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Multiplicity along embedded schemes and differential operators

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A Marta,
porque hoy es su cumpleaños.
¡Felicidades!

Soy una jarra llena de agua viva y agua muerta, basta que me incline un poco para que me rebosen los más bellos pensamientos, soy culto a pesar de mí mismo y ya no sé qué ideas son mías, surgidas propiamente de mí, y cuáles he adquirido leyendo, y es que durante estos treinta y cinco años me he amalgamado con el mundo que me rodea porque yo, cuando leo, de hecho no leo, sino que tomo una frase bella en el pico y la chupo como un caramelo, la sorbo como una copita de licor, la saboreo hasta que, como el alcohol, se disuelve en mí, la saboreo durante tanto tiempo que acaba no sólo penetrando mi cerebro y mi corazón, sino que circula por mis venas hasta las raíces mismas de los vasos sanguíneos.

Bohumil Hrabal, *Una soledad demasiado ruidosa.*
Traducción del checo: Monika Zgustova.

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Resumen

Sea X un esquema equidimensional excelente definido por ciertas ecuaciones en un medio ambiente regular V. La multiplicidad a lo largo de los puntos de X define una función semicontinua superiormente, digamos $\operatorname{mult}_X : X \to \mathbb{N}$, que nos da una medida de la complejidad de las singularidades de X. Por ejemplo, X es regular en un punto ξ si y solo si $\operatorname{mult}_X(\xi) = 1$. La función multiplicidad induce una estratificación de X en conjuntos localmente cerrados. Cada uno de estos conjuntos puede ser descrito localmente como los ceros de un ideal sobre V. A lo largo de este trabajo, buscaremos condiciones en el espacio ambiente V bajo las cuales podamos emplear operadores diferenciales de manera efectiva en la construcción de dichos ideales.

Desde el punto de vista de la resolución de singularidades, es importante comprender el comportamiento de la multiplicidad por explosiones. Recordemos que, si X es reducido, una secuencia de explosiones

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_m$$

define una resolución de singularidades de X si X_m es regular. Es decir, si la multiplicidad es constante e igual a 1 en los puntos de X_m. Denotemos por $\operatorname{máx} \operatorname{mult}_X$ al lugar de máxima multiplicidad de X. Un resultado de Dade nos dice que, si $X \leftarrow X_1$ es una explosión de X a lo largo de un centro cerrado, regular y equimúltiple, entonces $\operatorname{máx} \operatorname{mult}_X \geq \operatorname{máx} \operatorname{mult}_{X_1}$. De esta forma, el problema de resolución de singularidades se puede reducir al problema de reducción de la multiplicidad máxima de X. Para ser precisos, se trata de encontrar una secuencia de explosiones a lo largo centros cerrados, regulares y equimúltiples, digamos

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_r, \tag{1}$$

tal que $\operatorname{máx} \operatorname{mult}_X > \operatorname{máx} \operatorname{mult}_{X_r}$. Este problema está resuelto en el caso en que X es una variedad sobre un cuerpo de característica cero, pero permanece abierto en el caso de característica positiva.

Fijemos una inmersión de X en un medio ambiente V como anteriormente. Bajo ciertas hipótesis, podemos encontrar un álgebra de Rees $\mathcal{G}$ sobre V que

describa el estrato de multiplicidad máxima de X, incluso después de considerar explosiones. En el caso de que exista un álgebra $\mathcal{G}$ con tal propiedad, se puede reformular el problema de reducción de la multiplicidad máxima de X en términos de álgebras de Rees. Concretamente, en el caso de característica cero, podemos usar $\mathcal{G}$ para construir una secuencia de explosiones como la de (1) tal que $\max \operatorname{mult}_X > \max \operatorname{mult}_{X_r}$. En principio, el resultado de este proceso podría depender de la elección de $\mathcal{G}$, ya que distintas álgebras podrían inducir diferentes secuencias de explosiones sobre V. Para solventar este problema, nosotros elegiremos un representante canónico de entre toda la familia de álgebras de Rees sobre V que describen el lugar de multiplicidad máxima de X.

Para que el procedimiento que acabamos de describir funcione, es necesario que existan *suficientes* operadores diferenciales sobre V. Por ejemplo, esta condición se cumple cuando V es una variedad regular sobre un cuerpo perfecto. A lo largo de este trabajo estudiaremos condiciones sobre el espacio ambiente V que garanticen el funcionamiento efectivo del procedimiento anterior. En el caso de característica cero, requeriremos que la condición jacobiana débil se satisfaga en V mientras que, en el caso de característica positiva, requeriremos la existencia de ciertas p-bases.

Introducción

Sea $f(x)$ un polinomio en una variable sobre $\mathbb{C}$. La multiplicidad de una raíz de $f(x)$ se puede calcular evaluando las derivadas de $f(x)$. Concretamente, una raíz a de $f(x)$ tiene multiplicidad mayor o igual que n si y solo si las primeras $n-1$ derivadas de $f(x)$ se anulan en $x=a$. Es decir, si $\frac{\partial^i f}{\partial x^i}(a)=0$ para $i=1,\dots,n-1$. Este método funciona para cualquier polinomio definido sobre un cuerpo de característica cero. Sin embargo, falla en característica positiva. Por ejemplo, consideremos un número primo $p>0$ y el polinomio $g(x)=x^{p^2}-x^p\in\mathbb{F}_p[x]$. En este caso, $\frac{\partial g}{\partial x}(x)=0$ y, por tanto, las derivadas de $g(x)$ no nos sirven para calcular la multiplicidad de sus raíces.

Desde el punto de vista de la geometría algebraica, los puntos regulares de un esquema se asemejan a las raíces simples de un polinomio, mientras que los puntos singulares se asemejan a raíces múltiples. Además, a cada punto singular de un esquema le podemos asociar un número entero conocido como la multiplicidad. Nuestro propósito es usar operadores diferenciales para clasificar las singularidades de un esquema y analizar la estratificación definida por la multiplicidad. Dicho de otra forma, buscamos condiciones bajo las cuales podamos aplicar los métodos analíticos de manera efectiva al estudio de las singularidades de un esquema.

Contexto y motivación

El caso más sencillo en el que nos encontramos una situación como la anterior surge cuando consideramos el espacio afín sobre un cuerpo k. Un polinomio f en d variables sobre k define una hipersuperficie

$$H=\operatorname{Spec}(k[x_1,\dots,x_d]/\langle f\rangle)\subset\mathbb{A}^d_k.$$

Fijemos un punto racional $\zeta\in H$ con coordenadas $(a_1,\dots,a_d)$. Consideremos el desarrollo natural de Taylor del polinomio f en ζ, digamos

$$f(x_1,\dots,x_d)=\tilde{f}(x_1-a_1,\dots,x_d-a_d).$$

Dado que $\zeta \in H$, el término de grado cero de $\tilde{f}$ será cero. Además, H será regular en ζ si y solo si $\tilde{f}$ contiene un elemento no nulo de grado uno, lo cual ocurrirá si y solo si $\frac{\partial f}{\partial x_i}(\zeta) \neq 0$ para algún $i = 1, \ldots, d$.

Cuando k es un cuerpo perfecto, el criterio anterior funciona para todos los puntos de H, incluyendo aquellos que no son racionales ni cerrados. Así,

$$\operatorname{Sing}(H) = \mathbb{V}\left(\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right\rangle\right).$$

Sin embargo, esta descripción del lugar singular de H falla cuando k no es perfecto.

Ejemplo. Consideremos el cuerpo $k = \mathbb{F}_p(t)$, en donde $\mathbb{F}_p$ representa el cuerpo primo de característica $p > 0$ y t es un elemento trascendente. Sea Y la curva plana definida por el polinomio $g = x_1^p + tx_2^p$. Es decir,

$$Y = \operatorname{Spec}(k[x_1, x_2]/\langle g\rangle) \subset \mathbb{A}_k^2.$$

La curva Y es regular en todo punto salvo en el origen y, sin embargo,

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x_2} = 0.$$

Así pues, las derivadas parciales de g se anulan en todos los puntos de Y, incluyendo los regulares.

A pesar de todo, cuando k no es perfecto, podemos obtener más información utilizando las derivadas absolutas de $k[x_1, x_2]$. Estas son derivadas relativas a $\mathbb{F}_p$, en contraposición a las derivadas relativas a k. Por ejemplo, en el caso anterior, usando la derivada parcial $\frac{\partial}{\partial t}$ obtenemos la siguiente descripción del lugar singular de Y:

$$\operatorname{Sing}(Y) = \mathbb{V}\left(\left\langle g, \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial t}\right\rangle\right) = \mathbb{V}\left(\langle x_1^p, x_2^p\rangle\right).$$

El hecho de que en este caso las derivadas absolutas nos permitan obtener una descripción del lugar singular de Y no es casual, sino se debe a un criterio jacobiano más general del que nos ocuparemos más adelante (véase el Lema 5.4.1).

Estratificación de hipersuperficies

Hasta ahora hemos clasificado los puntos de una hipersuperficie en regulares y no regulares. En general, esta clasificación resulta demasiado basta, por lo que sería deseable tener un refinamiento apropiado de la misma. Sea $V = \operatorname{Spec}(S)$ un esquema afín regular y consideremos una hipersuperficie H definida por un elemento $f \in S$. Es decir,

$$H = \operatorname{Spec}(S/\langle f\rangle) \subset V.$$

En este caso, el orden de f en puntos de V induce un refinamiento natural del lugar singular de H.

Fijemos un punto $\xi \in V$ y denotemos por $\mathfrak{M}_\xi$ al ideal maximal de $\mathcal{O}_{V,\xi}$. El orden de f en ξ se define como

$$\nu_\xi(f) = \sup\{n \in \mathbb{N} \mid f \in \mathfrak{M}_\xi^n\}.$$

A continuación, consideremos los conjuntos de la forma

$$\{\xi \in V \mid \nu_\xi(f) \geq n\}\,. \tag{1}$$

Estos conjuntos poseen ciertas propiedades geométricas relacionadas con H. Por ejemplo, para $n = 1$ tenemos que

$$H = \{\xi \in V \mid \nu_\xi(f) \geq 1\}$$

y, para $n = 2$,

$$\mathrm{Sing}(H) = \{\xi \in V \mid \nu_\xi(f) \geq 2\}\,.$$

Cuando V es excelente, el estrato (1) resulta ser cerrado en V para todo $n \geq 0$. Uno de nuestros objetivos es encontrar ideales $I_n \subset S$, $n \in \mathbb{N}$, tales que

$$\mathbb{V}(I_n) = \{\xi \in V \mid \nu_\xi(f) \geq n\}\,.$$

Veremos que estos ideales se pueden construir como extensiones de $\langle f \rangle$ aplicando operadores diferenciales al elemento f.

Ejemplo. Sea f un polinomio en d variables sobre un cuerpo k y consideremos la hipersuperficie

$$H = \mathrm{Spec}(k[x_1, \ldots, x_d]/\langle f \rangle) \subset \mathbb{A}_k^d.$$

Cuando k es perfecto, el criterio jacobiano nos dice que

$$\mathrm{Sing}(H) = \left\{\xi \in \mathbb{A}_k^d \mid \nu_\xi(f) \geq 2\right\} = \mathbb{V}\left(\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right\rangle\right).$$

Ejemplo. Supongamos que k es un cuerpo de característica cero y fijemos $f \in k[x_1, \ldots, x_d]$ como en el ejemplo anterior. Para cada $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, sea $|\alpha| = \alpha_1 + \cdots + \alpha_d$, y

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_d} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Fijemos un punto racional $\zeta \in \mathbb{A}_k^d$. Atendiendo al desarrollo de Taylor de f en ζ, podemos ver que $\nu_\zeta(f) \geq n$ si y solo si $\frac{\partial^\alpha f}{\partial x^\alpha}(\zeta) = 0$ para todo $\alpha \in \mathbb{N}^d$ con $|\alpha| < n$. Además, cuando k es de característica cero, se puede probar que este criterio funciona para todo punto $\xi \in \mathbb{A}_k^d$ y que, por tanto,

$$\left\{\xi \in \mathbb{A}_k^d \mid \nu_\xi(f) \geq n\right\} = \mathbb{V}\left(\left\langle \frac{\partial^\alpha f}{\partial x^\alpha} \mid \alpha \in \mathbb{N}^d,\, |\alpha| < n \right\rangle\right)$$

para cada $n \in \mathbb{N}$.

En los ejemplos anteriores hemos descrito la estratificación definida por el orden de f mediante ciertos ideales. Para construir estos ideales, hemos utilizado derivadas y composiciones de derivadas. En general, este método falla en característica positiva. Para sortear esta dificultad, tendremos que recurrir a operadores diferenciales de orden superior que, en característica positiva, no tienen por qué provenir de composiciones de derivadas.

Lema (cf. [17, Cap. III, Lema 1.2.7])**.** *Sea k un cuerpo arbitrario y consideremos el anillo de polinomios $S = k[x_1, \ldots, x_d]$. Denotemos por $\mathrm{Diff}^{n-1}(S)$ al módulo de operadores diferenciales de orden a lo sumo $n-1$ de S sobre el cuerpo primo de k (o, equivalentemente, sobre $\mathbb{Z}$). Entonces, para cualquier $f \in S$,*

$$\left\{\xi \in \mathbb{A}^d_k \mid \nu_\xi(f) \geq n\right\} = \mathbb{V}\Big(\Big\langle \Delta(f) \mid \Delta \in \mathrm{Diff}^{n-1}(S) \Big\rangle\Big).$$

Nuestro objetivo es encontrar un clase más amplia de anillos en la cual podamos utilizar operadores diferenciales para para estratificar el lugar singular de un esquema. Para ello, seguiremos dos enfoques, dependiendo de la característica.

- *Característica cero.* Supongamos que S es un anillo regular definido sobre un cuerpo de característica cero. Diremos que S satisface la *condición jacobiana débil* si para cada primo $\mathfrak{p} \subset S$, tomando $d = \dim(S_{\mathfrak{p}})$, podemos encontrar elementos $y_1, \ldots, y_d \in \mathfrak{p}$ y derivadas $\delta_1, \ldots, \delta_d \in \mathrm{Der}(S)$ tales que la matriz $\big(\delta_i(y_j)\big)$ tenga determinante no nulo módulo $\mathfrak{p}$. En el Capítulo 5 mostraremos que, bajo estas hipótesis, dado $f \in S$, podemos describir los conjuntos de la forma (1) como ceros de ideales que construiremos aplicando derivadas a f (véase la Proposición 5.4.3).

- *Característica positiva.* Sea S un anillo regular sobre un cuerpo de característica $p > 0$ y supongamos que S admite una p-base sobre su cuerpo primo. Cuando estas condiciones se cumplan veremos que, dado $f \in S$, podemos describir los subconjuntos de la forma (1) como ceros de ciertos ideales que obtendremos aplicando operadores diferenciales a f (véase la Proposición 5.2.17 y la Proposición 5.4.7)

Las condiciones anteriores serán estudiadas en el Capítulo 5.

Multiplicidad en esquemas

Sea $(R, \mathfrak{M})$ un anillo local noetheriano de dimensión d. Para cada $n > 0$, el cociente $R/\mathfrak{M}^n$ es un anillo artiniano que, por tanto, tiene longitud finita visto como R-módulo. Denotemos por $\ell(R/\mathfrak{M}^n)$ a esta longitud. Es bien sabido que existe un polinomio de grado d con coeficientes racionales, digamos $P(x) \in \mathbb{Q}[x]$, tal que $\ell(R/\mathfrak{M}^n) = P(n)$ para todo n suficientemente grande. Además, el coeficiente principal de $P(x)$ es de la forma $\frac{e}{d!}$ para cierto $e \in \mathbb{N}$. A dicho número e lo llamamos la *multiplicidad* de R. Si R es un anillo local regular, entonces tiene multiplicidad 1. Un resultado de Nagata [27] nos dice que, si R es excelente y estrictamente equidimensional, entonces R es regular si y solo si tiene multiplicidad 1 (véase el Teorema B.0.14).

Sea X un esquema noetheriano excelente. La multiplicidad de X en un punto ξ se define como la multiplicidad del anillo local $\mathcal{O}_{X,\xi}$. De esta forma, podemos ver la multiplicidad a lo largo de los puntos de X como una función $\mathrm{mult}_X : X \to \mathbb{N}$. En el caso de una hipersuperficie, digamos $H = \mathrm{Spec}(S/\langle f\rangle)$, la multiplicidad de H en un punto ξ coincide con el orden de anulación de f en ξ, es decir, $\mathrm{mult}_H(\xi) = \nu_\xi(f)$.

A continuación, supongamos que X es un esquema equidimensional y excelente. El teorema de Nagata (Teorema B.0.14) nos dice que X es regular en un punto ξ si y solo si $\mathrm{mult}_X(\xi) = 1$. Por tanto, tenemos que

$$\mathrm{Sing}(X) = \{\xi \in X \mid \mathrm{mult}_X(\xi) \geq 2\}.$$

En general, la multiplicidad nos da una medida de la complejidad de las singularidades: cuando mayor es la multiplicidad en un punto, peor es la singularidad. Otro resultado de Nagata nos dice que, si ξ y η son dos puntos de X tales que $\xi \in \overline{\{\eta\}}$, entonces

$$\mathrm{mult}_X(\xi) \geq \mathrm{mult}_X(\eta).$$

Dade probó que, bajo estas mismas hipótesis, la función $\mathrm{mult}_X : X \to \mathbb{N}$ es semicontinua superiormente (véase [14] o [34, Observación 6.13]). Así pues, la multiplicidad estratifica X en conjuntos cerrados de la forma

$$\{\xi \in X \mid \mathrm{mult}_X(\xi) \geq n\}.$$

A lo largo del Capítulo 4 estudiaremos algunas propiedades naturales de esta estratificación. Prestaremos especial atención al estrato de máxima multiplicidad de X, al cual denotaremos por $\underline{\mathrm{Max}}\,\mathrm{mult}_X$, y analizaremos su comportamiento por explosiones. Además, construiremos un objeto algebraico intrínsecamente asociado a $\underline{\mathrm{Max}}\,\mathrm{mult}_X$ que contiene información relevante sobre dicho estrato (véase el Teorema 4.4.4).

Conexión con el problema de resolución de singularidades

Una resolución de singularidades de un esquema reducido e irreducible X consiste en un morfismo propio y birracional, digamos $X \leftarrow X'$, tal que X' sea regular. El avance más sobresaliente en este problema se lo debemos a Hironaka, que en 1964 probó que cualquier variedad definida sobre un cuerpo de característica cero admite una resolución de singularidades. Más concretamente, Hironaka demostró que, para cualquier variedad X definida sobre un cuerpo de característica cero, existe una secuencia de explosiones a lo largo de centros cerrados normalmente planos, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l,$$

tal que X_l es regular. La demostración de Hironaka es puramente existencial y utiliza la función de Hilbert-Samuel como invariante principal para encontrar los centros de las explosiones.

Retomando la función multiplicidad, Dade probó en 1960 que, si $X \xleftarrow{\pi_1} X_1$ es una explosión de un esquema excelente equidimensional a lo largo de un centro regular y equimúltiple, entonces $\operatorname{mult}_{X_1}(\xi) \leq \operatorname{mult}_X(\pi_1(\xi_1))$ para cada $\xi_1 \in X_1$ (véase [14]). Este resultado fue generalizado y simplificado más adelante por Orbanz [29]. Observemos que, en particular, este resultado implica que $\max \operatorname{mult}_{X_1} \leq \max \operatorname{mult}_X$. Esto nos conduce de manera natural a la siguiente pregunta: dado un esquema equidimensional X, ¿existe una secuencia de explosiones en centros regulares y equimúltiples, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_s, \tag{2}$$

tal que la máxima multiplicidad de X decrezca? Es decir, tal que $\max \operatorname{mult}_{X_l} < \max \operatorname{mult}_X$. Esta cuestión fue ya planteada por Hironaka en su famoso artículo de 1964 (véase [21, Cuestión D, p. 134]). En 2014, Villamayor resolvió este problema para el caso de esquemas equidimensionales de tipo finito sobre un cuerpo de característica cero (véase [34]).

Supongamos que, dado X en una cierta clase de esquemas, supiésemos construir una secuencia de explosiones como (2) tal que la máxima multiplicidad de X decreciese. Entonces, podríamos obtener una resolución de singularidades de X iterando este procedimiento. En general, este proceso será diferente del propuesto por Hironaka, dado que solo pedimos que los centros de las explosiones sean equimúltiples, pero no requerimos que sean normalmente planos. En este trabajo nos centraremos en el proceso de simplificación de las singularidades de X basado en la multiplicidad, extendiendo las técnicas de [34] a un contexto más general.

A continuación, veamos algunas de las diferencias existentes entre el enfoque de Hironaka y el que utilizamos en esta memoria. Para empezar, la demostración de Hironaka es puramente existencial. Por el contrario, nuestro enfoque es determinista: para refinar la multiplicidad, definiremos una serie de invariantes auxiliares que, en última instancia, determinarán los centros de las explosiones.

Por otro lado, la demostración de Hironaka utiliza la función de Hilbert-Samuel como invariante principal. En contraposición, nosotros utilizaremos la estratificación definida por la multiplicidad. En el caso de hipersuperficies, ambas estratificaciones coinciden pero, en general, son distintas. Recordemos que la función de Hilbert-Samuel es un invariante que toma valores en el conjunto $\mathbb{N}^{\mathbb{N}}$, mientras que la multiplicidad toma valores en $\mathbb{N}$. En este sentido, la multiplicidad resulta ser un invariante más geométrico e intuitivo que la función de Hilbert-Samuel.

También existen otras diferencias más profundas entre estos dos invariantes. Para refinar la estratificación definida por la función de Hilbert-Samuel, Hironaka considera inmersiones de X en un medio ambiente regular V. Por el contrario, el estudio de la multiplicidad ha estado históricamente más ligado a morfismos finitos de la forma $X \to V$, en lugar de inmersiones. Por ejemplo, dada una variedad X sobre un cuerpo perfecto k, la multiplicidad en un punto ξ de X se puede expresar en términos de morfismos finitos de la forma $X \to V$ definidos en un entorno de ξ (véase [10, Apéndice A, p. 185]). El empleo de morfismos finitos en el estudio de la multiplicidad se remonta incluso a los trabajos de Albanese (véase [3] o [25, Clase 1, §5]). En nuestro caso, la existencia de morfismos finitos apropiados de X en un esquema regular V va a ser clave para encontrar una buena descrpción del lugar de máxima multiplicidad de X.

Álgebras de Rees

Sea X un esquema equidimensional y excelente inmerso como un subesquema cerrado en un medio ambiente regular V. Tal y como hemos indicado anteriormente, la multiplicidad es una función semicontinua superiormente que estratifica X en conjuntos localmente cerrados. En particular, el estrato de máxima multiplicidad es cerrado en X. Las álgebras de Rees son unos objetos algebraicos definidos sobre V que nos permiten describir el lugar de máxima multiplicidad de X como un subconjunto de V y, en última instancia, refinarlo.

Observación 1. Fijada una inmersión $X \hookrightarrow V$, hay más ejemplos de funciones semicontinuas en X cuyo estrato de máximo valor puede ser descrito mediante un álgebra de Rees sobre V. Por ejemplo, la función de Hilbert-Samuel define una función semicontinua superiormente X. En la demostración de resolución de singularidades sobre cuerpos de característica cero de [15], dada una inmersión $X \hookrightarrow V$, las álgebras de Rees son utilizadas para describir el estrato máximo de la función de Hilbert-Samuel en X. Nosotros nos centraremos exclusivamente en el caso de la multiplicidad.

Observación 2. El papel que juegan las álgebras de Rees en este trabajo es análogo al que juegan los exponentes idealísticos de Hironaka [22]. Además, existe una transcripción directa del lenguaje de álgebras de Rees al de exponentes idealísticos y viceversa (véase [15]).

Sea $V = \operatorname{Spec}(S)$ un esquema afín. Un *álgebra de Rees* sobre V es una S-álgebra $\mathbb{N}$-graduada y finita generada de la forma

$$\mathcal{G} = S\left[f_1 W^{N_1}, \dots, f_r W^{N_r}\right] \subset S[W]. \tag{3}$$

Cuando V es regular, se define el *lugar singular*[1] del álgebra $\mathcal{G}$ como

$$\operatorname{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\}.$$

Si, además, V es excelente, entonces $\operatorname{Sing}_V(\mathcal{G})$ resulta ser un subconjunto cerrado de V (véase el Corolario B.0.18). Se puede comprobar que la definición de

[1]No debemos confundir el lugar singular de un álgebra de Rees $\mathcal{G}$ sobre un esquema regular V con el lugar singular de un esquema general X, consistente en los puntos no regulares de X.

$\operatorname{Sing}_V(\mathcal{G})$ no depende de la elección de los generadores de $\mathcal{G}$ en (3) y que, por tanto, $\operatorname{Sing}_V(\mathcal{G})$ es un conjunto intrínsecamente asociado a $\mathcal{G}$.

Un ágebra de Rees sobre un esquema arbitrario V será un subhaz de $\mathcal{O}_V[W]$ cuya restricción a cualquier abierto afín de V sea un álgebra de Rees en el sentido de la definición anterior. El lugar singular de un álgebra de Rees sobre un esquema no afín se obtiene pegando los lugares singulares de las correspondientes álgebras afines.

A continuación, introduciremos una noción de transformación para álgebras de Rees. Dada un álgebra $\mathcal{G}$ sobre un esquema regular V, una *transformación $\mathcal{G}$-permisible* consistirá en un cierto morfismo de esquemas regulares, digamos $V \xleftarrow{\varphi_1} V_1$, junto con una regla de transformación de $\mathcal{G}$ que produzca un álgebra de Rees sobre V_1, digamos $\mathcal{G}_1$. Al álgebra $\mathcal{G}_1$ la llamaremos el *transformado de $\mathcal{G}$ por φ_1*. Consideraremos dos tipos de transformaciones:

- *Explosiones permisibles.* En este caso, $V \xleftarrow{\varphi_1} V_1$ es la explosión de V a lo largo de un centro cerrado y regular contenido en $\operatorname{Sing}_V(\mathcal{G})$. Cuando $Y \subset \operatorname{Sing}_V(\mathcal{G})$ sea un centro cerrado y regular, diremos que Y es *$\mathcal{G}$-permisible.* Por el momento, para no complicar las cosas, omitiremos la regla de transformación de $\mathcal{G}$ por explosiones.

- *Morfismos lisos.* Este tipo de transformaciones vienen dadas por un morfismo liso $V \xleftarrow{\varphi_1} V_1$. En este caso, el transformado de $\mathcal{G}$ por φ_1 se define como el *pull-back* de $\mathcal{G}$ a V_1. Es decir, $\mathcal{G}_1 = \varphi_1^*(\mathcal{G})$.

Diremos que una secuencia de transformaciones sobre V, digamos

$$\begin{array}{ccccccccc} \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V = V_0 & \xleftarrow{\varphi_1} & V_1 & \xleftarrow{\varphi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\varphi_m} & V_m, \end{array}$$

es una *secuencia $\mathcal{G}$-permisible* si cada φ_i es una transformación $\mathcal{G}_{i-1}$-permisible y $\mathcal{G}_i$ es el transformado de $\mathcal{G}_{i-1}$ por φ_i. Si, además, todas las transformaciones anteriores son explosiones permisibles y $\operatorname{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$, entonces diremos que la secuencia anterior es una *resolución de $\mathcal{G}$.*

Consideremos un esquema equidimensional X provisto de una inmersión $X \hookrightarrow V$ en un medio ambiente regular. Nuestro objetivo es encontrar un álgebra de Rees $\mathcal{G}$ sobre V que "describa" el lugar de máxima multiplicidad de X. Más concretamente, buscamos un álgebra $\mathcal{G}$ cuyo lugar singular coincida con el estrato de máxima multiplicidad de X. Es decir,

$$\underline{\operatorname{Max}} \operatorname{mult}_X = \operatorname{Sing}_V(\mathcal{G}). \tag{4}$$

Observemos que, si se cumple la igualdad anterior, entonces un centro cerrado y regular $Y \subset \operatorname{Sing}_V(\mathcal{G})$ es un centro $\mathcal{G}$-permisible y viceversa. De esta forma, para cualquier centro $\mathcal{G}$-permisible Y tenemos un diagrama conmutativo

$$\begin{tikzcd} V & \operatorname{Bl}_Y(V) \arrow[l] \\ X \arrow[u, hook] & \operatorname{Bl}_Y(X), \arrow[l] \arrow[u, hook] \end{tikzcd}$$

donde las flechas verticales representan inmersiones cerradas. Sea $X_1 = \mathrm{Bl}_Y(X)$, $V_1 = \mathrm{Bl}_Y(V)$ y denotemos por $\mathcal{G}_1$ al transformado de $\mathcal{G}$ en V_1. El teorema de Dade [14] nos dice que

$$\text{máx mult}_{X_1} \leq \text{máx mult}_X .$$

En el caso de que se cumpla la igualdad en la expresión anterior, requeriremos que

$$\underline{\mathrm{Max}}\ \mathrm{mult}_{X_1} = \mathrm{Sing}_{V_1}(\mathcal{G}_1).$$

En otras palabras, pediremos que la igualdad de (4) se preserve por explosiones permisibles siempre que la máxima multiplicidad de X no decrezca.

Algo similar ocurre si consideramos morfismos lisos. Un morfismo liso $V \leftarrow V_1$ induce por cambio de base otro morfismo liso $X \leftarrow X_1$ y un diagrama conmutativo

$$\begin{tikzcd} V & V_1 \arrow[l] \\ X \arrow[u, hook] & X_1. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

Denotemos por $\mathcal{G}_1$ al transformado de $\mathcal{G}$ por $V \leftarrow V_1$. En este caso, también requeriremos que $\underline{\mathrm{Max}}\ \mathrm{mult}_{X_1} = \mathrm{Sing}_{V_1}(\mathcal{G}_1)$.

Definición 3. Fijemos un inmersión $X \hookrightarrow V$ como en la discusión anterior. Diremos que un álgebra de Rees $\mathcal{G}$ sobre V *representa* el estrato de máxima multiplicidad de X si se cumplen las siguientes tres condiciones:

i) $\mathrm{Sing}_V(\mathcal{G}) = \underline{\mathrm{Max}}\ \mathrm{mult}_X$.

ii) Cualquier secuencia $\mathcal{G}$-permisible en V, digamos

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m, \end{array}$$

induce una secuencia de explosiones a lo largo centros regulares equimúltiples y morfismos lisos en X, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

tal que

$$\text{máx mult}_X = \text{máx mult}_{X_1} = \cdots = \text{máx mult}_{X_{m-1}} \geq \text{máx mult}_{X_m},$$

y viceversa. Además, las secuencias anteriores se relacionan naturalmente por medio de un diagrama conmutativo

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_2 & & \mathcal{G}_m \\ V & V_1 \arrow[l] & V_2 \arrow[l] & \cdots \arrow[l] & V_m \arrow[l] \\ X \arrow[u, hook] & X_1 \arrow[l] \arrow[u, hook] & X_2 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & X_m. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

iii) Para cualesquiera secuencias como las de ii), requerimos que

$$\mathrm{Sing}_{V_i}(\mathcal{G}_i) = \underline{\mathrm{Max}}\ \mathrm{mult}_{X_i}$$

para $i = 1, \ldots, m-1$, y

$$\mathrm{Sing}_{V_m}(G_m) = \emptyset \iff \text{máx}\,\mathrm{mult}_{X_m} < \text{máx}\,\mathrm{mult}_X\,.$$

Además, si $\text{máx}\,\mathrm{mult}_{X_m} = \text{máx}\,\mathrm{mult}_X$, entonces pedimos que $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \underline{\mathrm{Max}}\ \mathrm{mult}_{X_m}$.

Observación 4. Fijada una inmersión de un esquema equidimensional X en un medio ambiente regular V, la existencia un álgebra $\mathcal{G}$ sobre V que represente el lugar de máxima multiplicidad de X no está garantizada.

En el caso particular en que X es una variedad definida sobre un cuerpo perfecto k, existe un procedimiento para construir (localmente en topología étale) una inmersión cerrada de X en una variedad regular, digamos $X \hookrightarrow V$, junto con un álgebra de Rees $\mathcal{G}$ sobre V que representa el estrato de máxima multiplicidad de X (véase [34, §7]). Además, para el caso en que k es de característica cero, existe un algoritmo de resolución de álgebras de Rees (véase [15]) que nos permite construir una secuencia de explosiones $\mathcal{G}$-permisibles en V, digamos

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

tal que $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. Por las condiciones impuestas a $\mathcal{G}$, esta secuencia induce una secuencia de explosiones a lo largo de centros regulares equimúltiples en X, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m, \tag{5}$$

tal que

$$\text{máx}\,\mathrm{mult}_{X_m} < \text{máx}\,\mathrm{mult}_X\,.$$

Si $\text{máx}\,\mathrm{mult}_{X_m} = 1$, entonces (5) es una resolución de singularidades de X. En caso contrario, podemos obtener una resolución de X iterando repetidamente el proceso anterior.

Observación 5. Dado un un esquema equidimensional X, existe una noción alternativa de representación del lugar de máxima multiplicidad de X que utiliza morfismos finitos en lugar de inmersiones. Más concretamente, en algunos casos es posible construir un morfismo finito de X en un esquema regular V, digamos $\beta : X \to V$, junto con un álgebra de Rees $\mathcal{G}$ sobre V, de tal forma que β envíe $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ de manera homeomorfa a su imagen en V y

$$\beta\left(\underline{\mathrm{Max}}\ \mathrm{mult}_X\right) = \mathrm{Sing}_V(\mathcal{G}).$$

Además, β se puede construir de tal forma que la igualdad anterior se preserve por explosiones y secuencias permisibles (formularemos esta propiedad de manera más precisa en el Lema 7.2.1). Cuando se den las condiciones anteriores diremos que $\mathcal{G}$ *representa el estrato de máxima multiplicidad de* X *a través de* $\beta : X \to V$. Esta noción alternativa de representación a través de morfismos finitos será tratada con más detalle en el Capítulo 7.

Representantes canónicos

Sea X un esquema equidimensional singular inmerso en un medio ambiente regular V. En la discusión anterior hemos introducido el concepto de representación del lugar de máxima multiplicidad de X por un álgebra de Rees $\mathcal{G}$ sobre V. Sin embargo, dada la inmersión $X \hookrightarrow V$, no hemos dicho cómo podemos construir una tal álgebra $\mathcal{G}$. Además, puede ocurrir (y de hecho ocurre) que haya muchas álgebras sobre V con la propiedad de representar el estrato de máxima multiplicidad de X. Esta observación nos conduce a la siguiente noción de equivalencia entre álgebras.

Definición 6. Sea V un esquema regular. Diremos que dos álgebras de Rees $\mathcal{G}$ y $\mathcal{G}'$ sobre V son *débilmente equivalentes* si se cumplen las siguientes condiciones:

i) $\operatorname{Sing}_V(\mathcal{G}) = \operatorname{Sing}_V(\mathcal{G}')$.

ii) Cualquier secuencia $\mathcal{G}$-permisible sobre V, digamos

$$V \longleftarrow V_1 \longleftarrow V_2 \longleftarrow \cdots \longleftarrow V_m,$$

es una secuencia $\mathcal{G}'$-permisible y viceversa.

iii) Para cualquier secuencia sobre V como en el punto anterior y para cualquier $i = 1, \ldots, m$ se ha de cumplir que

$$\operatorname{Sing}_{V_i}(\mathcal{G}_i) = \operatorname{Sing}_{V_i}(\mathcal{G}'_i),$$

en donde $\mathcal{G}_i$ y $\mathcal{G}'_i$ representan los transformados de $\mathcal{G}$ y $\mathcal{G}'$ en V_i respectivamente.

Observemos que la equivalencia débil es una relación de equivalencia en el conjunto de álgebras de Rees sobre V. De ahora en adelante denotaremos a la clase de equivalencia de un álgebra $\mathcal{G}$ por $\mathscr{C}_V(\mathcal{G})$.

Observación 7. Fijemos una inmersión $X \hookrightarrow V$ y supongamos que $\mathcal{G}$ y $\mathcal{G}'$ son dos $\mathcal{O}_V$-álgebras que representan el lugar de máxima multiplicidad de X. Entonces,

$$\operatorname{Sing}_V(\mathcal{G}) = \underline{\operatorname{Max}} \operatorname{mult}_X = \operatorname{Sing}_V(\mathcal{G}').$$

Además, por las condiciones impuestas a $\mathcal{G}$ y $\mathcal{G}'$, sabemos que la igualdad anterior se preserva por secuencias permisibles sobre V (en el sentido de la Definición 3, p. xxi). De esta forma, tenemos que $\mathcal{G}$ y $\mathcal{G}'$ son débilmente equivalentes. Recíprocamente, si $\mathcal{G}$ es un álgebra que representa el lugar de máxima multiplicidad de X y $\mathcal{G}'$ es débilmente equivalente $\mathcal{G}$, entonces $\mathcal{G}'$ también representa el lugar de máxima multiplicidad de X.

Consideremos una variedad X definida sobre un cuerpo de característica cero provista de una inmersión en una variedad regular, digamos $X \hookrightarrow V$. Bajo estas hipótesis, existe un método para construir un álgebra de Rees $\mathcal{G}$ sobre V que representa el lugar de máxima multiplicidad de X. Además, tal y como se indica en la página xxii, existe un algoritmo de resolución de álgebras sobre cuerpos de característica cero que, dada $\mathcal{G}$, produce una secuencia de explosiones en centros permisibles sobre V, digamos

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

tal que $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. Entonces, por las condiciones impuestas a $\mathcal{G}$, esta secuencia induce una secuencia de explosiones en centros regulares equimúltiples sobre X, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

tal que $\max \mathrm{mult}_{X_m} < \max \mathrm{mult}_X$. Es decir, que induce un proceso de reducción de la máxima multiplicidad de X. Una propiedad importante del algoritmo de resolución de álgebras que acabamos de mencionar es que, si dos álgebras $\mathcal{G}$ y $\mathcal{G}'$ son débilmente equivalentes, entonces la secuencia de explosiones sobre V inducida usando tanto $\mathcal{G}$ como $\mathcal{G}'$ es la misma, digamos

$$\begin{array}{ccccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m. \end{array}$$

En otras palabras, el resultado que produce el algoritmo de resolución de álgebras depende de la clase de equivalencia de $\mathcal{G}$, digamos $\mathscr{C}_V(\mathcal{G})$, pero no de la elección de ningún representante en particular de dicha clase. Esta propiedad resulta fundamental a la hora de probar que la resolución de singularidades inducida en X es intrínseca e independiente incluso de la inmersión de X en V.

Consideremos un esquema regular noetheriano V. En el Capítulo 6 probaremos que, bajo ciertas hipótesis adicionales, es posible encontrar un representante canónico para cada clase de equivalencia $\mathscr{C}$ de álgebras de Rees sobre V, digamos $\mathcal{G}^*$, de forma que $\mathscr{C} = \mathscr{C}_V(\mathcal{G}^*)$. Esta propiedad resulta de gran utilidad a la hora de globalizar argumentos locales del proceso de resolución de singularidades.

Resultados principales

La multiplicidad a lo largo de los puntos de un esquema inmerso define una estratificación en conjuntos localmente cerrados. En este trabajo presentaremos condiciones bajo las cuales podamos usar de manera efectiva operadores diferenciales para encontrar una descripción esta estratificación.

Nuestros resultados se pueden clasificar en cuatro grandes bloques que detallaremos a continuación:

- Estratificación definida por la multiplicidad.
- Condiciones diferenciales y multiplicidad en hipersuperficies.

- Representantes canónicos.
- Simplificación de puntos de multiplicidad n.

Estratificación definida por la multiplicidad

Este primer bloque, correspondiente al Capítulo 4, está dedicado al estudio de la estratificación definida por la multiplicidad en esquemas excelentes y equidimensionales. Los resultados de este bloque están publicados en [1].

Consideremos un esquema X excelente y equidimensional. Dade [14] demostró que la multiplicidad define una función semicontinua superiormente en X, digamos $\mathrm{mult}_X : X \to \mathbb{N}$. Al principio del Capítulo 4 damos una prueba alternativa de este teorema para el caso en el que X es un esquema equidimensional de tipo finito sobre un cuerpo perfecto k (véase la demostración del Teorema 4.2.6 y el Corolario 4.2.8).

Dado un esquema X excelente y equidimensional, comenzaremos discutiendo cierta compatibilidad entre la estratificación definida por la multiplicidad en X y en el correspondiente esquema reducido, al cual denotamos por X_{red}. Recordemos X y X_{red}, vistos como espacios topológicos, son homeomorfos. Asimismo, X_{red} está provisto de una inmersión cerrada de esquemas en X, digamos $X_{\mathrm{red}} \hookrightarrow X$. El Lema 4.3.2 nos dice que la estratificación inducida por la multiplicidad en ambos esquemas es localmente la misma. En particular, esto implica que X_{red} es regular si y solo si X es una unión disjunta de componentes irreducibles con multiplicidad constante. Además, veremos que los procesos de simplificación de la multiplicidad de X y X_{red} son equivalentes en cierto sentido: cualquier secuencia de explosiones a lo largo de centros regulares equimúltiples en X, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

induce una secuencia de explosiones a lo largo de centros regulares equimúltiples en X_{red}, digamos

$$X_{\mathrm{red}} \longleftarrow X_1' \longleftarrow X_2' \longleftarrow \cdots \longleftarrow X_m', \tag{6}$$

y viceversa, de manera que $X_i' \simeq (X_i)_{\mathrm{red}}$ para cada $i = 1, \ldots, m$ (véase la Proposición 4.3.8). En particular, la secuencia (6) es una resolución de singularidades de X_{red} si y solo si X_m es una unión disjunta de componentes irreducibles con multiplicidad constante.

En la última parte del Capítulo 4, dada una variedad irreducible X sobre un cuerpo perfecto k, construiremos un álgebra de Rees sobre X, digamos $\mathcal{G}_X$, canónicamente asociada al estrato de máxima multiplicidad de X (véase la Definición 4.4.1 y el Teorema 4.4.4). Nótese que $\mathcal{G}_X$ es un álgebra de Rees definida sobre un esquema singular, en contraposición a aquellas álgebras definidas sobre esquemas regulares. Aun así, este álgebra contiene información importante sobre la multiplicidad en X. En este trabajo nos limitamos a definir el álgebra $\mathcal{G}_X$ y a analizar algunas propiedades relacionadas con su construcción. Sin embargo, posteriormente hemos seguido estudiando otras propiedades de este álgebra (véase [2]).

Condiciones diferenciales y multiplicidad en hipersuperficies

En el segundo bloque, correspondiente al Capítulo 5, introduciremos condiciones en un esquema regular que nos permitan aplicar de manera efectiva operadores diferenciales al estudio de la estratificación definida por la multiplicidad en una hipersuperficie.

Consideremos un anillo regular S, un elemento no nulo $f \in S$ y la hipersuperficie $H = \operatorname{Spec}(S/\langle f \rangle)$ contenida en $V = \operatorname{Spec}(S)$. Recordemos que, para cada $\xi \in H$, la multiplicidad de H en ξ coincide con el orden de f en $\mathcal{O}_{V,\xi}$. Es decir,

$$\operatorname{mult}_H(\xi) = \nu_\xi(f)$$

(véase la Proposición A.0.14). Supongamos que S está definido sobre un cuerpo k. En función de la característica de k, requeriremos que S satisfaga una de las siguientes condiciones:

- Cuando k sea de característica cero, requeriremos que S satisfaga la condición jacobiana débil. Es decir, pediremos que, para cada ideal primo $\mathfrak{q} \subset S$ y para cada sistema regular de parámetros de $S_\mathfrak{q}$, digamos $x_1, \dots, x_d$, exista una colección de derivadas $\delta_1, \dots, \delta_d \in \operatorname{Der}(S)$ tales que la matriz cuadrada $(\delta_i(x_j))$ tenga determinante no nulo módulo $\mathfrak{q}$ (véase la Definición 5.1.3 y el Lema 5.1.6).

- Si k tiene característica $p > 0$, entonces pediremos que S admita una p-base absoluta, es decir, una p-base sobre $\mathbb{F}_p$ (el cuerpo primo de k). El problema de existencia de p-bases absolutas será tratado en la Sección 5.2. Mostraremos que una k-álgebra reducida que admite una p-base absoluta es diferencialmente lisa sobre $\mathbb{F}_p$ (véase la Proposición 5.2.17 y el Corolario 5.2.19). También analizaremos la estabilidad de esta propiedad por extensiones de S (véase el Lema 5.2.8 y el Lema 5.3.10). Además, probaremos que toda variedad regular definida sobre un cuerpo arbitrario k de característica $p > 0$ puede ser recubierta por cartas afines de la forma $\operatorname{Spec}(S)$, en donde S admite una p-base absoluta (Proposición 5.3.12).

En la Sección 5.4 mostraremos que, si S satisface cualquiera de las dos condiciones anteriores, entonces, para cualquier elemento no nulo $f \in S$ y para cualquier ideal primo $\mathfrak{q} \subset S$,

$$\nu_\mathfrak{q}(f) \geq n \iff \mathfrak{q} \subset \operatorname{Diff}^{n-1}(S)(f),$$

en donde $\operatorname{Diff}^{n-1}(S)(f)$ denota el ideal generado por todos los elementos de la forma $\Delta(f)$ con $\Delta \in \operatorname{Diff}^{n-1}(S)$. El caso de característica cero se trata en la Proposición 5.4.3 y el Corolario 5.4.4, mientras que el de característica positiva se aborda en la Proposición 5.4.7.

Los resultados anteriores tienen dos consecuencias inmediatas. Por un lado, nos permiten describir los estratos definidos por la multiplicidad en una hipersuperficie contenida en $V = \operatorname{Spec}(S)$. Para ser precisos, dado $f \in S$ con $f \neq 0$ y tomando $H = \operatorname{Spec}(S/\langle f\rangle)$, tenemos que

$$\{\xi \in H \mid \operatorname{mult}_H(\xi) \geq n\} = \mathbb{V}\left(\operatorname{Diff}^{n-1}(S)(f)\right)$$

(véase el Corolario 5.4.5 y el Corolario 5.4.8). En segundo lugar, podemos describir el lugar singular de un álgebra de Rees $\mathcal{G}$ sobre S como un subconjunto cerrado de V: si $\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right]$, entonces

$$\operatorname{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \mathbb{V}\left(\operatorname{Diff}^{N_i-1}(S)(f_i)\right) \subset V$$

(véase el Corolario 5.4.6 y el Corolario 5.4.9).

Representantes canónicos

Este bloque corresponde al Capítulo 6. Sea V un esquema regular noetheriano. Bajo ciertas condiciones en V, probaremos la existencia de un representante canónico para cada clase de álgebras de Rees débilmente equivalentes sobre V. Aquí, por *representante canónico* de una clase de equivalencia $\mathscr{C}$ entendemos un álgebra de Rees sobre V, digamos $\mathcal{G}^*$, tal que $\mathscr{C} = \mathscr{C}_V(\mathcal{G}^*)$ y $\mathcal{G} \subset \mathcal{G}^*$ para todo $\mathcal{G} \in \mathscr{C}$. En otras palabras, $\mathcal{G}^*$ es el álgebra más grande de su clase. Para garantizar la existencia de un representante canónico para cada clase de álgebras sobre V, necesitaremos que V satisfaga ciertas condiciones adicionales: cuando V esté definido sobre un cuerpo de característica cero, requeriremos que V satisfaga la condición jacobiana débil; si V está definido sobre un cuerpo de característica $p > 0$, entonces requeriremos la existencia de ciertas p-bases absolutas.

La existencia de representantes canónicos ya fue probada en [9, Teorema 3.11] para el caso en que V es una variedad regular sobre un cuerpo perfecto k. Nosotros extendemos este resultado a una clase más amplia de esquemas. Merece la pena destacar que hay importantes diferencias entre la prueba que presentamos en este trabajo y la de [9], muchos de cuyos argumentos se basan en la estructura relativa de V sobre k. Este enfoque, que funciona bien en la clase de variedades regulares definidas sobre un cuerpo perfecto, se queda corto cuando tratamos con esquemas más generales. Otra diferencia es que, mientras en [9] los casos de característica cero y característica positiva se tratan simultáneamente, nosotros los trataremos por separado.

Sea $\mathscr{C}$ una clase de álgebras de Rees débilmente equivalentes sobre V. Para construir el representante canónico de $\mathscr{C}$, digamos $\mathcal{G}^* \in \mathscr{C}$, partiremos de un representante arbitrario $\mathcal{G} \in \mathscr{C}$. Entonces, $\mathcal{G}^*$ se obtendrá mediante un proceso de saturación de $\mathcal{G}$ en dos pasos: primero saturamos $\mathcal{G}$ utilizando operadores diferenciales sobre V (aclararemos este paso más adelante) y luego tomamos la clausura entera del álgebra resultante (véase la Sección 3.5). Es importante mencionar que el resultado de este procedimiento es independiente de la elección de $\mathcal{G}$. Esta propiedad se demuestra en el Teorema 6.4.3 para el caso de característica cero y en el Teorema 6.6.7 para el de característica positiva.

Una aplicación: simplificación de puntos de multiplicidad n

Este bloque está dedicado a la simplificación del lugar de máxima multiplicidad de un esquema equidimensional por medio de explosiones a lo largo de centros regulares y equimúltiples. Para ser precisos, sea X un esquema excelente equidimensional sobre un cuerpo de característica cero con multiplicidad máxima n. Supongamos que X está provisto de un cierto morfismo finito en un esquema regular, digamos $\beta : X \to V$. En el Capítulo 7 veremos que, bajo ciertas condiciones adicionales en β, podemos encontrar una secuencia de explosiones a lo largo de centros cerrados, regulares y equimúltiples sobre X, digamos

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l,$$

tal que la máxima multiplicidad de X decrezca. Es decir, tal que $\max \operatorname{mult}_{X_l} < n$. Este resultado se conocía ya para el caso en que X es un esquema equidimensional de tipo finito sobre un cuerpo de característica cero (véase [34]). Nosotros extendemos dicho resultado a una clase más general de esquemas (siempre definidos sobre un cuerpo de característica cero).

El resultado principal de este bloque es el Teorema 7.1.1. La demostración de este teorema, que presentamos a lo largo de las Secciones 7.2 y 7.3, sigue el guión de la prueba de [34] y [15]. Sin embargo, para adaptar los argumentos a nuestras condiciones de partida (más generales que las de [34] y [15]), necesitaremos recurrir a las técnicas desarrolladas en los capítulos anteriores.

En la Sección 7.2 construiremos un álgebra de Rees $\mathcal{G}$ sobre V que represente el estrato de máxima multiplicidad de X a través del morfismo finito β (véase la Observación 5 en la p. xxii y el Lema 7.2.2). Este álgebra se puede construir localmente utilizando el procedimiento de [34] (véase el Lema 7.2.2). Para garantizar que todas estas construcciones locales son compatibles, requeriremos que V satisfaga la condición jacobiana débil. Entonces, en virtud del Teorema 6.4.6, deducimos que existe un álgebra de Rees (globalmente definida) sobre V, digamos $\mathcal{G}$, que representa el lugar de máxima multiplicidad de X (véase el Lema 7.2.1).

La segunda parte de la demostración del Teorema 7.1.1 consiste en probar que el algoritmo de resolución de álgebras de Rees puede ser aplicado en las condiciones que nosotros planteamos. Recordemos que una resolución de un álgebra $\mathcal{G}$ sobre un esquema regular V es una secuencia de explosiones $\mathcal{G}$-permisibles, digamos

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

tal que $\operatorname{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. El algoritmo de [15] está formulado para el caso en que V es una variedad regular sobre un cuerpo k de característica cero. En la Sección 7.3 nosotros extendemos este procedimiento al caso en que V es un esquema regular arbitrario sobre un cuerpo de característica cero que satisface la condición jacobiana débil. La principal dificultad con al que nos encontramos al tratar de generalizar el algoritmo es la existencia de hipersuperficies de contacto maximal (veáse la Definición 7.3.3). Esta cuestión se resuelve en el Lema 7.3.5.

Summary

Let X be an equidimensional excellent scheme given by some equations in a regular ambient space V. The multiplicity along points of X defines a function with values on the integers, say $\mathrm{mult}_X : X \to \mathbb{N}$, which measures the complexity of the singularities of X. For instance, X is regular at a point ξ if and only if $\mathrm{mult}_X(\xi) = 1$. This function defines a stratification of X into locally closed sets. Each of these sets can be locally described as the zeros of an ideal over the ambient space V. In this work we give conditions on V that ensure that one can effectively use differential operators in the construction of such ideals.

From the point of view of resolution of singularities, it is important to analyze the behavior of the multiplicity under blow-ups. Recall that, if X is reduced, then a sequence of blow-ups

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_m$$

defines a resolution of singularities of X if X_m is regular. That is, if the multiplicity is constantly equal to 1 along points of X_m. Let us denote by $\max \mathrm{mult}_X$ the maximum multiplicity of X. A result of Dade says that, if $Y \subset X$ is a closed regular equimultiple center and $X \leftarrow X_1$ represents the blow-up of X along Y, then $\max \mathrm{mult}_X \geq \max \mathrm{mult}_{X_1}$. Thus the problem of resolution of singularities can be reduced to that of lowering the maximum multiplicity of X. Namely, the aim is to find a sequence of blow-ups along closed regular equimultiple centers, say

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_r, \tag{1}$$

so that $\max \mathrm{mult}_X > \max \mathrm{mult}_{X_r}$. This problem has been solved for the case in which X is a variety over a field of characteristic zero, but it remains open in the case of positive characteristic.

Fix an embedding of X in V as above. Under suitable conditions, it is possible to find a Rees algebra $\mathcal{G}$ over V that describes the stratum of maximum multiplicity of X, even after blowing up. When such a $\mathcal{G}$ exists, the problem of lowering of the maximum multiplicity of X can be reformulated in terms of Rees algebras. More precisely, in the case of characteristic zero, the algebra $\mathcal{G}$

can be used to construct a sequence like (1) where $\max \operatorname{mult}_X > \max \operatorname{mult}_{X_r}$. In principle, the outcome of this process depends on the choice of $\mathcal{G}$, and different algebras could define different sequences of blow-ups on X. We will overcome this difficulty by constructing a canonical representative among the family of Rees algebras that describe the highest multiplicity locus of X.

The methods discussed above rely on the existence of *enough* differential operators over the ambient space V. For instance, this requirement is known to be met when V is a regular variety over a perfect field. In this work we will also explore conditions on the regular ambient space V so as to ensure that the previous methods can be applied. In the case of characteristic zero we will show that the previous conditions hold when V satisfies the weak Jacobian condition, whereas in positive characteristic we will impose on V the existence of suitable p-bases.

Chapter 1

Introduction

Let $f(x)$ be a polynomial in one variable over $\mathbb{C}$. One can compute the multiplicity of a root of $f(x)$ by evaluating the derivatives of $f(x)$. Namely, a root a has multiplicity greater than or equal to n if and only if the first $n-1$ derivatives of $f(x)$ vanish at $x=a$. That is, $\frac{\partial^i f}{\partial x^i}(a)=0$ for $i=1,\ldots,n-1$. Moreover, this method works for any polynomial defined over a field of characteristic zero. However, it fails in positive characteristic. For instance, let $p>0$ be a prime number, and consider the polynomial $g(x)=x^{p^2}-x^p\in\mathbb{F}_p[x]$. Here $\frac{\partial g}{\partial x}(x)=0$, so the derivatives of $g(x)$ do not help to compute the multiplicity of its roots.

From the point of view of algebraic geometry, the regular points of a scheme resemble the simple roots of a polynomial, while singular points resemble multiple roots. In addition, one can attach an integer to each singular point of a scheme, called the multiplicity. Our aim is to use differential methods to classify the singularities of a scheme and the stratification induced by the multiplicity. In other words, we look for conditions under which one can effectively apply analytic methods to the study of singularities.

1.1 Context and motivation

The simplest case in which the previous discussion applies arises when we consider the affine space over a field k. Let f be a polynomial in d variables over k, which defines a hypersurface, say

$$H=\operatorname{Spec}(k[x_1,\ldots,x_d]/\langle f\rangle)\subset\mathbb{A}^d_k.$$

Fix a rational point $\zeta\in H$. Assume that ζ has coordinates $(a_1,\ldots,a_d)$, and consider the natural Taylor expansion of f at ζ, say

$$f(x_1,\ldots,x_d)=\tilde{f}(x_1-a_1,\ldots,x_d-a_d).$$

Since $\zeta \in H$, the term of degree zero of $\tilde{f}$ is zero. Moreover, H is regular at ζ if and only if $\tilde{f}$ contains a non-zero term of degree one. An easy computation shows that the latter occurs if and only if $\frac{\partial f}{\partial x_i}(\zeta) \neq 0$ for some $i = 1, \ldots, d$.

When k is a perfect field, the previous criterion works for all points $\xi \in H$, including those points which are not rational, or closed. Thus

$$\operatorname{Sing}(H) = \mathbb{V}\left(\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right\rangle\right).$$

However, this description of the singular locus fails when k is non-perfect.

Example 1.1.1. Consider the field $k = \mathbb{F}_p(t)$, where $\mathbb{F}_p$ denotes the prime field of characteristic $p > 0$, and t represents a transcendental element. Let Y be the plane curve defined by the polynomial $g = x_1^p + tx_2^p$. That is,

$$Y = \operatorname{Spec}(k[x_1, x_2]/\langle g\rangle) \subset \mathbb{A}_k^2.$$

It turns out that Y is regular everywhere, except at the origin. However,

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x_2} = 0,$$

so the partial derivatives of g vanish along all points of Y, including those which are regular.

Nevertheless, when dealing with a non-perfect field, more information can be obtained by using absolute derivatives on $k[x_1, x_2]$. These are derivatives relative to $\mathbb{F}_p$, as opposed to those which are relative to k. For instance, using the partial derivative $\frac{\partial}{\partial t}$, we obtain a description of the singular locus of Y as follows:

$$\operatorname{Sing}(Y) = \mathbb{V}\left(\left\langle g, \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial t}\right\rangle\right) = \mathbb{V}\left(\langle x_1^p, x_2^p\rangle\right).$$

The fact that absolute derivatives enable us to describe the singular locus of Y is a consequence of a more general Jacobian criterion that we shall discuss in the following lines.

Stratification of hypersurfaces

So far we have classified the points of a hypersurface into regular, and singular. However, this classification is too coarse, and one would like to have further refinements. Assume again that H is a hypersurface embedded in a regular affine scheme $V = \operatorname{Spec}(S)$, defined by an element $f \in S$. That is,

$$H = \operatorname{Spec}(S/\langle f\rangle) \subset V.$$

A natural refinement of the singular locus of H is that given by the order of f at points of V.

Fix a point $\xi \in V$, and let $\mathfrak{M}_\xi$ denote the maximal ideal of $\mathcal{O}_{V,\xi}$. The order of f at ξ is defined by

$$\nu_\xi(f) = \sup\{n \in \mathbb{N} \mid f \in \mathfrak{M}_\xi^n\}.$$

This notion enables us to consider subsets in V of the form

$$\{\xi \in V \mid \nu_\xi(f) \geq n\}\,, \tag{1.1}$$

which have interesting geometric properties. For instance, observe that for $n = 1$ we have that

$$H = \{\xi \in V \mid \nu_\xi(f) \geq 1\}\,,$$

and for $n = 2$,

$$\mathrm{Sing}(H) = \{\xi \in V \mid \nu_\xi(f) \geq 2\}\,.$$

When V is excellent, the stratum (1.1) turns out to be closed for each n (see Corollary B.0.18). One of our objectives is to find ideals $I_n \subset S$ so that

$$\mathbb{V}(I_n) = \{\xi \in V \mid \nu_\xi(f) \geq n\}\,,$$

These ideals will be constructed as extensions of $\langle f \rangle$ by applying differential operators on f.

