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Nonlocal problems in partial differential equations.

Memoria para optar al título de Doctor en Matemáticas presentada por:

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A Damián,
por haberse olvidado de cómo se restan dos fracciones.
Estas páginas son tan suyas como mías.

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Contents

Agradecimientos **iii**

Notations **ix**

Introducción, resumen de resultados y conclusiones. **xiii**

Introducción: el operador Laplaciano fraccionario. xiii
Resumen de resultados y conclusiones. xviii
Problemas abiertos y cuestiones por estudiar. xxvii

Introduction, summary of contents and conclusions. **1**

Introduction: the fractional Laplacian operator. 1
Summary of contents and conclusions. 6
Open problems and further results. 15

I An elliptic nonlocal problem: variational and non variational methods. **17**

1 On some elliptic critical problems for the spectral fractional Laplacian operator. **19**

1.1 Introduction, preliminaries and functional settings. 19
1.2 Sublinear case: $0 < q < 1$. 28
1.2.1 The existence of the first solution. 29
1.2.2 The existence of the second solution. Variational techniques. 32
1.3 Linear case: $q = 1$. 55
1.4 Superlinear case: $1 < q < 2_s^* - 1$. 59
1.5 Regularity and concentration-compactness. 64

2 Elliptic critical problems for the fractional Laplacian operator in a bounded domain. **73**

2.1 Introduction, preliminaries and functional settings. 73

2.2 Sublinear case: $0 < q < 1$. 78
2.3 Superlinear case: $q > 1$. 93
2.4 Regularity result. 99

3 Some remarks on the solvability of non local elliptic problems with the Hardy potential. 103
3.1 Introduction, preliminaries and functional settings. 103
3.2 Existence of minimal solutions for $1 < p < p(\lambda, s)$. 108
3.3 Subcritical case: multiplicity of solutions. 116
3.4 Non-existence for $p \geq p(\lambda, s)$: complete blow up. 127
3.4.1 Fractional Picone's inequality. 131

II Regularity of non local minimal surfaces. 133

4 Bootstrap regularity for integro-differential equations. 135
4.1 Introduction, preliminaries and functional settings. 135
4.2 Proof of the main result: regularity of the solutions. 139
4.2.1 Toolbox. 141
4.2.2 Approximation by nicer kernels. 148
4.2.3 Smoothness of the approximate solutions. 152
4.2.4 Uniform estimates and conclusion of the proof for $k = 0$. 155
4.2.5 The induction argument. 158

5 Regularity of nonlocal minimal surfaces. 161
5.1 Introduction, preliminaries and functional settings. 161
5.2 Proof of the principal result: $\mathcal{C}^\infty$ smoothness. 165
5.2.1 Writing the equation in terms of the function u. 165
5.2.2 The regularity of the equation and conclusion. 168
5.2.3 Hölder regularity of A_r. 171

III A non local parabolic problem. 175

6 A Widder's type Theorem for the heat equation with nonlocal diffusion. 177
6.1 Introduction, preliminaries and functional settings. 177
6.2 Uniqueness for weak solutions. 183
6.3 Uniqueness for strong positive solutions. 191
6.3.1 A remark on viscosity solutions. 197
6.4 Other results. 198

Notations

General notations

Symbol	**Meaning**
$x = (x_1, x_2, ..., x_N)$	Element of $\mathbb{R}^N$
$r = \|x\| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_N^2)}$	Modulus of x
$\{e_1, e_2, \ldots, e_N\}$	The standard Euclidean basis
$D_i u = \partial_i u = \dfrac{\partial u}{\partial x_i} = u_{x_i}$	Partial derivative of u respect to x_i
$D_{ij} u = \partial_{ij} u = \dfrac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j}$	Second partial derivative of u respect to x_i and x_j
$Du = \nabla u = \left(\dfrac{\partial u}{\partial x_1}, \dfrac{\partial u}{\partial x_2}, \ldots, \dfrac{\partial u}{\partial x_N} \right)$	Gradient of u
$\Delta u = \mathrm{div}(\nabla u)$	Laplacian of u
$A_s u$	Spectral fractional Laplacian of u
$(-\Delta)^s u(x) = C(N,s)\mathrm{P.V} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{\|x-y\|^{N+2s}}\, dy$	Fractional Laplacian of u
$C(N,s)$	Normalized constant equal to $\left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{\|\xi\|^{N+2s}}\, d\xi \right)^{-1}$
$\frac{\partial w}{\partial \nu^s}$	$-\kappa_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}$
κ_s	Normalized constant equal to $\frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$
p'	Conjugate exponent of p, that satisfies $\frac{1}{p} + \frac{1}{p'} = 1$
$2_s^* = \dfrac{2N}{(N-2s)}$	Critical fractional Sobolev exponent
ρ_1	First eigenvalue of $(-\Delta)$ with zero Dirichlet condition
$\rho_{1,(-\Delta)^s}$	First eigenvalue of $(-\Delta)^s$ with zero Dirichlet condition

Symbol	**Meaning**
$\partial\Omega$	Boundary of Ω
$\Omega' \subset\subset \Omega$	Ω' Open subset of Ω with $\overline{\Omega'} \subset \Omega$
$\mathscr{C}_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$	Cylinder of Ω
$\partial_L \mathscr{C}_\Omega = \partial\Omega \times (0,\infty)$	Lateral boundary of $\mathscr{C}_\Omega$
B_R	Ball in $\mathbb{R}^N$ centered in the origin with radio R
$B_R(x_0)$	Ball in $\mathbb{R}^N$ centered in x_0 with radio R
sop u	Support of the function u
a.e.	Almost everywhere
$\lvert A\rvert$	Lebesgue measure of $A \subset \mathbb{R}^N$
$\lVert \cdot \rVert_X$	Norm in the space X
X'	Dual space of X
$\langle \cdot, \cdot \rangle$	Scalar product in $\mathbb{R}^N$ / duality X, X'
δ_{x_0}	Dirac's Delta in x_0
$\setminus$	Difference of sets
$v^+ = \max(v,0)$	Positive part of the function v
$v^- = \max(-v,0)$	Negative part of the function v
$\mathcal{C}(\Omega)$ ó $\mathcal{C}^0(\Omega)$	Continuous functions in Ω
$\mathcal{C}_0(\Omega)$	Continuous functions in Ω with compact support
$\mathcal{C}^{0,\beta}(\Omega) = \mathcal{C}^\beta(\Omega)$	Hölder continuous functions in Ω with exponent β
$\mathcal{C}^k(\Omega)$	Functions of class k in Ω
$\mathcal{C}^{k,\beta}(\Omega)$	Functions Hölder continuous of class k in Ω
$\mathcal{C}^k_0(\Omega)$	Functions in $\mathcal{C}^k(\Omega)$ with compact support
$\mathcal{C}^\infty(\Omega)$	Functions indefinitely differentiable in Ω
$\mathcal{C}^\infty_0(\Omega) = \mathcal{D}(\Omega)$	Functions in $\mathcal{C}^\infty(\Omega)$ with compact support
$\mathcal{D}'(\Omega)$	Dual space of $\mathcal{C}^\infty_0(\Omega)$, that is, the space of distributions
$\mathcal{S}(\mathbb{R}^N)$	Schwartz class of functions in $\mathbb{R}^N$
$L^p(\Omega)$, $1 \le p < \infty$	$\{u : \Omega \to \mathbb{R} \mid u$ measurable, $\int_\Omega \lvert u\rvert^p < \infty\}$
$L^\infty(\Omega)$	$\{u : \Omega \to \mathbb{R} \mid u$ measurable and $\exists C$ such that $\lvert u(x)\rvert \le C$ in a.e. $x \in \Omega$ $\}$
$L^{p'}(\Omega)$	Dual space of $L^p(\Omega)$
$\mathcal{L}^s(\mathbb{R}^N)$	$\{u : \mathbb{R}^N \to \mathbb{R}$ measurable : $\int_{\mathbb{R}^N} \frac{\lvert u(x)\rvert}{(1+\lvert x\rvert^{N+2s})}\,dx < \infty\}$

$H^1(\mathbb{R}^N)$	Completeness of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm $\|\phi\|_{H^1(\mathbb{R}^N)} = \|\phi\|_{L^2(\mathbb{R}^N)} + \|\nabla\phi\|_{L^2(\mathbb{R}^N)}$
$H^s(\mathbb{R}^N)$	Completeness of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm $\|\phi\|_{H^s(\mathbb{R}^N)} = \|(1+\|\xi\|^s)\widehat{\phi}(\xi)\|_{L^2(\mathbb{R}^N)}$
$\left(\frac{2}{S(N,s)C(N,s)}\right)^{1/2}$	Embedding Sobolev constant $\dot{H}^s(\mathbb{R}^N) \to L^{2_s^*}(\mathbb{R}^N)$
$(\kappa_s T(N,s))^{1/2}$	Trace constant $H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s}) \to L^{2_s^*}(\Omega)$
$\Lambda_{N,s}$	Fractional Hardy-Sobolev constant
$H(\Omega,s)$	$\left\{u = \sum a_j\varphi_j \in L^2(\Omega) \,:\, \|u\|_{H(\Omega)} = \left(\sum \rho_j^s a_j^2\right)^{1/2} < \infty\right\}$
$H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$	Completeness of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm $\|\phi\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} = \left(\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\|\nabla\phi\|^2\right)^{1/2}$
$X_0^s(\Omega)$	$\left\{u \in H^s(\mathbb{R}^N) : \quad u = 0 \quad a.e. \quad \mathbb{R}^N\right\}$

Introducción, resumen de resultados y conclusiones.

Introducción: el operador Laplaciano fraccionario.

Este trabajo está dedicado al estudio de varios problemas, elípticos y parabólicos, en Ecuaciones en Derivadas Parciales (EDP) que involucran al Laplaciano fraccionario y otros operadores no locales. La teoría de integrales singulares y operadores no locales más generales en espacios de Banach ha sido estudiada desde hace mucho tiempo desde el punto de vista del análisis harmónico y funcional. Resultados clásicos sobre la misma pueden encontrarse en los trabajos de, entre otros, S. Bochner [32], T. Kato [108], H. Komatsu [110], H. Landkof [114] y E. Stein [154]. Sin embargo en los últimos años se ha acrecentado el interés sobre esta teoría debido a las aplicaciones y conexiones con múltiples fenómenos del *mundo real* que tienen las estructuras no locales. De hecho los operadores no locales aparecen de manera natural en problemas de elasticidad [148], en el problema del obstáculo con membrana delgada [52], problemas de transición de fase [15, 50, 152], propagación de llamas [54], dislocación de cristales [167], materiales estratificados [135], ondas de agua [71, 72, 157] y fluidos quasi-geostróficos [62, 121] entre otros.

Como estos operadores están relacionados con los procesos de Lévy (ver la Introducción del Capítulo 4) y tiene muchas aplicaciones a las matemáticas financieras, han sido también estudiados desde un punto de vista probabilístico (ver por ejemplo [24, 33, 36, 103, 106]). Un enfoque sencillo y muy recomendable que ilustra como estas integrales singulares aparecen como límite continuo de procesos con saltos discretos aleatorios puede encontrarse en [169].

El ejemplo básico de operador no local linear es el Laplaciano fraccionario. Recordemos que si denotamos por Δ el Laplaciano en $\mathbb{R}^N$, $N \geq 1$, y para una función u de la clase de Schwartz $\mathcal{S}(\mathbb{R}^N)$ definimos su transformada de

Fourier como

$$\widehat{u}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi}\, dx, \tag{0.0.1}$$

entonces se tiene que

$$\widehat{(-\Delta)}u(\xi) = |\xi|^2 \widehat{u}(\xi), \quad \xi \in \mathbb{R}^N.$$

Por tanto, es natural definir para $0 < s < 1$, el Laplaciano fraccionario como

$$\widehat{(-\Delta)^s}u(\xi) = |\xi|^{2s} \widehat{u}(\xi), \quad \xi \in \mathbb{R}^N. \tag{0.0.2}$$

La primera observación que podemos hacer a partir de la definicíon anterior es que si $u \in \mathcal{S}$ y $0 < s < 1$, $(-\Delta)^s u$ no es necesariamente una función de la clase de Schwartz porque $|\xi|^{2s}$ introduce una singularidad, en el origen, en su transformada de Fourier.

Como es bien sabido (ver [114,154,156,169]) este operador también puede ser representado, para funciones adecuadas, como un valor principal en la forma

$$\begin{aligned} (-\Delta)^s u(x) &:= C(N,s) \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy \\ &:= C(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy. \end{aligned} \tag{0.0.3}$$

En la expresión anterior

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\, d\xi \right)^{-1} = \frac{4^s \Gamma\left(\frac{N}{2}+s\right)}{-\pi^{\frac{N}{2}} \Gamma(-s)}, \tag{0.0.4}$$

es una constante de normalización elegida para garantizar que (0.0.2) se verifica (ver [76, 151, 156, 169]). Más aún, (ver [76, Corollary 4.2]), se tiene que

$$\lim_{s \to 1^-} \frac{C(N,s)}{s(1-s)} = \frac{4N}{\omega_{N-1}} \quad \text{y} \quad \lim_{s \to 0^+} \frac{C(N,s)}{s(1-s)} = \frac{2}{\omega_{N-1}},$$

donde ω_{N-1} es la medida $(N-1)$ dimensional de la esfera unidad $\mathbb{S}^{N-1}$. Con estas propiedades asintóticas de la constante $C(N,s)$, en [76, Proposition 4.4] (ver también [156, Proposition 5.3]) los autores demuestran que

$$\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u \quad \text{y} \quad \lim_{s \to 0^+} (-\Delta)^s u = u, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^N),$$

donde la convergencia es entendida en el sentido de las distribuciones. Notar que los límites anteriores pueden ser obtenidos también, de manera formal, de (0.0.2).

Usando (0.0.3) se puede demostrar que

$$|(-\Delta)^s \phi(x)| \leq \frac{C}{1+|x|^{N+2s}}, \quad \text{para toda } \phi \in \mathcal{S}(\mathbb{R}^N). \tag{0.0.5}$$

Esto motiva la introducción del espacio

$$\mathcal{L}^s(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} : \quad \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\, dx < \infty\}, \tag{0.0.6}$$

dotado con la norma usual

$$\|u\|_{\mathcal{L}^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\, dx.$$

Entonces para $u \in \mathcal{L}^s$ y $\phi \in \mathcal{S}$, utilizando (0.0.5), podemos definir formalmente el producto de dualidad $\langle (-\Delta)^s u, \phi \rangle$ en sentido distribucional como

$$\langle (-\Delta)^s u, \phi \rangle := \int_{\mathbb{R}^N} u(-\Delta)^s \phi\, dx.$$

Volviendo de nuevo a la definición dada en (0.0.3) deducimos claramente que $(-\Delta)^s$ es un operador no local porque el valor de $(-\Delta)^s u(x)$ no depende unicamente del comportamiento de u en un entorno de x sino de su comportamiento en todo $\mathbb{R}^N$. Observemos como ejemplo que si u es una función no negativa con soporte compacto en, por ejemplo Ω, entonces $(-\Delta)^s u(x) \neq 0$ para todo punto $x \notin \Omega$. Destacamos aquí que, como veremos a lo largo de este trabajo, esta propiedad, intrínseca en todos los operadores que vamos a estudiar, crea complicaciones porque los métodos clásicos utilizados en los problemas en EDP locales no pueden ser aplicados para estudiar los problemas no lineales que nos ocupan. Para superar esta dificultad L. Caffarelli y L. Silvestre probaron en [55] que todo Laplaciano fraccionario puede ser caracterizado como un operador que transforma una condición de contorno de tipo Dirichlet en una de tipo Neumann utilizando un problema extendido que es de naturaleza local. Esta herramienta se puede aplicar, por ejemplo, para demostrar la desigualdad de Harnack para $(-\Delta)^s$. Usaremos esta técnica de extensión para resolver el problema tratado en el Capítulo 1.

Además de la no localidad, muchas otras propiedades de este operador se pueden encontrar en [151]. Una de estas propiedades que usaremos a menudo a lo largo de este trabajo es el hecho de que

$$|(-\Delta)^s \phi| \leq C, \quad \phi \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\mathbb{R}^N),\ \beta > 0. \tag{0.0.7}$$

Cuando $s \geq 1/2$ la anterior suposición de regularidad debe entenderse como $\phi \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,2s+\beta-1}(\mathbb{R}^N)$. Hacemos notar aquí que (0.0.7) puede ser obtenida también si se remplaza la condición $\phi \in L^\infty(\mathbb{R}^N)$ por la condición, más débil $\phi \in \mathcal{L}^s(\mathbb{R}^N)$ donde $\mathcal{L}^s(\mathbb{R}^N)$ el espacio definido en (0.0.6). Demostraremos (0.0.5) y (0.0.7) en las Secciones 2.2 y 2.1 del Capítulo 2 respectivamente.

Una definición equivalente del Laplaciano fraccionario para funciones regulares es la dada como la media de un segundo cociente incremental. De hecho un cambio de variable estándar transforma (0.0.3) en la expresión

$$(-\Delta)^s u(x) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}}\, dy. \qquad (0.0.8)$$

El argumento que presentaremos en la prueba de (0.0.7) (ver Section 2.1 del Capítulo 2), demuestra que esta expresión es muy útil para tratar la singularidad del núcleo en el caso $s \geq 1/2$.

Finalmente presentamos otra definición equivalente de $(-\Delta)^s$ que proviene da la teoría de semigrupos y que nos muestra la relación entre $(-\Delta)^s$ y el operador A_s, definido a continuación, que será objeto de estudio en el Capítulo 1. Este enfoque también permite extender la definición de potencias fraccionarias a una amplia clase de operadores positivos (ver [156]). En nuestro caso denotamos por $e^{-t(-\Delta)}$ al generador del semigrupo del calor asociado a $(-\Delta)$. Esto significa que, para una función u adecuada,

$$e^{-t(-\Delta)}u(x) = \int_{\mathbb{R}^N} G_t(x)u(x-z)\, dz = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4t}} u(x-z)\, dz,$$

es la solución única de

$$\begin{cases} v_t + (-\Delta)v = 0 & \text{en } \mathbb{R}^N \times (0,\infty), \\ v(x,0) = u(x) & \text{en } \mathbb{R}^N. \end{cases}$$

Entonces aplicando, formalmente, la transformada de Fourier (ver [156, Lemma 5.1]), se tiene que

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t(-\Delta)} - 1)u(x)\, \frac{dt}{t^{1+s}}, \qquad (0.0.9)$$

donde

$$\Gamma(-s) = -\frac{\Gamma(1-s)}{s} = \int_0^\infty (e^{-t} - 1)\frac{dt}{t^{1+s}} < 0. \qquad (0.0.10)$$

Presentamos a continuación el importante papel que tiene el Laplaciano fraccionario con respecto a los espacios de Sobolev en $\mathbb{R}^N$. Éste es resumido en los siguientes teoremas que son bien conocidos. El primero es el teorema de inmersión fraccionario, que usaremos de manera frecuente a lo largo de este trabajo, y el segundo es la desigualdad de Hardy-Sobolev fraccionaria que vamos a utilizar en el Capítulo 3.

Teorema 0.0.1. (Ver [7, Theorem 7.58], [76, Theorem 6.5], [117,153]) *Sean $0 < s < 1$ y $N > 2s$. Existe una constante $S(N,s)$ tal que, para toda función medible y con soporte compacto $u : \mathbb{R}^N \to \mathbb{R}$, se tiene*

$$S(N,s)\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leq \frac{2}{C(N,s)}\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)},$$

donde

$$2^*_s = \frac{2N}{N-2s}, \tag{0.0.11}$$

es el exponente de Sobolev fraccionario.

Teorema 0.0.2. (Ver [25,90,101]) *Sean u una función en $\mathcal{C}^\infty_0(\mathbb{R}^N)$ y $N > 2s$. Entonces $u/|x|^s \in L^2(\mathbb{R}^N)$ y*

$$\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2}{|x|^{2s}}\,dx \leq ||(-\Delta)^{s/2}u||^2_{L^2(\mathbb{R}^N)},$$

donde

$$\Lambda_{N,s} = 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}, \tag{0.0.12}$$

es una constante óptima que no se alcanza.

Terminamos esta sección destacando que, como la primera parte de este trabajo está dedicada al estudio de problemas de Dirichlet, usaremos con frecuencia la Teoría del Cálculo de Variaciones. Consideremos el problema linear

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \text{condición de } \textit{borde} \text{ en } \Omega \text{ igual a cero,} \end{cases}$$

donde el operador L está bien definido para elementos del espacio de Hilbert H. Entonces, por el Teorema de Lax Milgram sabemos que si f pertenece al espacio dual H^*, existe un único elemento $u_f \in H$ tal que

$$\langle u_f, v\rangle_H = f(v) \quad \text{para todo } v \in H. \tag{0.0.13}$$

Aquí $f(v) = \langle f, v\rangle_{H^*,H}$. Más aún, u_f es el minimizante del siguiente funcional

$$J(u) = \frac{1}{2}\langle Lu, u\rangle_{H^*,H} - f(u).$$

Esto es, (0.0.13) es la ecuación de Euler-Lagrange de J. Esta teoría puede ser aplicada a los operadores no locales que estudiaremos a lo largo de este trabajo. Por ejemplo se puede aplicar al problema de Dirichlet lineal que involucra al operador Laplaciano fraccionario espectral definido como

$$A_s u(x) := \sum \rho_j^s a_j \varphi_j(x), \quad x \in \Omega, \tag{0.0.14}$$

donde $u(x) = \sum a_j \varphi_j(x)$, $x \in \Omega$ y (φ_j, ρ_j) son las autofunciones y autovalores de $(-\Delta)$ en Ω con condicón de borde cero. Aquí el espacio de energía natural asociado a dicho operador es el espacio de Hilbert definido por

$$H(\Omega, s) := \left\{ u \in L^2(\Omega) \, : \, \|u\|_{H(\Omega,s)} = \|A_{s/2} u\|_{L^2(\Omega)} < \infty \right\}. \tag{0.0.15}$$

También podemos usar la teoría previa para tratar problemas de Dirichlet que contienen el operador Laplaciano fraccionario definido en (0.0.3) y su espacio de energía asociado

$$X_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ c.t.p. en } \mathbb{R}^N \setminus \Omega \right\},$$

dotado con la norma

$$\|u\|^2_{X_0^s(\Omega)} = \frac{2}{C(N,s)} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)}.$$

Aquí $C(N, s)$ es la constante de normalización dada en (0.0.4).

Sin embargo, en este trabajo consideraremos problemas de Dirichlet con ciertos operadores no locales pero con no linealidades en vez de problemas lineales. Por lo tanto la teoría previa de minimización no puede ser utilizada por lo que, como veremos, necesitaremos otros resultados de la teoría no lineal de puntos críticos. Daremos más detalles sobre los operadores anteriores y sus espacios de Sobolev asociados en las Sección 1.1 del Capítulo 1 y en la Sección 2.1 del Capítulo 2 respectivamente.

A lo largo de este trabajo usaremos el convenio habitual de que c y C denotan constantes positivas que pueden tener valores diferentes cuando las escribimos en distintos lugares, incluso en la misma línea. Algunas veces usaremos los paréntesis para indicar la dependencia explícita de estas constantes de ciertos parámetros particulares. Es decir, por ejemplo, $C(a, b, c)$ expresa que C puede depender de a, b y c.

Resumen de resultados y conclusiones.

Como hemos comentado, el objetivo de este trabajo es estudiar algunos problemas en EDP que involucran operadores no locales. La **Parte I** está dedicada a problemas de Dirichlet elípticos no locales con no linearidades generales de tipo cóncavo-convexo (Ver Capítulos 1-3). En el caso clásico (local), los problemas de tipo

$$\begin{cases} -\Delta u = f(u) & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{0.0.16}$$

para distintos tipos de funciones no lineales f, han sido el principal objetivo de estudio de un gran número de trabajos de investigación en los últimos treinta años. Ver, por ejemplo, [9,10,45,92,119]. Uno de los casos más importantes del problema (0.0.16) es el dado por la potencia cri'tica $f(u) = u^{\frac{N+2}{N-2}}$, $N > 2$. Es bien conocido que este problema no tiene soluciones positivas cuando el dominio es estrellado. En su trabajo pionero [45], Brezis y Nirenberg demostraron que, en contra de la intuición, perturbando el problema crítico con un pequeño término lineal se obtienen soluciones positivas. Posteriormente en [10], utilizando los resultados de concentración-compacidad de Lions [119], Ambrosetti, Brezis y Cerami demostraron algunos resultados de existencia y multiplicidad para perturbaciones sublineales del exponente crítico, entre otras.

Siguiendo estas ideas, en el **Capítulo 1** estudiaremos el efecto que tienen las perturbaciones de menor orden en la existencia de soluciones positivas del siguiente problema crítico que involucra al Laplaciano fraccionario definido mediante su descomposición espectral. Es decir, consideraremos el siguiente

$$(P_\lambda) = \begin{cases} A_s u = \lambda u^q + u^{\frac{N+2s}{N-2s}} & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \\ u > 0 & \text{en } \Omega, \end{cases}$$

donde $\Omega \subset \mathbb{R}^N$ es un dominio regular acotado, $N > 2s$, $\lambda > 0$, $0 < q < 2_s^* - 1 = \frac{N+2s}{N-2s}$ y A_s es el Laplaciano fraccionario espectral definido en (0.0.14).

Nuestros resultados principales en relación al Problema (P_λ) son los siguientes.

Teorema. 1.2.1 *Sea* $0 < q < 1$. *Entonces existe* $0 < \Lambda < \infty$ *tal que el problema* (P_λ)

1 no tiene solución para $\lambda > \Lambda$*;*

2 tiene una solución minimal de energía para $0 < \lambda < \Lambda$ *y, más aún, la familia de soluciones minimales es creciente con respecto a* λ*;*

3 si $\lambda = \Lambda$*, existe al menos una solución de energía;*

4 si $s \geq 1/2$*, existen al menos dos soluciones de energía para* $0 < \lambda < \Lambda$.

Teorema. 1.3.1 *Supongamos que $q = 1$, $0 < s < 1$ y $N \geq 4s$. Entonces el problema (P_λ)*

1 no tiene solución para $\lambda \geq \rho_1^s$;

2 tiene al menos una solución de energía para cada $0 < \lambda < \rho_1^s$. Aquí ρ_1 denota al primer autovalor del operador Laplaciano con condiciones de Dirichlet cero.

Teorema. 1.4.1 *Sean $1 < q < 2_s^* - 1$ y $0 < s < 1$. Entonces el problema (P_λ) admite al menos una solución de energía siempre que, o bien,*

- *$N > \frac{2s(q+3)}{q+1}$ para $\lambda > 0$, o*
- *$N \leq \frac{2s(q+3)}{q+1}$ y $\lambda > 0$ es suficientemente grande.*

Con respecto a la regularidad de las soluciones, demostraremos que éstas son acotadas y “clásicas” en el sentido de que tienen tanta regularidad como requiere la ecuación. Es decir, admiten al menos “$2s$ derivadas”. Más aún, en el caso $s = 1/2$, pertenecen a $\mathcal{C}^{1,q}(\overline{\Omega})$ o $\mathcal{C}^{2,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$, depende de si $0 < q < 1$ o $q \geq 1$, respectivamente. En efecto, tenemos la siguiente.

Proposición. 1.5.3 *Sea u una función no negativa perteneciente al espacio $H(\Omega, s)$ definido en* (0.0.15). *Si u es una solución de energía del problema*

$$\begin{cases} A_s u = f(x,u) & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases}$$

con f satisfaciendo

$$|f(x,t|) \leq C(1 + |t|^p), \quad (x,t) \in \Omega \times \mathbb{R}, \tag{0.0.17}$$

para algún $1 \leq p \leq 2_s^* - 1$ y $C > 0$, entonces $u \in L^\infty(\Omega)$.

Proposición. 1.5.5 *Sea u una solución de energía de (P_λ). Entonces se verifica que*

(i) Si $s = 1/2$ y $q \geq 1$ entonces $u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^{2,\gamma}(\overline{\Omega})$, para algún $0 < \gamma < 1$.

(ii) Si $s = 1/2$ y $q < 1$, entonces $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) Si $s < 1/2$, entonces $u \in \mathcal{C}^{2s}(\overline{\Omega})$.

(iv) Si $s > 1/2$, entonces $u \in \mathcal{C}^{1,2s-1}(\overline{\Omega})$.

Organización del Capítulo 1. En la Sección preliminar 1.1 describimos el marco funcional adecuado para el estudio del problema (P_λ). Además introducimos el problema (local) extendido, mediante una variable extra, que es equivalente al nuestro y que proporcionará algunas ventajas computacionales (ver la Observación 1.1.4). Posteriormente dedicamos las Secciones 1.2, 1.3 y 1.4 a las demostraciones de los Teoremas 1.2.1–1.4.1. Finalmente los resultados de regularidad, junto con el teorema de concentración-compacidad, serán probados en la Sección 1.5.

Note: Los resultados obtenidos en este capítulo están contenidos en [18].

En el **Capítulo 2** consideraremos el mismo problema que tratamos en el Capítulo 1 pero reemplazando el Laplaciano fraccionario espectral por el operador fraccionario definido en (0.0.3). Es decir, centraremos nuestra atención en el problema

$$(D_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{\frac{N+2s}{N-2s}} & \text{en } \Omega \\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{en } \Omega, \end{cases}$$

donde $\Omega \subset \mathbb{R}^N$ es un dominio regular y acotado, $N > 2s$, $\lambda > 0$, $0 < s < 1$ y $(-\Delta)^s$ es el Laplaciano fraccionario definido en (0.0.3). Consideraremos, como en el Capítulo 1, de manera separada los dos casos $0 < q < 1$ (potencia cóncava) y $1 < q < 2_s^* - 1$ (potencia convexa). El caso lineal $q = 1$ ha sido estudiado recientemente en [138–142] obteniéndose la existencia de soluciones no necesariamente positivas.

Bajo condiciones apropiadas probaremos la existencia y multiplicidad de la soluciones del problema (D_λ). Enunciamos a continuación los resultados principales de este capítulo.

Teorema. 2.2.1 *Supongamos que* $0 < q < 1$. *Entonces existe* $0 < \Lambda < \infty$ *tal que el problema* (D_λ)

1 no tiene solución para $\lambda > \Lambda$*;*

2 tiene una solución minimal de energía para todo $0 < \lambda < \Lambda$ *y, más aún, la familia de soluciones minimales es creciente con respecto a* λ*;*

3 si $\lambda = \Lambda$*, existe al menos una solución de energía;*

4 para $0 < \lambda < \Lambda$*, existen al menos dos soluciones de energía.*

En el marco convexo obtenemos el mismo resultado de existencia que en el Teorema 1.4.1 para el problema (D_λ). Esto es.

Teorema. 2.3.1 *Sea $1 < q < 2_s^* - 1$. Entonces el problema (D_λ) admite al menos una solución de energía siempre que, o bien*

- *$N > \frac{2s(q+3)}{q+1}$ y $\lambda > 0$, o*
- *$N \leq \frac{2s(q+3)}{q+1}$ y $\lambda > 0$ es suficientemente grande.*

Con respecto a la regularidad de las soluciones del problema (D_λ), tenemos la siguiente.

Proposición. 2.4.1 *Sea u una solución de energía no negativa del problema*

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \text{en } \Omega, \\ u &= 0 \quad \text{en } \mathbb{R}^N \setminus \Omega, \end{cases}$$

y supongamos que $|f(x,t)| \leq C(1+|t|^p)$, para algún $1 \leq p \leq 2_s^* - 1$. Entonces $u \in L^\infty(\Omega)$.

Por el resultado anterior y utilizando [130, Proposition 1.1] se sigue que las soluciones de (D_λ) pertenecen al espacio $\mathcal{C}^s(\mathbb{R}^N)$. Ver también [107] y [147]. Hacemos notar aquí que el Teorema 2.2.1 corresponde a la versión no local del resultado en [10] mientras que el Teorema 2.3.1 puede entenderse como el equivalente no local de los resultados obtenidos, en el marco estándar del Laplaciano, en [45, Subsecciones 2.3, 2.4 and 2.5]. Ver también [92, Teoremas 3.2 y 3.3] para el caso del operador p-Laplaciano. En particular, observar que cuando $s = 1$, caso que corresponde con el clásico del Laplaciano, se tiene que $2s(q+3)/(q+1) = 2(q+3)/(q+1) < 4$, debido a la elección de $q > 1$.

Organización del Capítulo 2. En la Sección 2.1 introducimos el marco funcional apropiado para el estudio del problema (D_λ) y el operador no local (0.0.3). Dedicaremos las Sección 2.2 a la demostración del Teorema 2.2.1 y la Sección 2.3 a la prueba del Teorema 2.3.1. Finalmente en la Sección 2.4 demostraremos la Proposición 2.4.1.

Nota: Los resultados contenidos en este capítulo pueden encontrarse en [19].

En el **Capítulo 3** continuamos con el estudio de problemas cóncavo-convexos no locales pero, en este caso, motivados por los artículos [41], [80] y [85], analizaremos también la interacción entre el potencial de Hardy-Leray (ver [116]) y el Laplaciano fraccionario. Es decir, consideraremos el siguiente problema de Dirichlet

$$(H_{\lambda,\mu}) = \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = \mu u^q + u^p & \text{en } \Omega \\ u = 0 & \text{en } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{en } \Omega, \end{cases}$$

donde $\Omega \subset \mathbb{R}^N$ es un dominio regular acotado, $N > 2s$ y $0 < \lambda < \Lambda_{N,s}$, siendo $\Lambda_{N,s}$ la constante definida en (0.0.12). Aquí $\mu > 0$, $0 < s < 1$, $0 < q < 1$ y $1 < p < p(\lambda, s)$, donde $p(\lambda, s)$ es el dado en (3.1.4).

Presentamos a continuación los resultados principales de este capítulo. En cuanto a la existencia de soluciones, se tiene el siguiente.

Teorema. 3.2.1 *Sean $0 < q < 1$ y $0 < \lambda < \Lambda_{N,s}$. Entonces existe $0 < \Upsilon < \infty$ tal que el problema $(H_{\lambda,\mu})$*

1 no tiene solución para $\mu > \Upsilon$;

2 para todo $0 < \mu < \Upsilon$, existe una solución minimal de energía si $1 < p \leq 2_s^* - 1$, una solución minimal débil en el caso $2_s^* - 1 < p < p(\lambda, s)$ y, más aún, la familia de soluciones minimales es creciente con respecto a μ;

3 si $\mu = \Upsilon$, existe al menos una solución débil;

4 si $1 < p < 2_s^* - 1$, existen al menos dos soluciones de energía para $0 < \mu < \Upsilon$.

Cuando p es mayor que el valor crítico $p(\lambda, s)$, obtenemos el siguiente resultado de no existencia.

Teorema. 3.4.3 *Supongamos que $0 < \lambda \leq \Lambda_{N,s}$. Sea $p \geq p(\lambda, s)$. Entonces hay explosión completa en el problema $(H_{\lambda,\mu})$.*

Organización del Capítulo 3. En la Sección 3.1 describimos el marco funcional adecuado para el estudio del problema $(H_{\lambda,\mu})$. Dedicaremos las Secciones 3.2 y 3.3 a la prueba del Teorema 3.2.1. Finalmente en la Sección 3.4 demostraremos el Teorema 3.4.3.

Nota: Los resultados de este capítulo pueden encontrarse en [21].

En la **Parte II** de este trabajo, estudiaremos operadores no locales más generales, en particular, los llamados operadores integro-diferenciales. Para ser consistentes con qué se entiende por un operador no local de manera general, introducimos la siguiente.

Definición 0.0.1. *([57, Definición 21]) Un operador no local I es una regla que asigna a una función u el valor $I(u, x)$ en todo punto x satisfaciendo las siguientes condiciones.*

- *$I(u, x)$ está bien definido siempre que $u \in \mathcal{C}^{1,1}(x) \cap L^1(\mathbb{R}^N, \omega)$.*

- *Si* $u \in \mathcal{C}^{1,1}(\Omega) \cap L^1(\mathbb{R}^N, w)$ *entonces* $I(u,x)$ *es una función de* x *continua en* Ω.

Típicamente nuestro peso ω *tendrá la forma* $\omega(y) = 1/(1+|y|^{N+2s})$, $0 < s < 1$.

En la definición anterior, $\mathcal{C}^{1,1}$ denota la familia de funciones que satisface la Definición 4.1.1.

El objetivo de esta segunda parte de este trabajo es probar un resultado de regularidad para operadores integro-diferenciales (ver Capítulo 4) para obtener posteriormente la regularidad de las superficies mínimas no locales que L. Caffarelli, J. M. Roquejoffre y O. Savin han introducido recientemente en [53] (ver Capítulo 5).

Para dar una idea intuitiva de lo que es una superficie mínima no local usaremos el concepto de "perímetro no local". Al igual que el perímetro clásico mide la variación total de la función característica de un conjunto E en un dominio fijo Ω, el perímetro no local mide la variación de dicha función característica pero dentro y fuera de este dominio fijo con respecto a un operador fraccionario.
Una *superficie mínima no local* es la frontera del conjunto E que minimiza el perímetro no local en un dominio fijo Ω con las "condiciones de frontera" que $E \cap (\mathbb{R}^N \setminus \Omega)$ prescribe.
Sorprendentemente, como veremos en la Introducción del Capítulo 5, como el perímetro no local está relacionado con la norma $H^{s/2}$ de la función caraterística χ_E, estas superficies se pueden obtener minimizandos estas normas. Precisamente, cuando $s < 1$ y E es razonablemente suave, $\|\chi_E\|_{H^{s/2}}$ es finito mientras que para $s = 1$ no es cierto.
Para establecer de manera más clara la relación de estos objetos con las superficies mínimas clásicas remarcamos aquí que, como se verá en el Capítulo 5, éstas tendrán *curvatura media no local* igual a cero. Más aún, cuando $s \to 1^-$ el perímetro no local aproxima al perḿetro clásico (ver [13, 60]).
Para obtener una idea intuitiva de las superficies mínimas no locales utilizando el concepto de curvatura media no local, recomendamos el reciente trabajo [1]. Ver también [170] donde el autor realiza un resumen de las principales propiedades y problemas donde aparecen dichas superficies.

Volviendo a la estructura de la Parte II, en el **Capítulo 4** desarrollaremos una"teoría de regularidad de Schauder" para las soluciones viscosas de una familia de ecuaciones integro-diferenciales formadas por una clase concreta de núcleos no invariantes por traslaciones. De manera específica, consideraremos núcleos $K = K(x,w) : \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ que satisfacen ciertas hipótesis generales. En lo que sigue, $1 < \sigma < 2$.

En primer lugar suponemos que K es próximo al núcleo del Laplaciano fraccionario, es decir

$$(4.2.1)\begin{cases} \text{existen } a_0, r_0 > 0 \text{ y } 0 < \eta < a_0/4 \text{ tal que} \\ \left|\dfrac{|w|^{N+\sigma}K(x,w)}{2-\sigma} - a_0\right| \le \eta, \qquad x \in B_1,\ w \in B_{r_0} \setminus \{0\}. \end{cases}$$

Además supondremos que

$$(4.2.2)\begin{cases} \text{existen } k \in \mathbb{N} \cup \{0\} \text{ y } C_k = C(k) > 0 \text{ tales que} \\ K \in \mathcal{C}^{k+1}\left(B_1 \times (\mathbb{R}^N \setminus \{0\})\right), \\ \|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{L^\infty(B_1)} \le \dfrac{C_k}{|w|^{N+\sigma+|\theta|}}, \\ \mu, \theta \in (\mathbb{N} \cup \{0\})^N,\ |\mu| + |\theta| \le k+1,\ w \in \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Observamos que usaremos $|\cdot|$ tanto para denotar la norma Euclídea de un vector como, siendo $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ un multiíndice, para denotar $|\alpha| := \alpha_1 + \cdots + \alpha_N$. Sin embargo el significado de $|\cdot|$ siempre se deducirá de manera clara del contexto en el que nos encontremos.

Por lo tanto, denotando a lo largo de toda la Parte II por

$$\delta u(x,w) := u(x+w) + u(x-w) - 2u(x), \tag{0.0.18}$$

el resultado principal de este capítulo se puede enunciar como sigue.

Teorema. 4.2.1 *Sean $1 < \sigma < 2$, $k \in \mathbb{N} \cup \{0\}$, y $u \in L^\infty(\mathbb{R}^N)$ una solución viscosa de la ecuación*

$$\int_{\mathbb{R}^N} K(x,w)\, \delta u(x,w) dw = f(x, u(x)) \qquad \text{en } B_1,$$

con $f \in \mathcal{C}^{k+1}(B_1 \times \mathbb{R})$. Supongamos que $K : B_1 \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ satisface las hipótesis (4.2.1) y (4.2.2) para el mismo valor de k.

Entonces, si η en (4.2.1) es lo suficientemente pequeño (cuyo tamaño es independiente de k), se tiene que $u \in \mathcal{C}^{k+\sigma+\alpha}(B_{1/2})$ para todo $\alpha < 1$, y

$$\|u\|_{\mathcal{C}^{k+\sigma+\alpha}(B_{1/2})} \le C\left(1 + \|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1 \times [-M.M])}\right),$$

donde $M = \|u\|_{L^\infty(B_1)}$ y $C > 0$ depende sólo de N, σ, k, C_k, y $\|f\|_{\mathcal{C}^{k+1}(B_1 \times \mathbb{R})}$.

Organización del Capítulo 4. En la Sección preliminar 4.1 introducimos la teoría de las ecuaciones integro-diferenciales y presentamos varios resultados de regularidad bien conocidos. Posteriormente dedicamos la

Sección 4.2 a la demostración del Teorema 4.2.1.

En el **Capítulo 5**, usando los resultados obtenidos en el Capítulo 4, y motivados por el hecho de que las (s)-superficies mínimas no locales aproximan a las clásicas cuando $s \to 1^-$, demostraremos que las superficies mínimas no locales son suaves. De hecho demostraremos que las (s)-superficies mínimas no locales que son $\mathcal{C}^1$ son de hecho $\mathcal{C}^\infty$. Para ello, a lo largo de este capítulo escribiremos $x \in \mathbb{R}^N$ como $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Más aún, dados $r > 0$ y $p \in \mathbb{R}^N$, definimos

$$K_r(p) := \{x \in \mathbb{R}^N \, : \, |x' - p'| < r \text{ y } |x_N - p_N| < r\}.$$

Para $p' \in \mathbb{R}^{N-1}$, consideraremos

$$B_r^{N-1}(p') := \{x' \in \mathbb{R}^{N-1} \, : \, |x' - p'| < r\},$$

$K_r := K_r(0)$, $B_r := B_r(0)$ y $B_r^{N-1} := B_r^{N-1}(0)$.

Con esta notación obtendremos el siguiente.

Teorema. 5.1.1 *Sean $0 < s < 1$ y ∂E una (s)-superficie mínima no local en K_R para algún $R > 0$. Supongamos que*

$$\partial E \cap K_R = \left\{(x', x_N) \, : \, x' \in B_R^{N-1} \text{ y } x_N = u(x')\right\},$$

para algua función $u : B_R^{N-1} \to \mathbb{R}$, tal que $u \in \mathcal{C}^{1,\alpha}(B_R^{N-1})$ para todo $\alpha < s$ y $u(0) = 0$. Entonces,

$$u \in \mathcal{C}^\infty(B_\rho^{N-1}) \quad \text{para todo } 0 < \rho < R.$$

Organización del Capítulo 5. En la Sección 5.1 introducimos la noción de (s)-superficie mínima no local con detalle y algunos resultados conocidos acerca de las mismas. En la Sección 5.2 demostramos el Teorema 5.1.1 utilizando el resultado de regularidad obtenido en la Sección 4.2.

Nota: Los resultados de estos dos capítulos están contenidos en [20].

Finalmente dedicamos la **Parte III** de este trabajo al estudio del problema parabólico más sencillo que involucra al Laplaciano fraccionario, es decir, la ecuación del calor no local. De manera más precisa, en el **Capítulo 6** extendemos algunos resultados clásicos de la ecuación del calor que demostró D.V. Widder en [172] en el marco de las ecuaciones de difusión no locales probando unicidad de las soluciones positivas acorde con los Principios de la Termodinámica. El resultado principal de este capítulo es el siguiente.

Teorema. 6.1.3 *Si $u \geq 0$ es una solución fuerte de*

$$u_t + (-\Delta)^s u = 0 \quad \text{para } (x,t) \in \mathbb{R}^N \times (0,T), \quad 0 < s < 1,$$

entonces

$$u(x,t) = \int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy.$$

Aquí

$$p^t(x) = \frac{1}{t^{N/2s}} p\left(\frac{x}{t^{1/2s}}\right),$$

y

$$p(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi - |\xi|^{2s}} d\xi,$$

es la solución de

$$\begin{cases} p^t + (-\Delta)^s p = 0 & \text{para } (x,t) \in \mathbb{R}^N \times (0,T), \\ p(x,0) = \delta_0(x) & \text{en } \mathbb{R}^N. \end{cases}$$

Organización del Capítulo 6. En la Sección 6.1 introduciremos el problema clásico de Widder y los distintos tipos de solución que utilizaremos en este capítulo. Después de ello, en la Sección 6.2 probaremos la unicidad para soluciones débiles que, a su vez, será el paso fundamental para obtener nuestro teorema de representación. Posteriormente en la Sección 6.3 demostraremos el resultado principal para soluciones fuertes. Empezaremos probando un lema de comparación que nos permitirá mostrar como cualquier solución positiva fuerte $u(x,t)$ es mayor o igual que la convolución de la traza $u(x,0)$ con el núcleo $p^t(x)$. Por un argumento de traslación probaremos que cualquier solución fuerte positiva es también una solución débil y, por el resultado de unicidad de la sección anterior, concluiremos la prueba del teorema. Finalmente en la última Sección 6.4, estableceremos un comportamiento puntual para las soluciones positivas que, a parte de tener interés por sí mismo, puede dar una prueba alternativa de nuestro teorema de representación.

Nota: Los resultados presentados en este capítulo están contenidos en [22].

Problemas abiertos y cuestiones por estudiar.

1. Un primer problema abierto de este trabajo es la demostración de la afirmación *4* del Teorema 1.2.1 para el caso $0 < s < 1/2$. Debido a la falta de regularidad, ver Proposición 1.5.5, no sabemos cómo separar las soluciones de manera correcta (ver el Lema 1.2.4 y también [67,73]).

Esta dificultad parace ser técnica. Hacemos notar aquí que la misma restricción en s aparece en el estudio de estimaciones uniformes L^∞ de tipo Gidas-Spruck ([96]) para el operador A_s que han sido estudiadas en [37, 162]. En ese caso la principal dificultad proviene de encontrar un teorema de tipo Liouville para $0 < s < 1/2$ en $\mathbb{R}^N_+$.

2. Dejamos abierto el rango $2s < N < 4s$ en el Teorema 1.3.1. Ver el caso especial $s = 1$ y $N = 3$ en [45]. Como observamos en [161], si $s = 1/2$ este rango es vacío. Cuando consideramos el problema de Dirichlet con $(-\Delta)^s$ en vez de con A_s, observamos que los autores dejan el mismo rango dimensional por resolver(ver [140, Theorem 4]). El motivo por el cual no hemos resuelto nuestro teorema para $N < 4s$ es porque no podemos probar la Proposición 1.3.2. De hecho la estimación obtenida en (1.2.60) cuando $N < 4s$ no es suficiente para obtener la conclusión de dicha proposición.

3. Una cuestión interesante que proponemos como un posible objetivo de estudio es obtener un resultado de multiplicidad establecido en la Observación 3.2.6 del Capítulo 3 para el caso crítico $p = 2_s^* - 1$. Una posible idea para resolver este problema podría ser aplicar el principio de concentración-compacidad de P. L. Lions [118,119], modificando los cálculos hechos en el Lema 2.2.10, y utilizar los minimizantes apropiados asociados al operador lineal $L(u) := (-\Delta)^s u - \lambda u/|x|^{2s}$. Para obtener dichos minimizantes se podría seguir las ideas dadas en [164]. Hacemos notar aquí que la existencia de un mínimo en el espacio de energía podría obtenerse haciendo un cambio de variable para *esconder* el potencial de Hardy en el operador fraccionario y así obtener la acotación de las soluciones. En nuestra opinión esto conllevaría un esfuerzo computacional bastante elevado.

4. Dejamos como problema abierto la existencia de, al menos, dos soluciones positivas del problema $(H_{\lambda,\mu})$ dado en el Capítulo 3 cuando μ pertenece a un intervalo adecuado $(0, \Upsilon)$, $\lambda = \Lambda_{N,s}$ y $1 < p < 2_s^* - 1$. Proponemos utilizar la desigualdad de Hardy-Sobolev mejorada, dada por ejemplo en [85] (ver también [90]), como una herramienta para resolver dicha cuestión. Hacemos hincapié aquí que en el caso local este tipo de resultados ha sido obtenido en [94].

5. En el Capítulo 5, un posible problema interesante que queda por resolver es la analiticidad de las superficies mínimas no locales estudiadas en el mismo.

6. Finalmente como problema abierto proponemos encontrar la mayor clase posible de soluciones viscosas positivas para las cuales el teorema de respresentación Teorema 6.1.3 se verifique. Más aún, en nuestra opinión, podría ser útil demostrar que el comportamiento puntual dado en el Lema 6.4.2 junto con algunos resultados de comparación, pueden proporcionar, como en el caso clásico, una prueba alternativa al resultado principal del Capítulo 6.

Introduction, summary of contents and conclusions.

Introduction: the fractional Laplacian operator.

This work is devoted to the study of several questions concerning elliptic and parabolic problems in Partial Differential Equations that involve the fractional Laplacian and other nonlocal operators. The theory of singular integrals and general nonlocal operators on Banach spaces has been treated at length in harmonic and functional analysis. Classical results about this topic can be found in the papers of S. Bochner [32], T. Kato [108], H. Komatsu [110], H. Landkof [114] and E. Stein [154] among others. However, only in recent years considerable attention has been given to the potential applications of these nonlocal structures and their connection with many real world phenomena. Indeed, non local operators naturally appear in elasticity problems [148], thin obstacle problem [52], phase transition [15, 50, 152], flames propagation [54], crystal dislocation [167], stratified materials [135], water waves [71, 72, 157], quasi-geostrophic flows [62, 121] and others.

Since these operators are related to Lévy processes (see the Introduction to Chapter 4) and have a lot of applications to mathematical finance, they have been also studied from a probabilistic point of view (see for instance [24, 33, 36, 103, 106]). A nice and simple approach that shows how singular integrals naturally appear as a continuous limit of discrete long jump random walks can be found in [169].

The basic example of a linear nonlocal operator is given by the fractional Laplacian. Let us recall that if we denote by Δ the Laplacian in $\mathbb{R}^N$, $N \geq 1$, and for a function u in the Schwartz's class $\mathcal{S}(\mathbb{R}^N)$ we define its Fourier transform as

$$\widehat{u}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi}\, dx, \tag{0.0.19}$$

then one has

$$\widehat{(-\Delta)u}(\xi) = |\xi|^2 \widehat{u}(\xi), \quad \xi \in \mathbb{R}^N.$$

From this, it becomes natural to define, for $0 < s < 1$, the fractional Laplacian as

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}(\xi), \quad \xi \in \mathbb{R}^N. \tag{0.0.20}$$

The first observation that we can do from this definition is that for $u \in \mathcal{S}$ and $0 < s < 1$, $(-\Delta)^s u$ is not necessarily a function in the Schwartz class because $|\xi|^{2s}$ introduces a singularity at the origin in its Fourier transform.

As is well known (see [114, 154, 156, 169]) this operator may be also represented, for suitable functions, as a principal value of the form

$$\begin{aligned}(-\Delta)^s u(x) \;&:=\; C(N,s)\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \\ &:=\; C(N,s)\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy. \end{aligned} \tag{0.0.21}$$

In the previous expression

$$C(N,s) := \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi\right)^{-1} = \frac{4^s\Gamma\left(\frac{N}{2}+s\right)}{-\pi^{\frac{N}{2}}\Gamma(-s)}, \tag{0.0.22}$$

is a normalizing constant chosen to guarantee that (0.0.20) is satisfied (see [76, 151, 156, 169]). Moreover, (see [76, Corollary 4.2]), we have

$$\lim_{s\to 1^-} \frac{C(N,s)}{s(1-s)} = \frac{4N}{\omega_{N-1}} \quad \text{and} \quad \lim_{s\to 0^+} \frac{C(N,s)}{s(1-s)} = \frac{2}{\omega_{N-1}},$$

where ω_{N-1} is the $(N-1)$ dimensional measure of the unit sphere $\mathbb{S}^{N-1}$. With these asymptotic properties of the constant $C(N,s)$, in [76, Proposition 4.4] (see also [156, Proposition 5.3]) the authors prove that

$$\lim_{s\to 1^-} (-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\to 0^+} (-\Delta)^s u = u, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^N),$$

where the convergence is given in the sense of distributions. Note that these previous limits may be also obtained formally from (0.0.20).

From (0.0.21) one can check that

$$|(-\Delta)^s \phi(x)| \le \frac{C}{1+|x|^{N+2s}}, \quad \text{for every } \phi \in \mathcal{S}(\mathbb{R}^N). \tag{0.0.23}$$

This motivates the introduction of the space

$$\mathcal{L}^s(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} : \quad \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\,dx < \infty\}, \tag{0.0.24}$$

endowed with the natural norm

$$\|u\|_{\mathcal{L}^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\, dx.$$

Then, if $u \in \mathcal{L}^s$ and $\phi \in \mathcal{S}$, using (0.0.23), we can formally define the duality product $\langle (-\Delta)^s u, \phi\rangle$ in the distributional sense as

$$\langle (-\Delta)^s u, \phi\rangle := \int_{\mathbb{R}^N} u(-\Delta)^s \phi\, dx.$$

Coming back to the definition given in (0.0.21) we clearly deduce that $(-\Delta)^s$ is a nonlocal operator because the value of $(-\Delta)^s u(x)$ does not depend only on the behavior of u in a neighborhood of x but on the whole $\mathbb{R}^N$. Observe as an example that if u is a nonnegative function with compact support in, say, Ω then $(-\Delta)^s u(x) \neq 0$ for every $x \notin \Omega$. We remark here that, as we will see along this work, this property, intrinsic to all the operators that we will study, creates complications because the classical local PDE methods cannot be applied to study nonlinear problems. To overcome this difficulty L. Caffarelli and L. Silvestre proved in [55] that every fractional Laplacian can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann type condition via an extension problem that is local in nature. This tool can be applied for example to prove Harnack's inequality for $(-\Delta)^s$. We will use the extension's technique to resolve the problem treated in Chapter 1.

In addition to the non locality, many other properties of this operator can be found in [151]. One of these properties that will be used often along this work is that

$$|(-\Delta)^s \phi| \leq C, \quad \phi \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\mathbb{R}^N),\ \beta > 0. \tag{0.0.25}$$

When $s \geq 1/2$ the previous regularity assumption must be understood as $\phi \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,2s+\beta-1}(\mathbb{R}^N)$. We note here that (0.0.25) can be also obtained replacing condition $\phi \in L^\infty(\mathbb{R}^N)$ by the weaker condition $\phi \in \mathcal{L}^s(\mathbb{R}^N)$ defined in (0.0.24).
We will prove (0.0.23) and (0.0.25) in Sections 2.2 and 2.1 of Chapter 2 respectively.

An equivalent definition of the fractional Laplacian for regular functions may be given as the average of a second order increment. In fact, a standard change of variables transforms (0.0.21) into the expression

$$(-\Delta)^s u(x) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}}\, dy. \tag{0.0.26}$$

The argument that we will present in the proof of (0.0.25) (see Section 2.1 of Chapter 2), shows that this expression is well fitted to treat the singularity of the kernel when $s \geq 1/2$.

Finally we introduce another equivalent definition of $(-\Delta)^s$ that comes from the theory of semigroups and that show the relationship with $(-\Delta)^s$ and the operator A_s, defined below, that will be the object of study in Chapter 1. This approach also allows to extend the definition of fractional powers to a wide generality of positive operators (see [156]). In our case, let $e^{-t(-\Delta)}$ denote the heat semigroup generator associated to $(-\Delta)$. This means that for suitable u,

$$e^{-t(-\Delta)}u(x) = \int_{\mathbb{R}^N} G_t(x)u(x-z)\,dz = \frac{1}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4t}}u(x-z)\,dz,$$

is the unique solution of

$$\begin{cases} v_t + (-\Delta)v = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ v(x,0) = u(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Then, a formal application of the Fourier transform (see [156, Lemma 5.1]), shows that

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{-t(-\Delta)} - 1)u(x)\,\frac{dt}{t^{1+s}}, \tag{0.0.27}$$

where

$$\Gamma(-s) = -\frac{\Gamma(1-s)}{s} = \int_0^\infty (e^{-t} - 1)\frac{dt}{t^{1+s}} < 0. \tag{0.0.28}$$

We now present the important role of the fractional Laplacian in connection with Sobolev spaces in $\mathbb{R}^N$. This is summarized in the following well-known theorems. The first one is the fractional embedding theorem that will be frequently used along this work and the second one is the fractional Hardy-Sobolev inequality that we will use in Chapter 3.

Theorem 0.0.2. (See [7, Theorem 7.58], [76, Theorem 6.5], [117, 153]) *Let $0 < s < 1$ and $N > 2s$. There exists a constant $S(N,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$S(N,s)\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \leq \frac{2}{C(N,s)}\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)},$$

where

$$2^*_s = \frac{2N}{N-2s}, \tag{0.0.29}$$

is the Sobolev critical exponent.

Theorem 0.0.3. (See [25, 90, 101]) *Let u be a function in $\mathcal{C}_0^\infty(\mathbb{R}^N)$ and $N > 2s$. Then $u/|x|^s \in L^2(\mathbb{R}^N)$ and*

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}}\,dx \le ||(-\Delta)^{s/2}u||^2_{L^2(\mathbb{R}^N)},$$

where

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}, \tag{0.0.30}$$

is an optimal constant that cannot be achieved.

We finish this section pointing out that, since the first part of this work is devoted to the study of Dirichlet problems, we will use frequently the Theory of Calculus of Variations. In the more abstract sense, let us consider the linear problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ \text{zero } \textit{boundary} \text{ condition in } \Omega, \end{cases}$$

where the operator L is well defined for elements in a Hilbert space H. Then, by the Lax-Milgram's Theorem, we know that if f belongs to the dual space H^* there exists a unique $u_f \in H$ such that

$$\langle u_f, v\rangle_H = f(v) \quad \text{for all } v \in H. \tag{0.0.31}$$

Here $f(v) = \langle f, v\rangle_{H^*,H}$. Moreover u_f is obtained as the minimizer of the functional

$$J(u) = \frac{1}{2}\langle Lu, u\rangle_{H^*,H} - f(u).$$

That is, (0.0.31) is the Euler-Lagrange equation associated to J. This theory can be applied to the nonlocal operators that we will study in this work. For example it can be applied to linear Dirichlet problems that involve the spectral fractional Laplacian defined as

$$A_s u(x) := \sum \rho_j^s a_j \varphi_j(x), \quad x \in \Omega, \tag{0.0.32}$$

where $u(x) = \sum a_j \varphi_j(x)$, $x \in \Omega$ and (φ_j, ρ_j) are the eigenfunctions and eigenvectors of $(-\Delta)$ in Ω with zero boundary data. Here the natural energy space associated to this operator is the Hilbert space defined as

$$H(\Omega, s) := \left\{u \in L^2(\Omega) \,:\, \|u\|_{H(\Omega,s)} = \|A_{s/2}u\|_{L^2(\Omega)} < \infty\right\}. \tag{0.0.33}$$

We can also use this theory to treat Dirichlet problems with the fractional Laplacian defined in (0.0.21) and the associated energy space

$$X_0^s(\Omega) := \left\{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

endowed with the norm

$$\|u\|^2_{X_0^s(\Omega)} = \frac{2}{C(N,s)} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)}.$$

Here $C(N,s)$ is the normalizing constant given in (0.0.22).

However, in this work we will also consider Dirichlet problems with certain nonlocal operators but with non linearities instead. Therefore we cannot use the previous theory of minimization and, as we will see, we need another results of nonlinear critical point theory. We will give more details about the previous operators and the associated Sobolev spaces in Section 1.1 of Chapter 1 and in Section 2.1 of Chapter 2 respectively. See also [145] where these two operators are compared.

All along this work, we will use the usual convention that c and C denote positive constants that could have different values at different places, even in the same line. Sometimes we will use parenthesis to indicate the explicit dependence of these constant on a particular parameter. Thus, for example, $C(a,b,c)$, expresses that C might depend on a, b and c.

Summary of contents and conclusions.

As we have said, the objective of this work is to study some problems in PDE that involve non local operators. **Part I** is dedicated to Dirichlet elliptic non local problems with a general concave-convex nonlinearity (See Chapters 1-3). In the classical (local) case, problems of the type

$$\left\{ \begin{array}{ll} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{0.0.34}$$

for different kind of nonlinearities f, have been the main subject of investigation in a large number of works in the last thirty years. See for example the list (far from complete) [9, 10, 45, 92, 119]. One of the most important cases of problem (0.0.34) is the one given by the critical power $f(u) = u^{\frac{N+2}{N-2}}$, $N > 2$. It is well known that this problem has no positive solutions provided the domain is star shaped. In the pioneering work [45], Brezis and Nirenberg showed that, contrary to intuition, the critical problem with small linear perturbations can provide positive solutions. After that, in [10], using the results on concentration-compactness of Lions [119], Ambrosetti, Brezis and Cerami proved some results on existence and multiplicity of solutions for a sublinear perturbation of the critical power, among others.

Following this spirit, in **Chapter 1** we study the effect of lower order perturbations in the existence of positive solutions to the following critical elliptic problem that involves the fractional Laplacian defined via the spectral decomposition. That is, we consider

$$(P_\lambda) = \begin{cases} A_s u = \lambda u^q + u^{\frac{N+2s}{N-2s}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $N > 2s$, $\lambda > 0$, $0 < q < 2_s^* - 1 = \frac{N+2s}{N-2s}$ and A_s is the spectral fractional Laplacian given in (0.0.32).

Our main results dealing with Problem (P_λ) are the following.

Theorem. 1.2.1 *Let $0 < q < 1$ and $0 < s < 1$. Then, there exists $0 < \Lambda < \infty$ such that the problem (P_λ)*

1 has no solution for $\lambda > \Lambda$;

2 has a minimal energy solution for any $0 < \lambda < \Lambda$ and, moreover, the family of minimal solutions is increasing with respect to λ;

3 if $\lambda = \Lambda$, there is at least one energy solution;

4 if $s \geq 1/2$, there are at least two energy solutions for $0 < \lambda < \Lambda$.

Theorem. 1.3.1 *Assume $q = 1$, $0 < s < 1$ and $N \geq 4s$. Then, the problem (P_λ)*

1 has no solution for $\lambda \geq \rho_1^s$;

2 has at least one energy solution for each $0 < \lambda < \rho_1^s$. Here ρ_1 is the first eigenvalue of the Laplace operator with zero Dirichlet condition.

Theorem. 1.4.1 *Let $1 < q < 2_s^* - 1$ and $0 < s < 1$. Then, problem (P_λ) admits at least one energy solution provided that either*

- *$N > \frac{2s(q+3)}{q+1}$ and $\lambda > 0$, or*
- *$N \leq \frac{2s(q+3)}{q+1}$ and $\lambda > 0$ is sufficiently large.*

With respect to the regularity of the solutions we will see that they are bounded and "classical" in the sense that they have as much regularity as is required in the equation. By this we mean that they have at least "$2s$ derivatives". Even more, if $s = 1/2$, they belong to $\mathcal{C}^{1,q}(\overline{\Omega})$ or $\mathcal{C}^{2,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$, depending on whether $0 < q < 1$ or $q \geq 1$, respectively. In fact we have the following.

Proposition. 1.5.3 *Let u be a nonnegative function of the space $H(\Omega, s)$ given in* (0.0.33)*. If u is an energy solution to the problem*

$$\begin{cases} A_s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

with f satisfying

$$|f(x,t|) \leq C(1+|t|^p), \quad (x,t) \in \Omega \times \mathbb{R}, \tag{0.0.35}$$

for some $1 \leq p \leq 2_s^* - 1$ and $C > 0$, then $u \in L^\infty(\Omega)$.

Proposition. 1.5.5 *Let u be an energy solution of (P_λ). Then the following hold*

(i) If $s = 1/2$ and $q \geq 1$ then $u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^{2,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$.

(ii) If $s = 1/2$ and $q < 1$, then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $s < 1/2$, then $u \in \mathcal{C}^{2s}(\overline{\Omega})$.

(iv) If $s > 1/2$, then $u \in \mathcal{C}^{1,2s-1}(\overline{\Omega})$.

Organization of Chapter 1. In the preliminary Section 1.1 we describe the appropriate functional setting for the study of problem (P_λ). Also we include the definition of a local equivalent problem, with the aid of an extra variable, which provides some advantages, (see Remark 1.1.4). Then we devote Sections 1.2, 1.3 and 1.4 to the proofs of Theorems 1.2.1–1.4.1. Finally the regularity results, together with a concentration-compactness theorem, are proved in Section 1.5.

Note: The results obtained in this chapter are contained in [18].

In **Chapter 2** we consider the same problem as in Chapter 1 but replacing the spectral Laplacian by the fractional operator given in (0.0.21). That is, we focus our attention in the following problem

$$(D_\lambda) = \begin{cases} (-\Delta)^s u = \lambda u^q + u^{\frac{N+2s}{N-2s}} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $N > 2s$, $\lambda > 0$, $0 < s < 1$ and $(-\Delta)^s$ is the fractional Laplace operator defined in (0.0.21). We will consider, as in Chapter 1, separately the two cases $0 < q < 1$ (concave power) and

$1 < q < 2_s^* - 1$ (convex power). The linear case $q = 1$ has been recently studied in [138–142] obtaining solutions that are not necessarily positive.

Under appropriate conditions we prove existence and multiplicity of solutions to Problem (D_λ). We now state the main results of this chapter.

Theorem. 2.2.1 *Assume* $0 < q < 1$. *Then, there exists* $0 < \Lambda < \infty$ *such that problem* (D_λ)

1 *has no solution for* $\lambda > \Lambda$;

2 *has a minimal energy solution for any* $0 < \lambda < \Lambda$ *and, moreover, the family of minimal solutions is increasing with respect to* λ;

3 *if* $\lambda = \Lambda$, *there exists at least one energy solution;*

4 *for* $0 < \lambda < \Lambda$, *there are at least two energy solutions.*

In the convex framework we obtain the same existence result as in Theorem 1.4.1 for the problem (D_λ). That is.

Theorem. 2.3.1 *Let* $1 < q < 2_s^* - 1$. *Then, problem* (D_λ) *admits at least one energy solution provided that either*

- $N > \frac{2s(q+3)}{q+1}$ *and* $\lambda > 0$, *or*
- $N \leq \frac{2s(q+3)}{q+1}$ *and* $\lambda > 0$ *is sufficiently large.*

With respect to the regularity of the solutions of (D_λ), we have the following.

Proposition. 2.4.1 *Let* u *be a nonnegative energy solution to the problem*

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and assume that $|f(x,t)| \leq C(1 + |t|^p)$, *for some* $1 \leq p \leq 2_s^* - 1$. *Then* $u \in L^\infty(\Omega)$.

From the previous result, by [130, Proposition 1.1] it follows that solutions of (D_λ) belong to $\mathcal{C}^s(\mathbb{R}^N)$. See also [107] and [147]. We remark here that Theorem 2.2.1 corresponds to the nonlocal version of the result in [10], while Theorem 2.3.1 may be seen as the nonlocal counterpart of the results obtained in the standard Laplace framework in [45, Subsection 2.3, 2.4 and 2.5]. See also [92, Theorem 3.2 and 3.3] for the case of the p-Laplacian operator. In particular, note that when $s = 1$, which corresponds to the classical case of Laplace, one has $2s(q+3)/(q+1) = 2(q+3)/(q+1) < 4$, due to the choice of $q > 1$.

Organization of Chapter 2. In Section 2.1 we introduce the appropriate functional framework for the study of problem (D_λ) and the non local operator (0.0.21). Then we devote Section 2.2 to the proof of Theorem 2.2.1 and Section 2.3 to the proof of Theorem 2.3.1. Finally in Section 2.4 we prove Proposition 2.4.1.

Note: The results contained in this chapter can be found in [19].

In **Chapter 3** we continue with the study of non local concave-convex problems but, in this case, motivated by the papers [41], [80] and [85], we also analize the interplay between the Hardy-Leray potential (see [116]) and the fractional Laplacian. That is, we consider the following Dirichlet problem

$$(H_{\lambda,\mu}) = \begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = \mu u^q + u^p & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, $N > 2s$ and $0 < \lambda < \Lambda_{N,s}$, with $\Lambda_{N,s}$ defined in (0.0.30). Here $\mu > 0$, $0 < s < 1$, $0 < q < 1$ and $1 < p < p(\lambda, s)$, where $p(\lambda, s)$ is given in (3.1.4).

We now present the main results of this chapter. Concerning the existence of solutions we get the following.

Teorema. 3.2.1 *Let $0 < q < 1$ and $0 < \lambda < \Lambda_{N,s}$. Then, there exists $0 < \Upsilon < \infty$ such that the problem $(H_{\lambda,\mu})$*

1 has no solution for $\mu > \Upsilon$;

2 for any $0 < \mu < \Upsilon$, there exists a minimal energy solution if $1 < p \leq 2^*_s - 1$, a minimal weak solution in the case $2^*_s - 1 < p < p(\lambda, s)$ and, moreover, the family of minimal solutions is increasing with respect to μ;

3 if $\mu = \Upsilon$, there is at least one weak solution;

4 if $1 < p < 2^*_s - 1$, there are at least two energy solutions for $0 < \mu < \Upsilon$.

