DEPARTAMENTO DE MATEMÁTICAS
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Supercritical elliptic and parabolic problems involving the Hardy-Leray potential. Solvability and qualitative properties

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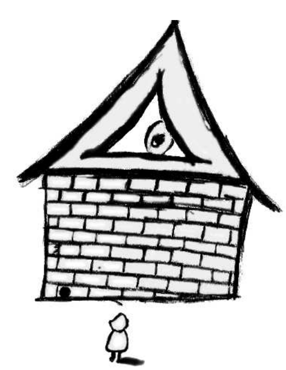
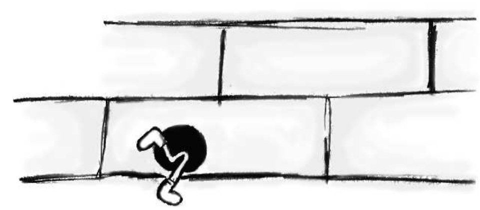
D. Ireneo Peral Alonso

a mis padres



Sé fuerte o sé inteligente pero sé algo en la tierra
y ten en mente un escondite por si empieza la guerra

Violadores del Verso - Información Planta Calle



Contents

Agradecimientos

Me senté en el tren y dije, ¿cómo se empieza un tema? pues digo, voy a empezar con "Hola, buenas noches venía a buscar a Lola.." y todo eso que sigue después.

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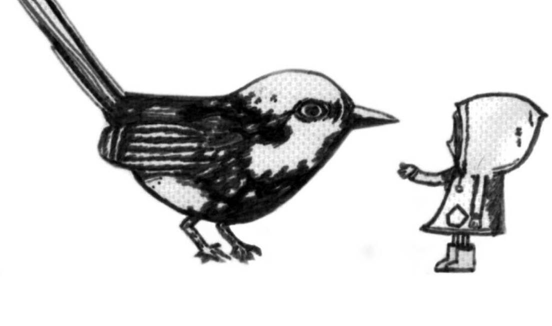
Por mí, por todos mis compañeros y por mí primero.



Chapter 1

Resumen en español

1 Introducción

Las *Ecuaciones en Derivadas Parciales* son una herramienta increíble y poderosa a través de la cual podemos estudiar el mundo y, si hay suerte, poder entenderlo mejor.

Muchas ecuaciones modelan comportamientos físicos que se suceden en la naturaleza y poder estudiar ese tipo de ecuaciones es un trabajo emocionante.

No todas las ecuaciones diferenciales están intrínsecamente relacionadas o modelan de manera directa comportamientos en nuestra naturaleza, pero el progreso en el estudio de este área puede significar un pequeño paso para conseguir futuros logros en este campo.

Es por ello que es importante continuar estudiando y entendiendo mejor las herramientas y los argumentos que en un futuro podrían ser esenciales para alcanzar algo más grande.

En esta memoria se estudian problemas y técnicas clásicas de *Ecuaciones en Derivadas Parciales*. En particular, se estudian problemas elípticos y parabólicos y la relación de éstos con el potencial de *Hardy-Leray*.

Durante los últimos 20 años la influencia del potencial de Hardy en el comportamiento de las ecuaciones elípticas y parabólicas ha sido estudiada ampliamente, véanse las siguientes referencias [20], [34], [33], [4], [5], [6], [7], [8], [9], [11].

Vamos a recordar la desigualdad de *Hardy-Leray* que utilizaremos asiduamente durante este trabajo.

Theorem 1. *(Desigualdad de Hardy-Leray). Sea $N \geq 3$, entonces*

$$\Lambda_N \int_{\mathbb{R}^N} \frac{|\phi|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 dx, \ \text{para todo } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{1.1}$$

con $\Lambda_N = \left(\frac{N-2}{2}\right)^2$ la constante óptima que no se alcanza en $\mathcal{D}^{1,2}(\mathbb{R}^N)$, la clausura de $\mathcal{C}_0^\infty(\mathbb{R}^N)$ respecto a la norma L^2 del gradiente.

Si $\Omega \in \mathbb{R}^N$ *es un dominio acotado tal que* $0 \in \Omega$*, obtenemos las mismas conclusiones en el espacio* $\mathcal{D}^{1,2}(\Omega)$*.*

Si $0 \in \partial\Omega$, la constante dependerá de $\partial\Omega$ y el alcanzarla de la geometría de la frontera en un entorno del 0. Véanse [55], [39], [59] y [60] para más detalle.

Una generalización de esta desigualdad es el siguiente Teorema.

Theorem 2. *(Desigualdad de Hardy-Leray generalizada).*

Sea $1 < p < N$ *y* $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$*. Entonces, se tiene*

$$\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx,$$

con $\Lambda_{N,p} = \left(\dfrac{N-p}{p}\right)^p$ *la constante óptima que no se alcanza.*

La constante no se alcanza en $\mathcal{D}^{1,p}(\Omega)$ *si* $\Omega \in \mathbb{R}^N$ *es un dominio acotado tal que* $0 \in \Omega$*.*

Véase [58] para algunas aplicaciones de esta desigualdad.

La ecuación de Euler correspondiente en este caso tiene un operador *quasilineal*, el *p-Laplaciano*, $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, el cual atañe cierta dificultad en el estudio de este modelo más general.

Las desigualdades anteriores son un caso particular del siguiente Teorema.

Theorem 3. *(Desigualdad de Caffarelli-Khon-Nirenberg) Sea* $u \in W_0^{1,p}(\Omega)$ *y* $1 < p < N$*, entonces existe una constante positiva* $C = C(N, p, \gamma)$ *tal que*

$$\left(\int_\Omega |u|^{p^*(\gamma)}|x|^\gamma dx\right)^{1/p^*(\gamma)} \leq C\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \tag{1.2}$$

con $p^*(\gamma) = \frac{p(N+\gamma)}{N-p}$*.*

Véanse los detalles en [36].

$p^*(\gamma)$ es el exponente crítico en la inclusión de $W_0^{1,p}(\Omega)$ en el espacio de Lebesgue pesado correspondiente.

Consecuentemente, $p^*(0) = p^*$ es el exponente crítico de Sobolev y $p^*(-p) = p$ corresponde con el exponente crítico de Hardy-Sobolev.

Los problemas

En este trabajo estudiaremos la solubilidad de algunos problemas *supercríticos* relacionados con las desigualdades de Hardy introducidas anteriormente. En particular, vamos a considerar los siguientes problemas elípticos y parabólicos.

Problemas elípticos

Empezaremos considerando el siguiente problema

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, \qquad u > 0 \quad \text{en } \Omega, \\ \\ \quad u = 0 \qquad\qquad \text{en } \partial\Omega, \end{cases} \tag{1.3}$$

con $p > 0$, $\Omega \subseteq \mathbb{R}^N$ un dominio acotado y $N \geq 3$.

Uno de los objetivos de esta memoria es enfatizar la importancia de la posición del origen respecto del dominio Ω y su influencia en la solubilidad del problema.

Los resultados previos sobre existencia de solución del problema (1.3) en función del exponente p pueden resumirse de la siguiente forma:

- En el caso **sub-lineal**, $0 < p < 1$, se puede probar la existencia de soluciones de energía independientemente de la posición del cero en el dominio, véase [4].

- El caso **lineal**, $p = 1$, ha sido estudiado en [31], los autores estudian la existencia de soluciones en función de un dato f en el lado derecho de la ecuación. La existencia de solución en el caso lineal con una perturbación de orden cero puede verse en [34].

  Si $0 \in \partial\Omega$, el problema ha sido estudiado en [55]. Considerando

  $$\mu(\Omega) = \inf\Big\{ \int_\Omega |\nabla\phi|^2 : \phi \in W_0^{1,2}(\Omega), \int_\Omega \frac{\phi^2}{|x|^2} = 1 \Big\}, \tag{1.4}$$

  los autores prueban que si $\mu(\Omega) < \mu(\mathbb{R}^N_+) = \dfrac{N^2}{4}$, entonces $\mu(\Omega)$ se alcanza y el problema tiene solución positiva. En caso contrario, $\mu(\Omega) \geq \mu(\mathbb{R}^N_+)$, no hay solución.

En esta memoria vamos a considerar principalmente el caso **supercrítico**, $p > 1$.

El problema semilineal,

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

con $0 \in \partial\Omega$ y $1 < p < \frac{N+2}{N-2}$ ha sido estudiado en [50]. Los autores utilizan un argumento perturbativo que relaciona la existencia de solución con la geometría del dominio. En particular, usando un argumento basado en una identidad de Pohozaev, se prueba que el problema no tiene solución de energía en dominios *estrellados* (respecto del cero). En este artículo también se prueba la existencia de soluciones de energía si el dominio tiene una geometría particular que detallaremos más adelante en este trabajo.

- En esta memoria vamos a considerar el caso $0 \in \partial\Omega$ en el siguiente problema perturbado de (1.5),

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda g(u), & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.6}$$

  con $\lambda > 0$, $p > 1$ y $g(u)$ un término sublineal.

- Con el objetivo de mostrar la influencia de la posición del polo con respecto al dominio en la existencia de soluciones, consideraremos también el caso $0 \in \Omega$ en el problema (1.5), con $\Omega \subset \mathbb{R}^N$ un dominio acotado y $p > 1$.

  Recordemos que si $0 \in \Omega$ y u es una solución distribucional del problema

$$\begin{cases} -\Delta u = \lambda \dfrac{u^p}{|x|^2}, & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.7}$$

  necesariamente, $u \equiv 0$ (véase [32] para más detalles). Con el objetivo de solventar esta obstrucción en la existencia de solución si $0 \in \Omega$, en este trabajo consideraremos el problema (1.7) con un término de absorción en el lado izquierdo de la ecuación,

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.8}$$

con $1 < p < 2$, $f \in L^1(\Omega)$ una función positiva y $\lambda > 0$.

La existencia de solución para el caso crítico, $p = 1$, ha sido estudiada en [9] para todo $\lambda > 0$ y para todo $f \in L^1(\Omega)$ con $f > 0$.

- El siguiente problema que vamos a considerar en esta memoria es el problema *supercrícito*,

$$\begin{cases} -\Delta_p u \equiv -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = \dfrac{u^q}{|x|^p}, & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.9}$$

con $0 \in \partial\Omega$ y $q > p-1$. El operador *p-Laplaciano* es una generalización no stándard del operador *Laplaciano*, este operador no es lineal y no se puede integrar dos veces por partes y puede ser degenerado (si $p > 2$) o singular (si $p < 2$) en el conjunto crítico $Z_u = \{x \in \Omega : \nabla u(x) = 0\}$.

Hay muchas referencias relacionadas con la existencia de solución en problemas con el *p-Laplaciano* como operador principal y con coeficientes singulares: [3], [5], [6], [13], [37], [38], [41], [57], [58], [61], [58], [69], [80], [81], [83], [84], [92].

- En este trabajo nos centraremos también en el estudio de una perturbación del problema (1.9),

$$\begin{cases} -\Delta_p u \equiv -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = \dfrac{u^q}{|x|^p} + \lambda g(u), & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{en } \partial\Omega. \end{cases} \tag{1.10}$$

con $0 \in \partial\Omega$, $q > p-1$, $\lambda > 0$ y $g(u)$ una perturbación subdifusiva en el sentido siguiente

$$\lim_{s\to 0} \frac{g(s)}{s^{p-1}} = \infty.$$

- Como en el caso semilineal, si $0 \in \Omega$ analizaremos también el problema (1.10) con un término de absorción,

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \lambda \dfrac{u^q}{|x|^p} + f, & u > 0 \quad \text{en } \Omega, \\ \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.11}$$

con $1 < p < N$, $q > p-1$, $f \in L^1(\Omega)$ una función positiva y $\lambda > 0$.

En este problema estudiaremos también la simetría de las soluciones de (1.11).

Hay muchas referencias en la literatura en las que se obtienen resultados de simetría para el operador *p-Laplaciano*, véanse por ejemplo: [46], [47], [48], [49], [73].

La dificultad mayor en este caso está en la no linearidad de la parte principal de la ecuación y la consecuente complicación para obtener un resultado fuerte de comparación.

Problemas parabólicos

La existencia de solución de la ecuación del calor con el potencial de *Hardy-Leray*,

$$\begin{cases} u_t - \Delta u &= \lambda \dfrac{u^p}{|x|^2} & \text{en } \Omega_T = \Omega \times (0,T), \\ u &> 0 & \text{en } \Omega_T, \\ u(x,0) &= u_0(x) \geq 0 & \text{en } \Omega, \\ u &= 0 & \text{en } \partial\Omega \times (0,T), \end{cases} \tag{1.12}$$

ha sido estudiada para el caso $0 \in \Omega$:

1. Si $p = 1$, Baras-Goldstein probaron en [20] que existe solución de (1.12) para un intervalo positivo del parámetro λ, más precisamente:
   - Si $\lambda \leq \Lambda_N \equiv \left(\dfrac{N-2}{2}\right)^2$, el problema (1.12) con $p = 1$ y dato f en el lado derecho de la ecuación, tiene una única solución si
   $$\int_\Omega |x|^{-\alpha_1} u_0(x)\,dx < \infty \quad \text{y} \quad \int_0^T \int_\Omega |x|^{-\alpha_1} f\,dx\,dt < \infty,$$
   con α_1 la raíz más pequeña del polinomio $\alpha^2 - (N-2)\alpha + \lambda = 0$.
   - Si $\lambda > \Lambda_N$, el problema (1.12) con $p = 1$ no tiene solución para el dato inicial $u_0 \geq 0$.
2. Si $p > 1$, Brezis-Cabré en [32] probaron un resultado de no existencia para soluciones distribucionales.

- En esta memoria vamos a considerar primero el problema parabólico cuando el polo está en la frontera del dominio, $0 \in \partial\Omega$. En concreto, estudiaremos el problema de evolución asociado a (1.5), es decir, el siguiente problema parabólico

$$
\begin{cases}
u_t - \Delta u = \lambda \dfrac{u^p}{|x|^2} & \text{en } \Omega_T = \Omega \times (0,T), \\
u > 0 & \text{en } \Omega_T, \\
u(x,0) = u_0(x) \geq 0 & \text{en } \Omega, \\
u = 0 & \text{en } \partial\Omega \times (0,T).
\end{cases} \tag{1.13}
$$

Consideraremos el caso *crítico*, $p = 1$ y *supercrítico*, $p > 1$.

- En el último capítulo de esta memoria, estudiaremos el problema (1.13) con $0 \in \Omega$ y con un término regularizante en el lado izquierdo de la ecuación, más particularmente, analizaremos la existencia de solución del siguiente problema,

$$
\begin{cases}
u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f & \text{en } \Omega_T = \Omega \times (0,T), \\
u > 0 & \text{en } \Omega_T, \\
u(x,0) = u_0(x) & \text{en } \Omega, \\
u = 0 & \text{en } \partial\Omega \times (0,T),
\end{cases} \tag{1.14}
$$

con $0 \in \Omega$, $1 < p < 3$, $f \in L^1(\Omega_T)$ una función positiva, $u_0 \in L^1(\Omega)$ y $\lambda > 0$.

2 Organización del trabajo

En esta Sección explicaremos más específicamente la estructura y contenido del trabajo.

Esta memoria está dividida en las siguientes tres partes:

PARTE I: *Problemas elípticos supercríticos con el polo en el interior del dominio.*

En esta primera parte estudiamos el efecto regularizante de algunas perturbaciones en el problema (1.3) con $0 \in \Omega$ y $p \geq 1$. Consideraremos el problema con el operador *Laplaciano* y el *p-Laplaciano*. Uno de los resultados más importantes de esta parte es el estudio de la simetría de las soluciones en el caso *quasilineal*, para ello utilizaremos el método del *Moving Plane*.

PARTE II: *Problemas elípticos supercríticos con el polo en la frontera del dominio.*

En esta segunda parte consideraremos el comportamiento del problema (1.3) cuando $0 \in \partial\Omega$. Al igual que en la primera parte, estudiaremos este problema con el operador *Laplaciano* y con el *p-Laplaciano*. En esta parte queremos enfatizar en la influencia de la posición relativa del polo con respecto al dominio para la existencia de solución.

PARTE III: *Problemas parabólicos críticos y supercríticos con respecto al potencial de Hardy.*

En esta última parte estudiamos el problema *parabólico* asociado a (1.3). En el primer Capítulo consideramos el caso $0 \in \partial\Omega$ y en el segundo trataremos de regularizar el problema si $0 \in \Omega$ con un término que depende del gradiente, evitando así la obstrución en la existencia de solución obtenida en [32].

A continuación detallaremos el contenido de cada capítulo de la memoria.

PARTE I: *Problemas elípticos supercríticos con el polo en el interior del dominio.*

- En el **Capítulo 3** estudiamos el problema *supercrítico* con el operador *Laplaciano* y $0 \in \Omega$.

  Más concretamente, vamos a considerar el siguiente problema

  $$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \ \text{ en } \Omega, \\ u = 0 & \text{en } \ \partial\Omega, \end{cases} \tag{1.15}$$

  con $\lambda > 0$, $p \geq 1$, $f \in L^1(\Omega)$, $f \geq 0$ y $N \geq 3$.

  Lo más destacable de este Capítulo es que en el caso $1 < p < 2$, en contraposición al resultado de no existencia para el problema sin perturbar, probamos que el término gradiente regulariza el problema y es posible hallar una solución para todo $\lambda > 0$. Nótese que para el caso $p \geq 2$ se puede obtener un resultado similar regularizando el problema con el término $|u|^{\beta-1}u|\nabla u|^2$ siendo $\beta > p - 2$.

  Los resultados de este Capítulo pueden verse en la segunda parte de [74].

- En el **Capítulo 4** consideramos el problema *supercrítico* con el operador *p-Laplaciano* y $0 \in \Omega$.

  En particular, estudiamos la existencia y las propiedades cualitativas de las soluciones del problema

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f \ \text{ en } \Omega \\ \quad u \geq 0 \ \text{ en } \Omega, \quad u = 0 \ \text{ en } \ \partial\Omega, \end{cases} \tag{1.16}$$

  donde Ω es un dominio acotado en $\mathbb{R}^N$ con $N \geq 3$ y tal que $0 \in \Omega$, $\vartheta > 0$, $p - 1 < q < p$, $f \geq 0$, $f \in L^1(\Omega)$ y $1 < p < N$.

  La **Sección 2** está dedicada a encontrar solución al problema (1.16). La existencia de solución enfatiza el hecho de que el término $|\nabla u|^p$ en el lado izquierdo de (1.16) es suficiente para obtener un resultado de *rotura de resonancia*.

  La **Sección 3** se centra en las propiedades cualitativas de las soluciones de (1.16), en particular probamos que, bajo ciertas condiciones en el dominio y en el dato f, la soluciones de (1.16) son simétricas. El argumento más importante utilizado en esta parte es el método del *Moving Plane*, que puede encontrarse en [87].

  Los resultados de este Capítulo pueden verse en [72].

PARTE II: *Problemas elípticos supercríticos con el polo en la frontera del dominio.*

- En el **Capítulo 5** consideraremos el siguiente problema *supercrítico* con el operador *Laplaciano* y con $0 \in \partial\Omega$;

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda u^q, \quad u \geq 0 \quad \text{en } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\quad \text{en } \partial\Omega, \end{cases} \tag{1.17}$$

  donde $p > 1$, $0 \leq q < 1$, $\Omega \subset \mathbb{R}^N$ con $N \geq 3$ y $\lambda > 0$ suficientemente pequeño.

  En este Capítulo probaremos la existencia de solución u en $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ para un intervalo de $\lambda > 0$ y sin ninguna restricción por arriba en el valor de p ni en la geometría del dominio, en contraste con los resultados obtenidos en [50].

La idea para la prueba del resultado de existencia es encontrar una supersolución de (1.17) para un intervalo positivo de λ, y obtener una subsolución para todo $\lambda > 0$. Probaremos que la supersolución y la subsolución están ordenadas y consideraremos los problemas iterados. Utilizando un principio de comparación obtendremos una sucesión acotada y ordenada, de manera que podemos pasar al límite y hallar una solución en $W_0^{1,2}(\Omega)$ al problema (1.17) para un intervalo de λ.

Los resultados de este Capítulo pueden verse en la primera parte de [74].

- En el **Capítulo 6** generalizamos el resultado del Capítulo anterior para el operador *p-Laplaciano* y también el resultado obtenido en [50], considerando en este caso el problema con una función $g(\lambda, x, u)$ en el lado derecho de la ecuación, es decir,

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} + g(\lambda, x, u) & \text{en } \Omega, \\ u \geq 0 & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{1.18}$$

con $1 < p < N$, $q > p - 1$ y $0 \in \partial\Omega$.

Este Capítulo está organizado de la siguiente manera:

  - En la **Sección 2** estudiamos el caso $g(\lambda, x, u) \equiv 0$. En la primera parte de la Sección, utilizando una identidad de Pohozaev obtenemos un resultado de no existencia para soluciones de energía en dominios *estrellados* con respecto al 0. En la segunda parte de esta Sección, probamos la existencia de soluciones de energía en un tipo de dominios *no estrellados* utilizando un enfoque variacional en una perturbación del *funcional de energía* asociado al problema (1.18).
  - En la **Sección 3** consideramos el caso $g(\lambda, x, u) = \lambda f(x) u^r$ en el problema (1.18). Probamos la existencia de solución u en $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ por el método de súper y subsolución. En este caso, la existencia de solución no depende de la geometría del dominio en contraste con el caso anterior, $g(\lambda, x, u) \equiv 0$.

Los resultados de este Capítulo pueden verse en [71].

PARTE III: *Problemas parabólicos críticos y supercríticos con respecto al potencial de Hardy.*

- En el **Capítulo 7** estudiamos la variación del problema (1.3) en el tiempo con $0 \in \partial\Omega$. En concreto, obtenemos un resultado de existencia para el problema parabólico siguiente

$$\left\{\begin{array}{cc} u_t - \Delta u = \lambda \dfrac{u^p}{|x|^2} & \text{en } \Omega_T, \\ u > 0 & \text{en } \Omega_T, \\ u(x,0) = u_0(x) > 0 & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega \times (0,T), \end{array}\right. \tag{1.19}$$

  donde $p \geq 1$ y $\Omega \subset \mathbb{R}^N$ es un dominio acotado con $0 \in \partial\Omega$.

  El principal objetivo de este Capítulo es mostrar la diferencia de comportamiento del problema (1.19) cuando $0 \in \partial\Omega$ y cuando $0 \in \Omega$.

  Más precisamente, el contenido de este Capítulo es el siguiente:

  - En la **Sección 2** estudiamos el caso lineal, $p = 1$, encontrando una única solución distribucional para todo $\lambda > 0$ y $u_0 \in L^1(\Omega)$. Es decir, probamos que no hay un resultado del tipo *Baras-Goldstein* si $0 \in \partial\Omega$.
  - En la **Sección 3** consideramos el caso *supercrítico*, $p > 1$. Probamos que existe una única solución bajo ciertas condiciones en el dato inicial.
  - En la **Sección 4** analizamos el caso *supercrítico* añadiendo un término de reacción μu^q, con $0 < q < 1$. Probaremos la existencia de solución para todo $\mu < \mu_0$.

  Los resultados de este Capítulo pueden verse en [19].

- En el **Capítulo 8** estudiamos la existencia de solución del siguiente problema *supercrítico* con $0 \in \Omega$,

$$\left\{\begin{array}{cc} u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f & \text{en } \Omega_T = \Omega \times (0,T), \\ u \geq 0 & \text{en } \Omega_T, \\ u(x,0) = u_0(x) > 0 & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega \times (0,T), \end{array}\right. \tag{1.20}$$

  donde $\Omega \subset \mathbb{R}^N$ con $N \geq 3$, es un dominio acotado que contiene al origen, $1 < p < 3$ y f es una función medible no negativa.

El principal objetivo de este Capítulo es tener una condición en p para obtener la existencia de solución del problema (1.20) para todo $f \in L^1(\Omega_T), u_0 \in L^1(\Omega)$ y para todo $\lambda > 0$.

Probaremos por un argumento de aproximación que si $p < 3$, el término de absorción $u|\nabla u|^2$ tiene un efecto regularizante en la ecuación y permite obtener un resultado de existencia.

Es importante notar que para $p > 3$ es suficiente incluir el término $|u|^{q-1}u|\nabla u|^2$ con $q > p - 2$ para regularizar el problema.

Los resultados de este Capítulo pueden verse en [1].

[1] *Esta memoria ha estado parcialmente subvencionada con los proyectos MTM2007-65018 y MTM2010-18128*

Chapter 2

Introduction

1 The Hardy potential: Presentation

Partial Differential Equations are a classic tool used to study models that attempt to understand better the world. Many physical phenomena in the nature are described by Partial Differential Equations and this study is important in order to predict qualitative or quantitative behaviors and to analyze observations. Nowadays, the application of this powerful tool has been extended to study models in Biology, Finances and Technology.

In this work we will study elliptic and also parabolic problems involving the Hardy-Leray potential. The Hardy-Leray potential is related to the analytical expression, for instance, of the *Heisenberg's uncertainty principle* in Quantum Mechanics, see for instance [56] and [21] among other references. It also appears as a borderline example for regularity, existence of eigenvalues and in the linearization of some *supercritical* semilinear elliptic problems.

It is worthy to point out that during the last 20 years, the influence of a Hardy potential in the behavior of Elliptic and Parabolic Equations has been widely studied in the literature, see for instance [20], [34], [33], [4], [5], [6], [7], [8], [9], [11].

More precisely, we recall the following Hardy's inequality can be stated as follows

Theorem 4. *(Hardy's inequality). Assume that $N \geq 3$, then*

$$\Lambda_N \int_{\mathbb{R}^N} \frac{|\phi|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 dx, \text{ for all } \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{2.1}$$

where $\Lambda_N = \left(\frac{N-2}{2}\right)^2$ is the optimal constant that is not reached in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, the closure of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ respect to the L^2-norm of the gradient.

Moreover, if $\Omega \in \mathbb{R}^N$ is a bounded domain such that $0 \in \Omega$, the same conclusion holds in $\mathcal{D}^{1,2}(\Omega)$.

When $0 \in \partial\Omega$ the inequality is different, for instance, the constant depends on $\partial\Omega$ and its attainability on the geometry of the boundary $\partial\Omega$ in a neighborhood of 0. See, for instance [55] and [39] and also [59], [60] for related problems.

A natural extension of the previous result is the corresponding Hardy's inequality to the homogeneous Sobolev space complection of the test functions with respect to the L^p-norm of the gradient.

More precisely, we have the following Theorem.

Theorem 5. *(Hardy's inequality generalized). Suppose* $1 < p < N$ *and* $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$. *Then, we have*

$$\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx,$$

with $\Lambda_{N,p} = \left(\dfrac{N-p}{p}\right)^p$ *the optimal constant which is not achieved.*

If $\Omega \in \mathbb{R}^N$ *is a bounded domain such that* $0 \in \Omega$, *the optimal constant is the same and also is not attained on* $\mathcal{D}^{1,p}(\Omega)$.

See [58] to find some applications for this inequality.

The fact that, in this case, the Euler equation is a quasilinear operator, the so called *p-Laplacian*, $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$, introduces non trivial difficulties in the study of this general model.

Notice that the previous inequalities are an extreme particular case of the following Theorem.

Theorem 6. *(Caffarelli-Khon-Nirenberg's inequality). Assume* $u \in W_0^{1,p}(\Omega)$ *with* $1 < p < N$. *Then, there exists a positive constant* $C = C(N, p, \gamma)$ *such that*

$$\left(\int_\Omega |u|^{p^*(\gamma)} |x|^\gamma dx\right)^{1/p^*(\gamma)} \leq C \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}, \tag{2.2}$$

where $p^*(\gamma) = \frac{p(N+\gamma)}{N-p}$.

See [36] for the details.

Here, $p^*(\gamma)$ is the critical exponent in the embedding of $W_0^{1,p}(\Omega)$ in the corresponding weighted Lebesgue space.

As a consequence, $p^*(0) = p^*$ is the classical critical Sobolev exponent and $p^*(-p) = p$ corresponds to the critical Hardy-Sobolev exponent.

The problems

In this memory we will study the solvability of some kind of *supercritical* problems related to the Hardy's inequalities above. More precisely, we study the following elliptic and parabolic problems.

Elliptic problems

We begin studying the problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

with $p > 0$, $\Omega \subseteq \mathbb{R}^N$ a bounded domain with smooth boundary and $N \geq 3$.

One of the goals of this work is to emphasize the role of the relative position of the pole with respect to the domain.

The previous results concerning existence of solution to problem (2.3) can be summarize as follows:

- In the **sub-linear** case, $0 < p < 1$, it is easy to prove solvability in the framework of the finite energy solutions with independence of the location of the pole respect to the domain, see [4].
- The **linear** case, $p = 1$, is also well understood, the solvability results are studied in [31] with respect to the summability of the data. The solvability of the linear case with a zero-order perturbation can be found in [34].

  If $0 \in \partial\Omega$, the problem has been studied in [55]. Setting

  $$\mu(\Omega) = \inf\Big\{ \int_\Omega |\nabla\phi|^2 : \phi \in W_0^{1,2}(\Omega), \int_\Omega \frac{\phi^2}{|x|^2} = 1 \Big\}, \tag{2.4}$$

  the authors show that if $\mu(\Omega) < \mu(\mathbb{R}^N_+) = \dfrac{N^2}{4}$, then $\mu(\Omega)$ is attained and the associated linear equation has a positive solution. In the opposite case, $\mu(\Omega) \geq \mu(\mathbb{R}^N_+)$, there is no solution to the linear problem. Moreover, the authors give a geometrical condition in order to have that $\mu(\Omega) < \frac{N^2}{4}$ and then the attainability of the best constant, $\mu(\Omega)$.

In this work we are going to focus mainly in the **supercritical** case, $p > 1$.

The semilinear problem,

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $0 \in \partial\Omega$ and $p > 1$, has been partially solved in [50] with a perturbative argument involving the shape of the domain and under the hypothesis $p < \frac{N+2}{N-2}$. Moreover, in [50], using a Pohozaev argument, it is proved that in a *starshaped* domain (with respect to 0) the problem has no solution of finite energy. However, using an involved perturbative method, the authors proved in the same paper that, if the domain has a suitable shape, then, there exists a solution of finite energy.

- In this work we are going to consider the following *supercritical* semilinear problem, which is a perturbation of (2.5),

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda g(u), & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

  with $\lambda > 0$, $p > 1$, $g(u)$ a sublinear term and $0 \in \partial\Omega$.

- In order to show the influence of the position of the pole with respect to the domain in the solvability of the problem, we also consider the problem assuming $0 \in \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p > 1$.

  It is known that if $0 \in \Omega$ and u is a distributional solution to the equation

$$\begin{cases} -\Delta u = \lambda \dfrac{u^p}{|x|^2}, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

  it is necessary that $u \equiv 0$. This result can be seen in [32]. The proof by Brezis-Cabré shows that there is a *local obstruction* in the existence of solution.

  In order to avoid this obstruction, in this work we will consider the problem

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

  with $0 \in \Omega$, $1 < p < 2$, $f \in L^1(\Omega)$ a positive function and $\lambda > 0$.

  If $p = 1$ (the critical problem), an existence result has been obtained in [9] for all $\lambda > 0$ and for all $f \in L^1(\Omega)$. That is, some kind of breaking of resonance is obtained for the critical case.

- The next step in this work is to consider the following quasilinear *supercritical* problem,

$$\begin{cases} -\Delta_p u \equiv -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = \dfrac{u^q}{|x|^p}, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

  with $0 \in \partial\Omega$ and $p^* - 1 > q > p - 1$. The *p-Laplacian* operator is not a standard generalization of the *Laplacian* case, this operator is non linear and we can not integrate twice by parts. Moreover, it can be degenerate or singular in the critical set $Z_u = \{x \in \Omega : \nabla u(x) = 0\}$, depending on if $p > 2$ or $p < 2$, respectively.

  There is an extensive literature regarding existence of solutions for problems involving the *p-Laplacian* operator with singular coefficients. We refer for example to the (far from being complete) list of references [3], [5], [6], [13], [37], [38], [41], [57], [58], [61], [58], [69], [80], [81], [83], [84], [92] and to the bibliographies therein.

- In this work we are also going to focus in the study of a perturbation of problem (2.9),

$$\begin{cases} -\Delta_p u \equiv -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = \dfrac{u^q}{|x|^p} + \lambda g(u), & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

  with $0 \in \partial\Omega$, $q > p - 1$, $\lambda > 0$ and $g(u)$ a sub-diffusive perturbation in the following sense

$$\lim_{s\to 0} \frac{g(s)}{s^{p-1}} = \infty.$$

- As in the semilinear case, we will analyze the *quasilinear* problem with some absorption term if $0 \in \Omega$,

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \lambda \dfrac{u^q}{|x|^p} + f, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

  with $1 < p < N$, $q > p - 1$, $f \in L^1(\Omega)$ a positive function and $\lambda > 0$. The existence of solution to problem (2.11) with $q = p - 1$ and the exponent of the gradient term equal to q has been studied in [75].

Another issue studied in this memory is some symmetry properties of the solution to the problem (2.11).

The precedents for this kind of results with the *p-Laplacian* operator can be found in the references [46], [47], [48], [49], [73] and the references therein. The main difficulty lies in the nonlinearity of the principal part and in the implication of it in the strong comparison result.

Parabolic problems

The heat equation with the Hardy-Leray potential, i.e, the problem

$$\left\{\begin{array}{rcll} u_t - \Delta u & = & \lambda \dfrac{u^p}{|x|^2} & \text{in } \Omega_T = \Omega \times (0,T), \\ u & > & 0 & \text{in } \Omega_T, \\ u(x,0) & = & u_0(x) \geq 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega \times (0,T), \end{array}\right. \tag{2.12}$$

has been studied in the case $0 \in \Omega$:

1. If $p = 1$, Baras-Goldstein in [20], proved that there exists a solution to (2.12) for a positive interval of the parameter λ. More precisely, we could summarize the results by Baras-Goldstein as follows

   - If $\lambda \leq \Lambda_N \equiv \left(\dfrac{N-2}{2}\right)^2$, problem (2.12), with $p = 1$ and a data f in the right hand side, has a unique global solution if

     $$\int_\Omega |x|^{-\alpha_1} u_0(x)\, dx < \infty \quad \text{and} \quad \int_0^T \int_\Omega |x|^{-\alpha_1} f\, dx\, dt < \infty,$$

     with α_1 the smallest root of $\alpha^2 - (N-2)\alpha + \lambda = 0$.
   - If $\lambda > \Lambda_N$, problem (2.12) with $p = 1$ has no (even local distributional) solution for $u_0 \geq 0$.

   Such *spectral-dependent* type of results are deeply associated to problems with the same principal part. See, for instance, [10], [11].

2. If $p > 1$, a strong local nonexistence result for solutions in a distributional sense was proved by Brezis-Cabré in [32].

- In this work we will try to obtain existence results when the pole is on the boundary of the domain, $0 \in \partial\Omega$. We will study the evolution problem associated to (2.5), that is, the following parabolic problem

$$\begin{cases} u_t - \Delta u &= \lambda \dfrac{u^p}{|x|^2} & \text{in } \Omega_T = \Omega \times (0,T), \\ u &> 0 & \text{in } \Omega_T, \\ u(x,0) &= u_0(x) \geq 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.13}$$

We consider the *critical* and *supercritical* parabolic problems with $0 \in \partial\Omega$, that is, $p = 1$ and $p > 1$ respectively.

- The last topic of this memory we study problem (2.13) with $0 \in \Omega$ and with some regularization term in the left hand side of the equation. That is, we analyze the existence of a solution for the following problem

$$\begin{cases} u_t - \Delta u + u|\nabla u|^2 &= \lambda \dfrac{u^p}{|x|^2} + f & \text{in } \Omega_T = \Omega \times (0,T), \\ u &> 0 & \text{in } \Omega_T, \\ u(x,0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.14}$$

  with $0 \in \Omega$, $1 < p < 3$, $f \in L^1(\Omega_T)$ a positive function, $u_0 \in L^1(\Omega)$ and $\lambda > 0$.

  The critical case $p = 1$ has been studied in [12].

2 Organization of the work

We have already described the different subjects studied in this work and in this Section we are going to explain more specifically the organization of the work.

This memory is divided in the following three parts:

PART I: *Supercritical elliptic problems with the pole inside the domain.*

In this part we study the regularizing effect of some perturbations in problem (2.3) with $0 \in \Omega$ and $p \geq 1$. We consider the problem with the *Laplace operator* and also with the *p-Laplace operator*. One of the main results is the study of symmetry properties of the solutions in the *quasilinear* case. In this Part we use the *Moving Plane Method* which in the context of the *p-Laplacian* operator is quite involved.

PART II: *Supercritical elliptic problems with the pole at the boundary of the domain.*

In this part we focus on the behavior of (2.3) when $0 \in \partial\Omega$. We study the problem with the *Laplace operator* and with the *p-Laplace operator*. This

Part emphasizes the influence of the relative position of the pole respect to the domain in the existence of solution.

PART III: *Critical and supercritical parabolic problems with respect to the Hardy potential.*

In this last Part we study some *parabolic* problems associated to (2.3). In the first Chapter we consider the case $0 \in \partial\Omega$ and in the second one we try to regularize the problem with a term dependent on the gradient in order to avoid the local obstruction obtained in [32] if $0 \in \Omega$.

We proceed to give some details of each Chapter of this work.

PART I: *Supercritical elliptic problems with the pole inside the domain.*

- In **Chapter 3** we study a *supercritical* elliptic problem with the *Laplace operator* as the main operator and $0 \in \Omega$.

  More precisely, we are going to consider the following problem
$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$
  where $\lambda > 0$, $p \geq 1$, $f \in L^1(\Omega)$, $f \geq 0$ and $N \geq 3$.

  The main feature in this Chapter is that if $1 < p < 2$, despite the local obstruction in the unperturbed problem, we prove that the square of the gradient regularizes the problem and we are able to obtain a solution for all $\lambda > 0$. This absorption term breaks down the lack of solvability killing the local obstruction in the existence when $0 \in \Omega$. Notice that for $p \geq 2$, we can obtain a similar result regularizing the problem with the term $|u|^{\beta-1}u|\nabla u|^2$, being $\beta > p - 2$.

  This result can be seen in the second part of [74].

- In **Chapter 4** we study the *supercritical* problem with the *p-Laplace operator* and $0 \in \Omega$.

  In particular, we study the existence and qualitative properties of the solutions to the problem
$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.16}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$ and such that $0 \in \Omega$, $\vartheta > 0$, $p-1 < q < p$, $f \geq 0$, $f \in L^1(\Omega)$ and $1 < p < N$.

Section 2 is dedicated to find a solution to (2.16). The existence of solution emphasizes the fact that the term $|\nabla u|^p$ on the left hand side of (2.16) is enough to get a *resonance breaking result*.

In **Section 3** we focus in the qualitative properties of the weak solutions to (2.16). In particular, we prove that, under some assumptions in the domain and in the data f, the solutions to (2.16) are symmetric. The main important tool of this part is the *Moving Plane Method*, that can be found in [87].

These results can be seen in [72].

PART II: *Supercritical elliptic problems with the pole at the boundary of the domain.*

- In **Chapter 5** we are going to consider the following *supercritical* problem with the *Laplacian* operator and $0 \in \partial\Omega$;

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda u^q, \quad u \geq 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.17}$$

where $p > 1$, $0 \leq q < 1$, $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ and $\lambda > 0$ small enough. We prove the existence of solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ for an interval of $\lambda > 0$ and without any restriction of the domain, in contrast with the previous results in [50] and without any condition from above on the value of p.

The main idea of this Chapter is to find a supersolution to (2.17) for an interval of λ and get a subsolution for all $\lambda > 0$. We prove that the supersolution and the subsolution are ordered and we consider the iterative problems. Using a comparison argument we get that the sequence of solutions of the iterative problems is bounded and is also ordered. Then, we are able to pass to the limit and to get a solution in $W_0^{1,2}(\Omega)$ to (2.17) for a positive interval of λ.

This result can be seen in the first part of [74].

- In **Chapter 6** we generalize the result of the previous Chapter for the *p-Laplace operator* and also the result in [50], being in this case the problem with a function $g(\lambda, x, u)$ in the right hand side of the equation, that is

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} + g(\lambda, x, u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.18}$$

with $1 < p < N$, $q > p - 1$ and $0 \in \partial\Omega$.

This Chapter is organized as follows:

- In **Section 2** we study the case $g(\lambda, x, u) \equiv 0$. First, by a Pohozaev's identity we deduce a nonexistence result of energy solutions in *starshaped* domains. Subsequently, we prove the existence of an energy solution for a convenient non-starshaped domain using a variational approach to a functional which is a perturbation of the *standard energy functional* associated to (2.18).
- In **Section 3** we deal with the term $g(\lambda, x, u) = \lambda f(x)u^r$. We show the existence of a solution u in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by the method of super- and subsolutions. In this case, the existence of solution does not depend on the geometry of the domain in contrast with the case $g(\lambda, x, u) \equiv 0$. Moreover, we characterize the *minimality of the solution* and some other comments are given.

This result can be seen in [71].

PART III: *Critical and supercritical parabolic problems with respect to the Hardy potential.*

- In **Chapter 7** we study the variation of the problem (2.3) on time, the *parabolic case* with $0 \in \partial\Omega$. The aim of this Chapter is to discuss the existence of solution to the following parabolic problem

$$\begin{cases} u_t - \Delta u = \lambda \dfrac{u^p}{|x|^2} & \text{in } \Omega_T, \\ u > 0 & \text{in } \Omega_T, \\ u(x, 0) = u_0(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{2.19}$$

where $p \geq 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with $N \geq 3$ and $0 \in \partial\Omega$.

The main goal in this Chapter is to show that the behavior of problem (2.19) when $0 \in \partial\Omega$ is essentially different from the one obtained when $0 \in \Omega$.

More precisely, the main new features in this Chapter are the following:

- In **Section 2** we study the linear case, $p = 1$, finding a unique distributional solution for all $\lambda > 0$ and $u_0 \in L^1(\Omega)$. That is, we prove that there is no a *Baras-Goldstein* type result if $0 \in \partial\Omega$.
- In **Section 3** we study the *supercritical* case, $p > 1$. We prove that there exists a unique solution under some condition on the initial data.
- In **Section 4** we analyze the *supercritical* parabolic problem adding a concave reaction term μu^q, with $0 < q < 1$. We prove the existence of a solution for all $\mu < \mu_0$.

This result can be seen in [19].

- In **Chapter 8** we discuss the existence of solution to the following *supercritical* parabolic problem

$$\begin{cases} u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f & \text{in } \Omega_T = \Omega \times (0,T), \\ u \geq 0 & \text{in } \Omega_T, \\ u(x,0) = u_0(x) > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.20}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, is a bounded domain that contains the origin, $0 \in \Omega$, $1 < p < 3$ and f is a measurable nonnegative function.

The main objective is to get a natural condition on p in order to obtain the existence of a solution to problem (2.20) for all $f \in L^1(\Omega_T), u_0 \in L^1(\Omega)$ and for all $\lambda > 0$.

We prove by an approximation argument that if $p < 3$, the absorption term $u|\nabla u|^2$ has a "regularizing" effect on the equation and allows us to get the existence of a solution for the largest possible class of data f, u_0 and for all $\lambda > 0$.

It is worthy to point out that, for $p > 3$, it is sufficient to regularize the problem with a quasilinear term of the form $|u|^{q-1}u|\nabla u|^2$ with $q > p - 2$.

This result can be seen in [1].

[2] *This memory has been partially supported by the proyects MTM2007-65018 and MTM2010-18128*

Part I

Supercritical elliptic problems with the pole inside the domain

Chapter 3

Regularization of a first order term in the semilinear model

1 Introduction

In this Chapter we are going to consider the solvability of the following problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, & u > 0 \quad \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $p \geq 1$, $\Omega \in \mathbb{R}^N$ a bounded domain and $N \geq 3$.

The behavior of the problem depends deeply on the situation of the pole with respect to $\overline{\Omega}$:

- It is clear that if $0 \in \mathbb{R}^N \setminus \overline{\Omega}$ and $p < \dfrac{N+2}{N-2}$ the problem has a positive solution by the classical *Mountain Pass Theorem* introduced by Ambrosetti and Rabinowitz in [16].
- In contrast, if $0 \in \Omega$ and $p > 1$, problem (3.1) has no solution, even in the weakest sense of distributional solution.

  Actually, (3.1) has no weak supersolution even locally, for a detailed proof see [32].

  Notice that to prove a nonexistence result for (3.1) in the sense of the energy solutions it is sufficient to argue by contradiction using the following Picone's inequality. In this way we get a contradiction with the Hardy's inequality.

  Theorem 7. *(Picone's inequality) Assume $u \in W_0^{1,2}(\Omega)$, $u \geq 0$ and $v \in W_0^{1,2}(\Omega)$, $-\Delta v \geq 0$ is a bounded Radon measure, $v_{|\partial\Omega} = 0$, $v \geq 0$ and not identically zero, then*

$$\int_\Omega |\nabla u|^2 dx \geq \int_\Omega \Big(\frac{|u|^2}{v}\Big)(-\Delta v) dx.$$

See [5] for the details of this Theorem.

- The intermediate case, $0 \in \partial\Omega$, has an extreme behavior and it will be considered in the second Part of this work.

The main result of this Chapter is to study the effect of a first order absorption term in the solvability of the problem (3.1) in the case $0 \in \Omega$, that is, we will study the following problem

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

if $1 \leq p < 2$, $\lambda > 0$ and $f \in L^1(\Omega)$. To be precise, in Section 2 we are going to prove the existence of a solution to problem (3.2) and, as in the case $p = 1$ in [9], we will be able to obtain a solution to the problem for all $\lambda > 0$.

The techniques used are: *i*) study the existence of solution to some approximated problems; *ii*) get a priori estimates and *iii*) pass to the limit.

In order to make easier the calculations, we first consider the case $f \in L^m(\Omega)$, with $m > \frac{N}{2}$ and then we pass to the limit.

In the last Sections of this Chapter we study some further results, for instance, the case $p = 2$ and $p \geq 2$ and the problem with a general exponent q for the gradient. In fact, in Chapter 8 of this memory, we will study carefully the existence of solutions to the parabolic problem associated to the case $p \geq 2$.

In this Chapter we are looking for solutions in the sense of the following definition.

Definition 1. *Let $u \in W_0^{1,2}(\Omega)$. We say that u is an energy solution to problem* (3.2) *if*

$$\int_\Omega < \nabla u, \nabla\phi > + \int_\Omega |\nabla u|^2 \phi = \lambda \int_\Omega \frac{u^p}{|x|^2}\phi + \int_\Omega f\phi \quad \forall \phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

We recall also the well known definition of a truncated function and the positive and negative part of a function.

Definition 2. *For a measurable function u, consider the k-truncation of u defined as*

$$T_k(u) = \begin{cases} u & \textit{if} \quad |u| < k; \\ k\dfrac{u}{|u|} & \textit{if} \quad |u| \geq k. \end{cases} \tag{3.3}$$

Definition 3. *We denote f^+ the positive part of the function f as $f^+ = \max\{f, 0\}$. We denote f^- the negative part of the function f as $f^- = \min\{f, 0\}$.*

All the results of this Chapter can be seen in the second part of the paper [74].

2 Existence result with $1 \leq p < 2$

The main existence result of this Section is the following.

Theorem 8. *Consider the problem*

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \ \ in\, \Omega, \\ u = 0 & on \ \ \partial\Omega, \end{cases} \tag{3.4}$$

with $1 \leq p < 2$ and assume that $f \in L^1(\Omega)$ is a positive function, then for all $\lambda > 0$ there exists a positive weak solution $u \in W_0^{1,2}(\Omega)$.

To prove Theorem 8 we proceed step by step. First we prove the result considering the data $f \in L^m(\Omega)$ with $m > \dfrac{N}{2}$ and then, the general case, $f \in L^1(\Omega)$, follows approximating the datum.

2.1 Existence result with $f \in L^m(\Omega)$

We consider first the truncated problem and we are going to find solution for this problem with a positive data $f \in L^m(\Omega)$.

Theorem 9. *Assume $1 \leq p < 2$ and $f \in L^m(\Omega)$, $m > \frac{N}{2}$, then, there exists a positive solution to the problem*

$$\begin{cases} -\Delta u_k + |\nabla u_k|^2 &=& \lambda \dfrac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} + f, \quad in \ \ \Omega, \\ u_k &=& 0 \qquad on \quad \partial\Omega, \\ u_k &\in& W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \qquad and \ u_k > 0. \end{cases} \tag{3.5}$$

Proof.

Since $f \geq 0$, $\phi \equiv 0$ is a subsolution to problem (3.5).

Consider the function ψ the solution to

$$\begin{cases} -\Delta\psi &= \frac{\lambda k}{|x|^2+\frac{1}{k}} + f \quad x\in\Omega,\\ \psi &= 0 \quad x\in\partial\Omega, \end{cases}$$

therefore, ψ is a supersolution to (3.5).

