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Parametric 2-dimensional L Systems and recursive fratal images: Mandelbrot set, Julia sets and biomorphs

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Abstract

L Systems have proved their expressive power. They have been used to represent the class of the initiator/iterator fractal curves (such as Sierpinski's gasket and von Koch's snowflake curve). Parametric L Systems, introduced by Prusinkiewicz and Lindenmayer, link real valued parameters to the symbols.

In this paper, parametric 0L systems are extended to n dimensions and used to represent a different class of classic fractals that includes objects such the Mandelbrot and Julia sets, or Pickover's biomorphs. [1]

Introduction: parametric 0L-systems

L-Systems have been used successfully to represent fractal curves in the initiator/iterator family [1] by providing them with a graphic interpretation, where all the lines and angles are reduced to integer multiples of the unit segment and angle. However, these systems have not been used previously to tackle a different family of fractals, which includes objects such as the Mandelbrot and Julia sets, or Pickover's biomorphs. These are hard to model by L-systems, because growth functions that capture continuous processes cannot be conveniently expressed in that way.

In a different context, with similar problems, Lindenmayer [2] proposed a generalization of L systems where numerical parameters can be attached to the system symbols. The definition of parametric 0L systems [7], [8] is reminded in the following paragraphs.

Parametric L systems [2] operate on *parametric words*, strings of *modules* consisting of *letters* with associated *parameters*. The letters belong to an *alphabet V* and the

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[2]The notation used here differs from [7] in a few symbols.

parameters belong to the set of *real numbers* $\Re$. A module with letter $A \in V$ and parameters $a_1, a_2, ..., a_n \in \Re$ is denoted $A(a_1, a_2, ..., a_n)$. Modules belong to the set $M = V \times \Re^*$, where $\Re^*$ is the set of all finite sequences of parameters. The se of all strings of modules and the set of all non-empty strings are denoted by $M^* = (V \times \Re^*)^*$ and $M^+ = (V \times \Re^*)^+$, respectively.

The real-valued *actual* parameters appearing in the words correspond with *formal* parameters used in the specification of L-system productions. If Σ is a set of formal parameters, then $c(\Sigma)$ denotes a *logical expression* with parameters from Σ, and $e(\Sigma)$ is an *arithmetic expression* with parameters from the same set. Both types of expressions consist of formal parameters and numeric constants, combined by the arithmetic operators +, –, $*$, /; the exponentiation operator with the usual representation x^n; the relational operators <, >, =; the logical operators ∨, ∧, ¬; and parenthesis (). Standard rules for syntactically correct expressions and operator precedence are applied. Relational and logical expressions evaluate to *false* or *true*. A logical statement specified as the empty string is assumed to have the value *true*. The sets of all correctly constructed logical and arithmetic expressions with parameters from Σ are noted $C(\Sigma)$ and $E(\Sigma)$ respectively.

A *parametric 0L system* is defined as an ordered quadruplet $G = (V, \Sigma, \omega, P)$; where *V* is the *alphabet* of the system, Σ is the *set of formal parameters*, $\omega \in M^+ = (V \times \Re^*)^+$ is a nonempty parametric word called the *axiom* and $P \subset (V \times \Sigma^*) \times C(\Sigma) \times (V \times E(\Sigma))^*$ is a finite *set of productions*. The symbols : and → are used to separate the three components of a production: the *predecessor,* the *condition* and the *successor*. For example, a production with predecessor $A(t)$, condition $t > 5$ and successor $B(t+1)CD(t^{0.5}, t-2)$ is written thus:

$$A(t) : t > 5 \rightarrow B(t+1)CD(t^{0.5}, t-2)$$

A production *matches* a module in a parametric word, if the letter in the module and the letter in the production predecessor are the same, the number of parameters in the module is equal to the number of formal parameters in the production predecessor, and the condition evaluates to *true* when the actual parameter values are substituted for the formal parameters in the production.

A matching production can be *applied* to the module, creating a string of modules specified by the production successor. The actual parameter values are substituted for the formal parameters in the same position. For example, the production above matches module $A(9)$, because the logical expression $t > 5$ is true for $t = 9$. The result of the application of the production to the module is the parametric word $B(10)CD(3,7)$. Derivation and language are defined for parametric L systems as for non-parametric L systems.

