta) \qquad (4.10)$$

with

$$I(E,\eta) = \frac{\sqrt{(E^2-m^2)(\eta^2+m^2-2E\eta)(\eta^2-3m^2-2\eta E)}}{m^2+\eta^2-2\eta E} \qquad (4.11)$$

This rate is proportional to the decay rate of a particle with mass η, $\Gamma_{M=\eta}$. We can give he approximate result of this integral if η is very close to or much bigger than $3m$:

$$\int_m^{\frac{\eta^2-3m^2}{2\eta}} dEI(E,\eta) \simeq 2\sqrt{3}m^2\epsilon^2 \text{ , if } \eta = 3m(1+\epsilon) \qquad (4.12)$$

$$\int_m^{\frac{\eta^2-3m^2}{2\eta}} dEI(E,\eta) \simeq \frac{\eta^2}{8} \text{ , if } \eta \gg m \tag{4.13}$$

From the application of Cutkosky's rules, the "setting sun" self-energy diagram has an imaginary part given by the corresponding decay rate, $\operatorname{Im}\Sigma(\eta^2) = \Gamma_{M=\eta}$, which is based in the identity [47]

$$\int_m^{\frac{\eta^2-3m^2}{2\eta}} dE\, I(E,\eta) = \frac{1}{4\pi^5}\operatorname{Im}\left((2\pi)^{3n}T_{234}\right)^{\text{fin}}\Big|_{n=4} \tag{4.14}$$

Since the processes in our model are directly related to the aforementioned diagrams, we can establish easily the unitarity relation

$$-2\text{Re}\langle \text{vac}|S^{(2)}|\text{vac}\rangle = \Gamma_{0\to 3} \tag{4.15}$$

In the previous calculation, we avoided to deal with the tadpole diagrams by taking the interaction operator to be normal ordered. If we choose to include those extra diagrams, the unitarity relation still holds. The tadpole contribution in (4.15) is then:

$$-2\text{Re}\langle \text{vac}|S^{(2)}|\text{vac}\rangle = \Gamma_{0\to 3} + \Gamma_{0\to 1} \tag{4.16}$$

Here the left member acquires a new term which is

$$\langle \text{vac}|S^{(2)}_{\text{tad}}|\text{vac}\rangle = -\frac{1}{8}\int dxdy\lambda(x^0)\lambda(y^0)G(x,x)G(y,y)G(x,y) \tag{4.17}$$

The closed loops $G(x,x)$ and $G(y,y)$ are constants, and they are always real (at least in dimensional regularization) and so, not very relevant for the calculation.

$$\begin{aligned}\langle \text{vac}|S^{(2)}_{\text{tad}}|\text{vac}\rangle &= -\frac{i}{8\cdot 16}G_0^2\int \frac{dxdydp}{(2\pi)^4}(2+2\cos\eta x^0)(2+2\cos\eta y^0)\frac{e^{ip(x-y)}}{D_p} = \\ &= -\frac{i(2\pi)^2V_3}{8\cdot 16}G_0^2\int dx^0dy^0dp^0(2+2\cos\eta x^0)(2+2\cos\eta y^0)\frac{e^{ip^0(x^0-y^0)}}{(p^0)^2-m^2+i\epsilon} = \\ &= -\frac{i(2\pi)^4V_4}{8\cdot 16}G_0^2\left(\frac{4}{m^2}+\frac{2}{\eta^2-m^2+i\epsilon}\right)\end{aligned} \tag{4.18}$$

In the other hand, the total decay rate $\Gamma_{0\to 1}$ is

$$\begin{aligned}\Gamma_{0\to 1} &= \frac{1}{3!^2}\sum_{1-\text{particle}}\int dxdy\lambda(x^0)\lambda(y^0)\langle \text{vac}|\phi(x)^3|1\rangle\langle 1|\phi(y)^3|\text{vac}\rangle = \frac{1}{4}G_0^2\int dxdy\lambda(x^0)\lambda(y^0)W(x,y) = \\ &= \frac{1}{64}G_0^2\int\frac{dxdyd\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}(2+2\cos\eta x^0)(2+2\cos\eta y^0)e^{-iE_{\mathbf{p}}(x^0-y^0)} = \\ &= \frac{(2\pi)^5V_4}{64}G_0^2\frac{1}{2m}\left(\delta(\eta-m)+\delta(\eta+m)\right)\end{aligned} \tag{4.19}$$

We have assumed $W(x,x) = G(x,x)$.

These new contributions to (4.15) should match each other. It is easy to check that they actually do by applying the Weierstrass theorem

$$\frac{i}{\eta^2 - m^2 + i\epsilon} = \frac{\pi}{2m}\left(\delta(\eta - m) + \delta(\eta + m)\right) + \ldots \tag{4.20}$$

where the dots indicate imaginary contributions. No infinities nor ambiguities have been encountered in our toy model.

4.1.1 Time dependent coupling and scale factor

Let us look for the cases in which a Minkowski space lagrangian with time-dependent coupling constants such as

$$L = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{m(t)^2}{2}\psi^2 - \frac{\lambda(t)}{6}\psi^3$$

is equivalent to a scalar field in a curved, conformally flat gravitational background

$$ds^2 = a(t)^2\eta_{\mu\nu}dx^\mu dx^\nu$$

namely

$$L = \sqrt{\|g\|}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{6}\phi^3\right) = \frac{a^2}{2}\left(\dot{\phi}^2 - (\phi')^2\right) - a^4\frac{m^2}{2}\phi^2 - a^4\frac{\lambda}{6}\phi^3$$

This is always possible, up to a total derivative, with the identifications

$$\begin{aligned}
&\psi = a\phi \\
&\lambda(t) \equiv \lambda a(t) \\
&m^2(t) \equiv m^2a^2 - \frac{\ddot{a}}{a}
\end{aligned} \tag{4.21}$$

It would also be interesting to find a mapping for the different vacuum states as well.

4.2 Kinetic equations

The effects shown in the previous section, allowed because of the non-conservation of energy, led to the production of particles in the initially empty space. In our very simplified model there are no conserved quantum numbers; but in general the set of produced particles must enjoy the quantum numbers (*id est*, charges) of the vacuum. An important physical question is towards which final state this instability leads to? Again, this question has many faces. In the general case in which a gravitational external field is present, the backreaction is surely important, but difficult to compute. It is not even clear that the usual procedure of solving again Einstein's equations with a second member given by the expectation value of some energy-momentum tensor [48] would be good enough for our purposes.

A second facet of this problem is the evolution of the particle (in general, conserved quantities) density. We can study this phenomenon by considering a spacetime box $V \times T$, with an initial number of particles (in our simplified model there is only one type of particles) N, do that the initial density is $n \equiv \frac{N}{V}$. [1]

It is well-known [49] that the transition *rate* $d\Gamma\,(\alpha \rightarrow \beta) \equiv \frac{dP(\alpha\rightarrow\beta)}{T}$ of a given process in which N_α particles in the initial state evolve into N_β particles in the final state depends on the three-space volume, V, as V^{1-N_α}. This clearly means that the vacuum decay terms in Boltzmann's equation (or whatever improvement thereof) will clearly dominate, because they are the only ones which are *extensive*, *id est*, proportional to the ordinary volume. Let us build up a simplified model for this equation. The essential thing is to capture the volume as well as the power of the distribution function itself. Taking into account the vacuum decay and absorption only, the balance equation in our fiducial spacetime box is of the schematic form

$$\frac{dN_{\mathbf{p}}}{dt} = \Gamma^{\mathbf{p}}_{03} - \Gamma^{\mathbf{p}}_{30}$$

Here $N_{\mathbf{p}}\, d\mathbf{p}$ represents the total number of particles with momentum between $\mathbf{p}$ and $\mathbf{p} + d\mathbf{p}$ in the fiducial volume V, and T is our fiducial time. The constructs $\Gamma^{\mathbf{p}}_{03}$ and $\Gamma^{\mathbf{p}}_{30}$ denote the amplitude for vacuum decay or annihilation to the vacuum when precisely one of the particles has momentum $\mathbf{p}$. The transition rate is simply related to the transition probability $dP(\alpha \rightarrow \beta)$ by $d\Gamma(\alpha \rightarrow \beta) = \frac{dP(\alpha\rightarrow\beta)}{T}$. General arguments implicate

$$d\Gamma(\alpha \rightarrow \beta) = (2\pi)^{3N_\alpha - 2} V^{1-N_\alpha} |M_{\beta\alpha}|^2 \delta^4(p_\alpha - p_\beta) d\beta$$

where the matrix M is intimately related to the matrix S:

$$S_{\beta\alpha} \equiv -2\pi i \delta^4(p_\alpha - p_\beta) M_{\beta\alpha}$$

[1] A finer description would be provided by the Wigner function, in which both momentum and position distributions are correlated, to the extent that this is compatible with Heisenberg's principle. This construct obeys suitable generalizations of Boltzmann's equation.

and

$$d\beta \equiv \prod_{i \in N_\beta} d\mathbf{p}_i$$

With this proviso, the mass dimension of $|M|^2$ is $8 - 3(N_\alpha + N_\beta)$.