Example 1.1.2. Let f be a polynomial in d variables over a field k, and consider the hypersurface

$$H = \mathrm{Spec}(k[x_1, \ldots, x_d]/\langle f \rangle) \subset \mathbb{A}^d_k.$$

If k is a perfect field, then the Jacobian criterion applies, and

$$\mathrm{Sing}(H) = \left\{\xi \in \mathbb{A}^d_k \mid \nu_\xi(f) \geq 2\right\} = \mathbb{V}\left(\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d}\right\rangle\right).$$

Example 1.1.3. Assume that k is a field of characteristic zero, and fix $f \in k[x_1, \ldots, x_d]$ as in the previous example. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, set $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1 + \cdots + \alpha_d} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Fix a rational point $\zeta \in \mathbb{A}^d_k$. Attending to the Taylor expansion of f at ζ, one readily checks that $\nu_\zeta(f) \geq n$ if and only if $\frac{\partial^\alpha f}{\partial x^\alpha}(\zeta) = 0$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| < n$. It can be proved that, when k is of characteristic zero, this criterion works for all points $\xi \in \mathbb{A}^d_k$, and hence

$$\left\{\xi \in \mathbb{A}^d_k \mid \nu_\xi(f) \geq n\right\} = \mathbb{V}\left(\left\langle \frac{\partial^\alpha f}{\partial x^\alpha} \mid \alpha \in \mathbb{N}^d,\, |\alpha| < n\right\rangle\right)$$

for all $n \in \mathbb{N}$.

In the previous examples, the stratification of the singular locus induced by the order of f is described by certain ideals, which make use of derivatives, and composition of derivatives. This procedure fails in positive characteristic. In this case, it is necessary to use differential operators of higher order, which might not be composition of derivatives.

Lemma 1.1.4 (cf. [17, Ch. III, Lemma 1.2.7]). *Let k be an arbitrary field, and consider the polynomial ring $S = k[x_1, \dots, x_d]$. Denote by $\mathrm{Diff}^{n-1}(S)$ the module of differential operators of order at most $n-1$ of S over the prime field (or equivalently over $\mathbb{Z}$). Then, for any $f \in S$, we have that*

$$\left\{ \xi \in \mathbb{A}^d_k \mid \nu_\xi(f) \geq n \right\} = \mathbb{V}\Big(\Big\langle \Delta(f) \mid \Delta \in \mathrm{Diff}^{n-1}(S) \Big\rangle \Big) .$$

Our aim is to study a wider class of rings in which differential operators can be used to stratify the singularities of a scheme. We will follow two different approaches, depending on the characteristic.

- *Characteristic zero.* Assume that S is a regular ring defined over a field of characteristic zero. We will say that S satisfies the *Weak Jacobian condition* if for each prime ideal $\mathfrak{p} \subset S$, setting $d = \dim(S_\mathfrak{p})$, one can find elements $y_1, \dots, y_d \in \mathfrak{p}$, and derivatives $\delta_1, \dots, \delta_d \in \mathrm{Der}(S)$, so that the matrix $\big(\delta_i(y_j)\big)$ has non-zero determinant modulo $\mathfrak{p}$. We will show that, under such conditions, and given $f \in S$, the closed subsets of (1.1) can be described by ideals obtained by applying derivatives, and compositions of derivatives to f (see Proposition 5.4.3).

- *Positive characteristic.* Let S be a regular ring over a field k of characteristic $p > 0$, and assume that S admits a p-basis over the prime field. When these conditions hold we will show that, given $f \in S$, the closed subsets defined in (1.1) can be described by ideals obtained by applying differential operators on f (see Proposition 5.2.17, and Proposition 5.4.7).

Both conditions will be studied in Chapter 5.

Multiplicity on schemes

Let $(R, \mathfrak{M})$ be a noetherian local ring of dimension d. For each $n > 0$, the quotient ring $R/\mathfrak{M}^n$ is artinian, and hence it has finite length when regarded as an R-module. Let $\ell(R/\mathfrak{M}^n)$ denote this length. It is well known that there exists a polynomial with rational coefficients of degree d, say $P(x) \in \mathbb{Q}[x]$, so that $\ell(R/\mathfrak{M}^n) = P(n)$ for n large enough. Moreover, the principal coefficient of $P(x)$ is of the form $\frac{e}{d!}$, for some $e \in \mathbb{N}$. The number e is called the *multiplicity* of R. If R is a regular local ring, then its multiplicity is 1. A theorem due to Nagata [27] asserts that, if R is a strictly equidimensional excellent local ring, then R is regular if and only it has multiplicity 1 (see Theorem B.0.14).

Let X be a noetherian scheme. The multiplicity of X at a point ξ is defined as that of the local ring $\mathcal{O}_{X,\xi}$. The multiplicity can be regarded as a function $\mathrm{mult}_X : X \to \mathbb{N}$. In the case of a hypersurface, say $H = \mathrm{Spec}(S/\langle f \rangle)$, the multiplicity of H at a point ξ coincides with the order of vanishing of f at ξ, i.e., $\mathrm{mult}_H(\xi) = \nu_\xi(f)$.

Next assume that X is equidimensional and excellent. Then it follows from Nagata's theorem (see Theorem B.0.14) that X is regular at ξ if and only if

$\text{mult}_X(\xi) = 1$, i.e.,

$$\text{Sing}(X) = \{\xi \in X \mid \text{mult}_X(\xi) \geq 2\}.$$

In general, the multiplicity serves as a measure of the complexity of a singularity: the higher is the multiplicity of X at a point, the worse is the singularity. Another result, also due to Nagata, asserts that if ξ, η are two points of X so that $\xi \in \overline{\{\eta\}}$, then

$$\text{mult}_X(\xi) \geq \text{mult}_X(\eta).$$

Furthermore, Dade proved that, under these hypotheses, $\text{mult}_X : X \to \mathbb{N}$ is upper semi-continuous (see [14], or [34, Remark 6.13]). Thus the multiplicity stratifies X into closed sets of the form

$$\{\xi \in X \mid \text{mult}_X(\xi) \geq n\}.$$

Along Chapter 4 we will study some natural properties of this stratification. We will draw particular attention to the stratum of maximum multiplicity of X, which we shall denote by $\underline{\text{Max}}\ \text{mult}_X$, and its behavior under blow-ups. Moreover, we will construct an intrinsic algebraic object attached to $\underline{\text{Max}}\ \text{mult}_X$, that encodes important information about this stratum (see Theorem 4.4.4).

Connection with the problem of resolution of singularities

Given a reduced and irreducible scheme X, a resolution of singularities of X is a proper and birational map $X \leftarrow X'$ so that X' is regular. The most notorious result on this problem is due to Hironaka [21], who proved in 1964 that any variety over a field of characteristic zero admits a resolution of singularities. Namely he showed that for any variety X defined over a field of characteristic zero, there exists a sequence of blow-ups along closed normally flat centers, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l,$$

so that X_l is regular. This proof was purely existential, and used the Hilbert-Samuel function as main invariant to find the centers of the blow-ups.

Going back to the multiplicity, Dade proved in 1960 that if $X \xleftarrow{\pi_1} X_1$ is the blow-up of an equidimensional excellent scheme along a closed regular equimultiple center, then $\text{mult}_{X_1}(\xi) \leq \text{mult}_X(\pi_1(\xi_1))$ for each $\xi_1 \in X_1$ (see [14]). This result was later generalized and simplified by Orbanz [29]. In particular, it ensures that $\max \text{mult}_{X_1} \leq \max \text{mult}_X$. Attending to this property, the following question arises naturally: given an equidimensional scheme X, can we find a sequence of blow-ups along closed regular equimultiple centers, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_s, \tag{1.2}$$

so that the maximum multiplicity of X drops? That is, so that $\max \text{mult}_{X_l} < \max \text{mult}_X$. This question was already posed by Hironaka in his celebrated article [21, Question D, p. 134]. In 2014, Villamayor solved this problem for the case of equidimensional schemes of finite type over a field of characteristic zero (see [34]).

Suppose that, for X in a certain class of schemes, one can find a sequence of blow-ups like (1.2), so that the maximum multiplicity of X drops. In this case, a resolution of X can be achieved by iterating this process. In general, such method will differ from Hironaka's approach, as the centers of the blow-ups are required to be equimultiple, but non-necessarily normally flat. In this work we will draw attention to the process of simplification of singularities based on the multiplicity, extending the techniques of [34] to a wider class of schemes.

Let us discuss some differences between Hironaka's approach, and that considered in this work. First, Hironaka's proof is purely existential. On the contrary, our method is deterministic: we will refine the stratification defined by the multiplicity with other invariants which ultimately enable us to determine the centers of the blow-ups.

On the other hand, Hironaka's proof uses the Hilbert-Samuel function as main invariant. By contrast, we will use the stratification given by the multiplicity. In the case of hypersurfaces, the stratifications defined by the multiplicity and the Hilbert-Samuel function coincide, but they differ in general. Recall that the Hilbert-Samuel function is an invariant that takes values in $\mathbb{N}^{\mathbb{N}}$. An advantage of the multiplicity is that it is more intuitive and geometrical.

There are also deeper differences between these invariants. In order to refine the stratification defined by the Hilbert-Samuel function on a variety X, Hironaka considers embeddings of X into a regular ambient space, say $X \hookrightarrow V$. On the contrary, the study of the multiplicity is linked to finite morphisms $X \to V$, rather than embeddings. For instance, for a variety X defined over a perfect field k, the multiplicity at a point ξ can be expressed in terms of finite morphisms $X \to V$ defined on a neighborhood of ξ (see [10, Appendix A, p. 185]). The use of finite morphisms to study the multiplicity also appears in the works of Albanese (see [3], or [25, Lect. 1, §5]). In our case, the existence of suitable finite morphism of X onto a regular scheme V will be essential to find a description of the maximum multiplicity stratum of X.

Rees algebras

Let X be an equidimensional excellent scheme embedded as a closed subscheme in a regular ambient space V. As we have indicated, the multiplicity is an upper semi-continuous function which therefore stratifies X into locally closed sets. In particular, the stratum of maximum multiplicity is closed in X. Rees algebras are algebraic objects over V that enable to describe the stratum of maximum multiplicity of X as a subset of V, and ultimately to refine it.

Remark 1.1.5. Given an immersion $X \hookrightarrow V$, there are other examples of upper semi-continuous functions on X whose stratum of maximum value can be described by a Rees algebra over V. For instance, the Hilbert-Samuel function along points of X defines an upper semi-continuous function on X. In the proof of resolution of singularities over fields of characteristic zero of [15], given a variety $X \hookrightarrow V$, Rees algebras are used to describe the maximum Hilbert-Samuel stratum of X. Here we shall just focus on the case in which the upper semi-continuous function is the multiplicity.

Remark 1.1.6. In this work, the role of Rees algebras parallels that of the idealistic exponents introduced by Hironaka [22]. Moreover, there is a direct translation from the language of Rees algebras to that of idealistic exponents, and vice-versa (see [15]).

Let $V = \mathrm{Spec}(S)$ be an affine scheme. A *Rees algebra* over V, or simply an $\mathcal{O}_V$*-Rees algebra*, is a finitely generated $\mathbb{N}$-graded S-algebra

$$\mathcal{G} = S\left[f_1 W^{N_1}, \dots, f_r W^{N_r}\right] \subset S[W]. \tag{1.3}$$

When V is regular, we define the *singular locus*[1] of $\mathcal{G}$ by

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\}.$$

If V is excellent, then $\mathrm{Sing}_V(\mathcal{G})$ turns out to be a closed subset of V (see Corollary B.0.18). It can be checked that the definition of $\mathrm{Sing}_V(\mathcal{G})$ does not depend on the choice of the generators in (1.3), and hence it is intrinsically attached to $\mathcal{G}$.

A Rees algebra over an arbitrary scheme V will be a subsheaf of $\mathcal{O}_V[W]$ which restricts to a Rees algebra as in the previous setting over any open affine subset of V. The singular locus of a Rees algebra over a non-affine scheme is obtained by patching the singular loci of the corresponding affine Rees algebras.

We now define a notion of transformation of Rees algebras. Given a Rees algebra $\mathcal{G}$ over a regular scheme V, a $\mathcal{G}$*-permissible transformation* will consist of a certain map of regular schemes $V \xleftarrow{\varphi_1} V_1$, together with a rule of transformation of $\mathcal{G}$, which produces an $\mathcal{O}_{V_1}$-Rees algebra $\mathcal{G}_1$. The latter will be called the *transform of* $\mathcal{G}$ *via* φ_1. There are two types of transformations:

- *Permissible blow-ups.* In this case, $V \xleftarrow{\varphi_1} V_1$ is the blow-up of V along a closed regular center contained in $\mathrm{Sing}_V(\mathcal{G})$. A closed regular center $Y \subset \mathrm{Sing}_V(\mathcal{G})$ is called a $\mathcal{G}$*-permissible center.* For clarity, we omit the rule of transformation of $\mathcal{G}$ under permissible blow-ups for the moment.

- *Smooth morphisms.* This type of transformations are given by a smooth morphism $V \xleftarrow{\varphi_1} V_1$, and the transform of $\mathcal{G}$ is defined as the pull-back of $\mathcal{G}$ to V_1. That is, $\mathcal{G}_1 = \varphi_1^*(\mathcal{G})$.

A sequence of transformations, say

$$\begin{array}{ccccccccc} \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V = V_0 & \xleftarrow{\varphi_1} & V_1 & \xleftarrow{\varphi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\varphi_m} & V_m, \end{array}$$

[1]The singular locus of a Rees algebra $\mathcal{G}$ over a regular scheme V should not be confused with the singular locus of a general scheme X, which consists of the non-regular points of X.

where each φ_i is a $\mathcal{G}_{i-1}$-permissible transformation, and $\mathcal{G}_i$ is the transform of $\mathcal{G}_{i-1}$ via φ_i, is called a *$\mathcal{G}$-permissible sequence*. In addition, if all the transformations in this sequence are permissible blow-ups and $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$, then the sequence is called a *resolution of $\mathcal{G}$*.

Fix an equidimensional scheme X, together with an immersion $X \hookrightarrow V$. Our aim is to find an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ that "describes" the highest multiplicity locus of X. First we will require that the singular locus of $\mathcal{G}$ coincide with the stratum of maximum multiplicity of X. Namely,

$$\underline{\mathrm{Max}}\ \mathrm{mult}_X = \mathrm{Sing}_V(\mathcal{G}). \tag{1.4}$$

Note that, under this assumption, a closed regular center $Y \subset \underline{\mathrm{Max}}\ \mathrm{mult}_X$ is a $\mathcal{G}$-permissible center, and vice-versa. Thus, given such a center Y, we have a natural commutative diagram

$$\begin{tikzcd} V & \mathrm{Bl}_Y(V) \arrow[l] \\ X \arrow[u, hook] & \mathrm{Bl}_Y(X), \arrow[l] \arrow[u, hook] \end{tikzcd}$$

where the vertical arrows represent closed immersions into regular schemes. Set $X_1 = \mathrm{Bl}_Y(X)$, $V_1 = \mathrm{Bl}_Y(V)$, and let $\mathcal{G}_1$ denote the transform of $\mathcal{G}$ on V_1. By Dade's theorem,

$$\max \mathrm{mult}_{X_1} \leq \max \mathrm{mult}_X .$$

If the equality holds in the latter expression, we will require that

$$\underline{\mathrm{Max}}\ \mathrm{mult}_{X_1} = \mathrm{Sing}_{V_1}(\mathcal{G}_1).$$

In other words, we will require (1.4) to be preserved by permissible blow-ups, whenever the maximum multiplicity of X does not drop.

A similar situation occurs when we consider smooth morphisms. A smooth morphism $V \leftarrow V_1$ induces, by base change, a smooth morphism $X \leftarrow X_1$, and a commutative diagram

$$\begin{tikzcd} V & V_1 \arrow[l] \\ X \arrow[u, hook] & X_1. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

Let $\mathcal{G}_1$ denote the transform of $\mathcal{G}$ on V_1. In this case we will require again that $\underline{\mathrm{Max}}\ \mathrm{mult}_{X_1} = \mathrm{Sing}_{V_1}(\mathcal{G}_1)$. More precisely:

Definition 1.1.7. Fix a closed immersion $X \hookrightarrow V$ as above. We will say that an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ *represents* the stratum of maximum multiplicity of X if the following three conditions hold:

i) $\mathrm{Sing}_V(\mathcal{G}) = \underline{\mathrm{Max}}\ \mathrm{mult}_X$.

ii) Any sequence of $\mathcal{G}$-permissible blow-ups and smooth transformations on V, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m, \end{array}$$

induces a sequence of blow-ups along closed regular equimultiple centers and smooth morphisms on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

so that

$$\max \operatorname{mult}_X = \max \operatorname{mult}_{X_1} = \cdots = \max \operatorname{mult}_{X_{m-1}} \geq \max \operatorname{mult}_{X_m},$$

and vice-versa. Moreover, these sequences are linked by a natural commutative diagram

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_2 & & \mathcal{G}_m \\ V & V_1 \arrow[l] & V_2 \arrow[l] & \cdots \arrow[l] & V_m \arrow[l] \\ X \arrow[u, hook] & X_1 \arrow[l] \arrow[u, hook] & X_2 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & X_m. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

iii) For any sequences as those in ii), we require that

$$\operatorname{Sing}_{V_i}(\mathcal{G}_i) = \underline{\operatorname{Max}} \operatorname{mult}_{X_i}$$

for $i = 1, \ldots, m-1$, and

$$\operatorname{Sing}_{V_m}(G_m) = \emptyset \iff \max \operatorname{mult}_{X_m} < \max \operatorname{mult}_X .$$

Moreover, if $\max \operatorname{mult}_{X_m} = \max \operatorname{mult}_X$, then we require that $\operatorname{Sing}_{V_m}(\mathcal{G}_m) = \underline{\operatorname{Max}} \operatorname{mult}_{X_m}$.

Remark 1.1.8. Given an embedding of an equidimensional scheme X in a regular ambient space V, it is not obvious that there exists an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ that represents the stratum of maximum multiplicity of X.

In the particular case in which X is a variety over a perfect field k, it is possible to construct (locally in étale topology) a closed immersion of X into a regular variety, say $X \hookrightarrow V$, together with an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ representing the stratum of maximum multiplicity of X (see [34, §7]). In addition, when k is of characteristic zero, there is an algorithm of resolution of Rees algebras [15] that produces $\mathcal{G}$-permissible sequence of blow-ups, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

where $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. By the conditions imposed on $\mathcal{G}$, the latter induces a sequence of blow-ups along closed regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m, \tag{1.5}$$

so that

$$\max \mathrm{mult}_{X_m} < \max \mathrm{mult}_X .$$

If $\max \mathrm{mult}_{X_m} = 1$, then (1.5) is a resolution of singularities of X. Otherwise, a resolution of X can be achieved by iterating this process finitely many times.

Remark 1.1.9. Let X be an equidimensional scheme. There is a different notion of representation of the maximum multiplicity locus of X using finite morphisms, rather than embeddings. Namely, in some cases it is possible to construct a finite morphism of X onto a regular scheme V, say $\beta : X \to V$, together with an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$, in such a way that $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ is mapped homeomorphically onto its image in V, and

$$\beta\left(\underline{\mathrm{Max}}\ \mathrm{mult}_X\right) = \mathrm{Sing}_V(\mathcal{G}).$$

Moreover, one can achieve that this identity is preserved by permissible blow-ups and sequences (the precise formulation of this condition can be found in Lemma 7.2.1). When these conditions are satisfied, we will say the stratum of maximum multiplicity of X is *represented by $\mathcal{G}$ via the finite morphism β : $X \to V$*. This notion of representation via finite morphisms will be discuss and clarified along Chapter 7.

Canonical representatives

Consider an embedding of a singular scheme X in a regular ambient space V. In the previous section we have discussed when an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ represents the maximum multiplicity locus of X. Note that, given $X \hookrightarrow V$, we have not indicated how to construct an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ with this property. In fact, there might be different Rees algebras over V that represent the maximum multiplicity of X. This fact leads to a notion of equivalence among Rees algebras so that two $\mathcal{O}_V$-Rees algebras that represent the maximum multiplicity locus of X will be equivalent in this sense.

Definition 1.1.10. Let V be a regular scheme. Two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ are said to be *weakly equivalent* if the following conditions hold:

i) $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Sing}_V(\mathcal{G}')$.

ii) Any $\mathcal{G}$-permissible sequence on V, say

$$V \longleftarrow V_1 \longleftarrow V_2 \longleftarrow \cdots \longleftarrow V_m,$$

is also a $\mathcal{G}'$-permissible sequence, and vice-versa.

iii) For any sequence as that in ii), if $\mathcal{G}_i$ and $\mathcal{G}'_i$ denote the transforms of $\mathcal{G}$ and $\mathcal{G}'$ on V_i respectively, we require that

$$\operatorname{Sing}_{V_i}(\mathcal{G}_i) = \operatorname{Sing}_{V_i}(\mathcal{G}'_i)$$

for $i = 1, \ldots, m$.

Note that weak equivalence defines an equivalence relation among Rees algebras over V. Hereafter we shall denote the equivalence class of an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ by $\mathscr{C}_V(\mathcal{G})$.

Remark 1.1.11. Fix an embedding $X \hookrightarrow V$, and assume the existence of two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ that represent the maximum multiplicity locus of X. By the conditions imposed on $\mathcal{G}$ and $\mathcal{G}'$, we have that

$$\operatorname{Sing}_V(\mathcal{G}) = \underline{\operatorname{Max}}\ \operatorname{mult}_X = \operatorname{Sing}_V(\mathcal{G}'),$$

and that these equalities are preserved by permissible sequences as formalized in Definition 1.1.7. Thus it follows that $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent. Conversely, one readily checks that if $\mathcal{G}$ represents the maximum multiplicity locus of X and $\mathcal{G}'$ is weakly equivalent to $\mathcal{G}$, then $\mathcal{G}'$ also represents the stratum of maximum multiplicity of X.

Consider a variety X over a field of characteristic zero, together with an embedding $X \hookrightarrow V$. In this case we will show how to construct a Rees algebra $\mathcal{G}$ over V that represents the highest multiplicity locus of X. As was mentioned at the end of the previous section, there is an algorithm of resolution of Rees algebras that, in characteristic zero, given $\mathcal{G}$, produces a sequence of $\mathcal{G}$-permissible blow-ups on V, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

so that $\operatorname{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. Then, by the conditions imposed on $\mathcal{G}$, the latter induces a sequence of blow-ups along closed regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

such that $\max\operatorname{mult}_{X_m} < \max\operatorname{mult}_X$, i.e., it induces a process that lowers the maximum multiplicity of X by blowing up along regular equimultiple centers. A key feature of the previous algorithm is that, if $\mathcal{G}$ and $\mathcal{G}'$ are two weakly equivalent Rees algebras over V, then the algorithm produces the same sequence of blow-ups on V, say

$$\begin{array}{ccccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m. \end{array}$$

In other words, the output of the algorithm only depends on the equivalence class of $\mathcal{G}$, say $\mathscr{C}_V(\mathcal{G})$, but not on the choice of a particular representative. This property is important in order to show that the resolution of singularities induced on X is intrinsic. In fact, it is even independent of the immersion $X \hookrightarrow V$.

For regular schemes V, with some additional conditions, it turns out given any equivalence class, say $\mathscr{C}$, there is a canonical choice of an algebra $\mathcal{G}^*$ so that $\mathscr{C} = \mathscr{C}_V(\mathcal{G}^*)$. This property will enable us to globalize quite easily local arguments in the process of resolution of singularities.

1.2 Main results

The multiplicity along points of a scheme stratifies it into locally closed sets, and the task is to find techniques that enable us to describe this stratification. In this work we present conditions under which differential operators provide an effective tool for this task, i.e., conditions that ensure that differential operators can be used to describe the stratification defined by the multiplicity.

The results contained in this work can be classified into four main blocks, to be discussed below:

- Stratification defined by the multiplicity.
- Differential conditions and multiplicity along hypersurfaces.
- Canonical representatives.
- Simplification of n-fold points.

Stratification defined by the multiplicity

This first block, corresponding to Chapter 4, is devoted to the study of the stratification defined by the multiplicity on equidimensional excellent schemes. The contents of this block are published in [1].

Consider an equidimensional excellent scheme X. As we mentioned in the previous section, Dade proved in [14] that the multiplicity defines an upper semi-continuous function on X, say $\mathrm{mult}_X : X \to \mathbb{N}$. At the beginning of Chapter 4 we give an alternative proof of this result for the case in which X is equidimensional of finite type over a perfect field k (see the proof of Theorem 4.2.6 and Corollary 4.2.8).

Given X, excellent and equidimensional as above, let X_{red} denote the underlying reduced scheme. Next we will discuss the natural compatibility of the stratification defined by the multiplicity on X, with that defined on X_{red}. Note that X and X_{red} are homeomorphic as topological spaces, and that there is a natural closed immersion of schemes $X_{\mathrm{red}} \hookrightarrow X$. Lemma 4.3.2 says that the stratification induced by the multiplicity on X and X_{red} respectively is essentially the same. In particular, this implies that X_{red} is regular if and only if X is a disjoint union of irreducible components having constant multiplicity. Moreover, we will show that the processes of simplification of the multiplicity on X and X_{red} are equivalent in the following sense: any sequence of blow-ups along closed regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

induces a sequence of blow-ups along regular equimultiple centers on X_{red}, say

$$X_{\mathrm{red}} \longleftarrow X'_1 \longleftarrow X'_2 \longleftarrow \cdots \longleftarrow X'_m, \tag{1.6}$$

and vice-versa, where $X'_i \simeq (X_i)_{\mathrm{red}}$ for $i = 1, \ldots, m$ (see Proposition 4.3.8). In particular, the sequence (1.6) is a resolution of singularities of X_{red} (i.e., X'_m is regular) if and only X_m is disjoint union of irreducible components having constant multiplicity.

Let X be an irreducible variety over a perfect field k. In the last part of Chapter 4 we construct a Rees algebra over X, say $\mathcal{G}_X$, that is canonically attached to the stratum of maximum multiplicity of X (see Definition 4.4.1 and Theorem 4.4.4). Note that here $\mathcal{G}_X$ is a Rees algebra defined over a singular scheme, as opposed to those which are defined over regular schemes. Nevertheless, it still encodes important information about the multiplicity on X. In this work we just give the definition of $\mathcal{G}_X$ and some basic properties related to its construction. However, we have pursued studying further properties of this algebra in [2].

Differential conditions and multiplicity along hypersurfaces

In the second block, which corresponds to Chapter 5, we introduce conditions on a regular scheme that ensure that one can effectively use differential operators to study the stratification defined by the multiplicity on a hypersurface.

Consider a regular ring S, a non-zero element $f \in S$, and the hypersurface $H = \mathrm{Spec}(S/\langle f \rangle)$ contained in $V = \mathrm{Spec}(S)$. Recall that, for $\xi \in H$, the multiplicity of H at ξ coincides with the order of f as an element of $\mathcal{O}_{V,\xi}$, i.e.,

$$\mathrm{mult}_H(\xi) = \nu_\xi(f)$$

(see Proposition A.0.14). Assume that S is an algebra over a field k. Then, depending on the characteristic of k, we will require that S satisfies one of the following conditions:

- When k has characteristic zero, we will require S to satisfy the weak Jacobian condition. This condition is defined and studied in Section 5.1. In short, it requires that for each prime ideal of S, say $\mathfrak{q}$, and for each regular system of parameters of $S_{\mathfrak{q}}$, say $x_1, \ldots, x_d$, there exists a collection of derivatives $\delta_1, \ldots, \delta_d \in \mathrm{Der}(S)$ so that the square matrix $(\delta_i(x_j))$ has non-zero determinant modulo $\mathfrak{q}$ (see Lemma 5.1.6).

- If k has characteristic $p > 0$, then we will require on S the existence of an absolute p-basis, that is, a p-basis of S over $\mathbb{F}_p$ (the prime field of k). The existence of absolute p-bases will be discussed in Section 5.2. We show that a k-algebra S with this property is differentially smooth over $\mathbb{F}_p$ (see Proposition 5.2.17 and Corollary 5.2.19). We also study the stability of this property under extensions of S (see Lemma 5.2.8 and Lemma 5.3.10). In addition, we prove that every regular variety over an arbitrary field k can be covered by affine charts of the form $\mathrm{Spec}(S)$, where S admits an absolute p-basis (Proposition 5.3.12).

In Section 5.4 we will show that, if S satisfies either of the previous conditions, then, for any non-zero element $f \in S$, and any prime ideal $\mathfrak{q} \subset S$,

$$\nu_{\mathfrak{q}}(f) \geq n \iff \mathfrak{q} \subset \mathrm{Diff}^{n-1}(S)(f),$$

where $\mathrm{Diff}^{n-1}(S)(f)$ represents the ideal generated by all the elements of the form $\Delta(f)$, with $\Delta \in \mathrm{Diff}^{n-1}(S)$. See Proposition 5.4.3 and Corollary 5.4.4 for the case of characteristic zero, and Proposition 5.4.7 for that of positive characteristic.

The previous results have two immediate consequences. Set $V = \mathrm{Spec}(S)$. First, it enables to describe the strata defined by the multiplicity on any hypersurface contained in V: for $f \in S$, $f \neq 0$, setting $H = \mathrm{Spec}(S/\langle f \rangle)$, we have that

$$\{\xi \in H \mid \mathrm{mult}_H(\xi) \geq n\} = \mathbb{V}\left(\mathrm{Diff}^{n-1}(S)(f)\right)$$

(see Corollary 5.4.5 and Corollary 5.4.8). Second, we can describe the singular locus of a Rees algebra $\mathcal{G}$ over S as a closed set in V: if $\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right]$, then

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \mathbb{V}\left(\mathrm{Diff}^{N_i - 1}(S)(f_i)\right)$$

(see Corollary 5.4.6 and Corollary 5.4.9).

Canonical representatives

This block corresponds to Chapter 6. Let V be a regular noetherian scheme. Here, under suitable conditions on V, we will construct a canonical representative for each class of weakly equivalent Rees algebras over V. More specifically, the *canonical representative* of an equivalence class $\mathscr{C}$ will be an $\mathcal{O}_V$-Rees algebra, say $\mathcal{G}^*$, so that $\mathscr{C} = \mathscr{C}_V(\mathcal{G}^*)$ and $\mathcal{G} \subset \mathcal{G}^*$ for all $\mathcal{G} \in \mathscr{C}$. In other words, $\mathcal{G}^*$ is the biggest Rees algebra of its class. To ensure the existence of a canonical representative of each equivalence class, some conditions will be required on V. When V is defined over a field of characteristic zero, we will require that V fulfills the weak Jacobian condition. When V is defined over a field of characteristic $p > 0$, we will require the existence of absolute p-bases.

The previous issue has already been addressed in [9, Theorem 3.11] for the case in which V is a regular variety over a perfect field k. Here we extend this result to a wider class of schemes. It is worth mentioning that there are important differences between our proof and that presented in [9], many of whose arguments are based on the relative structure of V over k. This approach works in the class of regular varieties over a perfect field, but it falls short for more general schemes. Moreover, while in [9] the cases of characteristic zero and positive characteristic are treated simultaneously, here we apply different strategies for each of them.

Let $\mathscr{C}$ be a class of weakly equivalent Rees algebras over V. In order to construct the canonical representative of $\mathscr{C}$, say $\mathcal{G}^* \in \mathscr{C}$, we choose an arbitrary element $\mathcal{G} \in \mathscr{C}$, and we obtain $\mathcal{G}^*$ by a certain process of saturation of $\mathcal{G}$. This process consists of two steps: first we saturate $\mathcal{G}$ using differential operators on V (a procedure to be clarified later), and then we take the integral closure of the resulting algebra (see Section 3.5). The previous process is independent of the choice of $\mathcal{G}$. This property is proved in Theorem 6.4.3 for the case of characteristic zero, and in Theorem 6.6.7 for that of positive characteristic.

An application: simplification of n-fold points

This block is devoted to the simplification of the multiplicity of a scheme by means of blow-ups along regular equimultiple centers. More precisely, let X be an equidimensional excellent scheme over a field of characteristic zero with maximum multiplicity n. Assume that X is endowed with a suitable finite morphism onto a regular scheme, say $\beta : X \to V$. Under these hypotheses, we will show that one can find a sequence of blow-ups along closed regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l,$$

so that the maximum multiplicity drops. That is, $\max \operatorname{mult}_{X_l} < n$. This result was already proved in [34] for the case in which X is equidimensional of finite type over a field of characteristic zero. Here we extend it to a more general class of schemes (over a field of characteristic zero) using the results obtained in the previous chapters.

The main result of this chapter is Theorem 7.1.1, whose proof carried out along Sections 7.2 and 7.3, and is patterned by those of [34] and [15] respectively. However, in order to adapt the arguments to our setting, we shall use the techniques developed in the previous chapters.

In Section 7.2 we will construct an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ that represents the highest multiplicity locus of X via the finite morphism β (see Remark 1.1.9 and Lemma 7.2.2). This algebra can be locally constructed using the arguments of [34] (see Lemma 7.2.2). In order to ensure that these local constructions patch, we will require V to satisfy the weak Jacobian condition. Then, by Theorem 6.4.6, we conclude that there exists a (globally defined) Rees algebra over V, say $\mathcal{G}$, with the prescribed property (see Lemma 7.2.1).

The second part of the proof of Theorem 7.1.1 consists on showing that the algorithm of resolution of Rees algebras discussed in [15] can be adapted to our setting. Recall that a resolution of a Rees algebra $\mathcal{G}$ over a regular scheme V is a sequence of $\mathcal{G}$-permissible blow-ups on V, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_m} & V_m, \end{array}$$

so that $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$. The algorithm in [15] is formulated for the case in which V is a regular variety over a field k of characteristic zero. In Section 7.3 we extend these arguments to the case in which V is an arbitrary scheme over a field of characteristic zero, which satisfies the weak Jacobian condition. The main difficulty that we must overcome in this more general setting is the existence of hypersurfaces of maximal contact (see Definition 7.3.3). This problem is addressed in Lemma 7.3.5.

1.3 Structure of this thesis

Chapter 1. Introduction

This is the chapter you are reading right now. Here we give an overview to the topics treated in this thesis, along with their historical context and motivation. Then we enumerate the main results that we have obtained, and we summarize the contents of each of the chapters.

Chapter 2. Differential operators

Here we collect all the definitions and results related to derivatives and differential operators to be used along this work. Most results of this chapter, which are rather classical, can be found in [20] or [26]. We will include some proofs when we consider that they clarify the forthcoming discussion.

Let k be an arbitrary ring, and S an arbitrary k-algebra. Along Sections 2.1 and 2.2 we discuss some properties of the differential operators of S relative to k. In Section 2.3 we review the notion of differential smoothness, and in Section 2.4 we focus on the differential operators of a polynomial ring. Finally, Section 2.5 states Giraud's Lemma, to be applied later for the study of Rees algebras.

Chapter 3. Rees algebras

Here we give a short introduction to Rees algebras. The purpose of this chapter is twofold. On the one hand, we collect results that appear scattered in the literature. On the other hand, we fix the notation to be used in the subsequent chapters (which may differ from that used in other works).

In Section 3.1 we present the notion of Rees algebra over a scheme. Section 3.2 is devoted to the study of algebras over regular schemes. Here we introduce the concepts of singular locus and permissible transformations of a Rees algebra. In Section 3.3 we discuss how a Rees algebra can be attached to the highest multiplicity locus of an embedded scheme. This will lead to the concept of *representation* of the maximum multiplicity of a scheme by a Rees algebra.

In Section 3.4 we discuss the notion of weak equivalence of Rees algebras over a regular scheme V, and we define the tree of permissible transformations of an algebra $\mathcal{G}$. In Sections 3.5 and 3.6 we introduce two operations on Rees algebras: integral closure and differential saturation. These two operations will play a key role in the construction of canonical representatives.

Chapter 4. Stratification defined by the multiplicity

In this chapter we analyze some natural properties of the stratification defined by the multiplicity on a scheme.

Let X be an equidimensional excellent scheme. Recall that the multiplicity along points of X can be regarded as a function $\mathrm{mult}_X : X \to \mathbb{N}$. In Section 4.2 we discuss the upper semi-continuity of mult_X. Let X_{red} denote the underlying reduced scheme of X. In Section 4.3 we study the natural compatibility between the stratification defined by the multiplicity on X, and that on X_{red}. Moreover we show that the processes of lowering the maximum multiplicity of X and X_{red} respectively are equivalent in some natural sense (see Proposition 4.3.8).

It is also interesting to compare the stratifications defined by the multiplicity on two schemes linked by a finite morphism, say $X' \to X$. In Section 4.4, given a variety X over perfect field k, we will construct an $\mathcal{O}_X$-Rees algebra that is canonically attached to the highest multiplicity locus of X, say $\mathcal{G}_X$ (see Definition 4.4.1 and Theorem 4.4.4). In Section 4.5 we will show that, given a suitable finite morphism $X' \to X$ of varieties over a perfect field, the pull-back of $\mathcal{G}_X$ is naturally included into $\mathcal{G}_{X'}$ (see Proposition 4.5.3).

Chapter 5. Differential conditions

Consider a regular ring S. Here we give conditions on S that enable us to use differential operators for the study of the stratification defined by the multiplicity on a hypersurface embedded in $\mathrm{Spec}(S)$.

We first assume that S is defined over a field k of characteristic zero. In Section 5.1 we review the weak Jacobian condition, which was originally introduced by Matsumura [26, §40]. A local criterion is given which enables us to check whether a regular ring S satisfies the previous condition (see Lemma 5.1.6).

Next suppose that S is defined over a field k of characteristic $p > 0$. Let $\mathbb{F}_p$ denote the prime field of k. In Section 5.2 we review the notion of p-basis, and we show that regular rings that admit an absolute p-basis are differentially smooth over $\mathbb{F}_p$ (see Proposition 5.2.17 and Corollary 5.2.19). In Section 5.3 we show that the existence of absolute p-bases is stable by regular extensions of finite type (Lemma 5.3.11). In particular, we prove that every regular variety over an arbitrary field k of characteristic $p > 0$ can be covered by affine charts of the form $\mathrm{Spec}(S)$, where S admits an absolute p-basis (see Proposition 5.3.12).

Assume finally that S is a regular ring over a field k which, according to the characteristic of k, satisfies one of the previous two conditions. Fix a non-zero element $f \in S$, and set $V = \mathrm{Spec}(S)$, and $H = \mathrm{Spec}(S/\langle f\rangle)$. Recall that the order of f along primes of S defines a stratification of H into locally closed sets, which coincides with that defined by the multiplicity on H. In Section 5.4 we show how differential operators can be used to describe the strata of H. More precisely, Proposition 5.4.3 and Proposition 5.4.7 say that, for $\xi \in V$,

$$\nu_{\xi}(f) \geq n \iff \xi \in \mathbb{V}\left(\mathrm{Diff}^{n-1}(S)(f)\right),$$

where $\mathrm{Diff}^{n-1}(S)(f)$ represents the ideal generated by all the elements of the form $\Delta(f)$, with $\Delta \in \mathrm{Diff}^{n-1}(S)$. As a consequence,

$$\{\xi \in H \mid \mathrm{mult}_H(\xi) \geq n\} = \mathbb{V}\left(\mathrm{Diff}^{n-1}(S)(f)\right).$$

Moreover, for an $\mathcal{O}_V$-Rees algebra $\mathcal{G} = S\left[f_1W^{N_1}, \ldots, f_rW^{N_r}\right]$, we have that

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \mathbb{V}\left(\mathrm{Diff}^{N_i-1}(S)(f_i)\right),$$

as we show in Corollary 5.4.6 and Corollary 5.4.9.

Chapter 6. Canonical representatives

This chapter is devoted to the construction of a canonical representative for each equivalence class of Rees algebras over a regular noetherian scheme V. This construction will be different for the case of characteristic zero and that of positive characteristic.

In Section 6.1, conditions will be given that enable us to check when a Rees algebra is the canonical representative of its class. These conditions, formulated in Lemma 6.1.1, are independent of the characteristic.

The construction of canonical representatives for the case of characteristic zero is carried out along Sections 6.2 and 6.4. Namely, the existence of canonical representatives is formulated in Theorem 6.4.3 and Theorem 6.4.6. The case of positive characteristic is addressed in Sections 6.5 and 6.6, and more precisely in Theorem 6.6.7 and Theorem 6.6.8.

Chapter 7. Simplification of n-fold points

Let X be an equidimensional excellent scheme over a field of characteristic zero with maximum multiplicity n. In this chapter we prove that, given a suitable finite morphism of X onto a regular scheme V, say $\beta : X \to V$, one can construct a sequence of blow-ups along closed regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l, \tag{1.7}$$

so that $\max \mathrm{mult}_{X_l} < n$. This result is stated in Theorem 7.1.1, and its proof is carried out along Sections 7.2 and 7.3.

In Section 7.2, making use of the finite morphism β, we construct a Rees algebra $\mathcal{G}$ over V that represents the maximum multiplicity locus of X. That is, $\mathcal{G}$ is an $\mathcal{O}_V$-algebra with the following property: any sequence of $\mathcal{G}$-permissible blow-ups on V, say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_r \\ V & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_r, \end{array}$$

induces a sequences of blow-ups along regular equimultiple centers on X, say

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_r,$$

where

$$\max \operatorname{mult}_X = \max \operatorname{mult}_{X_1} = \cdots = \max \operatorname{mult}_{X_{r-1}} \geq \max \operatorname{mult}_{X_r},$$

and a commutative diagram

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & & \mathcal{G}_r \\[-20pt] V & V_1 \arrow[l] & \cdots \arrow[l] & V_r \arrow[l] \\ X \arrow[u, hook] & X_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & X_r, \arrow[l] \arrow[u, hook] \end{tikzcd}$$

where $\underline{\operatorname{Max}} \operatorname{mult}_{X_i} = \operatorname{Sing}_{V_i}(\mathcal{G}_i)$ for $i = 1, \ldots, r-1$, and $\max \operatorname{mult}_{X_r} < n$ if and only if $\operatorname{Sing}_{V_r}(\mathcal{G}_r) = \emptyset$.

Next, applying the algorithm of resolution of Rees algebras to $\mathcal{G}$ we obtain a sequence of $\mathcal{G}$-permissible blow-ups on V, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_l \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_l, \end{array}$$

so that $\operatorname{Sing}_{V_r}(\mathcal{G}_r) = \emptyset$. Then, by the conditions imposed on $\mathcal{G}$, the latter induces a sequence of blow-ups along regular equimultiple centers on X like (1.7), where $\max \operatorname{mult}_{X_l} < n$.

Note that the argument of the previous discussion relies on the existence of an algorithm of resolution of Rees algebras over V. However, a priori, such an algorithm is not known to exist for the class of schemes of characteristic zero that we consider here. Section 7.3 is devoted to show that, in fact, the algorithm of resolution of algebras applies in our setting, when V is a regular scheme over a field of characteristic zero that satisfies the weak Jacobian condition. A crucial issue in this argument is the existence of hypersurfaces of maximal contact. This question is addressed in Lemma 7.3.5.

Appendices

Appendix A is devoted to the multiplicity of local rings. Here we review the definition and some important properties. We also review the graded algebra associated to a local ring and the tangent cone of a scheme at a point. Finally we prove that, given a regular local ring R and a non-zero element $f \in R$, the multiplicity of $R/\langle f \rangle$ coincides with the order of f.

In Appendix B we review the notions of excellent ring and scheme. Besides giving the definitions and some properties, here we state Dade's theorem, which says that the multiplicity along points of an equidimensional excellent scheme, say $\mathrm{mult}_X : X \to \mathbb{N}$, is an upper semi-continuous function which does not increase when blowing up along regular equimultiple centers. Then we use this result to show that, given a regular excellent ring S and a non-zero element $f \in S$, the order of f along primes of S is upper semi-continuous on $\mathrm{Spec}(S)$.

In Appendix C we give the definition and some basic properties of étale morphisms, and we introduce the concept of étale topology.

Chapter 2

Differential operators

2.1 Derivations

Definition 2.1.1. Let k be a ring, S a k-algebra, and M an S-module. A *derivation of S over k* is a k-linear map $\delta : S \to M$ which satisfies the Leibniz rule. That is, we require that

$$\delta(ab) = a\delta(b) + b\delta(a)$$

for all $a, b \in S$.

We shall denote by $\mathrm{Der}_k(S, M)$ the set of all derivatives of S over k into M. A simple computation shows that, if δ_1 and δ_2 are two k-linear derivatives from S into M, then $c_1\delta_1 + c_2\delta_2$ is again a k-linear derivative from S into M for all $c_1, c_2 \in S$. Thus $\mathrm{Der}_k(S, M)$ has a natural structure of S-module. We often consider derivatives from the ring S into itself. In this case we shall simply write $\mathrm{Der}_k(S) = \mathrm{Der}_k(S, S)$.

There is a universal characterization of the derivatives of a ring S. Consider the diagonal morphism $S \otimes_k S \to S$ given by $a \otimes b \mapsto ab$, and let $I_{S/k}$ denote its kernel. There is a natural short exact sequence

$$0 \longrightarrow I_{S/k} \longrightarrow S \otimes_k S \longrightarrow S \longrightarrow 0. \tag{2.1}$$

The *module of Kähler differentials of S over k* is defined by

$$\Omega^1_{S/k} = I_{S/k}/I^2_{S/k}.$$

Note that $\Omega^1_{S/k}$ has structure of $(S \otimes_k S)$-module. In addition, it is endowed with a natural structure of S-module to the left, and a natural structure of S-module to the right. Hereafter we will always consider $\Omega^1_{S/k}$ endowed with its structure of S-module to the left. That is, with the multiplication given by

$$a \cdot \overline{s \otimes s'} = \overline{as \otimes s'},$$

for $a, s, s' \in S$. The *universal derivation of S over k* is defined by

$$\begin{aligned} \delta_{S/k} : S &\longrightarrow \Omega^1_{S/k} \\ b &\longmapsto \overline{1 \otimes b - b \otimes 1}. \end{aligned}$$

Observe that $\delta_{S/k}$ satisfies the Leibniz rule: for all $a, b \in S$,

$$\begin{aligned} \delta_{S/k}(ab) &= \overline{1 \otimes ab - ab \otimes 1} \\ &= \overline{1 \otimes ab - ab \otimes 1} - \overline{1 \otimes a - a \otimes 1} \cdot \overline{1 \otimes b - b \otimes 1} \\ &= \overline{b \otimes a - ab \otimes 1} + \overline{a \otimes b - ab \otimes 1} \\ &= a \cdot \overline{1 \otimes b - b \otimes 1} + b \cdot \overline{1 \otimes a - a \otimes 1} = a \cdot \delta_{S/k}(b) + b \cdot \delta_{S/k}(a). \end{aligned}$$

Thus we see that $\delta_{S/k}$ is actually a derivation.

Proposition 2.1.2 ([26, §26.C, p. 182])**.** *For any derivation $\delta : S \to M$ of S with respect to k there exists a unique morphism of S-modules, say $\varphi : \Omega^1_{S/k} \to M$, so that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[d, "\delta_{S/k}"'] & M \\ \Omega^1_{S/k} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Corollary 2.1.3. $\mathrm{Der}_k(S, M) \simeq \mathrm{Hom}_S(\Omega^1_{S/k}, M)$.

Theorem 2.1.4 (First fundamental exact sequence [26, Theorem 57, p. 186])**.** *Let $k \to S \to S'$ be a chain of ring homomorphisms. Then there exists a natural exact sequence*

$$\Omega^1_{S/k} \otimes_S S' \xrightarrow{\varepsilon} \Omega^1_{S'/k} \longrightarrow \Omega^1_{S'/S} \longrightarrow 0.$$

In addition, ε has left inverse if and only if every derivation of S over k into an S'-module M' can be extended to a derivation $S' \to M'$.

Example 2.1.5. Let S be a k-algebra. Consider the S-algebra $S' = S[T]$, and an arbitrary derivation $\delta \in \mathrm{Der}_k(S)$. In this case δ can be extended to a derivation on S' which acts on the coefficients of the polynomials. Namely,

$$\delta(a_0 + a_1 T + \cdots + a_N T^N) = \delta(a_0) + \delta(a_1) T + \cdots + \delta(a_N) T^N.$$

A simple computation shows that $\delta : S' \to S'$ satisfies the Leibniz rule and then, according to the previous theorem, the first fundamental sequence splits in this case.

Theorem 2.1.6 (Second fundamental exact sequence [26, Theorem 58, p. 187])**.** *Let k be a ring, S a k-algebra, and $S' = S/J$ for some ideal $J \subset S$. Then there is a natural exact sequence*

$$J/J^2 \xrightarrow{\overline{d_{S/k}}} \Omega^1_{S/k} \otimes_S S' \longrightarrow \Omega^1_{S'/k} \longrightarrow 0.$$

The following result shows that a derivation on S is continuous with respect to any adic topology on S.

Lemma 2.1.7. *Let S be a noetherian k-algebra, $\delta : S \to M$ a derivation over k, and $J \subset S$ and ideal. Then, for all $N \geq 1$,*

$$\delta(J^N) \subset J^{N-1}M.$$

Proof. For $b_1, \ldots, b_N \in J$, Leibniz's rule yields

$$\delta(b_1 \cdot \ldots \cdot b_N) = \sum_{i=1}^{N} (b_1 \cdot \ldots \cdot b_{i-1})(b_{i+1} \cdot \ldots \cdot b_N) \cdot \delta(b_i) \in J^{N-1}M.$$

Since δ is linear and every element of J^N can be expressed as a sum of terms of the form $b_1 \cdot \ldots \cdot b_N$ with $b_1, \ldots, b_N \in J$, we conclude that $\delta(J^N) \subset J^{N-1}M$. $\square$

Corollary 2.1.8. *Under the hypotheses of the lemma, δ is continuous with respect to the J-adic topology.*

Corollary 2.1.9. *Let $(R, \mathfrak{M})$ be a local k-algebra, and $\delta : R \to R$ a derivation over k. Let $\widehat{R}$ denote the completion of R with respect to the $\mathfrak{M}$-adic topology. Then δ extends to a unique continuous derivation of $\widehat{R}$ over k, say $\delta : \widehat{R} \to \widehat{R}$.*

Proof. The extension exists and is unique by the continuity of δ. In addition, it inherits the Leibniz's rule, and hence it is a derivation. $\square$

2.2 Differential operators of higher order

Definition 2.2.1. Let $\Delta : S \to M$ be a k-linear map. The map Δ is a *differential operator of order* 0 if it is S-linear. We say that Δ *differential operator of order* $n > 0$ *over* k if

$$[b, \Delta] := bD(\square) - D(b \cdot \square)$$

is a differential operator of order $n - 1$ for all $b \in S$. The set of differential operators of S of order n over k is denoted by $\mathrm{Diff}^n_k(S, M)$. This set is usually considered with its natural structure of S-module.

Example 2.2.2. Let $\delta : S \to M$ be a derivation. For each $b \in S$, the map $[b, \delta] : S \to M$ satisfies

$$[b, \delta](c) = b\delta(c) - \delta(b \cdot c) = -\delta(b) \cdot c$$

for all c. This means that $[b, \delta]$ is S-linear, i.e., $[b, \delta]$ is a differential operator of order 0. Hence every derivation of S is a differential operator of order 1.

Remark 2.2.3. It follows from the definition that every differential operator of order n is also a differential operator of order $n+1$, or, in other words, that

$$\mathrm{Diff}_k^n(S, M) \subset \mathrm{Diff}_k^{n+1}(S, M).$$

As in the case of the derivations, one can also give a universal characterization of the differential operators of S over k. Consider the multiplication map $S \otimes_k S \to S$, and let $I_{S/k}$ denote its kernel as in (2.1). We define the module of principal parts of order n of S over k as

$$\mathrm{P}_{S/k}^n = (S \otimes_k S)/I_{S/k}^{n+1}.$$

We will always consider $\mathrm{P}_{S/k}^n$ endowed with its right S-module structure, unless we explicitly state the opposite. This module has a natural map attached to it,

$$\begin{aligned} d_{S/k}^n : S &\longrightarrow \mathrm{P}_{S/k}^n \\ b &\longmapsto (1 \otimes b), \end{aligned}$$

which is known as the universal operator of order n of S over k. Note that, with its right S-module structure, $\mathrm{P}_{S/k}^n$ is generated by the elements of the form $d_{S/k}^n(b)$, with $b \in S$.

Proposition 2.2.4 ([20, Proposition 16.8.4])**.** *The map $d_{S/k}^n : S \to \mathrm{P}_{S/k}^n$ is a differential operator of order n over k. Moreover, the pair $(\mathrm{P}_{S/k}^n, d_{S/k}^n)$ has the following universal property: for any differential operator $D : S \to M$ of order n over k, there exists a unique morphism of S-modules, say $\Phi : \mathrm{P}_{S/k}^n \to M$, making the following diagram commutative:*

$$\begin{tikzcd} S \arrow[r, "D"] \arrow[d, "d_{S/k}^n"'] & M . \\ \mathrm{P}_{S/k}^n \arrow[ur, "\Phi"'] & \end{tikzcd}$$

Corollary 2.2.5. $\mathrm{Diff}_k^n(S, M) \simeq \mathrm{Hom}_S(\mathrm{P}_{S/k}^n, M)$.

Proposition 2.2.6 (Localization, cf. [20, Proposition 16.8.6])**.** *Let S be a k-algebra, and $\Delta : S \to M$ a differential operator of order n over k. Then, for any multiplicative subset $\mathcal{U} \subset S$, there exists a unique differential operator $\Delta' : \mathcal{U}^{-1}S \to \mathcal{U}^{-1}M$ of order n over k which extends Δ.*

Proposition 2.2.7 ([20, Proposition 16.8.9])**.** *If $D_1 : S \to S$ and $D_2 : S \to S$ are differential operators of order n_1 and n_2 over k respectively, then $D_2 \circ D_1$ is a differential operator of order $n_1 + n_2$ over k.*

Lemma 2.2.8 (cf. [23, Lemma 3.1])**.** *Let S be a noetherian k-algebra, $\Delta : S \to M$ a differential operator of order n over k, and $J \subset S$ and ideal. Then, for any integer $N > n$,*

$$\Delta(J^N) \subset J^{N-n} M.$$

Proof. By Proposition 2.2.4, it suffices to show that

$$d^n_{S/k}(J^N) \subset J^{N-n}\,\mathrm{P}^n_{S/k}\,.$$

We will proceed by induction on N.

Fix $N > n$, and assume that the claim holds for all integers between n and N. Note that every element of J^N can be regarded as sum of terms of the form $b_1 \cdot \ldots \cdot b_N$, with $b_i \in J$. Then it suffices to check that

$$d^n_{S/k}(b_1 \cdot \ldots \cdot b_N) \in J^{N-n}\,\mathrm{P}^n_{S/k} \tag{2.2}$$

for all $b_1, \ldots, b_N \in J$.

Fix $b_1, \ldots, b_N \in J$. Since $N > n$, we have that $I^N_{S/k} \subset I^{n+1}_{S/k}$, and hence

$$\prod_{i=1}^{N}(1 \otimes b_i - b_i \otimes 1) \in I^{n+1}_{S/k}\,.$$

Moreover,

$$\prod_{i=1}^{N}(1 \otimes b_i - b_i \otimes 1) = \sum_{\Lambda \subset \{1,\ldots,N\}} (-1)^{|\Lambda|} \left[\left(\prod_{i\in\Lambda} b_i\right) \otimes \left(\prod_{i\notin\Lambda} b_i\right)\right].$$

By isolating the term corresponding to $\Lambda = \emptyset$ in the previous expression we get

$$d^n_{S/k}(b_1 \cdot \ldots \cdot b_N) = -\sum_{\substack{\Lambda \subset \{1,\ldots,N\}\\ \Lambda\neq\emptyset}} (-1)^{|\Lambda|} \overline{\left[\left(\prod_{i\in\Lambda} b_i\right) \otimes \left(\prod_{i\notin\Lambda} b_i\right)\right]}. \tag{2.3}$$

For each of the terms of right hand side of the previous expression there are two options. On the one hand, if $|\Lambda| \geq N - n$, we have that

$$\prod_{i\in\Lambda} b_i \in J^{N-n}.$$

On the other hand, if $|\Lambda| < N - n$, i.e., if $N - |\Lambda| > n$, we have that

$$\prod_{i\in\Lambda} b_i \in J^{|\Lambda|},$$

and, in this case, the inductive hypothesis yields

$$\overline{\left[1 \otimes \left(\prod_{i\notin\Lambda} b_i\right)\right]} = d^n_{S/k}\left(\prod_{i\notin\Lambda} b_i\right) \in J^{N-|\Lambda|-n}\,\mathrm{P}^n_{S/k}\,.$$

In either case we get

$$\overline{\left[\left(\prod_{i\in\Lambda} b_i\right) \otimes \left(\prod_{i\notin\Lambda} b_i\right)\right]} \in J^{N-n}\,\mathrm{P}^n_{S/k},$$

Hence (2.3) implies (2.2), which proves the assertion. □

Corollary 2.2.9. *Let S be a k-algebra and $J \subset S$ an ideal. Then any differential operator of S over k is continuous with respect to the J-adic topology.*

Corollary 2.2.10. *Let $(R, \mathfrak{M})$ be a local k-algebra, and $\Delta : R \to R$ a differential operator of order n over k. Let $\widehat{R}$ denote the completion of R with respect to the $\mathfrak{M}$-adic topology. Then Δ uniquely extends to a continuous differential operator, say $\Delta : \widehat{R} \to \widehat{R}$, of order n over k.*

2.3 Differential smoothness

There is natural notion of formal smoothness related to the existence of certain differential operators which act, somehow, like the partial derivatives on a polynomial ring. In the case of finite type morphisms, this notion coincides with that of smoothness, but there is not an equivalence in general. A detailed discussion on this topic can be found in [20, §16.10, p. 51].

Definition 2.3.1. Given an algebra S over a ring k, there is a natural surjective homomorphism of graded S-algebras

$$\Upsilon_{S/k} : \mathrm{Sym}_S(\Omega^1_{S/k}) \longrightarrow \mathrm{Gr}_{I_{S/k}}(\mathrm{P}_{S/k}) = \bigoplus_{n \geq 0} I^n_{S/k}/I^{n+1}_{S/k}.$$

The ring S is said to be *differentially smooth* over k if $\Omega^1_{S/k}$ is a projective S-module and $\Upsilon_{S/k}$ is an isomorphism.

Theorem 2.3.2 ([20, Theorem 16.11.2])**.** *Let S be a k-algebra. Suppose that there exists a set of elements $\{u_\lambda\}_{\lambda \in \Lambda} \subset S$ such that $\{\delta_{S/k}(u_\lambda)\}_{\lambda\in\Lambda}$ generates the S-module $\Omega^1_{S/k}$. Then the following conditions are equivalent:*

a) S is differentially smooth over k, and $\{\delta_{S/k}(u_\lambda)\}_{\lambda\in\Lambda}$ is a basis of $\Omega^1_{S/k}$.

b) There exists a family of differential operators of S over k, say $\{D^\beta \mid \beta \in \mathbb{N}^{\oplus\Lambda}\}$, such that

$$D^\beta(u^\alpha) = \binom{\alpha}{\beta} u^{\alpha-\beta} \quad \textit{for all} \quad \alpha \in \mathbb{N}^{\oplus\Lambda}.$$

In addition, when condition b) holds, the set of differential operators $\{D^\beta\}$ has the following properties:

i) For all $\beta, \beta' \in \mathbb{N}^{\oplus\Lambda}$,

$$D^\beta \circ D^{\beta'} = D^{\beta'} \circ D^\beta = \frac{(\beta+\beta')!}{\beta!\beta'!} D^{\beta+\beta'}.$$

ii) For any element $f \in S$, there exists a finite number of indexes $\beta \in \mathbb{N}^{\oplus\Lambda}$ so that $D^\beta(f) \neq 0$.

iii) For any differential operator $\Delta : S \to M$ *of order* N *over* k*,*

$$\Delta = \sum_{|\beta| \le N} \Delta(u^\beta) D^\beta.$$

Remark 2.3.3. In principle, the sum in iii) has infinitely many terms. However, for a particular element $f \in S$, we have

$$\Delta(f) = \sum_{|\beta| \le N} \Delta(u^\beta) D^\beta(f),$$

which is a finite sum because of ii).