Definition, n-dimensional parametric 0L systems

A parametric n-dimensional 0L system is a quadruplet (V, Σ, ω, P); where V is the alphabet of the system, Σ is the set of formal parameters, ω is a n-dimensional grid over the set of parametric symbols $V \times \Re^*$ and $P \subset (V \times \Sigma^*) \times C(\Sigma) \times E(\Sigma)$ is a finite *set of productions*.

Graphic representation

We shall represent recursive fractal images by means of parametric 0L systems, using the following additional data:

- The complex area under consideration (see figure 1), defined by its lower left corner $\left(c^{00} = c_x^{00} + c_y^{00} j\right)$ and its real and imaginary sizes (Δ_x, Δ_y).
- A convergence threshold (θ), used to determine if a function f diverges for an element (z) of the ball, i.e. after a number (m) of iterations of the function, the modulus of f is greater than the threshold $\left(|f(z_m)| > \theta\right)$.
- To generate an image, the complex area is spatially digitized with real and imaginary steps (ε_x, ε_y).
- The image is a matrix M_{i_x, i_y} where $i_x = \left\lfloor \frac{\Delta_x}{\varepsilon_x} \right\rfloor, i_y = \left\lfloor \frac{\Delta_y}{\varepsilon_y} \right\rfloor$

(Insert figure 1 here)

We are using 2-dimensional parametric 0L systems to model recursive fractal images, with a grid of the same size as the image being generated $\left(M_{i_x, i_y}\right)$. The 0L system needs only a symbol (p) that gets by means of parameters all the necessary information to determine if function f diverges at that point. Each derivation applies function f to every element in the grid in such a way that the parameters associated to the symbol are changed to the result of the next application of f. To keep the system as general as possible, three classes of parameters are identified: convergence parameters, color parameters and additional parameters.

- Convergence parameters determine if function f converges or diverges. Function (f) is applied to the previous value. The modulus (or the real and imaginary part) of the result are compared to the convergence threshold (θ).
- The way each point in the image is shown depends on the value of the color parameters. Each color may be represented by a number or by a 3-dimensional vector (red, blue and green components).
- Additional information may be included: sometimes the color of the points depends on the first iteration where f diverges.

A 2-dimensional parametric 0L systems that "looks" like the Mandelbrot set

The Mandelbrot set is the boundary between the convergence and divergence domains of the recursive complex function $z_{n+1} = f(z_n) = z_n^2 + c$ where $c = c_x + c_y j$ is taken from a ball in the complex plane and $z_0 = 0$ [3].

Let $\theta \in \Re$ be the convergence threshold, $\Delta_x, \Delta_y \in \Re$ the dimension of the ball, $c^{00} = c_x^{00} + c_y^{00} j$ the under left corner of the ball and $\varepsilon_x, \varepsilon_y \in \Re$ the real and imaginary steps as described in a previous section.

The 2-dimensional parametric 0L-system is

$$S = \left(V = \{p\}, \Sigma = \{x, y, d_1, ..., d_{n_\delta}, c_x, c_y, a_1, ..., a_m\}, \omega, P\right),$$

where

- $V = \{p\}$ is the alphabet of the system.
- $\Sigma = \{x, y, d_1, ..., d_{n_\delta}, c_x, c_y, a_1, ..., a_m\}$ is the set of formal parameters, where $x, y \in \Re$ are the real and imaginary components of the complex point (with initial value 0); $d_1, ..., d_{n_\delta} \in \Re$ is the color associated to the complex point in the graphical representation; $c_x, c_y \in \Re$ are the components of the complex point $c_x + c_y j$ associated to each position; $a_1, ..., a_m \in \Re$ are *m* additional parameters.

There is a mapping

$$\delta : \Re^* \to \Re^{n_\delta} \left| \delta(x, y, d_1, ..., d_{n_\delta}, c_x, c_y, a_1, ..., a_m) = (d_1', ..., d_{n_\delta}') \right.$$

that assigns a color to each point in the derived grid from the previous values of the parameters in the grid.