To prove Theorem 9 we consider a sequence of approximated problems that we solve by iteration and using a convenient comparison argument. We take as starting point $w_0 = 0$ and we consider iteratively the problem,

$$\begin{cases} -\Delta w_n + \dfrac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2} = \lambda \dfrac{T_k(w_{n-1}^p)}{|x|^2+\frac{1}{k}} + f \quad \text{in} \quad \Omega,\\ w_n = 0 \quad \text{on} \quad \partial\Omega, \quad w_n\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega),\\ w_n > 0. \end{cases} \tag{3.6}$$

Proposition 1. *There exists $w_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ solution to* (3.6)*.*

Moreover, $0 \le w_n \le \psi \quad \forall n\in\mathbb{N}$*.*

Proof. Let us consider the problem:

$$-\Delta w_n + g(x, w_n, \nabla w_n) = 0, \tag{3.7}$$

where

$$g(x,w_n,\nabla w_n) = \begin{cases} -\frac{\lambda k}{|x|^2+\frac{1}{k}} - f \qquad \text{if } w_n \ge \psi,\\ \frac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} - \lambda\dfrac{T_k(w_{n-1}^p)}{|x|^2+\frac{1}{k}} - f \qquad \text{if } 0\le w_n<\psi,\\ -f \qquad \text{if } w_n \le 0. \end{cases} \tag{3.8}$$

Using the Leray-Lions arguments, see it in [68], we can find solutions to the approximated problem (3.7) for each n and by classical regularity results such solution w_n belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

We are going to show that $w_n \ge 0$. Since $\phi\equiv 0$ is a subsolution of (3.7) and w_n is a solution we have

$$-\Delta w_n + g(x,w_n,\nabla w_n) + f \ge 0.$$

Using $-(w_n^-)$ as a test function in the last expression, one has

$$-\int_\Omega \nabla w_n\cdot\nabla w_n^- dx - \int_\Omega \Big(g(x,w_n,\nabla w_n)+f\Big)(w_n^-)dx \ge 0.$$

We define the following set,

$$R = \{x : x\in\Omega : w_n \le 0\},$$

therefore,

$$-\int_{\Omega} \nabla w_n \cdot \nabla w_n^- dx - \int_R \Big(g(x, w_n, \nabla w_n) - f\Big)(w_n^-) dx \geq 0.$$

Taking into account (3.8), $g(x, w_n, \nabla w_n) = f$ in R, then,

$$-\int_{\Omega} |\nabla w_n^-|^2 dx \geq 0.$$

Hence, we conclude that $w_n \geq 0$.

We want to prove now $w_n \leq \psi$.

Since ψ and w_n are respectively a supersolution and a solution of (3.7), we have

$$-\Delta w_n + \Delta \psi + g(x, w_n, \nabla w_n) - g(x, \psi, \nabla \psi) \leq 0.$$

Using $T_M([w_n - \psi]^+)$ with $M > 0$ as a test function in the last expression it follows

$$\begin{aligned} & \int_{\Omega} |\nabla T_M(w_n - \psi)^+|^2 dx \\ + & \int_{\Omega} \Big(g(x, w_n, \nabla w_n) + \frac{\lambda k}{|x|^2 + \frac{1}{k}} + f\Big) T_M([w_n - \psi]^+) dx \leq 0. \end{aligned}$$

We define the following sets,

$$R = \{x : x \in \Omega : \psi \leq w_n\},$$
$$R^M = \{x : x \in \Omega : 0 \leq w_n - \psi \leq M\}.$$

Thus,

$$T_M([w_n - \psi]^+) = 0 \ \text{ if } x \in \Omega - R \qquad \text{or} \qquad w_n^- = 0$$

and

$$\nabla T_M([w_n - \psi]^+) = 0 \ \text{ if } x \in \Omega - R^M \qquad \text{or} \qquad w_n^- = 0.$$

Therefore,

$$\begin{aligned} & \int_{R^M} |\nabla T_M(w_n - \psi)^+|^2 dx \\ - & \int_{R} \Big(g(x, w_n, \nabla w_n) + \frac{\lambda k}{|x|^2 + \frac{1}{k}} + f\Big) T_M([w_n - \psi]^+) dx \leq 0. \end{aligned}$$

By (3.8), we get

$$\int_{R^M} |\nabla (w_n - \psi)^+|^2 dx \leq 0 \quad \forall M \in \mathbb{R}^+.$$

Therefore, $w_n \leq \psi$ and we conclude. $\square$

2.2 Convergence of w_n in $W_0^{1,2}(\Omega)$

i) *Weak convergence of w_n to u_k in $W_0^{1,2}(\Omega)$*

For simplicity of typing, we are going to call $H_n(\nabla w_n) = \dfrac{|\nabla w_n|^2}{1+\frac{1}{n}|\nabla w_n|^2}$.

Taking w_n as a test function in the approximated problems (3.6), we obtain

$$\int_\Omega |\nabla w_n|^2 dx + \int_\Omega H_n(\nabla w_n) w_n dx = \lambda \int_\Omega \frac{T_k(w_{n-1}^p)}{|x|^2+\frac{1}{k}} w_n dx + \int_\Omega f w_n dx$$
$$\leq \lambda \int_\Omega k \frac{w_n}{|x|^2+\frac{1}{k}} dx + \int_\Omega f w_n dx \leq \lambda \int_\Omega k \frac{\psi}{|x|^2+\frac{1}{k}} dx + \int_\Omega f \psi dx.$$

That is,

$$\int_\Omega |\nabla w_n|^2 dx + \int_\Omega H_n(\nabla w_n) w_n dx \leq C(k, f, \Omega).$$

Since

$$\int_\Omega H_n(\nabla w_n) w_n dx \geq 0, \qquad \int_\Omega |\nabla w_n|^2 dx \leq C(k, f, \Omega).$$

Therefore, up to a subsequence, $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$.

Since $||w_n||_{L^\infty(\Omega)} \leq C$,

$$\int_\Omega w_n \varphi dx = \int_\Omega u_k \varphi dx \quad \text{for} \quad \varphi \in L^1(\Omega)$$

and then, $w_n \rightharpoonup u_k$ weakly-* in $L^\infty(\Omega)$, hence, $u_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

ii) *Strong convergence of w_n to u_k in $W_0^{1,2}(\Omega)$*

We want to prove that $w_n \to u_k$ strongly in $W_0^{1,2}(\Omega)$ to conclude that u_k solves the truncated problem (3.5).

Consider the function $\phi(s) = s e^{\frac{1}{4}s^2}$ which verifies $\phi'(s) - |\phi(s)| \geq \frac{1}{2}$. Taking $\phi(w_n - u_k)$ as a test function in (3.6) we get

$$\int_\Omega \nabla w_n \phi'(w_n - u_k) \nabla(w_n - u_k) dx + \int_\Omega H_n(\nabla w_n) \phi(w_n - u_k) dx$$
$$= \lambda \int_\Omega \frac{T_k(w_{n-1}^p)}{|x|^2+\frac{1}{k}} \phi(w_n - u_k) dx + \int_\Omega f \phi(w_n - u_k) dx.$$

Notice that

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)dx - \int_\Omega \nabla u_k \phi'(w_n - u_k)\nabla(w_n - u_k)dx$$
$$= \int_\Omega \phi'(w_n - u_k)|\nabla(w_n - u_k)|^2 dx,$$

then, the first term on the left hand side can be estimated as follows:

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)dx$$
$$= \int_\Omega \phi'(w_n - u_k)|\nabla(w_n - u_k)|^2 dx + \int_\Omega \nabla u_k \phi'(w_n - u_k)\nabla(w_n - u_k)dx$$
$$= \int_\Omega \phi'(w_n - u_k)|\nabla(w_n - u_k)|^2 dx + \int_\Omega \nabla u_k \phi'(w_n - u_k)\nabla w_n dx$$
$$- \int_\Omega \nabla u_k \phi'(w_n - u_k)\nabla u_k dx.$$

Since $w_n \rightharpoonup u_k$ weakly in $W_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)dx$$
$$= \int_\Omega \phi'(w_n - u_k)|\nabla(w_n - u_k)|^2 dx + \int_\Omega |\nabla u_k|^2 \phi'(w_n - u_k)dx$$
$$- \int_\Omega |\nabla u_k|^2 \phi'(w_n - u_k)dx.$$

Therefore,

$$\int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)dx = \int_\Omega |\nabla(w_n - u_k)|^2 \phi'(w_n - u_k)dx + o(1).$$

For the second term on the left hand side we have

$$\int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)dx \leq \int_\Omega |\nabla w_n|^2 |\phi(w_n - u_k)|dx$$
$$= \int_\Omega |\nabla w_n - \nabla u_k|^2 |\phi(w_n - u_k)|dx - \int_\Omega |\nabla u_k|^2 |\phi(w_n - u_k)|dx$$
$$+ 2\int_\Omega \nabla w_n \nabla u_k |\phi(w_n - u_k)|dx.$$

Since $w_n \rightharpoonup u_k$ in $W_0^{1,2}(\Omega)$ and $|\phi(w_n - u_k)| \to 0$ almost everywhere, we obtain

$$\int_\Omega |\nabla u_k|^2 |\phi(w_n - u_k)|dx \to 0 \quad \text{ as } n \to \infty$$

and also by the weak convergence,

$$\int_\Omega \nabla w_n \nabla u_k \phi(w_n - u_k)dx \to 0 \quad \text{ as } n \to \infty.$$

Then, passing to the limit as $n \to \infty$, we have the following estimation,

$$\int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)dx \leq \int_\Omega |\nabla w_n - \nabla u_k|^2 |\phi(w_n - u_k)|dx + o(1).$$

Notice that

$$\lambda \int_\Omega \frac{T_k(w_{n-1}^p)}{|x|^2 + \frac{1}{n}} \phi(w_n - u_k)dx + \int_\Omega f\phi(w_n - u_k)dx$$

go to zero as $n \to \infty$.

Hence,

$$\int_\Omega |\nabla(w_n - u_k)|^2 \phi'(w_n - u_k)dx - \int_\Omega |\nabla(w_n - u_k)|^2 \phi(w_n - u_k)dx + o(1)$$

$$\leq \int_\Omega \nabla w_n \phi'(w_n - u_k)\nabla(w_n - u_k)dx + \int_\Omega H_n(\nabla w_n)\phi(w_n - u_k)dx \leq o(1)$$

and, since $\phi'(s) - |\phi(s)| > \frac{1}{2}$ we conclude that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla w_n - \nabla u_k|^2 dx \\
&\quad \leq \int_\Omega (\phi'(w_n - u_k) - |\phi(w_n - u_k)|)|\nabla w_n - \nabla u_k|^2 dx \leq o(1),
\end{aligned}$$

whence, $w_n \to u_k$ in $W_0^{1,2}(\Omega)$. In particular, up to a subsequence, $H_n(\nabla w_n) \to |\nabla u_k|^2$ a.e. in Ω and since $w_n \to u_k$ in $W_0^{1,2}(\Omega)$, the equi-integrability follows, therefore, by Vitali's Theorem,

$$H_n(\nabla w_n) \to |\nabla u_k|^2 \text{ in } L^1(\Omega).$$

Since $-\Delta w_n \to -\Delta u_k$ in the sense of distributions, we conclude that u_k satisfies the problem

$$\begin{cases}
-\Delta u_k + |\nabla u_k|^2 = \lambda \dfrac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} + f & \text{in} \quad \Omega, \\
u_k = 0 & \text{on} \quad \partial\Omega, \\
u_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \quad \text{and} & u_k > 0.
\end{cases} \tag{3.9}$$

□

2.3 Pass to the limit when $k \to \infty$

In this Subsection we are going to study the convergence of the solutions to the truncated problem, u_k, to the solution u, in this way we will be able to prove the following result.

Theorem 10. *Consider* $1 \le p < 2$ *and assume that* $f \in L^m(\Omega)$, $m > \frac{N}{2}$, *is a positive function, then, for all* $\lambda > 0$, *there exists a positive solution* $u \in W_0^{1,2}(\Omega)$ *to problem* (3.4).

Proof. We need to analyze the convergence of $\{u_k\}$, the solutions to problems (3.9).

i) Weak convergence of $\{u_k\}$ *to* u *in* $W_0^{1,2}(\Omega)$.

Since $\{u_k\} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we can use u_k as a test function in the truncated problem (3.9). It follows that

$$\int_\Omega |\nabla u_k|^2 dx + \frac{4}{9}\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx \le \lambda \int_\Omega \frac{u_k^{p+1}}{|x|^2} dx + \int_\Omega f u_k dx. \quad (3.10)$$

Using Hölder's and Hardy-Leray's inequalities we obtain that

$$\begin{aligned} \int_\Omega \frac{u_k^{p+1}}{|x|^2} dx &\le \Big(\int_\Omega \frac{u_k^3}{|x|^2} dx\Big)^{\frac{p+1}{3}} \Big(\int_\Omega \frac{dx}{|x|^2}\Big)^{\frac{2-p}{3}} \\ &\le C\Lambda_N^{-\frac{p+1}{3}} \Big(\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx\Big)^{\frac{p+1}{3}}. \end{aligned}$$

Therefore, since $p < 2$, we obtain that for all $\varepsilon > 0$ there exists $C_\varepsilon = C_\varepsilon(p, N) > 0$ such that

$$\Big(\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx\Big)^{\frac{p+1}{3}} \le \varepsilon \int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx + C_\varepsilon. \quad (3.11)$$

On the other hand, setting $m' = 1 - \dfrac{1}{m}$,

$$\begin{aligned} \int_\Omega f u_k dx &\le ||f||_{L^m(\Omega)} ||u_k||_{L^{m'}(\Omega)} \\ &\le ||f||_{L^m(\Omega)} \Big(\big(\int_\Omega u_k^{2^*} dx\big)^{\frac{m'}{2^*}} |\Omega|^{1-\frac{m'}{2^*}} \Big)^{\frac{1}{m'}}. \end{aligned}$$

Therefore, using Sobolev's inequality, we get

$$\int_\Omega f u_k dx \le S\Big(\int_\Omega |\nabla u_k|^2 dx\Big)^{\frac{1}{2}} |\Omega|^{\frac{1}{m'} - \frac{1}{2^*}} ||f||_{L^m(\Omega)}.$$

Thus, for all $\varepsilon > 0$ there exists $D_\varepsilon = D_\varepsilon(m, N, \Omega, f)$ such that

$$\Big(\int_\Omega |\nabla u_k|^2 dx\Big)^{\frac{1}{2}} \le \varepsilon \int_\Omega |\nabla u_k|^2 dx + D_\varepsilon. \tag{3.12}$$

Hence, for a suitable small ε, from (3.10), (3.11), (3.12) we find a positive A such that,

$$\int_\Omega |\nabla u_k|^2 dx \le A \quad \text{and} \quad \int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx \le A.$$

Then, up to a subsequence,

$$u_k \rightharpoonup u \text{ and } u_k^{\frac{3}{2}} \rightharpoonup u^{\frac{3}{2}} \text{ weakly in } W_0^{1,2}(\Omega) \text{ and a.e.}$$

In order to prove that u solves the problem (3.4) we proceed showing that the truncated terms converge strongly in $L^1(\Omega)$.

ii) *Strong convergence in $L^1(\Omega)$ of the Hardy truncated potential.*

We deal with the truncation of the Hardy potential,

$$\int_\Omega \frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} dx \le \int_\Omega \frac{u_k^p}{|x|^2} dx \le \Big(\int_\Omega \frac{u_k^2}{|x|^2} dx\Big)^{\frac{p}{2}} \Big(\int_\Omega \frac{1}{|x|^2} dx\Big)^{\frac{2-p}{2}}.$$

Since $p < 2$ and thanks to the estimation of the gradient of u_k,

$$\int_\Omega \frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} dx \le C\Big(\int_\Omega |\nabla u_k|^2\Big)^{\frac{p}{2}} dx \le C.$$

It follows that $\dfrac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}}$ is bounded in $L^1(\Omega)$ and converges almost everywhere to $\dfrac{u^p}{|x|^2}$. In particular, by Fatou's lemma, $\dfrac{u^p}{|x|^2} \in L^1(\Omega)$.

To complete the proof we need to check the equi-integrability of the term.

Let $E \subset \Omega$ be a measurable set, then, as above, we have

$$\begin{aligned}\int_E \frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} dx &\le \int_E \frac{u_k^p}{|x|^2} dx \le \Big(\int_E \frac{u_k^2}{|x|^2} dx\Big)^{\frac{p}{2}} \Big(\int_E \frac{1}{|x|^2} dx\Big)^{\frac{2-p}{2}} \\ &\le C\Big(\int_E \frac{1}{|x|^2} dx\Big)^{\frac{2-p}{2}},\end{aligned}$$

where C is a positive constant independent of k.

The term $\int_E \frac{1}{|x|^2} dx$ is going smaller if $|E|$ is small, hence, by the absolutely continuity of the integral we can use Vitali's Theorem to obtain

$$\frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} \to \frac{u^p}{|x|^2} \text{ in } L^1(\Omega).$$

iii) Strong convergence in $L^1(\Omega)$ *of the square of the gradient.*

We need to prove also that $|\nabla u_k|^2 \to |\nabla u|^2$ in $L^1(\Omega)$.

To obtain this convergence we need some previous results concerning the truncated gradient term.

Lemma 1. *Let* u_k *be defined by* (3.9). *Then,*

$$\lim_{n\to\infty} \int_{\{u_k \ge n\}} |\nabla u_k|^2 dx = 0 \tag{3.13}$$

uniformly on k.

Proof. Consider the function $G_n(s) = s - T_n(s)$ and $\psi_{n-1}(s) = T_1(G_{n-1}(s))$. Notice that, $\psi_{n-1}(u_k)|\nabla u_k|^2 \ge |\nabla u_k|^2_{\chi\{u_k \ge n\}}$.

Using $\psi_{n-1}(u_k)$ as a test function in (3.9), we obtain

$$\begin{aligned}
&\int_{\{u_k \ge n\}} |\nabla u_k|^2 dx \\
&\le \int_\Omega |\nabla \psi_{n-1}(u_k)|^2 dx + \int_\Omega \psi_{n-1}(u_k)|\nabla u_k|^2 dx \\
&= \int_\Omega \lambda \frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} \psi_{n-1}(u_k) dx + \int_\Omega \psi_{n-1}(u_k) f dx.
\end{aligned}$$

Since $\{u_k\}$ is uniformly bounded in $W_0^{1,2}(\Omega)$, then, by Rellich Theorem, up to a subsequence, $\{u_k\}$ strongly converges in $L^p(\Omega)$, $\forall p < 2^*$ and almost everywhere. As a consequence,

$$\int_{\{n-1 < u_k < n\}} u_k dx \le \frac{1}{n-1} \int_\Omega u_k^2 dx \le \frac{C}{n-1}$$

and

$$\int_{\{u_k > n\}} u_k dx \le \frac{1}{n} \int_\Omega u_k^2 dx \le \frac{C}{n}.$$

Therefore,

$$|\{x \in \Omega : n-1 < u_k(x) < n\}| \to 0,$$
$$|\{x \in \Omega : u_k(x) > n\}| \to 0, \text{ uniformly on } k \text{ as } n \to \infty.$$

Thus,

$$\begin{aligned}
\int_{\{u_k \ge n\}} |\nabla u_k|^2 dx &\le \psi^p \lambda \int_{\{u_k \ge n-1\}} \psi_{n-1}(u_k) dx + \int_{\{u_k \ge n-1\}} f \psi_{n-1}(u_k) dx \\
&\le C \int_{\{u_k \ge n-1\}} u_k dx + ||f||_{L^{\frac{N}{2}}(\Omega} \Big(\int_{\{u_k \ge n\}} u_k^{\frac{N}{N-2}} dx \Big)^{\frac{N-2}{N}}.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty}\int_{\{u_k\geq n\}}|\nabla u_k|^2dx=0 \text{ uniformly on } k. \tag{3.14}$$

□

Lemma 2. *Consider $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ as above. Then*

$$T_n(u_k)\to T_n(u) \quad in \quad W_0^{1,2}(\Omega).$$

Proof. Consider the functions $G_n(s)=s-T_n(s)$ and $\phi(s)=se^{\frac{1}{4}s^2}$ which verifies

$$\phi'(s)-|\phi(s)|\geq\frac{1}{2}.$$

Take $\phi(T_n(u_k)-T_n(u))$ as a test function in (3.9),

$$\begin{aligned}&\int_\Omega \nabla u_k\phi'(T_n(u_k)-T_n(u))\nabla(T_n(u_k)-T_n(u))dx\\&\quad+\int_\Omega|\nabla u_k|^2\phi(T_n(u_k)-T_n(u))dx\\&=\int_\Omega(\lambda\frac{T_k(u_k^p)}{|x|^2+\frac{1}{k}}+f)\phi(T_n(u_k)-T_n(u))dx.\end{aligned}$$

To estimate the first term on the left hand side we proceed as follows,

$$\begin{aligned}&\int_\Omega \nabla u_k\phi'(T_n(u_k)-T_n(u))\nabla(T_n(u_k)-T_n(u))dx\\&=\int_\Omega\nabla T_n(u_k)\phi'(T_n(u_k)-T_n(u))\nabla(T_n(u_k)-T_n(u))dx\\&\quad+\int_\Omega\nabla G_n(u_k)\phi'(T_n(u_k)-T_n(u))\nabla(T_n(u_k)-T_n(u))dx\\&=\int_\Omega|\nabla T_n(u_k)-\nabla T_n(u)|^2\phi'(T_n(u_k)-T_n(u))dx\\&\quad+\int_\Omega\nabla T_n(u)\phi'(T_n(u_k)-T_n(u))\nabla(T_n(u_k)-T_n(u))dx\\&\quad+\int_\Omega\nabla G_n(u_k)\phi'(T_n(u_k)-T_n(u))\nabla T_n(u_k)dx\\&\quad-\int_\Omega\nabla G_n(u_k)\phi'(T_n(u_k)-T_n(u))\nabla T_n(u)dx.\end{aligned}$$

Since the supports of $\nabla G_n(u_k)$ and $\nabla T_n(u_k)$ are disjoint and the ones of $\nabla G_n(u_k)$ and $\nabla T_n(u)$ are almost disjoint, we get

$$\int_\Omega \nabla G_n(u_k)\phi'(T_n(u_k) - T_n(u))\nabla T_n(u_k)dx = 0$$
$$= \int_\Omega \nabla G_n(u_k)\phi'(T_n(u_k) - T_n(u))\nabla T_n(u)dx.$$

On the other hand, since $T_n(u_k) \rightharpoonup T_n(u)$ weakly in $W_0^{1,2}(\Omega)$, $\nabla T_n(u_k) \rightharpoonup \nabla T_n(u)$ in $L^2(\Omega)$ and

$$\int_\Omega \nabla T_n(u)\phi'(T_n(u_k) - T_n(u))\nabla(T_n(u_k) - T_n(u))dx$$
$$= \int_\Omega \nabla T_n(u)\phi'(T_n(u_k) - T_n(u))\nabla T_n(u_k)dx$$
$$- \int_\Omega |\nabla T_n(u)|^2\phi'(T_n(u_k) - T_n(u))dx.$$

Thus,

$$\int_\Omega \nabla T_n(u)\phi'(T_n(u_k) - T_n(u))\nabla(T_n(u_k) - T_n(u))dx \to 0, \text{ as } k \to \infty.$$

Therefore,

$$\int_\Omega \nabla u_k\phi'(T_n(u_k) - T_n(u))\nabla(T_n(u_k) - T_n(u))dx$$
$$= \int_\Omega |\nabla(T_n(u_k) - T_n(u))|^2\phi'(T_n(u_k) - T_n(u))dx + o(1).$$

Notice that we have $\phi(T_n(u_k) - T_n(u))_{\chi_{\{u_k \geq n\}}} = 0$ and

$$\int_\Omega |\nabla u_k|^2\phi(T_n(u_k) - T_n(u))dx = \int_{\{u_k \leq n\}} |\nabla u_k|^2\phi(T_n(u_k) - T_n(u))dx$$
$$+ \int_{\{u_k \geq n\}} |\nabla u_k|^2\phi(T_n(u_k) - T_n(u))dx.$$

Therefore,

$$\int_\Omega |\nabla u_k|^2\phi(T_n(u_k) - T_n(u))dx = \int_\Omega |\nabla T_n(u_k)|^2|\phi(T_n(u_k) - T_n(u))|dx$$
$$= \int_\Omega |\nabla T_n(u_k) - \nabla T_n(u)|^2|\phi(T_n(u_k) - T_n(u))|dx$$
$$- \int_\Omega |\nabla T_n(u)|^2|\phi(T_n(u_k) - T_n(u))|dx$$
$$+ 2\int_\Omega |\nabla T_n(u_k)\nabla T_n(u)|\phi(T_n(u_k) - T_n(u))|dx.$$

Since $\nabla T_n(u)\phi(T_n(u_k)-T_n(u)) \to 0$ in $L^2(\Omega)$, and $\nabla T_n(u_k) \rightharpoonup \nabla T_n(u)$ in $L^2(\Omega)$ we obtain

$$\int_\Omega |\nabla T_n(u)|^2 |\phi(T_n(u_k) - T_n(u))| dx \to 0 \text{ as } k \to \infty$$

and

$$\int_\Omega \nabla T_n(u_k) \nabla T_n(u) \phi(T_n(u_k) - T_n(u)) dx \to 0 \text{ as } k \to \infty.$$

Then, passing to the limit as $k \to \infty$, we have

$$\begin{aligned}
&\int_\Omega |\nabla u_k|^2 \phi'(T_n(u_k) - T_n(u)) dx \\
&= \int_\Omega |\nabla T_n(u_k) - \nabla T_n(u)|^2 |\phi(T_n(u_k) - T_n(u))| dx + o(1).
\end{aligned}$$

Notice that

$$\int_\Omega \Big(\lambda \frac{T_k(u_k^p)}{|x|^2 + \frac{1}{k}} + f\Big) \phi(T_n(u_k) - T_n(u)) dx$$

goes to zero as $k \to \infty$.

Hence, since $\phi'(s) - |\phi(s)| > \frac{1}{2}$ we conclude that

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |\nabla T_n(u_k) - \nabla T_n(u)|^2 dx \\
\leq &\int_\Omega (\phi'(T_n(u_k) - T_n(u)) - |\phi(T_n(u_k) - T_n(u))|) |\nabla T_n(u_k) - \nabla T_n(u)|^2 dx
\end{aligned}$$

with the last integral equal to $o(1)$, whence, we conclude that $T_n(u_k) \to T_n(u)$ strongly in $W_0^{1,2}(\Omega)$. □

To finish, we proceed to prove that

$$|\nabla u_k|^2 \to |\nabla u|^2 \text{ strongly in } L^1(\Omega).$$

Using Lemma 2, the sequence of the gradients converges a.e. In order to apply Vitali's Theorem again, we have to prove the equi-integrability of the term $|\nabla u_k|^2$.

Let $E \subset \Omega$ be a measurable set. Then,

$$\int_E |\nabla u_k|^2 dx \leq \int_E |\nabla T_n(u_k)|^2 dx + \int_{\{u_k \geq n\} \cap E} |\nabla u_k|^2 dx.$$

By Lemma 2, for every $n > 0$, $T_n(u_k) \to T_n(u)$ in $W_0^{1,2}(\Omega)$ therefore, $\int_E |\nabla T_n(u_k)|^2 dx$ is uniformly small for $|E|$ small enough.
Using Lemma 1, we obtain

$$\int_{\{u_k \ge n\} \cap E} |\nabla u_k|^2 dx \le \int_{\{u_k \ge n\}} |\nabla u_k|^2 dx \to 0$$

as $n \to \infty$ uniformly on k. Then, by Vitali's Theorem,

$$|\nabla u_k|^2 \to |\nabla u|^2 \ \text{ in } \ L^1(\Omega).$$

Therefore, in particular, we conclude that u is a distributional solution to the problem,

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \ge 0 \ \text{ in } \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.15}$$

It is worthwhile to point out that the equation is verified even in a stronger way, that is, testing with functions $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

□

2.4 Solving the problem with $L^1(\Omega)$ data.

Consider now the problem with the following approximation of the data $f_k = T_k(f)$, that is $f_k \uparrow f$ in $L^1(\Omega)$. Consider u_k the solution to

$$\begin{cases} -\Delta u_k + |\nabla u_k|^2 = \lambda \dfrac{u_k^p}{|x|^2} + f_k, & u_k \ge 0 \ \text{ in } \Omega, \\ u_k = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{3.16}$$

found in the Subsection 2.1.

We define the following function,

$$\Psi_n(s) = \int_0^s T_n(t)^{\frac{1}{2}} dt,$$

that explicitly is,

$$\Psi_n(s) = \begin{cases} \frac{2}{3} s^{\frac{3}{2}} \ \text{ if } s < n, \\ \frac{2}{3} n^{\frac{3}{2}} + (s-n) n^{\frac{1}{2}} \ \text{ if } s > n. \end{cases} \tag{3.17}$$

We also consider the following numerical estimate.

Lemma 3. *Fixed $p \in [1,2)$, $\forall \varepsilon > 0$, $\forall k > 0$, $\exists C_\varepsilon$ such that*

$$s^p\ T_n(s) \leq \varepsilon \Psi_n^2(s) + C_\varepsilon, \quad s \geq 0.$$

Proof.

- If $s \leq n$, the estimation would be

$$s^{p+1} \leq \varepsilon C s^3 + C_\varepsilon.$$

  Since $p \in [1,2)$, $p+1 \leq 3$ and the estimation follows.

- If $s > n$, the estimation would be

$$s^p n \leq \varepsilon (C n^{\frac{3}{2}} + (s-n) n^{\frac{1}{2}})^2 + C_\varepsilon \leq \tilde{C}\varepsilon (C n^3 + (s-n)^2 n) + C_\varepsilon.$$

  Since $p < 2$, the inequality follows.

$\square$

Taking $T_n(u_k)$ as a test function in the truncated problem, it follows that

$$\int_\Omega |\nabla T_n(u_k)|^2 dx + \int_\Omega |\nabla u_k|^2 T_n(u_k) dx = \lambda \int_\Omega \frac{u_k^p}{|x|^2} T_n(u_k) dx + \int_\Omega f_k T_n(u_k) dx.$$

Notice that, taking into account the definition (3.17),

$$\int_\Omega |\nabla \Psi_n(u_k)|^2 dx = \int_{\{u_k < n\}} |u_k^{\frac{1}{2}}|^2 |\nabla u_k|^2 dx + \int_{\{u_k > n\}} |\nabla u_k|^2 n dx.$$

Thus,

$$\int_\Omega |\nabla \Psi_n(u_k)|^2 dx = \int_\Omega |\nabla u_k|^2 T_n(u_k) dx.$$

Hence,

$$\int_\Omega |\nabla T_n(u_k)|^2 dx + \int_\Omega |\nabla \Psi_n(u_k)|^2 dx \leq \lambda \int_\Omega \frac{u_k^p}{|x|^2} T_n(u_k) dx + \int_\Omega f T_n(u_k) dx.$$

From Lemma 3 and by Poincaré's and Young's inequalities, we get

$$\begin{aligned} &\int_\Omega |\nabla T_n(u_k)|^2 dx + \int_\Omega |\nabla \Psi_n(u_k)|^2 dx \\ &\leq \varepsilon \frac{\lambda}{\Lambda_N} \int_\Omega |\nabla \Psi_n(u_k)|^2 dx + \int_\Omega \frac{C_\varepsilon}{|x|^2} dx + n||f||_{L^1(\Omega)}. \end{aligned}$$

Choosing $0 < \varepsilon \dfrac{\lambda}{\Lambda_N} < 1$, we get

$$\int_\Omega |\nabla T_n(u_k)|^2 dx + (1 - \varepsilon \frac{\lambda}{\Lambda_N}) \int_\Omega |\nabla \Psi_n(u_k)|^2 dx \le \int_\Omega \frac{C_\varepsilon}{|x|^2} dx + n||f||_{L^1(\Omega)}.$$

Therefore, for every $n > 0$ it follows that

$$\int_\Omega |\nabla T_n(u_k)|^2 dx \le C(\lambda, \varepsilon, \Omega, f, n) \quad \text{uniformly on } k \in \mathbb{N},$$

$$\int_\Omega |\nabla \Psi_n(u_k)|^2 dx \le C(\lambda, \varepsilon, \Omega, f, n) \quad \text{uniformly on } k \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \int_\Omega |\nabla u_k|^2 dx &\le \int_\Omega |\nabla T_n(u_k)|^2 dx + \int_{\Omega \cap \{u_k > n\}} |\nabla u_k|^2 dx \\ &\le \int_\Omega |\nabla T_n(u_k)|^2 dx + \int_\Omega |\nabla \Psi_n(u_k)|^2 dx \le C, \end{aligned}$$

where C is independent of k. Hence, up to a subsequence

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega).$$

We prove now in a similar way to the previous Subsection that

1. $\dfrac{u_k^p}{|x|^2} \to \dfrac{u^p}{|x|^2}$ in $L^1(\Omega)$.
2. $T_n(u_k) \to T_n(u)$ strongly in $W_0^{1,2}(\Omega)$, for all $n > 0$.
3. $\lim\limits_{n\to\infty} \displaystyle\int_{\{u_k \ge n\}} |\nabla u_k|^2 dx = 0$ uniformly on k.

As before, we deduce that $\dfrac{u_k^p}{|x|^2}$ is bounded in $L^1(\Omega)$ and converges a.e. to $\dfrac{u^p}{|x|^2}$. In order to apply Vitali's Theorem, we check the equi-integrability of $\dfrac{u_k^p}{|x|^2}$. In this way we get 1.

Notice that 2 and 3 are necessary to demonstrate the strong convergence of the gradients.

To get 2 we consider the function $G_n(s) = s - T_n(s)$ and $\phi(s) = s e^{\frac{1}{4}s^2}$, which verifies $\phi'(s) - |\phi(s)| \ge \frac{1}{2}$. Using $\phi(T_n(u_k) - T_n(u))$ as a test function in (3.16), we get

$$\int_\Omega \nabla u_k \phi'(T_n(u_k) - T_n(u))\nabla(T_n(u_k) - T_n(u))dx + \int_\Omega |\nabla u_k|^2 \phi(T_n(u_k) - T_n(u))dx$$
$$= \int_\Omega (\lambda \frac{u_k^p}{|x|^2} + f_k)\phi(T_n(u_k) - T_n(u))dx.$$

Estimating term by term as in Lemma 2, considering that $u_k \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ and using the assumption on $\phi(s)$ we conclude 2.

To get 3 we use the truncated function of $G_n(s)$, $\psi_{r-1}(s) = T_1(G_{r-1}(s))$ as a test function in (3.16), and proceed exactly as in the proof of Lemma 1.

Now we are able to prove that

$$|\nabla u_k|^2 \to |\nabla u|^2 \text{ strongly in } L^1(\Omega).$$

By 2, the sequence of the gradients converges a.e. In order to use Vitali's Theorem again, we need to prove the equi-integrability of $|\nabla u_k|^2$.

Let $E \subset \Omega$ be a measurable set. Then

$$\int_E |\nabla u_k|^2 dx \leq \int_E |\nabla T_n(u_k)|^2 dx + \int_{\{u_k \geq n\} \cap E} |\nabla u_k|^2 dx.$$

By 2, for every $n > 0$, we get the strong convergence $T_n(u_k) \to T_n(u)$ in $W_0^{1,2}(\Omega)$ and therefore, $\int_E |\nabla T_n(u_k)|^2 dx$ is uniformly small for $|E|$ small enough. Thanks to 3, we obtain that

$$\int_{\{u_k \geq n\} \cap E} |\nabla u_k|^2 dx \leq \int_{\{u_k \geq n\}} |\nabla u_k|^2 dx \to 0$$

as $n \to \infty$ uniformly on k.

Then, by Vitali's Theorem we obtain that

$$|\nabla u_k|^2 \to |\nabla u|^2 \quad \text{strongly in } L^1(\Omega).$$

Therefore, in particular we conclude that u is a distributional solution to the problem,

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

3 Existence with $p \geq 2$

In order to have the same homogeneity in the two sides of the equation, we need to add some term in the left hand side. We multiply the square of the gradient by a power of u.

$$\begin{cases} -\Delta u + u^{\beta}|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f, & u \geq 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

Notice that if we take u as a test function in equation (3.19), we obtain

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega u^{\beta+1}|\nabla u|^2 dx = \int_\Omega \lambda \frac{u^{p+1}}{|x|^2} dx + \int_\Omega f u dx.$$

Thus, it follows

$$\int_\Omega |\nabla u|^2 dx + C\int_\Omega |\nabla (u)^{\frac{\beta+1}{2}+1}|^2 dx = \int_\Omega \lambda \frac{u^{p+1}}{|x|^2} dx + \int_\Omega f u dx.$$

In order to get an estimation for the square of the gradient we need to establish that

$$\lambda \int_\Omega \frac{u^{p+1}}{|x|^2} dx \leq \int_\Omega |\nabla u^{\frac{p+1}{2}}|^2 dx.$$

Therefore, $\frac{\beta+1}{2} + 1 = \frac{p+1}{2}$ and this identity is true if β is, at least, $p-2$.

Hence, it is sufficient to have $\beta > p - 2$ for the existence of solution. In particular, if $p = 2$, to have existence it is sufficient with $\beta = 1$, in fact, in Chapter 8 we are going to study carefully the existence of solution to the parabolic problem associated to the problem (3.19) with $\beta = 1$ and $p = 2$.

Remark 1. *We point out that we can find an interval of λ where we can find a solution with the same regularization term, even if $p = 2$.*

Assume $f \in L^1(\Omega)$ a positive function and $0 < \lambda < \frac{4}{9}\Lambda_N$. Then, there exists a solution $u \in W_0^{1,2}(\Omega)$ to the problem

$$\begin{cases} -\Delta u + |\nabla u|^2 = \lambda \dfrac{u^2}{|x|^2} + f, & u \geq 0 \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Proof. *To simplify the calculations we first assume f in $L^m(\Omega)$ and $m > \frac{N}{2}$ as in the case with $1 \leq p < 2$.*

The same arguments as in Section 2 allow us to conclude the existence of a solution to the truncated problem

$$\begin{cases} -\Delta u_k + |\nabla u_k|^2 = \lambda T_k(\frac{u_k^2}{|x|^2}) + f \quad in \quad \Omega, \\ u_k = 0 \qquad on \qquad \partial\Omega, \\ u_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \quad and \quad u_k > 0. \end{cases}$$

In order to get the weak convergence in $W_0^{1,2}(\Omega)$ *of* u_k*, we use* u_k *as a test function in the last equation,*

$$\begin{aligned} \int_\Omega |\nabla u_k|^2 dx + \frac{4}{9}\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx &= \lambda \int_\Omega T_k(\frac{u_k^2}{|x|^2}) u_k dx + \int_\Omega f u_k dx \\ &\leq \lambda \int_\Omega \frac{(u_k^{\frac{3}{2}})^2}{|x|^2} dx + \int_\Omega f u_k dx. \end{aligned}$$

Then, applying Poincaré, Young, Hölder and Sobolev inequalities, we get

$$\alpha \int_\Omega |\nabla u_k|^2 dx + \frac{4}{9}\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx \leq \frac{\lambda}{\Lambda_N}\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx + C||f||_{L^{\frac{N}{2}}(\Omega)}.$$

Therefore,

$$\alpha \int_\Omega |\nabla u_k|^2 dx + (\frac{4}{9} - \frac{\lambda}{\Lambda_N})\int_\Omega |\nabla u_k^{\frac{3}{2}}|^2 dx \leq C||f||_{L^{\frac{N}{2}}(\Omega)}.$$

Then, follow the arguments in Section 2, we can prove that there exists a solution if $0 < \lambda < \frac{4}{9}\Lambda_N$. □

The optimality of this value of λ*, as far as we know, is unknown.*

4 Further results

In this Section we are going to study the existence of solution in relation with the exponent p and a general exponent q for the gradient.

Theorem 11. *Consider* $\frac{pN}{N-2+p} < q \leq 2$*,* $1 \leq p < 2$ *and* f *a positive function such that* $f \in L^1(\Omega)$*. Then, for all* $\lambda > 0$*, there exists a solution* $u \in W_0^{1,2}(\Omega)$ *obtained as limit of approximations to the problem*

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda \frac{u^p}{|x|^2} + f, \quad u \geq 0 \ \ in\ \Omega, \\ u = 0 \qquad\qquad on\ \ \partial\Omega. \end{cases} \tag{3.21}$$

Proof.

The same arguments of Section 2 allow us to conclude the existence of a solution to the truncated problems

$$-\Delta u_k + |\nabla u_k|^q = \lambda T_k(\frac{u_k^p}{|x|^2}) + T_k(f) \text{ in } \Omega, \quad u_k \in W_0^{1,2}(\Omega). \tag{3.22}$$

In order to analyze the convergence of $\{u_k\}$, we take the function $T_n(u_k)$ as a test function in the truncated problem, it follows that

$$\int_\Omega |\nabla T_n u_k|^2 dx + \int_\Omega |\nabla u_k|^q T_n u_k dx = \lambda \int_\Omega T_k(\frac{u_k^p}{|x|^2}) T_n u_k dx + \int_\Omega T_k(f) T_n u_k dx.$$

We define, as in Section 2, the function $\Psi_n(s) = \int_0^s T_n(t)^{\frac{1}{q}} dt$, that explicitly is,

$$\Psi_n(s) = \begin{cases} \frac{q}{q+1} s^{\frac{q+1}{q}} & \text{if } s < n \\ \frac{q}{q+1} n^{\frac{q+1}{q}} + (s-n) n^{\frac{1}{q}} & \text{if } s > n. \end{cases} \tag{3.23}$$

Then,

$$\int_\Omega |\nabla T_n u_k|^2 dx + \int_\Omega |\nabla \Psi_n u_k|^q dx = \lambda \int_\Omega T_k(\frac{u_k^p}{|x|^2}) T_n u_k dx + \int_\Omega T_k(f) T_n u_k dx$$

and

$$\lambda \int_\Omega T_k(\frac{u_k^p}{|x|^2}) T_n u_k dx + \int_\Omega T_k(f) T_n u_k dx \le n\lambda \int_\Omega \frac{u_k^p}{|x|^2} dx + n||f||_{L^1(\Omega)}.$$

Therefore,

$$\int_\Omega |\nabla T_n u_k|^2 dx + \int_\Omega |\nabla \Psi_n u_k|^q dx \le n\lambda \int_\Omega \frac{u_k^p}{|x|^2} dx + n||f||_{L^1(\Omega)}.$$

To estimate the first term on the right hand side we are going to use Hölder's inequality,

$$n\lambda \int_\Omega \frac{u_k^p}{|x|^2} dx \le n\lambda \Big(\int_\Omega (\frac{1}{|x|^2})^{\frac{qN}{qN - p(N-q)}} dx \Big)^{\frac{qN - p(N-q)}{qN}} \Big(\int_\Omega u_k^{\frac{qN}{N-q}} dx \Big)^{\frac{p(N-q)}{qN}}.$$

Notice that since $q > \frac{Np}{N+p-2}$:

$$2q < Nq - pN + pq \Leftrightarrow \frac{qN}{qN - p(N-q)} < \frac{N}{2}.$$

Thus,

$$\Big(\int_\Omega (\frac{1}{|x|^2})^{\frac{qN}{qN-p(N-q)}}dx\Big)^{\frac{qN-p(N-q)}{qN}} \qquad \text{is integrable.}$$

Since $\frac{p}{q} > 1$, we can apply Young's inequality and we also use Sobolev's inequality, then, we obtain

$$n\lambda \int_\Omega \frac{u_k^p}{|x|^2}dx \leq n\lambda\varepsilon C\int_\Omega |\nabla u_k|^q dx + n\lambda C(\varepsilon).$$

Therefore,

$$\int_\Omega |\nabla T_n u_k|^2 dx + \int_\Omega |\nabla \Psi_n u_k|^q dx \leq n\varepsilon\lambda C\int_\Omega |\nabla u_k|^q dx + \lambda n C(\varepsilon) + n||f||_{L^1(\Omega)}.$$

Notice that

$$\int_\Omega |\nabla u_k|^q dx \leq \int_\Omega |\nabla T_n u_k|^2 dx + n\int_{\{u_k\geq n\}} |\nabla u_k|^q dx + C_q|\Omega|.$$

Hence, for $\varepsilon > 0$ suitable small,

$$\big(1 - n\varepsilon\lambda C\big)\int_\Omega |\nabla u_k|^q dx \leq n||f||_{L^1(\Omega)} + C_q|\Omega|,$$

then, $u_k \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$ and $T_n(u_k) \rightharpoonup T_n u$ in $W_0^{1,2}(\Omega)$.

To get the strong convergence of $T_n(u_k)$ to $T_n(u)$ in $W_0^{1,2}(\Omega)$ we argue as in Section 2. Then, up to a subsequence the gradients converge a.e. and we can prove for the Vitali Lemma that $|\nabla u_k|^q \to |\nabla u|^q$ in $L^1(\Omega)$, therefore, u is a weak solution to the problem,

$$\begin{cases} -\Delta u + |\nabla u|^q = \lambda\dfrac{u^p}{|x|^2} + f, & u \geq 0 \text{ in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

with $\frac{pN}{N-2+p} < q \leq 2$ and $1 \leq p < 2$. $\square$

Chapter 4

Existence and qualitative properties for the p-Laplacian case

1 Introduction

In this Chapter we are going to study a perturbation of the following *quasilinear* problem

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $-\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $q > p-1$ and $0 \in \Omega$.

Notice that also in this quasilinear setting we are considering a *supercritical* problem, therefore the results in this Chapter can be understood as an extension of the ones obtained in Chapter 3 for the *Laplacian* operator.

If $0 \in \Omega$ and $q > p-1$, it is possible to prove by a direct way the nonexistence of solution in $W_0^{1,p}(\Omega)$ to the problem (4.1). The argument for this proof is by contradiction and it is based in a comparison result and a generalization of the standard Picone's inequality. We recall these two classical results for the reader's convenience.

Proposition 2. *(Weak Comparison Principle). Let f, g belonging to $L^1(\Omega)$ and u, v the unique entropy solutions to the problems*

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, let us suppose that $f \leq g$. Then,

$$u \leq v \qquad \text{in } \Omega. \tag{4.2}$$

Theorem 12. *(Picone inequality generalization)*

Assume $v \in W_0^{1,p}(\Omega)$ verifying

$$-\Delta_p v = \nu,$$

a positive bounded Radon measure, $v_{|\partial\Omega} = 0$ *and* $v > 0$*. Then, for all* $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega \frac{|u|^p}{v^{p-1}}\big(-\Delta_p v)\big) dx.$$

A detailed proof can be found in [6, Theorem 3.1].

Now we are able to prove the following nonexistence result for solutions $u \in W_0^{1,p}(\Omega)$.

Theorem 13. *Assume* $0 \in \Omega$*,* $N \geq 3$*,* $1 < p < N$ *and* $p - 1 < q$*. Then, problem* (4.1) *has no solution* $u \in W_0^{1,p}(\Omega)$*.*

Proof. We argue by contradiction, let u be a solution to the problem

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $-\Delta_p u = \dfrac{u^q}{|x|^p} \geq 0$, and by the *Maximum Strong Principle* for the *p-Laplacian* operator, it follows that $u > 0$ in Ω. Therefore, there exists c such that

$$c := \inf_{B_R} u > 0,$$

for some R small. Then,

$$-\Delta_p u = \frac{u^q}{|x|^p} \geq \frac{c^q}{|x|^p} = -\Delta_p v \quad \text{in } B_R. \tag{4.3}$$

Since $\dfrac{c^q}{|x|^p} \in L^1(\Omega)$, thanks a uniqueness result in [6], the problem

$$\begin{cases} -\Delta_p v = \dfrac{c^q}{|x|^p} & \text{in } B_R, \\ v \geq 0 & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R, \end{cases} \tag{4.4}$$

has a unique radial solution.

We write the $-\Delta_p(\cdot)$ operator in radial coordinates in $B_R \setminus \{0\}$ as

$$-r^{1-n}\partial_r(r^{n-1}|v_r|^{p-2}v_r),$$

with, as usual, $r = |x|$. Integrating (4.4), we get

$$r^{n-1}|v_r|^{p-2}v_r = -\int c^q r^{n-1-p} dr = -\frac{1}{n-p}c^q r^{n-p} + C.$$

Choosing $C = 0$, v_r has to be negative and we have

$$v(|x|) = \left(\frac{c^q}{n-p}\right)^{\frac{1}{p-1}} \log\left(\frac{R}{|x|}\right).$$

Therefore, by (4.3) and Proposition 2 we get

$$u(x) \geq \tilde{C} \log\left(\frac{R}{|x|}\right),$$

where $\tilde{C} > 0$ is big enough and $\tilde{C} = \tilde{C}(c, n, p, q)$.

A generalization of the standard Picone's inequality, see Theorem 12, allows us to get

$$\begin{aligned}\int_{B_R} |\nabla\phi|^p dx \quad &\geq \quad \int_{B_R} \frac{|\phi|^p}{u^{p-1}}\Big(-div(|\nabla u|^{p-2}\nabla u)\Big)dx = \int_{B_R} \frac{u^{q-p+1}|\phi|^p}{|x|^p} \\ &\geq \quad \tilde{C}^{q-p+1}\int_{B_R} \frac{|\phi|^p}{|x|^p}\left(\log\left(\frac{R}{|x|}\right)\right)^{q-p+1} dx,\end{aligned}$$

$\forall \phi \in C_0^\infty(B_R)$. This is a contradiction with the Hardy-Sobolev's inequality and it concludes the proof. □

Indeed, there is no solution to problem (4.1) even in a more general sense, *entropy* sense (see [22]). The proof of this result is a little more complicated and it is written quite detailed in [5].

We define the kind of solutions that we are going to consider in this Chapter.

Definition 4. *Assume $f \in L^1(\Omega)$. Let u be in $W_0^{1,p}(\Omega)$. We say that u is an energy solution to problem*

$$-\Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f \quad in\ \Omega,$$

if

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla\phi) + \int_\Omega |\nabla u|^p \phi = \vartheta \int_\Omega \frac{u^q}{|x|^p}\phi + \int_\Omega f\phi$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

The contents of this Chapter can be summarized as follows.

- As in the previous Chapter we will prove the regularizing effect due to the presence of the gradient term $|\nabla u|^p$ on the left hand side of the problem (4.1).

  More precisely, we study the existence and qualitative properties of weak positive solutions to the *supercritical* problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f \text{ in } \Omega, \\ \quad u \geq 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.5}$$

  where Ω is a bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, $\vartheta > 0$, $p-1 < q < p$, $f \geq 0$, $f \in L^1(\Omega)$ and $1 < p < N$. Notice that we prove the existence of a weak solution u to problem (4.5) for any $\vartheta > 0$ and for each $f \in L^1(\Omega)$, $f \geq 0$.