It is now clear that the increase in the density of particles of momentum $\mathbf{p}$ out of the vacuum in the fiducial volume V during the time interval T is

$$\Gamma_{03}^{\mathbf{p}} \equiv \int d\Gamma_{03}(\mathbf{p}, \mathbf{p}_2', \mathbf{p}_3')\, d\mathbf{p}_2'\, d\mathbf{p}_3'(1 + N_{\mathbf{p}})(1 + N_{\mathbf{p}_2})(1 + N_{\mathbf{p}_3}).$$

The decrease in the density of particles of momentum $\mathbf{p}$ due to annihilation to the vacuum (*id est*, the reverse process) in the same fiducial volume and time interval is proportional to the existing density cube of particles:

$$\Gamma_{30}^{\mathbf{p}} \equiv \int N_{\mathbf{p}} N_{\mathbf{p}_2} N_{\mathbf{p}_3}\, d\mathbf{p}_2\, d\mathbf{p}_3\, d\Gamma_{30}\,(\mathbf{p}, \mathbf{p}_2, \mathbf{p}_3)$$

The total (integrated) widths are given by

$$\Gamma_{03} = \frac{V\lambda^2}{3!64(2\pi)^3} \int_m^{\frac{\eta^2 - 3m^2}{2\eta}} dE\, I(E, \eta) \simeq \frac{V\lambda^2}{\sqrt{3}\,64(2\pi)^3} m^2 \epsilon^2 \tag{4.22}$$

where the last equality holds approximately when $\eta = 3m(1 + \epsilon)$.

$$\Gamma_{30} = \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 N_{\mathbf{p}_1} N_{\mathbf{p}_2} N_{\mathbf{p}_3} d\Gamma_{30}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) =$$
$$= \frac{\lambda^2 (2\pi)^4}{16 V^2} \int N_{\mathbf{p}_1} N_{\mathbf{p}_2} N_{\mathbf{p}_3} \frac{d\mathbf{p}_1}{2E_1} \frac{d\mathbf{p}_2}{2E_2} \frac{d\mathbf{p}_3}{2E_3} \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{4.23}$$

A toy model that embodies some of the characteristics of our integro-differential equation is

$$\frac{dn}{dt} = C\left(M^4 - \frac{n^3}{M^5}\right)$$

where C is a constant and we have saturated all dimensions with an average mass scale, M. It is clear that in any such model the density of particles will rapidly grow as

$$n \sim C M^4 t$$

until it becomes so big that

$$n \sim M^3$$

which happens in a characteristic time

$$\tau \equiv \frac{1}{CM}$$

Chapter 5

Conclusions and Outlook

In this work, we have examined the possible hints of an instability of de Sitter space given by the behavior of an interacting theory living on it. It is not clear whether such a phenomenon could be still present in a more realistic set, such as a quasi-de Sitter phase in the Universe. Still, the possibility of a runaway particle production has an intrinsic interest from the theoretical point of view, as valuable as the black hole evaporation.

The eternity proposal by Polyakov has been reformulated from a different perspective. The guiding principle for the quantum theory (the composition principle) has been enlighted in terms of the heat kernel formalism. In addition to this, a careful classification of invariant bifunctions for the generalized spheres has been presented. This allowed us to examine the properties of the propagators in the light of this new guiding principle. With these tools, we have derived the absence of any imaginary part in the effective potential for an interacting scalar in de Sitter to one loop order.

However, this effect is present in higher orders. We have shown that there is a connection between the imaginary part of the vacuum energy, and a dynamical mechanism for its decay. This is a consequence of the unitarity of the quantum theory. The unitarity rules, as in the flat case for the Cutkosky's rules, have been presented as a valuable tool to check the consistency of the different results regarding particle production and propagation amplitudes. A simple toy model in flat space has been introduced as an example of this technique. Also the particle production of the vacuum has been calculated (in an special case) and compared with previous results.

There are still open questions concerning the internal consistency of the theory of interacting fields in de Sitter and its possible consequences in cosmology. We can summarize some of the possible future developments:

- There is necessity of clear, well-defined observable quantities, that can be computed unambiguously. The scattering matrix relies on certain assumptions over the asymptotics of the space in order to define it. This assumptions do not seem to fit quite well with the eternal expansion of de Sitter. Computations of finite time interactions could be very enlightening.

- Related to the previous point, the behaviour of interaction amplitudes in de Sitter is very different from the flat case. As remarked in [32], the Fermi's Golden Rule works in some particular cases. However, this matter deserves a careful examination.

- In general, the problem of the backreaction of quantum systems over classical backgrounds is still poorly understood, even for toy models. The ultimate consequence of the imaginary part of the vacuum energy is that we cannot describe a de Sitter phase indefinitely in time, without taking into account this backreaction. This is completely analogous to the problem of describing black hole evaporation. This direction necessarily points to the "mother" theory of Quantum Gravity.

Work in these matters is in progress.

Conclusiones y Perspectivas

En este trabajo, hemos examinado las posibles indicaciones de una inestabilidad en De Sitter, dada por la dinámica de una teoría en interacción definida sobre el mismo. No está claro si este fenómeno podría estar presente en circunstancias más realistas, como una fase cuasi-De Sitter en el Universo. Aun así, la posibilidad de una producción de partículas exponencial tiene interés intrínseco desde un punto de vista teórico, tan valiosa como la evaporación de los agujeros negros.

La propuesta del "eternity test" de A. Polyakov ha sido reformulada desde una perspectiva diferente. El principio guía para la teoría cuántica (el principio de composición) ha sido clarificado en términos del formalismo de *heat kernel*. Además, se ha presentado una clasificación cuidadosa de las funciones escalares invariantes en las esferas generalizadas. Esto nos ha permitido examinar las propiedades de los propagadores a la luz de este nuevo principio guía. Con estas herraminetas, hemos demostrado la ausencia de una parte imaginaria en el potencial efectivo de un escalar en interacción en De Sitter a un *loop*.

Sin embargo este efecto está presente en órdenes superiores. Hemos mostrado que hay una conexión entre la parte imaginaria de la energía del vacío, y el mecanismo dinámico para su decaimiento. Esto es consecuencia de la unitariedad de la teoría cuántica. Hemos presentado estas reglas de unitariedad, como en el caso plano con las reglas de Cutkosky, como una herramienta valiosa para comparar distintos cálculos de producción de partículas y amplitudes de propagación. Hemos introducido un *toy model* sencillo como ejemplo de esta técnica. También se han mostrado cálculos de producción de partículas (en un caso especial) y comparado con resultados previos.

Hay aún cuestiones abiertas concernientes a la consistencia de la teoría de campos en interacción en De Sitter, y sus posibles consecuencias en cosmología. Podemos resumir algunos de los posibles futuros desarrollos:

- Es necesario encontrar observables que puedan ser calculados sin ambigüedad. La matriz de *scattering* necesita hipótesis complicadas de justificar para ser definida. Dichas hipótesis no parecen encajar con la expansión eterna de De

Sitter. Cálculos para interacciones a tiempo finito podrían ser muy esclarecedores.

- Relacionado con el punto anterior, el comportamiento de las amplitudes de interacción en De Sitter es muy diferente al caso plano. Como se enfatiza en [32], la regla de oro de Fermi funciona en algunos casos particulares. Sin embargo, esta cuestión merece ser estudiada en detalle.

- En general, el entendimiento del problema de la *backreaction* de sistemas cuánticos sobre fuentes clásicas es insatisfactorio. La consecuencia última de una parte imaginaria de la energía del vacío es que no podemos describir una fase De Sitter indefinidamente en el tiempo, sin tener esta *backreaction* en cuenta. Esto es completamente aálogo al problema de describir la evaporación de un agujero negro. Necesariamente, esta dirección apunta a la teoría "madre", la gravedad cuántica.

Actualmente estamos trabajando en estas cuestiones.

Appendix A

Spaces of constant curvature

A.1 Taxonomy of the complex sphere

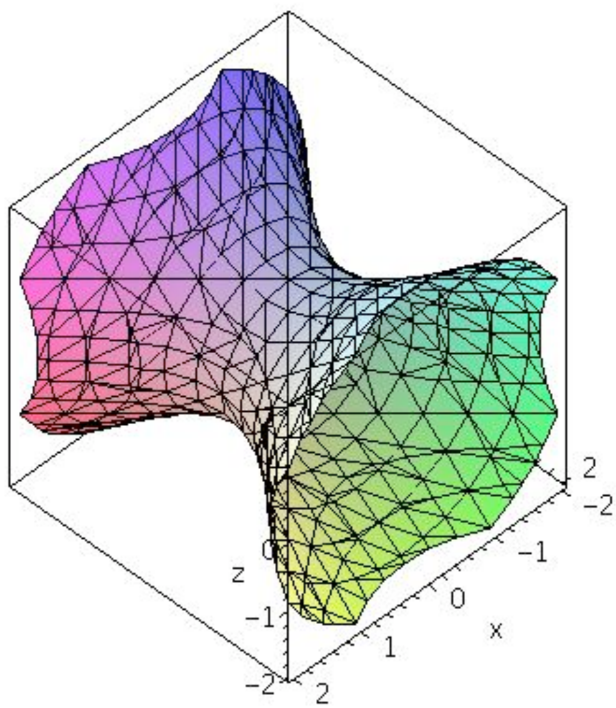

Figure A.1: A pictorial representation of Anti de Sitter ($X_0^2 + X_1^2 = l^2 + \vec{X}^2$ in $\mathbb{R}^{n+1}_{n-1}$).

The real sections of the complex sphere can be treated in an unified way. Let us choose coordinates in the embedding space in such a way that in the defining equation we have

$$X^2 = \sum_{A=0}^{n} \epsilon_A X_A^2 \equiv \eta_{AB} dX^A dX^B = \pm l^2 \tag{A.1}$$

on a flat space with metric $ds^2 = \eta_{AB} dX^A dX^B$. If we change in an arbitrary manifold $g_{AB} \to -g_{AB}$, then both Christoffels and Riemann tensor remain invariant, but the scalar curvature flips sign $R \to -R$. We can furthermore group together times and spaces, in such a way that

$$\eta_{AB} = (1^t, (-1)^s) \tag{A.2}$$

If we call $n + 1 \equiv t + s$, then this ambient space is Wolf's $\mathbb{R}_s^{n+1}$ where the subindex indicates the number of *spaces*.