- ω is a 2-dimensional grid over the set of parametric symbols $V \times \Re^*$ such that

$$\omega[k,l] = p\left(0, 0, d_1, ..., d_{n_\delta}, c_x^{00} + l * \varepsilon_x, c_y^{00} + k * \varepsilon_y, a_1, ..., a_m\right)$$
$$\forall k \in \left[0, \left\lfloor \frac{\Delta_x}{\varepsilon_x} \right\rfloor - 1\right] \cap Z, l \in \left[0, \left\lfloor \frac{\Delta_y}{\varepsilon_y} \right\rfloor - 1\right] \cap Z$$

where *[a,b]* denotes closed intervals, and $\lfloor x \rfloor$ the integer part of *x*.

- $P \subset (V \times \Sigma^*) \times C(\Sigma) \times E(\Sigma)$ is the finite *set of productions* defined as follows:

$$P = \left\{ \begin{matrix} p\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y, a_1, \ldots, a_m\right): \\ \phi_i\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y, a_1, \ldots, a_m\right) \rightarrow \\ p\left(\begin{matrix} c_x + x^2 - y^2, \\ c_y + 2xy, \\ \delta_i\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y, a_1, \ldots, a_m\right), \\ c_x, c_y, \\ \alpha_i^1\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y, a_1, \ldots, a_m\right), \\ \ldots \\ \alpha_i^m\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y, a_1, \ldots, a_m\right) \end{matrix} \right) \\ \forall i \in I \subseteq Z \end{matrix} \right\}$$

where

- $I \subseteq Z$ is a set of integer indexes.
- $\left\{\phi_i : \Re^* \rightarrow \{true, false\} \forall i \in I\right\}$ are the condition functions. There are #*I* conditions.
- $\left\{\delta_i : \Re^* \rightarrow \Re^{n_{\delta_i}} \forall i \in I\right\}$ are the color functions. The number of color functions is less than #*I*+1.
- $A = \left\{\alpha_i^j : \Re^* \rightarrow \Re \forall i \in I, \forall j \in [1, m] \cap \aleph\right\}$ are the functions that compute the next value of the additional parameters. Their number is less than #*I***m*.

The graphic representation is defined by the mapping

$$\varphi\left(p\left(x, y, d_1, \ldots, d_{n_\delta}, c_x, c_y\right)\right) = \left(d_1, \ldots, d_{n_\delta}\right)$$

It is easy to see that φ assigns to each word derived from *S* an approximation to the graphic representation of the Mandelbrot set. The graphic representation of the set is the grid obtained from φ when the depth of the derivation tends to $\propto$.

Example 1

In this example, $\Delta_x = 3.2, \Delta_y = 2.4, c^{00} = -2.4 - 1.2j$ ([4], page 9, figure 2).
The size of the image is 3200 pixels * 2400 pixels, so the steps used to digitize it are $\varepsilon_x = \varepsilon_y = \frac{1}{1000}$. The function is assumed to diverge at a point when the modulus of *f* is greater than θ=100, and a color depending on the derivation depth is assigned to the point, black color is used otherwise.

There are two additional parameters: a_1 keeps the derivation depth at which function *f* is considered to diverge: a_2 keeps track of the current derivation depth. If $a_1 \neq a_2$ *f* diverged in a previous step, and the color remains unchanged for that point in subsequent derivations.

The following algorithm is used to assign a color to each point:

- Black color ((0,0,0)) is initially assigned to every point.
- If $a_1 \neq a_2$, the color remains unchanged.
- When *f* diverges, black is used if $a_1 \in [0,16]$; if $a_1 \in (16,32]$,color $\delta_1(a_1) = \left(255, \left\lfloor \frac{165}{16} a_1 - 75 \right\rfloor, 0\right)$ is assigned to the point; otherwise, if $a_1 \in (32,200]$ color will be $\delta_2(a_1) = \left(255, \left\lfloor \frac{2005}{7} - \frac{55}{56} a_1 \right\rfloor, \left\lfloor \frac{155}{168} a_1 + \frac{1480}{21} \right\rfloor\right)$.

There are two additional parameter functions: if function *f* has diverged at a point, a_2 increases its value by 1, but a_1 remains unchanged. Otherwise, both parameters increase their values by 1.