  Taking into account that the data is a L^1-function, the solution is obtained as limit of approximation, to be short, *SOLA*, see [43], and by using the results in [45] we know that also is an *entropy solution* or *renormalized solution*. Notice that, since $|\nabla u|^p \in L^1(\Omega)$ we can conclude also that u is a solution in the sense of the Definition 4.

  Summarizing, the main existence result in this Chapter is the following

  Theorem 14. *Consider problem* (4.5) *with* $1 < p < N$, $p-1 < q < p$ *and assume that* $f \in L^1(\Omega)$ *is a positive function. Then, for all* $\vartheta > 0$ *there exists a weak solution* $u \in W_0^{1,p}(\Omega)$ *to* (4.5).

  The proof of this result has the following steps.

  (i) We first prove the existence of solution to the truncated problem

$$-\Delta_p u_k + |\nabla u_k|^p = \vartheta T_k(\frac{u_k^q}{|x|^p}) + T_k(f) \text{ in } \Omega, \quad u_k \in W_0^{1,p}(\Omega).$$

  This can be done solving the correspondent approximated problem and passing to the limit in $W_0^{1,p}(\Omega)$.

  (ii) We show that the sequence of solutions to the truncated problem converges weakly in $W_0^{1,p}(\Omega)$ to the solution of (4.5) and then we deduce the a.e. convergence of the gradients. Finally, we exploit it to deduce the strong convergence in $W_0^{1,p}(\Omega)$.

  (iii) We pass to the limit in the truncated problem and we obtain the existence of a solution to (4.5).

Let us remark that, because of the presence of the gradient term (which causes the existence of solutions), to pass to the limit in the truncated problem it is necessary to deduce the convergence of u_k (solutions of the truncated problem) in $W_0^{1,p}(\Omega)$. A convergence in $W_0^{1,q}(\Omega)$ (in the spirit of [28]), would be not sufficient to pass to the limit and get a weak formulation of the problem; because $q < p$ and $W_0^{1,q}(\Omega)$ is not enough, hence, we need to check the convergence in $W_0^{1,p}(\Omega)$.

- In Section 3 of this Chapter we deal with the study of the qualitative properties of solutions to (4.5). The main result is the following

Theorem 15. *Let $u \in C^1(\overline{\Omega} \setminus \{0\})$ be a weak solution to* (4.5). *Consider the domain Ω strictly convex w.r.t. the $\nu-$direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν, where*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

Moreover, assume $f \in C^1(\overline{\Omega} \setminus \{0\})$ to be non-decreasing w.r.t. the $\nu-$direction in the set

$$\Omega_0^\nu = \{x \in \Omega : x \cdot \nu < 0\}$$

and even w.r.t. T_0^ν. Then, u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the $\nu-$direction in Ω_0^ν.

Remark 2. *Notice that the extra regularity hypothesis on f is sufficient to have the corresponding regularity of the solution.*

Remark 3. *If Ω is a ball and f is radial, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$.*

We point out that Theorem 15 will be a consequence of a more general result, see Proposition 5 in Section 3, which states a monotonicity property of the solutions in general domains near strictly convex parts of the boundary. This can be useful for example in blow-up analysis. Also, it will be clear from the proof that the same technique could be applied to study the case of more general nonlinearities. Recall that we are only looking for positive solutions of (4.5), thus, we only consider the interval $[0, \infty)$ and since if u is near to zero, for some values of q, the nonlinearity can be not Lipschitz anymore, we note that the nonlinearity in problem (4.5) is in general locally Lipschitz continuous only in $(0\,,\,\infty)$.

The main ingredient in the proof of the symmetry result is the well known *Moving Plane Method* ([87]), that it was used in a clever way in the celebrated paper [62] for the semilinear nondegenerate case. Actually, the proof showed in this memory is more similar to the one in [23] and is based on the weak comparison principle in small domains. The *Moving Plane Method* was extended to the case of *p-Laplace* equations firstly in [47] for the case $1 < p < 2$ and later in [49] for the case $p \geq 2$.

To study the qualitative properties of the solutions to (4.5), we first need to include some summability results of the gradient and also a weighted Poincaré's inequality. These are, together with a weak comparison principle, the most important ingredients to prove the symmetry result using the *Moving Plane Method*.

The first crucial step is the proof of a weak comparison principle in small domains that we carry out in Proposition 4. This is based on some regularity results in the spirit of [49]. These results hold only away from the origin due to the presence of the Hardy potential in our problem. This will require more attention in the application of the *Moving Plane* procedure. Moreover, the presence of the gradient term $|\nabla u|^p$, leads to a proof of the weak comparison principle in small domains which makes use of the right choice of test functions.

We recall the following classical inequality which will be very useful in all the memory.

Lemma 4. *Let $\eta, \eta' \in \mathbb{R}^N$. There exists positive constants C_1, C_2 depending on p such that, $\forall \eta\,,\ \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$ and $\forall x \in \Omega$, it follows*

$$||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|, \tag{4.6}$$

$$[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] \geq C_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \tag{4.7}$$

Since $|\eta - \eta'| \leq |\eta| + |\eta'|$ by (4.6) and by (4.7) it follows

$$||\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'| \leq C_1|\eta - \eta'|^{p-1} \qquad \text{if} \quad 1 < p \leq 2 \tag{4.8}$$

and

$$[|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta'][\eta - \eta'] \geq C_2|\eta - \eta'|^p \qquad \text{if} \quad p \geq 2. \tag{4.9}$$

See the details, for example, in [46].

All the results of this Chapter can be seen in the paper [72].

2 Existence of an energy solution

We use the same arguments as in the previous Chapter to find a solution to the problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f \ \text{ in } \Omega \\ \quad u \geq 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.10}$$

We study the weak and the strong convergence of the approximated and truncated solutions, and then, we pass to the limit.

2.1 Existence of solution to the truncated problem

First of all we are going to study the existence of solution to the following truncated problem

$$-\Delta_p u_k + |\nabla u_k|^p = \vartheta T_k(\frac{u_k^q}{|x|^p}) + T_k(f) \ \text{ in } \ \Omega, \ \ u_k \in W_0^{1,p}(\Omega), \tag{4.11}$$

where $T_k(s) = \max\{\min\{k, s\}, -k\}$ and $k > 0$.

The idea is to find a solution first of the approximated problems and get the convergence of the solutions of these problems to the solution of (4.11).

Theorem 16. *There exists a positive solution to problem* (4.11).

Notice that $\phi \equiv 0$ is a subsolution to problem (4.11). Consider ψ the solution to

$$\begin{cases} -\Delta_p \psi = \vartheta \cdot k + T_k(f) & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

In fact, ψ turns to be a supersolution to (4.11).

To prove Theorem 16 we will consider a sequence of approximated problems that we solve by iteration and using some convenient comparison argument. We take as starting point $w_0 = 0$ and consider iteratively the following problem,

$$\begin{cases} -\Delta_p w_n + \dfrac{|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} = \vartheta T_k(\frac{w_{n-1}^q}{|x|^p}) + T_k(f) & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

Notice that the subsolution $\phi \equiv 0$ and the supersolution ψ to problem (4.11) are subsolution and supersolution to the problem (4.13) as well.

The poof of the next proposition follows using a comparison argument from [29].

Proposition 3. *There exists* $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *solution to* (4.13).

Moreover, $0 \leq w_n \leq \psi \quad \forall n \in \mathbb{N}$.

Proof. Let us consider the problem:

$$-\Delta_p w_n + g(x, w_n, \nabla w_n) = 0, \tag{4.14}$$

where the function $g(x, w_n, \nabla w_n)$ is defined by

$$\begin{cases} -\vartheta k - T_k(f) & \text{if } w_n \geq \psi, \\ \frac{|\nabla w_n|^p}{1+\frac{1}{n}|\nabla w_n|^p} - \vartheta T_k(\frac{w_{n-1}^q}{|x|^p}) - T_k(f) & \text{if } 0 \leq w_n < \psi, \\ -T_k(f) & \text{if } w_n \leq 0. \end{cases} \tag{4.15}$$

Using Leray-Lions arguments, see it in [68], we can find solutions to the approximated problem (4.14) for each n and by classical regularity results, such solution w_n belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

As first step we are going to show that $w_n \geq 0$.

Since w_n is a solution of (4.14) and 0 is a subsolution

$$-\Delta_p w_n + g(x, w_n, \nabla w_n) + T_k(f) \geq 0.$$

Using $-(w_n^-)$ as a test function in the last expression one has

$$-\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla w_n^- - \int_\Omega \Big(g(x, w_n, \nabla w_n) + T_k(f) \Big)(w_n^-) \geq 0.$$

We define the following set,

$$R = \{x : x \in \Omega : w_n \leq 0\},$$

therefore,

$$-\int_\Omega |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla w_n^- dx - \int_R \Big(g(x, w_n, \nabla w_n) + T_k(f) \Big)(w_n^-) dx \geq 0.$$

Taking into account (4.15), $g(x, w_n, \nabla w_n) = -T_k(f)$ in R, then,

$$-\int_\Omega |\nabla w_n^-|^p dx \leq 0.$$

Hence, we can wconclude $w_n \geq 0$.

Now we want to prove $w_n \leq \psi$.

Since ψ and w_n are respectively a supersolution and a solution of (4.14), we have

$$-\Delta_p w_n + \Delta_p \psi + g(x, w_n, \nabla w_n) - g(x, \psi, \nabla\psi) \leq 0$$

Using $T_M([w_n - \psi]^+)$ with $M \in \mathbb{R}^+$ as a test function in the last expression it follows

$$\begin{aligned}&\int_\Omega < |\nabla w_n|^{p-2}\nabla w_n - |\nabla\psi|^{p-2}\nabla\psi, \nabla T_M(w_n - \psi)^+ > dx\\ &+ \int_\Omega \Big(g(x, w_n, \nabla w_n) + \vartheta k + T_k(f)\Big) T_M([w_n - \psi]^+) dx \leq 0.\end{aligned}$$

We define the following sets,

$$R = \{x : x \in \Omega : \psi \leq w_n\},$$

$$R^M = \{x : x \in \Omega : 0 \leq w_n - \psi \leq M\}.$$

Thus,

$$T_M([w_n - \psi]^+) = 0 \text{ if } x \in \Omega - R \text{ or } w_n^- = 0$$

and

$$\nabla T_M([w_n - \psi]^+) = 0 \text{ if } x \in \Omega - R^M \text{ or } w_n^- = 0.$$

Therefore,

$$\begin{aligned}&\int_{R^M} < |\nabla w_n|^{p-2}\nabla w_n - |\nabla\psi|^{p-2}\nabla\psi, \nabla T_M(w_n - \psi)^+ > dx\\ &+ \int_R \Big(g(x, w_n, \nabla w_n) + \vartheta k + T_k(f)\Big) T_M([w_n - \psi]^+) dx \leq 0.\end{aligned}$$

By (4.15) and taking into account that $\nabla T_M(w_n - \psi)^+ = \nabla(w_n - \psi)^+$, we get

$$\int_{R^M} < |\nabla w_n|^{p-2}\nabla w_n - |\nabla\psi|^{p-2}\nabla\psi, \nabla(w_n - \psi)^+ > dx \leq 0 \quad \forall M \in \mathbb{R}^+.$$

From Lemma 4, we have

$$0 \geq \begin{cases} C_p \displaystyle\int_{R^M} \frac{|\nabla(w_n - \psi)^+|^2}{(|\nabla w_n| + |\nabla\psi|)^{2-p}} dx & \text{if } 1 < p < 2,\\ C_p \displaystyle\int_{R^M} |\nabla(w_n - \psi)^+|^p dx & \text{if } p \geq 2,\end{cases}$$

which imply in any case that $w_n \le \psi$. $\square$

Proof of Theorem 16: we proceed in two steps in order to study carefully the convergence of w_n.

Step 1: *Weak convergence of w_n in $W_0^{1,p}(\Omega)$.*

By simplicity we set

$$H_n(\nabla w_n) = \frac{|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p}. \tag{4.16}$$

Taking w_n as a test function in the approximated problems (4.13), we obtain

$$\begin{aligned}\int_\Omega |\nabla w_n|^p dx + \int_\Omega H_n(\nabla w_n) w_n dx &= \vartheta \int_\Omega T_k(\frac{w_{n-1}^q}{|x|^p}) w_n dx + \int_\Omega T_k(f) w_n dx \\ &\le \vartheta \int_\Omega k w_n dx + \int_\Omega f w_n dx.\end{aligned}$$

Since $w_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $f \in L^1(\Omega)$,

$$\int_\Omega |\nabla w_n|^p dx + \int_\Omega H_n(\nabla w_n) w_n dx \le \vartheta k ||\psi||_{L^\infty(\Omega)} + ||\psi||_{L^\infty(\Omega)} ||f||_{L^1(\Omega)}.$$

Therefore, there exists a positive constant $C(k, f, \psi, \vartheta, \Omega)$ such that

$$\int_\Omega |\nabla w_n|^p dx + \int_\Omega H_n(\nabla w_n) w_n dx \le C(k, f, \psi, \vartheta, \Omega).$$

Moreover, since $\int_\Omega H_n(\nabla w_n) w_n dx \ge 0$, we have

$$\int_\Omega |\nabla w_n|^p dx \le C(k, f, \psi, \vartheta, \Omega). \tag{4.17}$$

Therefore, up to a subsequence, $w_n \rightharpoonup u_k$ weakly in $W_0^{1,p}(\Omega)$ and, since $||w_n||_{L^\infty(\Omega)} < C$, $w_n \rightharpoonup u_k$ weakly-* in $L^\infty(\Omega)$, thus,

$$\int_\Omega w_n \varphi dx = \int_\Omega u_k \varphi dx; \quad \text{for} \quad \varphi \in L^1(\Omega).$$

Hence,

$$u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Step 2: *Strong convergence of w_n in $W_0^{1,p}(\Omega)$ and passing to the limit in* (4.11).

To get the strong convergence of w_n in $W_0^{1,p}(\Omega)$ first of all we notice that

$$||w_n - u_k||_{W_0^{1,p}(\Omega)} \leq ||(w_n - u_k)^+||_{W_0^{1,p}(\Omega)} + ||(w_n - u_k)^-||_{W_0^{1,p}(\Omega)}. \tag{4.18}$$

Thus, we proceed estimating each term on the right-hand side of (4.18).

Asymptotic behavior of $||(w_n - u_k)^+||_{W_0^{1,p}(\Omega)}$.

Chosing $(w_n - u_k)^+$ as a test function in (4.13) we obtain

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}(\nabla w_n, \nabla(w_n - u_k)^+)dx + \int_\Omega H_n(\nabla w_n)(w_n - u_k)^+ dx \\ &\quad = \vartheta \int_\Omega T_k(\frac{w_{n-1}^q}{|x|^p})(w_n - u_k)^+ dx + \int_\Omega T_k(f)(w_n - u_k)^+ dx. \end{aligned} \tag{4.19}$$

Since $w_n \rightharpoonup u_k$ in $W_0^{1,p}(\Omega)$, one has $w_n \to u_k$ a.e. in Ω and thus, $(w_n - u_k)^+ \to 0$ a.e. in Ω together with $(w_n - u_k)^+ \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ as well. Therefore, the right-hand side of (4.19) goes to zero when n goes to infinity.

Then, taking into account that $\int_\Omega H_n(\nabla w_n)(w_n - u_k)^+ dx \geq 0$, the expression (4.19) becomes

$$\begin{aligned} &\int_\Omega |\nabla w_n|^{p-2}\left(\nabla w_n, \nabla(w_n - u_k)^+\right)dx \\ &\leq \int_\Omega |\nabla w_n|^{p-2}\left(\nabla w_n, \nabla(w_n - u_k)^+\right)dx + \int_\Omega H_n(\nabla w_n)(w_n - u_k)^+ dx \\ &\leq o(1), \end{aligned} \tag{4.20}$$

as $n \to +\infty$.

Since by weak convergence;

$$\int_\Omega |\nabla u_k|^{p-2}\left(\nabla u_k, \nabla(w_n - u_k)^+\right)dx = o(1) \quad \text{as } n \to +\infty,$$

it follows

$$\int_\Omega \left(|\nabla w_n|^{p-2}\nabla w_n - |\nabla u_k|^{p-2}\nabla u_k, \nabla(w_n - u_k)^+\right)dx \leq o(1). \tag{4.21}$$

Then, from (4.21) and using Lemma 4, we have

$$o(1) = \begin{cases} C_1(p) \displaystyle\int_\Omega \frac{|\nabla(w_n - u_k)^+|^2}{(|\nabla w_n| + |\nabla u_k|)^{2-p}} & \text{if } 1 < p < 2, \\ \\ C_1(p) \displaystyle\int_\Omega |\nabla(w_n - u_k)^+|^p & \text{if } p \geq 2, \end{cases} \tag{4.22}$$

with $C_1(p)$ a positive constant depending on p. In any case, since for $1 < p < 2$ using Hölder's inequality one has

$$\begin{aligned} &\int_\Omega |\nabla(w_n - u_k)^+|^p dx \\ &\leq \left(\int_\Omega \frac{|\nabla(w_n - u_k)^+|^2}{(|\nabla w_n| + |\nabla u_k|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla w_n| + |\nabla u_k|)^p dx \right)^{\frac{2-p}{2}}, \end{aligned} \tag{4.23}$$

we obtain

$$||(w_n - u_k)^+||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n \to +\infty. \tag{4.24}$$

Asymptotic behavior of $||(w_n - u_k)^-||_{W_0^{1,p}(\Omega)}$**.**

Let us consider $e^{-w_n}[(w_n - u_k)^-]$ as a test function in (4.13),

$$\begin{aligned} &\int_\Omega e^{-w_n} |\nabla w_n|^{p-2} \left(\nabla w_n, \nabla (w_n - u_k)^- \right) dx \\ &+ \int_\Omega e^{-w_n} \left(\frac{|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} - |\nabla w_n|^p \right) (w_n - u_k)^- dx \\ &= \vartheta \int_\Omega e^{-w_n} T_k\left(\frac{w_{n-1}^q}{|x|^p}\right)(w_n - u_k)^- dx + \int_\Omega e^{-w_n} T_k(f)(w_n - u_k)^- dx. \end{aligned} \tag{4.25}$$

Recalling that $f = min(0, f)$, we want to point out that using this test function it follows

$$\int_\Omega e^{-w_n} \left(\frac{|\nabla w_n|^p}{1 + \frac{1}{n}|\nabla w_n|^p} - |\nabla w_n|^p \right) (w_n - u_k)^- dx \geq 0. \tag{4.26}$$

As above, since $(w_n - u_k)^- \to 0$ a.e. in Ω, the right-hand side of (4.25) tends to zero as n goes to infinity. Moreover, being $w_n \leq \psi$ (see Proposition 3), there exists γ such that $e^{-w_n} \geq \gamma > 0$ uniformly on n. Then, equation (4.25) states as

$$\gamma \int_\Omega |\nabla w_n|^{p-2} \left(\nabla w_n, \nabla (w_n - u_k)^- \right) dx \leq o(1). \tag{4.27}$$

Arguing in the same way as we have done from equation (4.20) to (4.24), we obtain

$$||(w_n - u_k)^-||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n \to +\infty. \tag{4.28}$$

From equation (4.18), by (4.24) and (4.28) it follows

$$||(w_n - u_k)||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as } n \to +\infty$$

and consequently, $\nabla w_n \to \nabla u_k$ a.e. in Ω. Then, by (4.16), $H_n(\nabla w_n) \to |\nabla u_k|^p$ a.e. in Ω and the equi-integrability follows. By Vitali's lemma,

$$H_n(\nabla w_n) \to |\nabla u_k|^p \quad \text{in } L^1(\Omega).$$

Hence, $u_k \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfies the problem in the following sense

$$\begin{aligned} &\int_\Omega |\nabla u_k|^{p-2}(\nabla u_k, \nabla\phi)dx + \int_\Omega |\nabla u_k|^p \phi dx \\ &= \vartheta \int_\Omega T_k(\frac{u_k^p}{|x|^p})\phi dx + \int_\Omega T_k(f)\phi dx, \end{aligned} \tag{4.29}$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and in this way we conclude the proof.

□

2.2 Passing to the limit and convergence to the solution

In this Subsection we are going to show that $u_k \to u$ strongly in $W_0^{1,p}(\Omega)$ in order to prove the existence of a solution u to problem (4.5) and to prove Theorem 14.

Proof of Theorem 14: We perform the proof in different steps in order to be clear.

Step 1: *Weak convergence of u_k in $W_0^{1,p}(\Omega)$.*

We start taking $T_n(u_k)$ as a test function in the truncated problem (4.11), in this way we obtain

$$\begin{aligned} &\int_\Omega |\nabla T_n(u_k)|^p dx + \int_\Omega |\nabla u_k|^p T_n(u_k)dx \\ &= \vartheta \int_\Omega T_k(\frac{u_k^q}{|x|^p})T_n(u_k)dx + \int_\Omega T_k(f)T_n(u_k)dx. \end{aligned}$$

Define
$$\Psi_n(s) = \int_0^s T_n(t)^{\frac{1}{p}} dt, \tag{4.30}$$
that is, explicitly,
$$\Psi_n(s) = \begin{cases} \frac{p}{p+1} s^{\frac{p+1}{p}} & \text{if } s < n, \\ \frac{p}{p+1} n^{\frac{p+1}{p}} + (s-n)n^{\frac{1}{p}} & \text{if } s \geq n. \end{cases} \tag{4.31}$$
Then,
$$\begin{aligned} &\int_\Omega |\nabla T_n(u_k)|^p dx + \int_\Omega |\nabla \Psi_n(u_k)|^p dx \\ &= \vartheta \int_\Omega T_k\Big(\frac{u_k^q}{|x|^p}\Big) T_n(u_k) dx + \int_\Omega T_k(f) T_n(u_k) dx \\ &\leq \vartheta \int_\Omega \frac{u_k^q}{|x|^p} T_n(u_k) dx + n||f||_{L^1(\Omega)}. \end{aligned} \tag{4.32}$$

We establish the following inequality.

For fixed $q \in [p-1, p)$, $\forall \varepsilon > 0$ and $\forall n > 0$, there exists C_ε such that
$$s^q T_n(s) \leq \varepsilon \Psi_n^p(s) + C_\varepsilon \quad s \geq 0. \tag{4.33}$$

By a straightforward calculation it is easy to check this expression;

- If $s < n$, (4.33) would be
$$s^{q+1} \leq \varepsilon C s^{p+1} + C_\varepsilon.$$
And, since $q < p$, the last inequality follows.
- If $s > n$, (4.33) would be
$$ns^q \leq (Cn^{p+1/p} + (s-n)n^{1/p})^p + C_\varepsilon \leq Cn^{p+1} + (s-n)^p n + C_\varepsilon.$$
And, since $q < p$, the last inequality follows.

Thanks to Hardy's inequality (see Theorem 5) and (4.33), equation (4.32) states as
$$\begin{aligned} &\int_\Omega |\nabla T_n(u_k)|^p dx + \int_\Omega |\nabla \Psi_n(u_k)|^p dx \\ &\leq \varepsilon \frac{\vartheta}{\Lambda_{N,p}} \int_\Omega |\nabla \Psi_n(u_k)|^p dx + \vartheta C_\varepsilon \int_\Omega \frac{dx}{|x|^p} + n||f||_{L^1(\Omega)}. \end{aligned}$$

Then, choosing $\varepsilon > 0$ such that $0 < \varepsilon \dfrac{\vartheta}{\Lambda_{N,p}} < 1$, for some positive C and since $p < N$, we get

$$\begin{aligned} &\int_\Omega |\nabla T_n(u_k)|^p dx + C \int_\Omega |\nabla \Psi_n(u_k)|^p dx \\ &\leq \vartheta C_\varepsilon \int_\Omega \frac{dx}{|x|^p} + n||f||_{L^1(\Omega)} \leq C(\vartheta, \varepsilon, f, p, n, \Omega). \end{aligned} \tag{4.34}$$

Fixed $n \geq 1$, by the definition (4.31) of Ψ_n and equation (4.34), one has

$$\begin{aligned} &\int_\Omega |\nabla u_k|^p dx \\ &\leq \int_\Omega |\nabla T_n(u_k)|^p dx + \int_{\Omega \cap \{u_k \geq n\}} |\nabla u_k|^p dx \\ &\leq \int_\Omega |\nabla T_n(u_k)|^p dx + \int_{\Omega \cap \{u_k \geq n\}} |\nabla u_k|^p dx + \int_{\Omega \cap \{u_k < n\}} |\nabla u_k|^p dx \\ &\leq \int_\Omega |\nabla T_n(u_k)|^p dx + \frac{1}{n} \int_\Omega |\nabla \Psi_n(u_k)|^p dx \leq C, \end{aligned} \tag{4.35}$$

uniformly on k.

Therefore, up to a subsequence it follows that $u_k \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and a.e.

Step 2: *Strong convergence in $L^1(\Omega)$ of the singular term.*

Since $p < q$, using Hölder's inequality and by the previous estimation for the gradient, we have

$$\begin{aligned} \int_\Omega T_k(\frac{u_k^q}{|x|^p}) \leq \int_\Omega \frac{u_k^q}{|x|^p} dx &\leq \left(\int_\Omega \frac{u_k^p}{|x|^p} dx \right)^{\frac{q}{p}} \left(\int_\Omega \frac{1}{|x|^p} dx \right)^{\frac{p-q}{p}} \\ &\leq C \left(\int_\Omega |\nabla u_k|^p dx \right)^{\frac{q}{p}} \leq C, \end{aligned} \tag{4.36}$$

with C a positive constant that does not depend on k. It follows that $T_k(\dfrac{u_k^q}{|x|^p})$ is bounded in $L^1(\Omega)$ and converges almost everywhere to $\dfrac{u^q}{|x|^p}$. In particular, Fatou's Lemma implies that $\dfrac{u^q}{|x|^p} \in L^1(\Omega)$.

Moreover, let $E \subset \Omega$ be a measurable set, by Fatou's Lemma we obtain

$$\int_E T_k(\frac{u_k^q}{|x|^p}) dx \leq \int_E \frac{u_k^q}{|x|^p} dx \leq \lim_{n \to +\infty} \int_E \frac{w_n^q}{|x|^p} dx \leq \int_E \frac{\psi^q}{|x|^p} dx \leq \delta(|E|),$$

uniformly on k, where $\lim_{s\to 0}\delta(s)=0$, w_n is as in the proof of Theorem 16 and ψ as in Proposition 3. Thus, from Vitali's Theorem it follows that

$$T_k(\frac{u_k^q}{|x|^p}) \to \frac{u^q}{|x|^p} \quad \text{in } L^1(\Omega). \tag{4.37}$$

Step 3: *Strong convergence of* $|\nabla u_k|^p \to |\nabla u|^p$ *in* $L^1(\Omega)$.

To show the strong convergence of the gradients we need some preliminary results as in the semilinear case, see Chapter 3.

We need first the following Lemma.

Lemma 5. *Let* u_k *be defined by* (4.11)*. Then*

$$\lim_{n\to\infty} \int_{\{u_k\geq n\}} |\nabla u_k|^p dx = 0 \tag{4.38}$$

uniformly on k.

Proof. Let us consider the truncated functions

$$G_n(s) = s - T_n(s), \quad \text{and} \quad \psi_{n-1}(s) = T_1(G_{n-1}(s)).$$

Notice that $\psi_{n-1}(u_k)|\nabla u_k|^p \geq |\nabla u_k|^p_{\chi\{u_k\geq n\}}$. Using $\psi_{n-1}(u_k)$ as a test function in (4.11) we get

$$\begin{aligned}
&\int_{\{u_k\geq n\}} |\nabla u_k|^p dx \\
&\leq \int_\Omega |\nabla \psi_{n-1}(u_k)|^p dx + \int_\Omega |\nabla u_k|^p \psi_{n-1}(u_k) dx \\
&= \int_\Omega \vartheta T_k(\frac{u_k^q}{|x|^p})\psi_{n-1}(u_k)dx + \int_\Omega T_k(f)\psi_{n-1}(u_k)dx \leq C.
\end{aligned} \tag{4.39}$$

Since $\{u_k\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$, then up to a subsequence, $\{u_k\}$ strongly converges in $L^p(\Omega)$ for $1 \leq p < p^* = \frac{Np}{N-p}$ and a.e. in Ω. Thus, we obtain that

$$\int_{\{n-1<u_k<n\}} u_k dx \leq \frac{1}{n-1}\int_\Omega u_k^2 dx \leq \frac{C}{n-1}$$

and

$$\int_{\{u_k>n\}} u_k dx \leq \frac{1}{n}\int_\Omega u_k^2 dx \leq \frac{C}{n}.$$

Therefore,

$$|\{x \in \Omega : n-1 < u_k(x) < n\}| \to 0 \quad \text{if } n \to \infty \quad \text{and}$$

$$|\{x \in \Omega, : u_k(x) > n\}| \to 0 \quad \text{if } n \to \infty,$$

uniformly on k.

By (4.39),

$$\begin{aligned}\int_{\{u_k \geq n\}} |\nabla u_k|^p dx &\leq k\psi \int_{\{u_k \geq n-1\}} \psi_{n-1}(u_k) dx + k \int_{\{u_k \geq n-1\}} \psi_{n-1}(u_k) dx \\ &\leq C \int_{\{u_k \geq n-1\}} u_k dx.\end{aligned}$$

Then, we have uniformly on k

$$\lim_{n \to \infty} \int_{\{u_k \geq n\}} |\nabla u_k|^p dx = 0. \tag{4.40}$$

□

Next Lemma shows the strong convergence in $W_0^{1,p}(\Omega)$ of the truncated terms.

Lemma 6. *Consider $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as above. Then, it holds uniformly on m,*

$$T_m(u_k) \to T_m(u) \text{ in } W_0^{1,p}(\Omega) \quad \text{for } k \to +\infty.$$

Proof. Notice that

$$\begin{aligned}&||T_m(u_k) - T_m(u)||_{W_0^{1,p}(\Omega)} \\ &\leq ||(T_m(u_k) - T_m(u))^+||_{W_0^{1,p}(\Omega)} + ||(T_m(u_k) - T_m(u))^-||_{W_0^{1,p}(\Omega)}.\end{aligned} \tag{4.41}$$

We are going to estimate the convergence of each term on the right-hand side of (4.41).

Asymptotic behavior of $||(T_m(u_k) - T_m(u))^+||_{W_0^{1,p}(\Omega)}$.

We take $(T_m(u_k) - T_m(u))^+$ as a test function in (4.11), obtaining

$$\begin{aligned}&\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+\right) dx \\ &+ \int_\Omega |\nabla u_k|^p (T_m(u_k) - T_m(u))^+ dx \\ &= \int_\Omega \left(\vartheta T_k\left(\frac{u_k^q}{|x|^p}\right) + T_k(f)\right) (T_m(u_k) - T_m(u))^+ dx.\end{aligned} \tag{4.42}$$

Since $T_m(u_k) \rightharpoonup T_m(u)$ in $W_0^{1,p}(\Omega)$ and $T_m(u_k) \to T_m(u)$ a.e. in Ω, we have $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ and $(T_m(u_k) - T_m(u))^+ \to 0$ a.e in Ω. Thus, the right-hand side of (4.42) can be written as

$$\begin{aligned}\Big|\int_\Omega \Big(\vartheta T_k(\frac{u_k^q}{|x|^p}) + T_k(f)\Big)(T_m(u_k) - T_m(u))^+ dx\Big| &\le \int_\Omega (\vartheta \frac{u_k^q}{|x|^p} + f)(2m)dx \\ &\le \int_\Omega (\vartheta \frac{\psi^q}{|x|^p} + f)(2m)dx,\end{aligned}$$

then, the right hand side is dominated by a function in $L^1(\Omega)$ independent on k, thus, by the dominated convergence and since $(T_m(u_k) - T_m(u))^+ \to 0$ a.e., tends to zero as k goes to infinity.

From (4.42) we have

$$\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+\right) dx \le o(1). \tag{4.43}$$

Let us define $\Omega_1 = \Omega \cap \{|u_k| \le m\}$ and $\Omega_2 = \Omega \cap \{|u_k| > m\}$.

We estimate the left hand side of (4.43) as

$$\begin{aligned}&\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+\right) dx = \\ &\int_{\Omega_1} \left(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\right) dx \\ &\quad + \int_{\Omega_1} \left(|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\right) dx \\ &\quad + \int_{\Omega_2} \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k) - T_m(u))^+\right) dx.\end{aligned} \tag{4.44}$$

Since $(T_m(u_k) - T_m(u))^+ \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$, the second term on the right hand side of (4.44) becomes

$$\begin{aligned}&\int_{\Omega_1} (|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+)dx \\ &\le \int_\Omega (|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+)dx \\ &\quad + \Big|\int_{\Omega_2} (|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+)dx\Big| \\ &\le o(1) + \int_{\Omega_2} |\nabla T_m(u)|^{p-1}\nabla(T_m(u_k) - T_m(u))^+)dx.\end{aligned}$$

Since $T_m(u_k) = m$ in the set $\Omega \cap \{u_k > m\}$, it follows

$$\begin{aligned}
&\int_{\Omega_1} (|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+)dx \\
&\leq o(1) + \int_{\Omega_2} |\nabla T_m(u)|^{p-1}\nabla u dx.
\end{aligned}$$

By Hölder's inequality and denoting χ_m the characteristic function of the set $\{x \in \Omega : |u_k| > m\}$,

$$\begin{aligned}
&\int_{\Omega_1} (|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+)dx \\
&\leq o(1) + C||u||^{p-1}_{W_0^{1,p}(\Omega)} ||\chi_m \nabla T_m(u)||_{L^p(\Omega)} \to 0 \quad \text{as} \quad k \to \infty,
\end{aligned}$$

by *Dominate Convergence Theorem*, since

$$\int_\Omega \chi_m |\nabla T_m(u)|^p dx \leq \int_\Omega |\nabla T_m(u)|^p dx$$

and since $u_k \to u$ a.e., $\nabla T_m(u) = 0$ in χ_m, thus, $\chi_m \nabla T_m(u) \to 0$ strongly in $(L^p(\Omega))^N$.

As above, the last term in (4.44) can be estimated as

$$\begin{aligned}
&\left| \int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \chi_m \nabla T_m(u)\right) dx \right| \\
&\leq C||u_k||^{p-1}_{W_0^{1,p}(\Omega)} ||\chi_m \nabla T_m(u)||_{L^p(\Omega)} \to 0,
\end{aligned} \tag{4.45}$$

as $k \to +\infty$.

We study now the first term in the right hand side of (4.44),

$$\begin{aligned}
&\int_{\Omega_1}\!\big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\big)dx \\
=& \int_{\Omega}\!\big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\big)dx \\
-& \int_{\Omega_2}\!\big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\big)dx \\
\leq& \int_{\Omega}\!\big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\big)dx \\
+& \left|\int_{\Omega_2}\!\big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k) - T_m(u))^+\big)dx\right|.
\end{aligned}$$

Considering that, by the *Dominated Convergence Theorem* and since $\nabla T_m(u_k) = 0$ in Ω_2, we have

$$\Big|\int_{\Omega_2}\Big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+\Big) dx\Big|$$
$$\leq \int_\Omega \chi_m |\nabla T_m(u)|^p dx \to 0 \quad \text{as} \quad k \to +\infty,$$

equation (4.44) becomes

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla (T_m(u_k) - T_m(u))^+\big)\, dx =$$
$$\int_\Omega \big(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k) - |\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla (T_m(u_k) - T_m(u))^+\big)\, dx$$
$$+\, o(1).$$

Finally, by Lemma 4, we obtain

$$\int_\Omega \big(|\nabla u_k|^{p-2}\nabla u_k, \nabla (T_m(u_k) - T_m(u))^+\big)\, dx \tag{4.46}$$
$$\geq \begin{cases} C_1(p) \displaystyle\int_\Omega \frac{|\nabla (T_m(u_k) - T_m(u))^+|^2}{(|\nabla T_m(u_k)| + |\nabla T_m(u|)^{2-p}} + o(1) & \text{if } 1 < p < 2, \\ C_1(p) \displaystyle\int_\Omega |\nabla (T_m(u_k) - T_m(u))^+|^p + o(1) & \text{if } p \geq 2, \end{cases}$$

with $C_1(p)$ a positive constant depending on p. Thanks to (4.43) it implies, as the calculus done in (4.23), that

$$||(T_m(u_k) - T_m(u))^+||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty. \tag{4.47}$$

Asymptotic behavior of $||(T_m(u_k) - T_m(u))^-||_{W_0^{1,p}(\Omega)}$**.**

We use $e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-$ as a test function in (4.11), obtaining

$$\begin{aligned}
&\int_\Omega e^{-T_m(u_k)}\left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right)dx \\
&\quad - \int_\Omega e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-(|\nabla u_k|^{p-2}\nabla u_k, \nabla T_m(u_k))dx \\
&\quad + \int_\Omega |\nabla u_k|^p e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^- dx \\
&= \int_\Omega e^{-T_m(u_k)}\left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right)dx \\
&\quad - \int_\Omega e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-(|\nabla u_k|^{p-2}\nabla u_k, \nabla T_m(u_k))dx \\
&\quad + \int_{\Omega_1} |\nabla u_k|^p e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^- dx \\
&\quad + \int_{\Omega_2} |\nabla u_k|^p e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^- dx \\
&= \int_\Omega \left(\vartheta T_k(\frac{u_k^q}{|x|^p}) + T_k(f)\right) e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^- dx.
\end{aligned} \tag{4.48}$$

In this case as well, since $(T_m(u_k) - T_m(u))^- \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$ and $(T_m(u_k) - T_m(u))^- \to 0$ a.e. in Ω, as in (4.44), the right hand side of (4.48) tends to zero as k goes to infinity.

Since $(\nabla T_m(u_k))\, \chi_m = 0$, the second term in the left hand side of (4.48), states as

$$\begin{aligned}
&- \int_\Omega e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-(|\nabla u_k|^{p-2}\nabla u_k, \nabla T_m(u_k))dx \\
&= - \int_{\Omega_1} e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-(|\nabla u_k|^{p-2}\nabla u_k, \nabla T_m(u_k))dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_\Omega e^{-T_m(u_k)}\left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right)dx \\
&+ \int_{\Omega_2} |\nabla u_k|^p e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^- dx = o(1).
\end{aligned} \tag{4.49}$$

We point out that

$$(T_m(u_k) - T_m(u))^- \chi_m = 0 \quad \text{and} \quad e^{-T_m(u_k)} \le e^{-m}.$$

Hence, (4.49) becomes

$$\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right) dx \leq C_m\, o(1) \tag{4.50}$$

as $k \to +\infty$, with C_m a positive constant depending on m.

The choice and use of $e^{-T_m(u_k)}(T_m(u_k)-T_m(u))^-$ as a test function allows us to simplify conveniently the equation (4.48) in order to obtain the desired result, the strong convergence. In fact, we proceed writing the left hand side of (4.50) as

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right) dx \\
=&\int_{\Omega_1}\left(|\nabla T_m(u_k)|^{p-2}\nabla T_m(u_k)-|\nabla T_m(u)|^{p-2}\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right)dx \\
&+\int_{\Omega_1} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) dx \\
&+\int_{\Omega_2} \left(|\nabla u_k|^{p-2}\nabla u_k, \nabla(T_m(u_k)-T_m(u))^-\right) dx \leq o(1).
\end{aligned} \tag{4.51}$$

Since $(T_m(u_k)-T_m(u))^- \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$ and $(T_m(u_k)-T_m(u))^- \to 0$ a.e. in Ω, the second term on the right hand side of (4.51) can be estimated as follows

$$\begin{aligned}
&\int_{\Omega_1} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) dx \\
&= \int_{\Omega} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) dx \\
&- \int_{\Omega_2} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) dx \\
&\leq o(1) - \int_{\Omega_2} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) dx.
\end{aligned} \tag{4.52}$$

By Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\Omega_1} |\nabla T_m(u)|^{p-2}\left(\nabla T_m(u), \nabla(T_m(u_k)-T_m(u))^-\right) \\
&\leq o(1) + C||u||^{p-1}_{W_0^{1,p}(\Omega)}||\chi_m \nabla T_m(u)||_{L^p(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty,
\end{aligned}$$

since by weak convergence the first term on the right hand side of (4.52) goes to zero, while the second one goes to zero using (4.35) and the fact that, for dominated convergence, $\chi_m \nabla T_m(u) \to 0$ strongly in $L^p(\Omega)$. Moreover, we observe that the last term in (4.51) is zero since $(T_m(u_k) - T_m(u))^- \chi_m = 0$. Finally, as above, by Lemma 4, equation (4.51) becomes

$$
\begin{aligned}
o(1) &\geq \int_\Omega \left(|\nabla u_k|^{p-2} \nabla u_k, \nabla (T_m(u_k) - T_m(u))^- \right) dx \\
&\geq \begin{cases} C_1(p) \displaystyle\int_\Omega \frac{|\nabla (T_m(u_k) - T_m(u))^-|^2}{(|\nabla T_m(u_k)| + |\nabla T_m(u|)^{2-p}} + o(1) & \text{if } 1 < p < 2, \\[2ex] C_1(p) \displaystyle\int_\Omega |\nabla (T_m(u_k) - T_m(u))^-|^p + o(1) & \text{if } p \geq 2, \end{cases}
\end{aligned}
\tag{4.53}
$$

with $C_1(p)$ a positive constant depending on p. By (4.50) and (4.53) (using (4.23) again), we get

$$
||(T_m(u_k) - T_m(u))^-||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty. \tag{4.54}
$$

From (4.41), (4.47) and (4.54) we have the desired result, i.e.

$$
||(T_m(u_k) - T_m(u))||_{W_0^{1,p}(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty.
$$

□

In order to be able to pass to the limit, we prove now that $|\nabla u_k|^p \to |\nabla u|^p$ strongly in $L^1(\Omega)$. By Lemma 6, the sequence of the gradients converges a.e. In order to use again Vitali's Theorem we need to prove the equi-integrability of $|\nabla u_k|^p$.

Let $E \subset \Omega$ be a measurable set, then

$$
\int_E |\nabla u_k|^p dx \leq \int_E |\nabla T_m(u_k)|^p dx + \int_{\{u_k \geq m\} \cap E} |\nabla u_k|^p dx.
$$

By Lemma 6, $T_m(u_k) \to T_m(u)$ in $W_0^{1,p}(\Omega)$ $\forall m > 0$ and therefore,

$\displaystyle\int_E |\nabla T_m(u_k)|^p dx$ is uniformly small for $|E|$ small enough.

Moreover, by Lemma 5 we obtain

$$
\int_{\{u_k \geq m\} \cap E} |\nabla u_k|^p dx \leq \int_{\{u_k \geq m\}} |\nabla u_k|^p dx \to 0 \quad \text{as } m \to \infty,
$$

uniformly on k. Then, Vitali's Theorem implies that

$$
|\nabla u_k|^p \to |\nabla u|^p \ \text{ strongly} \quad \text{in} \ \ L^1(\Omega). \tag{4.55}
$$

Step 4: *Passing to the limit in* (4.11).

Finally, since $||u_k - u||_{W_0^{1,p}(\Omega)} \to 0$ as $k \to +\infty$, we conclude that u is a distributional solution to the problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \vartheta \dfrac{u^q}{|x|^p} + f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ u = 0 & \text{in } \Omega. \end{cases}$$

In particular, we point out that the equation is verified even in a stronger way, that is

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \phi)dx + \int_\Omega |\nabla u|^p \phi dx = \vartheta \int_\Omega \frac{u^q}{|x|^p}\phi dx + \int_\Omega f\phi dx$$

for all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

$\square$

3 Qualitative properties: Symmetry of solutions

In this Section we are going to study a symmetry property of the solution u to (4.5). To study this qualitative property, we will need a weighted Poincare's inequality. First, we are going to recall the following definition.

Definition 5. *Assume $\tilde{\Omega} \subset\subset \Omega \setminus \{0\}$ and let $\rho \in L^1(\tilde{\Omega})$ and $1 \leq q < \infty$. The space $H_\rho^{1,q}(\tilde{\Omega})$ is defined as the completion of $C^1(\tilde{\Omega})$ (or $C^\infty(\tilde{\Omega})$) with the norm*

$$\|v\|_{H_\rho^{1,q}} = \|v\|_{L^q(\tilde{\Omega})} + \|\nabla v\|_{L^q(\tilde{\Omega},\rho)}, \tag{4.56}$$

where

$$\|\nabla v\|^q_{L^p(\tilde{\Omega},\rho)} := \int_{\tilde{\Omega}} \rho(x)|\nabla v(x)|^q dx.$$

We also recall that $H_\rho^{1,q}(\tilde{\Omega})$ may be equivalently defined as the space of functions with distributional derivatives represented by a function for which the norm defined in (4.56) is bounded. These two definitions are equivalent if the domain has piecewise regular boundary.
The space $H_{0,\rho}^{1,q}(\tilde{\Omega})$ is consequently defined as the completion of $C_c^1(\tilde{\Omega})$ (or $C_c^\infty(\tilde{\Omega})$), w.r.t. the norm (4.56).

A short, but quite complete, reference for weighted Sobolev spaces is in [65] and the references therein.

Theorem 17. *(Weighted Poincare's inequality). Let $p \geq 2$ and $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ be a solution of (4.5). Setting $\rho = |\nabla u|^{p-2}$ and $\tilde{\Omega} \subset\subset \Omega \setminus \{0\}$, we have that $H_0^{1,2}(\tilde{\Omega}, \rho)$ is continuously embedded in $L^q(\tilde{\Omega})$ for $1 \leq q < \hat{2}^*$ where*

$$\frac{1}{\hat{2}^*} = \frac{1}{2} - \frac{1}{N} + \frac{p-2}{p-1}\,\frac{1}{N}\,.$$

Consequently, since $\hat{2}^* > 2$, for $w \in H_0^{1,2}(\tilde{\Omega}, \rho)$ we have

$$\|w\|_{L^2(\tilde{\Omega})} \leqslant \mathcal{C}_S \|\nabla w\|_{L^2(\tilde{\Omega},\rho)} = \mathcal{C}_S \left(\int_{\tilde{\Omega}} \rho\, |\nabla w|^2 \right)^{\frac{1}{2}}, \tag{4.57}$$

with $\mathcal{C}_S = \mathcal{C}_S(\tilde{\Omega}) \to 0$ if $|\tilde{\Omega}| \to 0$.

See a detailed proof in [49].

Notice that Theorem 17 holds for $p \geq 2$. If $1 < p < 2$ and $|\nabla u|$ is bounded, $\frac{1}{\rho} = \frac{1}{|\nabla u|^{2-p}} \geq C$ and

$$c \int_{\tilde{\Omega}} \rho |\nabla u|^2 \geq \tilde{C} \int_{\tilde{\Omega}} |\nabla u|^2 \geq \int_{\tilde{\Omega}} u^2,$$

therefore, the weighted Poincaré's inequality (4.57) follows at once by the classic Poincaré's inequality.

In order to prove the symmetry of the solution we need the analysis of the regularity of the solution u that is summarized in the following Subsection.

3.1 Local regularity of solutions

Given any solution $u \in W_0^{1,p}(\Omega)$ to (4.5), the $C^{1,\alpha}_{loc}(\Omega \setminus \{0\})$ regularity of u follows by a classical regularity result, see [51, 93]. The arguments in [51, 93] do not work up to the origin, because the singularity of the potential. Moreover, if one assumes that the domain is smooth, the $C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ regularity up to the boundary follows by a result in [70].

The fact that the solutions to *p-Laplace* equations are not in general $C^2(\Omega)$, leads to the study of the summability properties of the second derivatives of the solutions. This fact is important for the study of some qualitative properties of these solutions. The results in [49] (and in [73] where appears a more general equation with a gradient term as in (4.5)) hold outside the singularity. In this direction we need some previous results.

To study the symmetry, one of the main ingredients is the *Moving Plane Method* by Alexandroff and Serrin. To use this geometrical argument we need the weighted Poincaré's inequality in Theorem 17 and, in order to obtain the weighted Poincaré's inequality, we need the following summability result for the gradient.

Theorem 18. *Assume* $1 < p < N$ *and consider* $u \in C^{1,\alpha}(\overline{\Omega}\setminus\{0\})$ *a solution to* (4.5)*, with* $f \in C^1(\overline{\Omega} \setminus \{0\})$*. Denoting* $u_i = \frac{\partial u}{\partial x_i}$*, we have*

$$\int_{\tilde{\Omega}} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^{\gamma}}\, dx \leqslant \mathcal{C} \qquad \forall\, i = 1, \ldots, N \quad , \tag{4.58}$$

for any $\tilde{\Omega} \subset\subset \Omega \setminus \{0\}$ *and uniformly for any* $y \in \tilde{\Omega}$*, with*

$$\mathcal{C} := \mathcal{C}\Big(p\,,\, \gamma\,,\, \beta\,,\, f\,,\, q\,,\, \vartheta\,,\, \|u\|_{L^\infty(\tilde{\Omega})}\,,\, \|\nabla u\|_{L^\infty(\tilde{\Omega})}\,,\, dist(\tilde{\Omega}, \{0\})\Big),$$

for $0 \leqslant \beta < 1$ *and* $\gamma < (N-2)$ *if* $N \geq 3$ *(*$\gamma = 0$ *if* $N = 2$*).*

If we also assume that f *is nonnegative in* Ω *then, it follows that, actually* $\vartheta \dfrac{u^q}{|x|^p} + f$ *is strictly positive in the interior of* Ω *and for any* $\tilde{\Omega} \subset\subset \Omega \setminus \{0\}$*, uniformly for any* $y \in \tilde{\Omega}$*, we have that*

$$\int_{\tilde{\Omega}} \frac{1}{|\nabla u|^t} \frac{1}{|x-y|^{\gamma}} dx \leqslant \mathcal{C}^*, \tag{4.59}$$

with $\max\{(p-2)\,,\, 0\} \leqslant t < p-1$ *and* $\gamma < (N-2)$ *if* $N \geq 3$ *(*$\gamma = 0$ *if* $N = 2$*). Moreover,* $\mathcal{C}^*$ *depends on* $\mathcal{C}$*.*

See [49], [73] for a detailed proof.