The standard nomenclature in Wolf's book [50] is

$$\begin{aligned} S_s^n &: X \in \mathbb{R}_s^{n+1},\, X^2 = l^2 \\ H_s^n &: X \in \mathbb{R}_{s+1}^{n+1},\, X^2 = -l^2 \end{aligned} \tag{A.3}$$

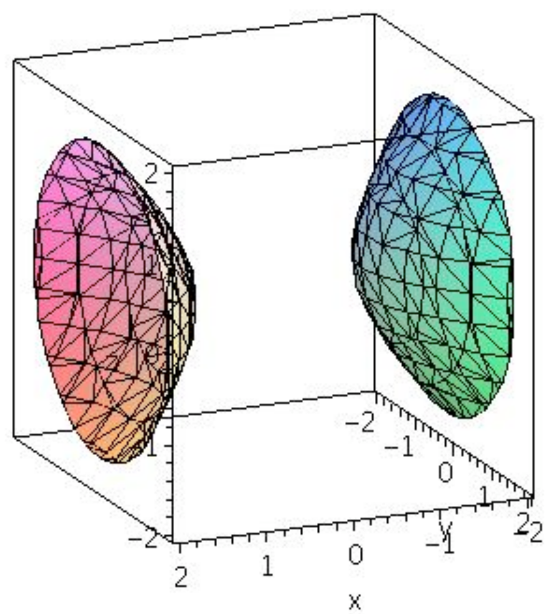

Figure A.2: A pictorial representation of euclidean Anti de Sitter ($X_0^2 + X_1^2 = l^2 + \vec{X}^2$ in $\mathbb{R}_{n-1}^{n+1}$).

The curvature scalar is given by:

$$R = \pm \frac{n(n-1)}{l^2} \tag{A.4}$$

and

$$
\begin{aligned}
R_{\mu\nu} &= \pm\frac{n-1}{l^2} g_{\mu\nu} \\
R_{\mu\nu\rho\sigma} &= \pm\frac{1}{l^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\rho}g_{\nu\sigma}\right)
\end{aligned} \tag{A.5}
$$

Please note that the curvature only depends on the sign on the second member, and not on the signs ϵ_A themselves.

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra $SO(n+1)$. The Killing vector fields are explicitly given (no sum in the definition) by

$$
L_{AB} \equiv \epsilon_A X^A \partial_B - \epsilon_B X^B \partial_A \equiv X_A \partial_B - X_B \partial_A \tag{A.6}
$$

The square of the corresponding Killing vector is

$$
L^2 = \epsilon_B X_A^2 + \epsilon_A X_B^2 \tag{A.7}
$$

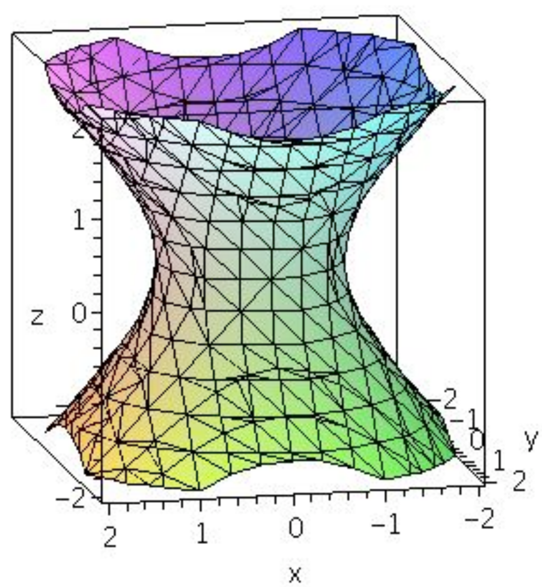

Figure A.3: A pictorial representation of de Sitter ($X_0^2 = -l^2 + \vec{X}^2$) in $\mathbb{R}_n^{n+1}$).

Our interest is concentrated on the euclidean and minkowskian cases:

- The sphere $S_n \equiv S_0^n \sim H_n^n$ is defined by $\vec{X}^2 = l^2$, with isometry group $SO(n+1)$.
- The euclidean Anti de Sitter (or euclidean de Sitter) $EAdS_n \equiv S_n^n \sim H_0^n$ is defined by $(X^0)^2 - \vec{X}^2 = l^2$, with isometry group $SO(1,n)$.
- The de Sitter space $dS_n \equiv H_{n-1}^n \sim S_1^n$ is defined by $(X^0)^2 - \vec{X}^2 = -l^2$, with isometry group $SO(1,n)$. In our conventions de Sitter has negative curvature, but positive cosmological constant.
- The Anti de Sitter space $AdS_n \equiv S_{n-1}^n \equiv H_1^n$ is defined by $(X^0)^2 + (X^1)^2 - \vec{X}^2 = l^2$, with isometry group $SO(2, n-1)$. For us AdS_n has positive curvature and negative cosmological constant.

A.1.1 Global coordinates

A very useful coordinate chart for these spaces is the one called *global* coordinates, wich nevertheless do not cover the full space in any case:

$$(X^A) = l\ (\cosh\tau\,\vec{u}_t(\Omega), \sinh\tau\,\vec{n}_s(\Omega')) \tag{A.8}$$

where $\vec{u}$ and $\vec{n}$ are unit vectors of both $t-1$ and $s-1$ dimensional spheres. This is for S_s^n spaces. For H_s^n spaces is simply:

$$(X^A) = l\ (\sinh\tau\,\vec{u}_{t-1}(\Omega), \cosh\tau\,\vec{n}_{s+1}(\Omega')) \tag{A.9}$$

Our convention for a unit vector of a $(n-1)$-dimensional sphere is:

$$\vec{u}_n(\Omega) = (\cos\theta_1, \sin\theta_1\cos\theta_2, \ldots, \sin\theta_1\ldots\sin\theta_{n-1}) \tag{A.10}$$

so that our convention for the "north pole" is:

$$S_s^n\!:\ \ N = (l, 0, \ldots)\ ;\ H_s^n\!:\ \ N = (\underbrace{0, \ldots}_{t-1}, l, 0, \ldots) \tag{A.11}$$

The invariant distance, that we call z, is defined as $z(X,Y) = \pm\frac{X\cdot Y}{l^2}$, where the sign is chosen to make $z(X,X) = 1$ in every space. In our cases of interest:

- Sphere: $X = l\,\vec{u}_n(\Omega)$, $z = \cos\theta_1$
- Euclidean Anti de Sitter: $X = l(\cosh\tau, \sinh\tau\vec{u}_{n-1}(\Omega))$, $z = \cosh\tau$
- de Sitter: $X = l(\sinh\tau, \cosh\tau\,\vec{u}_{n-1}(\Omega))$, $z = \cosh\tau\cos\theta_1$
- Anti de Sitter: $X = l(\cosh\tau\cos\theta, \cosh\tau\sin\theta, \sinh\tau\vec{u}_{n-2}(\Omega'))$, $z = \cosh\tau\cos\theta$

where we take $z \equiv z(X,N)$ with our previous conventions about the "North pole".

A.1.2 Projective coordinates

We shall further assume that $\epsilon_k = \pm 1$, that is, the choosen coordinate has the same sign for the metric as the second member in (A.3). We then define the south pole (i.e. $X^k = -l$) stereographic projection for $\mu \neq k$, as

$$x_S^\mu \equiv \frac{2l}{X^k + l} X^\mu \equiv \frac{X^\mu}{\Omega_S} \tag{A.12}$$

The equation of the surface then leads to

$$X^k = l(2\Omega_S - 1) \ ; \ \Omega_S = \frac{1}{1 \pm \frac{x_S^2}{4l^2}} \ ; \ x_S^2 \equiv \sum_{\mu \neq k} \epsilon_\mu \left(x_S^\mu\right)^2 \tag{A.13}$$

The metric in these coordinates is conformally flat:

$$ds^2 = \Omega_S^2 \eta_{\mu\nu} dx_S^\mu dx_S^\nu \tag{A.14}$$

We could have done projection from the North pole (for that we need that $X^k \neq l$). Uniqueness of the definition of X^k needs

$$\Omega_N + \Omega_S = 1 \tag{A.15}$$

and uniqueness of the definition of X^μ

$$x_N^\mu = \frac{\Omega_S}{\Omega_N} x_S^\mu = \pm \frac{4l^2}{x_S^2} x_S^\mu \tag{A.16}$$

The antipodal $\mathbb{Z}_2$ map $X^A \to -X^A$ is equivalent to a change of the reference pole in stereographic coordinates

$$x_N^\mu \leftrightarrow x_S^\mu \tag{A.17}$$

A.1.3 Poincaré coordinates

A generalization of Poincaré's metric for the half-plane can easily be obtained by introducing the horospheric coordinates. It will always be assumed that $\epsilon_0 = +1$, that is that X^0 is a time, and also that $\epsilon_n = -1$, that is X^n is a space, in our conventions. Otherwise (like in the all-important case of the sphere S_n) it it not possible to construct these coordinates.

$$\begin{aligned} \frac{l}{z} &\equiv X^- = X^n - X^0 \\ y^i &\equiv z X^i \end{aligned} \tag{A.18}$$

The promised generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum_1^{n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{A.19}$$

where the sign is the opposite to the one defined in (A.3), and the surfaces of constant z are sometimes called *horospheres*. This form of the metric is conformally flat in a manifest way.

- In **de Sitter space**, dS_n, z is a timelike coordinate, and its metric reads

  $$ds^2_{dS_n} = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j + l^2 dz^2}{z^2} \tag{A.20}$$

  The square of the Killing vectors M_{0A} (candidates to be timelike) are

  $$M_{0A}^2 = X_0^2 - X_A^2 = \sum_{B \neq A} X_B^2 - l^2 \tag{A.21}$$

  so they are timelike only outside the *horizon* defined as

  $$H_{0A} \equiv \sum_{B \neq A} X_B^2 = l^2 \tag{A.22}$$

  For example, the horizon corresponding to H_{0n} is

  $$\sum y_i^2 = l^2 z^2 \tag{A.23}$$

  This means that de Sitter space, dS_n is not globally static.