- P_{m_2} is the finite *set of productions* defined as follows:

$$P_{m_2} = \left\{ \begin{array}{c} p(x,y,d_r,d_g,d_b,c_x,c_y,a_1,a_2) : \phi_0(x,y,c_x,c_y,a_1,a_2) \rightarrow \\ p\begin{pmatrix} x,y,\delta_0, \\ c_x,c_y, \\ \alpha_0^1(a_1),\alpha_0^2(a_2) \end{pmatrix}, \\ p(x,y,d_r,d_g,d_b,c_x,c_y,a_1,a_2) : \phi_1(x,y,c_x,c_y,a_1,a_2) \rightarrow \\ p\begin{pmatrix} x,y, \\ \delta_1(a_1),c_x,c_y, \\ \alpha_1^1(a_1),\alpha_1^2(a_2) \end{pmatrix}, \\ p(x,y,d_r,d_g,d_b,c_x,c_y,a_1,a_2) : \phi_2(x,y,c_x,c_y,a_1,a_2) \rightarrow \\ p\begin{pmatrix} x,y, \\ \delta_2(a_1),c_x,c_y, \\ \alpha_2^1(a_1),\alpha_2^2(a_2) \end{pmatrix}, \\ p(x,y,d_r,d_g,d_b,c_x,c_y,a_1,a_2) : \phi_3(a_1,a_2) \rightarrow \\ p\begin{pmatrix} x,y, \\ \delta_I(d_r,d_g,d_b),c_x,c_y, \\ \alpha_3^1(a_1),\alpha_3^2(a_2) \end{pmatrix}, \\ p(x,y,d_r,d_g,d_b,c_x,c_y,a_1,a_2) : \phi_4(x,y,c_x,c_y,a_1,a_2) \rightarrow \\ p\begin{pmatrix} c_x + x^2 - y^2, c_y + 2xy, \\ \delta_I(d_r,d_g,d_b),c_x,c_y, \\ \alpha_4^1(a_1),\alpha_4^2(a_2) \end{pmatrix}, \end{array} \right\}$$

Figure 2 shows the image derived by this system from the axiom after 200 derivations.

(Insert figure 2 here)

L System that behaves like the Julia set

The Julia set is the boundary between the convergence and divergence domains of the recursive complex function $z_{n+1} = f(z_n) = z_n^2 + c$ where $c = c_x + c_y j$ is a fixed complex point and z_0 is taken from a ball in the complex plane [3].

As in the case of Mandelbrot set, the Julia set can be graphically represented by a 2-dimensional parametric 0L system, by appropriate adjustment of the parameters associated to a single symbol p. Examples of this may be provided if needed.

A 2-dimensional parametric 0L-system that “looks” like Pickover’s biomorphs

“Biomorph” is the name that Clifford A. Pickover gave to the shapes he discovered as a result of a programming bug when studying fractal properties of several complex functions. They are described in Pickover [5] and [6]. Our approach considers

Pickover's biomorphs as a particular case of Julia set.

Example 2

This example uses the complex recursive function $z_{n+1} = sin(z_n) + e^{z_n} + c$. The values of the parameters are the following:

$$\theta = 10, \Delta_x = \Delta_y = 3.75, c^{00} = c_x^{00} + c_y^{00} j = 4.5 + 4.5 j, \varepsilon_x = \varepsilon_y = \frac{1}{240} \in \Re,$$
$$c_x + c_y j = 0.5 + 0.5 j$$

The 2-dimensional parametric 0L-system is $S_{b_2} = (\{p\}, \{x, y, d\}, \omega_{b_2}, P_{b_2})$.

- There are two condition functions

$$\phi_1(x,y) = \begin{pmatrix} \left(\left|sin(x+yj) + e^{(x+yj)} + c\right| \geq \theta\right) \vee \\ \left(\left|real\left(sin(x+yj) + e^{(x+yj)} + c\right)\right| \geq \theta\right) \vee \\ \left(\left|imag\left(sin(x+yj) + e^{(x+yj)} + c\right)\right| \geq \theta\right) \end{pmatrix}$$
$$\phi_2(x,y) = \neg\phi_1(x,y)$$

- There are two color functions:

$$\delta_1(x,y) = \begin{cases} 1(white) \quad if & \begin{pmatrix} \begin{pmatrix} \left(\left|sin(x+yj) + e^{(x+yj)} + c\right| \geq \theta\right) \wedge \\ \left(\left|real(sin(x+yj) + e^{(x+yj)} + c)\right| \geq \theta\right) \wedge \\ \left(\left|imag(sin(x+yj) + e^{(x+yj)} + c)\right| \geq \theta\right) \wedge \end{pmatrix} \\ \vee \\ \begin{pmatrix} \left(\left|sin(x+yj) + e^{(x+yj)} + c\right| \geq \theta\right) \wedge \\ \left(\left|real(sin(x+yj) + e^{(x+yj)} + c)\right| < \theta\right) \wedge \\ \left(\left|imag(sin(x+yj) + e^{(x+yj)} + c)\right| < \theta\right) \wedge \end{pmatrix} \end{pmatrix} \\ 0(black) & otherwise \end{cases}$$

$\delta_2 = 1$, that is color white.