Remark 4. *Let* $Z_u = \{x \in \Omega :\ \nabla u(x) = 0\}$*. It is clear that* Z_u *is a closed set in* Ω *and moreover, by* (4.59) *it follows implicitly that the Lebesgue measure*

$$|Z_u| = 0,$$

provided that f *is nonnegative.*

Notice that if $|Z_u| \neq 0$, hence, there exists x such that $\nabla u(x) = 0$. Therefore, the integral in (4.59) would explode.

3.2 Previous statements and properties of the solutions

We precise some notations and statements to introduce the main arguments to study the qualitative property of the solution u.

Let ν be a direction in $\mathbb{R}^N$ with $|\nu| = 1$. For a real number λ we set the hyperplane

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\}. \tag{4.60}$$

Notice that $0 \in T_0^\nu$. Moreover, let us denote

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\}, \tag{4.61}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x\cdot\nu)\nu, \tag{4.62}$$

(which is the reflection trough the hyperplane T_λ^ν),

$$u_\lambda^\nu(x) = u(x_\lambda^\nu), \tag{4.63}$$

$$a(\nu) = \inf_{x\in\Omega} x\cdot\nu. \tag{4.64}$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$(\Omega_\lambda^\nu)' := R_\lambda^\nu(\Omega_\lambda^\nu) \tag{4.65}$$

and finally, for $\lambda > a(\nu)$ we denote

$$\lambda_1(\nu) = \sup\{\lambda : (\Omega_\lambda^\nu)' \subset \Omega\}. \tag{4.66}$$

Here below we are going to prove some useful results.

Lemma 7. *Assume $\vartheta > 0$ and $f \geq 0$. Consider $u \in W_0^{1,p}(\Omega)$ a nonnegative weak solution to problem* (4.5) *founded by Theorem 14. Then,*

$$\lim_{|x|\to 0} u(x) = +\infty.$$

Proof. We consider as a test function $\varphi = e^{-u}\psi$, with $\psi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, thus, $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Then, using φ as test function in (4.5) we obtain

$$\int_\Omega -e^{-u}\psi|\nabla u|^{p-1}\nabla u dx + \int_\Omega \nabla\psi e^{-u}|\nabla u|^{p-1}dx + \int_\Omega |\nabla u|^p e^{-u}\psi dx =$$

$$= \vartheta\int_\Omega \frac{u^q}{|x|^p}e^{-u}\psi dx + \int_\Omega f e^{-u}\psi dx.$$

Hence,

$$\int_\Omega \nabla\psi e^{-u}|\nabla u|^{p-1}dx \geq \vartheta\int_\Omega \frac{u^q}{|x|^p}e^{-u}\psi dx.$$

Therefore,

$$\int_\Omega |e^{-\frac{u}{p-1}}\nabla u|^{p-2}(e^{-\frac{u}{p-1}}\nabla u, \nabla\psi)dx \geq \vartheta\int_\Omega \frac{u^q}{|x|^p}\left(e^{-\frac{u}{p-1}}\right)^{p-1}\psi dx, \tag{4.67}$$

being $f(\cdot)$ nonnegative. Defining $v = 1 - e^{-\frac{u}{p-1}}$, it follows

$$|\nabla v|^{p-2} = \left|\frac{-e^{\frac{-u}{p-1}}}{p-1}\nabla u\right|^{p-2}.$$

From (4.67), we get

$$C_p\int_\Omega |\nabla v|^{p-2}(\nabla v, \nabla\psi)dx \geq \vartheta\int_\Omega \frac{u^q}{|x|^p}(1-v)^{p-1}\psi dx. \tag{4.68}$$

Let us consider now u_R the radial solution to the problem

$$\begin{cases} -\Delta_p u + |\nabla u|^p = \frac{C}{|x|^p} \text{ in } B_R \\ u \geq 0 \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R, \end{cases} \tag{4.69}$$

constructed as limit of the solutions, say $u_{R,k}$, to the truncated problems, in the same way as we did in Section 2 but setting here $\vartheta = 0$, with C, R some positive constants that we choose later. Moreover, for k fixed, since the right hand side is not depending on u, it is easy to check that the solution $u_{R,k}$ is unique. In particular, the reflected function $u^\nu_{R,k,\lambda}$ will be a solution too and, since $u_{R,k}$ is unique, $u_{R,k} = u^\nu_{R,k,\lambda}$, then, $u_{R,k}$ must be radial for all k. Finally, the strong convergence in $W^{1,p}_0(\Omega)$ (and thus, pointwise $u_R(x) = \lim_{k\to\infty} u_{R,k}(x)$) implies that the limit of $u_{R,k}$ will be radial too, then, $u_R(x) = u_R(|x|)$.

Then, by setting $\varphi = e^{-u_R}\psi$, $v_R = 1 - e^{-\frac{u_R}{p-1}}$ (as in equations (4.67) and (4.68)), we have

$$C_p \int_{B_R} |\nabla v_R|^{p-2}(\nabla v_R, \nabla\psi)dx = \int_{B_R} \frac{C}{|x|^p}(1 - v_R)^{p-1}\psi dx. \tag{4.70}$$

We note that by the regularity of u, the function v (resp. v_R) belongs to $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ (to $W^{1,p}_0(B_R) \cap L^\infty(B_R)$). Using (4.68) with $\psi = (v_R - v)^+$, R small such that $B_R \subset\subset \Omega$ and, since $u \geq 0$ in Ω; in particular, $u \geq 0$ on ∂B_R and $v \geq 0$ as well. Otherwise, by definition $u_R = 0$ on ∂B_R, thus, $v_R = 0$ on ∂B_R, therefore, $v_R < v$ on ∂B_R and $\psi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, thus

$$\begin{aligned} &C_p \int_{B_R} |\nabla v|^{p-2}(\nabla v, \nabla(v_R - v)^+)dx \\ &\geq \int_{B_R} \frac{u^q}{|x|^p}(1 - v)^{p-1}(v_R - v)^+ dx \\ &\geq \int_{B_R} \frac{C_R}{|x|^p}(1 - v)^{p-1}(v_R - v)^+ dx \end{aligned} \tag{4.71}$$

with $C_R = \inf_{B_R} u(x) > 0$ by the *strong maximum principle* and

$$\begin{aligned} &C_p \int_{B_R} |\nabla v_R|^{p-2}(\nabla v_R, \nabla(v_R - v)^+)dx \\ &= \int_{B_R} \frac{C_R}{|x|^p}(1 - v_R)^{p-1}(v_R - v)^+ dx, \end{aligned} \tag{4.72}$$

where, in (4.69), we choose $C = C_R$. Thus, subtracting (4.71) and (4.72) we obtain

$$\begin{aligned} &C_p \int_{B_R} (|\nabla v_R|^{p-2}\nabla v_R - |\nabla v|^{p-2}\nabla v, \nabla(v_R - v)^+)dx \\ &= \int_{B_R} \frac{C_R}{|x|^p}\Big((1 - v_R)^{p-1} - (1 - v)^{p-1}\Big)(v_R - v)^+ dx. \end{aligned} \tag{4.73}$$

On the set $B_R \cap \{v_R \geq v\}$, the right hand side of (4.73) is nonpositive and, therefore,

$$\int_{B_R} (|\nabla v_R|^{p-2}\nabla v_R - |\nabla v|^{p-2}\nabla v, \nabla(v_R - v)^+)dx \leq 0,$$

By Lemma 4, $\nabla(v_R - v)^+ = 0$, hence, $(v_R - v)^+ = C$ and, since $(v_R - v)^+ = 0$ on ∂B_R, $(v_R - v)^+ = 0$ in B_R, that is (using the definition of v and v_R and the monotonicity of $s = 1 - e^{-\frac{s}{p-1}}$),

$$1 - e^{-\frac{u}{p-1}} \geq 1 - e^{-\frac{u_R}{p-1}} \qquad \text{then} \qquad u \geq u_R. \tag{4.74}$$

We are going to study the qualitative behavior of u_R considering the test function $\varphi = e^{-u_R}\psi$, with $\psi = \psi(|x|)$ belonging to $W_0^{1,p}(B_R) \cap L^\infty(B_R)$. Then, by (4.69) we have

$$\int_0^R |\nabla u_R|^{p-2}(\nabla u_R, \psi')e^{-u_R}|x|^{N-1}dx - \int_0^R |\nabla u_R|^{p-2}(\nabla u_R, \psi)|\nabla u_R|e^{-u_R}dx$$
$$+ \int_0^R |\nabla u_R|^p e^{-u_R}\psi dx = \int_0^R \frac{c}{|x|^p} e^{-u_R}\psi|x|^{N-1}dx.$$

Therefore,

$$\int_0^R e^{-u_R}|u_R'|^{p-2}(u_R', \psi')\rho^{N-1}d\rho = \int_0^R C_R e^{-u_R}\psi\rho^{N-1-p}d\rho,$$

with $\rho = |x|$. By Hopf's Lemma, if $|x| = \rho \neq 0$, $\nabla u(x) \neq 0$, and by classical regularity results for the *Laplacian* operator we have $u_R \in C^2(\overline{B}_R \setminus \{0\})$ and thus, integrating by parts,

$$\int_0^R e^{-u_R}|u_R'|^{p-2}(u_R', \psi')\rho^{N-1}d\rho = \int_0^R (e^{-u_R}|u_R'|^{p-2}|u_R'|\psi\rho^{N-1})d\rho.$$

Therefore,

$$\int_0^R \psi(e^{-u_R}|u_R'|^{p-2}|u_R'|\rho^{N-1})d\rho - \int_0^R C_R e^{-u_R}\psi\rho^{N-1-p}d\rho = 0,$$

thus,

$$\int_0^R \psi(e^{-u_R}|u_R'|^{p-2}|u_R'|\rho^{N-1} - C_R e^{-u_R}\rho^{N-1-p})d\rho = 0.$$

Hence,

$$\left(e^{-u_R}|u_R'|^{p-2}(-u_R')\rho^{N-1}\right)' = C_R e^{-u_R}\rho^{N-1-p} \quad \forall\, \rho \neq 0. \tag{4.75}$$

Since $u_R(\rho)$ is positive and monotone decreasing w.r.t. ρ, we have the two following cases:

(i) either $\lim_{\rho\to 0} u_R(\rho) = C > 0$;

(ii) or $\lim_{\rho\to 0} u_R(\rho) = +\infty$.

If we assume the case (i), from the expression (4.75), we have

$$\left(e^{-u_R}|u_R'|^{p-2}(-u_R')\rho^{N-1}\right)'/(\rho^{N-p})' \to C \qquad \text{as} \qquad \rho \to 0,$$

for some positive constant C. Using L'Hôpital,

$$\left(e^{-u_R}|u_R'|^{p-2}(-u_R')\rho^{N-1}\right)/\rho^{N-p} \to C \qquad \text{as} \qquad \rho \to 0.$$

Since $u_R(\rho)$ is positive and by (i),

$$(-u_R')^{p-1} \to C/\rho^{p-1} \qquad \text{as} \qquad \rho \to 0.$$

Therefore, $-u_R' \geq C/\rho + o(1)$ for $\rho \to 0$, getting a contradiction with the case (i). Then, the case (ii) holds and together with (4.74) it concludes the proof. ☐

From now on we shall assume the following hypotheses:

(HP. 1) $f(x) \in C^1(\overline{\Omega} \setminus \{0\})$ and $f(x) \geq 0$;

(HP. 2) Monotonicity of $f(\cdot)$ in the $\nu-$direction:

$$f(x) \leq f(x_\lambda^\nu),\ \forall \lambda \in (a(\nu)\,, \lambda_1(\nu)).$$

Define $\phi_\rho(x) \in C_c^\infty(\Omega)$, $\phi \geq 0$ such that

$$\begin{cases} \phi \equiv 1 & \text{in } \Omega \setminus B_{2\rho}(0) \\ \phi \equiv 0 & \text{in } B_\rho(0) \\ |\nabla \phi| \leq \frac{C}{\rho} & \text{in } B_{2\rho}(0) \setminus B_\rho(0), \end{cases} \tag{4.76}$$

where $B_\rho(0)$ denotes the open ball centered in 0 and with radius $\rho > 0$.

Lemma 8. *Let $u \in C^1(\overline{\Omega} \setminus \{0\})$ be a solution to (4.5) and let us define the critical set*

$$Z_u = \{x \in \Omega :\ \nabla u(x) = 0\}.$$

Then, the set $\Omega \setminus Z_u$ does not contain any connected component $\mathcal{C}$ such that $\overline{\mathcal{C}} \subset \Omega$. Moreover, if we assume that Ω is a smooth bounded domain with connected boundary, it follows that $\Omega \setminus Z_u$ is connected.

Proof. To prove the Lemma we proceed by contradiction. Assume that such component exists, namely

$$\mathcal{C} \subset \Omega \quad \text{such that} \quad \partial\mathcal{C} \subset Z_u.$$

Recall that, by Remark 4, we have that

$$|Z_u| = 0.$$

Thus,

$$-\Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f(x) \quad \text{a.e. in } \Omega. \tag{4.77}$$

For all $\varepsilon > 0$, we define $J_\varepsilon : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ in the following way

$$J_\varepsilon(t) = \begin{cases} t & \text{if } t \geq 2\varepsilon \\ 2t - 2\varepsilon & \text{if } \varepsilon \leq t \leq 2\varepsilon \\ 0 & \text{if } 0 \leq t \leq \varepsilon. \end{cases} \tag{4.78}$$

We shall use

$$\Psi = e^{-u} \phi_\rho(x) \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} \chi_{\mathcal{C}} \tag{4.79}$$

as a test function in (4.77), where $\phi_\rho(x)$ is as in (4.76). Notice that the function Ψ does not have problems in the critical set Z_u because $\Psi = 0$ if $\nabla u(x) = 0$. We point out that Ψ is well defined in $\mathcal{C}$ and, since $\partial\mathcal{C} \subset Z_u$, Ψ is 0 on $\partial\mathcal{C}$. In particular, $\mathrm{supp}\Psi \subset \mathcal{C}$, which implies that $\Psi \in W_0^{1,p}(\mathcal{C})$. Integrating by parts we get

$$\begin{aligned} &\int_{\mathcal{C}} e^{-u} \left(|\nabla u|^{p-2} \nabla u, \nabla \left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} \right) \right) \phi_\rho dx \\ &+ \int_{\mathcal{C}} e^{-u} (|\nabla u|^{p-2} \nabla u, \nabla \phi_\rho) \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} dx \\ &- \int_{\mathcal{C}} e^{-u} |\nabla u|^p \phi_\rho \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} dx + \int_{\mathcal{C}} e^{-u} |\nabla u|^p \phi_\rho \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} dx \\ &= \vartheta \int_{\mathcal{C}} \frac{u^q}{|x|^p} e^{-u} \phi_\rho \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} dx + \int_{\mathcal{C}} f e^{-u} \phi_\rho \frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} dx, \end{aligned} \tag{4.80}$$

notice that we have used the fact that the boundary term in the integration is zero since $\partial\mathcal{C} \subset\ Z_u$ and the definition of Ψ. Remarkably, using the test function Ψ defined in (4.79), we are able to integrate on the boundary $\partial\mathcal{C}$ which could be not regular. We estimate the first term on the left hand side of (4.80), denoting $h_\varepsilon(t) = \dfrac{J_\varepsilon(t)}{t}$ and since $e^{-u} \leq 1$,

$$\begin{aligned}
&\left| \int_{\mathcal{C}} e^{-u} \left(|\nabla u|^{p-2} \nabla u, \nabla \left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} \right) \right) \phi_\rho dx \right| \\
&\leq C \int_{\mathcal{C}} |\nabla u|^{p-1} |h'_\varepsilon(|\nabla u|)| |\nabla(|\nabla u|)| \phi_\rho dx \qquad (4.81) \\
&\leq C \int_{\mathcal{C}} |\nabla u|^{p-2} \Big(|\nabla u| h'_\varepsilon(|\nabla u|) \Big) |D^2 u| |\phi_\rho dx.
\end{aligned}$$

We show now the following claim.

CLAIM.

(i) $|\nabla u|^{p-2} ||D^2 u|| \phi_\rho \in L^1(\mathcal{C})$ $\forall \rho > 0$;

(ii) $|\nabla u| h'_\varepsilon(|\nabla u|) \to 0$ a.e. in $\mathcal{C}$ as $\varepsilon \to 0$ and $|\nabla u| h'_\varepsilon(|\nabla u|) \leq C$ with C not depending on ε.

Let us prove (i).

By Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\mathcal{C}} |\nabla u|^{p-2} ||D^2 u|| \phi_\rho dx \\
&\leq C(\mathcal{C}) \left(\int_{\mathcal{C}} |\nabla u|^{2(p-2)} ||D^2 u||^2 \phi_\rho^2 dx \right)^{\frac{1}{2}} \qquad (4.82) \\
&= C \left(\int_{\mathcal{C}} |\nabla u|^{p-2-\beta} ||D^2 u||^2 \phi_\rho^2 |\nabla u|^{p-2+\beta} dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Notice that $\phi_\rho = 0$ in B_ρ and, taking into account that we are looking away from the singularity, $\phi_\rho^2 |\nabla u|^{p-2+\beta}$ is bounded since β can be any value with $0 \leq \beta < 1$,

$$\begin{aligned}
&\int_{\mathcal{C}} |\nabla u|^{p-2} ||D^2 u|| \phi_\rho dx \\
&\leq C ||\nabla u||_{L^\infty(\mathcal{C} \setminus B_\rho)}^{(p-2+\beta)/2} \left(\int_{\mathcal{C} \setminus B_\rho} |\nabla u|^{p-2-\beta} ||D^2 u||^2 dx \right)^{\frac{1}{2}} \leq C,
\end{aligned}$$

where we have used also Theorem 18 to conclude (i).

Let us prove (ii). Exploiting the definition (4.78), by straightforward calculation we obtain

$$h'_\varepsilon(t) = \begin{cases} 0 & \text{if } t \geq 2\varepsilon \\ \frac{2\varepsilon}{t^2} & \text{if } \varepsilon \leq t \leq 2\varepsilon \\ 0 & \text{if } 0 \leq t \leq \varepsilon, \end{cases}$$

and then, we have

$$|\nabla u| h'_\varepsilon(|\nabla u|) = \begin{cases} 0 & \text{if } \nabla u \geq 2\varepsilon \\ \frac{2\varepsilon}{|\nabla u|} & \text{if } \varepsilon \leq \nabla u \leq 2\varepsilon \\ 0 & \text{if } 0 \leq \nabla u \leq \varepsilon. \end{cases}$$

Thus,

$|\nabla u| h'_\varepsilon(|\nabla u|) \to 0$ a.e. for $\varepsilon \to 0$ in $\mathcal{C}$. Notice also that in $\varepsilon \leq \nabla u \leq 2\varepsilon$, $\frac{2\varepsilon}{|\nabla u|} \leq 2$; hence, $|\nabla u| h'_\varepsilon(|\nabla u|) \leq 2$.

Then, by the *Claim* (using *Dominated Convergence Theorem*) and equation (4.81) we have

$$\int_{\mathcal{C}} e^{-u} \left(|\nabla u|^{p-2} \nabla u, \nabla \left(\frac{J_\varepsilon(|\nabla u|)}{|\nabla u|} \right) \right) \phi_\rho dx \to 0 \quad \text{as } \varepsilon \to 0, \quad \forall \rho > 0.$$

Since $h_\varepsilon(t) \leq J_\varepsilon(t)$, by the *Dominated Convergence Theorem*, exploiting (4.78) and passing to the limit in (4.80), it follows

$$\int_{\mathcal{C}} e^{-u} (|\nabla u|^{p-2} \nabla u, \nabla \phi_\rho) dx = \vartheta \int_{\mathcal{C}} \frac{u^q}{|x|^p} e^{-u} \phi_\rho dx + \int_{\mathcal{C}} f e^{-u} \phi_\rho dx \quad \forall \rho > 0.$$

Then, using the definition of ϕ_ρ, we obtain

$$\int_{B_{2\rho} \setminus B_\rho} e^{-u} (|\nabla u|^{p-2} \nabla u, \nabla \phi_\rho) dx = \vartheta \int_{\mathcal{C}} \frac{u^q}{|x|^p} e^{-u} \phi_\rho dx + \int_{\mathcal{C}} f e^{-u} \phi_\rho dx. \tag{4.83}$$

Letting $\rho \to 0$ in (4.83), by Hölder's inequality and since $e^{-u} \leq C$, we estimate the left hand side as

$$\begin{aligned} &\left| \int_{B_{2\rho} \setminus B_\rho} e^{-u} (|\nabla u|^{p-2} \nabla u, \nabla \phi_\rho) dx \right| \\ &\leq C \left(\int_{B_{2\rho} \setminus B_\rho} |\nabla u|^p dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2\rho} \setminus B_\rho} |\nabla \phi_\rho|^p dx \right)^{\frac{1}{p}} \\ &\leq C \left((2\rho - \rho)^{\frac{N}{p-1}} (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p-1}} \right)^{\frac{p-1}{p}} (\frac{C}{\rho^p})^{\frac{1}{p}} \\ &= C \left(\rho^{\frac{N}{p}} ||u||_{W_0^{1,p}(\Omega)} \right) \frac{C}{\rho} = \left(\frac{\rho^N}{\rho^p} \right)^{\frac{1}{p}} \to 0, \qquad \text{as} \qquad \rho \to 0, \end{aligned}$$

where we used also that $|\nabla \phi_\rho| \leq \dfrac{C}{\rho}$ and $p < N$. On the other hand, for $\rho \to 0$, the right hand side of (4.83), by *Dominated Convergence Theorem*, becomes

$$\vartheta \int_{\mathcal{C}} \frac{u^q}{|x|^p} e^{-u} dx + \int_{\mathcal{C}} f e^{-u} dx > 0,$$

which is a contradiction.

If Ω is smooth, since the right hand side of (4.77) is positive, by Hopf's Lemma (see [82]), a neighborhood of the boundary belongs to a component $\mathcal{C}$ of $\Omega \setminus Z_u$. By what we have just proved above, a second component $\mathcal{C}'$ can not be contained compactly in Ω. Thus, $\Omega \setminus Z_u$ is connected. $\square$

3.3 Comparison principles

In order to use the *Moving Plane Method*, we need to prove the following proposition

Proposition 4 (*Weak Comparison Principle*)**.** *Let $\lambda < 0$ and $\tilde{\Omega}$ be a bounded domain such that $\tilde{\Omega} \subset\subset \Omega_\lambda^\nu$. Assume that $u \in C^1(\overline{\Omega} \setminus \{0\})$ is a solution to* (4.5) *such that $u \le u_\lambda^\nu$ on $\partial\tilde{\Omega}$. Then, there exists a positive constant $\delta = \delta\left(\lambda, dist(\tilde{\Omega}, \partial\Omega)\right)$ such that if we assume $|\tilde{\Omega}| \le \delta$, then it holds*

$$u \le u_\lambda^\nu \quad \text{in } \tilde{\Omega}.$$

Proof. We have (in the weak sense)

$$-\quad \Delta_p u + |\nabla u|^p = \vartheta \frac{u^q}{|x|^p} + f \qquad \text{in } \Omega, \tag{4.84}$$

$$-\quad \Delta_p u_\lambda^\nu + |\nabla u_\lambda^\nu|^p = \vartheta \frac{(u_\lambda^\nu)^q}{|x_\lambda^\nu|^p} + f_\lambda^\nu \qquad \text{in } \Omega, \tag{4.85}$$

where $f_\lambda^\nu(x) = f(x_\lambda^\nu)$.

Let us set $\phi_{\rho,\lambda}^\nu(x) = \phi_\rho(x_\lambda^\nu)$, with $\phi_\rho(\cdot)$ as in (4.76). By contradiction, we assume the statement false and we consider

(*i*) $e^{-u}(u - u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2\chi_{\tilde{\Omega}} \in W_0^{1,p}(\tilde{\Omega})$, as a test function in (4.84);

(*ii*) $e^{-u_\lambda^\nu}(u - u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2\chi_{\tilde{\Omega}} \in W_0^{1,p}(\tilde{\Omega})$, as a test function in (4.85).

Notice that, by Lemma 7 we have that $\lim\limits_{|x|\to 0} u(x) = +\infty$. This, together with the fact that $u \in L^\infty(\tilde{\Omega})$, implies that (see expression (4.62))

$$0_\lambda^\nu = R_\lambda^\nu(0) \notin \operatorname{supp}(u - u_\lambda^\nu)^+, \tag{4.86}$$

because if $0_\lambda^\nu \in \text{supp}(u-u_\lambda^\nu)^+$, $u > u_\lambda^\nu$ in $R_\lambda^\nu(0)$ but when x is near the 0, $u_\lambda^\nu = \infty$ thus, $u > \infty$, which is a contradiction with the fact that $u \in L^\infty(\tilde{\Omega})$.

Therefore, $R_\lambda^\nu(0) \notin \text{supp}(u-u_\lambda^\nu)^+$ which implies that $\text{supp}(u-u_\lambda^\nu)^+$ is away from 0 and ∇u is bounded in this support.

Then, if we subtract (in the weak formulation) (4.84) and (4.85), we get

$$\begin{aligned}
&\int_{\tilde{\Omega}} (e^{-u_\lambda^\nu} - e^{-u_\lambda^\nu} + e^{-u})(|\nabla u|^{p-2}\nabla u, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&+ \int_{\tilde{\Omega}} Ce^{-u}(|\nabla u|^{p-2}\nabla u, (u-u_\lambda^\nu)^+)|\nabla \phi_{\rho,\lambda}^\nu|\phi_{\rho,\lambda}^\nu dx \\
&- \int_{\tilde{\Omega}} e^{-u}(|\nabla u|^p, (u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx + \int_{\tilde{\Omega}} e^{-u}(|\nabla u|^p, (u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&- \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}(|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&- \int_{\tilde{\Omega}} e^{-u_\lambda^\nu} C(|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, (u-u_\lambda^\nu)^+)|\nabla \phi_{\rho,\lambda}^\nu|\phi_{\rho,\lambda}^\nu dx \\
&+ \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}(|\nabla u_\lambda^\nu|^p, (u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 - e^{-u_\lambda^\nu}(|\nabla u_\lambda^\nu|^p, (u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad = \vartheta \int_{\tilde{\Omega}} e^{-u}\frac{u^q}{|x|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \vartheta \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}\frac{(u_\lambda^\nu)^q}{|x_\lambda^\nu|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\qquad + \int_{\tilde{\Omega}} e^{-u} f(x)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \int_{\tilde{\Omega}} e^{-u_\lambda^\nu} f(x_\lambda^\nu)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx.
\end{aligned}$$

Groping the terms, we obtain

$$\begin{aligned}
&\int_{\tilde{\Omega}} e^{-u_\lambda^\nu}(|\nabla u|^2\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde{\Omega}} (e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^2\nabla u, \nabla(u-u_{\lambda^\nu})^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde{\Omega}} C(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla \phi_{\rho,\lambda}^\nu)(u-u_\lambda^\nu)^+\phi_{\rho,\lambda}^\nu dx \\
&= \vartheta \int_{\tilde{\Omega}} e^{-u}\frac{u^q}{|x|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \vartheta \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}\frac{(u_\lambda^\nu)^q}{|x_\lambda^\nu|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde{\Omega}} e^{-u} f(x)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \int_{\tilde{\Omega}} e^{-u_\lambda^\nu} f(x_\lambda^\nu)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\tilde\Omega} e^{-u_\lambda^\nu}(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde\Omega}(e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^{p-2}\nabla u, \nabla(u-u_{\lambda^\nu})^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde\Omega} C(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla\phi_{\rho,\lambda}^\nu)(u-u_\lambda^\nu)^+\phi_{\rho,\lambda}^\nu dx \\
&\geq \int_{\tilde\Omega} e^{-u_\lambda^\nu}(|\nabla u|^2\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad - \int_{\tilde\Omega}\Big|(e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^2\nabla u, \nabla(u-u_{\lambda^\nu})^+)(\phi_{\rho,\lambda}^\nu)^2\Big| dx \\
&\quad - \int_{\tilde\Omega}\Big|C(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla\phi_{\rho,\lambda}^\nu)(u-u_\lambda^\nu)^+\phi_{\rho,\lambda}^\nu\Big| dx,
\end{aligned}$$

we can write

$$\begin{aligned}
&\int_{\tilde\Omega} e^{-u_\lambda^\nu}(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\leq \int_{\tilde\Omega}\left|(e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^{p-2}\nabla u, \nabla(u-u_\lambda^\nu)^+)\right|(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + C\int_{\tilde\Omega}\left|(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla\phi_{\rho,\lambda}^\nu)\right|(u-u_\lambda^\nu)^+\phi_{\rho,\lambda}^\nu dx \\
&\quad + \vartheta\int_{\tilde\Omega} e^{-u}\frac{u^q}{|x|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \vartheta\int_{\tilde\Omega} e^{-u_\lambda^\nu}\frac{(u_\lambda^\nu)^q}{|x_\lambda^\nu|^p}(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx \\
&\quad + \int_{\tilde\Omega} e^{-u}f(x)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx - \int_{\tilde\Omega} e^{-u_\lambda^\nu}f(x_\lambda^\nu)(u-u_\lambda^\nu)^+(\phi_{\rho,\lambda}^\nu)^2 dx.
\end{aligned} \tag{4.87}$$

The term in the third line of (4.87) can be estimated using the definition of $\phi_{\rho,\lambda}^\nu$, Hölder's inequality, the condition $p<N$ and taking into account that if $u>u_\lambda^\nu$, $u-u_\lambda^\nu \leq ||u|| + ||u_\lambda^\nu|| \leq C||u||$,

$$\begin{aligned}
&C\int_{\tilde\Omega}\left|(e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla\phi_{\rho,\lambda}^\nu)\right|(u-u_\lambda^\nu)^+\phi_{\rho,\lambda}^\nu dx \\
&\leq C(||u||_{L^\infty(\Omega_\lambda^\nu)})\int_{\tilde\Omega}\left||\nabla u|^{p-1} + |\nabla u_\lambda^\nu|^{p-1}\right||\nabla\phi_{\rho,\lambda}^\nu|\phi_{\rho,\lambda}^\nu dx \\
&\leq C||u|_{L^\infty(\Omega_\lambda^\nu)}\Big[\Big(\int_{\tilde\Omega}|\nabla u|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_{\tilde\Omega}|\nabla\phi_{\rho,\lambda}^\nu|^p dx\Big)^{\frac{1}{p}} \\
&\qquad\qquad + \Big(\int_{\tilde\Omega}|\nabla u_\lambda^\nu|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_{\tilde\Omega}|\nabla\phi_{\rho,\lambda}^\nu|^p dx\Big)^{\frac{1}{p}}\Big] \\
&\leq C(||u||_{L^\infty(\Omega_\lambda^\nu)})\left(\int_{\tilde\Omega}(|\nabla u|^p + |\nabla u_\lambda^\nu|^p)dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2\rho}\setminus B_\rho}|\nabla\phi_{\rho,\lambda}^\nu|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $u, u_\lambda^\nu \in W^{1,p}(\tilde{\Omega})$ and by the definition of $\phi_{\rho,\lambda}^\nu$,

$$
\begin{aligned}
& C \int_{\tilde{\Omega}} \left| (e^{-u}|\nabla u|^{p-2}\nabla u - e^{-u_\lambda^\nu}|\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla \phi_{\rho,\lambda}^\nu) \right| (u-u_\lambda^\nu)^+ \phi_{\rho,\lambda}^\nu dx \\
& \leq C|B_{2\rho} - B_\rho|^{\frac{1}{p}} \left(\frac{C}{\rho}\right)^p \rho^{N-1} \to 0 \qquad \text{as} \qquad \rho \to 0.
\end{aligned}
$$

Notice that we are considering the set $\tilde{\Omega} \cap \{u \geq u_\lambda\}$ and, therefore, $|x| \geq |x_\lambda^\nu|$ and $\frac{1}{|x|} \leq \frac{1}{|x_\lambda^\nu|}$.

Recall also that since we are in this set, $-e^{-u_\lambda^\nu} \geq -e^{-u}$. Using the last estimation, equation (4.87) becomes

$$
\begin{aligned}
& \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
& \leq \int_{\tilde{\Omega}} \left| (e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^{p-2}\nabla u, \nabla(u-u_\lambda^\nu)^+) \right| (\phi_{\rho,\lambda}^\nu)^2 dx \\
& \quad + \vartheta \int_{\tilde{\Omega}} e^{-u} \left(\frac{u^q - (u_\lambda^\nu)^q}{|x|^p} \right) (u-u_\lambda^\nu)^+ (\phi_{\rho,\lambda}^\nu)^2 dx \\
& \quad + \int_{\tilde{\Omega}} e^{-u} \left(f(x) - f(x_\lambda^\nu)\right) (u-u_\lambda^\nu)^+ (\phi_{\rho,\lambda}^\nu)^2 dx + o(1).
\end{aligned}
$$

By (**hp. 2**), $f(x) \leq f(x_\lambda^\nu)$, it follows that the last term in the previous equation is negative.

Taking into account also that for $\lambda < 0$, the distance to any point to 0 is positive, that is, $|x| \geq C$ in Ω_λ^ν for some positive constant C, one has

$$
\begin{aligned}
& \int_{\tilde{\Omega}} e^{-u_\lambda^\nu}(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda^\nu|^{p-2}\nabla u_\lambda^\nu, \nabla(u-u_\lambda^\nu)^+)(\phi_{\rho,\lambda}^\nu)^2 dx \\
& \leq \int_{\tilde{\Omega}} \left| (e^{-u} - e^{-u_\lambda^\nu})(|\nabla u|^{p-2}\nabla u, \nabla(u-u_\lambda^\nu)^+) \right| (\phi_{\rho,\lambda}^\nu)^2 dx \\
& + C_3 \int_{\tilde{\Omega}} [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx + o(1),
\end{aligned} \tag{4.88}
$$

with $C_3 = C_3 \left(\lambda, \vartheta, ||u||_{L^\infty(\Omega_\lambda^\nu)}, dist(\tilde{\Omega}, \partial\Omega)\right)$.

We note that, in the last inequality, we have used the fact that the term $u^q - (u_\lambda^\nu)^q$ is locally Liptschitz continuous in $(0, +\infty)$, hence, $u^q - (u_\lambda^\nu)^q \leq C(u - u_\lambda^\nu)$ and the fact that, by strong maximum principle (see [82]), the solution u is strictly positive in $\tilde{\Omega}$.

Thus, since the term $(e^{-u} - e^{-u_\lambda^\nu})$ is locally Lipschitz continuous and since $\frac{1}{e^{u_\lambda^\nu}} \geq C$, from (4.88) and by Lemma 4, we get

$$\begin{aligned} & C_1 \int_{\tilde{\Omega}} (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\ & \leq C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-1} |\nabla (u-u_\lambda^\nu)^+| (u-u_\lambda^\nu)^+ (\phi_{\rho,\lambda}^\nu)^2 dx \\ & + C_3 \int_{\tilde{\Omega}} [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx + o(1), \end{aligned} \tag{4.89}$$

with $C_1 = C_1(p, ||u||_{L^\infty(\Omega_\lambda^\nu)})$ and $C_2 = C_2(||u||_{L^\infty(\Omega_\lambda^\nu)})$ positive constants.

We are going to split the proof in two cases.

Case: $p \geq 2$**.** Let us evaluate the terms on the right hand side of the inequality (4.89). Exploiting the weighted Young's inequality and taking into account that $u \in W_0^{1,p}(\tilde{\Omega})$, we get

$$\begin{aligned} & C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-1} |\nabla (u-u_\lambda^\nu)^+| (u-u_\lambda^\nu)^+ (\phi_{\rho,\lambda}^\nu)^2 dx \\ & \leq \varepsilon C_2 \int_{\tilde{\Omega}} |\nabla u|^{p-2} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx + \frac{C_2}{\varepsilon} \int_{\tilde{\Omega}} |\nabla u|^p [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\ & \leq \varepsilon C_2 \int_{\tilde{\Omega}} (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\ & + \tilde{C}_2 \int_{\tilde{\Omega}} [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx, \end{aligned} \tag{4.90}$$

with $\tilde{C}_2 = \tilde{C}_2(\varepsilon, ||u||_{L^\infty(\Omega_\lambda^\nu)}, ||\nabla u||_{L^\infty(\Omega_\lambda^\nu)})$ a positive constant. Since $p > 2$, we also used $|\nabla u|^{p-2} \leq (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2}$.

Thus, choosing ε sufficiently small such that $C_1 - \varepsilon C_2 \geq \tilde{C}_1 > 0$, using (4.90), equation (4.89) becomes

$$\begin{aligned} & \int_{\tilde{\Omega}} (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\ & \leq C \int_{\tilde{\Omega}} [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx + o(1), \end{aligned} \tag{4.91}$$

for some positive constant $C = \frac{\tilde{C}_2 + C_3}{\tilde{C}_1}$.

By weighted Poincaré's inequality (see Theorem 17) and the definition of $\phi^{\nu}_{\rho,\lambda}$, we get

$$\begin{aligned}
&C\int_{\tilde{\Omega}}[(u-u^{\nu}_{\lambda})^{+}]^2(\phi^{\nu}_{\rho,\lambda})^2dx\\
&\leq \tilde{C}C_p^2(\tilde{\Omega})\int_{\tilde{\Omega}}|\nabla u|^{p-2}|\nabla(u-u_{\lambda})^{+}|^2(\phi^{\nu}_{\rho,\lambda})^2dx\\
&\quad+C^*(||u||_{L^\infty(\Omega^{\nu}_{\lambda})},||\nabla u||_{L^\infty(\Omega^{\nu}_{\lambda})})\int_{B_{2\rho}\setminus B_\rho}|\nabla\phi^{\nu}_{\rho,\lambda}|^2dx+o(1)\\
&\leq \tilde{C}C_p^2(\tilde{\Omega})\int_{\tilde{\Omega}}(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}|\nabla(u-u_{\lambda})^{+}|^2(\phi^{\nu}_{\rho,\lambda})^2dx+o(1),
\end{aligned}\tag{4.92}$$

as before, since $N>p>2$, we have $|\nabla u|^{p-2}\leq(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}$ and

$$\int_{B_{2\rho}\setminus B_\rho}|\nabla\phi^{\nu}_{\rho,\lambda}|^2dx\to 0\quad\text{as }\rho\to 0.$$

Concluding, collecting the estimates (4.91) and (4.92) we get

$$\begin{aligned}
&\int_{\tilde{\Omega}}(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}|\nabla(u-u^{\nu}_{\lambda})^{+}|^2(\phi^{\nu}_{\rho,\lambda})^2dx\\
&\leq \tilde{C}C_p^2(\tilde{\Omega})\int_{\tilde{\Omega}}(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}|\nabla(u-u_{\lambda})^{+}|^2(\phi^{\nu}_{\rho,\lambda})^2dx+o(1).
\end{aligned}\tag{4.93}$$

Since (see Theorem 17) $C_p(\tilde{\Omega})$ goes to zero provided the Lebesgue measure of $\tilde{\Omega}$ goes to 0, if $|\tilde{\Omega}|\leq\delta$, with δ (depending on λ) sufficiently small, we assume $C_p(\tilde{\Omega})$ so small such that

$$\tilde{C}C_p^2(\tilde{\Omega})<1.$$

Notice that, since the definition of $\phi^{\nu}_{\rho,\lambda}$,

$$\begin{aligned}
&\int_{\tilde{\Omega}}(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}|\nabla(u-u^{\nu}_{\lambda})^{+}|^2(\phi^{\nu}_{\rho,\lambda})^2dx\\
&\leq\int_{\tilde{\Omega}}(|\nabla u|+|\nabla u^{\nu}_{\lambda}|)^{p-2}|\nabla(u-u^{\nu}_{\lambda})^{+}|^2dx.
\end{aligned}$$

Thus, letting $\rho\to 0$ in (4.93), by the Dominated Convergence Theorem we get the contradiction, showing that, actually, $(u-u^{\nu}_{\lambda})^{+}=0$ and then the thesis for $p\geq 2$. We point out that here ($p\geq 2$) we do not need to assume that $|\nabla u|$ is bounded.

Let us consider now the other interval of p.

Case: $1<p<2$**.** From (4.86) we infer that $|\nabla u|,|\nabla u^{\nu}_{\lambda}|\in L^\infty(\tilde{\Omega}\cap\{u\geq u^{\nu}_{\lambda}\})$ and, therefore we have that $(u-u^{\nu}_{\lambda})^{+}\in W^{1,2}(\tilde{\Omega}\cap\{u\geq u^{\nu}_{\lambda}\})$. Then, the conclusion follows using the classical Poincaré inequality: in fact, since $p < 2$, the term $(|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2}$ is bounded below being $|\nabla u|, |\nabla u_\lambda^\nu| \in L^\infty(\tilde\Omega \cap \{u \geq u_\lambda^\nu\})$. Then, equation (4.89) gives

$$\begin{aligned}
&C_1 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} (|\nabla u| + |\nabla u_\lambda^\nu|)^{p-2} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\
&\leq C_1 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 (\phi_{\rho,\lambda}^\nu)^2 dx \\
&\leq C_2 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+| (u-u_\lambda^\nu)^+ (\phi_{\rho,\lambda}^\nu)^2 dx \\
&+ C_3 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} [(u-u_\lambda^\nu)^+]^2 (\phi_{\rho,\lambda}^\nu)^2 dx + o(1).
\end{aligned} \tag{4.94}$$

By *Dominated Convergence Theorem* and by the definition of $\phi_{\rho,\lambda}^\nu$, the last expression states as

$$\begin{aligned}
&C_1 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx \\
&\leq C_2 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+| (u-u_\lambda^\nu)^+ dx + C_3 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} [(u-u_\lambda^\nu)^+]^2 dx + o(1)
\end{aligned}$$

and by weighted Young's inequality, arguing as above (see equation (4.90)),

$$\begin{aligned}
C_1 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx &\leq \varepsilon C_2 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx \\
&+ \frac{C_2}{\varepsilon} \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} ((u-u_\lambda^\nu)^+)^2 dx + C_3 \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} [(u-u_\lambda^\nu)^+]^2 dx + o(1).
\end{aligned}$$

For fixed small ε such that

$$C_1 - \varepsilon C_2 \geq \tilde C_1 > 0,$$

we have

$$\int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx \leq C \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} [(u-u_\lambda^\nu)^+]^2 dx, \tag{4.95}$$

with $C = \dfrac{C_2 + \varepsilon C_3}{\varepsilon \tilde C_1}$. The conclusion follows using the classical Poincaré's inequality in (4.95), i.e.

$$\int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx \leq C C_p^2(\tilde\Omega) \int_{\tilde\Omega\cap\{u\geq u_\lambda^\nu\}} |\nabla (u-u_\lambda^\nu)^+|^2 dx,$$

choosing $\delta = \delta(\lambda) \geq |\tilde\Omega|$ small such that $CC_p^2(\tilde\Omega) < 1$ and then, getting $(u-u_\lambda^\nu)^+ = 0$. $\square$

3.4 The moving plane method

In this last part of the Chapter, we complete the proof of Theorem 15. Now we are able to use the *Moving Plane Method*.

We refer to the notations and definitions of preliminaries, (equations (4.60) - (4.66)). To prove Theorem 15, we first need the following result

Proposition 5. *Let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$ be a solution to problem* (4.5)*. Set*

$$\lambda_1^0(\nu) := \min\{0\,,\, \lambda_1(\nu)\},$$

where $\lambda_1(\nu)$ is defined in (4.66)*. Then, for any $a(\nu) \leq \lambda \leq \lambda_1^0(\nu)$, we have*

$$u(x) \leq u_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu. \tag{4.96}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_1^0(\nu)$, we have

$$u(x) < u_\lambda^\nu(x), \quad \forall x \in \Omega_\lambda^\nu \setminus Z_{u,\lambda}, \tag{4.97}$$

where $Z_{u,\lambda} \equiv \{x \in \Omega_\lambda^\nu \,:\, \nabla u(x) = \nabla u_\lambda^\nu(x) = 0\}$. Finally,

$$\frac{\partial u}{\partial \nu}(x) \geq 0, \quad \forall x \in \Omega_{\lambda_1(\nu)}^\nu. \tag{4.98}$$

Proof. Let $a(\nu) < \lambda < \lambda_1^0(\nu)$ with λ sufficiently close to $a(\nu)$. By Hopf's Lemma, it follows that for a neighborhood of the boundary, $\nabla u(x) > 0$, thus, u and u_λ^ν are ordered on $\partial\Omega_\lambda^\nu$. Therefore, by Proposition 4 (since $|\Omega_\lambda^\nu|$ is small enough because λ is sufficient close to $a(\nu)$),

$$u - u_\lambda^\nu \leq 0 \quad \text{in} \quad \Omega_\lambda^\nu.$$

We now define the set where the functions are ordered,

$$\Lambda_0 = \{\lambda > a(\nu) : u \leq u_t \text{ in } \Omega_t^\nu \text{ for all } t \in (a(\nu), \lambda]\} \tag{4.99}$$

and

$$\lambda_0 = \sup \Lambda_0. \tag{4.100}$$

Notice that by continuity we obtain $u \leq u_{\lambda_0}^\nu$ in $\Omega_{\lambda_0}^\nu$. We have to show that $\lambda_0 = \lambda_1^0(\nu)$. Assume by contradiction that $\lambda_0 < \lambda_1^0(\nu) \leq 0$ and let $A_{\lambda_0} \subset \Omega_{\lambda_0}^\nu$ be an open set such that $Z_{u,\lambda_0} \cap \Omega_{\lambda_0}^\nu \subset A_{\lambda_0} \subset\subset \Omega$. Such set exists by Hopf's Lemma because by this Lemma there exists, at least, a neighborhood of $\partial\Omega$ such that $\nabla u(x) \neq 0$ and then, $x \notin Z_u$ and $A_{\lambda_0} \subset\subset \Omega$.

Notice also that, since $|Z_{u,\lambda_0}| = 0$ as we remarked above, we can take A_{λ_0} with measure arbitrarily small. Since we are working in $\Omega_{\lambda_0}^\nu$ which is away from 0, we have that the weight $1/|x|^p$ is not singular there. Moreover, in a neighborhood of the reflected point of the origin 0^ν_λ, we know, by Lemma 7, that $u < u^\nu_{\lambda_0}$. Since elsewhere $1/|x^\nu_\lambda|^p$ is not singular and $u, \nabla u, u^\nu_\lambda, \nabla u^\nu_\lambda$ are bounded, we can exploit the *strong comparison principle*, see e.g. [82, Theorem 2.5.2], to get that

$$u < u^\nu_{\lambda_0} \qquad \text{or} \qquad u \equiv u^\nu_{\lambda_0}$$

in any connected component of $\Omega^\nu_{\lambda_0} \setminus Z_u$. It follows that

- the case $u \equiv u^\nu_{\lambda_0}$ in some connected component $\overline{\mathcal{C}}$ of $\Omega_{\lambda_0} \setminus Z_{u,\lambda_0}$ is not possible, since if $u \equiv u^\nu_{\lambda_0}$, when we are close to Z_u, $\nabla u = \nabla u^\nu_{\lambda_0}$ and, by symmetry, it would imply the existence of *a local symmetry phenomenon* and consequently that $\Omega \setminus Z_{u,\lambda_0}$ would be not connected, in spite of the result showed in Lemma 8.

Note also that, since the domain is strictly convex, by Hopf's Lemma if u is near to the hyperplane T_{λ_0}, there exists a neighborhood of $\partial\Omega^\nu_{\lambda_0}$ such that $\nabla u(x) > 0$ and, then, there are monotony. If u is on $\partial\Omega^\nu_{\lambda_0}$ but far from the hyperplane $T^\nu_{\lambda_0}$, the Dirichlet condition (see e.g. [48]), implies that $u = 0$ at the boundary, but $u^\nu_{\lambda_0} > 0$ in the interior, then, we get that there exists a neighborhood $\mathcal{N}_{\lambda_0}$ of $\partial\Omega^\nu_{\lambda_0} \cap \partial\Omega$ where $u < u^\nu_{\lambda_0}$ in $\mathcal{N}_{\lambda_0}$.

We deduce that there exists a compact set K in $\Omega^\nu_{\lambda_0}$ such that

- $|\Omega^\nu_{\lambda_0} \setminus \big((K \setminus A_{\lambda_0}) \cup \mathcal{N}_{\lambda_0}\big)|$ is sufficiently small so that Proposition 4 can be used.
- $u^\nu_{\lambda_0} - u$ is positive in $(K \setminus A_{\lambda_0}) \cup \mathcal{N}_{\lambda_0}$, because $u^\nu_{\lambda_0} - u$ is positive in $\Omega^\nu_{\lambda_0} \setminus Z_u$ and K is a subset of $\Omega^\nu_{\lambda_0}$ and we just proved also that there exists a set $\mathcal{N}_{\lambda_0}$ where $u^\nu_{\lambda_0} - u$ is positive..

Therefore, since $K \setminus A_{\lambda_0}$ is a compact set, by continuity (and redefining $A_{\lambda_0+\varepsilon}$ as small as we want and $\mathcal{N}_{\lambda_0+\varepsilon}$, exploiting Hopf's Lemma) we find $\varepsilon > 0$ such that

- $|\Omega^\nu_{\lambda_0+\varepsilon} \setminus \big((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\big)|$ is sufficiently small so that Proposition 4 applies.
- $u^\nu_{\lambda_0+\varepsilon} - u$ is positive in $(K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}$.