2.4 Differential operators on a polynomial ring

Fix a ring k, non-necessarily a field. Let Λ be an arbitrary set of indexes, and consider the polynomial ring

$$S = k[T_\lambda \mid \lambda \in \Lambda] = k[T].$$

This section is devoted to the study of the differential operators of S over k.

Remark 2.4.1. We are particularly interested in the case in which the set Λ is infinite, i.e., when S is a polynomial ring in an infinite number of variables. This case will arise in Section 5.2, where we introduce the notion of p-basis. Note that a polynomial ring in an infinite number of variables is not noetherian.

Lemma 2.4.2. *Let* $S = k[T_\lambda \mid \lambda \in \Lambda]$ *be a polynomial ring as above. Then* $\Omega^1_{S/k}$ *is a free* S*-module with* $\{\delta_{S/k}(T_\lambda)\}$ *as a basis.*

Proof. On the one hand observe that, for each $F(T) \in S$,

$$\delta_{S/k}(F(T)) = \sum_\lambda \frac{\partial F}{\partial T_\lambda}(T) \cdot \delta_{S/k}(T_\lambda).$$

Hence $\{\delta_{S/k}(T_\lambda)\}$ spans $\Omega^1_{S/k}$. On the other hand, for each $\lambda \in \Lambda$ let $\Phi_\lambda : \Omega^1_{S/k} \to S$ denote the unique morphism of S-modules which makes the following diagram commutative (see Proposition 2.1.2):

$$\begin{tikzcd} S \arrow[r, "\frac{\partial}{\partial T_\lambda}"] \arrow[d, "\delta_{S/k}"'] & S. \\ \Omega^1_{S/k} \arrow[ur, "\Phi_\lambda"'] & \end{tikzcd}$$

As $\Phi_\lambda(T_\mu) = \delta_{\lambda\mu}$ (Kronecker's delta) for all $\lambda, \mu \in \Lambda$, it follows that $\{\delta_{S/k}(T_\lambda)\}$ is linearly independent over S, which proves the assertion. $\square$

Multi-index notation

Along this section we will continuously use the multi-index notation. Set $\mathbb{N}^{\oplus\Lambda} = \bigoplus_{\lambda\in\Lambda}\mathbb{N}$. An element $\beta \in \mathbb{N}^{\oplus\Lambda}$ is a tuple, say $(\beta_\lambda)_{\lambda\in\Lambda}$, whose entries are all zero except for a finite number of them. For $\beta \in \mathbb{N}^{\oplus\Lambda}$, we define the monomial

$$T^\beta = \prod_{\lambda\in\Lambda} T_\lambda^{\beta_\lambda}.$$

Observe that this is a monomial in a finite number of variables. The norm of the multi-index β is defined by

$$|\beta| = \sum_{\lambda\in\Lambda} \beta_\lambda \in \mathbb{N}.$$

With this notation, each polynomial $f(T) \in S = k[T]$ of degree N has a unique expression of the form

$$f(T) = \sum_{|\beta|\leq N} a_\beta T^\beta,$$

with $\{a_\beta\} \subset k$. For a couple of multi-indexes $\alpha, \beta \in \mathbb{N}^{\oplus\Lambda}$, let us define the factorial of β as

$$\beta! = \prod_{\lambda\in\Lambda} \beta_\lambda! \in \mathbb{N},$$

and the binomial coefficient $\binom{\alpha}{\beta} \in \mathbb{N}$ as

$$\binom{\alpha}{\beta} = \prod_{\lambda\in\Lambda}\binom{\alpha_\lambda}{\beta_\lambda} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

(and we set $\binom{\alpha}{\beta} = 0$ whenever $\beta \not\leq \alpha$). Finally, let us fix a basis of $\mathbb{N}^{\oplus\Lambda}$, say $\{\omega^{(\mu)} \mid \mu \in \Lambda\}$, where $\omega^{(\mu)} = (\delta_{\lambda\mu})_{\lambda\in\Lambda}$ (Kronecker's delta).

The Taylor expansion

Let $\{T\} = \{T_\lambda \mid \lambda \in \Lambda\}$ and $\{X\} = \{X_\lambda \mid \lambda \in \Lambda\}$ be two sets of variables. Consider the polynomial ring $S = k[T]$, where k is not necessarily a field. For $F(T) \in k[T]$, one can expand $F(T+X) \in k[T,X]$ as a polynomial in X with coefficients in S. That is,

$$F(X+T) = \sum_{\beta\in\mathbb{N}^{\oplus\Lambda}} F_\beta(T)X^\beta.$$

Definition 2.4.3. With the notation of the previous paragraph, for each $\beta \in \mathbb{N}^{\oplus\Lambda}$, we define the Taylor map $\mathrm{Tay}^\beta : S \to S$ by

$$\mathrm{Tay}^\beta\big(F(T)\big) = F_\beta(T), \quad F(T) \in S.$$

Lemma 2.4.4. Tay^β *is a differential operator of order* $|\beta|$ *of* S *over* k*.*

Lemma 2.4.5. *For $\alpha, \beta \in \mathbb{N}^{\oplus\Lambda}$,*

$$\operatorname{Tay}^{\beta}(T^{\alpha}) = \binom{\alpha}{\beta} T^{\alpha-\beta}.$$

Proof. In the expansion of $(T_\lambda + X_\lambda)^{\alpha_\lambda}$ as a polynomial on the variable X_λ, the coefficient of $X_\lambda^{\beta_\lambda}$ is $\binom{\alpha_\lambda}{\beta_\lambda} T_\lambda^{\alpha_\lambda - \beta_\lambda}$. Thus the coefficient of X^{α} in $(T+X)^{\beta}$ is

$$\prod_{\lambda \in \Lambda} \binom{\alpha_\lambda}{\beta_\lambda} T_\lambda^{\alpha_\lambda - \beta_\lambda} = \binom{\alpha}{\beta} T^{\alpha-\beta}.$$ □

Proposition 2.4.6. *The polynomial ring $S = k[T]$ is differentially smooth over k. In particular, the differential operators $\operatorname{Tay}^{\beta}$, $\beta \in \mathbb{N}^{\oplus\Lambda}$, have the following properties:*

i) For all $\beta, \beta' \in \mathbb{N}^{\oplus\Lambda}$,

$$\operatorname{Tay}^{\beta} \circ \operatorname{Tay}^{\beta'} = \operatorname{Tay}^{\beta} \circ \operatorname{Tay}^{\beta'} = \frac{(\beta+\beta')!}{\beta!\beta'!} \operatorname{Tay}^{\beta+\beta'}.$$

ii) Given a polynomial $F(T) \in S$, there exists a finite number of indexes $\beta \in \mathbb{N}^{\oplus\Lambda}$ such that $\operatorname{Tay}^{\beta}(F(T)) \neq 0$.

iii) For any differential operator $D : S \to M$ of order N over k,

$$D = \sum_{|\beta| \leq N} D(T^{\beta}) \operatorname{Tay}^{\beta}.$$

Proof. Recall that $\{\delta_{S/k}(T_\lambda)\}$ is a basis of $\Omega^1_{S/k}$ by Lemma 2.4.2. Then the result follows from Theorem 2.3.2 (note that S fulfills condition b) of the theorem by Lemma 2.4.4 and Lemma 2.4.5). □

Lemma 2.4.7. *For all $F_1(T), \dots, F_r(T) \in S$, and $\beta \in \mathbb{N}^{\oplus\Lambda}$, we have*

$$\operatorname{Tay}^{\beta}\big(F_1(T) \cdot \ldots \cdot F_r(T)\big) = \sum_{\alpha_1 + \cdots + \alpha_r = \beta} \operatorname{Tay}^{\alpha_1}\big(F_1(T)\big) \cdot \ldots \cdot \operatorname{Tay}^{\alpha_r}\big(F_r(T)\big).$$

Proof. Let $G(T) = F_1(T) \cdots F_r(T)$. Clearly,

$$G(T+X) = F_1(T+X) \cdot \ldots \cdot F_r(T+X).$$

Thus the result follows from the definition of $\operatorname{Tay}^{\beta}$. □

2.5 Giraud's lemma

In this section we consider a regular domain S. Giraud's lemma (Lemma 2.5.3) analyzes the behavior of the differential operators of S under blow-ups. As we will see in Section 3.6, this result plays a key role in the study of Rees algebras.

Let S be a regular domain over k, where k is not necessarily a field, and let $K = \operatorname{Frac}(S)$ denote the field of fractions of S. Let $\mathfrak{p} = \langle x_1, \dots, x_r \rangle \subset S$ be a prime ideal so that $S/\mathfrak{p}$ is regular. That is, $\mathbb{V}(\mathfrak{p})$ defines a regular center in $\operatorname{Spec}(S)$. Let S_1 denote the x_1-chart of the blow-up of S along $\mathfrak{p}$. Namely,

$$S_1 = S\left[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}\right] \subset K.$$

Consider a differential operator $\Delta : S \to S$ of order n over k. By Proposition 2.2.6, this extends to a differential operator $\Delta : K \to K$ of order n over k. However, the image of S_1 via Δ might not be contained in S_1, and thus $\Delta|_{S_1}$ does not necessarily define a morphism from S_1 into itself.

Example 2.5.1. Let $S = k[T_1, T_2]$, $\mathfrak{p} = \langle T_1, T_2 \rangle$, and $S_1 = \left[T_1, \frac{T_2}{T_1}\right]$. Consider the differential operator $\Delta = \frac{\partial}{\partial T_1} : S \to S$. Note that Δ is a derivation over k, and hence so is its extension to $K = k(T_1, T_2)$. Therefore, by the Leibniz's rule,

$$\Delta(T_2) = \Delta\left(T_1 \frac{T_2}{T_1}\right) = T_1 \Delta\left(\frac{T_2}{T_1}\right) + \frac{T_2}{T_1}\Delta(T_1) = T_1 \Delta\left(\frac{T_2}{T_1}\right) + \frac{T_2}{T_1}.$$

On the other hand, $\Delta(T_2) = \frac{\partial}{\partial T_1}(T_2) = 0$, which implies

$$\Delta\left(\frac{T_2}{T_1}\right) = \frac{-T_2}{T_1^2} \in K.$$

Thus the image of $\Delta|_{S_1} : S_1 \to K$ is not contained in S_1.

The next lemma is a technical result that we will use in the proof of Giraud's Lemma.

Lemma 2.5.2. *Let S be a k-algebra, and S' a finitely generated extension of S, say $S' = S[u_1, \dots, u_r]$. Then $\mathrm{P}^n_{S'/k}$ is generated by the elements of the form $d^n_{S'/k}(bu^\beta)$, with $b \in S$, and $\beta \in \mathbb{N}^r$, $|\beta| \leq n$.*

Proof. Recall that $\mathrm{P}^n_{S/k}$ is generated by the elements of the form $d^n_{S'/k}(b')$ with $b' \in S'$. Thus it suffices to show that, for each $b' \in S'$, the element $d^n_{S'/k}(b')$ belongs to the S'-submodule

$$\left\langle d^n_{S'/k}(bu^\gamma) \mid b \in S, \gamma \in \mathbb{N}^r, |\gamma| \leq n \right\rangle \subset \mathrm{P}^n_{S'/k}. \tag{2.4}$$

We will proceed in two steps: first we will address the case in which b' is a monomial of the form u^β with $\beta \in \mathbb{N}^r$, and then we will treat the general case.

Assume that $b' = u^\beta$ for some $\beta \in \mathbb{N}^r$. If $|\beta| \leq n$, there is nothing to prove. Otherwise, if $|\beta| > n$, we proceed by induction on $|\beta|$. Recall that

$$\mathrm{P}^n_{S'/k} = (S' \otimes_k S')/I^{n+1}_{S'/k}.$$

Moreover, observe that

$$\prod_{i=1}^{r} (1 \otimes u_i - u_i \otimes 1)^{\beta_i} \in I^{n+1}_{B/k},$$

and

$$\begin{aligned}\prod_{i=1}^{r} (1 \otimes u_i - u_i \otimes 1)^{\beta_i} &= \prod_{i=1}^{r} \sum_{\alpha_i \leq \beta_i} \binom{\beta_i}{\alpha_i} (-1)^{\beta_i - \alpha_i} \left(u_i^{\beta_i - \alpha_i} \otimes u_i^{\alpha_i} \right) \\ &= \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} (-1)^{|\beta - \alpha|} \big(u^{\beta - \alpha} \otimes u^\alpha \big).\end{aligned}$$

Thus, by isolating the term corresponding to $\alpha = \beta$ in the latter expression we get

$$d^n_{S'/k}(u^\beta) = \overline{(1 \otimes u^\beta)} = - \sum_{\alpha < \beta} \binom{\beta}{\alpha} (-1)^{|\beta - \alpha|} \overline{\big(u^{\beta - \alpha} \otimes u^\alpha \big)} \tag{2.5}$$

$$= - \sum_{\alpha < \beta} \binom{\beta}{\alpha} (-1)^{|\beta - \alpha|} u^{\beta - \alpha} \cdot d^n_{S'/k}(u^\alpha). \tag{2.6}$$

By the inductive hypothesis, $d^n_{S'/k}(u^\alpha)$ belongs to (2.4) for each $\alpha < \beta$. Hence $d^n_{S'/k}(u^\beta)$ also belongs to the submodule (2.4).

Now we proceed with the general case. Observe that every element $b' \in S'$ admits a polynomial expression of the form

$$b' = \sum_{|\beta| \leq N} b_\beta u^\beta,$$

for some $N \gg 0$. In addition, by (2.5), we may express each $d^n_{S'/k}(u^\beta)$ as

$$d^n_{S'/k}(u^\beta) = \sum_{|\alpha| \leq n} c_{\beta,\alpha} \cdot d^n_{S'/k}(u^\alpha),$$

where $\{c_{\beta,\alpha}\} \subset S$. In this way one readily checks that

$$\begin{aligned} d^n_{S'/k}(b') = \overline{(1 \otimes b')} &= \sum_{|\beta| \leq N} \overline{\big(1 \otimes b_\beta u^\beta\big)} \\ &= \sum_{|\beta| \leq N} \overline{\big(1 \otimes b_\beta\big)} \cdot d^n_{S'/k}(u^\beta) \\ &= \sum_{|\beta| \leq N} \sum_{|\alpha| \leq n} c_{\beta,\alpha} \overline{\big(1 \otimes b_\beta\big)} \cdot d^n_{S'/k}(u^\alpha) \\ &= \sum_{|\beta| \leq N} \sum_{|\alpha| \leq n} c_{\beta,\alpha} \overline{\big(1 \otimes b_\beta u^\alpha\big)} \\ &= \sum_{|\beta| \leq N} \sum_{|\alpha| \leq n} c_{\beta,\alpha} \cdot d^n_{S'/k}(b_\alpha u^\alpha), \end{aligned}$$

which clearly belongs to the submodule (2.4). □

Lemma 2.5.3 (Giraud's lemma)**.** *Let S be a regular domain over k, and $\mathfrak{p} = \langle x_1, \dots, x_r \rangle \subset S$, and S_1 be as at the beginning of the section. Set $K = \mathrm{Frac}(S)$. Consider a differential operator $\Delta : S \to S$ of order n over k. Then, for any $N \geq n$:*

i) The map $\Delta_1 : S_1 \to K$ given by $\Delta_1(f) = \Delta(x_1^N \cdot f)$ is a differential operator of order n over k.

ii) The image of Δ_1 is contained in $\langle x_1^{N-n} \rangle S_1$.

iii) There exists a map $x_1^{-(N-n)} \cdot \Delta_1 : S_1 \to S_1$ which is a differential operator on S_1 of order n over k.

Proof. According to Proposition 2.2.6, Δ extends to a differential operator $\Delta : K \to K$ of order n over k. Note that the multiplication in K by x_1^N can also be regarded differential operator of order 0 over k. Thus, by composition, we see that $\Delta(x_1^N \cdot \square) : K \to K$ is a differential operator of order n over k (Proposition 2.2.7). As Δ_1 is the restriction of $\Delta(x_1^N \cdot \square)$ to S_1, it follows that Δ_1 is a differential operator of order n over k.

Let us proceed with ii). According to Lemma 2.5.2 (taking $S' = S_1$, and $u_i = \frac{x_i}{x_1}$), we have that

$$\mathrm{P}^n_{S_1/k} = \left\langle d^n_{S_1/k}(bu^\beta) \mid b \in S, \, \beta \in \mathbb{N}^r, \, |\beta| \leq n \right\rangle.$$

Moreover, by Proposition 2.2.4, there exists a unique homomorphism of S_1-modules, say $\Phi_1 : \mathrm{P}^n_{S_1/k} \to S_1$, so that

$$\Delta_1 = \Phi_1 \circ d^n_{S_1/k}.$$

Thus, in order to prove ii), it suffices to show that

$$\Phi_1\big(d^n_{S_1/k}(bu^\beta)\big) = \Delta_1(bu^\beta) \in \langle x_1^{N-n} \rangle S_1$$

for all $b \in S$ and $\beta \in \mathbb{N}^r$ with $|\beta| \leq n$. To check this, observe that

$$x_1^N b u^\beta = x_1^N b \left(\frac{x_1}{x_1}\right)^{\beta_1} \cdots \left(\frac{x_r}{x_1}\right)^{\beta_r} = x_1^{N-|\beta|} b \cdot x_1^{\beta_1} \cdots x_r^{\beta_r} \in \mathfrak{p}^N.$$

Hence, by Lemma 2.2.8,

$$\Delta_1 \left(b u^\beta\right) = \Delta \left(x_1^N b u^\beta\right) = \Delta \left(x_1^{N-|\beta|} b \cdot x_1^{\beta_1} \cdots x_r^{\beta_r})\right) \in \mathfrak{p}^{N-n}.$$

Since $\mathfrak{p} S_1 = \langle x_1 \rangle S_1$, we conclude that the image of Δ_1 is contained in $\langle x_1^{N-n} \rangle S_1$, which proves ii).

Finally we proceed with iii). Since x_1 is a unit in S_1, the multiplication by $x_1^{-(N-n)}$ induces a well-defined map from the ideal $\langle x_1^{N-n} \rangle S_1$ into S_1. Note also that this map can be regarded as an isomorphism of S_1-modules. Thus it follows that $x_1^{-(N-n)} \cdot \Delta_1 : S_1 \to S_1$ is a differential operator of order n over k. $\square$

Chapter 3

Rees algebras

3.1 Definitions

Definition 3.1.1. Let B be a noetherian ring. We define a *Rees algebra over B*, or simply a B-Rees algebra, as a finitely generated $\mathbb{N}$-graded algebra over B, say

$$\mathcal{G} = \bigoplus_{n\in\mathbb{N}} J_n W^n \subset B[W],$$

such that $J_0 = B$, and J_n is an ideal of B for $n > 0$. Note that these conditions imply $J_m \cdot J_n \subset J_{m+n}$ for all $m, n \in \mathbb{N}$.

Remark 3.1.2. Let us stress the requirement of being finitely generated over B. A graded B-algebra which is not finitely generated will not be called a Rees algebra.

Usually the easiest way to work with a Rees algebra is by fixing a set of homogeneous generators, say $f_1W^{N_1}, \dots, f_rW^{N_r} \in \mathcal{G}$. In this case we write

$$\mathcal{G} = B\left[f_1W^{N_1}, \dots, f_rW^{N_r}\right].$$

Given another B-Rees algebra, say $\mathcal{K}$, there is a minimum Rees algebra which contains both $\mathcal{G}$ and $\mathcal{K}$. This algebra is called the *amalgamation* of $\mathcal{G}$ and $\mathcal{K}$, and it is denoted by $\mathcal{G} \odot \mathcal{K}$. If we also fix a set of generators of $\mathcal{K}$, say

$$\mathcal{K} = B\left[g_1W^{N'_1}, \dots, g_sW^{N'_s}\right],$$

then it is easy to check that

$$\mathcal{G} \odot \mathcal{K} = B\left[f_1W^{N_1}, \dots, f_rW^{N_r}, g_1W^{N'_1}, \dots, g_sW^{N'_s}\right].$$

Sheaves of Rees algebras

The notion of Rees algebra extends naturally to schemes. Consider a noetherian scheme V. A sheaf of Rees algebras over V is a subsheaf of $\mathcal{O}_V[W]$, say $\mathcal{G}$, so that $\Gamma(U, \mathcal{G})$ is a $\Gamma(U, \mathcal{O}_V)$-Rees algebra for each open subset $U \subset V$. In general we do not want to work with any kind of sheaves of Rees algebras, but only with those that are quasi-coherent.

Definition 3.1.3. A quasi-coherent sheaf of Rees algebras over a noetherian scheme V is called a *Rees algebra over* V, or simply an $\mathcal{O}_V$*-Rees algebra*.

Remark 3.1.4. Consider an affine scheme $V = \operatorname{Spec}(S)$. Observe that there is a natural correspondence between Rees algebras over S, and Rees algebras over V (i.e., quasi-coherent sheaves of Rees algebras over V). Thus, in this case, we shall abuse our notation and make no distinction between them. Furthermore, given an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$, and a set of generators of $\Gamma(V, \mathcal{G})$ over S, say $f_1 W^{N_1}, \ldots, f_r W^{N_r}$, we shall write

$$\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right] = \mathcal{O}_V\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right].$$

3.2 Rees algebras over regular schemes

Most of the time we will work with Rees algebras defined over regular schemes, as they provide a suitable tool to describe the singularities of a scheme.

Definition 3.2.1. Let $\mathcal{G}$ be a Rees algebra over a regular noetherian scheme V. We define the *singular locus*[1] of $\mathcal{G}$ as the set of points $\xi \in V$ so that $\nu_\xi(f) \geq N$ for every homogeneous element $fW^N \in \mathcal{G}_\xi$. We denote this set by $\operatorname{Sing}_V(\mathcal{G})$.

Note that the previous definition is compatible with localization. Namely, for an open subscheme $U \subset V$, we have that $\operatorname{Sing}_U(\mathcal{G}|_U) = \operatorname{Sing}_V(\mathcal{G}) \cap U$. In the case that V is affine, put $V = \operatorname{Spec}(S)$, and $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} I_n W^n$, one readily checks that

$$\operatorname{Sing}_V(\mathcal{G}) = \bigcap_{n \in \mathbb{N}} \{\xi \in V \mid \nu_\xi(I_n) \geq n\}.$$

The following Lemma provides an effective method to compute the singular locus of a Rees algebra defined over an affine scheme.

Lemma 3.2.2. *Let $V = \operatorname{Spec}(S)$ be a regular affine scheme, and consider an $\mathcal{O}_V$-Rees algebra, say*

$$\mathcal{G} = \mathcal{O}_V\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right].$$

Then

$$\operatorname{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\}.$$

[1]The singular locus of $\mathcal{G}$ should not be confused with the singular locus of a general scheme X, formed by the non-regular points of X.

Proof. Clearly $\xi \in \mathrm{Sing}_V(\mathcal{G})$ yields $\nu_\xi(f_i) \geq N_i$ for all i. Thus we just need to prove the converse.

Fix a point $\xi \in V$ so that $\nu_\xi(f_i) \geq N_i$ for $i = 1, \dots, r$. Let us express $\mathcal{G}$ as a direct sum of ideals, say $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} J_n W^n$. Note that

$$J_n = \langle f_1^{\alpha_1} \cdots f_r^{\alpha_r} \mid \alpha_1 N_1 + \cdots + \alpha_r N_r = n \rangle ,$$

and, by the assumption on ξ,

$$\nu_\xi(f_1^{\alpha_1} \cdots f_r^{\alpha_r}) \geq \alpha_1 N_1 + \cdots + \alpha_r N_r = n.$$

This implies that $\nu_\xi(J_n) \geq n$ for all n, and therefore $\xi \in \mathrm{Sing}_V(\mathcal{G})$. □

Corollary 3.2.3. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Rees algebras over a regular noetherian scheme V. Then*

$$\mathrm{Sing}_V(\mathcal{G} \odot \mathcal{G}') = \mathrm{Sing}_V(\mathcal{G}) \cap \mathrm{Sing}_V(\mathcal{G}').$$

Proposition 3.2.4. *Let V be a regular excellent scheme, and $\mathcal{G}$ an $\mathcal{O}_V$-Rees algebra. Then $\mathrm{Sing}_V(\mathcal{G})$ is a closed subset of V.*

Proof. It suffices to prove the claim in the affine case. Assume that $V = \mathrm{Spec}(S)$, and

$$\mathcal{G} = \mathcal{O}_V \left[f_1 W^{N_1}, \dots, f_r W^{N_r} \right].$$

The previous lemma implies that

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \{ \xi \in V \mid \nu_\xi(f_i) \geq N_i \} .$$

Since V is excellent, each of the sets $\{ \xi \in V \mid \nu_\xi(f_i) \geq N_i \}$ is closed in V by Corollary B.0.18. Hence $\mathrm{Sing}_V(\mathcal{G})$ is closed. □

There is also another set that one can naturally attach to a Rees algebra over a regular scheme: its zeros.

Definition 3.2.5. Let $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} \mathcal{I}_n W^n$ be a Rees algebra over regular noetherian scheme V. We define the *set of zeros of* $\mathcal{G}$ as

$$\mathrm{Zeros}_V(\mathcal{G}) := \bigcap_{n \geq 1} \mathbb{V}(\mathcal{I}_n).$$

Note that, by definition, $\mathrm{Zeros}_V(\mathcal{G})$ is an intersection of closed subsets of V, and hence it is closed. Moreover, when V is affine, put $V = \mathrm{Spec}(S)$, and $\mathcal{G} = S \left[f_1 W^{N_1}, \dots, f_r W^{N_r} \right]$, one readily checks that

$$\mathrm{Zeros}_V(\mathcal{G}) = \mathbb{V}(\langle f_1, \dots, f_r \rangle).$$

Observe that the set of zeros of an algebra always contains the singular locus of the algebra:

$$\mathrm{Sing}_V(\mathcal{G}) \subset \mathrm{Zeros}_V(\mathcal{G}).$$

In general, this inclusion is strict. In Chapter 6 we will study conditions on V and $\mathcal{G}$ that ensure that $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Zeros}_V(\mathcal{G})$ (see Lemma 6.2.6 and Lemma 6.6.1). This property will be used to construct a canonical representative for each class of Rees algebras over V (see Lemma 6.1.1).

Permissible transformations

Let $\mathcal{G}$ be a Rees algebra over a regular scheme V. A *$\mathcal{G}$-permissible transformation*, consists of a map of regular schemes $V \xleftarrow{\varphi_1} V_1$, together with a rule of transformation of $\mathcal{G}$ which produces an $\mathcal{O}_{V_1}$-Rees algebra $\mathcal{G}_1$. The algebra $\mathcal{G}_1$ is called the transform of $\mathcal{G}$ via φ. Thus, after taking a permissible transformation, we obtain a pair $(V_1, \mathcal{G}_1)$. There are three kinds of transformations, depending on the map φ. Let us first enumerate the types of maps that we will consider in this work:

- *Permissible blow-ups.* These are blow-ups of V along a closed regular center contained in $\mathrm{Sing}_V(\mathcal{G})$, say $V_1 = \mathrm{Bl}_Y(V)$. A regular center $Y \subset \mathrm{Sing}_V(\mathcal{G})$ is called a *$\mathcal{G}$-permissible center.*

- *Open restrictions.* These are restrictions to an open subscheme of V, say $V_1 \subset V$, with the condition that $\mathrm{Sing}_V(\mathcal{G}) \cap V_1 \neq \emptyset$.

- *Multiplication by an affine line.* In this case, $V_1 = V \times \mathbb{A}^1$.

As we mentioned above, each of these maps is associated with a rule of transformation of $\mathcal{G}$. When $V \xleftarrow{\varphi_1} V_1$ is either an open restriction or the multiplication by an affine line, the transform of $\mathcal{G}$ is simply the pull-back of $\mathcal{G}$ via φ_1. That is, $\mathcal{G}_1 = \varphi_1^*(\mathcal{G})$. The transform of $\mathcal{G}$ through permissible blow-ups is more complicated and we shall describe it in the following lines.

Transform of an algebra by blow-ups

Let $V \xleftarrow{\pi_1} V_1$ be a $\mathcal{G}$-permissible blow-up with center $Y \subset \mathrm{Sing}_V(\mathcal{G})$. The transform of $\mathcal{G}$ via π_1, say $\mathcal{G}_1$, will be defined locally. For $\xi \in V_1 \setminus \pi_1^{-1}(Y)$, we set $(\mathcal{G}_1)_\xi = \mathcal{G}_{\pi_1(\xi)}$ (recall that π_1 is an isomorphism locally at ξ). Next fix a point $\xi \in \pi_1^{-1}(Y)$, and consider an affine neighborhood of $\pi(\xi)$, say $\mathrm{Spec}(S) \subset V$. Fix a local set of generators of $\mathcal{G}$, say

$$\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right],$$

and suppose that Y is locally defined by a prime ideal $\mathfrak{p} = \langle x_1, \ldots, x_m \rangle \subset S$. Then the blow-up of S at $\mathfrak{p}$ can be covered by m affine charts of the form $\mathrm{Spec}(S_j)$, with

$$S_j = S\left[\frac{x_1}{x_j}, \ldots, \frac{x_m}{x_j}\right].$$

Let us assume $\xi \in \mathrm{Spec}(S_1)$. Since Y is $\mathcal{G}$-permissible and $\xi \in \pi_1^{-1}(Y)$, we have that $\pi_1(\xi) \in \mathrm{Sing}_V(\mathcal{G})$. Thus we may also assume without loss of generality that $f_i \in \mathfrak{p}^{N_i}$ for $i = 1, \ldots, r$. Hence, regarding f_i as an element of S_1, we have that

$$f_i \in \mathfrak{p}^{N_i} S_1 = \langle x_1^{N_i} \rangle S_1.$$

This enables us two express f_i as a product of the form

$$f_i = x_1^{N_i} \cdot f_i^*,$$

with $f_i^* \in S_1$. Note that, since x_1 is invertible in S_1, the element f_i^* is uniquely determined. Then we define the transform $\mathcal{G}$ locally at $\mathrm{Spec}(S_1)$ by

$$\mathcal{G}_1 = S_1 \left[f_1^* W^{N_1}, \ldots, f_r^* W^{N_r} \right].$$

Remark 3.2.6. In the previous discussion we have defined the transform of $\mathcal{G}$ through the blow-up $V \xleftarrow{\pi_1} V_1$ locally. However, it can be shown that all these local constructions patch over V_1, and thus that $\mathcal{G}_1$ is well-defined (see [10, §9.12, p. 118]).

Remark 3.2.7. In principle, the definition of f_i^* depends on the choice of x_1. However, one can check that the ideal $\langle f_i^* \rangle \subset S_1$ is well-defined. Hereafter we shall refer to the element $f_i^* W^{N_i} \in \mathcal{G}_1$ as the *weighted transform* of $f_i W^{N_i} \in \mathcal{G}$ via π_1.

Permissible sequences and resolution of algebras

Given a Rees algebra $\mathcal{G}$ over a regular scheme V, it is possible to concatenate a series of permissible transformations on V. A sequence of transformations

$$\begin{array}{ccccccccc} \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V = V_0 & \xleftarrow{\varphi_1} & V_1 & \xleftarrow{\varphi_2} & V_2 & \xleftarrow{\varphi_3} & \cdots & \xleftarrow{\varphi_m} & V_m, \end{array}$$

where each φ_i is a $\mathcal{G}_{i-1}$-permissible transformation, and each $\mathcal{G}_i$ represents the transform of $\mathcal{G}_{i-1}$ via φ_i, will be called a *$\mathcal{G}$-permissible sequence.*

Definition 3.2.8. Let V be a regular scheme and $\mathcal{G}$ an $\mathcal{O}_V$-Rees algebra. A *resolution*[2] *of* $\mathcal{G}$ consists on a sequence

$$\begin{array}{ccccccccc} \mathcal{G} = \mathcal{G}_0 & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \xleftarrow{\pi_3} & \cdots & \xleftarrow{\pi_m} & V_m \end{array}$$

where each π_i represents a $\mathcal{G}_{i-1}$-permissible blow-up and $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \emptyset$.

[2]The resolution of a Rees algebra over a regular scheme should not be confused with the resolution of singular scheme.

3.3 Local presentations and resolution of singularities

Fix an embedding of a singular equidimensional scheme X in a regular excellent scheme V. Consider a closed regular center $Y \subset \underline{\mathrm{Max}}\ \mathrm{mult}_X$ and let $X \leftarrow X_1$ denote the blow-up of X at Y. According to Dade's theorem (Theorem B.0.16), we have that $\max \mathrm{mult}_X \geq \max \mathrm{mult}_{X_1}$. Note also that Y can be regarded as a regular center in V. Thus there is a natural commutative diagram

$$\begin{tikzcd} V & V_1 \arrow[l] \\ X \arrow[u, hook] & X_1, \arrow[l] \arrow[u, hook] \end{tikzcd}$$

where V_1 represents the blow-up of V at Y. In this section we look for Rees algebras over V that describe the behavior of the highest multiplicity locus of X under blow-ups.

Definition 3.3.1. Let X be an excellent equidimensional scheme endowed with a closed immersion into a regular scheme V. We will say that an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ *represents the maximum multiplicity locus of* X, or simply that $\mathcal{G}$ *represents* $\underline{\mathrm{Max}}\ \mathrm{mult}_X$, if the following three conditions hold:

i) $\mathrm{Sing}_V(\mathcal{G}) = \underline{\mathrm{Max}}\ \mathrm{mult}_X$.

ii) Any $\mathcal{G}$-permissible sequence on V, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m, \end{array}$$

induces a sequence of blow-ups along closed regular equimultiple centers and smooth morphisms on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_m,$$

so that

$$\max \mathrm{mult}_X = \max \mathrm{mult}_{X_1} = \cdots = \max \mathrm{mult}_{X_{m-1}} \geq \max \mathrm{mult}_{X_m},$$

and vice-versa.

iii) For any sequences as those in ii), there is a natural commutative diagram

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_2 & & \mathcal{G}_m \\[-20pt] V & V_1 \arrow[l] & V_2 \arrow[l] & \cdots \arrow[l] & V_m \arrow[l] \\ X \arrow[u, hook] & X_1 \arrow[l] \arrow[u, hook] & X_2 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & X_m. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

We require on this diagram that

$$\mathrm{Sing}_{V_i}(\mathcal{G}_i) = \underline{\mathrm{Max}}\ \mathrm{mult}_{X_i}$$

for $i = 1, \ldots, m-1$, and

$$\mathrm{Sing}_{V_m}(G_m) = \emptyset \iff \max \mathrm{mult}_{X_m} < \max \mathrm{mult}_X .$$

Moreover, if $\max \mathrm{mult}_{X_m} = \max \mathrm{mult}_X$, then we require that $\mathrm{Sing}_{V_m}(\mathcal{G}_m) = \underline{\mathrm{Max}}\ \mathrm{mult}_{X_m}$.

Given an embedding of X in V as above and an $\mathcal{O}_V$-Rees algebra that represents $\underline{\mathrm{Max}}\ \mathrm{mult}_X$, the pair formed by the immersion $X \hookrightarrow V$ and $\mathcal{G}$ is called a (*global*) *presentation* of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$. In general, it is not possible to construct a global presentation of the maximum multiplicity locus of an arbitrary scheme X. However, given a point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}_X$, sometimes one can find such a presentation after restricting to a suitable neighborhood of X at ξ. These are called *local presentations* of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$.

Theorem 3.3.2 ([34, §7])**.** *Let X be a variety over a perfect field. Then $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ is locally representable in étale topology; that is, for each point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}_X$ there exists an étale neighborhood of X at ξ, say X', an embedding of X' in a smooth variety V, and an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ that represents $\underline{\mathrm{Max}}\ \mathrm{mult}_{X'}$ via $X' \hookrightarrow V$.*

3.4 Weak equivalence and the tree of permissible transformations

Fix an immersion of a singular scheme X in a regular ambient space V, and suppose that there exists an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ that represents $\underline{\mathrm{Max}}\ \mathrm{mult}_X$. In principle, there might be more algebras with this property. However, for any other Rees algebra $\mathcal{G}'$ representing $\underline{\mathrm{Max}}\ \mathrm{mult}_X$, one has that

$$\mathrm{Sing}_V(\mathcal{G}) = \underline{\mathrm{Max}}\ \mathrm{mult}_X = \mathrm{Sing}_V(\mathcal{G}').$$

Moreover, by conditions imposed on $\mathcal{G}$ and $\mathcal{G}'$, this equality is preserved by sequences of permissible transformations. This observation motivates the following definition.

Definition 3.4.1. Let V be a regular noetherian scheme. We will say that two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ are *weakly equivalent* if the following conditions hold:

i) $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Sing}_V(\mathcal{G}')$.

ii) Every $\mathcal{G}$-permissible sequence of transformations on V is also a $\mathcal{G}'$-permissible sequence, and vice-versa.

iii) For any $\mathcal{G}$ or $\mathcal{G}'$-permissible sequence of transformations on V, say

$$\begin{array}{ccccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m, \end{array}$$

we have $\operatorname{Sing}_{V_i}(\mathcal{G}_i) = \operatorname{Sing}_{V_i}(\mathcal{G}'_i)$ for $i = 1, \ldots, m$.

Remark 3.4.2. Fix an immersion of X into a regular noetherian scheme, say $X \hookrightarrow V$. Then, by definition, all the $\mathcal{O}_V$-Rees algebras that represent $\underline{\operatorname{Max}} \operatorname{mult}_X$ are weakly equivalent.

Lemma 3.4.3. *Let V be a regular excellent scheme. Then weak equivalence is an equivalence relation on the class of Rees algebras over V.*

Definition 3.4.4. Let $\mathcal{G}$ be a Rees algebra over a regular excellent scheme V. We denote the class of equivalence of $\mathcal{G}$ by $\mathscr{C}_V(\mathcal{G})$.

As we said in the introduction, Rees algebras can be used in the problem of lowering of the maximum multiplicity of a scheme (see the discussion on pages 6 to 12). From this point of view, it is interesting to define a canonical representative for each class of Rees algebras over V. This issue will be addressed in Chapter 6.

The tree of permissible transformations

Consider a pair $(V, \mathcal{G})$, and a $\mathcal{G}$-permissible sequence of transformations, say

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m. \end{array}$$

Note that, for each $i = 1, \ldots, m$, the singular locus of $\mathcal{G}_i$ determines a distinguished subset on V_i, namely $\operatorname{Sing}_{V_i}(\mathcal{G}_i)$. One could also consider simultaneously all the sequences of $\mathcal{G}$-permissible transformations, as well as the family of subsets determined by $\mathcal{G}$ and its transforms. This collection of transformations and subsets has a natural structure of tree, which we will call the *tree of $\mathcal{G}$-permissible transformations*, and will denote by $\mathscr{F}_V(\mathcal{G})$.

Lemma 3.4.5. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two Rees algebras over a regular excellent scheme V. Then $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if and only if $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$.*

Proof. If $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent, then it is clear that $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$. For the converse, assume that $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$. In this case we shall show that, for any sequence of transformations on V which is simultaneously $\mathcal{G}$ and $\mathcal{G}'$-permissible, say

$$\begin{array}{ccccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m, \end{array}$$

one has that $\mathrm{Sing}_{V_i}(\mathcal{G}_i) = \mathrm{Sing}_{V_i}(\mathcal{G}'_i)$ for $i = 1, \ldots, m$.

We proceed by contradiction: assume that $\mathrm{Sing}_{V_i}(\mathcal{G}_i) \neq \mathrm{Sing}_{V_i}(\mathcal{G}'_i)$. Suppose without loss of generality that $\mathrm{Sing}_{V_i}(\mathcal{G}_i) \not\subseteq \mathrm{Sing}_{V_i}(\mathcal{G}'_i)$, and fix a point $\xi \in \mathrm{Sing}(\mathcal{G}_i)$ so that $\xi \notin \mathrm{Sing}_{V_i}(\mathcal{G}'_i)$. Since V is excellent and V_i is a scheme of finite over V, Proposition B.0.8 says that V_i is excellent. Therefore, by Proposition B.0.11, there exists an open neighborhood of V_i at ξ, say $U \subset V_i$, so that $\overline{\{\xi\}} \cap U$ defines a closed regular subscheme. In this way we see that $\overline{\{\xi\}}$ defines locally a permissible center for $\mathcal{G}_i$, but not for $\mathcal{G}'_i$. Hence $\mathscr{F}_V(\mathcal{G}) \neq \mathscr{F}_V(\mathcal{G}')$, which is a contradiction. □

It is also interesting to study the case in which the tree of permissible transformations of an $\mathcal{O}_V$-Rees algebra $\mathcal{G}'$ is contained in that of $\mathcal{G}$.

Definition 3.4.6. Let V be a regular noetherian scheme, and $\mathcal{G}, \mathcal{G}'$ two $\mathcal{O}_V$-Rees algebras. We will say that $\mathscr{F}_V(\mathcal{G}') \subset \mathscr{F}_V(\mathcal{G})$ if the following conditions hold:

i) $\mathrm{Sing}_V(\mathcal{G}') \subset \mathrm{Sing}_V(\mathcal{G})$.

ii) Every $\mathcal{G}'$-permissible sequence of transformations is $\mathcal{G}$-permissible.

iii) For any $\mathcal{G}'$-permissible sequence of transformations, say

$$\begin{array}{ccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m\,, \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m \end{array}$$

we have $\mathrm{Sing}_{V_i}(\mathcal{G}'_i) \subset \mathrm{Sing}_{V_i}(\mathcal{G}_i)$ for each $i = 1, \ldots, m$.

Remark 3.4.7. Two Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ will be weakly equivalent if and only if $\mathscr{F}_V(\mathcal{G}') \subset \mathscr{F}_V(\mathcal{G})$ and $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}')$.

Lemma 3.4.8. *Let V be a regular noetherian scheme and $\mathcal{G}, \mathcal{G}'$ two $\mathcal{O}_V$-Rees algebras. Then $\mathcal{G} \subset \mathcal{G}'$ implies $\mathscr{F}_V(\mathcal{G}') \subset \mathscr{F}_V(\mathcal{G})$.*

Proof. This is a result of local nature, so let us assume that $V = \mathrm{Spec}(S)$, with

$$\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right],$$

and

$$\mathcal{G}' = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}, f_{r+1} W^{N_{r+1}}, \ldots, f_{r+s} W^{N_{r+s}}\right].$$

Then, by the definition of singular locus,

$$\begin{aligned} \mathrm{Sing}_V(\mathcal{G}') &= \bigcap_{i=1}^{r+s} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\} \\ &\subset \bigcap_{i=1}^{r} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\} = \mathrm{Sing}_V(\mathcal{G}). \end{aligned}$$

From this inclusion we deduce that every $\mathcal{G}'$-permissible transformation is $\mathcal{G}$-permissible.

Next fix a $\mathcal{G}'$-permissible transformation, say $V \overset{\pi_1}{\longleftarrow} V_1$, and denote by $\mathcal{G}_1$ and $\mathcal{G}_1'$ the transforms of $\mathcal{G}$ and $\mathcal{G}'$ respectively. We claim that $\mathcal{G}_1 \subset \mathcal{G}_1'$. In order to prove this assertion, we consider each of the three possible kinds of transformations. In case that π_1 is either an open restriction or the multiplication by an affine line, the claim is obvious. So suppose that π_1 is a permissible blow-up whose center is defined by a prime ideal

$$\mathfrak{p} = \langle x_1, \dots, x_e \rangle \subset S.$$

Recall that the blow-up of S at $\mathfrak{p}$ can be covered by e affine charts of the form $\mathrm{Spec}(S_1), \dots, \mathrm{Spec}(S_e)$, with

$$S_i = S\left[\frac{x_1}{x_i}, \dots, \frac{x_e}{x_i}\right].$$

Let us focus on the first of these charts, say $U_1 = \mathrm{Spec}(S_1)$. Then, following the construction of page 38, set

$$f_i = x_1^{N_i} \cdot f_i^*,$$

with $f_i^* \in S_1$ for $i = 1, \dots, r+s$ (i.e., $f_i^* W^{N_i}$ is the weighted transform of $f_i W^{N_i}$). By definition,

$$\mathcal{G}_1|_{U_1} = S_1\left[f_1^* W^{N_1}, \dots, f_r^* W^{N_r}\right],$$

and

$$\mathcal{G}_1'|_{U_1} = S\left[f_1^* W^{N_1}, \dots, f_r^* W^{N_r}, f_{r+1}^* W^{N_{r+1}}, \dots, f_{r+s}^* W^{N_{r+s}}\right],$$

which shows that $\mathcal{G}_1|_{U_1} \subset \mathcal{G}_1'|_{U_1}$. Repeating the same arguments for each of the affine charts that cover the blow-up of S at $\mathfrak{p}$ one readily checks that $\mathcal{G}_1 \subset \mathcal{G}_1'$. This proves the case of the blow-up.

As we have shown that after applying a $\mathcal{G}'$-permissible transformation to $\mathcal{G}$ and $\mathcal{G}'$ we have that $\mathcal{G}_1 \subset \mathcal{G}_1'$, the result follows by induction on the number of transformations. $\square$

Remark 3.4.9. In general, the converse of the previous lemma is false. For instance, consider the scheme $V = \mathrm{Spec}(S)$ with $S = k[x, y]$, and the Rees algebras $\mathcal{G} = S[xW]$, $\mathcal{H} = S[xW, yW]$, and $\mathcal{H}' = S[x^2W^2, yW]$. One can check that

$$\mathscr{F}_V(\mathcal{H}') = \mathscr{F}_V(\mathcal{H}) \subset \mathscr{F}_V(\mathcal{G}),$$

but nevertheless $\mathcal{G} \not\subset \mathcal{H}'$.

Definition 3.4.10. Let V a regular noetherian scheme and $\mathcal{G}, \mathcal{G}'$ two $\mathcal{O}_V$-Rees algebras. We define the *intersection tree* $\mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{G}')$ as the collection of sequences on V which are simultaneously $\mathcal{G}$-permissible and $\mathcal{G}'$-permissible.

Lemma 3.4.11. *Let $\mathcal{G}$ and $\mathcal{G}'$ two Rees algebras over a regular noetherian scheme V. Then, for any sequence of transformations on V which is simultaneously $\mathcal{G}$-permissible and $\mathcal{G}'$-permissible, say*

$$\begin{array}{ccccccccc} \mathcal{G}' & & \mathcal{G}'_1 & & \mathcal{G}'_2 & & & & \mathcal{G}'_m \,, \\ \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_m \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_m \end{array}$$

we have that

$$\operatorname{Sing}_{V_i}(\mathcal{G}_i \odot \mathcal{G}'_i) = \operatorname{Sing}_{V_i}(\mathcal{G}_i) \cap \operatorname{Sing}_{V_i}(\mathcal{G}'_i).$$

for $i = 1, \ldots, m$.

Proof. Set $\mathcal{K} = \mathcal{G} \odot \mathcal{G}'$. As $\operatorname{Sing}_V(\mathcal{K}) = \operatorname{Sing}_V(\mathcal{G}) \cap \operatorname{Sing}_V(\mathcal{G}')$ by Corollary 3.2.3, we have that $V \leftarrow V_1$ is a $\mathcal{K}$-permissible transformation. Let $\mathcal{K}_1$ denote the transform of $\mathcal{K}$. An easy computation shows that $\mathcal{K}_1 = \mathcal{G}_1 \odot \mathcal{G}'_1$. Then

$$\operatorname{Sing}_{V_1}(\mathcal{K}_1) = \operatorname{Sing}_{V_1}(\mathcal{G}_1 \odot \mathcal{G}'_1) = \operatorname{Sing}_{V_1}(\mathcal{G}_1) \cap \operatorname{Sing}_{V_i}(\mathcal{G}'_1),$$

and the result follows by induction on the number of transformations. $\square$

Corollary 3.4.12. *Under the hypotheses of the previous lemma,*

$$\mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{G}') = \mathscr{F}_V(\mathcal{G} \odot \mathcal{G}').$$

3.5 Integral closure

Let $\mathcal{G}$ be a Rees algebra defined over a normal domain B. Note that $\mathcal{G}$ can be regarded as a domain contained in $B[W]$. Then we define the *integral closure of* $\mathcal{G}$, denoted by $\overline{\mathcal{G}}$, as its normalization (regarded as a domain). Another B-Rees algebra $\mathcal{G}'$ will be said to be *integral over* $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{G}'$ and $\mathcal{G}' \subset \overline{\mathcal{G}}$.

Remark 3.5.1. Since B is normal, it follows that $B[W]$ is normal, and hence $\overline{\mathcal{G}} \subset B[W]$. That is, $\overline{\mathcal{G}}$ is a graded algebra over B. However, it could happen that $\overline{\mathcal{G}}$ is not finitely generated over B. Thus, in order ensure that $\overline{\mathcal{G}}$ is again a Rees algebra over B, we need to add some assumptions on either B or $\mathcal{G}$.

Lemma 3.5.2. *Let B be a normal domain and $\mathcal{G}$ a Rees algebra over B. If B is excellent, then $\overline{\mathcal{G}}$ is finitely generated over B (i.e., $\overline{\mathcal{G}}$ is a Rees algebra over B).*

Proof. Since B is excellent and $\mathcal{G}$ is finitely generated over B, we have that $\mathcal{G}$ is excellent by Proposition B.0.8. Hence $\overline{\mathcal{G}}$ is finite over $\mathcal{G}$ by Proposition B.0.12, and therefore it is finitely generated over B. $\square$

Lemma 3.5.3 (Localization)**.** *Let $\mathcal{G}$ be Rees algebra over an excellent normal domain B. Then, for any multiplicative subset $\mathcal{U} \subset B$,*

$$\mathcal{U}^{-1}\overline{\mathcal{G}} = \overline{\mathcal{U}^{-1}\mathcal{G}}.$$

Proof. By definition, $\overline{\mathcal{U}^{-1}\mathcal{G}}$ is the smallest $\mathcal{U}^{-1}B$ algebra which contains $\mathcal{U}^{-1}\mathcal{G}$ and is integrally closed. In this way one readily checks that $\overline{\mathcal{G}} \subset \overline{\mathcal{U}^{-1}\mathcal{G}}$, and hence $\mathcal{U}^{-1}\overline{\mathcal{G}} \subset \overline{\mathcal{U}^{-1}\mathcal{G}}$. On the other hand, $\mathcal{U}^{-1}\overline{\mathcal{G}}$ is integrally closed by [31, §III.C.1, Proposition 9, p. III-13]. Thus it follows that $\mathcal{U}^{-1}\overline{\mathcal{G}} = \overline{\mathcal{U}^{-1}\mathcal{G}}$. □

Lemma 3.5.4. *Let $\mathcal{G} = \bigoplus_{n\geq 0} I_n W^n$ be a Rees algebra over a normal domain B. If $\mathcal{G}$ is integrally closed, then $I_{n+1} \subset I_n$ for all n.*

Proof. Fix an element $f \in I_{n+1}$. Clearly fW^n satisfies the polynomial equation $P(fW^n) = 0$, where

$$P(T) = T^{n+1} - f \cdot (fW^{n+1})^n \in \mathcal{G}[T].$$

Note that this is a relation of integral dependence of fW^n over $\mathcal{G}$. Since $\mathcal{G}$ is integrally closed, it follows that $fW^n \in \mathcal{G}$. Hence $f \in I_n$. □

Lemma 3.5.5. *Let $\mathcal{G} \subset \mathcal{G}'$ be an inclusion of Rees algebra over a noetherian normal domain B. Assume that*

$$\mathcal{G}' = B\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right].$$

Then $\mathcal{G}'$ is integral over $\mathcal{G}$ if and only if each $f_i W^{N_i}$ is integral over $\mathcal{G}$.

Proof. Note that $\mathcal{G}$ can be regarded as a domain contained in $B[W]$, and $\mathcal{G}'$ as a $\mathcal{G}$-algebra generated by $f_1W^{N_1}, \ldots, f_rW^{N_r}$. Thus $\mathcal{G}'$ is integral over $\mathcal{G}$ if and only if each $f_iW^{N_i}$ is integral over $\mathcal{G}$. □

As we said at the beginning of Section 3.2, most of the time we will work with Rees algebras defined over regular rings and schemes. Recall that, if S is a regular domain, then it is normal, and hence all the previous discussion applies in this case.

Lemma 3.5.6. *Let S be a regular domain, $V = \mathrm{Spec}(S)$, and $\mathcal{G} \subset \mathcal{G}'$ an inclusion of S-Rees algebras. If $\mathcal{G}'$ is integral over $\mathcal{G}$, then $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Sing}_V(\mathcal{G}')$.*

Proof. Since $\mathcal{G} \subset \mathcal{G}'$, clearly $\mathrm{Sing}_V(\mathcal{G}') \subset \mathrm{Sing}_V(\mathcal{G})$. To prove the converse, let us proceed by contradiction: fix a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$, and assume that $\xi \notin \mathrm{Sing}_V(\mathcal{G}')$.

Since $\xi \notin \mathrm{Sing}_V(\mathcal{G}')$, one can find a homogeneous element $fW^N \in \mathcal{G}'$ such that $\nu_\xi(f) < N$. Set $n = \nu_\xi(f) < N$. By hypotheses, fW^N is integral over $\mathcal{G}$, i.e., f satisfies an equation of integral dependence of the form

$$f^d + a_1 f^{d-1} + \cdots + a_d = 0,$$

with $a_iW^{iN} \in \mathcal{G}$. As $\xi \in \mathrm{Sing}_V(\mathcal{G})$, we have that $\nu_\xi(a_i) \geq i$ for all i. Hence

$$\begin{aligned} \nu_\xi(a_i f^{d-i}) &= \nu_\xi(a_i) + \nu_\xi(f^{d-i}) \\ &\geq iN + (d-i)n \\ &> in + (d-i)n = nd, \end{aligned}$$

and therefore

$$\nu_\xi(f^d) = nd < \nu_\xi(a_i f^{d-i}),$$

for $i = 1, \dots, d$. This implies that

$$\nu_\xi(f^d + a_1 f^{d-1} + \cdots + a_d) = \nu_\xi(f^d) = nd,$$

which is a contradiction because we assumed $f^d + a_1 f^{d-1} + \cdots + a_d = 0$. $\square$

Lemma 3.5.7. *Let S be a regular domain, $V = \operatorname{Spec}(S)$, and $\mathcal{G} \subset \mathcal{G}'$ an inclusion of S-Rees algebras. If $\mathcal{G}'$ is integral over $\mathcal{G}$, then $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$ (i.e., $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent).*

Proof. According to Lemma 3.4.8 we have that $\mathscr{F}_V(\mathcal{G}') \subset \mathscr{F}_V(\mathcal{G})$. Hence it suffices to prove that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}')$.

Let us start by fixing generators for $\mathcal{G}$ and $\mathcal{G}'$, say

$$\mathcal{G} = S\left[f_1 W^{N_1}, \dots, f_r W^{N_r}\right],$$

and

$$\mathcal{G}' = S\left[f_1 W^{N_1}, \dots, f_{r+s} W^{N_{r+s}}\right].$$

By Lemma 3.5.6 we have that $\operatorname{Sing}_V(\mathcal{G}') = \operatorname{Sing}_V(\mathcal{G})$, and hence any $\mathcal{G}$-permissible transformation on V is also $\mathcal{G}'$-permissible. In this way, if we prove that for any $\mathcal{G}$-permissible transformation, say $V \xleftarrow{\varphi_1} V_1$, the transform of $\mathcal{G}'$, say $\mathcal{G}'_1$, is integral over the transform of $\mathcal{G}$, say $\mathcal{G}_1$, then the inclusion $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}')$ follows by induction on the number of transformations.

Fix a $\mathcal{G}$-permissible transformation $V \xleftarrow{\varphi_1} V_1$, and let $\mathcal{G}_1$ and $\mathcal{G}'_1$ denote the transforms of $\mathcal{G}$ and $\mathcal{G}'$ respectively. Let us check case by case that, whichever kind of transformation φ_1 is, $\mathcal{G}'_1$ is integral over $\mathcal{G}_1$.

Suppose that φ_1 is an open restriction or the multiplication by an affine line. In either case $\mathcal{G}_1$ and $\mathcal{G}'_1$ are the pull-back of $\mathcal{G}$ and $\mathcal{G}'$ to V_1 respectively. Then one readily checks that $\mathcal{G}'_1$ integral over $\mathcal{G}_1$.

On the other hand, suppose that φ_1 is the blow-up of V along a regular center $Y \subset \operatorname{Sing}_V(\mathcal{G})$ given by a prime ideal

$$\mathfrak{p} = \langle x_1, \dots, x_e \rangle \subset S.$$

Recall that, in this case, V_1 can be covered by affine charts of the form $\operatorname{Spec}(S_i)$ with

$$S_i = S\left[\frac{x_1}{x_i}, \dots, \frac{x_e}{x_i}\right].$$

Let us focus on x_1-chart, i.e., that given by $U_1 = \operatorname{Spec}(S_1)$. Then, locally at U_1, the algebras $\mathcal{G}_1$ and $\mathcal{G}'_1$ are defined by

$$\mathcal{G}_1 = S_1\left[f_1^* W^{N_1}, \dots, f_r^* W^{N_r}\right],$$

and

$$\mathcal{G}'_1 = S_1\left[f_1^* W^{N_1}, \dots, f_{r+s}^* W^{N_{r+s}}\right],$$

where $f_i^* W^{N_i}$ represents the weighted transform of $f_i W^{N_i}$ (see the discussion on p. 38). Fix $i \in \{r+1, \ldots, r+s\}$. As $f_i W^{N_i}$ is integral over $\mathcal{G}$, the element f_i satisfies an equation of integral dependence over $\mathcal{G}$ of the form

$$f_i^n + a_1 f_i^{n-1} + \cdots + a_n = 0,$$

where $a_j W^{n-j} \in \mathcal{G}$ for all j. Let $a_j^* W^{n-j} \in \mathcal{G}_1$ denote the weighted transform of $a_j W^{n-j}$ via φ_1. Then, regarding the elements of the previous equation as elements of S_1, we have that

$$x_1^{N_i} \cdot \left((f_i^*)^n + a_1^*(f_i^*)^{n-1} + \cdots + a_n^*\right) = f_i^n + a_1 f_i^{n-1} + \cdots + a_n = 0.$$

Since x_1 is invertible in S_1, this implies that

$$(f_i^*)^n + a_1^*(f_i^*)^{n-1} + \cdots + a_n^* = 0,$$

which gives an equation of integral dependence of $f_i^* W^{N_i}$ over $\mathcal{G}_1$. Finally, repeating the same argument for each $i = 1, \ldots, r+s$, we see that $\mathcal{G}_1'$ is integral over $\mathcal{G}_1$. □

3.6 Saturation by differentials

Definition 3.6.1. Let S be a regular ring, and $\mathcal{G} \subset \mathcal{G}'$ an inclusion of Rees algebras over S. We will say that $\mathcal{G}'$ is *differential relative to* $\mathcal{G}$ if there exist elements

$$f_1 W^{N_1}, \ldots, f_r W^{N_r} \in \mathcal{G}$$

and differential operators $\Delta_1, \ldots, \Delta_r$ with $\Delta_i \in \mathrm{Diff}^{n_i}(S)$ and $n_i < N_i$ so that

$$\mathcal{G}' = S\left[\Delta_1(f_1) W^{N_1 - n_1}, \ldots, \Delta_r(f_r) W^{N_r - n_r}\right]. \tag{3.1}$$

Remark 3.6.2. Note that condition (3.1) is equivalent to require

$$\mathcal{G}' = \mathcal{G} \odot S\left[\Delta_1(f_1) W^{N_1 - n_1}, \ldots, \Delta_r(f_r) W^{N_r - n_r}\right],$$

as for each $fW^N \in \mathcal{G}$, we may consider the differential operator $\mathrm{id} \in \mathrm{Diff}^0(S)$, and the element $fW^N = \mathrm{id}(f) W^{N-0}$.