- P_{b_2} is the finite *set of productions* defined as follows:

$$P_{b_2} = \left\{ \begin{array}{c} p(x,y,d): \\ \phi_1(x,y) \to \\ p\begin{pmatrix} x, y, \\ \delta_1(x,y) \end{pmatrix} \\ , \\ p(x,y,d): \\ \phi_2(x,y) \to \\ p\begin{pmatrix} real\left(sin(x+yj)+e^{(x+yj)}+c\right), \\ imag\left(sin(x+yj)+e^{(x+yj)}+c\right), \\ \delta_2 \end{pmatrix} \end{array} \right\}$$

Figure 3 shows the image derived from the axiom after 10 derivations.

(Insert figure 3 here)

Figure 4 shows another example, which uses the complex recursive function $z_{n+1} = z_n^5 + c$ with the following values for the parameters:

$$\theta = 10, \Delta_x = \Delta_y = 8, c^{00} = c_x^{00} + c_y^{00} j = -4 - 4j, \varepsilon_x = \varepsilon_y = \frac{1}{300} \in \Re,$$
$$c_x + c_y j = 0.05 + 0.75 j$$

The 2-dimensional parametric 0L system uses three color parameters plus two additional parameters.

(Insert figure 4 here)

Conclusion

L Systems have proved to be expressive enough to model initiator-iterator fractal images (Sierpinski gasket, von Koch snowflake, etc…), but it is not clear how they could handle fractals as those described in this paper.

We decided to use parametric L systems to represent the Mandelbrot and Julia sets, plus biomorphs, after several unsuccessful attempts to model these recursive fractals by means of non parametric L systems. The difficulties we have found are similar to those reported by Prusinkiewicz in his introduction to parametric L systems.

After studying the examples described in this paper and others we may provide, it can be observed that the alphabet of our L systems contains only a symbol. This is enough to carry all the information associated to the problem. Such a simple structure proves the expressive power of parametric n-dimensional L systems.

The procedure has been implemented in both APL and Java to generate the fractal images shown here. Languages that operate easily with matrices take advantage of the parametric L systems approach.