Since $u^\nu_{\lambda_0+\varepsilon} - u > 0$ in $K \setminus A_{\lambda_0+\varepsilon}$, then $u^\nu_{\lambda_0+\varepsilon} - u > 0$ on $\partial(K \setminus A_{\lambda_0+\varepsilon})$ because K is compact. Hence, $u^\nu_{\lambda_0+\varepsilon} - u \geq 0$ on $\partial\Big((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big)$. Since $\partial\Big(\Omega^\nu_{\lambda_0+\varepsilon} \setminus (K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big) = \partial\Big(K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big) \cup T^\nu_{\lambda_0+\varepsilon}$ and, since $u = u^{\nu}_{\lambda_0+\varepsilon}$ in $T^{\nu}_{\lambda_0+\varepsilon}$, $u \leq u^{\nu}_{\lambda_0+\varepsilon}$ on $\partial\Big(\Omega^{\nu}_{\lambda_0+\varepsilon} \setminus \Big((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big)\Big)$. By Proposition 4 it follows $u \leq u^{\nu}_{\lambda_0+\varepsilon}$ in $\Omega^{\nu}_{\lambda_0+\varepsilon} \setminus \Big((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big)$ and since $u < u^{\nu}_{\lambda_0+\varepsilon}$ in $(K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}$, $u \leq u^{\nu}_{\lambda_0+\varepsilon}$ in $\Omega^{\nu}_{\lambda_0+\varepsilon}$, what contradicts the assumption $\lambda_0 < \lambda^0_1(\nu)$. Therefore, $\lambda_0 \equiv \lambda^0_1(\nu)$ and the thesis is proved.

We point out that we are exploiting Proposition 4 in the set $\Omega^{\nu}_{\lambda_0+\varepsilon} \setminus \Big((K \setminus A_{\lambda_0+\varepsilon}) \cup \mathcal{N}_{\lambda_0+\varepsilon}\Big)$ which is bounded away from the boundary $\partial\Omega$ and then, the constant δ in the statement is uniformly bounded.

The proof of (4.97) follows by the strong comparison theorem applied as above; because since $\Omega^{\nu}_{\lambda} \setminus Z_{u,\lambda}$ has one connected component, $u < u^{\nu}_{\lambda}$ or $u \equiv u^{\nu}_{\lambda}$ in all $\Omega^{\nu}_{\lambda} \setminus Z_{u,\lambda}$, but the case $u \equiv u^{\nu}_{\lambda}$ is not possible (as we saw before), thus, $u < u^{\nu}_{\lambda}$ in $\Omega^{\nu}_{\lambda} \setminus Z_{u,\lambda}$ for any $\lambda < \lambda^0_1(\nu)$.

Finally, (4.98) follows considering $x_1 \leq x_2$ and taking the hyperplane which is in the middle of $u(x_1)$ and $u(x_2)$. By (4.96), $u(x) \leq u(x^{\nu}_{\lambda})$ in Ω^{ν}_{λ}, with $\lambda \leq \lambda^0_1$. Thus, taking $x = x_1$ and $x_2 = x^{\nu}_{1\lambda}$, $u(x_1) \leq u(x^{\nu}_{1\lambda}) = u(x_2)$ and then $u(x_1) \leq u(x_2)$. Hence, the monotonicity follows. □

Proof of Theorem 15:

Since Ω is strictly convex w.r.t. the $\nu-$direction and symmetric w.r.t. to (see equation (4.60))

$$T^{\nu}_0 = \{x \in \mathbb{R}^N : x \cdot \nu = 0\},$$

it follows by Proposition 5, being $\lambda_1(\nu) = 0 = \lambda^0_1(\nu)$ in this case, that

$$u(x) \leq u^{\nu}_{\lambda}(x) \ \text{ for } x \in \Omega^{\nu}_0.$$

In the same way, performing the *Moving Plane Method* in the opposite direction, $-\nu$, we obtain

$$u(x) \geq u^{\nu}_{\lambda}(x) \ \text{ for } x \in \Omega^{\nu}_0,$$

that is, u is symmetric and non decreasing w.r.t. the $\nu-$direction, since monotonicity follows by (4.98).

Finally, if Ω is a ball, repeating this argument along any direction, it follows that u is radially symmetric. The fact that $\dfrac{\partial u}{\partial r}(r) < 0$ for $r \neq 0$, follows by the Hopf's boundary Lemma which works in this case since the level sets are balls and the Hopf's boundary Lemma works in each level set and therefore, fulfill the interior sphere condition.

□

Part II

Supercritical elliptic problems with the pole at the boundary of the domain

Chapter 5

Regularization by a concave term

1 Introduction and some preliminaries

The existence of solution to the *supercritical* problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2}, \quad u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with $0 \in \partial\Omega$ and $1 < p < 2^* - 1$ has been studied in the paper [50]. The existence of solution in this case depends on the geometry of the domain. J. Dávila and I. Peral find solution to problem (5.1) for a specific type of domain and they get also a nonexistence result for a *starshaped* domains. In the next Chapter we will explain deeply this argument and we generalize it for the *p-Laplacian* operator.

In this Chapter we perturb problem (5.1) in order to get solutions without restriction on the domain. In this way, we avoid the lack of the existence that it has been introduced in [50].

To get this result we add a regularizing zero order term in the right hand side of problem (5.1). More precisely, we study the existence of solutions to the model problem

$$\begin{cases} -\Delta u = \dfrac{u^p}{|x|^2} + \lambda u^q, \quad u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

with $p \geq 1$, $0 < q < 1$ and $0 \in \partial\Omega$. We prove that for any smooth domain Ω there exists a solution for a positive interval of λ.

We use a Sattinger's monotonicity argument (see [85]) to get the solution. A nonexistence result for λ large is also studied in this Chapter.

To obtain the solution we are going to use the following classical existence and regularity results

Theorem 19. *Let $\rho \geq C$, $\frac{f(t)}{t}$ decreasing and consider Ω a bounded domain,* *then, the problem*

$$\begin{cases} -\Delta u = \rho(x) f(u) & in\, \Omega, \\ u = 0 & on\, \partial\Omega, \end{cases}$$

has a unique solution.

The existence is given by minimization and the proof of the uniqueness can be seen in [35].

Theorem 20. *Let Ω be a bounded domain in $\mathbb{R}$ and consider the problem*

$$\begin{cases} -\Delta u = f & in\, \Omega, \\ u = 0 & on\;\; \partial\Omega, \end{cases} \tag{5.3}$$

- *(i) If $f \in L^p(\Omega)$ with $1 < p < +\infty$, then,* (5.3) *has a unique weak solution $u \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$ such that*

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{L^p(\Omega)}$$

- *(ii) If Ω is of the class $C^{2,\alpha}$ and $f \in C^{0,\alpha}(\overline{\Omega})$, then, $u \in C^{2,\alpha}(\overline{\Omega})$.*

See details in [15].

We recall also some Hölder-regularity result for weak solutions.

Theorem 21. *Let Ω be a bounded open subset of $\mathbb{R}^N$, with a C^1 boundary. Assume $u \in W^{k,p}(\Omega)$.*

- *If $k < \frac{N}{p}$, then $u \in L^q(\Omega)$ where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{N}.$$

We have in addition the estimate

$$||u||_{L^q(\Omega)} \leq C||u||_{W^{k,p}(\Omega)},$$

with the constant C depending only on k, p, N and Ω.

- *If $k > \frac{N}{p}$, then $u \in C^{k-[\frac{N}{p}]-1,\alpha}(\overline{\Omega})$. In addition we have,*

$$||u||_{C^{k-[\frac{N}{p}]-1,\alpha}(\overline{\Omega})} \leq C||u||_{W^{k,p}(\Omega)},$$

with the constant C depending only on k, p, N and Ω.

We introduce the following Theorem that we will use to get the Lipschitz regularity up to the boundary. See details in [42].

Theorem 22. *Let Ω be a bounded subset of $\mathbb{R}^N$ with $N \geq 3$, such that $\partial\Omega \in C^{1,\alpha}(\Omega)$. Assume that $f \in L^{N,1}(\Omega)$, where $L^{N,1}(D)$ denotes the corresponding Lorentz space. Recall that*

$$\begin{array}{rcll} L^{q,q}(D) & = & L^q(D) & \text{for } q \in (1,\infty), \\ L^{q_1,\sigma_1}(D) & \subsetneq & L^{q_2,\sigma_2}(D) & \text{if } q_1 > q_2 \text{ and } \sigma_1, \sigma_2 \in (0,\infty]. \end{array} \tag{5.4}$$

Let u be a weak solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Then, u is Lipschitz continuous on $\overline{\Omega}$.

All the results in this Chapter can be seen in the first part of the paper [74].

2 Existence result for an interval of λ

The main existence result in this Chapter is the following

Theorem 2.1. *Let $0 \leq q < 1$ and $p > 1$. Then, there exists $\Lambda_0 > 0$, such that*

a) $\forall \lambda \in (0, \Lambda_0)$, problem (5.2) *admits a solution $u_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

b) If $\lambda > \Lambda_0$, problem (5.2) *has no solution.*

Moreover, if $\lambda_i < \Lambda_0$, $i = 1, 2$,

1. $u_{\lambda_i} \in C^{2,\alpha}(\Omega)$.

2. If $0 < \lambda_1 < \lambda_2 \leq \Lambda_0$, then $u_{\lambda_1} \leq u_{\lambda_2}$.

Proof. Consider $\Gamma \subset \partial\Omega$, a regular submanifold which is a neighborhood of the pole of the potential. We can assume, for instance, that $\Gamma = \partial\Omega \cap B_r(0)$ for $r > 0$ small in order to have Γ connected. We define the following function for $x \in \overline{\Omega}$,

$$d_\Gamma = dist(x, \Gamma).$$

We organize the proof in several steps in order to be clear.

Step 1. We start looking for a supersolution $\overline{u}_\lambda$ to (5.2).

Since $0 \in \Gamma \subset \partial\Omega$ and the boundary is a smooth manifold, we have that

$$\frac{d_\Gamma^p}{|x|^2} \in L^q(\Omega) \Leftrightarrow (p-2)q > N \Leftrightarrow q > \frac{N}{p-2}.$$

In particular, it holds for some $q > N$ (see [50]). Then, since $0 < q < 1$, by Theorem 20, there exists a unique solution w which verifies

$$\begin{cases} -\Delta w = \dfrac{d_\Gamma^p}{|x|^2} + w^q, \quad w \geq 0 & \text{in } \Omega, \\ w = d_\Gamma & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

By classical elliptic regularity theory, see Theorems 21 and 22, we have that $w \in C^{1,\alpha}(\Omega) \cap C^{0,1}(\overline{\Omega})$. Notice that the regularity of w implies the existence of a constant $C = C(\Omega, p, q)$ such that

$$\frac{w(x) - w(y)}{|x-y|} \leq C.$$

If $x \in \Omega$ and $y \in \partial\Omega$, $w(y) = d_\Gamma$ and

$$w(x) - d_\Gamma \leq \tilde{C} d_\Gamma,$$

thus,

$$w(x) \leq C d_\Gamma, \quad x \in \Omega.$$

Let T be a positive parameter and define $\overline{u}_\lambda = Tw$, thus

$$\begin{aligned} -\Delta \overline{u}_\lambda &= T(-\Delta w) = T\frac{d_\Gamma^p}{|x|^2} + Tw^q \\ &\geq T\frac{w^p}{C^p|x|^2} + Tw^q = T\frac{\overline{u}_\lambda^p}{T^pC^p|x|^2} + T\frac{\overline{u}_\lambda{}^q}{T^q}. \end{aligned}$$

In order to have a supersolution we need that $-\Delta \overline{u}_\lambda \geq \dfrac{\overline{u}_\lambda^p}{|x|^2} + \lambda \overline{u}_\lambda^q$.

Notice that, to get this, it is sufficient that

$$T \geq \lambda^{\frac{1}{1-q}} \text{ and } T \leq \left(\frac{1}{C^p}\right)^{\frac{1}{p-1}}.$$

Therefore, putting together both inequalities we observe that it is possible to find such supersolution $\overline{u}_\lambda$ for $\lambda \in (0, \Lambda)$, where $\Lambda = \left(\dfrac{1}{C^p}\right)^{\frac{1-q}{p-1}}$.

Notice that we have found a supersolution only in the interval $(0, \Lambda)$ and this result is *almost optimal* as we will see in Step 4 below.

STEP 2. Next we have to find a subsolution $\underline{u}$ to (5.2), such that $\underline{u}_\lambda \leq \overline{u}_\lambda$.

We consider $\underline{u}_\lambda$ the solution to the problem

$$\begin{cases} -\Delta \underline{u}_\lambda = \lambda \underline{u}_\lambda^q, & \underline{u}_\lambda \geq 0 \quad \text{in } \Omega, \\ \underline{u}_\lambda = 0 & \qquad\quad \text{on } \partial\Omega. \end{cases}$$

The existence (and also the uniqueness) of this function, since $0 < q < 1$, is given by Theorem 19.

It is obvious that $\underline{u}_\lambda$ is a subsolution of (5.2) and, by elliptic regularity results, see Theorems 20 and 21, $\underline{u}_\lambda \in \mathcal{C}^{2,\gamma}(\Omega) \cap \mathcal{C}^{1,\beta}(\overline{\Omega})$.

To prove that the supersolution and the subsolution are ordered we are going to use the argument used in [35] to prove the uniqueness in Theorem 19.

Notice that $\underline{u}_\lambda$ and $\overline{u}_\lambda$ are subsolution and supersolution respectively to the equation

$$-\Delta \underline{u}_\lambda = \lambda \underline{u}_\lambda^q, \quad \text{with} \quad 0 < q < 1$$

and $\underline{u}_\lambda \leq \overline{u}_\lambda$ on $\partial\Omega$.

That is,

$$\begin{cases} -\Delta \underline{u}_\lambda \leq \lambda \underline{u}_\lambda^q, & \underline{u}_\lambda \geq 0 \quad \text{in } \Omega, \\ \underline{u}_\lambda = 0 & \qquad\quad \text{on } \partial\Omega. \end{cases}$$

and

$$\begin{cases} -\Delta \overline{u}_\lambda \geq \lambda \overline{u}_\lambda^q, & \overline{u}_\lambda \geq 0 \quad \text{in } \Omega, \\ \overline{u}_\lambda = 0 & \qquad\quad \text{on } \partial\Omega. \end{cases}$$

We divide each equation by $\underline{u}_\lambda$ and $\overline{u}_\lambda$ respectively and we subtract the two expressions, it follows that

$$\int_\Omega \Big(\frac{-\Delta \underline{u}_\lambda}{\underline{u}_\lambda} + \frac{\Delta \overline{u}_\lambda}{\overline{u}_\lambda} \Big) dx \leq \int_\Omega \lambda (\underline{u}_\lambda^{q-1} - \overline{u}_\lambda^{q-1}) dx.$$

Multiplying the last expression by $(\underline{u}_\lambda^2 - \overline{u}_\lambda^2)^+$,

$$\int_\Omega -\Delta \underline{u}_\lambda \underline{u}_\lambda + \frac{\Delta \underline{u}_\lambda}{\underline{u}_\lambda} \overline{u}_\lambda^2 + \frac{\Delta \overline{u}_\lambda}{\overline{u}_\lambda} \underline{u}_\lambda^2 - \Delta \overline{u}_\lambda \overline{u}_\lambda dx \leq \int_\Omega \lambda (\underline{u}_\lambda^{q-1} - \overline{u}_\lambda^{q-1})(\underline{u}_\lambda^2 - \overline{u}_\lambda^2)^+ dx.$$

Integrating by parts and using Picone's inequality in the left hand side, see Theorem 7, we obtain

$$0 \leq \int_\Omega \lambda (\underline{u}_\lambda^{q-1} - \overline{u}_\lambda^{q-1})(\underline{u}_\lambda^2 - \overline{u}_\lambda^2)^+ dx.$$

Since $\lambda > 0$, $q < 1$ and taking into account that we are in the set $\{\underline{u}_\lambda > \overline{u}_\lambda\}$,

$$0 \leq \int_\Omega \lambda(\underline{u}_\lambda^{q-1} - \overline{u}_\lambda^{q-1})(\underline{u}_\lambda^2 - \overline{u}_\lambda^2)^+ dx \leq 0.$$

Thus, $(\underline{u}_\lambda^2 - \overline{u}_\lambda^2)^+ = 0$. Therefore, we can conclude that $\underline{u}_\lambda \leq \overline{u}_\lambda$.

STEP 3. We use an iteration argument as in [26]. Consider $\lambda \in (0, \Lambda)$, $u_0 \equiv \underline{u}_\lambda$ and u_k the solution to

$$\begin{cases} -\Delta u_k = \dfrac{u_{k-1}^p}{|x|^2} + \lambda u_{k-1}^q, \quad u_k \geq 0 & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Using the weak comparison principle as above, we prove by recurrence that

$$\underline{u}_\lambda \equiv u_0 \leq u_1 \leq ... \leq u_k \leq ... \leq \overline{u}_\lambda.$$

Therefore, we get a sequence of functions $\{u\}_k$ which is bounded from above and from below and it is ordered, thus, we can define $u_\lambda(x) = \lim\limits_{k\to\infty} u_k(x)$ with $x \in \Omega$.

Moreover, since $\dfrac{u_{k-1}^p}{|x|^2} + \lambda u_{k-1}^q \leq \dfrac{\overline{u_\lambda^p}}{|x|^2} + \lambda \overline{u_\lambda^q}$, by the *Dominated Convergence Theorem*, the right hand side of (5.7) converges to $\dfrac{u_\lambda^p}{|x|^2} + \lambda u_\lambda^q$ in $L^1(\Omega)$. Therefore, u_λ is a solution to problem (5.2) in a distributional sense. The right hand side of the equation (5.7) converges in $W^{-1,2}(\Omega)$ because, since this side converges to $\dfrac{u_\lambda^p}{|x|^2} + \lambda u_\lambda^q$ in $L^1(\Omega)$,

$$\int_\Omega (\frac{u_{k-1}^p}{|x|^2} + \lambda u_{k-1}^q) v dx \to \int_\Omega (\frac{u_\lambda^p}{|x|^2} + \lambda u_\lambda^q) v dx,$$

with $v \in W_0^{1,2}(\Omega)$.

Therefore, the continuity of the operator $-\Delta^{-1}$ implies that

$$u_k \to u_\lambda \text{ in } W_0^{1,2}(\Omega).$$

It is easy to check that this solution, u_λ, is a *minimal solution* for such λ.

Since if $\lambda_1 < \lambda_2$, the solution u_{λ_2} for λ_2 is a supersolution to the problem for λ_1 then, using the weak comparison argument as above, we conclude that $u_{\lambda_1} \leq u_{\lambda_2}$.

The regularity is now easy to obtain. Since $\overline{u}_\lambda \leq C d_\Gamma$, the right hand side of (5.2) belongs to some $L^r(\Omega)$, with $r > N$. The solution to (5.2), u_λ, verifies $u_\lambda \in L^\infty(\Omega)$ then, by elliptic regularity as above and a bootstrapping argument, $u_\lambda \in C^{0,1}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

STEP 4.- We will prove the following claim.

Claim.- There exists λ_0 such that $\forall \lambda \in [\lambda_0, \infty)$, problem (5.2) has no solution $u_\lambda \in W_0^{1,2}(\Omega)$.

To prove the claim we closely follow the arguments used in [26].

We proceed by contradiction. Consider v_1 such that

$$-\Delta v_1 = \lambda_1 v_1, \quad v_1 \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad v_1 > 0.$$

Suppose that for all λ there exists a solution to problem (5.2), $u_\lambda \in W_0^{1,2}(\Omega)$, with $u_\lambda > 0$. Then, near the zero, by Hopf's Lemma, there exists space between the solution u_λ and the boundary, so we can put the eigenfunction v_1 under the solution u_λ, hence, there exists $t > 0$ verifying $tv_1 \le u_\lambda$ in Ω. We define $\psi = tv_1$. Pick $\varepsilon > 0$ such that $\lambda_1 + \varepsilon < \lambda_2$, the second eigenvalue of the *Laplace* operator. Consider now $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$ and $c_\Omega = \inf\limits_{x\in\Omega} \dfrac{1}{|x|^2}$. If λ is such that $\lambda_1 + \varepsilon \le b_q \lambda^{\frac{p-1}{p-q}}$ where b_q satisfies

$$b_q \lambda^{\frac{p-1}{p-q}} u_\lambda \le \lambda u_\lambda^q + c_\Omega u_\lambda^p \le \lambda u_\lambda^q + \frac{u_\lambda^p}{|x|^2},$$

then,

$$-\Delta \psi = \lambda_1 \psi \le \mu \psi \le \mu u_\lambda \le (\lambda_1 + \varepsilon) u_\lambda \le b_q \lambda^{\frac{p-1}{p-q}} u_\lambda \le \lambda u_\lambda^q + \frac{u_\lambda^p}{|x|^2} = -\Delta u_\lambda.$$

That is, $\psi \le u_\lambda$ are subsolution and supersolution respectively to the problem

$$\begin{cases} -\Delta u &= \mu u \quad \text{in} \quad \Omega, \\ \quad u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.8}$$

with $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$. A standard iteration argument shows that problem (5.8) has positive solution, that is a contradiction with the isolation of the first eigenvalue of the *Laplacian*, λ_1. Then, there exists λ_0 such that $\forall \lambda \in [\lambda_0, \infty)$ the problem (5.2) has no solution.

FINAL STEP. Define $\Lambda_0 = \sup\{\lambda \,|\, \text{ problem (5.2) has a solution}\}$. According to the previous step, $\Lambda_0 < \infty$. Moreover, if $\lambda \in (0, \Lambda_0)$ we can find λ^* such that $\lambda < \lambda^*$ and problem (5.2) has a solution for λ^*. Such solution is a supersolution to problem (5.2) for λ. Then, we proceed as in Step 3 to find a solution for $\lambda \in (0, \Lambda_0)$.

In others words, we conclude that the set of $\lambda > 0$ for which there exists a solution to problem (5.2) is a bounded interval in the positive real line.

□

Remark 5. *It is worthy to point out that if* $1 < p < \frac{N+2}{N-2}$ *a perturbative argument as in [50] allows us to find a second solution to problem* (5.2) *in a conveniently thin dumbbell domains. We skip the details because can be found in [50].*

Chapter 6

Supercritical problem for the p-Laplacian: Solvability and regularization

1 Introduction

In this Chapter we are going to study the following problem with $0 \in \partial\Omega$

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} + g(\lambda, x, u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $q > p-1$, $1 < p < N$ and $\Omega \in \mathbb{R}^{\mathbb{N}}$ is a bounded domain with smooth boundary and $g(\lambda, x, u)$ as in one of the two cases:

(*i*) $g(\lambda, x, u) \equiv 0$ and $q < p^* - 1$;

(*ii*) $g(\lambda, x, u) = \lambda f(x) u^r$, with $\lambda > 0$, $f(x) \gneq 0$ belonging to $L^\infty(\Omega)$, $0 \leq r < p-1$ and any positive exponent $q > p-1$.

We are going to study the existence of solutions to (6.1) in the cases (*i*) and (*ii*) with different approach. We notice that since $q > p-1$, the problem to be considered is *supercritical*. We would like to point out that the regularization that produces the *sub-diffusive* term eliminates any condition on the size of q.

It is classical that if $0 < q < p-1$, problem (6.1) has a solution that does not depend on the location of the origin. If $g(\lambda, x, u) = \lambda u^r$ with $r \leq p^* - 1$, a variational solution can be found as a critical point of the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q+1}\int_\Omega \frac{u^{q+1}}{|x|^p} dx - \frac{\lambda}{r+1}\int_\Omega f u^{r+1} dx,$$

which is well defined in $u \in W_0^{1,p}(\Omega)$.

As a consequence, we will focus only in the case $q > p-1$.

We want to point out also that the case (*i*), $g(\lambda, x, u) \equiv 0$, exhibits a different behavior from the other one, the existence of solutions turns to be, as in the semilinear case, depending on the geometry of the domain and on the exponent q, while in the case (ii), $g(\lambda, x, u) = \lambda f(x)u^r \gneq 0$, the existence does not depend neither on the geometry of Ω or on the exponent q.

In the case (i), $g(\lambda, x, u) \equiv 0$, we get the existence result following the idea used in [50], but taking into account the intrinsic differences due to the *p-Laplacian* operator.

If $g(\lambda, x, u) \equiv 0$, the problem in general has no solution. For instance, if the domain is *starshaped* with respect to the pole, a suitable application of the Pohozaev's identity provides a non existence result in $W_0^{1,p}(\Omega)$. This result motivates, as in the semilinear case, see [50], to look for *dumbbell* type domains in which such obstruction does not exists.

To motivate the analysis and the existence result for the *dumbbell* type domains we first consider the problem in the following sets

$\Omega = B_1(1, 0, \ldots, 0) \cup B_1(3, 0, \ldots, 0)$ and let v be a solution to

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p}, & \text{in } B_1(3, 0, \ldots, 0), \\ u \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial B_1(3, 0, \ldots, 0). \end{cases}$$

This solution is obtained, for instance, using the classical *Mountain Pass Theorem*, see Theorem 25 below, since in this domain the weight is bounded and hence, the problem is *subcritical.* Then, we can extend this solution to 0 for the set $B_1(1, 0, \ldots, 0)$, in this way,

$$u(x) = \begin{cases} 0 & \text{if } x \in B_1(1, 0, \ldots, 0) \\ v(x) & \text{if } x \in B_1(3, 0, \ldots, 0) \end{cases}$$

is a solution to

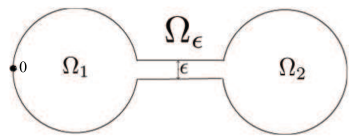
$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = B_1(1, 0, \ldots, 0) \cup B_1(3, 0, \ldots, 0)$.

The idea is to have connected domains but being not too far from the above case, this is why we join the sets $B_1(1, 0, \ldots, 0)$ and $B_1(3, 0, \ldots, 0)$ with a tiny tubular neighborhood C_ε. This perturbation of the domain allows us to obtain some type of connected domains, *dumbbell* domains, for which the problem has solution.

We can conjecture, then, that the result of existence could be obtained if the domain lies in the conditions of the following definition.

Definition 6. *We call Ω_ε a dumbbell domain if it is a domain with a smooth boundary of the form $\Omega_\varepsilon = \Omega_1 \cup C_\varepsilon \cup \Omega_2$, where Ω_1 and Ω_2 are smooth bounded domains such that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$ and C_ε is a region contained in a tubular neighborhood of radius less than $\varepsilon > 0$ around a curve joining Ω_1 and Ω_2.*



Even assuming a domain as in Definition 6, the proof is quite involved and perturbative in nature. A direct variational approach is not possible. Notice that if we consider the energy functional associated to (6.1) in a naif way,

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q+1}\int_\Omega \frac{u^{q+1}}{|x|^p}dx,$$

it is not well defined in $W_0^{1,p}(\Omega)$, just by the *supercritical* value of the power q.

Therefore, to handle the problem we will proceed as follows.

i) Truncating the functional in a convenient way:

$$E_{\delta,\varepsilon,\theta}(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p+\delta}dx.$$

ii) Penalizing such truncated functional in order to avoid that the mountain pass level goes to zero:

$$E_{\delta,\varepsilon,\theta}(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p+\delta}dx + \eta_\theta(I_\delta(u)).$$

iii) Considering a convenient supersolution in order to have control from above near the zero.

The main result for the case (i), $g(\lambda, x, u) = 0$, is the following

Theorem 23. *Assume that $1 < p < N$ and assume also*

($\flat$) *Ω_ε is a dumbbell domain;*

($\natural$) $0 \in \partial\Omega_1 \cap \partial\Omega_\varepsilon$;

($\sharp$) $p-1 < q < p^*-1$.

Then, there exists ε_0 such that if $0<\varepsilon<\varepsilon_0$, the problem

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} & \text{in } \Omega_\varepsilon, \\ u \geq 0 & \text{in } \Omega_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

has a solution $u \in W_0^{1,p}(\Omega_\varepsilon)$.

In the case (ii), $g(\lambda, x, u) = \lambda f(x)u^r$, we prove the existence of a solution using the method of super- and subsolutions, see [26], [85].

The presence of the concave term $\lambda f(x)u^r$ in the equation gives the existence of a solution that does not depend neither on the geometry of the domain or on the coefficient q. We generalize the result of Chapter 5 for the *p-Laplace* operator. The idea is to construct an appropriate super- and subsolution and then generate a monotone (and bounded) non decreasing sequence. Finally, we pass to the limit to conclude the existence. We summarize this result in the following Theorem

Theorem 24. *Let $0 \in \partial\Omega$, $q > p-1$, $1<p<N$, $0 \leq r < p-1$ and consider the problem*

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} + \lambda f(x)u^r & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

with $f \in L^\infty(\Omega)$, $f(x) \gneq 0$ and $\Omega \in \mathbb{R}^{\mathbb{N}}$ a smooth bounded domain with $N \geq 3$. Then, there exists a positive constant $\lambda_{\max}$ such that

(a) *$\forall\lambda \in (0, \lambda_{\max})$, problem (6.2) has a solution $u_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

(b) *If $\lambda > \lambda_{\max}$, problem (6.2) has no solution.*

Moreover, if $0 < \lambda_1 < \lambda_2 < \lambda_{\max}$ then, $u_{\lambda_1} \leq u_{\lambda_2}$.

For the reader's convenience we are going to recall the Definition of a *Palais-Smale* sequence and *Palais-Smale condition*, in order to recall also the classical Ambrosetti-Rabinowitz *Mountain Pass Theorem*, introduced in [16] which is an important tool in the proof of the existence of solution in the case (i), $g(\lambda, x, u) \equiv 0$.

Definition 7. *(Palais Smale sequence) Let X be a Banach space and let $J : X \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_k\}_k \subset X$ such that $\{J(u_k)\}_k$ is bounded in $\mathbb{R}$ and $J'(u_k) \to 0$ in X' as $k \to \infty$, is called a Palais-Smale sequence for J .*

Definition 8. *(Palais Smale condition) Let X be a Banach space and let $J : X \to \mathbb{R}$ be a differentiable functional. We say that J satisfies the Palais-Smale condition (shortly: J satisfies (PS)) if every Palais-Smale sequence for J has a converging subsequence (in X).*

Theorem 25. *(Mountain Pass Theorem) Let H be a Hilbert space, and let $J \in C^{1,1}(H)$ satisfying $J(0) = 0$. Assume that there exist positive numbers ρ and α such that*

- *$J(u) > \alpha$ if $||u|| = \rho$;*
- *There exists $v \in H$ such that $||v|| > \rho$ and $J(v) < 0$.*

Then, there exists a Palais-Smale sequence (see Definition 7) for J at a level $c \geq 0$. If J satisfies (PS) (see Definition 8), then, there exists a critical point at level c.

All the results in this Chapter can be seen in the paper [71].

2 The problem with $g(\lambda, x, u) \equiv 0$

In this Section we are interested in find some sufficient conditions for which the existence or nonexistence of solutions holds. In particular, we refer to *energy* solutions of (6.1) with $g(\lambda, x, u) \equiv 0$, namely

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

with $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \phi) = \int_\Omega \frac{u^q}{|x|^p}\phi \quad \forall \phi \in W_0^{1,p}(\Omega).$$

We shall see below in Proposition 6, that if the domain is *starshaped* with respect to the origin there are no energy solutions.

However, despite this negative result we show also in this Section that for a convenient non-starshaped domain there exists a solution to problem (6.3).

We introduce first the non-exisistence result of energy solutions to (6.3).

2.1 Nonexistence result of energy solutions in starshaped domains

First of all we are going to recall the Definition of *starshaped* domain.

Definition 9. *An open set Ω is called starshaped with respect to 0 provided for each $x \in \overline{\Omega}$, the line segment*

$$\{\lambda x \,|\, 0 \leq \lambda \leq 1\}$$

lies in $\overline{\Omega}$.

Notice that if $\partial\Omega$ is sufficiently regular then, denoting ν the outer unit normal, $x \cdot \nu \geq 0, \quad \forall x \in \partial\Omega$.

To use conveniently the Pohozaev's identity and to obtain the nonexistence result of energy solutions to (6.3), we need a-priori estimates for the gradient ∇u of these solutions.

Lemma 9. *Assume $N \geq 2$ and $1 < q < p^* - 1$ ($q > 1$ if $N = p$). If u is an energy solution to (6.3), then $u \in L^\infty(\Omega)$. Moreover, there exists some $C > 0$ such that*

$$|\nabla u(x)| \leq \frac{C}{|x|} \qquad \forall x \in \Omega.$$

Proof.

By a classical regularity result it is well known that, away from the origin, the solutions of *p-Laplace* equations are locally bounded, since the problem is *subcritical*, see [86]. Actually, by a classical regularity result of [70], the solutions are $C^{1,\alpha}_{loc}$ and the equation (6.3) has to be understood in the weak sense.

Suppose now $x_0 \in \Omega$ and $x_0 \neq 0$. Consider $r = \frac{|x_0|}{2}$ and define $v(y) = u(x_0 + ry)$ for $y \in \dfrac{(\Omega - x_0)}{r}$. Then, v satisfies

$$-\Delta_p v = \frac{r^p u^q(x_0 + ry)}{|x_0 + ry|^p} = \frac{r^p}{|x_0 + ry|^p} v^q, \qquad \text{in } \frac{(\Omega - x_0)}{r}. \tag{6.4}$$

Considering (6.4) in $\mathcal{D} := B_1(0) \cap \frac{\Omega - x_0}{r}$,

$$|x_0| - r|y| \leq |x_0 + ry| \leq |x_0| + r|y| = |x_0| + \frac{|x_0|}{2} = \frac{3}{2}|x_0|.$$

Since $y \in B_1(0), |y| \leq 1$ and

$$|x_0| - \frac{|x_0|}{2} \leq |x_0| - r|y| \leq |x_0 + ry| \leq \frac{3}{2}|x_0|,$$

therefore,

$$\frac{|x_0|}{2} \leq |x_0 + ry| \leq \frac{3}{2}|x_0| \quad \text{thus,} \quad \frac{2}{3}|x_0| \leq \frac{1}{|x_0 + ry|} \leq \frac{2}{|x_0|}.$$

Then,

$$\frac{(\frac{|x_0|}{2})^p}{|x_0 + ry|} \leq 1,$$

hence, the weight is uniformly bounded and it is smooth in this region. Using the results in [88] and in [95] we know that there exists a universal constant $C > 0$ such that $v(0) < C$ and then, $u \in L^\infty(\Omega)$. Using a regularity result in [70] we deduce that $v \in C^{1,\alpha}(\mathcal{D})$. Namely, there exists some universal constant C (not depending on v) such that $|\nabla v(0)| \leq C$. Since $|\nabla v| = r|\nabla u|$, we get $|\nabla u(x)| \leq C/|x|, \ \forall x \in \Omega$, being x_0 arbitrary.

□

Proposition 6. *Assume* $1 < p < N$ *and* $p - 1 < q < p^* - 1$*. Then, problem* (6.3) *has no energy solutions if* $0 \in \partial\Omega$ *and* Ω *is starshaped with respect to the origin.*

Proof. In this proof we are going to use a Pohozaev's identity.

We multiply (using a density argument) $-\Delta_p u = \dfrac{u^q}{|x|^p}$ by $< x, \nabla u >$ and integrate in $\Omega \setminus B_\rho(0)$, with $\rho > 0$. One has

$$\int_{\Omega\setminus B_{\rho(0)}} (-\Delta_p u) < x, \nabla u > dx = \int_{\Omega\setminus B_{\rho(0)}} -div(|\nabla u|^{p-2}\nabla u) < x, \nabla u > dx$$

$$= \int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx + \sum_{i=1}^N \int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^{p-2} < \nabla u, \nabla \frac{\partial u}{\partial x_i} > x_i dx$$

$$- \int_{\partial(\Omega\setminus B_{\rho(0)})} \nabla u |\nabla u|^{p-2} < x, \nabla u > \frac{\partial u}{\partial \nu} dx.$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega\setminus B_{\rho(0)}} (-\Delta_p u) < x, \nabla u > dx \\
&= \int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx + \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega\setminus B_{\rho(0)}} \left(\frac{\partial}{\partial x_i}|\nabla u|^p\right) x_i dx \\
&\quad - \int_{\partial(\Omega\setminus B_{\rho(0)})} \nabla u|\nabla u|^{p-2} < x, \nabla u > \frac{\partial u}{\partial \nu} dx \\
&= \int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx - \frac{N}{p}\int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx - \int_{\partial(\Omega\setminus B_{\rho(0)})} \nabla u|\nabla u|^{p-2} < x, \nabla u > \frac{\partial u}{\partial \nu} dx \\
&\quad + \int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{|\nabla u|^p}{p} < x, \nu > dx \\
&= \frac{p-N}{p}\int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx + \frac{1-p}{p}\int_{\partial(\Omega\setminus B_{\rho(0)})} |\nabla u|^p < x, \nu > dx
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\Omega\setminus B_{\rho(0)}} \frac{u^q}{|x|^p} < x, \nabla u > dx &= -\frac{1}{q+1}\int_{\Omega\setminus B_{\rho(0)}} u^{q+1} div\left(\frac{x}{|x|^p}\right) dx \\
&\quad + \frac{1}{q+1}\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x, \nu > dx.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
-\frac{u^{q+1}}{q+1} div\left(\frac{x}{|x|^p}\right) &= -\frac{u^{q+1}}{q+1}\left(\frac{div(x)}{|x|^p} + x div\left(\frac{1}{|x|^p}\right)\right) \\
&= -\frac{u^{q+1}}{q+1}\left(\frac{N}{|x|^p} - p\frac{x}{|x|^{p+1}}\right) = -\frac{u^{q+1}}{q+1}\left(\frac{N}{|x|^p} - p\frac{1}{|x|^p}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega\setminus B_{\rho(0)}} (-\Delta_p u) < x, \nabla u > dx \\
&= -\frac{N-p}{q+1}\int_{\Omega\setminus B_{\rho(0)}} \frac{u^{q+1}}{|x|^p} dx + \frac{1}{q+1}\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x, \nu > dx.
\end{aligned}
$$

Combining the last calculations, we get

$$
\begin{aligned}
&-\frac{N-p}{q+1}\int_{\Omega\setminus B_{\rho(0)}} \frac{u^{q+1}}{|x|^p} dx + \frac{1}{q+1}\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x, \nu > dx \\
&= \frac{1-p}{p}\int_{\partial(\Omega\setminus B_{\rho(0)})} |\nabla u|^p < x, \nu > dx + \frac{p-N}{p}\int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx.
\end{aligned}
$$

Multiplying equation (6.1) by u and integrating in $\Omega \setminus B_\rho(0)$ we have

$$\int_{\Omega\setminus B_{\rho(0)}} -u\Delta_p u dx = \int_{\Omega\setminus B_{\rho(0)}} \frac{u^q}{|x|^p} u dx.$$

Integrating this last expression by parts,

$$-\int_{\partial(\Omega\setminus B_{\rho(0)})} u|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}dx + \int_{\Omega\setminus B_{\rho(0)}} |\nabla u|^p dx = \int_{\Omega\setminus B_{\rho(0)}} \frac{u^{q+1}}{|x|^p}dx.$$

Therefore, substituting,

$$0 = (N-p)(\frac{1}{p} - \frac{1}{q+1})\int_{\Omega\setminus B_{\rho(0)}} \frac{u^{q+1}}{|x|^p}dx + (\frac{N-p}{p}) \int_{\partial(\Omega\setminus B_{\rho(0)})} u|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}dx$$
$$+\frac{1}{q+1}\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x,\nu > dx + (\frac{p-1}{p})\int_{\partial(\Omega\setminus B_{\rho(0)})} |\nabla u|^p < x,\nu > dx.$$

Since $u \equiv 0$ on $\partial\Omega$, $\partial(\Omega \setminus B_\rho(0)) = \partial B_\rho(0) \cap \Omega$, then,

$$\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x,\nu > dx = \int_{\partial B_{\rho(0)}\cap\Omega} \frac{u^{q+1}}{|x|^p} < x,\nu > dx.$$

Since $p < N$ and by Lemma 9,

$$\int_{\partial(\Omega\setminus B_{\rho(0)})} u|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}dx \le \int_{\partial(\Omega\setminus B_{\rho(0)})} C(\frac{C}{|x|})^{p-1}dx$$
$$= \int_{\partial(\Omega\setminus B_{\rho(0)})} C(\frac{C}{|\rho|})^{p-1}\rho^{N-2}dx = O(\rho^{N-p}) \to 0 \text{ as } \rho \to 0.$$

Since $|\nu| = 1$ and by Lemma 9, $u \in L^\infty(\Omega)$,

$$\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x,\nu > dx \le \int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} x dx \le C\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{x}{|x|^p}dx.$$

Since $p < N$ and being ρ^{N-2} the jacobian term in the boundary,

$$\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{u^{q+1}}{|x|^p} < x,\nu > dx$$
$$\le C\int_{\partial(\Omega\setminus B_{\rho(0)})} \frac{\rho^{N-2}}{\rho^{p-1}}d\rho = O(\rho^{N-p}) \to 0 \qquad \text{as} \qquad \rho \to 0.$$

In this way we get

$$0 = (N-p)(\frac{1}{p} + \frac{1}{q+1})\int_\Omega \frac{u^{q+1}}{|x|^p}dx + (\frac{p-1}{p})\int_{\partial\Omega} |\nabla u|^p < x,\nu > dx.$$

Therefore,

$$(N-p)(\frac{1}{p}+\frac{1}{q+1})\int_\Omega \frac{u^{q+1}}{|x|^p}dx = -(\frac{p-1}{p})\int_{\partial\Omega} |\nabla u|^p < x,\nu > dx.$$

Being Ω a *starshaped* domain w.r.t. the origin (i.e $< x,\nu > \geq 0$ on $\partial\Omega$) and $1 < p < N$, then,

$$(N-p)(\frac{1}{p}+\frac{1}{q+1})\int_\Omega \frac{u^{q+1}}{|x|^p}dx = 0.$$

Therefore, $u \equiv 0$ in Ω. □

2.2 Existence of energy solutions in dumbbell domains

In this Subsection we prove Theorem 23. We follow [50] to prove the existence result, taking into account the differences due to the nonlinearity and the change of regularity of solutions introduced by the *p-Laplacian* operator. We consider the truncated weight $\frac{1}{|x|^p+\delta}$ and we proceed in several steps in order to be clear in the proof.

We recall first some regularity results that we are going to use along this Section. A Stampacchia's type argument gives

Lemma 10. *Let $u \in W_0^{1,p}(\Omega)$ an energy solution to*

$$\begin{cases} -\Delta_p u = f & in\,\Omega \\ u = 0 & on\;\;\partial\Omega, \end{cases} \tag{6.5}$$

with $f \in L^r(\Omega)$, and $r > \frac{N}{p}$. Then, $u \in L^\infty(\Omega)$.

We also need some regularity up to the boundary, see [70] for the details. It is important to point out that the result for the regularity without reach the boundary has been studied in [93] and [51].

2.2.1 The truncated-penalized functional.

We first define the function $\eta \in C^1(\mathbb{R})$,

$$\begin{cases} \eta(s) = 0 & s \in [0,1], \\ 0 \leq \eta(s) \leq 2 \;\text{ and }\; \eta'(s) \geq 0 & s \in [1,2], \\ \eta(s) = s & s \in [2,\infty]. \end{cases} \tag{6.6}$$

Given $\theta > 0$, we define the function $\eta_\theta(t) = \eta(t/\theta),\ \ t \geq 0$ and we fix h such that

$$\min\left(p, (q-(p-1))\frac{N}{p}\right) < h < q+1.$$

Since in the hypothesis we are considering $q > p-1$, we only need to check that $(q-(p-1))\frac{N}{p} < q+1$. To do that we argue by contradiction. Suppose that

$$(q-(p-1))\frac{N}{p} \geq q+1,$$

hence,

$$q(\frac{N}{p}-1) \geq 1 + \frac{N}{p}(p-1).$$

Then,

$$q \geq \frac{N(p-1)+p}{N-p},$$

which is a contradiction with ($\sharp$) in the hypotheses.

Let $g \in C^1(\mathbb{R})$ be the function

$$\begin{cases} g(s) = 0, & s \leq 0 \\ 0 \leq g(s) \leq 1 \ \text{ and } \ 0 \leq g'(s) \leq h s^{(q-(p-1))\frac{N}{p}-1} & s \in [0,1], \\ g(s) = s^{(q-(p-1))\frac{N}{p}} & s \geq 1. \end{cases} \tag{6.7}$$

Given $\delta > 0$, we define

$$g_\delta(t) = \delta^{(q-(p-1))\frac{N}{p}} g(\frac{t}{\delta}).$$

For $\varepsilon, \delta, \theta > 0$ we define the penalized energy functional $E_{\delta,\varepsilon,\theta} : W_0^{1,p}(\Omega_\varepsilon) \to \mathbb{R}$ as follows

$$E_{\delta,\varepsilon,\theta}(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p + \delta} dx + \eta_\theta(I_\delta(u)),$$

where $u^+ = \max(u, 0)$ and $\eta_\theta(I_\delta(u))$ is the penalization of the energy functional. This term will be crucial to prove that the solution that we are looking for is not the trivial one. It is called penalization because we try to find a infimum and this term rises a little the functional.

Notice that the penalization is only defined in Ω_1 because is where the zero is located. The term $I_\delta(u)$ is given by the following expression

$$I_\delta(u) = \int_{\Omega_1} \frac{g_\delta(u+\delta)}{(|x|^p+\delta)^{\frac{N}{p}}} dx. \tag{6.8}$$

It is easy to verify the C^1 regularity of $E_{\delta,\varepsilon,\theta}$ since the C^1 regularity of $\eta(\cdot)$ and $g(\cdot)$ holds. Then, if u is a critical point of $E_{\delta,\varepsilon,\theta}$ it satisfies

$$\begin{cases} -\Delta_p u + a(x,u)g'_\delta(u+\delta) = \frac{(u^+)^q}{|x|^p+\delta} & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{6.9}$$

where $a(x,u)g'_\delta(u+\delta) = \eta'_\theta(I_\delta(u))I'_\delta(u)$ and by the definition of $I_\delta(u)$,

$$I'_\delta(u) = g'_\delta(u+\delta)\frac{\chi_{\Omega_1}(x)}{(|x|^p+\delta)^{\frac{N}{p}}}.$$

In this way,

$$a(x,u) = \eta'_\theta(I_\delta(u))\frac{\chi_{\Omega_1}(x)}{(|x|^p+\delta)^{\frac{N}{p}}},$$

with $\chi_{\Omega_1}(x)$ the characteristic function of Ω_1.

Multiplying (6.9) by the test function $(u+\delta)^-$, we get

$$\begin{aligned} &\int_{\Omega_\varepsilon} -div(|\nabla u|^{p-2}\nabla u)(u+\delta)^- dx + \int_{\Omega_\varepsilon} a(x,u)g'_\delta(u+\delta)(u+\delta)^- dx \\ &= \int_{\Omega_\varepsilon} \frac{(u^+)^q(u+\delta)^-}{|x|^p+\delta}dx. \end{aligned} \tag{6.10}$$

Integrating (6.10) by parts we have

$$\begin{aligned} &\int_{\Omega_\varepsilon} |\nabla u|^{p-1}\nabla(u+\delta)^- dx + \int_{\Omega_\varepsilon} a(x,u)g'_\delta(u+\delta)(u+\delta)^- dx \\ &= \int_{\Omega_\varepsilon} \frac{(u^+)^q(u+\delta)^-}{|x|^p+\delta}dx. \end{aligned}$$

Notice that if $u+\delta \leq 0$, $g_\delta(u+\delta) = 0$, as a consequence,

$$\int_{\Omega_\varepsilon} a(x,u)g'_\delta(u+\delta)(u+\delta)^- dx = 0 \quad \text{and} \quad \int_{\Omega_\varepsilon} \frac{(u^+)^q(u+\delta)^-}{|x|^p+\delta}dx \leq 0.$$

Therefore,

$$\int_{\Omega_\varepsilon} |\nabla u|^{p-1}\nabla(u+\delta)^- dx \leq 0.$$

Thus,

$$\int_{\Omega_\varepsilon} |\nabla(u+\delta)^-|^p dx \leq 0. \tag{6.11}$$

Then, we have that $(u+\delta)^- \equiv 0$, that is, a solution $u \in W_0^{1,p}(\Omega_\varepsilon)$ to (6.9) is not necessarily positive but satisfies $u \geq -\delta$ in Ω_ε.

2.2.2 Study of the functional $E_{\delta,\varepsilon,\theta}$ and the mountain pass geometry condition.

We study the functional for each $\delta,\varepsilon,\theta$ fixed in order to obtain the conditions for the *Mountain Pass Theorem* (see Theorem 25).