Lemma 3.6.3 (Localization)**.** *Let $\mathcal{G}'$ be a S-Rees algebra which is differential relative to $\mathcal{G}$. Then, for any multiplicative subset $\mathcal{U} \subset S$, the $\mathcal{U}^{-1}S$-Rees algebra $\mathcal{U}^{-1}\mathcal{G}'$ is differential relative to $\mathcal{G}$.*

Proof. Fix $f_1 W^{N_1}, \ldots, f_r W^{N_r} \in \mathcal{G}$ and $\Delta_1, \ldots, \Delta_r$ as in Definition 3.6.1. Since $\mathcal{U}^{-1}\mathcal{G}'$ has the same generators of $\mathcal{G}'$ and differential operators localize (see Proposition 2.2.6), the claim follows immediately. □

Lemma 3.6.4. *Let S be a regular ring, $V = \mathrm{Spec}(S)$, and $\mathcal{G} \subset \mathcal{G}'$ an extension of Rees algebras over S. If $\mathcal{G}'$ is differential relative to $\mathcal{G}$, then* $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Sing}_V(\mathcal{G}')$.

Proof. Since $\mathcal{G} \subset \mathcal{G}'$, clearly $\mathrm{Sing}_V(\mathcal{G}') \subset \mathrm{Sing}_V(\mathcal{G})$. To prove the converse, fix $f_1W^{N_1}, \ldots, f_rW^{N_r} \in \mathcal{G}$ and $\Delta_1, \ldots, \Delta_r$ as in Definition 3.6.1. That is,

$$\mathcal{G}' = \mathcal{G} \odot S\left[\Delta_1(f_1)W^{N_1-n_1}, \ldots, \Delta_r(f_r)W^{N_r-n_r}\right].$$

Note that, for any prime ideal $\mathfrak{p} \subset S$,

$$\nu_{\mathfrak{p}}(f_i) \geq N_i \implies \nu_{\mathfrak{p}}(\Delta_i(f_i)) \geq N_i - n_i$$

by Lemma 2.2.8. Hence $\mathrm{Sing}_V(\mathcal{G}) \subset \mathrm{Sing}_V(\mathcal{G}')$. □

Lemma 3.6.5. *Let $\mathcal{G} \subset \mathcal{G}'$ be an extension of Rees algebras over a regular ring S. Set $V = \mathrm{Spec}(S)$, and consider a $\mathcal{G}$-permissible transformation $V \xleftarrow{\varphi_1} V_1$. Let $\mathcal{G}_1$ and $\mathcal{G}_1'$ denote the transforms of $\mathcal{G}$ and $\mathcal{G}'$ via φ_1 respectively. If $\mathcal{G}'$ is differential relative to $\mathcal{G}$, then $\mathcal{G}_1'$ is differential relative to $\mathcal{G}_1$.*

Proof. Suppose that

$$\mathcal{G}' = S\left[h_1W^{N_1-n_1}, \ldots, h_rW^{N_r-n_r}\right],$$

with $h_i = \Delta_i(f_i)$ for some $f_iW^{N_i} \in \mathcal{G}$, and some $\Delta_i \in \mathrm{Diff}^{n_i}(S)$. If φ_1 represents an open restriction or the multiplication by an affine line, then the claim is trivial. Hence assume that φ_1 is the blow-up of V along a regular center $Y \subset \mathrm{Sing}_V(\mathcal{G})$ given by a prime ideal

$$\mathfrak{p} = \langle x_1, \ldots, x_e\rangle \subset S.$$

Next recall that V_1 can be covered by e affine charts of the form $\mathrm{Spec}(S_j)$, with

$$S_j = S\left[\frac{x_1}{x_j}, \ldots, \frac{x_e}{x_j}\right].$$

We shall focus on the first of these charts, say $U_1 = \mathrm{Spec}(S_1)$. By definition we have that, locally at U_1,

$$\mathcal{G}_1' = S_1\left[h_1^*W^{N_1-n_1}, \ldots, h_r^*W^{N_r-n_r}\right],$$

where $h_i^*W^{N_i-n_i}$ represents the weighted transform of $h_iW^{N_i-n_i}$ via φ_1 (see the discussion on p. 38). Recall that the element $h_i^* \in S_1$ satisfies the equation

$$h_i = x_1^{N_i-n_i} \cdot h_i^*.$$

In addition, Giraud's lemma (Lemma 2.5.3) ensures that there exists a differential operator $\Delta_i^* \in \mathrm{Diff}^{n_i}(S_1)$ so that

$$\Delta_i^*(g) = x_1^{-(N_i-n_i)} \cdot \Delta_i\left(x_1^{N_i} \cdot g\right)$$

for $g \in S_1$. In this way, if $f_i^*W^{N_i} \in \mathcal{G}_1$ denotes the weighted transform of $f_iW^{N_i} \in \mathcal{G}$ via φ_1, then we have that

$$\begin{aligned} h_i^* &= x_1^{-(N_i-n_i)} \cdot h_i \\ &= x_1^{-(N_i-n_i)} \cdot \Delta_i(f_i) \\ &= x_1^{-(N_i-n_i)} \cdot \Delta_i\left(x_1^{N_i} \cdot f_i^*\right) = \Delta_i^*(f_i^*), \end{aligned}$$

which shows that $\mathcal{G}_1'$ is differential relative to $\mathcal{G}_1$. □

Proposition 3.6.6. *Let S be a regular ring, $V = \operatorname{Spec}(S)$, and $\mathcal{G} \subset \mathcal{G}'$ an extension of Rees algebras over S. If $\mathcal{G}'$ is differential relative to $\mathcal{G}$, then $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$ (i.e., $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent).*

Proof. It follows easily from Lemma 3.6.4 and Lemma 3.6.5 by induction on the number of transformations. □

Differential saturation

In the following lines we introduce the notion of differential saturation for a Rees algebra. This concept will play a key role along Chapter 6, in the construction of the canonical representative of a given class of Rees algebras.

Definition 3.6.7. A Rees algebra $\mathcal{G}$ over regular ring S is said to be *differentially saturated* if for every $fW^N \in \mathcal{G}$, every integer $n < N$, and every differential operator $\Delta \in \operatorname{Diff}^n(S)$ one has that $\Delta(f)W^{N-n} \in \mathcal{G}$.

Given a Rees algebra $\mathcal{G}$ over a regular ring S, there is a natural way to construct an algebra over S which contains $\mathcal{G}$ and is differentially saturated. Namely this can be achieved by adding to $\mathcal{G}$ all the elements of the form $\Delta(f)W^{N-n} \in \mathcal{G}$ with $fW^N \in \mathcal{G}$, $n < N$, and $\Delta \in \operatorname{Diff}^n(S)$. The algebra obtained by this process is called the *differential saturation of* $\mathcal{G}$, and we shall denote it by $\operatorname{Diff}(\mathcal{G})$.

Remark 3.6.8. Note that, in general, $\operatorname{Diff}(\mathcal{G})$ could not be finitely generated over S. In other words, we do not know whether $\operatorname{Diff}(\mathcal{G})$ is a Rees algebra.

Let $V = \operatorname{Spec}(S)$ be a regular affine variety over a perfect field k. For Rees algebras defined over S, we will be turn our attention to the differential saturation with respect to the relative structure of V over k. That is, we will just consider differential operators of S relative k.

Definition 3.6.9. Let S be a regular algebra of finite type over a perfect field k. We will say that a Rees algebra $\mathcal{G}$ over S is *differentially saturated* (*with respect to k*) if for every $fW^N \in \mathcal{G}$, every integer $n < N$, and every differential operator $\Delta \in \operatorname{Diff}^n_k(S)$ one has that $\Delta(f)W^{N-n} \in \mathcal{G}$.

Theorem 3.6.10 ([33, Theorem 3.4])**.** *Let $\mathcal{G}$ be a Rees algebra over a regular algebra S of finite type over a perfect field k. Then there exists a minimum Rees algebra over S which contains $\mathcal{G}$ and is differentially saturated with respect to k.*

Hereafter we shall refer to the Rees algebra of the theorem as the *differential saturation of $\mathcal{G}$ with respect to k*, and we shall denote it by $\operatorname{Diff}_k(\mathcal{G})$.

Remark 3.6.11. Let S be a regular algebra of finite over a perfect field k as in the previous theorem. [33, Theorem 2.9] provides a criterion to check whether a Rees algebra over S is differentially saturated with respect to k. Namely, assume that

$$\mathcal{G} = \left[f_1W^{N_1}, \dots, f_rW^{N_r}\right].$$

Then $\mathcal{G}$ is differentially saturated with respect to k if and only if for each $i = 1, \ldots, r$, each integer $n < N_i$, and each differential operator $\Delta \in \mathrm{Diff}_k^n(S)$, one has that

$$\Delta(f_i)W^{N_i - n} \in \mathcal{G}.$$

Note also that, since S is a finitely generated k-algebra, $\mathrm{Diff}_k^n(S)$ is a finitely generated S-module for each $n \geq 0$. Thus the previous criterion enables us to give an explicit construction of $\mathrm{Diff}_k(\mathcal{G})$ for any Rees algebra $\mathcal{G}$ over S.

Theorem 3.6.12 ([9, Theorem 3.10]). *Let S be a regular algebra of finite type over a perfect field k, and $\mathcal{G}, \mathcal{G}'$ two Rees algebras S. Set $V = \mathrm{Spec}(S)$. Then $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}')$ if and only if $\overline{\mathrm{Diff}_k(\mathcal{G}')} \subset \overline{\mathrm{Diff}_k(\mathcal{G})}$.*

Corollary 3.6.13. *Let S be a regular algebra of finite type over a perfect field k, and $\mathcal{G}, \mathcal{G}'$ two Rees algebras S. Then $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if and only if $\overline{\mathrm{Diff}_k(\mathcal{G})} = \overline{\mathrm{Diff}_k(\mathcal{G}')}$.*

3.7 Restriction to closed subschemes

Let $\mathcal{G}$ be a Rees algebra over a regular noetherian scheme V, and consider a closed regular subscheme $Z \subset V$. We define the *restriction of $\mathcal{G}$ to Z* as

$$\mathcal{G}|_Z = \mathcal{G} \otimes_{\mathcal{O}_V} \mathcal{O}_Z.$$

Note that $\mathcal{G}|_Z$ is a Rees algebra over Z. In this section we study the relation between $\mathscr{F}_V(\mathcal{G})$ and $\mathscr{F}_Z(\mathcal{G}|_Z)$.

Remark 3.7.1. Let $Z \subset V$ be closed immersion of regular noetherian schemes. There is a natural $\mathcal{O}_V$-Rees algebra that one can attach to the embedding of Z in V, say

$$\mathcal{Z} = \mathcal{O}_V[\mathcal{I}(Z)W].$$

Here $\mathcal{I}(Z)$ represents the ideal of definition of Z in V. Observe that, by definition, $Z = \mathrm{Sing}_V(\mathcal{Z})$. Moreover, this equality is preserved by permissible sequences as follows: any $\mathcal{Z}$-permissible sequence of blow-ups and smooth morphisms on V, say

$$\begin{array}{cccccc} \mathcal{Z} & & \mathcal{Z}_1 & & & \mathcal{Z}_m \\ V & \longleftarrow & V_1 & \longleftarrow & \cdots \longleftarrow & V_m, \end{array}$$

induces a sequence of blow-ups along regular centers and smooth morphisms on Z, say

$$Z \longleftarrow Z_1 \longleftarrow \cdots \longleftarrow Z_m,$$

which is linked to the previous one by a natural commutative diagram

$$\begin{tikzcd} \mathcal{Z} & \mathcal{Z}_1 & & \mathcal{Z}_m \\[-20pt] V & V_1 \arrow[l] & \cdots \arrow[l] & V_m \arrow[l] \\ Z \arrow[u, hook] & Z_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & Z_m, \arrow[l] \arrow[u, hook] \end{tikzcd}$$

where each $\mathcal{Z}_i$ coincides with the natural Rees algebra attached to embedding of Z_i in V_i. In particular, $Z_i = \mathrm{Sing}_{V_i}(\mathcal{Z}_i)$ for $i = 1, \dots, m$.

Proposition 3.7.2. *Let $\mathcal{G}$ be a Rees algebra over a regular noetherian scheme V, and let Z be a closed regular subscheme of V. Following the notation of Remark 3.7.1 above, let $\mathcal{Z} = \mathcal{O}_V[\mathcal{I}(Z)W]$ denote the natural Rees algebra attached to the embedding of Z in V. Then*

$$\mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}) \subset \mathscr{F}_Z(\mathcal{G}|_Z).$$

Proof. Set $\mathcal{G}' = \mathcal{G}|_Z$. Observe that for any point $\xi \in Z$, and for any element $f \in \mathcal{O}_{V,\xi}$, one has that

$$\nu_{V,\xi}(f) \leq \nu_{Z,\xi}(f).$$

Hence

$$\mathrm{Sing}_V(\mathcal{G}) \cap Z \subset \mathrm{Sing}_Z(\mathcal{G}').$$

In addition, attending to this inclusion, any transformation on V which is simultaneously $\mathcal{G}$-permissible and $\mathcal{Z}$-permissible, say

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{G}_1 \\ V & \longleftarrow & V_1, \end{array}$$

induces a $\mathcal{G}'$-permissible transformation on Z, say

$$\begin{array}{ccc} \mathcal{G}' & & \mathcal{G}'_1 \\ Z & \longleftarrow & Z_1 \end{array}$$

which is linked to the previous one by a natural commutative diagram

$$\begin{tikzcd} V & V_1 \arrow[l] \\ Z \arrow[u, hook] & Z_1. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

Here $\mathcal{G}_1$ represents the transform of $\mathcal{G}$ via $V \leftarrow V_1$, and $\mathcal{G}'_1$ that of $\mathcal{G}'$ via $Z \leftarrow Z_1$. An easy computation shows that $\mathcal{G}'_1 = \mathcal{G}_1|_{Z_1}$. Thus the result follows by induction on the number of transformations. $\square$

Chapter 4

Stratification defined by the multiplicity

Along this chapter we study some natural properties of the stratification defined by the multiplicity on an equidimensional excellent scheme.

Recall that the multiplicity along points of an equidimensional excellent scheme X can be regarded as a function $\mathrm{mult}_X : X \to \mathbb{N}$. Sections 4.1 and 4.2 are devoted to the study of the upper semi-continuity of this function for the case in which X is an equidimensional scheme of finite type over a perfect field k. In Section 4.3 we show that there is a natural compatibility between the process of lowering of the multiplicity of an equidimensional scheme X and that of the underlying reduced scheme, say X_{red}.

The main result of this chapter is Theorem 4.4.4, where we prove that there exists a canonical $\mathcal{O}_X$-Rees algebra, say $\mathcal{G}_X$, attached to the stratum of maximum multiplicity of a variety X defined over a perfect field k. In Section 4.5 we analyze the behavior of this algebra under finite extensions.

4.1 Local presentations on varieties

In this section we review a method which enables us to construct local presentations of the multiplicity for equidimensional reduced schemes defined over a perfect field k. In particular, this includes the case of varieties over k. The details of this construction can be found in [10, Appendix A] and [34].

Proposition 4.1.1 ([10, Proposition 31.1])**.** *Let X be an equidimensional reduced scheme of finite type over a perfect field k. For any closed point $\xi \in X$ it is possible to construct an étale affine neighborhood of X at ξ, say $X \leftarrow \mathrm{Spec}(B)$,* *and a regular subalgebra of B, say S, such that $S \subset B$ is a finite extension of generic rank $n = \mathrm{mult}_X(\xi)$.*

Corollary 4.1.2. *Let X be an equidimensional reduced scheme of finite type over a perfect field k. Then every closed point of X is locally a point of maximum multiplicity on X.*

Proof. Let $\xi \in X$ be a closed point and consider B and $S \subset B$ as in the proposition. By Zariski's formula for finite morphisms (Theorem A.0.2), the multiplicity on $\mathrm{Spec}(B)$ is bounded by the generic rank of $S \subset B$ which, by assumption, is equal to $\mathrm{mult}_X(\xi)$. Since $X \leftarrow \mathrm{Spec}(B)$ is étale, it is open and preserves the multiplicity (see Theorem C.0.8 and Lemma C.0.9). Hence there exists a Zariski open neighborhood of X at ξ where the multiplicity is bounded by $\mathrm{mult}_X(\xi)$. □

Theorem 4.1.3 (cf. [34, §7])**.** *Let X be an equidimensional reduced scheme of finite type over a perfect field k. For any closed point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}_X$, it is possible to construct an étale neighborhood of X at ξ, say $X \leftarrow X'$, an embedding of X' in a regular ambient space, say $X' \hookrightarrow V$, and an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ so that $\mathcal{G}$ represents the stratum of maximum multiplicity of X' via the embedding in V.*

Remark 4.1.4. In the situation of the theorem, we usually write $X \hookrightarrow V$, it being understood that the immersion is defined in an étale neighborhood of X.

Remark 4.1.5. Let us give a sketch of the construction of the theorem. Given a closed point $\xi \in X$, construct an étale neighborhood $X \leftarrow \mathrm{Spec}(B)$ and a regular subalgebra $S \subset B$ as in Proposition 4.1.1, and set $X' = \mathrm{Spec}(B)$. Recall that $S \subset B$ is a finite extension whose generic rank coincides with $\mathrm{mult}_X(\xi)$. Since B is finite over S, one can find elements $\theta_1, \dots, \theta_r \in B$ so that $B = S[\theta_1, \dots, \theta_r]$. This presentation of B induces a surjective map

$$S[T_1, \dots, T_r] \longrightarrow B = S[\theta_1, \dots, \theta_r] \tag{4.1}$$

given by $T_i \mapsto \theta_i$, where $T_1, \dots, T_r$ represent variables. We will take

$$V = \mathrm{Spec}(S[T_1, \dots, T_r]),$$

and the immersion $X' \hookrightarrow V$ will be that induced by (4.1).

Next we proceed with the construction of $\mathcal{G}$. Let $K = \mathrm{Frac}(S)$ denote the field of fractions of S, and consider the minimal polynomial of each θ_i over K, say $f_i(T_i) \in K[T_i]$. Let N_i denote the degree of $f_i(T_i)$. As S is normal, it can be shown that the coefficients of $f_i(T_i)$ lie in S (see [34, Lemma 5.2]). That is, $f_i(T_i) \in S[T_i]$. Then we claim that the Rees algebra

$$\mathcal{G} = S\left[f_1(T_1)W^{N_1}, \dots, f_r(T_r)W^{N_r}\right]$$

represents the stratum of maximum multiplicity of X' via the closed immersion in $X' \hookrightarrow V$ (see [34, §7]).

Corollary 4.1.6. *Let X be an equidimensional reduced scheme of finite type over a perfect field k. For any closed point $\xi \in X$ there exists an open neighborhood of X at ξ, say $U \subset X$, so that the set*

$$\{\zeta \in U \mid \mathrm{mult}_X(\zeta) = \mathrm{mult}_X(\xi)\}$$

is closed in U.

Proof. Let $X \leftarrow X'$, $X' \hookrightarrow V$, and $\mathcal{G}$ be as in Theorem 4.1.3. We claim that the image of X' in X, which we shall denote by U, is an open neighborhood of ξ with the required properties.

Let us prove the claim. Set $n = \mathrm{mult}_X(\xi)$, let $F_n(X)$ and $F_n(X')$ denote the set of points of multiplicity exactly equal to n of X and X' respectively. Note that, with this notation,

$$F_n(X) \cap U = \{\zeta \in U \mid \mathrm{mult}_X(\zeta) = \mathrm{mult}_X(\xi)\}.$$

By the conditions imposed on $\mathcal{G}$, we have that $F_n(X') = \mathrm{Sing}_V(\mathcal{G})$. Hence $F_n(X')$ is a closed subset X' by Proposition 3.2.4. Let φ denote the map from X' to X, that is, $\varphi : X' \to X$. Since étale morphisms are open (Theorem C.0.8) and preserve multiplicity (Lemma C.0.9), we have that the sets $U = \varphi(X')$ and

$$U \setminus F_n(X) = \varphi\left(X' \setminus F_n(X')\right)$$

are open. As $U \cap F_n(X)$ is the complement of $U \setminus F_n(X)$ in U, it follows that $U \cap F_n(X)$ is closed in U. □

4.2 Upper semi-continuity

Proposition 4.2.1 ([13, Lemma 1.34])**.** *Let X be a noetherian topological space, and let $(\Lambda, \leq)$ be a totally ordered set. Consider a function $f : X \to \Lambda$, and suppose that the following conditions hold:*

1) For all $\zeta, \eta \in X$, $\zeta \in \overline{\{\eta\}}$ implies $f(\zeta) \geq f(\eta)$.

2) For each $\eta \in X$, there exists a non-empty open set $U \subset \overline{\{\eta\}}$ such that $f(\zeta) = f(\eta)$ for all $\zeta \in U$.

Then f is upper semi-continuous.

Lemma 4.2.2. *Let X be an irreducible scheme of finite type over a field k. If a closed subset $F \subset X$ contains all the closed points of X, then $F = X$.*

Proof. Let η denote the generic point of X. It suffices to check that $\eta \in F$. Assume without loss of generality that $X = \mathrm{Spec}(B)$, and $F = \mathbb{V}(I)$ for a suitable ideal $I \subset B$. Let $\mathfrak{q}$ denote the minimal prime ideal of B, which corresponds to the generic point η. Note that, by hypothesis, both I and $\mathfrak{q}$ are contained in all the maximal ideals of B. Therefore, according to the zero-point theorem of Hilbert [26, Theorem 25, p. 93], we have $\sqrt{I} = \mathfrak{q}$. Hence $\eta \in F$. □

Proposition 4.2.3. *Let X be a scheme of finite type over a field k, and $f : X \to \Lambda$ a function into a well-ordered set $(\Lambda, \leq)$. Let $\mathfrak{X}$ denote the set of closed points of X, endowed with the induced topology. Then f is upper semi-continuous if and only if the following conditions hold:*

i) $f|_{\mathfrak{X}} : \mathfrak{X} \to \Lambda$ *is upper semi-continuous.*

ii) For each $\xi \in \mathfrak{X}$, there exists a Zariski open neighborhood of ξ in X, say $U \subset X$, such that the set $\{\zeta \in U \mid f(\zeta) = f(\xi)\}$ is closed in U.

Proof. If f is upper semi-continuous, it clearly satisfies i) and ii). In order to prove the converse, assume that i) and ii) hold. We will show that, in this case, f satisfies conditions 1) and 2) of Proposition 4.2.1.

For property 1), fix two points $\eta, \zeta \in X$ so that $\zeta \in \overline{\{\eta\}}$. By Lemma 4.2.4 below, it is possible to find a closed point $\xi \in \overline{\{\zeta\}}$ such that $f(\zeta) = f(\xi)$. Since $\xi \in \overline{\{\zeta\}} \subset \overline{\{\eta\}}$, Lemma 4.2.4 implies $f(\eta) \leq f(\xi) = f(\zeta)$. Therefore 1) holds.

To check 2), fix $\eta \in X$. According to Lemma 4.2.4, one can find a non-empty open subset of $\overline{\{\eta\}}$, say $U \subset \overline{\{\eta\}}$, so that $f(\xi) = f(\eta)$ for every closed point $\xi \in U$. Next, take an arbitrary point $\zeta \in U$. Observe that $\overline{\{\zeta\}} \cap U$ should contain, at least, a closed point, say $\xi \in \overline{\{\zeta\}} \cap U$. Thus, by property i) and the choice of U,

$$f(\eta) \leq f(\zeta) \leq f(\xi) = f(\eta),$$

which implies that f is constantly equal to $f(\eta)$ on U. □

Lemma 4.2.4. *Let $f : X \to \Lambda$ is a function under the hypotheses of Proposition 4.2.3, and assume that it satisfies conditions i) and ii). Fix an arbitrary point $\eta \in X$. Then $f(\xi) \geq f(\eta)$ for every closed point $\xi \in \overline{\{\eta\}}$. Moreover there exists a non-empty open subset $U \subset \overline{\{\eta\}}$ such that $f(\xi) = f(\eta)$ for every closed point $\xi \in U$.*

Proof. Suppose that $X = \overline{\{\eta\}}$, and let $\lambda = \min\{f(\xi) \mid \xi \in \mathfrak{X}\}$. In virtue of i), take a non-empty open subset of X, say $U \subset X$, such that $f(\xi) = \lambda$ for every closed point $\xi \in U$. According to ii), we may assume that U has been chosen so that the subset $F = \{\zeta \in U \mid f(\zeta) = \lambda\}$ is closed in U. Note that U can be regarded as an irreducible scheme of finite type over k whose generic point is η. Thus Lemma 4.2.2 applied to $F \subset U$ yields $\eta \in F$. Hence $f(\eta) = \lambda$, and the result follows. □

Remark 4.2.5. The requirement on the well order of $(\Lambda, \leq)$ on Proposition 4.2.3 is essential. Consider the following example. Let

$$X = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec}(\mathbb{Q}[T]).$$

Denote by η the generic point of X, i.e., that corresponding to the ideal $\langle 0 \rangle \subset \mathbb{Q}[T]$, and by $\mathfrak{X} = X \setminus \{\eta\}$ the set of closed points of X. Note that $\mathfrak{X}$ is countable.

Thus there exists a bijection $n : \mathfrak{X} \to \mathbb{N}$. Then consider the function $f : X \to (\mathbb{Q}, \leq)$ defined by

$$f(\xi) = \begin{cases} 1 + \frac{1}{n(\xi)+1} & \text{if } \xi \in \mathfrak{X}, \\ 0 & \text{if } \xi = \eta. \end{cases}$$

It is clear that f satisfies conditions i) and ii) of Proposition 4.2.3. However it is not upper semi-continuous because the set

$$\{\xi \in X \mid f(\xi) \geq 1\} = \mathfrak{X}$$

is not closed in X.

Theorem 4.2.6 (Dade [14]). *Let X be a strictly equidimensional scheme of finite type over a perfect field k. The multiplicity function on X, $\operatorname{mult}_X : X \to \mathbb{N}$, is upper semi-continuous.*

This theorem was first proved by Dade in [14] under more general conditions than the previous ones. Alternatively, it can be regarded as a consequence of the upper semi-continuity of the Hilbert-Samuel function, proved by Bennet in [8] (a simplified version can be found in [13, Theorem 1.33, p. 25]). In our proof, all arguments remain within the class of algebras of finite type over k.

Proof. Let us check that $\operatorname{mult}_X : X \to \mathbb{N}$ verifies the hypotheses of Proposition 4.2.3. First, $(\mathbb{N}, \leq)$ is well-ordered. Moreover mult_X satisfies condition i) by Corollary 4.1.2. Finally condition ii) follows from Corollary 4.1.6. Thus Proposition 4.2.3 yields the theorem. $\square$

So far, we have discussed the upper semi-continuity of the multiplicity over equidimensional schemes of finite type over a field. Next we will show that, given an equidimensional noetherian scheme X, the study of the multiplicity on X can be reduced to that of its irreducible components.

Fix a point $\xi \in X$, and consider the local ring $R = \mathcal{O}_{X,\xi}$. Let $\mathfrak{m}$ be the maximal ideal of R, and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ its minimal primes. Observe that $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are in correspondence with the irreducible components of X containing ξ. Let $e_{\mathfrak{m}}(R)$ denote the multiplicity of R, and $l(R_{\mathfrak{q}_i})$ the length of $R_{\mathfrak{q}_i}$. From the additivity of the multiplicity (Lemma A.0.1), it follows that

$$e_{\mathfrak{m}}(R) = \sum_{i=1}^{s} l(R_{\mathfrak{q}_i}) e_{\mathfrak{m}}(R/\mathfrak{q}_i). \tag{4.2}$$

Denote by $Y_1, \dots, Y_r$ the irreducible components of X endowed with their reduced structure, and by η_i the generic point of Y_i for $i = 1, \dots, r$. According to (4.2),

$$\operatorname{mult}_X(\xi) = \sum_{\xi \in Y_i} l(\mathcal{O}_{X,\eta_i}) \operatorname{mult}_{Y_i}(\xi),$$

and since each $\mathcal{O}_{X,\eta_i}$ is artinian, $l(\mathcal{O}_{X,\eta_i}) = \operatorname{mult}_X(\eta_i)$.

Note also that, for $i = 1, \dots, r$, the function $\operatorname{mult}_{Y_i} : Y_i \to \mathbb{N}$ can be trivially extended to an upper semi-continuous one defined on X by setting $\operatorname{mult}_{Y_i}(\zeta) = 0$ for $\zeta \in X \setminus Y_i$. Let us abuse of notation and call this extension $\operatorname{mult}_{Y_i} : X \to \mathbb{N}$.

Proposition 4.2.7. *Consider a noetherian equidimensional scheme X, and denote by $Y_1, \dots, Y_r$ its irreducible components endowed with their reduced structure. For $i = 1, \dots, r$, let η_i be the generic point of Y_i. Then*

$$\operatorname{mult}_X = \sum_{i=1}^{r} \operatorname{mult}_X(\eta_i) \cdot \operatorname{mult}_{Y_i}.$$

Corollary 4.2.8. *Let X be an equidimensional scheme of finite type over a perfect field k. The multiplicity function over X is upper semi-continuous.*

Proof. Let us use the notation of Proposition 4.2.7, and set $\alpha_i = \operatorname{mult}_X(\eta_i)$ for $i = 1, \dots, r$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} \{\xi \in X \mid \operatorname{mult}_X(\xi) \geq n\} &= \left\{ \xi \in X \mid \sum_{i=1}^{r} \alpha_i \operatorname{mult}_{Y_i}(\xi) \geq n \right\} \\ &= \bigcup_{\sum \alpha_i n_i \geq n} \left(\bigcap_{i=1}^{r} \{\xi \in X \mid \operatorname{mult}_{Y_i}(\xi) \geq n_i\} \right). \end{aligned}$$

By Theorem 4.2.6, $\operatorname{mult}_{Y_i} : X \to \mathbb{N}$ is upper semi-continuous for $i = 1, \dots, r$. Thus, the last expression is a finite union of closed subsets of X, and hence it is closed. □

4.3 Simplification of non-reduced schemes

Along this section, X will be an equidimensional excellent scheme, and X_{red} will denote the underlying reduced scheme. Proposition 4.2.7 applied to X_{red} says that

$$\operatorname{mult}_{X_{\text{red}}} = \sum_{i=1}^{r} \operatorname{mult}_{Y_i},$$

where $Y_1, \dots, Y_r$ denote the irreducible components of X endowed with their reduced structure. Thus the following property is straight forward.

Proposition 4.3.1. *Under the previous hypotheses, X_{red} is regular if and only if the irreducible components of X are disjoint, and each of them has constant multiplicity.*

The next criterion shows that the stratification of X defined by the multiplicity does not depend on its infinitesimal structure.

Lemma 4.3.2. *Let $f_1, \dots, f_r : X \to \mathbb{R}$ be upper semi-continuous functions. For any $\alpha_1, \dots, \alpha_r \in \mathbb{R}_{>0}$, the level sets of the functions $f_1 + \dots + f_r$ and $\alpha_1 f_1 + \dots + \alpha_r f_r$ are locally the same.*

Proof. Fix a point $\xi \in X$. Let $n = (f_1 + \cdots + f_r)(\xi)$, and $n' = (\alpha_1 f_1 + \cdots + \alpha_r f_r)(\xi)$. Since $f_1, \ldots, f_r$ are upper semi-continuous, there exists a neighborhood of ξ, say $U \subseteq X$, so that $f_i(\zeta) \leq f_i(\xi)$ for all $\zeta \in U$, and $i = 1, \ldots, r$. Thus,

$$\begin{aligned} &\big\{\zeta \in U \mid (f_1 + \cdots + f_r)(\zeta) = n\big\} \\ &\qquad = \big\{\zeta \in U \mid f_1(\zeta) = f_1(\xi), \ldots, f_r(\zeta) = f_r(\xi)\big\} \\ &\qquad = \big\{\zeta \in U \mid (\alpha_1 f_1 + \cdots + \alpha_r f_r)(\zeta) = n'\big\}. \end{aligned}$$

□

The process of simplification

Another aspect to consider is the lowering of the multiplicity by means of blow-ups. Namely, when the multiplicity is not constant along the irreducible components of X, the objective is to find a sequence of blow-ups along closed regular equimultiple centers, say

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} X_n \tag{4.3}$$

so that $\max \mathrm{mult}_{X_n} < \max \mathrm{mult}_X$. If X is reduced and X_n is regular, this sequence is a resolution of singularities of X. In the non-reduced case, X_n will be non-reduced, and therefore it cannot be regular.

Definition 4.3.3. We will say that a sequence like (4.3) is a *simplification of* X if the multiplicity is constant along the irreducible components of X_n.

Remark 4.3.4. Note that, if (4.3) is a simplification of X, then the scheme X_n cannot be "improved" any more, in the sense that we already obtain a regular scheme by taking reduction (see Proposition 4.3.1).

Proposition 4.3.5. *Let X be an equidimensional excellent scheme, and let Y be a closed regular subscheme of X. Then Y defines an equimultiple center in X if and only if it defines an equimultiple center in X_{red}.*

Proof. Denote by η the generic point of Y. First, suppose that Y defines an equimultiple permissible center for X. By assumption, for each point $\xi \in Y$ we have that $\mathrm{mult}_X(\xi) = \mathrm{mult}_X(\eta)$. Moreover, since every neighborhood of ξ contains η, Lemma 4.3.2 ensures that $\mathrm{mult}_{X_{\mathrm{red}}}(\xi) = \mathrm{mult}_{X_{\mathrm{red}}}(\eta)$. Thus $Y \subset X_{\mathrm{red}}$ defines an equimultiple permissible center for X_{red}. The converse follows similarly. □

Recall that X and X_{red} are naturally linked by a closed immersion of schemes, say $X_{\mathrm{red}} \hookrightarrow X$. In what follows, we present a series of technical results intended to prove that this relation is preserved by sequences of blow-ups along closed regular centers.

Consider a noetherian ring B, and an ideal $J = \langle x_1, \ldots, x_r \rangle \subset B$. The blow-up of B at J, which we will denote by $\mathrm{Bl}_J(B)$, can be covered by r affine charts associated to the elements $x_1, \ldots, x_r$. The chart corresponding to x_i, also called the x_i-chart of $\mathrm{Bl}_J(B)$, can be expressed as $\mathrm{Spec}(B_i)$, with $B_i = [B[JW]_{x_iW}]_0$. It is not hard to see that the points of the x_i-chart of $\mathrm{Bl}_J(B)$ are in one-to-one correspondence with the homogeneous primes of $B[JW]$ which do not contain x_iW.

In addition, consider another ideal $I \subset B$ which contains and is integral over J. Assume that $I = \langle x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}\rangle$, with $x_{r+1}, \ldots, x_{r+s}$ integral over J, and let $\mathrm{Spec}(B'_i)$ denote x_i-chart of $\mathrm{Bl}_I(B)$ for $i = 1, \ldots, r+s$.

Lemma 4.3.6. *Under the previous hypotheses, $\mathrm{Bl}_I(B)$ can be covered by the r affine charts corresponding to the elements $x_1, \ldots, x_r$.*

Proof. Fix $j \in \{r+1, \ldots, r+s\}$. Since x_j is integral over $J = \langle x_1, \ldots, x_r\rangle$, the element x_jW must be integral over $JW \subset B[IW]$. Thus, if $\mathfrak{p} \subset B[IW]$ is a homogeneous prime not containing x_jW, then $x_iW \notin \mathfrak{p}$ for some $i \in \{1, \ldots, r\}$. In other words, every point of $\mathrm{Spec}(B'_j)$ must be contained in one of the charts $\mathrm{Spec}(B'_1), \ldots, \mathrm{Spec}(B'_r)$. $\square$

Proposition 4.3.7. *Let B be a noetherian ring, and $I \subset B$ an ideal containing the nilradical of B. Denote by I' the image of I in B_{red}. Then there is a natural commutative diagram, say*

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longleftarrow & \mathrm{Bl}_I(B) \\ \uparrow & & \uparrow \\ \mathrm{Spec}(B_{\mathrm{red}}) & \longleftarrow & \mathrm{Bl}_{I'}(B_{\mathrm{red}}), \end{array}$$

which induces an isomorphism $\mathrm{Bl}_{I'}(B_{\mathrm{red}}) \simeq (\mathrm{Bl}_I(B))_{\mathrm{red}}$.

Proof. Fix a set of generators of $I' \subset B_{\mathrm{red}}$, say $x'_1, \ldots, x'_r$, and a collection of preimages of these elements, say $x_1, \ldots, x_r \in I$. Observe that $I = \langle x_1, \ldots, x_r\rangle + N$, where N denotes the nilradical of B. In particular, note that I must be integral over $\langle x_1, \ldots, x_r\rangle$. Thus, by the previous lemma, $\mathrm{Bl}_I(B)$ can be covered by the affine charts corresponding to $x_1, \ldots, x_r$.

Let $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(B'_i)$ be the affine charts of $\mathrm{Bl}_I(B)$ and $\mathrm{Bl}_{I'}(B_{\mathrm{red}})$ associated to x_i and x'_i respectively. By definition, there is a natural homomorphism

$$B_i = \big[B[IW]_{x_iW}\big]_0 \longrightarrow B'_i = \big[B_{\mathrm{red}}[I'W]_{x'_iW}\big]_0.$$

One can readily check that this morphism is surjective, and that its kernel coincides with the nilradical of B_i. Therefore, $B'_i \simeq (B_i)_{\mathrm{red}}$. Finally, since all these homomorphisms are compatible, they induce a morphism $\mathrm{Bl}_I(B) \leftarrow \mathrm{Bl}_{I'}(B_{\mathrm{red}})$ which makes the above diagram commutative. $\square$

The next result shows the equivalence between the process of simplification of X and that of resolution of X_{red}.

Proposition 4.3.8. *Let X be an equidimensional excellent scheme and let X_{red} denote the underlying reduced scheme. Every sequence of blow-ups along closed regular equimultiple centers on X, say*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_m, \tag{4.4}$$

induces a sequence of blow-ups along closed regular equimultiple centers on X_{red}, say

$$X_{\mathrm{red}} = X'_0 \longleftarrow X'_1 \longleftarrow \cdots \longleftarrow X'_m, \tag{4.5}$$

and vice-versa, and these sequences are linked by a natural commutative diagram

$$\begin{tikzcd} X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_m \arrow[l] \\ X'_0 \arrow[u, "\rho_0"] & X'_1 \arrow[l] \arrow[u, "\rho_1"] & \cdots \arrow[l] & X'_m, \arrow[l] \arrow[u, "\rho_m"] \end{tikzcd}$$

where $\rho_1, \ldots, \rho_m$ represent a closed immersions, and $X'_i \simeq (X_i)_{\mathrm{red}}$ via ρ_i for $i = 1, \ldots, m$. In particular, (4.5) *is a resolution of singularities of X_{red} if and only if the composition* (4.4) *is a simplification of X in the sense of Definition 4.3.3.*

Proof. By Proposition 4.3.5, any closed regular equimultiple center in X defines a closed regular equimultiple center in X_{red}, and vice-versa. In addition, given such a center Y, one has that

$$\mathrm{Bl}_Y(X_{\mathrm{red}}) \simeq (\mathrm{Bl}_Y(X))_{\mathrm{red}}$$

by Proposition 4.3.7. Thus the result follows by induction on the number of blow-ups.

The last claim concerning the resolution of X_{red} and the simplification of X follows from Proposition 4.3.1. □

4.4 Canonical Rees algebra attached to the multiplicity

Throughout this section, X will denote a variety over a perfect field k. Next we will define a Rees algebra over X which, in contrast to those algebras attached to a local presentation of mult_X, turns out to be intrinsic to the variety.

Definition 4.4.1. For a point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}_X$, consider a local presentation of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ in a neighborhood of ξ given by a closed immersion $X \hookrightarrow V$ and a Rees algebra $\mathcal{G}$. We define the $\mathcal{O}_X$-Rees algebra $\mathcal{G}_X$ as

$$\mathcal{G}_X = \mathscr{D}\!\mathit{iff}_k(\mathcal{G})|_X.$$

Remark 4.4.2. The immersion $X \hookrightarrow V$ is defined on an étale neighborhood of ξ, not on X itself. Hence $\mathcal{G}_X$ is just defined locally in étale topology.

At first sight, it may seem that the definition of $\mathcal{G}_X$ depends on the immersion $X \hookrightarrow V$ and the algebra $\mathcal{G}$. However, we will show that it is intrinsic to X. More precisely, we will prove that, regardless of the local presentation chosen, $\mathcal{G}_X$ is well defined up to integral closure.

Lemma 4.4.3 (cf. [10, Lemma 29.1])**.** *Let X be a variety over a perfect field k. Suppose that, for a local presentation of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ given by an embedding $X \hookrightarrow V$ and a Rees algebra $\mathcal{G}$ over V, we have a commutative diagram as follows,*

$$\begin{tikzcd} & V' \arrow[d, "\beta"] \\ X \arrow[ur, hook, "\rho"] \arrow[r, hook] & V, \end{tikzcd}$$

where ρ represents a closed immersion, and β is a smooth morphism. Then there exists an $\mathcal{O}_{V'}$-Rees algebra $\mathcal{G}'$ which defines a local presentation of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ via ρ, and such that $\mathcal{G}'|_X = \mathcal{G}|_X$. Moreover, if $\mathcal{G}$ is differentially saturated with respect to k, the same holds for $\mathcal{G}'$.

Theorem 4.4.4. *For a variety X defined over a perfect field k, the Rees algebra $\mathcal{G}_X$ is well defined up to integral closure, i.e., it does not depend on the choice of $\mathcal{G}$ or the immersion $X \hookrightarrow V$.*

Proof. Consider two local presentations of $\underline{\mathrm{Max}}\ \mathrm{mult}_X$ given by immersions $X \hookrightarrow V_1$ and $X \hookrightarrow V_2$, and Rees algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Assume without loss of generality that $\mathcal{G}_1 = \mathscr{D}\!\mathit{iff}_k(\mathcal{G}_1)$, and $\mathcal{G}_2 = \mathscr{D}\!\mathit{iff}_k(\mathcal{G}_2)$. Then define $V' = V_1 \times_{\mathrm{Spec}(k)} V_2$. Observe that V' is a smooth variety endowed with two natural smooth morphisms: $V' \to V_1$, and $V' \to V_2$. Moreover, by the universal property of the fibered product, there is a closed immersion $X \hookrightarrow V'$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] \arrow[drrr] & & V' \arrow[dl] \arrow[dr] & \\ & V_1 & & V_2. \end{tikzcd}$$

Note that, for $i = 1, 2$, the varieties V' and V_i, together with the algebra $\mathcal{G}_i$, are under the hypotheses of Lemma 4.4.3. Hence there exists a differentially saturated $\mathcal{O}_{V'}$-Rees algebra, say $\mathcal{G}'_i$, such that

$$\mathcal{G}_i|_X = (\mathcal{G}'_i)|_X.$$

Since both $(V', \mathcal{G}'_1)$ and $(V', \mathcal{G}'_2)$ represent $\underline{\mathrm{Max}}\ \mathrm{mult}_X$, Corollary 3.6.13 yields $\overline{\mathcal{G}'_1} = \overline{\mathcal{G}'_2}$, and therefore $\overline{(\mathcal{G}'_1|_X)} = \overline{(\mathcal{G}'_2|_X)}$. $\square$

4.5 Finite extensions

Consider a finite extension of domains of finite type over a perfect field k, say $B \subset B'$. Denote by K and K' the fields of fractions of B and B' respectively. Suppose that $B \subset B'$ has generic rank n, i.e, that $[K' : K] = n$. Consider also the morphism $\varphi : X' \to X$ induced by the inclusion $B \subset B'$.

Proposition 4.5.1. *Under the previous hypotheses we have that*

$$\max \operatorname{mult}_{X'} \leq n \cdot \max \operatorname{mult}_X .$$

In addition, if the equality holds in the latter expression, then

$$\varphi(\underline{\operatorname{Max}} \operatorname{mult}_{X'}) \subset \underline{\operatorname{Max}} \operatorname{mult}_X .$$

Proof. Fix a prime ideal $\mathfrak{q} \subset B'$, and set $\mathfrak{p} = \mathfrak{q} \cap B$. Denote by $\mathfrak{q}_1 = \mathfrak{q}, \mathfrak{q}_2, \ldots, \mathfrak{q}_l$ the preimages of $\mathfrak{p}$ by φ (by abuse of notation, we shall identify the prime ideals of B and B' with the points of X and X' respectively). Zariski's formula for finite morphisms (Theorem A.0.2) says that

$$e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})[K' : K] = \sum_{i=1}^{l} e_{B'_{\mathfrak{q}_i}}(\mathfrak{p}B'_{\mathfrak{q}_i})[k(\mathfrak{q}_i) : k(\mathfrak{p})],$$

where $k(\mathfrak{p})$, and $k(\mathfrak{q}_1), \ldots, k(\mathfrak{q}_l)$ denote the residue fields of $\mathfrak{p}$, and $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ respectively. Recall that $\mathfrak{p}B'_{\mathfrak{q}} \subset \mathfrak{q}B'_{\mathfrak{q}}$ implies $e_{B'_{\mathfrak{q}}}(\mathfrak{p}B'_{\mathfrak{q}}) \geq e_{B'_{\mathfrak{q}}}(\mathfrak{q}B'_{\mathfrak{q}})$. In this way, as all the terms in the previous equation are positive, we deduce that

$$e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})[K' : K] \geq e_{B'_{\mathfrak{q}}}(\mathfrak{p}B'_{\mathfrak{q}}) \geq e_{B'_{\mathfrak{q}}}(\mathfrak{q}B'_{\mathfrak{q}}).$$

Thus, substituting $e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}}) = \operatorname{mult}_X(\mathfrak{p})$ and $e_{B'_{\mathfrak{q}}}(\mathfrak{q}B'_{\mathfrak{q}}) = \operatorname{mult}'_X(\mathfrak{q})$, we obtain

$$n \cdot \operatorname{mult}_X(\mathfrak{p}) \geq \operatorname{mult}_{X'}(\mathfrak{q}).$$

In particular, for the case in which $\mathfrak{q} \in \underline{\operatorname{Max}} \operatorname{mult}_{X'}$ this inequality yields

$$\max \operatorname{mult}_{X'} = \operatorname{mult}_{X'}(\mathfrak{q}) \leq n \cdot \operatorname{mult}_X(\mathfrak{p}) \leq n \cdot \max \operatorname{mult}_X .$$

Finally, if the equality holds in the latter expression, we must have $\operatorname{mult}_X(\mathfrak{p}) = \max \operatorname{mult}_X$. Hence $\varphi(\mathfrak{q}) = \mathfrak{p} \in \underline{\operatorname{Max}} \operatorname{mult}_X$. □

Proposition 4.5.2. *In addition to the previous hypotheses, let us assume that* $\max \operatorname{mult}_{X'} = n \cdot \max \operatorname{mult}_X$. *Then φ maps the closed set* $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$ *homeomorphically onto its image in* $\underline{\operatorname{Max}} \operatorname{mult}_X$. *Moreover, if* $Y \subset \underline{\operatorname{Max}} \operatorname{mult}_{X'}$ *is a regular closed subset, so is* $\varphi(Y) \subset \underline{\operatorname{Max}} \operatorname{mult}_X$, *and vice-versa.*

Proof. After replacing X and X' by suitable étale neighborhoods, we may assume that there exists a regular domain, say $S \subset B$, such that $S \subset B$ is a finite extension of generic rank $m = \max \operatorname{mult}_X$ (i.e., with $[K : K_0] = m$, where K_0 denotes the field of fractions of S). Then we have a commutative diagram

$$\begin{tikzcd} X' = \operatorname{Spec}(B') \arrow[d, "\varphi"] \arrow[dr, "\psi'"] & \\ X = \operatorname{Spec}(B) \arrow[r, "\psi"] & \operatorname{Spec}(S). \end{tikzcd}$$

Note that $S \subset B'$ is also a finite extension of generic rank $mn = \max \operatorname{mult}_{X'}$.

Under the previous hypotheses, it can be proved that ψ maps $\underline{\operatorname{Max}} \operatorname{mult}_X$ homeomorphically to its image in $\operatorname{Spec}(S)$ (see Corollary 5.9, p. 346 [34]). In addition, a closed subset $Y \subset \underline{\operatorname{Max}} \operatorname{mult}_X$ is regular if and only if so is $\psi(Y) \subset \psi(\underline{\operatorname{Max}} \operatorname{mult}_X)$ (Proposition 6.3, p. 349 [34]). And similar properties hold for ψ' and $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$. Hence the proposition follows from the commutativity of the diagram above. $\square$

Proposition 4.5.3. *Let $\varphi : X' \to X$ be a finite and dominant morphism of varieties over a perfect field k. Suppose that φ has generic rank n, and that $\max \operatorname{mult}_{X'} = n \cdot \max \operatorname{mult}_X$. Then the pull-back of the Rees algebra $\mathcal{G}_X$ to X' is a subalgebra of $\mathcal{G}_{X'}$.*

Proof. We shall begin by constructing local presentations of $\underline{\operatorname{Max}} \operatorname{mult}_X$ and $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$ as in Remark 4.1.5. Suppose that $X = \operatorname{Spec}(B)$, $X' = \operatorname{Spec}(B')$, and replace them by suitable étale neighborhoods if necessary. Then consider a regular domain $S \subset B$ as in the proof of the previous proposition. Assume that $B = S[\theta_1, \dots, \theta_r]$, and $B' = S[\theta_1, \dots, \theta_r, \theta_{r+1}, \dots, \theta_{r+l}]$ with θ_i integral over S for $i = 1, \dots, r+l$. Let $f_i(T_i) \in S[T_i]$ be the minimal polynomial of θ_i over S, and denote by n_i the degree of $f_i(T_i)$. Then recall that the closed immersion

$$X \hookrightarrow V = \operatorname{Spec}(S[T_1, \dots, T_r]),$$

together with the Rees algebra

$$\mathcal{G} = \mathcal{O}_V \left[f_1(T_1) W^{N_1}, \dots, f_r(T_r) W^{N_r} \right],$$

is a local presentation of $\underline{\operatorname{Max}} \operatorname{mult}_X$. Similarly, the immersion

$$X' \hookrightarrow V' = \operatorname{Spec}(S[T_1, \dots, T_r, T_{r+1}, \dots, T_{r+l}]),$$

together with

$$\mathcal{G}' = \mathcal{O}_{V'} \left[f_1(T_1) W^{N_1}, \dots, f_{r+l}(T_{r+l}) W^{N_{r+l}} \right],$$

is a local presentation of $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$.

Note that, by construction, there is a smooth morphism $\beta : V' \to V$. Since $f_1(T_1) W^{N_1}, \dots, f_r(T_r) W^{N_r} \in \mathcal{G}'$, it follows that $\beta^*(\mathcal{G}) \subset \mathcal{G}_{V'}$. Moreover, since β is smooth, every differential operator on V can be lifted to a differential operator in V'. Hence $\beta^*(\mathscr{D}\!\mathit{iff}_k(\mathcal{G})) \subset \mathscr{D}\!\mathit{iff}_k(\mathcal{G}')$. Finally, from the commutativity of the diagram

$$\begin{tikzcd} V' \arrow[d, "\beta"] & X' \arrow[l, hook'] \arrow[d, "\varphi"] \\ V & X, \arrow[l, hook'] \end{tikzcd}$$

we deduce that $\varphi^*(\mathcal{G}_X) \subseteq \mathcal{G}_{X'}$. $\square$

Example 4.5.4. Let us use the same notation as above. Suppose that $X = \operatorname{Spec}(B)$ is the curve defined by

$$B = k[\bar{u}, \bar{v}] = k[u, v]/\langle v^2 - u^5 \rangle,$$

where u, v represent variables, and let $\bar{u}, \bar{v}$ denote their residue classes modulo $v^2 - u^5$. Let $\xi \in X$ be the point defined by the ideal $\langle \bar{u}, \bar{v} \rangle$. Observe that ξ is the only singular point of X, and $\operatorname{mult}_X(\xi) = 2$. In addition, consider the ring

$$B' = k\left[\bar{u}, \frac{\bar{v}}{\bar{u}}\right] = k\left[u, \frac{v}{u}\right] \Big/ \left\langle \left(\frac{v}{u}\right)^2 - u^3 \right\rangle,$$

and the curve $X' = \operatorname{Spec}(B')$, which corresponds to the $\bar{u}$-chart of the blow-up of X along $\langle \bar{u}, \bar{v} \rangle$. Note that $\frac{\bar{v}}{\bar{u}}$ is integral over B, since it satisfies the equation $(\frac{\bar{v}}{\bar{u}})^2 - \bar{u}^3 = 0$. Hence, $B \subset B'$ is a finite extension of generic rank 1. Moreover, one readily checks that

$$\max \operatorname{mult}_{X'} = 1 \cdot \max \operatorname{mult}_X = 2.$$

Therefore, X and X' are under the hypotheses of Proposition 4.5.3.

Next, let us construct local presentations of $\underline{\operatorname{Max}} \operatorname{mult}_X$ and $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$ following the procedure of Remark 4.1.5. Taking $S = k[\bar{u}] \subset B$, we have $B = S[\theta_1]$, and $B' = S[\theta_2]$, where $\theta_1 = \bar{v}$, and $\theta_2 = \frac{\bar{v}}{\bar{u}}$. The minimal polynomials of θ_1 and θ_2 over S are $f_1(T_1) = T_1^2 - \bar{u}^5$, and $f_2(T_2) = T_2^2 - \bar{u}^3$ respectively. Hence a local presentation of $\underline{\operatorname{Max}} \operatorname{mult}_X$ is given by the variety $V = \operatorname{Spec}(S[T_1])$, together with the algebra

$$\mathcal{G} = \mathcal{O}_V[(T_1^2 - \bar{u}^5)W^2].$$

Similarly, the variety $V' = \operatorname{Spec}(S[T_2])$, together with

$$\mathcal{G}' = \mathcal{O}_{V'}\left[(T_2^2 - \bar{u}^3)W^2\right],$$

defines a local presentation of $\underline{\operatorname{Max}} \operatorname{mult}_{X'}$.

To obtain $\mathcal{G}_X$ and $\mathcal{G}_{X'}$ we shall first compute the differential saturation of $\mathcal{G}$ and $\mathcal{G}'$. Assume that $\operatorname{char}(k) \neq 2, 3, 5$. In virtue of Remark 3.6.11,

$$\begin{aligned} \mathscr{D}\!\mathit{iff}_k(\mathcal{G}) &= \mathcal{O}_V\left[T_1W, \bar{u}^4W, (T_1^2 - \bar{u}^5)W, (T_1^2 - \bar{u}^5)W^2\right] \\ &= \mathcal{O}_V\left[T_1W, \bar{u}^4W, (T_1^2 - \bar{u}^5)W^2\right], \end{aligned}$$

and

$$\mathscr{D}\!\mathit{iff}_k(\mathcal{G}') = \cdots = \mathcal{O}_{V'}\left[T_2 W, \bar{u}^2 W, (T_2^2 - \bar{u}^3)W^2\right].$$

Then, restricting these algebras to X and X' respectively, we obtain

$$\mathcal{G}_X = \mathcal{O}_X\left[\bar{v}W, \bar{u}^4 W\right],$$

and

$$\mathcal{G}_{X'} = \mathcal{O}_{X'}\left[\frac{\bar{v}}{\bar{u}} W, \bar{u}^2 W\right].$$

Since $\bar{v}W = \bar{u} \cdot \frac{\bar{v}}{\bar{u}} W$ and $\bar{u}^4 W = \bar{u}^2 \cdot \bar{u}^2 W$, one readily checks that the pull-back of $\mathcal{G}_X$ is contained in $\mathcal{G}_{X'}$.

Chapter 5

Differential conditions

Let S be a regular ring. Recall that, for an element $f \in S$ and a prime ideal $\mathfrak{p} \subset S$, the order of f at $\mathfrak{p}$ is defined by

$$\nu_{\mathfrak{p}}(f) = \max \{n \in \mathbb{N} \mid f \in \mathfrak{p}^n S_{\mathfrak{p}}\}.$$

When S is a polynomial ring in a finite number of variables over a field k, say $S = k[T_1, \ldots, T_r]$, there is a Jacobian criterion saying that

$$\nu_{\mathfrak{p}}(f) \geq N \iff \mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p}, \tag{5.1}$$

where

$$\mathrm{Diff}^{N-1}(S)(f) = \left\langle \Delta(f) \mid \Delta \in \mathrm{Diff}^{N-1}(S) \right\rangle$$

(see Lemma 5.4.1). Note that here k is not necessarily a perfect field, and that Δ runs over all the absolute differential operators on S, not only those which are relative to k.

In this chapter we introduce conditions on a general regular ring S so as to ensure that (5.1) holds for any $f \in S$ and any $\mathfrak{p} \in \mathrm{Spec}(S)$. In the case that S is defined over a field of characteristic zero, this will be the weak Jacobian condition. On the other hand, if S is a regular ring defined over a field of characteristic $p > 0$, then we will require that S admits a p-basis over the prime field $\mathbb{F}_p$.

The chapter is organized as follows: in Section 5.1 we introduce the weak Jacobian condition; Section 5.2 is devoted to the study of p-bases and differential operators; along Section 5.3 we discuss some issues related to p-bases and regular systems of parameters on local rings; finally, in Section 5.4 we give two extensions of the Jacobian criterion mentioned above (see Proposition 5.4.3 and Proposition 5.4.7).

5.1 The weak Jacobian condition

Let S be a regular ring and $\mathfrak{p} \subset S$ a prime ideal. Given elements $x_1, \ldots, x_s \in S$, and derivatives $\delta_1, \ldots, \delta_r \in \mathrm{Der}(S)$, we shall denote the $(r \times s)$-matrix $(\delta_i x_j)$ by

$$\mathrm{Jac}(x_1, \ldots, x_s; \delta_1, \ldots, \delta_r).$$

Similarly, the matrix $(\delta_i x_j \text{ modulo } \mathfrak{p})$ will be denoted by

$$\mathrm{Jac}(x_1, \ldots, x_s; \delta_1, \ldots, \delta_r)(\mathfrak{p}).$$

Note that $S/\mathfrak{p} \subset k(\mathfrak{p}) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. Thus the latter can be regarded as a matrix over the field $k(\mathfrak{p})$.

Lemma 5.1.1. *Let S be a regular ring, $\mathfrak{p} \subset S$ a prime ideal, and $\delta_1, \ldots, \delta_r \in \mathrm{Der}(S)$ a collection of derivatives. Consider two collections of elements in $\mathfrak{p}$, say $x_1, \ldots, x_s \in \mathfrak{p}$ and $y_1, \ldots, y_t \in \mathfrak{p}$. If $\langle x_1, \ldots, x_s \rangle = \langle y_1, \ldots, y_t \rangle$, then*

$$\operatorname{rank} \mathrm{Jac}(x_1, \ldots, x_s; \delta_1, \ldots, \delta_r)(\mathfrak{p}) = \operatorname{rank} \mathrm{Jac}(y_1, \ldots, y_t; \delta_1, \ldots, \delta_r)(\mathfrak{p}).$$

Proof. Assume without loss of generality that $S = S_{\mathfrak{p}}$. By the Leibniz's rule, each δ_i induces a $k(\mathfrak{p})$-linear map $\overline{\delta}_i : \mathfrak{p}/\mathfrak{p}^2 \to k(\mathfrak{p})$, which maps an element $\overline{f} \in \mathfrak{p}/\mathfrak{p}^2$ to $\overline{\delta_i f} \in k(\mathfrak{p})$ (here $\overline{f}$ represents the class of f in $\mathfrak{p}/\mathfrak{p}^2$, and $\overline{\delta_i f}$ that of $\delta_i f$ in $S/\mathfrak{p}$). In this way, $\delta_1, \ldots, \delta_r$ induce an application of $k(\mathfrak{p})$-vector spaces, say $\hat{\delta} : \mathfrak{p}/\mathfrak{p}^2 \to k(\mathfrak{p})^r$, given by $\overline{f} \mapsto \left(\overline{\delta_1 f}, \ldots, \overline{\delta_r f}\right)$ for $f \in \mathfrak{p}$.