When p is greater than the critical value $p(\lambda, s)$, we obtain the following non existence result.

Theorem. 3.4.3 *Assume that $0 < \lambda \leq \Lambda_{N,s}$. Let $p \geq p(\lambda, s)$. Then, there exists complete blow up of the problem $(H_{\lambda,\mu})$.*

Organization of Chapter 3. In Section 3.1 we describe the appropriate functional setting for the study of problem $(H_{\lambda,\mu})$. Then we devote Sections 3.2 and 3.3 to the proof of Theorem 3.2.1. Finally in Section 3.4 we prove Theorem 3.4.3.

Note: The results contained in this chapter can be found in [21].

In **Part II** of this work we study more general non local operators, particularly, the so called integro-differential operators. To be consistent with what we consider is a general non local operator, we introduce the following.

Definition 0.0.4. *([57, Definition 21]) A nonlocal operator I is a rule that assigns to a function u a value $I(u,x)$ at every point x satisfying the following assumptions.*

- *$I(u,x)$ is well defined as long as $u \in \mathcal{C}^{1,1}(x) \cap L^1(\mathbb{R}^N, \omega)$.*
- *If $u \in \mathcal{C}^{1,1}(\Omega) \cap L^1(\mathbb{R}^N, w)$, then $I(u,x)$ is continuous in Ω as a function of x.*

Typically our weight ω will be of the form $\omega(y) = 1/(1+|y|^{N+2s})$, $0<s<1$.

The class of functions $\mathcal{C}^{1,1}(x)$ will be properly defined in Section 4.1.

The objective of this part of the work is to prove a regularity result for these integro-differential operators (see Chapter 4) in order to improve the regularity of non local minimal surfaces that L. Caffarelli, J. M. Roquejoffre and O. Savin have recently introduced in [53] (see Chapter 5).

To give an intuitive idea of what we mean by nonlocal minimal surfaces we use the concept of "nonlocal perimeter". As the classical perimeter measures the total variation of the characteristic function of a set E in a fixed domain Ω, a nonlocal perimeter measures the variation of this characteristic function inside and outside this fixed domain with respect to a fractional operator.

A *nonlocal minimal surface* is then the boundary of a set E that minimizes this nonlocal perimeter inside a fixed domain Ω with the "boundary condition" that $E \cap (\mathbb{R}^N \setminus \Omega)$ is prescribed.

Surprisingly, as we will see in the Introduction of Chapter 5, since the nonlocal perimeter is related with the $H^{s/2}$ norm of the characteristic function χ_E, these surfaces can be attained minimizing this norm. Precisely, when $s<1$ and E is reasonably smooth, $\|\chi_E\|_{H^{s/2}}$ becomes finite whereas for $s=1$ this is not true.

To establish more clearly the relation of these objects with the classical minimal surfaces we remark here that, as we will discuss in Chapter 5, they have

a *nonlocal mean curvature* equal to zero. Moreover when $s \to 1^-$ the nonlocal perimeter approaches the classical one (see [13, 60]).
To give an intuitive definition to nonlocal minimal surfaces using the notion of mean curvature and its geometric meaning, we refer to the recent work [1]. See also [170] where the author makes a summary of the main properties and problems where these surfaces appear.

Coming back to the structure of Part II, in **Chapter 4** we will develop a "Schauder regularity theory" for viscosity solutions of a family of linear integro-differential equations that involves a special class of kernels not invariant under translations. Specifically we will consider kernels $K = K(x, w) : \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ satisfying some general structural assumptions. In the following, $1 < \sigma < 2$.

First of all, we suppose that K is close to an autonomous kernel of fractional Laplacian type, namely

$$(4.2.1)\begin{cases} \text{there exist } a_0, r_0 > 0 \text{ and } 0 < \eta < a_0/4 \text{ such that} \\ \left| \dfrac{|w|^{N+\sigma}K(x,w)}{2-\sigma} - a_0 \right| \le \eta, \qquad x \in B_1,\ w \in B_{r_0} \setminus \{0\}. \end{cases}$$

Moreover, we assume that

$$(4.2.2)\begin{cases} \text{there exist } k \in \mathbb{N} \cup \{0\} \text{ and } C_k = C(k) > 0 \text{ such that} \\ K \in \mathcal{C}^{k+1}\left(B_1 \times (\mathbb{R}^N \setminus \{0\})\right), \\ \|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{L^\infty(B_1)} \le \dfrac{C_k}{|w|^{N+\sigma+|\theta|}}, \\ \mu, \theta \in (\mathbb{N} \cup \{0\})^N,\ |\mu| + |\theta| \le k+1,\ w \in \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Observe that we use $|\cdot|$ both to denote the Euclidean norm of a vector and, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, to denote $|\alpha| := \alpha_1 + \cdots + \alpha_N$. However, the meaning of $|\cdot|$ will always be clear from the context.

Then, setting along this Part II

$$\delta u(x, w) := u(x + w) + u(x - w) - 2u(x), \qquad (0.0.36)$$

the principal result of this chapter reads as follows.

Theorem. 4.2.1 *Fix $1 < \sigma < 2$, $k \in \mathbb{N} \cup \{0\}$, and let $u \in L^\infty(\mathbb{R}^N)$ be a viscosity solution of the equation*

$$\int_{\mathbb{R}^N} K(x, w)\, \delta u(x, w) dw = f(x, u(x)) \qquad \textit{inside } B_1,$$

with $f \in \mathcal{C}^{k+1}(B_1 \times \mathbb{R})$. Assume that $K : B_1 \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ satisfies assumptions (4.2.1) *and* (4.2.2) *for the same value of k.*

Then, if η *in* (4.2.1) *is sufficiently small (the smallness being independent of* k*), we have* $u \in \mathcal{C}^{k+\sigma+\alpha}(B_{1/2})$ *for any* $\alpha < 1$*, and*

$$\|u\|_{\mathcal{C}^{k+\sigma+\alpha}(B_{1/2})} \leq C\left(1 + \|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1\times[-M,M])}\right),$$

where $M = \|u\|_{L^\infty(B_1)}$ *and* $C > 0$ *depends only on* N*,* σ*,* k*,* C_k*, and* $\|f\|_{\mathcal{C}^{k+1}(B_1\times\mathbb{R})}$*.*

Organization of Chapter 4. In the preliminary Section 4.1 we introduce the theory of integro-differential equations and we present some well-known regularity results. Then we devote Section 4.2 to the proof of Theorem 4.2.1.

In **Chapter 5**, using the results obtained in Chapter 4, and motivated by the fact that (s)-minimal surfaces approach the classical ones when $s \to 1^-$, we will prove that nonlocal minimal surfaces are smooth. In fact we get that $\mathcal{C}^1$ (s)-minimal surfaces are of class $\mathcal{C}^\infty$. For that we write, here and along this chapter, $x \in \mathbb{R}^N$ as $x = (x', x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$. Moreover, given $r > 0$ and $p \in \mathbb{R}^N$, we define

$$K_r(p) := \{x \in \mathbb{R}^N \,:\, |x'-p'| < r \text{ and } |x_N - p_N| < r\}.$$

For $p' \in \mathbb{R}^{N-1}$, we set

$$B_r^{N-1}(p') := \{x' \in \mathbb{R}^{N-1} \,:\, |x'-p'| < r\}.$$

We also consider $K_r := K_r(0)$, $B_r := B_r(0)$ and $B_r^{N-1} := B_r^{N-1}(0)$.

With this notation, we get the following.

Theorem. 5.1.1 *Take* $0 < s < 1$*, and let* ∂E *be an* (s)*-minimal surface in* K_R *for some* $R > 0$*. Assume that*

$$\partial E \cap K_R = \left\{(x', x_N) \,:\, x' \in B_R^{N-1} \text{ and } x_N = u(x')\right\},$$

for some $u : B_R^{N-1} \to \mathbb{R}$*, with* $u \in \mathcal{C}^{1,\alpha}(B_R^{N-1})$ *for any* $\alpha < s$ *and* $u(0) = 0$*. Then,*

$$u \in \mathcal{C}^\infty(B_\rho^{N-1}) \quad \text{for every } 0 < \rho < R.$$

Organization of Chapter 5. In Section 5.1 we introduce the notion of (s)-nonlocal minimal surfaces with detail and some well-known facts about them. Then in Section 5.2 we prove Theorem 5.1.1 using the bootstrap regularity result obtained in Section 4.2.

Note: The results of these two chapters are contained in [20].

Finally we devote **Part III** of this work to study the most simple parabolic problem that involves the fractional Laplacian operator, that is, the non local heat equation. Specifically in **Chapter 6** we extend some classical results for the heat equation by D.V. Widder in [172] to the nonlocal diffusion framework proving uniqueness in the setting of positive solutions according with the Principles of Thermodynamics. The main result of this chapter is the following.

Theorem. 6.1.3 *If $u \geq 0$ is a strong solution (see Definition 6.1.7) of*

$$u_t + (-\Delta)^s u = 0 \quad \textit{for } (x,t) \in \mathbb{R}^N \times (0,T), \quad 0 < s < 1,$$

then

$$u(x,t) = \int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy.$$

Here,

$$p^t(x) = \frac{1}{t^{N/2s}} p\left(\frac{x}{t^{1/2s}}\right),$$

and

$$p(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi - |\xi|^{2s}} d\xi,$$

is the solution of

$$\begin{cases} p^t + (-\Delta)^s p = 0 & \textit{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ p(x,0) = \delta_0(x) & \textit{in } \mathbb{R}^N. \end{cases}$$

Organization of Chapter 6. In Section 6.1 we will introduce the classical problem of Widder and the different type of solutions that will be used in the chapter. After that, in Section 6.2 we prove a uniqueness result for weak solutions (see Definition 6.1.5) that, in turn, will be the key step to obtain our representation theorem. Next, in Section 6.3, we prove the main result for strong solutions. We start by proving a comparison lemma that will allow us to show that every positive strong solution $u(x,t)$ is bigger than or equal to the convolution of the trace $u(x,0)$ with the kernel $p^t(x)$. By a scaling argument we will prove that any positive strong solution is also a weak solution and then, by the uniqueness result of the previous section, we will conclude with the proof of the theorem. Finally, in the last section, Section 6.4, we establish the pointwise behavior of positive strong solution that has an interest on its own and that could provide an alternative proof to our representation result.

Note: The results presented in this chapter are contained in [22].

Open problems and further results.

1. It remains an open problem the statement *4* of Theorem 1.2.1 for the case $0 < s < 1/2$. The reason for it is that, due to the lack of regularity, we do not know how to separate the solutions in the appropriate way (see Lemma 1.2.4 and also [67,73]). This problem seems to be technical. We note here that the same restriction on s appeared in the study of an uniform L^∞ estimate of Gidas-Spruck type ([96]) for the operator A_s that have been studied in [37,162]. In that case the main difficulty was to find a Liouville-type theorem for $0 < s < 1/2$ in $\mathbb{R}^N_+$.

2. In Theorem 1.3.1 we have left open the range $2s < N < 4s$. See the special case $s = 1$ and $N = 3$ in [45]. As one can observe in [161], if $s = 1/2$ this range is empty. When the Dirichlet problem is considered with $(-\Delta)^s$ instead of A_s, the same range remains open (see [140, Theorem 4]). One of the reasons why we leave this range open is because for $N < 4s$ we cannot prove Proposition 1.3.2. In fact the estimate obtained in (1.2.60) when $N < 4s$ is not enough to get the conclusion of the proposition in that case.

3. We propose as an interesting unsolved question the study of the result of multiplicity of solutions stated in the Remark 3.2.6 of Chapter 3 for the critical case $p = 2^*_s - 1$. A possible way to solve this problem would be to apply the Concentration Compactness Principle of P. L. Lions [118,119], modifying the computations specified in Lemma 2.2.10, and to use the appropriate minimizers associated to the lineal operator $L(u) := (-\Delta)^s u - \lambda u/|x|^{2s}$. To get these minimizers one should follow, perhaps, the ideas given in [164]. We notice that the existence of a minimum in the energy space could be obtained doing a change of variable to *hide* the Hardy potential in the operator in order to get the boundedness of the solutions. In our opinion this will involve a considerable computational effort.

4. We leave as an open problem the existence of at least two positive solutions to the problem $(H_{\lambda,\mu})$ given in Chapter 3 when μ belongs to a suitable bounded interval $(0, \Upsilon)$, $\lambda = \Lambda_{N,s}$ and $1 < p < 2^*_s - 1$. We believe that the improved Hardy-Sobolev inequality, given for instance in [85] (see also [90]), could be used as a tool to solve this question. We remark here that in the local case this result was obtained in [94].

5. In Chapter 5, we mention as an interesting unsolved problem the analyticity of the non local minimal surfaces studied there.

6. Finally, we consider an interesting problem to find the largest class of positive viscosity solutions for which the representation property given in Theorem 6.1.3 holds. In order to do that, it would be useful to prove that the pointwise behavior obtained in Lemma 6.4.2, together with some comparison arguments, suffices as in the classical case to give an alternative proof of the main result in Chapter 6.

Part I

An elliptic nonlocal problem: variational and non variational methods.

Chapter 1

On some elliptic critical problems for the spectral fractional Laplacian operator.

1.1 Introduction, preliminaries and functional settings.

Recently, several studies have been performed for classical elliptic equations with the Laplacian operator substituted by one of its fractional powers defined through the spectral decomposition. In particular, in [161] is studied the following problem

$$\left\{\begin{array}{ll} A_{1/2}u = \lambda u + u^{\frac{N+1}{N-1}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{array}\right. \tag{1.1.1}$$

That is, the author has considered the analogue case to the problem in [45] but with the square root of the Laplacian, defined spectrally in Ω with zero Dirichlet boundary conditions, instead of the classical Laplacian operator. Prior to this study, the authors in [51] proved that there is no solution in the case $\lambda = 0$ for Ω star shaped. Moreover it has been proved in [37], using a generalized Pohozaev identity, that the previous problem (1.1.1) with a general operator A_s, $0 < s < 1$, has no solution for $\lambda = 0$ whenever Ω is a starshaped domain (see also [63] for the existence of positive and multiple sign changing solutions when $s = 1/2$ and Ω is annular-shaped). This is the reason why in this chapter we are interested in the following perturbations of the critical power case for different fractional powers of the Laplacian,

$$(P_\lambda) = \begin{cases} A_s u = f_\lambda(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a regular bounded domain. Here, we are under the hypotheses that $N > 2s$,

$$f_\lambda(u) := \lambda u^q + u^{2_s^*-1}, \quad \lambda > 0, \quad 0 < q < 2_s^* - 1, \quad 0 < s < 1, \tag{1.1.2}$$

and 2_s^* is the fractional critical Sobolev exponent given in (0.0.29).

As we mentioned before, for the definition of the fractional Laplacian operator we follow some recent ideas of [51], together with other results from [37] and [55]. The powers A_s of the positive Laplace operator $(-\Delta)$, in a bounded domain Ω with zero Dirichlet boundary data, are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. Indeed, if L is a positive linear operator with discrete spectrum (φ_j, ρ_j) in Ω, the action of L on a function

$$u(x) = \sum \langle u, \varphi_j\rangle \varphi_j(x), \quad x \in \Omega,$$

is given by its action on each eigenfunction. That is

$$Lu(x) = \sum \langle u, \varphi_j\rangle L\varphi_j(x) = \sum \rho_j \langle u, \varphi_j\rangle \varphi_j(x), \quad x \in \Omega.$$

Then is clear how to define L^s, $0 < s < 1$:

$$L^s u(x) = \sum \rho_j^s \langle u, \varphi_j\rangle \varphi_j(x), \quad x \in \Omega.$$

Moreover using the formula

$$\eta^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\eta} - 1) \frac{dt}{t^{1+s}}, \; 0 < s < 1, \; \eta > 0, \tag{1.1.3}$$

where $\Gamma(-s)$ was defined in (0.0.28), it follows that

$$\begin{aligned} L^s u(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \sum (e^{-t\rho_j} - 1)\langle u, \varphi_j\rangle \varphi_j \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL} - 1) u(x) \, \frac{dt}{t^{1+s}}. \end{aligned}$$

Here

$$e^{-tL} u(x) := \sum e^{-t\rho_j} \langle u, \varphi_j\rangle \varphi_j(x), \quad x \in \Omega,$$

is the solution of the diffusion equation

$$\begin{cases} v_t + Lv = 0 & \text{in } \Omega \times (0,\infty), \\ v(x,0) = u(x) & \text{on } \Omega, \\ v(x,t) = 0 & \text{in } \partial\Omega \times (0,\infty). \end{cases}$$

See the clear relationship with (0.0.27).

Hence, if we consider (φ_j, ρ_j) the eigenfunctions and eigenvectors of $(-\Delta)$ in Ω with zero Dirichlet boundary data, then (φ_j, ρ_j^s) are the eigenfunctions and eigenvectors of A_s, also with Dirichlet boundary conditions. In fact, as we commented in the Introduction of this work, the fractional Laplacian A_s is well defined in the space of functions

$$H(\Omega, s) = \left\{ u = \sum a_j \varphi_j \in L^2(\Omega) \,:\, \|u\|^2_{H(\Omega,s)} = \sum \rho_j^s a_j^2 < \infty \right\}, \quad (1.1.4)$$

and, as a consequence,

$$A_s u = \sum \rho_j^s a_j \varphi_j \quad \text{with} \quad \|u\|_{H(\Omega,s)} = \|A_{s/2} u\|_{L^2(\Omega)}. \quad (1.1.5)$$

To understand the definition of $H(\Omega, s)$ we introduce the well know fractional Sobolev space $H^s(\mathbb{R}^N) = (H^1(\mathbb{R}^N), L^2(\mathbb{R}^N))_{[1-s]}$, $0 < s < 1$, as follows

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}, \quad (1.1.6)$$

with the usual norm given by

$$\|u\|_{H^s(\mathbb{R}^N)} = ||(1 + |\xi|^s)\hat{u}(\xi)\|_{L^2(\mathbb{R}^N)}.$$

Note that, by Plancherel's formula, this norm is equivalent to the so called Gagliardo norm, see for example [76, Proposition 3.4],

$$\|u\|_{L^2(\mathbb{R}^N)} + \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}. \quad (1.1.7)$$

Given a regular bounded domain $\Omega \subseteq \mathbb{R}^N$ we define the space $H^s(\Omega)$, $0 < s < 1$ as the family of functions $u \in L^2(\Omega)$ for which the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}, \quad (1.1.8)$$

is finite. Also we define $H_0^s(\Omega)$ the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the previous norm (1.1.8). That is,

$$H_0^s(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

By [120, Theorem 11.1] it is known that $H_0^s(\Omega) = H^s(\Omega)$ when $0 < s \le 1/2$. In the case $1/2 < s < 1$ the inclusion $H_0^s(\Omega) \subseteq H^s(\Omega)$ is strict.

The space defined in (1.1.4) is the interpolation space $(H_0^1(\Omega), L^2(\Omega))_{[1-s]}$ (see for example [7,120,163]). Moreover in [120, Chapter2] the authors proved that

$$(H_0^1(\Omega), L^2(\Omega))_{[1-s]} = \begin{cases} H_0^s(\Omega) & \text{for } 0 < s < 1,\ s \neq 1/2, \\ H_{00}^{1/2}(\Omega) & \text{if } s = 1/2, \end{cases}$$

where

$$H_{00}^{1/2}(\Omega) = \left\{ u \in H^{1/2}(\Omega) : \int_\Omega \frac{u^2(x)}{\text{dist}(x, \partial\Omega)}\, dx < \infty \right\}.$$

Therefore we obtain that

$$H(\Omega, s) = \begin{cases} H^s(\Omega) & \text{for } 0 < s < 1/2, \\ H_{00}^{1/2}(\Omega) & \text{if } s = 1/2, \\ H_0^s(\Omega) & \text{for } 1/2 < s < 1. \end{cases}$$

In all the cases, denoting $H'(\Omega, s)$ the topological dual of $H(\Omega, s)$, we have that $A_s : H(\Omega, s) \to H'(\Omega, s)$ is an isometric isomorphism. The inverse operator is denoted by A_{-s}.

We now go back to our problem (P_λ). To define correctly the energy formulation of the problem, since we are looking for positive solutions, in what follows we consider the next Dirichlet problem

$$(P_\lambda^+) = \begin{cases} A_s u = f_\lambda(u_+) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega\,. \end{cases}$$

Note that, by the Maximum Principle given in [64, Lemma 2.3] and [84], if u is a solution of (P_λ^+) then $u > 0$ in Ω and therefore is also a solution of (P_λ). Since the above definition of the fractional Laplacian allows to integrate by parts in the proper spaces, we are interested to find functions that satisfy the following.

Definition 1.1.1. *We say that $u \in H(\Omega, s)$ is a energy solution of (P_λ^+) if the identity*

$$\int_\Omega A_{s/2} u A_{s/2} \varphi\, dx = \int_\Omega f_\lambda(u_+) \varphi\, dx \tag{1.1.9}$$

holds for every function $\varphi \in H(\Omega, s)$.

Since $f_\lambda(u) = \lambda u^q + u^{2_s^*-1}$, the right-hand side of (1.1.9) is well defined (for the details, see (1.1.30)). Indeed $\varphi \in H(\Omega, s) \hookrightarrow L^{2_s^*}(\Omega)$ and $f(u) \in L^{\frac{2N}{N+2s}}(\Omega) \hookrightarrow H'(\Omega, s)$, whence $u \in H(\Omega, s)$.

Associated to problem (P_λ^+) we consider the energy functional

$$\mathcal{I}_{s,\lambda}(u) = \frac{1}{2}\int_\Omega \left|A_{s/2}u\right|^2 \, dx - \int_\Omega F_\lambda(u)\, dx\,,$$

where $F_\lambda(u) = \int_0^u f_\lambda(\eta)\, d\eta$. In our case it reads

$$\mathcal{I}_{s,\lambda}(u) = \frac{1}{2}\int_\Omega \left|A_{s/2}u\right|^2 \, dx - \int_\Omega \left(\frac{\lambda}{q+1}u_+^{q+1} + \frac{1}{2_s^*}u_+^{2_s^*}\right) dx. \qquad (1.1.10)$$

This functional is well defined in $H(\Omega, s)$, and moreover, by the standard variational theory, the critical points of $\mathcal{I}_{s,\lambda}$ correspond to solutions to (P_λ).

We now include the main ingredients of a recently developed technique used in order to deal with fractional powers of the Laplacian that we will use in this chapter.

Motivated by the work of Caffarelli and Silvestre [55], several authors have considered an equivalent definition of the operator A_s in a bounded domain with zero Dirichlet boundary data by means of an auxiliary variable. See [37, 48, 49, 51, 64, 156]. To explain this equivalent definition, associated to the bounded domain Ω, let us consider the cylinder

$$\mathscr{C}_\Omega = \Omega \times (0, \infty) \subset \mathbb{R}_+^{N+1}.$$

The points in $\mathscr{C}_\Omega$ are denoted by (x, y). The lateral boundary of the cylinder will be denoted by $\partial_L\mathscr{C}_\Omega = \partial\Omega \times (0, \infty)$. Now, for a function $u \in H(\Omega, s)$, we define the *s-harmonic extension* $w = \mathrm{E}_s(u)$ to the cylinder $\mathscr{C}_\Omega$ as the unique solution to the problem

$$\left\{\begin{array}{ll} \operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathscr{C}_\Omega, \\ w = 0 & \text{on } \partial_L\mathscr{C}_\Omega, \\ w = u & \text{on } \Omega \times \{y = 0\}, \end{array}\right. \qquad (1.1.11)$$

that belongs to the Hilbert space

$$H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s}) = \overline{\mathcal{C}_0^\infty(\Omega \times (0, \infty))}^{\|\cdot\|_{H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})}}.$$

Here,

$$\|z\|_{H_{0,L}^1(\mathscr{C}_\Omega, y^{1-2s})} = \left(\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla z|^2\right)^{1/2},$$

and κ_s is the normalization constant given by

$$\kappa_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}. \qquad (1.1.12)$$

With this constant we have that the extension operator is an isometry between $H(\Omega, s)$ and $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$. That is

$$\| \mathrm{E}_s(\psi)\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} = \|\psi\|_{H(\Omega,s)}\,, \quad \psi \in H(\Omega, s). \tag{1.1.13}$$

Indeed we have the following result, whose proof we add for completeness.

Lemma 1.1.2. *If* $u = \sum a_j\varphi_j \in H(\Omega, s)$ *then*

$$\mathrm{E}_s(u)(x, y) = \sum a_j\varphi_j(x)\psi(\rho_j^{1/2}y) \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}),$$

where $\psi(s){>}0$ *solves the problem*

$$\begin{cases} \psi'' + \dfrac{(1-2s)}{y}\psi' &= \psi, \qquad y > 0,\\ -\displaystyle\lim_{y\to 0^+} y^{1-2s}\psi'(y) &= \dfrac{1}{\kappa_s},\\ \psi(0) &= 1. \end{cases} \tag{1.1.14}$$

One has also,

$$\|A_s u\|_{H'(\Omega,s)} = \|A_{s/2}u\|_{L^2(\Omega)} = \|u\|_{H(\Omega,s)} = \| \mathrm{E}_s(u)\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})}. \tag{1.1.15}$$

Proof. Let us define

$$v(x, y) := \sum a_j\varphi_j(x)\psi(\rho_j^{1/2}y).$$

Then it is clear that

$$v(x, 0) = u(x) \text{ for } x \in \Omega \quad \text{and } v(x, y) = 0 \text{ when } (x, y) \in \partial_L\mathscr{C}_\Omega. \tag{1.1.16}$$

Using the orthogonality of the family $\{\varphi_j\}$, (1.1.14) and

$$\int_\Omega \varphi_j^2 = 1, \qquad \int_\Omega |\nabla\varphi_j|^2 = \rho_j, \tag{1.1.17}$$

we have

$$\begin{aligned} \|v\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} &= \kappa_s \int_0^\infty y^{1-2s} \int_\Omega \Big(\sum a_j^2|\nabla\varphi_j(x)|^2\psi^2(\rho_j^{1/2}y) \\ &+ a_j^2\rho_j\varphi_j(x)^2(\psi'(\rho_j^{1/2}y))^2\Big)\, dx\, dy \\ &= \kappa_s \int_0^\infty y^{1-2s} \sum a_j^2\rho_j \left(\psi^2(\rho_j^{1/2}y) + (\psi'(\rho_j^{1/2}y))^2\right) dy \\ &= \kappa_s \sum a_j^2\rho_j^s \int_0^\infty \eta^{1-2s}\left(\psi^2(\eta) + (\psi'(\eta))^2\right) d\eta. \end{aligned} \tag{1.1.18}$$

Since ψ satisfies the problem (1.1.14), then it can be shown that it is a combination of Bessel functions, see [115]. More precisely, ψ satisfies the following asymptotic behaviour

$$\begin{aligned} &\psi(\eta) \sim 1 - c_1\eta^{2s}, && \text{for } \eta \to 0, \\ &\psi(s) \sim c_2\eta^{\frac{2s-1}{2}}e^{-\eta}, && \text{for } \eta \to \infty, \end{aligned} \tag{1.1.19}$$

where

$$c_1(s) = \frac{2^{1-2s}\Gamma(1-s)}{2s\Gamma(s)}, \quad c_2(s) = \frac{2^{\frac{1-2s}{2}}\pi^{1/2}}{\Gamma(s)}.$$

Then, by (1.1.12), (1.1.14) and (1.1.19), we obtain

$$\int_0^\infty \left(\psi^2(\eta) + ((\psi')(\eta))^2\right)\eta^{1-2s}\,d\eta = -\lim_{\eta\to 0}\eta^{1-2s}\psi'(\eta)\psi(\eta) = \frac{1}{\kappa_s}.$$

Hence, since $u \in H(\Omega, s)$, from (1.1.17) and (1.1.18) it follows that

$$\|v\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} = \sum a_j^2\rho_j^s = \sum(a_j\rho_j^{s/2})^2 = \|A_{s/2}u\|^2_{L^2(\Omega)} = \|u\|^2_{H(\Omega,s)} < \infty.$$

That is, we have obtained (1.1.15) and that

$$v \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}). \tag{1.1.20}$$

Finally using that ψ is a solution of (1.1.14) clearly we also get that

$$\operatorname{div}(y^{1-2s}\nabla v) = 0 \quad \text{in } \mathscr{C}_\Omega. \tag{1.1.21}$$

Thus, by (1.1.16), (1.1.20) and (1.1.21), we conclude that $v = \mathrm{E}_s(u)$. □

Observe that for any function $\varphi \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$, we have that the trace operator

$$tr_\Omega : H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}) \to H(\Omega, s),$$

is linear and bounded, that is

$$\|\varphi(\cdot, 0)\|_{H(\Omega,s)} \le \|\varphi\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})}. \tag{1.1.22}$$

Then,

$$H(\Omega, s) = \{u = tr_\Omega v : \; v \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})\}.$$

See [51, Proposition 2.1] for the case $s = 1/2$ and [64, Proposition 2.1] when $s \neq 1/2$.

Note that from (1.1.13) and (1.1.22) we deduce that $E_s(u)$ is the solution of the problem

$$\min\{\|v\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} : \quad v \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}), \quad tr_\Omega(v) = u\}. \tag{1.1.23}$$

The relevance of the extension function $w = \mathrm{E}_s(u)$ is that it is related to the fractional Laplacian of the original function u. Indeed, from Lemma 1.1.2 we can easily deduce the following formula:

$$-\lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y) = \frac{1}{\kappa_s}A_s u(x). \tag{1.1.24}$$

See also [37, 48, 51, 55, 64, 156].

When $\Omega = \mathbb{R}^N$, the above Dirichlet-Neumann procedure (1.1.11)–(1.1.24) provides a formula for the fractional Laplacian in the whole space equivalent to that obtained from Fourier Transform, see [55]. In that case, the s-harmonic extension and the fractional Laplacian have explicit expressions in terms of the Poisson and the Riesz kernels, respectively. That is,

$$w(x,y) = P^s_y * u(x) = d(N,s)y^{2s}\int_{\mathbb{R}^N}\frac{u(\eta)}{(|x-\eta|^2+y^2)^{\frac{N+2s}{2}}}\,d\eta\,, \tag{1.1.25}$$

$$(-\Delta)^s u(x) = C(N,s)\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy. \tag{1.1.26}$$

Here $C(N,s)$ is the normalized constant given in (0.0.22). Moreover the constants in (1.1.24), (1.1.25) and (1.1.26) clearly satisfy the identity $2sd(N,s)\kappa_s = C(N,s)$. In that case the corresponding functional spaces are well defined on the homogeneous fractional Sobolev space

$$\dot{H}^s(\mathbb{R}^N) = \{u \in \mathcal{S} : |\xi|^s\hat{u}(\xi) \in L^2(\mathbb{R}^N)\},\ s > 0,$$

and the weighted Sobolev space $H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$.

Throughout this chapter we will use the following notation,

$$L_s w := -\operatorname{div}(y^{1-2s}\nabla w), \qquad \frac{\partial w}{\partial \nu^s} := -\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}\,.$$

With the above notation and (1.1.24), we can reformulate our problem (P^+_λ) as

$$(\overline{P}^+_\lambda) = \begin{cases} L_s w = 0 & \text{in } \mathscr{C}_\Omega \\ w = 0 & \text{on } \partial_L\mathscr{C}_\Omega \\ \dfrac{\partial w}{\partial \nu^s} = \lambda w^q_+ + w^{2^*_s-1}_+ & \text{in } \Omega\times\{y=0\}. \end{cases}$$

We introduce the following.

Definition 1.1.3. *An energy solution to problem* $(\overline{P}_\lambda^+)$ *is a function* $w \in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ *such that for any* $\varphi \in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$*,*

$$\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx\, dy = \int_\Omega f_\lambda(w_+(x,0))\varphi \, dx. \tag{1.1.27}$$

As we have seen before, for any energy solution $w \in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ to problem $(\overline{P}_\lambda^+)$, the function $u = w(\cdot, 0)$, defined in the sense of traces, belongs to the space $H(\Omega, s)$ and is an energy solution to problem (P_λ^+). The converse is also true. Therefore, both formulations are equivalent.

Moreover if w is an enery solution of (P_λ^+), either $w > 0$ in $\overline{\mathscr{C}_\Omega}$ or $w \equiv 0$. In fact, from the classical strong maximum principle for strictly elliptic operators we know that w cannot vanish at an interior point. Furthermore, by the strong maximum principle given in [84] (see also [64, Lemma 2.6]), or by the Hopf principle given in [48, Proposition 4.11], w cannot vanish when $y = 0$. Therefore if w is a nontrivial solution of $(\overline{P}_\lambda^+)$ then, its trace $u := tr_\Omega w$ will be strictly positive in Ω. Therefore u is going to be also a solution of (P_λ).

We also introduce the associated energy functional to the problem $(\overline{P}_\lambda^+)$:

$$\mathcal{I}_{s,\lambda}^*(w) = \frac{\kappa_s}{2} \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla w|^2 \, dx\, dy - \int_\Omega \left(\frac{\lambda}{q+1} w_+^{q+1} + \frac{1}{2_s^*} w_+^{2_s^*} \right) dx. \tag{1.1.28}$$

Clearly, critical points of $\mathcal{I}_{s,\lambda}^*$ in $H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ correspond to critical points of $\mathcal{I}_{s,\lambda}$ in $H(\Omega, s)$. Even more, minima of $\mathcal{I}_{s,\lambda}^*$ also correspond to minima of $\mathcal{I}_{s,\lambda}$, see Section 1.2.

In what follows we will omit the term *energy* when working with solutions that satisfy Definition 1.1.1 or Definition 1.1.3

Remark 1.1.4. *In the sequel of this chapter, and in view of the above equivalence, we will use both formulations of the problem, in* Ω *or in* $\mathscr{C}_\Omega$*, whenever we may take some advantage. In particular, we will use the extension version* $(\overline{P}_\lambda)$ *when dealing with the fractional operator acting on products of functions. This difficulty appears in the proof of the concentration-compactness result, Theorem 1.5.6, among others.*

Another tool that will be very useful in what follows is the trace inequality. See for instance [37, Theorem 2.1]. That is, for any $1 \le r \le 2_s^*$, $N > 2s$, and $v \in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ there exists $C = C(N, 2s, r, \Omega) > 0$ such that

$$\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla v(x,y)|^2 \, dx\, dy \ge C \left(\int_\Omega |v(x,0)|^r \, dx \right)^{2/r}. \tag{1.1.29}$$

By (1.1.15), this is equivalent to the fractional Sobolev inequality

$$\int_\Omega |A_{s/2}u|^2\,dx \ge C\kappa_s \left(\int_\Omega |u|^r\,dx\right)^{2/r}, \tag{1.1.30}$$

where κ_s is defined in (1.1.12) and $u \in H(\Omega, s)$. The analogous result for the Laplacian operator, instead of A_s, could be found in [82] and [119].

When $r = 2_s^*$, the best constant in (1.1.29), that is not achieved in any bounded domain, will be denoted by $T(N,s)$. Its exact value, that is independent of the domain, is

$$T(N,s) = \frac{2\pi^s \Gamma(\frac{N+2s}{2})\Gamma(1-s)(\Gamma(\frac{N}{2}))^{\frac{2s}{N}}}{\Gamma(s)\Gamma(\frac{N-2s}{2})(\Gamma(N))^{\frac{2s}{N}}}. \tag{1.1.31}$$

See for instance [37, 156]. In the case $\Omega = \mathbb{R}^N$, the previous constant is achieved when $U = v(\cdot, 0)$ takes the form

$$U(x) = U_\varepsilon(x) := (\varepsilon + |x|^2)^{-\frac{N-2s}{2}}, \tag{1.1.32}$$

with $\varepsilon > 0$ arbitrary and $v = \mathrm{E}_s(U)$. See [70, Theorem1.1] and [37, 66, 81, 91, 117]. When $s = 1$, that is we are in the classical case of the Laplacian operator, this result was obtained by G. Talenti in [159]. Consequently the best constant in (1.1.30), when $\Omega = \mathbb{R}^N$ and $r = 2_s^*$, is $\kappa_s T(N,s)$. The function given in (1.1.32) will be used in Sections 1.2, 1.3 and 1.4.

1.2 Sublinear case: $0 < q < 1$.

Previously to study with detail the critical problem (P_λ) when $0 < q < 1$, we remark here that the associated subcritical problem, that is when the right hand side f_λ of the problem (P_λ) is equal to $\lambda u^q + u^p$, $1 < p < 2_s^* - 1$, has been studied in [37, Section 5.3]. In this paper the authors used an argument due by Alama [14] adapted for the spectral fractional Laplacian using the extension tool.

The aim of this Section is to prove the following

Theorem 1.2.1. *Let $0 < q < 1$. Then, there exists $0 < \Lambda < \infty$ such that the problem (P_λ)*

1. *has no solution for $\lambda > \Lambda$;*
2. *has a minimal energy solution for any $0 < \lambda < \Lambda$ and, moreover, the family of minimal solutions is increasing with respect to λ;*
3. *if $\lambda = \Lambda$, there is at least one energy solution;*
4. *if $s \ge 1/2$, there are at least two energy solutions for $0 < \lambda < \Lambda$.*

1.2.1 The existence of the first solution.

First of all we have

Lemma 1.2.2. *Let* Λ *be defined by*

$$\Lambda = \sup \{\lambda > 0 \;:\; \textit{Problem } (P_\lambda) \textit{ has solution}\}.$$

Then $0 < \Lambda < \infty$.

Proof. Following the ideas given in [31], we consider (ρ_1^s, φ_1) the first eigenvalue and the corresponding positive eigenfunction of the spectral fractional Laplacian in Ω with zero Dirichlet condition. Let $u > 0$ be a solution of (P_λ^+). Using φ_1 as a test function in this problem we have that

$$\int_\Omega \left(\lambda u^q + u^{2_s^*-1}\right) \varphi_1 \, dx = \rho_1^s \int_\Omega u\varphi_1 \, dx. \tag{1.2.1}$$

Since there exist positive constants c_0, c_1 such that $\lambda t^q + t^{2_s^*-1} > c_0 \lambda^{c_1} t$, for any $t > 0$ we obtain from (1.2.1) that $c_0\lambda^{c_1} < \rho_1^s$ which implies

$$\Lambda < \infty. \tag{1.2.2}$$

To prove $\Lambda > 0$, following the ideas given in [31], see also [10, 92], we use the sub- and supersolution technique to construct a solution for λ small enough.

Step 1: Searching for a supersolution.

Let $\omega > 0$ be the solution of the problem

$$\begin{cases} A_s\omega(x) = 1 & x \in \Omega, \\ \omega(x) = 0 & x \in \partial\Omega. \end{cases}$$

First of all we note that, by Proposition 1.5.3, or Proposition 1.5.1, $\omega \in L^\infty(\Omega)$. Using the homogeneity of the operator A_s, if we define $\omega_\lambda(x) = \lambda\omega(x)$, it is clear that ω_λ solves the problem

$$\begin{cases} A_s v(x) = \lambda & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega \end{cases}$$

and that

$$||\omega_\lambda||_{L^\infty(\Omega)} = c_2\lambda \quad \text{with} \quad c_2 = ||\omega_1||_{L^\infty(\Omega)}. \tag{1.2.3}$$

Whence we obtain that there exists $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0]$ then

$$0 \le \overline{u}_\lambda := T(\lambda)\omega_\lambda \text{ is a supersolution of } (P_\lambda^+) \text{ for some } T(\lambda) > 0. \tag{1.2.4}$$

Indeed, we have to prove

$$T(\lambda)\lambda \geq \lambda(T(\lambda)\omega_\lambda)^q + (T(\lambda)\omega_\lambda)^{2_s^*-1},$$

for some $T(\lambda) > 0$ and $\lambda \in (0, \lambda_0]$. By (1.2.3) it is enough to find $T(\lambda)$ such that

$$T(\lambda)\lambda \geq \lambda(T(\lambda)c_2\lambda)^q + (T(\lambda)c_2\lambda)^{2_s^*-1}. \tag{1.2.5}$$

Note that (1.2.5) is equivalent to

$$1 \geq F_\lambda(T(\lambda)), \quad \text{whit} \quad F_\lambda(T(\lambda)) = (T(\lambda))^{q-1}(c_2\lambda)^q + (T(\lambda)\lambda)^{2_s^*-2}c_2^{2_s^*-1}.$$

Taking the derivative with respect to T, we get that

$$\frac{dF_\lambda}{dT} = 0 \Leftrightarrow T(\lambda) = T_0(\lambda) = \left(\frac{(1-q)\lambda^q}{(2_s^*-2)c_2^{2_s^*-1-q}\lambda^{2_s^*-2}}\right)^{\frac{1}{2_s^*-1-q}}.$$

So $T_0(\lambda)$ is a minimum for the function $F_\lambda(T(\lambda))$. Choosing

$$\lambda \leq \lambda_0(T_0) := \left(c_2\left(\left(\frac{1-q}{2_s^*-2}\right)^{\frac{q-1}{2_s^*-1-q}} + \left(\frac{1-q}{2_s^*-2}\right)^{\frac{2_s^*-2}{2_s^*-1-q}}\right)\right)^{\frac{-1}{2_s^*-2}}, \tag{1.2.6}$$

we have that $F_\lambda(T_0(\lambda)) \leq 1$. Therefore (1.2.5) is obtained and consequently (1.2.4) follows.

Step 2: Searching for a subsolution.

Let φ_1 be the positive eigenfunction associated to ρ_1^s with $||\varphi_1||_{L^\infty(\Omega)} = 1$. Let $c_3 > 0$ a constant such that

$$\lambda c_3 > \rho_1^s. \tag{1.2.7}$$

Then there exists a positive constant $\eta > 0$ such that if $0 < t < \eta$ then

$$t\rho_1^s\varphi_1 \leq \lambda(t\varphi_1)^q \leq \lambda(t\varphi_1)^q + (t\varphi_1)^{2_s^*-1}.$$

Hence $0 \leq \underline{u} := t\varphi_1$ is a subsolution of (P_λ^+).

Step3: Comparison of sub and supersolutions.

Before using the Sattinger method ([132]) we have to check that there exits a subsolution $\underline{u}$ such that $\underline{u} \leq \overline{u}_\lambda$ defined in (1.2.4). To obtain it we will use a family of subsolutions in the form $\underline{u}_k := t_k\varphi_1$ where $\{t_k\}_{k\in\mathbb{N}}$ verified that

$$t_k \in (0, \eta)\,,\ t_k \overset{k\to\infty}{\longrightarrow} 0. \tag{1.2.8}$$

Let λ_0 and c_3 given in (1.2.6) and (1.2.7) respectively. For all $\lambda \in (\rho_1^s/c_3, \lambda_0]$, by (1.2.8) there exist $k_0 \in \mathbb{N}$ such that, if $k \geq k_0$ then

$$t_k \rho_1^s \varphi_1(x) \leq \lambda T(\lambda), \quad x \in \Omega.$$

Then

$$A_s \underline{u}_k(x) \leq A_s \overline{u}_\lambda(x), \quad x \in \Omega.$$

Therefore, by the comparison principle given in [64, Lemma 2.5] we conclude.

Step4: Applying the Sattinger method.

Set $\lambda \in (\rho_1^s/c_3, \lambda_0]$ and $k \geq k_0$ where k_0 was given in *Step 3* and λ_0 and c_3 are given in (1.2.6) and (1.2.7) respectively. We define

$$\overline{u} := \overline{u}_\lambda$$

and

$$\underline{u} := t_{k_0} \varphi_1.$$

Since by *Step 3* $\overline{u} \geq \underline{u}$, we can apply the Sattinger method. Indeed we consider

$$u_0 := \underline{u} \tag{1.2.9}$$

and a nonnegative sequence of functions $\{u_k\}_{k \geq 1}$ in $H(\Omega, s) \cap L^\infty(\Omega)$ of solutions to the iterated problems

$$(P_k) = \begin{cases} A_s u_k = f_\lambda(u_{k-1}) & \text{in } \Omega, \\ u_k = 0 & \text{in } \partial\Omega\,. \end{cases}$$

By comparison it can be checked that

$$\underline{u} \leq u_1 \leq ... \leq \overline{u}.$$

Hence, we can define, up to subsequence,

$$0 \leq u_\lambda := \lim_{k \to \infty} u_k \quad \text{in } L^1(\Omega). \tag{1.2.10}$$

Moreover, since $\overline{u} \in L^\infty(\Omega)$,

$$\begin{aligned} \|A_{s/2} u_k\|^2_{L^2(\Omega)} &= \lambda \int_\Omega u_k u_{k-1}^q \, dx + \int_\Omega u_k u_{k-1}^{2_s^*-1} \, dx \\ &\leq \lambda \int_\Omega \overline{u}^{q+1} \, dx + \int_\Omega \overline{u}^{2_s^*-1} \, dx \\ &\leq C. \end{aligned}$$

Therefore, since $H(\Omega, s)$ is a Hilbert space, up to a subsequence, we have $u_k \rightharpoonup u_\lambda$ in $H(\Omega, s)$. Hence we can pass to the limit in the iterated problems to conclude that u_λ defined in (1.2.10) is a bounded solution of (P_λ^+) and, consequently to (P_λ). That is

$$\Lambda > 0. \tag{1.2.11}$$

Then, by (1.2.2) and (1.2.11) we ended the proof. □

In addition, we have the following.

Lemma 1.2.3. (P_λ) *admits at least one minimal solution for* $0 < \lambda \le \Lambda$. *Also, for* $0 < \lambda < \Lambda$ *the family of minimal solutions is increasing with respect to* λ.

Proof. Since $\Lambda > 0$, we can find a solution for a value of λ as close as we want to Λ. Denote this value by $\widetilde{\lambda}$ and by $u_{\widetilde{\lambda}}$ the associated minimal solution. Then, for all $\lambda < \widetilde{\lambda}$, we get that $u_{\widetilde{\lambda}}$ is a supersolution for the problems (P_λ). Furthermore, for every λ, φ_1 can be modified to a subsolution to (P_λ) as previously. Following the same procedure as before, we conclude that there exists a solution u_λ for all $\lambda \in (0, \widetilde{\lambda})$, and therefore for all $\lambda \in (0, \Lambda)$.

Even more, by construction, these solutions are minimal and, as a consequence of the comparison principle, are increasing with respect to λ.

To prove existence of solution in the extremal value $\lambda = \Lambda$, the idea, like in [10], consists on passing to the limit as $\lambda_n \nearrow \Lambda$ on the sequence $\{u_n\} = \{u_{\lambda_n}\} \ge 0$, where u_{λ_n} is the minimal solution of (P_λ) with $\lambda = \lambda_n$. Denote by $\mathcal{I}_{s,\lambda_n}$ the associated functional. Following the proof of [10, Lemma 3.5 and Theorem 2.1], we get that $\mathcal{I}_{s,\lambda_n}(u_n) < 0$. Hence

$$\begin{aligned} 0 &> \mathcal{I}_{s,\lambda_n}(u_n) - \frac{1}{2_s^*}\langle (\mathcal{I}_{s,\lambda_n})'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|u_n\|_{H(\Omega,s)}^2 - \lambda_n\left(\frac{1}{q+1} - \frac{1}{2_s^*}\right)\int_\Omega u_n^{q+1}dx. \end{aligned}$$

Therefore, since $q + 1 < 2_s^*$, by (1.1.30), there exits a constant $C > 0$ such that $\|u_n\|_{H(\Omega,s)} \le C$. As a consequence, there exists a subsequence weakly convergent to some u_Λ in $H(\Omega, s)$. By comparison, $u_\Lambda \ge u_\lambda > 0$ in Ω, for any $0 < \lambda < \Lambda$, so one gets easily that u_Λ is a nontrivial solution to (P_λ) with $\lambda = \Lambda$. □

1.2.2 The existence of the second solution. Variational techniques.

Having proved the first three items in Theorem 1.2.1, we focus in the sequel on proving the fourth statement. That is, the existence of the second solution fo $s \geq 1/2$.

Although for λ small enough the functional $\mathcal{I}_{s,\lambda}$ satisfies the geometry of the Mountain Pass Theorem (MPT for short) by Ambrosetti and Rabinowitz given in [11], we know that we cannot obtain a Palais Smale ((PS) for short) condition without any assumption related to the unicity of the minimal solution. In particular, even for λ small we cannot get inmediately, as occurs in the subcritical case, a second solution applying the MPT. See Section 3.3 of Chapter 3 to see this difference with the subcritical case.

The proof of statement *4* of Theorem 1.2.1 uses a contradiction argument, inspired by [10], and is divided into several steps. Using the minimal solutions given in Subsection 1.2.1, we first show that the functional $\mathcal{I}_{s,\lambda}$ has a local minimum. In the next step, in order to find a second solution, we assume that this local minimum is the only critical point of the functional and then we prove a local PS condition. This condition will be denoted by $(\text{PS})_c$, with a constant c that has to be under a critical level related with the best fractional critical Sobolev constant given in (1.1.31) for the extended functional $\mathcal{I}^*_{s,\lambda}$. Also we will find a path with energy under this critical level localizing the Sobolev minimizers of the Trace/Sobolev inequalities at the possible Dirac Deltas. These Deltas are obtained by the concentration-compactness result in Theorem 1.5.6 inspired in the classical result by P.L. Lions in [119]. Applying the MPT, given [11] and its refinement given in [95], we obtain a contradiction.

We begin proving the first step. That is, checking that $\mathcal{I}_{s,\lambda}$ has a positive minimum. First of all we have the next separation Lemma in the $\mathcal{C}^1$-topology. To avoid confusions with the notation we clarify here that, in the next three results we will denote by $\mathcal{C}^1_0(\Omega)$ the family of functions in $\mathcal{C}^1(\Omega)$ that are equal to zero in $\partial\Omega$.

Lemma 1.2.4. *Let* $0 < \lambda_1 < \lambda_0 < \lambda_2 < \Lambda$. *Let* z_{λ_1}, z_{λ_0} *and* z_{λ_2} *be the corresponding minimal solutions to* (P_λ), $\lambda = \lambda_1$, λ_0 *and* λ_2 *respectively. If* $X = \{z \in \mathcal{C}^1_0(\Omega) \,|\, z_{\lambda_1} \leq z \leq z_{\lambda_2}\}$, *then there exists* $\varepsilon > 0$ *such that*

$$\{z_{\lambda_0}\} + \varepsilon B_1 \subset X,$$

where B_1 *is the unit ball in* $\mathcal{C}^1_0(\Omega)$.

Proof. Since $s \geq 1/2$, by Proposition 1.5.5, if $0 < \lambda < \Lambda$ we have that any solution u of (P_λ) belongs to $\mathcal{C}^{1,\gamma}(\overline{\Omega})$ for some positive γ. Therefore if we define $\overline{z} := z_{\lambda_2} - z_{\lambda_0} > 0$ and $\underline{z} := z_{\lambda_0} - z_{\lambda_1} > 0$, since f_{λ_i}, $i = 0, 1, 2$ is an increasin function, by Hopf's Lemma (see [162, Lemmas 1.2 and 3.5]) we get that exist C_1, $C_2 > 0$ such that

$$\overline{z}(x) \geq C_1\delta(x) \quad \text{and} \quad \underline{z}(x) \geq C_2\delta(x), \quad x \in \Omega.$$

Thus,

$$z_{\lambda_0}(x) \le z_{\lambda_2}(x) - C_1\delta(x) \quad \text{and} \quad z_{\lambda_0}(x) \ge z_{\lambda_1}(x) + C_2\delta(x), \quad x \in \Omega.$$

Hence, taking $\varepsilon < \min\{C_1, C_2\}$, for any $x \in \Omega$, we conclude that

$$z_{\lambda_1}(x)+\varepsilon\delta(x) \le z_{\lambda_1}(x)+C_2\delta(x) \le z_{\lambda_0}(x) \le z_{\lambda_2}(x)-C_1\delta(x) \le z_{\lambda_2}(x)-\varepsilon\delta(x).$$

That is $\{z_{\lambda_0}\} + \varepsilon B_1 \subset X$. □

With this result, and following the ideas given in [10], we now obtain a local minimum of the functional $\mathcal{I}_{s,\lambda}$ in $\mathcal{C}_0^1(\Omega)$, as a first step, to obtain a local minimum in $H(\Omega, s)$.

Lemma 1.2.5. *For all $\lambda \in (0, \Lambda)$ the problem (P_λ) has a solution u_0 which is in fact a local minimum of the functional $\mathcal{I}_{s,\lambda}$ in the $\mathcal{C}^1$-topology.*

Proof. Given $0 < \lambda_1 < \lambda < \lambda_2 < \Lambda$, let z_{λ_1} and z_{λ_2} be the minimal solutions of (P_{λ_1}) and (P_{λ_2}) respectively. Since z_{λ_1} and z_{λ_2} are properly ordered, then

$$\begin{cases} A_s(z_{\lambda_2} - z_{\lambda_1}) > 0 & \text{in } \Omega, \\ z_{\lambda_2} - z_{\lambda_1} = 0 & \text{on } \partial\Omega. \end{cases}$$

We set

$$f_T(x,\eta) = \begin{cases} f_\lambda(z_{\lambda_1}(x)) & \text{if } \eta \le z_{\lambda_1}, \\ f_\lambda(\eta) & \text{if } z_{\lambda_1} < \eta < z_{\lambda_2}, \\ f_\lambda(z_{\lambda_2}(x)) & \text{if } z_{\lambda_2} \le \eta, \end{cases}$$

$$F_T(x,z) = \int_0^z f_T(x,\eta)\, d\eta$$

and

$$\mathcal{I}_{s,\lambda}^T(z) = \frac{1}{2}\|z\|_{H(\Omega,s)}^2 - \int_\Omega F_T(x,z)dx.$$

Standard calculation of weak lower semi-continuity shows that $\mathcal{I}_{s,\lambda}^T$ achieves its global minimum at some $u_0 \in H(\Omega, s)$, that is

$$\mathcal{I}_{s,\lambda}^T(u_0) \le \mathcal{I}_{s,\lambda}^T(z), \quad z \in H(\Omega, s). \tag{1.2.12}$$

Moreover it holds

$$\begin{cases} A_s u_0 = f_T(x, u_0) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $z_{\lambda_1} < u_0 < z_{\lambda_2}$, by Lemma 1.2.4, it follows that $\{u_0\} + \varepsilon B_1 \subseteq X$ for $0 < \varepsilon$ small enough. Let now z satisfy

$$\|z - u_0\|_{\mathcal{C}^1_0(\Omega)} \le \frac{\varepsilon}{2}.$$

Then, for $x \in \Omega$,

$$z(x) \le u_0(x) + \frac{\varepsilon}{2}\delta(x) \le z_{\lambda_2}(x) - \frac{\varepsilon}{2}\delta(x) < z_{\lambda_2}(x)$$

and

$$z(x) \ge u_0(x) - \frac{\varepsilon}{2}\delta(x) \ge z_{\lambda_1}(x) + \frac{\varepsilon}{2}\delta(x) > z_{\lambda_1}(x).$$

Therefore, for every z such that $\|z - u_0\|_{\mathcal{C}^1_0(\Omega)} \le \frac{\varepsilon}{2}$, the previous inequalities give that $\mathcal{I}^T_{s,\lambda}(z) - \mathcal{I}_{s,\lambda}(z)$ is zero. Hence, by (1.2.12) we obtain that

$$\mathcal{I}_{s,\lambda}(z) = \mathcal{I}^T_{s,\lambda}(z) \ge \mathcal{I}^T_{s,\lambda}(u_0) = \mathcal{I}_{s,\lambda}(u_0), \quad z \in \mathcal{C}^1_0(\Omega), \text{ with } \|z - u_0\|_{\mathcal{C}^1_0(\Omega)} \le \frac{\varepsilon}{2}.$$

That is, u_0 is a local minimum of $\mathcal{I}_{s,\lambda}$ in the $\mathcal{C}^1$-topology. □

Finally, to show that we have obtained the desired minimum in $H^s_0(\Omega)$, we now check that the result by Brezis and Nirenberg in [46] is also valid in our context.

Proposition 1.2.6. *Let $z_0 \in H(\Omega, s)$ be a local minimum of $\mathcal{I}_{s,\lambda}$ in $\mathcal{C}^1_0(\Omega)$, this means that, there exists $r_1 > 0$ such that*

$$\mathcal{I}_{s,\lambda}(z_0) \le \mathcal{I}_{s,\lambda}(z_0 + z), \qquad z \in \mathcal{C}^1_0(\Omega) \text{ with } \|z\|_{\mathcal{C}^1_0(\Omega)} \le r_1. \tag{1.2.13}$$

Then z_0 is a local minimum of $\mathcal{I}_{s,\lambda}$ in $H(\Omega, s)$, that is, there exists $r_2 > 0$ so that

$$\mathcal{I}_{s,\lambda}(z_0) \le \mathcal{I}_{s,\lambda}(z_0 + z), \qquad z \in H(\Omega, s) \text{ with } \|z\|_{H(\Omega,s)} \le r_2.$$

Proof. Let z_0 be as in (1.2.13) and set, for $\varepsilon > 0$,

$$B_\varepsilon(z_0) = \left\{ z \in H(\Omega, s) : \; \|z - z_0\|_{H(\Omega,s)} \le \varepsilon \right\}.$$

Now, we argue by contradiction and we suppose that for every $\varepsilon > 0$ we have

$$\min_{v \in B_\varepsilon(z_0)} \mathcal{I}_{s,\lambda}(v) < \mathcal{I}_{s,\lambda}(z_0). \tag{1.2.14}$$

We pick $v_\varepsilon \in B_\varepsilon(z_0)$ such that $\min\limits_{v \in B_\varepsilon(z_0)} \mathcal{I}_{s,\lambda}(v) = \mathcal{I}_{s,\lambda}(v_\varepsilon)$. The existence of v_ε comes from a standard argument of weak lower semi-continuity. We want to prove that

$$v_\varepsilon \to z_0 \quad \text{in} \quad \mathcal{C}^1_0(\Omega) \quad \text{as} \quad \varepsilon \searrow 0, \tag{1.2.15}$$

because this will imply that there are $z \in \mathcal{C}_0^1(\Omega)$, arbitrarily close to z_0 in the metric of $\mathcal{C}_0^1(\Omega)$ (in fact, $z = v_\varepsilon$ for some ε), such that

$$\mathcal{I}_{s,\lambda}(z) < \mathcal{I}_{s,\lambda}(z_0).$$

This contradicts our hypothesis (1.2.13).

Let $0 < \varepsilon \ll 1$. We note that the Euler-Lagrange equation satisfied by v_ε involves a Lagrange multiplier ξ_ε in such a way that

$$\langle \mathcal{I}'_{s,\lambda}(v_\varepsilon), \varphi \rangle_{H'(\Omega,s),H(\Omega,s)} = \xi_\varepsilon \langle v_\varepsilon, \varphi \rangle_{H(\Omega,s)}, \quad \varphi \in H(\Omega, s). \tag{1.2.16}$$

Since v_ε is a minimum of $\mathcal{I}_{s,\lambda}$, we have

$$\xi_\varepsilon = \frac{\langle \mathcal{I}'_{s,\lambda}(v_\varepsilon), v_\varepsilon \rangle}{\|v_\varepsilon\|^2_{H(\Omega,s)}} \leq 0. \tag{1.2.17}$$

Note that by (1.2.16), v_ε satisfies the problem

$$\begin{cases} A_s v_\varepsilon = \frac{1}{1-\xi_\varepsilon} f_\lambda((v_\varepsilon)_+) := f_\lambda^\varepsilon(v_\varepsilon) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly $\|v_\varepsilon\|_{H(\Omega,s)} \leq C$, thus, by Proposition 1.5.3, this implies that $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C$. Moreover, by (1.2.17) it follows that $\|f_\lambda^\varepsilon(v_\varepsilon)\|_{L^\infty(\Omega)} \leq C$. Therefore, following the proof of Proposition 1.5.5, we get that

$$\|v_\varepsilon\|_{\mathcal{C}^{1,r}(\overline{\Omega})} \leq C, \qquad \text{for } r = \min\{q, 2s-1\} \text{ and } C \text{ independent of } \varepsilon.$$

By Ascoli-Arzelá Theorem, and the fact that $v_\varepsilon \to z_0$ in $H(\Omega, s)$, we obtain that there exists a subsequence, still denoted by v_ε, such that $v_\varepsilon \to z_0$ uniformly in $\mathcal{C}_0^1(\Omega)$ as $\varepsilon \searrow 0$, that is, (1.2.15) is proved. □

Lemma 1.2.5 and Proposition 1.2.6 provide us with a local minimum of $\mathcal{I}_{s,\lambda}$ in $H(\Omega, s)$, which will be denoted by u_0. We now perform a translation in order to simplify the calculations. That is we consider the functions

$$\begin{aligned} g_\lambda(x,\eta) &= \begin{cases} \lambda(u_0+\eta)^q - \lambda u_0^q + (u_0+\eta)^{2_s^*-1} - u_0^{2_s^*-1} & \text{if } \eta \geq 0, \\ 0 & \text{if } \eta < 0, \end{cases} \\ &= \lambda(u_0+\eta_+)^q - \lambda u_0^q + (u_0+\eta_+)^{2_s^*-1} - u_0^{2_s^*-1}, \end{aligned} \tag{1.2.18}$$

$$G_\lambda(x,u) = \int_0^u g_\lambda(x,\eta)\, d\eta, \tag{1.2.19}$$

and the energy functional

$$\widetilde{\mathcal{I}}_{s,\lambda}(u) = \frac{1}{2}\|u\|^2_{H(\Omega,s)} - \int_\Omega G_\lambda(x,u)dx. \tag{1.2.20}$$

Since $u \in H(\Omega, s)$, G_λ is well defined and bounded from below. Consider the moved problem

$$(\widetilde{P}_\lambda) = \begin{cases} A_s u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard variational theory, we know that if $\widetilde{u} \not\equiv 0$ is a critical point of $\widetilde{\mathcal{I}}_{s,\lambda}$ then it is a solution of $(\widetilde{P}_\lambda)$. By the Strong Maximum Principle (see [64, Lemma 2.3] and [84] or Remark 4.2 of [48]), we must have $\widetilde{u} > 0$. Therefore $u = u_0 + \widetilde{u}$ will be a second solution of (P_λ) for the sublinear case. Thus our next objective is to study the existence of these non-trivial critical points for $\widetilde{\mathcal{I}}_{s,\lambda}$.

As we have mentioned in Remark 1.1.4, there are some points where it is difficult to work directly with the spectral fractional Laplacian. For that reason in what follows we consider the extended problem $(\overline{P}^+_\lambda)$. To begin with that problem it is necessary to prove that local minima of the functional $\mathcal{I}_{s,\lambda}$ correspond to local minima of the extended functional $\mathcal{I}^*_{s,\lambda}$.

Proposition 1.2.7. *A function $u_0 \in H(\Omega, s)$ is a local minimum of $\mathcal{I}_{s,\lambda}$ if and only if $w_0 = \mathrm{E}_s(u_0) \in H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ is a local minimum of $\mathcal{I}^*_{s,\lambda}$.*

Proof. On one hand let $u_0 \in H(\Omega, s)$ be a local minimum of $\mathcal{I}_{s,\lambda}$. Suppose, by contradiction, that $w_0 = \mathrm{E}_s(u_0)$ is not a local minimum for the extended functional $\mathcal{I}^*_{s,\lambda}$. Then by (1.1.13) and (1.1.22), we have that, for any $\varepsilon > 0$, there exists $w_\varepsilon \in H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$, with $\|w_0 - w_\varepsilon\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} < \varepsilon$, such that

$$\mathcal{I}_{s,\lambda}(u_0) = \mathcal{I}^*_{s,\lambda}(w_0) > \mathcal{I}^*_{s,\lambda}(w_\varepsilon) \geq \mathcal{I}_{s,\lambda}(z_\varepsilon),$$

where $z_\varepsilon = w_\varepsilon(\cdot, 0) \in H(\Omega, s)$ satisfies $\|u_0 - z_\varepsilon\|_{H(\Omega,s)} < \varepsilon$.

On the other hand, let $w_0 \in H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ be a local minimum of $\mathcal{I}^*_{s,\lambda}$ such that $w_0(\cdot, 0) = u_0$. Then by the definition of the s-harmonic extension, see (1.1.23), $w_0 = E_s(u_0)$. Therefore, (1.1.13) clearly implies that u_0 is a minimum of $\mathcal{I}_{s,\lambda}$, so we conclude. □

Considering now the moved functional

$$\widetilde{\mathcal{I}}^*_{s,\lambda}(w) = \frac{1}{2}\|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} - \int_\Omega G_\lambda(w(x,0))dx,$$

with G_λ defined in (1.2.19), we can prove the next.

Lemma 1.2.8. *$w=0$ is a local minimum of $\widetilde{\mathcal{I}}^*_{s,\lambda}$ in $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$.*

Proof. The proof follows the lines of [10, Lemma 4.2]. Let w belong to $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$, then

$$G_\lambda(x,w(x,0)) = F(u_0+w_+(x,0)) - F(u_0) - f_\lambda(u_0)w_+(x,0). \qquad (1.2.21)$$

Therefore

$$\begin{aligned}\widetilde{\mathcal{I}}^*_{s,\lambda}(w) &= \frac{1}{2}\|w_+\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \frac{1}{2}\|w_-\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} - \int_\Omega G_\lambda(x,w(x,0))\,dx \\ &= \frac{1}{2}\|w_+\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \frac{1}{2}\|w_-\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} \\ &- \int_\Omega F(u_0+w_+(x,0))\,dx + \int_\Omega F(u_0)\,dx + \int_\Omega f_\lambda(u_0)w_+(x,0)\,dx.\end{aligned}$$

On the other hand by Proposition 1.2.7

$$\begin{aligned}\mathcal{I}^*_{s,\lambda}(w_0+w_+) &= \frac{1}{2}\|w_0+w_+\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} - \int_\Omega F(u_0+w_+(x,0))\,dx \\ &= \frac{1}{2}\|w_0\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \frac{1}{2}\|w_+\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} \\ &+ \kappa_s\int_{\mathscr{C}_\Omega} y^{1-2s}\nabla w_0\nabla w_+ dx\,dy - \int_\Omega F(u_0+w_+(x,0))\,dx \\ &= \frac{1}{2}\|w_0\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \frac{1}{2}\|w_+\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} \\ &+ \int_\Omega f_\lambda(u_0)w_+(x,0)\,dx - \int_\Omega F(u_0+w_+(x,0))\,dx.\end{aligned}$$

Finally, since w_0 is a local minimum of $\mathcal{I}^*_{s,\lambda}$, we have that

$$\begin{aligned}\widetilde{\mathcal{I}}^*_{s,\lambda}(w) &= \mathcal{I}^*_{s,\lambda}(w_0+w_+) - \frac{1}{2}\|w_0\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \frac{1}{2}\|w_-\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} \\ &+ \int_\Omega F(u_0)\,dx \\ &\geq \mathcal{I}^*_{s,\lambda}(w_0+w_+) - \mathcal{I}^*_{s,\lambda}(w_0) \\ &\geq 0 = \widetilde{\mathcal{I}}^*_{s,\lambda}(0),\end{aligned}$$

provided $\|w\|_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} < \varepsilon$. Then $w=0$ is a local minimum of $\widetilde{\mathcal{I}}^*_{s,\lambda}$ in $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$. □

Note that as a consequence of Proposition 1.2.7 and Lemma 1.2.8, we also obtain that $u=0$ is a local minimum of $\widetilde{\mathcal{I}}_{s,\lambda}$ in $H(\Omega,s)$.

As we said in the introduction of this subsection, to prove the existence of the second solution of (P_λ), we will proceed by contradiction. For that, if we assume that $v = 0$ is the unique critical point of $\widetilde{\mathcal{I}}^*_{s,\lambda}$, using an extension of a concentration-compactness result by Lions, see Theorem 1.5.6, then we will prove that $\widetilde{\mathcal{I}}^*_{s,\lambda}$ satisfies a local $(\mathrm{PS})_c$ condition for c under a critical level

$$c^* = \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}}. \tag{1.2.22}$$

Here κ_s and $T(N,s)$ are defined in (1.1.12) and (1.1.31) respectively. First, in order to apply Theorem 1.5.6, since we are considering the unbounded domain $\mathscr{C}_\Omega$, we need the next.