- What one would want to call **Euclidean anti de Sitter**, $EAdS_n$, has got all its coordinates spacelike, and *positive* curvature. To be specific

  $$ds^2_{EAdS_n} = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{A.24}$$

- Finally, when the metric is given by

  $$ds^2_{AdS_n} = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{A.25}$$

  (where as usual, $\eta_{ij} \equiv \mathrm{diag}(1, -1^{n-2})$) this is the **Anti de Sitter**, AdS_n. In this case there is a globally defined timelike Killing vector field, namely M_{01}

  $$M_{01}^2 = X_0^2 + X_1^2 = l^2 + \sum_{A>1} X_A^2 \tag{A.26}$$

  that is everywhere positive. This means that Anti de Sitter space is globally static, as opposed to de Sitter.

A.1.4 Conformal Invariance

Let us be very explicit with the definition of Poincaré coordinates: Let us denote

$$x^2 \equiv y^2 \mp l^2 z^2 \equiv \sum \epsilon_i y_i^2 \mp l^2 z^2 \tag{A.27}$$

Then

$$\begin{aligned} X^0 &= \frac{l^2 - x^2}{2lz} \\ X^n &= -\frac{l^2 + x^2}{2lz} \\ X^i &= \frac{y^i}{z}\,(i = 1 \ldots n-1) \end{aligned} \tag{A.28}$$

This is a legitimate change of coordinates as long as we keep the radius l itself as one of the coordinates.

Conversely,

$$\begin{aligned} y^i &= \frac{X^i}{X^0 - X^n} l \\ z &= \frac{l}{X^0 - X^n} \\ l^2 &= \mp \left(X_0^2 - X_n^2 + \epsilon_i X_i^2\right) \end{aligned} \tag{A.29}$$

Some useful formulas:

$$\begin{aligned} \frac{\partial}{\partial X_0} &= -\frac{z}{l} y^i \partial_i - \frac{z^2}{l}\partial_z \mp \frac{l^2 - x^2}{lz}\partial_{l^2} \\ \frac{\partial}{\partial X_n} &= \frac{z}{l} y^i \partial_i + \frac{z^2}{l}\partial_z \mp \frac{l^2 + x^2}{lz}\partial_{l^2} \\ \frac{\partial}{\partial X_i} &= z\partial_i \mp 2\frac{\epsilon_i y_i}{z}\partial_{l^2} \end{aligned} \tag{A.30}$$

The full isometry group is some noncompact form of $SO(n+1)$. In Poincare coordinates, there is a $ISO(n-1)$ manifest isometry group not involving the horographic coordinate. It will be important for us to understand all isometries in Poincaré coordinates. Let us work out the non-explicit generators:

$$L_{0n} \equiv X^0 \partial_n + X_n \partial_0 = y^i \partial_i + z\partial_z$$

$$L_{0i} = X^0 \partial_i - \epsilon_i X_i \partial_0 = \sum_j \frac{(l^2 - x^2)\,\delta_{ij} + 2\epsilon_i y_i y_j}{2l}\partial_j + \epsilon_i y^i \frac{z}{l}\partial_z$$

$$L_{ni} = -X^n \partial_i - \epsilon_i X_i \partial_n = \sum_j \frac{(l^2 + x^2)\,\delta_{ij} - 2\epsilon_i y_i y_j}{2l} \partial_j - \epsilon_i y^i \frac{z}{l} \partial_z$$

Translations of the y^i correspond to the combination:

$$k_i \equiv l \frac{\partial}{\partial y^i} = -\left(L_{ni} + L_{oi}\right) \tag{A.31}$$

All spaces we are considering here, which in Poincaré coordinates enjoy the metric

$$ds^2 = \frac{\sum_{i=1}^{i=n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{A.32}$$

are obviously *scale invariant*

$$\begin{aligned} y_i &\rightarrow \lambda\, y_i \\ z &\rightarrow \lambda\, z \end{aligned} \tag{A.33}$$

This corresponds in Weierstrass coordinates to the Lorentz transformation in the plane $(X^0 X^n)$

$$\begin{aligned} (X')^0 &= \frac{(\lambda^2 + 1)\, X^0 + (\lambda^2 - 1)\, X^n}{2\lambda} \\ (X')^n &= \frac{(\lambda^2 - 1)\, X^0 + (\lambda^2 + 1)\, X^n}{2\lambda} \end{aligned} \tag{A.34}$$

id est,

$$\begin{aligned} X^- &\rightarrow \lambda X^- \\ X^+ &\rightarrow \frac{X^+}{\lambda} \end{aligned} \tag{A.35}$$

(This ought to be more or less obvious already from the previous formula for the generator L_{0n}). Not only that, but also they are invariant under *inversions*, *id est*,

$$\begin{aligned} y_i &\rightarrow \frac{y_i}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \\ z &\rightarrow \frac{z}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \end{aligned} \tag{A.36}$$

Inversions in Weierstrass coordinates look even simpler; just exchange the two light-cone coordinates in the aforementioned plane $(X^0 X^n)$:

$$X^+ \leftrightarrow X^- \tag{A.37}$$

The remaining isometries are the somewhat nasty combinations

$$L_{0i} - L_{ni} = \sum_j \frac{(-x^2)\,\delta_{ij} + 2\epsilon_i y_i y_j}{l}\partial_j + 2\epsilon_i y^i \frac{z}{l}\partial_z \tag{A.38}$$

We are now in a position to study the little group H of a given point (which can always be rotated to

$$P \equiv \left(\vec{y} = \vec{0}, z = 1\right) \tag{A.39}$$

We know that then the space will be isomorphic to $SO(n+1)/H$. The translational isometries must be generated by the n generators

$$\begin{aligned} &L_{ni} + L_{0i} \\ &L_{0n} \end{aligned} \tag{A.40}$$

It seems then that

$$\begin{aligned} H^+ &= \{L_{ij}, L_{ni}\} \\ H^- &= \{L_{ij}, L_{0i}\} \end{aligned} \tag{A.41}$$

The number of not compact generators is equal to the number of times in the coordinates y^i in the + case, and the number of times plus one in the minus case. This seems to imply that

$$\begin{aligned} AdS_n &= SO(2, n-1)/SO(1, n-1) \\ EAdS_n &= SO(1, n)/SO(n) \\ dS_n &= SO(1, n)/SO(1, n-1) \\ EdS_n &= SO(n, 1)/SO(n) \end{aligned} \tag{A.42}$$

Euclidean anti de Sitter $EAdS_n$ is just de Sitter dS_n with imaginary radius. Euclidean de Sitter EdS_n is Euclidean anti de Sitter dS_n with negative ambient metric.

A.2 Conformal structure

- $\mathbf{dS_n}$ From the global coordinates in de Sitter (cf. A.1.1), we can define $\cos T = \frac{1}{\cosh \tau}$ where $-\pi/2 \leq T \leq \pi/2$ so it yields

$$ds^2 = \frac{l^2}{\cos^2 T}\left(dT^2 - d\Omega^2_{n-1}\right) \tag{A.43}$$

which is conformal to a piece of $\mathbb{R} \times S_{n-1}$, which is the Einstein static universe to study conformal structure. The piece is a slab in the timelike direction, but otherwise including the full three-sphere at each time. The fact that conformal infinity is spacelike means that there are both particle and event horizons.

- $\mathbf{AdS_n}$ The same change of coordinates from the global chart can be used, $\cos \rho = \frac{1}{\cosh \tau}$, where $\rho \in (0, \pi/2)$. The space is again conformal to a piece of half Einstein' s static universe:

$$ds^2 = \frac{l^2}{\cos^2 \rho}\left(d\theta^2 - d\rho^2 - \sin^2 \rho d\Omega^2_{n-2}\right) = \frac{l^2}{\cos^2 \rho}\left(d\theta^2 - d\Omega^2_{n-1}\right) \tag{A.44}$$

If we want to eliminate the closed timelike lines, one can consider the covering space $-\infty \leq \theta \leq \infty$. The slab of $\mathbb{R} \times S_{n-1}$ to which AdS_n is conformal to includes now the full timelike direction, but only an hemisphere at each particular time. Null and spacelike infinity can be considered as the timelike surfaces $\rho = 0$ and $\rho = \pi/2$. This implies that there are no Cauchy surfaces.

A.3 Poincaré patch

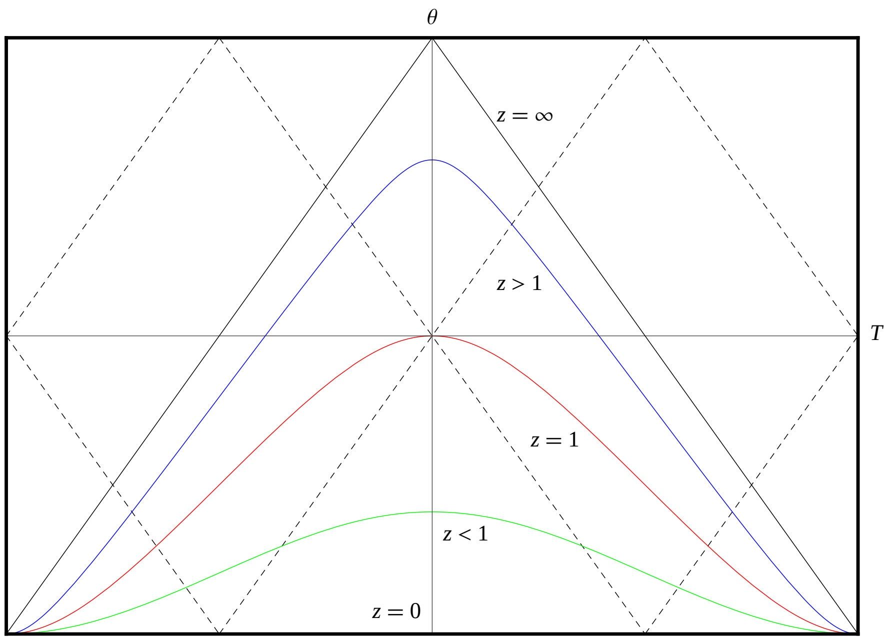

Figure A.4: Conformal structure of dS_n. The coloured lines are z =const. surfaces in Poincaré coordinates.