Next consider the $k(\mathfrak{p})$-vector space spanned by the classes of $x_1, \ldots, x_s$ in $\mathfrak{p}/\mathfrak{p}^2$, say $\mathbb{E}_1 = \langle \overline{x}_1, \ldots, \overline{x}_s \rangle \subset \mathfrak{p}/\mathfrak{p}^2$. Note that

$$\operatorname{rank} \mathrm{Jac}(x_1, \ldots, x_s; \delta_1, \ldots, \delta_r)(\mathfrak{p})$$

is precisely the dimension of the $k(\mathfrak{p})$-subspace $\hat{\delta}(\mathbb{E}_1) \subset k(\mathfrak{p})^r$. Similarly, set $\mathbb{E}_2 = \langle \overline{y}_1, \ldots, \overline{y}_t \rangle \subset \mathfrak{p}/\mathfrak{p}^2$, and observe that

$$\operatorname{rank} \mathrm{Jac}(y_1, \ldots, y_t; \delta_1, \ldots, \delta_r)(\mathfrak{p})$$

coincides with the dimension of $\hat{\delta}(\mathbb{E}_2) \subset k(\mathfrak{p})^r$. Since $\langle x_1, \ldots, x_s \rangle = \langle y_1, \ldots, y_t \rangle$, we have that $\mathbb{E}_1 = \mathbb{E}_2$, and thus the result follows. $\square$

According to the previous lemma, for any ideal $I \subset \mathfrak{p}$, we can define unambiguously

$$\operatorname{rank} \mathrm{Jac}(I; \delta_1, \ldots, \delta_r)(\mathfrak{p}) := \operatorname{rank} \mathrm{Jac}(x_1, \ldots, x_s; \delta_1, \ldots, \delta_r)(\mathfrak{p}),$$

where $x_1, \ldots, x_s$ is any collection of generators of I. In addition, for an arbitrary subset $\mathcal{D} \subset \mathrm{Der}(S)$, we shall define

$$\operatorname{rank} \mathrm{Jac}(I; \mathcal{D})(\mathfrak{p})$$

as the supremum of $\operatorname{rank} \mathrm{Jac}(I; \delta_1, \ldots, \delta_r)(\mathfrak{p})$, where $\{\delta_1, \ldots, \delta_r\}$ runs over the finite subsets of $\mathcal{D}$.

Lemma 5.1.2. *Let S be a regular ring, and $\mathcal{D} \subset \mathrm{Der}(S)$ a subset of derivatives. Then, for each prime $\mathfrak{p} \subset S$,*

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{p}; \mathcal{D})(\mathfrak{p}) \leq \dim(S_{\mathfrak{p}}).$$

Proof. According to the proof Lemma 5.1.1, the rank of $\mathrm{Jac}(\mathfrak{p}; \mathcal{D})(\mathfrak{p})$ is bounded by that of $\mathfrak{p}/\mathfrak{p}^2$ over $k(\mathfrak{p})$, which coincides with $\dim(S_{\mathfrak{p}})$ by the regularity of S. □

Definition 5.1.3 (Weak Jacobian condition, cf. [26, §40.F])**.** Let S be a regular ring and $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivatives. We shall say that $\mathcal{D}$ satisfies the *weak Jacobian condition* (WJ) on S, or equivalently that S satisfies WJ relative to $\mathcal{D}$, if

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{p}; \mathcal{D})(\mathfrak{p}) = \dim(S_{\mathfrak{p}})$$

for each prime ideal $\mathfrak{p} \subset S$. When $\mathrm{Der}(S)$ satisfies WJ on S, we shall simply say that S satisfies WJ. For a regular noetherian scheme V, we shall say that WJ holds on V if there exists an open affine covering of V, say $V = \bigcup \mathrm{Spec}(S_i)$, so that each S_i satisfies WJ.

Remark 5.1.4 (Localization). Let S be a regular ring, and $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivations. Consider an arbitrary localization of S, say $S' = \mathcal{U}^{-1}S$, where $\mathcal{U}$ denotes a multiplicative subset of S (for instance, $S' = S_f$ for some $f \in S$, or $S' = S_{\mathfrak{q}}$ for some prime $\mathfrak{q} \subset S$). Observe that $\mathcal{D} \subset \mathcal{U}^{-1}\mathcal{D}$, and $\mathfrak{p} \subset \mathfrak{p}S'$ for all primes $\mathfrak{p} \subset S$. In this way, if $\mathcal{D}$ satisfies the weak Jacobian condition on S, one readily checks that $\mathcal{U}^{-1}\mathcal{D} \subset \mathrm{Der}(S')$ is a S'-module that satisfies the weak Jacobian condition on S'.

The following results (Lemma 5.1.5 and Lemma 5.1.6) are inspired by the ideas of Matsumura (see [26, §40]).

Lemma 5.1.5. *Let $(R, \mathfrak{M})$ be a regular local ring of dimension d, and $\mathcal{D} \subset \mathrm{Der}(R)$ a submodule of derivations satisfying*

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{M}; \mathcal{D})(\mathfrak{M}) = \dim(R).$$

Then, for each regular system of parameters of R, say $x_1, \ldots, x_d$, there exist derivatives $\delta_1, \ldots, \delta_d \in \mathcal{D}$ so that $\delta_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all $1 \leq i, j \leq d$.

Proof. By assumption there are derivatives $\delta_1^*, \ldots, \delta_d^* \in \mathcal{D}$ so that

$$\operatorname{rank} \mathrm{Jac}(x_1, \ldots, x_d; \delta_1^*, \ldots, \delta_d^*)(\mathfrak{M}) = d.$$

Hence the square matrix

$$M = \mathrm{Jac}(x_1, \ldots, x_d; \delta_1^*, \ldots, \delta_d^*) \in \mathrm{Mat}_{d \times d}(R)$$

has non-zero determinant modulo $\mathfrak{M}$. This implies that $\det(M)$ is a unit in R, and therefore the matrix M is invertible in $\mathrm{Mat}_{d\times d}(R)$. Let M^{-1} denote the inverse of M, and set

$$\begin{pmatrix}\delta_1\\ \vdots\\ \delta_d\end{pmatrix} = M^{-1}\begin{pmatrix}\delta_1^*\\ \vdots\\ \delta_d^*\end{pmatrix}.$$

Then one readily checks that

$$\mathrm{Jac}(x_1,\dots,x_d;\delta_1,\dots,\delta_d) = M^{-1}M = I_{d\times d},$$

which means that $\delta_i(x_j) = \delta_{ij}$ for all $1 \le i,j \le d$. □

Lemma 5.1.6. *Let S be a regular ring and fix a submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$. For each prime ideal $\mathfrak{p} \subset S$, the following conditions are equivalent:*

i) $\operatorname{rank}\mathrm{Jac}(\mathfrak{p};\mathcal{D})(\mathfrak{p}) = \dim(S_\mathfrak{p})$.

ii) For each regular system of parameters of $S_\mathfrak{p}$, say $x_1,\dots,x_d$, there exist derivatives $\delta_1,\dots,\delta_d \in \mathcal{D}_\mathfrak{p}$ so that $\delta_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all $1 \le i,j \le d$.

iii) There exists a regular system of parameters of $S_\mathfrak{p}$, say $x_1,\dots,x_d$, and derivatives $\delta_1,\dots,\delta_d \in \mathcal{D}_\mathfrak{p}$, so that $\delta_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all $1 \le i,j \le d$.

Proof. Lemma 5.1.5 yields i) ⇒ ii), and ii) ⇒ iii) is trivial. Thus we just need to prove iii) ⇒ i).

Fix a prime ideal $\mathfrak{p} \subset S$, and suppose that iii) holds. Since $\delta_1,\dots,\delta_d \in \mathcal{D}_\mathfrak{p}$, we should have

$$\delta_i = a_{i1}\delta_1^* + \cdots + a_{ir}\delta_r^*,$$

for some global derivatives $\delta_1^*,\dots,\delta_r^* \in \mathcal{D}$ and some coefficients $a_{ij} \in S_\mathfrak{p}$. Set $A = (a_{ij}) \in \mathrm{Mat}_{d\times r}(S_\mathfrak{p})$. Clearly,

$$\mathrm{Jac}(x_1,\dots,x_d;\delta_1,\dots,\delta_d) = A\cdot\mathrm{Jac}(x_1,\dots,x_d;\delta_1^*,\dots,\delta_r^*).$$

Thus we see that the rank of the left Jacobian matrix (modulo $\mathfrak{p}S_\mathfrak{p}$) is bounded by that of the right Jacobian matrix (modulo $\mathfrak{p}S_\mathfrak{p}$), i.e.,

$$\begin{aligned}\dim(S_\mathfrak{p}) &= \operatorname{rank}\mathrm{Jac}(x_1,\dots,x_d;\delta_1,\dots,\delta_d)(\mathfrak{p}S_\mathfrak{p})\\ &\le \operatorname{rank}\mathrm{Jac}(x_1,\dots,x_d;\delta_1^*,\dots,\delta_r^*)(\mathfrak{p}S_\mathfrak{p}).\end{aligned}$$

Choose a set of generators of $\mathfrak{p}$ over S, say $\mathfrak{p} = \langle f_1,\dots,f_s\rangle$. This is also a set of generators of $\mathfrak{p}S_\mathfrak{p}$ over $S_\mathfrak{p}$. Thus, by Lemma 5.1.1, we have that

$$\begin{aligned}\dim(S_\mathfrak{p}) &\le \operatorname{rank}\mathrm{Jac}(x_1,\dots,x_d;\delta_1^*,\dots,\delta_r^*)(\mathfrak{p}S_\mathfrak{p})\\ &= \operatorname{rank}\mathrm{Jac}(f_1,\dots,f_s;\delta_1^*,\dots,\delta_r^*)(\mathfrak{p}S_\mathfrak{p}) \le \operatorname{rank}\mathrm{Jac}(\mathfrak{p};\mathcal{D})(\mathfrak{p}),\end{aligned}$$

which implies $\dim(S_\mathfrak{p}) = \operatorname{rank}\mathrm{Jac}(\mathfrak{p};\mathcal{D})(\mathfrak{p})$ by Lemma 5.1.2. □

Proposition 5.1.7. *Let S be a regular algebra of finite type over a perfect field k. Then S satisfies the weak Jacobian condition relative to* $\mathrm{Der}_k(S)$.

Proof. Fix a prime ideal $\mathfrak{q} \subset S$. Since $S/\mathfrak{q}$ is a finite type domain over k, there should be a maximal ideal containing $\mathfrak{q}$, say $\mathfrak{q} \subset \mathfrak{m} \subset S$, so that $S_{\mathfrak{m}}/\mathfrak{q}S_{\mathfrak{m}}$ is regular (this follows from Proposition B.0.8, Proposition B.0.11, and the Zero-point theorem of Hilbert [26, Theorem 25, p. 93]). Hence it is possible to find a regular system of parameters of $S_{\mathfrak{m}}$, say $x_1, \dots, x_d$, so that $\mathfrak{q}S_{\mathfrak{m}} = \langle x_1, \dots, x_e \rangle S_{\mathfrak{m}}$, where $e = \dim(S_{\mathfrak{q}})$. Then $x_1, \dots, x_e$ is a regular system of parameters of $S_{\mathfrak{q}}$.

Next consider the subalgebra $k[x_1, \dots, x_d] \subset S_{\mathfrak{m}}$, which can be regarded as a polynomial ring contained in $S_{\mathfrak{m}}$. By Lemma [4, Theorem V.3.2, p. 91], $S_{\mathfrak{m}}$ is flat over $k[x_1, \dots, x_d]$. Thus we see that

$$k[x_1, \dots, x_d]_{\langle x_1, \dots, x_d \rangle} \longrightarrow S_{\mathfrak{m}}$$

is an étale morphism of local rings (see Corollary C.0.4). Then the partial derivatives $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_e}$ can be uniquely extended to local derivatives on $S_{\mathfrak{m}}$ (see Lemma C.0.5), and, by localization, on $S_{\mathfrak{q}}$. Denote these extensions by $\delta_1, \dots, \delta_e \in \mathrm{Der}_k(S_{\mathfrak{q}})$ respectively. By definition,

$$\delta_i(x_j) = \delta_{ij} \quad \text{(Kronecker's delta)}$$

for all $1 \leq i, j \leq e$. Note also that $\mathrm{Der}_k(S_{\mathfrak{q}}) \simeq \mathrm{Der}_k(S) \otimes_S S_{\mathfrak{q}}$ by [20, Proposition 16.8.6, p. 41]. Hence Lemma 5.1.6 yields

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{q}, \mathrm{Der}_k(S))(\mathfrak{q}) = \dim(S_{\mathfrak{q}}).$$

Repeating this argument for each prime ideal $\mathfrak{q} \subset S$, we see that S satisfies the weak Jacobian condition relative to $\mathrm{Der}_k(S)$. □

Theorem 5.1.8 ([26, Theorem 97, p. 286])**.** *Let R be a power series ring over a field k, say $R = k[[x_1, \dots, x_d]]$. Then the weak Jacobian condition holds on R.*

Theorem 5.1.9 ([26, Theorem 101, p. 291])**.** *Let S be a regular ring over a field of characteristic zero. If the weak Jacobian condition holds on S, then S is excellent.*

Theorem 5.1.10 ([26, Theorem 103, p. 292])**.** *Let S be a regular ring. If $S[T_1, \dots, T_n]$ satisfies the weak Jacobian condition for all $n \geq 0$, then S is excellent.*

5.2 Differential operators in positive characteristic: p-bases

Let k be an arbitrary field k of characteristic $p > 0$. In this section we review the notion of p-basis of a ring S over k and we discuss some properties related to it.

Definition 5.2.1. Let k be a ring over $\mathbb{F}_p$, and S an arbitrary k-algebra. We will say that a finite set $\{b_1, \ldots, b_r\} \subset S$ is *p-independent over k* if the set of monomials $\{b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r} \mid 0 \leq \alpha_i < p\}$ is linearly independent over the subring $S^p[k]$. A set $\mathcal{B} \subset S$ is said to be *p-independent over k* if each finite subset of $\mathcal{B}$ is so. The set $\mathcal{B}$ is called a *p-basis of S over k* if it is p-independent over k, and

$$S = S^p[k][\mathcal{B}].$$

An *absolute p-basis* of S is defined as a p-basis of S over the prime field $\mathbb{F}_p$.

Remark 5.2.2. If $\mathcal{B}$ is a p-basis of S over k, then S can be regarded as a free $S^p[k]$-module with the set of monomials $\{b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r} \mid 0 \leq \alpha_i < p\}$ as a basis.

Remark 5.2.3 (Summary of p-bases). In general, given an arbitrary ring S of characteristic $p > 0$ and subring $k \subset S$, there is no a p-basis of S over k. However:

- If K/k is a field extension, then K always admits a p-basis over k (see Example 5.2.4 below).
- Any p-basis of K over k can be extended to a p-basis of the polynomial ring $K[T_1, \ldots, T_n]$ over k (see Lemma 5.2.8).
- In spite of the previous property, we will show that, in general, the power series ring $K[[T_1, \ldots, T_n]]$ does not admit a p-basis over k (see Example 5.3.3).

Along Section 5.3 we will also discuss some properties related to the existence of p-bases on regular regular rings. More precisely we will show that:

- If a noetherian domain S admits an absolute p-basis, then S is regular and excellent (see Remark 5.2.15 and Proposition 5.3.6).
- There are also examples of regular excellent local rings which do not admit an absolute p-basis. Indeed, we will exhibit a complete regular local ring of dimension 1 which does not have an absolute p-basis (see Example 5.3.3).
- Let k be an arbitrary field of characteristic $p > 0$. Then any regular variety over k can be covered by affine charts of the form $\mathrm{Spec}(S)$ so that S admits an absolute p-basis (see Proposition 5.3.12).

Example 5.2.4. In the case of a field extension, say K/k, one can always find a p-basis of K over k. Indeed, it can be shown that a p-independent set $\mathcal{B} \subset K$ is a p-basis of K over k if and only if it is a maximal p-independent set. Thus, by Zorn's lemma, K admits a p-basis. However, these arguments do not apply for arbitrary rings. For instance, consider the ring $\mathbb{F}_2[x, y]$, where x, y represent variables. It can be checked that $\{x, xy\}$ is a maximal p-independent set over $\mathbb{F}_2$, but it is not a p-basis of $\mathbb{F}_2[x, y]$ over $\mathbb{F}_2$.

Lemma 5.2.5. *If a set $\mathcal{B} \subset S$ is p-independent over k, then*

$$S^p[k][\mathcal{B}] \simeq S^p[k][T_b \mid b \in \mathcal{B}]/\langle T_b^p - b^p \mid b \in \mathcal{B}\rangle,$$

where $\{T_b\}_{b\in\mathcal{B}}$ represent variables.

Proof. There is a natural homomorphism of S^p-algebras from the polynomial ring $S^p[k][T_b \mid b \in \mathcal{B}]$ to S, which maps the variable T_b to b for each $b \in \mathcal{B}$. Obviously the ideal generated by the elements of the form $T_b^p - b^p$ is contained in the kernel of this map. Finally, since $S^p[k][T_b \mid b \in \mathcal{B}]/\langle T_b^p - b^p \mid b \in \mathcal{B}\rangle$ is a free $S^p[k]$-module with the monomials of the form $T_{b_1}^{\alpha_1} \cdot \ldots \cdot T_{b_r}^{\alpha_r}$, $0 \leq \alpha_i < p$, as a basis, the claim holds. □

Lemma 5.2.6 (Localization)**.** *Let k be a ring over $\mathbb{F}_p$ and let S be an arbitrary k-algebra which has a p-basis over k, say $\mathcal{B}$. Consider a multiplicative subset $\mathcal{U} \subset S$ which does not contain nilpotent elements. Then the image of $\mathcal{B}$ in $\mathcal{U}^{-1}S$ is a p-basis of $\mathcal{U}^{-1}S$ over k.*

Proof. Set $\mathcal{U}^p = \{u^p \mid u \in \mathcal{U}\}$, which is a multiplicative subset of $S^p[k]$. Observe that

$$\mathcal{U}^{-1}S = (\mathcal{U}^p)^{-1}S = S \otimes_{S^p[k]} (\mathcal{U}^p)^{-1}(S^p[k]).$$

Since S is a free $S^p[k]$-module with the set of monomials of the form $b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r}$ with $b_i \in \mathcal{B}$ and $0 \leq \alpha_i < p$ as basis, the image of these monomials is also a basis of $\mathcal{U}^{-1}S$ over $(\mathcal{U}^p)^{-1}(S^p[k])$. Hence the image of $\mathcal{B}$ is a p-basis of $\mathcal{U}^{-1}S$ over k. □

Remark 5.2.7. Let k, S and $\mathcal{B}$ be as in the previous lemma. Suppose that one of the monomials $b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r}$ has a zero-divisor in S, say s. Then, from the definition of p-independence, it follows that $s^p = 0$, i.e., s is nilpotent.

Lemma 5.2.8. *Let k be a ring over $\mathbb{F}_p$ and let S be an arbitrary k-algebra which has a p-basis over k, say $\mathcal{B}$. Consider the polynomial ring $S' = S[T_1, \ldots, T_n]$. Then the set $\mathcal{B}' = \mathcal{B} \cup \{T_1, \ldots, T_n\}$ is a p-basis of S' over k.*

Proof. Observe that $(S')^p[k] = S^p[k][T_1^p, \ldots, T_n^p]$. Since S is a free $S^p[k]$ module with the monomials of the form $b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r}$, $0 \leq \alpha_i < p$, as a basis, it follows easily that S' is a free $(S')^p[k]$-module with the monomials of the form $b_1^{\alpha_1} \cdot \ldots \cdot b_r^{\alpha_r} T_1^{\beta_1} \cdot \ldots \cdot T_n^{\beta_n}$, $0 \leq \beta < p$, as a basis. Hence $\mathcal{B}'$ is a p-basis of S' over k. □

p-bases and derivatives

Definition 5.2.9. Let S be an arbitrary k-algebra and suppose that $\Omega^1_{S/k}$ is a free S-module. A subset $\mathcal{B} \subset S$ is called a *differential basis* of $\Omega^1_{S/k}$ if the set $d_{S/k}(\mathcal{B}) = \{d_{S/k}(b) \mid b \in \mathcal{B}\}$ is a basis of $\Omega^1_{S/k}$.

Lemma 5.2.10. *If $\mathcal{B}$ is a p-basis of S over k, then $\Omega^1_{S/k}$ is a free S-module with the set $\mathcal{B}$ as a differential basis.*

Proof. Using the notation of Lemma 5.2.5, set $A_0 = S^p[k]$, $A = A_0[T_b \mid b \in \mathcal{B}]$, and $J = \langle T_b^p - b^p \mid b \in \mathcal{B}\rangle$. Observe that, in this way, $S = A/J$, and $\Omega^1_{S/k} = \Omega^1_{S/A_0}$. Then, by Theorem 2.1.6, there is a natural exact sequence

$$J/J^2 \xrightarrow{\overline{d_{A/A_0}}} \Omega^1_{A/A_0} \otimes_A S \longrightarrow \Omega^1_{S/k} \longrightarrow 0.$$

Since $T_b^p \in A^p$, one readily checks that J/J^2 is mapped to zero via $\overline{d_{A/A_0}}$. Thus we have an isomorphism

$$\Omega^1_{A/A_0} \otimes_A S \xrightarrow{\sim} \Omega^1_{S/k},$$

which maps the class of $d_{A/A_0}(T_b)$ to $d_{S/k}(b)$ for each $b \in \mathcal{B}$. Hence $\Omega^1_{S/k}$ is a free module with $\{d_{S/k}(b) \mid b \in \mathcal{B}\}$ as a basis. □

Remark 5.2.11. Let k be a ring over $\mathbb{F}_p$ and let S be an arbitrary k-algebra which admits a p-basis over k, say $\mathcal{B}$. In virtue of Lemma 5.2.10, any set-theoretical map $\varphi_0 : \mathcal{B} \to S$ induces a unique k-derivation, say $\delta_0 : S \to S$, such that $\delta_0(b) = \varphi_0(b)$ for $b \in \mathcal{B}$. Thus, for each $b \in \mathcal{B}$, there exists a unique derivative of S over k, which we shall denote by $\frac{\partial}{\partial b} : S \to S$, satisfying $\frac{\partial}{\partial b}(c) = \delta_{bc}$ (Kronecker's delta) for all $c \in \mathcal{B}$.

Next, consider another derivative $\delta \in \mathrm{Der}_k(S)$. Since $\Omega^1_{S/k} = \bigoplus_{b\in\mathcal{B}} S\, d_{S/k}(b)$, one readily checks that

$$\delta = \sum_{b\in\mathcal{B}} \delta(b)\frac{\partial}{\partial b}.$$

Note that, in general, this is an infinite sum. However, for a given element $f \in S$, as $\Omega^1_{S/k}$ is a free module with $\{d_{S/k}(b) \mid b \in \mathcal{B}\}$ as a basis, there is a finite number of elements $b \in \mathcal{B}$ so that $\frac{\partial}{\partial b}(f) \neq 0$. Thus, if we denote these elements by $b_1, \ldots, b_s$, we have that

$$\delta(f) = \sum_{i=1}^{s} \delta(b_i)\frac{\partial}{\partial b_i}(f).$$

Theorem 5.2.12 (Tyc [32, Theorem 1])**.** *Let k be a ring over $\mathbb{F}_p$, and S a noetherian k-algebra. A subset $\mathcal{B} \subset S$ is a p-basis of S over k if and only if $\Omega^1_{S/k}$ is a free S-module with $\mathcal{B}$ as a differential basis.*

Remark 5.2.13. This result was first proved by Kimura and Niitsuma for the case in which S is a regular local ring over $\mathbb{F}_p$ (see [24]). In fact, the proof of Tyc is strongly based on that of Kimura and Niitsuma. Along this chapter, and specially on Section 5.3, we will use several ideas introduced in the works of Kimura and Niitsuma, and Tyc.

Remark 5.2.14. In general, given k and S as in Theorem 5.2.12, and provided that $\Omega^1_{S/k}$ is a free S-module, we do not know whether S admits a p-basis over

k. This happens because, as $\Omega^1_{S/k}$ might be an infinite module, the condition of having a differential basis is stronger than that of being free. Only in the case that $\Omega^1_{S/k}$ is finite we can ensure that S has a p-basis over k if and only if $\Omega^1_{S/k}$ is free.

Remark 5.2.15. Suppose that S is a reduced noetherian ring over $\mathbb{F}_p$ that has an absolute p-basis. Then, by [32, Theorem 2], S is smooth[1] over $\mathbb{F}_p$. This implies that, for each prime ideal $\mathfrak{q} \subset S$, the local ring $S_\mathfrak{q}$ is smooth over $\mathbb{F}_p$, and therefore $S_\mathfrak{q}$ is formally smooth over $\mathbb{F}_p$ for the topology induced by its maximal ideal. Hence $S_\mathfrak{q}$ is regular by [26, Proposition 28.M, p. 207]. In this way we see that, if a reduced noetherian ring S has an absolute p-basis, then S is regular.

Absolute p-bases and differential operators

Consider an arbitrary $\mathbb{F}_p$-algebra S. In this section will draw attention to the absolute differential operators of S under the assumption that S admits an absolute p-basis. Recall that these are differential operators and p-bases defined over $\mathbb{F}_p$ and that, in this situation, we shall abbreviate the universal pairs $\left(\Omega^1_{S/\mathbb{F}_p}, d_{S/\mathbb{F}_p}\right)$ and $\left(\mathrm{P}^n_{S/\mathbb{F}_p}, d^n_{S/\mathbb{F}_p}\right)$ by $\left(\Omega^1_S, d_S\right)$ and $\left(\mathrm{P}^n_S, d^n_S\right)$ respectively.

Lemma 5.2.16. *Let k be a ring over $\mathbb{F}_p$, A an arbitrary k-algebra, and M an A-module. Consider a differential operator $\Delta : A \to M$ of order n over k. Then Δ is A^{p^e}-linear for all $p^e > n$.*

Proof. Let $\Phi \in \mathrm{Hom}_A(\mathrm{P}^n_{A/k}, M)$ be as in Proposition 2.2.4, i.e., Φ is the unique homomorphism of A-modules so that $\Phi \circ d^n_{A/k} = \Delta$. Recall that $\mathrm{P}^n_{A/k} = (A \otimes_k A)/I^{n+1}_{A/k}$, where $I_{A/k}$ denotes the kernel of the multiplication map $A \otimes_k A \to A$. Fix two elements $a, f \in A$. Since $n < p^e$, we have $I^{p^e}_{A/k} \subset I^n_{A/k}$. Hence,

$$(1 \otimes a - a \otimes 1)^{p^e} = (1 \otimes a^{p^e} - a^{p^e} \otimes 1) \in I^n_{A/k},$$

and therefore $\overline{1 \otimes a^{p^e}} = \overline{a^{p^e} \otimes 1}$ in $\mathrm{P}^n_{A/k}$. Thus,

$$\Delta(a^{p^e} f) = \Phi\big(\overline{1 \otimes a^{p^e} f}\big) = \Phi\big(\overline{a^{p^e} \otimes f}\big) = a^{p^e}\Delta(f). \qquad \square$$

Proposition 5.2.17. *Let S be a reduced algebra over $\mathbb{F}_p$ which has an absolute p-basis, say $\mathcal{B}$. Then, for each $\beta \in \mathbb{N}^{\oplus\mathcal{B}}$, there exists a differential operator of order $|\beta|$ over $\mathbb{F}_p$, say $D^{[\mathcal{B};\beta]}_S : S \to S$, so that*

$$D^{[\mathcal{B};\beta]}_S(\mathcal{B}^\alpha) = \binom{\alpha}{\beta}\mathcal{B}^{\alpha-\beta} \tag{5.2}$$

for $\alpha \in \mathbb{N}^{\oplus\mathcal{B}}$.

[1] Here smooth means formally smooth for the discrete topology (see [26, §28.D]).

Proof. Fix $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$ and choose $e > 0$ so that $p^e > |\beta|$. By definition of p-basis, we have that

$$S \simeq S^p\big[T_b \mid b \in \mathcal{B}\big]/\big\langle T_b^p - b^p \mid b \in \mathcal{B}\big\rangle,$$

where $\{T_b\}_{b\in\mathcal{B}}$ represent variables. Since S is reduced, $S \simeq S^p$ via the Frobenius isomorphism. Thus, by induction on e, we get

$$S \simeq S^{p^e}\big[T_b \mid b \in \mathcal{B}\big]/\big\langle T_b^{p^e} - b^{p^e} \mid b \in \mathcal{B}\big\rangle.$$

Set $A = S^{p^e}\big[T_b \mid b \in \mathcal{B}\big]$, and $J = \big\langle T_b^{p^e} - b^{p^e} \mid b \in \mathcal{B}\big\rangle$. Regarding A as a polynomial ring over S^{p^e}, and according to Lemma 2.4.5, there is a S^{p^e}-linear differential operator $\mathrm{Tay}^\beta : A \to A$ so that

$$\mathrm{Tay}^\beta(T^\alpha) = \binom{\alpha}{\beta} T^{\alpha-\beta}$$

for $\alpha \in \mathbb{N}^{\oplus \mathcal{B}}$. We will show that Tay^β induces a natural differential operator $D_S^{[\mathcal{B};\beta]} : S \to S$ of order $|\beta|$ over $\mathbb{F}_p$ via the quotient map $A \to S \simeq A/J$.

By composing Tay^β with the map $A \to S$, we obtain a differential operator $\overline{\mathrm{Tay}^\beta} : A \to S$. Note that $\overline{\mathrm{Tay}^\beta}(T^\alpha) = \binom{\alpha}{\beta}\mathcal{B}^{\alpha-\beta}$ for $\alpha \in \mathbb{N}^{\oplus \mathcal{B}}$. In order to see that $\overline{\mathrm{Tay}^\beta}$ induces a differential operator $D_S^{[\mathcal{B};\beta]} : S \to S$, we shall verify that it annihilates the ideal $J \subset A$.

Recall that Tay^β is a differential operator of order $|\beta| < p^e$ over S^{p^e}. Then it is simultaneously S^{p^e}-linear and A^{p^e}-linear (see Lemma 5.2.16). In this way, for each $a \in A$ and $b \in \mathcal{B}$, we have that

$$\mathrm{Tay}^\beta\left(a \cdot \left(T_b^{p^e} - b^{p^e}\right)\right) = \mathrm{Tay}^\beta(a) \cdot \left(T_b^{p^e} - b^{p^e}\right) \in J.$$

This implies that $\overline{\mathrm{Tay}^\beta}$ annihilates J, and hence it induces a differential operator $D_S^{[\mathcal{B};\beta]} : S \to S$ as required. □

Remark 5.2.18. At first sight, it may seem that the definition $D_S^{[\mathcal{B};\beta]}$ depends on the choice of e. However, as it will be shown in Corollary 5.2.19 below, there exists a unique differential operator on S of order $|\beta|$ over $\mathbb{F}_p$ satisfying condition (5.2). Thus $D_S^{[\mathcal{B};\beta]}$ is well-defined and unique. Hereafter, when there is no risk of confusion, we shall simply write $D^{[\mathcal{B};\beta]}$ instead of $D_S^{[\mathcal{B};\beta]}$.

Corollary 5.2.19. *Let S be a reduced ring over $\mathbb{F}_p$. Suppose that S admits an absolute p-basis, say $\mathcal{B}$. Then S is differentially smooth over $\mathbb{F}_p$ and the differential operators $D^{[\mathcal{B};\beta]}$ defined in Proposition 5.2.17 have the following properties:*

i) For all $\beta, \beta' \in \mathbb{N}^{\oplus \mathcal{B}}$,

$$D^{[\mathcal{B};\beta]} \circ D^{[\mathcal{B};\beta']} = D^{[\mathcal{B};\beta']} \circ D^{[\mathcal{B};\beta]} = \frac{(\beta+\beta')!}{\beta!\beta'!} D^{[\mathcal{B};\beta+\beta']}.$$

ii) For any element $f \in S$, there exists a finite number of indexes $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$ so that $D^{[\mathcal{B};\beta]}(f) \neq 0$.

iii) For any differential operator $\Delta : S \to M$ of order N (over $\mathbb{F}_p$),

$$\Delta = \sum_{|\beta| \leq N} \Delta(b^\beta) D^{[\mathcal{B};\beta]}.$$

(note that, although β runs over a possibly infinite set, $\Delta(f)$ is a finite sum for each $f \in S$ by the previous condition).

Proof. It follows from Lemma 5.2.10, the definition of the differential operators $D^{[\mathcal{B};\beta]}$, and Theorem 2.3.2. □

Corollary 5.2.20. *Let S, $\mathcal{B}$, and $\{D^{[\mathcal{B};\beta]}\}_{\beta \in \mathbb{N}^{\oplus \mathcal{B}}}$ be as in Proposition 5.2.17. Then, for every $f_1, \dots, f_r \in S$,*

$$D^{[\mathcal{B};\beta]}(f_1 \cdot \dots \cdot f_r) = \sum_{\alpha_1 + \dots + \alpha_r = \beta} D^{[\mathcal{B};\alpha_1]}(f_1) \cdot \dots \cdot D^{[\mathcal{B};\alpha_r]}(f_r).$$

Proof. Recall that, in the proof of Proposition 5.2.17, $D^{[\mathcal{B};\beta]}$ was constructed from the differential operator $\operatorname{Tay}^\beta : A \to A$, where A represents the polynomial ring $S^{p^e}[T_b \mid b \in \mathcal{B}]$. Thus the claim follows from Lemma 2.4.7. □

Remark 5.2.21. Let S be a ring over $\mathbb{F}_p$ which admits an absolute p-basis, say $\mathcal{B}$, and consider the polynomial ring $S' = S[T_1, \dots, T_n]$. The set $\mathcal{T} = \{T_1, \dots, T_n\}$ is a p-basis of S' over S, and, according to Lemma 5.2.8, the set $\mathcal{B}' = \mathcal{B} \cup \mathcal{T}$ is an absolute p-basis of S'. Observe that we have a natural isomorphism $\mathbb{N}^{\oplus \mathcal{B}'} \simeq \mathbb{N}^{\mathcal{T}} \oplus \mathbb{N}^{\oplus \mathcal{B}}$. Thus any $\beta' \in \mathbb{N}^{\oplus \mathcal{B}'}$ can be uniquely decomposed as $\beta' = \alpha + \beta$, with $\alpha \in \mathbb{N}^{\mathcal{T}}$ and $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$. Regarding α and β as elements of $\mathbb{N}^{\oplus \mathcal{B}'}$, condition i) of Corollary 5.2.19 says that

$$D_{S'}^{[\mathcal{B}';\beta']} = D_{S'}^{[\mathcal{B}';\beta]} \circ D_{S'}^{[\mathcal{B}';\alpha]}.$$

Since $D_{S'}^{[\mathcal{B}';\alpha]}$ acts linearly on the monomials on $\mathcal{B}$, we have that $D_{S'}^{[\mathcal{B}';\alpha]}$ is S-linear. Hence, regarding S' as a polynomial ring over S on the variables $T_1, \dots, T_n$, and following the notation of Definition 2.4.3, one readily checks that, for $\alpha \in \mathbb{N}^{\mathcal{T}}$,

$$D_{S'}^{[\mathcal{B}';\alpha]} = \operatorname{Tay}^\alpha.$$

Note also that, for $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$,

$$\left(D_{S'}^{[\mathcal{B}';\beta]} \right)\Big|_S = D_S^{[\mathcal{B};\beta]}.$$

In this way, for every $f \in S$ and $\beta' = \alpha + \beta$, we have that

$$D_{S'}^{[\mathcal{B}';\beta']}(f) = \left(D_{S'}^{[\mathcal{B}';\beta]} \circ \operatorname{Tay}^\alpha \right)(f) = D_{S'}^{[\mathcal{B}';\beta]}(f) = D_S^{[\mathcal{B};\beta]}(f).$$

5.3 Absolute p-bases on regular rings

Along this section we explore some properties of those regular rings of characteristic $p > 0$ which admit an absolute p-basis (i.e., a p-basis over $\mathbb{F}_p$). The main results are Proposition 5.3.1 and Proposition 5.3.12.

Regular systems of parameters

The proof of the following result, although not explicitly stated, is contained in that of [24, Theorem].

Proposition 5.3.1. *Let $(R, \mathfrak{M}, k)$ be a regular local ring of characteristic $p > 0$ which admits an absolute p-basis. Then there exists a set $\mathcal{B}_0 \subset R$ with the following properties:*

i) The image of $\mathcal{B}_0$ in the residue field k is an absolute p-basis of k.

ii) For any regular system of parameters of R, say $x_1, \ldots, x_d$, the set

$$\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \ldots, x_d\}$$

is an absolute p-basis of R.

Proof. Fix an arbitrary p-basis of R, say $\mathcal{A}$. Then Ω^1_R is a free R-module with $\{d_R(a) \mid a \in \mathcal{A}\}$ as a basis. Recall that, by Theorem 2.1.6, there is a natural exact sequence

$$\mathfrak{M}/\mathfrak{M}^2 \longrightarrow \Omega^1_R \otimes_R k \longrightarrow \Omega^1_k \longrightarrow 0\ .$$

Hence one can find a subset $\mathcal{B}_0 \subset \mathcal{A}$ so that the image of $\{d_R(b) \mid b \in \mathcal{B}_0\}$ in Ω^1_k is a basis of Ω^1_k. We claim that $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \ldots, x_d\}$ is an absolute p-basis of R.

Note that $\Omega^1_{R/R^p[\mathcal{B}_0]}$ is a free module with a differential basis given by the set $\mathcal{A} \setminus \mathcal{B}_0$. According to [24, Lemma 5],

$$|\mathcal{A} \setminus \mathcal{B}_0| = \operatorname{rank}_k \mathfrak{M}/\mathfrak{M}^2 = d.$$

Hence $\Omega^1_{R/R^p[\mathcal{B}_0]}$ is a finite module, and therefore R is finite over $R^p[\mathcal{B}_0]$ by [16, Proposition 1, p 1160]. In addition, by [24, Lemma 4], there is a natural isomorphism of k-modules, say

$$\mathfrak{M}/\mathfrak{M}^2 \simeq \Omega^1_{R/R^p[\mathcal{B}_0]} \otimes_R k,$$

where the class of x_i in $\mathfrak{M}/\mathfrak{M}^2$ is identified with that of $d_{R/R^p[\mathcal{B}_0]}(x_i)$. By Nakayama's lemma, $\{x_1, \ldots, x_d\}$ is a differential basis of $\Omega^1_{R/R^p[\mathcal{B}_0]}$, and thus [26, §38.G, Proposition, p. 276] implies that $\{x_1, \ldots, x_d\}$ is a p-basis of R over $R^p[\mathcal{B}_0]$.

Finally, since $\mathcal{B}_0$ is a p-basis of $R^p[\mathcal{B}_0]$ over R^p and $\{x_1, \ldots, x_d\}$ is a p-basis of R over $R^p[\mathcal{B}_0]$, we conclude that $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \ldots, x_d\}$ is a p-basis of R over R^p. That is, $\mathcal{B}$ is an absolute p-basis of R. $\square$

Corollary 5.3.2. *Under the hypotheses of Proposition 5.3.1, the set $\mathcal{B}_0$ is algebraically independent over $\mathbb{F}_p$. Moreover, there is an inclusion $\mathbb{F}_p(\mathcal{B}_0) \subset R$, and this is a quasi-coefficient field of R. That is, the completion of R with respect to its maximal ideal, say $\widehat{R}$, contains a unique coefficient field which extends $\mathbb{F}_p(\mathcal{B}_0)$.*

Proof. Since the image of $\mathcal{B}_0$ in k is a p-basis of k over $\mathbb{F}_p$, and $\mathbb{F}_p$ is perfect, the set $\mathcal{B}_0$ is algebraically independent over $\mathbb{F}_p$ (see [26, Theorem 89, p. 272]). Hence $\mathbb{F}_p[\mathcal{B}_0] \cap \mathfrak{M} = \emptyset$, and therefore $\mathbb{F}_p(\mathcal{B}_0) \subset R$.

As $\mathbb{F}_p[\mathcal{B}_0] \cap \mathfrak{M} = \emptyset$, we may think that $\mathbb{F}_p(\mathcal{B}_0) \subset k$. Then k is a formally étale extension of $\mathbb{F}_p(\mathcal{B}_0)$ (see [26, §38.E, p. 273]), and hence it can be lifted to a unique coefficient field of $\widehat{R}$. □

Example 5.3.3. Here we exhibit a regular local ring of characteristic $p > 0$ which does not admit an absolute p-basis. Consider a field k of charactersitic $p > 0$ so that $[k : k^p] = \infty$. We claim that the ring $R = k[[x]]$ does not admit an absolute p-basis.

We shall prove the claim by contradiction. Assume that $\mathcal{B}$ is an absolute p-basis of R. By Theorem 5.2.12, there is a subset $\mathcal{B}_0 \subset \mathcal{B}$ so that $\mathcal{B}_0 \cup \{x\}$ is an absolute p-basis of R. Moreover, according to Corollary 5.3.2, the set $\mathcal{B}_0$ is algebraically independent over $\mathbb{F}_p$, and there is a unique coefficient field $k_0 \subset R$ which extends $\mathbb{F}_p(\mathcal{B}_0)$. Thus, by Cohen's structure theorem, $R = k_0[[x]]$.

Note that, as $[k : k^p] = \infty$, the set $\mathcal{B}_0$ is infinite. Fix an infinite countable subset $\{a_i\}_{i\in\mathbb{N}} \subset \mathcal{B}_0$ and consider the element

$$f = \sum_{i=0}^{\infty} a_i x^i \in k_0[[x]].$$

Observe that, by construction, the coefficients of f (regarded as power series in $k_0[[x]]$) generate an infinite extension of k_0^p. On the other hand, note that $R^p = k_0^p[[x^p]]$. Since $\mathcal{B}_0 \cup \{x\}$ is a p-basis of R, it should follow that

$$f \in R^p\,[a_0, \ldots, a_m, x] = (k_0^p[a_0, \ldots, a_m])\,[[x]]$$

for some $m \gg 0$, which is clearly impossible. Thus we conclude that $R = k[[x]]$ does not admit an absolute p-basis.

Remark 5.3.4. Suppose that k if a field of characteristic $p > 0$. If $[k : k^p] < \infty$ and $\mathcal{B}_0$ is an absolute p-basis of k, one can check that $\mathcal{B}_0 \cup \{x\}$ is a p-basis of $k[[x]]$. Thus, attending to Example 5.3.3 above, we see that the power series ring $k[[x]]$ admits an absolute p-basis if and only if $[k : k^p] < \infty$.

A criterion of excellence

Lemma 5.3.5. *Let S be a regular ring over $\mathbb{F}_p$ which admits an absolute p-basis. Then S satisfies the weak Jacobian condition.*

Proof. Fix a prime ideal $\mathfrak{p} \subset S$, and a regular system of parameters of $S_{\mathfrak{p}}$, say $x_1, \ldots, x_d$. We need to check that

$$\operatorname{rank} \operatorname{Jac}(\mathfrak{p}, \operatorname{Der}(S))(\mathfrak{p}) = \dim(S_{\mathfrak{p}}).$$

To this end we will use the equivalence of Lemma 5.1.6. Namely, we will show that there exist local derivatives

$$\delta_1, \ldots, \delta_d \in \operatorname{Der}(S)_{\mathfrak{p}} \subset \operatorname{Der}(S_{\mathfrak{p}}),$$

such that $\delta_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all i, j.

Since S admits a p-basis, the same holds for $S_{\mathfrak{p}}$ (see Lemma 5.2.6). Hence $x_1, \ldots, x_d$ can be extended to a p-basis of $S_{\mathfrak{p}}$ of the form $\mathcal{B}^* = \mathcal{B}_0^* \cup \{x_1, \ldots, x_d\}$ (see Proposition 5.3.1). This implies that $dx_1, \ldots, dx_d$ are part of a basis of $\Omega^1_{S_{\mathfrak{p}}}$, and therefore one can find local derivatives $\delta_1^*, \ldots, \delta_d^* \in \operatorname{Der}(S_{\mathfrak{p}})$ such that $\delta_i^*(x_j) = \delta_{ij}$ for all i, j.

Next consider an absolute p-basis of S, say $\mathcal{B}$. Using the notation of Remark 5.2.11, for each $b \in \mathcal{B}$ we have a global derivative $\frac{\partial}{\partial b} : S \to S$ satisfying $\frac{\partial}{\partial b}(c) = \delta_{bc}$ (Kronecker's delta) for all $c \in \mathcal{B}$. By localization, the set $\mathcal{B}$ is also an absolute p-basis of $S_{\mathfrak{p}}$, and each $\frac{\partial}{\partial b}$ induces a local derivative on $S_{\mathfrak{p}}$. Abusing our notation, we shall denote this derivative by $\frac{\partial}{\partial b} : S_{\mathfrak{p}} \to S_{\mathfrak{p}}$. According to Remark 5.2.11, one can find a finite subset of $\mathcal{B}$, say $\{b_1, \ldots, b_s\}$, such that

$$\delta_i^*(x_j) = \sum_{k=1}^{s} \delta_i^*(b_k) \frac{\partial}{\partial b_k}(x_j)$$

for all i, j. Then, for $i = 1, \ldots, d$, set

$$\delta_i = \sum_{k=1}^{s} \delta_i^*(b_k) \frac{\partial}{\partial b_k} \in \operatorname{Der}(S_{\mathfrak{p}}).$$

Since $\frac{\partial}{\partial b_1}, \ldots, \frac{\partial}{\partial b_s}$ can be regarded as global derivatives on S and $\delta_1, \ldots, \delta_d$ are defined as finite combinations of these derivatives with coefficients in $S_{\mathfrak{p}}$, we have that $\delta_1, \ldots, \delta_d \in \operatorname{Der}(S)_{\mathfrak{p}}$. Moreover, by construction, $\delta_i(x_j) = \delta_{ij}$ for all i, j. Thus we conclude that the weak Jacobian condition holds on S. □

Proposition 5.3.6. *Let S be a regular ring over $\mathbb{F}_p$ which admits an absolute p-basis. Then S is excellent.*

Proof. By Lemma 5.2.8, every polynomial ring over S, say $S[T_1, \ldots, T_n]$, admits an absolute p-basis. Hence $S[T_1, \ldots, T_n]$ satisfies the weak Jacobian condition by Lemma 5.3.5, and, in this way, Theorem 5.1.10 implies that S is excellent. □

Regular embeddings and p-bases

Consider a regular ring S' of characteristic $p > 0$ presented as the quotient of another regular ring S, say $S' = S/J$. The aim of the forthcoming discussion is to determine whether S' admits an absolute p-basis provided that S does. This issue has particular interest when S is a polynomial ring over an arbitrary field k, say $S = k[T_1, \dots, T_n]$, and $\operatorname{Spec}(S')$ defines a regular variety embedded in $\mathbb{A}^n_k$ (see Proposition 5.3.12). We ignore whether the results presented below were previously known.

Lemma 5.3.7. *Let $(R, \mathfrak{M}, k)$ be a regular local ring of characteristic $p > 0$. Suppose that R has an absolute p-basis of the form $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \dots, x_d\}$, where $x_1, \dots, x_d$ is a regular system of parameters of R (cf. Proposition 5.3.1). Fix $e \in \{1, \dots, d\}$, and set $J = \langle x_1, \dots, x_e \rangle$ and $R' = R/J$. Then the image of $\mathcal{B}_0 \cup \{x_{e+1}, \dots, x_d\}$ in R', which we shall denote by $\mathcal{B}'$, is an absolute p-basis of R'.*

Proof. By Theorem 2.1.6, there is a natural exact sequence

$$J/J^2 \xrightarrow{\overline{d_R}} \Omega^1_R \otimes_R R' \longrightarrow \Omega^1_{R'} \longrightarrow 0.$$

Since Ω^1_R is a free module with the set $\mathcal{B}$ as a differential basis and the image of J/J^2 in $\Omega^1_R \otimes_R R'$ is the submodule spanned by $\overline{d_R(x_1)}, \dots, \overline{d_R(x_e)}$, it follows that $\Omega^1_{R'}$ is a free module with $\mathcal{B}'$ as a differential basis. Hence $\mathcal{B}'$ is an absolute p-basis of R' by Theorem 5.2.12. $\square$

Remark 5.3.8. Recall that, according to Proposition 5.2.17, there is a collection of differential operators on R attached to the p-basis $\mathcal{B}$, say $\{D_R^{[\mathcal{B};\beta']}\}$, which behave like the Taylor operators on a polynomial ring. Similarly, there is a collection of differential operators on R' attached to the p-basis $\mathcal{B}'$, say $\{D_{R'}^{[\mathcal{B}';\beta']}\}$. Next we show that there is a natural compatibility between these two families. This compatibility will be used in some of the the proofs of Chapter 6.

Under the same hypotheses of Lemma 5.3.7, consider an index $\beta' \in \mathbb{N}^{\oplus \mathcal{B}'}$. Note that we have a natural isomorphism $\mathbb{N}^{\oplus \mathcal{B}} \simeq \mathbb{N}^{\oplus \mathcal{B}'} \oplus \mathbb{N}^e$, where each coordinate of $\mathbb{N}^e$ corresponds to one of the elements $x_1, \dots, x_e$. Thus β' can also be regarded as an element of $\mathbb{N}^{\oplus \mathcal{B}}$. Let $\pi : R \to R'$ denote the natural quotient map. Then we claim that the following diagram is commutative:

$$\begin{tikzcd} R \arrow[r, "D_R^{[\mathcal{B};\beta']}"] \arrow[d, "\pi"] & R \arrow[d, "\pi"] \\ R' \arrow[r, "D_{R'}^{[\mathcal{B}';\beta']}"] & R'. \end{tikzcd}$$

In order to check the claim, set $\Delta = \pi \circ D_R^{[\mathcal{B};\beta']}$ and $\Delta' = D_{R'}^{[\mathcal{B}';\beta']} \circ \pi$. Both Δ and Δ' are absolute differential operators of order $|\beta'|$ from R to R'. For a given $\alpha \in \mathbb{N}^{\oplus \mathcal{B}}$, put $\alpha = \alpha_1 \oplus \alpha_2$, with $\alpha_1 \in \mathbb{N}^{\oplus \mathcal{B}'}$ and $\alpha_2 \in \mathbb{N}^e$. Note that, if $\alpha_2 \neq 0$, then $\mathcal{B}^\alpha \in J$, and one readily checks that

$$\Delta(\mathcal{B}^\alpha) = 0 = \Delta'(\mathcal{B}^\alpha).$$

Otherwise, if $\alpha \in \mathbb{N}^{\oplus \mathcal{B}'}$, we have that

$$\Delta(\mathcal{B}^{\alpha}) = \binom{\alpha}{\beta'} \pi\left(\mathcal{B}^{\alpha-\beta'}\right) = \binom{\alpha}{\beta'} \left(\mathcal{B}'\right)^{\alpha-\beta'} = \Delta'(\mathcal{B}^{\alpha}).$$

In this way, Corollary 5.2.19 iii) implies that $\Delta = \Delta'$, which proves the commutativity of the diagram.

Proposition 5.3.9. *Let R be a regular local ring of characteristic $p > 0$ that has an absolute p-basis. Then every regular quotient of R, say $R' = R/J$, admits an absolute p-basis.*

Proof. Since R and R' are regular, one can find a regular system of parameters of R, say $x_1, \ldots, x_d$, so that $J = \langle x_1, \ldots, x_e \rangle$. By Proposition 5.3.1, these elements can be extended to a p-basis of R over $\mathbb{F}_p$, say $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \ldots, x_d\}$. Then the result follows from Lemma 5.3.7. $\square$

Lemma 5.3.10. *Let S be a regular ring which admits a p-basis over $\mathbb{F}_p$, and let $S' = S/J$ be a regular quotient of S. Then, for each prime ideal $\mathfrak{q} \subset S'$, there exists an element $g \in S' \setminus \mathfrak{q}$ so that S'_g admits an absolute p-basis.*

Proof. According to Theorem 5.2.12, it suffices to find an element $g \in S' \setminus \mathfrak{q}$ so that $\Omega^1_{S'_g} = \Omega^1_{S'} \otimes_{S'} S'_g$ has a differential basis.

Fix an absolute p-basis of S, say $\mathcal{B}$, and consider the exact sequence

$$J/J^2 \longrightarrow \Omega^1_S \otimes_S S' \longrightarrow \Omega^1_{S'} \longrightarrow 0.$$

Suppose that $J = \langle f_1, \ldots, f_r \rangle$. Using the notation of Remark 5.2.11, take

$$\mathcal{B}_0 = \left\{ b \in \mathcal{B} \mid \frac{\partial}{\partial b}(f_1) = \cdots = \frac{\partial}{\partial b}(f_r) = 0 \right\}.$$

Let $\mathcal{B}'_0$ denote the image of $\mathcal{B}_0$ in S'. Note that, as $d_S(\mathcal{B})$ forms a basis of Ω^1_S, we have that $|\mathcal{B} \setminus \mathcal{B}_0| < \infty$. Thus $\Omega^1_{S'}$ can be expressed as a direct sum, say $\Omega^1_{S'} = M_0 \oplus M_1$, where M_0 is the free submodule generated by $d_{S'}(\mathcal{B}'_0)$, and M_1 is a finite S'-module.

Consider the subring $A = S^p[\mathcal{B}_0] \subset S$, and its image in S', say $A' = (S')^p[\mathcal{B}'_0]$. From the previous discussion it follows that each derivative from A' into S' over $(S')^p$ can be extended (possibly non-uniquely) to an absolute derivative of S'. In this way we have a split short exact sequence (see Theorem 2.1.4)

$$0 \longrightarrow \Omega^1_{A'/(S')^p} \otimes_{A'} S' \xrightarrow{\iota} \Omega^1_{S'} \longrightarrow \Omega^1_{S'/A'} \longrightarrow 0, \tag{5.3}$$

where the image of ι coincides with the submodule M_0. Hence $\Omega^1_{S'/A'}$ is a finite S'-module. Next, for a given left inverse of ι, we get an isomorphism

$$\Omega^1_{S'} \simeq \Omega^1_{S'/A'} \oplus \left(\Omega^1_{A'/(S')^p \otimes_{A'} S'}\right),$$

and localizing at $\mathfrak{q}$ we obtain

$$\Omega^1_{S'_{\mathfrak{q}}} \simeq \Omega^1_{S'_{\mathfrak{q}}/A'} \oplus \left(\Omega^1_{A'/(S')^p \otimes_{A'} S'_{\mathfrak{q}}}\right).$$

Since $S'_{\mathfrak{q}}$ is a regular quotient of a regular local ring that has an absolute p-basis, $S'_{\mathfrak{q}}$ admits an absolute p-basis by Proposition 5.3.9. This implies that $\Omega^1_{S'_{\mathfrak{q}}}$ is a free module. Therefore $\Omega^1_{S'_{\mathfrak{q}}/A'}$ is projective, and, as it is a finite module over a local ring, it should be free (see [4, Lemma III.5.8, p. 60]). Then one can find elements $b'_1, \dots, b'_s \in S'_{\mathfrak{q}}$ so that $d_{S'_{\mathfrak{q}}/A'}(b'_1), \dots, d_{S'_{\mathfrak{q}}/A'}(b'_s)$ form a basis of $\Omega^1_{S'_{\mathfrak{q}}/A'}$ (see also [32, Corollary of Theorem 1]).

Next, choose $g \in S' \setminus \mathfrak{q}$ such that $b'_1, \dots, b'_s \in S'_g$, and the module $\Omega^1_{S'_g/A'} = \Omega^1_{S'/A'} \otimes_{S'} S'_g$ is free with $d_{S'_g/A'}(b'_1), \dots, d_{S'_g/A'}(b'_s)$ as a basis. Localizing (5.3) at g we get a short exact sequence

$$0 \longrightarrow \Omega^1_{A'/(S')^p} \otimes_{A'} S'_g \longrightarrow \Omega^1_{S'_g} \longrightarrow \Omega^1_{S'_g/A'} \longrightarrow 0.$$

This sequence is split. Thus it is not hard to check that the set $\mathcal{B}' = \mathcal{B}'_0 \cup \{b'_1, \dots, b'_s\}$ is a differential basis of $\Omega^1_{S'_g}$. Hence, by Theorem 5.2.12, $\mathcal{B}'$ is an absolute p-basis of S'_g. $\square$

Lemma 5.3.11. *Let $V' \to V$ be a finite type morphism of regular noetherian schemes of characteristic $p > 0$. Assume that V can be covered by open affine charts of the form* $\mathrm{Spec}(S)$ *so that S admits an absolute p-basis. Then V' can also be covered by open affine charts of the form* $\mathrm{Spec}(S')$ *so that S' admits an absolute p-basis.*

Proof. Observe that V' can be covered by affine charts of the form

$$\mathrm{Spec}(S[T_1, \dots, T_r]/J),$$

where $\mathrm{Spec}(S)$ is an affine chart of V, and J is an ideal of the polynomial ring $S[T_1, \dots, T_r]$. Then the result follows from Lemma 5.2.8 and the previous lemma. $\square$

Proposition 5.3.12. *Let V be a regular variety over an arbitrary field k of characteristic $p > 0$. For each $\xi \in V$, there exists an open affine neighborhood of V at ξ, say* $\mathrm{Spec}(S) \subset V$*, so that S admits an absolute p-basis.*

Remark 5.3.13. The author ignores whether this result was previously known.

Proof. By assumption V is endowed with a finite type morphism $V \to \mathrm{Spec}(k)$. Since every field admits an absolute p-basis, the claim follows from the previous lemma. $\square$

5.4 Order of elements and differentials

Let S be regular ring. For an element $f \in S$ and a prime ideal $\mathfrak{p} \subset S$, the order of f at $\mathfrak{p}$ is defined by

$$\nu_{\mathfrak{p}}(f) = \max\{n \in \mathbb{N} \mid f \in \mathfrak{p}^n S_{\mathfrak{p}}\}.$$

In addition, for $n \geq 0$, we define the ideal $\mathrm{Diff}^{N-1}(S)(f)$ by

$$\mathrm{Diff}^n(S)(f) = \langle \Delta(f) \mid \Delta \in \mathrm{Diff}^n(S) \rangle,$$

where $\mathrm{Diff}^n(S)$ represents the module of absolute differential operators of of order at most n of S. Note that, by Lemma 2.2.8,

$$\nu_{\mathfrak{p}}(f) \geq N \implies \mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p},$$

However, the converse is false in general.

Lemma 5.4.1 ([17, Ch. III, Lemma 1.2.7])**.** *Let S be a polynomial ring in a finite number of variables over a field k, say $S = k[T_1, \dots, T_r]$. Then, for each element $f \in S$ and each prime $\mathfrak{p} \subset S$, the following conditions are equivalent:*

i) $\nu_{\mathfrak{p}}(f) \geq N$.

ii) $\mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p}$.

The aim of this section is to extend the criterion above to a wider class of rings using the conditions introduced in the previous sections.

The case of characteristic zero

Next we extend Lemma 5.4.1 to the case in which S is a regular ring defined over a field of characteristic zero which satisfies the weak Jacobian condition.