Lemma 1.2.9. *Let* $\{z_n\} \subseteq H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ *be such that*

$$\|z_n\|^2_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} \leq M, \tag{1.2.23}$$

for some $M > 0$ *and*

$$(\mathcal{I}^*_{s,\lambda})'(z_n) \to 0. \tag{1.2.24}$$

Then the sequence $\{y^{1-2s}|\nabla z_n|^2\}_{n\in\mathbb{N}}$ *is tight, i.e., for any* $\eta > 0$ *there exists* $\tau_0 > 0$ *such that*

$$\int_{\{y>\tau_0\}}\int_\Omega y^{1-2s}|\nabla z_n|^2 dxdy \leq \eta, \quad n \in \mathbb{N}. \tag{1.2.25}$$

Proof. The proof of this Lemma follows some arguments of [12, Lema 2.2] and [118, 119]. Since $\{z_n\} \subseteq H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ then, for every $\delta > 0$, there exists, for each $n \in \mathbb{N}$, τ_n such that

$$\int_{\{y>\tau_n\}}\int_\Omega y^{1-2s}|\nabla z_n|^2 dx\, dy \leq \delta. \tag{1.2.26}$$

Let $\varepsilon > 0$ be fixed, to be specified later, z be the weak limit of $\{z_n\}$ in $H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ and $r > 0$ be such that

$$\int_{\{y>r\}}\int_\Omega y^{1-2s}|\nabla z|^2 dx\, dy < \varepsilon.$$

Let now $j = \left[\frac{M}{\kappa_s\varepsilon}\right]$ be the integer part of $\frac{M}{\kappa_s\varepsilon}$ and consider $I_k = \{y \in \mathbb{R}^+ : r+k \leq y \leq r+k+1\}$, $k = 0, 1, \ldots, j$. By (1.2.23), we clearly obtain that

$$\sum_{k=0}^{j}\int_{I_k}\int_\Omega y^{1-2s}|\nabla z_n|^2 dx\, dy \leq \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla z_n|^2 dx\, dy \leq \frac{M}{\kappa_s} \leq \varepsilon(j+1).$$

Therefore there exists $k_0 \in \{0, \ldots, j\}$ such that, up to a subsequence,

$$\int_{I_{k_0}} \int_\Omega y^{1-2s} |\nabla z_n|^2 dx\, dy \leq \varepsilon, \quad n \in \mathbb{N}. \tag{1.2.27}$$

Let $\chi \geq 0$ be the following regular nondecreasing cut-off function

$$\chi(y) = \begin{cases} 0 & \text{if } y \leq r + k_0, \\ 1 & \text{if } y > r + k_0 + 1. \end{cases}$$

Define $v_n(x,y) = \chi(y) z_n(x,y)$. Since $v_n(x,0) = 0$ and $v_n(x,y) = z_n(x,y)$ if $y > r + k_0 + 1$, it follows that

$$\begin{aligned} \frac{|\langle (\mathcal{I}^*_{s,\lambda})'(z_n) - (\mathcal{I}^*_{s,\lambda})'(v_n), v_n \rangle|}{\kappa_s} &= \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla(z_n - v_n), \nabla v_n \rangle dx\, dy \\ &= \int_{\{y > r + k_0\}} \int_\Omega y^{1-2s} \langle \nabla(z_n - v_n), \nabla v_n \rangle dx\, dy \\ &= \int_{I_{k_0}} \int_\Omega y^{1-2s} \langle \nabla(z_n - v_n), \nabla v_n \rangle dx\, dy. \end{aligned} \tag{1.2.28}$$

Moreover by the Cauchy-Schwartz inequality, (1.2.27) and the compact inclusion $H^1_0(I_{k_0} \times \Omega, y^{1-2s})$ into $L^2(I_{k_0} \times \Omega, y^{1-2s})$, we have

$$\begin{aligned} & \left(\int_{I_{k_0}} \int_\Omega y^{1-2s} |\nabla v_n|^2\, dx\, dy \right)^{\frac{1}{2}} \\ = \; & \left(\int_{I_{k_0}} \int_\Omega y^{1-2s} |z_n \nabla \chi + \chi \nabla z_n|^2\, dx\, dy \right)^{\frac{1}{2}} \\ \leq \; & \left(C \int_{I_{k_0}} \int_\Omega y^{1-2s} |\nabla z_n|^2\, dx\, dy + 2 \int_{I_{k_0}} \int_\Omega y^{1-2s} \langle \nabla z_n, z_n \rangle|\, dx\, dy \right)^{\frac{1}{2}} \\ \leq \; & \left(C\varepsilon + 2 \left(\int_{I_{k_0}} \int_\Omega y^{1-2s} |z_n|^2\, dx\, dy \right)^{\frac{1}{2}} \left(\int_{I_{k_0}} \int_\Omega y^{1-2s} |\nabla z_n|^2\, dx\, dy \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ \leq \; & C\varepsilon^{\frac{1}{2}}. \end{aligned} \tag{1.2.29}$$

Similarly we also get that

$$\left(\int_{I_{k_0}} \int_\Omega y^{1-2s} |\nabla(z_n - v_n)|^2 dx\, dy \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} \tag{1.2.30}$$

Then, from (1.2.28), (1.2.29), (1.2.30) and the Cauchy-Schwartz inequality again, we get that

$$\begin{aligned}
&|\langle(\mathcal{I}^*_{s,\lambda})'(z_n) - (\mathcal{I}^*_{s,\lambda})'(v_n), v_n\rangle| \\
\leq\ & \kappa_s \left(\int_{I_{k_0}}\int_\Omega y^{1-2s}|\nabla(z_n - v_n)|^2 dx\, dy\right)^{\frac{1}{2}} \left(\int_{I_{k_0}}\int_\Omega y^{1-2s}|\nabla v_n|^2 dx\, dy\right)^{\frac{1}{2}} \\
\leq\ & C\,\kappa_s\,\varepsilon. \qquad (1.2.31)
\end{aligned}$$

On the other hand, by (1.2.24),

$$|\langle(\mathcal{I}^*_{s,\lambda})'(z_n), v_n\rangle| \leq C\,\kappa_s\,\varepsilon + o(1). \qquad (1.2.32)$$

So by (1.2.31) and (1.2.32), there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\begin{aligned}
\int_{\{y>r+k_0+1\}}\int_\Omega y^{1-2s}|\nabla z_n|^2 dx\, dy &\leq \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla v_n|^2 dx\, dy \\
&= \frac{\langle(\mathcal{I}^*_{s,\lambda})'(v_n), v_n\rangle}{\kappa_s} \\
&\leq C\,\varepsilon.
\end{aligned}$$

Then, for any $n \in \mathbb{N}$ there exits $\tau_0 = \max\{r + k_0 + 1, \tau_1, \ldots, \tau_{n_0}\}$, with τ_i, $i = 1, \ldots, n_0$ given in (1.2.26), such that

$$\int_{\{y>\tau_0\}}\int_\Omega y^{1-2s}|\nabla z_n|^2 dx\, dy \leq \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla v_n|^2 dx\, dy \leq \eta,$$

for every $\eta > 0$. □

Now we are able in the situation to prove the compactness property of the extended moved functional.

Lemma 1.2.10. *If $v = 0$ is the only critical point of $\widetilde{\mathcal{I}}^*_{s,\lambda}$ in $H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$ then $\widetilde{\mathcal{I}}^*_{s,\lambda}$ satisfies a local (PS)$_c$ condition below the critical level c^* defined in* (1.2.22).

Proof. Let $\{w_n\}$ be a PS sequence for $\widetilde{\mathcal{I}}^*_{s,\lambda}$ satisfying

$$\widetilde{\mathcal{I}}^*_{s,\lambda}(w_n) \to c < c^* \quad \text{and} \quad (\widetilde{\mathcal{I}}^*_{s,\lambda})'(w_n) \to 0. \qquad (1.2.33)$$

Then, there exist two positive constants C_1 and C_2 such that

$$\begin{aligned} c+o(1) &= \widetilde{\mathcal{I}}^*_{s,\lambda}(w_n) - \frac{1}{2^*_s}\langle (\widetilde{\mathcal{I}}^*_{s,\lambda})'(w_n), w_0+(w_n)_+\rangle \\ &\geq \left(\frac{1}{2}-\frac{1}{2^*_s}\right)\|w_n\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} \\ &- \lambda\left(\frac{1}{q+1}-\frac{1}{2^*_s}\right)\int_\Omega (w_0+(w_n)_+)^{q+1}\,dx \\ &\geq C_1\|w_n\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} \\ &- C_2\left(\|w_0\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} + \|(w_n)_+\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})}\right)^{q+1}. \end{aligned}$$

That is, $\{w_n\}$ is uniformly bounded in $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$. By the hypothesis that $v=0$ is the unique critical point of $\widetilde{\mathcal{I}}^*_{s,\lambda}$, it follows that, up to a subsequence,

$$\begin{aligned} w_n &\rightharpoonup 0 && \text{weakly in } H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}), \\ w_n(\cdot,0) &\to 0 && \text{strongly in } L^r(\Omega),\quad 1\leq r<2^*_s, \\ w_n(\cdot,0) &\to 0 && \text{a.e. in } \Omega. \end{aligned} \tag{1.2.34}$$

Also, since w_0 is a critical point of $\mathcal{I}^*_{s,\lambda}$, we have that

$$\begin{aligned} \mathcal{I}^*_{s,\lambda}(z_n) &= \widetilde{\mathcal{I}}^*_{s,\lambda}(w_n) + \mathcal{I}^*_{s,\lambda}(w_0) \\ &+ \lambda\int_\Omega\left(\frac{(w_0+(w_n)_+)^{q+1}}{q+1} + w_0^q(w_n-(w_n)_+) - \frac{(w_0+w_n)_+^{q+1}}{q+1}\right)dx \\ &+ \int_\Omega\left(\frac{(w_0+(w_n)_+)^{2^*_s}}{2^*_s} + w_0^{2^*_s-1}(w_n-(w_n)_+) - \frac{(w_0+w_n)_+^{2^*_s}}{2^*_s}\right)dx \\ &\leq \widetilde{\mathcal{I}}^*_{s,\lambda}(w_n) + \mathcal{I}^*_{s,\lambda}(w_0), \end{aligned} \tag{1.2.35}$$

where

$$z_n := w_n + w_0.$$

Moreover for every $\varphi\in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$,

$$\begin{aligned} \langle(\mathcal{I}^*_{s,\lambda})'(z_n),\varphi\rangle &= \langle(\widetilde{\mathcal{I}}^*_{s,\lambda})'(w_n),\varphi\rangle \\ &+ \int_\Omega\left(\lambda(w_0+(w_n)_+)^q + (w_0+(w_n)_+)^{2^*_s-1}\right)\varphi\,dx \\ &- \int_\Omega\left(\lambda(w_0+w_n)_+^q + (w_0+w_n)_+^{2^*_s-1}\right)\varphi\,dx. \end{aligned} \tag{1.2.36}$$

Then, by (1.2.33), (1.2.34) and (1.2.36) we obtain that

$$(\mathcal{I}_{s,\lambda}^*)'(z_n) \to 0. \tag{1.2.37}$$

Since $\{z_n\}$ is uniformly bounded in $H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$, up to a subsequence,

$$\begin{array}{rcll} z_n & \rightharpoonup & z & \text{weakly in } H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s}) \\ z_n(\cdot,0) & \to & z(\cdot,0) & \text{strong in } L^r(\Omega), \quad 1 \le r < 2_s^*, \\ z_n(\cdot,0) & \to & z(\cdot,0) & \text{a.e. in } \Omega. \end{array} \tag{1.2.38}$$

Note that since we are assuming that $v = 0$ is the unique critical point of $\widetilde{\mathcal{I}}_{s,\lambda}^*$ then, $z = w_0$.

Since $\{z_n\}$ satisfies (1.2.23) and (1.2.24), applying Lemma 1.2.9 we get that $\{y^{1-2s}|\nabla z_n|^2\}_{n\in\mathbb{N}}$ is tight and consequently $\{y^{1-2s}|\nabla (z_n)_+|^2\}_{n\in\mathbb{N}}$ is also tight. Therefore by Theorem 1.5.6, up to a subsequence, there exist an index set I, at most countable, a sequence of points $\{x_k\}_{k\in I} \subset \Omega$, and nonnegative real numbers μ_k, ν_k, such that

$$y^{1-2s}|\nabla (z_n)_+|^2 \to \mu \ge y^{1-2s}|\nabla w_0|^2 + \sum_{k\in I} \mu_k \delta_{x_k} \tag{1.2.39}$$

and

$$|(z_n)_+(\cdot,0)|^{2_s^*} \to \nu = |w_0(\cdot,0)|^{2_s^*} + \sum_{k\in I} \nu_k \delta_{x_k}, \tag{1.2.40}$$

in the sense of measures. Also we have relation

$$\mu_k \ge T(N,s)\nu_k^{\frac{2}{2_s^*}}, \qquad \text{for every } k \in I. \tag{1.2.41}$$

We fix $k_0 \in I$, and let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+^{N+1})$ be a non increasing cut-off function satisfying

$$\phi = \begin{cases} 1 & \text{in } B_1^+(x_{k_0}), \\ 0 & \text{in } B_2^+(x_{k_0})^c. \end{cases}$$

Set now $\phi_\varepsilon(x,y) := \phi(x/\varepsilon, y/\varepsilon)$. Clearly

$$|\nabla \phi_\varepsilon| \le \frac{C}{\varepsilon}. \tag{1.2.42}$$

We denote $\Gamma_{2\varepsilon} = B_{2\varepsilon}^+(x_{k_0}) \cap \{y = 0\}$. Taking $\phi_\varepsilon (z_n)_+ \in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ as a test function in (1.2.37), since

$$\int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla z_n, \nabla(\phi_\varepsilon (z_n)_+)\rangle dx\, dy = \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla (z_n)_+, \nabla(\phi_\varepsilon (z_n)_+)\rangle dx\, dy,$$

we have

$$
\begin{aligned}
&\lim_{n\to\infty} \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla (z_n)_+, \nabla \phi_\varepsilon \rangle (z_n)_+ dx\, dy \\
= \;& \lim_{n\to\infty} \left(\int_{\Gamma_{2\varepsilon}} (\lambda |(z_n)_+|^{q+1} + |(z_n)_+|^{2_s^*}) \phi_\varepsilon \, dx \right. \\
- \;& \left. \kappa_s \int_{B^+_{2\varepsilon}(x_{k_0})} y^{1-2s} |\nabla (z_n)_+|^2 \phi_\varepsilon \, dx\, dy \right).
\end{aligned}
$$

By (1.2.38), (1.2.39) and (1.2.40) we get

$$
\begin{aligned}
&\lim_{n\to\infty} \kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla (z_n)_+, \nabla \phi_\varepsilon \rangle (z_n)_+ \, dx\, dy \\
= \;& \lambda \int_{\Gamma_{2\varepsilon}} w_0^{q+1} \phi_\varepsilon \, dx + \int_{\Gamma_{2\varepsilon}} \phi_\varepsilon \, d\nu - \kappa_s \int_{B^+_{2\varepsilon}(x_{k_0})} \phi_\varepsilon \, d\mu. \qquad (1.2.43)
\end{aligned}
$$

On the other hand, using Theorem 1.6 in [84], with $w = y^{1-2s} \in A_2$ and $k = 1$, by (1.2.42) we obtain that

$$
\begin{aligned}
\left(\int_{B^+_{2\varepsilon}(x_{k_0})} y^{1-2s} |\nabla \phi_\varepsilon|^2 |(z_n)_+|^2 dx\, dy \right)^{1/2} &\leq \frac{C}{\varepsilon} \left(\int_{B^+_{2\varepsilon}(x_{k_0})} y^{1-2s} |(z_n)_+|^2 dx\, dy \right)^{1/2} \\
&\leq C \left(\int_{B^+_{2\varepsilon}(x_{k_0})} y^{1-2s} |\nabla (z_n)_+|^2 dx\, dy \right)^{1/2}.
\end{aligned}
$$

Since $(z_n)_+ \in H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$, the last expression goes to zero as $\varepsilon \to 0$. Therefore

$$
\begin{aligned}
0 &\leq \lim_{n\to\infty} \left| \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla (z_n)_+, \nabla \phi_\varepsilon \rangle (z_n)_+ dx\, dy \right| \\
&\leq \lim_{n\to\infty} \left(\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla (z_n)_+|^2 dx\, dy \right)^{1/2} \left(\int_{B^+_{2\varepsilon}(x_{k_0})} y^{1-2s} |\nabla \phi_\varepsilon|^2 |(z_n)_+|^2 dx\, dy \right)^{1/2} \\
&\longrightarrow 0, \quad \text{when } \varepsilon \to 0. \qquad (1.2.44)
\end{aligned}
$$

Hence, by (1.2.43) and (1.2.44), it follows that

$$
\lim_{\varepsilon\to 0} \left[\lambda \int_{\Gamma_{2\varepsilon}} w_0^{q+1} \phi_\varepsilon \, dx + \int_{\Gamma_{2\varepsilon}} \phi_\varepsilon \, d\nu - \kappa_s \int_{B^+_{2\varepsilon}(x_{k_0})} \phi_\varepsilon \, d\mu \right] = \nu_{k_0} - \kappa_s \mu_{k_0} = 0.
$$

Therefore, by (1.2.41), we get that

$$
\nu_{k_0} = 0 \qquad \text{or} \qquad \nu_{k_0} \geq (\kappa_s T(N,s))^{\frac{N}{2s}}.
$$

Suppose that $\nu_{k_0} \neq 0$. Since $\langle (\mathcal{I}^*_{s,\lambda})'(w_0), w_0 \rangle = 0$, by (1.2.33), (1.2.35) and (1.2.37) we obtain that

$$\begin{aligned} c + \mathcal{I}^*_{s,\lambda}(w_0) &\geq \lim_{n\to\infty} \left(\mathcal{I}^*_{s,\lambda}(z_n) - \frac{1}{2}\langle (\mathcal{I}^*_{s,\lambda})'(z_n), z_n \rangle \right) \\ &\geq \lambda \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega w_0^{q+1} dx + \frac{s}{N} \int_\Omega w_0^{2^*_s} dx + \frac{s}{N} \nu_{k_0} \\ &\geq \mathcal{I}^*_{s,\lambda}(w_0) + \frac{s}{N} (\kappa_s T(N,s))^{N/2s} \\ &= \mathcal{I}^*_{s,\lambda}(w_0) + c^*. \end{aligned}$$

This gives a contradiction with (1.2.33), and since k_0 was arbitrary, $\nu_k = 0$ for all $k \in I$. As a consequence, $(w_n)_+ \to 0$ in $L^{2^*_s}(\Omega)$. We finish in the standard way: convergence of $(w_n)_+$ in $L^{2^*_s}(\Omega)$ implies convergence of $f((w_n)_+)$ in $L^{\frac{2N}{N+2s}}(\Omega)$, and finally by using the continuity of the inverse operator A_{-s}, we obtain convergence of w_n in $H(\Omega, s)$. Note that the argument we have used reflects the fact that the composition of a compact operator with a continuous operator is compact. $\square$

Now it remains to show that we can obtain a local $(\mathrm{PS})_c$ sequence for $\widetilde{\mathcal{I}}^*_{s,\lambda}$ under the critical level $c = c^*$. To do that we will use the family of minimizers to the Trace-Sobolev inequalities given in (1.1.29)-(1.1.30). The reason why we use these functions is because we have raised the problem (P_λ) as a perturbation of the critical problem

$$(P_*) = \begin{cases} A_s u = u^{2^*_s - 1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Therefore it seems natural to expect that the solutions of (P_λ) are closer to the solutions of (P_*). But remember that, by Pohozaev's identity we can assure that (P_*) has not solution when Ω is a star-shaped domain. However when $\Omega = \mathbb{R}^N$ the problem (P_*) has solution (see [70, Theorem 1.1] and [37, 81, 117]). That is, there exists a family of functions U_ε given in (1.1.32) for which the best constant in the inclusion $H^s(\mathbb{R}^N) \subseteq L^{2^*_s}(\mathbb{R}^N)$ is achieved. Moreover this family of functions is the unique positive solution of (P_*), with $\Omega = \mathbb{R}^N$, except dilations and translations. Therefore, in what follows we will consider

$$u_\varepsilon(x) := U_\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right) = \frac{\varepsilon^{(N-2s)/2}}{(|x|^2 + \varepsilon^2)^{(N-2s)/2}}. \qquad (1.2.45)$$

Observe that u_ε is also a minimizer of (1.1.30) because, by (1.1.26),

$$\begin{aligned}\left\|(-\Delta)^{s/2}\left(U_\varepsilon\left(\frac{x}{\sqrt{\varepsilon}}\right)\right)\right\|^2_{L^2(\mathbb{R}^N)} &= \left\|\varepsilon^{-\frac{s}{2}}\left((-\Delta)^{s/2}U_\varepsilon\right)\left(\frac{x}{\sqrt{\varepsilon}}\right)\right\|^2_{L^2(\mathbb{R}^N)}\\ &= \varepsilon^{\frac{N}{2}-s}\|(-\Delta)^{s/2}U_\varepsilon\|^2_{L^2(\mathbb{R}^N)}\\ &= \kappa_s T(N,s)\varepsilon^{\frac{N}{2}-s}\|U_\varepsilon\|^2_{L^{2^*_s}(\mathbb{R}^N)}\\ &= \kappa_s T(N,s)\|u_\varepsilon\|^2_{L^{2^*_s}(\mathbb{R}^N)}.\end{aligned}$$

Moreover

$$\begin{aligned}\|u_\varepsilon\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)} &= \int_{\mathbb{R}^N}\frac{\varepsilon^N}{(|x|^2+\varepsilon^2)^N}\,dx\\ &\le \int_{\{0<|x|<\varepsilon\}}\varepsilon^{-N}+\int_{\{|x|>\varepsilon\}}\frac{\varepsilon^N}{|x|^{2N}}\,dx := K_1, \quad (1.2.46)\end{aligned}$$

is a constant independent of ε.

We note that the functions given in (1.2.45) do not have compact support but when ε is small their mass is concentrated in $x=0$. Therefore although there are not solutions of (P_*), if we multiple them by a positive cut off function, to adapt them to the domain Ω, and we take ε small enough, it is expected that these functions are a good approximations to the solutions of (P_*).

We remark that, except for the cases $s=1/2$ and $s=1$, $w_\varepsilon = E_s(u_\varepsilon)$ does not have an explicit expression. Indeed for the case $s=1/2$, by a result of J. Escobar in [82] that uses differential geometry, we know that

$$w_\varepsilon(x,y) = \frac{\varepsilon^{\frac{N-1}{2}}}{\left(|x|^2+(y+\varepsilon)^2\right)^{\frac{N-1}{2}}}.$$

If $s\neq 1/2$, we could think that the function

$$z(x,y) := \frac{\varepsilon^{\frac{N-2s}{2}}}{\left(|x|^2+(y+\varepsilon)^2\right)^{\frac{N-2s}{2}}},$$

provides the associated minimizer. However $z\neq w_\varepsilon$ when $s\neq 1/2$ because z

is not a s-harmonic function. Indeed, if $(x, y) \in \mathscr{C}_\Omega$, then

$$\begin{aligned}
& L_s z(x,y) \\
= \; & -y^{1-2s}(\Delta_{x,y} z + \frac{1-2s}{y} z_y)(x,y) \\
= \; & -y^{1-2s}\left(\frac{\varepsilon^{\frac{N-2s}{2}}(2s-N)(2s-1)}{(|x|^2+(y+\varepsilon)^2)^{\frac{N-2s}{2}+1}} + \frac{1-2s}{y} \frac{\varepsilon^{\frac{N-2s}{2}}(2s-N)(y+\varepsilon)}{(|x|^2+(y+\varepsilon)^2)^{\frac{N-2s}{2}+1}} \right) \\
= \; & \frac{y^{-2s}\varepsilon^{\frac{N-2s}{2}+1}(1-2s)(N-2s)}{(|x|^2+(y+\varepsilon)^2)^{\frac{N-2s}{2}+1}}. \qquad (1.2.47)
\end{aligned}$$

Since, for every $(x, y) \in \mathscr{C}_\Omega$, $0 < s < 1$ and $N > 2s$, from (1.2.47) we deduce that $L_s z = 0$ if and only if $s = 1/2$. In fact $L_s z < 0$ when $s > 1/2$ and $L_s z > 0$ if $s < 1/2$.

The fact that w_ε is not explicit, is an extra difficulty that we have to overcome. We take into account that the family u_ε is self-similar, that is, $u_\varepsilon(x) = \varepsilon^{\frac{2s-N}{2}} u_1(x/\varepsilon)$ and the fact that the Poisson kernel (1.1.25) is also self-similar

$$P_y^s(x) = \frac{1}{y^N} P_1^s\left(\frac{x}{y}\right), \qquad (1.2.48)$$

where

$$P_1^s(z) = \frac{1}{(1+|z|^2)^{\frac{N+2s}{2}}}.$$

This gives easily that the family w_ε satisfies

$$w_\varepsilon(x,y) = \varepsilon^{\frac{2s-N}{2}} w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \qquad (1.2.49)$$

Here

$$w_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) = \left(P^s_{\frac{y}{\varepsilon}}(\cdot) * u_1(\cdot)\right)\left(\frac{x}{\varepsilon}\right).$$

We will denote $P^s = P_1^s$. Also, we will write $w_{1,2s}$ ($u_{1,2s}$) instead of w_1 (u_1) to emphasize the dependence on the parameter s. Before taking a cut off function to adapt the minimizers to our domain Ω, we are going to prove some useful properties of that family of extended functions that we present as follows.

Lemma 1.2.11. *With the above notation one has*

$$|\nabla w_{1,2s}(x,y)| \le \frac{C}{y} w_{1,2s}(x,y), \quad s > 0, \; (x,y) \in \mathbb{R}^{N+1}_+, \qquad (1.2.50)$$

$$|\nabla w_{1,2s}(x,y)| \le C w_{1,2s-1}(x,y), \quad s > 1/2, \; (x,y) \in \mathbb{R}^{N+1}_+. \qquad (1.2.51)$$

Moreover if

$$(x, y) \in \mathcal{C}_{1/\varepsilon} := \left\{ (x, y) \in \mathbb{R}^{N+1}_{+} : \frac{1}{2\varepsilon} \le r_{xy} \le \frac{1}{\varepsilon} \right\},$$

where $r_{xy} := |(x, y)| = (|x|^2 + y^2)^{\frac{1}{2}}$, *then*

$$w_{1,2s}(x, y) \le C\varepsilon^{N-2s}, \quad s > 0. \tag{1.2.52}$$

Proof. Differentiating with respect to each variable x_i, $i = 1, \dots, N$, and the variable y, by Young's inequality it follows that

$$\begin{aligned} |\partial_{x_i} w_{1,2s}(x, y)| &\le \int_{\mathbb{R}^N} \frac{(N + 2s) y^{2s} |x - z|}{(y^2 + |x - z|^2)^{\frac{N+2s}{2}+1} (1 + |z|^2)^{\frac{N-2s}{2}}} \, dz \\ &\le \frac{N + 2s}{2y} \int_{\mathbb{R}^N} \frac{y^{2s}}{(y^2 + |x - z|^2)^{\frac{N+2s}{2}} (1 + |z|^2)^{\frac{N-2s}{2}}} \, dz \\ &= \frac{C}{y} w_{1,2s}(x, y). \end{aligned} \tag{1.2.53}$$

Also since $2s|x - z|^2 - Ny^2 \le C(|x - z|^2 + y^2)$, we obtain

$$\begin{aligned} |\partial_y w_{1,2s}(x, y)| &= \left| \int_{\mathbb{R}^N} \frac{y^{2s-1}(2s|x - z|^2 - Ny^2)}{(y^2 + |x - z|^2)^{\frac{N+2s}{2}+1} (1 + |z|^2)^{\frac{N-2s}{2}}} \, dz \right| \\ &\le C \int_{\mathbb{R}^N} \frac{y^{2s-1}}{(y^2 + |x - z|^2)^{\frac{N+2s}{2}} (1 + |z|^2)^{\frac{N-2s}{2}}} \, dz \\ &= \frac{C}{y} w_{1,2s}(x, y). \end{aligned} \tag{1.2.54}$$

By (1.2.53) and (1.2.54) we deduce (1.2.50).

To obtain (1.2.51) we transfer the derivative to the function $u_{1,2s}$ because it has the worst decay at infinity. Since $u_{1,2s}(z) = (1 + |z|^2)^{-\frac{N-2s}{2}}$, by (1.2.48) and doing the change of variable $\tilde{z} = \dfrac{x-z}{y}$, it follows that

$$
\begin{aligned}
|\partial_y w_{1,2s}(x,y)| &= \left|\partial_y \left(\int_{\mathbb{R}^N} \frac{1}{y^N} P_1^s \left(\frac{x-z}{y} \right) u_{1,2s}(z) dz \right) \right| \\
&= \left|\partial_y \left(\int_{\mathbb{R}^N} P_1^s(\tilde{z}) u_{1,2s}(x - y\tilde{z})\, d\tilde{z} \right) \right| \\
&= \left| \int_{\mathbb{R}^N} P_1^s(\tilde{z}) \left\langle \tilde{z}, \nabla u_{1,2s}(x - y\tilde{z}) \right\rangle \, d\tilde{z} \right| \\
&= \left| \int_{\mathbb{R}^N} \frac{1}{y^N} P_1^s \left(\frac{x-z}{y} \right) \left\langle \frac{x-z}{y}, \nabla u_{1,2s}(z) \right\rangle \, dz \right| \\
&\le (N-2s) \int_{\mathbb{R}^N} \frac{1}{y^N} P_1^s \left(\frac{x-z}{y} \right) \frac{|x-z||z|}{y(1+|z|^2)^{\frac{N-2s}{2}+1}} \, dz \\
&\le (N-2s) \int_{\mathbb{R}^N} \frac{y^{2s-1}}{(y^2+|x-z|^2)^{\frac{N+2s-1}{2}} (1+|z|^2)^{\frac{N-2s+1}{2}}} \, dz \\
&= C w_{1,2s-1}(x,y). \qquad (1.2.55)
\end{aligned}
$$

Doing the same calculations in the variables x_i for $i = 1, \ldots, N$, we obtain

$$
\begin{aligned}
|\partial_{x_i} w_{1,2s}(x,y)| &= \left|\partial_{x_i} \left(\int_{\mathbb{R}^N} P_1^s(\tilde{z}) u_{1,2s}(x - y\tilde{z})\, d\tilde{z} \right) \right| \\
&\le \int_{\mathbb{R}^N} P_1^s(\tilde{z}) |\nabla u_{1,2s}|(x - y\tilde{z}) d\tilde{z} \\
&= \int_{\mathbb{R}^N} \frac{1}{y^N} P_1^s \left(\frac{x-z}{y} \right) |\nabla u_{1,2s}|(z)\, dz \\
&\le (N-2s) \int_{\mathbb{R}^N} \frac{y^{2s}}{(y^2+|x-z|^2)^{\frac{N+2s}{2}}} \frac{|z|}{(1+|z|^2)^{\frac{N-2s}{2}+1}} \, dz \\
&\le C w_{1,2s-1}(x,y). \qquad (1.2.56)
\end{aligned}
$$

Then (1.2.51) follows from (1.2.55) and (1.2.56).

Finally we will prove (1.2.52). Let $(x,y) \in \mathcal{C}_{1/\varepsilon}$ and $s > 0$. Since

$$
(y^2 + |x-z|^2)^{1/2} = C(y^2 + |x|^2)^{1/2}, \qquad \text{for } |z| < 1/4\varepsilon
$$

and P^s_y is a summability kernel, we obtain that

$$\begin{aligned} w_{1,2s}(x,y) &= \int_{|z|<\frac{1}{4\varepsilon}} P^s_y(x-z)u_{1,2s}(z)dz + \int_{|z|>\frac{1}{4\varepsilon}} P^s_y(x-z)u_{1,2s}(z)\,dz \\ &\leq C\varepsilon^{N+2s}y^{2s}\int_{|z|<\frac{1}{4\varepsilon}} \frac{dz}{|z|^{N-2s}} + C\varepsilon^{N-2s}\int_{\mathbb{R}^N} P^s_y(z)\,dz \\ &\leq Cy^{2s}\varepsilon^N + C\varepsilon^{N-2s} \\ &\leq C\varepsilon^{N-2s}. \end{aligned}$$

□

Let us now introduce a non increasing cut-off function $\phi_0 \in \mathcal{C}^\infty(\mathbb{R}_+)$, satisfying

$$\phi_0(\eta) = \begin{cases} 1 & \text{if } 0 \leq \eta \leq \frac{1}{2}, \\ 0 & \text{if } \eta \geq 1. \end{cases}$$

Assume without loss of generality that $0 \in \Omega$. We then define, for some fixed $r > 0$ small enough so that $\overline{B}^+_r \subseteq \overline{\mathcal{C}}_\Omega$, the function

$$\phi(x,y) = \phi_r(x,y) = \phi_0\left(\frac{r_{xy}}{r}\right), \tag{1.2.57}$$

where r_{xy} was defined in Lemma 1.2.11. Note that $\phi w_\varepsilon \in H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$. In this way we get the next.

Lemma 1.2.12. *With the above notation, for ε small enough and $C > 0$ that may depends on r, the family $\{\phi w_\varepsilon\}$ and its trace at $\{y = 0\}$, $\{\phi u_\varepsilon\}$, satisfy*

$$\|\phi w_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} \leq \|w_\varepsilon\|^2_{H^1(\mathbb{R}^{N+1}_+, y^{1-2s})} + C\varepsilon^{N-2s}, \tag{1.2.58}$$

and

$$\|\phi u_\varepsilon\|^{p+1}_{L^{p+1}(\Omega)} \geq C\varepsilon^{N-\left(\frac{N-2s}{2}\right)(p+1)}, \quad \text{if} \quad N > 2s\left(1+\frac{1}{p}\right). \tag{1.2.59}$$

In particular, for some positive constants C and $\tilde{C}$,

$$\|\phi u_\varepsilon\|^2_{L^2(\Omega)} \geq \begin{cases} C\varepsilon^{2s} & \text{if } N > 4s, \\ C\varepsilon^{2s}\log(1/\varepsilon) & \text{if } N = 4s, \\ C\varepsilon^{N-2s} - \tilde{C}\varepsilon^{2s} & \text{if } N < 4s. \end{cases} \tag{1.2.60}$$

Proof. First of all, since $\operatorname{supp}\phi$ and $\operatorname{supp}\nabla\phi$ are contained in $\mathscr{C}_\Omega$, the product ϕw_ε satisfies

$$\begin{aligned}\|\phi w_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} &= \|\phi w_\varepsilon\|^2_{H^1(\mathbb{R}^{N+1}_+, y^{1-2s})}\\ &= \kappa_s\int_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\phi\nabla w_\varepsilon|^2+|w_\varepsilon\nabla\phi|^2+2\langle w_\varepsilon\nabla\phi,\phi\nabla w_\varepsilon\rangle)dx\,dy\\ &\le \|w_\varepsilon\|^2_{H^1(\mathbb{R}^{N+1}_+, y^{1-2s})} + \kappa_s\int_{\mathscr{C}_\Omega} y^{1-2s}|w_\varepsilon\nabla\phi|^2 dxdy\\ &\quad + 2\kappa_s\int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon\nabla\phi,\phi\nabla w_\varepsilon\rangle dx\,dy. \qquad (1.2.61)\end{aligned}$$

To estimate the second term of the right hand side we take $r > 0$ such that $\mathcal{C}_r = \{r/2 \le r_{xy} \le r\} \subset \mathscr{C}_\Omega$. Then by (1.2.52),

$$\begin{aligned}\int_{\mathscr{C}_\Omega} y^{1-2s}|w_\varepsilon\nabla\phi|^2\,dx\,dy &\le C\int_{\mathcal{C}_r} y^{1-2s}w_\varepsilon^2(x,y)\,dx\,dy\\ &= \varepsilon^{2s-N}\int_{\mathcal{C}_r} y^{1-2s}w_{1,2s}^2\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\,dx\,dy\\ &= C\varepsilon^{N-2s}, \qquad (1.2.62)\end{aligned}$$

where C may depends of $r > 0$. For the remaining term we note that by (1.2.49) we get

$$\begin{aligned}&\int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon\nabla\phi,\phi\nabla w_\varepsilon\rangle dx\,dy\\ &\le C\int_{\mathcal{C}_r} y^{1-2s}|w_\varepsilon(x,y)||\nabla w_\varepsilon(x,y)|dx\,dy\\ &= C\varepsilon^{-N+2s-1}\int_{\mathcal{C}_r} y^{1-2s}\left|w_{1,2s}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|\left|\nabla w_{1,2s}\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right|dx\,dy\\ &= C\varepsilon\int_{\mathcal{C}_{\frac{r}{\varepsilon}}} y^{1-2s}|w_{1,2s}(x,y)|\,|\nabla w_{1,2s}(x,y)|dx\,dy. \qquad (1.2.63)\end{aligned}$$

If $s < 1/2$, from (1.2.50), (1.2.52) and (1.2.63), it follows that

$$\begin{aligned}\int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon\nabla\phi,\phi\nabla w_\varepsilon\rangle dx\,dy &\le C\varepsilon^{1+2(N-2s)}\int_{\mathcal{C}_{\frac{r}{\varepsilon}}} y^{-2s}dx\,dy\\ &= C\varepsilon^{N-2s}. \qquad (1.2.64)\end{aligned}$$

To obtain the similar estimate for $s > 1/2$ we use (1.2.51). Indeed by this estimate, together with (1.2.52) and (1.2.63) we get that

$$\begin{aligned} \int_{\mathscr{C}_\Omega} y^{1-2s}\langle w_\varepsilon \nabla\phi, \phi\nabla w_\varepsilon\rangle dx\, dy &\leq C\varepsilon^{2(1+N-2s)}\int_{\mathcal{C}_{\frac{r}{\varepsilon}}} y^{1-2s} dx\, dy \\ &= C\varepsilon^{N-2s}. \end{aligned} \tag{1.2.65}$$

Note that for $s = 1/2$, as w_ε is explicit, we can obtain the same estimate directly.

Then, by (1.2.61)-(1.2.65), we have proved (1.2.58).

Finally we show that (1.2.59) holds. Let $N > 2s\left(1+\dfrac{1}{p}\right)$. Denoting by $\alpha := -(N-(N-2s)(p+1)) > 0$, for some positive constant $C > 0$, we get that

$$\begin{aligned} \int_\Omega |\phi u_\varepsilon(x)|^{p+1}\, dx &\geq \int_{|x|<r/2} |u_\varepsilon|^{p+1}\, dx \\ &= \varepsilon^{\left(\frac{N-2s}{2}\right)(p+1)} \int_{|x|<r/2} \frac{dx}{\left(|x|^2+\varepsilon^2\right)^{\frac{(N-2s)(p+1)}{2}}} \\ &= C\varepsilon^{-\left(\frac{N-2s}{2}\right)(p+1)} \int_0^{r/2} \frac{\rho^{N-1}}{\left(1+\left(\frac{\rho}{\varepsilon}\right)^2\right)^{\frac{(N-2s)(p+1)}{2}}}\, d\rho \\ &= C\varepsilon^{N-\left(\frac{N-2s}{2}\right)(p+1)} \int_0^{r/2\varepsilon} \frac{t^{N-1}}{(1+t^2)^{\frac{(N-2s)(p+1)}{2}}}\, dt \\ &\geq C\varepsilon^{N-\left(\frac{N-2s}{2}\right)(p+1)} \int_1^{r/2\varepsilon} t^{N-1-(N-2s)(p+1)}\, dt \\ &= C\frac{\varepsilon^{N-\left(\frac{N-2s}{2}\right)(p+1)}}{\alpha}\left(1-\left(\frac{2\varepsilon}{r}\right)^\alpha\right) \\ &\geq C\varepsilon^{N-\left(\frac{N-2s}{2}\right)(p+1)}. \end{aligned}$$

Finally we observe that (1.2.60) follows taking $p = 1$ in the previous estimate. □

With the above properties in mind, we define the family of functions

$$\eta_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi u_\varepsilon\|_{L^{2_s^*}(\Omega)}} > 0. \tag{1.2.66}$$

The we have.

Lemma 1.2.13. *There exists $\varepsilon > 0$ small enough such that*

$$\sup_{t\geq 0} \widetilde{\mathcal{I}}^*_{s,\lambda}(t\eta_\varepsilon) < c^*. \tag{1.2.67}$$

Proof. Assume $N \geq 4s$. We make use that for some $\mu > 0$,

$$(a+b)^p \geq a^p + b^p + \mu a^{p-1}b, \quad a,b \geq 0, \quad p > 1. \tag{1.2.68}$$

Therefore $g(x,\eta) \geq \eta_+^{2_s^*-1} + \mu w_0^{2_s^*-2}\eta_+$, and thus

$$G(w) \geq \frac{1}{2_s^*}w_+^{2_s^*} + \frac{\mu}{2}w_+^2 w_0^{2_s^*-2}. \tag{1.2.69}$$

This implies that

$$\widetilde{\mathcal{I}}_{s,\lambda}^*(t\eta_\varepsilon) \leq \frac{t^2}{2}\|\eta_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} - \frac{t^{2_s^*}}{2_s^*} - \frac{t^2}{2}\mu\int_\Omega w_0^{2_s^*-2}\eta_\varepsilon^2 dx.$$

Since $w_0 \geq a_0 > 0$ in $\operatorname{supp}(\eta_\varepsilon)$ we get

$$\widetilde{\mathcal{I}}_{s,\lambda}^*(t\eta_\varepsilon) \leq \frac{t^2}{2}\|\eta_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} - \frac{t^{2_s^*}}{2_s^*} - \frac{t^2}{2}\widetilde{\mu}\|\eta_\varepsilon\|^2_{L^2(\Omega)}. \tag{1.2.70}$$

On the other hand we note that

$$\begin{aligned}\|\phi u_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\Omega)} &= \|\phi u_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} \\ &= \|u_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} + \int_{\mathbb{R}^N}(\phi^{2_s^*}-1)\frac{\varepsilon^N}{(|x|^2+\varepsilon^2)^N}\,dx \\ &= \|u_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} - C\varepsilon^N. \end{aligned} \tag{1.2.71}$$

Therefore, by (1.2.46), (1.2.58) and (1.2.71) we have

$$\begin{aligned}\|\eta_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} &\leq \frac{\|w_\varepsilon\|^2_{H^1(\mathbb{R}^{N+1}_+,y^{1-2s})} + C\varepsilon^{N-2s}}{\left(\|u_\varepsilon\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} - C\varepsilon^N\right)^{\frac{2}{2_s^*}}} \\ &= \kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s}. \end{aligned} \tag{1.2.72}$$

Moreover, since, by (1.2.46),

$$\|\phi u_\varepsilon\|^2_{L^{2_s^*}(\Omega)} = \|\phi u_\varepsilon\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leq \|u_\varepsilon\|^2_{L^{2_s^*}(\mathbb{R}^N)} = K_1,$$

whit $K_1 > 0$ is a constant independent of ε, from (1.2.60) we obtain that

$$\|\eta_\varepsilon\|^2_{L^2(\Omega)} \geq \begin{cases} \widetilde{C}\varepsilon^{2s} & \text{if } N > 4s, \\ \widetilde{C}\varepsilon^{2s}\log(1/\varepsilon) & \text{if } N = 4s. \end{cases} \tag{1.2.73}$$

Then from (1.2.70), (1.2.72) and (1.2.73), we get

$$\widetilde{\mathcal{I}}^*_{s,\lambda}(t\eta_\varepsilon) \le \frac{t^2}{2}(\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s}) - \frac{t^{2^*_s}}{2^*_s} - \frac{t^2}{2}\widetilde{\mu}\widetilde{C}\varepsilon^{2s} := g(t). \tag{1.2.74}$$

It is clear that $\lim_{t\to\infty} g(t) = -\infty$, and $\sup_{t\ge 0} g(t)$ is attained at some $t_\varepsilon \ge 0$. If $t_\varepsilon = 0$ then

$$\sup_{t\ge 0} \widetilde{\mathcal{I}}^*_{s,\lambda}(t\eta_\varepsilon) \le \sup_{t\ge 0} g(t) = g(0) = 0,$$

so (1.2.67) is trivially obtained. Therefore, we consider $t_\varepsilon > 0$. Differentiating the above function we get

$$0 = g'(t_\varepsilon) = t_\varepsilon(\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s}) - t_\varepsilon^{2^*_s-1} - t_\varepsilon\widetilde{\mu}\widetilde{C}\varepsilon^{2s}, \tag{1.2.75}$$

which implies

$$t_\varepsilon \le (\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s})^{\frac{1}{2^*_s-2}}. \tag{1.2.76}$$

Moreover for $\varepsilon > 0$ small enough, we have

$$t_\varepsilon \ge \overline{c} > 0. \tag{1.2.77}$$

Indeed from (1.2.75),

$$t_\varepsilon^{2^*_s-2} = \kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s} - \widetilde{\mu}\tilde{C}\varepsilon^{2s} \ge \overline{c} > 0,$$

provided ε is small. On the other hand, the function

$$t \mapsto \frac{t^2}{2}(\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s}) - \frac{t^{2^*_s}}{2^*_s}$$

is increasing on $[0, (\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s})^{\frac{1}{2^*_s-2}}]$. Whence, by (1.2.74), (1.2.76) and (1.2.77), we obtain

$$\sup_{t\ge 0} g(t) = g(t_\varepsilon) \le \frac{s}{N}(\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s})^{\frac{N}{2s}} - \overline{C}\varepsilon^{2s}, \tag{1.2.78}$$

for some $\overline{C} > 0$. Since for some $\underline{C} > 0$,

$$(\kappa_s T(N,s) + \widehat{C}\varepsilon^{N-2s})^{\frac{N}{2s}} = (\kappa_s T(N,s))^{\frac{N}{2s}} + \underline{C}\varepsilon^{N-2s} + O(\varepsilon^{N-2s})$$

and $N > 4s$, from (1.2.78) we conclude that

$$g(t_\varepsilon) \le \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}} + C\varepsilon^{N-2s} - \overline{C}\varepsilon^{2s} < \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}} = c^*.$$

Hence from (1.2.74) we obtain (1.2.67) for $N > 4s$.

If $N = 4s$ the same conclusion follows because, doing as before, we obtain

$$g(t_\varepsilon) \le \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}} + C\varepsilon^{2s} - \overline{C}\varepsilon^{2s}\log\frac{1}{\varepsilon} < c^*.$$

The last case $2s < N < 4s$ follows by using the estimate (1.2.68) which gives, for some $\mu' > 0$,

$$G(w) \ge \frac{1}{2_s^*} w_+^{2_s^*} + \mu' w_0 w_+^{2_s^*-1}. \tag{1.2.79}$$

Arguing in a similar way, by (1.2.79) jointly with (1.2.59) for $p+1 = 2_s^* - 1$, we finish the proof. □

Proof of Theorem 1.2.1-3. To finish the last statement in Theorem 1.2.1, in view of the previous results, we look for a path with energy below the critical level c^*. We consider $M_\varepsilon > 0$ big enough such that $\widetilde{\mathcal{I}}^*_{s,\lambda}(M_\varepsilon \eta_\varepsilon) < \widetilde{\mathcal{I}}^*_{s,\lambda}(0) = 0$. Also, by Lemma 1.2.8, there exists $\alpha > 0$ such that if $\|u\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} = \alpha$ then $\widetilde{\mathcal{I}}^*_{s,\lambda}(u) \ge \widetilde{\mathcal{I}}^*_{s,\lambda}(0)$. We define

$$\Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0,1], H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})) :\ \gamma(0) = 0,\, \gamma(1) = M_\varepsilon \eta_\varepsilon\}$$

and the minimax value

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \sup_{0\le t\le 1} \widetilde{\mathcal{I}}^*_{s,\lambda}(\gamma(t)). \tag{1.2.80}$$

By the arguments above $c_\varepsilon \ge \widetilde{\mathcal{I}}^*_{s,\lambda}(0) = \max\{\widetilde{\mathcal{I}}^*_{s,\lambda}(0), \widetilde{\mathcal{I}}^*_{s,\lambda}(M_\varepsilon\eta_\varepsilon)\}$. Also, by Lemma 1.2.13, for $\varepsilon \ll 1$, we obtain that

$$c_\varepsilon \le \sup_{0\le t\le 1} \widetilde{\mathcal{I}}^*_{s,\lambda}(tM_\varepsilon\eta_\varepsilon) = \sup_{t\ge 0} \widetilde{\mathcal{I}}^*_{s,\lambda}(t\eta_\varepsilon) < c^*.$$

Therefore by Lemma 1.2.10 and the MPT [11] if $c_\varepsilon > \widetilde{\mathcal{I}}^*_{s,\lambda}(0)$, or the corresponding refinement given in [95] if the minimax level is equal to $\widetilde{\mathcal{I}}^*_{s,\lambda}(0)$, we obtain the existence of a nontrivial solution to $(\widetilde{P}_\lambda)$. This is a contradiction with the assumption that $v = 0$ is the unique critical point of $\widetilde{\mathcal{I}}^*_{s,\lambda}$. Hence, there exist at least two solutions of (P_λ) for $s \ge 1/2$.

1.3 Linear case: $q = 1$.

In this section we will prove the following

Theorem 1.3.1. *Assume $q = 1$, $0 < s < 1$ and $N \geq 4s$. Then the problem* (P_λ)

1 has no solution for $\lambda \geq \rho_1^s$;

2 has at least one energy solution for each $0 < \lambda < \rho_1^s$. Here ρ_1 is the first eigenvalue of the Laplace operator with zero Dirichlet condition.

The proof of the previous result follows the ideas of [45]. Note that for $s = 1/2$, where the minimizers given in (1.2.49) are explicit, this result has ben recently proven in [161]. In our case, since we consider all the range $0 < s < 1$, as above, the principal difficulty is that these minimizers are not explicit.
The first part of the proof of that theorem is a straightforward computation.
Proof of Theorem 1.3.1 *1.* Let φ_1 be the first positive eigenfunction of A_s in Ω with zero Dirichlet condition. We have

$$\int_\Omega A_{s/2}uA_{s/2}\varphi_1\,dx = \int_\Omega \rho_1^s u\varphi_1\,dx.$$

Moreover, taking $\varphi_1 \in H(\Omega, s)$ as a test function in (P_λ),

$$\int_\Omega A_{s/2}uA_{s/2}\varphi_1\,dx = \int_\Omega (\lambda u + u^{2_s^*-1})\varphi_1\,dx > \int_\Omega \lambda u\varphi_1\,dx.$$

This clearly implies $\lambda < \rho_1^s$.

To prove the second part of Theorem 1.3.1 we introduce the following Rayleigh quotient

$$Q_\lambda(w) = \frac{\|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} - \lambda\|w(\cdot,0)\|^2_{L^2(\Omega)}}{\|w(\cdot,0)\|^2_{L^{2_s^*}(\Omega)}}$$

and we define

$$S_\lambda = \inf\{Q_\lambda(w) \mid \ w \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})\}. \tag{1.3.1}$$

Then we have the following.

Proposition 1.3.2. *Assume $0 < \lambda < \rho_1^s$. If $N \geq 4s$ then $S_\lambda < \kappa_s T(N,s)$ where κ_s and $T(N,s)$ are defined in* (1.1.12) *and* (1.1.31) *respectively.*

Proof. Let $\phi = \phi_r$ be the cut-off function defined in (1.2.57) and consider $\phi(x) := \phi(x,0)$. Let w_ε be defined as in (1.2.49). Taking r sufficiently small we can use $\phi w_\varepsilon \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$ as a test function in Q_λ. Consider $N > 4s$.

Then, taking ε small enough, by (1.2.46), (1.2.58), (1.2.60), and (1.2.71) we obtain that

$$\begin{aligned} Q_\lambda(\phi w_\varepsilon) &\leq \frac{\kappa_s \displaystyle\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w_\varepsilon|^2\,dx\,dy - \lambda\widetilde{C}\varepsilon^{2s} + C\varepsilon^{N-2s}}{(K_1 - C\varepsilon^N)^{2/2^*_s}} \\ &\leq \kappa_s T(N,s) - \lambda\widetilde{C}\varepsilon^{2s} + C\varepsilon^{N-2s} \\ &< \kappa_s T(N,s). \end{aligned}$$

If $N = 4s$, a similar computation proves that for ε small enough,

$$Q_\lambda(\phi w_\varepsilon) \leq \kappa_s T(N,s) - \lambda\widetilde{C}\varepsilon^{2s}\log(1/\varepsilon) + C\varepsilon^{2s} < \kappa_s T(N,s),$$

which finishes the proof. □

Note that Recall now the Brezis-Lieb Lemma given in [44, Theorem 1].

Lemma 1.3.3. *Let* $1 \leq q < \infty$*,* Ω *be an open set and* $\{u_n\}$ *be a sequence weakly convergent in* $L^q(\Omega)$ *and a.e. convergent in* Ω*. Then*

$$\lim_{n\to\infty}(\|u_n\|^q_{L^q(\Omega)} - \|u_n - u\|^q_{L^q(\Omega)}) = \|u\|^q_{L^q(\Omega)}.$$

This property allows us to prove the following.

Proposition 1.3.4. *Assume* $0 < \lambda < \rho_1^s$*. Then the infimum* S_λ *defined in* (1.3.1) *is achieved in a nonnegative function.*

Proof. First, since $\lambda < \rho_1$ we have that $S_\lambda > 0$. Let us take a minimizing sequence of S_λ, $\{w_m\} \subset H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$. That is a sequence such that

$$\lim_{m\to\infty} \frac{\|w_m\|^2_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)}}{\|w_m(\cdot,0)\|^2_{L^{2^*_s}(\Omega)}} = S_\lambda. \tag{1.3.2}$$

Without loss of generality we consider $w_m \geq 0$ (otherwise we take $|w_m|$ instead of w_m) and $\|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} = 1$. Clearly this implies that

$$\|w_m\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} \leq C,$$

then there exists a subsequence, still denoted by $\{w_m\}$, satisfying

$$\begin{array}{rcll} w_m & \rightharpoonup & w & \text{weakly in } H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s}), \\ w_m(\cdot,0) & \to & w(\cdot,0) & \text{strongly in } L^q(\Omega),\ 1 \leq q < 2^*_s, \\ w_m(\cdot,0) & \to & w(\cdot,0) & \text{a.e in } \Omega. \end{array} \tag{1.3.3}$$

A simple calculation, using the weak convergence, gives that

$$\begin{aligned}\|w_m\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} &= \|w_m - w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})}\\ &+ 2\kappa_s \int_{\mathscr{C}_\Omega} y^{1-2s}\langle \nabla w, \nabla w_m - \nabla w\rangle dx\,dy\\ &= \|w_m - w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})}\\ &+ o(1). \end{aligned} \tag{1.3.4}$$

Since, by Lemma 1.3.3, we have that

$$\|(w_m - w)(\cdot,0)\|_{L^{2^*_s}(\Omega)} = \|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} - \|w(\cdot,0)\|_{L^{2^*_s}(\Omega)} + o(1),$$

we can consider that $\|(w_m - w)(\cdot,0)\|_{L^{2^*_s}(\Omega)} \le 1$ for m big enough. Moreover since $\|w_m(\cdot,0)\|_{L^{2^*_s}(\Omega)} \le 1$ and $w_m(\cdot,0) \to w(\cdot,0)$ a.e., we get that $\|w(\cdot,0)\|_{L^{2^*_s}(\Omega)} \le 1$. Hence by (1.3.3) and (1.3.4) it follows that

$$\begin{aligned} Q_\lambda(w_m) &= \|w_m\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)}\\ &= \|w_m - w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} + \|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} - \lambda\|w_m(\cdot,0)\|^2_{L^2(\Omega)}\\ &+ o(1)\\ &\ge \kappa_s T(N,s)\|(w_m - w)(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + S_\lambda\|w(\cdot,0)\|^2_{L^{2^*_s}(\Omega)} + o(1)\\ &\ge \kappa_s T(N,s)\|(w_m - w)(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)} + S_\lambda\|w(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)} + o(1). \end{aligned}$$

By Lemma 1.3.3 again, this leads to

$$\begin{aligned} Q_\lambda(w_m) &\ge (\kappa_s T(N,s) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)} + S_\lambda\|w_m(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)}\\ &+ o(1)\\ &= (\kappa_s T(N,s) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)} + S_\lambda + o(1). \end{aligned}$$

Then, by (1.3.2) and the previous inequality we obtain that

$$o(1) + S_\lambda \ge (\kappa_s T(N,s) - S_\lambda)\|(w_m - w)(\cdot,0)\|^{2^*_s}_{L^{2^*_s}(\Omega)} + S_\lambda + o(1).$$

Thus by Proposition 1.3.2

$$w_m(\cdot,0) \to w(\cdot,0) \qquad \text{in } L^{2^*_s}(\Omega). \tag{1.3.5}$$

Finally, by a standard lower semi-continuity argument, we conclude that w is a minimizer for Q_λ. Indeed if we consider

$$\widetilde{Q}_\lambda(v) := \|v\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\,y^{1-2s})} - \lambda\|v(\cdot,0)\|^2_{L^2(\Omega)},\; v \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s}),$$

we obtain that

$$\widetilde{Q}_\lambda(\underline{\lim}_{m\to\infty} w_m) \le \underline{\lim}_{m\to\infty} \widetilde{Q}_\lambda(w_m) = S_\lambda \|w_m(\cdot,0)\|_{L^{2_s^*}(\Omega)}^{2_s^*}. \tag{1.3.6}$$

Moreover

$$\widetilde{Q}_\lambda(\underline{\lim}_{m\to\infty} w_m) = \widetilde{Q}_\lambda(w) \ge S_\lambda \|w(\cdot,0)\|_{L^{2_s^*}(\Omega)}^{2_s^*}. \tag{1.3.7}$$

Hence, by (1.3.5), (1.3.6) and (1.3.7) we conclude that $S_\lambda = Q_\lambda(w)$. □

Proof of Theorem 1.3.1 2. By Proposition 1.3.4 there exists an s-harmonic function $w \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$, such that $\|w(\cdot,0)\|^2_{L^{2_s^*}(\Omega)} = 1$ and

$$\|w\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} - \lambda \|w(\cdot,0)\|^2_{L^2(\Omega)} = S_\lambda.$$

So we get a solution of (P_λ).

1.4 Superlinear case: $1 < q < 2_s^* - 1$.

In this section we discuss the problem (P_λ) in the convex setting $q > 1$. That is, we will prove the following.

Theorem 1.4.1. *Let $1 < q < 2_s^* - 1$ and $0 < s < 1$. Then, problem (P_λ) admits at least one energy solution provided that either*

- $N > \frac{2s(q+3)}{q+1}$ *and $\lambda > 0$ or*
- $N \le \frac{2s(q+3)}{q+1}$ *and $\lambda > 0$ is sufficiently large.*

To prove the previous result, the only difficult part is to show that we have a $(\mathrm{PS})_c$ sequence under the critical level $c = c^*$. This follows the same type of computations like in Lemma 1.2.13, with the estimate (1.2.59) instead of (1.2.60).

First of all it is easy to check the geometry of the functional fo every $\lambda > 0$. That is we have the following

Proposition 1.4.2. *Assume $\lambda > 0$ and $1 < q < 2_s^* - 1$. Then there exist $\alpha > 0$ and $\beta > 0$ such that*

a) For any $w \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$ with $||w||_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} = \alpha$ it results that $\mathcal{I}^*_{s,\lambda}(w) \ge \beta$.

b) There exists a nonnegative function $w_1 \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$ such that $||w_1||_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} > \alpha$ and $\mathcal{I}^*_{s,\lambda}(w_1) < \beta$.

Proof. a) By the Trace inequality given in (1.1.29), since $q+1<2_s^*$, one can check that

$$\mathcal{I}_{s,\lambda}^*(w) \geq g(||w||_{H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})}),$$

where $g(t)=C_1t^2-\lambda C_2 t^{q+1}-C_3t^{2_s^*}$, for some positive constants C_1, C_2 and C_3. Therefore there will exist $\alpha>0$ such that $\beta := g(\alpha) > 0$. Then $\mathcal{I}_{s,\lambda}^*(w)\geq \beta$ for $w\in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ with $||w||_{H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})}=\alpha$.

b) Fix a positive function $w_0\in H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$ such that $\|w_0\|_{H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})}=1$ and consider $t>0$. Since $2_s^*>2$, it follows that

$$\lim_{t\to\infty}\mathcal{I}_{s,\lambda}^*(tw_0)=-\infty.$$

Then, there exists t_0 large enough, such that, defining $w_1 := t_0w_0$, $||w_1||_{H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})}>\alpha$ and $\mathcal{I}_{s,\lambda}^*(w_1)<\beta$.

□

By a similar argument, it follows that

$$\lim_{t\to 0^+}\mathcal{I}_{s,\lambda}^*(tw_0)=0. \tag{1.4.1}$$

Let us check now that we have the compactness properties of $\mathcal{I}_{s,\lambda}^*$.

Proposition 1.4.3. *Let $\lambda>0$ and $1<q<2_s^*-1$. Then, the functional $\mathcal{I}_{s,\lambda}^*$ satisfies the $(PS)_c$ condition at any level c, provided $c<c^*$, where c^* is defined in* (1.2.22)*.*

Proof. Let $\{w_n\}$ be a PS sequence for $\mathcal{I}_{s,\lambda}^*$ verifying

$$\mathcal{I}_{s,\lambda}^*(w_n)\to c<c^* \tag{1.4.2}$$

and

$$(\mathcal{I}_{s,\lambda}^*)'(w_m)\to 0. \tag{1.4.3}$$

Doing the same as in the beginning of the proof of the Lemma 1.2.10 we obtain that $\{w_m\}$ is uniformly bounded in $H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s})$. Therefore

$$w_n \rightharpoonup w \text{ in } H_{0,L}^1(\mathscr{C}_\Omega,\, y^{1-2s}),\quad w_n(\cdot,0)\to w(\cdot,0) \text{ in } L^r(\Omega),\, 1\leq r<2_s^*$$

and

$$w_n(\cdot,0)\to w(\cdot,0) \text{ a.e. in } \Omega.$$

Moreover we also obtain that

$$y^{1-2s}|\nabla(w_n)_+|^2\to\mu\geq y^{1-2s}|\nabla w_+|^2+\sum_{k\in I}\mu_k\delta_{x_k},\mu_k>0$$

and
$$|(w_n)_+(\cdot,0)|^{2_s^*} \to \nu = |w_+(\cdot,0)|^{2_s^*} + \sum_{k\in I} \nu_k \delta_{x_k}, \nu_k > 0$$
where
$$\nu_k = 0 \qquad \text{or} \qquad \nu_k \geq (\kappa_s T(N,s))^{N/2s}, \, k \in I.$$
Suppose that there exists $k_0 \in I$ such that $\nu_{k_0} \neq 0$. Therefore, by (1.4.2) and (1.4.3), since $2 < q+1$, it follows that

$$\begin{aligned} c &= \lim_{n\to\infty} \left(\mathcal{I}_{s,\lambda}^*(w_n) - \frac{1}{2}\langle (\mathcal{I}_{s,\lambda}^*)'(w_n), w_n\rangle \right) \\ &= \lim_{n\to\infty} \left(\lambda \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega (w_n)_+^{q+1}(x,0)dx + \frac{s}{2N} \int_\Omega (w_n)_+^{2_s^*}(x,0)dx \right) \\ &\geq \lim_{n\to\infty} \frac{s}{2N} \int_\Omega (w_n)_+^{2_s^*}(x,0)dx \\ &\geq \frac{s}{N}(\kappa_s T(N,s))^{N/2s} \end{aligned}$$

which implies a contradiction with the hypothesis (1.4.2). Thus $\nu_k = \mu_k = 0$ for every $k \in \mathbb{N}$. That is we get $(w_n)_+(\cdot,0) \to w_+(\cdot,0)$ in $L^{2_s^*}(\Omega)$ so, by the continuity of the inverse of the spectral fractional Laplacian, we obtain the desired conclusion. □

By Proposition 1.4.2 and (1.4.1) we get that $\mathcal{I}_{s,\lambda}^*$ satisfies the geometric features required by the MPT (see [11]). Moreover, by Proposition 1.4.3 the functional $\mathcal{I}_{s,\lambda}^*$ verifies the $(\mathrm{PS})_c$ at any level c, provided $c < c^*$, where c^* is given in (1.2.22).

Now, as in the concave case, we show that we could find a path with energy below the critical level c^*. That is we have the following.

Proposition 1.4.4. *Let $\lambda > 0$, $1 < q < 2_s^* - 1$, c^* given in (1.2.22) and η_ε be the function defined in (1.2.66). Then, there exists $\varepsilon > 0$ small enough such that*
$$\sup_{t\geq 0} \mathcal{I}_{s,\lambda}^*(t\eta_\varepsilon) < c^*,$$
provided

- $N > 2s\left(\frac{3+q}{1+q}\right)$ *and $\lambda > 0$ or*

- $N \leq 2s\left(\frac{3+q}{1+q}\right)$ *and $\lambda > \lambda_s$, for a suitable $\lambda_s > 0$.*

Proof. Let $N > 2s\left(\frac{3+q}{1+q}\right)$.

First of all note that since $q > 1$ we get that $N > 2s\left(1+\dfrac{1}{q}\right)$. Therefore by (1.2.59) it follows that

$$\|\eta_\varepsilon\|^{q+1}_{L^{q+1}(\Omega)} \geq \widetilde{C}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)}, \tag{1.4.4}$$

for some $\widetilde{C}$. Then by (1.2.72) and (1.4.4) for any $t \geq 0$ and $\varepsilon > 0$ small enough we obtain

$$\begin{aligned} \mathcal{I}^*_{s,\lambda}(t\eta_\varepsilon) &= \frac{t^2}{2}\|\eta_\varepsilon\|^2_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} - \frac{t^{2^*_s}}{2^*_s} - \lambda\frac{t^{q+1}}{q+1}\int_\Omega \eta_\varepsilon^{q+1}\,dx \\ &\leq \frac{t^2}{2}(\kappa_s T(N,s) + C\varepsilon^{N-2s}) - \frac{t^{2^*_s}}{2^*_s} - \widetilde{C}\lambda\frac{t^{q+1}}{q+1}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)} \\ &:= g(t). \end{aligned} \tag{1.4.5}$$

It is clear that

$$\lim_{t\to\infty} g(t) = -\infty,$$

therefore $\sup_{t\geq 0} g(t)$ is attained at some $t_{\varepsilon,\lambda} := t_\varepsilon \geq 0$. As we mentioned in the proof of Lemma 1.2.13 we may suppose $t_\varepsilon > 0$. Differentiating the function $g(t)$ and equaling to zero, we obtain that

$$0 = g'(t_\varepsilon) = t_\varepsilon(\kappa_s T(N,s) + C\varepsilon^{N-2s}) - t_\varepsilon^{2^*_s-1} - \widetilde{C}\lambda t_\varepsilon^q \varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)}. \tag{1.4.6}$$

Hence

$$t_\varepsilon < (\kappa_s T(N,s) + C\varepsilon^{N-2s})^{\frac{1}{2^*_s-2}}.$$

Moreover we have that for $\varepsilon > 0$ small enough

$$t_\varepsilon \geq \hat{c} > 0. \tag{1.4.7}$$

Indeed, from (1.4.6) it follows that

$$t_\varepsilon^{2^*_s-2} + \widetilde{C}\lambda t_\varepsilon^{q-1}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)} = \kappa_s T(N,s) + C\varepsilon^{N-2s} \geq \hat{c} > 0.$$

Also, since the function

$$t \mapsto \frac{t^2}{2}(\kappa_s T(N,s) + C\varepsilon^{N-2s}) - \frac{t^{2^*_s}}{2^*_s}$$

is increasing on $[0, (\kappa_s T(N,s)+C\varepsilon^{N-2s})^{\frac{1}{2^*_s-2}}]$, by (1.4.5) and (1.4.7), we obtain

$$\begin{aligned} \sup_{t\geq 0} g(t) = g(t_\varepsilon) &\leq \frac{s}{N}(\kappa_s T(N,s) + C\varepsilon^{N-2s})^{\frac{N}{2s}} - \overline{C}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)} \\ &\leq \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}} + C\varepsilon^{N-2s} - \overline{C}\varepsilon^{\left(\frac{N+2s}{2}\right)(q+1)}, \end{aligned} \tag{1.4.8}$$

for some $\overline{C} > 0$. Finally, since $N > 2s\left(\dfrac{3+q}{1+q}\right)$, from (1.4.5) and (1.4.8) we conclude that

$$\sup_{t\geq 0} \mathcal{I}^*_{s,\lambda}(t\eta_\varepsilon) \leq g(t_\varepsilon) < \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}}.$$

Consider now the case $N \leq 2s\left(\frac{3+q}{1+q}\right)$. Arguing exactly as in the case $N > 2s\left(\frac{3+q}{1+q}\right)$, we get that

$$t_{\varepsilon,\lambda}^{2_s^*-2} + \tilde{C}\lambda t_{\varepsilon,\lambda}^{q-1}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)} = (\kappa_s T(N,s) + C\varepsilon^{N-2s}), \tag{1.4.9}$$

whit $t_{\varepsilon,\lambda} > 0$ the point where the $\sup_{t\geq 0} g(t)$ is attained. We claim that

$$t_{\varepsilon,\lambda} \to 0 \quad \text{when} \quad \lambda \to +\infty. \tag{1.4.10}$$

To see this assume that $\overline{\lim_{\lambda\to\infty}}\, t_{\varepsilon,\lambda} = \ell > 0$. Then, passing to the limit when $\lambda \to +\infty$ in (1.4.9) we would get $(\kappa_s T(N,s) + C\varepsilon^{N-2s}) = +\infty$ which is a contradiction. Therefore, by (1.4.5) and (1.4.10) we obtain that

$$\begin{aligned} 0 &\leq \sup_{t\geq 0} \mathcal{I}^*_{s,\lambda}(t\eta_\varepsilon) \\ &\leq g(t_{\varepsilon,\lambda}) \\ &= \frac{t_{\varepsilon,\lambda}^2}{2}(\kappa_s T(N,s) + C\varepsilon^{N-2s}) - \frac{t_{\varepsilon,\lambda}^{2_s^*}}{2_s^*} - C\lambda\frac{t_{\varepsilon,\lambda}^{q+1}}{q+1}\varepsilon^{N-\left(\frac{N-2s}{2}\right)(q+1)} \\ &\leq \frac{t_{\varepsilon,\lambda}^2}{2}(\kappa_s T(N,s) + C\varepsilon^{N-2s}) - \frac{t_{\varepsilon,\lambda}^{2_s^*}}{2_s^*} \to 0, \end{aligned}$$

when $\lambda \to \infty$. Then

$$\lim_{\lambda\to+\infty} \sup_{t\geq 0} \mathcal{I}^*_{s,\lambda}(t\eta_\varepsilon) = 0,$$

which easily yields the desired conclusion for the case $N \leq 2s\left(\frac{3+q}{1+q}\right)$. □

We conclude now the proof of Theorem 1.4.1. We define

$$\Gamma_\varepsilon = \{\gamma \in \mathcal{C}([0,1], H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})) : \gamma(0) = 0, \gamma(1) = M_\varepsilon \eta_\varepsilon\},$$

for some $M_\varepsilon > 0$ big enough such that $\mathcal{I}^*_{s,\lambda}(M_\varepsilon\eta_\varepsilon) < 0$. We observe that for every $\gamma \in \Gamma_\varepsilon$ the function $t \to \|\gamma(t)\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})}$ is continuous in $[0,1]$. Therefore for α given in Proposition 1.4.2, since

$$\|\gamma(0)\|_{H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})} = 0 < \alpha,$$

and, for every M_ε big enough,

$$\|\gamma(1)\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} = \|M_\varepsilon \eta_\varepsilon\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} > \alpha,$$

there exists $t_0 \in (0,1)$ such that $\|\gamma(t_0)\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})} = \alpha$. As a consequence,

$$\sup_{0\le t\le 1} \mathcal{I}^*_{s,\lambda}(\gamma(t)) \ge \mathcal{I}^*_{s,\lambda}(\gamma(t_0)) \ge \inf_{\|v\|_{H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})}=\alpha} \mathcal{I}^*_{s,\lambda}(v) \ge \beta > 0,$$

where β is the positive value given in Proposition 1.4.2. Hence

$$c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \sup_{0\le t\le 1} \mathcal{I}^*_{s,\lambda}(\gamma(t)) > 0.$$

Then, by Proposition 1.4.4, Proposition 1.4.3 and the MPT given in [11] we conclude that the functional $\mathcal{I}^*_{s,\lambda}$ admits a critical point $u \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$, provided $N > 2s\left(\frac{3+q}{1+q}\right)$ and $\lambda > 0$ or $N \le 2s\left(\frac{3+q}{1+q}\right)$ and $\lambda > \lambda_s$, for a suitable $\lambda_s > 0$. Moreover, since $\mathcal{I}^*_{s,\lambda}(w) = c_\varepsilon \ge \beta > 0$ and $\mathcal{I}^*_{s,\lambda}(0) = 0$, the function w is not the trivial one. This concludes the proof of Theorem 1.4.1. □

1.5 Regularity and concentration-compactness.

We begin this section with some results about the boundedness and regularity of solutions. The next result, that we include here for the readers convenience, can be found also in [37, Proposition 5.3].