Lemma 11. *Fix $\varepsilon,\theta,\delta>0$. Then, $E_{\delta,\varepsilon,\theta}:W_0^{1,p}(\Omega_\varepsilon)\to\mathbb{R}$ is C^1 and satisfies the Palais-Smale condition (see Definition 8).*

Proof. Let u_n be a *Palais-Smale* sequence (see Definition 7) in $W_0^{1,p}(\Omega_\varepsilon)$ such that $E_{\delta,\varepsilon,\theta}\le C$ and $E'_{\delta,\varepsilon,\theta}\to 0$ in $W^{-1,p'}(\Omega_\varepsilon)$. Therefore,

$$
\begin{aligned}
&C+o(1)||u_n||_{W_0^{1,p}(\Omega_\varepsilon)}\ge E_{\delta,\varepsilon,\theta}(u_n)-\frac{1}{q+1}E'_{\delta,\varepsilon,\theta}(u_n)u_n\\
&=\frac{1}{p}\int_{\Omega_\varepsilon}|\nabla u_n|^p dx-\frac{1}{q+1}\int_{\Omega_\varepsilon}\frac{(u_n^+)^{q+1}}{|x|^p+\delta}dx+\eta_\theta(I_\delta(u_n))-\frac{1}{q+1}\int_{\Omega_\varepsilon}|\nabla u_n|^{p-1}\nabla u_n dx\\
&\quad+\frac{1}{q+1}\int_{\Omega_\varepsilon}\frac{(u_n^+)^{q+1}}{|x|^p+\delta}dx-\frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1}\frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\\
&=\frac{1}{p}\int_{\Omega_\varepsilon}|\nabla u_n|^p dx+\eta_\theta(I_\delta(u_n))-\frac{1}{q+1}\int_{\Omega_\varepsilon}|\nabla u_n|^p dx\\
&\quad-\frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1}\frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\\
&=(\frac{1}{p}-\frac{1}{q+1})\int_{\Omega_\varepsilon}|\nabla u_n|^p dx+\eta_\theta(I_\delta(u_n))-\frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1}\frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\\
&=(\frac{1}{p}-\frac{1}{q+1})||u_n||^p_{W_0^{1,p}(\Omega_\varepsilon)}+\eta_\theta(I_\delta(u_n))-\frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1}\frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx.
\end{aligned}
$$

We are going to prove the following claim

$$
g'_\delta(u_n+\delta)u_n\le hg_\delta(u_n+\delta),\quad\forall u_n\in\mathbb{R}. \tag{6.12}
$$

We consider the following three cases:

→ If $u_n\ge 0$, by definition, $g_\delta(u_n+\delta)=(u_n+\delta)^{(q-(p-1))\frac{N}{p}}$ and

$$
g'_\delta(u_n+\delta)=(q-(p-1))\frac{N}{p}(u_n+\delta)^{(q-(p-1))\frac{N}{p}-1}.
$$

Therefore,

$$
g'_\delta(u_n+\delta)(u_n+\delta)=(q-(p-1))\frac{N}{p}(u_n+\delta)^{(q-(p-1))\frac{N}{p}}\le hg_\delta(u_n+\delta).
$$

→ If $-\delta \le u_n \le 0$, we obtain

$$g'_\delta(u_n+\delta)u_n \le 0 \le hg_\delta(u_n+\delta).$$

→ If $u_n \le -\delta$, (6.12) is satisfied because

$$g'_\delta(u_n+\delta)u_n = 0 = hg_\delta(u_n+\delta).$$

By the previous claim we have

$$\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx \le \int_{\Omega_1} h\frac{g_\delta(u_n+\delta)}{(|x|^p+\delta)^{\frac{N}{p}}}dx \le hI_\delta(u_n). \tag{6.13}$$

We consider now

(*i*) Assume $I_\delta(u_n) \ge 2\theta$, then $\eta'_\theta(I_\delta(u_n)) = 1$ and by the definition of $\eta_\theta(I_\delta(u_n))$, it follows

$$\begin{aligned}&\eta_\theta(I_\delta(u_n)) - \frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\\ &= \frac{1}{\theta}\left(I_\delta(u_n) - \frac{1}{q+1}\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\right)\end{aligned}$$

and by (6.13) and since $h < q+1$, we get

$$\begin{aligned}&\eta_\theta(I_\delta(u_n)) - \frac{1}{q+1}\eta'_\theta(I_\delta(u_n))\int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}}dx\\ &\ge \frac{1}{\theta}(1-\frac{h}{q+1})I_\delta(u_n) \ge 0.\end{aligned}$$

Therefore,

$$C + o(1)||u_n||_{W_0^{1,p}(\Omega_\varepsilon)} \ge (\frac{1}{p} - \frac{1}{q+1})||u_n||^p_{W_0^{1,p}(\Omega_\varepsilon)}$$

and for all n,

$$||u_n||_{W_0^{1,p}(\Omega_\varepsilon)} \le C,$$

with C independent on n.

(ii) If $I_\delta(u_n) \leq \theta$, then $\eta_\theta(I_\delta(u_n)) = 0 = \eta'_\theta(I_\delta(u_n))$ and

$$\eta_\theta(I_\delta(u_n)) - \frac{1}{q+1}\eta'_\theta(I_\delta(u_n)) \int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}} dx = 0,$$

hence, as in the case (i),

$$||u_n||_{W_0^{1,p}(\Omega_\varepsilon)} \leq C \ \ \forall n.$$

(iii) If $\theta \leq I_\delta(u_n) \leq 2\theta$, by (6.13),

$$\begin{aligned}&\eta_\theta(I_\delta(u_n)) - \frac{1}{q+1}\eta'_\theta(I_\delta(u_n)) \int_{\Omega_1} \frac{g'_\delta(u_n+\delta)u_n}{(|x|^p+\delta)^{\frac{N}{p}}} dx \\ &\geq -\frac{C}{\theta} I_\delta(u_n) \geq -C.\end{aligned}$$

Therefore,

$$||u_n||_{W_0^{1,p}(\Omega_\varepsilon)} \leq C \ \ \forall n.$$

So, we can conclude

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ in } W_0^{1,p}(\Omega_\varepsilon), \\ u_n &\to u \ \text{ in } L^p(\Omega_\varepsilon) \ \text{ for } 1 \leq p < p^*. \end{aligned}$$

Taking a subsequence, u_n converges weakly to u in $W_0^{1,p}(\Omega_\varepsilon)$.

Then, u_n^{q+1} and $u_n^{(q-(p-1))\frac{N}{p}}$ converge strongly in $L^1(\Omega_\varepsilon)$, namely

$$\frac{(u^+)^q}{|x|^p+\delta} - a(x,u)g'_\delta(u+\delta) \subset W^{-1,p'}(\Omega_\varepsilon)$$

because

$$\int_{\Omega_\varepsilon} \left(\frac{(u_n^+)^q}{|x|^p+\delta}\right)\varphi dx \to \int_{\Omega_\varepsilon} \left(\frac{(u^+)^q}{|x|^p+\delta}\right)\varphi dx$$

with $\varphi \in W_0^{1,p}(\Omega_\varepsilon)$, and

$$\int_{\Omega_\varepsilon} a(x,u_n)g'_\delta(u_n+\delta)\varphi dx \to \int_{\Omega_\varepsilon} a(x,u)g'_\delta(u+\delta)\varphi dx$$

with $\varphi \in W_0^{1,p}(\Omega_\varepsilon)$.

In the first case, since $|x|^p + \delta \geq \delta$, we only need to check that the following term is going to zero

$$\int_{\Omega_\varepsilon} \frac{(u_n^+)^q - (u^+)^q}{|x|^p+\delta}\varphi dx \leq \int_{\Omega_\varepsilon} \frac{(u_n^+)^q - (u^+)^q}{\delta}\varphi dx.$$

Using Hölder's inequality and since $q\frac{p^*}{p^*-1} \le p^*$ (recall that $q \le p*-1$ and $u_n \to u$ in $L^p(\Omega_\varepsilon)$ for $1 \le p < p^*$) and using also that $\varphi \in W_0^{1,p}(\Omega_\varepsilon)$,

$$\int_{\Omega_\varepsilon} \frac{(u_n^+)^q - (u^+)^q}{|x|^p+\delta}\varphi dx \le \Big(\int_{\Omega_\varepsilon} \varphi^{p^*} dx\Big)^{\frac{1}{p^*}} \Big(\int_{\Omega_\varepsilon} \big((u_n^+)^q - (u^+)^q\big)^{\frac{p^*}{p^*-1}} dx\Big)^{\frac{p^*-1}{p^*}}$$
$$\le \Big(\int_{\Omega_\varepsilon} \varphi^{p^*} dx\Big)^{\frac{1}{p^*}} C(p^*) \Big(\int_{\Omega_\varepsilon} (u_n^+)^{q\frac{p^*}{p^*-1}} - (u^+)^{q\frac{p^*}{p^*-1}} dx\Big)^{\frac{p^*-1}{p^*}} \le o(1).$$

For the second convergence, in order to be able to pass to the limit, we only need to prove that

$$u_n^{q-(p-1)\frac{N}{p}-1}\varphi \in L^1(\Omega_\varepsilon).$$

Using Hölder's inequality, we get

$$\int_{\Omega_\varepsilon} u_n^{q-(p-1)\frac{N}{p}-1}\varphi dx \le \Big(\int_{\Omega_\varepsilon} \varphi^{p^*} dx\Big)^{\frac{1}{p^*}} \Big(\int_{\Omega_\varepsilon} u_n^{(q-(p-1)\frac{N}{p}-1)(\frac{p^*}{p^*-1})} dx\Big)^{\frac{p^*-1}{p^*}}$$

and since, by the hypothesis of h, we know that $q-(p-1)\frac{N}{p}-1 < q$, thus, $(q-(p-1)\frac{N}{p}-1)(\frac{p^*}{p^*-1}) < p^*$, hence, as before, since $u_n \to u$ in $L^p(\Omega_\varepsilon)$ for $1 \le p < p^*$ and $\varphi \in W_0^{1,p}(\Omega_\varepsilon)$,

$$u_n^{q-(p-1)\frac{N}{p}-1}\varphi \in L^1(\Omega_\varepsilon).$$

Since u_n is a *Palais-Smale* sequence, and by the continuity of $-\Delta_p^{-1}(\cdot)$, one has

$$u_n = (-\Delta_p^{-1})\left(\frac{(u_n^+)^q}{|x|^p+\delta} - a(x,u_n)g'_\delta(u_n+\delta) + y_n\right) \xrightarrow{n\uparrow\infty}$$
$$(-\Delta_p^{-1})\left(\frac{(u^+)^q}{|x|^p+\delta} - a(x,u)g'_\delta(u+\delta)\right) = u \quad \text{in} \quad W_0^{1,p}(\Omega_\varepsilon).$$

□

The mountain pass geometry condition is a consequence of the following observation: if $\|u\|_{W_0^{1,p}(\Omega_\varepsilon)}$ is small enough by definitions (6.6) and (6.7) one has $E_{\delta,\varepsilon,\theta}(u) = \tilde{E}_{\delta,\varepsilon}(u)$ where

$$\tilde{E}_{\delta,\varepsilon}(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx,$$

is the functional without penalization.

Let $v > 0$ be a solution to $-\Delta_p u = \dfrac{u^q}{|x|^p}$ in Ω_2 with $u = 0$ on $\partial\Omega_2$, obtained for example by the Mountain Pass Theorem, since the weight is bounded because 0 is away from Ω_2 and the problem is subcritical. Fix $s > 0$ large enough such that

$$E_{\delta,\varepsilon,\theta}(sv) = \frac{1}{p}s^p \int_{\Omega_2} |\nabla v|^p dx - \frac{1}{q+1}s^{q+1} \int_{\Omega_2} \frac{v^{q+1}}{|x|^p + \delta} dx < 0.$$

Consider $c_{\delta,\varepsilon,\theta} = \inf_\gamma \max_{t\in[0,1]} E_{\delta,\varepsilon,\theta}(\gamma(t))$, where the infimum ranges over all continuous paths $\gamma : [0,1] \to W_0^{1,p}(\Omega_\varepsilon)$ such that $\gamma(0) = 0$ and $\gamma(1) = sv$.

Lemma 12. *Since the functional $E_{\delta,\varepsilon,\theta}$ satisfies the assumptions of the Mountain Pass Lemma, there exists a critical point u of $E_{\delta,\varepsilon,\theta}$ with critical value $c_{\delta,\varepsilon,\theta}$. Moreover, there exists a constant C independent of $\varepsilon, \theta, \delta$ such that*

$$c_{\delta,\varepsilon,\theta} \le C.$$

Proof. Since the weight $\dfrac{1}{|x|^p + \delta}$ is bounded, there exists $\rho > 0$ such that

$$\begin{aligned} &\inf \{\tilde{E}_{\delta,\varepsilon}(u) \ : \ \|u\|_{W_0^{1,p}(\Omega_\varepsilon)} = \rho\} > \\ &\inf \{\frac{1}{p}\rho^p - \frac{1}{q+1} \int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p + \delta}\} > 0 \end{aligned}$$

and also since $\eta_\theta(I_\delta(u)) > 0$,

$$\begin{aligned} &\inf \{E_{\delta,\varepsilon,\theta}(u) \ : \ \|u\|_{W_0^{1,p}(\Omega_\varepsilon)} = \rho\} \\ &> \inf \{\frac{1}{p}\rho^p - \frac{1}{q+1} \int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p + \delta} + \eta_\theta(I_\delta(u))\} > 0. \end{aligned}$$

Consider $\gamma(t) = t(sv)$ and notice that $\max_{t\in[0,1]} E_{\delta,\varepsilon,\theta}(t(sv))$ is bounded uniformly on $\delta, \varepsilon, \theta$ because v is a solution in Ω_2 and the penalization is only in Ω_1, so the maximum does not depend on $\delta, \varepsilon, \theta$. Thus, since $c_{\delta,\varepsilon,\theta} = \inf_\delta \max_{t\in[0,1]} E_{\delta,\varepsilon,\theta}(\gamma(t))$, $c_{\delta,\varepsilon,\theta} \le \max_{t\in[0,1]} E_{\delta,\varepsilon,\theta}(\gamma(t)) \le C$.

Hence, we get the upper bound for $c_{\delta,\varepsilon,\theta}$.

□

Next lemma proves that the *mountain pass* level $c_{\delta,\epsilon,\theta}$ admits a uniform bound from below away from zero. This will be important in the limit process in order to find a positive ($u \not\equiv 0$) solution.

Lemma 13. *There exists $\theta_0 > 0$ and $c_0 > 0$ independent of $\varepsilon, \theta, \delta$ such that*

$$c_{\delta,\varepsilon,\theta} \geq c_0,$$

for $0 < \theta \leq \theta_0$.

Proof. Since $0 \in \partial\Omega_1$, in $\Omega_\varepsilon \setminus \Omega_1$ the weight $\frac{1}{|x|^p+\delta}$ is uniformly bounded on ε, δ and we can fix $\rho > 0$ independently on ε, δ such that, by Hölder's inequality,

$$\begin{aligned}
&\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon\setminus\Omega_1} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx \\
&\geq \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1} C\Big(\int_{\Omega_\varepsilon\setminus\Omega_1} ((u^+)^{q+1})^{\frac{p^*}{q+1}} dx\Big)^{\frac{q+1}{p^*}} |\Omega_\varepsilon \setminus \Omega_1|^{(\frac{q+1}{p^*})'} \\
&= \frac{1}{p}||u||^p_{W_0^{1,p}(\Omega_\varepsilon)} - \frac{1}{q+1} C_\Omega ||u||^{q+1}_{L^{p^*}(\Omega_\varepsilon\setminus\Omega_1)}.
\end{aligned}$$

Using Sobolev's inequality,

$$\begin{aligned}
&\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon\setminus\Omega_1} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx \\
&\geq \frac{1}{p}||u||^p_{W_0^{1,p}(\Omega_\varepsilon)} - \frac{1}{q+1} S C_\Omega ||u||^{q+1}_{W_0^{1,p}(\Omega_\varepsilon\setminus\Omega_1)}.
\end{aligned}$$

Since $q+1 > p$, if $||u||_{W_0^{1,p}(\Omega_\varepsilon)}$ is small enough, $||u||^p_{W_0^{1,p}(\Omega_\varepsilon)} \geq ||u||^{q+1}_{W_0^{1,p}(\Omega_\varepsilon\setminus\Omega_1)}$.

Therefore,

$$\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon\setminus\Omega_1} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx \geq C_p \|u\|^p_{W_0^{1,p}(\Omega_\varepsilon)}, \quad \forall \|u\|_{W_0^{1,p}(\Omega_\varepsilon)} \leq \rho.$$

If we focus now only in Ω_1, by Hölder's inequality

$$\int_{\Omega_1} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx \leq \left(\int_{\Omega_1} (u^+)^{p^*} dx\right)^{\frac{N-p}{N}} \left(\int_{\Omega_1} \frac{(u^+)^{(q-(p-1))\frac{N}{p}}}{(|x|^p+\delta)^{\frac{N}{p}}} dx\right)^{\frac{p}{N}}. \tag{6.14}$$

Using Sobolev's inequality we obtain

$$\int_{\Omega_1} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx \leq C\|u\|^p_{W_0^{1,p}(\Omega_\varepsilon)} I_\delta(u)^{\frac{p}{N}},$$

where C depends on Ω_1, p, N. By this expression and the calculation in $\Omega_\varepsilon \setminus \Omega_1$, it follows that

$$E(u) \geq C_p\|u\|^p_{W_0^{1,p}(\Omega_\varepsilon)} - C\rho^p I_\delta(u)^{\frac{p}{N}} + \eta_\theta(I_\delta(u)), \quad \forall \|u\|_{W_0^{1,p}(\Omega_\varepsilon)} \leq \rho. \tag{6.15}$$

Consider now the function $h(t) = \eta_\theta(t) - C\rho^p t^{\frac{p}{N}}$. If $t \geq 2\theta$,

$$h(t) = t^{\frac{p}{N}} \left(\frac{t^{\frac{N-p}{N}}}{\theta} - C\rho^p \right) \geq t^{\frac{p}{N}} \left(2^{\frac{N-p}{N}} \theta^{-\frac{p}{N}} - C\rho^p \right).$$

Fixing $\rho > 0$, we take $\theta_0 > 0$ small, thus

$$2^{\frac{N-p}{N}} \theta^{-\frac{p}{N}} - C\rho^p \geq 1 \quad \text{for } 0 < \theta \leq \theta_0. \tag{6.16}$$

Therefore,

$$h(t) \geq t^{\frac{p}{N}}, \quad \text{for } t \geq 2\theta. \tag{6.17}$$

For $0 \leq t \leq 2\theta$ we obtain

$$h(t) \geq -C\rho^p t^{\frac{p}{N}} \geq -C\rho^p (2\theta)^{\frac{p}{N}}. \tag{6.18}$$

Let $\gamma : [0,1] \to W_0^{1,p}(\Omega_\varepsilon)$ be continuous such that $\gamma(0) = 0$ and $\gamma(1) = sv$. We take $\rho > 0$ small so that $s\|v\|_{W_0^{1,p}(\Omega_\varepsilon)} > \rho$. Let t^* be defined by

$$t^* = \min\{\ t \in [0,1]\ :\ \|\gamma(t)\|_{W_0^{1,p}(\Omega_\varepsilon)} \geq \rho \text{ or } I_\delta(\gamma(t)) \geq 1\ \}.$$

t^* is well defined and satisfies the properties $\|\gamma(t)\|_{W_0^{1,p}(\Omega_\varepsilon)} \leq \rho$, $I_\delta(\gamma(t)) \leq 1$ for $0 \leq t \leq t^*$ because of the definition of t^* and one of the following cases: either $\|\gamma(t^*)\|_{W_0^{1,p}(\Omega_\varepsilon)} = \rho$ or $I_\delta(\gamma(t^*)) = 1$.

Assume first that $\|\gamma(t^*)\|_{W_0^{1,p}(\Omega_\varepsilon)} = \rho$. Then, using (6.15), (6.17) and (6.18) we get

$$E(\gamma(t^*)) \geq C_p \|\gamma(t^*)\|^p_{W_0^{1,p}(\Omega_\varepsilon)} - C\rho^p (2\theta)^{\frac{p}{N}} = C_p\rho^p - C\rho^p(2\theta)^{\frac{p}{N}}.$$

Choosing θ_0 smaller it follows

$$C_p\rho^p - C\rho^p(2\theta)^{\frac{p}{N}} \geq \tilde{C}_p\rho^p, \quad \text{for } 0 < \theta \leq \theta_0, \tag{6.19}$$

for some positive constant $\tilde{C}_p$ independent of $\varepsilon, \theta, \delta$. Thus,

$$E(\gamma(t^*)) \geq \tilde{C}_p\rho^p.$$

Suppose now that $I_\delta(\gamma(t^*)) = 1$. Since $I_\delta(\gamma(t^*)) \geq 2\theta$, we may also assume that $\theta_0 \leq \frac{1}{2}$. Then, by (6.15) it follows

$$E(\gamma(t^*)) \geq C_p\|\gamma(t^*)\|^p_{W_0^{1,p}(\Omega_\varepsilon)} - C\rho^p I_\delta(\gamma(t^*))^{\frac{p}{N}} + \eta_\theta(I_\delta(\gamma(t^*))),$$

for all $\|\gamma(t^*)\|_{W_0^{1,p}(\Omega_\varepsilon)} \leq \rho$.

By (6.17),

$$E(\gamma(t^*)) \geq C_p \|\gamma(t^*)\|^p_{W_0^{1,p}(\Omega_\varepsilon)} + I_\delta(\gamma(t^*))^{\frac{p}{N}} \geq C_p \|\gamma(t^*)\|^p_{W_0^{1,p}(\Omega_\varepsilon)} + 1 \geq 1.$$

It follows that the mountain pass level $c_{\delta,\varepsilon,\theta}$ satisfies

$$c_{\delta,\varepsilon,\theta} \geq \min\left(\tilde{C}_p \rho^p, 1\right) \tag{6.20}$$

provided $0 < \theta \leq \theta_0$ and $0 < \theta_0 \leq \frac{1}{2}$ is such that (6.16) and (6.19) hold. □

Remark 6. *The inequality* (6.14) *motivates the importance of the penalization term. We need this term to obtain the uniform bound from below* (6.20) *away from zero. In Subsection 2.2.6 we shall see that this control will be necessary to pass to the limit and then to reach a nontrivial solution to* (6.3)*.*

2.2.3 Uniform estimates for the *mountain pass* critical points.

In order to prove that the penalization is small enough when we pass to the limit, we need to check some uniform estimates of the critical points.

We claim that there exists C independent on $\delta, \theta, \varepsilon$ such that for all $\delta > 0$, $\theta > 0$ and $\varepsilon > 0$, if $u \in W_0^{1,p}(\Omega_\varepsilon)$ is a mountain pass critical point of the energy functional $E_{\delta,\varepsilon,\theta}(u)$, then

$$\|u\|_{W_0^{1,p}(\Omega_\varepsilon)} \leq C \tag{6.21}$$

and

$$I_\delta(u) \leq C\theta. \tag{6.22}$$

The argument is the same as in the proof of Lemma 11. Indeed, since $E_{\delta,\varepsilon,\theta}(u) \leq C$ we have

$$\begin{aligned} C &\geq E_{\delta,\varepsilon,\theta}(u) - \frac{1}{q+1} E'_{\delta,\varepsilon,\theta}(u)u \\ &= \left(\frac{1}{p} - \frac{1}{q+1}\right) \|u\|^p_{W_0^{1,p}(\Omega_\varepsilon)} + \eta_\theta(I_\delta(u)) - \frac{h}{q+1}\eta'_\theta(I_\delta(u)) I_\delta(u). \end{aligned}$$

(i) If $I_\delta(u) \geq 2\theta$

$$\eta_\theta(I_\delta(u)) - \frac{h}{p+1}\eta'_\theta(I_\delta(u)) I_\delta(u) = \frac{1}{\theta}\left(1 - \frac{h}{q+1}\right) I_\delta(u)$$

and we deduce (6.21) as in Lemma 11.

Notice also that

$$\begin{aligned} C &\geq C_{p,q}||u||^p_{W_0^{1,p}(\Omega_\varepsilon)} + \frac{1}{\theta}\left(1 - \frac{h}{q+1}\right) I_\delta(u) \\ &\geq \frac{1}{\theta}\left(1 - \frac{h}{q+1}\right) I_\delta(u) \geq \frac{1}{\theta}\tilde{C} I_\delta(u). \end{aligned}$$

Therefore, $I_\delta(u) \leq C\theta$ and (6.22) holds with C independent on $\delta, \theta, \varepsilon$.

(ii) If $I_\delta(u) \leq \theta$ we obtain the same conclusion because

$$\eta_\theta(I_\delta(u)) - \frac{h}{p+1}\eta'_\theta(I_\delta(u)) I_\delta(u) = 0.$$

(iii) If $\theta \leq I_\delta(u) \leq 2\theta$,

$$\begin{aligned} &\eta_\theta(I_\delta(u)) - \frac{h}{p+1}\eta'_\theta(I_\delta(u)) I_\delta(u) \\ &\geq -\frac{h}{p+1}\eta'_\theta(I_\delta(u)) I_\delta(u) \geq -\frac{C}{\theta} I_\delta(u) \geq -C, \end{aligned}$$

then,

$$C \geq C_{p,q}||u||^p_{W_0^{1,p}(\Omega_\varepsilon)} - C,$$

proving (6.21) and concluding the proof.

2.2.4 A local supersolution.

In order to control the *mountain pass* solutions close to the singularity and to be able to pass to the limit, we are going to construct an appropriate supersolution to be above the solution u and to control this function.

Fix $r_0 > 0$ small and define a set near the singularity,

$$D = \{x \in \Omega_\varepsilon \ : |x| < r_0\},\ \Gamma_1 = \partial\Omega_\varepsilon \cap \{|x| < r_0\} \text{ and } \Gamma_2 = \Omega_\varepsilon \cap \{|x| = r_0\}.$$

Since we assume that the curve that joins Ω_1 and Ω_2 along which runs C_ε is fixed and $0 \in \partial\Omega_1 \cap \partial\Omega_\varepsilon$, if we take $r_0 > 0$ small, D is independent of ε.

Consider $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$ with Γ_1 defined as above and $p-1 < q < p^*-1$. Let ζ be defined as the solution to

$$\begin{cases} -\Delta_p \zeta = \frac{d_{\Gamma_1}^q}{|x|^p} & \text{in } D \\ \zeta = 0 \quad \text{on } \Gamma_1, \quad \zeta = d_{\Gamma_1} & \text{on } \Gamma_2. \end{cases} \tag{6.23}$$

We want to check the regularity of the term in the right hand side of (6.23),

$$\Big(\int_D \big(\frac{d_{\Gamma_1}^q}{|x|^p} \big)^r \Big)^{\frac{1}{r}} < \infty.$$

The function $d_{\Gamma_1}^q/|x|^p$ belongs to $L^r(D)$ for any $1 \le r < \frac{N}{p-q}$ if $q < p$ and for any $r \ge 1$ if $q \ge p$. In both cases there exists $r > N > \frac{N}{p}$ such that $d_{\Gamma_1}^q/|x|^p \in L^r(D)$. The solution ζ to (6.23) is bounded by Lemma 10 and by classical regularity results, see [93] and [51], belongs to $C^{1,\alpha}(D)$. To construct the desired supersolution, we need a control from above of ζ, up to the boundary, by the distance function $d_{\Gamma_1}(x)$. Obviously, since in the case $q \ge p$ the weight is bounded, $\zeta \in C^{1,\alpha}(\overline{D})$ by a regularity result in [70] and, moreover, $\frac{\partial \zeta}{\partial \nu} < 0$ by [94], ν denoting the outer unit normal. If $q < p$, we need to be away from the origin, and the regularity will be $C^{1,\alpha}(\overline{D \setminus B_\rho(0)})$. For our purposes we need the boundedness of the gradient of the solution to (6.23) up to the boundary, that is, the global Lipschitz continuity of the solution. In the paper [42], the authors established the minimal assumptions on the integrability of the data and on the regularity of the boundary to get the boundedness of the gradient of some class of quasilinear elliptic equations.

In particular ζ turns to be Lipschitz continuous on $\overline{D}$ when $d_{\Gamma_1}^q/|x|^p \in L^{N,1}(D)$ (see in [42] a quasilinear version of the Theorem 22), where $L^{N,1}(D)$ denotes the corresponding Lorentz space. We recall the following embedding,

$$\begin{aligned} L^{q,q}(D) &= L^q(D) && \text{for } q \in (1, \infty), \\ L^{q_1,\sigma_1}(D) &\subsetneq L^{q_2,\sigma_2}(D) && \text{if } q_1 > q_2 \text{ and } \sigma_1, \sigma_2 \in (0, \infty]. \end{aligned} \tag{6.24}$$

Then, for every $q > p-1$, there exists $r > N$ such that $d_{\Gamma_1}^q/|x|^p \in L^r(D)$ and, since $r > N$, by the embedding (6.24) it follows that $d_{\Gamma_1}^q/|x|^p \in L^{N,1}(D)$. Thus, $\zeta \in C^{0,1}(\overline{D}) \cap C^{1,\alpha}(\overline{D \setminus B_\rho(0)})$.

Hence, there exists some constant $C > 0$ such that $\zeta \le C d_{\Gamma_1}$. Setting $\lambda_0 = C^{-\frac{q}{q-(p-1)}} > 0$ and defining $w = \lambda \zeta$, such that

$$-\Delta_p(w) = -\Delta_p(\lambda \zeta) = \lambda^{p-1}(-\Delta_p(\zeta)) = \lambda^{p-1} \frac{d_{\Gamma_1}^q}{|x|^p}.$$

Since $\zeta^q \leq C^q d_{\Gamma_1}^q$, $-\Delta_p(\lambda\zeta) \geq \lambda^q \frac{\zeta^q}{|x|^p}$, and w satisfies

$$\begin{cases} -\Delta_p w \geq \frac{w^q}{|x|^p} & \text{in } D \\ w > 0 & \text{in } D \\ w = 0 \quad \text{on } \Gamma_1, \quad w \geq \lambda d_{\Gamma_1} & \text{on } \Gamma_2, \end{cases} \tag{6.25}$$

for any $0 \leq \lambda \leq \lambda_0$ and, furthermore, $w(x) \leq C d_{\Gamma_1}(x)$ for some constant C. In the sequel we fix $\lambda = \lambda_0$ and $w = \lambda_0 \zeta$.

2.2.5 Comparison and control of the penalization.

We are going to show that, when θ goes to zero, any *mountain pass critical point* u of $E_{\delta,\varepsilon,\theta}$ satisfies

$$u \leq w \quad \text{in } D.$$

Indeed, as $I_\delta(u) \leq C\theta$, see (6.22), by the energy estimate (6.21), the classical L^∞ and $\mathcal{C}^\beta$-estimates give us that for any K compact, $K \subset (\Omega_1 \cup \Gamma_1) \setminus \{0\}$,

$$\|u\|_{L^\infty(K)} \to 0 \quad \text{as } \theta \to 0 \quad \text{uniformly on } \varepsilon, \delta.$$

Then, by bootstrapping, $\|u\|_{\mathcal{C}^{1,\beta}(K)} \leq C$ uniformly on ε, δ. Thus, there is $\theta_1 > 0$ independent of ε, δ, such that for $0 < \theta \leq \theta_1$ we obtain

$$u \leq \lambda d_{\Gamma_1} \text{ on } \Gamma_2.$$

From (6.9) we have

$$-\Delta_p u \leq \frac{u^q}{|x|^p + \delta} \quad \text{in } D,$$

and therefore

$$-\Delta_p u - (-\Delta_p w) \leq \frac{u^q - w^q}{|x|^p + \delta} \quad \text{in } D.$$

Multiplying the equation by $(u - w)^+$ and integrating on D we obtain

$$\begin{aligned} &\int_D < |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla(u-w)^+ > dx \\ &\leq \int_D \frac{u^q - w^q}{|x|^p + \delta}(u-w)^+ dx. \end{aligned} \tag{6.26}$$

- If $p > 2$, thanks to Lemma 4, the left hand side of (6.26) becomes

$$\begin{aligned} &C_p \int_D |\nabla(u-w)^+|^p dx \\ &\leq \int_D < |\nabla u|^{p-2}\nabla u - |\nabla w|^{p-2}\nabla w, \nabla(u-w)^+ > dx, \end{aligned} \tag{6.27}$$

for some positive constant C_p depending on p. Using Lagrange's Theorem, the right hand side of (6.26) satisfies

$$\begin{aligned}&\int_D \frac{u^q-w^q}{|x|^p+\delta}(u-w)^+dx \le C\int_D[(u-w)^+]^2\frac{u^{q-1}}{|x|^p+\delta}dx\\ &=\int_D[(u-w)^+]^2\frac{u^{[q-(p-1)]+(p-2)}}{|x|^p+\delta}dx,\end{aligned}\tag{6.28}$$

since $q>1$ in this case.
Being $\frac{(N-p)(p-2)}{Np}+\frac{p}{N}+2\frac{N-p}{Np}=1$, by Hölder's inequality we obtain

$$\begin{aligned}&\int_D[(u-w)^+]^2\frac{u^{[q-(p-1)]+(p-2)}}{|x|^p+\delta}dx\\ &\le\Big(\int_D[(u-w)^+]^{\frac{Np}{N-p}}dx\Big)^{\frac{2(N-p)}{Np}}\Big(\int_D\frac{u^{[q-(p-1)]\frac{N}{p}}}{(|x|^p+\delta)^{\frac{N}{p}}}dx\Big)^{\frac{p}{N}}\Big(\int_D u^{\frac{pN}{N-p}}dx\Big)^{\frac{(N-p)(p-2)}{Np}}.\end{aligned}\tag{6.29}$$

By Sobolev's inequality and by the definition of $I_\delta(u)$, we get

$$\begin{aligned}&\int_D[(u-w)^+]^2\frac{u^{[q-(p-1)]+(p-2)}}{|x|^p+\delta}dx\\ &\le C||\nabla(u-w)^+||^2_{L^p(D)}(I_\delta(u))^{\frac{p}{N}}S||u||^{p-2}_{W_0^{1,p}(D)}.\end{aligned}$$

Therefore, by (6.27), (6.28), (6.29) and taking into account the estimation in (6.21), one has

$$\int_D|\nabla(u-w)^+|^pdx\le C||\nabla(u-w)^+||^2_{L^p(\Omega)}(I_\delta(u))^{\frac{p}{N}},$$

that is

$$||\nabla(u-w)^+||_{L^p(\Omega)}\le C(I_\delta(u))^{\frac{p}{N(p-2)}}.\tag{6.30}$$

- If $p\le 2$, by classical estimates which characterize the *p-Laplacian* operator (see Lemma 4), we get

$$\begin{aligned}&C_p\int_D\frac{|\nabla(u-w)^+|^2}{(|\nabla u|+|\nabla w|)^{2-p}}dx\\ &\le\int_D<|\nabla u|^{p-2}\nabla u-|\nabla w|^{p-2}\nabla w,\nabla(u-w)^+>dx,\end{aligned}\tag{6.31}$$

with $C_p=C_p(p)$ a positive constant.

Being

$$\int_D |\nabla(u-w)^+|^p dx = \int_D \frac{|\nabla(u-w)^+|^p}{(|\nabla u| + |\nabla w|)^{\frac{(2-p)p}{2}}} (|\nabla u| + |\nabla w|)^{\frac{(2-p)p}{2}} dx,$$

by Hölder's inequality it follows

$$\int_D |\nabla(u-w)^+|^p dx \le \Big(\int_D \frac{|\nabla(u-w)^+|^2}{(|\nabla u| + |\nabla w|)^{(2-p)}} dx\Big)^{\frac{p}{2}} \Big(\int_D (|\nabla u| + |\nabla w|)^p dx\Big)^{\frac{2-p}{2}}.$$

Hence,

$$\int_D \frac{|\nabla(u-w)^+|^2}{(|\nabla u| + |\nabla w|)^{(2-p)}} dx \ge \Big(\frac{\int_D |\nabla(u-w)^+|^p}{(\int_D (|\nabla u| + |\nabla w|)^p)^{\frac{2-p}{2}}} dx\Big)^{\frac{2}{p}}. \quad (6.32)$$

We estimate the right hand side of (6.26) as

$$\begin{aligned}&\int_D \frac{u^q - w^q}{|x|^p + \delta}(u-w)^+ dx \\ &\le C \int_D (u-w)^+ \frac{u^q}{|x|^p + \delta} dx = C \int_D (u-w)^+ \frac{u^{[q-(p-1)]+(p-1)}}{|x|^p + \delta} dx.\end{aligned} \quad (6.33)$$

Since $\frac{(N-p)(p-1)}{Np} + \frac{p}{N} + \frac{N-p}{Np} = 1$, by Hölder's inequality we get in this case

$$\begin{aligned}&\int_D (u-w)^+ \frac{u^{[q-(p-1)]+(p-1)}}{|x|^p + \delta} dx \\ &\le \Big(\int_D [(u-w)^+]^{\frac{Np}{N-p}} dx\Big)^{\frac{N-p}{Np}} \Big(\int_D \frac{u^{[q-(p-1)]\frac{N}{p}}}{(|x|^p + \delta)^{\frac{N}{p}}} dx\Big)^{\frac{p}{N}} \Big(\int_D u^{\frac{Np}{N-p}} dx\Big)^{\frac{(N-p)(p-1)}{Np}}\end{aligned}$$

and then, using (6.31), (6.32), (6.33) and the estimation for the norm of u, we have

$$\Big(\frac{\int_D |\nabla(u-w)^+|^p}{(\int_D (|\nabla u| + |\nabla w|)^p)^{\frac{2-p}{2}}} dx\Big)^{\frac{2}{p}} \le C ||\nabla(u-w)^+||_{L^p(\Omega)} (I_\delta(u))^{\frac{p}{N}}. \quad (6.34)$$

If $||\nabla(u-w)^+||_{L^p(\Omega)} = 0$ we are done. Suppose that $||\nabla(u-w)^+||_{L^p(\Omega)} > 0$. From (6.34) we have

$$||\nabla(u-w)^+||_{L^p(\Omega)} \le C(I_\delta(u))^{\frac{p}{N}}. \quad (6.35)$$

By (6.30) and (6.35) we get the contradiction since $I_\delta(u) \to 0$ as $\theta \to 0$, because the left hand side of (6.30) and (6.35) is strictly positive and the right hand side is going to zero.

The following lemma shows that for ε small enough, the contribution due to the penalization is actually zero. This is a necessary result to pass to the limit and to have a solution of our problem.

Lemma 14. *There exists $\varepsilon_0 > 0$ such that*

$$I_\delta(u_{\varepsilon,\delta}) < \theta \quad \text{for all } 0 < \varepsilon \le \varepsilon_0 \text{ and all } 0 < \delta \le \varepsilon_0.$$

If we prove that, we would have

$$E_{\delta,\varepsilon,\theta}(u_{\varepsilon,\delta}) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+)^{q+1}}{|x|^p+\delta} dx.$$

Proof. We argue by contradiction. Assume that there are sequences of positive numbers $\varepsilon_n \to 0$, $\delta_n \to 0$ such that $I_{\delta_n}(u_{\varepsilon_n,\delta_n}) \ge \theta$. Let us write $u_n = u_{\varepsilon_n,\delta_n}$. By (6.21), for some subsequence, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega_1)$ weakly and, by Rellich's Theorem, it converges strongly in $L^r(\Omega_1)$ with $r < p^*$; in particular, since $q < p^* - 1$ by hypothesis, $u_n \to u$ strongly in $L^{q+1}(\Omega_1)$. Moreover, $u \le w$ in D as we saw before.

Let us show that

$$I_{\delta_n}(u_n) \to I_0(u) \quad \text{as } n \to \infty, \tag{6.36}$$

where

$$I_{\delta_n}(u_n) = \int_{\Omega_1} \frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} dx \quad \text{and} \quad I_0 = \int_{\Omega_1} \frac{(u^+)^{(q-(p-1))\frac{N}{p}}}{|x|^N} dx.$$

Notice that $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \to \dfrac{(u^+)^{(q-(p-1))\frac{N}{p}}}{|x|^N}$ pointwise in Ω_1 and since we are far from zero, $\dfrac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}}$ is uniformly bounded in $\Omega_1 \setminus D$. Since $u \le w$ in D and by the definition of $g_{\delta_n}(u_n+\delta_n)$, the following inequalities hold

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \le \frac{g_{\delta_n}(w+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \le C\frac{(w+\delta_n)^{(q-(p-1))\frac{N}{p}}}{(|x|^p+\delta_n)^{\frac{N}{p}}}.$$

Since $w \le C|x|$,

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \le C\frac{(|x|+\delta_n)^{(q-(p-1))\frac{N}{p}}}{(|x|^p+\delta_n)^{\frac{N}{p}}}.$$

For δ_n small,

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \le C\frac{(|x|+\delta_n^{\frac{1}{p}})^{(q-(p-1))\frac{N}{p}}}{(|x|+\delta_n^{\frac{1}{p}})^N} \le C(|x|+\delta_n^{\frac{1}{p}})^{(q-(p-1))\frac{N}{p}-N}.$$

If $(q-(p-1))\frac{N}{p} - N \ge 0$ this quantity is uniformly bounded and if $(q-(p-1))\frac{N}{p} - N < 0$, one has

$$\frac{g_{\delta_n}(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}} \le \frac{C}{(|x|+\delta_n^{\frac{1}{p}})^{N-(q-(p-1))\frac{N}{p}}} \le C|x|^{(q-(p-1))\frac{N}{p}-N},$$

which is integrable because since $q > p-1$, $-(q-(p-1))\frac{N}{p} + N < N$. By the *Dominated Convergence Theorem* (since $|x|^{(q-(p-1))\frac{N}{p}-N}$ is independent on n), we deduce the validity of (6.36). As a consequence of (6.36) and (6.22), we have

$$I_0(u) \le C\theta. \tag{6.37}$$

We claim that u satisfies

$$\begin{cases} -\Delta_p u + (q-(p-1))\frac{N}{p}\eta_\theta'(I_0(u))\frac{\chi_{[u>0]}}{|x|^N}u^{(q-(p-1))\frac{N}{p}-1} \le \frac{(u^+)^q}{|x|^p} & \text{in } \Omega_1, \\ u = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{6.38}$$

Consider $\varphi \in \mathcal{C}^1(\overline{\Omega}_1)$, $\varphi \ge 0$ with $\varphi = 0$ on $\partial\Omega_1 \cap \partial\Omega_\varepsilon$. Multiplying (6.9) by φ and integrating by parts in Ω_1 it yields

$$\begin{aligned} &-\Big|\varphi|\nabla u_n|^{p-2}\nabla u_n\Big|_{\partial\Omega_1} + \int_{\Omega_1} < |\nabla u_n|^{p-2}\nabla u_n, \nabla\varphi > dx \\ &+ \eta_\theta'(I_{\delta_n}(u_n))\int_{\Omega_1}\frac{g_{\delta_n}'(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}}\varphi dx = \int_{\Omega_1}\frac{(u_n^+)^q}{|x|^p+\delta_n}\varphi dx. \end{aligned}$$

Since $u \equiv 0$ in $\partial\Omega_1$, we only need to evaluate the first term of the last expression in the intersection of the boundary of C_ε and the boundary of Ω_1, where the measure is ε^{N-1}. Since we are away from zero, $u_n \in C^1$ and then, $\nabla u_n = C$, thus,

$$\begin{aligned} &\int_{\Omega_1} < |\nabla u_n|^{p-2}\nabla u_n, \nabla\varphi > dx - C\varepsilon^{N-1} + \eta_\theta'(I_{\delta_n}(u_n))\int_{\Omega_1}\frac{g_{\delta_n}'(u_n+\delta_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}}\varphi dx \\ &= \int_{\Omega_1}\frac{(u_n^+)^q}{|x|^p+\delta_n}\varphi dx. \end{aligned}$$

We remark that the term $C\varepsilon^{N-1}$ comes from the integration by parts and C is a constant that does not depend on ε. In fact, away from the origin, problem (6.9) has $C^{1,\alpha}_{loc}$ solutions. As before,

$$\int_{\Omega_1} \frac{(u_n^+)^q}{|x|^p+\delta_n}\varphi dx \to \int_{\Omega_1} \frac{(u^+)^q}{|x|^p}\varphi dx \quad \text{as } n\to\infty.$$

Using Fatou's lemma we conclude (6.38),

$$(q-(p-1))\frac{N}{p}\int_{\Omega_1}\frac{\chi_{[u>0]}u^{(q-(p-1))\frac{N}{p}-1}}{|x|^N}\varphi dx \le \int_{\Omega_1}\frac{g'_{\delta_n}(u_n)}{(|x|^p+\delta_n)^{\frac{N}{p}}}\varphi dx.$$

Multiplying (6.38) by u^+, integrating on Ω_1 and using Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned}
&\int_{\Omega_1}|\nabla u^+|^p dx\\
&\le \int_{\Omega_1}\frac{(u^+)^{q+1}}{|x|^p}dx \le C\left(\int_{\Omega_1}(u^+)^{p^*}dx\right)^{\frac{N-p}{N}}\left(\int_{\Omega_1}\frac{(u^+)^{(q-(p-1))\frac{N}{p}}}{|x|^N}dx\right)^{\frac{p}{N}} + C\varepsilon^{N-1}\\
&\le C\int_{\Omega_1}|\nabla u^+|^p I_0(u)^{\frac{p}{N}}dx + C\varepsilon^{N-1}
\end{aligned}$$

and by (6.37) we have

$$\int_{\Omega_1}|\nabla u^+|^p dx \le C\theta^{\frac{p}{N}}\int_{\Omega_1}|\nabla u^+|^p dx + C\varepsilon^{N-1}.$$

For a fixed $\theta>0$ sufficiently small we conclude that $u^+\equiv 0$ in Ω_1, since we can choose ε small as we like. Therefore, $I_0(u)=0$, which is a contradiction with $I_0(u)=\lim_{n\to\infty} I_{\delta_n}(u_n)\ge\theta$.

2.2.6 The end of the proof.

To conclude the proof we pass to the limit. In this way we are able to get a solution to (6.3) for ε small enough.

If $0<\varepsilon\le\varepsilon_0$ and $0<\delta\le\varepsilon_0$ the *mountain pass solution* $u_{\varepsilon,\delta}$ of Lemma 12 satisfies

$$\begin{cases} -\Delta_p u = \frac{(u^+)^q}{|x|^p+\delta} & \text{in } \Omega_\varepsilon\\ u=0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{6.39}$$

Then, $u_{\varepsilon,\delta}\ge 0$.

Since by definition, $E(0)=0$ and recalling that, by Lemma 13,

$$E_{\delta,\varepsilon,\theta}(u_{\varepsilon,\delta}) \ge c_0 > 0,$$

thus, $u_{\varepsilon,\delta} \not\equiv 0$ and by the strong maximum principle [94], $u_{\varepsilon,\delta} > 0$ in Ω_ε.

Now, for fixed $0 < \varepsilon \le \varepsilon_0$ we let $\delta \to 0$. Since $u_{\varepsilon,\delta_n} \le w$ we can apply the *Dominated Convergence Theorem* to show that

$$\int_{\Omega_\varepsilon} \frac{u^q_{\varepsilon,\delta_n}}{|x|^p + \delta_n}\varphi dx \to \int_{\Omega_\varepsilon} \frac{u^q_\varepsilon}{|x|^p}\varphi dx \quad \text{as } \delta_n \to 0 \quad \text{for any } \varphi \in \mathcal{C}^1(\overline{\Omega}_\varepsilon).$$

Thus, by the continuity of $-\Delta_p^{-1}(\cdot)$, u_ε satisfies

$$\begin{cases} -\Delta_p u = \frac{u^q}{|x|^p} & \text{in } \Omega_\varepsilon \\ u \ge 0 & \text{in } \Omega_\varepsilon \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Multiplying (6.39) by u_{ε,δ_n} we find that,

$$\int_{\Omega_\varepsilon} (-\Delta_p u_{\varepsilon,\delta_n}) u_{\varepsilon,\delta_n} dx = \int_{\Omega_\varepsilon} u_{\varepsilon,\delta_n} \frac{(u^+_{\varepsilon,\delta_n})^q}{|x|^p + \delta_n} dx.$$

Then,

$$\int_{\Omega_\varepsilon} |\nabla u_{\varepsilon,\delta_n}|^p dx = \int_{\Omega_\varepsilon} \frac{(u^+_{\varepsilon,\delta_n})^{q+1}}{|x|^p + \delta_n} dx.$$

By definition we know that $E_{\delta_n,\varepsilon,\theta}(u_{\varepsilon,\delta_n})$ satisfies

$$E_{\delta_n,\varepsilon,\theta}(u_{\varepsilon,\delta_n}) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u_{\varepsilon,\delta_n}|^p dx - \frac{1}{q+1}\int_{\Omega_\varepsilon} \frac{(u^+_{\varepsilon,\delta_n})^{q+1}}{|x|^p + \delta_n} dx.$$

Therefore,

$$E_{\delta_n,\varepsilon,\theta}(u_{\varepsilon,\delta_n}) = \left(\frac{1}{p} - \frac{1}{q+1}\right)\int_{\Omega_\varepsilon} \frac{u^{q+1}_{\varepsilon,\delta_n}}{|x|^p + \delta_n} dx$$

and, by *Dominated Convergence Theorem*, using the fact that $u_{\varepsilon,\delta_n} \le w$ in D, we see that

$$E_{\delta_n,\varepsilon,\theta}(u_{\varepsilon,\delta_n}) \to \left(\frac{1}{p} - \frac{1}{q+1}\right)\int_{\Omega_\varepsilon} \frac{u^{q+1}_\varepsilon}{|x|^p} dx \quad \text{as } n \to \infty.$$

Since $E_{\delta_n,\varepsilon,\theta}(u_{\varepsilon,\delta_n}) \ge c_0 > 0$ by Lemma 13 we deduce that $u_\varepsilon > 0$. This concludes the proof of Theorem 23.

□

Remark 7. *For the same reasons as in subsection 2.2.4, since $u_{\varepsilon,\delta_n} \le w$ uniformly on ε and δ, the energy solution u to* (6.3) *belongs to $C^{0,1}(\overline{\Omega_\varepsilon}) \cap C^{1,\alpha}(\Omega_\varepsilon)$.*

3 The problem with $g(\lambda, x, u) = \lambda f(x) u^r$

In this Section we are goint to prove Theorem 24, the aim is to study the existence of solution to the problem

$$\begin{cases} -\Delta_p u = \dfrac{u^q}{|x|^p} + \lambda f(x) u^r & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.40}$$

where $1 < p < N$, $q > p-1$, $f \gneq 0$ belongs to $L^\infty(\Omega)$, $0 \leq r < p-1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$. The aim is to find a solution that does not depend on the geometry of the domain. The idea is to construct a super- and subsolution ordered and then, by iteration, to get a non decreasing sequence of solutions $\{u_k\}_{k\geq 1}$ uniformly bounded by the supersolution. Notice that we do not assume any bound (from above) for the exponent q.

This result can be considered as a generalization of the one in Chapter 5. In the same way, we include the concave term in the equation in order to avoid the geometry restriction for the existence of solution.

To prove Theorem 24, we proceed step by step in order to be clear.