Lemma 5.4.2. *Let $(R, \mathfrak{M}, k)$ be a regular local ring over a field of characteristic zero, and let $\mathcal{D} \subset \mathrm{Der}(R)$ be a submodule of derivatives satisfying*

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{M}; \mathcal{D})(\mathfrak{M}) = \dim(R).$$

Then, for each $f \in \mathfrak{M}$, there exists a derivation $\delta \in \mathcal{D}$ such that

$$\nu_{\mathfrak{M}}(\delta(f)) = \nu_{\mathfrak{M}}(f) - 1.$$

Proof. Fix a regular system of parameters of R, say $x_1, \dots, x_d$, and choose derivatives $\delta_1, \dots, \delta_d \in \mathrm{Der}(R)$ as in Lemma 5.1.5 so that $\delta_i(x_j) = \delta_{ij}$ (Kronecker's delta) for $1 \leq i, j \leq d$. Since R is regular, its associated graded ring is isomorphic to a polynomial ring over k, say

$$\mathrm{Gr}_{\mathfrak{M}}(R) = \bigoplus_{n \in \mathbb{N}} \mathfrak{M}^n / \mathfrak{M}^{n+1} \simeq k[X_1, \dots, X_d].$$

Here $X_1, \dots, X_d$ represent variables and each X_i corresponds to the initial part of x_i in $\mathrm{Gr}_{\mathfrak{M}}(R)$ respectively. We claim that each δ_i induces a k-linear map on $\mathrm{Gr}_{\mathfrak{M}}(R)$, say

$$\hat{\delta}_i : \mathrm{Gr}_{\mathfrak{M}}(R) \longrightarrow \mathrm{Gr}_{\mathfrak{M}}(R),$$

as follows: for a homogeneous element $G \in \mathrm{Gr}_{\mathfrak{M}}(R)$ of degree $n > 0$, choose $g \in \mathfrak{M}^n$ so that $\mathrm{In}(g) = G$, and define $\hat{\delta}_i(G)$ as the class of $\delta_i(g)$ in $\mathfrak{M}^{n-1}/\mathfrak{M}^n$; for $u \in k$, set $\hat{\delta}_i(u) = 0$. Note that $\hat{\delta}_i$ is well-defined because $\delta_i(\mathfrak{M}^{n+1}) \subset \mathfrak{M}^n$ for $n > 0$. To check that $\hat{\delta}_i$ is k-linear, fix an element $u \in k$ and a homogeneous polynomial $G \in \mathrm{Gr}_{\mathfrak{M}}(R)$ of degree $n > 0$. Then choose a representative of u in R, say a (note that a is unit in R, unless $u = 0$), and an element $g \in \mathfrak{M}^n$ so that $G = \mathrm{In}(g)$. By the Leibniz's rule,

$$\delta(af) \equiv a\delta(f) \text{ modulo } \mathfrak{M}^n.$$

This implies that $\hat{\delta}(uG) = u\hat{\delta}(G)$, and hence $\hat{\delta}_i$ is k-linear. Finally observe that $\hat{\delta}$ inherits the Leibniz's rule from δ. Thus we conclude that $\hat{\delta}$ is a derivation with respect to k.

Once we know that $\hat{\delta}_i \in \mathrm{Der}_k(k[X_1, \dots, X_d])$, it follows immediately that $\hat{\delta}_i = \frac{\partial}{\partial X_i}$, since $\hat{\delta}_i(X_j) = \delta_{ij}$ (Kronecker's delta) for each $j = 1, \dots, d$.

In order to prove the lemma, suppose that f has order N and set $F = \mathrm{In}(f) \in \mathfrak{M}^N/\mathfrak{M}^{N+1}$. Note that F can be regarded as a homogeneous polynomial of degree N in the variables $X_1, \dots, X_d$. Then, since k has characteristic zero, one can always find a partial derivative $\frac{\partial}{\partial X_i}$ so that $\frac{\partial}{\partial X_i}(F)$ is homogeneous of degree $N-1$, i.e., so that

$$0 \neq \frac{\partial}{\partial X_i}(F) = \hat{\delta}_i(F) \in \mathfrak{M}^{N-1}/\mathfrak{M}^N.$$

Hence, by the definition of $\hat{\delta}_i$, it follows that $\delta_i(f) \in \mathfrak{M}^{N-1}$ and $\delta_i(f) \notin \mathfrak{M}^N$. That is, $\nu_{\mathfrak{M}}(\delta_i(f)) = N - 1 = \nu_{\mathfrak{M}}(f) - 1$. $\square$

Proposition 5.4.3. *Let S be a regular ring over a field of characteristic zero, and let $\mathcal{D} \subset \mathrm{Der}(S)$ be a submodule of derivatives satisfying the weak Jacobian condition on S. Then for any prime $\mathfrak{p} \subset S$ and any element $f \in \mathfrak{p}$ there exists a derivation $\delta \in \mathcal{D}$ such that*

$$\nu_{\mathfrak{p}}(\delta(f)) = \nu_{\mathfrak{p}}(f) - 1.$$

Proof. Set $N = \nu_{\mathfrak{p}}(f)$. According to the previous lemma, there exists a derivative $\delta^* \in \mathcal{D}_{\mathfrak{p}}$ such that $\nu_{\mathfrak{p}S_{\mathfrak{p}}}(\delta^*(f)) = N - 1$. Note that

$$\delta^* = a_1\delta_1 + \cdots + a_r\delta_r$$

for some $\delta_1, \dots, \delta_r \in \mathcal{D}$, and $a_1, \dots, a_r \in S_{\mathfrak{p}}$. If $\nu_{\mathfrak{p}}(\delta_i(f)) \geq N$, i.e., if $\delta_i(f) \in \mathfrak{p}^N S_{\mathfrak{p}}$ for all i, we would have

$$\delta^*(f) = a_1\delta_1(f) + \cdots + a_r\delta_r(f) \in \mathfrak{p}^N S_{\mathfrak{p}},$$

which is a contradiction. Hence $\nu_{\mathfrak{p}}(\delta_i(f)) = N - 1$ for some $i \in \{1, \dots, r\}$. $\square$

Corollary 5.4.4. *Let S be a regular ring over a field of characteristic zero satisfying the weak Jacobian condition. For any element $f \in S$ and any prime ideal $\mathfrak{p} \subset S$ the following conditions are equivalent:*

i) $\nu_{\mathfrak{p}}(f) \geq N$.

ii) $\mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p}$.

Proof. i) $\Rightarrow$ ii) follows from Lemma 2.2.8, so we just need to prove the converse. We proceed by contradiction: assume that $\mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p}$ and $\nu_{\mathfrak{p}}(f) = n$ with $n < N$. Then, by the previous lemma, there exist derivatives $\delta_1, \ldots, \delta_n \in \mathrm{Der}(S)$ so that

$$\nu_{\mathfrak{p}}\big((\delta_n \circ \cdots \circ \delta_1)(f)\big) = 0,$$

i.e., so that $(\delta_n \circ \cdots \circ \delta_1)(f) \notin \mathfrak{p}$. Recall that a composition of n derivatives is a differential operator of order n. Since $n \leq N-1$, we have

$$(\delta_n \circ \cdots \circ \delta_1) \in \mathrm{Diff}^n(S) \subset \mathrm{Diff}^{N-1}(S),$$

and therefore $\mathrm{Diff}^{N-1}(S)(f) \not\subset \mathfrak{p}$. □

Proposition 5.4.3 has two immediate consequences concerning the stratification defined by the multiplicity on a hypersurface and the singular locus of a Rees algebra.

Corollary 5.4.5. *Let S be a regular ring over a field of characteristic zero satisfying the weak Jacobian condition. Set $V = \mathrm{Spec}(S)$. For any hypersurface*

$$H = \mathrm{Spec}(S/\langle f \rangle) \subset V,$$

and any integer $n \geq 1$, we have that

$$\{\xi \in H \mid \mathrm{mult}_H(\xi) \geq n\} = \mathbb{V}\left(\mathrm{Diff}^{n-1}(S)(f)\right).$$

Proof. Since $\mathrm{mult}_H(\xi) = \nu_\xi(f)$ for each $\xi \in H$ (see Lemma A.0.14), the result follows from Corollary 5.4.4. □

Corollary 5.4.6. *Let S be a regular ring over a field of characteristic zero satisfying the weak Jacobian condition. Set $V = \mathrm{Spec}(S)$. For any S-Rees algebra*

$$\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right],$$

we have that

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \mathbb{V}\left(\mathrm{Diff}^{N_i - 1}(S)(f_i)\right).$$

The case of positive characteristic

The following result extends Lemma 5.4.1 to the case in which S is a regular ring of characteristic $p > 0$ that admits an absolute p-basis.

Proposition 5.4.7. *Let S be a regular ring over a field of characteristic $p > 0$ which has an absolute p-basis. Then, for any element $f \in S$ and any prime ideal $\mathfrak{p} \subset S$ the following conditions are equivalent:*

i) $\nu_{\mathfrak{p}}(f) \geq N$.

ii) $\mathrm{Diff}^{N-1}(S)(f) \subset \mathfrak{p}$.

Proof. ii) $\Rightarrow$ i) follows Lemma 2.2.8. For the converse we proceed by contradiction. Namely, assume that ii) holds and that $\nu_{\mathfrak{p}}(f) = n$ with $n < N$.

Let $\widehat{S_{\mathfrak{p}}}$ denote the completion of $S_{\mathfrak{p}}$ with respect to its maximal ideal. Observe that f has the same order regarded as an element of $S_{\mathfrak{p}}$, or as an element of $\widehat{S_{\mathfrak{p}}}$. In addition, given a regular system of parameters of $S_{\mathfrak{p}}$, say $x_1, \ldots, x_d$, Cohen's structure theorem [12, Theorem 9, p. 72] implies that $\widehat{S_{\mathfrak{p}}}$ is isomorphic to the power series ring $k(\mathfrak{p})[[x_1, \ldots, x_d]]$, where $k(\mathfrak{p})$ represents the residue field of $S_{\mathfrak{p}}$. Since f has order n, there should a differential operator of order n over $k(\mathfrak{p})$, say $\Delta : \widehat{S_{\mathfrak{p}}} \to \widehat{S_{\mathfrak{p}}}$, so that $\Delta(f) \in k(\mathfrak{p})$. In particular, $\Delta(f) \notin \mathfrak{p}\widehat{S_{\mathfrak{p}}}$.

Next, fix an absolute p-basis of S, say $\mathcal{B}$, and consider the natural morphism from S into $\widehat{S_{\mathfrak{p}}}$, say $\iota : S \to \widehat{S_{\mathfrak{p}}}$. By composition, $(\Delta \circ \iota)$ is a differential operator of S into $\widehat{S_{\mathfrak{p}}}$ of order n over $\mathbb{F}_p$. Thus, by Corollary 5.2.19 ii) and iii), there exists a finite number of indexes, put $\beta_1, \ldots, \beta_r \in \mathbb{N}^{\oplus \mathcal{B}}$, with $|\beta_i| \leq n$, so that

$$(\Delta \circ \iota)(f) = \sum_{i=1}^{r} \Delta(\mathcal{B}^{\beta_i}) D^{[\mathcal{B};\beta_i]}(f).$$

Since $\Delta(f) \notin \mathfrak{p}\widehat{S_{\mathfrak{p}}}$, it follows that $D^{[\mathcal{B};\beta_i]}(f) \notin \mathfrak{p}$ for some $i \in \{1, \ldots, r\}$. Therefore $\mathrm{Diff}^{N-1}(S)(f) \not\subset \mathfrak{p}$. □

Corollary 5.4.8. *Let S be a regular ring over a field of characteristic $p > 0$ which has an absolute p-basis. Set $V = \mathrm{Spec}(S)$. Then, for any hypersurface*

$$H = \mathrm{Spec}(S/\langle f \rangle) \subset V,$$

and any integer $n \geq 1$,

$$\{\xi \in H \mid \mathrm{mult}_H(\xi) \geq n\} = \mathbb{V}\left(\mathrm{Diff}^{n-1}(S)(f)\right).$$

Proof. As $\mathrm{mult}_H(\xi) = \nu_\xi(f)$ for $\xi \in H$ (see Proposition A.0.14), the result is a consequence of Proposition 5.4.7. □

Corollary 5.4.9. *Let S be a regular ring over a field of characteristic $p > 0$ which has an absolute p-basis. Set $V = \mathrm{Spec}(S)$. Then, for any S-Rees algebra*

$$\mathcal{G} = S\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right],$$

we have that

$$\mathrm{Sing}_V(\mathcal{G}) = \bigcap_{i=1}^{r} \mathbb{V}\left(\mathrm{Diff}^{N_i-1}(S)(f_i)\right).$$

Chapter 6

Canonical representatives of Rees algebras

Let V be a regular noetherian scheme and consider a Rees algebra $\mathcal{G}$ over V. In this section we give methods that, under suitable conditions on V, enable us to find the canonical representative of the class of $\mathcal{G}$, say $\mathscr{C}_V(\mathcal{G})$ (see Section 3.4). Recall that, for us, the canonical representative of $\mathscr{C}_V(\mathcal{G})$ is a Rees algebra $\mathcal{G}^* \in \mathscr{C}_V(\mathcal{G})$ so that $\mathcal{G}' \subset \mathcal{G}^*$ for all $\mathcal{G}' \in \mathscr{C}_V(\mathcal{G})$.

Remark 6.0.1. The canonical representative is well-defined whenever $\mathscr{C}_V(\mathcal{G})$ has a maximum with respect to the inclusion. However, given an arbitrary Rees algebra $\mathcal{G}$ over V, it is not clear whether such a maximum exists. More precisely, one can check that

$$\varinjlim_{\mathcal{H} \in \mathscr{C}_V(\mathcal{G})} \mathcal{H}$$

is a graded algebra, but it might not be finitely generated over $\mathcal{O}_V$.

The existence of canonical representatives has already been proved for the case in which V is a regular variety defined over a perfect field.

Theorem 6.0.2 (cf. [9, Theorem 3.11])**.** *Let V be a regular variety over a perfect field k and let $\mathcal{G}$ be a Rees algebra over V. Then $\overline{\mathscr{D}iff_k(\mathcal{G})}$ is the canonical representative of $\mathscr{C}_V(\mathcal{G})$.*

Here we try to generalize the previous result to a wider class of schemes. Namely, we will extend the theorem to the following cases:

- When V is a regular scheme defined over a field of characteristic zero that satisfies the weak Jacobian condition (see Theorem 6.4.3 and Theorem 6.4.6).

- When V is a regular scheme over a field of characteristic $p > 0$ that can be covered by open affine charts of the form $\mathrm{Spec}(S)$, where S admits an absolute p-basis (see Theorem 6.6.7 and Theorem 6.6.8).

Although the proofs presented in this chapter are inspired on that of [9, Theorem 3.11], there are important differences between them. In particular, many of the techniques used in [9], which are suitable for working with varieties defined over a perfect field, fall short when dealing with more general schemes, and thus they should be replaced by new arguments.

6.1 Geometrical conditions

Turning to the general case, consider a regular excellent ring S and set $V = \mathrm{Spec}(S)$. Given two Rees algebras $\mathcal{G}$ and $\mathcal{K}$ over S, one has that

$$\mathcal{K} \subset \mathcal{G} \implies \mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K}).$$

However, the converse is false in general. The following lemma provides a partial result towards this direction.

Lemma 6.1.1. *Let S be a regular excellent ring, set $V = \mathrm{Spec}(S)$, and let $\mathcal{G}$ and $\mathcal{K}$ be two Rees algebras over S with the following properties:*

a) $\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Zeros}_V(\mathcal{G})$.

b) $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$.

In addition, suppose that these properties are stable under finite type morphisms of regular schemes. Namely, assume that for any morphism of finite type from a regular scheme Z to V, say $\varphi : Z \to V$, the following conditions hold:

a*) $\mathrm{Sing}_Z(\varphi^*(\mathcal{G})) = \mathrm{Zeros}_Z(\varphi^*(\mathcal{G}))$.

b*) $\mathscr{F}_Z(\varphi^*(\mathcal{G})) \subset \mathscr{F}_Z(\varphi^*(\mathcal{K}))$.

Then $\mathcal{K} \subset \overline{\mathcal{G}}$.

Remark 6.1.2. In the following sections we will introduce strong conditions on $\mathcal{G}$ in such a way that, whenever a) and b) hold for some algebra $\mathcal{K}$, then a*) and b*) hold as well. Moreover, when this occurs, $\overline{\mathcal{G}}$ will be the canonical representative of the class of $\mathcal{G}$.

The proof of Lemma 6.1.1 presented below is based on the ideas of [9, §7].

Proof. Set $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$ and $\mathcal{K} = \bigoplus_{n\in\mathbb{N}} J_n W^n$. Note that, for a suitable choice of $N > 0$, the Rees algebras $\mathcal{G}$ and $\mathcal{K}$ are integral over $S[I_N W^N]$ and $S[J_N W^N]$ respectively. Thus we may assume without loss of generality that $\mathcal{G} = S[I_N W^N]$ and $\mathcal{K} = S[J_N W^N]$. In this way,

$$\mathcal{K} \subset \overline{\mathcal{G}} \iff J_N \subset \overline{I_N}.$$

As the inclusion on the right hand side is an inclusion of ideals, it can be tested on the normalized blow-up of S along I_N, say $X = \overline{\mathrm{Bl}_{I_N}(S)}$. More precisely, $J_N \subset \overline{I_N}$ if and only if $J_N\mathcal{O}_X \subset I_N\mathcal{O}_X$. In addition, since $I_N\mathcal{O}_X$ is a principal ideal and X is normal, the latter will occur if and only if $J_N\mathcal{O}_{X,\eta} \subset I_N\mathcal{O}_{X,\eta}$ for each codimension 1 point $\eta \in X$ (see [31, Ch. III, Proposition 9. p. III-13]). Summarizing, in order to prove the lemma it suffices to show that $J_N\mathcal{O}_{X,\eta} \subset I_N\mathcal{O}_{X,\eta}$ for each codimension 1 point $\eta \in X$.

Fix a point η of codimension 1 in X. Since X is normal, it is regular in codimension 1. Hence $\eta \in \mathrm{Reg}(X)$. Moreover, Proposition B.0.8 and Proposition B.0.12 imply that X is an excellent scheme of finite type over V. Therefore $\mathrm{Reg}(X)$ is open in X by Proposition B.0.11.

In virtue of the previous observations, consider a regular open subscheme $Z \subset X$ so that $\eta \in Z$. Note that Z can also be regarded as a regular scheme endowed with a natural morphism of finite type to V, say $\varphi : Z \to V$. Under these hypotheses, conditions a*) and b*) say that

$$\mathrm{Sing}_Z\big(\varphi^*(\mathcal{G})\big) = \mathrm{Zeros}_Z\big(\varphi^*(\mathcal{G})\big)$$

and

$$\mathscr{F}_Z\big(\varphi^*(\mathcal{G})\big) \subset \mathscr{F}_Z\big(\varphi^*(\mathcal{K})\big).$$

In addition, with this notation,

$$\varphi^*(\mathcal{G})_\eta = \mathcal{O}_{Z,\eta}\big[(I_N\mathcal{O}_{Z,\eta})W^N\big]$$

and

$$\varphi^*(\mathcal{K})_\eta = \mathcal{O}_{Z,\eta}\big[(J_N\mathcal{O}_{Z,\eta})W^N\big].$$

Thus the inclusion $J_N\mathcal{O}_{Z,\eta} \subset I_N\mathcal{O}_{Z,\eta}$ follows from Lemma 6.1.3 below. $\square$

Lemma 6.1.3 (cf. [9, Proposition 5.3])**.** *Let $(R, \mathfrak{M})$ be a regular local ring of dimension 1 and consider two Rees algebras over R of the form $\mathcal{K} = R\left[IW^N\right]$ and $\mathcal{G} = R\left[JW^N\right]$, where I and J denote two non-zero ideals of R. Set $U = \mathrm{Spec}(R)$ and assume that $\mathrm{Sing}_U(\mathcal{G}) = \mathrm{Zeros}_U(\mathcal{G})$. Then*

$$\mathscr{F}_U(\mathcal{G}) \subset \mathscr{F}_U(\mathcal{K}) \iff I \subset J.$$

Proof. If $I \subset J$, then it is clear that $\mathscr{F}_U(\mathcal{G}) \subset \mathscr{F}_U(\mathcal{K})$. Conversely, suppose that $\mathscr{F}_U(\mathcal{G}) \subset \mathscr{F}_U(\mathcal{K})$. Since R is a regular noetherian local ring of dimension 1, it is a discrete valuation ring. Then it follows that either $J = R$, or J is a power of $\mathfrak{M}$, say $J = \mathfrak{M}^n$. If $J = R$, the claim is trivial. Otherwise, if $J = \mathfrak{M}^n$, the condition $\mathrm{Sing}_U(\mathcal{G}) = \mathrm{Zeros}_U(\mathcal{G})$ yields $n > N$ and, in this case, the result follows from [9, Proposition 5.3]. $\square$

Equivalence between conditions a) and a*)

Note that, in principle, condition a*) of Lemma 6.1.1 is stronger than condition a). The following result shows that, in fact, condition a) implies a*). Thus both conditions turn out to be equivalent.

Lemma 6.1.4. *Let V be a regular noetherian scheme and let $\mathcal{G}$ be an $\mathcal{O}_V$-Rees algebra satisfying*

$$\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Zeros}_V(\mathcal{G})$$

Then, for any morphism of finite type from a regular scheme Z to V, say $\varphi : Z \to V$, we have that

$$\mathrm{Sing}_Z\big(\varphi^*(\mathcal{G})\big) = \mathrm{Zeros}_Z\big(\varphi^*(\mathcal{G})\big).$$

Proof. By the local nature of the claim, we may assume that both V and Z are affine, say $V = \mathrm{Spec}(S)$ and $Z = \mathrm{Spec}(B)$, where φ is given by a ring homomorphism $S \to B$. Since B is of finite type over S, it can be regarded as the quotient of a polynomial ring over S, say $B = S[T_1, \dots, T_m]/J$. Thus, setting $V' = S[T_1, \dots, T_m]$, we obtain a commutative diagram as follows:

$$\begin{tikzcd} & & V' \arrow[dl, "\beta"'] & \mathcal{G}' = \beta^*(\mathcal{G}) \\ \mathcal{G} & V & Z \arrow[l, "\varphi"'] \arrow[u, hook] & \varphi^*(\mathcal{G}) = \mathcal{G}'|_Z. \end{tikzcd}$$

Since β is a smooth morphism and $\mathcal{G}' = \beta^*(\mathcal{G})$, one readily checks that

$$\mathrm{Sing}_{V'}(\mathcal{G}') = \varphi^{-1}\big(\mathrm{Sing}_V(\mathcal{G})\big) = \varphi^{-1}\big(\mathrm{Zeros}_V(\mathcal{G})\big) = \mathrm{Zeros}_{V'}(\mathcal{G}').$$

Next, fix a set of generators of $\mathcal{G}$ over S, say $\mathcal{G} = S\big[f_1 W^{N_1}, \dots, f_r W^{N_r}\big]$, and let $\overline{f}_1, \dots, \overline{f}_r$ denote the images of $f_1, \dots, f_r$ in B respectively. Observe that $\varphi^*(\mathcal{G}) = B\big[\overline{f}_1 W^{N_1}, \dots, \overline{f}_r W^{N_r}\big]$. Then

$$\begin{aligned} \mathrm{Sing}_Z\big(\varphi^*(\mathcal{G})\big) \subset \mathrm{Zeros}_Z\big(\varphi^*(\mathcal{G})\big) &= \mathbb{V}_Z\big(\langle \overline{f}_1, \dots, \overline{f}_r \rangle\big) \\ &= \mathbb{V}_{V'}\big(\langle f_1, \dots, f_r \rangle\big) \cap Z \\ &= \mathrm{Zeros}_{V'}\big(\mathcal{G}'\big) \cap Z \\ &= \mathrm{Sing}_{V'}\big(\mathcal{G}'\big) \cap Z \\ &\subset \mathrm{Sing}_Z\big(\mathcal{G}'|_Z\big) = \mathrm{Sing}_Z\big(\varphi^*(\mathcal{G})\big). \end{aligned}$$

Since the first and the last term of this chain coincide, all the terms in the middle are equal to them. Hence $\mathrm{Sing}_Z\big(\varphi^*(\mathcal{G})\big) = \mathrm{Zeros}_Z\big(\varphi^*(\mathcal{G})\big)$. □

On conditions b) and b*)

Remark 6.1.5. Consider two Rees algebras $\mathcal{G}$ and $\mathcal{K}$ over a regular excellent ring S so that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$, where $V = \mathrm{Spec}(S)$. That is, $\mathcal{K}$ and $\mathcal{G}$ are two Rees algebras satisfying condition b) of Lemma 6.1.1. Note that any morphism of finite type $\varphi : Z \to V$ as that in Lemma 6.1.1 can be locally regarded as an affine morphism of the form

$$\mathrm{Spec}(S) \longleftarrow \mathrm{Spec}(S[T_1, \dots, T_n]/J), \tag{6.1}$$

where $T_1, \ldots, T_n$ denote variables. Following this idea, consider all the diagrams of the form

$$\begin{tikzcd}
 & V' = \operatorname{Spec}(S[T_1,\ldots,T_n]) \arrow[dl, "\beta"'] \\
V = \operatorname{Spec}(S) & Z = \operatorname{Spec}(S[T_1,\ldots,T_n]/J), \arrow[u, hook]
\end{tikzcd}$$

where Z represents a closed regular subscheme of V', and let

$$\mathcal{Z} = \mathcal{O}_{V'}[JW]$$

denote the $\mathcal{O}_{V'}$-Rees algebra naturally attached to the immersion of Z in V' (see Remark 3.7.1). Next let us assume that the Rees algebra $\mathcal{G}$ has following property: for each diagram as above,

$$\mathscr{F}_Z\left(\beta^*(\mathcal{G})|_Z\right) = \mathscr{F}_{V'}\left(\beta^*(\mathcal{G})\right) \cap \mathscr{F}_{V'}(\mathcal{Z}). \tag{6.2}$$

Then we claim that condition b) implies condition b*).

Indeed, let us assume without loss of generality that φ is an affine morphism like (6.1) and consider the diagram

$$\begin{tikzcd}
 & V' = \operatorname{Spec}(S[T_1,\ldots,T_n]) \arrow[dl, "\beta"'] \\
V = \operatorname{Spec}(S) & Z = \operatorname{Spec}(S[T_1,\ldots,T_n]/J). \arrow[u, hook] \arrow[l, "\varphi"']
\end{tikzcd}$$

Since β can be regarded as a $\mathcal{G}$-permissible transformation on V, we have that

$$\mathscr{F}_{V'}\left(\beta^*(\mathcal{G})\right) \subset \mathscr{F}_{V'}\left(\beta^*(\mathcal{K})\right).$$

In this way, one readily checks that

$$\begin{aligned}
\mathscr{F}_Z\big(\varphi^*(\mathcal{G})\big) &= \mathscr{F}_Z\big(\beta^*(\mathcal{G})|_Z\big) \\
&= \mathscr{F}_{V'}\big(\beta^*(\mathcal{G})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_{V'}\big(\beta^*(\mathcal{K})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_Z\big(\beta^*(\mathcal{K})|_Z\big) \\
&= \mathscr{F}_Z\big(\varphi^*(\mathcal{K})\big).
\end{aligned}$$

Remark 6.1.6. Note that condition (6.2) only involves the Rees algebra $\mathcal{G}$, but not $\mathcal{K}$. In the following sections we will construct a Rees algebra $\mathcal{G}$ which satisfies (6.2) for any morphism $\varphi : Z \to V$ as above (see Lemma 6.4.2 and Lemma 6.6.6). In this way, whenever condition b) holds for some algebra $\mathcal{K}$, condition b*) will hold as well.

6.2 Saturation by derivatives

Let S be a regular ring. In this section, given a Rees algebra $\mathcal{G}$ over S and a submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$, we construct a certain saturation of $\mathcal{G}$ with respect to the derivatives of $\mathcal{D}$. This saturation, which we shall denote by $\mathcal{D}(\mathcal{G})$, will be used in the construction of canonical representatives in the case of characteristic zero (see Section 6.4).

Remark 6.2.1. Recall that, by Theorem 5.1.9, if a noetherian ring S defined over a field of characteristic zero satisfies the weak Jacobian condition relative to a submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$, then S is excellent.

Definition 6.2.2. For an ideal $I \subset S$ and a submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$, we shall define the ideal $\mathcal{D}(I) \subset S$ by

$$\mathcal{D}(I) = I + \langle \delta(f) \mid \delta \in \mathcal{D}, f \in I \rangle.$$

Remark 6.2.3. Note that $\mathcal{D}(I)$ might be a non-proper ideal of S. In general, we have strict inclusions $I \subset \mathcal{D}(I) \subset \mathcal{D}(\mathcal{D}(I))$. However, it is also possible that for some ideals $I = \mathcal{D}(I)$.

Remark 6.2.4. Suppose that $I = \langle f_1, \ldots, f_s \rangle$ and $\mathcal{D} = \langle \delta_\lambda \mid \lambda \in \Lambda \rangle$. Then

$$\mathcal{D}(I) = I + \langle \delta_\lambda(f_i) \mid \lambda \in \Lambda, 1 \leq i \leq s \rangle.$$

Definition 6.2.5. We will say that a Rees algebra $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$ over a noetherian ring S is *saturated with respect to a submodule of derivatives* $\mathcal{D} \subset \mathrm{Der}(S)$, or simply that $\mathcal{G}$ is *$\mathcal{D}$-saturated*, if $\mathcal{D}(I_n) \subset I_{n-1}$ for each $n > 0$.

Lemma 6.2.6. *Let S be a regular ring over a field of characteristic zero and let $\mathcal{G}$ be a Rees algebra over S. Set $V = \mathrm{Spec}(S)$. Suppose that $\mathcal{G}$ is $\mathcal{D}$-saturated for some submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$ satisfying the weak Jacobian condition on S. Then*

$$\mathrm{Sing}_V(\mathcal{G}) = \mathrm{Zeros}_V(\mathcal{G}).$$

Remark 6.2.7. This result says that $\mathcal{G}$ satisfies condition a) of Lemma 6.1.1.

Proof. The inclusion $\mathrm{Sing}_V(\mathcal{G}) \subset \mathrm{Zeros}_V(\mathcal{G})$ always holds. To prove the converse, fix a point $\xi \in V$ which does not belong to $\mathrm{Sing}_V(G)$. By definition, this means that there exists a homogeneous element $f_N W^N \in \mathcal{G}$ so that $\nu_\xi(f_N) < N$. Set $n = \nu_\xi(f_N)$. According to Proposition 5.4.3, it is possible to find a derivative $\delta \in \mathcal{D}$ such that $\nu_\xi(\delta(f_N)) = n - 1$. Set $f_{N-1} = \delta(f_N)$. By the $\mathcal{D}$-saturation of $\mathcal{G}$, we have that $f_{N-1}W^{N-1} \in \mathcal{G}$. Thus it follows by induction that $\mathcal{G}$ contains an element $f_{N-n}W^{N-n}$ such that $\nu_\xi(f_{N-n}) = 0$. That is, f_{N-n} is a unit in $\mathcal{O}_{V,\xi}$. Hence $\xi \notin \mathrm{Zeros}_V(\mathcal{G})$.

As the previous argument shows that $\xi \notin \mathrm{Sing}_V(\mathcal{G})$ implies $\xi \notin \mathrm{Zeros}_V(\mathcal{G})$, it follows that $\mathrm{Zeros}_V(\mathcal{G}) \subset \mathrm{Sing}_V(\mathcal{G})$. $\square$

Lemma 6.2.8 (Localization)**.** *Let S be a noetherian ring, $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivatives, and $\mathcal{G}$ a $\mathcal{D}$-saturated Rees algebra over S. Then, for any multiplicative subset $\mathcal{U} \subset S$, the Rees algebra $\mathcal{U}^{-1}\mathcal{G}$ (defined over $\mathcal{U}^{-1}S$) is saturated with respect to $\mathcal{U}^{-1}\mathcal{D} \subset \mathrm{Der}(\mathcal{U}^{-1}S)$.*

Proof. Suppose that $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$. Then $\mathcal{U}^{-1}\mathcal{G} = \bigoplus_{n\in\mathbb{N}}(\mathcal{U}^{-1}I_n)W^n$, and we have to check that $(\mathcal{U}^{-1}\mathcal{D})(\mathcal{U}^{-1}I_{n+1}) \subset \mathcal{U}^{-1}I_n$ for all $n > 0$.

Fix an integer $N > 0$, and generators of I_{N+1} and $\mathcal{D}$ over S respectively, say $I_{N+1} = \langle f_1, \dots, f_s\rangle$, and $\mathcal{D} = \langle \delta_\lambda \mid \lambda \in \Lambda\rangle$. Observe that the same elements generate $\mathcal{U}^{-1}I_{N+1}$ and $\mathcal{U}^{-1}\mathcal{D}$ over $\mathcal{U}^{-1}S$ respectively. Therefore, in virtue of Remark 6.2.4,

$$(\mathcal{U}^{-1}\mathcal{D})(\mathcal{U}^{-1}I_{N+1}) = (\mathcal{U}^{-1}I_{N+1}) + \langle \delta_\lambda(f_i)\rangle(\mathcal{U}^{-1}S) = \mathcal{U}^{-1}\mathcal{D}(I_{N+1}).$$

Since $\mathcal{G}$ is $\mathcal{D}$-saturated, we have $\mathcal{D}(I_{N+1}) \subset I_N$. Hence the equality above yields $(\mathcal{U}^{-1}\mathcal{D})(\mathcal{U}^{-1}I_{N+1}) \subset \mathcal{U}^{-1}I_N$. □

Definition 6.2.9. Let S be a noetherian ring, $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivatives, and $\mathcal{G}$ a Rees algebra over S. We define the *saturation of $\mathcal{G}$ with respect to $\mathcal{D}$*, which we shall denote by $\mathcal{D}(\mathcal{G})$, as the minimum $\mathcal{D}$-saturated S-Rees algebra containing $\mathcal{G}$.

Lemma 6.2.10. *The saturation of $\mathcal{G}$ with respect to $\mathcal{D}$ is finitely generated over S (that is, $\mathcal{D}(\mathcal{G})$ is a Rees algebra over S).*

Proof. Suppose that $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$ is generated on degree lower than or equal to N, i.e., $\mathcal{G} = S\left[I_1 W, \dots, I_N W^N\right]$. Set $J_N = I_N$, and, for each $n < N$, $J_n = I_n + \mathcal{D}(J_{n+1})$. Then take $\mathcal{G}' = S\left[J_1 W, \dots, J_N W^N\right]$. Note that any $\mathcal{D}$-saturated Rees algebra over S containing $\mathcal{G}$ should also contain $\mathcal{G}'$. We claim that $\mathcal{G}'$ is saturated with respect to $\mathcal{D}$, and hence that it is the minimum Rees algebra containing $\mathcal{G}$ with this property. This shows that $\mathcal{D}(\mathcal{G}) = \mathcal{G}'$.

In order to prove the claim, choose a set of homogeneous generators of $\mathcal{G}'$, say $\mathcal{G}' = S\left[f_1 W^{N_1}, \dots, f_r W^{N_r}\right]$, with $N_1, \dots, N_r \leq N$. Let us denote by J_n the homogeneous part of degree n of $\mathcal{G}'$, i.e., $\mathcal{G}' = \bigoplus_{n\in\mathbb{N}} J_n W^n$. Next fix $m \geq 2$. Observe that J_m is generated by elements of the form $f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}$, with $\alpha_1 N_1 + \cdots + \alpha_r N_r = m$. Moreover, for $\delta \in \mathcal{D}$, by the Leibniz's rule we have

$$\delta\left(f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}\right) = \sum_{i=1}^{r} \alpha_i \cdot f_1^{(\alpha_1 - \delta_{i1})} \cdot \ldots \cdot f_r^{(\alpha_r - \delta_{ir})} \cdot \delta(f_i),$$

where δ_{ij} represents the Kronecker's delta. By definition, $\mathcal{D}(J_{N_i}) \subset J_{N_i - 1}$, and hence $\delta(f_i) \in J_{N_i - 1}$. Thus we see that $\delta\left(f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}\right) \in J_{m-1}$. By Remark 6.2.4, this implies that $\mathcal{D}(J_m) \subset J_{m-1}$, and, repeating this argument for each $m \geq 2$, we conclude that $\mathcal{G}'$ is $\mathcal{D}$-saturated. □

Remark 6.2.11. From the construction of the $\mathcal{D}$-saturation of $\mathcal{G}$, it follows that $\mathcal{D}(\mathcal{G})$ is differential relative to $\mathcal{G}$ (see Definition 3.6.1). Thus, if S is regular and $V = \mathrm{Spec}(S)$, we have $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{D}(\mathcal{G}))$.

Lemma 6.2.12. *Let S be a regular algebra over a field of characteristic zero, and let $S' = S[T_1, \ldots, T_n]$ be a polynomial ring over S. Then each submodule $\mathcal{D} \subset \mathrm{Der}(S)$ satisfying the weak Jacobian condition on S can be extended to a submodule $\mathcal{D}' \subset \mathrm{Der}(S')$ with the following properties:*

i) $\mathcal{D}'$ satisfies the weak Jacobian condition on S'.

ii) If $\mathcal{G}$ is a $\mathcal{D}$-saturated Rees algebra over S, then $\mathcal{G}' = \mathcal{G}S'$ is saturated with respect to $\mathcal{D}'$.

Proof. If we proof the case $n = 1$, then the general one follows by induction. Hence assume that $n = 1$.

Note that each derivative $\delta \in \mathrm{Der}(S)$ can be extended to a derivative on $S' = S[T_1]$ which acts on the coefficients of T_1 (see [26, Example 26.J]). Let us denote this extension by $\delta' : S' \to S'$. Set

$$\mathcal{D}' = \left\langle \delta' \mid \delta \in \mathcal{D} \right\rangle S[T_1] \oplus S[T_1] \frac{\partial}{\partial T_1}.$$

Next we will show that $\mathcal{D}'$ satisfies conditions i) and ii).

To check i), we shall use the equivalence of Lemma 5.1.6. Fix a prime ideal $\mathfrak{q} \subset S'$. Set $\mathfrak{p} = \mathfrak{q} \cap S$, and $k(\mathfrak{p}) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. Note that

$$S'_{\mathfrak{q}}/\mathfrak{p}S'_{\mathfrak{q}} = (S_{\mathfrak{p}}[T_1])_{\mathfrak{q}}/\mathfrak{p}(S_{\mathfrak{p}}[T_1])_{\mathfrak{q}} \simeq k(\mathfrak{p})[T_1]_{\overline{\mathfrak{q}}},$$

where $\overline{\mathfrak{q}}$ represents the image of $\mathfrak{q}$ in $k(\mathfrak{p})[T_1]$. Consider a regular system of parameters of $S_{\mathfrak{p}}$, say $x_1, \ldots, x_d$, and choose derivatives $\delta_1, \ldots, \delta_d \in \mathcal{D}_{\mathfrak{p}}$ as in Lemma 5.1.6 ii). Let $\delta'_1, \ldots, \delta'_d$ denote their corresponding extensions to $S_{\mathfrak{p}}[T_1]$. Since $k(\mathfrak{p})[T_1]$ is a polynomial ring in one variable over a field, there are two options:

- If $\overline{\mathfrak{q}} = \langle 0 \rangle$, it follows that $x_1, \ldots, x_d$ is a regular system of parameters of $(S[T_1])_{\mathfrak{q}}$, and we have that $\delta_i^{(1)}(x_j) = \delta_{ij}$ (Kronecker's delta) for all $1 \leq i, j \leq d$.
- Otherwise, $\overline{\mathfrak{q}} = \langle \overline{G}(T_1) \rangle$ for some monic polynomial $\overline{G}(T_1) \in k(\mathfrak{p})[T_1]$. Consider an arbitrary lift of $\overline{G}(T_1)$ to $S_{\mathfrak{p}}[T_1]$, say $G(T_1)$, and set $x_{d+1} = G(T_1)$. Observe that $x_1, \ldots, x_d, x_{d+1}$ is a regular system of parameters of $(S[T_1])_{\mathfrak{q}}$. Since $k(\mathfrak{p})$ is a field of characteristic zero, $k(\mathfrak{q})$ is a finite separable extension of $k(\mathfrak{p})$. Hence $\frac{\partial}{\partial T_1}(\overline{G})$, and in turn $\frac{\partial}{\partial T_1}(G) = \frac{\partial}{\partial T_1}(x_{d+1})$, are units in $k(\mathfrak{p})[T_1]_{\overline{\mathfrak{q}}}$ and $(S[T_1])_{\mathfrak{q}}$ respectively. Set $\delta'_{d+1} = \left(\frac{\partial}{\partial T_1}(x_{d+1})\right)^{-1} \frac{\partial}{\partial T_1}$. In this way, $\delta'_i(x_j) = \delta_{ij}$ (Kronecker's delta) for all $1 \leq i, j \leq d+1$.

In either case, condition iii) of Lemma 5.1.6 is fulfilled at $\mathfrak{q}$. Thus we conclude that $\mathcal{D}'$ satisfies the weak Jacobian condition on S'.

For condition ii), fix a set of generators of $\mathcal{D}$, say $\mathcal{D} = \langle \delta_\lambda \mid \lambda \in \Lambda \rangle$. Observe that $\{\delta'_\lambda \mid \lambda \in \Lambda\} \cup \{\frac{\partial}{\partial T_1}\}$ is a set of generators of $\mathcal{D}'$. Since $\frac{\partial}{\partial T_1}(f) = 0$ for all $f \in S$, Remark 6.2.4 implies that, if $\mathcal{G}$ is $\mathcal{D}$-saturated Rees algebra over S, then $\mathcal{G}' = \mathcal{G}S'$ is saturated with respect to $\mathcal{D}'$. □

Lemma 6.2.13. *Let S be a regular ring, and let $S' = S/J$ be a regular quotient of S. Then any submodule of derivatives $\mathcal{D} \subset \mathrm{Der}(S)$ satisfying the weak Jacobian condition on S induces a canonical submodule of derivatives $\mathcal{D}' \subset \mathrm{Der}(S')$ with the following properties:*

i) *$\mathcal{D}'$ satisfies the weak Jacobian condition on S'.*

ii) *For each $\mathcal{D}$-saturated Rees algebra over S, say $\mathcal{G}$, the S'-Rees algebra $\mathcal{G}' = \mathcal{G}S'$ is saturated with respect to $\mathcal{D}'$.*

Proof. Along this proof, we shall denote the class of an element $f \in S$ modulo J by $\overline{f}$. Let us start by constructing $\mathcal{D}'$. Suppose that a derivative $\delta \in \mathcal{D}$ maps the ideal J into itself, i.e., $\delta(J) \subset J$. Then δ induces a natural derivative on S', say $\overline{\delta} : S' \to S'$, which maps an element $\overline{f} \in S'$ to $\overline{\delta}(\overline{f}) = \overline{\delta(f)}$. We shall define $\mathcal{D}' \subset \mathrm{Der}(S')$ as the submodule

$$\left\langle \overline{\delta} : S' \to S' \mid \delta \in \mathcal{D},\ \delta(J) \subset J \right\rangle .$$

Next we will show that $\mathcal{D}'$ fulfils i) and ii).

In order to check i), i.e., that $\mathcal{D}'$ satisfies the weak Jacobian condition on S', we need to verify that, for each prime $\mathfrak{q} \subset S'$,

$$\operatorname{rank} \mathrm{Jac}(\mathfrak{q}; \mathcal{D}')(\mathfrak{q}) = \dim S'_{\mathfrak{q}}. \tag{6.3}$$

To this end, we will make use of the equivalence on Lemma 5.1.6. Fix $\mathfrak{q} \subset S'$, and let $\mathfrak{p} \subset S$ denote the preimage of $\mathfrak{q}$ in S (i.e., $\mathfrak{p}$ is the unique prime of S containing J and satisfying $\mathfrak{q} = \mathfrak{p}/J$). Since S' is regular, one can choose a regular system of parameters of $S_{\mathfrak{p}}$, say $x_1, \ldots, x_d$, such that $JS_{\mathfrak{p}} = \langle x_1, \ldots, x_e \rangle S_{\mathfrak{p}}$. Note that, in this case, $\overline{x}_{e+1}, \ldots, \overline{x}_d$ is a regular system of parameters of $S'_{\mathfrak{q}}$. In virtue of Lemma 5.1.6 ii), consider local derivatives on $S_{\mathfrak{p}}$, say $\delta_1^*, \ldots, \delta_d^* \in \mathcal{D}_{\mathfrak{p}}$, satisfying $\delta_i^*(x_j) = \delta_{ij}$ (Kronecker's delta) for all i, j. On this proof we will just focus on $\delta_{e+1}^*, \ldots, \delta_d^*$. By definition of $\mathcal{D}_{\mathfrak{p}}$, one can find elements $u_{e+1}, \ldots u_d \in S \setminus \mathfrak{p}$ so that $u_i \delta_i^*$ is a global derivative on S, and $u_i \delta_i^* \in \mathcal{D}$. Fix $i \in \{e+1, \ldots, d\}$, and set $\delta_i = u_i \delta_i^*$. As

$$\delta_i^*(x_1) = \cdots = \delta_i^*(x_e) = 0,$$

it follows that

$$\delta_i^*(JS_{\mathfrak{p}}) \subset \langle x_1, \ldots, x_e \rangle S_{\mathfrak{p}} = JS_{\mathfrak{p}}.$$

Since u_i is a unit in $S_{\mathfrak{p}}$, the latter inclusion yields $\delta_i(J) \subset (J)$, and thus δ_i induces a natural derivative on S', say $\overline{\delta}_i \in \mathcal{D}'$. Now, using $\overline{\delta}_i$, we can construct a local derivative on $S'_{\mathfrak{q}}$, say $\overline{\delta}_i^* = (\overline{u}_i)^{-1} \cdot \overline{\delta}_i \in \mathcal{D}'_{\mathfrak{q}}$, which clearly satisfies

$$\overline{\delta}_i^*(\overline{x}_j) = (\overline{u}_i)^{-1} \cdot \overline{\delta}_i(\overline{x}_j) = (\overline{u}_i)^{-1} \cdot \overline{u}_i \cdot \overline{\delta_i^*(x_j)} = \delta_{ij} \quad \text{(Kronecker's delta)},$$

for all $e+1 \leq i, j \leq d$. Therefore Lemma 5.1.6 yields (6.3), and hence $\mathcal{D}'$ satisfies the weak Jacobian condition on S'.

For condition ii), assume that $\mathcal{G} = \bigoplus_{n\in\mathbb{N}} I_n W^n$. Observe that, by definition, $\mathcal{G}' = \bigoplus_{n\in\mathbb{N}} (I_n S')W^n$. Since $\mathcal{G}$ is $\mathcal{D}$-saturated, $\mathcal{D}(I_n) \subset \mathcal{D}(I_{n-1})$ for all $n > 1$, and then Remark 6.2.4 implies that

$$\begin{aligned}
\mathcal{D}'\big(I_n S'\big) &= \Big\langle \overline{\delta}(\overline{f}) \in S' \mid \delta \in \mathcal{D},\ \delta(J) \subset J,\ \overline{f} \in \big(I_n S'\big) \Big\rangle \\
&= \Big\langle \overline{\delta(f)} \in S' \mid \delta \in \mathcal{D},\ \delta(J) \subset J,\ f \in I_n \Big\rangle \\
&\subset \Big\langle \overline{\delta(f)} \in S' \mid \delta \in \mathcal{D},\ f \in I_n \Big\rangle \\
&= (\mathcal{D}(I_n))S' \subset I_{n-1}S'.
\end{aligned}$$

This proves that $\mathcal{G}'$ is $\mathcal{D}'$-saturated. $\square$

6.3 Formal retractions

The results presented in this section are of technical nature. They will be used in the proofs of existence of canonical representatives in both zero and positive characteristic (see Lemma 6.4.1 and Lemma 6.6.4 respectively).

Definition 6.3.1. Let R be a regular local ring, and $R' = R/\mathfrak{q}$ a regular quotient of R. Let $\pi : R \to R'$ denote the natural quotient map, and $\hat{\pi} : \widehat{R} \to \widehat{R'}$ the induced morphism between the completions. A ring inclusion $\varepsilon : R' \hookrightarrow R$ is called a *retraction* of π if $\pi \circ \varepsilon$ is the identity map on R'. A retraction of $\hat{\pi}$ is also called a *formal retraction* of π.

Transformations and regular morphisms

Definition 6.3.2. Let Z be a regular noetherian scheme. A morphism $Z \xleftarrow{\varphi_1} Z_1$ will be called a *permissible transformation on* Z is it is either a smooth morphism or the blow-up of Z along a closed regular center.

Remark 6.3.3. Let $\mathcal{G}$ be an arbitrary Rees algebra over a regular scheme Z. Then any $\mathcal{G}$-permissible transformation is also permissible transformation on Z. Similarly, any $\mathcal{G}$-permissible sequence on Z, say

$$Z \xleftarrow{\varphi_1} Z_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_s} Z_s,$$

is a sequence of permissible transformations on Z.

Lemma 6.3.4. *Let Z be a noetherian scheme, and $Z \xleftarrow{\rho} Z^*$ a regular morphism. Consider a regular closed subscheme $Y \subset Z$. Then:*

i) $Y^* = Y \times_Z Z^*$ is a regular closed subscheme of Z^*.

ii) $\mathrm{Bl}_{Y^*}(Z^*)$ *is naturally isomorphic to* $\mathrm{Bl}_Y(Z) \times_Z Z^*$. *As a consequence, there is a canonical commutative diagram*

$$\begin{array}{ccc} Z^* & \longleftarrow & \mathrm{Bl}_{Y^*}(Z^*) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_1} \\ Z & \longleftarrow & \mathrm{Bl}_Y(Z), \end{array}$$

where ρ *and* ρ_1 *are regular morphisms.*

Proof. By base change, the natural morphism $Y \leftarrow Y^*$ is regular. Since regular morphisms preserve regularity, i) follows.

For ii), let us fix affine charts, say $\mathrm{Spec}(S) \subset Z$, and $\mathrm{Spec}(S^*) \subset Z^*$. Assume that ρ is locally defined by a regular ring homomorphism $S \to S^*$. Let $\mathfrak{p}$ denote the ideal of definition of Y locally at $\mathrm{Spec}(S)$, and $\mathfrak{p}^* = \mathfrak{p}S^*$ that of Y^* locally at $\mathrm{Spec}(S^*)$. Recall that $\mathrm{Bl}_{\mathfrak{p}}(S) = \mathrm{Proj}(S[\mathfrak{p}W])$, and $\mathrm{Bl}_{\mathfrak{p}^*}(S^*) = \mathrm{Proj}(S^*[\mathfrak{p}^*W])$. Since $S \to S^*$ is regular, S^* is flat over S, and therefore $\mathfrak{p}^* \simeq \mathfrak{p} \otimes_S S^*$. Thus we get $S^*[\mathfrak{p}^*W] \simeq S[\mathfrak{p}W] \otimes_S S^*$, which proves that $\mathrm{Bl}_{Y^*}(Z^*) \simeq \mathrm{Bl}_Y(Z) \times_Z Z^*$. Finally, the morphism ρ_1 is regular because regularity is preserved by base change. $\square$

Lemma 6.3.5. *Let* $Z \leftarrow Z^*$ *be regular morphism of noetherian schemes. Then any sequence of permissible transformations on* Z*, say*

$$Z \xleftarrow{\varphi_1} Z_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_s} Z_s,$$

induces by base change a sequence of permissible transformations on Z^**, say*

$$Z^* \xleftarrow{\varphi_1^*} Z_1^* \xleftarrow{\varphi_2^*} \cdots \xleftarrow{\varphi_s^*} Z_s^*,$$

where $Z_i^* = Z_i \times_Z Z^*$ *for* $i = 1, \dots, s$*, and a Cartesian commutative diagram as follows,*

$$\begin{array}{ccccccc} Z^* & \longleftarrow & Z_1^* & \longleftarrow & \cdots & \longleftarrow & Z_s^* \\ \downarrow & & \downarrow & & & & \downarrow \\ Z & \longleftarrow & Z_1 & \longleftarrow & \cdots & \longleftarrow & Z_s. \end{array}$$

where the vertical arrows represent regular morphisms.

Proof. Set $Z_0 = Z$ and $Z_0^* = Z^*$. Each transformation φ_i can be of one of the following types: a permissible blow-up, an open restriction, or the multiplication by an affine line. Let us consider each of these three cases separately.

- Assume that φ_i is the blow-up of Z_{i-1} along a regular center $Y_{i-1} \subset Z_{i-1}$. In this case we define φ_i^* as the blow-up of Z_{i-1}^* along $Y_{i-1}^* = Y_{i-1} \times_{Z_{i-1}} Z_{i-1}^*$. According to Lemma 6.3.4 Y_{i-1}^* is a regular center in Z_{i-1}^* and there is a natural commutative diagram

$$\begin{tikzcd} Z^*_{i-1} \arrow[d] & \mathrm{Bl}_{Y^*_{i-1}}(Z^*_{i-1}) \arrow[l, "\varphi^*_i"'] \arrow[d] \\ Z_{i-1} & \mathrm{Bl}_{Y_{i-1}}(Z_{i-1}), \arrow[l, "\varphi_i"'] \end{tikzcd}$$

where $\mathrm{Bl}_{Y^*_{i-1}}(Z^*_{i-1}) \simeq \mathrm{Bl}_{Y_{i-1}}(Z_{i-1}) \times_{Z_{i-1}} Z^*_{i-1}$. Hence $Z^*_i \simeq Z_i \times_{Z_{i-1}} Z^*_{i-1}$.

- In case that Z_i is an open subscheme of Z_{i-1}, we have that $Z^*_i = Z_i \times_{Z_{i-1}} Z^*_{i-1}$ is clearly an open subscheme of Z^*_{i-1}.
- Finally, if φ_i is the multiplication by an affine line, i.e., if $Z_i = Z_{i-1} \times \mathbb{A}^1$, simply take $Z^*_i = Z^*_{i-1} \times \mathbb{A}^1 = Z^*_{i-1} \times_{Z_{i-1}} Z_i$ and φ^*_i as the obvious morphism.

In either case we have shown that $Z^*_i \simeq Z_i \times_{Z_{i-1}} Z^*_{i-1}$. Thus, by induction on i, we see that

$$Z^*_i \simeq Z_i \times_{Z_{i-1}} Z^*_{i-1} \simeq Z_i \times_{Z_{i-1}} (Z_{i-1} \times_Z Z^*) \simeq Z_i \times_Z Z^*. \qquad \square$$

Lemma 6.3.6. *Let $H \hookrightarrow V$ be a closed immersion of regular excellent schemes and consider a sequence of permissible transformations on H, say*

$$H \longleftarrow H_1 \longleftarrow \cdots \longleftarrow H_s. \tag{6.4}$$

Fix a point $\xi_s \in H_s$. For $i = 1, \dots, s-1$, let ξ_i denote the image of ξ_s in H_i and, similarly, let ξ denote the image of ξ_s in H. Set $H^* = \mathrm{Spec}(\widehat{\mathcal{O}_{H,\xi}})$ and $V^* = \mathrm{Spec}(\widehat{\mathcal{O}_{V,\xi}})$. Then:

i) *Sequence* (6.4) *induces a sequence of permissible transformations on V, say*

$$V \longleftarrow V_1 \longleftarrow \cdots \longleftarrow V_s, \tag{6.5}$$

and a natural commutative diagram, say

$$\begin{tikzcd} V & V_1 \arrow[l] & \cdots \arrow[l] & V_s \arrow[l] \\ H \arrow[u, hook] & H_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & H_s, \arrow[l] \arrow[u, hook] \end{tikzcd} \tag{6.6}$$

where the vertical arrows represent closed immersions.

ii) *Sequence* (6.5) *induces a natural sequence of permissible transformations on V^*, say*

$$V^* \longleftarrow V^*_1 \longleftarrow \cdots \longleftarrow V^*_s,$$

where $V^*_i = V_i \times_V V^*$ for $i = 1, \dots, s$. Similarly, (6.4) *induces a natural permissible sequence on H^*, say*

$$H^* \longleftarrow H^*_1 \longleftarrow \cdots \longleftarrow H^*_s, \tag{6.7}$$

where $H^*_i = H_i \times_H H^*$ for $i = 1, \dots, s$.

iii) The four sequences of transformations mentioned above are linked by a natural commutative diagram as follows,

$$\begin{tikzcd}
 & V^* \arrow[rr, dash] \arrow[dl] & & V_1^* \arrow[dl] & & \cdots \arrow[ll] & & V_s^* \arrow[ll] \arrow[dl] \\
V & & V_1 \arrow[ll] & & \cdots \arrow[ll] & & V_s \arrow[ll] & \\
 & H^* \arrow[uu, hook] \arrow[dl] & & H_1^* \arrow[ll] \arrow[uu, hook] \arrow[dl] & & \cdots \arrow[ll] & & H_s^* \arrow[ll] \arrow[uu, hook] \arrow[dl] \\
H \arrow[uu, hook] & & H_1 \arrow[ll] \arrow[uu, hook] & & \cdots \arrow[ll] & & H_s \arrow[ll] \arrow[uu, hook] &
\end{tikzcd} \tag{6.8}$$

where the vertical arrows represent closed immersions.

iv) All the slanted arrows of the diagram above represent regular morphisms.

v) For each $i = 1, \dots, s$, there exists a unique point $\xi_i^* \in H_i^*$ which maps to $\xi_i \in H_i$ (and clearly $\xi_i^* \mapsto \xi_{i-1}^*$ for all $i > 1$).

Proof. i) is clear. Property ii) follows from Lemma 6.3.5 after replacing $Z \leftarrow Z^*$ by $V \leftarrow V^*$ and $H \leftarrow H^*$ respectively.

For iii), observe that $H^* = H \times_V V^*$ and, as a consequence,

$$H_i^* = H_i \times_H H^* = H_i \times_H (H \times_V V^*) = H_i \times_V V^*.$$

In this way, (6.7) can be regarded as the base change of (6.4) via $V \leftarrow V^*$ (instead of $H \leftarrow H^*$). Similarly, the rear grid of diagram (6.8) can be regarded as the base change of the front grid via $V \leftarrow V^*$ (note that the front grid coincides with diagram (6.6)).

Property iv) follows from the fact that $V_i \leftarrow V_i^*$ can also be regarded as the base change of $V \leftarrow V^*$ via $V \leftarrow V_i$. Since V is excellent, the morphism $V \leftarrow V^*$ is regular. Moreover, regular morphisms are preserved by base change. Thus we conclude that $V_i \leftarrow V_i^*$ is regular. Replacing V, V^*, V_i, V_i^* by H, H^*, H_i, H_i^* respectively on the previous argument, we also see that $H_i \leftarrow H_i^*$ is regular.

To prove v), recall that $H_i^* = H_i \times_H H^*$, where $H^* = \mathrm{Spec}(\widehat{\mathcal{O}_{H,\xi}})$, and observe that

$$\left(\widehat{\mathcal{O}_{H,\xi}} \otimes_{\mathcal{O}_{H,\xi}} k(\xi)\right) \simeq k(\xi).$$

Thus one readily checks that ξ_i^* is uniquely determined. $\square$

Remark 6.3.7. In general, the ring $\mathcal{O}_{V_i^*,\xi_i^*}$ (resp. $\mathcal{O}_{H_i^*,\xi_i^*}$) does not coincide with the completion of $\mathcal{O}_{V_i,\xi_i}$ (resp. $\mathcal{O}_{H_i,\xi_i}$). However, in some aspects, they have a very similar behavior. From the proof of v), it follows that $\xi_i^* \in V_i^*$ is rational over $\xi_i \in V_i$, i.e., $k(\xi_i^*) = k(\xi_i)$. Moreover, since $\mathcal{O}_{V_i,\xi_i} \to \mathcal{O}_{V_i^*,\xi_i^*}$ is a regular morphism and ξ_i^* is the unique point of $\mathrm{Spec}(\mathcal{O}_{V_i^*,\xi_i^*})$ mapping to ξ_i, we deduce that

$$\mathcal{O}_{V_i^*,\xi_i^*} \otimes_{\mathcal{O}_{V_i,\xi_i}} k(\xi_i) = k(\xi_i^*).$$

Note that this is equivalent to saying that $\mathfrak{M}_{V_i,\xi_i}\mathcal{O}_{V_i^*,\xi_i^*} = \mathfrak{M}_{V_i^*,\xi_i^*}$. As a consequence of this equality and the faithful flatness of the local homomorphism $\mathcal{O}_{V_i,\xi_i} \to \mathcal{O}_{V_i^*,\xi_i^*}$ (which follows from its regularity), one can see that

$$(\mathfrak{M}_{V_i,\xi_i})^N = (\mathfrak{M}_{V_i^*,\xi_i^*})^N \cap \mathcal{O}_{V_i,\xi_i}$$

for all $N \geq 0$. Thus, for each $f \in \mathcal{O}_{V_i,\xi_i}$, one has that $\nu_{V_i,\xi_i}(f) = \nu_{V_i^*,\xi_i^*}(f)$.