We consider the problem given by

$$\begin{cases} A_s u = g(x) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.5.1}$$

We also introduce the extension problem. That is,

$$\begin{cases} L_s w = 0 & \text{in } \mathscr{C}_\Omega, \\ w = 0 & \text{in } \partial_L \mathscr{C}_\Omega, \\ \frac{\partial w}{\partial \nu^s} = g(x) & \text{in } \Omega \times \{y=0\}. \end{cases} \tag{1.5.2}$$

The following regularity result holds true

Proposition 1.5.1. *Let $u \in H(\Omega, s)$ be the solution to problem* (1.5.1) *and let $g \in L^r(\Omega)$ for some $r > \frac{N}{2s}$. Then, $u \in L^\infty(\Omega)$.*

Proof. The proof follows the well-known Moser's iterative technique (see [97, Theorem 8.15]). Without loss of generality, we may assume that the solution w of (1.5.2) is non-negative, to simplify the notation. The general case follows in a similar way taking $|w|$ instead of w. Consider $\beta \geq 1$, $K \geq k$ and define the following functions

$$H(z) = \begin{cases} z^\beta - k^\beta & \text{if } z \in [k, K], \\ \beta K^{\beta-1}(z-K) + (K^\beta - k^\beta) & \text{if } z > K, \end{cases}$$

and $v = w + k$, $\nu = v(x,0)$. We choose as test function

$$\varphi = G(v) = \int_k^v |H'(\eta)|^2 \, d\eta, \qquad \text{with} \quad \nabla\varphi = |H'(v)|^2 \nabla v.$$

Observe that, since $|H'(v)| \leq \beta K^{\beta-1} = C$, φ is an admissible test function for problem (1.5.2), i.e.,

$$\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla\varphi|^2 dx dy < \infty.$$

Then, we obtain

$$\begin{aligned} \int_{\mathscr{C}_\Omega} y^{1-2s} \langle \nabla w, \nabla\varphi \rangle \, dx\, dy &= \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla v|^2 |H'(v)|^2 \, dx\, dy \\ &= \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla H(v)|^2 \, dx\, dy \\ &\geq C \|H(\nu)\|^2_{L^{2_s^*}(\Omega)}, \end{aligned} \tag{1.5.3}$$

where the last inequality is a consequence of the Trace one given in (1.1.29). Moreover, since H' is increasing, we get that

$$G(t) = \int_k^t |H'(\eta)|^2 \, d\eta \leq t|H'(t)|^2 = tG'(t),$$

so

$$\begin{aligned} \int_\Omega g(x)\varphi(x,0)\, dx &= \int_\Omega g(x) G(\nu)\, dx \\ &\leq \int_\Omega g(x) \nu G'(\nu)\, dx \\ &\leq \int_\Omega g(x) |\nu^{\frac{1}{2}} H'(\nu)|^2 \, dx \\ &\leq \|g\|_{L^r(\Omega)} \|\nu^{\frac{1}{2}} H'(\nu)\|^2_{L^{\frac{2r}{r-1}}(\Omega)}. \end{aligned} \tag{1.5.4}$$

Inequality (1.5.3), together with (1.5.2) and (1.5.4), leads to

$$\|H(\nu)\|_{L^{2^*_s}(\Omega)} \leq \left(\frac{1}{C}\|g\|_{L^r(\Omega)}\right)^{1/2} \|\nu^{1/2}H'(\nu)\|_{L^{\frac{2r}{r-1}}(\Omega)}. \tag{1.5.5}$$

By choosing $k = 0$ and passing to the limit as $K \to \infty$ in the definition of H, the inequality (1.5.5) becomes

$$\|\nu^{\beta}\|_{L^{2^*_s}(\Omega)} \leq C\beta\|\nu^{\beta-\frac{1}{2}}\|_{L^{\frac{2r}{r-1}}(\Omega)},$$

where C is a positive constant. Hence, for all $\beta \geq 1$ we have that

$$\nu \in L^{\frac{2r(\beta-\frac{1}{2})}{r-1}}(\Omega) \quad \Rightarrow \quad \nu \in L^{\frac{2N\beta}{N-2s}}(\Omega)\,.$$

Observe that we have obtained a better integrability since

$$\frac{2N\beta}{N-2s} > \frac{2r(\beta-\frac{1}{2})}{r-1},$$

for all $\beta > 1$ if and only if $r > \frac{N}{2s}$. The conclusion follows now, as in [97], by an iteration argument, starting with the exponent $\beta = \frac{N(r-1)}{r(N-2s)} + \frac{1}{2} > 1$. This gives $\nu \in L^{\infty}(\Omega)$. □

Remark 1.5.2. *Following the proof of Proposition 1.5.1 one also proves that the solution of problem* (1.5.2) *is in* $L^{\infty}(\mathscr{C}_{\Omega})$.

Now we obtain the following result that is essentially based on an argument used, in the classical case, by Trudinger in [168] for Yamabe's Problem. The main point is to use some nonlinear test functions in the line of the classical Moser method. See also [43].

Proposition 1.5.3. *Let* u *a nonnegative function of the space* $H(\Omega, s)$. *If* u *is an energy solution to the problem*

$$\begin{cases} A_s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

with f *satisfying*

$$|f(x,t|) \leq C(1+|t|^p), \quad (x,t) \in \Omega \times \mathbb{R}, \tag{1.5.6}$$

for some $1 \leq p \leq 2^*_s - 1$ *and* $C > 0$, *then* $u \in L^{\infty}(\Omega)$.

Proof. We define

$$a(x) := \frac{|f(x, u(x))|}{1 + |u(x)|}\,.$$

Since $1 \le p \le 2_s^* - 1$, it is clear that for any $x \in \Omega$

$$0 \le a(x) \le C(1 + |u(x)|^{p-1}) \in L^{\frac{N}{2s}}(\Omega). \tag{1.5.7}$$

Given $T > 0$ we define

$$w_T = w - (w - T)_+, \quad u_T(\cdot) = w_T(\cdot, 0)\,,$$

where v_+ denotes the positive part of v, that is $v_+ := \max\{v, 0\}$. For $\beta \ge 0$, we have

$$\begin{aligned}&\int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla(w w_T^{\beta})|^2\,dx\,dy\\ =&\int_{\mathscr{C}_\Omega} y^{1-2s} w_T^{2\beta}|\nabla w|^2\,dx\,dy + (2\beta+\beta^2)\int_{\{w\le T\}} y^{1-2s} w^{2\beta}|\nabla w|^2\,dx\,dy.\end{aligned} \tag{1.5.8}$$

We also note that

$$\begin{aligned}&\int_{\mathscr{C}_\Omega} y^{1-2s}\langle\nabla w, \nabla(w w_T^{2\beta})\rangle\,dx\,dy\\ =&\int_{\mathscr{C}_\Omega} y^{1-2s} w_T^{2\beta}|\nabla w|^2\,dx\,dy + 2\beta\int_{\{w\le T\}} y^{1-2s} w^{2\beta}|\nabla w|^2\,dx\,dy.\end{aligned} \tag{1.5.9}$$

On the other hand, using $w w_T^{\beta} \in H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$ as a test function in problem (1.5.2) we obtain

$$\begin{aligned}\kappa_s\int_{\mathscr{C}_\Omega} y^{1-2s}\langle\nabla w, \nabla(w w_T^{2\beta})\rangle\,dx\,dy \;&=\; \int_\Omega f(x,u) u u_T^{2\beta}\,dx\\ &\le\; 2\int_\Omega a(x)(1+u^2)u_T^{2\beta}\,dx.\end{aligned} \tag{1.5.10}$$

Thus, by (1.5.8)-(1.5.10) we have

$$\int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla(w w_T^{\beta})|^2\,dx\,dy \le C\int_\Omega a(x)(1+u^2)u_T^{2\beta}\,dx,$$

which, by (1.1.29), implies that

$$\|u u_T^{\beta}\|^2_{L^{2_s^*}(\Omega)} \le \tilde{C}\int_\Omega a(x)(1+u^2)u_T^{2\beta}\,dx, \tag{1.5.11}$$

where C and $\tilde{C}$ are positive constants depending only on N, s, β, and $|\Omega|$.

To estimate the term on the right-hand side in (1.5.11) we take β to be the maximum exponent such that $u^{\beta+1} \in L^2(\Omega)$, that is, $(\beta+1)/2 = 2_s^*$. We want to prove that $\|uu_T^\beta\|_{L^{2_s^*}(\Omega)}$ is bounded by a constant that is independent of T. Therefore we consider, without loss of generality, that

$$\|uu_T^\beta\|_{L^{2_s^*}(\Omega)} \geq 1. \tag{1.5.12}$$

Then we obtain that

$$\begin{aligned}\int_\Omega a(x)\, u^2 u_T^{2\beta}\, dx &= T_0 \int_{\{a<T_0\}} u^2 u_T^{2\beta} + \int_{\{a\geq T_0\}} a(x) u^2 u_T^{2\beta} \\ &= C_1 T_0 + \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{2s}{N}} \|uu_T^\beta\|^2_{L^{2_s^*}(\Omega)}. \end{aligned}\tag{1.5.13}$$

Similarly, using (1.5.12) and the fact that $|u_T| \leq u$, we get

$$\begin{aligned}\int_\Omega a(x)\, u_T^{2\beta}\, dx &\leq C_2 T_0 + \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{2s}{N}} \left(\int_{\{a\geq T_0\}} u_T^{2_s^*\beta}\right)^{\frac{2}{2_s^*}} \\ &\leq C_2 T_0 \\ &\quad + \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{2s}{N}} \left(\left(\left(\int_{\{a\geq T_0\}} u_T^{2_s^*(\beta+1)}\right)^{\frac{\beta}{\beta+1}} |\{a \geq T_0\}|^{\frac{1}{\beta+1}}\right)^{\frac{2}{2_s^*}} \\ &\leq C_2 T_0 + \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{2s}{N}} |\{a \geq T_0\}|^{\frac{2}{2_s^*(\beta+1)}} \|u_T^{\beta+1}\|^{\frac{2\beta}{\beta+1}}_{L^{2_s^*}(\Omega)} \\ &\leq C_2 T_0 + \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{2s}{N}} |\{a \geq T_0\}|^{\frac{2}{2_s^*(\beta+1)}} \|uu_T^\beta\|^2_{L^{2_s^*}(\Omega)} \end{aligned}\tag{1.5.14}$$

Here C_1 and C_2 are positive constants independent of T, that depend on β. Since

$$\lim_{T_0\to\infty} \varepsilon(T_0) := \lim_{T_0\to\infty} \left(\int_{\{a\geq T_0\}} a(x)^{\frac{N}{2s}}\right)^{\frac{N}{2s}}$$

and

$$\lim_{T_0\to\infty} \tilde{\varepsilon}(T_0) := \lim_{T_0\to\infty} |\{a \geq T_0\}|^{\frac{2}{2_s^*(\beta+1)}},$$

from (1.5.11), using (1.5.13) and (1.5.14), it follows that

$$\|uu_T^\beta\|^2_{L^{2_s^*}(\Omega)} \leq \tilde{C}\, T_0(C_1 + C_2) + \tilde{C}\, \|uu_T^\beta\|^2_{L^{2_s^*}(\Omega)} \varepsilon(T_0)(1 + \tilde{\varepsilon}(T_0)).$$

Therefore, taking T_0 big enough such that $\tilde{C}\, \varepsilon(T_0)(1+\tilde{\varepsilon}(T_0)) = 1/2$, we obtain

$$\|uu_T^\beta\|^2_{L^{2_s^*}(\Omega)} \leq K,$$

where $K := 2\tilde{C}\, T_0(C_1 + C_2)$ is a constant independent of T but dependent of β.

Finally, letting $T \to \infty$ we conclude that $u^{\beta+1} \in L^{2_s^*}(\Omega)$. Therefore, by (1.5.6), in a finite number of steps we get that $f(\cdot, u) \in L^r(\Omega)$ for some $r > \frac{N}{2s}$, and so the assertion of Proposition 1.5.3 comes from Proposition 1.5.1. □

Remark 1.5.4. *As in Remark 1.5.2, following the proof of Proposition 1.5.3, one also proves that the solution of problem* (1.5.2) *is in* $L^\infty(\mathscr{C}_\Omega)$.

Now we characterize the regularity of the solutions of (P_λ) for the whole range of exponents. That is, we will prove the following.

Proposition 1.5.5. *Let u be an energy solution of* (P_λ)*. Then the following hold*

(i) If $s = 1/2$ and $q \geq 1$ then $u \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^{2,\gamma}(\overline{\Omega})$, for some $0 < \gamma < 1$.

(ii) If $s = 1/2$ and $q < 1$ then $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) If $s < 1/2$ then $u \in \mathcal{C}^{2s}(\overline{\Omega})$.

(iv) If $s > 1/2$ then $u \in \mathcal{C}^{1,2s-1}(\overline{\Omega})$.

Proof. First we observe that, by Proposition 1.5.3, we have $u \in L^\infty(\Omega)$ and also $f_\lambda(u) \in L^\infty(\Omega)$.

(i) Applying [51, Proposition 3.1 (iii)], we get that $u \in \mathcal{C}^\gamma(\overline{\Omega})$, for some $\gamma < 1$. Since $q \geq 1$ then $f_\lambda(u) \in \mathcal{C}^\gamma(\overline{\Omega})$, so, again by [51, Proposition 3.1 (iv)], it follows that $u \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$ for some $\gamma < 1$. That is, $f_\lambda(u) \in \mathcal{C}^{1,\tilde{\gamma}}(\overline{\Omega})$, for some $\tilde{\gamma} > 0$, and therefore we conclude by [51, Proposition 3.1 (v)]. The interior regularity is clear iterating this process.

(ii) As before we have $u \in \mathcal{C}^\gamma(\overline{\Omega})$, for some $\gamma < 1$. Therefore $f_\lambda(u) \in \mathcal{C}^{q\gamma}(\overline{\Omega})$. It follows that $u \in \mathcal{C}^{1,q\gamma}(\overline{\Omega})$, which gives $f_\lambda(u) \in \mathcal{C}^q(\overline{\Omega})$. Finally this implies $u \in \mathcal{C}^{1,q}(\overline{\Omega})$.

(iii) By [64, Lemma 2.10] we obtain that $u \in \mathcal{C}^\gamma(\overline{\Omega})$ for all $\gamma \in (0, 2s)$. This implies that $f_\lambda(u) \in \mathcal{C}^r(\overline{\Omega})$ for every $r < \min\{q2s, 2s\}$. Therefore by [64, Lemma 2.9 and Lemma 2.11], we get that $u \in \mathcal{C}^{2s}(\overline{\Omega})$.

(iv) Since $s > 1/2$, we can write problem (P_λ) as follows

$$\left\{\begin{array}{ll} A_{\frac{1}{2}} u = v & \text{in } \Omega, \\ A_{\frac{2s-1}{2}} v = f_\lambda(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.5.15}$$

Reasoning as before, we obtain the desired regularity using [64, Lemma 2.9 and Lemma 2.11] and [51, Proposition 3.1].

□

We end the last section of this chapter adapting to our setting a concentration-compactness result by P.L. Lions [119, Lemma 2.3], used in the proof of Lemma 1.2.10. We recall that a related concentration-compactness result for the fractional Laplacian, not to the spectral one, has been recently obtained in [126]. We will use this result in the next chapter.

Theorem 1.5.6. *Let $\{w_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$, such that the sequence $\{y^{1-2s}|\nabla w_n|^2\}_{n\in\mathbb{N}}$ is tight. Let $u_n = tr(w_n)$ and $u = tr(w)$. Let μ, ν be two non negative measures such that*

$$y^{1-2s}|\nabla w_n|^2 \to \mu \qquad \text{and} \qquad |u_n|^{2^*_s} \to \nu, \quad \text{as } n \to \infty \tag{1.5.16}$$

in the sense of measures. Then there exist an at most countable set I and points $\{x_i\}_{i\in I} \subset \Omega$ such that

1. $\nu = |u|^{2^*_s} + \sum_{k\in I} \nu_k \delta_{x_k}$, $\nu_k > 0$,
2. $\mu \geq y^{1-2s}|\nabla w|^2 + \sum_{k\in I} \mu_k \delta_{x_k}$, $\mu_k > 0$,
3. $\mu_k \geq T(N,s)\nu_k^{\frac{2}{2^*_s}}$.

Proof. Let $\varphi \in \mathcal{C}^\infty_0(\overline{\mathscr{C}_\Omega})$. By (1.1.29) with $r = 2^*_s$ it follows that

$$T(N,s)\left(\int_\Omega |\varphi u_n|^{2^*_s} dx\right)^{2/2^*_s} \leq \int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla(\varphi w_n)|^2 dx\, dy. \tag{1.5.17}$$

Let $K^* := K_1 \times K_2 \subseteq \overline{\mathscr{C}}_\Omega$ be the support of φ and suppose first that the weak limit of w_n in $H^1_{0,L}(\mathscr{C}_\Omega, y^{1-2s})$, that we will call w, is equal to zero. Then we get that

$$\begin{aligned}
\int_{\mathscr{C}_\Omega} y^{1-2s}|\nabla(\varphi w_n)|^2 dx\, dy &= \int_{K^*} y^{1-2s}|\nabla(\varphi w_n)|^2 dx\, dy \\
&= \int_{K^*} y^{1-2s}|w_n|^2|\nabla\varphi|^2 dx\, dy \\
&+ \int_{K^*} y^{1-2s}|\varphi|^2|\nabla w_n|^2 dx\, dy \\
&+ 2\int_{K^*} y^{1-2s} w_n\varphi\langle\nabla\varphi, \nabla w_n\rangle dx\, dy.
\end{aligned} \tag{1.5.18}$$

Since K^* is a bounded domain, and y^{1-2s} is an A_2 weight, we have the compact inclusion

$$\dot{H}^1(K^*, y^{1-2s}) \hookrightarrow L^r(K^*, y^{1-2s})\,,\ 1 \le r < \frac{2(N+1)}{N-1}\,,\ 0 < s < 1.$$

Therefore, for a suitable subsequence, we get that

$$\lim_{n\to\infty} \int_{K^*} y^{1-2s} |w_n|^2 |\nabla\varphi|^2 dx\,dy = 0. \tag{1.5.19}$$

By the weak convergence, given by hypothesis, we also obtain

$$\lim_{n\to\infty} \int_{K^*} y^{1-2s} w_n \varphi \langle \nabla\varphi, \nabla w_n\rangle dx\,dy = 0. \tag{1.5.20}$$

Hence, by (1.5.16), (1.5.19) and (1.5.20), from (1.5.18) we conclude that

$$\lim_{n\to\infty} \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla(\varphi w_n)|^2 dx\,dy = \int_{\mathscr{C}_\Omega} |\varphi(x,y)|^2 d\mu.$$

Then, from (1.5.16) and (1.5.17) we get

$$T(N,s)\left(\int_\Omega |\varphi|^{2_s^*} d\nu\right)^{2/2_s^*} \le \int_{\mathscr{C}_\Omega} |\varphi|^2 d\mu, \quad \varphi \in \mathcal{C}_0^\infty(\overline{\mathcal{C}}_\Omega). \tag{1.5.21}$$

If now $w \neq 0$, we apply the above result to the function $v_n = w_n - w$. Indeed if

$$y^{1-2s}|\nabla v_n|^2 \to d\tilde{\mu} \qquad \text{and} \qquad |v_n(\cdot,0)|^{2_s^*} \to d\tilde{\nu}, \quad \text{as } n\to\infty,$$

it follows that

$$T(N,s)\left(\int_\Omega |\varphi|^{2_s^*} d\tilde{\nu}\right)^{2/2_s^*} \le \int_{\mathscr{C}_\Omega} |\varphi|^2 d\tilde{\mu}\,, \quad \varphi \in \mathcal{C}_0^\infty(\overline{\mathcal{C}}_\Omega),$$

therefore by [119], for some sequence of points $\{x_k\}_{k\in I} \subset \Omega$, we have

$$d\tilde{\nu} = \sum_{k\in I} \tilde{\nu}_k \delta_{x_k}\,, \qquad d\tilde{\mu} \ge \sum_{k\in I} \tilde{\mu}_k \delta_{x_k}\,,$$

with $\tilde{\mu}_k \ge T(N,s)\tilde{\nu}_k^{2/2_s^*}$. Hence, by the Brezis-Lieb Lemma stated in Lemma 1.3.3, we obtain

$$d\nu = |u|^{2_s^*} + \sum_{k\in I} \tilde{\nu}_k \delta_{x_k}.$$

On the other hand, for every test function φ we have

$$\begin{aligned}\int_{\mathscr{C}_\Omega} y^{1-2s}\varphi|\nabla w_n|^2 dx\,dy &= \int_{\mathscr{C}_\Omega} y^{1-2s}\varphi|\nabla w|^2 dx\,dy \\ &+ \int_{\mathscr{C}_\Omega} y^{1-2s}\varphi|\nabla (w_n-w)|^2 dx\,dy \\ &+ 2\int_{\mathscr{C}_\Omega} y^{1-2s}\varphi\langle\nabla (w_n-w),\nabla w\rangle dx\,dy.\end{aligned}$$

Since $w_n \rightharpoonup w$ in $H^1_{0,L}(\mathscr{C}_\Omega,\, y^{1-2s})$, taking limits when $n\to\infty$ we get that

$$\begin{aligned}\int_{\mathscr{C}_\Omega}\varphi d\mu &= \int_{\mathscr{C}_\Omega} y^{1-2s}\varphi|\nabla w|^2 dx\,dy + \int_{\mathscr{C}_\Omega}\varphi d\tilde{\mu} \\ &\geq \int_{\mathscr{C}_\Omega} y^{1-2s}\varphi|\nabla w|^2 dx\,dy + \int_{\mathscr{C}_\Omega} y^{1-2s}\varphi\sum_{k\in I}\tilde{\mu}_k\delta_{x_k} dx\,dy.\end{aligned}$$

That is,

$$\mu \geq |\nabla w|^2 + \sum_{k\in I}\tilde{\mu}_k\delta_{x_k},$$

with $\tilde{\mu}_k \geq T(N,s)\tilde{\nu}_k^{2_s^*/2}$. So we obtain the desired conclusion. □

Chapter 2

Elliptic critical problems for the fractional Laplacian operator in a bounded domain.

2.1 Introduction, preliminaries and functional settings.

The aim of this chapter is to study the same equation treated in Chapter 1, with the operator A_s replaced by the fractional Laplace operator $(-\Delta)^s$ defined by the Riesz potential as in (0.0.21) and (0.0.26). We recall that for $0 < s < 1$ and an appropriate function u, we set for $x \in \mathbb{R}^N$

$$\begin{aligned}(-\Delta)^s u(x) &= C(N,s)\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \\ &= \frac{C(N,s)}{2}\text{P.V.}\int_{\mathbb{R}^N} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}\,dy,\end{aligned}$$

where $C(N,s)$ is the normalized constant given in (0.0.22).

Before to introduce the Dirichlet problem that we will study along this chapter, first we will prove a useful property that we announced in (0.0.25) in the Introduction. That is, we have the following.

Proposition 2.1.1. *Set $\beta > 0$ and take $\phi \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\mathbb{R}^N)$ (or $\mathcal{C}^{1,2s+\beta-1}(\mathbb{R}^N)$ if $s \geq 1/2$). Then there exists $C > 0$ such that*

$$|(-\Delta)^s \phi(x)| \leq C, \quad x \in \mathbb{R}^N.$$

Proof. Let us first consider $0 < s < 1/2$. Then

$$\begin{aligned} |(-\Delta)^s \phi(x)| &= C(N,s)\text{P.V.}\left|\int_{\mathbb{R}^N} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy\right| \\ &\leq C(N,s)\int_{\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|}{|x-y|^{N+2s}}\,dy \\ &\leq C(N,s)\|\phi\|_{\mathcal{C}^{\beta+2s}}\int_{\{y:\,|x-y|<1\}} \frac{1}{|x-y|^{N-\beta}}\,dy \\ &+ 2C(N,s)\|\phi\|_{L^\infty(\mathbb{R}^N)}\int_{\{y:\,|x-y|\geq 1\}} \frac{1}{|x-y|^{N+2s}}\,dy \\ &= C(N,s,\phi) < \infty. \end{aligned}$$

We now take $1/2 \leq s < 1$. Manipulating the principal value, we get that there exists $C_1 > 0$ such that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\left|\int_{\{y:\,\varepsilon<|x-y|<1\}} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy\right| \\ =\ &\lim_{\varepsilon\to 0}\left|\int_{\{z:\,\varepsilon<|z|<1\}} \frac{\phi(x)-\phi(x-z)}{|z|^{N+2s}}\,dz\right| \\ =\ &\lim_{\varepsilon\to 0}\left|\int_{\{z:\,\varepsilon<|z|<1\}} \frac{\phi(x)-\phi(x-z)+z\cdot\nabla\phi(x)}{|z|^{N+2s}}\,dz\right| \\ \leq\ &\int_{\{z:\,|z|<1\}} \frac{|\phi(x)-\phi(x-z)+z\cdot\nabla\phi(x)|}{|z|^{N+2s}}\,dz \\ \leq\ &C\int_{\{z:\,|z|<1\}} \frac{\|\phi\|_{\mathcal{C}^{1,2s+\beta-1}}}{|z|^{N-\beta}}\,dz \leq C_1(s,\beta,\phi). \end{aligned}$$

From here it follows that

$$\begin{aligned} |(-\Delta)^s \phi(x)| &\leq C(N,s)\left(\lim_{\varepsilon\to 0}\left|\int_{\{y:\,\varepsilon<|x-y|<1\}} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\,dy\right|\right. \\ &+ \left.\int_{\{y:\,|x-y|\geq 1\}} \frac{|\phi(x)-\phi(y)|}{|x-y|^{N+2s}}\,dy\right) \\ &\leq C(N,s)C_1 \\ &+ 2C(N,s)\|\phi\|_{L^\infty(\mathbb{R}^N)}\int_{\{y:\,|x-y|\geq 1\}} \frac{1}{|x-y|^{N+2s}}\,dy \\ &< \tilde{C}(N,s,\phi). \end{aligned}$$

□

We now consider the following Dirichlet problem with convex-concave nonlinearities

$$(D_\lambda) = \begin{cases} (-\Delta)^s u = f_\lambda(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain satisfying the exterior ball condition, $N > 2s$ and f_λ is given in (1.1.2). We recall that a domain Ω satisfies the exterior ball condition if there exists a positive radius ρ^* such that all the points on $\partial\Omega$ can be touched by some exterior ball of radius ρ^*. These hypotheses will remain and will not be specified again in what follows.

We point out here that, as it happens in Chapter 1, if $p \geq 2_s^* - 1$ and Ω is a starshaped domain, the only solution of the critical problem

$$\begin{cases} (-\Delta)^s u = u^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

is the trivial one. This result follows by an argument of Pohozaev type (see [131] and [86, Corollary 1.3]). As in Chapter 1, this fact motivates the term u^q, $q < 1$ in (D_λ). Other recent works related to the Dirichlet problem for the fractional Laplacian with semilinear perturbations are [140,141,144,146].

In what follows we denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space defined in (1.1.6) endowed with the norm in (1.1.7), while $X_0^s(\Omega)$ is the function space defined as

$$X_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}. \tag{2.1.1}$$

We refer to [140, 144] and the references therein, for a general definition of $X_0^s(\Omega)$ and its properties even with kernels different from the fractional Laplacian. In $X_0^s(\Omega)$ we can consider the following norm

$$\begin{aligned} \|v\|_{X_0^s(\Omega)} &= \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}}\, dx\, dy \right)^{1/2} \\ &= \left(\int_Q \frac{|v(x) - v(y)|^2}{|x - y|^{N+2s}}\, dx\, dy \right)^{1/2}, \end{aligned} \tag{2.1.2}$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega)$. It is easy to prove that $X_0^s(\Omega)$ may also be defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the metric of (2.1.2), that is.

$$X_0^s(\Omega) := \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{X_0^s(\Omega)}}, \tag{2.1.3}$$

(see [88,123]). We also recall that $\big(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)}\big)$ is a Hilbert space, with scalar product

$$\langle u,v\rangle_{X_0^s(\Omega)} = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy\,. \tag{2.1.4}$$

See for instance [7, Theorem 5], [120] and [144, Lemma 7]. Moreover,

$$(-\Delta)^s : X_0^s(\Omega) \to X^{-s}(\Omega),$$

and its inverse, are continuous operators.

Observe that by [76, Proposition 3.6] we have the following identity

$$\|u\|^2_{X_0^s(\Omega)} = \frac{2}{C(N,s)}\|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^N)}, \quad u\in X_0^s(\Omega). \tag{2.1.5}$$

Then it is easy to check that for u, $\varphi \in X_0^s(\Omega)$,

$$\frac{2}{C(N,s)}\int_{\mathbb{R}^N} u(x)(-\Delta)^s\varphi(x)\,dx = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy. \tag{2.1.6}$$

Therefore, in particular, $(-\Delta)^s$ is selfadjoint in $X_0^s(\Omega)$. Also, in this context, the Sobolev constant is given by

$$S(N,s) := \inf_{v\in H^s(\mathbb{R}^N)\setminus\{0\}} Q_{N,s}(v) > 0, \tag{2.1.7}$$

where

$$Q_{N,s}(v) := \frac{\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{\left(\displaystyle\int_{\mathbb{R}^N}|v(x)|^{2^*_s}dx\right)^{2/2^*_s}}, \quad v\in H^s(\mathbb{R}^N),$$

is the associated Rayleigh quotient (see Theorem (0.0.2)). As can be seen in [7, Theorem 7.58], the constant $S(N,s)$ is well defined. Moreover by (2.1.5) one has

$$S(N,s) = \frac{2}{C(N,s)}\kappa_s T(N,s), \tag{2.1.8}$$

where κ_s and $T(N,s)$ are defined in (1.1.12) and (1.1.31) respectively.

Similarly to Chapter 1, to define correctly the weak formulation of problem (D_λ), and taking into account that we are looking for positive solutions, in what follows we consider the next Dirichlet problem

$$(D_\lambda^+) = \begin{cases} (-\Delta)^s u = f_\lambda(u_+) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega\,. \end{cases}$$

Note that, by the Maximum Principle [151, Proposition 2.2.8], if u is a solution of (D_λ^+) then $u > 0$ in Ω and therefore is also a solution of (D_λ). Therefore we can introduce the following definition .

Definition 2.1.2. *We say that* $u \in X_0^s(\Omega)$ *is an* energy *solution of* (D_λ^+) *if, for every* $\varphi \in X_0^s(\Omega)$,

$$\frac{C(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy = \lambda\int_\Omega f_\lambda(u_+)\varphi\,dx.$$

We remark that, as occurs in Chapter 1, by Theorem 0.0.2, the previous equality is finite. In this chapter we will omit the term *energy* when referring to solutions that satisfy Definition 2.1.2. To find solutions of (D_λ), as in Chapter 1 we will use a variational approach. Hence, we will associate a suitable functional to our problem. More precisely, the equation given (D_λ^+) is the Euler–Lagrange equation associated to the functional $\mathcal{J}_{s,\lambda} : X_0^s(\Omega) \to \mathbb{R}$ defined as follows

$$\mathcal{J}_{s,\lambda}(u) = \frac{C(N,s)}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_\Omega\left(\frac{\lambda}{q+1}u_+^{q+1} + \frac{1}{2_s^*}u_+^{2_s^*}\right)dx.$$

Note that $\mathcal{J}_{s,\lambda}$ is $\mathcal{C}^1$ and, clearly, its critical points correspond to solutions of (D_λ^+).

In both cases, $q < 1$ and $q > 1$, we will use, as in Chapter 1, the Mountain Pass Theorem (MPT for short) of [11]. In order to do that, we will show that $\mathcal{J}_{s,\lambda}$ satisfies a compactness property and has suitable geometrical features. The fact that the functional has the suitable geometry is easy to check. Observe that the embedding $X_0^s(\Omega) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is not compact (see [7]). This is even true when the nonlocal operator has a more general kernel (see [140, Lemma 9-b)]). Hence, the difficulty to apply MPT lies on proving a local Palais–Smale (PS for short) condition at level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$). Moreover, since the PS condition does not hold globally, we have to prove that the Mountain Pass critical level of $\mathcal{J}_{s,\lambda}$ lies below the threshold of application of the $(\mathrm{PS})_c$ condition.

In the concave setting, $q < 1$, as in Section 1.2 of Chapter 1, we prove, in Section 2.2, the existence of at least two positive solutions for an admissible small range of λ. We first show that we have a solution that is a local minimum for the functional $\mathcal{J}_{s,\lambda}$. In the next step, in order to find a second solution, we suppose that this local minimum is the only critical point of the functional, and then we prove a local $(\mathrm{PS})_c$ condition for c under a critical level related with the best fractional critical Sobolev constant given in (2.1.8).

Also we find a path under this critical level localizing the Sobolev minimizers at the possible concentration on Dirac Deltas. These Deltas are obtained by the concentration-compactness result in [126, Theorem 1.5] inspired in the classical result by P.L. Lions in [118, 119]. Applying the MPT given in [11] and its refined version given in [95], we will reach a contradiction.

In Section 2.3 we treated the case $q > 1$. We also apply the MPT to obtain the existence of at least one solution for (D_λ) for suitable values of λ depending on the dimension N. As before, we prove a local $(\text{PS})_c$ condition in a appropriate range related with the constant $S(N, s)$ defined on (2.1.8). The strategy to obtain a solution follows the ideas given in [45] (see also [158,174]) adapted to our nonlocal functional framework.

The linear case $q = 1$, when the right hand side of the equation is equal to $\lambda u + |u|^{2_s^*-2}u$, was treated in [138–142]. In these works the authors studied also nonlinearities more general, but also symmetric, than those given by the power critical function as well as the existence of solutions not necessarily positive.

2.2 Sublinear case: $0 < q < 1$.

We begin this sections noting that the technique that we present to prove the existence of at least two nontrivial solutions can also be applied for the sublinear subcritical case. That is when the right hand side of (D_λ) is equal to $\lambda u^q + u^p$ with $1 < p < 2_s^* - 1$ and $0 < q < 1$. We remark here that, for this subcritical case, the existence of solutions could also be obtained applying the Alama's tool that we present in Section 3.3 of Chapter 3.

As we said in the previous section, the objective here is to prove the following.

Theorem 2.2.1. *Assume $0 < q < 1$. Then, there exists $0 < \Lambda < \infty$ such that problem (D_λ)*

1 has no solution for $\lambda > \Lambda$;

2 has a minimal energy solution for any $0 < \lambda < \Lambda$ and, moreover, the family of minimal solutions is increasing with respect to λ;

3 if $\lambda = \Lambda$ there exists at least one energy solution;

4 for $0 < \lambda < \Lambda$ there are at least two energy solutions.

To prove the previous theorem we need some results that we present as follows. Firstly, by standard arguments, it can be proved the following comparison lemma.

Lemma 2.2.2 (Comparison principle for energy solutions). *Let $u \in H^s(\mathbb{R}^N)$ and $v \in H^s(\mathbb{R}^N)$ be solutions to the problems*

$$\begin{cases} (-\Delta)^s u = f_1 \ \textit{in } \Omega, \\ u = g_1 \ \textit{in } \mathbb{R}^N \setminus \Omega. \end{cases} \qquad \begin{cases} (-\Delta)^s v = f_2 \ \textit{in } \Omega, \\ v = g_2 \ \textit{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

respectively. If $f_1 \leq f_2$ and $g_1 \leq g_2$ then $u(x) \leq v(x)$, for all $x \in \mathbb{R}^N$.

Proof. Let us define a function $w = u - v$. Then, since $(-\Delta)^s$ is a linear operator, w solves the problem

$$\begin{cases} (-\Delta)^s w = f_1 - f_2 \text{ in } \Omega, \\ w = g_1 - g_2 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Consider $w_+ = \max\{w, 0\}$ as a test function in the previous problem. Therefore,

$$\begin{aligned} \frac{C(N,s)}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(w(x) - w(y))(w_+(x) - w_+(y))}{|x-y|^{N+2s}} \, dx \, dy &= \int_\Omega (f_1 - f_2) w_+ \, dx \\ &\leq 0. \end{aligned} \tag{2.2.1}$$

Note that

$$\text{if} \quad w(x) \geq w(y), \quad \text{then} \quad w_+(x) \geq w_+(y).$$

Likewise,

$$\text{if} \quad w(x) \leq w(y), \quad \text{we also have} \quad w_+(x) \leq w_+(y).$$

Thus,

$$(w(x) - w(y))(w_+(x) - w_+(y)) \geq 0$$

for all $x, y \in \mathbb{R}^N$. Then from (2.2.1) we deduce that,

$$(w(x) - w(y))(w_+(x) - w_+(y)) = 0.$$

Therefore $w(x) - w(y) = 0$ or $w_+(x) - w_+(y) = 0$ for all $x, y \in \mathbb{R}^N$. In both cases we get that $w_+(x)$ =cte. Since $w_+ = 0$ in $\mathbb{R}^N \setminus \Omega$, we conclude that $w_+(x) = 0$ for all $x \in \mathbb{R}^N$, and consequently $w(x) \leq 0$. That is, $u(x) \leq v(x)$. □

By this previous lemma, Proposition 2.4.1 and following the ideas given in Lemma 1.2.2, we have the next result.

Lemma 2.2.3. *Let $0 < q < 1$ and let Λ be defined by*

$$\Lambda := \sup\{\lambda > 0 \ : \ \textit{problem } (D_\lambda) \textit{ has solution}\}. \tag{2.2.2}$$

Then, $0 < \Lambda < \infty$. The critical concave problem (D_λ) has at least one solution for every $0 < \lambda \le \Lambda$. Moreover, for $0 < \lambda < \Lambda$ we get a family of minimal solutions increasing with respect to λ.

We remark here that in the proof of this previous lemma, to construct the subsolution, instead of (ρ_1, φ_1) introduced in the proof of Lemma 1.2.2, we consider $(\rho_{1,(-\Delta)^s}, \varphi_{1,(-\Delta)^s})$ where $\varphi_1 := \varphi_{1,(-\Delta)^s}$ in that case solve the eigenvalue problem

$$(D1) = \begin{cases} (-\Delta)^s\varphi_1 = \rho_{1,(-\Delta)^s}\varphi_1 \text{ in } \Omega, \\ \varphi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2.3}$$

We also note that, by Proposition 2.4.1 (or [141, Proposition 4]) and [144, Proposition 9], we can assure that $0 \le \varphi_1 \in X_0^s(\Omega) \cap L^\infty(\Omega)$.

By Lemma 2.2.3 we easily deduce statements *1-3* of Theorem 2.2.1. Hence, in the sequel we focus on proving statement *4* of that theorem, that is on the existence of a second solution for (D_λ) and $0 < s < 1$.

As we said in the introduction of this chapter, to find the existence of the second solution, we first show that the minimal solution given by Lemma 2.2.3 is a local minimum for the functional $\mathcal{J}_{s,\lambda}$. For that, following the ideas given in Lemma 1.2.4 we establish a separation lemma now in the topology of the class

$$\mathcal{C}_s(\Omega) := \left\{ w \in \mathcal{C}^0(\overline{\Omega}) : \ \|w\|_{\mathcal{C}_s(\Omega)} := \left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} < \infty \right\}, \tag{2.2.4}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Indeed we have the following separation result.

Lemma 2.2.4. *Let $0 < \lambda_1 < \lambda_0 < \lambda_2 < \Lambda$. Let z_{λ_1}, z_{λ_0} and z_{λ_2} be the corresponding minimal solutions to (D_λ), for $\lambda = \lambda_1$, λ_0 and λ_2 respectively. If*

$$Z = \{z \in \mathcal{C}_s(\Omega) |\, z_{\lambda_1} \le z \le z_{\lambda_2}\},$$

then there exists $\varepsilon > 0$ such that

$$\{z_{\lambda_0}\} + \varepsilon B_1 \subset Z,$$

with $B_1 = \{w \in \mathcal{C}^0(\overline{\Omega}) : \ \|\frac{w}{\delta^s}\|_{L^\infty(\Omega)} < 1\}$.

Proof. Let u be an arbitrary solution of (D_λ) for $0 < \lambda < \Lambda$. By [130, Proposition 1.1], we get that there exists a positive constant C such that

$$u(x) \le C\delta(x)^s,\ x \in \Omega.$$

Then, doing as in the proof of Lemma 1.2.4, using the Hopf's Lemma given in [130, Lemma 3.2] (see also [54, Proposition 2.7]), we conclude. □

As we know, the previous result is the first step to obtain a local minimum of $\mathcal{J}_{s,\lambda}$ in $X_0^s(\Omega)$. Previously we need the following.

Lemma 2.2.5. *For all $\lambda \in (0, \Lambda)$ the problem (D_λ) has a solution u_0 which is in fact a local minimum of the functional $\mathcal{J}_{s,\lambda}$ in the $\mathcal{C}_s$-topology.*

Proof. The proof follows in a similar way as that in Lemma 1.2.5. Here we have to consider the non local operator $(-\Delta)^s$ instead of A_s and the space $\mathcal{C}_s(\Omega)$ instead of $\mathcal{C}_0^1(\Omega)$. We omit the details. □

To prove that we already have the desired minimum in the space $X_0^s(\Omega)$ we now prove that the result obtained in [45] is also valid in our setting. The proof of the following Proposition follows, closely, the proof of Proposition 1.2.6.

Proposition 2.2.6. *Let $z_0 \in X_0^s(\Omega)$ be a local minimum of $\mathcal{J}_{s,\lambda}$ in $\mathcal{C}_s(\Omega)$, this means that, there exists $r_1 > 0$ such that*

$$\mathcal{J}_{s,\lambda}(z_0) \le \mathcal{J}_{s,\lambda}(z_0 + z), \qquad z \in \mathcal{C}_s(\Omega) \text{ with } \|z\|_{\mathcal{C}_s(\Omega)} \le r_1. \tag{2.2.5}$$

Then z_0 is a local minimum of $\mathcal{J}_{s,\lambda}$ in $X_0^s(\Omega)$, that is, there exists $r_2 > 0$ so that

$$\mathcal{J}_{s,\lambda}(z_0) \le \mathcal{J}_{s,\lambda}(z_0 + z), \qquad z \in X_0^s(\Omega) \text{ with } \|z\|_{X_0^s(\Omega)} \le r_2.$$

Proof. Let z_0 be as in (2.2.5) and

$$B_\varepsilon(z_0) = \left\{ z \in X_0^s(\Omega) : \ \|z - z_0\|_{X_0^s(\Omega)} \le \varepsilon \right\}$$

for any $\varepsilon > 0$.

As in Proposition 1.2.6, we argue by contradiction and we suppose that for any $\varepsilon > 0$

$$\exists\, v_\varepsilon \in B_\varepsilon(z_0) \quad \text{such that} \qquad \min_{v \in B_\varepsilon(z_0)} \mathcal{J}_{s,\lambda}(v) = \mathcal{J}_{s,\lambda}(v_\varepsilon) < \mathcal{J}_{s,\lambda}(z_0)\,. \tag{2.2.6}$$

The existence of v_ε comes from a standard argument of weak lower semi-continuity. We want to prove that

$$v_\varepsilon \to z_0 \quad \text{in} \quad \mathcal{C}_s(\Omega) \quad \text{as} \quad \varepsilon \searrow 0\,. \tag{2.2.7}$$

because this will implies a contradiction with (2.2.5).

Let $0 < \varepsilon \ll 1$. Doing the same as in (1.2.16)-(1.2.17) with the space $X_0^s(\Omega)$ instead of $H(s,\Omega)$, we get that v_ε satisfies

$$\begin{cases} (-\Delta)^s v_\varepsilon = \frac{1}{1-\xi_\varepsilon} f_\lambda(v_\varepsilon) =: f_\lambda^\varepsilon((v_\varepsilon)_+) & \text{in } \Omega \\ v_\varepsilon = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with the Lagrange multiplier

$$\xi_\varepsilon \leq 0. \tag{2.2.8}$$

Since $\|v_\varepsilon\|_{X_0^s(\Omega)} \leq C$, by Proposition 2.4.1 there exists a constant $C_1 > 0$ independent of ε such that $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq C_1$. Moreover, by (2.2.8), it follows that $\|f_\lambda^\varepsilon(v_\varepsilon)\|_{L^\infty(\Omega)} \leq C$. Therefore, by [130, Proposition 1.1], we get that $\|v_\varepsilon\|_{\mathcal{C}^s(\overline{\Omega})} \leq C_2$, for some C_2 independent of ε.

Thus, by the Ascoli-Arzelá Theorem there exists a subsequence, still denoted by v_ε, such that $v_\varepsilon \to z_0$ uniformly as $\varepsilon \searrow 0$. Moreover, by [130, Theorem 1.2], we obtain that for a suitable positive constant C

$$\left\| \frac{v_\varepsilon - z_0}{\delta^s} \right\|_{L^\infty(\Omega)} \leq C \sup_\Omega \left| f_\lambda^\varepsilon(v_\varepsilon) - f_\lambda(z_0) \right| \to 0 \text{ as } \varepsilon \searrow 0,$$

that is (2.2.7) is proved.

□

Lemma 2.2.5 and Proposition 2.2.6 provide us a local minimum in $X_0^s(\Omega)$ that we will denote u_0. Now, as in Section 1.2 in Chapter 1, we make a translation in order to simplify the calculations. That is, for $0 < \lambda < \Lambda$, we consider the energy functional $\widetilde{\mathcal{J}}_{s,\lambda} : X_0^s(\Omega) \to \mathbb{R}$ given by

$$\widetilde{\mathcal{J}}_{s,\lambda}(u) = \frac{C(N,s)}{4} \|u\|^2_{X_0^s(\Omega)} - \int_\Omega G_\lambda(x,u)dx, \tag{2.2.9}$$

where G_λ is given in (1.2.19). Also we define the translate problem

$$(\widetilde{D}_\lambda) = \begin{cases} (-\Delta)^s u = g_\lambda(x,u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We note here that, doing as in Lemma 1.2.8 we get that

$$u = 0 \text{ is a local minimum of } \widetilde{\mathcal{J}}_{s,\lambda} \text{ in } X_0^s(\Omega). \tag{2.2.10}$$

Moreover we know that if $\widetilde{u} \not\equiv 0$ is a critical point of $\widetilde{\mathcal{J}}_{s,\lambda}$ then it is a solution of $(\widetilde{D}_\lambda)$ and, by the Maximum Principle, given in [151, Proposition 2.2.8], this implies that $\widetilde{u} > 0$. Therefore $u = u_0 + \widetilde{u} > 0$ will be a second solution of (D_λ).

Hence, as we did in Section 1.2 in Chapter 1, in order to prove statement 4 of Theorem 2.2.1, it is enough to study the existence of a non-trivial critical point for $\widetilde{\mathcal{J}}_{s,\lambda}$. Then our objective now, as in Lemma 1.2.10, is to prove that, assuming that we have a unique critical point, the functional $\widetilde{\mathcal{J}}_{s,\lambda}$ satisfies a local PS condition (see Lemma 2.2.10). The main tool for proving this fact is an application of the concentration-compactness principle by Lions in [118, 119] for nonlocal fractional operators, given in [126, Theorem 1.5]. In order to do that we need some technical auxiliary results, related to the behavior of the fractional Laplacian of a product of two functions. We start with the following.

Lemma 2.2.7. *Let ϕ be a regular functions that satisfies*

$$|\phi(x)| \le \frac{\tilde{C}}{1+|x|^{N+s}},\ x \in \mathbb{R}^N \tag{2.2.11}$$

and

$$|\nabla\phi(x)| \le \frac{\tilde{C}}{1+|x|^{N+s+1}},\ x \in \mathbb{R}^N, \tag{2.2.12}$$

for some $\tilde{C} > 0$. Let $B : X_0^{s/2}(\Omega) \times X_0^{s/2}(\Omega) \to \mathbb{R}$ be the bilinear form defined by

$$B(f,g)(x) := C(N,s) \int_{\mathbb{R}^N} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{N+s}}\,dy,\ x \in \mathbb{R}^N. \tag{2.2.13}$$

Then, for every $0 < s < 1$, there exist positive constants C_1 and C_2, that depend on N and s, such that for any $x \in \mathbb{R}^N$

$$|(-\Delta)^{s/2}\phi(x)| \le \frac{C_1}{1+|x|^{N+s}}$$

and

$$|B(\phi,\phi)(x)| \le \frac{C_2}{1+|x|^{N+s}}.$$

Proof. Let

$$I(x) := \int_{\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|}{|x-y|^{N+s}}\,dy.$$

For any $x \in \mathbb{R}^N$, it is clear that,

$$|(-\Delta)^{s/2}\phi(x)| \le C(N,s)I(x)$$

and, since $|\phi(x)| \le \tilde{C}$,

$$|B(\phi,\phi)(x)| \le 2C(N,s)\tilde{C}I(x).$$

Hence it suffices to prove that, for every $x \in \mathbb{R}^N$,

$$I(x) \leq \frac{C}{1+|x|^{N+s}}, \tag{2.2.14}$$

for a suitable positive constante C.
Since ϕ is a regular function, for $|x|<1$ we obtain that,

$$\begin{aligned} I(x) &\leq \|\nabla\phi\|_{L^\infty(\mathbb{R}^N)} \int_{|y|<2} \frac{dy}{|x-y|^{N+s-1}} + C\int_{|y|\geq 2} \frac{dy}{|y|^{N+s}} \\ &\leq C \leq \frac{C}{1+|x|^{N+s}}. \end{aligned} \tag{2.2.15}$$

Let now $|x| \geq 1$. Then

$$I(x) := I_{A_1}(x) + I_{A_2}(x) + I_{A_3}(x), \tag{2.2.16}$$

where

$$I_{A_i}(x) := \int_{A_i} \frac{|\phi(x)-\phi(y)|}{|x-y|^{N+s}}\, dy,\ i=1,2,3,$$

with

$$A_1 := \left\{ y:\ |x-y| \leq \frac{|x|}{2} \right\}, \quad A_2 := \left\{ y:\ |x-y| > \frac{|x|}{2},\ |y| \leq 2|x| \right\}$$

and

$$A_3 := \left\{ y:\ |x-y| > \frac{|x|}{2},\ |y| > 2|x| \right\}.$$

Therefore, since for $|x| \geq 1$ and $y \in A_1$, it follows that $|\phi(x)-\phi(y)| \leq |\nabla\phi(\xi)||x-y|$ with $\frac{|x|}{2} \leq |\xi| \leq \frac{3}{2}|x|$ by (2.2.12), we obtain that

$$I_{A_1}(x) \leq \frac{C}{|x|^{N+s+1}} \int_{A_1} \frac{dy}{|x-y|^{N+s-1}} \leq C|x|^{-(N+2s)}. \tag{2.2.17}$$

Using now that, for any $x,\ y \in \mathbb{R}^N$ it holds true the following inequality,

$$|\phi(x)| + |\phi(y)| \leq \frac{C}{1+\min\{|x|^{N+s}, |y|^{N+s}\}},$$

we get

$$I_{A_2}(x) \leq \frac{C}{|x|^{N+s}} \int_{A_2} \frac{dy}{(1+|y|^{N+s})} \leq C|x|^{-(N+s)} \tag{2.2.18}$$

and

$$I_{A_3}(x) \le \frac{C}{|x|^{N+s}} \int_{A_3} \frac{dy}{|y|^{N+s}} \le C|x|^{-(N+2s)}. \qquad (2.2.19)$$

The last estimate has been obtained using that if $(x, y) \in A_3$, then $|x - y| \ge |y|/2$. Then, by (2.2.17)-(2.2.19), from (2.2.16) we obtain that

$$I(x) \le C|x|^{-(N+s)} \le \frac{C}{1 + |x|^{N+s}}, \quad |x| \ge 1. \qquad (2.2.20)$$

Hence, by (2.2.15) and (2.2.20), we get (2.2.14) as wanted. □

We remark here that the previous Lemma in particular proves (0.0.23) that we annouced in the Introduction of this work. To establish the next auxiliary results we consider a radially nonincreasing cut-off function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $\chi_{B_1(x_0)} \le \phi \le \chi_{B_2(x_0)}$ for some $x_0 \in \mathbb{R}^N$, and set, for $\varepsilon > 0$,

$$\phi_\varepsilon(x) := \phi(x/\varepsilon). \qquad (2.2.21)$$

Now we have the following

Lemma 2.2.8. *Let* $\{z_m\}$ *be an uniformly bounded sequence in* $X_0^s(\Omega)$ *and* ϕ_ε *the function defined in* (2.2.21)*. Then*

$$\lim_{\varepsilon \to 0} \lim_{m \to \infty} \left| \int_{\mathbb{R}^N} z_m(x)(-\Delta)^{s/2}\phi_\varepsilon(x)(-\Delta)^{s/2} z_m(x)\, dx \right| = 0. \qquad (2.2.22)$$

Proof. First of all note that, as a consequence of the fact that $\{z_m\}$ is uniformly bounded in the reflexive space $X_0^s(\Omega)$, say by M, we get that there exists $z \in X_0^s(\Omega)$, such that, up to a subsequence,

$$\begin{array}{lll} z_m \rightharpoonup z & \text{weakly in } X_0^s(\Omega), & \\ z_m \to z & \text{strongly in } L^r(\Omega), \quad 1 \le r < 2_s^*, & (2.2.23) \\ z_m \to z & \text{a.e. in } \Omega. & \end{array}$$

Also by Proposition 2.1.1 it is clear that

$$|(-\Delta)^{s/2}\phi_\varepsilon(x)| = \varepsilon^{-s} \left| \left((-\Delta)^{s/2}\phi\right)\left(\frac{x}{\varepsilon}\right) \right| \le C\varepsilon^{-s}. \qquad (2.2.24)$$

Therefore defining

$$I_1 := \left| \int_{\mathbb{R}^N} z_m(x)(-\Delta)^{s/2}\phi_\varepsilon(x)(-\Delta)^{s/2} z_m(x)\, dx \right|,$$

from (2.2.24) and the fact that $\|z_m\|_{X_0^s(\Omega)} < M$, we get

$$\begin{aligned} I_1 &\leq \|(-\Delta)^{s/2} z_m\|_{L^2(\mathbb{R}^N)} \|z_m(-\Delta)^{s/2}\phi_\varepsilon\|_{L^2(\Omega)} \\ &\leq M\|(z_m - z)(-\Delta)^{s/2}\phi_\varepsilon\|_{L^2(\Omega)} + M\|z(-\Delta)^{s/2}\phi_\varepsilon\|_{L^2(\Omega)} \\ &\leq C\varepsilon^{-s}\|z_m - z\|_{L^2(\Omega)} + M\|z(-\Delta)^{s/2}\phi_\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{2.2.25}$$

Let us now recall that since $\|z\|_{X_0^s(\Omega)} \leq M$ then $\|z\|_{L^{2_s^*}(\Omega)} \leq C$, that is $z^2 \in L^{\frac{N}{N-2s}}(\Omega)$. Hence, for every $\rho > 0$ there exits $\eta \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\|z^2 - \eta\|_{L^{\frac{N}{N-2s}}(\Omega)} \leq \rho. \tag{2.2.26}$$

Then, by (2.2.24), (2.2.26) and the Hölder inequality with $p = N/N - 2s$ we obtain that

$$\begin{aligned} \|z(-\Delta)^{s/2}\phi_\varepsilon\|^2_{L^2(\Omega)} &\leq \int_{\mathbb{R}^N} |z^2(x) - \eta(x)||(-\Delta)^{s/2}\phi_\varepsilon(x)|^2\,dx \\ &+ \int_{\mathbb{R}^N} |\eta(x)||(-\Delta)^{s/2}\phi_\varepsilon(x)|^2\,dx \\ &\leq \|z^2 - \eta\|_{L^{\frac{N}{N-2s}}(\Omega)} \|(-\Delta)^{s/2}\phi_\varepsilon\|^2_{L^{\frac{N}{s}}(\mathbb{R}^N)} \\ &+ \|\eta\|_{L^\infty(\Omega)} \|(-\Delta)^{s/2}\phi_\varepsilon\|^2_{L^2(\mathbb{R}^N)} \\ &\leq \rho\varepsilon^{-2s}\left(\int_{\mathbb{R}^N} \left|\left((-\Delta)^{s/2}\phi\right)\left(\frac{x}{\varepsilon}\right)\right|^{\frac{N}{s}}\,dx\right)^{\frac{2s}{N}} \\ &+ C\varepsilon^{-2s}\int_{\mathbb{R}^N} \left|\left((-\Delta)^{s/2}\phi\right)\left(\frac{x}{\varepsilon}\right)\right|^2\,dx \\ &\leq \rho\left(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi(z)|^{\frac{N}{s}}\,dz\right)^{\frac{2s}{N}} \\ &+ C\varepsilon^{N-2s}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi(z)|^2\,dz \\ &\leq C\rho + C\varepsilon^{N-2s}. \end{aligned} \tag{2.2.27}$$

Hence using (2.2.23), from (2.2.25), (2.2.27) and the fact that $N > 2s$, it follows that

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty} I_1 \leq \lim_{\varepsilon\to 0} C\left(\rho + \varepsilon^{N-2s}\right)^{\frac{1}{2}} = C\rho^{\frac{1}{2}}.$$

Since $\rho > 0$ is fix but arbitrarily small we conclude the proof of Lemma 2.2.8. $\square$

Also we have the following.

Lemma 2.2.9. *With the same assumptions of Lemma 2.2.8 we have that*

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\left|\int_{\mathbb{R}^N}(-\Delta)^{s/2}z_m(x)B(z_m,\phi_\varepsilon)(x)\,dx\right| = 0, \tag{2.2.28}$$

where B is defined in (2.2.13)*.*

Proof. Let

$$I_2 := \left|\int_{\mathbb{R}^N}(-\Delta)^{s/2}z_m(x)B(z_m,\phi_\varepsilon)(x)\,dx\right|.$$

Since $\|z_m\|_{X_0^s(\Omega)} \le M$, then

$$\begin{aligned} I_2 &\le M\|B(z_m,\phi_\varepsilon)\|_{L^2(\mathbb{R}^N)} \\ &\le M\|B(z_m - z,\phi_\varepsilon)\|_{L^2(\mathbb{R}^N)} + M\|B(z,\phi_\varepsilon)\|_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{2.2.29}$$

where z is, as in Lemma 2.2.8, the weak limit of the sequence $\{z_m\}$ in $X_0^s(\Omega)$. We estimate each of the summands in the previous inequality. Let

$$\psi(x) := \frac{1}{1+|x|^{N+s}} \quad \text{and} \quad \psi_\varepsilon(x) := \psi\left(\frac{x}{\varepsilon}\right). \tag{2.2.30}$$

By Lemma 2.2.7 applied to ϕ, we note that

$$B(\phi_\varepsilon,\phi_\varepsilon)(x) \le \varepsilon^{-s}B(\phi,\phi)\left(\frac{x}{\varepsilon}\right) \le C\varepsilon^{-s}\psi_\varepsilon(x) \le C\varepsilon^{-s}. \tag{2.2.31}$$

Therefore, by Cauchy-Schwarz inequality, the fact that $\{z_m\}$ is uniformly bounded in $X_0^s(\Omega)$ and (2.2.31), it follows that

$$\begin{aligned} \|B(z_m - z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N} B(z_m - z, z_m - z)(x)B(\phi_\varepsilon,\phi_\varepsilon)(x)\,dx \\ &\le C\varepsilon^{-s}\int_{\mathbb{R}^N} B(z_m - z, z_m - z)(x)\,dx \\ &= C\varepsilon^{-s}\|z_m - z\|^2_{X_0^{\frac{s}{2}}(\Omega)} \\ &= C\varepsilon^{-s}\int_{\mathbb{R}^N}(z_m - z)(x)(-\Delta)^{s/2}(z_m - z)(x)\,dx \\ &\le C\varepsilon^{-s}\|z_m - z\|_{L^2(\Omega)}\|(-\Delta)^{s/2}(z_m - z)\|_{L^2(\mathbb{R}^N)} \\ &\le C\varepsilon^{-s}\|z_m - z\|_{L^2(\Omega)}. \end{aligned} \tag{2.2.32}$$

On the other hand, for a suitable function f, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} z^2(x)(-\Delta)^{s/2}f(x)\,dx &= \int_{\mathbb{R}^N} f(x)(-\Delta)^{s/2}z^2(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x)\left(2z(x)(-\Delta)^{s/2}z(x)\right. \\ &- \left. B(z,z)(x)\right)\,dx. \end{aligned} \tag{2.2.33}$$

Then, arguing as in (2.2.32), by (2.2.31) and applying (2.2.33) with $f := \psi_\varepsilon(x)$, we get

$$\begin{aligned}
\|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} &\leq \int_{\mathbb{R}^n} B(z,z)(x)B(\phi_\varepsilon,\phi_\varepsilon)(x)\,dx \\
&\leq C\varepsilon^{-s}\int_{\mathbb{R}^N} B(z,z)(x)\psi_\varepsilon(x)\,dx \\
&\leq C\varepsilon^{-s}\int_{\mathbb{R}^N}\big(-z^2(x)(-\Delta)^{s/2}\psi_\varepsilon(x) \\
&+ 2z(x)\psi_\varepsilon(x)(-\Delta)^{s/2}z(x)\big)\,dx \\
&:= I_{2,1}+I_{2,2}.
\end{aligned} \tag{2.2.34}$$

We now estimate $I_{2,1}$ and $I_{2,2}$ separately. Let $\rho>0$. On one hand, since ψ also satisfies (2.2.24), by Lemma 2.2.7 applied to ψ, it follows that

$$\begin{aligned}
|I_{2,1}| &\leq C\varepsilon^{-2s}\int_{\mathbb{R}^N} z^2(x)\left|\left((-\Delta)^{s/2}\psi\right)\left(\frac{x}{\varepsilon}\right)\right|\,dx \\
&\leq C\varepsilon^{-2s}\int_{\mathbb{R}^N} z^2(x)\psi\left(\frac{x}{\varepsilon}\right)\,dx \\
&\leq C\varepsilon^{-2s}\int_{\mathbb{R}^N}(z^2-\eta)(x)\psi_\varepsilon(x)\,dx \\
&+ \varepsilon^{-2s}\int_{\mathbb{R}^N}\eta(x)\psi_\varepsilon(x)\,dx,
\end{aligned} \tag{2.2.35}$$

where $\eta\in\mathcal{C}^\infty_0(\Omega)$ is the function that satisfies (2.2.26). Then using Hölder inequality, (2.2.26) and (2.2.35) we obtain

$$\begin{aligned}
|I_{2,1}| &\leq \rho\varepsilon^{-2s}\,\|\psi_\varepsilon\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} + \varepsilon^{-2s}\|\eta\|_{L^\infty(\mathbb{R}^N)}\,\|\psi_\varepsilon\|_{L^1(\mathbb{R}^N)} \\
&\leq \rho\|\psi\|_{L^{\frac{N}{2s}}(\mathbb{R}^N)} + \varepsilon^{N-2s}\|\eta\|_{L^\infty(\mathbb{R}^N)}\|\psi\|_{L^1(\mathbb{R}^N)}.
\end{aligned} \tag{2.2.36}$$

On the other hand, since $z\in X^s_0(\Omega)$,

$$|I_{2,2}| \leq C\varepsilon^{-s}\|(-\Delta)^{s/2}z\|_{L^2(\mathbb{R}^N)}\,\|z\psi_\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{-s}\,\|z\psi_\varepsilon\|_{L^2(\Omega)}\,. \tag{2.2.37}$$

Therefore, by (2.2.26), we get

$$\begin{aligned}
|I_{2,2}|^2 &\leq C\varepsilon^{-2s}\left(\int_\Omega |z^2-\eta|(x)\,|\psi_\varepsilon(x)|^2\,dx + \int_{\mathbb{R}^N}\eta\,|\psi_\varepsilon(x)|^2\,dx\right) \\
&\leq C\varepsilon^{-2s}\left(\rho\|\psi_\varepsilon\|^2_{L^{\frac{N}{s}}(\mathbb{R}^N)} + \|\eta\|_{L^\infty(\mathbb{R}^N)}\,\|\psi_\varepsilon\|^2_{L^2(\mathbb{R}^N)}\right) \\
&\leq C\rho\|\psi\|^2_{L^{\frac{N}{s}}(\mathbb{R}^N)} + C\varepsilon^{N-2s}\|\eta\|_{L^\infty(\mathbb{R}^N)}\|\psi\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned} \tag{2.2.38}$$

Then by (2.2.36) and (2.2.38), from (2.2.34), it follows that

$$\|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} \leq C\left(\rho + \rho^{\frac{1}{2}}\right) + \tilde{C}\left(\varepsilon^{N-2s} + \varepsilon^{\frac{N-2s}{2}}\right). \tag{2.2.39}$$

Hence from (2.2.23), (2.2.32) and (2.2.39), since $N > 2s$, we obtain

$$\begin{aligned} & \lim_{\varepsilon\to 0}\lim_{m\to\infty} \|B(z_m - z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} + \|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} \\ \leq\ & \lim_{\varepsilon\to 0} C\left(\rho^{\frac{1}{2}} + \varepsilon^{\frac{N-2s}{2}}\right) \\ =\ & C\rho^{\frac{1}{2}}. \end{aligned}$$

Thus, since ρ is an arbitraty positive value,

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty} \|B(z_m - z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} + \|B(z,\phi_\varepsilon)\|^2_{L^2(\mathbb{R}^N)} = 0. \tag{2.2.40}$$

Finally, by (2.2.29) and (2.2.40), we conclude that

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty} |I_2| = 0.$$

□

Now we can prove the compactness result for the translated functional $\widetilde{\mathcal{J}}_{s,\lambda}$ defined in (2.2.9).

Lemma 2.2.10. *If $u = 0$ is the only critical point of $\widetilde{\mathcal{J}}_{s,\lambda}$ in $X_0^s(\Omega)$ then $\widetilde{\mathcal{J}}_{s,\lambda}$ satisfies a local Palais Smale condition below the critical level c^* given in* (1.2.22).