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Figures

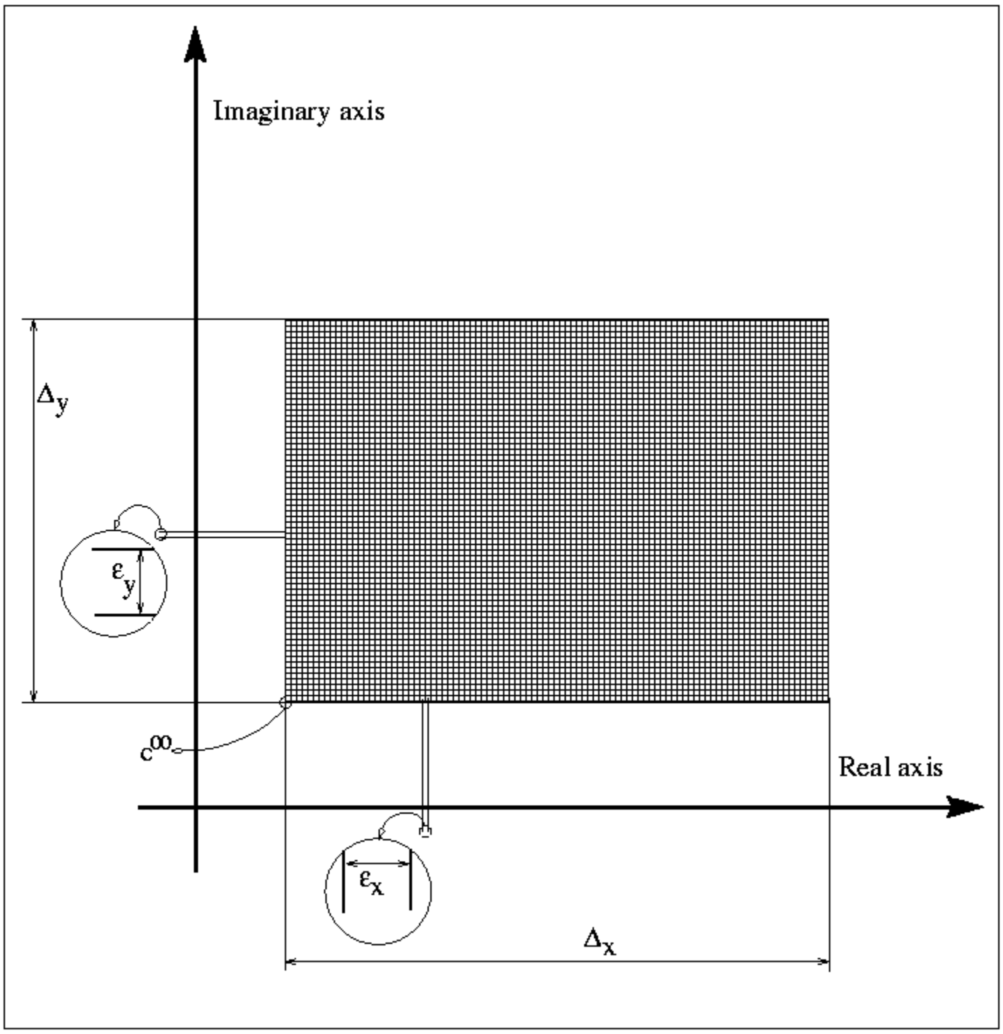

Figure 1. Graphical parameters