Step 1: Construction of a supersolution $\overline{u}$ to (6.40). Consider $d_\Gamma = dist(x, \Gamma)$, with $x \in \overline{\Omega}$, $0 \in \Gamma \subset \partial\Omega$ and Γ a regular submanifold of the boundary. Let w_1 be the solution to

$$\begin{cases} -\Delta_p w = \dfrac{d_\Gamma^q}{|x|^p} + f(x) w^r & \text{in } \Omega \\ w \geq 0 & \text{in } \Omega \\ w = d_\Gamma & \text{on } \partial\Omega, \end{cases} \tag{6.41}$$

Since $f \in L^\infty(\Omega)$, $q > p-1$ and $0 \leq r < p-1$, by standard regularity theory [42], [51],[93], we have $w_1 \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$ and thus, $w_1 \leq C d_\Gamma$, where $C = C(\Omega, p, q, r, f)$ is a positive constant.

Let $\overline{u}_\lambda = T w_1$, with T a positive parameter. Then,

$$\begin{aligned} -\Delta_p \overline{u}_\lambda &= T^{p-1} \frac{d_\Gamma^q}{|x|^p} + T^{p-1} f(x) w_1^r \geq T^{p-1} \frac{w_1^q}{C^q |x|^p} + T^{p-1} f(x) w_1^r \\ &= T^{p-1} \frac{\overline{u}_\lambda^q}{T^q C^q |x|^p} + \frac{T^{p-1}}{T^r} f(x) \overline{u}_\lambda^r. \end{aligned}$$

In order to get a supersolution we want that $-\Delta_p \overline{u}_\lambda \geq \dfrac{\overline{u}_\lambda^q}{|x|^p} + \lambda f(x) \overline{u}_\lambda^r$.

Therefore, $\exists T = T(\lambda)$ such that if

$$T \geq \lambda^{\frac{1}{p-1-r}} \qquad \text{and} \qquad T \leq \left(\frac{1}{C^q}\right)^{\frac{1}{q-(p-1)}},$$

$\overline{u}_\lambda$ is a supersolution to (6.40).

Hence, there exists $\Lambda > 0$ such that $\forall \lambda \in (0, \Lambda]$ we have a supersolution $\overline{u}_\lambda \in C^{0,1}(\overline{\Omega}) \cap C^1(\Omega)$, with $\Lambda = C^{\frac{-q(p-1-r)}{q-(p-1)}}$. Let us denote $\overline{u} = \overline{u}_\lambda$.

Step 2: Construction of a subsolution $\underline{u}$ to (6.40).

To construct the subsolution of (6.40) we are going to use the following Theorem

Theorem 26. *Let ρ a non-negative bounded function such that $\rho(x) \neq 0$, the function $\frac{f(t)}{t^{p-1}}$ decreasing and consider Ω a bounded domain. Then, the problem*

$$\begin{cases} -\Delta_p u = \rho(x) f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution.

The existence result is given by minimization and the proof of the uniqueness can be seen in [5].

Therefore, we can consider v be the (unique) solution to

$$\begin{cases} -\Delta_p v = \lambda f(x) v^r & \text{in } \Omega \\ v \geq 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.42}$$

Actually, it is easy to see that v is, in fact, a subsolution of (6.40) and, by regularity results, see [70], belongs to $C^{1,\beta}(\overline{\Omega})$. Let us recall $v = \underline{u}$.

Step 3: Comparison and iteration argument. To prove that $\overline{u} \geq \underline{u}$ in Ω, we note that $-\Delta_p \overline{u} \geq \lambda f(x) \overline{u}^r$, being $\overline{u}$ a supersolution to (6.40). We argue as in the comparison proof in [5]. Thanks to the definition of $\underline{u}$, one can write

$$\frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} + \frac{\Delta_p \underline{u}}{\underline{u}^{p-1}} \geq \lambda f(x) \left(\frac{\overline{u}^r}{\overline{u}^{p-1}} - \frac{\underline{u}^r}{\underline{u}^{p-1}}\right).$$

Multiplying the last expression by $(\underline{u}^p - \overline{u}^p)^+$ we get that

$$\int_\Omega \left(\frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} + \frac{\Delta_p \underline{u}}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx \geq \lambda f(x) \int_\Omega \left(\frac{\overline{u}^r}{\overline{u}^{p-1}} - \frac{\underline{u}^r}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx$$
$$= \lambda f(x) \int_{\Omega \cap \{\underline{u} \geq \overline{u}\}} \left(\frac{\overline{u}^r}{\overline{u}^{p-1}} - \frac{\underline{u}^r}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx. \tag{6.43}$$

Integrating by parts, the left hand side of (6.43) becomes

$$\int_\Omega \left(\frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} + \frac{\Delta_p \underline{u}}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx$$
$$= \int_{\Omega \cap \{\underline{u} \geq \overline{u}\}} \left(p\frac{\underline{u}^{p-1}}{\overline{u}^{p-1}} |\nabla \overline{u}|^{p-2} \langle \nabla \overline{u}, \nabla \underline{u} \rangle - (p-1)\frac{\underline{u}^p}{\overline{u}^p} |\nabla \overline{u}|^p - |\nabla \overline{u}|^p\right) dx$$
$$+ \int_{\Omega \cap \{\underline{u} \geq \overline{u}\}} \left(p\frac{\overline{u}^{p-1}}{\underline{u}^{p-1}} |\nabla \underline{u}|^{p-2} \langle \nabla \underline{u}, \nabla \overline{u} \rangle - (p-1)\frac{\overline{u}^p}{\underline{u}^p} |\nabla \underline{u}|^p - |\nabla \underline{u}|^p\right) dx.$$

Since $\overline{u} > 0$ and $\underline{u} > 0$, by a generalized Picone's inequality (see Theorem 12), we have

$$0 \geq \int_\Omega \left(\frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} + \frac{\Delta_p \underline{u}}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx. \tag{6.44}$$

We point out that $g(t) = \frac{t^r}{t^{p-1}} \downarrow$ since $r < p-1$. Then, as a consequence of $f \gneq 0$, the following term turns to be non-negative in the set $\Omega \cap \{\underline{u} \geq \overline{u}\}$

$$\lambda \int_{\Omega \cap \{\underline{u} \geq \overline{u}\}} f(x) \left(\frac{\overline{u}^r}{\overline{u}^{p-1}} - \frac{\underline{u}^r}{\underline{u}^{p-1}}\right)(\underline{u}^p - \overline{u}^p)^+ dx \geq 0. \tag{6.45}$$

Equations (6.44) and (6.45) imply that $(\underline{u} - \overline{u})^+ \equiv 0$ in Ω. Therefore, $\underline{u} \leq \overline{u}$.

We are going to define now the iterative problems in order to pass to the limit and to get the desired solution.

Let u_1 be the solution to

$$\begin{cases} -\Delta_p u_1 = \frac{\underline{u}^q}{|x|^p} + \lambda f(x) \underline{u}^r & \text{in } \Omega \\ u_1 \geq 0 & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.46}$$

Since $\underline{u}$ is a subsolution, one has

$$-\Delta_p \underline{u} \leq \frac{\underline{u}^q}{|x|^p} + \lambda f(x) \underline{u}^r = -\Delta_p u_1 \leq \frac{\overline{u}^q}{|x|^p} + \lambda f(x) \overline{u}^r \leq -\Delta_p \overline{u}.$$

By the *weak comparison principle*, as we saw before, we get $\underline{u} \leq u_1 \leq \overline{u}$. Therefore, using the Sattinger method (see in [85]), we construct a sequence $\{u_k\}_{k\geq 1}$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} -\Delta_p u_{k+1} = \frac{u_k^q}{|x|^p} + \lambda f(x) u_k^r & \text{in } \Omega \\ u_{k+1} \geq 0 & \text{in } \Omega \\ u_{k+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.47}$$

with $\underline{u} \leq u_1 \leq u_2 \leq ... \leq u_k \leq ... \leq \overline{u}$, $\forall k \geq 1$. In particular, $\forall x \in \Omega$, $\{u_k(x)\}_{k\geq 1}$ is a non decreasing sequence which is bounded and therefore, it converges. Thus, we are able to define the limit $u_\lambda(x) = \lim_{k\to\infty} u_k(x)$, $\forall x \in \Omega$. By *Dominated Convergence Theorem*, the right hand side of (6.47) converges to $\frac{u_\lambda^q}{|x|^p} + \lambda f(x) u^r$ in $L^1(\Omega)$, $\forall q > 0$. Finally, we pass to the limit in (6.47) using the continuity of $-\Delta_p^{-1}(\cdot)$.

The solution u_λ is a *minimal solution* in the sense that any other solution $\tilde{u}$ to (6.40) verifies $u_\lambda \leq \tilde{u}$. To prove that, it is needed to repeat the above argument using $\tilde{u}$ as a supersolution.

Step 4: Nonexistence for λ large. We show the following claim:

There exists $\tilde{\lambda}$ such that, $\forall \lambda \in [\tilde{\lambda}, \infty)$, problem (6.40) *has no solution $u \in W_0^{1,p}(\Omega)$.*

Consider the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda f(x)^{\frac{q-(p-1)}{q}} |u|^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.48}$$

We are going to proceed by contradiction. Let us suppose that $u_\lambda \in W_0^{1,p}(\Omega)$ is a solution to problem (6.40). Let us see that in weak sense the following inequality holds,

$$\begin{aligned} &-\Delta_p u_\lambda = \frac{u_\lambda^q}{|x|^p} + \lambda f(x) u_\lambda^r \geq C u_\lambda^q + \lambda f(x) u_\lambda^r \\ &\geq (\lambda_1 + \varepsilon) f(x)^{\frac{q-(p-1)}{q}} u_\lambda^{p-1} \qquad \text{in } \Omega, \end{aligned} \tag{6.49}$$

where $C = \inf_{x\in\Omega} \dfrac{1}{|x|^p}$ and λ_1 is the first eigenvalue of problem (6.48). Notice that in the points where $f(\cdot)$ is equal to zero, (6.49) is true for every λ since

$q > p-1$. On the other hand, by straightforward calculations in the set $\{x \in \Omega : f(x) > 0\}$, it is possible to show that there exists λ such that

$$\min_{t>0} \Phi_\lambda(t) := Ct^{q-(p-1)} + \lambda f(x) t^{r+1-p} \geq (\lambda_1 + \varepsilon) f(x)^{\frac{q-(p-1)}{q}}.$$

We now fix $\varepsilon > 0$ small such that $\lambda_1 + \varepsilon < \lambda_2$, the second eigenvalue of (6.48) and consider $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$. Moreover, by Hopf's Lemma, near the zero there exist space between the solution u_λ and the boundary, so we can put the eigenfunction φ_1 under the solution u_λ, hence, there exists $\delta > 0$ small enough such that $v_1 = \delta^{1/(p-1)}\varphi_1 \leq u_\lambda$, where φ_1 be the first positive eigenfunction to (6.48) associated with the eigenvalue λ_1. Thus, using (6.49) we have

$$\begin{aligned} -\Delta_p v_1 = \lambda_1 f(x)^{\frac{q-(p-1)}{q}} v_1^{p-1} \leq \mu f(x)^{\frac{q-(p-1)}{q}} v_1^{p-1} \leq \\ \mu f(x)^{\frac{q-(p-1)}{q}} u_\lambda^{p-1} \leq (\lambda_1 + \varepsilon) f(x)^{\frac{q-(p-1)}{q}} u_\lambda^{p-1} \leq -\Delta_p u_\lambda, \end{aligned}$$

that is, by *weak comparison principle*, $v_1 \leq u_\lambda$ and moreover u_λ, v_1 are super- and subsolution to

$$\begin{cases} -\Delta_p u = \mu f(x)^{\frac{q-(p-1)}{q}} u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.50}$$

with $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$.

A standard iteration argument proves that problem (6.50) has a positive solution, which is a contradiction with the isolation of λ_1 (see [17]) being ε arbitrary.

Step 5: The maximal interval of existence.

We define

$$\lambda_{\max} = \sup\{\lambda \in \mathbb{R}^+ \ : \ (6.40) \ \text{ has a non trivial solution}\}.$$

Obviously, previous steps imply $\lambda_{\max} > 0$ and $\lambda_{\max} < \infty$. Moreover if $\lambda \in (0, \lambda_{\max})$, we can find λ^* such that $\lambda < \lambda^*$ and problem (6.40) has a solution for λ^*, which is a supersolution to (6.40) for λ. As in Step 3 we find a solution for such λ.

At the end, notice that for $0 < \lambda_1 < \lambda_2 \leq \Lambda$ we have that the solution u_{λ_2} is a supersolution to problem (6.40) for λ_1. Hence, $-\Delta_p u_{\lambda_1} \leq -\Delta_p u_{\lambda_2}$ and by the *weak comparison principle* we have $u_{\lambda_1} \leq u_{\lambda_2}$.

$\square$

Remark 8. *Since* $\overline{u} \le C d_\Gamma$, *the right hand side of* (6.40) *belongs to some* $L^r(\Omega)$ *with* $r > N$. *Then, by Lemma 10, the solution* u_λ *of* (6.40) *belongs to* $L^\infty(\Omega)$ *and hence* $u \in C^{0,1}(\overline{\Omega}) \cap C^{1,\alpha}(\Omega)$.

In the next proposition we show that the minimal solution u_λ is, in fact, the minimizer of the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q+1}\int_\Omega \frac{u^q}{|x|^p} dx - \frac{\lambda}{r+1}\int_\Omega f(x)u^{r+1}dx, \qquad (6.51)$$

defined on

$$K = \{v \in W_0^{1,p}(\Omega) : \underline{u} \le v \le u_\lambda\}. \qquad (6.52)$$

Thanks to the supersolution $\overline{u}$ (as in Step 1 above), the functional (6.51) is well defined in the closed and convex set K. Then, there exists $u \in K$ such that

$$J(u) = \min_{v \in K} J(v).$$

Proposition 7. *The minimal solution* u_λ *is the minimizer of* $J(v)$.

Proof. By the definition of K, it is sufficient to prove that $u_\lambda \le u$.

Let u_1 be the solution to (6.47) corresponding to $k = 1$. Let us define $v = u + (u_1 - u)^+$ which belongs to K. Then, by definition of minimizer u we have

$$\int_\Omega < |\nabla u|^{p-2}\nabla u, \nabla(u_1 - u)^+ > dx \ \ \ge \int_\Omega \Big(\frac{u^q}{|x|^p} + \lambda f(x)u^r\Big)(u_1 - u)^+ dx$$

and since u_1 verifies (6.46),

$$\int_\Omega < |\nabla u_1|^{p-2}\nabla u_1, \nabla(u_1 - u)^+ > dx \ \ = \int_\Omega \Big(\frac{\underline{u}^q}{|x|^p} + \lambda f(x)\underline{u}^r\Big)(u_1 - u)^+ dx.$$

Therefore, subtracting the last two expressions, we obtain

$$\begin{aligned} &\int_\Omega < |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u|^{p-2}\nabla u, \nabla(u_1 - u)^+ > dx \\ &\le \int_\Omega (\frac{\underline{u}^q}{|x|^p} - \frac{u^q}{|x|^p})(\underline{u} - u)^+ dx. \end{aligned} \qquad (6.53)$$

Since,

$$\int_\Omega (\frac{\underline{u}^q}{|x|^p} - \frac{u^q}{|x|^p})(\underline{u} - u)^+ dx \le 0$$

and by in Lemma 4, $(\underline{u} - u)^+ = 0$.

An induction argument allows us to prove that $u_k \le u$, $\forall k \in \mathbb{N}$. Since $u_\lambda(x) = \lim_{k\to\infty} u_k(x)$ (see Step 3 above), we conclude $u_\lambda \equiv u$. ☐

Part III

Critical and supercritical parabolic problems with respect to the Hardy potential

Chapter 7

Critical and supercritial parabolic problems with $0 \in \partial\Omega$

1 Introduction

In this Chapter we are going to study the following parabolic problem

$$\begin{cases} u_t - \Delta u &= \lambda \dfrac{u^p}{|x|^2} & \text{in } \Omega_T = \Omega \times (0,T), \\ u &> 0 & \text{in } \Omega_T, \\ u(x,0) &= u_0(x) \geq 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{7.1}$$

where $p \geq 1$ and $\Omega \subset I\!R^N$ is a bounded domain such that $N \geq 3$ and $0 \in \partial\Omega$.

One of the main goals in this Chapter will be to emphasize the contrast with the case $0 \in \Omega$, which behaves in a deep different way.

If $0 \in \Omega$, Baras and Goldstein proved in [20] that if $p = 1$ and $\lambda > \Lambda_N$, the problem does not have distributional solution. More precisely, they established the following result

Theorem 27. *(Baras–Goldstein's Theorem)*

Consider the initial value problem with Dirichlet boundary data,

$$(P) \quad \begin{cases} u_t - \Delta u = \lambda \dfrac{u}{|x|^2} + g & \textit{if } x \in \Omega \subset I\!R^N,\ N \geq 3,\ t > 0,\ \lambda \in I\!R, \\ u(x,0) = f(x) & \textit{if } x \in \Omega,\ f \in L^2(\Omega), \\ u(x,t) = 0 & \textit{if } x \in \partial\Omega,\ t > 0, \end{cases}$$

where Ω is a domain such that $0 \in \Omega$. Then,

(i) *If $\lambda \leq \Lambda_N$, the problem (P) has a unique global solution if*

$$\int_\Omega |x|^{-\alpha_1} u(x,0)\, dx < \infty \quad \textit{and} \quad \int_0^T \int_\Omega |x|^{-\alpha_1} g \, dx\, dt < \infty,$$

with α the smallest root of $\alpha^2 - (N-2)\alpha + \lambda = 0$.

(ii) *If $\lambda > \Lambda_N$, the problem (P) has no local solution if $f > 0$.*

Indeed, if v_n is the solution of the truncated problem for $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$,

$$\begin{cases} u_t - \Delta u = \min\left\{n, \lambda \dfrac{u}{|x|^2}\right\} & \text{if } x \in \Omega \subset \mathbb{R}^N,\ N \geq 3,\ t > 0, \\ u(x,0) = f(x) & \text{if } x \in \Omega, \quad f \in L^2(\Omega), \\ u(x,t) = 0 & \text{if } x \in \partial\Omega,\ t > 0, \end{cases}$$

then, $\lim_{n\to\infty} v_n(x,t) = \infty$, for every $(x,t) \in \Omega \times (0,\infty)$.

Moreover, in the supercritical problem, $p > 1$, independently of the value of $\lambda > 0$, a nonexistence result in distributional sense is obtained in [32], where it is also proved an instantaneous and complete blow-up result.

We prove in this Chapter that if $0 \in \partial\Omega$ and $p = 1$ there is no such Baras-Goldstein type result. Indeed, we find a unique global solution to (7.1) without restriction in the parameter λ and for all initial data in $L^1(\Omega)$.

We also prove in Section 3 that if $0 \in \partial\Omega$ and $p > 1$, the problem

$$\begin{cases} u_t - \Delta u &= \dfrac{u^p}{|x|^2} & \text{in } \Omega_T = \Omega \times (0,T), \\ u &= 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) & \text{in } \Omega, \end{cases} \tag{7.2}$$

has a bounded solution provided that $\bar{u} \in L^\infty(\Omega)$ is a suitable supersolution to the stationary problem (5.1) and $0 \leq u_0 \leq \bar{u}$. Moreover, this solution is unique in $L^\infty(\Omega_T)$. Notice that, in the parabolic problem we do not have any restriction on the shape of the domain in contrast with the Elliptic case (see the previous Part of this work and [50]).

We recall the wellknown Gronwall's inequality and some preliminary results that we are going to use in this Chapter.

Lemma 15. *(Gronwall's inequality) Let $\eta(.)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies a.e. the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t). \tag{7.3}$$

Where $\phi(t)$ and $\psi(t)$ are nonnegative summable functions on [0, T], then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right] \quad \text{for all } \ 0 \leq t \leq T. \tag{7.4}$$

See, for instance, [53].

Lemma 16. *Let Ω be a bounded domain such that $0 \in \partial\Omega$ and assume that $0 < q < 1$. Then, the problem*

$$\begin{cases} -\Delta w = \dfrac{w^q}{|x|^2} & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.5}$$

has a unique positive solution w such that $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

See [4] (even if $0 \in \Omega$) for a proof of existence of solution in the energy space. Notice that the uniqueness is a consequence of Brezis-Kamin comparison argument in [35]. The boundedness can be obtained considering a suitable $\mu > 0$ for which we have a solution u to the problem

$$\begin{cases} -\Delta u = \frac{u^q}{|x|^2} + \mu u^r & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $1 < r < \frac{N+2}{N-2}$.

In a similar way to [50] (see Lemma 2.2 in such reference), we define $v(y) = u(|x_0| + ry)$ with $r = \frac{|x_0|}{2}$ and $y \in \frac{(\Omega - x_0)}{r}$ such that the problem

$$-\Delta v = \frac{r^2 v^q}{(|x_0| + ry)^2} + \mu r^2 v^r.$$

Using the Gidas-Spruck estimates (see Theorem 1.1 in [63]), there exists a universal constant $C > 0$ such that, in particular, $v(0) \leq C$. Since u is a supersolution to problem (7.5), using the comparison argument by Brezis-Kamin in [35] we get that $w(x) \leq u(x)$ and then, $w \in L^\infty(\Omega)$.

The following Theorem is proved in [24], see also [79].

Theorem 28. *Suppose that $F \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$, then, the problem*

$$\begin{cases} u_t - \Delta u = F \ \text{ in } \Omega_T, \\ u = 0 \ \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \ \text{ in } \Omega, \end{cases} \tag{7.6}$$

has one and only one entropy solution $u \in C([0,T], L^1(\Omega))$, moreover, $u \in L^s(0,T; W_0^{1,s}(\Omega))$ for all $s < \frac{N+2}{N+1}$.

We point out that in this case the solution called *entropy solution* is equivalent to *solution obtained as limit of approximations* and to *distributional solution*. Thus, we will consider distributional solution obtained as limit of approximated problems that also provides this regularity.

All the results in this Chapter can be seen in the paper [19].

2 The critical problem: $p = 1$

2.1 Existence result

The main existence result of this Section is the following.

Theorem 29. *Let $\Omega \subset I\!\!R^N$ be a bounded domain and $0 \in \partial\Omega$. Assume that $u_0 \in L^1(\Omega)$ is a nonnegative function. Then, for all $\lambda > 0$, the problem*

$$\begin{cases} u_t - \Delta u = \lambda \dfrac{u}{|x|^2} & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{7.7}$$

has a unique solution u such that $\dfrac{u}{|x|^2} \in L^1(\Omega_T)$ and $u \in L^\sigma(0,T;W_0^{1,\sigma}(\Omega))$ for all $\sigma < \frac{N+2}{N+1}$.

Proof. Fixed $u_0 \in L^1(\Omega)$, consider a sequence of nonnegative functions $\{u_{0n}\}_n \subset L^\infty(\Omega)$ such that u_{0n} is increasing in n and $u_{0n} \uparrow u_0$ as $n \to \infty$ in $L^1(\Omega)$.

Consider $u_1 \equiv 0$, and define by recurrence, $u_n \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$, with $n > 1$, as the unique positive solution to the approximated problem

$$\begin{cases} (u_n)_t - \Delta u_n = \lambda \dfrac{u_{n-1}}{|x|^2 + \frac{1}{n}} & \text{in } \Omega_T, \\ u_n > 0 & \text{in } \Omega_T, \\ u(x,0) = u_{0n}(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{7.8}$$

The existence and the uniqueness of u_n and the fact that $\{u_n\}_n$ is an increasing sequence, are a consequence of the elementary results for the heat equation, see Theorem 28.

Thanks to Theorem 16, we can set $0 < q < \frac{1}{2}$ and consider w the unique positive solution to

$$\begin{cases} -\Delta w = \dfrac{w^q}{|x|^2} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.9}$$

Fixed $\lambda > 0$, for all $\varepsilon > 0$ there exists $C > 0$ such that,

$$\lambda < C|x|^2 + \varepsilon w^{q-1}(x) \text{ for all } x \in \Omega. \tag{7.10}$$

Indeed, in a neighborhood of 0 at the boundary $w = 0$ and $\forall K \subset\subset \Omega$ compact, $|x|^2 > C_1$, then it is sufficient to choose C large enough.

Using w as a test function in (7.8) and taking into account the expression (7.10), we get

$$\int_\Omega (u_n)_t w dx + \int_\Omega u_n(-\Delta w) dx \le C \int_\Omega u_n w dx + \varepsilon \int_\Omega \frac{w^q u_n}{|x|^2} dx.$$

By the definition of w and choosing $\varepsilon << 1$, it follows that

$$\int_\Omega (u_n)_t w dx + (1-\varepsilon) \int_\Omega u_n \frac{w^q}{|x|^2} dx \le C \int_\Omega u_n w dx.$$

Since $u_n, w > 0$,

$$\frac{d}{dt} \int_\Omega u_n w dx \le C \int_\Omega u_n w dx.$$

Applying Gronwall's inequality again(see Lemma 15),

$$\int_\Omega u_n w dx \le e^{CT} [\int_\Omega u_0 w dx].$$

Thus,

$$\int_\Omega (u_n)_t w dx + (1-\varepsilon) \int_\Omega u_n \frac{w^q}{|x|^2} dx \le e^{CT} [\int_\Omega u_0 w dx].$$

Integrating the last expression on time,

$$\int_\Omega u_n(x,T) w dx + (1-\varepsilon) \int_0^T \int_\Omega u_n \frac{w^q}{|x|^2} dx dt \le (T + \bar{C}) e^{CT} \int_\Omega u_0 w dx. \quad (7.11)$$

Therefore, taking into account that $u_0 \in L^1(\Omega)$ and $w \in L^\infty(\Omega)$,

$$\int_0^T \int_\Omega u_n \frac{w^q}{|x|^2} dx dt \le C_1(T) \text{ and } \int_\Omega u_n(x,T) w dx dt \le C_2(T).$$

Since $\{u_n\}_n$ is increasing in n, using the *Monotone Convergence Theorem*, we get the existence of a measurable function u such that $\dfrac{u_n w^q}{|x|^2} \to \dfrac{u w^q}{|x|^2}$ strongly in $L^1(\Omega_T)$, and then, $u_n \uparrow u$ strongly in $L^1_{loc}(\Omega_T)$.

We claim that $\{\dfrac{u_n}{|x|^2}\}_n$ is bounded in $L^1(\Omega_T)$. Indeed, we consider $\psi \in W_0^{1,2}(\Omega)$ as the unique positive bounded solution to the problem

$$-\Delta \psi = \frac{1}{|x|^2 + \varepsilon} \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \quad (7.12)$$

We define $\varphi = \psi^{\frac{1}{1-q}}$, and by a direct computation we obtain

$$-\Delta \varphi = \Big(\frac{1}{1-q}\Big) \psi^{\frac{q}{q-1}} (-\Delta \psi) = \Big(\frac{1}{1-q}\Big) \frac{\varphi^q}{|x|^2 + \varepsilon}.$$

Therefore,

$$-\Delta\varphi \leq \Big(\frac{1}{1-q}\Big)\frac{\varphi^q}{|x|^2}.$$

Hence, since $-\Delta w = \dfrac{w^q}{|x|^2}$ and $q < 1$, we can use the same arguments as in the proof of the uniqueness to Theorem 19 (that we used also in Chapter 5).

Therefore, get that $\varphi \leq Cw$ and then

$$\psi \leq Cw^{1-q}. \tag{7.13}$$

Using ψ as a test function in (7.8) we deduce that

$$\begin{aligned} &\frac{d}{dt}\int\limits_\Omega u_n\psi dx + \int\limits_\Omega u_n(-\Delta\psi)dx \\ &= \lambda\int\limits_\Omega \frac{u_{n-1}\psi}{|x|^2+\frac{1}{n}}dx \leq C\lambda\int\limits_\Omega \frac{u_n w^{1-q}}{|x|^2}dx. \end{aligned} \tag{7.14}$$

Therefore,

$$\frac{d}{dt}\int_\Omega u_n\psi dx + \int_\Omega \frac{u_n}{|x|^2+\varepsilon}dx \leq C\lambda\int_\Omega \frac{u_n w^{1-q}}{|x|^2}dx.$$

Taking into account that $q < \frac{1}{2}$ and using the fact that w is bounded, we obtain that $w^{1-q} \leq Cw^q$ because

$$1 \leq \frac{Cw^q}{w^{1-q}} \Leftrightarrow 1 \leq Cw^{2q-1}.$$

Since $q < \frac{1}{2}$, $1 \leq \frac{C}{w^a}$, with $a > 0$, then, $w \leq C$.

Thus, thanks to (7.14),

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_n\psi dx + \int_\Omega \frac{u_n}{|x|^2+\varepsilon}dx = \lambda\int_\Omega \frac{u_{n-1}\psi}{|x|^2+\frac{1}{n}}dx \\ &\leq C\lambda\int_\Omega \frac{u_n w^q}{|x|^2}dx \leq C(T). \end{aligned} \tag{7.15}$$

Integrating on time, we get

$$\int_\Omega u_n(x,T)\psi dx + \int_0^T\int_\Omega \frac{u_n}{|x|^2+\varepsilon}dxdt \leq C(T).$$

Hence, using Fatou's Lemma we can pass to the limit in ε and the claim follows. Therefore, we obtain that $\dfrac{u_n}{|x|^2} \in L^1(\Omega_T)$. The equation holds in a distributional sense and also as a *limit of approximations*. By Theorem 28 we get that $u \in \mathcal{C}((0,T),L^1(\Omega))$ and $u \in L^\sigma(0,T;W_0^{1,\sigma}(\Omega))$ for all $\sigma < \dfrac{N+2}{N+1}$.

Notice that estimate (7.11) ensures that u is globally defined on time.

It is clear that u is the minimal solution to problem (7.7), this follows considering another solution v, in particular, v is a supersolution and, using the classical comparison result as before, one can get that $u \le v$.

Let us prove now the uniqueness result.

We argue by contradiction. Suppose v is another solution. Since u is a minimal solution, $v \ge u$. Let us define the function $z = v - u$, then $\dfrac{z}{|x|^2} \in L^1(\Omega_T)$ and z satisfies

$$\begin{cases} z_t - \Delta z = \lambda \dfrac{z}{|x|^2} & \text{in } \Omega_T, \\ z(x,0) = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{7.16}$$

Using w, the solution of (7.5), as a test function in (7.16) and following the same computation as in the proof of the existence result, we get

$$\int_\Omega z_t w dx + (1-\varepsilon)\int_\Omega z \frac{w^q}{|x|^2} dx \le C \int_\Omega zw dx,$$

where $\varepsilon << 1$. Since $z, w \ge 0$,

$$\frac{d}{dt}\int_\Omega zw dx \le C \int_\Omega zw dx.$$

Using Gronwall's inequality as before, we obtain that $z(x,t)w(x)dx \le 0$ for all $t > 0$. Therefore, $z \equiv 0$ and the uniqueness follows. $\square$

Concerning the regularity of u we want to point out the following remarks. We consider $\mu(\Omega)$ as the Hardy constant for Ω defined in as

$$\mu(\Omega) = \inf\Big\{ \int_\Omega |\nabla \phi|^2 : \phi \in W_0^{1,2}(\Omega), \int_\Omega \frac{\phi^2}{|x|^2} = 1 \Big\}.$$

- If $\lambda < \mu(\Omega)$ and $u_0 \in L^2(\Omega)$,

$$\int_0^T \int_\Omega u_{nt} u_n dx dt + \int_0^T \int_\Omega u_n(-\Delta u_n) dx dt = \lambda \int_0^T \int_\Omega \frac{u_n^2}{|x|^2} dx dt.$$

  Thus,

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_n^2(x,T) dx + \int_0^T \int_\Omega |\nabla u_n|^2 dx dt \\ &\le \frac{\lambda}{\mu(\Omega)} \int_0^T \int_\Omega |\nabla u_n|^2 dx dt + \frac{1}{2}\int_\Omega u_0^2 dx, \end{aligned}$$

then,

$$\frac{1}{2}\int_\Omega u_n^2(x,T)dx + (1-\frac{\lambda}{\mu(\Omega)})\int_0^T\int_\Omega |\nabla u_n|^2 dxdt \le C.$$

Since $\lambda < \mu(\Omega)$, $\quad (1-\frac{\lambda}{\mu(\Omega)}) \ge 0$ and $u \in L^2(0,T;W_0^{1,2}(\Omega))$.
Notice that if $u_0 \in L^\infty(\Omega)$, we can find a supersolution $\overline{u} \in L^\infty(\Omega_T)$ such that $u_0 \le \overline{u}$ and we can prove that $u \in L^\infty(\Omega_T)$.

- If $\lambda \ge \mu(\Omega)$, we claim $\displaystyle\int_0^T\int_\Omega |\nabla u|^2 w dxdt < \infty$. To see that we use $u_n w$ as a test function in (7.8), obtaining

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_n^2 w dx + \int_\Omega |\nabla u_n|^2 w dx + \frac{3}{2}\int_\Omega \frac{u_n^2 w^q}{|x|^2}dxdt \le \lambda \int_\Omega \frac{u_n^2 w}{|x|^2}dxdt.$$

By (7.10), we have that

$$\lambda\frac{wu_n^2}{|x|^2} < Cwu_n^2 + \varepsilon\frac{w^q u_n^2}{|x|^2},$$

thus,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega u_n^2 w dx + \int_\Omega |\nabla u_n|^2 w dx + \frac{3}{2}\int_\Omega \frac{u_n^2 w^q}{|x|^2}dxdt\\ &\le \lambda\int_\Omega Cwu_n^2 dx + \int_\Omega \varepsilon\frac{w^q u_n^2}{|x|^2}dx,\end{aligned}$$

and choosing $\varepsilon < \frac{3}{2}$, it yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_n^2 w dx + \int_\Omega |\nabla u_n|^2 w dx + (\frac{3}{2}-\varepsilon)\int_\Omega \frac{u_n^2 w^q}{|x|^2}dxdt \le C\int_\Omega u_n^2 w dx.$$

Gronwall's inequality allows us to conclude that

$$\int_\Omega u_n^2(x,t) w dx \le \Big(\int_\Omega u_{0n}^2 w dx\Big)e^{Ct}, \qquad t > 0.$$

Therefore,

$$\int_0^T\int_\Omega |\nabla u_n|^2 w dxdt \le C(T).$$

Hence, the claim follows. Notice that, an iteration argument allows us to prove that, if $u_0 \in L^\infty(\Omega)$, $u \in L^\infty(D)$ for any compact set D in Ω_T.

In the next Subsection we are going to analyze the asymptotic behavior of the solution to problem (7.7) obtained in Theorem 29.

2.2 Asymptotic behavior

Theorem 30. *Let u be the solution to problem* (7.7) *found in Theorem 29, then*

1. *If $\lambda < \mu(\Omega)$, $u(x,t) \to 0$ in $L^1(\Omega)$ as $t \to \infty$*
2. *If $\lambda > \mu(\Omega)$, $u(x,t) \to \infty$ in $L^1(\delta(x)dx, \Omega)$ as $t \to \infty$,*

where $\mu(\Omega)$ is, as above, the Hardy constant for Ω defined in as

$$\mu(\Omega) = \inf\Big\{ \int_\Omega |\nabla \phi|^2 : \phi \in W_0^{1,2}(\Omega), \int_\Omega \frac{\phi^2}{|x|^2} = 1 \Big\},$$

and $\delta(x) = \min_{y \in \partial\Omega} \{|x-y|\}$.

Proof. Let u be the very weak solution to problem (7.7), then $\dfrac{u}{|x|^2} \in L^1(\Omega_T)$ for all $T < \infty$. We split the proof in the two cases:

Case 1: $\lambda < \mu(\Omega)$. We use u as a test function in (7.7),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega \frac{u^2}{|x|^2} dx \leq \frac{\lambda}{\mu(\Omega)} \int_\Omega |\nabla u|^2 dx.$$

Since $\lambda < \mu(\Omega)$ and by Poincaré's inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + C(1 - \frac{\lambda}{\mu(\Omega)}) \int_\Omega u^2 dx \leq 0,$$

therefore, we reach that setting $\gamma := C(1 - \dfrac{\lambda}{\mu(\Omega)})$,

$$\int_\Omega u^2 dx \leq e^{-\gamma t} \int_\Omega u_0^2 dx \qquad \text{and then} \qquad \lim_{t\to\infty} \int_\Omega u(x,t) dx = 0.$$

Case 2: $\lambda > \mu(\Omega)$. Let ρ_n be the positive eigenfunction of the eigenvalue problem

$$\begin{cases} -\Delta \rho_n = \mu_n \dfrac{\rho_n}{|x|^2 + \frac{1}{n}} & \text{in } \Omega, \\ \rho_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.17}$$

We assume ρ_n normalized, $||\rho_n||_\infty = 1$. In the limit, ρ verifies

$$\begin{cases} -\Delta \rho = D \dfrac{\rho}{|x|^2} & \text{in } \Omega, \\ \rho = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.18}$$

Multiplying this equation by ρ, we get

$$\int_\Omega |\nabla \rho|^2 dx = D \int_\Omega \frac{\rho^2}{|x|^2} dx$$

and since ρ is an eigenfunction, we know that $D = \mu(\Omega)$. Therefore, $\mu_n \downarrow \mu(\Omega)$ as $n \to \infty$. Since $\lambda > \mu(\Omega)$, there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, $\mu_n < \lambda$. For n_0 fixed, we denote ρ_{n_0} by ρ and μ_{n0} by μ. Using ρ as a test function in (7.7), we get

$$\frac{d}{dt}\int_\Omega u\rho dx + \mu \int_\Omega \frac{u\rho}{|x|^2 + \frac{1}{n_0}} dx = \lambda \int_\Omega \frac{u\rho}{|x|^2} dx.$$

Taking into account that $\dfrac{1}{|x|^2} \geq \dfrac{1}{|x|^2 + \frac{1}{n_0}} \geq C$ in Ω, it follows

$$\frac{d}{dt}\int_\Omega u\rho dx \geq C(\lambda - \mu) \int_\Omega u\rho dx.$$

Integrating in both sides of the last expression and considering $\mu < \lambda$, we obtain that, for some $c > 0$,

$$Y(t) \geq Y_0 e^{(\lambda-\mu)ct} \qquad \text{where} \qquad Y(t) = \int_\Omega u(x,t)\rho dx.$$

Thus, $Y(t) \to \infty$ as $t \to \infty$ and by Hopf's lemma, we know that the distance from any point x to a point y in the boundary is strictly positive, so there is space between the function $\rho(x)$ and this distance, i.e., $\rho(x) \geq c\delta(x)$. Then, we conclude. $\square$

3 The supercritical problem: $p > 1$

In this Section we are interested in the super-linear case, $p > 1$, that corresponds to the *supercritical* case with respect to the Sobolev embedding with the Hardy weight, see Theorem 6. Without loss of generality we can assume $\lambda = 1$.

3.1 Existence result

The main result in this Section is the following.

Theorem 31. *Assume that* Ω *is a smooth bounded domain such that* $0 \in \partial\Omega$ *and let* $p > 1$. *Then, there exists* $\bar{u} \in L^\infty(\Omega)$ *such that if* $0 \le u_0 \le \bar{u}$, *the problem*

$$\begin{cases} u_t - \Delta u &= \dfrac{u^p}{|x|^2} \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \end{cases} \tag{7.19}$$

has a unique positive solution $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$, *for all* $T > 0$.

Proof. We start the proof finding a supersolution to (7.19). Consider $d_{\Gamma_1}(x) = dist(x, \Gamma_1)$, with $x \in \overline{\Omega}$, where Γ_1 is a regular submanifold of the boundary and $0 \in \Gamma_1 \subset\subset \partial\Omega$. Let ζ be defined as the solution to

$$\begin{cases} -\Delta\zeta = \dfrac{d^p_{\Gamma_1}}{|x|^2} & \text{in } \Omega, \\ \zeta = d_{\Gamma_1} & \text{on } \partial\Omega. \end{cases} \tag{7.20}$$

The function $d^p_{\Gamma_1}/|x|^2$ belongs to $L^r(\Omega)$ for any $1 \le r < \frac{N}{2-p}$ if $p < 2$ and for any $r \ge 1$ if $p \ge 2$. In both cases there exists $r > N$ such that $d^p_{\Gamma_1}/|x|^2 \in L^r(\Omega)$. Thus, using the classical regularity theory, as in Chapter 5, we conclude that ζ, the solution of (7.20), is bounded and, moreover, $\zeta \in C^{1,\alpha}(\overline{\Omega})$. Then, by Hopf's Lemma we conclude that, there exists some constant $C > 0$ such that $\zeta \le C d_{\Gamma_1}$.

Setting $T = C^{-\frac{p}{p-1}} > 0$ and defining $\overline{u} = T\zeta$, by a direct calculation we find that

$$-\Delta\overline{u} = T(-\Delta\zeta) = \frac{T d^p_{\Gamma_1}}{|x|^2} \ge \frac{T\zeta^p}{|x|^2 C^p} = \frac{T^p T^{1-p}\zeta^p}{|x|^2 C^p} = \frac{\overline{u}^p}{|x|^2}.$$

Therefore,

$$\begin{cases} -\Delta\overline{u} \ge \dfrac{\overline{u}^p}{|x|^2} & \text{in } \Omega, \\ \overline{u} > 0 & \text{in } \Omega, \\ \overline{u} = T d_{\Gamma_1} & \text{on } \partial\Omega. \end{cases} \tag{7.21}$$

Hence, $\overline{u}$ is a supersolution to (7.19) if $u_0 \le \overline{u}$.

To find a subsolution to problem (7.19) it is sufficient to consider $\underline{u}$ the solution to the linear problem,

$$\begin{cases} \underline{u}_t - \Delta\underline{u} = 0 & \text{in } \Omega_T, \\ \underline{u}(x,0) = u_0(x) \le \overline{u} & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{7.22}$$

Furthermore, by the weak comparison principle, see Lemma 18, we get easily that $\overline{u}(x) \geq \underline{u}(x,t)$, $\forall (x,t) \in \Omega_T$.

Consider now the following iterative approximation: We set $u^0 = \underline{u}$ and for $k \geq 1$ and define u^k as the unique solution to the problem,

$$\left\{\begin{array}{rcl} u_t^k - \Delta u^k = \dfrac{(u^{k-1})^p}{|x|^2 + \frac{1}{k}} & & \text{in } \Omega_T, \\ u^k(x,0) = u_0(x) \leq \overline{u} & & \text{in } \Omega, \\ u^k = 0 & & \text{on } \partial\Omega \times (0,T). \end{array}\right. \tag{7.23}$$

The existence and the uniqueness of this solution follows by Theorem 28. By recurrence and using Lemma 18, we get

$$\underline{u} \leq u^1 \leq u^2 \leq ... \leq u^k \leq u^{k+1} \leq .. \leq \overline{u}.$$

Therefore, $\{u^k\}_{k \in N}$ is an ordered increasing sequence. This fact allows us to define u by $\lim\limits_{k\to\infty} u^k(x,t) = u(x,t)$. Since $\frac{(u^{k-1})^p}{|x|^2+\frac{1}{k}} \leq \frac{\overline{u}^p}{|x|^2}$, by *Dominated Convergence Theorem*, u is a solution to (7.19) in the distributional sense. Moreover

$$u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T), \quad \frac{u^p}{|x|^2} \in L^1(\Omega_T).$$

As in the previous Subsection, to prove the uniqueness we argue by contradiction. Consider $v \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$ a second positive solution to problem (7.19). By the construction of u, we have that $u \leq v$. Defining $\overline{v} = v - u$, it follows

$$\left\{\begin{array}{rcl} \overline{v}_t - \Delta \overline{v} = \dfrac{v^p - u^p}{|x|^2} \leq p||v||_{L^\infty(\Omega)}^{p-1} \dfrac{\overline{v}}{|x|^2} & & \text{in } \Omega_T, \\ \overline{v}(x,0) = 0 & & \text{in } \Omega, \\ \overline{v} = 0 & & \text{on } \partial\Omega \times (0,T). \end{array}\right. \tag{7.24}$$

We conclude using the same argument as in the proof of the uniqueness in the linear case. That is, taking w defined by (7.5) as a test function in (7.24) and using Gronwall's inequality, we obtain that $\overline{v} = 0$, hence $v = u$.

□

Remark 9. *Notice that in contrast with the elliptic case, see [50] and Chapter 6, in Theorem 31 we do not need any restriction on the shape of the domain Ω.*

4 Further results

Consider the following parabolic problem

$$\begin{cases} u_t - \Delta u = \dfrac{u^p}{|x|^2} + \mu u^q & \text{in } \Omega \times (0, T(\mu)), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T(\mu)), \end{cases} \tag{7.25}$$

with Ω a smooth bounded domain, $0 \in \partial\Omega$ and $0 < q < 1 < p$.

A similar problem without the Hardy potential was studied in [40]. In Chapter 5 we show that the associated stationary problem has a nontrivial solution independently of the shape of the domain Ω. More precisely, we prove the existence of $\mu_0 > 0$ such that the problem

$$\begin{cases} -\Delta z = \dfrac{z^p}{|x|^2} + \mu z^q & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.26}$$

has at least a positive solution for $\mu \le \mu_0$ and it has not a positive solution for $\mu > \mu_0$.

As a consequence we can formulate the following result.

Proposition 8. *Assume that $0 \in \partial\Omega$, $0 < q < 1 < p$ and $\mu_0 > \mu > 0$, then the following problem*

$$\begin{cases} u_t - \Delta u = \dfrac{u^p}{|x|^2} + \mu u^q & \textit{in } \Omega_T, \\ u(x,0) = u_0(x) \le \bar{v} & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega \times (0,T), \end{cases} \tag{7.27}$$

admits a solution $u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$, where $\bar{v}$ is a positive solution to (7.26)*.*

Chapter 8

Regularization of a first order term

1 Introduction and some preliminaries

In this last Chapter we are going to consider $0 \in \Omega$. In [32] has been proved that if u satisfies the following inequality

$$u_t - \Delta u \geq \lambda \frac{u^2}{|x|^2} \text{ in } \mathcal{D}'(\Omega \setminus \{0\}) \times (0,T), \tag{8.1}$$

there is no solution but the trivial one, $u \equiv 0$. Moreover, the authors proved that a instantaneous and complete blow-up happens for related equations.

The main goal of this Chapter is to analyze how a first order absorption term regularizes the *supercritical* term with respect to the Hardy potential avoiding the restriction on the existence of solution obtained in [32]. We will prove also the existence of a solution for the largest class of initial and source data. More precisely, we will study the following parabolic problem,

$$\begin{cases} u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f & \text{in } \Omega_T = \Omega \times (0,T), \\ u = 0 \text{ on } \partial\Omega \times (0,T) \quad \text{and} \quad u \geq 0 & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{8.2}$$

where $\Omega \subset I\!R^N$ is a bounded domain with $0 \in \Omega$, $\lambda > 0$, $f \in L^1(\Omega_T)$ a positive function, the initial data $u_0 \in L^1(\Omega)$ and $1 < p < 3$.

This Chapter is organized as follows:

- In Section 2 we prove the main existence result. We begin considering the regular case, namely $f \in L^\infty(\Omega_T)$ and $u_0 \in L^\infty(\Omega)$. Truncating the gradient term and the reaction term $\dfrac{u^p}{|x|^2}$, we are able to get the existence of a minimal solution. The main difficulty to reach the general case, $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$, is to pass to the limit in the gradient term, to face this difficulty we use a particular "exponential" function term introduced in [30], in this way we get the existence of a solution to (8.2). Taking into account that the data is a L^1-function, the solution is obtained as limit of solution to approximated problems, see [43].

- A partial result on the asymptotic behavior of the solution is given in Section 3. More precisely, we are able to prove that if $f \equiv 0$, there exists a suitable positive constant C, such that if $\lambda < C$, $u(x,t) \to 0$ as $t \to \infty$, a.e. in Ω.

In this Chapter we are looking for *distributional solutions* and we will give some regularity information in each case. In the case of the heat equation with integrable data, all the usual concepts of solution that appear in the literature coincide. See, for instance [2] for the proof of the corresponding uniqueness result, even in a more general framework.

We recall some classical results that we are going to use along this Chapter.

Theorem 32. *(Compactness result) Consider the sequences $\{F_n\}_n, \{u_{n0}\}_n$ be such that $F_n \in L^\infty(\Omega_T)$ and $u_{n0} \in L^1(\Omega)$. Assume that $||F_n||_{L^1(\Omega_T)} + ||u_{no}||_{L^1(\Omega)} \leq C$.*

Let u_n be the unique solution to the problem

$$\begin{cases} u_{nt} - \Delta u_n = F_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_{n0}(x) & \text{in } \Omega. \end{cases} \tag{8.3}$$

Then, there exists a measurable function u such that $u_n \rightharpoonup u$ weakly in $L^s(0,T;W_0^{1,s}(\Omega)), \forall s < \frac{N+2}{N+1}$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $L^2(0,T;W_0^{1,2}(\Omega))$, moveover, up to a subsequence, $\nabla u_n \to \nabla u$ a.e. in Ω_T. If, in addition, $F_n \to F$ strongly in $L^1(\Omega_T)$ and $u_{n0} \to u_0$ strongly in $L^1(\Omega)$, then $T_k(u_n) \to T_k(u)$ strongly in $L^2(0,T;W_0^{1,2}(\Omega))$ and u is an entropy solution to (8.3) with data (F, u_0).

See, for instance [76].

Definition 10. *Since $F_n \to F$ strongly in $L^1(\Omega_T)$ and $u_{n0} \to u_0$ strongly in $L^1(\Omega)$, the solution obtained in the previous result is called solution obtained as limit of approximation.*

The following compactness result in L^1 can be found in [89], Corollary 4.

Theorem 33. *(Compactness result in L^1) Let u_n be a sequence bounded in $L^q(0,T;W^{1,q}(\Omega))$ such that u_{nt} is bounded in $L^1(\Omega_T) + L^{s'}(0,T;W^{-1,s'}(\Omega))$ with $q, s > 1$, then u_n is relatively strongly compact in $L^1(\Omega_T)$, that is, up to subsequences, u_n strongly converges in $L^1(\Omega_T)$ to some function $u \in L^1(\Omega_T)$.*

We recall also the following maximum principle, proved in [11].