Proof. Let $\{u_m\}$ be a PS sequence for $\widetilde{\mathcal{J}}_{s,\lambda}$ verifying

$$\widetilde{\mathcal{J}}_{s,\lambda}(u_m) \to c < c^* \qquad \text{and} \qquad \widetilde{\mathcal{J}}'_{s,\lambda}(u_m) \to 0. \tag{2.2.41}$$

The argument presented at the beginning of the proof of Lemma 1.2.10, shows that $\{u_m\}$ is uniformly bounded in $X_0^s(\Omega)$. Moreover taking

$$z_m := u_m + u_0,$$

we have that

$$\mathcal{J}_{s,\lambda}(z_m) \leq \widetilde{\mathcal{J}}_{s,\lambda}(u_m) + \mathcal{J}_{s,\lambda}(u_0) \tag{2.2.42}$$

and

$$\mathcal{J}'_{s,\lambda}(z_m) \to 0. \tag{2.2.43}$$

Since $\{z_m\}$ is uniformly bounded in $X_0^s(\Omega)$ and $u = 0$ is the unique critical point of $\mathcal{J}_{s,\lambda}$, up to a subsequence, we get that

$$\begin{aligned} z_m &\rightharpoonup u_0 && \text{weakly in } X_0^s(\Omega), \\ z_m &\to u_0 && \text{strongly in } L^r(\Omega), \quad 1 \le r < 2_s^*, \\ z_m &\to u_0 && \text{a.e. in } \Omega. \end{aligned} \tag{2.2.44}$$

Moreover, since the positive part is a Lipschitz function, with constant equal to one, we know that

$$\|(z_m)_+\|_{X_0^s(\Omega)} \le \|z_m\|_{X_0^s(\Omega)}.$$

Hence $\{(z_m)_+\}$ is also uniformly bounded in $X_0^s(\Omega)$ and, since $u_0 > 0$, $\{(z_m)_+\}$ has the same convergence properties as $\{z_m\}$ given above. Then, by (2.1.3), applying [126, Theorem 1.5] we have that there exist an index set $I \subseteq \mathbb{N}$, a sequence of points $\{x_k\}_{k\in I} \subset \Omega$, and nonnegative real numbers μ_k, ν_k, such that

$$|(-\Delta)^{s/2}(z_m)_+|^2 \to \mu \ge |(-\Delta)^{s/2}u_0|^2 + \sum_{k\in I} \mu_k \delta_{x_k} \tag{2.2.45}$$

and

$$|(z_m)_+|^{2_s^*} \to \nu = |u_0|^{2_s^*} + \sum_{k\in I} \nu_k \delta_{x_k}, \tag{2.2.46}$$

in the sense of measures, with

$$\nu_k \le \left(\frac{C(N,s)}{2} S(N,s)\right)^{-\frac{2_s^*}{2}} \mu_k^{\frac{2_s^*}{2}} = (\kappa_s T(N,s))^{-\frac{2_s^*}{2}} \mu_k^{\frac{2_s^*}{2}}, \tag{2.2.47}$$

for every $k \in I$. Here δ_{x_k} denotes the Dirac delta at x_k, while κ_s and $T(N,s)$ are the constants defined in (1.1.12) and (1.1.31) respectively. We fix $k_0 \in I$, and we consider $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ a radially nonincreasing cut-off function

$$\phi = \begin{cases} 1 & \text{in } B_1(x_{k_0}), \\ 0 & \text{in } B_2(x_{k_0})^c. \end{cases} \tag{2.2.48}$$

Set now

$$\phi_\varepsilon(x) = \phi(x/\varepsilon). \tag{2.2.49}$$

Then, using $\phi_\varepsilon(z_m)_+$ as a test function in (2.2.43), by (2.1.6) and the fact that

$$\int_{\mathbb{R}^N} (\phi_\varepsilon(z_m)_+)(-\Delta)^s z_m \, dx \ge \int_{\mathbb{R}^N} (\phi_\varepsilon(z_m)_+)(-\Delta)^s (z_m)_+ \, dx,$$

we have that

$$\begin{aligned} 0 &\geq \lim_{m\to\infty}\left(\int_{\mathbb{R}^N}(\phi_\varepsilon(z_m)_+)(-\Delta)^s(z_m)_+\,dx\right.\\ &- \left.\left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}((z_m)_+)^{q+1}\phi_\varepsilon\,dx+\int_{B_{2\varepsilon}(x_{k_0})}((z_m)_+)^{2_s^*}\phi_\varepsilon\,dx\right)\right). \end{aligned}$$

Hence, denoting by

$$A(\phi_\varepsilon,(z_m)_+)(x,y):=(\phi_\varepsilon(x)-\phi_\varepsilon(y))((z_m)_+(x)-(z_m)_+(y)),\ x,\ y\in\mathbb{R}^N,$$

it follows that

$$\begin{aligned} &\lim_{m\to\infty}\left(\int_{\mathbb{R}^N}(z_m)_+(x)(-\Delta)^{s/2}(z_m)_+(x)(-\Delta)^{s/2}\phi_\varepsilon(x)\,dx\right.\\ &- \left.C(N,s)\int_{\mathbb{R}^N}(-\Delta)^{s/2}(z_m)_+(x)\int_{\mathbb{R}^N}\frac{A(\phi_\varepsilon,(z_m)_+)(x,y)}{|x-y|^{N+s}}\,dx\,dy\right)\\ &\leq \lim_{m\to\infty}\left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}((z_m)_+)^{q+1}\phi_\varepsilon\,dx+\int_{B_{2\varepsilon}(x_{k_0})}((z_m)_+)^{2_s^*}\phi_\varepsilon\,dx\right.\\ &- \left.\int_{B_{2\varepsilon}(x_{k_0})}((-\Delta)^{s/2}(z_m)_+)^2\phi_\varepsilon\,dx\right). \end{aligned}$$

Therefore by (2.2.44), (2.2.45) and (2.2.46) we get

$$\begin{aligned} &\lim_{\varepsilon\to0}\lim_{m\to\infty}\left(\int_{\mathbb{R}^N}(z_m)_+(x)(-\Delta)^{s/2}(z_m)_+(x)(-\Delta)^{s/2}\phi_\varepsilon(x)\,dx\right.\\ &- \left.C(N,s)\int_{\mathbb{R}^N}(-\Delta)^{s/2}(z_m)_+(x)\int_{\mathbb{R}^N}\frac{A(\phi_\varepsilon,(z_m)_+)(x,y)}{|x-y|^{N+s}}\,dx\,dy\right)\\ &\leq \lim_{\varepsilon\to0}\left(\lambda\int_{B_{2\varepsilon}(x_{k_0})}u_0^{q+1}\phi_\varepsilon\,dx\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\nu-\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\mu\right). \quad (2.2.50) \end{aligned}$$

Since ϕ is a regular function with compact support, it is clear that ϕ satisfies the hypotheses of Lemma 2.2.7. Therefore by Lemma 2.2.8 and Lemma 2.2.9 applied to the sequence $\{(z_m)_+\}$, it follows that the left hand side of (2.2.50) goes to zero. That is, we obtain that

$$\lim_{\varepsilon\to0}\left(\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\nu+\lambda\int_{B_{2\varepsilon}(x_{k_0})}|u_0|^{q+1}\phi_\varepsilon\,dx-\int_{B_{2\varepsilon}(x_{k_0})}\phi_\varepsilon\,d\mu\right)=\nu_{k_0}-\mu_{k_0}\geq0.$$

Thus, from (2.2.47), we have that

$$\nu_{k_0}=0 \qquad \text{or} \qquad \nu_{k_0}\geq(\kappa_s T(N,s))^{\frac{N}{2s}}.$$

As in the proof of Proposition 1.2.10, we get a contradiction with (2.2.41). Since k_0 was arbitrary, we deduce that $\nu_k = 0$ for all $k \in I$. As a consequence, we obtain that $(u_m)_+ \to 0$ in $L^{2_s^*}(\Omega)$. Note that, since u_m is equal to zero outside Ω, we have that $(u_m)_+ \to 0$ in $L^{2_s^*}(\mathbb{R}^N)$. We finish in the standard way: convergence of u_m in $L^{2_s^*}(\mathbb{R}^N)$ implies convergence of $f((u_m)_+)$ in $L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$, and finally by using the continuity of the inverse operator $(-\Delta)^{-s}$, we obtain convergence of u_m in $X_0^s(\Omega)$. $\square$

In Lemma 2.2.10 we have proved that if $u = 0$ is the only critical point of the functional $\widetilde{\mathcal{J}}_{s,\lambda}$, then $\widetilde{\mathcal{J}}_{s,\lambda}$ verifies the $(\mathrm{PS})_c$ condition at any level c, provided c stays below the level c^* defined in (1.2.22).

Now, we want to show that we can obtain a local $(\mathrm{PS})_c$ sequence for $\widetilde{\mathcal{J}}_{s,\lambda}$ under the critical level c^*. For this, assume, without loss of generality, that $0 \in \Omega$. As we explain in Section 1.2 of Chapter 1, we know that the infimum in (2.1.7) is attained at the function

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2s)/2}}{(|x|^2 + \varepsilon^2)^{(N-2s)/2}}, \quad \varepsilon > 0, \tag{2.2.51}$$

that is

$$\|(-\Delta)^{s/2} u_\varepsilon\|^2_{L^2(\mathbb{R}^N)} = \kappa_s T(N,s) \|u_\varepsilon\|^2_{L^{2_s^*}(\mathbb{R}^N)}.$$

Consider now, as in Chapter 1, the family of truncated functions

$$\eta_\varepsilon := \frac{\phi u_\varepsilon}{\|\phi u_\varepsilon\|_{L^{2_s^*}(\Omega)}}, \tag{2.2.52}$$

where

$$\phi(x) = \phi_0\left(\frac{|x|}{r}\right), \quad x \in \mathbb{R}^N,$$

with r a small positive number such that $\overline{B}_r \subseteq \overline{\Omega}$ and $\phi_0 \in \mathcal{C}^\infty(\mathbb{R}_+)$, satisfying

$$\phi_0(\eta) = \begin{cases} 1 & \text{if } 0 \le \eta \le \frac{1}{2}, \\ 0 & \text{if } \eta \ge 1. \end{cases}$$

Following the same procedure as in Lemma 1.2.13, using the estimate given in [140, Proposition 21] instead of the estimate (1.2.58) given in Lemma 1.2.12, we have the following

Lemma 2.2.11. *There exists $\varepsilon > 0$ small enough such that*

$$\sup_{t \ge 0} \widetilde{\mathcal{J}}_{s,\lambda}(t\eta_\varepsilon) < c^*.$$

Therefore, to prove statement *4* in Theorem 2.2.1, as in the end of Section 1.2 in Chapter 1, we use (2.2.10), Lemma 2.2.10 and Lemma 2.2.11.

2.3 Superlinear case: $q > 1$.

The existence of at least one solution of (D_λ) for the convex subcritical case is straightforward. In fact this existence follows proving the good geometry of the functional using the ideas given in the proof of Proposition 1.4.2 and the Theorem 0.0.2 instead of the Trace inequality given in (1.1.29). The conclusion follows using the compactness property of the functional, see for example Proposition 3.3.3, and applying the MPT given in [11].

Therefore we focus in the critical case, that is, in the proof of the following.

Theorem 2.3.1. *Let $1 < q < 2_s^* - 1$. Then, problem (D_λ) admits at least one energy solution provided that either*

- $N > \frac{2s(q+3)}{q+1}$ *and* $\lambda > 0$ *or*
- $N \leq \frac{2s(q+3)}{q+1}$ *and* $\lambda > 0$ *is sufficiently large.*

As we said, even for the critical case it is easy to check the good geometry of the functional $\mathcal{J}_{s,\lambda}$. Now we show that the functional $\mathcal{J}_{s,\lambda}$ satisfies the PS condition in a suitable energy range involving the best fractional critical Sobolev constant $S(N,s)$ given in (2.1.8). The proof that we will present has been done following the ideas of [158] and [174] , adapted in order to take into account the nonlocal nature of the fractional Laplace operator. We mention here that, alternatively, Proposition 2.3.2 could be proved using the concentration-compactness theory. That is, using the same arguments performed in the proof of Lemma 2.2.10.

Proposition 2.3.2. *Assume $\lambda > 0$ and $1 < q < 2_s^* - 1$. Then, the functional $\mathcal{J}_{s,\lambda}$ satisfies the $(PS)_c$ condition provided $c < c^*$, where c^* is given in* (1.2.22).

Proof. Let $\{u_m\}$ be a $(\mathrm{PS})_c$ sequence for $\mathcal{J}_{s,\lambda}$ in $X_0^s(\Omega)$, that is

$$\mathcal{J}_{s,\lambda}(u_m) \to c \tag{2.3.1}$$

and

$$\mathcal{J}'_{s,\lambda}(u_m) \to 0 \tag{2.3.2}$$

as $m \to \infty$. First of all by (2.3.1) and (2.3.2) there exists $M > 0$ such that

$$\|u_m\|_{X_0^s(\Omega)} \leq M. \tag{2.3.3}$$

In order to prove our result we proceed by steps.

Claim 1. *There exists* $u_\infty \in X_0^s(\Omega)$ *such that* $\langle \mathcal{J}'_{s,\lambda}(u_\infty), \varphi\rangle = 0$ *for any* $\varphi \in X_0^s(\Omega)$ *.*

Proof. By (2.3.3) and the fact that $X_0^s(\Omega)$ is a reflexive space, up to a subsequence, still denoted by u_m, there exists $u_\infty \in X_0^s(\Omega)$ such that $u_m \rightharpoonup u_\infty$ weakly in $X_0^s(\Omega)$, that is, for any $\varphi \in X_0^s(\Omega)$,

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} &\frac{(u_m(x)-u_m(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \to \\ &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_\infty(x)-u_\infty(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy.\end{aligned} \tag{2.3.4}$$

Moreover by the Sobolev embedding theorem given in Theorem 0.0.2 we also have that, up to a subsequence

$$u_m \rightharpoonup u_\infty \quad \text{weakly in } L^{2_s^*}(\Omega), \tag{2.3.5}$$

$$u_m \to u_\infty \quad \text{in } L^r(\Omega), \quad 1 \le r < 2_s^* - 1 \tag{2.3.6}$$

and

$$u_m \to u_\infty \quad \text{a.e. in } \Omega. \tag{2.3.7}$$

Also, since the positive part is a Lipschitz function, by the above convergences we also know that

$$(u_m)_+^{2_s^*-1} \rightharpoonup (u_\infty)_+^{2_s^*-1} \quad \text{weakly in } L^{2_s^*/(2_s^*-1)}(\Omega), \tag{2.3.8}$$

$$(u_m)_+ \to (u_\infty)_+ \quad \text{in } L^{q+1}(\Omega) \quad \text{and} \quad (u_m)_+ \to (u_\infty)_+ \quad \text{a.e. in } \Omega. \tag{2.3.9}$$

Therefore, since $(2_s^*/(2_s^*-1))' = 2_s^*$, by (2.3.8), we obtain that

$$\int_\Omega (u_m)_+^{2_s^*-1}\varphi\,dx \to \int_\Omega (u_\infty)_+^{2_s^*-1}\varphi\,dx, \quad \varphi \in X_0^s(\Omega). \tag{2.3.10}$$

Similarly, by (2.3.9), for every $\varphi \in X_0^s(\Omega)$ we get

$$\int_\Omega (u_m)_+^q\varphi\,dx \to \int_\Omega (u_\infty)_+^q\varphi\,dx. \tag{2.3.11}$$

Then, by (2.3.2), (2.3.4), (2.3.10) and (2.3.11) we conclude

$$\begin{aligned}&\frac{C(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_\infty(x)-u_\infty(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \\ -\ &\lambda\int_\Omega (u_\infty)_+^q\varphi\,dx - \int_\Omega (u_\infty)_+^{2_s^*-1}\varphi\,dx = 0,\end{aligned}$$

for any $\varphi \in X_0^s(\Omega)$. ∎

Claim 2. *The following equality holds:*

$$\begin{aligned} \mathcal{J}_{s,\lambda}(u_m) &= \mathcal{J}_{s,\lambda}(u_\infty) + \frac{C(N,s)}{4}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} \\ &- \frac{1}{2_s^*}\int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2_s^*}\,dx + o(1). \end{aligned}$$

Proof. First of all since the sequence u_m is bounded in $X_0^s(\Omega)$ and in $L^{2_s^*}(\Omega)$, by (2.3.9) and the Brezis-Lieb Lemma given in Lemma 1.3.3, we get

$$\|u_m\|^2_{X_0^s(\Omega)} = \|u_m - u_\infty\|^2_{X_0^s(\Omega)} + \|u_\infty\|^2_{X_0^s(\Omega)} + o(1), \tag{2.3.12}$$

$$\|(u_m)_+\|^{2_s^*}_{L^{2_s^*}(\Omega)} = \|(u_m)_+(x) - (u_\infty)_+(x)\|^{2_s^*}_{L^{2_s^*}(\Omega)} + \|(u_\infty)_+\|^{2_s^*}_{L^{2_s^*}(\Omega)} + o(1) \tag{2.3.13}$$

and

$$\|(u_m)_+\|_{L^{q+1}(\Omega)} \to \|(u_\infty)_+\|_{L^{q+1}(\Omega)}. \tag{2.3.14}$$

Therefore by (2.3.12), (2.3.13) and (2.3.14) we deduce that

$$\begin{aligned} \mathcal{J}_{s,\lambda}(u_m) &= \frac{C(N,s)}{4}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} + \frac{C(N,s)}{4}\|u_\infty\|^2_{X_0^s(\Omega)} \\ &- \frac{\lambda}{q+1}\int_\Omega ((u_\infty)_+)^{q+1}dx - \frac{1}{2_s^*}\int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2_s^*}\,dx \\ &- \frac{1}{2_s^*}\int_\Omega ((u_\infty)_+)^{2_s^*}\,dx + o(1) \\ &= \mathcal{J}_{s,\lambda}(u_\infty) + \frac{C(N,s)}{4}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} \\ &- \frac{1}{2_s^*}\int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2_s^*}\,dx + o(1), \end{aligned}$$

which gives the desired assertion. □

Claim 3. *The following equality holds:*

$$\begin{aligned} \frac{C(N,s)}{2}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} &= \int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2_s^*}dx + o(1) \\ &\le \int_\Omega |(u_m)(x) - (u_\infty)(x)|^{2_s^*}dx + o(1). \end{aligned}$$

Proof. First of all, note that, as a consequence of (2.3.6), (2.3.8) and (2.3.13), we get

$$\begin{aligned}
& \int_\Omega \left(((u_m)_+)^{2_s^*-1}(x) - ((u_\infty)_+)^{2_s^*-1}(x)\right)(u_m(x) - u_\infty(x))\, dx \\
= & \int_\Omega ((u_m)_+)^{2_s^*}\, dx - \int_\Omega ((u_\infty)_+)^{2_s^*-1} u_m\, dx \\
- & \int_\Omega ((u_m)_+)^{2_s^*-1} u_\infty\, dx + \int_\Omega ((u_\infty)_+)^{2_s^*}\, dx \\
= & \int_\Omega ((u_m)_+)^{2_s^*}\, dx - \int_\Omega ((u_\infty)_+)^{2_s^*}\, dx + o(1) \\
= & \int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2_s^*}\, dx + o(1).
\end{aligned} \tag{2.3.15}$$

Furthermore, (2.3.6), (2.3.11) and (2.3.14) give

$$\begin{aligned}
& \int_\Omega (((u_m)_+)^q(x) - ((u_\infty)_+)^q(x))(u_m(x) - u_\infty(x))\, dx \\
= & \int_\Omega ((u_m)_+)^{q+1}\, dx - \int_\Omega ((u_\infty)_+)^q u_m\, dx \\
- & \int_\Omega ((u_m)_+)^q u_\infty\, dx + \int_\Omega ((u_\infty)_+)^{q+1}\, dx \\
= & \; o(1).
\end{aligned} \tag{2.3.16}$$

Then, by (2.3.2), Claim 1, (2.3.15) and (2.3.16), we conclude that

$$\begin{aligned}
o(1) &= \langle \mathcal{J}'_{s,\lambda}(u_m), u_m - u_\infty \rangle \\
&= \langle \mathcal{J}'_{s,\lambda}(u_m) - \mathcal{J}'_{s,\lambda}(u_\infty), u_m - u_\infty \rangle \\
&= \frac{C(N,s)}{2} \|u_m - u_\infty\|^2_{X_0^s(\Omega)} \\
&- \lambda \int_\Omega (((u_m)_+)^q(x) - ((u_\infty)_+)^q(x))(u_m(x) - u_\infty(x))\, dx \\
&- \int_\Omega \left(((u_m)_+)^{2_s^*-1}(x) - ((u_\infty)_+)^{2_s^*-1}(x)\right)(u_m(x) - u_\infty(x))\, dx \\
&= \frac{C(N,s)}{2} \|u_m - u_\infty\|^2_{X_0^s(\Omega)} - \int_\Omega |((u_m)_+)(x) - ((u_\infty)_+)(x)|^{2_s^*}\, dx + o(1).
\end{aligned}$$

□

Now, we can finish the proof of Proposition 2.3.2 .

By Claim 3 we know that

$$\begin{aligned} & \frac{C(N,s)}{4}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} - \frac{1}{2^*_s}\int_\Omega |(u_m)_+(x) - (u_\infty)_+(x)|^{2^*_s} dx \\ = \; & \frac{s}{N}\frac{C(N,s)}{2}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} + o(1). \end{aligned} \tag{2.3.17}$$

Then, by Claim 2, (2.3.1) and (2.3.17), for $m \to +\infty$, we obtain

$$\mathcal{J}_{s,\lambda}(u_\infty) + \frac{s}{N}\frac{C(N,s)}{2}\|u_m - u_\infty\|^2_{X_0^s(\Omega)} = \mathcal{J}_{s,\lambda}(u_m) + o(1) = c + o(1). \tag{2.3.18}$$

Also, by (2.3.3), up to a subsequence we can assume that

$$\|u_m - u_\infty\|^2_{X_0^s(\Omega)} \to L \geq 0. \tag{2.3.19}$$

Then, as a consequence of Claim 3,

$$\int_\Omega |u_m(x) - u_\infty(x)|^{2^*_s}\, dx \to \widetilde{L} \geq \frac{C(N,s)}{2}L.$$

Therefore, by definition of the constant $S(N,s)$ we have

$$L \geq (\widetilde{L})^{2/2^*_s} S(N,s) \geq L^{2/2^*_s} S(N,s)\left(\frac{C(N,s)}{2}\right)^{\frac{2}{2^*_s}},$$

so that, by (2.1.8),

$$L = 0 \quad \text{or} \quad L \geq \frac{2}{C(N,s)}(\kappa_s T(N,s))^{\frac{N}{2s}}.$$

We will prove that the case $L \geq \frac{2}{C(N,s)}(\kappa_s T(N,s))^{\frac{N}{2s}}$ cannot hold. Indeed, taking $\varphi = u_\infty \in X_0^s(\Omega)$ as a test function in Claim 1, we have that

$$\frac{C(N,s)}{2}\|u_\infty\|^2_{X_0^s(\Omega)} = \lambda\int_\Omega ((u_\infty)_+)^{q+1}\, dx + \int_\Omega ((u_\infty)_+)^{2^*_s} dx.$$

That is,

$$\begin{aligned} \mathcal{J}_{s,\lambda}(u_\infty) \; &= \; \lambda\left(\frac{1}{2} - \frac{1}{q+1}\right)\|((u_\infty)_+)\|^{q+1}_{L^{q+1}(\Omega)} \\ & + \; \frac{s}{N}\,\|((u_\infty)_+)\|^{2^*_s}_{L^{2^*_s}(\Omega)} \geq 0\,, \end{aligned} \tag{2.3.20}$$

thanks to the positivity of λ and the fact that $q>1$. Therefore, if $L \geq \frac{2}{C(N,s)}(\kappa_s T(N,s))^{\frac{N}{2s}}$, then, by (2.3.18), (2.3.19) and (2.3.20) we get

$$c = \mathcal{J}_{s,\lambda}(u_\infty) + \frac{s}{N}\frac{C(N,s)}{2}L \geq \frac{s}{N}\frac{C(N,s)}{2}L \geq \frac{s}{N}(\kappa_s T(N,s))^{\frac{N}{2s}},$$

which contradicts the fact that $c < c^*$. Thus $L = 0$ and so, by (2.3.19), we obtain that

$$\|u_m - u_\infty\|_{X_0^s(\Omega)} \to 0.$$

□

Then as in Proposition 1.4.4 using the estimate in [140, Proposition 21] instead of (1.2.58) we get

Proposition 2.3.3. *Let $\lambda > 0$, $1 < q < 2_s^* - 1$, c^* given in (1.2.22) and η_ε be the function defined in (2.2.52). Then, there exists $\varepsilon > 0$ small enough such that*

$$\sup_{t \geq 0} \mathcal{J}_{s,\lambda}(t\eta_\varepsilon) < c^*,$$

provided

- $N > 2s\left(\frac{3+q}{1+q}\right)$ *and $\lambda > 0$ or*
- $N \leq 2s\left(\frac{3+q}{1+q}\right)$ *and $\lambda > \lambda_s$, for a suitable $\lambda_s > 0$.*

Since $\mathcal{J}_{s,\lambda}$ satisfies the geometric features required by the MPT given in [11] and, by Proposition 2.3.2, the functional $\mathcal{J}_{s,\lambda}$ satisfies the $(\mathrm{PS})_c$ condition at any level c, provided $c < c^*$, we conclude the proof of Theorem 2.3.1.

Remark 2.3.4. *Note that for the case $s = 1$ we have obtained that*

- *For $N \geq 4$, exists a solution for (D_λ) for every $\lambda > 0$.*
- *For $N < 4$ and $q < \dfrac{6-N}{N-2}$ exists a solution for $\lambda > 0$ big enough.*

This coincide when the result obtained in [45] and [92] for the p-Laplacian operator.

2.4 Regularity result.

We present the proof of the regularity result about the solutions of (D_λ) used in the proof of Lemma 2.2.6. That is, we will prove the following.

Proposition 2.4.1. *Let u be a nonnegative energy solution to the problem*

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and assume that $|f(x,t)| \le C(1+|t|^p)$, for some $1 \le p \le 2_s^* - 1$. Then $u \in L^\infty(\Omega)$.

Note that the equivalent result for the spectral fractional Laplacian was given in Proposition 1.5.5. We point out here that in the proof of that proposition we used the extension tool associated to problem (P_λ) and here we will not use it.

Proof. The proof uses standard techniques for the fractional Laplacian, in particular the following inequality: If φ is a convex function, then

$$(-\Delta)^s \varphi(u) \le \varphi'(u)\,(-\Delta)^s u.$$

This follows from the fact that for φ convex one has

$$\varphi(v) - \varphi(w) \le \varphi'(v)(v-w), \quad v,\, w \in \mathbb{R}.$$

Let us define, for $\beta \ge 1$ and $T > 0$ large,

$$\varphi(t) = \varphi_{T,\beta}(t) = \begin{cases} \max(t,0)^\beta, & \text{if } \ t < T \\ \beta T^{\beta-1}(t-T) + T^\beta, & \text{if } \ t \ge T. \end{cases}$$

Observe that $\varphi(u) \in X_0^s(\Omega)$ since φ is Lipschitz with constant $K = \beta T^{\beta-1}$ and, therefore,

$$\begin{aligned} \|\varphi(u)\|_{X_0^s(\Omega)} &= \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2} \\ &\le \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{K^2 |u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2} = K\|u\|_{X_0^s(\Omega)}. \end{aligned}$$

By (2.1.5) and the Sobolev embedding theorem we have

$$\begin{aligned} \int_\Omega \varphi(u)(-\Delta)^s \varphi(u) &= \frac{C(N,s)}{2} \|\varphi(u)\|_{X_0^s(\Omega)}^2 \\ &\ge \kappa_s T(N,s) \|\varphi(u)\|_{L^{2_s^*}(\Omega)}^2. \end{aligned} \tag{2.4.1}$$

On the other hand, since $0 \le \varphi$ is convex, and $\varphi(u)\varphi'(u) \in X_0^s(\Omega)$,

$$\int_\Omega \varphi(u)(-\Delta)^s\varphi(u) \le \int_\Omega \varphi(u)\varphi'(u)\,(-\Delta)^s u \le C\int_\Omega \varphi(u)\varphi'(u)\,\left(1+u^{2_s^*-1}\right).$$

From (2.4.1) and the previous inequality we get the following basic estimate:

$$\|\varphi(u)\|^2_{L^{2_s^*}(\Omega)} \le C\int_\Omega \varphi(u)\varphi'(u)\,\left(1+u^{2_s^*-1}\right). \tag{2.4.2}$$

Since $u\,\varphi'(u) \le \beta\,\varphi(u)$ and $\varphi'(u) \le \beta\,(1+\varphi(u))$, then

$$\begin{aligned}\varphi(u)\varphi'(u)\,\left(1+u^{2_s^*-1}\right) &\le \beta\varphi(u)(1+\varphi(u)) + \beta(\varphi(u))^2u^{2_s^*-2}\\ &\le C\beta(1+(\varphi(u))^2) + \beta(\varphi(u))^2u^{2_s^*-2}.\end{aligned}$$

Therefore (2.4.2) becomes

$$\left(\int_\Omega (\varphi(u))^{2_s^*}\right)^{2/2_s^*} \le C\,\beta\left(1+\int_\Omega (\varphi(u))^2 + \int_\Omega (\varphi(u))^2u^{2_s^*-2}\right). \tag{2.4.3}$$

It is important to point out here that since $\varphi(u)$ grows linearly, both sides of (2.4.3) are finite.

Claim: Let β_1 such that $2\beta_1 = 2_s^*$. Then $u \in L^{\beta_1 2_s^*}$.

To see this, we take R large to be determined later. Then, Hölder's inequality with $p = \beta_1 = 2_s^*/2$ and $p' = 2_s^*/(2_s^*-2)$ gives

$$\begin{aligned}\int_\Omega (\varphi(u))^2\,u^{2_s^*-2} &= \int_{\{u\le R\}} (\varphi(u))^2\,u^{2_s^*-2} + \int_{\{u>R\}} (\varphi(u))^2\,u^{2_s^*-2}\\ &\le \int_{\{u\le R\}} (\varphi(u))^2\,R^{2_s^*-2}\\ &\quad + \left(\int_\Omega (\varphi(u))^{2_s^*}\right)^{2/2_s^*}\left(\int_{\{u>R\}} u^{2_s^*}\right)^{(2_s^*-2)/2_s^*}.\end{aligned}$$

By the Monotone Convergence Theorem, we may take R so that

$$\left(\int_{\{u>R\}} u^{2_s^*}\right)^{(2_s^*-2)/2_s^*} \le \frac{1}{2\,C\,\beta_1}.$$

In this way, the second term above is absorbed by the left hand side of (2.4.3) to get for $\varphi = \varphi_{T,\beta_1}$,

$$\left(\int_\Omega (\varphi(u))^{2_s^*}\right)^{2/2_s^*} \le 2\,C\,\beta_1\left(1+\int_\Omega (\varphi(u))^2 + \int_{\{u\le R\}} (\varphi(u))^2\,R^{2_s^*-2}\right).$$

Using that $\varphi_{T,\beta_1}(u) \leq u^{\beta_1}$ in the right hand side of the previous inequality and letting $T \to \infty$ in the left hand side, since $2\beta_1 = 2_s^*$, we obtain

$$\left(\int_\Omega u^{2_s^*\beta_1}\right)^{2/2_s^*} \leq 2\,C\,\beta_1\left(1 + \int_\Omega u^{2_s^*} + R^{2_s^*-2}\int_\Omega u^{2_s^*}\right) < \infty.$$

This proves the Claim.

We now go back to inequality (2.4.3) and we use as before that $\varphi_{T,\beta}(u) \leq u^\beta$ in the right hand side and we take $T \to \infty$ in the left hand side. Then,

$$\left(\int_\Omega u^{2_s^*\beta}\right)^{2/2_s^*} \leq C\,\beta\left(1 + \int_\Omega u^{2\beta} + \int_\Omega u^{2\beta+2_s^*-2}\right).$$

Since $\displaystyle\int_\Omega u^{2\beta} \leq |\Omega| + \int_\Omega u^{2\beta+2_s^*-2}$, we get the following recurrence formula

$$\left(\int_\Omega u^{2_s^*\beta}\right)^{2/2_s^*} \leq 2\,C\,\beta\,(1+|\Omega|)\left(1 + \int_\Omega u^{2\beta+2_s^*-2}\right).$$

Therefore,

$$\left(1 + \int_\Omega u^{2_s^*\beta}\right)^{\frac{1}{2_s^*(\beta-1)}} \leq C_\beta^{\frac{1}{2(\beta-1)}}\left(1 + \int_\Omega u^{2\beta+2_s^*-2}\right)^{\frac{1}{2(\beta-1)}}, \tag{2.4.4}$$

where $C_\beta = 4\,C\,\beta\,(1+|\Omega|)$.
For $m \geq 1$ we define β_{m+1} inductively so that $2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m$, that is

$$\beta_{m+1} - 1 = \frac{2_s^*}{2}(\beta_m - 1) = \left(\frac{2_s^*}{2}\right)^m(\beta_1 - 1).$$

Hence from (2.4.4) it follows that

$$\left(1 + \int_\Omega u^{2_s^*\beta_{m+1}}\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq C_{\beta_{m+1}}^{\frac{1}{2(\beta_{m+1}-1)}}\left(1 + \int_\Omega u^{2_s^*\beta_m}\right)^{\frac{1}{2_s^*(\beta_m-1)}},$$

with $C_{m+1} := C_{\beta_{m+1}} = 4\,C\,\beta_{m+1}\,(1+|\Omega|)$. Therefore, defining

$$A_m := \left(1 + \int_\Omega u^{2_s^*\beta_m}\right)^{\frac{1}{2_s^*(\beta_m-1)}}, \quad m \geq 1,$$

by the Claim proved before, and using a limiting argument, we conclude that there exists $C_0 > 0$, independent of $m > 1$, such that

$$A_{m+1} \leq \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \leq C_0 A_1. \tag{2.4.5}$$

This implies that

$$\|u\|_{L^\infty(\Omega)} \leq C_0 A_1. \tag{2.4.6}$$

Indeed, suppose that (2.4.6) is not true, that is, there exits

$$M > C_0 A_1 \quad \text{ such that } |\{u > M\}| \neq 0. \tag{2.4.7}$$

Then, by (2.4.5), for every $m \geq 1$, we obtain that

$$\left(M^{2_s^* \beta_m} |\{u > M\}|\right)^{\frac{1}{2_s^*(\beta_m - 1)}} \leq C_0 A_1,$$

that is

$$M^{\frac{\beta_m}{\beta_m - 1}} |\{u > M\}|^{\frac{1}{2_s^*(\beta_m - 1)}} \leq C_0 A_1.$$

Taking the limit when $m \to \infty$ this implies that $M \leq C_0 A_1$ which is a contradiction with (2.4.7), so (2.4.6) is proved. $\square$

Chapter 3

Some remarks on the solvability of non local elliptic problems with the Hardy potential.

3.1 Introduction, preliminaries and functional settings.

During the last twenty years or so a great effort has been devoted to understanding the role of the Hardy-Leray potential in the solvability of elliptic and parabolic problems in, both, the linear and nonlinear settings. The evolution of the research on elliptic problems involving the *inverse square potential* can be found in the references, [39, 41, 80, 93, 94] as well as in [2–6], among others.

The Hardy potential appears as a pure analytical subject in one dimension. To the best of our knowledge, the first time in which the Hardy-Leray potential arises in dimension $N \geq 3$ is in the seminal paper by J. Leray about the Navier-Stokes equations, [116]. From the point of view of the applications, the *inverse square potential* appears for instance as a borderline case in Quantum Mechanics, in some elliptic problems with supercritical reaction terms that are models in Combustion Theory or in Astrophysics (see [65, 125, 128]).

As is well known the classical Hardy-Leray inequality, states

$$\Lambda_N \int_{\mathbb{R}^N} \frac{\phi^2(x)}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{3.1.1}$$

where $\Lambda_N = \left(\frac{N-2}{2}\right)^2$ is the optimal constant. This constant is never attained in the sense that equality in (3.1.1) only happens for the trivial case $\phi \equiv 0$.

Notice that $|x|^{-2} \in L^p_{loc}(\mathbb{R}^N)$ for all $p < \frac{N}{2}$, in fact the inverse square potential is in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{2}}(\mathbb{R}^N)$. This is the analytical motivation to the peculiar behavior of the Hardy potential in its interaction with the differential operators. A classical extension of the Hardy inequality (3.1.1) in terms of the Fourier transform can be written as follows,

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx \le \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{u}^2 d\xi, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^N), \tag{3.1.2}$$

where $0 < s < 1$, the Fourier's transform is given in (0.0.19) and $\Lambda_{N,s}$ was given in (0.0.30). See [25, 90, 155]. Taking into account the behavior of the Fourier transform with respect to the homogeneity (see (0.0.20)), the *Heisenberg uncertainty principle* can be related to the Hardy Sobolev inequality for the pseudodifferential operator $(-\Delta)^{s/2}$, $0 < s < 1$, in the form announced in Theorem 0.0.3. That is,

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx \le ||(-\Delta)^{s/2} u||^2_{L^2(\mathbb{R}^N)}, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{3.1.3}$$

In a more general setting, beyond the Hilbertian framework, we find the references [89, 91] where an improved inequality is also proved. Notice that

$$\Lambda_{N,s} \to \left(\frac{N-2}{2}\right)^2 \text{ as } s \to 1.$$

As we have commented in the introduction of this work, inspired by the papers [41], [80] and [85] our objective here is to study the interplay between the Hardy potential, the solvability of the Dirichlet problem for the nonlocal operator $(-\Delta)^s$ and a concave term. In particular, we will analyze the existence of non trivial solutions for the following problem

$$(H_{\lambda,\mu}) = \begin{cases} (-\Delta)^s u - \lambda \dfrac{u}{|x|^{2s}} = \mu u^q + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where, as in Chapter 2, $\Omega \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain of $\mathbb{R}^N$ satisfying the exterior ball condition. We also consider $0 \in \Omega$, $0 < s < 1$, $N > 2s$, $\mu > 0$, $0 < q < 1$ and $\lambda < \Lambda_{N,s}$. Here $p > 1$ and is smaller than an upper bound that we will explain right after. These hypotheses will remain and will not be specified again in what follows. As in Chapter 2, the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ (not only in $\partial\Omega$) is necessary due to the non local character of the operator.

Recently M. M. Fall, in [85], has extended for the non local case some results given by Brezis-Dupaigne-Tesei in [41]. He analyzes in detail the threshold of the power p to have solvability when $\mu = 0$ and studies the Dirichlet-Neumann operator associated to the problem using the extension given by L. Caffarelli and L. Silvestre in [55]. We will study directly the problem $(H_{\lambda,\mu})$ without appealing to the harmonic extension.

To establish the upper bound for p we follow the ideas of [41] developed in [85] in the non local setting. We look for a radial solution to the problem

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = w^p.$$

In particular, if we choose $w = A|x|^{\frac{2s-N}{2}+\beta}$, with A a positive constant, we have

$$A\gamma_\beta |x|^{-2s+\frac{2s-N}{2}+\beta} - \lambda A|x|^{-2s+\frac{2s-N}{2}+\beta} = A^p|x|^{(\frac{2s-N}{2}+\beta)p},$$

where

$$\gamma_\beta := \frac{2^{2s}\Gamma(\frac{N+2s+2\beta}{4})\Gamma(\frac{N+2s-2\beta}{4})}{\Gamma(\frac{N-2s-2\beta}{4})\Gamma(\frac{N-2s+2\beta}{4})}.$$

Hence, in order to have homogeneity, we need that

$$\frac{2s-N}{2} + \beta = \frac{-2s}{p-1},$$

and therefore, the equation becomes

$$\gamma_\beta - \lambda = A^{p-1}.$$

Since $A > 0$, we need $\gamma_\beta - \lambda > 0$. Indeed, since the map

$$\begin{array}{rcl} \gamma : [0, \frac{N-2s}{2}) & \mapsto & (0, \Lambda_{N,s}] \\ \beta & \mapsto & \gamma_\beta \end{array}$$

is decreasing, see [74], there exists an unique α_λ such that $\gamma_{\alpha_\lambda} = \lambda$. Thus $\gamma_\beta - \lambda > 0$ is equivalent to $\alpha_\lambda > \beta$. Therefore we choose

$$\alpha_\lambda > \frac{-2s}{p-1} + \frac{N-2s}{2},$$

or equivalently,

$$p < \frac{N+2s-2\alpha_\lambda}{N-2s-2\alpha_\lambda} := p(\lambda, s). \tag{3.1.4}$$

Then, for $p < p(\lambda, s)$ we will be able to construct a radial supersolution for the Dirichlet problem $(H_{\lambda,\mu})$, just modifying the w found above. Hence this bound for p will be the threshold for the existence also for the Dirichlet problem.

In this chapter we follow the same notation as in Chapter 2. That is we will consider the Hilbert space

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm given in (2.1.2).

We point out here that, by (2.1.5), if $0 \in \Omega$, we can rewrite the Hardy inequality (3.1.3) as

$$\frac{C(N,s)}{2} \int_Q \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \geq \Lambda_{N,s} \int_\Omega \frac{u^2}{|x|^{2s}}\, dx,\ u \in X_0^s(\Omega), \tag{3.1.5}$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega)$. By scaling we can prove that the optimal constant is independent of the domain containing the origin.

Also as in the previous chapter, in order to describe correctly the energy formulation of the problem $(H_{\lambda,\mu})$, we will work in all this chapter with the problem

$$(H^+_{\lambda,\mu}) = \begin{cases} (-\Delta)^s u - \lambda \dfrac{u_+}{|x|^{2s}} = \mu u_+^q + u_+^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

By Lemma 2.2.2 and the Maximum Principle given in [151, Proposition 2.2.8], as occurs in Chapter 2, the solutions of the previous problem are strictly positive in Ω and so satisfy also the problem $(H_{\lambda,\mu})$. Taking into account (2.1.6), we introduce the following.

Definition 3.1.1. *Let $1 < p \leq 2_s^* - 1$. We say that $u \in X_0^s(\Omega)$ is an* energy *solution of $(H^+_{\lambda,\mu})$ if, for every $\varphi \in X_0^s(\Omega)$, the following condition hold:*

$$\begin{aligned} &\frac{1}{2} C(N,s) \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\, dx\, dy - \lambda \int_\Omega \frac{u_+\varphi}{|x|^{2s}}\, dx \\ = \ &\mu \int_\Omega u_+^q \varphi\, dx + \int_\Omega u_+^p \varphi\, dx. \end{aligned} \tag{3.1.6}$$

As usual, if considering positive test functions, the previous equality is satisfied for with $\geq$ (respectively $\leq$) and $u \geq 0$ (respectively $\leq$) in $\mathbb{R}^N \setminus \Omega$, we say that u is a supersolution (respectively subsolution) of $(H^+_{\lambda,\mu})$.

If $1 < p \leq 2_s^* - 1$ the problem $(H^+_{\lambda,\mu})$ is variational in nature, i.e., solutions in the sense of Definition 3.1.1 correspond to critical points of the functional

$\mathcal{F}_{s,\lambda,\mu} : X_0^s(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}\mathcal{F}_{s,\lambda,\mu}(u) &:= \frac{1}{4}C(N,s)\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &- \frac{\lambda}{2}\int_\Omega \frac{(u_+)^2}{|x|^{2s}}\,dx - \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx - \frac{1}{p+1}\int_\Omega u_+^{p+1}\,dx.\end{aligned}$$

That is, the problem $(H^+_{\lambda,\mu})$ represents the Euler-Lagrange equation associated to the functional $\mathcal{F}_{s,\lambda,\mu}$. Observe that, since $p \le 2_s^* - 1$, $u_+^p \in L^{\frac{2N}{N+2s}}(\Omega)$, so, in particular, the right hand side in (3.1.6) is finite.

If $p > 2_s^* - 1$, we are concerned with a supercritical problem and there is no possible variational formulation. In this supercritical context we need the following definition of solution.

Definition 3.1.2. *We say that $u \in L^1(\Omega)$ is a* weak *solution of $(H_{\lambda,\mu})$ if $u \ge 0$ a.e., $u = 0$ in $\mathbb{R}^N \setminus \Omega$, and satisfies*

$$\int_\Omega \left(\lambda\frac{u}{|x|^{2s}} + u^p + \mu u^q\right)\delta^s\,dx < \infty \tag{3.1.7}$$

and

$$\int_{\mathbb{R}^N} u(-\Delta)^s\varphi\,dx = \int_\Omega \left(\lambda\frac{u}{|x|^{2s}} + u^p + \mu u^q\right)\varphi\,dx,$$

for all $\varphi \in \mathcal{C}^{2s+\beta}(\Omega) \cap \mathcal{C}^s(\overline{\Omega})$, $\beta > 0$, with $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$ and $\delta(x) :=$ dist$(x, \partial\Omega)$.

Note that the right hand side makes sense because, since $\varphi \in \mathcal{C}^s(\overline{\Omega})$, it follows that $|\varphi(x)| \le C\delta^s(x)$.

We remark here that if $\mu = 0$ and $p < 2_s^* - 1$ is possible to find a variational solution using the classical Mountain Pass Lemma (MPL) introduced by A. Ambrosetti and P. Rabinowitz in [11] (see Section 3.3). However if $p \ge 2_s^* - 1$, $\mu = 0$ and Ω is a star shaped domain, the only solution in $X_0^s(\Omega)$ is the trivial one. This result follows by an argument of Pohozaev type (see [86, Corollary 1.3]). As in the previous chapters, this fact motivates the term u^q, $q < 1$ in this chapter.

In Section 3.2 we deal with the existence of at least one solution for the whole range $1 < p < p(\lambda, s)$, where $p(\lambda, s)$ is the threshold to have existence of a positive radial weak solution of the equation in the whole $\mathbb{R}^N$ (see (3.1.4)). Proceeding by a monotonicity argument, similar to the one used in the previous chapters, if $1 < p \le 2_s^* - 1$ we reach a finite energy solution, while if $2_s^* - 1 < p < p(\lambda, s)$ we find a weak solution.

In Section 3.3 we prove the existence of at least two energy solutions of $(H_{\lambda,\mu})$ in the subcritical case, that is, $1 < p < 2_s^* - 1$. To prove the existence of the second solution, in Subsection 3.3, we generalize an argument due to Alama (see [14] for the classical case, and [37] for the case of the spectral fractional Laplacian using the s-harmonic extension). This generalization is not immediate because, due to the nonlocal behavior of our operator, the bounded support of the test functions is not preserved.

In Section 3.4 we will study the non-existence of solution for the case $p \geq p(\lambda, s)$. In fact, using the non local version of [39, Lemma 3.2] (see Lemma 3.4.2), we will prove that the solutions to the truncated problems blow up for every x in Ω. In this section we also prove a fractional Picone's inequality.

3.2 Existence of minimal solutions for $1 < p < p(\lambda, s)$.

In this section we will prove the first three items of the following.

Theorem 3.2.1. *Let $0 < q < 1$ and $0 < \lambda < \Lambda_{N,s}$. Then, there exists $0 < \Upsilon < \infty$ such that the problem $(H_{\lambda,\mu})$*

1 has no solution for $\mu > \Upsilon$;

2 for any $0 < \mu < \Upsilon$, there exists a minimal energy solution if $1 < p \leq 2_s^* - 1$, a minimal weak solution in the case $2_s^* - 1 < p < p(\lambda, s)$ and, moreover, the family of minimal solutions is increasing with respect to μ;

3 if $\mu = \Upsilon$, there is at least one weak solution;

4 if $1 < p < 2_s^* - 1$, there are at least two energy solutions for $0 < \mu < \Upsilon$.

The fourth statement of the previous result will be proved in the next section.

To prove the assertions *1-3* of the previous theorem for the supercritical case, that is when $2_s^* - 1 < p < p(\lambda, s)$, we will need some regularity results that we present as follows.

Lemma 3.2.2 (Comparison principle for weak solutions). *Consider a non-negative function $f \in L^1(\Omega, \delta^s(x)\, dx)$. If $v \in L^1(\Omega)$ is a weak solution of the* *problem*

$$\begin{cases} (-\Delta)^s v = f \ \ in\ \Omega, \\ v \geq 0 \ \ in\ \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2.1}$$

then

$$v \geq 0 \ \ a.e.\ \ in\ \Omega. \tag{3.2.2}$$

Proof. We consider a nonnegative function $F \in \mathcal{C}_0^\infty(\Omega)$ and we define φ_F the solution of

$$\begin{cases} (-\Delta)^s \varphi_F = F \geq 0 \text{ in } \Omega, \\ \varphi_F = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By the Maximum Principle, [151, Proposition 2.2.8], $\varphi_F \geq 0$ in $\mathbb{R}^N$. Also [130, Proposition 1.1 and Proposition 1.4] imply that $\varphi_F \in \mathcal{C}^{2s+\beta}(\Omega) \cap \mathcal{C}^s(\overline{\Omega})$. Therefore considering φ_F as a test function in (3.2.1), we obtain that

$$\int_\Omega vF\,dx + \int_{\mathbb{R}^N \setminus \Omega} v(-\Delta)^s \varphi_F\,dx = \int_\Omega f\varphi_F\,dx \geq 0. \tag{3.2.3}$$

Since $\varphi_F = 0$ in $\mathbb{R}^N \setminus \Omega$ and $\varphi_F \geq 0$ in $\mathbb{R}^N$ then

$$(-\Delta)^s \varphi_F(x) \leq 0, \quad \text{ if } x \in \mathbb{R}^N \setminus \Omega. \tag{3.2.4}$$

Hence by (3.2.3), (3.2.4) and the fact that $v \geq 0$ in $\mathbb{R}^N \setminus \Omega$, we conclude that

$$\int_\Omega vF\,dx \geq 0.$$

Then (3.2.2) follows. □

Also we have the next.

Lemma 3.2.3. *Take* $f \in L^1(\Omega, \delta^s(x)\,dx)$*. Then there exists a unique* $v \in L^1(\Omega)$ *that is a weak solution of the problem*

$$(D_f) = \begin{cases} (-\Delta)^s v = f \ \ in\ \Omega, \\ v = 0 \ \ in\ \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover

$$\|v\|_{L^1(\Omega)} \leq C\|f\|_{L^1(\Omega, \delta^s(x)\,dx)}. \tag{3.2.5}$$

Proof. The proof follows closely the arguments given in [40, Lemma 1]. Let us assume $f \geq 0$ (otherwise we write $f = f_+ - f_-$). We define the nondecreasing family of functions

$$f_k(x) := \min\{k, f(x)\},\ k \in \mathbb{Z} \cup \{0\}.$$

By the Monotone Convergence Theorem we clearly get that

$$\int_\Omega f_k \delta^s(x)\, dx \to \int_\Omega f \delta^s(x)\, dx. \tag{3.2.6}$$

Let us now consider v_k the weak solution of

$$(D_{f_k}) = \begin{cases} (-\Delta)^s v_k = f_k \text{ in } \Omega, \\ v_k = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By Lemma 3.2.2 it follows that $\{v_k\}$ is also a monotone nondecreasing sequence of positive functions. Moreover considering $\xi_1 \in \mathcal{C}^{2s+\beta}(\Omega) \cap \mathcal{C}^s(\overline{\Omega})$ the solution to the linear problem

$$(D_{\xi_1}) = \begin{cases} (-\Delta)^s \xi_1 = 1 \text{ in } \Omega, \\ \xi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2.7}$$

we get that

$$\int_\Omega v_k - v_l\, dx = \int_\Omega (f_k - f_l)\xi_1\, dx \leq C \int_\Omega (f_k - f_l)\delta^s(x)\, dx,\ k \geq l. \tag{3.2.8}$$

Therefore by (3.2.6) and (3.2.8),

$$\{v_k\} \quad \text{is a Cauchy sequence in } L^1(\Omega). \tag{3.2.9}$$

Consequently there exists $v \in L^1(\Omega)$ such that

$$v_k \to v, \quad \text{in } L^1(\Omega) \text{ and } v_k \to v, \text{ a.e. } \Omega \text{ in a monotone fashion.}$$

By the Monotone Convergence Theorem, passing to the limit in (D_{f_k}) we obtain that v is a weak solution of (D_f), and (3.2.5) follows taking ξ_1 as a test function in (D_f).

To prove the uniqueness we consider $v_1 \leq v_2$ two weak solutions of (D_f) and we define $w := v_2 - v_1$. Following the proof of Lemma 3.2.2 we easily conclude that $w = 0$ a.e. Ω. □

Consider now $1 < p < p(\lambda, s)$. By Lemma 3.2.2 and Lemma 3.2.3, we can already prove the existence results of this section.

Proposition 3.2.4. *Let* Υ *be defined by*

$$\Upsilon = \sup\{\mu > 0 : \text{ problem } (H_{\lambda,\mu}) \text{ has a solution}\}. \tag{3.2.10}$$

Then $0 < \Upsilon < \infty$.

Proof. We want to construct a well ordered subsolution and supersolution to the problem $(H^+_{\lambda,\mu})$. For the first one, consider $\varphi_1 \in X_0^s(\Omega) \cap L^\infty(\Omega)$ the nonnegative solution of the eigenvalue problem (2.2.3). Hence, taking t small enough, we have that, in Ω,

$$(-\Delta)^s(t\varphi_1) = \rho_{1,(-\Delta)^s} t\varphi_1 \le \mu(t\varphi_1)^q \le \mu(t\varphi_1)^q + (t\varphi_1)^p + \lambda \frac{t\varphi_1}{|x|^{2s}}.$$

Thus, $0 \le \underline{u} := t\varphi_1$ is a subsolution of $(H^+_{\lambda,\mu})$. To build the supersolution, we need to deal with the subcritical and the supercritical case separately.

(i) Subcritical and critical cases: $1 < p \le 2^*_s - 1$.

We look for a supersolution of the form $w(x) := A|x|^{-\beta} \ge 0$ where $A > 0$ and β is a positive real parameter that satisfies

$$\beta < \frac{N - 2s}{2}.$$

Since $p \le 2^*_s - 1$, for this value of β one has

$$p\beta < \beta + 2s \tag{3.2.11}$$

and

$$\beta(p + 1) < N. \tag{3.2.12}$$

The condition (3.2.11) and an appropriate choice of A clearly imply that

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} \ge w^p, \quad \text{ in } \Omega. \tag{3.2.13}$$

Let $\varsigma := \inf_\Omega w > 0$. For μ small enough, taking $\overline{u} := C_1 w$ with $0 < C_1 < 1$ a suitable constant such that

$$\varsigma^{p-q} \ge \mu \frac{1}{C_1^{1-q}(1 - C_1^p)}, \tag{3.2.14}$$

it follows that

$$(-\Delta)^s \overline{u} - \lambda \frac{\overline{u}}{|x|^{2s}} \ge \overline{u}^p + \mu \overline{u}^q.$$

Thus, we have obtained a positive supersolution of $(H^+_{\lambda,\mu})$ for $1 < p \leq 2^*_s - 1$. Moreover, by (3.2.11) and (3.2.12) we also have

$$\overline{u} \in L^{p+1}(\Omega) \quad \text{and} \quad \frac{\overline{u}^2}{|x|^{2s}} \in L^1(\Omega). \tag{3.2.15}$$

Choosing the parameter t of $\underline{u} := t\varphi_1$ small enough, it yields $\overline{u} \geq \underline{u}$. Now, since $p \leq 2^*_s - 1$, we can build a nonnegative sequence $\{u_k\}$ in $X^s_0(\Omega)$ of solutions to the iterated problems

$$(H_k) = \begin{cases} (-\Delta)^s u_k = \lambda \frac{u_{k-1}}{|x|^{2s}} + \mu u^q_{k-1} + u^p_{k-1} & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

for $k \geq 1$ and $u_0 := \underline{u}$. By Lemma 2.2.2 it can be checked that

$$\underline{u} \leq u_1 \leq ... \leq \overline{u}.$$

Hence, we can define, up to a subsequence, $u_\mu := \lim_{k\to\infty} u_k$ in $L^1(\Omega)$. Moreover, by (3.2.15),

$$\begin{aligned} \|(-\Delta)^{s/2}u_k\|^2_{L^2(\mathbb{R}^N)} &= \lambda \int_\Omega \frac{u_k u_{k-1}}{|x|^{2s}}\,dx + \int_\Omega u_k u^p_{k-1}\,dx + \mu \int_\Omega u_k u^q_{k-1}\,dx \\ &\leq \lambda \int_\Omega \frac{\overline{u}^2}{|x|^{2s}}\,dx + \int_\Omega \overline{u}^{p+1}\,dx + \mu \int_\Omega \overline{u}^{q+1}\,dx \\ &\leq C. \end{aligned} \tag{3.2.16}$$

Therefore, up to a subsequence again, we know that $u_k \rightharpoonup u_\mu$ in $X^s_0(\Omega)$. Hence, since $p \leq 2^*_s - 1$, by the Dominated Convergence Theorem, we can pass to the limit in the iterated problems to conclude that $0 \leq u_\mu$ is a minimal energy solution of $(H^+_{\lambda,\mu})$ and, consequently, of $(H_{\lambda,\mu})$.

(ii) Supercritical case: $2^*_s - 1 < p < p(\lambda, s)$.

In this case, we follow the ideas of [80]. First, as we said in the Introduction, in [85] M. M. Fall builds a radial function $u(x) := A|x|^{\frac{-2s}{p-1}} \geq 0$, with A a positive constant, satisfying

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = u^p \text{ in } \mathbb{R}^N.$$

Since $p > 2^*_s - 1 > \dfrac{N}{N-2s}$,

$$u \in L^p_{loc}(\mathbb{R}^N), \quad \text{and} \quad \frac{u}{|x|^{2s}} \in L^1_{loc}(\mathbb{R}^N). \tag{3.2.17}$$

Taking $\overline{u} = C_1 u$, with $C_1 > 0$ a suitable constant (see (3.2.14)) we get that

$$\begin{cases} (-\Delta)^s \overline{u} - \lambda \frac{\overline{u}}{|x|^{2s}} \geq \mu \overline{u}^q + \overline{u}^p \text{ in } \Omega, \\ \overline{u} > 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover by (3.2.17), $\overline{u} \geq 0$ satisfies (3.1.7). Hence, by Lemma 3.2.3 we can define $\{u_k\}$ to be the weak solutions to (H_k), and we will prove by induction that, in fact,

$$u_k \in L^1(\mathbb{R}^N) \text{ and } 0 \leq \underline{u} \leq u_{k-1} \leq u_k \leq \overline{u} \text{ a.e. } \Omega, \text{ for every } k \in \mathbb{N}.$$

For $\underline{u}$ there is nothing to prove. Suppose the result true up to order $k-1$, that is,

$$\underline{u} \leq u_{j-1} \leq u_j \leq \overline{u} \quad \text{ for } j \leq k-1 \text{ a.e. in } \Omega.$$

Then

$$(-\Delta)^s u_k = \lambda \frac{u_{k-1}}{|x|^{2s}} + \mu u_{k-1}^q + u_{k-1}^p \leq \lambda \frac{\overline{u}}{|x|^{2s}} + \mu \overline{u}^q + \overline{u}^p.$$

Hence, by (3.2.17), since the right hand side in the previous inequality is in $L^1(\Omega)$, by Lemma 3.2.3 u_k will be in $L^1(\Omega)$ too. Moreover, by the induction hypothesis,

$$(-\Delta)^s (u_k - u_{k-1}) = \lambda \frac{(u_{k-1} - u_{k-2})}{|x|^{2s}} + \mu (u_{k-1}^q - u_{k-2}^q) + (u_{k-1}^p - u_{k-2}^p) \geq 0,$$

and

$$(-\Delta)^s (\overline{u} - u_k) = \lambda \frac{(\overline{u} - u_{k-1})}{|x|^{2s}} + (\overline{u}^p - u_{k-1}^p) + \mu (\overline{u}^q - u_{k-1}^q) \geq 0.$$

Therefore, by (3.2.2), we have that

$$0 \leq \underline{u} \leq u_{k-1} \leq u_k \leq \overline{u} \quad \text{a.e. } \Omega.$$

By a standard monotone convergence argument we conclude that $\{u_k\}$ converges in $L^1(\mathbb{R}^N)$ to a weak nonnegative solution u_μ of $(H_{\lambda,\mu})$ for $2_s^* - 1 < p < p(\lambda, s)$.

Observe that for μ small enough, we have built a minimal solution in both subcritical, critical and supercritical case. That is, we have proved that $\Upsilon > 0$.

To finish the proof we need to check that $\Upsilon < \infty$. Consider $1 < p \leq 2_s^* - 1$ and the eigenvalue problem with the Hardy potential given by

$$(D2) = \begin{cases} (-\Delta)^s \phi_1 - \lambda \frac{\phi_1}{|x|^{2s}} = \lambda_1 \phi_1 \text{ in } \Omega, \\ \phi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Note that, since $\lambda < \Lambda_{N,s}$ this problem is well defined and, following the same ideas as in the proof of [146, Lemma 9-b)], we also know that $\phi_1 \in X_0^s(\Omega)$. Suppose that u is a solution to problem $(H^+_{\lambda,\mu})$. Then, we know that $u > 0$ in Ω. Taking ϕ_1 as a test function in this problem we get that

$$\begin{aligned} &\frac{1}{2}C(N,s)\int_Q \frac{(\phi_1(x)-\phi_1(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{\phi_1 u}{|x|^{2s}}\,dx \\ = \;& \mu\int_\Omega u^q\phi_1\,dx + \int_\Omega u^p\phi_1\,dx. \end{aligned}$$

Using now that ϕ_1 is a solution of $(D2)$ it follows that

$$\int_\Omega (\mu u^q + u^p)\phi_1\,dx = \lambda_1\int_\Omega u\phi_1\,dx. \tag{3.2.18}$$

If $2^*_s - 1 < p < p(\lambda,s)$, we consider $\varphi_1 \geq 0$, solution to (2.2.3), as a test function in $(H^+_{\lambda,\mu})$. Then

$$\begin{aligned} \int_\Omega u(-\Delta)^s\varphi_1\,dx &= \int_\Omega \left(\lambda\frac{u}{|x|^{2s}} + \mu u^q + u^p\right)\varphi_1\,dx \\ &\geq \int_\Omega (\mu u^q + u^p)\,\varphi_1\,dx. \end{aligned} \tag{3.2.19}$$

Moreover, since φ_1 is also an energy bounded solution of the eigenvalue problem (2.2.3) that belongs to $\mathcal{C}^{2s+\beta}(\Omega)\cap\mathcal{C}^s(\overline{\Omega})$ (see [130, Proposition 1.1 and Proposition 1.4]), then φ_1 is also a classical solution. Hence from (3.2.19) we also get

$$\rho_{1,(-\Delta)^s}\int_\Omega u\varphi_1\,dx \geq \int_\Omega (u^p + \mu u^q)\,\varphi_1\,dx. \tag{3.2.20}$$

Since there exist structural positive constants c_0, c_1 such that

$$t^p + \mu t^q > c_0\mu^{c_1}t, \text{ for every } t > 0,$$

we obtain from (3.2.18) and (3.2.20) that $\Upsilon < \infty$ for $p < p(\lambda,s)$. □

Proposition 3.2.5. *Problem $(H_{\lambda,\mu})$ has at least one solution for every $0 < \mu < \Upsilon$. In fact, the sequence $\{u_\mu\}$ of minimal solutions is increasing with respect to μ. If $\mu = \Upsilon$ the problem $(H_{\lambda,\mu})$ admits at least one weak solution.*

Proof. Doing the same procedure as in the proof of Lemma 1.2.2, we conclude that there exists a solution u_μ for all $\mu \in (0,\overline{\mu})$, and therefore for all $\mu \in (0,\Upsilon)$. Moreover $u_\mu < u_{\overline{\mu}}$ if $\mu < \overline{\mu}$.

For the case $\mu = \Upsilon$, the idea, as in [80, Proposition 2.1], consists on passing to the limit when $\mu_n \nearrow \Upsilon$ on the sequence $\{u_n\} = \{u_{\mu_n}\} \geq 0$, where u_{μ_n} is the minimal solution of $(H^+_{\lambda,\mu})$ (and of $(H_{\lambda,\mu})$) with $\mu = \mu_n$. Consider the solution φ_1 to the eigenvalue problem $(D1)$, defined in (2.2.3), as a test function in $(H^+_{\lambda,\mu})$. Since $\operatorname{supp} u_n \subseteq \Omega$ we get that

$$\rho_{1,(-\Delta)^s} \int_\Omega u_n \varphi_1 \, dx = \lambda \int_\Omega \frac{u_n \varphi_1}{|x|^{2s}} \, dx + \mu \int_\Omega u_n^q \varphi_1 \, dx + \int_\Omega u_n^p \varphi_1 \, dx. \quad (3.2.21)$$

By Young's inequality, for $\varepsilon > 0$, it follows that

$$\rho_{1,(-\Delta)^s} \int_\Omega u_n \varphi_1 \, dx \leq \rho_{1,(-\Delta)^s} \left(\frac{\varepsilon}{p} \int_\Omega u_n^p \varphi_1 \, dx + \frac{p-1}{p\varepsilon^{\frac{1}{p-1}}} \int_\Omega \varphi_1 \, dx \right).$$

Then from (3.2.21) we obtain that

$$\begin{aligned} & \lambda \int_\Omega \frac{u_n \varphi_1}{|x|^{2s}} \, dx + \mu \int_\Omega u_n^q \varphi_1 \, dx + \frac{p - \varepsilon \rho_{1,(-\Delta)^s}}{p} \int_\Omega u_n^p \varphi_1 \, dx \\ \leq \; & \rho_{1,(-\Delta)^s} \frac{p-1}{p\varepsilon^{\frac{1}{p-1}}} \int_\Omega \varphi_1 \, dx \leq C, \end{aligned}$$

with $C > 0$ independent of n. Therefore, by Hopf's Lemma (see [54] or [130, Lemma 3.2]) we can conclude that

$$\lambda \int_\Omega \frac{u_n \delta^s}{|x|^{2s}} \, dx + \mu \int_\Omega u_n^q \delta^s \, dx + \int_\Omega u_n^p \delta^s \, dx \leq C, \quad (3.2.22)$$

where $\delta(x) := dist(x, \partial\Omega)$, $x \in \Omega$.

Using ξ_1, given in (3.2.7), as a test function of $(H^+_{\lambda,\mu})$, since $\operatorname{supp} u_n \subseteq \Omega$, by [130, Proposition 1.1] and (3.2.22), we have that

$$\begin{aligned} \int_\Omega u_n \, dx &= \lambda \int_\Omega \frac{u_n \xi_1}{|x|^{2s}} \, dx + \mu \int_\Omega u_n^q \xi_1 \, dx + \int_\Omega u_n^p \xi_1 \, dx \\ &\leq C \left(\lambda \int_\Omega \frac{u_n \delta^s}{|x|^{2s}} \, dx + \mu \int_\Omega u_n^q \delta^s \, dx + \int_\Omega u_n^p \delta^s \, dx \right) \\ &\leq C. \end{aligned}$$

Hence $\{u_n\}$ converges in $L^1(\Omega)$ to a limit $u_\Upsilon \geq 0$. Then, since $\{u_n\}$ is an increasing sequence that is uniformly bounded in $L^1(\Omega)$, by the Monotone Convergence Theorem, taking the limit when $n \to \infty$, we conclude that u_Υ is actually a weak solution of (H_Υ). □

Remark 3.2.6. *1. Propositions 3.2.4 and 3.2.5 can be easily reproduced in the case $q = 0$, that is, considering a function f with appropriate growth conditions instead of the concave term u^q. In particular this gives the same existence results given in [80] in the nonlocal framework.*

2. Note that the procedure done in Proposition 3.2.5 to obtain a solution in the extremal value $\mu = \Upsilon$ is different from the ones done in Lemma 1.2.2 and Lemma 2.2.3. This is because, following the ideas given in [10, Lemma 3.5 and Theorem 2.1], we cannot assert that $\mathcal{F}_{s,\lambda,\mu}(u_n) < 0$. This comes from the fact that we do not guarantee that the function $a(x) := \mu q u_\mu^{q-1} + p u_\mu^{p-1}$ is in $L^r(\Omega)$, for $r > N/2s$. Observe also that this is the reason why we obtain a weak solution in the extremal value. In fact, if we could prove that $\mathcal{F}_{s,\lambda,\mu}(u_n) < 0$, we would obtain that $\|u_n\|_{X_0^s(\Omega)} \leq C$ and this would imply that $u_\Upsilon \in X_0^s(\Omega)$.

3.3 Subcritical case: existence of at least two solutions. Variational techniques.

In this section we will consider $1 < p < 2_s^* - 1$. Since Propositions 3.2.4 and Proposition 3.2.5 prove the first three items of Theorem 3.2.1, we will prove in this section statement *4* of this theorem taking advantage of the variational structure of $(H_{\lambda,\mu})$.

We will use minimization to find the first solution, and the MPT to guarantee the existence of the second one. In order to use this last result, we need to check some conditions concerning to the geometry and the compactness of the functional. By Theorem 0.0.2, the Hardy inequality given in (3.1.5) and following the ideas given in the proof of Proposition 1.4.2, it is easy to check that, for μ small enough, the functional $\mathcal{F}_{s,\lambda,\mu}$ has the good geometry. That is, we have the following

Proposition 3.3.1. *There exist $\alpha > 0$ and $\beta > 0$ such that*

a) $\mathcal{F}_{s,\lambda,\mu}(u) \geq \beta$, for any $u \in X_0^s(\Omega)$ with $||u||_{X_0^s(\Omega)} = \alpha$ and μ small enough.

b) There exists $u_1 \in X_0^s(\Omega)$ positive such that $||u_1||_{X_0^s(\Omega)} > \alpha$ and $\mathcal{F}_{s,\lambda,\mu}(u_1) < \beta$.

Also we obtain that

$$\lim_{t\to 0^+} \mathcal{F}_{s,\lambda,\mu}(tu_0) = 0^-. \tag{3.3.1}$$

Now we need to check that the functional $\mathcal{F}_{s,\lambda,\mu}$ satisfies the (PS) condition. First we prove the following

Proposition 3.3.2. *Let $\{u_n\}$ be a bounded sequence in $X_0^s(\Omega)$ such that $\mathcal{F}'_{s,\lambda,\mu}(u_n) \to 0$ as $n \to \infty$. Then there exists $u \in X_0^s(\Omega)$ such that, up to a subsequence, $||u_n - u||_{X_0^s(\Omega)} \to 0$ when $n \to \infty$.*

Proof. Since $\{u_n\}$ is uniformly bounded in the Hilbert space $X_0^s(\Omega)$ with the norm $\|\cdot\|_{X_0^s(\Omega)}$, then there exists $C > 0$ such that

$$\|u_n\|_*^2 := \frac{C(N,s)}{2}\|u_n\|^2_{X_0^s(\Omega)} - \lambda \int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx \leq \frac{C(N,s)}{2}\|u_n\|^2_{X_0^s(\Omega)} \leq C.$$

Since, by (3.1.5), $\|\cdot\|_*^2$ is a norm equivalent to $\|\cdot\|^2_{X_0^s(\Omega)}$, we consider $X_0^s(\Omega)$ endowed with this norm and hence, up to a subsequence, there exists $u \in X_0^s(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } X_0^s(\Omega) \text{ with the norm } \|\cdot\|_*, \tag{3.3.2}$$

$$u_n \to u \text{ in } L^r(\mathbb{R}^N),\ 1 \leq r < 2_s^*,\ u_n \to u \text{ a.e. in } \mathbb{R}^N. \tag{3.3.3}$$

Therefore, since $\mathcal{F}'_{s,\lambda,\mu}(u_n) \to 0$ and $q+1 < p+1 < 2_s^*$, by (3.3.3) it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{C(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx\right)\\
=&\ \lim_{n\to\infty}\left(\mu\int_\Omega (u_n)_+^{q+1}\,dx + \int_\Omega (u_n)_+^{p+1}\,dx\right)\\
=&\ \mu\int_\Omega u_+^{q+1}\,dx + \int_\Omega u_+^{p+1}\,dx\\
=&\ f_{\lambda,\mu}(u).
\end{aligned} \tag{3.3.4}$$

Similarly we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{C(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{N+2s}}\,dx\,dy - \lambda\int_\Omega \frac{(u_n)_+u}{|x|^{2s}}\,dx\right)\\
=&\ f_{\lambda,\mu}(u).
\end{aligned} \tag{3.3.5}$$

Hence, from (3.3.2), (3.3.4) and (3.3.5) we get

$$\lim_{n\to\infty}\|u_n\|_*^2 = \|u\|_*^2.$$

Consequently, by (3.3.2), we conclude that $\lim\limits_{n\to\infty}||u_n - u||_*^2 = 0$. Finally, by the equivalence of norms, we conclude that

$$\lim_{n\to\infty}||u_n - u||^2_{X_0^s(\Omega)} = 0.$$

□

Now we can prove the PS condition. That is,

Proposition 3.3.3. *Let u_n be a sequence in $X_0^s(\Omega)$, and $c \in \mathbb{R}$ such that*

$$\mathcal{F}_{s,\lambda,\mu}(u_n) \to c, \tag{3.3.6}$$

$$\mathcal{F}'_{s,\lambda,\mu}(u_n) \to 0. \tag{3.3.7}$$

Then, up to a subsequence, there exists $u = \lim\limits_{n\to\infty} u_n$ in $X_0^s(\Omega)$.