Lifting of retractions

Lemma 6.3.8. *Assume the same hypotheses of Lemma 6.3.6. Suppose that a fixed retraction of the quotient map $\widehat{\mathcal{O}_{V,\xi}} \to \widehat{\mathcal{O}_{H,\xi}}$ is given, say*

$$\varepsilon : \widehat{\mathcal{O}_{H,\xi}} \hookrightarrow \widehat{\mathcal{O}_{V,\xi}}.$$

Then, for each $i = 1, \ldots, s$, the map ε can be lifted to a unique retraction of the quotient map $\mathcal{O}_{V_i^*,\xi_i^*} \to \mathcal{O}_{H_i^*,\xi_i^*}$, say

$$\varepsilon_i : \mathcal{O}_{H_i^*,\xi_i^*} \hookrightarrow \mathcal{O}_{V_i^*,\xi_i^*},$$

such that the following diagram commutes:

$$\begin{tikzcd}
\widehat{\mathcal{O}_{V,\xi}} \arrow[r] & \mathcal{O}_{V_1^*,\xi_1^*} \arrow[r] & \cdots \arrow[r] & \mathcal{O}_{V_s^*,\xi_s^*} \\
\widehat{\mathcal{O}_{H,\xi}} \arrow[r] \arrow[u, hook, "\varepsilon"] & \mathcal{O}_{H_1^*,\xi_1^*} \arrow[r] \arrow[u, hook, "\varepsilon_1"] & \cdots \arrow[r] & \mathcal{O}_{H_s^*,\xi_s^*}. \arrow[u, hook, "\varepsilon_s"]
\end{tikzcd}$$

Proof. Set $V_0^* = \mathrm{Spec}(\widehat{\mathcal{O}_{V,\xi}})$, $H_0^* = \mathrm{Spec}(\widehat{\mathcal{O}_{H,\xi}})$, and $\varepsilon_0 = \varepsilon$. Let ξ_0^* denote the closed point of H_0^* and V_0^*. With this notation, it suffices check that, whenever ε_{i-1} exists, it can be lifted to a unique retraction $\varepsilon_i : \mathcal{O}_{H_i^*,\xi_i^*} \hookrightarrow \mathcal{O}_{V_i^*,\xi_i^*}$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_{V_{i-1}^*,\xi_{i-1}^*} \arrow[r] & \mathcal{O}_{V_i^*,\xi_i^*} \\
\mathcal{O}_{H_{i-1}^*,\xi_{i-1}^*} \arrow[r] \arrow[u, hook, "\varepsilon_{i-1}"] & \mathcal{O}_{H_i^*,\xi_i^*}. \arrow[u, hook, "\varepsilon_i"]
\end{tikzcd}$$

If this condition holds, then the lemma follows by induction.

In the case that $H_{i-1}^* \leftarrow H_i^*$ (and hence $V_{i-1}^* \leftarrow V_i^*$) is either an open restriction, or the multiplication by an affine line, the claim is trivial. The case of permissible blow-ups follows from Lemma 6.3.9 below. □

Lemma 6.3.9. *Let $H^* \subset V^*$ be a closed immersion of regular noetherian schemes. Consider a proper regular center $Y^* \subset H^*$, the blow-ups of V^* and H^**

along Y^* respectively, say $V_1^* = \mathrm{Bl}_{Y^*}(V^*)$ and $H_1^* = \mathrm{Bl}_{Y^*}(H^*)$, and the induced commutative diagram:

$$\begin{tikzcd} V^* & V_1^* \arrow[l] \\ H^* \arrow[u, hook, "\sigma"] & H_1^*. \arrow[l] \arrow[u, hook, "\sigma_1"'] \end{tikzcd}$$

Fix a point $\xi_1^* \in H_1^*$, and let ξ^* denote its image in H^*. Then, each retraction of the quotient map $\pi : \mathcal{O}_{V^*,\xi^*} \to \mathcal{O}_{H^*,\xi^*}$, say

$$\varepsilon : \mathcal{O}_{H^*,\xi^*} \hookrightarrow \mathcal{O}_{V^*,\xi^*},$$

induces a canonical retraction of $\pi_1 : \mathcal{O}_{V_1^*,\xi_1^*} \to \mathcal{O}_{H_1^*,\xi_1^*}$, say

$$\varepsilon_1 : \mathcal{O}_{H_1^*,\xi_1^*} \hookrightarrow \mathcal{O}_{V_1^*,\xi_1^*},$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_{V^*,\xi^*} \arrow[r] & \mathcal{O}_{V_1^*,\xi_1^*} \\ \mathcal{O}_{H^*,\xi^*} \arrow[u, hook, "\varepsilon"] \arrow[r] & \mathcal{O}_{H_1^*,\xi_1^*}. \arrow[u, hook, "\varepsilon_1"'] \end{tikzcd}$$

Proof. Let us assume without loss of generality that $V^* = \mathrm{Spec}(\mathcal{O}_{V^*,\xi^*})$, and $H^* = \mathrm{Spec}(\mathcal{O}_{H^*,\xi^*})$, and denote by $\beta : V^* \to H^*$ the morphism of schemes induced by ε. Let $\mathfrak{p}, \mathfrak{q} \subset \mathcal{O}_{V^*,\xi^*}$ be the ideals of definition of Y^* and H^* inside V^* respectively. Since $\mathfrak{q} \subset \mathfrak{p}$, one can easily find elements $x'_1, \ldots, x'_r \in \mathcal{O}_{H^*,\xi^*}$ so that $\mathfrak{p} = \langle \varepsilon(x'_1), \ldots, \varepsilon(x'_r) \rangle + \mathfrak{q}$. Note that, in this case, $\langle x'_1, \ldots, x'_r \rangle \subset \mathcal{O}_{H^*,\xi^*}$ is the ideal of definition of Y^* inside H^*.

Set $x_i = \varepsilon(x'_i)$ for $i = 1, \ldots, r$. The condition $\xi_1^* \in H_1^*$ ensures that ξ_1^*, regarded as a point of $V_1^* = \mathrm{Bl}_{Y^*}(V^*)$, belongs to one of the $x_1, \ldots, x_r$-charts of V_1^*. Thus

$$\varepsilon(\langle x'_1, \ldots, x'_r \rangle)\mathcal{O}_{V_1^*,\xi_1^*} = \langle x_1, \ldots, x_r \rangle \mathcal{O}_{V_1^*,\xi_1^*}$$

is an invertible ideal of $\mathcal{O}_{V_1^*,\xi_1^*}$. Hence, by the universal property of the blow-up, there exists a unique morphism $\beta_1 : \mathrm{Spec}(\mathcal{O}_{V_1^*,\xi_1^*}) \to H_1^* = \mathrm{Bl}_{Y^*}(H^*)$ such that the following diagram commutes:

$$\begin{tikzcd} H^* & H_1^* \arrow[l] \\ V^* \arrow[u, "\beta"] & \mathrm{Spec}(\mathcal{O}_{V_1^*,\xi_1^*}). \arrow[l] \arrow[u, "\beta_1"'] \end{tikzcd}$$

Since β_1 maps the closed point of $\mathrm{Spec}(\mathcal{O}_{V_1^*,\xi_1^*})$ to $\xi_1^* \in H_1^*$, this morphism factors through $\mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*})$. That is, we have a commutative diagram

$$\begin{tikzcd} H^* & H_1^* \arrow[l] & \mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*}). \arrow[l] \\ V^* \arrow[u, "\beta"] & \mathrm{Spec}(\mathcal{O}_{V_1^*,\xi_1^*}) \arrow[l] \arrow[u, "\beta_1"'] \arrow[ur, "\beta'_1"'] & \end{tikzcd}$$

Let σ'_1 denote the closed immersion of schemes induced by the quotient map $\mathcal{O}_{V_1^*,\xi_1^*} \to \mathcal{O}_{H_1^*,\xi_1^*}$. The composition of β'_1 with σ'_1 gives a natural morphism from $\mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*})$ into itself:

$$\begin{tikzcd} & & \mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*}). \\ \mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*}) \arrow[r, hook, "\sigma'_1"] \arrow[urr, bend left, "\beta'_1\circ\sigma'_1"] & \mathrm{Spec}(\mathcal{O}_{V_1^*,\xi_1^*}) \arrow[ur, "\beta'_1"'] & \end{tikzcd}$$

By the universal property of the blow-up, $(\beta'_1 \circ \sigma'_1)$ should be the identity map on $\mathrm{Spec}(\mathcal{O}_{H_1^*,\xi_1^*})$, i.e., $(\sigma'_1)^\# \circ (\beta'_1)^\#$ is the identity homomorphism on $\mathcal{O}_{H_1^*,\xi_1^*}$. This shows that $\varepsilon_1 = (\beta'_1)^\#$ is a retraction of $(\sigma'_1)^\# : \mathcal{O}_{V_1^*,\xi_1^*} \to \mathcal{O}_{H_1^*,\xi_1^*}$. $\square$

6.4 Canonical representatives in characteristic zero

Let V be a regular scheme defined over a field of characteristic zero and let $\mathcal{G}$ be a Rees algebra over V. In this section we prove that, if the weak Jacobian condition holds on V, then it is possible to find a canonical representative of the class of $\mathcal{G}$, say $\mathscr{C}_V(\mathcal{G})$ (see Theorem 6.4.3 and Theorem 6.4.6).

Lemma 6.4.1. *Let S be a regular ring over a field of characteristic zero, $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivatives satisfying the weak Jacobian condition on S, and $\mathcal{G}$ a $\mathcal{D}$-saturated Rees algebra over S. Fix a regular hypersurface $H = \mathrm{Spec}(S/J) \subset V$, and let $\mathcal{H} = S[JW]$ denote the Rees algebra attached to the immersion of H in V (see Remark 3.7.1). Then the following conditions hold:*

i) $\mathrm{Sing}_H(\mathcal{G}|_H) = \mathrm{Sing}_V(\mathcal{G}) \cap H$.

ii) $\mathscr{F}_H(\mathcal{G}|_H) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H})$.

Proof. Let us start with i). The inclusion $\mathrm{Sing}_V(\mathcal{G}) \cap H \subset \mathrm{Sing}_H(\mathcal{G}|_H)$ always holds. To prove the converse we shall show that, for each $\xi \in H$,

$$\xi \in \mathrm{Sing}_H(\mathcal{G}|_H) \implies \xi \in \mathrm{Sing}_V(\mathcal{G}).$$

Fix a point $\xi \in \mathrm{Sing}_H(\mathcal{G}|_H)$ and a homogeneous element $fW^N \in \mathcal{G}$. Since H is a regular hypersurface of V, there is a regular system of parameters of $\mathcal{O}_{V,\xi}$, say $x_1, \dots, x_d$, such that $\mathcal{O}_{H,\xi} = \mathcal{O}_{V,\xi}/\langle x_1 \rangle$. Then, according to Lemma 5.1.6, one can find a derivative $\delta_1 \in \mathcal{D}_\xi$ satisfying $\delta_1(x_j) = \delta_{1j}$ (Kronecker's delta) for all j. Under these hypotheses, [36, Lemma 4, p. 526] claims that there exists a formal retraction of the quotient map $\mathcal{O}_{V,\xi} \to \mathcal{O}_{H,\xi}$, say

$$\varepsilon : \widehat{\mathcal{O}_{H,\xi}} \hookrightarrow \widehat{\mathcal{O}_{V,\xi}}, \tag{6.9}$$

and an isomorphism, say $\widehat{\mathcal{O}_{V,\xi}} \simeq \widehat{\mathcal{O}_{H,\xi}}[[x_1]]$, so that δ_1 vanishes at $\widehat{\mathcal{O}_{H,\xi}}$. In virtue of this isomorphism, consider the expansion of f as a power series in x_1:

$$f = \sum_{i=0}^{\infty} \varepsilon(a_i) x_1^i, \quad a_i \in \widehat{\mathcal{O}_{H,\xi}}.$$

Regarding f as an element of $\widehat{\mathcal{O}_{V,\xi}}$, we have that

$$f \equiv \varepsilon(a_0) + \varepsilon(a_1)x_1 + \cdots + \varepsilon(a_{N-1})x_1^{N-1} \text{ modulo } \langle x_1 \rangle^N. \tag{6.10}$$

Thus, applying δ_1 iteratively to this expression, we get

$$\frac{1}{n!}\delta_1^n(f) \equiv a_n \text{ modulo } \langle x_1 \rangle.$$

In this way, since $\mathcal{G}_\xi$ is $\mathcal{D}_\xi$-saturated (see Lemma 6.2.8), we see that $a_n W^{N-n} \in \mathcal{G}_\xi|_H$ for $n = 0, \ldots, N-1$.

Since $\xi \in \operatorname{Sing}_H(\mathcal{G}|_H)$, it follows from the previous discussion that $\nu_{H,\xi}(a_n) \geq N - n$ for all $0 \leq n < N$. In addition,

$$\begin{aligned} \nu_{H,\xi}(a_n) \geq N - n &\implies a_n \in (\mathfrak{M}_{H,\xi})^{(N-n)}\widehat{\mathcal{O}_{H,\xi}} \\ &\implies \varepsilon(a_n) \in (\mathfrak{M}_{V,\xi})^{(N-n)}\widehat{\mathcal{O}_{V,\xi}}. \end{aligned}$$

Thus, in virtue of (6.10), we have that $f \in (\mathfrak{M}_{V,\xi})^N \widehat{\mathcal{O}_{V,\xi}}$. Since $\mathcal{O}_{V,\xi} \to \widehat{\mathcal{O}_{V,\xi}}$ is a faithfully flat morphism, this implies that

$$f \in (\mathfrak{M}_{V,\xi})^N = \mathcal{O}_{V,\xi} \cap (\mathfrak{M}_{V,\xi})^N \widehat{\mathcal{O}_{V,\xi}},$$

i.e., $\nu_{V,\xi}(f) \geq N$. Finally, as we can repeat the same argument for each homogeneous element $fW^N \in \mathcal{G}$, we conclude that $\xi \in \operatorname{Sing}_V(\mathcal{G})$. Therefore i) holds.

Let us proceed with ii). Recall that, by Proposition 3.7.2, we have $\mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H}) \subset \mathscr{F}_H(\mathcal{G}|_H)$. Hence any sequence of $(\mathcal{G} \odot \mathcal{H})$-permissible transformations on V, say

$$\begin{array}{ccccccc} \mathcal{G} & & \mathcal{G}_1 & & & & \mathcal{G}_s \\ \mathcal{H} & & \mathcal{H}_1 & & & & \mathcal{H}_s \\ V & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_s, \end{array} \tag{6.11}$$

induces a sequence of $\mathcal{G}|_H$-permissible transformations on H, say[1]

$$\begin{array}{ccccccc} \mathcal{G}|_H & & \mathcal{G}'_1 & & & & \mathcal{G}'_s \\ H & \longleftarrow & H_1 & \longleftarrow & \cdots & \longleftarrow & H_s, \end{array} \tag{6.12}$$

and a commutative diagram as follows:

$$\begin{tikzcd} V & V_1 \arrow[l] & \cdots \arrow[l] & V_s \arrow[l] \\ H \arrow[u, hook] & H_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & H_s. \arrow[l] \arrow[u, hook] \end{tikzcd}$$

In order to prove ii), we shall show that, for every sequence like (6.11), and each $\xi_s \in H_s$,

$$\xi_s \in \operatorname{Sing}_{H_s}(\mathcal{G}'_s) \implies \xi_s \in \operatorname{Sing}_{V_s}(\mathcal{G}_s).$$

[1] Here $\mathcal{G}'_i$ denotes the transform of $\mathcal{G}|_H$ on H_i which, in general, differs from $\mathcal{G}_i|_{H_i}$.

Fix a sequence like (6.11), and a point $\xi_s \in \operatorname{Sing}_{H_s}(\mathcal{G}'_s)$. Let $\xi \in H$ denote the image of ξ_s through (6.12). Note that $\xi \in \operatorname{Sing}_H(\mathcal{G}|_H)$ necessarily. Then fix a homogeneous element $fW^N \in \mathcal{G}$ and proceed as in i). Namely, consider an element $x_1 \in \mathcal{O}_{V,\xi}$ such that $\mathcal{O}_{H,\xi} = \mathcal{O}_{V,\xi}/\langle x_1 \rangle$, a formal retraction $\varepsilon : \widehat{\mathcal{O}_{H,\xi}} \hookrightarrow \widehat{\mathcal{O}_{V,\xi}}$ as in (6.9), and coefficients $a_0, \dots, a_{N-1} \in \widehat{\mathcal{O}_{H,\xi}}$ so that

$$f \equiv \varepsilon(a_0) + \varepsilon(a_1)x_1 + \cdots + \varepsilon(a_{N-1})x_1^{N-1} \text{ modulo } \langle x_1 \rangle^N,$$

as in (6.10). Now we are under the hypotheses of Lemma 6.3.6 and Lemma 6.3.8. In short, if we set $V^* = \operatorname{Spec}(\widehat{\mathcal{O}_{V,\xi}})$ and $H^* = \operatorname{Spec}(\widehat{\mathcal{O}_{H,\xi}})$, then Lemma 6.3.6 states that there exists a commutative diagram, say

$$\begin{tikzcd}
& V^* \arrow[dl] & & V_1^* \arrow[ll, dash] \arrow[dl] & & \cdots \arrow[ll] & & V_s^* \arrow[ll] \arrow[dl] \\
V & & V_1 \arrow[ll] & & \cdots \arrow[ll] & & V_s \arrow[ll] & \\
& H^* \arrow[uu, hook] \arrow[dl] & & H_1^* \arrow[ll] \arrow[uu, hook] \arrow[dl] & & \cdots \arrow[ll] & & H_s^* \arrow[ll] \arrow[uu, hook] \arrow[dl] \\
H \arrow[uu, hook] & & H_1 \arrow[ll] \arrow[uu, hook] & & \cdots \arrow[ll] & & H_s, \arrow[ll] \arrow[uu, hook] &
\end{tikzcd}$$

where the vertical arrows are closed immersions, and the diagonal arrows represent regular morphisms. Furthermore, there exists a unique point in H_s^*, say $\xi_s^* \in H_s^*$, which maps to $\xi_s \in H_s$. Then, by Lemma 6.3.8, the retraction $\varepsilon : \widehat{\mathcal{O}_{H,\xi}} \hookrightarrow \widehat{\mathcal{O}_{V,\xi}}$ can be lifted to a unique retraction of the quotient map $\mathcal{O}_{V_s^*,\xi_s^*} \to \mathcal{O}_{H_s^*,\xi_s^*}$, say

$$\varepsilon_s : \mathcal{O}_{H_s^*,\xi_s^*} \hookrightarrow \mathcal{O}_{V_s^*,\xi_s^*},$$

which makes the following diagram commutative:

$$\begin{tikzcd}
\widehat{\mathcal{O}_{V,\xi}} \arrow[r] & \mathcal{O}_{V_s^*,\xi_s^*} \\
\widehat{\mathcal{O}_{H,\xi}} \arrow[r] \arrow[u, hook, "\varepsilon"] & \mathcal{O}_{H_s^*,\xi_s^*}. \arrow[u, hook, "\varepsilon_s"']
\end{tikzcd}$$

Let $\tilde{f}W^N$ and $\tilde{x}_1W$ denote the weighted transforms of $fW^N \in \mathcal{G}$ and $x_1W \in \mathcal{H}$ via 6.11 respectively. Similarly, denote by $\tilde{a}_0W^N, \dots, \tilde{a}_{N-1}W$ the weighted transforms of $a_0W^N, \dots, a_{N-1}W \in \mathcal{G}|_H$ via (6.12). Using this notation, we deduce from (6.10) that

$$\tilde{f} \equiv \varepsilon_s(\tilde{a}_0) + \varepsilon_s(\tilde{a}_1)(\tilde{x}_1) + \cdots + \varepsilon_s(\tilde{a}_{N-1})(\tilde{x}_1)^{N-1} \text{ modulo } \langle \tilde{x}_1 \rangle^N. \tag{6.13}$$

The final argument is analogous to that of i). Suppose that $\xi_s \in \operatorname{Sing}_{H_s}(\mathcal{G}'_s)$. Then it follows that $\nu_{H_s,\xi_s}(\tilde{a}_n) \geq N - n$ for all n. Moreover,

$$\begin{aligned}
\nu_{H_s,\xi_s}(\tilde{a}_n) \geq N - n &\implies \tilde{a}_n \in (\mathfrak{M}_{H_s,\xi_s})^{N-n} \\
&\implies \tilde{a}_n \in (\mathfrak{M}_{H_s^*,\xi_s^*})^{N-n} \\
&\implies \varepsilon_s(\tilde{a}_n) \in (\mathfrak{M}_{V_s^*,\xi_s^*})^{N-n}.
\end{aligned}$$

In virtue of (6.13), this yields $\tilde{f} \in (\mathfrak{M}_{V_s^*,\xi_s^*})^N$, and hence, by Remark 6.3.7, we have that

$$\nu_{V_s,\xi_s}(\tilde{f}) = \nu_{V_s^*,\xi_s^*}(\tilde{f}) \geq N.$$

Repeating the same argument for each homogeneous element $fW^N \in \mathcal{G}$, we conclude that $\xi_s \in \mathrm{Sing}_{V_s}(\mathcal{G}_s)$. Thus ii) holds. □

Lemma 6.4.2. *Let S be a regular ring over a field of characteristic zero, $\mathcal{D} \subset \mathrm{Der}(S)$ a submodule of derivatives satisfying the weak Jacobian condition on S, and $\mathcal{G}$ a $\mathcal{D}$-saturated Rees algebra over S. Set $V = \mathrm{Spec}(S)$. For any closed regular subscheme $Z = \mathrm{Spec}(S/J) \subset V$, if $\mathcal{Z} = S[JW]$ denotes the Rees algebra associated to the embedding of Z in V (see Remark 3.7.1), then*

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}).$$

Proof. First of all, observe that the previous equality can be checked locally. Namely, it suffices to see that, for each point $\xi \in Z$, there exists an open neighborhood of ξ, say $U \subset V$, so that

$$\mathscr{F}_{Z\cap U}\big((\mathcal{G}|_U)|_Z\big) = \mathscr{F}_U\big(\mathcal{G}|_U\big) \cap \mathscr{F}_U\big(\mathcal{Z}|_U\big).$$

Fix $\xi \in Z$. Since Z is regular, one can find a regular system of parameters of $\mathcal{O}_{V,\xi}$, say $x_1, \ldots, x_d$, so that $J\mathcal{O}_{V,\xi} = \langle x_1, \ldots, x_e\rangle \mathcal{O}_{V,\xi}$, where e denotes the codimension of Z. Hence, for a suitable choice of $f \in S$, we have that $x_1, \ldots, x_d \in S_f$, with $JS_f = \langle x_1, \ldots, x_e\rangle S_f$. Recall that, in virtue of Remark 5.1.4, $\mathcal{D}_f$ satisfies the weak Jacobian condition on S_f, and $\mathcal{G}_f$ is a $\mathcal{D}_f$-saturated S_f-Rees algebra by Lemma 6.2.8. Thus we may assume without loss of generality that $S = S_f$, $\mathcal{D} = \mathcal{D}_f$, and $\mathcal{G} = \mathcal{G}_f$. Since $\mathcal{O}_{V,\xi}/\langle x_1\rangle$ is regular, we may also assume that $S/\langle x_1\rangle$ is regular. Now we proceed by induction on e.

In the case $e = 1$, i.e., when Z is a regular hypersurface, the claim follows from Lemma 6.4.1.

For $e > 1$, consider the regular hypersurface defined by x_1 on V, say $H = \mathrm{Spec}(S/\langle x_1\rangle)$. Let $\mathcal{H} = S[x_1W]$ denote the Rees algebra associated to the immersion of H in V. According to Lemma 6.4.1,

$$\mathscr{F}_H(\mathcal{G}|_H) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H}).$$

In addition, by Lemma 6.2.13, the Rees algebra $\mathcal{G}|_H$ is saturated with respect to some submodule of derivatives, say $\mathcal{D}' \subset \mathrm{Der}(S/\langle x_1\rangle)$, satisfying the weak Jacobian condition on $S/\langle x_1\rangle$. Note also that $(\mathcal{G}|_H)|_Z = \mathcal{G}|_Z$. Thus, by the inductive hypothesis,

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_H(\mathcal{G}|_H) \cap \mathscr{F}_H(\mathcal{Z}|_H).$$

Since $\mathscr{F}_H(\mathcal{Z}|_H) = \mathscr{F}_V(\mathcal{Z})$, gathering these identities we finally get

$$\begin{aligned}\mathscr{F}_Z(\mathcal{G}|_Z) &= \mathscr{F}_H(\mathcal{G}|_H) \cap \mathscr{F}_H(\mathcal{Z}|_H)\\ &= \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H}) \cap \mathscr{F}_V(\mathcal{Z})\\ &= \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}).\end{aligned}$$

□

Theorem 6.4.3. *Let S be a regular ring over a field of characteristic zero, set $V = \operatorname{Spec}(S)$, and let $\mathcal{G}$ be a Rees algebra over S. Suppose that $\mathcal{G}$ is $\mathcal{D}$-saturated for some submodule of derivatives $\mathcal{D} \subset \operatorname{Der}(S)$ satisfying the weak Jacobian condition on S. Then, for any Rees algebra $\mathcal{K}$ over S,*

$$\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K}) \iff \mathcal{K} \subset \overline{\mathcal{G}}.$$

Proof. Recall that $\mathscr{F}_V(\overline{\mathcal{G}}) = \mathscr{F}_V(\mathcal{G})$ by Lemma 3.5.7. Thus $\mathcal{K} \subset \overline{\mathcal{G}} \Rightarrow \mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G})$ is trivial.

To prove the converse, let us assume that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$. Then we claim that $\mathcal{G}$ and $\mathcal{K}$ satisfy conditions a), a*), b), and b*) of Lemma 6.1.1. In this way, we see that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}) \Rightarrow \mathcal{K} \subset \mathcal{G}$.

Let us check that $\mathcal{G}$ and $\mathcal{K}$ fulfill the conditions of Lemma 6.1.1. On the one hand, as $\mathcal{G}$ is $\mathcal{D}$-saturated, it satisfies condition a) by Lemma 6.2.6, and hence it also satisfies a*) by Lemma 6.1.4. On the other hand, b) holds by assumption. Thus it just remains to check that $\mathcal{G}$ and $\mathcal{K}$ satisfy condition b*).

Fix a morphism $\varphi : Z \to V$ as in b*). That is, φ is a morphism of finite type from a regular scheme Z to V. Following the ideas of Remark 6.1.5, observe that φ can be locally regarded as an affine morphism of the form

$$\operatorname{Spec}(S) \longleftarrow \operatorname{Spec}(S[T_1, \dots, T_n]/J),$$

where J represents an ideal contained in the polynomial ring $S[T_1, \dots, T_n]$. Thus we get a commutative diagram

$$\begin{tikzcd}
 & V' = \operatorname{Spec}(S[T_1, \dots, T_n]) \arrow[dl, "\beta"'] \\
V = \operatorname{Spec}(S) & Z = \operatorname{Spec}(S[T_1, \dots, T_n]/J). \arrow[l, "\varphi"'] \arrow[u, hook]
\end{tikzcd}$$

Since β is a smooth morphism and $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$, we have that

$$\mathscr{F}_{V'}(\beta^*(\mathcal{G})) \subset \mathscr{F}_{V'}(\beta^*(\mathcal{K})).$$

In this way, applying Lemma 6.4.2, one readily checks that

$$\begin{aligned}
\mathscr{F}_Z\big(\varphi^*(\mathcal{G})\big) &= \mathscr{F}_Z\big(\beta^*(\mathcal{G})|_Z\big) \\
&= \mathscr{F}_{V'}\big(\beta^*(\mathcal{G})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_{V'}\big(\beta^*(\mathcal{K})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_Z\big(\beta^*(\mathcal{K})|_Z\big) \\
&= \mathscr{F}_Z\big(\varphi^*(\mathcal{K})\big).
\end{aligned}$$

This proves that $\mathcal{G}$ and $\mathcal{K}$ satisfy condition b*) of Lemma 6.1.1 as required. □

Corollary 6.4.4. *Let S, $V = \operatorname{Spec}(S)$, $\mathcal{D} \subset \operatorname{Der}(S)$, and $\mathcal{G}$ be as in the previous theorem. Fix a non-zero element $f \in S$, and consider the open subscheme $V' = \operatorname{Spec}(S_f) \subset V$. Then, for any Rees algebra $\mathcal{K}$ over S_f,*

$$\mathscr{F}_{V'}(\mathcal{G}_f) \subset \mathscr{F}_{V'}(\mathcal{K}) \iff \mathcal{K} \subset \overline{\mathcal{G}}_f.$$

Proof. By Lemma 3.5.3, we have that $\overline{\mathcal{G}}_f = \overline{(\mathcal{G}_f)}$. Thus it suffices to see that the ring S_f, the Rees algebra $\mathcal{G}_f$, and the submodule $\mathcal{D}_f \subset \operatorname{Der}(S_f)$ are under the hypotheses of the theorem. Namely, that $\mathcal{D}_f$ satisfies the weak Jacobian condition on S_f, and that $\mathcal{G}_f$ is $\mathcal{D}_f$-saturated. The first of these conditions follows from Remark 5.1.4, and the second from Lemma 6.2.8. □

Corollary 6.4.5. *Let S be a regular ring over a field of characteristic zero satisfying the weak Jacobian condition, and let $\mathcal{G}$ be a Rees algebra over S. Set $V = \operatorname{Spec}(S)$. Then, for any Rees algebra $\mathcal{K}$ over S,*

$$\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K}) \iff \mathcal{K} \subset \overline{\operatorname{Der}(S)(\mathcal{G})}.$$

Theorem 6.4.6. *Let V be a regular noetherian scheme over a field of characteristic zero satisfying the weak Jacobian condition, and let $\mathcal{G}$ be a Rees algebra over V. Then there exists a canonical representative of $\mathscr{C}_V(\mathcal{G})$, say $\mathcal{G}^*$, such that any other algebra of $\mathscr{C}_V(\mathcal{G})$ is contained in $\mathcal{G}^*$.*

Proof. [2] Consider an affine covering of V, say $V = \bigcup_{i=1}^r U_i$, with $U_i = \operatorname{Spec}(S_i)$, where each S_i satisfies the weak Jacobian condition. For $i = 1, \dots, r$, set $G_i = \Gamma(U_i, \mathcal{G})$, which is an S_i-Rees algebra. Then define $G_i^* = \overline{\operatorname{Der}(S_i)(G_i)}$, and $\mathcal{G}_i^* = (G_i^*)^\sim$. Observe that, by Proposition 3.6.6 and Lemma 3.5.7, $\mathscr{F}_{U_i}(G_i^*) = \mathscr{F}_{U_i}(G_i)$, and hence $\mathscr{F}_{U_i}(\mathcal{G}_i^*) = \mathscr{F}_{U_i}(\mathcal{G}|_{U_i})$. We claim that $\mathcal{G}_1^*, \dots, \mathcal{G}_r^*$ induce a Rees algebra over V, say $\mathcal{G}^*$, which is the canonical representative of $\mathscr{C}_V(\mathcal{G})$.

Let us start by checking that $\mathcal{G}^*$ is well-defined. For this, we need to verify that $\mathcal{G}_i^*|_{U_i \cap U_j} = \mathcal{G}_j^*|_{U_i \cap U_j}$ for all i, j. Fix i and j. Observe that $U_i \cap U_j$ is an open subset of U_i that can be covered by affine charts of the form $U_{i,f} := \operatorname{Spec}\big((S_i)_f\big)$, with $f \in S_i$. Since $\mathcal{G}_j^*$ is a quasi-coherent sheaf over U_j, and

$$\mathscr{F}_{U_{i,f}}\big((G_i^*)_f\big) = \mathscr{F}_{U_{i,f}}\big(\mathcal{G}_i^*|_{U_{i,f}}\big) = \mathscr{F}_{U_{i,f}}\big(\mathcal{G}|_{U_{i,f}}\big) = \mathscr{F}_{U_{i,f}}\big(\mathcal{G}_j^*|_{U_{i,f}}\big),$$

Corollary 6.4.4 implies that

$$\Gamma(U_{i,f}, \mathcal{G}_j^*) \subset \Gamma(U_{i,f}, \mathcal{G}_i^*) = (G_i^*)_f.$$

Hence $(\mathcal{G}_j^*)|_{U_{i,f}} \subset (\mathcal{G}_i^*)|_{U_{i,f}}$. Applying the same argument on each open subset of the form $U_{i,f} \subset U_i \cap U_j$, we see that $(\mathcal{G}_j^*)|_{U_i \cap U_j} \subset (\mathcal{G}_i^*)|_{U_i \cap U_j}$. Similarly, swapping the roles of i and j, we see that $(\mathcal{G}_i^*)|_{U_i \cap U_j} \subset (\mathcal{G}_j^*)|_{U_i \cap U_j}$, and therefore $(\mathcal{G}_i^*)|_{U_i \cap U_j} = (\mathcal{G}_j^*)|_{U_i \cap U_j}$. Thus $\mathcal{G}_1^*, \dots, \mathcal{G}_r^*$ induce a quasi-coherent sheaf of Rees algebras (i.e., a Rees algebra) over V.

By construction, it is clear that $\mathscr{F}_V(\mathcal{G}^*) = \mathscr{F}_V(\mathcal{G})$. To check that it is the canonical representative of $\mathscr{C}_V(\mathcal{G})$, consider another Rees algebra $\mathcal{K}$ over V with $\mathscr{F}_V(\mathcal{K}) = \mathscr{F}_V(\mathcal{G})$. Observe that $\mathcal{K}$ is a quasi-coherent sheaf over V. Thus Corollary 6.4.4 yields

$$\Gamma(U_i, \mathcal{K}) \subset \Gamma(U_i, \mathcal{G}_i^*) = G_i^*.$$

Thus $\mathcal{K}|_{U_i} \subset \mathcal{G}_i^*$, and therefore $\mathcal{K} \subset \mathcal{G}^*$. □

[2] Along this proof and that of Corollary 6.4.7 we shall make distinction between Rees algebras defined over a ring S, and quasi-coherent sheaves of Rees algebras over $\operatorname{Spec}(S)$ (see Section 3.1). Namely, we shall use roman letters to denote the first ones, and calligraphic letters for the second ones.

Corollary 6.4.7. *Let V be a regular noetherian scheme over a field of characteristic zero satisfying the weak Jacobian condition, and let $\mathcal{G}$ be a Rees algebra over V. Let $\mathcal{G}^*$ denote the canonical representative of $\mathscr{C}_V(\mathcal{G})$ (see Theorem 6.4.6). Then, for any open subscheme $U \subset V$, the $\mathcal{O}_U$-Rees algebra $\mathcal{G}^*|_U$ is the canonical representative of $\mathscr{C}_U(\mathcal{G}|_U)$.*

Proof. Set $\mathcal{K} = \mathcal{G}|_U$, and let $\mathcal{K}^*$ denote the canonical representative of $\mathscr{C}_V(\mathcal{K})$. With this notation, we need to show that $\mathcal{K}^* = \mathcal{G}^*|_U$. Consider an affine chart of U of the form $\mathrm{Spec}(S)$, where S is a ring satisfying the weak Jacobian condition. Note that $\mathrm{Spec}(S)$ is also an affine chart of V. Put $G = \Gamma\big(\mathrm{Spec}(S), \mathcal{G}\big)$, and $G^* = \overline{\mathrm{Der}(S)(G)}$. According to the proof Theorem 6.4.6, we have $\mathcal{G}^*|_{\mathrm{Spec}(S)} = (G^*)^\sim$, and, for the same reason, $\mathcal{K}^*|_{\mathrm{Spec}(S)} = (G^*)^\sim$. That is, $\mathcal{G}^*|_{\mathrm{Spec}(S)} = \mathcal{K}^*|_{\mathrm{Spec}(S)}$. Since U can be covered by affine charts of the form $\mathrm{Spec}(S)$, with S satisfying the weak Jacobian condition, we conclude that $\mathcal{K}^* = \mathcal{G}^*|_U$. □

6.5 Differential saturation in positive characteristic

Recall that, given a Rees algebra $\mathcal{G}$ defined over a regular ring S, we define the (absolute) differential saturation of $\mathcal{G}$, say $\mathrm{Diff}(\mathcal{G})$, as the algebra obtained by adding to $\mathcal{G}$ all the elements of the form $\Delta(f)W^{N-n}$, where $fW^N \in \mathcal{G}$ and $\Delta : S \to S$ is a differential operator of order at most n (see Definition 3.6.7). In general, $\mathrm{Diff}(\mathcal{G})$ is not finitely generated. In this section we prove that, if S has an absolute p-basis (i.e., a p-basis over $\mathbb{F}_p$), then $\mathrm{Diff}(\mathcal{G})$ is finitely generated as a graded algebra over S.

Remark 6.5.1. Let us insist on the fact that $\mathrm{Diff}(\mathcal{G})$ is an intrinsic object constructed from $\mathcal{G}$, whose definition does not depend on the choice of any particular p-basis of S.

Remark 6.5.2. Recall that, according to Lemma 5.3.6, if a regular ring S defined over a field of characteristic $p > 0$ admits an absolute p-basis, then S is excellent.

Lemma 6.5.3. *Let S be a regular ring over $\mathbb{F}_p$ which admits a p-basis, say $\mathcal{B}$. Consider the family of differential operators $D^{[\mathcal{B};\beta]}$, with $\beta \in \mathbb{N}^{\oplus\mathcal{B}}$, as in Proposition 5.2.17. Then a Rees algebra over S, say $\mathcal{G} = S\left[f_1W^{N_1}, \dots, f_rW^{N_r}\right]$, is differentially saturated if and only if, for each $i = 1, \dots, r$, each $\beta \in \mathbb{N}^{\oplus\mathcal{B}}$, and each integer n with $|\beta| \leq n < N_i$, we have that*

$$D^{[\mathcal{B};\beta]}(f_i)W^{N_i-n} \in \mathcal{G}. \tag{6.14}$$

Proof. The "only if" part of the proof is trivial. For the converse, we shall show that, for any element $fW^N \in \mathcal{G}$, and for any differential operator $\Delta \in \mathrm{Diff}^n(S)$ with $n < N$, one has that $\Delta(f)W^{N-n} \in \mathcal{G}$. In order to check this property, we shall start with a particular case, and then we will address the general one.

First note that $D^{[\mathcal{B};0]}$ is the identity map on S. Thus it follows from condition (6.14) that $f_iW^{m_i} \in \mathcal{G}$ for all $m_i \leq N_i$. In this way, setting $\mathcal{G} = \bigoplus_{m\in\mathbb{N}} I_mW^m$, one readily checks that

$$I_{m+1} \subset I_m \tag{6.15}$$

for all $m \in \mathbb{N}$.

Next consider a homogeneous element $hW^N \in \mathcal{G}$ of the form

$$hW^N = (af_{i_1} \cdot \ldots \cdot f_{i_s})W^N,$$

with $a \in S$, and $i_1, \ldots, i_s \in \{1, \ldots, r\}$, with $N_{i_1} + \cdots + N_{i_s} = N$ (and possibly many of the i_j repeated). Note that every homogeneous element of degree N of $\mathcal{G}$ is a sum of terms of this form. In the following lines we shall show that, for any multi-index $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$, and for any integer $n \in \mathbb{N}$ so that $|\beta| \leq n < N$, one has that $D^{[\mathcal{B};\beta]}(h)W^{N-n} \in \mathcal{G}$.

Fix a multi-index $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$ and an integer $n \in \mathbb{N}$ with $|\beta| \leq n < N$ as above. According to Corollary 5.2.20,

$$D^{[\mathcal{B};\beta]}(h) = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_s=\beta} D^{[\mathcal{B};\alpha_0]}(a) \cdot D^{[\mathcal{B};\alpha_1]}(f_{i_1}) \cdot \ldots \cdot D^{[\mathcal{B};\alpha_s]}(f_{i_s}). \tag{6.16}$$

Then, for a collection of multi-indexes $\alpha_0, \alpha_1, \ldots, \alpha_s \in \mathbb{N}^{\oplus \mathcal{B}}$ with $\alpha_0 + \alpha_1 + \cdots + \alpha_s = \beta$, set

$$N'_j = \max\left\{0, N_{i_j} - |\alpha_j|\right\}$$

for $j = 1, \ldots, s$. Under these hypotheses, condition (6.14) implies that

$$D^{[\mathcal{B};\alpha_j]}\left(f_{i_j}\right)W^{N'_j} \in \mathcal{G}. \tag{6.17}$$

Since

$$\sum_{j=1}^{s} N'_j \geq \sum_{j=1}^{s} \left(N_{i_j} - |\alpha_j|\right) = N - |\beta| \geq N - n,$$

it follows from (6.15) and (6.17) that

$$\left(D^{[\mathcal{B};\alpha_1]}(f_{i_1}) \cdot \ldots \cdot D^{[\mathcal{B};\alpha_s]}(f_{i_s})\right) W^{N-n} \in \mathcal{G}.$$

In this way, we deduce from (6.16) that

$$D^{[\mathcal{B};\beta]}(h)W^{N-n} \in \mathcal{G}. \tag{6.18}$$

Next we address the general case: we shall show that, for any element $fW^N \in \mathcal{G}$, and for any differential operator $\Delta \in \mathrm{Diff}^n(S)$ with $n < N$, one has that $\Delta(f)W^{N-n} \in \mathcal{G}$. Since differential operators are linear and every homogeneous element of degree N of $\mathcal{G}$ can be expressed as a sum of terms of the form (6.16), condition (6.18) implies that

$$D^{[\mathcal{B};\beta]}(f)W^{N-n} \in \mathcal{G}$$

for all $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$ with $|\beta| \leq n$. In addition, by Corollary 5.2.19 iii), we have that

$$\Delta(f) = \sum_{|\beta| \leq n} \Delta(\mathcal{B}^{\beta}) \cdot D^{[\mathcal{B};\beta]}(f).$$

Recall that, by Corollary 5.2.19 ii), this is always a finite sum. Thus we conclude that

$$\Delta(f)W^{N-n} = \sum_{|\beta|\leq n} \Delta(\mathcal{B}^{\beta}) \cdot D^{[\mathcal{B};\beta]}(f)W^{N-n}$$

belongs to $\mathcal{G}$. □

Proposition 6.5.4. *Let S be a noetherian ring over $\mathbb{F}_p$ which admits an absolute p-basis, and let $\mathcal{G}$ be a Rees algebra over S. Then the differential saturation of $\mathcal{G}$, which we denote by* $\mathrm{Diff}(\mathcal{G})$*, is finitely generated over S, i.e., it is a Rees algebra over S.*

Proof. Fix a set of generators of $\mathcal{G}$, say $\mathcal{G} = S\left[f_1W^{N_1}, \ldots, f_rW^{N_r}\right]$, and an absolute p-basis of S, say $\mathcal{B}$. Let $\mathcal{G}'$ be the Rees algebra obtained by adding to $\mathcal{G}$ all the elements of the form

$$D^{[\mathcal{B};\beta]}(f_i)W^{N_i-n} \in \mathcal{G},$$

with $\beta \in \mathbb{N}^{\oplus\mathcal{B}}$ and $|\beta| \leq n < N_i$. By Corollary 5.2.19 ii), only finitely many of the previous elements are non-zero. Hence $\mathcal{G}'$ is finitely generated over S, i.e., it is a Rees algebra over S.

By construction, the Rees algebra $\mathcal{G}'$ is differential relative to $\mathcal{G}$. Moreover, using Lemma 6.5.3 and Corollary 5.2.19 i), one readily checks that $\mathcal{G}'$ is differentially saturated. Thus $\mathcal{G}' = \mathrm{Diff}(\mathcal{G})$. □

Corollary 6.5.5 (Localization)**.** *Let S be a noetherian domain over $\mathbb{F}_p$ which admits an absolute p-basis, and let $\mathcal{G}$ be a differentially saturated Rees algebra over S. Then, for any multiplicative subset $\mathcal{U} \subset S$, the $\mathcal{U}^{-1}S$-Rees algebra $\mathcal{U}^{-1}\mathcal{G}$ is differentially saturated.*

Proof. Fix an absolute p-basis of S, say $\mathcal{B}$, and a set of generators of $\mathcal{G}$, say $\mathcal{G} = S\left[f_1W^{N_1}, \ldots, f_rW^{N_r}\right]$. Observe that $f_1W^{N_1}, \ldots, f_rW^{N_r}$ generate $\mathcal{U}^{-1}\mathcal{G}$, and, in virtue of Lemma 5.2.6, the set $\mathcal{B}$ is a p-basis of $\mathcal{U}^{-1}S$. Thus, using the fact that $\mathcal{G}$ is differentially saturated and Lemma 6.5.3, one readily checks that $\mathcal{U}^{-1}\mathcal{G}$ is differentially saturated. □

Lemma 6.5.6. *Let S be a regular ring over $\mathbb{F}_p$ that admits an absolute p-basis. Consider the polynomial ring $S' = S[T_1, \ldots, T_n]$. Then:*

i) S' admits an absolute p-basis.

ii) For any differentially saturated Rees algebra $\mathcal{G}$ over S, the S'-Rees algebra $\mathcal{G}' = \mathcal{G}S'$ is differentially saturated.

Proof. Fix an absolute p-basis of S, say $\mathcal{B}$. By Lemma 5.2.8, the set $\mathcal{B}' = \mathcal{B} \cup \{T_1, \ldots, T_m\}$ is an absolute p-basis of S'. Thus i) holds.

To check ii), fix a set of generators of $\mathcal{G}$, say $\mathcal{G} = S\left[f_1W^{N_1}, \ldots, f_rW^{N_r}\right]$. Observe that $\mathbb{N}^{\oplus\mathcal{B}'} = \mathbb{N}^{\oplus\mathcal{B}} \oplus \mathbb{N}^m$. Fix $\beta' \in \mathbb{N}^{\oplus\mathcal{B}'}$, and put $\beta' = \beta \oplus \alpha$, with

$\beta \in \mathbb{N}^{\oplus \mathcal{B}}$ and $\alpha \in \mathbb{N}^m$. Following the notation of Proposition 5.2.17, consider the differential operator on S associated to $\mathcal{B}$ and β, say

$$D^{[\mathcal{B};\beta]} : S \longrightarrow S,$$

and that on S' associated to $\mathcal{B}'$ and β', say

$$D^{[\mathcal{B}';\beta']} : S' \longrightarrow S'.$$

Arguing as in Remark 5.2.21, one can check that

$$D^{[\mathcal{B}';\beta']}(f_i) = D^{[\mathcal{B};\beta]}(f_i)$$

for $i = 1, \dots, r$. Since $\mathcal{G}$ is differentially saturated we have that, for any n with $|\beta| \leq n \leq N_i$,

$$D^{[\mathcal{B};\beta]}(f_i)W^{N_i-n} \in \mathcal{G}.$$

Therefore, for any n with $|\beta'| \leq n \leq N_i$, we have that

$$D^{[\mathcal{B}';\beta']}(f_i)W^{N_i-n} \in \mathcal{G}.$$

As $\mathcal{G}'$ is generated by $f_1W^{N_1}, \dots, f_rW^{N_r}$, it follows from Lemma 6.5.3 that $\mathcal{G}'$ is differentially saturated. □

Lemma 6.5.7. *Let R be a regular local ring of characteristic $p > 0$ which admits an absolute p-basis. Consider a regular quotient of R, say $R' = R/J$. Then:*

i) R' admits an absolute p-basis.

ii) For any differentially saturated Rees algebra $\mathcal{G}$ over R, the R'-Rees algebra $\mathcal{G}' = \mathcal{G}R'$ is differentially saturated.

Proof. Since R' is regular, there exists a regular system of parameters of R, say $x_1, \dots, x_d$, so that $J = \langle x_1, \dots, x_e \rangle$, where e represents the codimension of R'. In addition, by Proposition 5.3.1, R admits a p-basis of the form $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \dots, x_d\}$, where $\mathbb{F}_p(\mathcal{B}_0)$ is a quasi-coefficient field of R (see Corollary 5.3.2). Under these hypotheses, Lemma 5.3.7 ensures that the image of $\mathcal{B}_0 \cup \{x_{e+1}, \dots, x_d\}$ in R', which we shall denote by $\mathcal{B}'$, is a p-basis of R'. Thus i) holds.

For ii), fix a set of generators of $\mathcal{G}$, say $\mathcal{G} = R\left[f_1W^{N_1}, \dots, f_rW^{N_r}\right]$. By assumption we have that, for every $\beta \in \mathbb{N}^{\oplus \mathcal{B}}$, and each integer n, with $|\beta| \leq n < N_i$,

$$D^{[\mathcal{B};\beta]}(f_i)W^{N_i-n} \in \mathcal{G}. \tag{6.19}$$

Let us denote the images of $f_1, \dots, f_r$ in R' by $\overline{f_1}, \dots, \overline{f_r}$ respectively. With this notation, $\mathcal{G}' = R'\left[\overline{f_1}W^{N_1}, \dots, \overline{f_r}W^{N_r}\right]$. Then (6.19), together with the second part of Lemma 5.3.7, imply that, for every $\beta' \in \mathbb{N}^{\oplus \mathcal{B}'}$, and every n with $|\beta'| \leq n < N_i$,

$$D^{[\mathcal{B}';\beta']}\left(\overline{f_i}\right)W^{N_i-n} \in \mathcal{G}'.$$

Hence $\mathcal{G}'$ is differentially saturated by Lemma 6.5.3. □

6.6 Canonical representatives in positive characteristic

Lemma 6.6.1. *Let S be a regular ring over $\mathbb{F}_p$ which has an absolute p-basis, and let $\mathcal{G}$ be a Rees algebra over S. Set $V = \operatorname{Spec}(S)$. If $\mathcal{G}$ is differentially saturated, then*

$$\operatorname{Sing}_V(\mathcal{G}) = \operatorname{Zeros}_V(\mathcal{G}).$$

Remark 6.6.2. This result says that $\mathcal{G}$ fulfills condition a) of Lemma 6.1.1.

Proof. Fix a point $\xi \in V$. Clearly $\xi \in \operatorname{Sing}_V(\mathcal{G})$ implies $\xi \in \operatorname{Zeros}_V(\mathcal{G})$. Conversely, assume that $\xi \notin \operatorname{Sing}_V(\mathcal{G})$. Then there should be a homogeneous element $fW^N \in \mathcal{G}$ with $\nu_\xi(f) < N$. Put $\mathcal{G} = \bigoplus_{i\in\mathbb{N}} I_i W^i$, and let $\mathfrak{p} \subset S$ denote the prime ideal corresponding to ξ. According to Proposition 5.4.7, $\operatorname{Diff}^{N-1}(f) \not\subseteq \mathfrak{p}$, and, since $\mathcal{G}$ is differentially saturated, $\operatorname{Diff}^{N-1}(f) \subset I_1$. Therefore $\xi \notin \operatorname{Zeros}(\mathcal{G})$. □

Remark 6.6.3. Let R be a regular local ring of characteristic $p > 0$ which admits an absolute p-basis. Fix a regular system of parameters of R, say $x_1, \dots, x_d$, and a p-basis of R as in Proposition 5.3.1, say $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \dots, x_d\}$. Recall that, by Corollary 5.3.2, $\mathbb{F}_p(\mathcal{B}_0)$ can be extended to a unique coefficient field of $\widehat{R}$, say k_0. Consider the regular quotient ring $R' = R/\langle x_1\rangle$. Observe that k_0 can also be regarded as a coefficient field of $\widehat{R'} = \widehat{R}/\langle x_1\rangle$. Then, by Cohen's structure theorem [12, Theorem 9, p. 72], we have isomorphisms $\widehat{R} \simeq k_0[[x_1, \dots, x_d]]$ and $\widehat{R'} \simeq k_0[[x_2, \dots, x_d]]$. In this way, we see that there is a natural retraction of the quotient map $\varepsilon : \widehat{R'} \hookrightarrow \widehat{R}$ given by the inclusion of power series rings $k_0[[x_2, \dots, x_d]] \subset k_0[[x_1, x_2, \dots, x_d]]$. Thus $\widehat{R} \simeq \widehat{R'}[[x_1]]$ and every element of $\widehat{R}$ admits an expansion of the form $f = \sum_{i=0}^{\infty} \varepsilon(a_i)x_1^i$ with $a_i \in \widehat{R'}$ for $i \in \mathbb{N}$.

Next, consider the multi-index $\beta = (n\delta_{x_1 b})_{b\in\mathcal{B}} \in \mathbb{N}^{\oplus\mathcal{B}}$, where $\delta_{x_1 b}$ represents Kronecker's delta (i.e., β is the multi-index whose entries are all zero, except that corresponding to x_1, which is equal to n). Here we shall denote the differential operator $D^{[\mathcal{B};\beta]} \in \operatorname{Diff}^n(R)$ by $D^{[\mathcal{B};x_1^n]}$. Since $D^{[\mathcal{B};x_1^n]}$ annihilates every monomial in the elements of $\mathcal{B} \cup \{x_2, \dots, x_d\}$, it follows that $D^{[\mathcal{B};x_1^n]}$ vanishes at the subring $k_0[[x_2, \dots, x_d]] \subset \widehat{R}$. In this way, for each element $f \in \widehat{R}$, setting $f = \sum_{i=0}^{\infty} \varepsilon(a_i)x_1^i$ with $a_i \in \widehat{R'}$, one readily checks that

$$D^{[\mathcal{B};x_1^n]}(f) \equiv a_n \text{ modulo } \langle x_1\rangle\widehat{R}.$$

Lemma 6.6.4. *Let S be a regular local ring of characteristic $p > 0$ which admits an absolute p-basis. Set $V = \operatorname{Spec}(S)$, and consider a regular hypersurface $H = \operatorname{Spec}(S/J) \subset V$. Let $\mathcal{H} = S[JW]$ denote the Rees algebra attached to the immersion of H in V (see Remark 3.7.1). Then, for any differentially saturated Rees algebra over S, say $\mathcal{G}$, the following conditions hold:*

i) $\operatorname{Sing}_H(\mathcal{G}|_H) = \operatorname{Sing}_V(\mathcal{G}) \cap H$.

ii) $\mathscr{F}_H(\mathcal{G}|_H) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H})$.

Proof. We start by i). The inclusion $\operatorname{Sing}_V(\mathcal{G}) \cap H \subset \operatorname{Sing}_H(\mathcal{G}|_H)$ always holds. For the converse, we shall show that, for each $\xi \in H$,

$$\xi \in \operatorname{Sing}_H(\mathcal{G}|_H) \implies \xi \in \operatorname{Sing}_V(\mathcal{G}).$$

Fix a point $\xi \in \operatorname{Sing}_H(\mathcal{G}|_H)$ and a homogeneous element $fW^N \in \mathcal{G}$. Set $R = \mathcal{O}_{V,\xi}$, $\mathfrak{M} = \mathfrak{M}_{V,\xi}$, $R' = \mathcal{O}_{H,\xi}$, and $\mathfrak{M}' = \mathfrak{M}_{H,\xi}$. Since H is a regular hypersurface in V, there should be a regular system of parameters of R, say $x_1, \ldots, x_d$, so that $R' = R/\langle x_1 \rangle$. Note that R is a localization of S. Then, by Lemma 5.2.6, R admits a p-basis, and, in virtue of Corollary 6.5.5, $\mathcal{G}_\xi$ is a differentially saturated algebra over R.

In virtue of Proposition 5.3.1, consider a p-basis of R of the form $\mathcal{B} = \mathcal{B}_0 \cup \{x_1, \ldots, x_d\}$, and let us proceed as in Remark 6.6.3. Namely, observe that f, regarded as an element of $\widehat{R}$, has a an expansion of the form $f = \sum_{i=0}^{\infty} \varepsilon(a_i) x_1^i$ with $a_i \in \widehat{R'}$ for $i \in \mathbb{N}$. Hence

$$f \equiv \varepsilon(a_0) + \varepsilon(a_1)x_1 + \cdots + \varepsilon(a_{N-1})x_1^{N-1} \text{ modulo } \langle x_1 \rangle^N \widehat{R}. \tag{6.20}$$

In addition, using the notation of Remark 6.6.3,

$$D^{[\mathcal{B};x_1^n]}(f) \equiv a_n \text{ modulo } \langle x_1 \rangle \widehat{R}$$

for all $n < N$. Since $\mathcal{G}_\xi$ is differentially saturated, this implies that $a_n W^{N-n} \in \mathcal{G}_\xi R'$ for $n < N$.

Since $\xi \in \operatorname{Sing}_H(\mathcal{G}|_H)$, it follows from the previous discussion that $\nu_{H,\xi}(a_n) \geq N - n$ for all $0 \leq n < N$. Moreover,

$$\begin{aligned} \nu_{H,\xi}(a_n) \geq N - n &\implies a_n \in (\mathfrak{M}')^{(N-n)} \widehat{R'} \\ &\implies \varepsilon(a_n) \in \mathfrak{M}^{(N-n)} \widehat{R}. \end{aligned}$$

In this way, it follows from (6.20) that $f \in \mathfrak{M}^N \widehat{R}$. As $R \to \widehat{R}$ is a faithfully flat morphism, we have that $\mathfrak{M}^N = R \cap \mathfrak{M}^N \widehat{R}$, and therefore $f \in \mathfrak{M}^N$. That is, $\nu_{V,\xi}(f) \geq N$.

Repeating the previous argument for each homogeneous element $fW^N \in \mathcal{G}$, we conclude that $\xi \in \operatorname{Sing}_V(\mathcal{G})$. Therefore i) holds.

The proof of ii) is analogous to that of Lemma 6.4.1 ii) (it only requires to replace the expansion of (6.10) by that of (6.20)). □

Lemma 6.6.5. *Let R be a regular local ring of characteristic $p > 0$ which admits an absolute p-basis. Set $U = \operatorname{Spec}(R)$, and consider a regular closed subscheme $Z = \operatorname{Spec}(R/J) \subset U$. Let $\mathcal{Z} = S[JW]$ denote the Rees algebra attached to the immersion of Z in U (see Remark 3.7.1). Then, for any differentially saturated Rees algebra over R, say $\mathcal{G}$, we have that*

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_U(\mathcal{G}) \cap \mathscr{F}_U(\mathcal{Z}).$$

Proof. Consider a regular system of parameters of R, say $x_1, \dots, x_d$, so that $J = \langle x_1, \dots, x_e \rangle$, where e denotes the codimension of Z. We proceed by induction on e.

If $e = 1$, the scheme Z is a regular hypersurface of U and the result follows from Lemma 6.6.4.

Otherwise, if $e > 1$, consider the regular hypersurface $H = \operatorname{Spec}(R/\langle x_1 \rangle)$, together with the Rees algebra attached to its immersion in V, say $\mathcal{H} = R[x_1 W]$. According to Lemma 6.6.4,

$$\mathscr{F}_H(\mathcal{G}|_H) = \mathscr{F}_U(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H}).$$

In addition, by Lemma 6.5.7, the quotient ring $R/\langle x_1 \rangle$ admits an absolute p-basis, and $\mathcal{G}|_H$ is differentially saturated algebra over $R/\langle x_1 \rangle$. Note also that $(\mathcal{G}|_H)|_Z = \mathcal{G}_Z$. Thus, by the inductive hypothesis,

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_H(\mathcal{G}|_H) \cap \mathscr{F}_H(\mathcal{Z}|_H).$$

Finally, as $\mathscr{F}_H(\mathcal{Z}|_H) = \mathscr{F}_U(\mathcal{Z})$, combining the previous identities we get

$$\begin{aligned} \mathscr{F}_Z(\mathcal{G}|_Z) &= \mathscr{F}_H(\mathcal{G}|_H) \cap \mathscr{F}_H(\mathcal{Z}|_H) \\ &= \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{H}) \cap \mathscr{F}_V(\mathcal{Z}) \\ &= \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}). \end{aligned}$$ □

Lemma 6.6.6. *Let S be a regular ring over a field of characteristic $p > 0$ which admits an absolute p-basis. Set $V = \operatorname{Spec}(S)$, and consider a regular closed subscheme $Z = \operatorname{Spec}(S/J) \subset V$. Then, for any differentially saturated Rees algebra over S, say $\mathcal{G}$, we have that*

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}),$$

where $\mathcal{Z} = S[JW]$ represents the Rees algebra attached t the embedding of Z in V.