Lemma 17. *(Maximum principle)*

Assume that Ω *is a bounded regular domain, and let* $h(x,t)$ *be a measurable function such that* $|h| \in L^{2r_0}([0,T]; L^{2p_0}(\Omega))$ *where* p_0, $r_0 > 1$ *and* $\dfrac{N}{2p_0} + \dfrac{1}{r_0} < 1$. *Assume that* $w(x,t) \geq 0$ *verifies,*

i) $w \in \mathcal{C}((0,T); L^1(\Omega)) \cap L^{r_1}([0,T]; W_0^{1,p_1}(\Omega))$, *where* $r_1, p_1 \geq 1$ *such that* $\frac{N}{2p_1} + \frac{1}{r_1} > \frac{N+1}{2}$,

ii) w *is a subsolution to problem*

$$\left\{\begin{array}{lcll} w_t - \Delta w & \leq & |h||\nabla w| & \text{in} \quad \Omega_T, \\ w(x,t) & = & 0 \quad \text{on} & \partial\Omega \times (0,T), \\ w(x,0) & = & 0 \quad \text{in} & \Omega. \end{array}\right. \tag{8.4}$$

Then, $w \equiv 0$.

We will apply the previous Lemma to get the following Comparison Principle.

Lemma 18. *(Comparison Principle) Consider* $H(x,t,s)$ *a Caratheodory function with* $(x,t,s) \in \Omega \times (0,T) \times I\!R$ *such that* $H(x,t,\cdot) \in \mathcal{C}^1(I\!R^N)$ *for all* $(x,t) \in \Omega_T$ *and*

$$|H(x,t,s_1) - H(x,t,s_2)| \leq h(x,t)|s_1 - s_2|,$$

where h *be a measurable function such that* $|h| \in L^{2r_0}([0,T]; L^{2p_0}(\Omega))$ *where* p_0, $r_0 > 1$ *and* $\dfrac{N}{2p_0} + \dfrac{1}{r_0} < 1$. *Let* u, $v \in \mathcal{C}((0,T); L^1(\Omega)) \cap L^p((0,T); W_0^{1,p}(\Omega))$, *for some* $p > 1$, *be such that* $|u_t - \Delta u| \in L^1(\Omega_T)$, $|v_t - \Delta v| \in L^1(\Omega_T)$ *and*

$$\begin{cases} u_t - \Delta u \geq H(x,t,\nabla u) + f & \text{in} \quad \Omega_T, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

$$\begin{cases} v_t - \Delta v \leq H(x,t,\nabla v) + f & \text{in} \quad \Omega_T, \\ v(x,0) = v_0(x) & \text{in} \quad \Omega, \end{cases} \tag{8.5}$$

where $f \in L^1(\Omega_T)$, $u_0, v_0 \in L^1(\Omega)$ *and* $v_0(x) \leq u_0(x) \quad$ *in* $\quad \Omega$.

Then, $v \leq u \quad$ *in* $\quad \Omega_T$.

See [11] for a proof of this Lemma.

All the results in this Chapter can be seen in the paper [1].

2 Existence result

The main existence result of this Section is the following.

Theorem 34. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$. Consider $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$ be such that $f, u_0 \geq 0$ and $(f, u_0) \neq (0,0)$. Assume that $p < 3$, then, for all $\lambda > 0$, the problem*

$$\begin{cases} u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} + f(x,t) & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{8.6}$$

has a nonnegative entropy solution u with $u^{\frac{3}{2}}(x,t) \in L^2(0,T,W_0^{1,2}(\Omega))$.

The proof of Theorem 34 will be given in several steps in order to be clear. The main idea is to show the existence for "regular data" and then get the general existence result using some compactness arguments.

Theorem 35. *Assume that $f \in L^\infty(\Omega_T)$ and $u_0 \in L^\infty(\Omega)$, with $f, u_0 \geq 0$. Then, for all $n, m \in \mathbb{N} \setminus \{0\}$, the problem*

$$\begin{cases} v_t - \Delta v + T_m(v) \dfrac{|\nabla v|^2}{1 + \frac{1}{m}|\nabla v|^2} = \lambda \dfrac{T_n(v)^p}{|x|^2 + \frac{1}{n}} + f & \text{in } \Omega_T, \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{8.7}$$

has a minimal nonnegative bounded solution $v_{m,n} \in L^2(0,T;W_0^{1,2}(\Omega))$.

Proof.

We follow by approximation. Fixed $n \geq 1$, for $i \geq 0$ we define the sequence $\{v_i\}_i$ with $v_0 \equiv 0$ and v_i defined as the solution of the problem

$$\begin{cases} v_{it} - \Delta v_i + T_m(v_i) \dfrac{|\nabla v_i|^2}{1 + \frac{1}{m}|\nabla v_i|^2} = \lambda \dfrac{(T_n(v_{i-1})^+)^p}{|x|^2 + \frac{1}{n}} + f & \text{in } \Omega_T, \\ v_i = 0 & \text{on } \partial\Omega \times (0,T), \\ v_i(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{8.8}$$

Notice that the existence of v_i follows using the classical existence result in [68]. Notice also that since $f \geq 0$, 0 is a subsolution to (8.8).

Using $-(v_i)^-$ as a test function in (8.8) (recall that $(v_i)^- \leq 0$) and in the equation verified by 0 and subtracting the two expressions, we obtain

$$-\int_0^T\int_\Omega v_{it}(v_i)^- dxdt - \int_0^T\int_\Omega |\nabla(v_i)^-|^2 dxdt - \int_0^T\int_\Omega T_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2}(v_i)^- dxdt$$
$$\geq \lambda \int_0^T\int_\Omega -(v_i)^- \frac{(T_n(v_{i-1})^+)^p}{|x|^2+\frac{1}{n}} \geq 0.$$

Therefore, multiplying the last expression by (-1),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^T\int_\Omega (v_i)^{-2} dxdt + \int_0^T\int_\Omega |\nabla(v_i)^-|^2 dxdt\\ &-\int_0^T\int_\Omega T_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2}|v_i| dxdt \leq 0.\end{aligned}$$

Since $-T_m(v_i) \geq -v_i$ and being in the set $v_i \leq 0$, $-v_i = -(-|v_i|)$, then, the second term in the last expression is positive too. Thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^T\int_\Omega (v_i)^{-2} dxdt + \int_0^T\int_\Omega |\nabla(v_i)^-|^2 dxdt \leq 0.$$

Hence, since the derivative is negative, the function is decreasing, but, since the starting point $v_i(x,0)$ is nonegative, $(v_i^-)^2 = 0$ and also

$$\int_0^T\int_\Omega |\nabla(v_i)^-|^2 dxdt \leq 0.$$

Therefore, $(v_i^-)^2 = 0$ and then, we conclude that $v_i \geq 0$.

Consider $\mathcal{V}_n$ the unique bounded positive solution to the problem

$$\begin{cases} \mathcal{V}_{nt} - \Delta\mathcal{V}_n = \lambda\dfrac{n^p}{|x|^2+\frac{1}{n}} + f & \text{in } \Omega_T,\\ \mathcal{V}_n = 0 & \text{on } \partial\Omega\times(0,T),\\ \mathcal{V}_n(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{8.9}$$

Considering $(v_i - \mathcal{V}_n)^+$ as a test function in (8.8) and in (8.9) and subtracting the both expressions, we get

$$\begin{aligned}&\int_0^T\int_\Omega (v_{it} - \mathcal{V}_{nt})(v_i - \mathcal{V}_n)^+ dxdt + \int_0^T\int_\Omega |\nabla(v_i - \mathcal{V}_n)^+|^2 dxdt\\ &+\int_0^T\int_\Omega T_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2}(v_i - \mathcal{V}_n)^+ dxdt\\ &=\lambda\int_0^T\int_\Omega \Big(\frac{(T_n(v_{i-1})^+)^p}{|x|^2+\frac{1}{n}} - \frac{n^p}{|x|^2+\frac{1}{n}}\Big)(v_i - \mathcal{V}_n)^+ dxdt.\end{aligned}$$

Since

$$\frac{(T_n(v_{i-1})^+)^p}{|x|^2+\frac{1}{n}} \leq \frac{n^p}{|x|^2+\frac{1}{n}},$$

the last term in the previous expression in negative, then,

$$\frac{1}{2}\frac{d}{dt}\int_0^T\int_\Omega((v_i-\mathcal{V}_n)^+)^2dxdt+\int_0^T\int_\Omega|\nabla(v_i-\mathcal{V}_n)^+|^2dxdt$$
$$+\int_0^T\int_\Omega T_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2}(v_i-\mathcal{V}_n)^+dxdt \leq 0.$$

Since $v_i \geq 0$, $T_m(v_i) \geq 0$ and $v_i(x,0)=\mathcal{V}_n(x,0)$, hence,

$$\frac{1}{2}\int_\Omega(((v_i-\mathcal{V}_n)(x,T))^+)^2dx+\int_0^T\int_\Omega|\nabla(v_i-\mathcal{V}_n)^+|^2dxdt \leq 0.$$

Arguing as before, $(v_i-\mathcal{V}_n)^+=0$ and, then, $v_i \leq \mathcal{V}_n$.

We claim that $v_i \leq v_{i+1}$ for all $i \geq 0$. Since $v_i \geq 0$ and $v_0=0$, then $v_1 \geq v_0$. Let us prove that $v_2 \geq v_1$. Using (8.8) and the monotony of $T_n(s)$, it follows that

$$v_{2t}-\Delta v_2+T_m(v_2)\frac{|\nabla v_2|^2}{1+\frac{1}{m}|\nabla v_2|^2} \geq v_{1t}-\Delta v_1+T_m(v_1)\frac{|\nabla v_1|^2}{1+\frac{1}{m}|\nabla v_1|^2}.$$

Now we set

$$H_m(s)=\frac{s^2}{1+\frac{1}{m}s^2},$$

then

$$(v_1-v_2)_t-\Delta(v_1-v_2)+T_m(v_1)(H_m(|\nabla v_1|)-H_m(|\nabla v_2|))$$
$$+(T_m(v_1)-T_m(v_2))H_m(|\nabla v_2|) \leq 0.$$

Since $(T_m(v_1)-T_m(v_2))H_m(|\nabla v_2|) \geq 0$ in the set $\{v_1 \geq v_2\}$, using the comparison principle in Lemma 18, there results that $v_1 \geq v_2$. Hence, the claim follows using an induction argument.

Thus, $0 \leq v_1 \leq v_i \leq \mathcal{V}_n$.

Taking the function v_i as a test function in (8.8),

$$\int_0^T\int_\Omega v_{it}v_idxdt+\int_0^T\int_\Omega|\nabla v_i|^2dxdt+\int_0^T\int_\Omega v_iT_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2}dxdt$$
$$=\lambda\int_0^T\int_\Omega v_i\frac{(T_n(v_{i-1})^+)^p}{|x|^2+\frac{1}{n}}dxdt+\int_0^T\int_\Omega v_ifdxdt.$$

Since $v_i \le \mathcal{V}_n$ and $f \in L^\infty(\Omega_T)$,

$$\int_0^T \int_\Omega v_{it} v_i dxdt + \int_0^T \int_\Omega |\nabla v_i|^2 dxdt + \int_0^T \int_\Omega v_i T_m(v_i) \frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2} dxdt \le C.$$

Therefore, $\{v_i\}_i$ is bounded in the space $L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.

Hence, we get the existence of $v \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$ such that $v_i \rightharpoonup v$ weakly in $L^2(0,T;W_0^{1,2}(\Omega))$ and $v_i \uparrow v$ strongly in $L^s(\Omega_T)$ for all $s \ge 1$. To show that v solves (8.7) we have just to prove that

$$H_m(|\nabla v_i|) \to H_m(|\nabla v|) \text{ strongly in } L^1(\Omega_T).$$

We define the following function,

$$F_i = T_m(v_i)\frac{|\nabla v_i|^2}{1+\frac{1}{m}|\nabla v_i|^2} + \lambda \frac{(T_n(v_{i-1}))^p}{|x|^2+\frac{1}{n}} + f,$$

since F_i is bounded in $L^\infty(\Omega)$, using Theorem 32, if follows that $\nabla v_i \to \nabla v_{m,n}$ a.e. in Ω_T. Thus, since $H_m(|\nabla v_i|) \le |\nabla v_i|$, the dominated convergence theorem allows us to conclude. □

Under the same hypotheses on f and u_0, we want to pass to the limit as $m \to \infty$ for n fixed.

More precisely, we are going to prove the following result

Theorem 36. *Assume $\lambda > 0$ and let $v_{m,n}$ be the minimal solution to the problem*

$$\begin{cases} v_t - \Delta v + T_m(v)\dfrac{|\nabla v|^2}{1+\frac{1}{m}|\nabla v|^2} = \lambda \dfrac{T_n(v)^p}{|x|^2+\frac{1}{n}} + f & \text{in } \Omega_T, \\ v = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{8.10}$$

Then, $v_{m,n} \to v_n$, as $m \to \infty$, strongly in $L^2(0,T;W_0^{1,2}(\Omega))$, where v_n solves the truncated problem

$$\begin{cases} v_{nt} - \Delta v_n + v_n|\nabla v_n|^2 = \lambda \dfrac{T_n(v_n)^p}{|x|^2+\frac{1}{n}} + f & \text{in } \Omega_T, \\ v_n = 0 & \text{on } \partial\Omega \times (0,T), \\ v_n(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{8.11}$$

and $v_n \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$.

Proof.

The existence of $v_{m,n}$ follows using Theorem 35, furthermore, we know that $v_{m,n} \le \mathcal{V}_n$ for all $m \ge 1$. Using $v_{m,n}$ as a test function in (8.10), we obtain

$$\int_0^T\int_\Omega (v_{m,n})_t v_{m,n}dxdt + \int_0^T\int_\Omega |\nabla v_{m,n}|^2 dxdt + \int_0^T\int_\Omega v_{m,n}\frac{T_m(v_{m,n})|\nabla v_{m,n}|^2}{1+\frac{1}{m}|\nabla v_{m,n}|^2}dxdt$$
$$\leq \lambda \int_0^T \int_\Omega \frac{v_{m,n}^{p+1}}{|x|^2}dxdt + ||f||_{L^1(\Omega_T)}||v_{m,n}||_{L^\infty(\Omega_T)}. \tag{8.12}$$

Therefore, integrating the first term on time and taking into account that $v_{m,n} \leq \mathcal{V}_n$,

$$\frac{1}{2}\int_\Omega v_{m,n}^2(x,T)dx + \int_0^T \int_\Omega |\nabla v_{m,n}|^2 dxdt + \int_0^T \int_\Omega v_{m,n}\frac{T_m(v_{m,n})|\nabla v_{m,n}|^2}{1+\frac{1}{m}|\nabla v_{m,n}|^2}dxdt$$
$$\leq \lambda C + \frac{1}{2}\int_\Omega u_0^2 dx. \tag{8.13}$$

Thus, since $u_0(x) \in L^\infty(\Omega)$ and $T_n(v_{m,n}) \to v_{m,n}$ as $m \to \infty$, there exists constant A such that,

$$\int_\Omega v_{m,n}^2(x,T)dx \leq A \quad \text{and} \quad \int_0^T\int_\Omega |\nabla v_{m,n}|^2 dxdt \leq A. \tag{8.14}$$

Then, up to a subsequence,

$$v_{m,n} \rightharpoonup v_n \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega)).$$

Therefore, the sequence $\{v_{m,n}\}_m$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$. Hence, we get the existence of $v_n \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\Omega_T)$ such that $v_{m,n} \rightharpoonup v_n$, as $m \to \infty$, weakly in $L^2(0,T;W_0^{1,2}(\Omega))$, in particular $v_{m,n}^{\frac{3}{2}} \in L^2(0,T;W_0^{1,2}(\Omega))$.

Since $\Delta v_{m,n} - T_m(v_{m,n})\dfrac{|\nabla v_{m,n}|^2}{1+\frac{1}{m}|\nabla v_{m,n}|^2} + \lambda\dfrac{T_n(v_{m,n})^p}{|x|^2+\frac{1}{n}} + f$ is in $L^1(\Omega_T) + L^2(0,T;W^{-1,2}(\Omega))$, by Theorem 33, $v_{m,n} \to v_n$ strongly in $L^1(\Omega_T)$. Then, since the strong convergence in $L^1(\Omega_T)$ and since the both functions are in $L^\infty(\Omega_T)$, we obtain that the term

$$\int_0^T\int_\Omega (v_{m,n}-v_n)^s dxdt = \int_0^T\int_\Omega (v_{m,n}-v_n)(v_{m,n}-v_n)^{s-1}dxdt$$

is going to zero as $m \to \infty$ for all $s \geq 1$. Therefore, $v_{m,n} \to v_n$ strongly in $L^s(\Omega_T)$, for all $s \geq 1$.

We need to check now the strong convergence of the truncated terms. We study first the Hardy potential truncation,

$$\int_0^T \int_\Omega \lambda \frac{T_n(v_{m,n})^p}{|x|^2+\frac{1}{n}}dxdt \leq \int_0^T \int_\Omega \lambda \frac{v_{m,n}^p}{|x|^2+\frac{1}{n}}dxdt.$$

Using Hölder's inequality and since $p<3$,

$$\int_0^T \int_\Omega \lambda \frac{T_n(v_{m,n})^p}{|x|^2+\frac{1}{n}}dxdt \leq \lambda \Big(\int_0^T \int_\Omega \frac{v_{m,n}^3}{|x|^2}dxdt\Big)^{\frac{p}{3}} \Big(\int_0^T \int_\Omega \frac{1}{|x|^2}dxdt\Big)^{\frac{3-p}{3}}.$$

By Hardy's inequality, see Theorem 4, we get

$$\int_0^T \int_\Omega \lambda \frac{T_n(v_{m,n})^p}{|x|^2+\frac{1}{n}}dxdt \leq \frac{\lambda}{\lambda_N}C\int_0^T \int_\Omega |\nabla v_{m,n}^{\frac{3}{2}}|^2dxdt.$$

Since $v_{m,n}^{\frac{3}{2}} \in L^2(0,T;W_0^{1,2}(\Omega))$, we have the term bounded in $L^1(\Omega_T)$. The strong convergence in $L^1(\Omega_T)$ follows by Vitali's Theorem. Thus, $\dfrac{T_n(v_{m,n})^p}{|x|^2+\frac{1}{n}} \to \dfrac{v_n^p}{|x|^2}$ in $L^1(\Omega_T)$.

To conclude, we have to prove also that $T_m(v_{m,n})|\nabla v_{m,n}|^2 \to v_n|\nabla v_n|^2$ strongly in $L^1(\Omega_T)$.

Let us consider the Landes regularizer defined by the following expression

$$v_{n,\nu}(x,t) = \int_{-\infty}^t \nu \bar{v}_n(x,s)\chi_{(0,T)}(s)e^{\nu(s-t)}ds,$$

where

$$\bar{v}_n(x,s) = \begin{cases} v_n(x,s) & \text{if } t\in[0,T], \\ 0 & \text{if } t\notin[0,T]. \end{cases} \tag{8.15}$$

then, $v_{n,\nu}(x,0)=0$ and $v_{n,\nu}$ converges to v_n strongly in $L^2(0,T;W_0^{1,2}(\Omega))$ as ν tends to infinity. Moreover, we have

$$(v_\nu)_t = \nu(v-v_\nu), \quad \text{i.e.,} \quad <(v_\nu)', w> = \nu\int_0^T\int_\Omega (v-v_\nu)wdxdt,$$

for all $w\in L^2(0,T;W_0^{1,2}(\Omega))$.

Since $v_n \in L^\infty(\Omega_T)$, $||v_{n,\nu}||_\infty \leq ||v_n||_\infty \equiv C_n$ and $v_{n,\nu}\to v_n$ strongly in $L^s(\Omega_T)$ for all $s\geq 1$.

Let us define $\phi(s) = se^{\alpha s^2}$ where $\alpha > C_n^2$, this function verifies that $\phi'(s) - C_n|\phi(s)| \geq \frac{1}{2}$.

Using $\phi(v_{m,n} - v_{n,\nu})$ as a test function in (8.10), we obtain

$$
\begin{aligned}
&< (v_{m,n})_t, \phi(v_{m,n} - v_{n,\nu}) > + \int_0^T \int_\Omega \nabla v_{m,n} \phi'(v_{m,n} - v_{n,\nu}) \nabla (v_{m,n} - v_{n,\nu}) dxdt \\
&+ \int_0^T \int_\Omega T_m(v_{m,n}) \frac{|\nabla v_{m,n}|^2}{1 + \frac{1}{m}|\nabla v_{m,n}|^2} \phi(v_{m,n} - v_{n,\nu}) dxdt \\
&= \int_0^T \int_\Omega (\lambda \frac{(T_n(v_{m,n}))^p}{|x|^2 + \frac{1}{n}} + f) \phi(v_{m,n} - v_{n,\nu}) dxdt.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&< (v_{m,n})_t, \phi(v_{m,n} - v_{n,\nu}) >= \\
&< (v_{m,n} - v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) > + < (v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) > .
\end{aligned}
$$

It is clear that

$$
< (v_{m,n} - v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) >= \int_\Omega \int_0^T \frac{d}{dt} \Big(\int_0^{v_{m,n} - v_{n,\nu}} \phi(s) ds \Big) dtdx.
$$

Therefore,

$$
\begin{aligned}
&< (v_{m,n} - v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) >= \int_\Omega \Big| \int_0^{v_{m,n} - v_{n,\nu}} \phi(s) ds \Big|_0^T dx \\
&= \int_\Omega [\overline{\phi}(v_{m,n} - v_{n,\nu})]_0^T dx \geq o(\nu, m),
\end{aligned}
$$

where $\overline{\phi}(s) = \int_0^s \phi(\sigma) d\sigma$.

By the definition of $(v_{n,\nu})_t$, we have

$$
\begin{aligned}
&< (v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) >= \nu \int_0^T \int_\Omega (v_n - v_{n,\nu}) \phi(v_{m,n} - v_{n,\nu}) dxdt \\
&= \nu \int_0^T \int_\Omega v_n \phi(v_{m,n} - v_{n,\nu}) dxdt - \nu \int_0^T \int_\Omega v_{n,\nu} \phi(v_{m,n} - v_{n,\nu}) dxdt.
\end{aligned}
$$

Since $v_n \in L^\infty(\Omega_T)$ and by the strong convergence of $v_{n\nu}$ to v_n in $L^2(0, T; W_0^{1,2}(\Omega))$, we have

$$
< (v_{n,\nu})_t, \phi(v_{m,n} - v_{n,\nu}) >= o(\nu, m).
$$

Therefore,

$$
< (v_{m,n})_t, \phi(v_{m,n} - v_{n,\nu}) >= o(\nu, m).
$$

Using the fact that $v_{m,n} \leq \mathcal{V}_n$, we get easily that

$$
\int_0^T \int_\Omega (\lambda \frac{(T_n(v_{m,n}))^p}{|x|^2 + \frac{1}{n}} + f) \phi(v_{m,n} - v_{n,\nu}) dxdt = o(\nu, m).
$$

Thus, we only need to study these two terms

$$I_1 = \int_0^T \int_\Omega \nabla v_{m,n} \phi'(v_{m,n} - v_{n,\nu}) \nabla (v_{m,n} - v_{n,\nu}) dxdt$$

and

$$I_2 = \int_0^T \int_\Omega T_m(v_{m,n}) \frac{|\nabla v_{m,n}|^2}{1 + \frac{1}{m}|\nabla v_{m,n}|^2} \phi(v_{m,n} - v_{n,\nu}) dxdt.$$

We can write I_1 as

$$\begin{aligned}
I_1 &= \int_0^T \int_\Omega \nabla v_{m,n} \phi'(v_{m,n} - v_{n,\nu}) \nabla (v_{m,n} - v_{n,\nu}) dxdt \\
&- \int_0^T \int_\Omega \nabla v_{n,\nu} \phi'(v_{m,n} - v_{n,\nu}) \nabla (v_{m,n} - v_{n,\nu}) dxdt \\
&+ \int_0^T \int_\Omega \nabla v_{n,\nu} \phi'(v_{m,n} - v_{n,\nu}) \nabla (v_{m,n} - v_{n,\nu}) dxdt.
\end{aligned}$$

The weak convergence of $v_{m,n}$ and the definition of $v_{n,\nu}$ imply that

$$I_1 = \int_0^T \int_\Omega |\nabla (v_{m,n} - v_{n,\nu})|^2 \phi'(v_{m,n} - v_{n,\nu}) dxdt + o(\nu, m).$$

On the other hand, we have

$$\begin{aligned}
|I_2| &\leq C_n \int_0^T \int_\Omega |\nabla v_{m,n}|^2 |\phi(v_{m,n} - v_{n,\nu})| dxdt \\
&\leq C_n \int_0^T \int_\Omega |\nabla (v_{m,n} - v_{n,\nu})|^2 |\phi(v_{m,n} - v_{n,\nu})| dxdt \\
&- C_n \int_0^T \int_\Omega |\nabla v_{n,\nu}|^2 |\phi(v_{m,n} - v_{n,\nu})| dxdt \\
&+ 2C_n \int_0^T \int_\Omega |\nabla v_{m,n} \nabla v_{n,\nu}| |\phi(v_{m,n} - v_{n,\nu})| dxdt.
\end{aligned}$$

Since the last two terms in the last expression are going to zero as $m \to \infty$, we obtain

$$|I_2| \leq C_n \int_0^T \int_\Omega |\nabla (v_{m,n} - v_{n,\nu})|^2 |\phi(v_{m,n} - v_{n,\nu})| dxdt,$$

where $C_n = ||\mathcal{V}_n||_\infty$. Combining the above estimates, it follows that

$$\int_0^T \int_\Omega (\phi'(v_{m,n} - v_{n,\nu}) - C_n |\phi(v_{m,n} - v_{n,\nu})|) |\nabla (v_{m,n} - v_{n,\nu})|^2 dxdt \leq o(\nu, m).$$

Since $\phi'(s) - C_n|\phi(s)| \geq \frac{1}{2}$, $\displaystyle\int_0^T \int_\Omega |\nabla (v_{m,n} - v_{n,\nu})|^2 dxdt = o(\nu, m).$

As a consequence, we conclude that $|\nabla v_{m,n}|^2 \to |\nabla v_n|^2$ strongly in $L^1(\Omega_T)$ and then the result follows because, since $v_{m,n}$ is bounded we can use the *Dominated Convergence Theorem* knowing that $T_m(v_{m,n}) \to v_n$ in $L^s(\Omega_T)$, for all $s \geq 1$, thus, $T_m(v_{m,n})|\nabla v_{m,n}|^2 \to v_n|\nabla v_n|^2$ in $L^1(\Omega_T)$. $\square$

We are now able to prove Theorem 34

Proof of Theorem 34. Let $f_n = T_n(f)$ and $u_{n0} = T_n(u_0)$. Define u_n as a nonnegative solution to the following approximated problem

$$\begin{cases} (u_n)_t - \Delta u_n + u_n|\nabla u_n|^2 = \lambda \dfrac{T_n(u_n)^p}{|x|^2 + \frac{1}{n}} + f_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{8.16}$$

The existence of u_n follows using Theorems 35 and 36.

Using $T_k(u_n)$ as a test function in (8.16), it follows that

$$\begin{aligned} &\int_0^T\int_\Omega (u_n)_t T_k(u_n)dxdt + \int_0^T\int_\Omega |\nabla T_k(u_n)|^2 dxdt \\ &+ \int_0^T\int_\Omega T_k(u_n)u_n|\nabla u_n|^2 dxdt \leq k\lambda \int_0^T\int_\Omega \frac{u_n^p}{|x|^2}dxdt + k||f||_{L^1(\Omega_T)}. \end{aligned} \tag{8.17}$$

Notice that

$$\int_0^T\int_\Omega (u_n)_t T_k(u_n)dxdt = \int_\Omega \Theta_k(u_n)(x,T)dx - \int_\Omega \Theta_k(u_{n0})(x)dx,$$

where $\Theta_k(s) = \displaystyle\int_0^s T_k(\tau)\,d\tau$ and verifies that $\Theta(s) \leq ks$. In the same way we have

$$\int_0^T\int_\Omega T_k(u_n)u_n|\nabla u_n|^2 dxdt = \int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt$$

where, for $s \geq 0$,

$$\Psi_k(s) = \int_0^s (\sigma T_k(\sigma))^{\frac{1}{2}} d\sigma. \tag{8.18}$$

Notice that, in particular,

$$\Psi_k(s) = \begin{cases} \frac{1}{2}s^2 & \text{if } s \leq k, \\ \frac{1}{2}k^2 + \frac{2}{3}k^{\frac{1}{2}}(s^{\frac{3}{2}} - k^{\frac{3}{2}}) & \text{if } s > k. \end{cases} \tag{8.19}$$

For $s < 0$, we set that $\Psi_k(s) \equiv \Psi_k(-s)$. It is clear that, for $s \geq 0$,

$$\Psi_k^2(s) \geq \frac{2}{3}s^3 - C(k). \tag{8.20}$$

Hence, we can conclude that

$$\int_\Omega \Theta_k(u_n)(x,T)dx + \int_0^T\int_\Omega |\nabla T_k(u_n)|^2 dxdt + \int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt \leq$$
$$k\lambda \int_0^T\int_\Omega \frac{u_n^p}{|x|^2} dxdt + k(||f||_{L^1(\Omega_T)} + ||u_0||_{L^1(\Omega)}).$$

Using Hardy Sobolev's inequality and by the inequality (8.20), we reach that

$$\int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt \geq \Lambda_N \int_0^T\int_\Omega \frac{\Psi_k^2(u_n)}{|x|^2} dxdt \geq \Lambda_N \int_0^T\int_\Omega \frac{u_n^3}{|x|^2} dxdt - C_2,$$

where $C_2 > 0$ independent on n.

Using the fact that $p < 3$, we get

$$k\lambda \int_0^T\int_\Omega \frac{u_n^p}{|x|^2} dxdt \leq k\lambda \Big(\int_0^T\int_\Omega \frac{u_n^3}{|x|^2} dxdt\Big)^{\frac{p}{3}} \Big(\int_0^T\int_\Omega \frac{1}{|x|^2} dxdt\Big)^{\frac{3-p}{3}}$$
$$\leq \varepsilon k\lambda \int_0^T\int_\Omega \frac{u_n^3}{|x|^2} dxdt + C(\varepsilon)k\lambda.$$

Hence,

$$\int_\Omega \Theta_k(u_n)(x,T)dx + \int_0^T\int_\Omega |\nabla T_k(u_n)|^2 dxdt + \int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt$$
$$\leq \frac{Ck\lambda\varepsilon}{\Lambda_N} \int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt + k(||f||_{L^1(\Omega_T)} + ||u_0||_{L^1(\Omega)}).$$

Therefore,

$$\int_\Omega \Theta_k(u_n)(x,T)dx + \int_0^T\int_\Omega |\nabla T_k(u_n)|^2 dxdt + \tilde{C}\int_0^T\int_\Omega |\nabla \Psi_k(u_n)|^2 dxdt$$
$$\leq C(k, ||f||_{L^1(\Omega_T)}, ||u_0||_{L^1(\Omega)}).$$

Combining the above estimates, we conclude that the sequence $\{u_n\}_n$ is bounded in $L^2(0,T,W_0^{1,2}(\Omega))$, hence, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{and} \quad \Psi_k(u_n) \rightharpoonup \Psi_k(u) \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega)).$$

Arguing as in the proof of Theorem 36 and since $p < 3$, using Vitali's Lemma we prove that

$$\lambda \frac{T_n(u_n)^p}{|x|^2 + \frac{1}{n}} + T_n(f) \to \lambda \frac{u^p}{|x|^2} + f \text{ strongly in } L^1(\Omega_T). \tag{8.21}$$

In order to conclude we have to show that $u_n|\nabla u_n|^2 \to u|\nabla u|^2$ strongly in $L^1(\Omega_T)$. To prove that we need some previous results.

We claim that

$$\lim_{k\to\infty}\int_0^T\int_{\{u_n\geq k\}} u_n|\nabla u_n|^2 dxdt = 0 \text{ uniformly on } n. \tag{8.22}$$

To prove the claim we take $\psi_{k-1}(u_n)$ as a test function in (8.16), where $\psi_{k-1}(s) = T_1(G_{k-1}(s))$ and $G_k(s) = s - T_k(s)$,

$$\begin{aligned}&\int_0^T\int_\Omega u_{nt}\psi_{k-1}(u_n)dxdt + \int_0^T\int_\Omega |\nabla\psi_{k-1}(u_n)|^2 dxdt\\ &+\int_0^T\int_\Omega \psi_{k-1}(u_n)u_n|\nabla u_n|^2 dxdt \leq\\ &\int_0^T\int_{\{u_n\geq k-1\}}\lambda\frac{u_n^p}{|x|^2}dxdt + \int_0^T\int_{\{u_n\geq k-1\}} f dxdt.\end{aligned}$$

Since $u_n \in L^2(\Omega_T)$,

$$\begin{gathered}|\{(x,t)\in\Omega_T : k-1 < u_n(x,t) < k\}| \to 0,\\ |\{(x,t)\in\Omega_T : u_n(x,t) > k\}| \to 0, \text{ uniformly on } n \text{ as } k\to\infty.\end{gathered} \tag{8.23}$$

Thus, using (8.21), we get

$$\lim_{k\to\infty}\Big(\int_0^T\int_{\{u_n\geq k-1\}}\lambda\frac{u_n^p}{|x|^2}dxdt+\int_0^T\int_{\{u_n\geq k-1\}} f dxdt\Big) = 0 \text{ uniformly on } n.$$

On the other hand, we have

$$\int_0^T\int_\Omega u_{nt}\psi_{k-1}(u_n)dxdt = \int_\Omega \overline{\psi}_k(u_n(x,T))dx - \int_\Omega \overline{\psi}_k(u_{0n}(x))dx,$$

where $\overline{\psi}_k(s) = \int_0^s \psi_k(\sigma)d\sigma$. In particular,

$$\overline{\psi}_k(s) = \begin{cases} 0 & \text{if } s\leq k-1,\\ \frac12(s-(k-1))^2 & \text{if } k-1\leq s\leq k,\\ \frac12 + (s-k) & \text{if } s>k.\end{cases}$$

Notice that,

$$\begin{aligned}&\int_0^T\int_\Omega u_{nt}\psi_{k-1}(u_n)dxdt + \int_0^T\int_{\{u_n\geq k\}} u_n|\nabla u_n|^2 dxdt\\ &\leq \int_0^T\int_\Omega u_{nt}\psi_{k-1}(u_n)dxdt + \int_0^T\int_\Omega |\nabla\psi_{k-1}(u_n)|^2 dxdt\\ &\quad+\int_0^T\int_\Omega \psi_{k-1}(u_n)u_n|\nabla u_n|^2 dxdt.\end{aligned}$$

Combining the above estimates and the definition of $\overline{\psi}_k(s)$, we conclude that

$$\int_0^T \int_{\{u_n \geq k\}} u_n |\nabla u_n|^2 dxdt$$
$$\leq \int_\Omega \overline{\psi}_k(u_{0n}(x))dx + \lambda \int_0^T \int_{\{u_n \geq k-1\}} \frac{u_n^p}{|x|^2} dxdt + \int_0^T \int_{\{u_n \geq k-1\}} f dxdt.$$

Hence, by (8.23) and since $u_0 \in L^1(\Omega)$,

$$\lim_{k\to\infty} \int_0^T \int_{\{u_n \geq k\}} u_n |\nabla u_n|^2 dxdt = 0 \text{ uniformly on } n$$

and the claim follows.

We are going to prove now the following claim.

Let u_n be as above, then,

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^2(0,T;W_0^{1,2}(\Omega)), \text{ for all } k > 0. \tag{8.24}$$

As in the proof of Theorem 36 we consider the Landes regularization of $T_k(u)$, $(T_k(u))_\nu$, defined as

$$\frac{d(T_k(u))_\nu}{dt} = \nu(T_k(u) - (T_k(u))_\nu),$$

then,

$$(T_k(u))_\nu \to T_k(u) \;\text{ strongly in }\; L^2(0,T;W_0^{1,2}(\Omega)) \;\text{ and a.e. in }\; \Omega_T$$

and

$$||T_k(u)_\nu||_{L^\infty(\Omega_T)} \leq k, \quad \forall k > 0.$$

We define again the function $\phi(s) = se^{\alpha s^2}$ for some $\alpha > k^2$, verifying $\phi'(s) - k|\phi(s)| \geq \frac{1}{2}$.

Using $\phi(T_n(u_k) - (T_n(u))_\nu)$ as a test function in (8.16), it follows that

$$\begin{aligned}
&< (u_n)_t, \phi(T_k(u_n) - (T_k(u))_\nu) > \\
&+ \int_0^T \int_\Omega \nabla u_n \phi'(T_k(u_n) - (T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu) dxdt \\
&+ \int_0^T \int_\Omega u_n |\nabla u_n|^2 \phi(T_k(u_n) - (T_k(u))_\nu) dxdt \\
&= \int_0^T \int_\Omega (\lambda \frac{T_k(u_n)^p}{|x|^2 + \frac{1}{n}} + f_n) \phi(T_k(u_n) - (T_k(u))_\nu) dxdt.
\end{aligned}$$

Notice that, by the definition of $G_k(s)$,

$$< (u_n)_t, \phi(T_k(u_n) - (T_k(u))_\nu) >=$$

$$< (T_k(u_n) - (T_k(u))_\nu)_t, \phi(T_k(u_n) - (T_k(u))_\nu) >$$

$$+ < (T_k(u))_\nu)_t, \phi(T_k(u_k) - (T_k(u))_\nu) >$$

$$+ < (G_k(u_n))_t, \phi(T_k(u_n) - (T_k(u))_\nu) > .$$

It is clear that

$$\begin{aligned}
&< (T_k(u_n) - (T_k(u))_\nu)_t, \phi(T_k(u_n) - (T_k(u))_\nu) > \\
&= \int_\Omega [\overline{\phi}(T_k(u_n) - (T_k(u))_\nu)]_0^T dx \geq 0,
\end{aligned}$$

where $\overline{\phi}(s) = \int_0^s \phi(\sigma) d\sigma$.

Notice that, taking into account the support of the function $G_k(s)$,

$$< (G_k(u_n))_t, \phi(T_k(u_n) - (T_k(u))_\nu) >=< (G_k(u_n))_t, \phi(k - (T_k(u))_\nu) >,$$

then, using a variation of Lemma 3.1. in [25], it follows that

$$< (G_k(u_n))_t, \phi(T_k(u_n) - (T_k(u))_\nu) >\geq o(\nu, n).$$

In the same way, using the definition of $(T_k(u))_\nu$, we reach that

$$< (T_k(u))_\nu)_t, \phi(T_k(u_k) - (T_k(u))_\nu) >\geq o(\nu, n).$$

We set

$$J_1 \equiv \int_0^T \int_\Omega \nabla u_n \phi'(T_k(u_u) - (T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu) dxdt,$$

then, we can write this term as follows

$$\begin{aligned}
J_1 &= \int_0^T \int_\Omega \nabla T_k(u_n) \phi'(T_k(u_n) - (T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu) dxdt \\
&+ \int_0^T \int_\Omega \nabla G_k(u_n) \phi'(T_k(u_n) - (T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu) dxdt \\
&= \int_0^T \int_\Omega |\nabla T_k(u_n) - \nabla (T_k(u))_\nu|^2 \phi'(T_k(u_n) - (T_k(u))_\nu) dxdt \\
&+ \int_0^T \int_\Omega \nabla (T_k(u))_\nu \phi'(T_k(u_k) - (T_k(u))_\nu) \nabla (T_k(u_n) - (T_k(u))_\nu) dxdt \\
&+ \int_0^T \int_\Omega \nabla G_k(u_n) \phi'(T_k(u_n) - (T_k(u))_\nu) \nabla T_k(u_n) dxdt \\
&- \int_0^T \int_\Omega \nabla G_k(u_n) \phi'(T_k(u_n) - (T_k(u))_\nu) \nabla (T_k(u))_\nu dxdt.
\end{aligned}$$

Since the supports of $\nabla G_k(u_n)$ and $\nabla T_k(u_n)$ are disjoint and using the weak convergence of $\{G_k(u_n)\}_n$ and the strong convergence of $\{(T_k(u_n))_\nu\}_{n,\nu}$, it follows that

$$\begin{aligned}&\Big|\int_0^T\int_\Omega \nabla(T_k(u))_\nu\phi'(T_k(u_k)-(T_k(u))_\nu)\nabla(T_k(u_n)-(T_k(u))_\nu)dxdt\\&+\int_0^T\int_\Omega \nabla G_k(u_n)\phi'(T_k(u_n)-(T_k(u))_\nu)\nabla T_k(u_n)dxdt\\&-\int_0^T\int_\Omega \nabla G_k(u_n)\phi'(T_k(u_n)-(T_k(u))_\nu)\nabla(T_k(u))_\nu dxdt\Big|\\&=o(\nu,n).\end{aligned}$$

Thus,

$$J_1=\int_0^T\int_\Omega |\nabla T_k(u_n)-\nabla(T_k(u))_\nu|^2\phi'(T_k(u_n)-(T_k(u))_\nu)dxdt+o(\nu,n).$$

We deal now with the term

$$J_2=\int_0^T\int_\Omega u_n|\nabla u_n|^2\phi(T_k(u_n)-T_k(u)_\nu)dxdt.$$

Notice that $\phi(T_k(u_n)-T_k(u)_\nu)_{\chi_{\{u_n\geq k\}}}\geq 0$, then we have

$$\begin{aligned}J_2 &= \int_0^T\int_{\{u_n\leq k\}} u_n|\nabla u_n|^2\phi(T_k(u_n)-T_k(u)_\nu)dxdt\\&+\int_0^T\int_{\{u_n\geq k\}} u_n|\nabla u_n|^2\phi(T_k(u_n)-T_k(u)_\nu)dxdt\\&\geq \int_0^T\int_\Omega T_k(u_n)|\nabla T_k(u_n)|^2|\phi(T_k(u_n)-T_k(u)_\nu)|dxdt\\&\geq -k\int_0^T\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)_\nu|^2|\phi(T_k(u_n)-T_k(u)_\nu)|dxdt-o(\nu,n).\end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned}&\int_0^T\int_\Omega (\phi'(T_k(u_n)-T_k(u)_\nu)-k|\phi(T_k(u_n)-T_k(u)_\nu)|)|\nabla T_k(u_n)-\nabla T_k(u)_\nu|^2dxdt\\&\leq o(\nu,n).\end{aligned}$$

Recall that $\phi'(s)-k|\phi(s)|>\frac{1}{2}$, hence,

$$\frac{1}{2}\int_0^T\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)_\nu|^2dxdt\leq o(\nu,n).$$

And since

$$\begin{aligned}&\int_0^T\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2dxdt\\&\leq C\Big(\int_0^T\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)_\nu|^2dxdt+\int_0^T\int_\Omega |\nabla T_k(u)_\nu-\nabla T_k(u)|^2dxdt\Big),\end{aligned}$$

we conclude that $T_k(u_n) \to T_k(u)$ strongly in $L^2(0,T;W_0^{1,2}(\Omega))$.

To finish we are going to prove that

$$u_n|\nabla u_n|^2 \to u|\nabla u|^2 \text{ strongly in } L^1(\Omega_T).$$

It is clear that $u_n|\nabla u_n|^2 \to u|\nabla u|^2$ a.e. in Ω_T. We are going to use Vitali's Lemma in order to prove the strong convergence of this term in $L^1(\Omega_T)$. Let $E \subset \Omega_T$ be a measurable set, then

$$\begin{aligned}
&\int_0^T \int_E u_n|\nabla u_n|^2 dxdt \\
&= \int_0^T \int_{E\cap\{u_n\le k\}} u_n|\nabla u_n|^2 dxdt + \int_0^T \int_{E\cap\{u_n>k\}} u_n|\nabla u_n|^2 dxdt \\
&\le \int_0^T \int_E T_k(u_n)|\nabla T_k(u_n)|^2 dxdt + \int_0^T \int_{\{u_n>k\}} u_n|\nabla u_n|^2 dxdt.
\end{aligned}$$

From (8.22) we get the existence of k_0, independent of n such that for $k \ge k_0$,

$$\int_0^T \int_{\{u_n>k\}} u_n|\nabla u_n|^2 dxdt \le \frac{\varepsilon}{2}.$$

Using the strong convergence of $\{T_k(u_n)\}_n$, we get the existence of $\delta > 0$ such that if $|E| \le \delta$, then, $\displaystyle\int_0^T \int_E T_k(u_n)|\nabla T_k(u_n)|^2 dxdt \le \frac{\varepsilon}{2}$ for $n \ge n_0$. Hence, we conclude that $\displaystyle\int_0^T \int_E u_n|\nabla u_n|^2 dxdt \le \varepsilon$.

Therefore, by Vitali's Lemma, $u_n|\nabla u_n|^2 \to u|\nabla u|^2$ strongly in $L^1(\Omega_T)$ and u solves (8.6).

Taking into account that

$$u|\nabla u|^2 + \lambda\frac{u^p}{|x|^2} + f(x,t) \in L^1(\Omega_T) \text{ and } u_0 \in L^1(\Omega),$$

u is an *entropy* solution to (8.6).

Remark 10. *It is worthy to point out that for $p > 3$ it is sufficient to regularize with a quasilinear term of the form $|u|^{q-1}u|\nabla u|^2$ $q > p-2$. The proof of this general result is an straightforward change on the proof of existence for problem* (8.6)*. We omit it to be short.*

3 Asymptotic behavior

In this Section we are going to analyze the asymptotic behavior of the solution to problem (8.6), obtained in Theorem 34, under certain hypotheses on the data.

Theorem 37. *Assume that $f \equiv 0$ and $u_0 \in L^1(\Omega)$ is such that $u_0 \gneqq 0$. Let u be the solution to*

$$\begin{cases} u_t - \Delta u + u|\nabla u|^2 = \lambda \dfrac{u^p}{|x|^2} & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{8.25}$$

Then, there exists $\bar{\lambda} > 0$ such that if $\lambda \leq \bar{\lambda}$, $u(x,t) \to 0$ as $t \to \infty$ a.e. in Ω.

Proof.

Let us define $H(s) = \int_0^s e^{-\frac{1}{2}\sigma^2} d\sigma$ and we also set $v_k = D_k(u)$, where $D_k(s) = H(s)H'(T_k(s))$, it is clear that $v_k \in L^2(0,T; W_0^{1,2}(\Omega)) \cap L^\infty(\Omega)$. Using v_k as a test function in problem (8.25), it follows that

$$\frac{d}{dt}\int\limits_\Omega \bar{D}_k(u)dx + \int\limits_\Omega D_k'(u)|\nabla u|^2 dx + \int\limits_\Omega D_k(u)u|\nabla u|^2 dx = \lambda \int\limits_\Omega \frac{u^p D_k(u)}{|x|^2} dx,$$

where $\bar{D}_k(s) = \int_0^s D_k(\sigma)d\sigma$. Notice that

$$\int\limits_\Omega D_k'(u)|\nabla u|^2 dx = \int_{\{u<k\}} H'(u)H'(u)|\nabla u|^2 dx + \int_{\{u<k\}} H(u)H''(u)|\nabla u|^2 dx$$
$$+ \int_{\{u\geq k\}} (H(u)H'(k))'|\nabla u|^2 dx.$$

Therefore,

$$\int\limits_\Omega D_k'(u)|\nabla u|^2 dx = \int\limits_\Omega H'(u)H'(T_k(u))|\nabla u|^2 dx + \int_{\{u<k\}} H(u)H''(u)|\nabla u|^2 dx.$$

Thus,

$$\int\limits_\Omega D_k'(u)|\nabla u|^2 dx + \int\limits_\Omega D_k(u)u|\nabla u|^2 dx =$$

$$\int\limits_\Omega H'(u)H'(T_k(u))|\nabla u|^2 dx + \int_{\{u\geq k\}} H'(k)H(u)u|\nabla u|^2 dx.$$

On the other hand, we have

$$s^p D_k(s) = s^p H(s)H'(s) \leq CH^2(s).$$

Since $u|\nabla u|^2, \dfrac{u^p}{|x|^2} \in L^1(\Omega_T)$, letting $k \to \infty$, combining the above estimates and using Hardy's inequality, it follows

$$\frac{d}{dt}\int_\Omega \mathcal{V}(x,t)dx + \int_\Omega |\nabla H(u)|^2 dx \le C\lambda \int_\Omega \frac{H^2(u)}{|x|^2}dx \le \frac{C\lambda}{\Lambda_N}\int_\Omega |\nabla H(u)|^2 dx.$$

where $\mathcal{V} = \frac{1}{2}H^2(u)$. If $\frac{C\lambda}{\Lambda_N} < 1$, there results that

$$\frac{d}{dt}\int_\Omega \mathcal{V}(x,t)dx + (1 - \frac{C\lambda}{\Lambda_N})\int_\Omega |\nabla H(u)|^2 dx \le 0.$$

Thus, using Poincaré inequality

$$\frac{d}{dt}\int_\Omega \mathcal{V}(x,t)dx + 2\lambda_1(1 - \frac{C\lambda}{\Lambda_N})\int_\Omega \mathcal{V}(x,t)dx \le 0.$$

Hence

$$\int_\Omega \mathcal{V}(x,t)dx \le e^{-2\lambda_1(1-\frac{C\lambda}{\Lambda_N})t}\int_\Omega \mathcal{V}(x,0)dx \le Ce^{-2\lambda_1(1-\frac{C\lambda}{\Lambda_N})t}.$$

Therefore, $\mathcal{V}(x,t) \to 0$ strongly in $L^1(\Omega)$ as $t \to \infty$ and then, the result follows.

$\square$

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