Proof. By (3.3.6) and (3.3.7) it follows that

$$\mathcal{F}_{s,\lambda,\mu}(u_n) - \frac{1}{p+1}\langle \mathcal{F}'_{s,\lambda,\mu}(u_n), u_n\rangle = c + o(1).$$

Hence, since $p+1 > 2 > q+1$, by the Sobolev embedding theorem given in Theorem 0.0.2 and (3.1.5), we obtain

$$\begin{aligned} c + o(1) &= \frac{C(N,s)}{2}\left(\frac{1}{2} - \frac{1}{p+1}\right)||u_n||^2_{X_0^s(\Omega)} - \lambda\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \frac{(u_n)_+^2}{|x|^{2s}}\,dx \\ &- \mu\left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_\Omega (u_n)_+^{q+1}\,dx \\ &\geq C_1||u_n||^2_{X_0^s(\Omega)} - C_2||u_n||^{q+1}_{X_0^s(\Omega)}, \end{aligned}$$

with C_1 and C_2 positive constants. Therefore there exists $C > 0$ such that $||u_n||_{X_0^s(\Omega)} \leq C$. Applying the previous proposition we conclude the strong convergence in the space $X_0^s(\Omega)$. □

Now we can state the following existence theorem.

Theorem 3.3.4. *For μ small enough, the problem $(H_{\lambda,\mu})$ has at least two solutions.*

Proof. We construct the first one by minimization. As we saw in Proposition 3.3.1, there exists $\alpha > 0$ such that $\mathcal{F}_{s,\lambda,\mu}(u) \geq \beta > 0$ for all $u \in X_0^s(\Omega)$ with $\|u\|_{X_0^s(\Omega)} = \alpha$. Thus we can choose

$$\alpha_1 = \left\{\inf_{\alpha\in\mathbb{R}} \alpha : \mathcal{F}_{s,\lambda,\mu}(u) > 0 \text{ for all } u \in X_0^s(\Omega) \text{ with } \|u\|_{X_0^s(\Omega)} = \alpha\right\}.$$

We know that $\alpha_1 > 0$, because near the origin the functional is negative. We choose now $\alpha_2 > \alpha_1$ so close that $\mathcal{F}_{s,\lambda,\mu}(u)$ is non decreasing for u with $\alpha_1 \leq ||u||_{X_0^s(\Omega)} \leq \alpha_2$. We define now a smooth function τ as

$$\tau(t) := \begin{cases} 1, \ t \leq \alpha_1, \\ 0, \ t \geq \alpha_2, \end{cases}$$

and we consider the truncated functional

$$\begin{aligned}\overline{\mathcal{F}}_{s,\lambda,\mu}(u) &:= \frac{C(N,s)}{4}\|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|x|^{2s}}\,dx \\ &- \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx - \frac{\tau(\|u\|_{X_0^s(\Omega)})}{p+1}\int_\Omega u_+^{p+1}\,dx.\end{aligned}$$

By definition,

$$\overline{\mathcal{F}}_{s,\lambda,\mu}(u) = \mathcal{F}_{s,\lambda,\mu}(u) \quad \text{whenever} \quad \|u\|_{X_0^s(\Omega)} \leq \alpha_1$$

and

$$\overline{\mathcal{F}}_{s,\lambda,\mu}(u) = \frac{C(N,s)}{4}\|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{u_+^2}{|x|^{2s}}\,dx - \frac{\mu}{q+1}\int_\Omega u_+^{q+1}\,dx,$$

whenever $\|u\|_{X_0^s(\Omega)} \geq \alpha_2$. Note that, by Theorem 0.0.2 and (3.1.5), since $q+1<2$ the functional $\overline{\mathcal{F}}_{s,\lambda,\mu}$ is coercive. The lower semicontinuity is given because $X_0^s(\Omega)$ is a Hilbert space. Therefore we can assert that there exists a minimum u_0 of $\overline{\mathcal{F}}_{s,\lambda,\mu}$ with negative energy, that is also a minimum of $\mathcal{F}_{s,\lambda,\mu}$. Hence, we have already found the first solution to $(H^+_{\lambda,\mu})$ and, therefore, of $(H_{\lambda,\mu})$.

For the second one, as we have proved, for μ small enough the functional $\mathcal{F}_{s,\lambda,\mu}$ has the suitable geometry (Propositions 3.3.1 and (3.3.1)) and satisfies the PS condition (Proposition 3.3.3). If we consider

$$\Gamma := \{\gamma \in \mathcal{C}^0([0,1], X_0^s(\Omega)) : \ \gamma(0) = u_0,\ \gamma(1) = u_1\},$$

where u_1 is the point found in Proposition 3.3.1, and

$$C := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} \mathcal{F}_{s,\lambda,\mu}(\gamma(t)),$$

the MPT in [11] gives us a solution $u \in X_0^s(\Omega)$ satisfying

$$\mathcal{F}_{s,\lambda,\mu}(u) = C \geq \beta > 0.$$

Here β is specified in Proposition 3.3.1. Note that this solution and the one obtained before are different because the previous one had negative energy. Therefore, for μ small enough, problem $(H^+_{\lambda,\mu})$ and consequently, $(H_{\lambda,\mu})$ has at least two solutions. □

Now we want to see that in fact our problem $(H^+_{\lambda,\mu})$ has two solutions for every $\mu \in (0, M)$. As we said in the introduction of this chapter, with this purpose we will generalize a result by Alama to check that the minimal solution obtained in Proposition 3.2.4 is a local minimum. This will allow us to apply the MPT. Due to the difficult computations involved, this proof is presented in detail. Before to prove the following result, we clarify that, in order to avoid cumbersome notation, along the next proof we use the symbol of positive or negative part of a function either as subscript or superscript.

Theorem 3.3.5. *Let $1 < p < 2_s^* - 1$. Then for $0 < \mu < \Upsilon$, where Υ is defined in* (3.2.10)*, the problem* $(H_{\lambda,\mu})$ *has at least two energy solutions.*

Proof. Let $\mu_0 \in (0, \Upsilon)$ and take μ_1 such that $\mu_0 < \mu_1 < \Upsilon$. Then, by Proposition 3.2.5, we can consider w_{μ_0} and w_{μ_1} the minimal solutions to the problems (H_{λ,μ_0}) and (H_{λ,μ_1}) respectively, which satisfy $w_{\mu_0} < w_{\mu_1}$. Now we define

$$W = \{w \in X_0^s(\Omega) : 0 \leq w \leq w_{\mu_1}\}.$$

Since W is a closed convex set of $X_0^s(\Omega)$, we know that $\mathcal{F}_{s,\lambda,\mu_0}$ is bounded from below and semicontinuous in W, and hence there exists $\underline{w} \in W$ such that $\mathcal{F}_{s,\lambda,\mu_0}(\underline{w}) = \inf_{w \in W} \mathcal{F}_{s,\lambda,\mu_0}(w)$. Let $w_0 \in X_0^s(\Omega)$ be a positive solution to

$$\begin{cases} (-\Delta)^s w_0 - \lambda \frac{w_0}{|x|^{2s}} = \mu_0 w_0^q & \text{in } \Omega, \\ w_0 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Note that, since $\lambda < \Lambda_{N,s}$, the existence is given by minimization. Then, for $0 < \varepsilon << \mu_0$, we have that $\mathcal{F}_{s,\lambda,\mu_0}(\varepsilon w_0) < 0$ because the term with power $q+1$ dominates over the quadratic terms. Taking ε small enough, since $\varepsilon w_0 \in W$, we get that $\underline{w} \neq 0$ and $\mathcal{F}_{s,\lambda,\mu_0}(\underline{w}) < 0$. Following the idea of the proof of [158, Theorem 2.4], adapted to the nonlocal framework, we also have that $\underline{w}$ is a solution to the problem (H_{λ,μ_0}).

Hence, we have two possible cases. If $\underline{w} \neq w_{\mu_0}$, then we have finished because we have found two different solutions of (H_{λ,μ_0}). Otherwise, if $\underline{w} = w_{\mu_0}$ and we prove that $\underline{w}$ is a local minimum of $\mathcal{F}_{s,\lambda,\mu_0}$, then we obtain a second solution applying the MPT given in [11], or its refinement given in [95] which is a contradiction. Therefore $\underline{w}$ is not the unique solution of $(H_{\lambda,\mu})$, and the conclusion follows.

Therefore our goal now is to prove that $\underline{w}$ is a local minimum of the functional $\mathcal{F}_{s,\lambda,\mu_0}$.

Let us argue by contradiction. Suppose $\underline{w}$ is not a local minimum of $\mathcal{F}_{s,\lambda,\mu_0}$ in the space $X_0^s(\Omega)$. Then there exists a sequence $\{v_n\} \subseteq X_0^s(\Omega)$ such that

$$\|v_n - \underline{w}\|_{X_0^s(\Omega)} \to 0 \quad \text{and} \quad \mathcal{F}_{s,\lambda,\mu_0}(v_n) < \mathcal{F}_{s,\lambda,\mu_0}(\underline{w}). \tag{3.3.8}$$

Let w_{μ_1} be the minimal solution associated to μ_1. Define

$$w_n := (v_n - w_{\mu_1})^+$$

and

$$0 \le z_n(x) := \begin{cases} 0, \ v_n(x) \le 0, \\ v_n(x), \ 0 \le v_n(x) \le w_{\mu_1}(x), \\ w_{\mu_1}(x), \ w_{\mu_1}(x) \le v_n(x). \end{cases}$$

Note that w_n and z_n belong to the energy space $X_0^s(\Omega)$. Consider now the sets given by

$$T_n := \{x : \ z_n(x) = v_n(x)\}, \quad \tilde{T}_n = T_n \cap \Omega$$

and

$$S_n := \{x : \ v_n(x) \ge w_{\mu_1}(x)\}, \quad \tilde{S}_n = S_n \cap \Omega.$$

Note that

$$z_n(x) = w_{\mu_1}(x), \, x \in S_n \tag{3.3.9}$$

$$z_n(x) = v_n^+(x), \, x \in S_n^c := \mathbb{R}^N \setminus S_n. \tag{3.3.10}$$

Define now

$$F_{\mu_0}(t) = \frac{1}{p+1} t_+^{p+1} + \frac{\mu_0}{q+1} t_+^{q+1}. \tag{3.3.11}$$

Thus,

$$\int_\Omega F_{\mu_0}(v_n) = \int_{\tilde{T}_n} F_{\mu_0}(v_n) + \int_{\tilde{S}_n} F_{\mu_0}(v_n) = \int_{\tilde{T}_n} F_{\mu_0}(z_n) + \int_{\tilde{S}_n} F_{\mu_0}(v_n).$$

By simplicity, let's denote

$$V_n(x,y) := \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}}, \quad V_n^+(x,y) := \frac{(v_n^+(x) - v_n^+(y))^2}{|x-y|^{N+2s}},$$

$$V_n^-(x,y) := \frac{(v_n^-(x) - v_n^-(y))^2}{|x-y|^{N+2s}}, \quad Z_n(x,y) := \frac{(z_n(x) - z_n(y))^2}{|x-y|^{N+2s}},$$

and

$$W_n(x,y) := \frac{(w_n(x) - w_n(y))^2}{|x-y|^{N+2s}}.$$

Hence, we have

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} V_n(x,y)\,dx\,dy &= \int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^+(x,y)\,dx\,dy + \int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^-(x,y)\,dx\,dy\\ &+ 2\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v_n^+(x)-v_n^+(y))(-v_n^-(x)+v_n^-(y))}{|x-y|^{N+2s}}\,dx\,dy\\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^+(x,y)\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} V_n^-(x,y)\,dx\,dy\\ &+ 4\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{v_n^+(x)v_n^-(y)}{|x-y|^{N+2s}}\,dx\,dy. \qquad (3.3.12)\end{aligned}$$

Also it is clear that

$$\int_\Omega \frac{v_n^2}{|x|^{2s}}dx = \int_\Omega \frac{(v_n^+)^2}{|x|^{2s}}\,dx + \int_\Omega \frac{(v_n^-)^2}{|x|^{2s}}\,dx. \qquad (3.3.13)$$

Then from (3.3.11), (3.3.12) and (3.3.13) it follows

$$\begin{aligned}\mathcal{F}_{s,\lambda,\mu_0}(v_n) &\geq \frac{C(N,s)}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N} V_n(x,y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega \frac{v_n^2}{|x|^{2s}}\,dx - \int_\Omega F_{\mu_0}(v_n)dx\\ &\geq \frac{C(N,s)}{4}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^+(x,y)\,dx\,dy + \int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^-(x,y)\,dx\,dy\right)\\ &- \frac{\lambda}{2}\int_\Omega \frac{(v_n^+)^2}{|x|^{2s}}\,dx - \frac{\lambda}{2}\int_\Omega \frac{(v_n^-)^2}{|x|^{2s}}\,dx\\ &- \int_{\tilde{T}_n} F_{\mu_0}(z_n)\,dx - \int_{\tilde{S}_n} F_{\mu_0}(v_n)\,dx. \qquad (3.3.14)\end{aligned}$$

From (3.3.10) we get

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^+(x,y)\,dx\,dy &= \int_{S_n\times S_n} V_n^+(x,y)\,dx\,dy\\ &+ \int_{S_n^c\times S_n^c} Z_n(x,y)\,dx\,dy\\ &+ 2\int_{S_n}\int_{S_n^c} V_n^+(x,y)\,dx\,dy. \qquad (3.3.15)\end{aligned}$$

Also, since

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} Z_n(x,y)\,dx\,dy &= \int_{S_n\times S_n} Z_n(x,y)\,dx\,dy\\ &+ \int_{S_n^c\times S_n^c} Z_n(x,y)\,dx\,dy + 2\int_{S_n}\int_{S_n^c} Z_n(x,y)\,dx\,dy,\end{aligned}$$

we obtain from (3.3.15)

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} V_n^+(x,y)\,dx\,dy &= \int_{S_n\times S_n} V_n^+(x,y)\,dx\,dy + 2\int_{S_n}\int_{S_n^c} V_n^+(x,y)\,dx\,dy\\ &+ \int_{\mathbb{R}^N\times\mathbb{R}^N} Z_n(x,y)\,dx\,dy - 2\int_{S_n}\int_{S_n^c} Z_n(x,y)\,dx\,dy\\ &- \int_{S_n\times S_n} Z_n(x,y)\,dx\,dy. \qquad (3.3.16)\end{aligned}$$

Moreover, using the same argument,

$$\int_\Omega \frac{(v_n^+)^2}{|x|^{2s}}\,dx = \int_{\tilde{S_n}} \frac{(v_n^+)^2}{|x|^{2s}}\,dx + \int_\Omega \frac{z_n^2}{|x|^{2s}}\,dx - \int_{\tilde{S_n}} \frac{z_n^2}{|x|^{2s}}\,dx. \qquad (3.3.17)$$

Therefore by (3.3.14), (3.3.16) and (3.3.17) we get that

$$\begin{aligned}\mathcal{F}_{s,\lambda,\mu_0}(v_n) &\geq \frac{C(N,s)}{4}\left(1-\frac{\lambda}{\Lambda_{N,s}}\right)\|v_n^-\|^2_{X_0^s(\Omega)} + \mathcal{F}_{s,\lambda,\mu_0}(z_n)\\ &+ \frac{C(N,s)}{4}\int_{S_n\times S_n}\frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ &+ \frac{C(N,s)}{2}\int_{S_n}\int_{S_n^c}\frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ &- \frac{\lambda}{2}\int_{\tilde{S_n}}\frac{(v_n^+)^2-z_n^2}{|x|^{2s}}\,dx\\ &+ \int_{\tilde{S_n}} F_{\mu_0}(z_n)-F_{\mu_0}(v_n)\,dx. \qquad (3.3.18)\end{aligned}$$

As $w_n(x) = v_n(x) - w_{\mu_1}(x)$ when $x \in S_n$, using (3.3.9) we obtain that

$$\begin{aligned}&\int_{S_n\times S_n}\frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ =& \int_{S_n\times S_n}\frac{(w_n(x)+w_{\mu_1}(x)-w_n(y)-w_{\mu_1}(y))^2-(w_{\mu_1}(x)-w_{\mu_1}(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ =& \int_{S_n\times S_n}\frac{(w_n(x)-w_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\\ +& \int_{S_n\times S_n}\frac{2(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy. \qquad (3.3.19)\end{aligned}$$

and

$$\begin{aligned}\int_{\tilde{S_n}}\frac{(v_n^+)^2-z_n^2}{|x|^{2s}}\,dx &= \int_{\tilde{S_n}}\frac{(w_n+w_{\mu_1})^2-w_{\mu_1}^2}{|x|^{2s}}\,dx\\ &= \int_{\tilde{S_n}}\frac{w_n^2+2w_nw_{\mu_1}}{|x|^{2s}}\,dx. \qquad (3.3.20)\end{aligned}$$

Also from (3.3.9) and (3.3.10) it follows that

$$\begin{aligned}
&\int_{S_n}\int_{S_n^c} \frac{(v_n^+(x)-v_n^+(y))^2-(z_n(x)-z_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
= &\int_{S_n}\int_{S_n^c} \frac{(v_n^+(x)-w_n(y)-w_{\mu_1}(y))^2-(v_n^+(x)-w_{\mu_1}(y))^2}{|x-y|^{N+2s}}\,dx\,dy \\
= &\int_{S_n}\int_{S_n^c} \frac{w_n^2(y)-2w_n(y)(v_n^+(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy. \qquad (3.3.21)
\end{aligned}$$

Furthermore, since $\operatorname{supp} w_n = S_n$, then

$$\begin{aligned}
\int_{\mathbb{R}^N\times\mathbb{R}^N} W_n(x,y)\,dx\,dy &= \int_{S_n\times S_n} W_n(x,y)\,dx\,dy \\
&+ 2\int_{S_n}\int_{S_n^c} \frac{w_n^2(y)}{|x-y|^{N+2s}}\,dx\,dy. \qquad (3.3.22)
\end{aligned}$$

Using (3.3.19), (3.3.20), (3.3.21) and (3.3.22), from (3.3.18) we get that

$$\begin{aligned}
\mathcal{F}_{s,\lambda,\mu_0}(v_n) \geq\ & \frac{C(N,s)}{4}\left(\left(1-\frac{\lambda}{\Lambda_{N,s}}\right)\|v_n^-\|^2_{X_0^s(\Omega)}+\|w_n\|^2_{X_0^s(\Omega)}\right)+\mathcal{F}_{s,\lambda,\mu_0}(z_n) \\
+\ & \frac{C(N,s)}{2}\int_{S_n\times S_n} \frac{(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
-\ & C(N,s)\int_{S_n}\int_{S_n^c} \frac{w_n(y)(v_n^+(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
-\ & \frac{\lambda}{2}\int_{\tilde{S}_n} \frac{w_n^2+2w_nw_{\mu_1}}{|x|^{2s}}\,dx \\
+\ & \int_{\tilde{S}_n} F_{\mu_0}(z_n)-F_{\mu_0}(v_n)\,dx. \qquad (3.3.23)
\end{aligned}$$

Since $v_n^+(x)\leq w_{\mu_1}(x)$, for $x\in S_n^c$, using that $\operatorname{supp} w_n = S_n$, from (3.3.23) it follows that

$$\begin{aligned}
\mathcal{F}_{s,\lambda,\mu_0}(v_n) \geq\ & \frac{C(N,s)}{4}\left(\left(1-\frac{\lambda}{\Lambda_{N,s}}\right)\|v_n^-\|^2_{X_0^s(\Omega)}+\|w_n\|^2_{X_0^s(\Omega)}\right)+\mathcal{F}_{s,\lambda,\mu_0}(z_n) \\
+\ & \frac{C(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy \\
-\ & \frac{\lambda}{2}\int_{\tilde{S}_n} \frac{w_n^2(x)+2w_nw_{\mu_1}(x)}{|x|^{2s}}\,dx \\
+\ & \int_{\tilde{S}_n} F_{\mu_0}(z_n)-F_{\mu_0}(v_n)\,dx. \qquad (3.3.24)
\end{aligned}$$

Since w_{μ_1} is a supersolution of (H_{λ,μ_0}), testing in that problem with the function w_n, we obtain that

$$\frac{C(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(w_n(x)-w_n(y))(w_{\mu_1}(x)-w_{\mu_1}(y))}{|x-y|^{N+2s}}\,dx\,dy-\lambda\int_\Omega\frac{w_n w_{\mu_1}}{|x|^{2s}}\,dx$$
$$\geq \int_{\tilde{S}_n} w_n F'_{\mu_0}(w_{\mu_1})\,dx. \tag{3.3.25}$$

Now, from (3.3.24), by (3.1.5) and (3.3.25) we have that

$$\begin{aligned}
\mathcal{F}_{s,\lambda,\mu_0}(v_n) &\geq \frac{C(N,s)}{4}\left(1-\frac{\lambda}{\Lambda_{N,s}}\right)\|v_n^-\|^2_{X_0^s(\Omega)}+\frac{C(N,s)}{4}\|w_n\|^2_{X_0^s(\Omega)}\\
&+\mathcal{F}_{s,\lambda,\mu_0}(z_n)-\frac{\lambda}{2}\int_{\tilde{S}_n}\frac{w_n^2}{|x|^{2s}}\,dx\\
&+\int_{\tilde{S}_n}(F_{\mu_0}(w_{\mu_1})-F_{\mu_0}(w_{\mu_1}+w_n)+w_nF'_{\lambda_0}(w_{\mu_1}))\,dx\\
&\geq \frac{C(N,s)}{4}\|w_n\|^2_{X_0^s(\Omega)}+\mathcal{F}_{s,\lambda,\mu_0}(z_n)-\frac{\lambda}{2}\int_{\tilde{S}_n}\frac{w_n^2}{|x|^{2s}}\,dx\\
&+\mu_0\int_{\tilde{S}_n}\frac{w_{\mu_1}^{q+1}-(w_{\mu_1}+w_n)^{q+1}}{q+1}+w_nw_{\mu_1}^q\,dx\\
&+\int_{\tilde{S}_n}\frac{w_{\mu_1}^{p+1}-(w_{\mu_1}+w_n)^{p+1}}{p+1}+w_nw_{\mu_1}^p\,dx.
\end{aligned}$$

Since $0<q+1<2$ it follows that

$$0\leq\frac{1}{q+1}\left((w_{\mu_1}+w_n)^{q+1}-w_{\mu_1}^{q+1}\right)-w_nw_{\mu_1}^q\leq\frac{q}{2}\frac{w_n^2}{w_{\mu_1}^{1-q}}. \tag{3.3.26}$$

On the other hand, using that w_{μ_1} is a solution of $(H_{\lambda,\,\mu_1})$, since

$$|w_n(x)-w_n(y)|^2\geq(w_{\mu_1}(x)-w_{\mu_1}(y))\left(\frac{w_n^2}{w_{\mu_1}}(x)-\frac{w_n^2}{w_{\mu_1}}(y)\right),$$

we obtain from (3.3.26), that

$$\begin{aligned}
\frac{C(N,s)}{2}\|w_n\|^2_{X_0^s(\Omega)} &\geq \int_\Omega\left(\lambda\frac{w_{\mu_1}}{|x|^{2s}}+\mu_1w_{\mu_1}^q\right)\frac{w_n^2}{w_{\mu_1}}\,dx\\
&\geq \lambda\int_\Omega\frac{w_n^2}{|x|^{2s}}\,dx+\mu_0\int_\Omega\frac{w_n^2}{w_{\mu_1}^{1-q}}\,dx.
\end{aligned}\tag{3.3.27}$$

Hence from (3.3.26) and (3.3.27) we get

$$\begin{aligned} &\frac{q}{2}\left(\frac{C(N,s)}{2}\|w_n\|^2_{X_0^s(\Omega)} - \lambda\int_\Omega \frac{w_n^2}{|x|^{2s}}\,dx\right) \\ \geq\ & \lambda_0 \int_{\tilde{S}_n} \frac{(w_{\mu_1}+w_n)^{q+1} - w_{\mu_1}^{q+1}}{q+1} - w_n w_{\mu_1}^q\,dx. \end{aligned} \tag{3.3.28}$$

Moreover, since $p+1>2$, in $\tilde{S}_n$, we have that

$$\begin{aligned} 0 &\leq \frac{1}{p+1}\left((w_{\mu_1}+w_n)^{p+1} - w_{\mu_1}^{p+1}\right) - w_{\mu_1}^p w_n \\ &\leq C(p)(w_{\mu_1}^{p-1}w_n^2 + w_n^{p+1}). \end{aligned} \tag{3.3.29}$$

Therefore, from (3.3.26), by (3.3.28) and (3.3.29) it follows that

$$\begin{aligned} \mathcal{F}_{s,\lambda,\mu_0}(v_n) &\geq (1-q)\left(\frac{1}{4}C(N,s)\|w_n\|^2_{X_0^s(\Omega)} - \frac{\lambda}{2}\int_\Omega \frac{w_n^2}{|x|^{2s}}\,dx\right) \\ &+ \mathcal{F}_{s,\lambda,\mu_0}(z_n) + \int_{\tilde{S}_n} C(p)(-w_{\mu_1}^{p-1}w_n^2 - w_n^{p+1})\,dx, \\ &\geq C_1\|w_n\|^2_{X_0^s(\Omega)} + \mathcal{F}_{s,\lambda,\mu_0}(z_n), \\ &+ \int_{\tilde{S}_n} C(p)(-w_{\mu_1}^{p-1}w_n^2 - w_n^{p+1})\,dx, \end{aligned} \tag{3.3.30}$$

where

$$C_1 = (1-q)\frac{C(N,s)}{4}\left(1-\frac{\lambda}{\Lambda_{N,s}}\right) > 0.$$

What remains to prove now is that

$$\lim_{n\to\infty} |\tilde{S}_n| = 0. \tag{3.3.31}$$

Let $\epsilon, \delta > 0$, and define

$$A_n = \{x\in\Omega:\ v_n(x)\geq w_{\mu_1}(x) \text{ and } w_{\mu_1} > \underline{w}+\delta\}$$

$$B_n = \{x\in\Omega:\ v_n(x)\geq w_{\mu_1}(x) \text{ and } w_{\mu_1} \leq \underline{w}+\delta\}.$$

Since

$$\begin{aligned} 0 &= |\{x\in\Omega: w_{\mu_1}(x) < \underline{w}\}| = \left|\bigcap_{j=1}^{\infty}\{x\in\Omega: w_{\mu_1}(x) < \underline{w}+\frac{1}{j}\}\right| \\ &= \lim_{j\to\infty}\left|\{x\in\Omega: w_{\mu_1}(x) < \underline{w}+\frac{1}{j}\}\right|, \end{aligned} \tag{3.3.32}$$

then, for j_0 large enough and $\delta < \frac{1}{j_0}$ we have

$$|\{x \in \Omega : w_{\mu_1}(x) < \underline{w} + \delta\}| \leq \frac{\epsilon}{2}.$$

Therefore $|B_n| \leq \frac{\epsilon}{2}$. Moreover, by (3.4.6),

$$\lim_{n\to\infty} \|v_n - \underline{w}\|_{X_0^s(\Omega)} = 0.$$

Then, by Theorem 0.0.2 and Hölder inequality, it follows that

$$\lim_{n\to\infty} \|v_n - \underline{w}\|_{L^2(\Omega)} = 0.$$

That is, for $n \geq n_0$ large enough we get that

$$\frac{\delta^2\varepsilon}{2} \geq \int_\Omega |v_n - \underline{w}|^2 \, dx \geq \int_{A_n} |v_n - \underline{w}|^2 \, dx \geq \delta^2 |A_n|.$$

Therefore $|A_n| \leq \dfrac{\varepsilon}{2}$, for $n \geq n_0$. Since $\widetilde{S}_n \subset B_n \cup A_n$ we conclude that $|\widetilde{S}_n| \leq \varepsilon$ for $n \leq n_0$. Hence (3.3.31) follows.

Since $p + 1 < 2_s^*$, by (3.3.31), Hölder inequality and Theorem 0.0.2, we obtain

$$\int_{\tilde{S}_n} w_n^{p+1} + w_{\mu_1}^{p-1} w_n^2 \, dx \leq o(1) \left(\|w_n\|^2_{X_0^s(\Omega)} + \|w_n\|^{p+1}_{X_0^s(\Omega)} \right).$$

From (3.3.30) we conclude that

$$\mathcal{F}_{s,\lambda,\mu_0}(v_n) \geq C_1 \|w_n\|^2_{X_0^s(\Omega)} + \mathcal{F}_{s,\lambda,\mu_0}(z_n) - o(1) \left(\|w_n\|^2_{X_0^s(\Omega)} + \|w_n\|^{p+1}_{X_0^s(\Omega)} \right).$$

Hence, for n large enough, since $z_n \in W$ and $\underline{w}$ was the infimum of $\mathcal{F}_{s,\lambda,\mu_0}$ over W, this implies

$$\mathcal{F}_{s,\lambda,\mu_0}(v_n) \geq \mathcal{F}_{s,\lambda,\mu_0}(z_n) \geq \mathcal{F}_{s,\lambda,\mu_0}(\underline{w}),$$

which is a contradiction with the hypothesis (3.3.8). Hence $\underline{w}$ is a minimum. □

3.4 Non-existence for $p \geq p(\lambda, s)$: complete blow up.

Let Ω be an open bounded set of $\mathbb{R}^N$. In [85, Theorem 0.2] it is proved, using an extension tool, that for $0 < \lambda \leq \Lambda_{N,s}$ and $p \geq p(\lambda, s)$, there does not exist a positive $u \in \mathcal{L}^s \cap L^p_{loc}(B \setminus \{0\})$ in any ball B centered at the origin such that

$$(-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \geq u^p \text{ in } \mathcal{D}'(B \setminus \{0\}),$$

where $\mathcal{L}^s$ is defined in (0.0.24). We remark here that this non existence result could be also obtained using a generalization of the Picone's identity (see Subsection 3.4.1 below) and [85, Lemma 3.2]. As a byproduct of the existence and non-existence results of the previous sections we are able to prove complete blow-up given by the following.

Definition 3.4.1. ([80, Definition 0.1]) *Let $\{a_n(x)\}$, $\{f_n(u)\}$ and $\{g_n(u)\}$ be increasing sequences of bounded smooth functions converging pointwise respectively to $|x|^{2s}$, u^q and u^p. Let u_n be the minimal nonnegative solution of*

$$\begin{cases} (-\Delta)^s u_n - \lambda a_n T_n(u_n) = \mu f_n(u_n) + g_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

where

$$T_n(x) := \begin{cases} x & \text{if } |x| \leq n, \\ n\frac{x}{|x|} & \text{if } |x| > n. \end{cases} \tag{3.4.1}$$

We say that there is a complete blow-up in $(H_{\lambda,\mu})$ if, given $\{a_n(x)\}$, $\{f_n(u)\}$, $\{g_n(u)\}$ and $\{u_n\}$ as above, then

$$\frac{u_n(x)}{\delta^s(x)} \to \infty \quad \text{uniformly in } \Omega,$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

We will see that the solutions u_n of the truncated problems considered in this chapter satisfy that $u_n(x) \to \infty$ when $n \to \infty$ and $x \in \Omega$. Before proving this result, that corresponds with Theorem 3.4.3, we need the following auxiliary lemma that is a generalization of [39, Lemma 3.2].

Lemma 3.4.2. *Let $F(x,u) \geq 0$ in $L^\infty(\Omega)$, and let u be the solution of*

$$(D_F) = \begin{cases} (-\Delta)^s u = F(x,u) \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then,

$$\frac{u(x)}{\delta^s(x)} \geq C \int_\Omega F(z,u)\delta^s(z)\,dz, \quad x \in \Omega, \tag{3.4.2}$$

where $\delta(z) = dist(z, \partial\Omega)$ and C is a constant depending only on Ω.

Proof. First of all we will prove (3.4.2) for points that belong to a, fix but arbitrary, compact set $K \subset \Omega$. Let $x_0 \in K$. There exits $r > 0$ such that $r \leq dist(x_0, \partial\Omega)$ for every $x_0 \in K$. Then by [151, Proposition 2.2.6 and Proposition 2.2.2], we have

$$u(x_0) \geq \int_{\mathbb{R}^N} u(z)\gamma_r(z - x_0)\, dz = \int_{\Omega} u(z)\gamma_r(z - x_0)\, dz > 0,\ x_0 \in K.$$

Here $\gamma_r = (-\Delta)^s \Gamma_r$ where Γ_r is a $\mathcal{C}^{1,1}$ function that matches outside the ball $B(0, r)$ with the fundamental solution $\Phi := C|x|^{2s-N}$ and that is a paraboloid inside this ball. Therefore there exist a positive constant $c > 0$ such that $u(x_0) > c$ for every $x_0 \in K$. That is

$$u(x_0) > M \int_{\Omega} u(z)\, dz, \quad x_0 \in K, \tag{3.4.3}$$

where

$$M = c\left(\int_{\Omega} u(z)\, dz\right)^{-1} > 0.$$

Consider now ξ_1, the solution of the problem (D_{ξ_1}) given in (3.2.7), as a test function in (D_F). By (3.4.3) we obtain

$$\begin{aligned} u(x_0) &\geq M \int_{\Omega} u(z)\, dz \\ &= M \int_{\Omega} F(z, u)\xi_1(z)\, dz, \quad x_0 \in K. \end{aligned}$$

Then, by Hopf's Lemma ([54] or [130, Lemma 3.2]),

$$u(x_0) \geq C \int_{\Omega} F(z, u)\delta^s(z)\, dz, \quad x_0 \in K.$$

Moreover, since $c_1 \leq \delta^s(x_0) \leq C_2$ for $x_0 \in K$, then there exists $\tilde{C} > 0$ such that

$$\frac{u(x_0)}{\delta^s(x_0)} \geq \tilde{C} \int_{\Omega} F(z, u)\delta^s(z) dz\,, \quad x_0 \in K. \tag{3.4.4}$$

Take now w satisfying

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } \Omega \setminus K, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ w = 1 & \text{in } K. \end{cases}$$

We define

$$v(x) = \frac{u(x)}{\tilde{C} \displaystyle\int_{\Omega} F(z, u)\delta^s(z)\, dz}, \quad x \in \mathbb{R}^N,$$

where $\tilde{C}$ was given in (3.4.4). Therefore

$$\begin{cases} (-\Delta)^s v \geq 0 & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ v \geq 1 & \text{in } K. \end{cases}$$

By the maximum principle, $v(x) \geq w(x)$ for $x \in \Omega \setminus K$. Then, since by Hopf's Lemma we get that $w(x) \geq C\delta^s(x)$, it follows that

$$\frac{u(x_0)}{\delta^s(x_0)} \geq \overline{C} \int_\Omega F(z,u)\delta^s(z)\,dz, \quad x_0 \in \Omega \setminus K, \tag{3.4.5}$$

for some $\overline{C} > 0$. Hence, by (3.4.4) and (3.4.5), we obtain the desired estimate given in (3.4.2). □

Now we can prove the following.

Theorem 3.4.3. *Assume that* $0 < \lambda \leq \Lambda_{N,s}$. *Let* $p \geq p(\lambda, s)$. *Then there exists complete blow up of the problem* $(H_{\lambda,\mu})$.

Proof. We argue by contradiction. Consider the positive minimal solution u_n to the truncated problem

$$(H_n) = \begin{cases} (-\Delta)^s u_n = \lambda a_n(x) T_n(u_n) + \mu f_n(u_n) + g_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,, \end{cases}$$

where

$$a_n(x) := T_n\left(\frac{1}{|x|^{2s}}\right), \quad f_n(u) := T_n(u_+^q), \quad g_n(u) := T_n(u_+^p)$$

and T_n is the truncated function defined in (3.4.1). Note that we can assume that this minimal solution exists because, since

$$\lambda a_n(x) T_n(u_n) + \mu f_n(u_n) + g_n(u_n) \leq C_n,$$

we can consider, for a suitable $c > 0$, the functions $\overline{u} := C_n \xi_1$ and $0 \leq \underline{u} := c\varphi_1$ as a well ordered super and subsolution of (H_n) respectively. Here φ_1 is the nonnegative first eigenfunction of the fractional Laplacian defined in (2.2.3) and ξ_1 is given in (3.2.7).

We suppose that

$$\int_\Omega (\lambda a_n(x) T_n(u_n) + \mu f_n(u_n) + g_n(u_n))\delta^s(x)dx \leq C < \infty,\ n \in \mathbb{N}, \tag{3.4.6}$$

with C independent of n. Using ξ_1 as a test function in problem (H_n), by [130, Proposition 1.1], we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} u_n &= \int_{\mathbb{R}^N} u_n(-\Delta)^s \xi_1 \\ &= \lambda \int_\Omega a_n T_n(u_n)\xi_1 + \mu \int_\Omega f_n(u_n)\xi_1 + \int_\Omega g_n(u_n)\xi_1 \\ &\leq C.\end{aligned}$$

Hence, up to a subsequence, $\{u_n\}$ converges in $L^1(\Omega)$ to a positive limit u. Then, since $\lambda a_n(x)T_n(u_n) + \mu f_n(u_n) + g_n(u_n)$ increases to $\lambda \dfrac{u}{|x|^{2s}} + \mu u^q + u^p$ in Ω, (3.4.6) also gives

$$a_n(x)T_n(u_n) + \mu f_n(u_n) + g_n(u_n) \nearrow \frac{u}{|x|^{2s}} + \mu u^q + u^p \quad \text{in } L^1(\Omega, \delta^s(x)\, dx),$$

again by monotone convergence. Then we can pass to the limit in (H_n) obtaining a positive weak solution of the problem

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = \mu u^q + u^p & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

But this is a contradiction with [85, Theorem 0.2], and therefore

$$\lim_{n\to\infty} \int_\Omega (\lambda a_n(x)T_n(u_n) + \mu f_n(u_n) + g_n(u_n))\delta^s(x)dx = \infty.$$

We conclude applying Lemma 3.4.2. □

3.4.1 Fractional Picone's inequality.

To finish this chapter, we present here an extension of a well-known inequality, that in the case of regular functions and the Laplacian operator was obtained by Picone in [129] (see also [3] and [8] for an extension to positive Radon measures and the p-Laplacian with $p > 1$).

Theorem 3.4.4. Picone Inequality. *Consider $u, v \in X_0^s(\Omega)$ with $u \geq 0$. Assume that $(-\Delta)^s u \geq 0$ restricted to Ω represents a positive Borel measure. Then,*

$$\int_\Omega \frac{(-\Delta)^s u}{u} v^2\, dx \leq \frac{C(N,s)}{2} \|v\|^2_{X_0^s(\Omega)}.$$

Proof. Let us recall first that, by (2.1.4) and (2.1.6), if $u, v \in X_0^s(\Omega)$, we have

$$\begin{aligned} \langle u, v\rangle_{X_0^s(\Omega)} &= \frac{2}{C(N,s)}\int_{\mathbb{R}^N} u(-\Delta)^s v = \frac{2}{C(N,s)}\int_{\mathbb{R}^N} v(-\Delta)^s u \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy. \end{aligned} \tag{3.4.7}$$

This means that, writing $A(u,v)(x,y) = (u(x)-u(y))(v(x)-v(y))$, then

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{A(u,v)(x,y)}{|x-y|^{N+2s}}\,dx\,dy = \langle u, v\rangle_{X_0^s(\Omega)}.$$

Define $v_k = T_k v$ given in (3.4.1) and $\widetilde{u} = u + \eta$, for $\eta > 0$. If we set $w = \dfrac{v_k^2}{\widetilde{u}}$, then one can easily check that $w \in X_0^s(\Omega)$. We will prove that, for every $k, \eta > 0$,

$$\frac{2}{C(N,s)}\int_\Omega (-\Delta)^s u \frac{v_k^2}{\widetilde{u}} \le \|v_k\|^2_{X_0^s(\Omega)}. \tag{3.4.8}$$

Note that, since $\|v_k\|^2_{X_0^s(\Omega)} \le \|v\|^2_{X_0^s(\Omega)}$, taking $k \to \infty$ and $\eta \to 0$ in the previous inequality we would obtain our result by monotone convergence.

From (3.4.7), prove (3.4.8) is equivalent to obtain that

$$\langle u, w\rangle_{X_0^s(\Omega)} \;\le\; \|v_k\|^2_{X_0^s(\Omega)}. \tag{3.4.9}$$

Inequality (3.4.9), in turns, follows from the trivial pointwise estimate

$$A(u,w)(x,y) \le A(v_k,v_k)(x,y), \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

which we now show. Observe that

$$\begin{aligned} A(u,w)(x,y) &= (u(x)-u(y))(w(x)-w(y)) \\ &= (\widetilde{u}(x)-\widetilde{u}(y))\left(\frac{v_k^2(x)}{\widetilde{u}(x)} - \frac{v_k^2(y)}{\widetilde{u}(y)}\right) \\ &= v_k^2(x) + v_k^2(y) - v_k^2(x)\frac{\widetilde{u}(y)}{\widetilde{u}(x)} - v_k^2(y)\frac{\widetilde{u}(x)}{\widetilde{u}(y)}. \end{aligned}$$

Then, putting $\alpha^2 = \frac{\widetilde{u}(y)}{\widetilde{u}(x)}$ and using that $2ab \le a^2 + b^2$, we get

$$\begin{aligned} A(u,w)(x,y) &= v_k^2(x) + v_k^2(y) - (v_k(x)\alpha)^2 - (v_k(y)/\alpha)^2 \\ &\le v_k^2(x) + v_k^2(y) - 2v_k(x)v_k(y) \\ &= A(v_k,v_k)(x,y). \end{aligned}$$

□

Part II

Regularity of non local minimal surfaces.

Chapter 4

Bootstrap regularity for integro-differential equations.

4.1 Introduction, preliminaries and functional settings.

The aim of this chapter is to prove a regularity result for some nonlocal linear elliptic equations. Recall that the model example for a linear elliptic equation is given by the Laplace equation.

$$\Delta u(x) = 0 \quad \text{in } \Omega.$$

Roughly speaking, we can say that elliptic equations are those which *have similar properties to the above equation.* The most natural, coordinate independent, definition of the Laplace operator may be

$$\Delta u(x) = \lim_{r \to 0} \frac{c}{r^{N+2}} \int_{B_r} (u(x+y) - u(x))\, dy, \quad x \in \Omega.$$

A simple, although rather uninteresting, example of a nonlocal equation would be the following non infinitesimal relationship

$$\int_{B_r} (u(x+y) - u(x))\, dy = 0, \quad x \in \Omega,$$

stated for a fixed $r > 0$.

The equation tells us that the value $u(x)$ is equal to the average of u in the ball $B_r(x)$. A more general integral equation is given by a weighted version of the above, that is,

$$\int_{\mathbb{R}^N}(u(x+y)-u(x))K(y)dy=0, \quad x\in\Omega,$$

where $K:\mathbb{R}^N\to\mathbb{R}$ is a non negative kernel. The equation shows that $u(x)$ is a weighted average of the values of u in the neighborhood of x. This is true in some sense for all elliptic equations, but it is most apparent for integro-differential ones.

As we know, (see Chapters 2 and 3), in the Dirichlet problem associated to this type of equations, the boundary values have to be prescribed in the whole complement of the domain and not only in the boundary. That is,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}(u(x+y)-u(x))K(y)\,dy=0, & x\in\Omega,\\ u(x)=g(x), \quad x\in\mathbb{R}^N\setminus\Omega. \end{cases}$$

Similar to the Laplace equation that comes from Brownian motion, integro-differential equations are derived from discontinuous stochastic processes, more precisely, from Levy processes with jumps. We remark here that a Levy process is an important type of stochastic processes, that is, a family of $\mathbb{R}^N$-valued random variables each indexed by a positive number $t\geq 0$. Roughly speaking a Levy proces is a random trajectory, that generalizes the concept of Brownian motion, and that may contain jump discontinuities. More precisely jumps from a point x to $x+y$, with $y\in A$ for some set $A\subseteq\mathbb{R}^N$, follow a Poisson process whose intensity is related with the Levy measure μ. In our case μ is given by

$$\mu(A)=\int_A K(y)\,dy, \quad A\subseteq\mathbb{R}^N.$$

The kernel K represents then the frequency of jumps in each direction. The small jumps may happen more often than large ones. In fact, small jumps may happen infinitely often and still have a well defined stochastic process. This means that the kernels K may have a singularity at the origin. The exact assumption one has to make is the standar Levy-Khintchine condition given by

$$\int_{\mathbb{R}^N}K(y)\min\{1,|y|^2\}\,dy<\infty.$$

Note that this guarantees that for $u\in\mathcal{C}^2$ near the origin and bounded at infinity, the operator makes sense.

These type of processes, that is, Levy processes, are well understood and studied in probability. The associated generating operator is given by

$$Iu(x) = \int_{\mathbb{R}^N} (u(x+y) - u(x) - y \cdot \nabla u(x)\chi_{B_1}(y))K(y)\,dy.$$

Assuming that $K(y) = K(-y)$, this expression simplifies to

$$Iu(x) = \text{P.V.} \int_{\mathbb{R}^N} (u(x+y) - u(x))K(y)\,dy,$$

or, equivalently, to

$$\begin{aligned} Iu(x) &= \frac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy \\ &:= \frac{1}{2}\int_{\mathbb{R}^N} \delta u(x,y)K(y)\,dy \end{aligned}$$

These equations are linear. However, an optimal control problem for jump processes leads to the non linear integro-differential Bellman equation

$$Iu(x) := \sup_{\alpha} \int_{\mathbb{R}^N} (u(x+y) - u(x))K^{\alpha}(y)\,dy = 0, \quad x \in \Omega.$$

Another possibility is to consider a problem with two parameters, which are controlled by two competitive players. This is the non linear integro-differential Isaacs equation.

$$Iu(x) := \inf_{\beta}\ \sup_{\alpha} \int_{\mathbb{R}^N} (u(x+y) - u(x))K^{\alpha\beta}(y)\,dy = 0, \quad x \in \Omega.$$

The difference of the two previous operators is the convexity and they have been recently studied in, for example, [56,57]. See also [150] for more details.

Other contexts in which integral equations arise are for example population dynamics, kinetic models ([124]), nonlocal electrostatics ([102, 105]), nonlocal image processing and fluid mechanics (see for instance [62,69,149]).

Following the classical case, a natural ellipticity condition for linear integro-differential operators would be to impose that the kernel is comparable to that of the fractional Laplacian. The condition could be

$$c(N,s)\frac{\lambda}{|y|^{N+s}} \le K(y) \le c(N,s)\frac{\Lambda}{|y|^{N+s}}, \quad \text{with } K(y) = K(-y).$$

But, proceeding as in the theory of fully non linear equations, other ellipticity conditions are possible (see [56] for a definition that involves the non local

Pucci maximal operators). Note that, with this assumptions in the kernel K, if $u \in \mathcal{C}^{1,1}(x) \cap \mathcal{L}^{s/2}(\mathbb{R}^N)$ then the integro-differential operator is well defined at x. When u is less regular we interpret the operator in a viscosity sense (see Definition 4.1.2 below). Here $\mathcal{C}^{1,1}(x)$ is a family of functions defined as follows.

Definition 4.1.1. *([56]) A function φ is said to be $\mathcal{C}^{1,1}$ at the point x_0, and we write $\varphi \in \mathcal{C}^{1,1}(x_0)$, if there are a vector $v \in \mathbb{R}^N$ and a positive number M such that*

$$|\varphi(x_0 + y) - \varphi(x_0) - v \cdot y| \le M|y|^2 \quad \text{for } |y| \text{ small enough.}$$

A function φ is $\mathcal{C}^{1,1}$ in a set Ω if $\varphi \in \mathcal{C}^{1,1}(x)$ for every $x \in \Omega$.

We also remark here that we can recover second order elliptic operators as limits of integral ones. In fact, given any bounded, even, positive function $a : \mathbb{R}^N \to \mathbb{R}$, the family of operators

$$L_s u(x) = (2-s) \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x)) \frac{a(y/|y|)}{|y|^{N+s}} dy, \quad 0 < s < 2,$$

defines in the limit $s \to 2^-$ a second order linear elliptic operator (possibly degenerate). This can be checked for any fixed $u \in \mathcal{C}^2$ by a straightforward computation using the second order Taylor expansion.

This fact has motivated the study of the non local version of some important regularity theorems such as the Krylov-Safonov's Theorem (KST), [112, 113], and Evans-Krylov's Theorem (EKT), [83, 111]. These results, adapted to integro-differential equations, involves the notion of viscosity solutions. Roughly speaking, the continuous function u satisfies $Iu \ge f$ in B_1 in the viscosity sense (u is subsolution) if the inequality holds at all points $y \in B_1$ where u admits a smooth tangent function by above. Similarly, it is possible to define the notion of supersolution. More precisely we have the following.

Definition 4.1.2. *([56–58]) A function $u : \mathbb{R}^N \to \mathbb{R}$, upper (lower) semi continuous in $\overline{\Omega}$, is a subsolution (supersolution) to $Iu = f$ in Ω in the viscosity sense, and we write $Iu \ge f$ ($Iu \le f$) in Ω, if every time we are in the situation that*

x_0 is a point in Ω,

$\mathcal{O}$ is a neighborhood of x_0 in Ω,

$\varphi : \mathbb{R}^N \to \mathbb{R}$ is some function such that $\varphi \in \mathcal{C}^{1,1}(x_0)$,

$\varphi(x_0) = u(x_0)$, *and*

$\varphi(y) > u(y)$ $(\varphi(y) < u(y))$ *for every* $y \in \mathcal{O} \setminus \{x_0\}$,

then $Iv(x_0) \geq f(x_0)$ $(Iv(x_0) \leq f(x_0))$, *with*

$$v(x) := \begin{cases} \varphi, & in \quad \mathcal{O}, \\ u, & in \quad \mathbb{R}^N \setminus \mathcal{O}. \end{cases}$$

A viscosity solution is a function u *for which both* $Iu \leq f$ *and* $Iu \geq f$ *hold in* Ω.

The idea of the definition is to translate the difficulty of pointwise evaluation of the operator I into a simple action on a smooth test function φ. In this way, the function u is only required to be continuous. More precisely, upper (lower) semicontinuous for the inequality $Iu \geq 0$ $(Iu \leq 0)$. The function φ is a test function "touching u from above (below)" at x_0. Aldo the previous definition focuses in the behavior around the point x_0, we are implicity assuming that u is integrable at infinity with respect to the weight determinated by the kernel of I.

As we mentioned before, using the previous notion of viscosity solution L. Caffarelli and L. Silvestre have recently proved in [56, Theorem 12.1], the non local version of the KST for fully nonlinear, elliptic and invariant by translations equations. Later the same authors extended the $\mathcal{C}^{1,\alpha}$ regularity to linear, and non linear, equations with variable coefficients ([57, Theorem 6.1 and Theorem 6.3]). Finally in [58, Theorem 1.1] they proved the non local version of the EKT for invariant convex non local fully non linear equations leaving open the case where the kernels have a dependence on x. Another type of regularity results for integro-differential equations can be found in [23].

Our objective in this chapter is to improve the recent result of [57] for linear equations that involves a special family of kernels not invariant under translations in order to, in the next chapter, prove the regularity theorem for the non local minimal surfaces (see Theorem 5.1.1). This is, in fact, the statement of Theorem 4.2.1.

4.2 Proof of the main result: regularity of the solutions.

As we said in the Introduction of this work, along this chapter we will consider a family of kernel that are not invariant by translation. More precisely they satisfy that

$$\begin{cases} \text{there exist } a_0, r_0 > 0 \text{ and } 0 < \eta < a_0/4 \text{ such that} \\ \left| \dfrac{|w|^{N+\sigma} K(x,w)}{2-\sigma} - a_0 \right| \le \eta, \qquad x \in B_1,\, w \in B_{r_0} \setminus \{0\}. \end{cases} \tag{4.2.1}$$

Also we assume that

$$\begin{cases} \text{there exist } k \in \mathbb{N} \cup \{0\} \text{ and } C_k = C(k) > 0 \text{ such that} \\ K \in \mathcal{C}^{k+1}\left(B_1 \times (\mathbb{R}^N \setminus \{0\})\right), \\ \|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{L^\infty(B_1)} \le \dfrac{C_k}{|w|^{N+\sigma+|\theta|}}, \\ \mu, \theta \in (\mathbb{N} \cup \{0\})^N,\ |\mu| + |\theta| \le k+1,\ w \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{4.2.2}$$

Under the above assumptions, in this section we will prove the principal result of this chapter. That is, we will give the proof of the following.

Theorem 4.2.1. *Fix $1 < \sigma < 2$, $k \in \mathbb{N} \cup \{0\}$, and let $u \in L^\infty(\mathbb{R}^N)$ be a viscosity solution of the equation*

$$\int_{\mathbb{R}^N} K(x,w)\, \delta u(x,w) dw = f(x, u(x)) \qquad \textit{inside } B_1, \tag{4.2.3}$$

with $f \in \mathcal{C}^{k+1}(B_1 \times \mathbb{R})$. Assume that $K : B_1 \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ satisfies (4.2.1) *and* (4.2.2) *for the same value of k.*

Then, if η in (4.2.1) *is sufficiently small (the smallness being independent of k), we have $u \in \mathcal{C}^{k+\sigma+\alpha}(B_{1/2})$ for any $\alpha < 1$, and*

$$\|u\|_{\mathcal{C}^{k+\sigma+\alpha}(B_{1/2})} \le C\left(1 + \|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1 \times [-M,M])}\right), \tag{4.2.4}$$

where $M = \|u\|_{L^\infty(B_1)}$ and $C > 0$ depends only on N, σ, k, C_k, and $\|f\|_{\mathcal{C}^{k+1}(B_1 \times \mathbb{R})}$.

As customary, when $1 < \sigma + \alpha < 2$ (resp. $\sigma + \alpha > 2$), by (4.2.4) we mean that $u \in \mathcal{C}^{k+1, \sigma+\alpha-1}(B_{1/2})$ (resp. $u \in \mathcal{C}^{k+2, \sigma+\alpha-1}(B_{1/2})$). To avoid any issue, we will always implicitly assume that α is chosen different from $2-\sigma$, so that $\sigma + \alpha \ne 2$.

Before starting with the proof of the previous result, we note that if in (4.2.2) one replaces the $\mathcal{C}^{k+1}$-regularity of K with the $\mathcal{C}^{k,\beta}$-assumption

$$\|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{\mathcal{C}^{0,\beta}(B_1)} \le \frac{C_k}{|w|^{N+\sigma+|\theta|}}, \tag{4.2.5}$$

for all $|\mu| + |\theta| \le k$, then we obtain the following.

Theorem 4.2.2. *Let $1 < \sigma < 2$, $k \in \mathbb{N} \cup \{0\}$, and take $u \in L^\infty(\mathbb{R}^N)$ a viscosity solution of equation* (4.2.3) *with $f \in \mathcal{C}^{k,\beta}(B_1 \times \mathbb{R})$. Assume that $K : B_1 \times (\mathbb{R}^N \setminus \{0\}) \to (0, +\infty)$ satisfies assumptions* (4.2.1) *and* (4.2.5) *for the same value of k.*

Then, if η in (4.2.1) *is sufficiently small (the smallness being independent of k), we have $u \in \mathcal{C}^{k+\sigma+\alpha}(B_{1/2})$ for any $\alpha < \beta$, and*

$$\|u\|_{\mathcal{C}^{k+\sigma+\alpha}(B_{1/2})} \leq C \left(1 + \|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1 \times [-M,M])}\right),$$

where $M = \|u\|_{L^\infty(B_1)}$, $C > 0$ depends only on n, σ, k, C_k, and $\|f\|_{\mathcal{C}^{k,\beta}(B_1 \times \mathbb{R})}$.

The proof of Theorem 4.2.2 is essentially the same as the one of Theorem 4.2.1, the only difference being that instead of differentiating the equations (see for instance the argument in Section 4.2.4) one should use incremental quotients. Although this does not introduce any major additional difficulties, it makes the proofs longer and more tedious. Hence, since the proof of Theorem 4.2.1 already contains all the main ideas to prove also Theorem 4.2.2, we will show the details of the proof only for Theorem 4.2.1. The core of this proof is the step $k = 0$, which will be proved in several steps.

4.2.1 Toolbox.

We collect here some preliminary observations on scaled Hölder norms, covering arguments, and differentiation of integrals that will play an important role in the proof of Theorem 4.2.1. This material is mainly technical, and the expert reader may go directly to Section 4.2.2.

Scaled Hölder norms and coverings.

Given $m \in \mathbb{N}$, $0 < \alpha < 1$, $x \in \mathbb{R}^N$, and $r > 0$, we define the $\mathcal{C}^{m,\alpha}$-norm of a function u in $B_r(x)$ as

$$\|u\|_{\mathcal{C}^{m,\alpha}(B_r(x))} := \sum_{|\gamma| \leq m} \|D^\gamma u\|_{L^\infty(B_r(x))} + \sum_{|\gamma| = m} \sup_{y \neq z \in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha}.$$

For our purposes it is also convenient to look at the following classical rescaled version of the norm:

$$\begin{aligned} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_r(x))} \quad &:= \quad \sum_{j=0}^{m} \sum_{|\gamma|=j} r^j \|D^\gamma u\|_{L^\infty(B_r(x))} \\ &+ \quad \sum_{|\gamma|=m} r^{m+\alpha} \sup_{y \neq z \in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y - z|^\alpha}. \end{aligned}$$

This scaled norm behaves nicely under covering, as the next observation points out.

Lemma 4.2.3. *Let $m \in \mathbb{N}$, $0 < \alpha < 1$, $\rho > 0$, and $x \in \mathbb{R}^N$. Fix $0 < \lambda < 1$, and suppose that $B_\rho(x)$ is covered by finitely many balls $\{B_{\lambda\rho/2}(x_k)\}_{k=1}^{N_o}$. Then, there exists $C_o > 0$, depending only on λ, α and m, such that*

$$\|u\|^*_{\mathcal{C}^{m,\alpha}(B_\rho(x))} \leq C_o \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\rho}(x_k))}.$$

Proof. We first observe that, if $j \in \{0, \dots, m\}$ and $|\gamma| = j$,

$$\begin{aligned} \rho^j \|D^\gamma u\|_{L^\infty(B_\rho(x))} &\leq \lambda^{-j} (\lambda\rho)^j \max_{k=1,\dots,N_o} \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\ &\leq \lambda^{-m} \sum_{k=1}^{N_o} (\lambda\rho)^j \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\ &\leq \lambda^{-m} \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\rho}(x_k))}. \end{aligned}$$

Now, let $|\gamma| = m$. We claim that

$$\rho^{m+\alpha} \sup_{y \neq z \in B_\rho(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} \leq 2\lambda^{-(m+\alpha)} \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\rho}(x_k))}.$$

To check this, we take $y, z \in B_\rho(x)$ with $y \neq z$ and we distinguish two cases. If $|y - z| < \lambda\rho/2$ we choose $k_o \in \{1, \dots, N_o\}$ such that $y \in B_{\lambda\rho/2}(x_{k_o})$. Then $|z - x_{k_o}| \leq |z - y| + |y - x_{k_o}| < \lambda\rho$, which implies $y, z \in B_{\lambda\rho}(x_{k_o})$, therefore

$$\begin{aligned} \rho^{m+\alpha} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} &\leq \rho^{m+\alpha} \sup_{\tilde{y} \neq \tilde{z} \in B_{\lambda\rho}(x_{k_o})} \frac{|D^\gamma u(\tilde{y}) - D^\gamma u(\tilde{z})|}{|\tilde{y}-\tilde{z}|^\alpha} \\ &\leq \lambda^{-(m+\alpha)} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\rho}(x_{k_o}))}. \end{aligned}$$

Conversely, if $|y - z| \geq \lambda\rho/2$, recalling that $0 < \alpha < 1$, we have

$$\begin{aligned} \rho^{m+\alpha} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} &\leq 2\lambda^{-\alpha} \rho^m \|D^\gamma u\|_{L^\infty(B_\rho(x))} \\ &\leq 2\lambda^{-\alpha} \rho^m \sum_{k=1}^{N_o} \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\ &\leq 2\lambda^{-(m+\alpha)} \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\rho}(x_k))}. \end{aligned}$$

This proves the claim and concludes the proof. □

Scaled norms behave also nicely in order to go from local to global bounds, as the next result shows.

Lemma 4.2.4. *Let* $m \in \mathbb{N}$, $0 < \alpha < 1$, *and* $u \in \mathcal{C}^{m,\alpha}(B_1)$. *Suppose that there exist* $0 < \mu < 1/2$ *and* $\mu < \nu \leq 1$ *for which the following holds: for any* $\epsilon > 0$ *there exists* $\Lambda_\epsilon > 0$ *such that, for any* $x \in B_1$ *and any* $0 < r \leq 1 - |x|$, *we have*

$$\|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} \leq \Lambda_\epsilon + \epsilon \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\nu r}(x))}. \tag{4.2.6}$$

Then there exist constants ϵ_o, $C > 0$, *depending only on* N, m, μ, ν, *and* α, *such that*

$$\|u\|_{\mathcal{C}^{m,\alpha}(B_\mu)} \leq C\Lambda_{\epsilon_o}.$$

Proof. First of all, since $0 < r < 1$, we observe that

$$\|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} \leq \|u\|_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} \leq \|u\|^*_{\mathcal{C}^{m,\alpha}(B_1)} = \|u\|_{\mathcal{C}^{m,\alpha}(B_1)},$$

which implies that

$$M := \sup_{\substack{x \in B_1 \\ r \in (0, 1-|x|]}} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} < \infty.$$

We now use a covering argument: pick $0 < \lambda \leq 1/2$ to be chosen later, and fixed any $x \in B_1$ and $0 < r \leq 1 - |x|$ we cover $B_{\mu r}(x)$ with finitely many balls $\{B_{\lambda\mu r/2}(x_k)\}_{k=1}^{N_o}$, with $x_k \in B_{\mu r}(x)$ and some N_o depending only on λ and the dimension N. We now observe that, since $\mu < 1/2$,

$$|x_k| + r/2 \leq |x_k - x| + |x| + r/2 \leq \mu r + |x| + r/2 < r + |x| \leq 1. \tag{4.2.7}$$

Hence, since $\lambda \leq 1/2$, we can use (4.2.6) (with $x = x_k$ and r scaled to λr) to obtain

$$\|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\mu r}(x_k))} \leq \Lambda_\epsilon + \epsilon \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\nu r}(x_k))}.$$

Then, using Lemma 4.2.3 with $\rho := \mu r$ and $\lambda = \mu/(2\nu)$, and recalling (4.2.7) and the definition of M, we get

$$\begin{aligned}
\|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} &\leq C_o \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\mu r}(x_k))} \\
&\leq C_o N_o \Lambda_\epsilon + C_o \epsilon \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\lambda\nu r}(x_k))} \\
&= C_o N_o \Lambda_\epsilon + C_o \epsilon \sum_{k=1}^{N_o} \|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r/2}(x_k))} \\
&\leq C_o N_o \Lambda_\epsilon + \epsilon C_o N_o M.
\end{aligned}$$

Therefore,

$$M \le C_o N_o \Lambda_\epsilon + \epsilon C_o N_o M,$$

so that, by choosing $\epsilon_o := 1/(2C_o N_o)$,

$$M \le 2C_o N_o \Lambda_{\epsilon_o}.$$

Thus we have proved that

$$\|u\|^*_{\mathcal{C}^{m,\alpha}(B_{\mu r}(x))} \le 2C_o N_o \Lambda_{\epsilon_o} \qquad x \in B_1,\ 0 < r \le 1 - |x|,$$

and the desired result follows by setting $x = 0$ and $r = 1$. □

Differentiating integral functions.

In the proof of Theorem 4.2.1 we will need to differentiate under the integral sign smooth functions that are either supported near the origin or far from it. This purpose will be accomplished in Lemmas 4.2.7 and 4.2.8, after some technical bounds that are needed in order to use the Dominated Convergence Theorem.

Recalling the notation introduced in (0.0.36), we get the following.

Lemma 4.2.5. *Let $r > r' > 0$, $v \in \mathcal{C}^3(B_r)$, $x \in B_{r'}$ and $h \in \mathbb{R}$ with $|h| < (r - r')/2$. Then, for any $w \in \mathbb{R}^N$ with $|w| < (r - r')/2$, we have*

$$|\delta v(x + he_1, w) - \delta v(x, w)| \le |h|\,|w|^2 \|v\|_{\mathcal{C}^3(B_r)}.$$

Proof. Fixed $x \in B_{r'}$ and $|w| < (r-r')/2$, for every $(r'-r)/2 \le h \le (r-r')/2$ we set

$$g(h) := v(x + he_1 + w) + v(x + he_1 - w) - 2v(x + he_1).$$

Then

$$\begin{aligned} |g(h) - g(0)| &\le |h| \sup_{|\xi|\le|h|} |g'(\xi)| \\ &\le |h| \sup_{|\xi|\le|h|} \big|\partial_1 v(x + \xi e_1 + w) + \partial_1 v(x + \xi e_1 - w) - 2\partial_1 v(x + \xi e_1)\big|. \end{aligned}$$

Noticing that since $|x + \xi e_1 \pm w| \le r' + |h| + |w| < r$, a second order Taylor expansion of $\partial_1 v$ with respect to the variable w gives

$$\big|\partial_1 v(x+\xi e_1+w)+\partial_1 v(x+\xi e_1-w)-2\partial_1 v(x+\xi e_1)\big| \le |w|^2 \|\partial_1 v\|_{\mathcal{C}^2(B_r)}. \quad (4.2.8)$$

Therefore

$$|\delta v(x + he_1, w) - \delta v(x, w)| = |g(h) - g(0)| \le |h|\,|w|^2 \|v\|_{\mathcal{C}^3(B_r)},$$

as desired. □

Lemma 4.2.6. *Let $r > r' > 0$, $v \in W^{1,\infty}(\mathbb{R}^N)$, $h \in \mathbb{R}$. Then, for any $w \in \mathbb{R}^N$,*

$$|\delta v(x + he_1, w) - \delta v(x, w)| \le 4|h| \|\nabla v\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. It suffices to proceed as in the proof of Lemma 4.2.5, but replacing (4.2.8) with the following trivial estimate:

$$\left|\partial_1 v(x + \xi e_1 + w) + \partial_1 v(x + \xi e_1 - w) - 2\partial_1 v(x + \xi e_1)\right| \le 4\|\partial_1 v\|_{L^\infty(\mathbb{R}^N)}.$$

□

Lemma 4.2.7. *Let $\ell \in \mathbb{N}$, $0 < r < 2$, K satisfy (4.2.2), and $U \in \mathcal{C}_0^{\ell+2}(B_r)$. Let $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{N}^N$ with $|\gamma| \le \ell \le k + 1$. Then*

$$\begin{aligned} \partial_x^\gamma \int_{\mathbb{R}^N} K(x, w)\, \delta U(x, w)\, dw &= \int_{\mathbb{R}^N} \partial_x^\gamma \left(K(x, w)\, \delta U(x, w)\right)\, dw \\ &= \sum_{\substack{1 \le i \le N \\ 0 \le \lambda_i \le \gamma_i \\ \lambda = (\lambda_1, \dots, \lambda_N)}} \binom{\gamma_1}{\lambda_1} \dots \binom{\gamma_N}{\lambda_N} \int_{\mathbb{R}^N} \partial_x^\lambda K(x, w)\, \delta(\partial_x^{\gamma-\lambda} U)(x, w)\, dw \end{aligned} \tag{4.2.9}$$

for any $x \in B_r$.