- $\mathbf{dS_n}$

  If we call u_n the n-th component of the unit vector $\vec{u}$, then there is a critical value of the parameter τ such that

$$\tanh \tau(u) = u_n(\Omega) \tag{A.45}$$

which is such that

$$\tau < \tau(u) \Rightarrow z > 0 \tag{A.46}$$

and

$$z \to \pm\infty \Leftrightarrow \tau \to \tau(n)^{\mp} \tag{A.47}$$

This means that at any given value of τ only those points on the sphere that obey

$$u_n(\Omega) \geq \tanh \tau \tag{A.48}$$

can be represented in Poincaré coordinates. For example, when $\tau = \infty$, that is $T = \pi/2$, $\tanh \tau = 1$, so that only the North pole ($n = 1$) can be covered. At the other extreme, when, $\tau = -\infty$, that is $T = -\pi/2$, $\tanh \tau = -1$, we can cover the full sphere.

On the other hand, it is clear that

$$z \to 0^{\pm} \Leftrightarrow \tau \to \mp\infty \tag{A.49}$$

There is a discontinuity at $\tau(n)$ which depends on the point in de Sitter space.

- $\mathbf{AdS_n}$

As in the previous case, it is clear that the region $1/z = 0$ corresponds to

$$u_{n-1}(\Omega)\sin\rho = \cos\theta \tag{A.50}$$

and the region $z > 0$ to

$$u_{n-1}(\Omega)\sin\rho > \cos\theta \tag{A.51}$$

The region

$$z = 0 \tag{A.52}$$

is dubbed the *boundary* (of the Poincaré patch) of AdS and corresponds to

$$\rho = \pi/2 \tag{A.53}$$

Finally

$$z = \infty \tag{A.54}$$

is usually called the *horizon* and corresponds to (A.50)

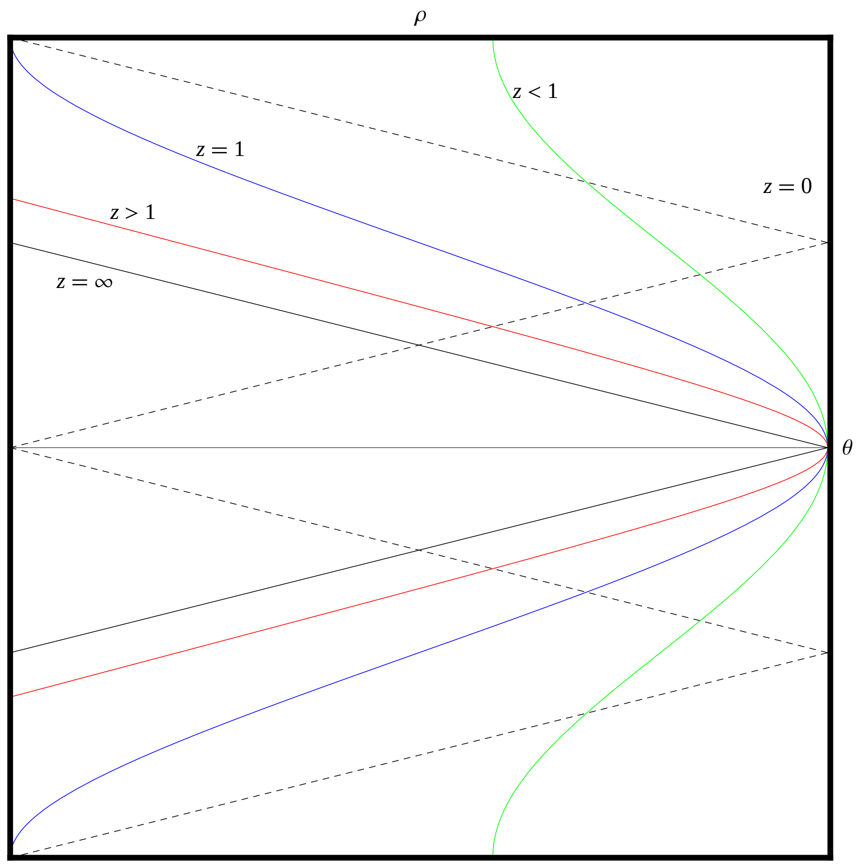

Figure A.5: Conformal structure of AdS_n. The coloured lines are z =const. surfaces in Poincaré coordinates.

Appendix B

Quantum Field Theory in Curved Spacetime

We detail in this appendix the standard and well known way to define the quantum theory of a free scalar field over a curved spacetime. Nothing here is substantially different from the classic references [34] [51].

B.1 Free scalar field

The action of a real scalar field in a curved spacetime M with metric g is:

$$S[\phi] = \int_M d^n x \sqrt{g}\, \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi R \phi^2 \right) \tag{B.1}$$

where ξ is an *a priori* non-minimal coupling to the curvature. The field equation, the Klein-Gordon equation, is easily derived:

$$0 = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, g^{\mu\nu} \partial_\nu \phi \right) \tag{B.2}$$

In order to have a well defined wave propagation, the most general class of target spacetimes for our scalar field has to have a clear and global separation between "space" and "time". This is the case for the so-called **globally hyperbolic** spaces, i.e. those that possess an $n-1$ dimensional Cauchy surface Σ. The technical definition of a Cauchy surface is not as important as the following theorem:

Theorem A globally hyperbolic spacetime M is topologically equivalent to $\mathbb{R} \times \Sigma$. Furthermore, the "time" function $t : M \to \mathbb{R}$ given by (the first component of) this homeomorphism can be chosen to be a $\mathcal{C}^\infty$ function whose gradient is non-vanishing everywhere.

Thus, this theorem clearly establishes this expected separation. Our spacetime can be seen as foliated into "copies" of the Cauchy surface, Σ_t. The Cauchy problem is well defined in every globally hyperbolic space, in the sense that there is always a solution to the Klein-Gordon equation with given initial conditions defined over a certain Cauchy surface. In addition to this, we also get a symplectic structure over the space of all fields, given by:

$$(\phi, \varphi) = -i \int_{\Sigma_t} dy \sqrt{g_\Sigma}\, n^\mu \;(\phi\, \partial_\mu \varphi^* - \partial_\mu \phi\, \varphi^*) \tag{B.3}$$

where the integration is taken over any Cauchy surface, and n is the only unit, future-directed vector field orthogonal to the surfaces Σ_t. This product is of course independent of t.

This structure allows us to construct a set of basic modes, a basis that spans the whole space of real solutions. For example, a real solution of the Klein-Gordon equation in Minkowski space can always be expressed as a combination of plane waves:

$$\phi(t, \mathbf{x}) = \int d\mathbf{k} \left\{ \alpha(\mathbf{k}) \frac{e^{i\mathbf{k}\mathbf{x}} e^{-i\omega_k t}}{(2\pi)^{\frac{n-1}{2}} \sqrt{2\omega_k}} + \alpha(\mathbf{k})^* \frac{e^{-i\mathbf{k}\mathbf{x}} e^{i\omega_k t}}{(2\pi)^{\frac{n-1}{2}} \sqrt{2\omega_k}} \right\} \tag{B.4}$$

We define in general a basis of modes $\{u_i\}$ as a set which is complete, in the sense that every real solution of the Klein-Gordon equation ϕ can be expressed as

$$\phi = \alpha_i u_i + \alpha_i^* u_i^* \tag{B.5}$$

and that is orthogonal with respect to the Klein-Gordon product:

$$(u_i, u_j) = -(u_i^*, u_j^*) = \delta_{ij} \;,\; (u_i, u_j^*) = 0 \tag{B.6}$$

Now that the basic modes are established, the quantization can take place. We interpret these modes as the basic excitations of the theory, so we postulate a field operator $\hat{\phi}(x)$ and a vacuum state $|\text{vac}\rangle_u$, with the following properties:

$$\hat{\phi}(x) = \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i(x)^* \tag{B.7}$$

$$\hat{a}_i |\text{vac}\rangle_u = 0,\ \forall i \tag{B.8}$$

Of course the basis is highly non unique, and the space of solutions can be split into "positive frequency" and "negative frequency" subspaces in many different ways. This means that we can describe the quantum theory with respect to many spectrums of basic excitations. For example, assuming that $\{v_i\}$ is another modes basis, there is relation between the new and the old modes, because of the completeness of both sets:

$$v_i = \alpha_{ij} u_j + \beta_{ij} u_j^* \tag{B.9}$$

where α and β are the **Bogoliubov coefficients**, obtained from:

$$\alpha_{ij} = (v_i, u_j) \ , \ \beta_{ij} = -(v_i, u_j^*) \tag{B.10}$$

Of course, these coefficients should obey some constraints if the v's are orthogonal. From (B.6) we obtain

$$\alpha\alpha^\dagger - \beta\beta^\dagger = \mathbb{I} \ , \ \alpha\beta^\dagger = \beta\alpha^\dagger \tag{B.11}$$

These coefficients can be used to transform also the quantum states. From the expresion for the quantum field

$$\hat{\phi}(x) = \hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i(x)^* = \hat{b}_i v_i(x) + \hat{b}_i^\dagger v_i(x)^* \tag{B.12}$$

we obtain that $\hat{a}_i = \alpha_{ji}\hat{b}_j - \beta_{ji}^*\hat{b}_j$. From this formula we can compute quantities associated to the overlap between different vacua, as for example:

$$_v\langle \mathrm{vac}| \sum_i \hat{a}_i^\dagger \hat{a}_i |\mathrm{vac}\rangle_v = \sum_j |\beta_{ji}|^2 \tag{B.13}$$

B.2 Interacting scalar field

In order to develop some sort of scattering theory for interacting field, well-behaved asymptotic regions are assumed. This implies that near the future (past) infinity, the field operator behaves as a free field, $\hat{\phi}_{\mathrm{out}}$ ($\hat{\phi}_{\mathrm{in}}$).