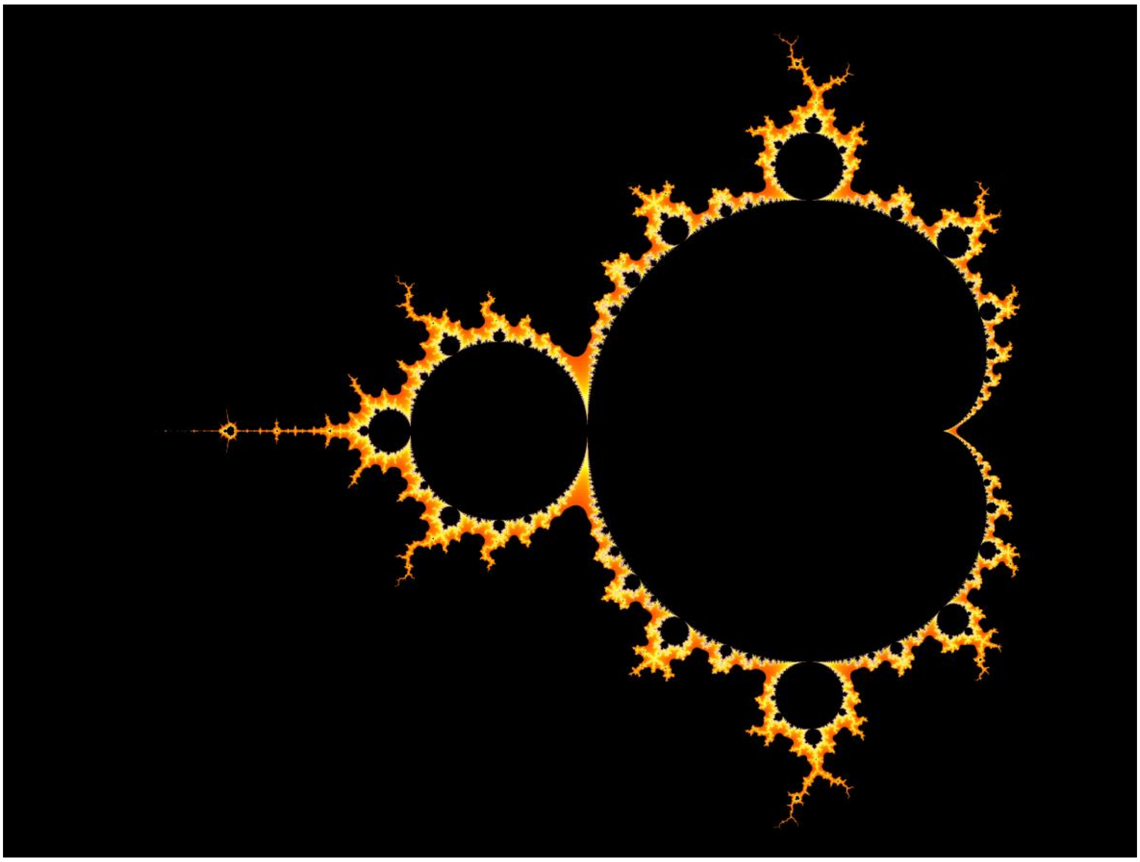

Figure 2. Graphical appearance of the Mandelbrot example

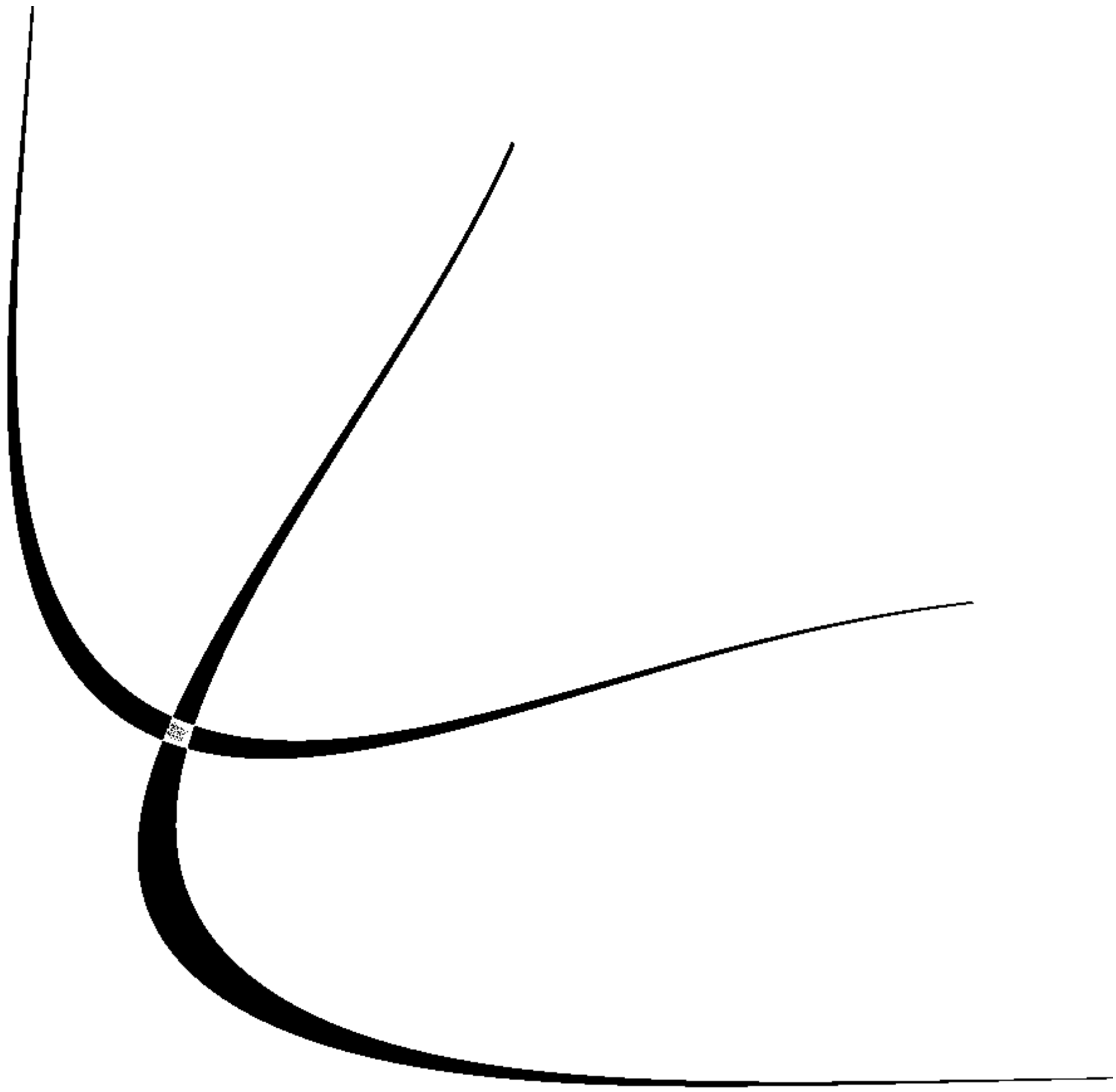