Proof. The second equality follows from the standard product derivation formula, so we focus on the proof of the first identity. The proof is by induction over $|\gamma|$. If $|\gamma| = 0$ the result is trivially true, so we consider the inductive step. We take x with $r' := |x| < r$, we suppose that $|\gamma| \le \ell - 1$ and, by inductive hypothesis, we know that

$$g_\gamma(x) := \partial_x^\gamma \int_{\mathbb{R}^N} K(x, w)\, \delta U(x, w)\, dw = \int_{\mathbb{R}^N} \theta(x, w)\, dw$$

with

$$\theta(x, w) := \sum_{\substack{1 \le i \le N \\ 0 \le \lambda_i \le \gamma_i \\ \lambda = (\lambda_1, \dots, \lambda_N)}} \binom{\gamma_1}{\lambda_1} \dots \binom{\gamma_N}{\lambda_N} \partial_x^\lambda K(x, w)\, \delta(\partial_x^{\gamma-\lambda} U)(x, w)\, dw.$$

By (4.2.2), if $0 < |h| < (r - r')/2$ then

$$|\partial_x^\lambda K(x + he_1, w) - \partial_x^\lambda K(x, w)| \le C_{|\lambda|+1} |h|\, |w|^{-N-\sigma}. \tag{4.2.10}$$

Moreover, if $|w| < (r - r')/2$, we can apply Lemma 4.2.5 with $v := \partial_x^{\gamma-\lambda} U$ and obtain

$$|\delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \delta(\partial_x^{\gamma-\lambda} U)(x, w)| \le |h|\, |w|^2 \|U\|_{\mathcal{C}^{|\gamma-\lambda|+3}(B_r)}. \tag{4.2.11}$$

On the other hand, by Lemma 4.2.6 we get

$$|\delta(\partial_x^{\gamma-\lambda}U)(x+he_1,w)-\delta(\partial_x^{\gamma-\lambda}U)(x,w)|\leq 4\,|h|\,\|\partial_x^{\gamma-\lambda}U\|_{\mathcal{C}^1(\mathbb{R}^N)}.$$

All in all,

$$\begin{aligned} & |\delta(\partial_x^{\gamma-\lambda}U)(x+he_1,w)-\delta(\partial_x^{\gamma-\lambda}U)(x,w)| \\ \leq\ & |h|\,\|U\|_{\mathcal{C}^{|\gamma-\lambda|+3}(\mathbb{R}^N)}\min\{4,|w|^2\}. \end{aligned} \tag{4.2.12}$$

Analogously, a simple Taylor expansion provides also the bound

$$|\delta(\partial_x^{\gamma-\lambda}U)(x,w)|\leq \|U\|_{\mathcal{C}^{|\gamma-\lambda|+2}(\mathbb{R}^N)}\min\{4,|w|^2\}. \tag{4.2.13}$$

Hence, (4.2.2), (4.2.10), (4.2.12), and (4.2.13) give

$$\begin{aligned} & \left|\partial_x^\lambda K(x+he_1,w)\,\delta(\partial_x^{\gamma-\lambda}U)(x+he_1,w)-\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}U)(x,w)\right| \\ \leq\ & \left|\partial_x^\lambda K(x+he_1,w)\ \big(\delta(\partial_x^{\gamma-\lambda}U)(x+he_1,w)-\delta(\partial_x^{\gamma-\lambda}U)(x,w)\big)\right| \\ +\ & \left|\big(\partial_x^\lambda K(x+he_1,w)-\partial_x^\lambda K(x,w)\big)\,\delta(\partial_x^{\gamma-\lambda}U)(x,w)\right| \\ \leq\ & C|h|\,\min\{|w|^{-N-\sigma},|w|^{2-N-\sigma}\}, \end{aligned}$$

with $C>0$ depending only on ℓ, C_ℓ, given in (4.2.2), and $\|U\|_{\mathcal{C}^{\ell+2}(\mathbb{R}^N)}$. As a consequence,

$$|\theta(x+he_1,w)-\theta(x,w)|\leq C_2|h|\,\min\{|w|^{-N-\sigma},|w|^{2-N-\sigma}\},$$

and, by the Dominated Convergence Theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^N}\partial_{x_1}\theta(x,w)\,dw &= \lim_{h\to 0}\int_{\mathbb{R}^N}\frac{\theta(x+he_1,w)-\theta(x,w)}{h}\,dw \\ &= \lim_{h\to 0}\frac{g_\gamma(x+he_1)-g_\gamma(x)}{h} \\ &= \partial_{x_1}g_\gamma(x), \end{aligned}$$

which proves (4.2.9) with γ replaced by $\gamma+e_1$. Analogously one could prove the same result with γ replaced by $\gamma+e_i$, $i=1,\dots,N$, concluding the inductive step. □

The differentiation under the integral sign in (4.2.9) may also be obtained under slightly different assumptions, as the next result points out.

Lemma 4.2.8. *Let $\ell\in\mathbb{N}$, $0<r<R$. Let $U\in\mathcal{C}^{\ell+1}(\mathbb{R}^N)$ with $U=0$ in B_R. Let $\gamma=(\gamma_1,\dots,\gamma_N)\in\mathbb{N}^N$ with $|\gamma|\leq\ell$. Then* (4.2.9) *holds true for any $x\in B_r$.*

Proof. If $x \in B_r$, $w \in B_{(R-r)/2}$ and $|h| \le (R-r)/2$, we have that $|x+w+he_1| < R$ and so $\delta U(x+he_1, w) = 0$. In particular

$$\delta U(x+he_1, w) - \delta U(x,w) = 0,$$

for small h when $w \in B_{(R-r)/2}$. This formula replaces (4.2.11), and the rest of the proof goes on as the one of Lemma 4.2.7. □

Integral computations.

Here we collect some integral computations which will be used in the proof of Theorem 4.2.1.

Lemma 4.2.9. *Let $v : \mathbb{R}^N \to \mathbb{R}$ be smooth and with all its derivatives bounded and K satisfy* (4.2.2). *Let $x \in B_{1/4}$, and γ, $\lambda \in \mathbb{N}^N$, with $\gamma_i \ge \lambda_i$ for any $i \in \{1, \dots, N\}$ and $|\gamma| \le k+1$. Then there exists a constant $C' > 0$, depending only on N and σ, such that*

$$\left| \int_{\mathbb{R}^N} \partial_x^\lambda K(x,w)\, \delta(\partial_x^{\gamma-\lambda} v)(x,w)\, dw \right| \le C'\, C_{|\gamma|}\, \|v\|_{\mathcal{C}^{|\gamma-\lambda|+2}(\mathbb{R}^N)}. \tag{4.2.14}$$

Furthermore, if

$$v = 0 \text{ in } B_{1/2}, \tag{4.2.15}$$

we have

$$\left| \int_{\mathbb{R}^N} \partial_x^\lambda K(x,w)\, \delta(\partial_x^{\gamma-\lambda} v)(x,w)\, dw \right| \le C'\, C_{|\gamma|}\, \|v\|_{L^\infty(\mathbb{R}^N)}. \tag{4.2.16}$$

Proof. By (4.2.2) and (4.2.13) (with $U = v$),

$$\begin{aligned} & \int_{\mathbb{R}^N} \left|\partial_x^\lambda K(x,w)\right| \, \left|\delta(\partial_x^{\gamma-\lambda} v)(x,w)\right| \, dw \\ \le\ & C_{|\lambda|} \left(\|v\|_{\mathcal{C}^{|\gamma-\lambda|+2}(\mathbb{R}^N)} \int_{B_2} |w|^{-N-\sigma+2}\, dw \right. \\ +\ & \left. 4\|v\|_{\mathcal{C}^{|\gamma-\lambda|}(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_2} |w|^{-N-\sigma}\, dw \right), \end{aligned}$$

which proves (4.2.14).

We now prove (4.2.16). For this we notice that, thanks to (4.2.15), $v(x+w)$ and $v(x-w)$, and also their derivatives, are equal to zero if x and w lie in $B_{1/4}$. Hence, by an integration by parts, for $x \in B_{1/4}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \partial_x^\lambda K(x,w)\, \delta(\partial_x^{\gamma-\lambda} v)(x,w)\, dw \\
= &\int_{\mathbb{R}^N} \partial_x^\lambda K(x,w)\, \partial_w^{\gamma-\lambda} \left[v(x+w) - v(x-w)\right]\, dw \\
= &\int_{\mathbb{R}^N \setminus B_{1/4}} \partial_x^\lambda K(x,w)\, \partial_w^{\gamma-\lambda} \left[v(x+w) - v(x-w)\right]\, dw \\
= &\ (-1)^{|\gamma-\lambda|} \int_{\mathbb{R}^N \setminus B_{1/4}} \partial_x^\lambda \partial_w^{\gamma-\lambda} K(x,w)\ \left[v(x+w) - v(x-w)\right]\, dw.
\end{aligned}$$

Consequently, by (4.2.2),

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \partial_x^\lambda K(x,w)\, \delta(\partial_x^{\gamma-\lambda} v)(x,w)\, dw \right| \\
&\leq 2C_{|\gamma|}\, \|v\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_{1/4}} |w|^{-N-\sigma-|\gamma-\lambda|}\, dw,
\end{aligned}$$

proving (4.2.16). □

4.2.2 Approximation by nicer kernels.

In what follows, it will be convenient to approximate the solution u of (4.2.3) with smooth functions u_ε obtained by solving equations similar to (4.2.3), but with kernels K_ε which coincide with the fractional Laplacian in a neighborhood of the origin. Indeed, this will allow us to work with smooth functions, ensuring that in our computations all integrals converge. We will then prove uniform estimates on u_ε, which will give the desired $\mathcal{C}^{\sigma+\alpha}$-bound on u by letting $\varepsilon \to 0$.

To simplify the notation, up to multipling both K and f by $1/a_0$, we assume without loss of generality that the constant a_0 in (4.2.1) is equal to 1.

Let $\eta \in \mathcal{C}^\infty(\mathbb{R}^N)$ satisfy

$$\eta = \begin{cases} 1 & \text{in } B_{1/2}, \\ 0 & \text{in } \mathbb{R}^N \setminus B_{3/4}, \end{cases}$$

and, for given $\varepsilon, \delta > 0$, set

$$\eta_\varepsilon(w) := \eta\left(\frac{w}{\varepsilon}\right), \quad \text{and} \quad \widetilde{\eta}_\delta(x) := \delta^{-N} \eta\left(\frac{x}{\delta}\right).$$

Then we define

$$K_\varepsilon(x,w) := \eta_\varepsilon(w)\frac{2-\sigma}{|w|^{N+\sigma}} + (1-\eta_\varepsilon(w))\widetilde{K}_\varepsilon(x,w), \tag{4.2.17}$$

where

$$\widetilde{K}_\varepsilon(x,w) := K(x,w) * \Big(\widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w)\Big), \tag{4.2.18}$$

and

$$L_\varepsilon v(x) := \int_{\mathbb{R}^N} K_\varepsilon(x,w)\, \delta v(x,w) dw. \tag{4.2.19}$$

We also define

$$f_\varepsilon(x) := f(x,u(x)) * \widetilde{\eta}_\varepsilon(x). \tag{4.2.20}$$

Note that we get a family $f_\varepsilon \in \mathcal{C}^\infty(B_1)$ such that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon = f \text{ uniformly in } B_{3/4}.$$

Finally, we define $u_\varepsilon \in L^\infty(\mathbb{R}^N)\cap\mathcal{C}(\mathbb{R}^N)$ as the unique (see for instance [17, Theorem 1] and the argument used in the proof of [28, Theorem 3.2]) solution to the following linear problem

$$\begin{cases} L_\varepsilon u_\varepsilon = f_\varepsilon(x) & \text{in } B_{3/4}, \\ u_\varepsilon = u & \text{in } \mathbb{R}^N \setminus B_{3/4}. \end{cases} \tag{4.2.21}$$

It is easy to check that the kernels K_ε satisfy (4.2.1) and (4.2.2) with constants independent of ε. This can be easily checked using the definition of $\widetilde{K}_\varepsilon$. Indeed, for example to prove (4.2.2) with $|\theta| = 0$, by the presence of the term $(1-\eta_\varepsilon(w))$ which vanishes for $|w| \le \varepsilon/2$, one only needs to check that

$$\int_{\mathbb{R}^N} |w-z|^{-N-\sigma}\widetilde{\eta}_{\varepsilon^2}(z)\, dz \le C|w|^{-N-\sigma} \qquad \text{for } |w| \ge \varepsilon/2,$$

which is easy to prove.

Also, since K satisfies assumption (4.2.2) with $k = 0$ and the convolution parameter ε^2 in (4.2.17) is much smaller than ε, the operators L_ε converge to the operator associated to K in the weak sense introduced in [57, Definition 22]. Indeed, let v a smooth function satisfying

$$|v| \le M \quad \text{in } \mathbb{R}^N \text{ for some } M > 0, \tag{4.2.22}$$

and

$$|v(w) - v(x) - (w-x)\cdot\nabla v(x)| \le M|x-w|^2, \quad w \in B_1(x). \tag{4.2.23}$$

Then, from (4.2.1), (4.2.2), (4.2.22) and (4.2.23), it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\eta_\varepsilon(w)\frac{2-\sigma}{|w|^{N+\sigma}} + (1-\eta_\varepsilon(w))(K(x,w) * \widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w)) - K(x,w)\right| \\ &\qquad\qquad \times|\delta v(x,w)|\, dw \\ \leq\ & \int_{\mathbb{R}^N} \left(\eta_\varepsilon(w)\left|\frac{2-\sigma}{|w|^{N+\sigma}} - K(x,w)\right|\right. \\ +\ & \left.(1-\eta_\varepsilon(w))\,|K(x,w) * \widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w) - K(x,w)|\right) \times |\delta v(x,w)|\, dw \\ \leq\ & \int_{B_\varepsilon} C|w|^{2-N-\sigma} \\ +\ & \int_{\mathbb{R}^N\setminus B_\varepsilon} \left|K(x,w) * \widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w) - K(x,w)\right| |\delta v(x,w)|\, dw \\ \leq\ & C\varepsilon^{2-\sigma} + I, \end{aligned} \tag{4.2.24}$$

with

$$I := \int_{\mathbb{R}^N\setminus B_\varepsilon} \left|K(x,w) * \widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w) - K(x,w)\right| |\delta v(x,w)|\, dw.$$

By (4.2.2), (4.2.22), and the fact that $\sigma > 1$, we have

$$\begin{aligned} I\ =\ & \int_{\mathbb{R}^N\setminus B_\varepsilon}\int_{B_{3/4}}\int_{B_{3/4}} \left|K(x-\varepsilon^2 y, w-\varepsilon^2\tilde{w})\eta(y)\eta(\tilde{w}) - K(x,w)\right|\, dy\, d\tilde{w} \\ &\qquad\qquad \times|\delta v(x,w)|\, dw \\ \leq\ & \int_{\mathbb{R}^N\setminus B_\varepsilon} \frac{C\varepsilon^2}{|w|^{N+1+\sigma}}\, |\delta v(x,w)|\, dw \\ \leq\ & C\int_{B_1\setminus B_\varepsilon} \frac{\varepsilon^2}{|w|^{N-1+\sigma}}\, dw + C\int_{\mathbb{R}^N\setminus B_1} \frac{\varepsilon^2}{|w|^{N+1+\sigma}}\, dw \\ \leq\ & C(\varepsilon^{3-\sigma} + \varepsilon^2). \end{aligned}$$

Combining this estimate with (4.2.24), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\eta_\varepsilon(w)\frac{2-\sigma}{|w|^{N+\sigma}} + (1-\eta_\varepsilon(\omega))(K(x,w) * \widetilde{\eta}_{\varepsilon^2}(x)\widetilde{\eta}_{\varepsilon^2}(w)) - K(x,w)\right| \\ &\qquad\qquad \times|\delta v(x,w)|\, dw \\ \leq\ & C\varepsilon^{2-\sigma}, \end{aligned}$$

where C depends of M and σ. Since $\sigma < 2$ we conclude that

$$\|L_\varepsilon - L\| \to 0 \quad \text{as } \varepsilon \to 0, \text{ where } \|\cdot\| \text{ was defined in [57, Definition 22].}$$

Thanks to this fact, we can repeat almost word by word the proof of [57, Lemma 7] to obtain the uniform convergence

$$u_\varepsilon \to u \quad \text{on } \mathbb{R}^N \qquad \text{as } \varepsilon \to 0. \tag{4.2.25}$$

Note that, in order to use the argument in the proof of [57, Lemma 7] one needs to know that the functions u_ε are equicontinuous, which is a consequence of [57, Lemmas 2 and 3]. To be precise, to apply [57, Lemma 3] one would need the kernels to satisfy the bounds $\frac{(2-\sigma)\lambda}{|w|^{N+\sigma}} \le K_*(x,w) \le \frac{(2-\sigma)\Lambda}{|w|^{N+\sigma}}$ for all $w \neq 0$, while in our case the kernel K (and so also K_ε) satisfies

$$\frac{(2-\sigma)\lambda}{|w|^{N+\sigma}} \le K(x,w) \le \frac{(2-\sigma)\Lambda}{|w|^{N+\sigma}}, \quad |w| \le r_0, \tag{4.2.26}$$

with $\lambda := a_0 - \eta$, $\Lambda := a_0 + \eta$, and $r_0 > 0$ (observe that, by our assumptions in (4.2.1), $\lambda \ge 3a_0/4$).

However this is not a big problem: if $v \in L^\infty(\mathbb{R}^N)$ satisfies

$$\int_{\mathbb{R}^N} K_*(x,w)\,\delta v(x,w)\,dw = f(x) \qquad \text{in } B_{3/4},$$

for some kernel satisfying (4.2.2) and $\frac{(2-\sigma)\lambda}{|w|^{N+\sigma}} \le K_*(x,w) \le \frac{(2-\sigma)\Lambda}{|w|^{N+\sigma}}$ only for $|w| \le r_0$, we define

$$K'(x,w) := \zeta(w)K_*(x,w) + (2-\sigma)\frac{1-\zeta(w)}{|w|^{N+\sigma}},$$

with ζ a smooth cut-off function supported inside B_{r_0}, to get

$$\begin{aligned} &\int_{\mathbb{R}^N} K'(x,w)\,\delta v(x,w)\,dw = f(x) \\ &+ \int_{\mathbb{R}^N} (1-\zeta(w))\left(-K_*(x,w) + \frac{2-\sigma}{|w|^{N+\sigma}}\right)\delta v(x,w)\,dw. \end{aligned}$$

Since $1-\zeta(w) = 0$ near the origin, by assumption (4.2.2), the second integral is uniformly bounded as a function of x, so [57, Lemma 3] applied to K' gives the desired equicontinuity.

As we said before, the uniqueness for the boundary problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} K(x,w)\,\delta v(x,w)\,dw = f(x,u(x)) & \text{in } B_{3/4}, \\ v = u, \quad \text{on } \mathbb{R}^N \setminus B_{3/4}. \end{cases}$$

follows by a standard comparison principle argument (see the proof of [28, Theorem 3.2]).

4.2.3 Smoothness of the approximate solutions.

We prove now that the functions u_ε defined in the previous section are of class $\mathcal{C}^\infty$ inside a small ball, whose size is uniform with respect to ε: namely, there exists $r \in (0, 1/4)$ such that, for any $m \in \mathbb{N}^N$

$$\|D^m u_\varepsilon\|_{L^\infty(B_r)} \le C, \tag{4.2.27}$$

for some positive constant $C = C(m, \sigma, \varepsilon, \|u\|_{L^\infty(\mathbb{R}^N)}, \|f\|_{L^\infty(B_1\times\mathbb{R})})$. For this, we observe that by (4.2.17)

$$\frac{2-\sigma}{|w|^{N+\sigma}} = K_\varepsilon(x,w) - (1-\eta_\varepsilon(w))\widetilde{K}_\varepsilon(x,w) + (1-\eta_\varepsilon(w))\,\frac{2-\sigma}{|w|^{N+\sigma}}.$$

Then, for any $x \in B_{1/4}$,

$$\begin{aligned}
-\frac{2(2-\sigma)}{C(N,\frac{\sigma}{2})}(-\Delta)^{\sigma/2}u_\varepsilon(x) &= \int_{\mathbb{R}^N} \frac{2-\sigma}{|w|^{N+\sigma}}\delta u_\varepsilon(x,w)dw \\
&= f_\varepsilon(x) - \int_{\mathbb{R}^N} (1-\eta_\varepsilon(w))\widetilde{K}_\varepsilon(x,w)\,\delta u_\varepsilon(x,w)dw \\
&+ \int_{\mathbb{R}^N} (1-\eta_\varepsilon(w))\frac{2-\sigma}{|w|^{N+\sigma}}\,\delta u_\varepsilon(x,w)dw,
\end{aligned}$$

where $C\left(N,\frac{\sigma}{2}\right)$ was introduced in (0.0.22). Then, for any $x \in B_{1/4}$ it follows that

$$\begin{aligned}
&(-\Delta)^{\sigma/2}u_\varepsilon(x) \\
= \;& D\left(N,\frac{\sigma}{2}\right)\left[f_\varepsilon(x) + \int_{\mathbb{R}^N} (1-\eta_\varepsilon(w))\left(\frac{2-\sigma}{|w|^{N+\sigma}} - \widetilde{K}_\varepsilon(x,w)\right)\delta u_\varepsilon(x,w)dw\right] \\
=: \;& D\left(N,\frac{\sigma}{2}\right)[f_\varepsilon(x) + h_\varepsilon(x)] \qquad (4.2.28) \\
=: \;& D\left(N,\frac{\sigma}{2}\right) g_\varepsilon(x).
\end{aligned}$$

with $D\left(N,\frac{\sigma}{2}\right) := -\dfrac{C\left(N,\frac{\sigma}{2}\right)}{2(2-\sigma)}$.

Making some changes of variables we can rewrite h_ε as follows:

$$\begin{aligned}
h_\varepsilon(x) &= \int_{\mathbb{R}^N}(1-\eta_\varepsilon(w-x))\left(\frac{2-\sigma}{|w-x|^{N+\sigma}} - \widetilde{K}_\varepsilon(x,w-x)\right)u_\varepsilon(w)dw \\
&+ \int_{\mathbb{R}^N}(1-\eta_\varepsilon(x-w))\left(\frac{2-\sigma}{|w-x|^{N+\sigma}} - \widetilde{K}_\varepsilon(x,x-w)\right)u_\varepsilon(w)dw \\
&- 2u_\varepsilon(x)\int_{\mathbb{R}^N}(1-\eta_\varepsilon(w))\left(\frac{2-\sigma}{|w|^{N+\sigma}} - \widetilde{K}_\varepsilon(x,w)\right)dw.
\end{aligned} \tag{4.2.29}$$

We now notice that "the function h_ε is locally as smooth as u_ε", is the sense that for any $m \in \mathbb{N}$ and $U \subset B_{1/4}$ open we have

$$\|h_\varepsilon\|_{C^m(U)} \leq C(\varepsilon, m)\left(1 + \|u_\varepsilon\|_{C^m(U)}\right), \tag{4.2.30}$$

for some constant $C(\varepsilon, m) > 0$. To see this observe that, in the first two integrals, the variable x appears only inside η_ε and in the kernel $\widetilde{K}_\varepsilon$, and η_ε is equal to 1 near the origin. Hence, since $\widetilde{K}_\varepsilon$ is smooth, see (4.2.18), the first two integrals are smooth functions of x. The third term is clearly as regular as u_ε because the third integral is smooth by the same reason as before. This proves (4.2.30).

We will now prove that the functions u_ε belong to $\mathcal{C}^\infty(B_{1/5})$, with

$$\|u_\varepsilon\|_{\mathcal{C}^m(B_{1/4-r_m})} \leq C(r_1, m, \sigma, \varepsilon, \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}, \|f\|_{L^\infty(B_1\times[-M,M])}), \tag{4.2.31}$$

for any $m \in \mathbb{N}$, where $r_m := 1/20 - 25^{-m}$ and $M = \|u\|_{L^\infty(B_1)}$. Note that, for every $m \in \mathbb{N}$,

$$1/4 - r_m > 1/5, \tag{4.2.32}$$

To show (4.2.31), we begin by observing that, since $1 < \sigma < 2$, by (4.2.28), (4.2.30), and [57, Theorem 61], we have that $u_\varepsilon \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}^{1,\beta}(B_{1/4-r_1})$ for any $\beta < \sigma - 1$ and, for $M = \|u\|_{L^\infty(B_1)}$,

$$\|u_\varepsilon\|_{\mathcal{C}^{1,\beta}(B_{1/4-r_1})} \leq C(\varepsilon)\left(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1\times[-M,M])}\right). \tag{4.2.33}$$

As already observed on page 151, the fact that the kernel satisfies (4.2.26) only for w small is not a problem, and one can easily check that [57, Theorem 61] still holds in our setting.

Now, to get a bound on higher derivatives, the idea would be to differentiate (4.2.28) and use again (4.2.30) and [57, Theorem 61]. However we do not have $\mathcal{C}^1$ bounds on the function u_ε outside $B_{1/4-r_1}$, and therefore we can not apply directly this strategy to obtain the $\mathcal{C}^{2,\alpha}$ regularity of the function u_ε.

To avoid this problem we follow the localization argument given in [56, Theorem 13.1]. That is, we take $\nu > 0$ small (to be chosen) and we consider a smooth cut-off function

$$\vartheta := \begin{cases} 1 & \text{in } B_{1/4-(1+\nu)r_1}, \\ 0 & \text{on } \mathbb{R}^N \setminus B_{1/4-r_1}. \end{cases}$$

For fixed $e \in \mathbb{S}^{N-1}$ and $|h| < \nu r_1$ we define

$$v(x) := \frac{u_\varepsilon(x + eh) - u_\varepsilon(x)}{|h|}. \tag{4.2.34}$$

We note that the function $v(x)$ is uniformly bounded in $B_{1/4-(1+\nu)r_1}$ because $u \in \mathcal{C}^1(B_{1/4-r_1})$. We now write $v(x) = v_1(x) + v_2(x)$, with

$$v_1(x) := \frac{\vartheta u_\varepsilon(x+eh) - \vartheta u_\varepsilon(x)}{|h|}$$

and

$$v_2(x) := \frac{(1-\vartheta)u_\varepsilon(x+eh) - (1-\vartheta)u_\varepsilon(x)}{|h|}.$$

By (4.2.33) it is clear that

$$v_1 \in L^\infty(\mathbb{R}^N)$$

and, since $|h| < \nu r_1$, that

$$v_1 = v \qquad \text{inside } B_{1/4-(1+2\nu)r_1}. \tag{4.2.35}$$

Moreover, for $x \in B_{1/4-(1+2\nu)r_1}$, using (4.2.20), (4.2.28) and (4.2.30) we get

$$\begin{aligned} \left|(-\Delta)^{\sigma/2} v_1(x)\right| &= \left|(-\Delta)^{\sigma/2} v(x) - (-\Delta)^{\sigma/2} v_2(x)\right| \\ &= \left|\frac{g_\varepsilon(x+eh) - g_\varepsilon(x)}{|h|} - (-\Delta)^{\sigma/2} v_2(x)\right| \\ &\le C(\varepsilon)\Big(1 + \|u_\varepsilon\|_{\mathcal{C}^1(B_{1/4-r_1})}\Big) + \left|(-\Delta)^{\sigma/2} v_2(x)\right|. \end{aligned} \tag{4.2.36}$$

Now, let us denote by $K_o(y) := \frac{C\left(N,\frac{\sigma}{2}\right)}{|y|^{N+\sigma}}$ the kernel of the fractional Laplacian defined in (0.0.21). Since for $x \in B_{1/4-(1+2\nu)r_1}$ and $|h| < \nu r_1$ we have that $(1-\vartheta)u_\varepsilon(x \pm eh) = 0$, then $v_2(x) = 0$. Therefore, from a change of variable, it follows that

$$\begin{aligned} |(-\Delta)^{\sigma/2} v_2(x)| &\le \left|\int_{\mathbb{R}^N} (v_2(x+y) + v_2(x-y) - 2v_2(x))K_o(y)\, dy\right| \\ &\le \left|\int_{\mathbb{R}^N} \frac{(1-\vartheta)u_\varepsilon(x+y+eh) - (1-\vartheta)u_\varepsilon(x+y)}{|h|} K_o(y)\, dy\right| \\ &+ \left|\int_{\mathbb{R}^N} \frac{(1-\vartheta)u_\varepsilon(x-y+eh) - (1-\vartheta)u_\varepsilon(x-y)}{|h|} K_o(y)\, dy\right| \\ &\le \int_{\mathbb{R}^N} (1-\vartheta)|u_\varepsilon(x+y)| \left|\frac{K_o(y-eh) - K_o(y)}{|h|}\right| dy \\ &+ \int_{\mathbb{R}^N} (1-\vartheta)|u_\varepsilon(x-y)| \left|\frac{K_o(y-eh) - K_o(y)}{|h|}\right| dy \\ &\le \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_{\nu r_1}} \frac{1}{|y|^{N+\sigma+1}}\, dy \\ &\le C\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}, \quad \text{for } x \in B_{1/4-(1+2\nu)r_1} \text{ and } C = C(\sigma, r_1). \end{aligned}$$

Therefore, by (4.2.36) we obtain

$$\left|(-\Delta)^{\sigma/2} v_1(x)\right| \le C(\varepsilon)\left(1+\|u_\varepsilon\|_{\mathcal{C}^1(B_{1/4-r_1})}+\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}\right),\ x\in B_{1/4-(1+2\nu)r_1},$$

and we can apply [57, Theorem 52] to get that $v_1 \in \mathcal{C}^{1,\beta}(B_{1/4-r_2})$ for any $\beta < \sigma - 1$, with

$$\|v_1\|_{\mathcal{C}^{1,\beta}(B_{1/4-r_2})} \le C(\varepsilon)\Big(1+\|v_1\|_{L^\infty(\mathbb{R}^N)}+\|u_\varepsilon\|_{\mathcal{C}^1(B_{1/4-r_1})}+\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}\Big),$$

provided $\nu > 0$ was chosen sufficiently small so that $r_2 > (1+2\nu)r_1$. By (4.2.33), (4.2.34) and (4.2.35) this implies that $u_\varepsilon \in \mathcal{C}^{2,\beta}(B_{1/4-r_2})$, with

$$\begin{aligned}\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/4-r_2})} &\le C(\varepsilon)\Big(1+\|u_\varepsilon\|_{\mathcal{C}^1(B_{1/4-r_1})}+\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}\Big)\\ &\le C(\varepsilon)\Big(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\Big),\end{aligned}$$

for $M = \|u\|_{L^\infty(B_1)}$. Iterating this argument we obtain (4.2.31), as desired.

4.2.4 Uniform estimates and conclusion of the proof for $k = 0$.

Knowing now that the functions u_ε defined by (4.2.21) are smooth inside $B_{1/5}$ (see (4.2.31) and (4.2.32)), our goal is to obtain a-priori bounds independent of ε.

By [57, Theorem 61] applied to u, we have that $u \in \mathcal{C}^{1,\beta}(B_{1-R_1})$ for any $\beta < \sigma - 1$ and $R_1 > 0$, with

$$\|u\|_{\mathcal{C}^{1,\beta}(B_{1-R_1})} \le C\left(\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right), \tag{4.2.37}$$

where $M = \|u\|_{L^\infty(B_1)}$. Then, for any ε sufficiently small, $f_\varepsilon \in \mathcal{C}^1(B_{1/2})$ with

$$\begin{aligned}\|f_\varepsilon\|_{\mathcal{C}^1(B_{1/2})} &\le C'\left(1+\|u\|_{\mathcal{C}^1(B_{1-R_1})}\right)\\ &\le C'C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right),\end{aligned} \tag{4.2.38}$$

where $C' > 0$ depends on $\|f\|_{C^1(B_1\times\mathbb{R})}$ only.

Consider a cut-off function

$$\widetilde{\vartheta} := \begin{cases} 1 & \text{in } B_{1/7},\\ 0 & \text{on } \mathbb{R}^N \setminus B_{1/6}.\end{cases}$$

Then, recalling (4.2.21), we write the equation satisfied by u_ε as

$$f_\varepsilon(x) = \int_{\mathbb{R}^N} K_\varepsilon(x,w)\,\delta(\widetilde{\vartheta}u_\varepsilon)(x,w)dw + \int_{\mathbb{R}^N} K_\varepsilon(x,w)\,\delta((1-\widetilde{\vartheta})u_\varepsilon)(x,w)dw.$$

Differentiating the previous equality, say in direction e_1, we obtain (recall Lemmas 4.2.7 and 4.2.8)

$$\begin{aligned}\partial_{x_1} f_\varepsilon(x) &= \int_{\mathbb{R}^N} K_\varepsilon(x,w)\delta(\partial_{x_1}(\widetilde{\vartheta} u_\varepsilon))(x,w)dw \\ &+ \int_{\mathbb{R}^N} \partial_{x_1}\Big[K_\varepsilon(x,w)\delta((1-\widetilde{\vartheta})u_\varepsilon)(x,w)\Big]\,dw \\ &+ \int_{\mathbb{R}^N} \partial_{x_1} K_\varepsilon(x,w)\delta(\widetilde{\vartheta} u_\varepsilon)(x,w)dw.\end{aligned}$$

It is convenient to rewrite this equation as

$$\int_{\mathbb{R}^N} K_\varepsilon(x,w)\delta(\partial_{x_1}(\widetilde{\vartheta} u_\varepsilon))(x,w)dw = A_1 - A_2 - A_3,$$

with

$$\begin{aligned}A_1 &:= \partial_{x_1} f_\varepsilon(x), \\ A_2 &:= \int_{\mathbb{R}^N} \partial_{x_1} K_\varepsilon(x,w)\delta(\widetilde{\vartheta} u_\varepsilon)(x,w)dw, \text{ and} \\ A_3 &:= \int_{\mathbb{R}^N} \partial_{x_1}\Big[K_\varepsilon(x,w)\delta((1-\widetilde{\vartheta})u_\varepsilon)(x,w)\Big]\,dw.\end{aligned}$$

We claim that

$$\|A_1 - A_2 - A_3\|_{L^\infty(B_{1/14})} \le C\Big(1 + \|u\|_{L^\infty(\mathbb{R}^N)} + \|u_\varepsilon\|_{\mathcal{C}^2(B_{1/6})}\Big) \tag{4.2.39}$$

with C depending only on $\|f\|_{\mathcal{C}^1(B_1\times\mathbb{R})}$. To prove this, we first observe that by (4.2.38),

$$\|A_1\|_{L^\infty(B_{1/14})} \le C\left(1 + \|u\|_{L^\infty(\mathbb{R}^N)}\right). \tag{4.2.40}$$

Also, since $|\partial_{x_1}\widetilde{K}_\varepsilon(x,w)| \le C|w|^{-(N+\sigma)}$, by (4.2.14) (used with $\gamma = \lambda := (1,0,\ldots,0)$ and $v := \widetilde{\vartheta} u_\varepsilon$) we get

$$\|A_2\|_{L^\infty(B_{1/14})} \le C\|\widetilde{\vartheta} u_\varepsilon\|_{\mathcal{C}^2(\mathbb{R}^N)} \le C\|u_\varepsilon\|_{\mathcal{C}^2(B_{1/6})}, \tag{4.2.41}$$

where we used that $\widetilde{\vartheta}$ is supported in $B_{1/6}$.

Moreover, since $(1-\widetilde{\vartheta})u_\varepsilon = 0$ inside $B_{1/7}$, we can use (4.2.16) with $v := (1-\widetilde{\vartheta})u_\varepsilon$ to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \partial_{x_1} K_\varepsilon(x,w)\,\delta((1-\widetilde{\vartheta})u_\varepsilon)(x,w)\,dw\right| \\ +\;&\left|\int_{\mathbb{R}^N} K_\varepsilon(x,w)\,\partial_{x_1}\delta((1-\widetilde{\vartheta})u_\varepsilon)(x,w)\,dw\right| \\ \le\;& C\|(1-\widetilde{\vartheta})u_\varepsilon\|_{L^\infty(\mathbb{R}^N)},\end{aligned} \tag{4.2.42}$$

for any $x \in B_{1/14}$. Since $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}\right)$, by (4.2.42) we obtain that

$$\|A_3\|_{L^\infty(B_{1/14})} \le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}\right). \tag{4.2.43}$$

Then (4.2.40)-(4.2.43) imply (4.2.39).

Since $\partial_{x_1}(\widetilde{\vartheta u_\varepsilon})$ is bounded on the whole of $\mathbb{R}^N$, by (4.2.39) and [57, Theorem 61] we obtain that $\partial_{x_1}(\widetilde{\vartheta u_\varepsilon}) \in \mathcal{C}^{1,\beta}(B_{1/14-R_2})$ for any $R_2>0$, with

$$\|\partial_{x_1}(\widetilde{\vartheta u_\varepsilon})\|_{\mathcal{C}^{1,\beta}(B_{1/14-R_2})} \le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|u_\varepsilon\|_{\mathcal{C}^2(B_{1/6})}\right),\ 0<\beta<\sigma-1,$$

which implies

$$\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/15})} \le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|u_\varepsilon\|_{\mathcal{C}^2(B_{1/6})}\right). \tag{4.2.44}$$

To end the proof we need to reabsorb the $\mathcal{C}^2$-norm on the right hand side. To do this, we observe that by standard interpolation inequalities (see for instance [97, Lemma 6.35]), for every $0<\nu<1$ there exists $C_\nu = C(\nu)>0$ such that

$$\|u_\varepsilon\|_{\mathcal{C}^2(B_{1/6})} \le \nu\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/5})} + C_\nu\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)}. \tag{4.2.45}$$

Hence, by (4.2.44) and (4.2.45) we obtain

$$\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/15})} \le C_\nu(1+\|u\|_{L^\infty(\mathbb{R}^N)}) + C\nu\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/5})}. \tag{4.2.46}$$

To conclude, one needs to apply the above estimates at every point inside $B_{1/5}$ at every scale: for any $x \in B_{1/5}$, let $r>0$ be a radius such that $B_r(x) \subset B_{1/5}$. Then we consider

$$v^x_{\varepsilon,r}(y) := u_\varepsilon(x+ry), \tag{4.2.47}$$

and we observe that $v^x_{\varepsilon,r}$ solves an analogous equation as the one solved by u_ε with the kernel given by

$$K^x_{\varepsilon,r}(y,z) := r^{N+\sigma}K_\varepsilon(x+ry, rz)$$

and with right hand side

$$F^x_{\varepsilon,r}(y) := r^\sigma \int_{\mathbb{R}^N} f(x+ry-\tilde{x}, u(x+ry-\tilde{x}))\widetilde{\eta}_\varepsilon(\tilde{x})d\tilde{x}.$$

We now observe that the kernels $K^x_{\varepsilon,r}$ satisfy assumptions (4.2.1) and (4.2.2) uniformly with respect to ε, r, and x. Moreover, for $|x|+r \le 1/5$, and ε small, we have

$$\|F^x_{\varepsilon,r}\|_{\mathcal{C}^1(B_{1/2})} \le r^\sigma C(1+\|u\|_{\mathcal{C}^1(B_{3/4})}),$$

with $C>0$ depending on $\|f\|_{\mathcal{C}^1(B_1\times\mathbb{R})}$ only. Hence, by (4.2.37) this implies

$$\|F^x_{\varepsilon,r}\|_{\mathcal{C}^1(B_{1/2})}\le r^\sigma C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right),$$

where $M=\|u\|_{L^\infty(B_1)}$. Thus, applying (4.2.46) to $v^x_{\varepsilon,r}$ (by the discussion we just made, the constants are all independent of ε, r, and x) and scaling back, we get

$$\|u_\varepsilon\|^*_{\mathcal{C}^{2,\beta}(B_{r/15}(x))}\le C_\nu\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right)+C\nu\|u_\varepsilon\|^*_{\mathcal{C}^{2,\beta}(B_{r/5}(x))}.$$

Using now Lemma 4.2.4 inside $B_{1/5}$ with $\mu=1/15$, $\nu=1/5$, $m=2$, and $\Lambda_\nu=C_\nu(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])})$, we conclude

$$\|u_\varepsilon\|_{\mathcal{C}^{2,\beta}(B_{1/75})}\le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right).$$

This implies

$$\|u\|_{\mathcal{C}^{2,\beta}(B_{1/75})}\le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right),$$

by letting $\varepsilon\to 0$ (see (4.2.25)). Since $\beta<\sigma-1$, this is equivalent to

$$\|u\|_{\mathcal{C}^{\sigma+\alpha}(B_{1/75})}\le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right),\quad \text{for any } \alpha<1,$$

and $M=\|u\|_{L^\infty(B_1)}$. A standard covering/rescaling argument completes the proof of Theorem 4.2.1 in the case $k=0$.

4.2.5 The induction argument.

We already proved Theorem 4.2.1 in the case $k=0$.

We now show by induction that, for any $k\ge 1$ and every $1<\sigma<2$, $0<\alpha<1$,

$$\|u\|_{\mathcal{C}^{k+\sigma+\alpha}(B_{1/2^{3k+4}})}\le C(k)\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right),\quad (4.2.48)$$

for $M=\|u\|_{L^\infty(B_1)}$ and some constant $C(k)>0$: by a standard covering/rescaling argument, this proves (4.2.4) and so Theorem 4.2.1. As we shall see, the argument is more or less identical to the case $k=0$.

By a slight abuse of notation, we define the cut-off function

$$\widetilde{\vartheta}:=\begin{cases}1 & \text{in } B_{1/2^{3k+5}},\\ 0 & \text{on } \mathbb{R}^N\setminus B_{1/2^{3k+4}}.\end{cases}$$

Using Lemmas 4.2.7 and 4.2.8, we differentiate the equation $k+1$ times according to the following computation: first we observe that, since by induction hypothesis (4.2.48) is true for $k-1$ and we can choose $2-\sigma<\alpha<1$ so that $\sigma+\alpha>2$, we deduce that $f_\varepsilon \in \mathcal{C}^{k+1}(B_{1/2^{3k+4}})$ with

$$\begin{aligned} \|f_\varepsilon\|_{\mathcal{C}^{k+1}(B_{1/2^{3k+4}})} &\leq C\left(1+\|u\|_{\mathcal{C}^{k+1}(B_{1/2^{3k+4}})}\right) \\ &\leq C\left(\|u\|_{L^\infty(\mathbb{R}^N)}+\|f\|_{L^\infty(B_1\times[-M,M])}\right), \end{aligned} \quad (4.2.49)$$

where $M=\|u\|_{L^\infty(B_1)}$ and with $C>0$ depending on $\|f\|_{\mathcal{C}^{k+1}(B_1\times\mathbb{R})}$ only. Now we take $\gamma\in(\mathbb{N}\cup\{0\})^N$ with $|\gamma|=k+1$ and we differentiate the equation to obtain

$$\begin{aligned} &\partial^\gamma f_\varepsilon(x) \\ &= \sum_{\substack{1\leq i\leq N\\ 0\leq\lambda_i\leq\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_N)}} \binom{\gamma_1}{\lambda_1}\dots\binom{\gamma_N}{\lambda_N}\int_{\mathbb{R}^N}\partial_x^\lambda K_\varepsilon(x,w)\,\delta(\partial_x^{\gamma-\lambda}(\widetilde{\vartheta}u_\varepsilon))(x,w)\,dw \\ &+ \sum_{\substack{1\leq i\leq N\\ 0\leq\lambda_i\leq\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_N)}} \binom{\gamma_1}{\lambda_1}\dots\binom{\gamma_N}{\lambda_N}\int_{\mathbb{R}^N}\partial_x^\lambda K_\varepsilon(x,w)\,\delta(\partial_x^{\gamma-\lambda}(1-\widetilde{\vartheta})u_\varepsilon)(x,w)\,dw. \end{aligned}$$

Then, we isolate the term with $\lambda=0$ in the first sum:

$$\int_{\mathbb{R}^N} K_\varepsilon(x,w)\,\delta(\partial_x^\gamma(\widetilde{\vartheta}u_\varepsilon))(x,w)\,dw = A_1-A_2-A_3,$$

with

$$\begin{aligned} A_1 &:= \partial^\gamma f_\varepsilon(x), \\ A_2 &:= \sum_{\substack{1\leq i\leq N\\ 0\leq\lambda_i\leq\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_N)\neq 0}} \binom{\gamma_1}{\lambda_1}\dots\binom{\gamma_N}{\lambda_N}\int_{\mathbb{R}^N}\partial_x^\lambda K_\varepsilon(x,w)\,\delta(\partial_x^{\gamma-\lambda}(\widetilde{\vartheta}u_\varepsilon))(x,w)\,dw \\ A_3 &:= \sum_{\substack{1\leq i\leq N\\ 0\leq\lambda_i\leq\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_N)}} \binom{\gamma_1}{\lambda_1}\dots\binom{\gamma_N}{\lambda_N}\int_{\mathbb{R}^N}\partial_x^\lambda K_\varepsilon(x,w)\,\delta(\partial_x^{\gamma-\lambda}(1-\widetilde{\vartheta})u_\varepsilon)(x,w)\,dw. \end{aligned}$$

We claim that

$$\|A_1-A_2-A_3\|_{L^\infty(B_{1/2^{3k+6}})}\leq C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)}+\|u_\varepsilon\|_{\mathcal{C}^{k+2}(B_{1/2^{3k+4}})}\right), \quad (4.2.50)$$

with a constant that depends only on $\|f\|_{\mathcal{C}^{k+1}(B_1\times\mathbb{R})}$. Indeed, by the fact that $|\gamma-\lambda|\leq k$, and the definition of $\widetilde{\vartheta}$, we see that

$$\begin{aligned} \|A_2\|_{L^\infty(B_{1/2^{3k+6}})} &\leq C(k)\,\|\widetilde{\vartheta}u_\varepsilon\|_{\mathcal{C}^{k+2}(\mathbb{R}^N)} \\ &\leq C(k)\,\|u_\varepsilon\|_{\mathcal{C}^{k+2}(B_{1/2^{3k+4}})}. \end{aligned} \quad (4.2.51)$$

Furthermore, since $(1-\widetilde{\vartheta})u_\varepsilon = 0$ inside $B_{1/2^{3k+5}}$, we can use (4.2.16) with $v := (1-\widetilde{\vartheta})u_\varepsilon$ to obtain

$$\|A_3\|_{L^\infty(B_{1/2^{3k+6}})} \le C\|u\|_{L^\infty(\mathbb{R}^N)}.$$

This last estimate, (4.2.49), and (4.2.51) allow us to conclude the validity of (4.2.50).

Now, by [57, Theorem 61] applied to $\partial_x^\gamma(\widetilde{\vartheta}u_\varepsilon)$ we get

$$\|u_\varepsilon\|_{\mathcal{C}^{\sigma+k+\alpha}(B_{1/2^{3k+7}})} \le C\left(1+\|u_\varepsilon\|_{\mathcal{C}^{k+2}(B_{1/2^{3k+4}})} + \|u\|_{L^\infty(\mathbb{R}^N)}\right),\ 0<\alpha<1,$$

with $C>0$ depending on $\|f\|_{\mathcal{C}^{k+1}(B_1\times\mathbb{R})}$. Note that the previous inequality is the analogous of (4.2.44) with $\sigma+\alpha = 2+\beta$ where $0<\beta<\sigma-1$. Hence, arguing as in the case $k=0$ (see the argument after (4.2.44)) we conclude that

$$\|u_\varepsilon\|_{\mathcal{C}^{\sigma+k+\alpha}(B_{1/2^{3(2k+1)+5}})} \le C\left(1+\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(B_1\times[-M,M])}\right),$$

where $M = \|u\|_{L^\infty(B_1)}$. Then taking the limit when $\varepsilon\to 0$, and using a covering argument, we prove (4.2.48) concluding the proof of Theorem 4.2.1.

Chapter 5

Regularity of nonlocal minimal surfaces.

5.1 Introduction, preliminaries and functional settings.

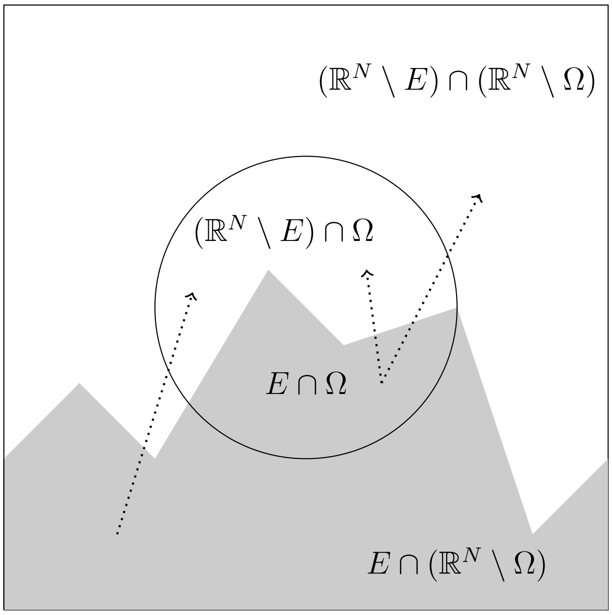
Classically, minimal surfaces, or surfaces with zero mean curvature, arise in physical situations where two phases interact and the energy of this interaction is proportional to the area of interface. Motivated by the structure of interphases arising in phase transition models with long range interactions, that is when two particles on different phases contribute with a non trivial amount to the total energy even if they are away from the interface, (see [59, 127, 133, 134, 136]), L. Caffarelli, J. M. Roquejoffre and O. Savin introduced in [53] a nonlocal version of minimal surfaces. These objects are obtained by minimizing a "nonlocal perimeter" inside a fixed domain Ω.
Roughly speaking, if we consider a measurable set $E \subseteq \mathbb{R}^N$ the main idea of its fractional, or nonlocal, perimeter inside Ω is that every point of E interacts with every point outside E giving rise to a functional that we want to minimize. In this minimization we have to take into account "the boundary datum" that will imply that there are some interactions that will not contribute to the minimization procedure. More precisely, for $0 < s < 1$ and two sets $A, B \subset \mathbb{R}^N$ with disjoint interiors, we define the interaction term

$$L(A, B) := \int_A \int_B \frac{dx\, dy}{|x - y|^{N+s}}.$$

Then, the nonlocal (s)-perimeter of E inside Ω is defined by

$$\begin{aligned}\operatorname{Per}(E,\Omega,s) &:= L\left(E\cap\Omega,(\mathbb{R}^N\setminus E)\cap\Omega\right)\\ &+ L\left(E\cap\Omega,(\mathbb{R}^N\setminus E)\cap(\mathbb{R}^N\setminus\Omega)\right)\\ &+ L\left(E\cap(\mathbb{R}^N\setminus\Omega),(\mathbb{R}^N\setminus E)\cap\Omega\right). \qquad (5.1.1)\end{aligned}$$

See the following figure where we represent the set E in gray and Ω as a circle:



We remark here that, formally,

$$\begin{aligned}\operatorname{Per}(E,\Omega,s) &= L(E,\mathbb{R}^N\setminus E)-L(E\setminus\Omega,(\mathbb{R}^N\setminus E)\cap(\mathbb{R}^N\setminus\Omega))\\ &= \frac{\|\chi_E\|^2_{H^{s/2}(\mathbb{R}^N)}}{2}-L(E\setminus\Omega,(\mathbb{R}^N\setminus E)\cap(\mathbb{R}^N\setminus\Omega)).(5.1.2)\end{aligned}$$

It is worth pointing out the assumption that $0<s<1$ is needed to define a proper interaction between a set and its complement. For example, an easy computation shows that even for an simple set E like the unit ball, the term $L(E,\mathbb{R}^N\setminus E)$ diverges to infty when $s\in(-\infty,0]\cup[1,\infty)$.

The previous functional (5.1.1) gives rise to a minimization problem on the family of sets which coincide with E outside Ω. That is, one can say that E is (s)-minimal in Ω if $\operatorname{Per}(E,\Omega,s)\le\operatorname{Per}(F,\Omega,s)$ for every measurable set F such that $F\setminus\Omega=E\setminus\Omega$. We note here that, by [53, Section 3], we can assure that the nonlocal perimeter has the necessary compactness and semicontinuity properties to guarantee the existence of these (s)-minimizers (see [53, Theorem 3.2]).

As was mentioned previously, the *nonlocal* (s)*-minimal surfaces* correspond to the boundary of the (s)-minimizers of the above functional (5.1.1) with the "boundary condition" that $E \cap (\mathbb{R}^N \setminus \Omega)$ is prescribed. By (5.1.2), we get that, surprisingly, these surfaces can be attained my minimizing the $H^{s/2}$ norm of the characteristic function χ_E. More recisely, when $s < 1$ and E is reasonably smooth, $\|\chi_E\|_{H^{s/2}}$ becomes finite whereas for $s = 1$ this is not true. That is, we cannot obtain classical minimal surfaces as sets minimizing the $H^{1/2}$ norm.

Moreover, in [53, Theorem 5.1], it is proved that the Euler-Lagrange equation corresponding to the functional (5.1.1), and satisfied by the (s)-minimizers, is the following:

$$H_s(x) := \int_{\mathbb{R}^N} \frac{\chi_E(y) - \chi_{\mathbb{R}^N \setminus E}(y)}{|x-y|^{N+s}} \, dy = 0, \quad x \text{ on } \partial E. \tag{5.1.3}$$

The scalar quantity $H_s(x)$ is called the *nonlocal mean curvature* of E (or of ∂E) at x. From the geometric point of view, the fact that $H_s(x)$ is equal to zero implies that an average of E, centered at a given point of ∂E, is adjusted by the average of its complementary. That is, while the standard mean curvature measures "mean deviation from flatness" at the infinitesimal scale, the nonlocal mean curvature takes into account all scales. We remark that if ∂E is $\mathcal{C}^2$ in a neighborhood of x, then $H_s(x)$ is well-defined in the principal value sense. On the other hand, if ∂E is not smooth, it satisfies the equation in a suitable viscosity sense (in particular, the equation is satisfied in the classical sense at every point where ∂E is $\mathcal{C}^2$). The Euler-Lagrange equation in the viscosity sense means that at every point x where ∂E has a tangent $\mathcal{C}^2$ surface included in E (respectively $\mathbb{R}^N \setminus E$) we have $\geq$ (respectively $\leq$) in (5.1.3).

We also remark here that (5.1.3) says that $(-\Delta)^{s/2}(\chi_E - \chi_{\mathbb{R}^N \setminus E}) = 0$ along ∂E.

As we observed, (s)-minimal surfaces have vanishing (s)-mean curvature, as occurs analogously in the classical case with minimal surfaces and the mean curvature. To make the analogy even stronger, we recall that, suitably renormalized, the (s)-perimeter approaches, when $s \to 1^-$, the classical perimeter, with good geometric and functional analytic properties (see [13, 60]). That is, s-minimal surfaces approach the classical ones, both in a geometric sense and in a Γ-convergence framework, with uniform estimates as $s \to 1^-$.

With respect to regularity, in the pioneering work [53] it is proved that "flat (s)-minimal surface" are $\mathcal{C}^{1,\alpha}$ hypersurfaces for all $\alpha < s$. In particular, when s is sufficiently close to 1, they inherit some nice regularity properties from the classical minimal surfaces.

However in all the previous literature people only focused on the $\mathcal{C}^{1,\alpha}$ regularity of these objects, and higher regularity was left as an open problem. In this chapter we address this issue, and we prove that $\mathcal{C}^{1,\alpha}$ (s)-minimal surfaces are indeed $\mathcal{C}^{\infty}$. In fact this is the statement of the following.

Theorem 5.1.1. *Take $0 < s < 1$, and let ∂E be an (s)-minimal surface in K_R for some $R > 0$. Assume that*

$$\partial E \cap K_R = \left\{(x', x_N) \,:\, x' \in B_R^{N-1} \text{ and } x_N = u(x')\right\}, \tag{5.1.4}$$

for some $u : B_R^{N-1} \to \mathbb{R}$, with $u \in \mathcal{C}^{1,\alpha}(B_R^{N-1})$ for any $\alpha < s$ and $u(0) = 0$. Then,

$$u \in \mathcal{C}^{\infty}(B_\rho^{N-1}) \quad \text{for every } 0 < \rho < R.$$

The previous theorem combined with Corollary 1 in [137] gives the following regularity result in the plane.

Corollary 5.1.2. *Let $N = 2$. Then, for $0 < s < 1$, any (s)-minimal surface is a smooth embedded curve of class $\mathcal{C}^{\infty}$.*

By [53, Theorem 2.4], the previous corollary and Theorem 5.1.1 we get that if E is an (s)-minimizer on B_1, then $\partial E \cap B_{1/2}$ is, with the possible exception of a closed set Σ of finite $(N-3)$ Hausdorff dimension, a $\mathcal{C}^{\infty}$-hypersurface around each of its points. Observe that, since we expect that ∂E has $(N-1)$ dimension, Σ is somehow negligible inside ∂E.

Moreover, the regularity result of Theorem 5.1.1 combined with [53, Theorem 6.1] and [61, Theorems 1, 3, 4, 5], implies also the following results.

Corollary 5.1.3. *Fix $0 < s_o < 1$. Let $s_o < s < 1$ and ∂E be a (s)-minimal surface in B_R for some $R > 0$. There exists $\varepsilon_\star > 0$, possibly depending on N, s_o and α, but independent of s and R, such that if*

$$\partial E \cap B_R \subseteq \{|x \cdot e_N| \le \varepsilon_\star R\}$$

then $\partial E \cap B_{R/2}$ is a $\mathcal{C}^{\infty}$-graph in the e_N-direction.

Corollary 5.1.4. *There exists $0 < \varepsilon_N < 1$ such that if $1 - \varepsilon_N < s < 1$, then:*

- *If $N \le 7$, any (s)-minimal surface is of class $\mathcal{C}^{\infty}$;*
- *If $N = 8$, any (s)-minimal surface is of class $\mathcal{C}^{\infty}$ except, at most, at countably many isolated points.*

More generally, in dimension $N \geq 9$ there exists $0 < \varepsilon_N < 1$ such that if $1 - \varepsilon_N < s < 1$ then any (s)-minimal surface is of class $\mathcal{C}^\infty$ outside a closed set Σ of Hausdorff dimension $N - 8$.

In the proof of the previous corollary nothing is known about ε_N, except when $N = 2$ because from Corollary 6.3.8 we deduce that $\varepsilon_2 = 1$. That is, except in the case $N = 2$, no explicit bound is available. We recall here that when $s \to 0^+$ the (s)-minimal sets are related with the minimizers of the Lebesgue measure (see [77]) for which no regularity is possible. This makes the regularity of (s)-minimal sets when s is close to 0 more difficult to prove than the case when s is close to 1.

5.2 Proof of the principal result: $\mathcal{C}^\infty$ smoothness.

The idea of the proof of Theorem 5.1.1 is to write the fractional minimal surface equation in a suitable form so that we can apply Theorem 4.2.1.

5.2.1 Writing the equation in terms of the function u.

The first step in our proof, using a *vertical integration* near the origin, consists in writing the (s)-minimal surface functional in terms of the function u, which (locally) parametrizes the boundary of a set E. More precisely, we assume that u parameterizes $\partial E \cap K_R$ and that, without loss of generality, $E \cap K_R$ is contained in the hypograph of u. That is,

$$E \cap K_R = \{(x', x_N) : \, x' \in B_R^{N-1} \quad \text{and} \quad u(x') \geq x_N\}.$$

Moreover, since by assumption $u(0) = 0$ and u is of class $\mathcal{C}^{1,\alpha}$, up to rotating the system of coordinates, so that $\nabla u(0) = 0$, and reducing the size of R, we can also assume that

$$\partial E \cap K_R \subset B_R^{N-1} \times [-R/8, R/8]. \tag{5.2.1}$$

Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be an even function satisfying

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leq 1/4, \\ 0 & \text{if } |t| \geq 1/2, \end{cases}$$

and define the smooth cut-off functions

$$\zeta_R(x') := \varphi(|x'|/R), \qquad \eta_R(x) := \varphi(|x'|/R)\varphi(|x_n|/R).$$

Observe that

$$\zeta_R = 1 \quad \text{in } B^{N-1}_{R/4}, \qquad \zeta_R = 0 \quad \text{outside } B^{N-1}_{R/2},$$

$$\eta_R = 1 \quad \text{in } K_{R/4}, \qquad \eta_R = 0 \quad \text{outside } K_{R/2}.$$

From (5.1.3) we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} \eta_R(y-x)\frac{\chi_E(y)-\chi_{\mathbb{R}^N\setminus E}(y)}{|x-y|^{N+s}}\,dy \\ = &\int_{\mathbb{R}^N} (\eta_R(y-x)-1)\frac{\chi_E(y)-\chi_{\mathbb{R}^N\setminus E}(y)}{|x-y|^{N+s}}\,dy, \end{aligned} \tag{5.2.2}$$

in the viscosity sense for $x \in \partial E \cap K_R$. We study each of the sides of this equation separately.

First of all we claim that, for any

$$x = (x', u(x')) \in \partial E \cap \left(B^{N-1}_{R/2} \times [-R/8, R/8]\right),$$

$$\begin{aligned} &\int_{\mathbb{R}^N} \eta_R(y-x)\frac{\chi_E(y)-\chi_{\mathbb{R}^N\setminus E}(y)}{|x-y|^{N+s}}\,dy \\ = \;& 2\int_{\mathbb{R}^{N-1}} F\left(\frac{u(x'-w')-u(x')}{|w'|}\right)\frac{\zeta_R(w')}{|w'|^{N-1+s}}\,dw', \end{aligned} \tag{5.2.3}$$

where

$$F(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{(N+s)/2}}.$$

Indeed, since η_R is even, writing $y = x - w$ we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \eta_R(y-x)\frac{\chi_E(y)-\chi_{\mathbb{R}^N\setminus E}(y)}{|x-y|^{N+s}}\,dy \\ = &\int_{\mathbb{R}^N} \eta_R(w)\frac{\chi_E(x-w)-\chi_{\mathbb{R}^N\setminus E}(x-w)}{|w|^{N+s}}\,dw \qquad (5.2.4) \\ = &\int_{\mathbb{R}^{N-1}} \zeta_R(w')\left[\int_{-R/4}^{R/4} \frac{\chi_E(x-w)-\chi_{\mathbb{R}^N\setminus E}(x-w)}{(1+(w_N/|w'|)^2)^{(N+s)/2}}\,dw_N\right]\frac{dw'}{|w'|^{N+s}}, \end{aligned}$$

where the last equality follows from the fact that $\varphi(|w_N|/R) = 1$ for $|w_N| \leq R/4$ and that, by (5.2.1) and by symmetry, the contributions of $\chi_E(x-w)$ and $\chi_{\mathbb{R}^N\setminus E}(x-w)$ outside $\{|w_N| \leq R/4\}$ cancel each other.

We now compute the inner integral: using the change variables $t := w_N/|w'|$ we have

$$\begin{aligned}
&\int_{-R/4}^{R/4} \frac{\chi_E(x-w)}{(1+(w_N/|w'|)^2)^{(N+s)/2}}\,dw_N \\
=\;& \int_{u(x')-u(x'-w')}^{R/4} \frac{1}{(1+(w_N/|w'|)^2)^{(N+s)/2}}\,dw_N \\
=\;& |w'| \int_{(u(x')-u(x'-w'))/|w'|}^{R/(4|w'|)} \frac{1}{(1+t^2)^{(N+s)/2}}\,dt \\
=\;& |w'| \left[F\left(\frac{R}{4|w'|}\right) - F\left(\frac{u(x')-u(x'-w')}{|w'|}\right)\right].
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\int_{-R/4}^{R/4} \frac{\chi_{\mathbb{R}^N\setminus E}(x-w)}{(1+(w_N/|w'|)^2)^{(N+s)/2}}\,dw_N \\
=\;& |w'| \left[F\left(\frac{u(x')-u(x'-w')}{|w'|}\right) - F\left(-\frac{R}{4|w'|}\right)\right].
\end{aligned}$$

Therefore, since F is odd, we immediately get that

$$\int_{-R/4}^{R/4} \frac{\chi_E(x-w)-\chi_{\mathbb{R}^N\setminus E}(x-w)}{(1+(w_N/|w'|)^2)^{(N+s)/2}}\,dw_N = 2|w'|F\left(\frac{u(x'-w')-u(x')}{|w'|}\right),$$

which together with (5.2.4) proves (5.2.3).

Let us point out that to justify these computations in a pointwise fashion one would need $u \in \mathcal{C}^{1,1}(x)$ (in the sense of Definition 4.1.1). However, by using the viscosity definition it is immediate to check that (5.2.3) holds in the viscosity sense (since one only needs to verify it at points where the graph of u can be touched with paraboloids).

Now we focus on the right hand side of the equation (5.2.1). Let us define the function

$$\Psi_R(x) := \int_{\mathbb{R}^N} [1-\eta_R(y-x)]\,\frac{\chi_E(y)-\chi_{\mathbb{R}^N\setminus E}(y)}{|x-y|^{N+s}}\,dy. \tag{5.2.5}$$

Since $1-\eta_R(y-x)$ vanishes in a neighborhood of $\{x=y\}$, it is immediate to check, by induction over $|\alpha|$, that the function $\psi_R(z) := \dfrac{1-\eta_R(z)}{|z|^{N+s}}$ is of class $\mathcal{C}^\infty$, with

$$|\partial^\alpha \psi_R(z)| \le \frac{C(|\alpha|)}{1+|z|^{N+s}}, \qquad \forall\,\alpha \in \mathbb{N}^N.$$

Hence, since $1/(1+|z|^{N+s}) \in L^1(\mathbb{R}^N)$ we deduce that

$$\Psi_R \in \mathcal{C}^\infty(\mathbb{R}^N), \text{ with all its derivatives uniformly bounded.} \tag{5.2.6}$$

Consequently, by (5.2.3) and (5.2.5) we deduce that u is a viscosity solution of

$$\int_{\mathbb{R}^{N-1}} F\left(\frac{u(x'-w')-u(x')}{|w'|}\right)\frac{\zeta_R(w')}{|w'|^{N-1+s}}\,dw' = -\frac{\Psi_R(x',u(x'))}{2},$$

inside $B^{N-1}_{R/2}$. Since F is odd, we can add the term $F\left(-\nabla u(x')\cdot\frac{w'}{|w'|}\right)$ inside the integral in the left hand side, so the equation actually becomes

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}}\left[F\left(\frac{u(x'-w')-u(x')}{|w'|}\right)-F\left(-\nabla u(x')\cdot\frac{w'}{|w'|}\right)\right]\frac{\zeta_R(w')}{|w'|^{N-1+s}}\,dw'\\ = \;& -\frac{\Psi_R(x',u(x'))}{2}.\end{aligned} \tag{5.2.7}$$

5.2.2 The regularity of the equation and conclusion.

We would like to apply the regularity result from Theorem 4.2.2, exploiting (5.2.6) to bound the right hand side of (5.2.7). To this aim, using the Fundamental Theorem of Calculus, we rewrite the left hand side in (5.2.7) as

$$\int_{\mathbb{R}^{N-1}}\big(u(x'-w')-u(x')+\nabla u(x')\cdot w'\big)\frac{a(x',-w')\zeta_R(w')}{|w'|^{N+s}}\,dw', \tag{5.2.8}$$

where

$$a(x',-w'):=\int_0^1\left(1+\left(t\frac{u(x'-w')-u(x')}{|w'|}-(1-t)\nabla u(x')\cdot\frac{w'}{|w'|}\right)^2\right)^{\frac{-(N+s)}{2}}dt.$$

Now, we claim that

$$\int_{\mathbb{R}^{N-1}}\delta u(x',w')\,K_R(x',w')\,dw' = -\Psi_R(x',u(x'))+A_R(x'), \tag{5.2.9}$$

where

$$K_R(x',w') := \frac{[a(x',w')+a(x',-w')]\zeta_R(w')}{2|w'|^{(N-1)+(1+s)}},$$

and

$$A_R(x') := \int_{\mathbb{R}^{N-1}}[u(x'-w')-u(x')+\nabla u(x')\cdot w']\frac{[a(x',w')-a(x',-w')]\zeta_R(w')}{|w'|^{N+s}}\,dw'.$$

To prove (5.2.9) we introduce a short-hand notation: we define

$$u^\pm(x',w') := u(x'\pm w') - u(x') \mp \nabla u(x')\cdot w', \quad a^\pm(x',w') := a(x',\pm w')\frac{\zeta_R(w')}{|w'|^{N+s}},$$

while the integration over $\mathbb{R}^{N-1}$, possibly in the principal value sense, will be denoted by $I[\cdot]$. With this notation, and recalling (5.2.8), it follows that (5.2.7) can be written

$$-\frac{\Psi_R}{2} = I[u^- a^-]. \tag{5.2.10}$$

By changing w' with $-w'$ in the integral given by I, we see that

$$I[u^+a^+] = I[u^-a^-].$$

Consequently (5.2.10) can be rewritten as

$$-\frac{\Psi_R}{2} = I[u^+a^+]. \tag{5.2.11}$$

Notice also that

$$u^+ + u^- = \delta u, \qquad I[u^+(a^+ - a^-)] = I[u^-(a^- - a^+)]. \tag{5.2.12}$$

Hence, by (5.2.10)-(5.2.12), we obtain

$$\begin{aligned} -\Psi_R &= I[u^+a^+] + I[u^-a^-] \\ &= \frac{1}{2}I[(u^+ + u^-)(a^+ + a^-)] + \frac{1}{2}I[(u^+ - u^-)(a^+ - a^-)] \\ &= \frac{1}{2}I[\delta u\,(a^+ + a^-)] + \frac{1}{2}I[(u^+ - u^-)(a^+ - a^-)] \\ &= \frac{1}{2}I[\delta u\,(a^+ + a^-)] - I[u^-(a^+ - a^-)], \end{aligned}$$

which proves (5.2.9).