Now four different vacua can be defined as usual in an S-matrix approach

$$\begin{array}{ll} \hat{a}_i^{\mathrm{in}}|\mathrm{vac}\rangle_{\mathrm{in}}^u = 0 & \hat{a}_i^{\mathrm{out}}|\mathrm{vac}\rangle_{\mathrm{out}}^u = 0 \\ \hat{b}_i^{\mathrm{in}}|\mathrm{vac}\rangle_{\mathrm{in}}^v = 0 & \hat{b}_i^{\mathrm{out}}|\mathrm{vac}\rangle_{\mathrm{out}}^v = 0 \end{array}$$

using two different expansions for each field:

$$\hat{\phi}_{\mathrm{in,out}}(x) = \hat{a}_i^{\mathrm{in,out}} u_i(x) + (\hat{a}_i^{\mathrm{in,out}})^\dagger u_i(x)^* = \hat{b}_i^{\mathrm{in,out}} v_i(x) + (\hat{b}_i^{\mathrm{in,out}})^\dagger v_i(x)^* \tag{B.14}$$

The two sets of modes are related by a Bogoliubov transformation, so that a given vacuum contains particles as defined with respect to a different vacuum (that is, using a different definition of positive frequency). When interactions are taken into account *even with the same definition of positive frequency* the vacuum is not stable

$$|{}^{u,v}_{\mathrm{out}}\langle \mathrm{vac}|\mathrm{vac}\rangle^{u,v}_{\mathrm{in}}| \neq 1 \tag{B.15}$$

In practice both effects (that is, particle creation due to the external gravitational field at zero coupling as reported, for example, in [52], and the effects of the interaction) compete, and in order to separate them one has to study specific channels as well as their dependence on the coupling.

The *physical* matrix elements to calculate in a $\alpha \to \beta$ are referred to the appropriate notion of particles ${}_{\text{out}}^{\;2}\langle\beta|\alpha\rangle_{\text{in}}^{1}$, i.e. choosing the modes that correspond to the appropiate notion of free particles for each asymptotic (past and future) region.

In order to separate the true interaction effects from the creation of particles due to the gravitational field, we need to focus in the ${}_{\text{out}}\langle\beta|\alpha\rangle_{\text{in}}$ matrix elements, referred to a common notion of vacuum, which are related by a Bogoliubov transformation to the previous ones.

For this matrix elements we can define the S-matrix as $S^{\dagger}|\alpha\rangle_{\text{in}} = |\beta\rangle_{\text{out}}$, which has the familiar interaction representation form

$$S = T\exp\{-i\int H_I(\phi)\}$$

Appendix C

Spherical harmonics

- **The n-dimensional sphere**. The simplest way of getting eigenfunctions of the Laplace operator in the sphere is Helgason's (confer [28]). Consider the following harmonic polynomial in $\mathbb{R}^{n+1}$

$$f_{a,\lambda} \equiv (\vec{a}.\vec{x})^{\lambda} \tag{C.1}$$

with $\vec{a} \in \mathbb{C}$, $\vec{a}^2 = 0$.

Now we know that the full laplacian in $\mathbb{R}^{n+1}$ is

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S_n} \tag{C.2}$$

This yields

$$\Delta_{\mathbb{R}^{n+1}} f_{a,\lambda} = 0 = \frac{\lambda^2 + (n-1)\lambda}{r^2} f_{a,\lambda} + \frac{1}{r^2}\Delta_{S_n} f_{a,\lambda} \tag{C.3}$$

so that the eigenvalues of the Laplacian in the sphere S_n are

$$-\lambda(\lambda + n - 1) \tag{C.4}$$

It is more or less equivalent to start from traceless homogeneous polynomials

$$P \equiv \sum P_{(i_1 \ldots i_k)} x^{i_1} \ldots x^{i_k} \tag{C.5}$$

The number of such animals is the number of symmetric polynomials in n variables of degree λ minus the number of symmetric polynomials of degree $\lambda - 2$:

$$d(\lambda) = \binom{\lambda + n - 1}{\lambda} - \binom{\lambda + n - 3}{\lambda - 2} = \frac{(n + 2\lambda - 2)\,(\lambda + n - 3)!}{\lambda!\,(n-2)!} \tag{C.6}$$

- If we represent by μ an appropiate collection of indices, then we first build harmonic polynomials such that

$$\int_{S_n} d\Omega h^*_{\lambda'\mu'} h_{\lambda\mu} = \delta_{\lambda\lambda'}\delta_{\mu\mu'} r^{\lambda+\lambda'} \tag{C.7}$$

The hyperspherical harmonics are then defined by

$$h_{\lambda\mu} \equiv r^\lambda Y_{\lambda\mu} \tag{C.8}$$

and are normalized in such a way that

$$\int_{S_n} d\Omega Y^*_{\lambda'\mu'} Y_{\lambda\mu} = \delta_{\lambda\lambda'}\delta_{\mu\mu'} \tag{C.9}$$

- Gegenbauer polynomials are generalizations of Legendre polynomials, in the sense that

$$\frac{1}{|\vec{x}-\vec{x'}|^{n-2}} = \frac{1}{r_>^{n-2}\left(1+\left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\hat{x}.\hat{x}'\right)^{\frac{n-2}{2}}} = \frac{1}{r_>^{n-2}}\sum_{\lambda=0}^{\infty}\left(\frac{r_<}{r_>}\right)^\lambda C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) \tag{C.10}$$

Let us now prove the sum rule for hyperspherical harmonics. For concreteness, let us assume that

$$\begin{aligned} r &\equiv |\vec{x}_<| \\ r' &\equiv |\vec{x}_>| \end{aligned} \tag{C.11}$$

Then it is a fact of life that

$$\Delta\frac{1}{|\vec{x}-\vec{x}'|^{n-2}} = 0 = \sum_{\lambda=0}^{\infty}\frac{1}{(r')^{\lambda+n-2}}\Delta\left(r^\lambda C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right)\right) \tag{C.12}$$

Imposing term by term vanishing leads to

$$\left(\frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_{S^{n-1}}\right)\left(r^\lambda C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right)\right) = 0 \tag{C.13}$$

which conveys the fact that

$$\Delta_{S_{n-1}} C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) = -\lambda\left(\lambda+n-2\right) C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) \tag{C.14}$$

Since the hyperspherical harmonics are by assumption a complete set of eigenfunctions,

$$C_\lambda^{\frac{n-2}{2}}\left(\hat{x}.\hat{x}'\right) = \sum_\mu a_{\lambda\mu}\left(\vec{x}'\right) Y_{\lambda\mu}\left(\hat{x}\right) \tag{C.15}$$

where

$$a_{\lambda\mu}(\vec{x}') = \int_{\hat{x}} C_\lambda^{\frac{n-2}{2}}(\hat{x}.\hat{x}')\, Y^*_{\lambda\mu}(\hat{x}) = \frac{2(n-2)\pi^{n/2}}{\Gamma(n/2)\,(2\lambda+n-2)} Y^*_{\lambda\mu}(\hat{x}') \tag{C.16}$$

This is related to the degeneracy $d(\lambda)$ of hyperspherical harmonics in the following way. Choosing $\hat{x} = \hat{x}'$, the sum rule leads to

$$C_\lambda^{\frac{n-2}{2}}(1) = K_\lambda \sum_\mu Y^*_{\lambda\mu}(\vec{x}')\, Y_{\lambda\mu}(\hat{x}) \tag{C.17}$$

Integrating now over the unit sphere

$$C_\lambda^{\frac{n-2}{2}}(1)\, V(S_{n-1}) = K_\lambda \sum_\mu 1 = K_\lambda d(\lambda) \tag{C.18}$$

The result is

$$d(\lambda) = \frac{(n+2\lambda-2)\,(\lambda+n-3)!}{\lambda!\,(n-2)!} \tag{C.19}$$

- Let us now become more specific and perform some computations in gory detail. The metric on S_n is

$$ds_n^2 = d\theta_n^2 + \sin^2\,\theta_n d\theta_{n-1}^2 + \ldots + \sin^2\,\theta_n\,\sin^2\,\theta_{n-1}\ldots\sin^2\,\theta_2 d\theta_1^2 \tag{C.20}$$

id est, in a recurrent form

$$\begin{aligned} ds_1^2 &= d\theta_1^2 \\ ds_n^2 &= d\theta_n^2 + \sin^2\theta_n\, ds_{n-1}^2 \end{aligned} \tag{C.21}$$

This corresponds to polar coordinates in $\mathbb{R}^n$

$$\begin{aligned} X_{n+1} &= \cos\,\theta_n \\ X_n &= \sin\,\theta_n\,\cos\,\theta_{n-1} \\ &\ldots \\ X_2 &= \sin\,\theta_n\,\sin\,\theta_{n-1}\ldots\cos\,\theta_1 \\ X_1 &= \sin\,\theta_n\,\sin\,\theta_{n-1}\ldots\sin\,\theta_1 \end{aligned} \tag{C.22}$$

Spherical harmonics have been constructed quite explicitly by Higuchi [7], are such that

$$\Delta_n Y_{j_n\ldots j_1}(\theta_n\ldots\theta_1) = -j_n(j_n+n-1)Y_{j_n\ldots j_1}(\theta_n\ldots\theta_1) \tag{C.23}$$