Figure 3. Graphical appearance of the first biomorph example

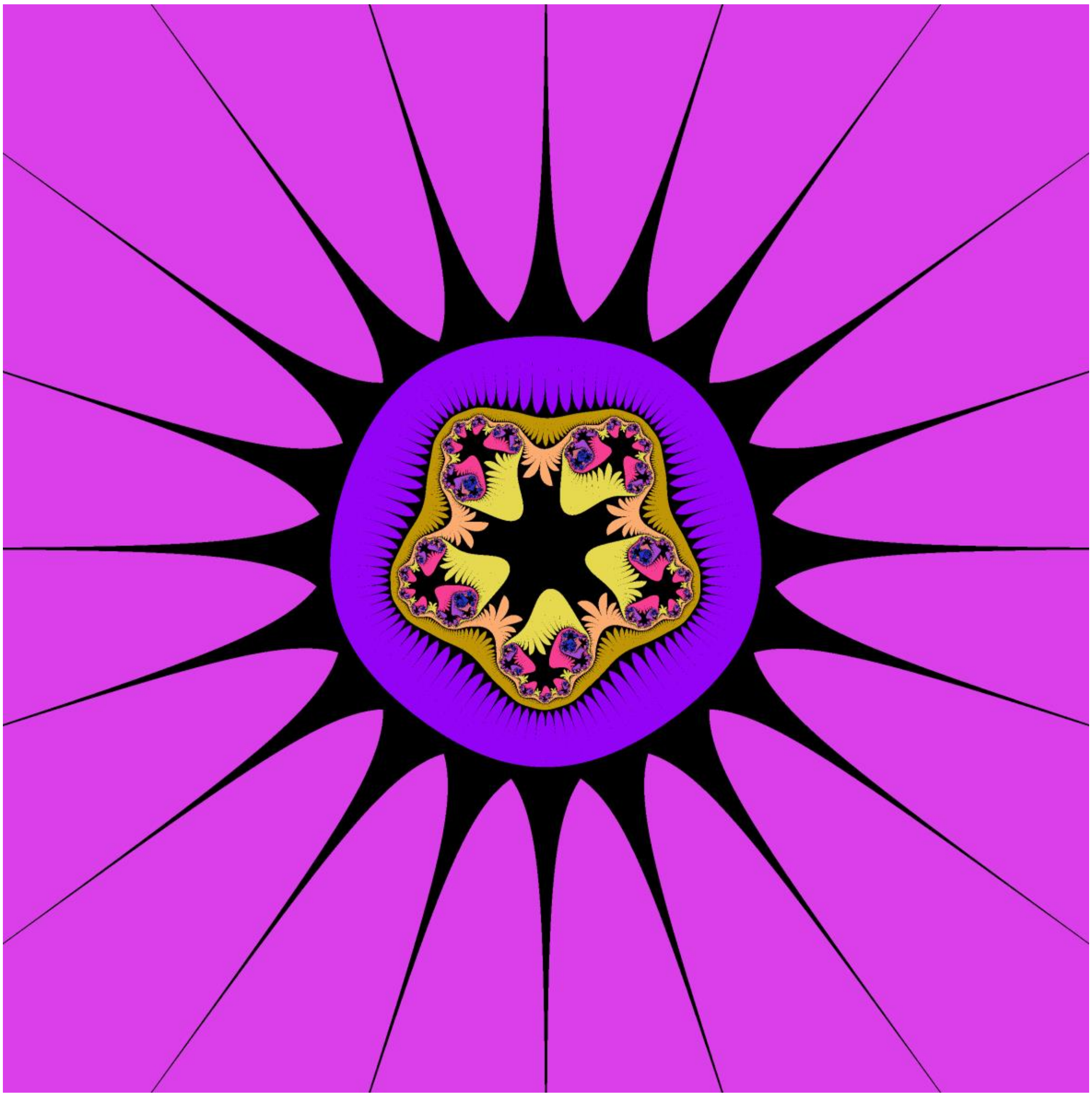
Figure 4. A second biomorph example