Now, to conclude the proof of Theorem 5.1.1 it suffices to apply Theorem 4.2.2 iteratively: more precisely, let us start by assuming that $u \in \mathcal{C}^{1,\beta}(B^{N-1}_{2r})$ for some $r \le R/2$ and any $\beta < s$. Then, by the discussion above we get that u solves

$$\int_{\mathbb{R}^{N-1}} \delta u(x',w')\,K_r(x',w')\,dw' = -\Psi_r(x',u(x')) + A_r(x') \qquad \text{in } B^{N-1}_r.$$

Moreover, one can easily check that the regularity of u implies that the assumptions of Theorem 4.2.2 with $k = 0$ are satisfied with $\sigma := 1 + s$,

$a_0 := 1/(1-s)$ and $r_0 = R/4$. Observe that (4.2.5) holds since $\|u\|_{\mathcal{C}^{1,\beta}(B_{2r}^{N-1})}$. Furthermore, for $|w'| \leq r/2$ and $|x'| \leq r$,

$$\begin{aligned} & |(u(x'-w') - u(x') + \nabla u(x') \cdot w') \,(a(x',w') - a(x',-w'))| \\ \leq \;& C|w'|^{2\beta+1}. \end{aligned} \tag{5.2.13}$$

To prove the previous inequality, we define

$$p(\tau) := \frac{1}{(1+\tau^2)^{\frac{N+s}{2}}}. \tag{5.2.14}$$

With this notation

$$a(x',-w') = \int_0^1 p\bigg(t\frac{u(x'-w') - u(x')}{|w'|} - (1-t)\nabla u(x') \cdot \frac{w'}{|w'|}\bigg)\,dt. \tag{5.2.15}$$

Let us now consider

$$\mathcal{A}(x',w') := a(x',w') - a(x',-w'), \tag{5.2.16}$$

and

$$\mathcal{A}_*(x',w') := a(x',w') - p\bigg(+\nabla u(x') \cdot \frac{w'}{|w'|}\bigg).$$

Since p is even, we get that

$$\mathcal{A}(x',w') = \mathcal{A}_*(x',w') - \mathcal{A}_*(x',-w'). \tag{5.2.17}$$

Therefore, since $|p'(t)| \leq C$, by (5.2.15) and the fact that $u \in \mathcal{C}^{1,\beta}(B_{2r}^{N-1})$, it follows that

$$\begin{aligned} & |\mathcal{A}_*(x',-w')| \\ \leq \;& \int_0^1 \int_0^1 \bigg|\frac{d}{d\lambda} p\bigg(\lambda t \frac{u(x'-w') - u(x')}{|w'|} - [\lambda(1-t) + (1-\lambda)]\nabla u(x') \cdot \frac{w'}{|w'|}\bigg)\bigg|\,d\lambda\,dt \\ \leq \;& \int_0^1 t\frac{|U(x',w')|}{|w'|}\bigg(\int_0^1 \bigg|p'\bigg(\lambda t\frac{U(x',w')}{|w'|} - \nabla u(x') \cdot \frac{w'}{|w'|}\bigg)\bigg|\,d\lambda\bigg)\,dt \\ \leq \;& C|w'|^{\beta}, \end{aligned} \tag{5.2.18}$$

for all $|w'| \leq r/2$, where

$$U(x',w') := u^-(x',w') = u(x'-w') - u(x') + \nabla u(x') \cdot w'. \tag{5.2.19}$$

Estimating $\mathcal{A}_*(x',w')$ in the same way, by (5.2.16)-(5.2.18), we get

$$|a(x',w') - a(x',-w')| \leq |w'|^{\beta}, \tag{5.2.20}$$

for $x' \in B_r^{N-1}$ and $w' \in B_{r/2}^{N-1}$. Since $u \in \mathcal{C}^{1,\beta}(B_{2r}^{N-1})$, by the previous inequality we obtain (5.2.13). Observe that (5.2.13) implies that the integral defining A_r is convergent by choosing $\beta > s/2$. Furthermore, a tedious computation (which we postpone to Subsection 5.2.3 below) shows that

$$A_r \in \mathcal{C}^{2\beta-s}(B_r^{N-1}). \tag{5.2.21}$$

Hence, by Theorem 4.2.2 with $k = 0$ we deduce that $u \in \mathcal{C}^{1,2\beta}(B_{r/2}^{N-1})$. But then this implies that $A_r \in \mathcal{C}^{4\beta-s}(B_{r/4}^{N-1})$ and so by Theorem 4.2.2 again $u \in \mathcal{C}^{1,4\beta}(B_{r/8}^{N-1})$ for all $\beta < s$. Note that, once we know that $\|u\|_{\mathcal{C}^{k,\beta}(B_{2r}^{N-1})}$ is bounded for some $k \geq 2$ and $\beta \in (0,1]$, for any $|\gamma| \leq k-1$ we get, exactly as in the case $k = 0$, that for $|w'| \leq r/2$,

$$\left|\partial_x^\gamma \big([u(x'-w') - u(x') + \nabla u(x') \cdot w'] \, [a(x',w') - a(x',-w')] \big)\right| \leq C|w'|^{2\beta+1}.$$

Hence

$$\partial_x^\gamma A_r(x) = \int_{\mathbb{R}^{N-1}} \partial_x^\gamma \big(U(x',w')\mathcal{A}(x',w') \big) \frac{\zeta_r(w')}{|w'|^{N+s}} \, dw',$$

is well define for $\beta > s/2$. One can also prove that $A_r \in \mathcal{C}^{k-1,2\beta-s}(B_r^{N-1})$. Therefore we can iterate this argument infinitely many times to get that $u \in \mathcal{C}^m(B_{\lambda^m r}^{N-1})$ for some $\lambda > 0$ small, for every $m \in \mathbb{N}$. Then, by a simple covering argument we obtain that $u \in \mathcal{C}^m(B_\rho^{N-1})$ for any $\rho < R$ and $m \in \mathbb{N}$, that is, u is of class $\mathcal{C}^\infty$ inside B_ρ for any $\rho < R$. This completes the proof of Theorem 5.1.1.

5.2.3 Hölder regularity of A_r.

We now prove (5.2.21), that is,

$$\text{if } u \in \mathcal{C}^{1,\beta}(B_{2r}^{N-1}) \text{ then } A_r \in \mathcal{C}^{2\beta-s}(B_r^{N-1}) \text{ for } r \leq R/2.$$

For this we observe that, by (5.2.17) and (5.2.19),

$$A_r(x') = \int_{\mathbb{R}^{N-1}} U(x',w') \frac{\mathcal{A}(x',w')}{|w'|^{N+s}} \zeta_r(w') \, dw'.$$

To prove the desired Hölder condition of the function $A_r(x')$, we first note that

$$U(x',w') = \int_0^1 \big[\nabla u(x') - \nabla u(x'-tw')\big] dt \cdot w'.$$

Since $u \in \mathcal{C}^{1,\beta}(B_R^{N-1})$ and $2r \leq R$, we get, for x', $y' \in B_r^{N-1}$ and $w' \in B_{r/2}^{N-1}$,

$$|U(x',w') - U(y',w')| \leq C \, \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\} \tag{5.2.22}$$

and

$$|U(x',w')| \le C|w'|^{\beta+1}. \tag{5.2.23}$$

Therefore, from (5.2.22) and (5.2.23) it follows that, for any x', $y' \in B_r^{N-1}$,

$$\begin{aligned}
& |A_r(x') - A_r(y')| \\
= & \left| \int_{\mathbb{R}^{N-1}} \big(U(x',w')\mathcal{A}(x',w') - U(y',w')\mathcal{A}(y',w')\big) \frac{\zeta_r(w')}{|w'|^{N+s}}\, dw' \right| \\
\le & \; C \int_{\mathbb{R}^{N-1}} \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\} \frac{|\mathcal{A}(x',w')|}{|w'|^{N+s}} \zeta_r(w')\, dw' \\
+ & \; C \int_{\mathbb{R}^{N-1}} |w'|^{\beta+1} \frac{|\mathcal{A}(x',w') - \mathcal{A}(y',w')|}{|w'|^{N+s}} \zeta_r(w')\, dw' \\
=: & \; I_1(x',y') + I_2(x',y').
\end{aligned} \tag{5.2.24}$$

On one hand, by (5.2.16) and (5.2.20) we get, for any $\beta > s/2$,

$$\begin{aligned}
I_1(x',y') & \le C \int_{\mathbb{R}^{N-1}} \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\} |w'|^{\beta-N-s} \zeta_r(w')\, dw' \\
& \le C|x'-y'|^\beta \int_{|x'-y'|}^{r/2} t^{\beta-s-1} dt + \int_0^{|x'-y'|} t^{2\beta-s-1}\, dt \\
& \le C|x'-y'|^{2\beta-s}.
\end{aligned} \tag{5.2.25}$$

On the other hand, to estimate I_2 we note that

$$\begin{aligned}
|\mathcal{A}(x',w') - \mathcal{A}(y',w')| & \le |\mathcal{A}_*(x',w') - \mathcal{A}_*(y',w')| \\
& + |\mathcal{A}_*(y',-w') - \mathcal{A}_*(x',-w')|.
\end{aligned} \tag{5.2.26}$$

Hence, arguing as in (5.2.18) we have

$$\begin{aligned}
& |\mathcal{A}_*(x',-w') - \mathcal{A}_*(y',-w')| \\
\le & \int_0^1 t \frac{|U(x',w')|}{|w'|} \int_0^1 \left| p'\!\left(\lambda t \frac{U(x',w')}{|w'|} - \nabla u(x') \cdot \frac{w'}{|w'|} \right) \right. \\
- & \left. p'\!\left(\lambda t \frac{U(y',w')}{|w'|} - \nabla u(y') \cdot \frac{w'}{|w'|} \right) \right| d\lambda\, dt \\
+ & \int_0^1 t \frac{|U(x',w') - U(y',w')|}{|w'|} \int_0^1 \left| p'\!\left(\lambda t \frac{U(y',w')}{|w'|} - \nabla u(y') \cdot \frac{w'}{|w'|} \right) \right| d\lambda\, dt \\
=: & \; I_{2,1}(x',y') + I_{2,2}(x',y').
\end{aligned} \tag{5.2.27}$$

We bound each of these integrals separately. First, since $|p'(t)| \le C$, it follows immediately from (5.2.22) that

$$I_{2,2}(x',y') \le C \min\{|x'-y'|^\beta, |w'|^\beta\}. \tag{5.2.28}$$

On the other hand, by (5.2.22), (5.2.23) and the fact that $u \in \mathcal{C}^{1,\beta}(B_{2r}^{N-1})$ and p' is uniformly Lipschitz, we get

$$\begin{aligned} I_{,1}(x',y') &\leq C|w'|^\beta \left(\frac{|U(x',w')-U(y',w')|}{|w'|} + |\nabla u(x') - \nabla u(y')|\right) \\ &\leq C|w'|^\beta \left(\min\{|x'-y'|^\beta, |w'|^\beta\} + |x'-y'|^\beta\right) \\ &\leq C|w'|^\beta |x'-y'|^\beta. \end{aligned} \tag{5.2.29}$$

Then, assuming without loss of generality $r \leq 1$, so that also $|x'-y'| \leq 1$ and $|w'| \leq 1/2$, by (5.2.27), (5.2.28) and (5.2.29) it follows that

$$\begin{aligned} |\mathcal{A}_*(x',-w') - \mathcal{A}_*(y',-w')| &\leq C\Big(|w'|^\beta |x'-y'|^\beta + \min\{|x'-y'|^\beta, |w'|^\beta\}\Big) \\ &\leq C\min\{|x'-y'|^\beta, |w'|^\beta\}. \end{aligned} \tag{5.2.30}$$

Since $|\mathcal{A}_*(y',w') - \mathcal{A}_*(x',w')|$ is bounded in the same way, by (5.2.26), we have

$$|\mathcal{A}(x',w') - \mathcal{A}(y',w')| \leq C\min\{|x'-y'|^\beta, |w'|^\beta\}.$$

By arguing as in (5.2.25), we get that, for any $s/2 < \beta < s$,

$$\begin{aligned} I_2(x',y') &\leq C\int_{\mathbb{R}^{N-1}} |w'|^{\beta+1} \frac{\min\{|x'-y'|^\beta, |w'|^\beta\}}{|w'|^{N+s}} \zeta_r(w')dw' \\ &\leq C|x'-y'|^{2\beta-s}. \end{aligned} \tag{5.2.31}$$

Finally, by (5.2.24), (5.2.25) and (5.2.31), we conclude that

$$|A_r(x') - A_r(y')| \leq C|x'-y'|^{2\beta-s}, \quad x',\, y' \in B_r^{N-1},$$

as desired.

Part III

A non local parabolic problem.

Chapter 6

A Widder's type Theorem for the heat equation with nonlocal diffusion.

6.1 Introduction, preliminaries and functional settings.

The heat equation has the hyperplane $t = 0$ as a characteristic surface and this causes that the Cauchy problem with initial data on $t = 0$ is not well posed in general. There is however a perfect match between the Principles of Thermodynamics (see [87]) and the model of transfer of heat given by such an equation. This is reflected into the fact that the initial temperature evolves in time as the convolution with a kernel, giving rise to an average and smoothing out in this way the potential effect of sharp thermal differences in agreement with the entropy effect of the Second Principle of Thermodynamics. On the other hand, uniqueness holds true under a positivity assumption on the function, in agreement this time with the so called Third Principle of Thermodynamics, according to which temperatures are always positive if measured in the Kelvin scale. In this sense D. V. Widder, following the ideas of Täcklind and Tychonoff in [160, 166], proved that there cannot be a positive solution of the heat equation that vanishes at time zero. Moreover he obtained the following classical result:

Theorem 6.1.1. *([172, Theorem 6]) Assume that* $u : \mathbb{R}^N \times [0, T) \subset \overline{\mathbb{R}^N_+} \to \mathbb{R}$ *is so that*

$$u(x, t) \geq 0, \quad u \in \mathcal{C}(\mathbb{R}^N \times [0, T)), \quad u_t, u_{x_i x_i} \in \mathcal{C}(\mathbb{R}^N \times (0, T)),$$

and satisfies

$$u_t(x,t) - \Delta u(x,t) = 0, \quad (x,t) \in \mathbb{R}^N \times (0,T),$$

in the classical sense. Then

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(y,0) e^{\frac{-|x-y|^2}{4t}}\, dy.$$

References [78, 98, 160, 165, 166] show how, in the 1920's and 1930's, previously to the work of Widder, a number of authors have dealt with the question of the uniqueness by imposing various restrictions on the behavior of the function $u(x,t)$ in portions of the xt-plane near $x = \pm\infty$. Widder's representation theorem has had a wide range of applications in the work of many other authors. See for instance [100] and the references therein.

A fundamental step to obtain this seminal result is the following.

Lemma 6.1.2. *([172, Theorem B]) Let $u \in \mathcal{C}(\mathbb{R}^N \times (0,T))$ be a strong solution of*

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = 0 & \text{in } \mathbb{R}^N, \end{cases}$$

such that $|u(x,t)| \leq a e^{b|x|^2}$ for some positive constants a and b and $(x,t) \in \mathbb{R}^N \times (0,T)$. Then $u(x,t) = 0$ for every $(x,t) \in \mathbb{R}^N \times (0,T)$.

Note that this lemma follows in the spirit of the well-known uniqueness lemma for bounded solutions, changing this boundedness property by a growth condition.

In this chapter we obtain a similar result for the *nonlocal heat equation*

$$u_t + (-\Delta)^s u = 0 \qquad \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \quad 0 < s < 1, \tag{6.1.1}$$

in which the diffusion is given by a power of the Laplacian. This is precisely the statement of the following.

Theorem 6.1.3. *If $u \geq 0$ is a strong solution of*

$$u_t + (-\Delta)^s u = 0 \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \quad 0 < s < 1, \tag{6.1.2}$$

then

$$u(x,t) = \int_{\mathbb{R}^N} p^t(x-y) u(y,0)\, dy.$$

Here,

$$p^t(x) = \frac{1}{t^{N/2s}} p\left(\frac{x}{t^{1/2s}}\right), \tag{6.1.3}$$

and

$$p(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi - |\xi|^{2s}} d\xi, \tag{6.1.4}$$

is the solution of

$$\begin{cases} p^t + (-\Delta)^s p = 0 & \textit{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ p(x,0) = \delta_0(x) & \textit{in } \mathbb{R}^N. \end{cases}$$

Also, from the previous theorem we trivially deduce the next.

Corollary 6.1.4. *If u is a bounded strong solution of* (6.1.2) *then*

$$u(x,t) = \int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy.$$

It is worthy to point out that for every $0 < s \leq 1$ the operator in (6.1.1) does not satisfies the so called *Hadamard condition*, that is, the Cauchy problem is ill posed (see [99] for more details). Indeed if we consider $u(x,t) = e^{\lambda t + ix\cdot w}$ as a solution of the *nonlocal heat equation* we obtain that

$$\lambda = |h|^{2s} > C(1 + \log(1 + |h|)), \quad C > 0.$$

This follows from the fact that

$$(-\Delta)^s e^{\lambda t + ix\cdot w} = |w|^{2s} e^{\lambda t + ix\cdot w},$$

as the following computations shows:

$$\begin{aligned} (-\Delta)^s(e^{ix\cdot w}) &= C(N,s)\text{P.V.}\int_{\mathbb{R}^N} \frac{e^{ix\cdot w} - e^{iy\cdot w}}{|x-y|^{N+2s}}\,dy \\ &= C(N,s)e^{ix\cdot w}\text{P.V.}\int_{\mathbb{R}^N} \frac{1 - e^{iz\cdot w}}{|z|^{N+2s}}\,dz \\ &= C(N,s)e^{ix\cdot w}\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \int_{\mathbb{S}^{N-1}} \frac{1-\cos(t\theta\cdot w)}{t^{1+2s}}\,d\sigma(\theta)\,dt \\ &= C(N,s)|w|^{2s}e^{ix\cdot w}\int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{1-\cos(t\theta\cdot w')}{t^{1+2s}}\,d\sigma(\theta)\,dt \\ &= |w|^{2s}e^{ix\cdot w}. \end{aligned}$$

Observe that, by rotation invariance,

$$(C(N,s))^{-1} = \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{1-\cos(t\theta\cdot w')}{t^{1+2s}}\,d\sigma(\theta)\,dt = \int_{\mathbb{R}^N} \frac{1-\cos(z_1)}{|z|^{N+2s}}\,dz,$$

independently of w'.

As a consequence we face here a problem similar to the one of the heat equation. For that reason, in order to get the proof of Theorem 6.1.3, it would be natural to look for a similar result as Lemma 6.1.2, that is, to get the uniqueness of the solutions under a pointwise condition related to the decay of the fundamental solution. However we will observe that, consistently with the non local behavior of the problem, it is necessary to ask for a boundedness in the norm instead of a pointwise boundedness. See condition *i)* of Definition 6.1.5 and Theorem 6.2.1. Proposition 6.4.2 shows how to obtain the natural pointwise condition for *strong* solutions that are s-subharmonic.

We also remark here that if we consider the problem

$$u_t + (-\Delta)^s u = 0 \qquad \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \quad 0 < s < 1,$$

with initial datum $u(x,0) = u_0(x)$ assuming, say, $u_0 \in \mathcal{C}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$, then a solution is obtained by the convolution with the kernel given in (6.1.3)-(6.1.4). That is, a solution is given by

$$u(x,t) = \int_{\mathbb{R}^N} p^t(x-y) u_0(y)\, dy. \tag{6.1.5}$$

We look here for a class of solutions of the fractional parabolic equation, such that the unilateral sign condition $u(x,t) \geq 0$ implies that u, necessarily, is of the form (6.1.5) with $u_0(x)$ replaced by the trace $u(x,0)$. This is the type of extension that we propose for the classical result of Widder to the nonlocal equation in (6.1.1). There exists another type of uniqueness results for this equation. See for example [109, Proposition 8.1] where the author obtained the uniqueness for solutions in $u(\cdot,t) \in \mathcal{C}_0(\mathbb{R}^N)$ when $u_0(x) \in \mathcal{C}_0(\mathbb{R}^N)$.

Along this chapter we will describe several interpretations of what *solution to equation* (6.1.1) means, and consider weak, viscosity and strong solutions, in a sense that we now make more precise.

Let us recall that, $u(\cdot,t) \in \mathcal{L}^s(\mathbb{R}^N)$ if

$$\int_{\mathbb{R}^N} \frac{|u(x,t)|}{1+|x|^{N+2s}}\, dx < \infty,$$

(see (0.0.24)). Then we have the following.

Definition 6.1.5. *We say that $u(x,t)$ is a weak solution of the fractional heat problem*

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

if the following conditions hold:

i) $u \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$ *for every* $T' < T$.

ii) $u \in \mathcal{C}((0,T), L^1_{loc}(\mathbb{R}^N))$.

iii) For every test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N \times [0,T))$ *and* $0 < T' < T$ *one has that*

$$\int_0^{T'} \int_{\mathbb{R}^N} [-u(x,t)\varphi_t(x,t) + u(x,t)(-\Delta)^s \varphi(x,t)]\, dx\, dt =$$

$$\int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\, dx - \int_{\mathbb{R}^N} u(x,T')\varphi(x,T')\, dx. \tag{6.1.6}$$

Condition *ii*) is imposed so that the right hand side of equality (6.1.6) makes sense for every T'. Notice that the continuity would follow in any case from the left hand side of (6.1.6) due to the integrability condition in *i*).

Consider now

$$\begin{aligned}\mathbb{C}_p^{1,2}(\mathbb{R}^N \times (0,T)) \;=\; \Big\{ f : \mathbb{R}^N \times (0,T) \to \mathbb{R} \,|\quad & f_t \in \mathcal{C}(\mathbb{R}^N \times (0,T)),\\ & f_{x_i,x_j} \in \mathcal{C}(\mathbb{R}^N \times (0,T)) \text{ and}\\ & \sup_{t\in(0,T)} |f(x,t)| \le C(1+|x|)^p \Big\}.\end{aligned}$$

Definition 6.1.6. *A function* $u \in \mathcal{C}(\mathbb{R}^N \times (0,T))$ *is a* viscosity subsolution (resp. supersolution) *of*

$$u_t + (-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^N \times (0,T), \tag{6.1.7}$$

if for all $(x,t) \in \mathbb{R}^N \times (0,T)$ *and* $\varphi \in \mathbb{C}_p^{1,2}(\mathbb{R}^N \times (0,T))$ *such that* $u - \varphi$ *attains a local maximum (minimum) at* (x,t) *one has*

$$\varphi_t(x,t) + (-\Delta)^s \varphi(x,t)) \le 0 \qquad (\textit{resp. } \ge).$$

We say that $u \in \mathcal{C}(\mathbb{R}^N \times (0,T))$ *is a* viscosity solution *of* (6.1.7) *in* $\mathbb{R}^N \times (0,T)$ *if it is both a viscosity subsolution and supersolution.*

See [104] for more details about this type of solutions.

Finally we introduce the notion of strong solutions of the fractional heat equation. For that we remark here that, by Proposition 2.1.1 if, for some $\beta > 0$, $u \in \mathcal{L}^s(\mathbb{R}^N) \cap \mathcal{C}^{2s+\beta}(\mathbb{R}^N)$ (or $\mathcal{C}^{1,2s+\beta-1}(\mathbb{R}^N)$ if $s > 1/2$), then $(-\Delta)^s u$ is well defined as the principal value given in (0.0.21) and defines a continuous function.

In this chapter, and following standard procedures in the theory of singular integrals, we will define hereafter the principal value as the two-sided limit

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon \to 0} \int_{\{y \,|\, \varepsilon < |x-y| < 1/\varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy.$$

We will use the same notation for this extended operator. When $u \in \mathcal{L}^s(\mathbb{R}^N)$ this definition coincides with the usual one.

Definition 6.1.7. *We say that $u(x,t)$ is a strong solution of the fractional heat equation*

$$u_t + (-\Delta)^s u = 0 \quad \textit{for } (x,t) \in \mathbb{R}^N \times (0,T),$$

if the following conditions hold:

i) $u_t \in \mathcal{C}(\mathbb{R}^N \times (0,T))$.

ii) $u \in \mathcal{C}(\mathbb{R}^N \times [0,T))$.

iii) The equation is satisfied pointwise for every $(x,t) \in \mathbb{R}^N \times (0,T)$, that is,

$$u_t(x,t) + C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x,t) - u(y,t)}{|x-y|^{N+2s}}\, dy = 0.$$

Note that if $u(x,t)$ is a strong solution then

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x,t) - u(y,t)}{|x-y|^{N+2s}}\, dy \in \mathcal{C}(\mathbb{R}^N \times (0,T)).$$

Observe also that condition $ii)$ implies $u \in \mathcal{C}((0,T), L^1_{loc}(\mathbb{R}^N))$. Therefore, if $u(x,t)$ is a strong solution of the fractional heat equation satisfying $u(x,t) \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$ for every $T' < T$, then

$$u(x,t) \text{ is a weak solution of the fractional heat equation.} \tag{6.1.8}$$

As a byproduct of our results, we will see that this holds true for non-negative strong solutions (see the forthcoming Corollary 6.3.6).

Now for a given polynomial $\overline{P}$ with real coefficients we consider the differential operator $\overline{P}(D)$. Then one has

$$\overline{P}(D)p^t(x) = \int_{\mathbb{R}^N} e^{ix\cdot\xi}\overline{P}(i\xi)e^{-t|\xi|^{2s}}\, d\xi,$$

where p^t is the kernel given in (6.1.3) and (6.1.4). Since $e^{-t|\xi|^{2s}}$ is a tempered distribution we deduce that $p^t \in \mathcal{C}^\infty(\mathbb{R}^N \times (0,\infty))$ (see for instance [34] and [79] for more details). In particular, and as a consequence of Theorem 6.1.3, we get that if u is a non negative strong solution of the fractional heat equation then $u \in \mathcal{C}^\infty(\mathbb{R}^N \times (0,T))$.

6.2 Uniqueness for weak solutions.

To begin with, we prove a uniqueness result for weak solutions with vanishing initial condition. The proof is quite complicated and involves many fine integral estimates. The nonlocal feature of the problem also makes localization and cutoff arguments much harder than in the classical case. The question of uniqueness for another type of weak solutions belonging to the class $L^2(\mathbb{R}^N \times (0,T)) \cap L^2((0,T) : H^s(\mathbb{R}^N))$, that is energy solutions, has been studied in [75, Theorem 6.1].

Our uniqueness result for general weak solutions is the following.

Theorem 6.2.1. *Set $T > 0$, $0 < s < 1$ and let u be a weak solution of the fractional heat equation*

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{6.2.1}$$

Then $u(x,t) = 0$ for every $t \in (0,T)$ and a.e. $x \in \mathbb{R}^N$.

Proof. We must show that $u(x,t_0) = 0$ for an arbitrary $t_0 \in (0,T)$ and $x \in \mathbb{R}^N$. For this, we fix $R_0 > 0$ and $\theta \in \mathcal{C}_0^\infty(B_{R_0})$ and we will prove that

$$\int_{\mathbb{R}^N} u(x,t_0)\theta(x)\,dx = 0.$$

For any $t \in [0,t_0)$, we define

$$\varphi(x,t) := (\theta(\cdot) * p^{t_0-t}(\cdot))(x).$$

Therefore

$$\widehat{\varphi}(\xi,t) = \widehat{\theta}(\xi)e^{-(t_0-t)|\xi|^{2s}} = C(\xi)e^{t|\xi|^{2s}}.$$

Since

$$\widehat{\varphi}_t(\xi,t) = |\xi|^{2s}\widehat{\varphi}(\xi,t),$$

we have that

$$\begin{cases} \varphi_t - (-\Delta)^s \varphi = 0 & \text{for } (x,t) \in \mathbb{R}^N \times [0,t_0), \\ \varphi(x,t_0) = \theta(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{6.2.2}$$

By [109] (see also [30, 35, 47]) we know that

$$\frac{1}{C}\frac{1}{1+|x|^{N+2s}} \le p(x) \le \frac{C}{1+|x|^{N+2s}}. \tag{6.2.3}$$

Then we claim that

$$|\varphi(x,t)| = |\theta(x) * p^{t_0-t}(x)| \leq \frac{C_1}{1+|x|^{N+2s}}, \tag{6.2.4}$$

where C_1 depends of N, $2s$, R_0, $M := \|\theta\|_{L^\infty(B_{R_0})}$ and t_0.
Indeed, considering without lost of generality that $2R_0 > 1$, we will distinguish two cases:

When $|x| \leq 2R_0$, using that p^t is a summability kernel in L^1 we have that

$$\begin{aligned} \left|\int_{\mathbb{R}^N} p^{t_0-t}(y)\theta(x-y)\,dy\right| &\leq M\int_{\mathbb{R}^N} p^{t_0-t}(y)\,dy = M \\ &\leq M\,\frac{1+(2R_0)^{N+2s}}{1+|x|^{N+2s}} \\ &\leq \frac{c_1}{1+|x|^{N+2s}}. \end{aligned} \tag{6.2.5}$$

where $c_1 = c_1(N,\, 2s,\, R_0,\, M)$.

Consider now $|x| > 2R_0$. Note that from (6.1.3) and (6.2.3), we obtain that

$$p^t(y) \leq \frac{C}{t^{\frac{N}{2s}}\left(1+\frac{|y|^{N+2s}}{t^{\frac{N+2s}{2s}}}\right)} \leq \frac{Ct}{|y|^{N+2s}}. \tag{6.2.6}$$

Then, since for $|x-y| \leq R_0$ it follows that $|y| \geq \frac{|x|}{2}$, we have

$$p^{t_0-t}(y) \leq \frac{2^{N+2s}C(t_0-t)}{|x|^{N+2s}}.$$

As a consequence,

$$\begin{aligned} \left|\int_{\mathbb{R}^N} p^{t_0-t}(y)\theta(x-y)\,dy\right| &= \left|\int_{|x-y|\leq R_0} p^{t_0-t}(y)\theta(x-y)\,dy\right| \\ &\leq 2^{N+2s}CM\,|B_{R_0}|\frac{t_0-t}{|x|^{N+2s}} \\ &\leq 2^{N+2s}2CM\,|B_{R_0}|\frac{t_0}{1+|x|^{N+2s}} \\ &\leq \frac{c_2}{1+|x|^{N+2s}}, \end{aligned} \tag{6.2.7}$$

where $c_2 = c_2(N,\, 2s,\, R_0,\, M,\, t_0)$ and $|B_{R_0}|$ denotes, as usual, the Lebesgue measure of the ball B_{R_0}. Hence, (6.2.4) follows from (6.2.5) and (6.2.7).

Applying (6.2.4) to the derivatives of $\theta \in \mathcal{C}_0^\infty(B_{R_0})$, we also have

$$|\nabla\varphi(x,t)| = |\nabla\theta(x) * p^{t_0-t}(x)| \le \frac{C_2}{1+|x|^{N+2s}}, \tag{6.2.8}$$

where C_2 depends on N, $2s$, R_0, $M' := \|\nabla\theta\|_{L^\infty(B_{R_0})}$ and t_0.

Then, from (6.2.4) and the fact that $u \in L^1([0,t_0], \mathcal{L}^s(\mathbb{R}^N))$ we deduce that

$$M_2 := \int_0^{t_0}\int_{\mathbb{R}^N} |u(x,t)\varphi(x,t)|\,dx\,dt < \infty. \tag{6.2.9}$$

Let now $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that

$$\chi_{B_{1/2}} \le \phi \le \chi_{B_1}. \tag{6.2.10}$$

For $R > 2R_0$ we define

$$\phi_R(x) := \phi\left(\frac{x}{R}\right), \quad x \in \mathbb{R}^N,$$

and

$$\psi(x,t) := \varphi(x,t)\phi_R(x).$$

As was discussed in Chapter 2, recall that, for suitable f and g, we have

$$(-\Delta)^s(fg)(x) = f(x)(-\Delta)^s g(x) + g(x)(-\Delta)^s f(x) - B(f,g)(x),$$

where $B(f,g)$ is the bilinear form given by

$$B(f,g)(x) := C(N,s)\int_{\mathbb{R}^N} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{N+2s}}\,dy.$$

Applying this formula, for a fixed t, to the functions φ and ϕ_R we obtain from (6.2.2)

$$(-\Delta)^s\psi = \varphi(-\Delta)^s\phi_R + \phi_R\varphi_t - B(\varphi,\phi_R). \tag{6.2.11}$$

Moreover, by the terminal condition in (6.2.2), $(u\psi)(x,t_0) = u(x,t_0)\theta(x)\phi_R(x)$.

Thus, considering ψ as a test function in (6.2.1) we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} u(x,t_0)\theta(x)\phi_R(x)\,dx\right| \qquad (6.2.12)\\
= &\left|\int_0^{t_0}\int_{\mathbb{R}^N} \left[u\varphi_t(x,t)\phi_R(x) - u(-\Delta)^s(\psi(x,t))\right]\,dx\,dt\right|\\
= &\left|\int_0^{t_0}\int_{\mathbb{R}^N} \left[uB(\varphi,\phi_R)(x,t) - u\varphi(x,t)(-\Delta)^s\phi_R(x)\right]\,dx\,dt\right|\\
\le &\int_0^{t_0}\int_{\mathbb{R}^N} |u\varphi(x,t)|\,|(-\Delta)^s\phi_R(x)|\,dx\,dt\\
+ &\int_0^{t_0}\int_{\mathbb{R}^N} |u(x,t)|\,|B(\varphi,\phi_R)(x,t)|\,dx\,dt.
\end{aligned}$$

Since θ is supported in B_{R_0} and $R_0 < R/2$, and recalling (6.2.10), we conclude that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} u(x,t_0)\theta(x)\,dx\right| \le &\int_0^{t_0}\int_{\mathbb{R}^N} |u\varphi(x,t)|\,|(-\Delta)^s\phi_R(x)|\,dx\,dt\\
+ &\int_0^{t_0}\int_{\mathbb{R}^N} |u(x,t)|\,|B(\varphi,\phi_R)(x,t)|\,dx\,dt\\
=: &\; I_1(R) + C(N,s)\,I_2(R). \qquad (6.2.13)
\end{aligned}$$

It remains to show that

$$\lim_{R\to\infty} I_1(R) + I_2(R) = 0.$$

Indeed, since by Proposition 2.1.1,

$$|(-\Delta)^s\phi_R(x)| = R^{-2s}\left|((-\Delta)^s\phi)\left(\frac{x}{R}\right)\right| \le C_0R^{-2s},$$

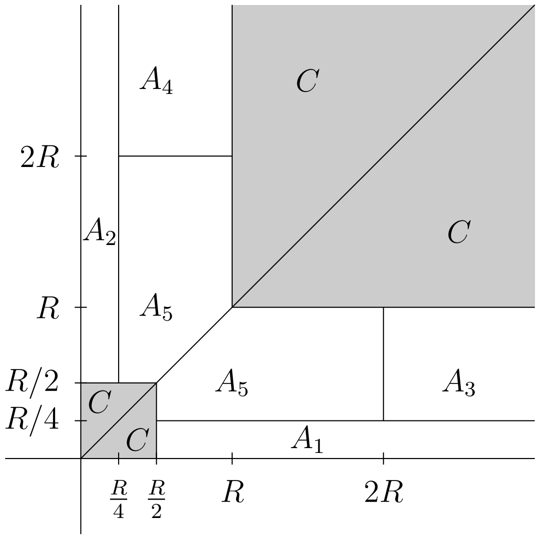
then, using (6.2.9), it follows that

$$I_1(R) \le C_0R^{-2s}\int_0^{t_0}\int_{\mathbb{R}^N} |u\varphi(x,t)|\,dx\,dt \le C_0M_2R^{-2s}.$$

Therefore

$$\lim_{R\to\infty} I_1(R) = 0. \qquad (6.2.14)$$

We now proceed with the estimate of $I_2(R)$. For this we split $\mathbb{R}^N \times \mathbb{R}^N$ into six domains suitably described by the radii $R/4$, $R/2$, R and $2R$ and represented (for $N = 1$) in the following picture:



Therefore

$$\mathbb{R}^{2N} = \left(\bigcup_{k=1}^{5} A_k\right) \cup C,$$

where

$$A_1 := \{(x,y) :\ |x| > R/2,\ |y| \le R/4\},\ A_2 := \{(x,y) :\ |x| \le R/4,\ |y| > R/2\},$$

$$A_3 := \{(x,y) :\ |x| \ge 2R,\ R/4 < |y| < R\},$$
$$A_4 := \{(x,y) :\ R/4 < |x| < R,\ |y| \ge 2R\},$$
$$A_5 := \{(x,y) :\ R/4 < |x| < 2R,\ R/4 < |y| < 2R\} \setminus C$$

and

$$C := \{(x,y) :\ |x| \le R/2,\ |y| \le R/2\} \cup \{(x,y) :\ |x| \ge R, |y| \ge R\}.$$

From (6.2.10), we know that $\phi_R(x) - \phi_R(y) = 0$ if $(x,y) \in C$, and so

$$\begin{aligned} I_2(R) &= \int_0^{t_0} \int_{\mathbb{R}^N} |u(x,t)| \left| \int_{\mathbb{R}^N} \frac{(\varphi(x,t) - \varphi(y,t))\,(\phi_R(x) - \phi_R(y))}{|x-y|^{N+2s}}\, dy\, dx \right| dt \\ &\le \sum_{k=1}^{5} I_2^{A_k}(R), \end{aligned} \tag{6.2.15}$$

where

$$I_2^{A_k}(R) = \int_0^{t_0} \int_{A_k} |u(x,t)| \frac{|\varphi(x,t) - \varphi(y,t)|\,|\phi_R(x) - \phi_R(y)|}{|x-y|^{N+2s}}\, dy\, dx\, dt,$$

for $k = 1, \dots, 5$.

We estimate each of these five integrals separately. For $(x,y) \in A_1$ we get that $|x-y| \geq C|x|$. Moreover by (6.2.4) it follows that

$$|\varphi(x,t)| + |\varphi(y,t)| \leq \frac{C}{1+|y|^{N+2s}}.$$

Therefore

$$\begin{aligned} I_2^{A_1}(R) &\leq \int_0^{t_0} \int_{|x|>R/2} \frac{|u(x,t)|}{|x|^{N+2s}} \int_{|y|\leq R/4} \frac{C}{1+|y|^{N+2s}} \, dy\, dx\, dt \\ &\leq C \int_0^{t_0} \int_{|x|>R/2} \frac{|u(x,t)|}{|x|^{N+2s}} \, dx\, dt. \end{aligned} \tag{6.2.16}$$

Following the same ideas, since for $(x,y) \in A_2$ we obtain that

$$|\varphi(x,t)| + |\varphi(y,t)| \leq \frac{C}{1+|x|^{N+2s}}, \quad \text{and} \quad |x-y| \geq C|y|, \tag{6.2.17}$$

then

$$\begin{aligned} I_2^{A_2}(R) &\leq \int_0^{t_0} \int_{|x|\leq R/4} \frac{|u(x,t)|}{1+|x|^{N+2s}} \int_{|y|>R/2} \frac{C}{|y|^{N+2s}} \, dy\, dx\, dt \\ &\leq CR^{-2s} \int_0^{t_0} \int_{|x|\leq R/4} \frac{|u(x,t)|}{1+|x|^{N+2s}} \, dx\, dt. \end{aligned} \tag{6.2.18}$$

Also, since for $(x,y) \in A_3$,

$$|\varphi(x,t)| + |\varphi(y,t)| \leq \frac{C}{|y|^{N+2s}}, \quad \text{and} \quad |x-y| \geq C|x|,$$

then

$$I_2^{A_3}(R) \leq CR^{-2s} \int_0^{t_0} \int_{|x|\geq 2R} \frac{|u(x,t)|}{|x|^{N+2s}} \, dx\, dt. \tag{6.2.19}$$

Similarly, using again the good decay of φ and the fact that $|x-y| \geq C|y|$ for every $(x,y) \in A_4$, we obtain that

$$I_2^{A_4}(R) \leq CR^{-2s} \int_0^{t_0} \int_{R/4<|x|<R} \frac{|u(x,t)|}{|x|^{N+2s}} \, dx\, dt. \tag{6.2.20}$$

Then, using the Monotone Convergence Theorem and the fact that $u \in L^1([0,t_0], \mathcal{L}^s(\mathbb{R}^N))$, from (6.2.16), (6.2.18), (6.2.19) and (6.2.20) it follows that

$$\lim_{R\to\infty} I_2^{A_1}(R) = \lim_{R\to\infty} I_2^{A_2}(R) = \lim_{R\to\infty} I_2^{A_3}(R) = \lim_{R\to\infty} I_2^{A_4}(R) = 0. \tag{6.2.21}$$

To estimate $I_2^{A_5}(R)$ we will treat separately the cases $0 < s < 1/2$ and $1/2 \le s < 1$.

We start with the case $0 < s < 1/2$. Let $(x,y) \in A_5$. Since in A_5 the roles of x and y are symmetric, we deduce from (6.2.4) that, in this case,

$$|\varphi(x,t)| + |\varphi(y,t)| \le \frac{C}{|x|^{N+2s}}. \tag{6.2.22}$$

Also

$$|\phi_R(x) - \phi_R(y)| \le \frac{C}{R}|x-y|. \tag{6.2.23}$$

Thus

$$I_2^{A_5}(R) \le \frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}} \int_{\frac{R}{4}\le|y|\le 2R} \frac{1}{|x-y|^{N+2s-1}}\, dy\, dx\, dt. \tag{6.2.24}$$

By the change of variables $\tilde{y} := x - y$, it follows from (6.2.24) that

$$\begin{aligned} I_2^{A_5}(R) &\le \frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}} \int_{\frac{R}{4}\le|x-\tilde{y}|\le 2R} \frac{1}{|\tilde{y}|^{N+2s-1}}\, d\tilde{y}\, dx\, dt \\ &\le \frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}} \int_{|\tilde{y}|\le 4R} \frac{1}{|\tilde{y}|^{N+2s-1}}\, d\tilde{y}\, dx\, dt. \\ &\le CR^{-2s}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}}\, dx\, dt. \end{aligned} \tag{6.2.25}$$

Therefore, using that $u \in L^1([0,t_0], \mathcal{L}^s(\mathbb{R}^N))$, we conclude that

$$\lim_{R\to\infty} I_2^{A_5}(R) = 0, \qquad \text{when } 0 < s < 1/2. \tag{6.2.26}$$

We consider now the case $1/2 \le s < 1$. By (6.2.8), we get that

$$|\varphi(x,t) - \varphi(y,t)| \le \frac{C}{1+|z|^{N+2s}}|x-y|, \tag{6.2.27}$$

for some z in the segment joining x and y. We define the set

$$\mathcal{Q} := \left\{(x,y) \in A_5 : |x-y| \le \frac{R}{100}\right\}.$$

Note that, if $(x,y) \in \mathcal{Q}$ then every point z lying on the segment from x to y satisfies $|z| \ge C|x|$. Hence, (6.2.27) and the previous estimate for ϕ_R in

(6.2.23), gives that

$$\begin{aligned}
&\int_0^{t_0}\int_{(x,y)\in\mathcal{Q}}|u(x,t)|\frac{|(\phi_R(x)-\phi_R(y))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\,dy\,dx\,dt\\
\leq\ &\int_0^{t_0}\int_{(x,y)\in\mathcal{Q}}|u(x,t)|\frac{C}{R|x|^{N+2s}|x-y|^{N+2s-2}}\,dy\,dx\,dt\\
\leq\ &\frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\leq|x|\leq 2R}\frac{|u(x,t)|}{|x|^{N+2s}}\int_{\frac{R}{4}\leq|y|\leq 2R}\frac{1}{|x-y|^{N+2s-2}}\,dy\,dx\,dt\\
\leq\ &\frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\leq|x|\leq 2R}\frac{|u(x,t)|}{|x|^{N+2s}}\int_{\frac{R}{4}\leq|x-\tilde{y}|\leq 2R}\frac{1}{|\tilde{y}|^{N+2s-2}}\,d\tilde{y}\,dx\,dt\\
\leq\ &CR^{1-2s}\int_0^{t_0}\int_{\frac{R}{4}\leq|x|\leq 2R}\frac{|u(x,t)|}{|x|^{N+2s}}\,dx\,dt. \qquad (6.2.28)
\end{aligned}$$

On the other hand, if $(x,y)\in A_5\setminus\mathcal{Q}$ we have that

$$|x-y|>\frac{R}{100}\geq C|y|. \qquad (6.2.29)$$

Then by (6.2.22) and (6.2.29) it follows that

$$\begin{aligned}
&\int_0^{t_0}\int_{(x,y)\in A_5\setminus\mathcal{Q}}|u(x,t)|\frac{|(\phi_R(x)-\phi_R(y))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\,dy\,dx\,dt\\
\leq\ &\frac{C}{R}\int_0^{t_0}\int_{(x,y)\in A_5\setminus\mathcal{Q}}|u(x,t)|\frac{|(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s-1}}\,dy\,dx\,dt\\
\leq\ &\frac{C}{R}\int_0^{t_0}\int_{(x,y)\in A_5\setminus\mathcal{Q}}\frac{|u(x,t)|}{|x|^{N+2s}}\frac{1}{|y|^{N+2s-1}}\,dy\,dx\,dt\\
\leq\ &\frac{C}{R}\int_0^{t_0}\int_{\frac{R}{4}\leq|x|\leq 2R}\frac{|u(x,t)|}{|x|^{N+2s}}\int_{\frac{R}{4}\leq|y|\leq 2R}\frac{1}{|y|^{N+2s-1}}\,dy\,dx\,dt\\
\leq\ &CR^{-2s}\int_0^{t_0}\int_{\frac{R}{4}\leq|x|\leq 2R}\frac{|u(x,t)|}{|x|^{N+2s}}\,dx\,dt. \qquad (6.2.30)
\end{aligned}$$

Therefore, from (6.2.28) and (6.2.30)

$$\begin{aligned} I_2^{A_5}(R) \le & \int_0^{t_0}\int_{(x,y)\in\mathcal{Q}} |u(x,t)| \frac{|(\phi_R(x)-\phi_R(y))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\,dy\,dx\,dt \\ + & \int_0^{t_0}\int_{(x,y)\in A_5\setminus\mathcal{Q}} |u(x,t)| \frac{|(\phi_R(x)-\phi_R(y))(\varphi(x,t)-\varphi(y,t))|}{|x-y|^{N+2s}}\,dy\,dx\,dt \\ \le & \; CR^{1-2s}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}}\,dx\,dt \\ + & \; CR^{-2s}\int_0^{t_0}\int_{\frac{R}{4}\le|x|\le 2R} \frac{|u(x,t)|}{|x|^{N+2s}}\,dx\,dt. \end{aligned} \tag{6.2.31}$$

Since $u \in L^1([0,t_0],\mathcal{L}^s(\mathbb{R}^N))$, and using the Monotone Convergence Theorem when $s = 1/2$, we obtain

$$\lim_{R\to\infty} I_2^{A_5}(R) = 0, \qquad \text{whenever } 1/2 \le s < 1. \tag{6.2.32}$$

Putting together (6.2.15), (6.2.21), (6.2.26) and (6.2.32) it follows that

$$\lim_{R\to\infty} I_2(R) = 0, \qquad \text{when } 0 < s < 1. \tag{6.2.33}$$

Therefore, from (6.2.13), by (6.2.14) and (6.2.33) we conclude that

$$\int_{\mathbb{R}^N} u(x,t_0)\theta(x)\,dx = 0,$$

for an arbitrary $\theta \in \mathcal{C}_0^\infty(B_{R_0})$, as wanted. □

6.3 Uniqueness for strong positive solutions.

In this section we will establish the representation of the positive strong solutions of the fractional heat equation as the Poisson integral of the initial value. That is, we prove here Theorem 6.1.3. We will need some preliminaries results. First of all, we establish that, among all possible positive solutions of the fractional heat equation, the minimal one is given by a formula that involves the convolution with the fractional heat kernel (see Lemma 6.3.2). To prove it we will use the following.

Lemma 6.3.1 (A maximum principle). *Set $D_T := \Omega\times(0,T)$ and let $v(x,t) \in \mathcal{C}(\overline{\Omega}\times[0,T])$ a strong solution of the problem*

$$\begin{cases} v_t + (-\Delta)^s v \le 0 & \text{for } (x,t)\in D_T, \\ v(x,t) \le 0 & \text{in } \big(\mathbb{R}^N\times[0,T)\big)\setminus D_T. \end{cases} \tag{6.3.1}$$

Then $v \le 0$ in $\overline{\Omega}\times[0,T)$.

Proof. Fixing an arbitrary $T' \in (0,T)$, we define

$$v(x_0,t_0) := \max_{\overline{\Omega}\times[0,T']} v(x,t).$$

Our goal is to show that $v(x_0,t_0) \le 0$. The proof is by contradiction, assuming that

$$v(x_0,t_0) > 0. \tag{6.3.2}$$

If this is the case, then (x_0,t_0) cannot lie in $(\partial\overline{\Omega}\times[0,T))\cup(\Omega\times\{0\})$, since $v \le 0$ there, thanks to the boundary conditions in (6.3.1). As a consequence, (x_0,t_0) lies in $\Omega\times(0,T']$ and then $v_t(x_0,t_0)\ge 0$. Therefore the equation in (6.3.1) implies that

$$\begin{aligned} 0 \;\ge\; & (-\Delta)^s v(x_0,t_0) = C(N,s)\text{P.V.}\int_{\mathbb{R}^N} \frac{v(x_0,t_0)-v(y,t_0)}{|y-x_0|^{N+2s}}\,dy \\ =\; & C(N,s)\left(\text{P.V.}\int_{\Omega}\frac{v(x_0,t_0)-v(y,t_0)}{|y-x_0|^{N+2s}}\,dy + \int_{\mathbb{R}^N\setminus\Omega}\frac{v(x_0,t_0)-v(y,t_0)}{|y-x_0|^{N+2s}}\,dy\right) \\ \ge\; & C(N,s)\text{P.V.}\int_{\mathbb{R}^N\setminus\Omega}\frac{v(x_0,t_0)-v(y,t_0)}{|y-x_0|^{N+2s}}\,dy. \end{aligned}$$

Since $v(y,t_0)\le 0$ for $y\in\mathbb{R}^N\setminus\Omega$, thanks to (6.3.1), we obtain that the latter integrand is strictly positive, due to (6.3.2), and this is a contradiction. □

Now we are able to prove that positive strong solutions are upper bounds for the kernel convolutions.

Lemma 6.3.2. *Let $(x,t)\in\mathbb{R}^N\times(0,T)$. If $u(x,t)\ge 0$ is a strong solution of the fractional heat equation then*

$$I := \int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy \le u(x,t), \tag{6.3.3}$$

where $p^t(x)$ is the function defined in (6.1.3) *and* (6.1.4).

Proof. First of all, observe that the integral $I = I(x,t)$ exists, for is given by the integration of the product of two (measurable) positive functions, although we do not know a priori that I is finite. However, this will be a consequence of our result that gives the inequality $I \le u(x,t)$ for every $(x,t)\in\mathbb{R}^N\times[0,T)$. To this aim, for $x\in\mathbb{R}^N$, we let, by a slight abuse of notation,

$$\phi_R(x) := \begin{cases} 1, & |x|\le R-1, \\ R-|x|, & R-1\le|x|\le R, \\ 0, & |x|>R. \end{cases} \tag{6.3.4}$$

We define

$$v_R(x,t) := \int_{\mathbb{R}^N} p^t(x-y)\phi_R(y)u(y,0)\,dy = (p^t(\cdot) * \phi_R u(\cdot,0))(x).$$

Then

$$\begin{cases} \dfrac{\partial v_R}{\partial t} + (-\Delta)^s v_R = 0 & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ v_R(x,t) \geq 0 & \text{for } (x,t) \in \mathbb{R}^N \times (0,T), \\ v_R(x,0) = \phi_R(x)u(x,0) & \text{in } \mathbb{R}^N. \end{cases}$$

Let $|x| > R$. Since $u(x,t) \in \mathcal{C}(\mathbb{R}^N \times [0,T))$, we can define the real number

$$M_R := \sup_{|y|<R} u(y,0) < \infty.$$

By (6.2.6) we have that

$$\begin{aligned} 0 \leq v_R(x,t) &\leq M_R \int_{B_R} p^t(x-y)\,dy \\ &\leq CM_R \int_{B_R} \frac{T}{|x-y|^{N+2s}}\,dy \\ &\leq C(T,M_R) \int_{B_R} \frac{dy}{||x|-R|^{N+2s}} \\ &= C(T,M_R,N)\frac{R^N}{||x|-R|^{N+2s}}, \end{aligned}$$

for any $(x,t) \in (\mathbb{R}^N \setminus B_R) \times (0,T)$.

Then, for every $\varepsilon > 0$ it follows that

$$0 \leq v_R(x,t) \leq \varepsilon, \qquad \text{for } |x| \geq \rho,\ t \in (0,T), \tag{6.3.5}$$

where

$$\rho = R + \left(\frac{C(T,M_R,N)R^N}{\varepsilon}\right)^{\frac{1}{N+2s}} > 0.$$

Moreover, we have that

$$v_R(x,0) = \phi_R(x)u(x,0) \leq u(x,0) \leq \varepsilon + u(x,0), \text{ for any } |x| \leq \rho, \tag{6.3.6}$$

and, since $u(x,t) \geq 0$ in $\mathbb{R}^N \times [0,T)$, from (6.3.5), we also obtain

$$0 \leq v_R(x,t) \leq \varepsilon \leq \varepsilon + u(x,t), \text{ for any } |x| \geq \rho, t \in [0,T). \tag{6.3.7}$$

Consider now the cylinder

$$D_{\rho,T} = B_\rho \times (0,T).$$

We define the function

$$w(x,t) := v_R(x,t) - u(x,t) - \varepsilon.$$

Then, by (6.3.6) and (6.3.7), we get that $w(x,t) \leq 0$ in $\mathbb{R}^N \times [0,T) \setminus D_{\rho,T}$. Therefore, since $u(x,t)$ is a strong solution of the fractional heat equation in $\mathbb{R}^N \times [0,T)$, applying Lemma 6.3.1 in $D_{\rho,T}$ to the function $w(x,t)$ we have that

$$v_R(x,t) \leq \varepsilon + u(x,t), \quad \text{for } |x| \leq \rho \text{ and } t \in [0,T).$$

Hence, from (6.3.7), it follows that

$$v_R(x,t) \leq \varepsilon + u(x,t), \quad \text{for } x \in \mathbb{R}^N \text{ and } t \in [0,T).$$

Since ε is fixed but arbitrary, the previous inequality implies that

$$v_R(x,t) \leq u(x,t), \text{ for every } x \in \mathbb{R}^N \text{ and } t \in [0,T).$$

Finally, by the Monotone Convergence Theorem, as $\lim_{R\to\infty} \phi_R = 1$, we conclude that

$$0 \leq v(x,t) = \lim_{R\to\infty} v_R(x,t) = \int_{\mathbb{R}^N} p^t(x-y)u(y,0)dy \leq u(x,t). \qquad \square$$

By a simple time translation, we obtain from Lemma 6.3.2 the following.

Corollary 6.3.3. *Let $0<\tau<T$ and $(x,t) \in \mathbb{R}^N \times (0, T-\tau)$. If $u(x,t) \geq 0$ is a strong solution of the fractional heat equation then*

$$\int_{\mathbb{R}^N} p^t(x-y)u(y,\tau)\, dy \leq u(x,t+\tau). \tag{6.3.8}$$

As a consequence, for every $x \in \mathbb{R}^N$ and $t \in (0,T-\tau)$ we have

$$\int_0^{T-t}\int_{\mathbb{R}^N} p^t(x-y)u(y,\tau)\, dy\, d\tau \leq \int_0^{T-t} u(x,t+\tau)d\tau. \tag{6.3.9}$$

The proof follows directly from the previous results and will be omited.

Moreover we have the following

Corollary 6.3.4. *Let $(x,t) \in \mathbb{R}^N \times (0,T)$. If $u(x,t) \geq 0$ is a strong solution of the fractional heat equation then, $u(\cdot,t) \in \mathcal{L}^s(\mathbb{R}^N)$.*

Proof. Let $0 < T' < T$. Taking $t = T - T'$ in (6.3.8), from (6.2.3), we get that

$$\begin{aligned}\frac{T-T'}{C}\int_{\mathbb{R}^N} \frac{u(y,\tau)}{(T-T')^{\frac{N+2s}{2s}} + |x-y|^{N+2s}}\,dy &\leq \int_{\mathbb{R}^N} p^{T-T'}(x-y)u(y,\tau)\,dy \\ &\leq u(x, T-T'+\tau) \\ &< \infty, \quad 0<\tau<T'. \qquad (6.3.10)\end{aligned}$$

Let

$$C(T) := \min\left\{1, \frac{1}{(T-T')^{\frac{N+2s}{2s}}}\right\}.$$

Then, since

$$\frac{1}{(T-T')^{\frac{N+2s}{2s}} + |y|^{N+2s}} \geq C(T)\frac{1}{1+|y|^{N+2s}},$$

taking $x = 0$ in (6.3.10), we conclude that

$$\int_{\mathbb{R}^N} \frac{|u(y,\tau)|}{(1+|y|^{N+2s})}\,dy < \infty, \quad 0<\tau<T'. \qquad \square$$

Remark 6.3.5. *Clearly, the same argument as in the previous proof gives, by (6.3.3), that $u(\cdot,0) \in \mathcal{L}^s(\mathbb{R}^N)$. Therefore, as was mentioned in the introduction of this chapter, if we define*

$$\widetilde{p}(x,t) := \int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy, \qquad (6.3.11)$$

then, using the decay of the kernel p^t, and its derivatives, we get that $\tilde{p}(x,t) \in \mathcal{C}^\infty(\mathbb{R}^N \times (0,T))$.

We also have

Corollary 6.3.6. *Let $(x,t) \in \mathbb{R}^N \times (0,T)$. If $u(x,t) \geq 0$ is a strong solution of the fractional heat equation, then $u(\cdot,t) \in L^1([0,T'],\mathcal{L}^s(\mathbb{R}^N))$ for every $0 < T' < T$.*

Proof. Take an arbitrary $0 < T' < T$. Doing as in the proof of Corollary 6.3.4 we get that

$$\tilde{C}(t)\int_{\mathbb{R}^N} \frac{u(y,\tau)}{1+|y|^{N+2s}}\,dy \leq u(0,t+\tau), \quad 0<\tau<T-t,$$

where

$$\tilde{C}(t) = \frac{t}{C} \min\left\{1, \frac{1}{t^{\frac{N+2s}{2s}}}\right\}.$$

That is

$$\begin{aligned}\int_0^{T'-t} \tilde{C}(t) \int_{\mathbb{R}^N} \frac{u(y,\tau)}{1+|y|^{N+2s}}\, dy\, d\tau \;&\leq\; \int_0^{T'-t} u(0, t+\tau)\, d\tau \\ &\leq\; \int_0^{T'} u(0,\tau)\, d\tau = c(T') < \infty.\end{aligned}$$

Let now $T' < T'' < T$. Doing the same as before we get

$$\int_0^{T''-t} \tilde{C}(t) \int_{\mathbb{R}^N} \frac{u(y,\tau)}{1+|y|^{N+2s}}\, dy\, d\tau \leq c(T'') < \infty.$$

Finally taking $0 < t = T'' - T' < T$ we obtain

$$0 < \tilde{C}(T'' - T') \int_0^{T'} \int_{\mathbb{R}^N} \frac{u(y,\tau)}{1+|y|^{N+2s}}\, dy\, d\tau \leq c(T'') < \infty.$$

That is, $u \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$ for every $0 < T' < T$. □

Note that, as we announced in Section 6.1 (see (6.1.8)), from Corollary 6.3.6 we can assert that if $u(x,t) \geq 0$ is a strong solution of the fractional heat equation then u is also a weak solution of the same equation.

Now we are in the situation to prove our main result:

Proof of Theorem 6.1.3. By Corollary 6.3.6 we get that $u \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$ for every $0 < T' < T$. Moreover by Lemma 6.3.2 and the previous Corollary, we also have that $\widetilde{p} \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$ where $\widetilde{p}$ was defined in (6.3.11). Define now the function

$$w(x,t) := u(x,t) - \widetilde{p}(x,t) \geq 0,\ (x,t) \in \mathbb{R}^N \times [0,T).$$

It is clear that w is a strong solution of the fractional heat equation. Moreover, as $w \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N))$, then $w(x,t)$ is also a solution in the weak sense with zero initial datum. Therefore applying Theorem 6.2.1, using the continuity of the function w, we conclude that $w(x,t) = 0$ for every $(x,t) \in \mathbb{R}^N \times [0,T)$. □

6.3.1 A remark on viscosity solutions.

As was said at the beginning of this chapter, it is natural to consider viscosity solutions of the fractional heat equation. Our purpose here is to describe some cases in which a positive viscosity solution has the unique representation in terms of the kernel p^t.

Proposition 6.3.7. *Let $\{u_n\}$ be a sequence of non-negative, strong solutions of the fractional heat equation converging, uniformly over compact sets, to a given function u. Then $u \geq 0$ is a viscosity solution of* (6.1.1) *satisfying*

$$\int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy \leq u(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,T) \tag{6.3.12}$$

and

$$u \in L^1([0,T'], \mathcal{L}^s(\mathbb{R}^N)) \quad \textit{for every } 0 < T' < T. \tag{6.3.13}$$

That is, the conclusions of Lemma 6.3.2 and Corollary 6.3.6 are satisfied.

Proof. By Lemma 6.3.2 it follows that

$$\int_{\mathbb{R}^N} p^t(x-y)u_n(y,0)\,dy \leq u_n(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,T).$$

Applying the Fatou Lemma we obtain (6.3.12). Therefore, doing as in the proof of the Corollary 6.3.6 we conclude (6.3.13). Note also that, by the comparison principle (Corolary 2.1.6 of [151]), $u_n \geq 0$ is a viscosity solution of (6.1.1) for every $n \in \mathbb{N}$. Therefore, since u is the uniform limit over compact sets of viscosity solutions, we get that u is also a viscosity solution of (6.1.1). □

Notice that to conclude the equality in (6.3.12) we would need to pass to the limit in the conclusion of Theorem 6.1.3. In order to obtain this we introduce a monotonicity condition over the sequence $\{u_n\}$. That is, we have the following.

Proposition 6.3.8. *Let $\{u_n\}$ be a monotone sequence of non-negative, strong solutions of the fractional heat equation converging, uniformly over compact sets, to a given function u. Then $u \geq 0$ is a strong, and viscosity, solution of* (6.1.1) *satisfying*

$$\int_{\mathbb{R}^N} p^t(x-y)u(y,0)\,dy = u(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,T). \tag{6.3.14}$$

Proof. Since, for every $n \in \mathbb{N}$, $u_n \geq 0$ satisfies Theorem 6.1.3, by the Monotone Convergence Theorem we obtain (6.3.14). Clearly this implies that $u \geq 0$ is a strong solution of (6.1.1). □

6.4 Other results.

Motivated by the results obtained in the classical case to prove Theorem 6.1.1, we present in this subsection two lemmas. As we remarked in the introduction of this work, it would be interesting to provide an alternative proof of Theorem 6.1.3, so the idea is to present them as a possible tool to obtain this different demonstration.

First of all we note that, as in the local case (see [173]), given a solution u, one can define the *enthalpy* term

$$v(x,t) := \int_0^t u(x,\tau)\,d\tau, \tag{6.4.1}$$

which is also a solution of the fractional heat equation. Indeed we have the following result.

Lemma 6.4.1. *Let $(x,t) \in \mathbb{R}^N \times [0,T)$. If $u(x,t)$ is a strong solution of the fractional heat equation with vanishing initial condition satisfying*

$$\int_0^t \int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|}{|x-y|^{N+2s}}\,dy\,dt = C(x) < \infty,$$

then the enthalpy term defined in (6.4.1) *is also a strong solution of the fractional heat equation. Moreover, if u is positive the function v is increasing in t for x fixed and s-subharmonic as a function of x.*

Proof. We will give a direct proof that does not require the full extend of Theorem 6.1.3. Let $(x,t) \in \mathbb{R}^N \times [0,T)$. First of all note that $v(x,t)$ satisfies the conditions $i) - ii)$ of the Definition 6.1.7. Therefore, since $u(x,t)$ is a strong solution of the fractional heat equation, by Fubini's Theorem and the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}
& C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{v(x,t) - v(y,t)}{|x-y|^{N+2s}}\,dy \\
= \;& C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \int_0^t \frac{u(x,\tau) - u(y,\tau)}{|x-y|^{N+2s}}\,d\tau\,dy \\
= \;& \int_0^t (-\Delta)^s u(x,\tau)\,d\tau \\
= \;& -\int_0^t u_\tau(x,\tau)\,d\tau \\
= \;& -u(x,t) \\
= \;& -v_t(x,t),
\end{aligned}$$

for any $(x,t) \in \mathbb{R}^N \times (0,T]$. Then, $v(x,t)$ satisfies the fractional heat equation in the strong sense. Also if $u \geq 0$ from the previous calculations we deduce that

$$-(-\Delta)^s v(x,t) = v_t(x,t) = u(x,t) \geq 0.$$

So v is s-subharmonic as a function of x and increasing in t for x fixed. □

Lemma 6.4.1 shows that the enthalpy belongs to the special class of positive strong solutions that are also s-subharmonic. This class naturally satisfies a polynomial estimate, as shown by the following result.

Lemma 6.4.2. *For $(x,t) \in \mathbb{R}^N \times [0,T)$, let $u(x,t) \geq 0$ be such that*

i) $u(x,t)$ is an s-subharmonic function with respect to the variable x,

ii) $u(x,t)$ is a strong solution of the fractional heat equation.

Then, $u(x,t) \leq C(t)(1+|x|^{N+2s})$.

Proof. Since u is an s-subharmonic function with respect to the variable x for t fixed, then u is increasing in time.
Let now $0 < t_1 < T$ and $0 < t_0 < T - t_1$. By Corollary 6.3.3 we have that

$$\int_{\mathbb{R}^N} p^t(x-y)u(y,t_1)dy \leq u(x,t+t_1), \text{ for any } 0 < t < T - t_1.$$

Therefore

$$M_{t_0} := \int_{\mathbb{R}^N} p^{t_0}(y)u(y,t_1)dy \leq u(0,t_0+t_1) < \infty. \tag{6.4.2}$$

Our objective is to show that

$$|u(x,t_1)| \leq C(t_1)(1+|x|^{N+2s}). \tag{6.4.3}$$

Once this is done, using that u is increasing in time, we would get

$$0 \leq u(x,t) \leq u(x,t_1) \leq C(t_1)(1+|x|^{N+2s}) \quad \text{for every } (x,t) \in \mathbb{R}^N \times (0,t_1).$$

But, since t_0 and t_1 are fixed but arbitrary, we would conclude that

$$|u(x,t)| \leq C(t)(1+|x|^{N+2s}).$$

So, we are left to showing that (6.4.3) is true. Note that from Corollary 6.3.4 we have that $u \in \mathcal{L}^s(\mathbb{R}^N)$. Then, since u is s-subharmonic, by [151, Proposition 2.2.6] u also satisfies the following mean value property:

$$u(x,t) \leq \int_{\mathbb{R}^N} \gamma_\lambda(x-y)u(y,t)dy, \tag{6.4.4}$$

for every $x \in \Omega \subseteq \mathbb{R}^N$ and $\lambda \leq \text{dist}(x, \partial\Omega)$. Here

$$\gamma_\lambda(x) := (-\Delta)^s \Gamma_\lambda(x),$$

was introduced in the proof of Lemma 3.4.2. Recall that

$$\Gamma_\lambda(x) := \frac{1}{\lambda^{N-2s}} \Gamma\left(\frac{x}{\lambda}\right),$$

with Γ is a $C^{1,1}$ function that coincides with $\Phi(x) := c|x|^{2s-N}$ (the fundamental solution of $(-\Delta)^s$), outside the ball B_λ and with a paraboloid inside this ball. From (6.4.4) we have

$$\begin{aligned} u(x,t_1)(1+|x|^{N+2s})^{-1} &\leq (1+|x|^{N+2s})^{-1} \int_{\mathbb{R}^N} \gamma_\lambda(y) u(x-y,t_1)\, dy \\ &= (1+|x|^{N+2s})^{-1} \int_{\{y\colon |y| \geq \lambda\}} \gamma_\lambda(y) u(x-y,t_1)\, dy \\ &+ (1+|x|^{N+2s})^{-1} \int_{\{y\colon |y| \leq \lambda\}} \gamma_\lambda(y) u(x-y,t_1)\, dy \\ &:= I_1(x) + I_2(x). \end{aligned} \tag{6.4.5}$$

Choosing

$$\lambda = \frac{|x|}{4} + 1, \tag{6.4.6}$$

if $y \in \{y : |y| \geq \lambda\}$, we have that $|y| \geq C_0(1 + |x - y|)$ whit $C_0 = 1/5$. Therefore, by [151, Proposition 2.2.3], we obtain

$$\begin{aligned} I_1(x) &\leq C(1+|x|^{N+2s})^{-1} \int_{\{y\colon |y| \geq \lambda\}} \frac{u(x-y,t_1)}{|y|^{N+2s}}\, dy \\ &\leq C(1+|x|^{N+2s})^{-1} \int_{\mathbb{R}^N} \frac{u(x-y,t_1)}{(1+|x-y|)^{N+2s}}\, dy \\ &\leq C \|u(\cdot,t_1)\|_{\mathcal{L}^s(\mathbb{R}^N)} := C_1(t_1). \end{aligned} \tag{6.4.7}$$

Moreover, since γ_λ is the fractional laplacian of a bounded $C^{1,1}$ function, we have that γ_λ is continuous and, in particular, uniformly bounded in the compact set $\{y: |y| \le \lambda\}$. Then, by (6.2.3) and (6.4.6), it follows that

$$\begin{aligned}
I_2(x) &\le C(1+|x|^{N+2s})^{-1}\int_{\{y:\,|y|\le\lambda\}} u(x-y,t_1)\,dy\\
&\le C(1+|x|^{N+2s})^{-1}\int_{\{z:\,|x-z|\le\lambda\}} u(z,t_1)\,dz\\
&\le C(1+|x|^{N+2s})^{-1}\int_{\{z:\,|z|\le 2|x|+1\}} u(z,t_1)p_1(z)\frac{1}{p_1(z)}\,dz\\
&\le C(1+|x|^{N+2s})^{-1}(1+(2|x|+1)^{N+2s})\int_{\{z:\,|z|\le 2|x|+1\}} u(z,t_1)p_1(z)\,dz\\
&\le CM_1(1+|x|^{N+2s})^{-1}(1+(2|x|+1)^{N+2s})\\
&\le \widetilde{C}(N,s)u(0,1+t_1) := C_2(t_1), \qquad (6.4.8)
\end{aligned}$$

where M_1 was given in (6.4.2). By (6.4.5), (6.4.7) and (6.4.8), we obtain (6.4.3) and we conclude the proof. □

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