We shall explicitly write down the laplacian in a moment. They are orhonormal with respect to the induced riemannian measure

$$d\Omega_n \equiv \sqrt{|g|} d\theta_1 \wedge \ldots d\theta_n = d\theta_1 \ldots d\theta_n \sin^{n-1}\theta_n \, \sin^{n-2}\, \theta_{n-1} \ldots \sin\, \theta_2 \tag{C.24}$$

The laplacian is easily found to be

$$\Delta_{S_n} = \left(\frac{\partial^2}{\partial\theta_n^2} + (n-1)\cot\, \theta_n \frac{\partial}{\partial\theta_n}\right) + \frac{1}{\sin^2\theta_n}\left(\frac{\partial^2}{\partial\theta_{n-1}^2} + (n-2)\cot\, \theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}\right) + \ldots$$
$$+\frac{1}{\sin^2\theta_n\, \sin^2\theta_{n-1} \ldots \sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2} \tag{C.25}$$

Another useful recurrence

$$d\Omega_n = \sin^{n-1}\theta_n d\theta_n d\Omega_{n-1} \tag{C.26}$$

and

$$V(S_{n-1}) = \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{C.27}$$

To be specific,

$$\int d\Omega_n Y_{j_n \ldots j_1}(\theta_n \ldots \theta_1) Y^*_{j'_n \ldots j'_1}(\theta_n \ldots \theta_1) = \delta_{j_n, j'_n} \ldots \delta_{j_n, j'_n} \tag{C.28}$$

- It is obvious that any function on the sphere can be expanded

$$f(\Omega) = \sum_{j_n \ldots j_1} C_{j_n \ldots j_1} Y_{j_n \ldots j_1}(\theta_n \ldots \theta_1) =$$
$$\sum_{j_n \ldots j_1} \int d\Omega' Y^*_{j_n \ldots j_1}(\theta'_n \ldots \theta'_1) f(\theta'_n \ldots \theta'_1) Y_{j_n \ldots j_1}(\theta_n \ldots \theta_1)$$

which means

$$\sum_{j_n \ldots j_1} Y^*_{j_n \ldots j_1}(\theta'_n \ldots \theta'_1) Y_{j_n \ldots j_1}(\theta_n \ldots \theta_1) \equiv \delta(\Omega - \Omega') \tag{C.29}$$

where by definition

$$\int d\Omega' \delta(\Omega - \Omega') f(\theta') = f(\theta) \tag{C.30}$$

whence in a somewhat symbolic form,

$$\delta(\Omega - \Omega') = \delta(\theta'_1 - \theta_1) \ldots \delta(\theta'_n - \theta_n) \sin^{-(n-1)}\theta'_n\, \sin^{-(n-2)}\, \theta'_{n-1} \ldots \sin^{-1}\, \theta'_2 \tag{C.31}$$

Now we can expand this function, as any other function, in series of Gegenbauer polynomials

$$\delta(\Omega - \Omega') = \sum_j d_j C_j^\nu(\cos\,\theta_n) \tag{C.32}$$

Let us choose our reference frame in such a way that

$$\Omega \cdot \Omega' \equiv \cos\,\theta_n \tag{C.33}$$

id est, Ω' is pointing towards the North pole.

On functions constant on S_{n-1},

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \sin^{n-1}\,\theta_n \, d\theta_n \tag{C.34}$$

and, denoting $x \equiv \cos\,\theta_n$

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left(1 - x^2\right)^{\frac{n-2}{2}} dx \tag{C.35}$$

as well as

$$\delta(\Omega) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(\theta_n) \frac{1}{\sin^{n-1}\,\theta_n} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(1-x)(1-x^2)^{\frac{2-n}{2}} \tag{C.36}$$

We can now integrate the two sides of the equation (C.32) against $C_{j'}^\nu(x)(1-x)^{\nu-1/2}$. The orthogonality property

$$\int_{-1}^{1} dx C_j^\nu(x) C_{j'}^\nu(x)(1-x^2)^{\nu-1/2} = \delta_{jj'} \frac{2^{1-2\nu}\pi\Gamma(j+2\nu)}{j!(\nu+j)\Gamma(\nu)^2} \tag{C.37}$$

then implies

$$d_j \frac{2^{1-2\nu}\pi\Gamma(j+2\nu)}{j!(\nu+j)\Gamma(\nu)^2} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{-1}^{1} dx C_j^\nu(x)(1-x^2)^{1-n/2} \delta(x-1)(1-x^2)^{\nu-1/2} \tag{C.38}$$

The member of the right converges when $\nu = \frac{n-1}{2}$. Given in addition the fact that

$$C_j^\nu(1) = \frac{\Gamma(j+2\nu)}{j!\,\Gamma(2\nu)} \tag{C.39}$$

we can write

$$d_j = \frac{\Gamma(\frac{n}{2})(j+\frac{n-1}{2})\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)\pi^{\frac{n+1}{2}} 2^{3-n}} = \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} \tag{C.40}$$

(using $\Gamma(2x) = 2^{1-2x}\sqrt{\pi}\Gamma(x+\frac{1}{2})/\Gamma(x)$) as well as

$$\delta(\Omega - \Omega') = \sum_j \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos\,\theta_n) \tag{C.41}$$

$$\sum_{j_n \dots j_1} Y^*_{j_n \dots j_1}(\theta'_n = 0 \dots \theta'_1) Y_{j_n \dots j_1}(\theta_n \dots \theta_1) = \sum_j \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos\,\theta_n) \tag{C.42}$$

If we employ the notation $j \equiv j_n$ and $\vec{m} \equiv (j_{n-1} \dots j_1)$, then the preceding formula presumably means that

$$\sum_{\vec{m}} Y^*_{j \dots \vec{m}}(\Omega_z) Y_{j \dots \vec{m}}(\Omega) = \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos\,\theta_n) \tag{C.43}$$

- We begin by defining some eigenfunctions of the differential operator:

$$D \equiv \frac{\partial^2}{\partial\theta^2} + (N-1)\cot\,\theta \frac{\partial}{\partial\theta} - \frac{j\,(j+N-2)}{\sin^2\,\theta} \tag{C.44}$$

such that

$$D\bar{P}^j_{Nk}(\theta) = -k\,(k+N-1)\,\bar{P}^j_{Nk}(\theta) \tag{C.45}$$

The form we are going to need is

$$\left(\frac{\partial^2}{\partial\theta^2} + (N-1)\cot\,\theta \frac{\partial}{\partial\theta}\right) \bar{P}^j_{Nk}(\theta) = \left(\frac{j\,(j+N-2)}{\sin^2\,\theta} - k\,(k+N-1)\right) \bar{P}^j_{Nk}(\theta) \tag{C.46}$$

To be specific,

$$\bar{P}^j_{Nk}(\theta) \equiv c^j_{Nk}\,(\sin\,\theta)^{-\frac{N-2}{2}}\, P^{-\left(j+\frac{N-2}{2}\right)}_{k+\frac{N-2}{2}}(\cos\,\theta) \tag{C.47}$$

where $P^\mu_\nu(z)$ are Legendre functions , and the normalization is given by

$$c^j_{Nk} \equiv \sqrt{\frac{2k+N-1}{2} \frac{(k+j+N-2)!}{(k-j)!}} \tag{C.48}$$

The differential equation that Legendre functions $P^\mu_\nu(z)$ are solutions of is given by

$$Lw(z) \equiv \left(1-z^2\right) \frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left(\nu\,(\nu+1) - \frac{\mu^2}{1-z^2}\right) w = 0 \tag{C.49}$$

Changing variables $z = \cos\theta$ this reads

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\,\theta\frac{\partial}{\partial\theta} - \frac{\mu^2}{\sin^2\,\theta}\right) w\left(\cos\,\theta\right) = -\nu\left(\nu+1\right) w\left(\cos\,\theta\right) \tag{C.50}$$

and using this it is not difficult to actually prove the basic equation (C.45).

The harmonics themselves are given by:

$$Y_{j_n\ldots j_1}(\theta_n,\ldots,\theta_1) \equiv \prod_{m=2}^{n} \bar{P}^{j_{m-1}}_{m\,j_m}\left(\theta_m\right)\frac{1}{\sqrt{2\pi}}e^{ij_1\theta_1} \tag{C.51}$$

It is actually easy to check. From the expression for the laplacian, the operator acting on θ_1, just leads to

$$-\frac{j_1^2}{\sin^2\theta_n\ldots\sin^2\theta_2} \tag{C.52}$$

Next, the operator acting on θ_2, corresponding to $N=2$,$k=j_2$ and $j=j_1$, yields

$$\frac{j_1^2}{\sin^2\theta_n\ldots\sin^2\theta_2} - \frac{j_2(j_2+1)}{\sin^2\theta_n\ldots\sin^2\theta_3} \tag{C.53}$$

Next, the operator acting on θ_3, which corresponds to $N=3$, $k=j_3$ and $j=j_2$, gives

$$\frac{j_2(j_2+1)}{\sin^2\theta_n\ldots\sin^2\theta_3} - \frac{j_3(j_3+2)}{\sin^2\theta_n\ldots\sin^2\theta_4} \tag{C.54}$$

After all pairwise cancellations, we are left with the last term, corresponding to $N=n$, $k=j_n$ and $j=j_{n-1}$, yielding the eigenvalue

$$-j_n(j_n+n-1) \tag{C.55}$$

- We can now employ the expansion ([8], formula 8.534)

$$e^{im\rho\cos\,\phi} = 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)i^k(m\rho)^{-\nu}J_{\nu+k}(m\rho)C_k^\nu(\cos\,\phi) \tag{C.56}$$

and using our expansion of the Gegenbauer polynomials in terms of spherical harmonics,

$$e^{iz\Omega.\Omega'} = 2^{n/2-1}\Gamma(n/2-1)\sum_{k=0}^{\infty}(n/2-1+k)i^k(z)^{-(n/2-1)}J_{n/2-1+k}(z)$$
$$C_{k,n}\sum_{\vec{m}}Y^*_{k\ldots\vec{m}}(\Omega)Y_{k\ldots\vec{m}}(\Omega') \tag{C.57}$$

where $C_{l,n}$ are apropiate constants.

Bibliography

[1] E. Alvarez and R. Vidal, "Eternity and the cosmological constant," JHEP **0910**, 045 (2009) [arXiv:0907.2375 [hep-th]].

[2] E. Alvarez, R. Vidal, "Comments on the vacuum energy decay," JCAP **1011** (2010) 043. [arXiv:1004.4867 [hep-th]].

[3] E. Alvarez, R. Vidal, "Very Special (de Sitter) Relativity," Phys. Rev. **D77** (2008) 127702. [arXiv:0803.1949 [hep-th]].

[4] E. Alvarez, R. Vidal, "Weyl transverse gravity (WTDiff) and the cosmological constant," Phys. Rev. **D81** (2010) 084057. [arXiv:1001.4458 [hep-th]].

[5] M. Spradlin, A. Strominger, A. Volovich, "Les Houches lectures on de Sitter space,"

[hep-th/0110007].

[6] L. F. Abbott and S. Deser, "Stability Of Gravity With A Cosmological Constant", Nucl. Phys. B **195** (1982) 76.

[7] A. Higuchi, "SYMMETRIC TENSOR SPHERICAL HARMONICS ON THE N SPHERE AND THEIR APPLICATION TO THE DE SITTER GROUP SO(N,1)", J. Math. Phys. **28** (1987) 1553 [Erratum-ibid. **43** (2002) 6385].

[8] Gradshteyn S., Ryzhik I., "Table of integrals, series and products", Academic Press, 2007

[9] N. A. Chernikov and E. A. Tagirov, "Quantum theory of scalar fields in de Sitter space-time", Annales Poincare Phys. Theor. A **9** (1968) 109.

[10] E. Joung, J. Mourad and R. Parentani, "Group theoretical approach to quantum fields in de Sitter space. I: The principle series", JHEP **0608** (2006) 082 [arXiv:hep-th/0606119].

[11] K. Kirsten and J. Garriga, “Massless minimally coupled fields in de Sitter space: O(4) symmetric states versus de Sitter invariant vacuum”, Phys. Rev. D **48** (1993) 567 [arXiv:gr-qc/9305013].

[12] B. Allen and A. Folacci, “THE MASSLESS MINIMALLY COUPLED SCALAR FIELD IN DE SITTER SPACE”, Phys. Rev. D **35**, 3771 (1987).

[13] N. P. Myhrvold, “Runaway Particle Production In De Sitter Space,” Phys. Rev. D **28** (1983) 2439.

[14] G. W. Gibbons, S. W. Hawking, Phys. Rev. **D15** (1977) 2738-2751.

[15] B. Allen, “Vacuum States In De Sitter Space”, Phys. Rev. D **32**, 3136 (1985).

[16] M. B. Einhorn and F. Larsen, “Interacting quantum field theory in de Sitter vacua,” Phys. Rev. D **67** (2003) 024001 [arXiv:hep-th/0209159].

[17] N. C. Tsamis and R. P. Woodard, “Reply to ‘Can infrared gravitons screen Λ?”’, Phys. Rev. D **78** (2008) 028501 [arXiv:0708.2004 [hep-th]].

[18] J. Garriga and T. Tanaka, “Can infrared gravitons screen Λ?”, Phys. Rev. D **77** (2008) 024021 [arXiv:0706.0295 [hep-th]].

[19] A. M. Polyakov, “De Sitter Space and Eternity,” Nucl. Phys. B **797** (2008) 199 [arXiv:0709.2899 [hep-th]].
“Decay of Vacuum Energy,” Nucl. Phys. B **834**, 316 (2010) [arXiv:0912.5503 [hep-th]].

[20] A. M. Polyakov, “GAUGE FIELDS AND STRINGS”, *CHUR, SWITZERLAND: HARWOOD (1987) 301 P. (CONTEMPORARY CONCEPTS IN PHYSICS, 3)*

[21] Mark Kac, “Can one hear the shape of a drum?” Amer. Math Monthly 73 (1966)1

[22] M. Berger, ”Geometry of the spectrum I” Proceedings of the symposia on pure mathematics, vol 27 (1975) 129.

[23] A. O. Barvinsky and D. V. Nesterov, “Effective equations in quantum cosmology,” Nucl. Phys. B **608**, 333 (2001) [arXiv:gr-qc/0008062].

[24] C. Grosche and F. Steiner, “PATH INTEGRALS ON CURVED MANIFOLDS”, Z. Phys. C **36** (1987) 699. “THE PATH INTEGRAL ON THE PSEUDOSPHERE”, Annals Phys. **182** (1988) 120.

[25] R. Camporesi, “Harmonic analysis and propagators on homogeneous spaces”, Phys. Rept. **196** (1990) 1.

[26] B. Allen and T. Jacobson, "Vector Two Point Functions In Maximally Symmetric Spaces", Commun. Math. Phys. **103** (1986) 669.

[27] L. B. Szabados, "Quasi-Local Energy-Momentum and Angular Momentum in GR: A Review Article", Living Rev. Rel. **7** (2004) 4.

[28] S. Helgason,"Groups and geometric analysis" (Academic press,1984)

[29] I. G. Avramidi, "The heat kernel on symmetric spaces via integrating over the group of isometries", Phys. Lett. B **336** (1994) 171 [arXiv:hep-th/9509079].

[30] N-Limic,J.Niederle and R. Raczca, "Eigenfunction expansions associated with the second-order invariant operator on hyperboloids and cones,III" J. Math. Phys. 8 (1967) 1079.

[31] J. Hadamard, " Theorème sur les séries entières" Acta Math. 22 (1898)55.

[32] J. Bros, H. Epstein and U. Moschella, "Particle decays and stability on the de Sitter universe," arXiv:0812.3513 [hep-th].
J. Bros, H. Epstein and U. Moschella, "Lifetime of a massive particle in a de Sitter universe," JCAP **0802** (2008) 003 [arXiv:hep-th/0612184].

[33] O. Nachtmann, Comm. Math. Phys. 6(1967)1.

[34] N. D. Birrell and P. C. W. Davies, "Quantum Fields In Curved Space", *Cambridge, Uk: Univ. Pr. (1982) 340p*

[35] M. B. Einhorn and D. R. T. Jones, "The effective potential, the renormalisation group and vacuum stability," JHEP **0704** (2007) 051 [arXiv:hep-ph/0702295].

[36] I.M. Gel'fand and G.E. Shilov, "Generalized Functions" (Academic Press,New York 1964)

[37] A. Higuchi, "Decay of the free-theory vacuum of scalar field theory in de Sitter spacetime in the interaction picture," Class. Quant. Grav. **26** (2009) 072001.

[38] J. F. Donoghue and T. Torma, "Infrared behavior of graviton-graviton scattering," Phys. Rev. D **60** (1999) 024003 [arXiv:hep-th/9901156].

[39] E. T. Akhmedov, "Real or Imaginary? (On pair creation in de Sitter space)," arXiv:0909.3722 [hep-th].
E. T. Akhmedov, P. V. Buividovich and D. A. Singleton, "De Sitter space and perpetuum mobile," arXiv:0905.2742 [gr-qc].
E. T. Akhmedov, A. Roura and A. Sadofyev, "Classical radiation by free-falling charges in de Sitter spacetime," arXiv:1006.3274 [gr-qc].

[40] M. J. G. Veltman, "Unitarity and causality in a renormalizable eld theory with unstable particles", Physica 29, 186 (1963).

[41] J. C. Collins, "Renormalization. An Introduction To Renormalization, The Renormalization Group, And The Operator Product Expansion," *Cambridge, Uk: Univ. Pr. (1984) 380p*

[42] R. L. Jaffe, "The Casimir effect and the quantum vacuum," Phys. Rev. D **72**, 021301 (2005) [arXiv:hep-th/0503158].

[43] L. Parker and D. J. Toms, "Renormalization Group Analysis Of Grand Unified Theories In Curved Space-Time," Phys. Rev. D **29**, 1584 (1984).

[44] G. 't Hooft, "Under the spell of the gauge principle," Adv. Ser. Math. Phys. **19**, 1 (1994).

[45] S. R. Coleman and E. J. Weinberg, "Radiative Corrections As The Origin Of Spontaneous Symmetry Breaking," Phys. Rev. D **7**, 1888 (1973).

[46] J. S. Dowker and R. Critchley, "Scalar Effective Lagrangian In De Sitter Space," Phys. Rev. D **13**, 224 (1976).

[47] F. A. Berends and J. B. Tausk, "On the numerical evaluation of scalar two loop selfenergy diagrams," Nucl. Phys. B **421** (1994) 456.

[48] G. Perez-Nadal, A. Roura and E. Verdaguer, "Backreaction from weakly and strongly non-conformal fields in de Sitter spacetime," PoS **QG-PH**, 034 (2007).

[49] S. Weinberg, "The Quantum theory of fields. Vol. 1: Foundations," *Cambridge, UK: Univ. Pr. (1995) 609 p*

[50] J.A. Wolf, "Spaces of constant curvature", 1974, Publish or Perish.

[51] Wald R., "General Relativity", University of Chicago Press, 1984.
Wald R., "Quantum field theory in curved spacetime and black hole thermodynamics", University of Chicago Press, 1994.

[52] E. Mottola, "Particle Creation In De Sitter Space", Phys. Rev. D **31**, 754 (1985).