Proof. Fix a prime ideal $\mathfrak{p} \subset S$. Set $U = \operatorname{Spec}(S_{\mathfrak{p}})$ and $Y = \operatorname{Spec}(S_{\mathfrak{p}}/JS_{\mathfrak{p}}) \subset U$. By Lemma 5.2.6, $S_{\mathfrak{p}}$ admits a p-basis over $\mathbb{F}_p$ and, by Corollary 6.5.5, $\mathcal{G}_{\mathfrak{p}}$ is a differentially saturated algebra over $S_{\mathfrak{p}}$. Note also that $\mathcal{Z}_{\mathfrak{p}}$ coincides with the Rees algebra attached to the immersion of Y in U. In this way, Lemma 6.6.5 says that

$$\mathscr{F}_Y(\mathcal{G}_{\mathfrak{p}}|_Y) = \mathscr{F}_U(\mathcal{G}_{\mathfrak{p}}) \cap \mathscr{F}_U(\mathcal{Z}_{\mathfrak{p}}).$$

Repeating this argument for each prime ideal $\mathfrak{p} \subset S$, we conclude that

$$\mathscr{F}_Z(\mathcal{G}|_Z) = \mathscr{F}_V(\mathcal{G}) \cap \mathscr{F}_V(\mathcal{Z}).$$ □

Theorem 6.6.7. *Let S be a regular ring over $\mathbb{F}_p$ that admits an absolute p-basis and set $V = \operatorname{Spec}(S)$. Then, for any couple of Rees algebras over S, say $\mathcal{G}$ and $\mathcal{K}$,*

$$\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K}) \iff \mathcal{K} \subset \overline{\operatorname{Diff}(\mathcal{G})}.$$

Proof. Since $\mathcal{G} \subset \overline{\mathrm{Diff}(\mathcal{G})}$, the implication $\mathcal{K} \subset \overline{\mathrm{Diff}(\mathcal{G})} \Rightarrow \mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$ is clear. Next we proceed with the converse.

Suppose that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$. Note that, by Lemma 3.6.4,

$$\mathscr{F}_V\left(\mathrm{Diff}(\mathcal{G})\right) = \mathscr{F}_V(\mathcal{G}).$$

Thus we may assume without loss of generality that $\mathcal{G} = \mathrm{Diff}(\mathcal{G})$ and, in this way, the problem is reduced to check that $\mathcal{K} \subset \overline{\mathcal{G}}$. To this end, it suffices to show that $\mathcal{G}$ and $\mathcal{K}$ are under the hypotheses of Lemma 6.1.1. That is, that $\mathcal{G}$ and $\mathcal{K}$ fulfill conditions a), a*), b), and b*) of the lemma.

Since $\mathcal{G} = \mathrm{Diff}(\mathcal{G})$, condition a) follows from Lemma 6.6.1, and then a*) follows from Lemma 6.1.4. On the other hand, condition b) is satisfied by assumption, and it just remains to check that b*) holds. That is, we need to verify that, for any morphism of finite type from a regular scheme Z to V, say $\varphi : Z \to V$, there is an inclusion

$$\mathscr{F}_Z\left(\varphi^*(\mathcal{G})\right) \subset \mathscr{F}_Z\left(\varphi^*(\mathcal{K})\right). \tag{6.21}$$

Fix $\varphi : Z \to V$ as above. Following the strategy of Remark 6.1.5, note that φ is locally given by an affine morphism of the form

$$\mathrm{Spec}(S) \longleftarrow \mathrm{Spec}(S[T_1, \ldots, T_n]/J).$$

Here $T_1, \ldots, T_n$ represent variables and J is an ideal contained in the polynomial ring $S[T_1, \ldots, T_n]$. Observe that there is a commutative diagram

$$\begin{tikzcd}
 & V' = \mathrm{Spec}(S[T_1, \ldots, T_n]) \arrow[dl, "\beta"'] \\
V = \mathrm{Spec}(S) & Z = \mathrm{Spec}(S[T_1, \ldots, T_n]/J). \arrow[l, "\varphi"] \arrow[u, hook]
\end{tikzcd}$$

Since β is a smooth morphism and $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{K})$, we have that

$$\mathscr{F}_{V'}\left(\beta^*(\mathcal{G})\right) \subset \mathscr{F}_{V'}\left(\beta^*(\mathcal{K})\right).$$

In this way, applying Lemma 6.6.6, one readily checks that

$$\begin{aligned}
\mathscr{F}_Z\big(\varphi^*(\mathcal{G})\big) &= \mathscr{F}_Z\big(\beta^*(\mathcal{G})|_Z\big) \\
&= \mathscr{F}_{V'}\big(\beta^*(\mathcal{G})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_{V'}\big(\beta^*(\mathcal{K})\big) \cap \mathscr{F}_{V'}\big(\mathcal{Z}\big) \\
&\subset \mathscr{F}_Z\big(\beta^*(\mathcal{K})|_Z\big) \\
&= \mathscr{F}_Z\big(\varphi^*(\mathcal{K})\big).
\end{aligned}$$

This proves (6.21), and thus the result follows. □

Theorem 6.6.8. *Let V be a regular excellent scheme over $\mathbb{F}_p$ that has an affine covering of the form $V = \bigcup_{i=1}^m \mathrm{Spec}(S_i)$, where each S_i admits an absolute p-basis. Then, for every Rees algebra $\mathcal{G}$ over V, there exists a canonical representative of $\mathscr{C}_V(\mathcal{G})$, say $\mathcal{G}^*$, such that any other Rees algebra of $\mathscr{C}_V(\mathcal{G})$ is contained in $\mathcal{G}^*$.*

Proof. [3] For $i = 1, \dots, m$, set $U_i = \mathrm{Spec}(S_i) \subset V$, and $G_i = \Gamma(U_i, \mathcal{G})$, which is a Rees algebra over S_i. Set $G_i^* = \overline{\mathrm{Diff}(G_i)}$, and $\mathcal{G}_i^* = (G_i^*)^{\sim}$. Recall that G_i^* is finitely generated over S_i (see Proposition 6.5.4, and Lemma 3.5.2), and $\mathscr{F}_{U_i}(\mathcal{G}_i) = \mathscr{F}_{U_i}(\mathcal{G}_i^*)$ (see Proposition 3.6.6, and Lemma 3.5.7). We claim that $\mathcal{G}_1^*, \dots, \mathcal{G}_m^*$ induce a Rees algebra over V that is the canonical representative of $\mathscr{C}_V(\mathcal{G})$.

Let us start by checking that $\mathcal{G}^*$ is a well-defined Rees algebra over V. To this end, we shall verify that $\mathcal{G}_i^*|_{U_i \cap U_j} = \mathcal{G}_j^*|_{U_i \cap U_j}$ for all i, j. Fix i and j. Note that $\mathcal{G}_i^*|_{U_i \cap U_j}$ and $\mathcal{G}_j^*|_{U_i \cap U_j}$ are two quasi-coherent subsheaves of $\mathcal{O}_{U_i \cap U_j}[W]$. In this way, given a point $\xi \in U_i \cap U_j$, Corollary 6.5.5 and Lemma 3.5.3 imply that

$$(\mathcal{G}_i^*)_\xi = \overline{\mathrm{Diff}(\mathcal{G}_\xi)} = (\mathcal{G}_j^*)_\xi.$$

Since this equality holds at each point of $\in U_i \cap U_j$, it follows that $\mathcal{G}_i^*|_{U_i \cap U_j} = \mathcal{G}_j^*|_{U_i \cap U_j}$, which proves that $\mathcal{G}^*$ is well-defined.

In order to see that $\mathcal{G}^*$ is the canonical representative of the class of $\mathcal{G}$, we shall show that, for any Rees algebra $\mathcal{K} \in \mathscr{C}_V(\mathcal{G})$, one has that $\mathcal{K} \subset \mathcal{G}^*$. Indeed, if $\mathcal{K} \in \mathscr{C}_V(\mathcal{G})$, Theorem 6.6.7 says that

$$\Gamma(U_i, \mathcal{K}) \subset \overline{\mathrm{Diff}(G_i)} = \Gamma(U_i, \mathcal{G}^*).$$

Since $\mathcal{K}$ and $\mathcal{G}^*$ are quasi-coherent sheaves and U_i is an affine subset of V, this implies that $\mathcal{K}|_{U_i} \subset \mathcal{G}^*|_{U_i}$. Hence $\mathcal{K} \subset \mathcal{G}^*$. □

Remark 6.6.9. From the previous proof it follows that $(\mathcal{G}^*)_\xi = \overline{\mathrm{Diff}(\mathcal{G}_\xi)}$ for every $\xi \in V$.

Corollary 6.6.10. *Let V, $\mathcal{G}$, and $\mathcal{G}^*$ be as in the previous theorem. Then, for any open subscheme $U \subset V$, the Rees algebra $\mathcal{G}^*|_U$ is the canonical representative of $\mathscr{C}_U(\mathcal{G}|_U)$.*

Proof. It follows from the same arguments exhibited in the proof of the theorem. □

Corollary 6.6.11. *Let V be a regular variety over an arbitrary field k of any characteristic, and let $\mathcal{G}$ be a Rees algebra over V. Then there is a canonical representative of $\mathscr{C}_V(\mathcal{G})$, say $\mathcal{G}^*$, so that any other algebra of $\mathscr{C}_V(\mathcal{G})$ is contained in $\mathcal{G}^*$.*

Proof. If k has characteristic zero, the result follows from Proposition 5.1.7 and Theorem 6.4.6. The case of positive characteristic follows from Proposition 5.3.12 and Theorem 6.6.8. □

[3] Along this proof we shall make distinction between Rees algebras defined over a ring S, and quasi-coherent sheaves of Rees algebras over $\mathrm{Spec}(S)$.

Chapter 7

Simplification of n-fold points

Consider an equidimensional excellent scheme X defined over a field of characteristic zero. Assume that X has maximum multiplicity n. In this chapter we prove that, given a suitable finite morphism from X to a regular scheme V, it is possible to construct a sequence of blow-ups along closed regular equimultiple centers, say

$$X \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \longleftarrow \cdots \xleftarrow{\pi_l} X_l,$$

so that the maximum multiplicity of X drops. That is, so that $\max \mathrm{mult}_{X_l} < \max \mathrm{mult}_X$. This result was already known for varieties defined over a field of characteristic zero (see [34]). Our aim is to generalize it to a more general class of schemes. The precise statement of this result is formulated in Theorem 7.1.1.

7.1 The statement

Before formulating the main theorem, let us fix some notation. Given an equidimensional noetherian scheme X, we shall denote the set of points of multiplicity n of X by $F_n(X)$. By a *F_n-permissible transformation* of X, say $X \leftarrow X_1$, we will understand the blow-up of X along a closed regular center contained in $F_n(X)$, or an open restriction, or the multiplication of X by an affine line, say $X_1 = X \times \mathbb{A}^1$ (compare with the definition of permissible transformation for a Rees algebra, on p. 38). A sequence of transformations, say

$$X \xleftarrow{\varphi_1} X_1 \xleftarrow{\varphi_2} X_2 \longleftarrow \cdots \xleftarrow{\varphi_l} X_l,$$

will be called a *F_n-permissible sequence* if each φ_i is a F_n-permissible transformation.

Theorem 7.1.1. *Let X be an equidimensional scheme endowed with a finite and dominant morphism $\beta : X \to V$, where V is an irreducible regular scheme over a field of characteristic zero satisfying the weak Jacobian condition* (*see Definition 5.1.3*). *Suppose that β has generic rank n. Then:*

i) $\max \operatorname{mult}_X \leq n$.

In addition, if the equality holds in i):

ii) $\underline{\operatorname{Max}} \operatorname{mult}_X$ *is a closed subset of X that is mapped homeomorphically onto its image in V via β. Moreover, a closed subset $Y \subset \underline{\operatorname{Max}} \operatorname{mult}_X$ is regular if and only if $\beta(Y)$ is so.*

iii) If the multiplicity is not constant along X, then one can construct a sequence of blow-ups along closed regular equimultiple centers, say

$$X \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \longleftarrow \cdots \xleftarrow{\pi_l} X_l,$$

so that $\max \operatorname{mult}_{X_l} < n$.

Some ideas behind the theorem

Consider a finite projection $\beta : X \to V$ of generic rank n as in Theorem 7.1.1. Recall that, under these hypotheses, Zariski's formula for finite morphisms (Theorem A.0.2) says that

$$\operatorname{mult}_X(\xi) \leq n \cdot \operatorname{mult}_V(\beta(\xi)),$$

for all $\xi \in X$. Since V is regular, this yields $\max \operatorname{mult}_X \leq n$. Zariski's formula also implies that, if ξ is a n-fold point of X, it is the unique point in the fiber of $\beta(\xi)$. Thus $F_n(X)$ is mapped injectively to its image in V. Moreover, it can be shown that the bijection induced by β between $F_n(X)$ and its image in V is a homeomorphism and that a closed subscheme $Y \subset F_n(X)$ is regular if and only if $\beta(Y)$ is so (see [34, Proposition 6.3, p. 349]). This proves i) and ii).

From ii) it follows that any regular center $Y \subset F_n(X)$ induces a regular center on V, say $\beta(Y) \subset V$. Denote by $X \leftarrow X_1$ and $V \leftarrow V_1$ the corresponding blow-ups along Y and $\beta(Y)$ respectively. In general, finite maps are not preserved by blow-ups. However, in this case there is a natural commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & X_1 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} \\ V & \longleftarrow & V_1, \end{array}$$

where β_1 is finite of generic rank n (see [2, §3]). In particular, $\beta_1 : X_1 \to V_1$ is again under the hypotheses of Theorem 7.1.1. Thus, applying an inductive argument, one can see that any sequence of F_n-permissible blow-ups on X, say

$$X \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots \longleftarrow X_l,$$

induces a natural sequence of blow-ups along regular centers on V, and a commutative diagram

$$\begin{tikzcd} X & X_1 \arrow[l] & X_2 \arrow[l] & \cdots \arrow[l] & X_l \arrow[l] \\ V \arrow[u, leftarrow, "\beta"] & V_1 \arrow[l] \arrow[u, leftarrow, "\beta_1"] & V_2 \arrow[l] \arrow[u, leftarrow, "\beta_2"] & \cdots \arrow[l] & V_l, \arrow[l] \arrow[u, leftarrow, "\beta_l"] \end{tikzcd}$$

where each β_i is finite of generic rank n and maps $F_n(X_i)$ homeomorphically onto its image in V_i.

In order to prove iii) we will construct a Rees algebra $\mathcal{G}$ over V with the following properties (see Lemma 7.2.1):

- $F_n(X)$ is homeomorphic to $\mathrm{Sing}_V(\mathcal{G})$ via β.
- Any local sequence of $\mathcal{G}$-permissible transformations on V, say

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_2 & & \mathcal{G}_l \\ V & V_1 \arrow[l] & V_2 \arrow[l] & \cdots \arrow[l] & V_l, \arrow[l] \end{tikzcd}$$

  induces a F_n-permissible sequence X, and a commutative diagram

$$\begin{tikzcd} X & X_1 \arrow[l] & X_2 \arrow[l] & \cdots \arrow[l] & X_l \arrow[l] \\ V \arrow[u, leftarrow, "\beta"] & V_1 \arrow[l] \arrow[u, leftarrow, "\beta_1"] & V_2 \arrow[l] \arrow[u, leftarrow, "\beta_2"] & \cdots \arrow[l] & V_l, \arrow[l] \arrow[u, leftarrow, "\beta_l"] \end{tikzcd}$$

  where each β_i is a finite and dominant morphism of generic rank n.
- The set $F_n(X_i)$ is homeomorphic to $\mathrm{Sing}_{V_i}(\mathcal{G}_i)$ via β_i for $i = 1, \dots, l$ (in particular, $\mathrm{Sing}_{V_l}(\mathcal{G}_l) = \emptyset$ if and only if $\max \mathrm{mult}_{X_l} < n$).

Then we shall show that, given such $\mathcal{G}$, there is an algorithm that produces a sequence of $\mathcal{G}$-permissible blow-ups, say

$$\begin{tikzcd} \mathcal{G} & \mathcal{G}_1 & \mathcal{G}_2 & & \mathcal{G}_l \\ V & V_1 \arrow[l, "\pi_1"'] & V_2 \arrow[l, "\pi_2"'] & \cdots \arrow[l] & V_l, \arrow[l, "\pi_l"'] \end{tikzcd}$$

so that $\mathrm{Sing}_{V_l}(\mathcal{G}_l) = \emptyset$. Such an algorithm is known to exist for the case in which V is a regular variety over a field of characteristic zero (see [15]). In Section 7.3 we extend it to a more general class of schemes defined over a field of characteristic zero (see Theorem 7.3.9).

Remark 7.1.2. In Sections 3.3 and 4.1 we gave representations of the set of n-fold points of X by means of local embeddings in a regular ambient space. By contrast, in this section we intend to use finite projections onto a regular scheme instead of embeddings. An advantage of this method is that it does not increase the original dimension of the problem. Namely, given a d-dimensional scheme X, we try to find a presentation of $F_n(X)$ in a regular scheme V of dimension d, and not higher. The idea of projecting the set of n-fold points of X was introduced in [34]. For further discussion see [2] and [10].

7.2 Representation via finite morphisms

Lemma 7.2.1. *Let $\beta : X \to V$ be a finite morphism as in Theorem 7.1.1. Assume that β has generic rank $n = \max \mathrm{mult}_X$. Then there exists a Rees algebra over V, say $\mathcal{G}$, that represents $F_n(X)$ in the following sense:*

i) *$F_n(X)$ is homeomorphic to $\mathrm{Sing}_V(\mathcal{G})$ via β.*

ii) *Any local sequence of $\mathcal{G}$-permissible transformations on V, say*

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_l \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & \cdots & \longleftarrow & V_l, \end{array}$$

induces a sequence of F_n-permissible transformations on X, and a commutative diagram as follows,

$$\begin{CD} X @<<< X_1 @<<< X_2 @<<< \cdots @<<< X_l \\ @V{\beta}VV @V{\beta_1}VV @V{\beta_2}VV @. @V{\beta_l}VV \\ V @<<< V_1 @<<< V_2 @<<< \cdots @<<< V_l, \end{CD}$$

where each β_i is a finite and dominant morphism of generic rank n (i.e., β_i is under the hypotheses of Theorem 7.1.1).

iii) *For $i = 1, \dots, l$, the set $F_n(X_i)$ is homeomorphic to $\mathrm{Sing}_{V_i}(\mathcal{G}_i)$.*

In particular, a resolution of $\mathcal{G}$ induces a lowering of the maximum multiplicity of X.

Let us start by proving the result in the affine case. The general statement is proved right after the affine one.

Lemma 7.2.2 (cf. [34, Theorem 6.8, p. 352])**.** *Let S be a regular domain over a field of characteristic zero, and B a finite equidimensional extension of S. Put $X = \mathrm{Spec}(B)$, $V = \mathrm{Spec}(S)$, and let $\beta : X \to V$ denote the morphism induced by the inclusion of algebras $S \subset B$. Assume that β has generic rank $n = \max \mathrm{mult}_X$. Then there exists a Rees algebra over S, say $\mathcal{G}$, satisfying conditions i), ii) and iii) of Lemma 7.2.1.*

Proof. Since β has generic rank n, and this is also the maximum multiplicity of X, we know that β maps $F_n(X)$ homeomorphically into its image in V. Moreover, this homeomorphism is preserved by permissible blow-ups and local sequences (see the discussion on p. 120).

Next fix a presentation of B over S, say $B = S[\theta_1, \dots, \theta_r]$, with $\theta_1, \dots, \theta_r$, with $\theta_1, \dots, \theta_r$ integral over S. This presentation induces a surjective homomorphism, say

$$S[T_1, \dots, T_r] \longrightarrow B = S[\theta_1, \dots, \theta_r],$$

and a closed immersion of X into the regular ambient space

$$V' = \mathrm{Spec}(S[T_1, \dots, T_r]).$$

Let $f_i(T_i)$ denote the minimum polynomial of θ_i over $K = \mathrm{Frac}(S)$. Put $n_i = \deg(f_i(T_i))$, and let us assume without loss generality that $n_i \geq 2$ for all i. Since S is normal, $f_i(T_i)$ has coefficients in S, i.e., $f_i(T_i) \in S[T_i]$. Then, by [34, Proposition 5.7, p. 343], we have that the Rees algebra

$$\mathcal{G}' = \mathcal{O}_{V'}\left[f_1(T_1)W^{n_1}, \dots, f_r(T_r)W^{n_r}\right]$$

represents the set of n-fold points of X in the sense of Section 3.3 (via the closed immersion $X \hookrightarrow V'$). That is,

$$\mathrm{Sing}_{V'}(\mathcal{G}') = F_n(X),$$

and this equality is preserved by permissible blow-ups and local sequences.

When S is defined over a field of characteristic zero, one can use the coefficients of $f_1(T_1), \dots, f_r(T_r)$ to construct an elimination algebra of $\mathcal{G}'$ on V, say $\mathcal{G}$ (see [34, Theorem 3.5, p. 332]). Then, by [34, Theorem 6.8, p. 352], it follows that $\mathcal{G}$ represents the set of n-fold points of X in the sense of the Lemma. □

Proof of Lemma 7.2.1. Fix an affine covering of V, say $V = \bigcup_{i=1}^{m} U_i$, with $U_i = \mathrm{Spec}(S_i)$. Note that $\beta^{-1}(U_i)$ is also an affine subset of X, put $\beta^{-1}(U_i) = \mathrm{Spec}(B_i)$, and β is locally given by the finite extension $S_i \subset B_i$ at this chart. According to Lemma 7.2.2, we can construct an $\mathcal{O}_{U_i}$-Rees algebra satisfying conditions i), ii) and iii) locally at U_i. The let $\mathcal{G}_i^*$ denote the canonical representative of $\mathscr{C}_{U_i}(\mathcal{G}_i)$, the class of equivalence of $\mathcal{G}_i$ (which exists by Theorem 6.4.6). We claim that $\mathcal{G}_1^*, \dots, \mathcal{G}_m^*$ patch, defining a Rees algebra over V that satisfies conditions i), ii) and iii).

In order to verify the claim, we need to check that $\mathcal{G}_i^*|_{U_i \cap U_j} = \mathcal{G}_j^*|_{U_i \cap U_j}$ for all i, j. Fix i and j. Observe that, by properties i), ii) and iii), $\mathcal{G}_i^*|_{U_i \cap U_j}$ and $\mathcal{G}_j^*|_{U_i \cap U_j}$ have the same tree of transformations, i.e.,

$$\mathscr{F}_{U_i \cap U_j}\left(\mathcal{G}_i^*|_{U_i \cap U_j}\right) = \mathscr{F}_{U_i \cap U_j}\left(\mathcal{G}_j^*|_{U_i \cap U_j}\right).$$

By Corollary 6.4.7, we have that $\mathcal{G}_i^*|_{U_i \cap U_j}$, regarded as a Rees algebra over $U_i \cap U_j$ is the canonical representaive of its class. And the same holds for $\mathcal{G}_j^*|_{U_i \cap U_j}$. Hence $\mathcal{G}_i^*|_{U_i \cap U_j} = \mathcal{G}_j^*|_{U_i \cap U_j}$. □

7.3 Resolution of Rees algebras

In this section we review the algorithm of resolution of algebras. A detailed description of the algorithm can be found in [15] and, using the language of idealistic exponents, also in [11], [35], or [7]. We shall pay special attention to the existence of hypersurfaces of maximal contact. In the case of varieties over a field of characteristic zero, this is proved by using differential operators over the ground field. By contrast, we will prove the existence of hypersurfaces of maximal contact by using the weak Jacobian condition (see Lemma 7.3.5). The rest of the algorithm works exactly as in the case of varieties.

Hironaka's order function

Let V be a regular scheme, and $\mathcal{G} = \bigoplus_{j\in\mathbb{N}} \mathcal{I}_j W^j$ an $\mathcal{O}_V$-Rees algebra. *Hironaka's order function* is defined by

$$\begin{array}{rcl} \mathrm{ord}_{\mathcal{G}} : V & \longrightarrow & \mathbb{Q} \\ \xi & \longmapsto & \mathrm{ord}_{\mathcal{G}}(\xi) = \inf\left\{ \dfrac{\nu_\xi(\mathcal{I}_j)}{j} \mid j \geq 1 \right\}. \end{array}$$

Given a set of generators of $\mathcal{G}_\xi$, say

$$\mathcal{G}_\xi = \mathcal{O}_{V,\xi}\left[f_1 W^{N_1}, \ldots, f_r W^{N_r}\right],$$

an easy computation shows that

$$\mathrm{ord}_{\mathcal{G}}(\xi) = \min\left\{ \frac{\nu_\xi(f_1)}{N_1}, \ldots, \frac{\nu_\xi(f_r)}{N_r} \right\}.$$

Thus we see that $\mathrm{ord}_{\mathcal{G}}(\xi)$ is actually a rational number. In fact, if $\mathcal{G}$ is generated in degree lower than d, the image of $\mathrm{ord}_{\mathcal{G}}$ is contained in $\frac{1}{d!}\mathbb{N}$. Another consequence of the definition is that, for $\xi \in V$,

$$\xi \in \mathrm{Sing}_V(\mathcal{G}) \iff \mathrm{ord}_{\mathcal{G}}(\xi) \geq 1.$$

The following result, known as Hironaka's trick, says that the function $\mathrm{ord}_{\mathcal{G}}$ does not depend on the algebraic structure of $\mathcal{G}$, but on its tree of permissible transformations, say $\mathscr{F}_V(\mathcal{G})$. Recall that, given two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{G}'$, sometimes we have an inclusion of trees, say $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{G}')$. When the inclusion holds in both directions, we say that $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent. The next result shows that, if two $\mathcal{O}_V$-Rees algebras are weakly equivalent, then they define the same order function on V.

Theorem 7.3.1 (cf. [7, Theorem 2.21, p. 156])**.** *Let V be a regular scheme, and $\mathcal{G}, \mathcal{G}'$ two $\mathcal{O}_V$-Rees algebras. If $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent, then* $\mathrm{ord}_{\mathcal{G}}(\xi) = \mathrm{ord}_{\mathcal{G}'}(\xi)$ *for every $\xi \in V$.*

Remark 7.3.2. In the setting in which V is a regular scheme over a field of characteristic zero satisfying the weak Jacobian condition, we can give an alternative proof of this theorem using the results of Chapter 6. Indeed, fix a point $\xi \in V$. Since $\mathscr{F}_V(\mathcal{G}) = \mathscr{F}_V(\mathcal{G}')$, Corollary 6.4.5 implies that

$$\overline{\mathrm{Der}(\mathcal{O}_{V,\xi})(\mathcal{G}_\xi)} = \overline{\mathrm{Der}(\mathcal{O}_{V,\xi})(\mathcal{G}'_\xi)}.$$

Following the ideas of Lemma 3.5.6 and Lemma 3.6.4 one can show that the order of $\mathcal{G}$ at ξ coincides with that of $\overline{\mathrm{Der}(\mathcal{O}_{V,\xi})(\mathcal{G}_\xi)}$ at ξ. Similarly, the order of $\mathcal{G}'$ at ξ coincides with that of $\overline{\mathrm{Der}(\mathcal{O}_{V,\xi})(\mathcal{G}'_\xi)}$ at ξ. Then $\mathrm{ord}_{\mathcal{G}}(\xi) = \mathrm{ord}_{\mathcal{G}'}(\xi)$.

Hypersurfaces of maximal contact

Consider a regular scheme V, a Rees algebra $\mathcal{G}$ over V, and a regular hypersurface $H \subset V$. Recall that there is a natural Rees algebra attached to the immersion of H in V (see Remark 3.7.1). Let us denote this algebra by $\mathcal{H} = \mathcal{O}_V[\mathcal{I}(H)W]$.

Definition 7.3.3. Under the previous hypotheses, H is said to be a *hypersurface of maximal contact* of $\mathcal{G}$ if $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{H})$.

Example 7.3.4. Set $\mathcal{G} = \bigoplus_{i\in\mathbb{N}} \mathcal{J}_i W^i$. A situation in which maximal contact occurs is when $\mathcal{J}_1$ contains an element of order exactly equal to 1. More precisely, suppose that there exists a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$ and an element $x_1 \in \mathcal{J}_1$ so that $\nu_\xi(x_1) = 1$ (note that this implies $\mathrm{ord}_{\mathcal{G}}(\xi) = 1$). Then x_1 defines a regular hypersurface locally at ξ, say $H \subset V$. Since $x_1W \in \mathcal{G}$, it follows that $\mathscr{F}_V(\mathcal{G})$ is locally contained in the tree of permissible transformations of $\mathcal{H} = \mathcal{O}_V[x_1W]$. Hence H is locally a hypersurface of maximal contact.

Lemma 7.3.5. *Let V be a regular scheme over a field of characteristic zero satisfying the weak Jacobian condition, and let $\mathcal{G}$ be a Rees algebra over V. Consider a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$. If $\mathrm{ord}_{\mathcal{G}}(\xi) = 1$, then $\mathcal{G}$ has a hypersurface of maximal contact locally at ξ.*

Proof. Assume that ξ belongs to a suitable open affine chart $\mathrm{Spec}(S) \subset V$, so that the weak Jacobian condition holds on S. Since $\mathrm{ord}_{\mathcal{G}}(\xi) = 1$, there must be a homogeneous element $fW^N \in \mathcal{G}_\xi$ with $\nu_\xi(f) = N$. In virtue of Proposition 5.4.3, one can find $N-1$ derivatives, say $\delta_1, \dots, \delta_{N-1} \in \mathrm{Der}(S)$, so that $(\delta_{N-1} \circ \cdots \circ \delta_1)(f)$ has order exactly equal to 1 at ξ. Put $x_1 = (\delta_{N-1} \circ \cdots \circ \delta_1)(f)$. Since $\nu_\xi(x_1) = 1$, the ideal $\langle x_1 \rangle$ defines a regular hypersurface locally at ξ. In addition, by Giraud's lemma (3.6.6), $\mathcal{G} \odot \mathcal{O}_V[x_1W]$ is weakly equivalent to $\mathcal{G}$. Thus $\langle x_1 \rangle$ defines a hypersurface of maximal contact locally at ξ. □

The following definition generalizes the concept of maximal contact to higher dimension.

Definition 7.3.6. Let V be a regular scheme, and $\mathcal{G}$ an $\mathcal{O}_V$-Rees algebra. We define the *codimensional type* of $\mathcal{G}$ at a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$ as the maximum codimension of a locally regular closed subscheme passing through ξ, say $Z \subset V$, so that

$$\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{Z}),$$

where $\mathcal{Z} = \mathcal{O}_V[\mathcal{I}(Z)W]$ represents the Rees algebra attached to the immersion of Z into V (see Remark 3.7.1).

Remark 7.3.7. Fix a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$. Set $d = \dim(\mathcal{O}_{V,\xi})$, and let e denote the codimensional type of $\mathcal{G}$ at ξ. Observe that, locally at ξ, a closed subscheme $Z \subset V$ as in the Definition is always defined by part of a regular system of parameters of $\mathcal{O}_{V,\xi}$. Put $\mathcal{O}_{Z,\xi} = \mathcal{O}_{V,\xi}/\langle x_1, \dots, x_e \rangle$, with $x_1, \dots, x_d$ a regular system of parameters of $\mathcal{O}_{V,\xi}$. Each of these $x_1, \dots, x_e$ defines a hypersurface of maximal contact locally at ξ, say $H_i = \mathrm{Spec}(\mathcal{O}_{V,\xi}/\langle x_i \rangle)$. Thus we see that the codimensional type of $\mathcal{G}$ at ξ coincides with the maximum number of hypersurfaces of maximal contact of $\mathcal{G}$ passing through ξ, and having normal crossings among them. Note also that, in characteristic zero, this number coincides with Hironaka's τ-invariant (see [6, §4]).

Example 7.3.8. Assume that $\mathcal{G}$ has codimensional type exactly equal to the dimension of V. In this case, $\mathrm{Sing}_V(\mathcal{G})$ consists on finitely many closed points, and the resolution of $\mathcal{G}$ is achieved by iteratively blowing up V at these points.

Resolution of algebras

The following theorem extends the algorithm of resolution of Rees algebras discussed in [15] to our setting, in which we consider Rees algebras over schemes of characteristic zero satisfying the weak Jacobian condition.

Theorem 7.3.9. *Let V be a regular scheme over a field of characteristic zero satisfying the weak Jacobian condition. Then every Rees algebra over V admits a resolution. Namely, given an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$, there exists a sequence of $\mathcal{G}$-permissible blow-ups, say*

$$\begin{array}{ccccccccc} \mathcal{G} & & \mathcal{G}_1 & & \mathcal{G}_2 & & & & \mathcal{G}_l \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & \longleftarrow & \cdots & \xleftarrow{\pi_l} & V_l, \end{array}$$

so that $\mathrm{Sing}_V(\mathcal{G}) = \emptyset$. Moreover, the choice of the center of each π_i is algorithmic.

Sketch of the proof. The idea is to use the codimensional type of $\mathcal{G}$ to find a refinement of Hironaka's order function that, at the end, leads to an algorithmic resolution of $\mathcal{G}$. Namely, for each e less than or equal to the codimensional type of $\mathcal{G}$ at a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$, one can define an invariant $\text{w-ord}^{(e)}_{\mathcal{G}}(\xi) \in \mathbb{Q}$ (see [15, Definition 7.9], and Remark 7.3.10 below). Then $\text{w-ord}^{(e)}_{\mathcal{G}}$ can be further refined with another invariant called $t^{(e)}_{\mathcal{G}}$[1] (see [15, Definition 7.13]).

Next assume that $\mathcal{G}$ has codimensional type exactly equal to e at a point $\xi \in \mathrm{Sing}_V(\mathcal{G})$. The key feature of the invariant $t^{(e)}_{\mathcal{G}}$ is that one can construct a new Rees algebra over V, say $\widehat{\mathcal{G}}$, so that $\mathrm{Sing}_V(\widehat{\mathcal{G}}) = \underline{\mathrm{Max}}\, t^{(e)}_{\mathcal{G}}$, and whose resolution leads to a lowering of $\max t^{(e)}_{\mathcal{G}}$ (see [15, §7, p. 79]). Moreover, the codimensional type of $\widehat{\mathcal{G}}$ is greater than or equal to $e+1$ (see [15, §7.15]). Thus, using an inductive argument on the codimensional type, one can find an algorithmic resolution of $\mathcal{G}$. $\square$

[1] In fact, the $t^{(e)}_{\mathcal{G}}$-invariant is defined for a *basic object* of the form $(V, \mathcal{G}, E)$, where E is a collection of regular hypersurfaces attached to $\mathcal{G}$. These hypersurfaces play a role in the process of monomialization of $\mathcal{G}$ (see [11, Remark 15.23, p. 426]).

Remark 7.3.10. The invariants w-$\mathrm{ord}_{\mathcal{G}}^{(e)}$ and $t_{\mathcal{G}}^{(e)}$ are defined by taking a regular closed subscheme $Z \subset V$ of codimension e, so that $\mathscr{F}_V(\mathcal{G}) \subset \mathscr{F}_V(\mathcal{Z})$, where $\mathcal{Z}$ denotes the Rees algebra attached to the immersion of Z into V (see Definition 7.3.6). Thus, in order to verify that w-$\mathrm{ord}_{\mathcal{G}}^{(e)}$ and $t_{\mathcal{G}}^{(e)}$ are well-defined, one needs to show that they do not depend on the choice of Z. To check this, one can see that these two invariants can be retrieved from Hironaka's order function, $\mathrm{ord}_{\mathcal{G}}$. Since $\mathrm{ord}_{\mathcal{G}}$ depends only on the tree of transformations of $\mathcal{G}$ by Hironaka's trick (Theorem 7.3.1), it follows that w-$\mathrm{ord}_{\mathcal{G}}^{(e)}$ and $t_{\mathcal{G}}^{(e)}$ are well-defined (cf. [7, §5.2, p. 181]).

Remark 7.3.11. The construction of w-$\mathrm{ord}_{\mathcal{G}}^{(e)}$ and $t_{\mathcal{G}}^{(e)}$ is local. Thus we need an argument to show that the procedure described above globalizes. This can be found in [35].

Remark 7.3.12. Note that the process of resolution exhibited above is algorithmic, as the centers of the blow-ups are completely determined by the invariants. This and other properties of the algorithm of resolution are discussed in [7].

Appendix A

Multiplicity of local rings

Let $(R, \mathfrak{M})$ be a noetherian local ring of dimension d, and $\mathfrak{a} \subset R$ a $\mathfrak{M}$-primary ideal. Note $R/\mathfrak{a}^n$ is artinian for all $n \geq 0$. Hence $R/\mathfrak{a}^n$ has finite length when regarded as an R-module (see [5, Proposition 6.8, p. 77]). The *Hilbert-Samuel function of R with respect to* $\mathfrak{a}$ is the function defined by

$$\begin{aligned} \chi_{\mathfrak{a}}^R : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto \ell(R/\mathfrak{a}^n), \end{aligned}$$

where $\ell(R/\mathfrak{a}^n)$ denotes the length of $R/\mathfrak{a}^n$. An important property of $\chi_{\mathfrak{a}}^R$ is that there exists a polynomial of degree d in n with rational coefficients, say

$$P_{\mathfrak{a}}(R, n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0 \in \mathbb{Q}[n],$$

such that

$$\chi_{\mathfrak{a}}^R(n) = P_{\mathfrak{a}}(R, n) \quad \text{for} \quad n \gg 0.$$

The latter is known as the *Hilbert-Samuel polynomial* of R with respect to $\mathfrak{a}$ (see [31, §II.B.4, p. II-25]). Samuel proved that $c_d = \frac{e}{d!}$ for some integer $e \in \mathbb{N}$. The number e is called the *multiplicity of R with respect to* $\mathfrak{a}$, and it is denoted by $e_{\mathfrak{a}}(R)$. For $\mathfrak{a} = \mathfrak{M}$, we shall simply write $e(R) = e_{\mathfrak{M}}(R)$, and we call this number the *multiplicity of R*. It follows easily from the definition above that, given an inclusion $\mathfrak{a} \subset \mathfrak{b}$ of $\mathfrak{M}$-primary ideals, $e_{\mathfrak{a}}(R) \geq e_{\mathfrak{b}}(R)$.

Some properties of the multiplicity

We begin by formulating the following additive property of the multiplicity.

Lemma A.0.1 ([31, §V.A.2, p. V-2])**.** *Let $(R, \mathfrak{M})$ be a complete noetherian local ring of dimension d, and let $\mathfrak{a}$ be a $\mathfrak{M}$-primary ideal. Let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_m\}$ be the set* *of minimal primes of R of depth d. Then*

$$e_{\mathfrak{a}}(R) = \sum_{i=1}^{m} \ell(R_{\mathfrak{q}_i}) e_{\mathfrak{a}}(R/\mathfrak{q}_i).$$

Theorem A.0.2 (Zariski's formula [37, §VIII.10, Corollary 1, p. 299])**.** *Let $(S, \mathfrak{n})$ be a noetherian local domain, and consider a finite extension $S \subset B$. Observe that B is a semi-local ring with a finite number of primes over $\mathfrak{n}$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Moreover, for each $i = 1, \ldots, r$, the extended ideal $\mathfrak{n}B_{\mathfrak{p}_i}$ is a $\mathfrak{p}_i B_{\mathfrak{p}_i}$-primary ideal. Set $K = \mathrm{Frac}(S)$, and $L = B \otimes_S K$. Denote by $k(\mathfrak{n})$ the residue field of S, and by $k(\mathfrak{p}_1), \ldots, k(\mathfrak{p}_r)$ those of $B_{\mathfrak{p}_1}, \ldots, B_{\mathfrak{p}_r}$ respectively. Then*

$$e_{\mathfrak{n}}(S)[L : K] = \sum_{i=1}^{r} e_{\mathfrak{n}B_{\mathfrak{p}_i}}(B_{\mathfrak{p}_i})[k(\mathfrak{p}_i) : k(\mathfrak{n})].$$

The next result is historically the first step towards the study of the upper semi-continuity of the multiplicity along the primes of a ring.

Theorem A.0.3 (Nagata [27], [28, Theorem 40.1])**.** *Let $(R, \mathfrak{M})$ be a noetherian local ring, and $\mathfrak{p} \subset R$ a prime ideal. If $\dim(R) = \dim(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p})$, and $\widehat{R}/\mathfrak{p}\widehat{R}$ is reduced (i.e., $\mathfrak{p}$ is analytically unramified in Nagata's terminology), then $e(R) \geq e(R_{\mathfrak{p}})$.*

Remark A.0.4. Recall that the multiplicity of an equidimensional scheme X at a point ξ, say $\mathrm{mult}_X(\xi)$, is defined as that of the local ring $\mathcal{O}_{X,\xi}$. It is worth noting that the previous Theorem does not imply that the multiplicity is upper semi-continuous on X. For the latter to hold, it is also required that for each $\xi \in X$, there exists a non-empty open subset $U \subset \overline{\{\xi\}}$ so that $\mathrm{mult}_X(\zeta) = \mathrm{mult}_X(\xi)$ for all $\zeta \in U$.

Another interesting result due to Nagata relates the multiplicity with the study of singularities.

Theorem A.0.5 (Nagata [27], [28, Theorem 40.6])**.** *Let R be a noetherian local ring so that $\widehat{R}$ is equidimensional and does not have embedded primes (i.e., R is unmixed in Nagata's terminology). Then R is regular if and only if $e(R) = 1$.*

The tangent cone

Let $(R, \mathfrak{M}, k)$ be a noetherian local ring of dimension d. We define the *graded algebra* associated to R as

$$\mathrm{Gr}_{\mathfrak{M}}(R) = \bigoplus_{i=0}^{\infty} \mathfrak{M}^i/\mathfrak{M}^{i+1} = R/\mathfrak{M} \oplus \mathfrak{M}/\mathfrak{M}^2 \oplus \mathfrak{M}^2/\mathfrak{M}^3 \oplus \cdots .$$

Note that $\mathrm{Gr}_{\mathfrak{M}}(R)$ has a natural structure of k-algebra, and that it is finitely generated on degree one. Moreover, the homogeneous elements of degree one of

$\mathrm{Gr}_{\mathfrak{M}}(R)$ generate a homogeneous maximal ideal which we shall denote by

$$\mathrm{Gr}^{+}_{\mathfrak{M}}(R) = \bigoplus_{i=1}^{\infty} \mathfrak{M}^i/\mathfrak{M}^{i+1}.$$

Theorem A.0.6 (cf. [5, Theorem 11.14])**.** *R and $\mathrm{Gr}_{\mathfrak{M}}(R)$ have the same Krull dimension.*

Definition A.0.7. Let f be a non-zero element of R, and let n be the biggest integer so that $f \in \mathfrak{M}^n$. That is, $f \in \mathfrak{M}^n$ and $f \notin \mathfrak{M}^{n+1}$. We define the *initial part* of f in $\mathrm{Gr}_{\mathfrak{M}}(R)$, denoted by $\mathrm{In}(f)$, as the homomorphic image of f in $\mathfrak{M}^n/\mathfrak{M}^{n+1}$.

Remark A.0.8. Given elements $f, g \in R$, in general, $\mathrm{In}(f+g) \neq \mathrm{In}(f) + \mathrm{In}(g)$. Thus $\mathrm{In} : R \to \mathrm{Gr}_{\mathfrak{M}}(R)$ is not a ring homomorphism.

Definition A.0.9. Let X be a noetherian scheme, fix a point $\xi \in X$, and let $(\mathcal{O}_{X,\xi}, \mathfrak{M}_\xi)$ denote the local ring of X at ξ. We define the *tangent cone of X at ξ* by

$$\mathbb{T}_{X,\xi} = \mathrm{Spec}\left(\mathrm{Gr}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})\right).$$

Remark A.0.10. Consider a collection of elements $z_1, \dots, z_l \in \mathfrak{M}_\xi$ whose residue classes form a basis of $\mathfrak{M}_\xi/\mathfrak{M}_\xi^2$ as a $k(\xi)$-vector space. Then $\mathrm{In}(z_1), \dots, \mathrm{In}(z_l)$ are homogeneous elements of degree 1 that generate $\mathrm{Gr}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})$ as a $k(\xi)$-algebra. Thus we have a surjective homomorphism

$$k[Z_1, \dots, Z_l] \longrightarrow \mathrm{Gr}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})$$

given by $Z_i \mapsto \mathrm{In}(z_i)$, where $Z_1, \dots, Z_l$ represent variables. This homomorphism induces an embedding of $\mathbb{T}_{X,\xi}$ in the affine space $\mathbb{A}^l_k$. By construction, the origin of $\mathbb{A}^l_k$ coincides with the closed point of $\mathbb{T}_{X,\xi}$ associated to the maximal ideal $\mathrm{Gr}^{+}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})$. For this reason, we shall refer to this closed point as *the origin of* $\mathbb{T}_{X,\xi}$.

Recall that the multiplicity of a noetherian scheme X at a point ξ is defined as that of the local ring $\mathcal{O}_{X,\xi}$.

Proposition A.0.11. *Let X be a noetherian scheme. Then, for each $\xi \in X$, the multiplicity of X at ξ coincides with that of $\mathbb{T}_{X,\xi}$ at the origin.*

Proof. Let $(S, \mathfrak{N})$ denote the local ring of $\mathbb{T}_{X,\xi}$ at the origin (i.e., S is the localization of $\mathrm{Gr}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})$ at the maximal ideal $\mathrm{Gr}^{+}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi})$). We shall show that

$$\ell\big(\mathcal{O}_{X,\xi}/\mathfrak{M}_\xi^{n+1}\big) = \ell\big(S/\mathfrak{N}^{n+1}\big) \tag{A.1}$$

for $n \in \mathbb{N}$. According to the definition of the multiplicity of a local ring (p. 129), this suffices to prove that $e(\mathcal{O}_{X,\xi}) = e(S)$.

Observe that, for each $i \in \mathbb{N}$, we have a short exact sequence

$$0 \longrightarrow \mathfrak{M}_\xi^i/\mathfrak{M}_\xi^{i+1} \longrightarrow \mathcal{O}_{X,\xi}/\mathfrak{M}_\xi^{i+1} \longrightarrow \mathcal{O}_{X,\xi}/\mathfrak{M}_\xi^i \longrightarrow 0.$$

From the additivity of the length, it follows by induction that

$$\ell\big(\mathcal{O}_{X,\xi}/\mathfrak{M}_\xi^{n+1}\big) = \sum_{i=0}^{n} \ell\big(\mathfrak{M}_\xi^i/\mathfrak{M}_\xi^{i+1}\big).$$

On the other hand, it is not hard to check that

$$S/\mathfrak{N}^{n+1} \simeq \bigoplus_{i=0}^{n} \mathfrak{M}_\xi^i/\mathfrak{M}_\xi^{i+1},$$

and hence

$$\ell\big(S/\mathfrak{N}^{n+1}\big) = \sum_{i=0}^{n} \ell\big(\mathfrak{M}_\xi^i/\mathfrak{M}_\xi^{i+1}\big).$$

In this way we see that (A.1) holds, and thus the result follows. □

Multiplicity of hypersurfaces

Let $(R, \mathfrak{M}, k)$ be a regular local ring of dimension d, and fix a regular system of parameters of R, say $z_1, \dots, z_d$. As the residue classes of $z_1, \dots, z_d$ generate $\mathfrak{M}/\mathfrak{M}^2$ as a k-vector space, one readily checks that $\mathrm{In}(z_1), \dots, \mathrm{In}(z_d)$ generate $\mathrm{Gr}_{\mathfrak{M}}(R)$ as a k-algebra (see Remark A.0.10). Thus we have a surjective homomorphism

$$k[Z_1, \dots, Z_d] \longrightarrow \mathrm{Gr}_{\mathfrak{M}}(R)$$

given by $Z_i \mapsto \mathrm{In}(z_i)$, where $Z_1, \dots, Z_d$ represent variables. Since $\mathrm{Gr}_{\mathfrak{M}}(R)$ has the same dimension as R (Theorem A.0.6), we deduce that the kernel of the previous map is 0. That is, $\mathrm{Gr}_{\mathfrak{M}}(R) \simeq k[Z_1, \dots, Z_d]$, where the variable Z_i is identified with $\mathrm{In}(z_i)$.

Next consider and ideal $J \subset \mathfrak{M}$, and the local ring $R' = R/J$. Let $\mathfrak{M}' = \mathfrak{M}/J$ denote the maximal ideal of R'. Note that, for each $n \in \mathbb{N}$, there is a natural k-linear map of $\mathfrak{M}^n/\mathfrak{M}^{n+1} \to (\mathfrak{M}')^n/(\mathfrak{M}')^{n+1}$. Hence there is a natural map of graded k-algebras

$$\Phi \,:\, \mathrm{Gr}_{\mathfrak{M}}(R) \longrightarrow \mathrm{Gr}_{\mathfrak{M}'}(R').$$

This map is obviously surjective, and one can check that

$$\ker(\Phi) = \langle \mathrm{In}(f) \mid f \in J \rangle.$$

Remark A.0.12. Fix a set of generators of J, say $J = \langle f_1, \dots, f_r \rangle$. Then

$$\langle \mathrm{In}(f_1), \dots, \mathrm{In}(f_r) \rangle \subset \ker(\Phi).$$

but, in general, this inclusion is strict.

Lemma A.0.13. *In the previous setting, assume that J is principal, i.e., $J = \langle f \rangle$ for some $f \in R$. Then* $\ker(\Phi) = \langle \mathrm{In}(f) \rangle$.

Proof. Fix the same notation as above. In addition, let $\varphi : R \to R' = R/\langle f \rangle$ denote the natural quotient map, and set $n = \nu_{\mathfrak{M}}(f)$.

Since Φ is a morphism of graded algebras, $\ker(\Phi)$ should be a homogeneous ideal, i.e., $\ker(\Phi)$ is generated by homogeneous elements. In this way, in order to prove the claim, it suffices to show that, for each homogeneous element $G \in \ker(\Phi)$, we have that $G \in \langle \mathrm{In}(f) \rangle$.

Fix a homogeneous element $G \in \ker(\Phi)$, $G \neq 0$. Assume that G has degree N. Then $G = \mathrm{In}(g)$ for some $g \in R$, with $\nu_{\mathfrak{M}}(g) = N$. Observe that $\Phi(G)$ is the class of $\varphi(g)$ in $(\mathfrak{M}')^N/(\mathfrak{M}')^{N+1}$. Since $\Phi(G) = 0$, it follows that $\varphi(g) \in (\mathfrak{M}')^{N+1}$, i.e., $g \in \mathfrak{M}^{N+1} + \langle f \rangle$. Suppose that $g = g' + af$ for some $g' \in \mathfrak{M}^{N+1}$, and $a \in R$. Then $g - g' = af$. Since $g' \in \mathfrak{M}^{N+1}$, it follows that $\nu_{\mathfrak{M}}(g - g') = \nu_{\mathfrak{M}}(g) = N$, and hence $\mathrm{In}(g - g') = \mathrm{In}(g)$. On the other hand, as R is regular, we have that $\mathrm{In}(af) = \mathrm{In}(a)\,\mathrm{In}(f)$. Therefore

$$G = \mathrm{In}(g) = \mathrm{In}(g - g') = \mathrm{In}(af) = \mathrm{In}(a)\,\mathrm{In}(f),$$

which shows that $G \in \langle \mathrm{In}(f) \rangle$. □

Proposition A.0.14. *Let S be a regular ring, and fix a non-zero element $f \in S$. Set $V = \mathrm{Spec}(S)$, and consider the hypersurface*

$$H = \mathrm{Spec}(S/\langle f \rangle) \subset V.$$

Then, for each $\xi \in H$,

$$\mathrm{mult}_H(\xi) = \nu_\xi(f).$$

Proof. Let $(R, \mathfrak{M}, k)$ and $(R', \mathfrak{M}', k)$ denote the local rings of V and H at ξ respectively. Observe that $R' = R/\langle f \rangle$. Fix a regular system of parameters of R, say $z_1, \ldots, z_d$. Then recall that $\mathrm{Gr}_{\mathfrak{M}}(R)$ is isomorphic to the polynomial ring $k[Z_1, \ldots, Z_d]$, where we identify the variable Z_i with $\mathrm{In}(z_i)$.

According to Lemma A.0.13, $\mathrm{Gr}_{\mathfrak{M}'}(R') = \mathrm{Gr}_{\mathfrak{M}}(R)/\langle \mathrm{In}(f) \rangle$, i.e.,

$$\mathrm{Gr}_{\mathfrak{M}'}(R') \simeq k[Z_1, \ldots, Z_d]/\langle \mathrm{In}(f) \rangle,$$

where $\mathrm{In}(f)$ is identified with a homogeneous polynomial of degree $n = \nu_{\mathfrak{M}}(f)$. In this way,

$$\mathbb{T}_{X,\xi} = \mathrm{Spec}\left(\mathrm{Gr}_{\mathfrak{M}'}(R')\right)$$

can be regarded as a closed subvariety of $\mathbb{A}^d_k$ defined by a homogeneous polynomial of degree n. Hence $\mathbb{T}_{X,\xi}$ has multiplicity $n = \nu_{\mathfrak{M}}(f)$ at the origin. Finally, as the multiplicity of X at ξ coincides with that of $\mathbb{T}_{X,\xi}$ at the origin (Proposition A.0.11), we conclude that $\mathrm{mult}_X(\xi) = \nu_{\mathfrak{M}}(f) = \nu_\xi(f)$. □

Appendix B

Excellent schemes

Let X be an integral noetherian scheme. A resolution of singularities of X is a proper and birational map $X \leftarrow X'$ so that X' is regular. In general, not every integral noetherian scheme admits a resolution of singularities. This observation led Grothendieck to introduce the notion of excellent schemes. From this perspective, excellent schemes can be regarded as a suitable category in which resolution of singularities could be achieved.

In this appendix we review the definitions of excellent rings and schemes, along with some of their properties. Most of the results of this appendix are taken from [18, §7.8].

Preliminary notions

Definition B.0.1. A noetherian ring B is said to be *catenary* if for each pair of prime ideals $\mathfrak{p} \subset \mathfrak{q} \subset B$, every saturated chain of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$ has the same length. B is said to be *universally catenary* if every algebra of finite type over B is catenary.

Proposition B.0.2 ([18, Proposition 5.6.4])**.** *Every ring that is a quotient of a regular noetherian ring is universally catenary.*

Definition B.0.3. An algebra B over a field k is said to be geometrically regular over k if $B \otimes_k k'$ is regular for every finite field extension k' of k.

Example B.0.4. A field extension K/k is separable[1] if and only if K is geometrically regular over k.

[1] We say that a field extension K/k is separable if $K \otimes_k k'$ is reduced for every finite field extension k' of k.

Example B.0.5. Let B be an algebra of finite type over a field k. By definition, B is smooth over k if and only if it is geometrically regular over k.

Definition B.0.6. A ring homomorphism $\varphi : B \to B'$ is said to be *regular* if the following conditions hold:

i) φ is flat.

ii) The fibers of φ are geometrically regular, i.e., for each prime ideal $\mathfrak{p} \subset B$, the algebra $B' \otimes_B k(\mathfrak{p})$ is geometrically regular over $k(\mathfrak{p})$ (where $k(\mathfrak{p})$ denotes the residue field of $B_{\mathfrak{p}}$).

Next we present the definition of excellent rings and schemes. For further discussion and motivation we refer to [18, §7.8, p. 214].

Definition B.0.7. Let B be a noetherian ring, and $X = \mathrm{Spec}(B)$. The ring B is said to be *excellent* if satisfies the following conditions:

i) B is universally catenary.

ii) For each prime ideal $\mathfrak{p} \subset B$, the morphism $B_{\mathfrak{p}} \to \widehat{B_{\mathfrak{p}}}$ is regular.

iii) For each prime ideal $\mathfrak{p} \subset B$, and each purely inseparable finite extension of $K = \mathrm{Frac}(B/\mathfrak{p})$, say K', there exists a finite extension $B/\mathfrak{p}$, say $B' \subset K'$, such that $K' = \mathrm{Frac}(B')$, and $\mathrm{Reg}(B')$ contains a non-empty open subset of $\mathrm{Spec}(B')$.

An affine scheme $X = \mathrm{Spec}(B)$ is *excellent* if B is excellent. A noetherian scheme is said to be *excellent* if it can be covered by excellent affine charts.

Properties

Attending to the previous definition, one readily checks that every field is excellent: a field is universally catenary by Proposition B.0.2, and conditions ii) and iii) are in this case. One can also check that the localization of an excellent ring is again excellent. The following results tell us about other families of schemes that are excellent.

Proposition B.0.8 ([18, Proposition 7.8.6 (i)])**.** *If X is an excellent scheme, then every scheme of finite type over X is excellent.*

Corollary B.0.9. *Every scheme of finite over a field k is excellent. In particular, every variety over k is excellent.*

Proposition B.0.10 ([18, Scholium 7.8.3 (iii)])**.** *A noetherian complete local ring is excellent.*

There are many other properties that can be derived from conditions i), ii), and iii) of the definition. One can find a good summary in [18, Scholie 7.8.3, p. 214]. Here we recall three of them, which are used along this work.

Proposition B.0.11. *If X is an excellent scheme, then* $\mathrm{Reg}(X)$*, the set of regular points of X, is an open subset of X.*

Proposition B.0.12. *Let B be an excellent domain with field of fractions K, and let L be a finite field extension of K. Then the normalization of B in L is a finite B-module.*

Proposition B.0.13. *Let R be an excellent local ring, and let $\widehat{R}$ denote the completion of R with respect to its maximal ideal. Then:*

i) *R is reduced is and only if $\widehat{R}$ is so.*

ii) *R is equidimensional if and only if $\widehat{R}$ is so.*

iii) *R is strictly equidimensional*[2] *if and only if $\widehat{R}$ is so.*

Multiplicity on excellent schemes

Let X be an equidimensional noetherian scheme. Recall that the multiplicity of X at a point ξ is defined as that of the local ring $\mathcal{O}_{X,\xi}$ with respect to its maximal ideal. Thus the multiplicity on X can be regarded as function $\mathrm{mult}_X : X \to \mathbb{N}$.

Theorem B.0.14 (cf. Nagata [27])**.** *Let X be a strictly equidimensional excellent scheme. Then X is regular at a point ξ if and only if* $\mathrm{mult}_X(\xi) = 1$*.*

Proof. By definition, X is regular at ξ if and only if $\mathcal{O}_{X,\xi}$ is a regular local ring. Since $\mathcal{O}_{X,\xi}$ is strictly equidimensional, the same holds for $\widehat{\mathcal{O}_{X,\xi}}$ (see Proposition B.0.13). Thus the result follows from Theorem A.0.5. □

Theorem B.0.15 (cf. Nagata [27])**.** *Let X be an equidimensional excellent scheme, and consider two points $\xi, \eta \in X$. If $\xi \in \overline{\{\eta\}}$, then* $\mathrm{mult}_X(\xi) \geq \mathrm{mult}_X(\eta)$*.*

Proof. Set $R = \mathcal{O}_{X,\xi}$, and let $\mathfrak{p} \subset R$ denote the prime ideal associated to η. Since R is equidimensional and catenary, we have that $\dim(R) = \dim(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p})$. Moreover, by Proposition B.0.13, the ring $\widehat{R}/\mathfrak{p}\widehat{R} = \widehat{R/\mathfrak{p}}$ is reduced. Hence the result follows from Theorem A.0.3. □

Theorem B.0.16 (Dade [14])**.** *Let X be an equidimensional excellent scheme. Then:*

i) $\mathrm{mult}_X : X \to \mathbb{N}$ *is upper semi-continuous.*

ii) *If $X \xleftarrow{\pi} X_1$ is the blow-up of X along a closed regular equimultiple center, then* $\mathrm{mult}_{X_1}(\xi_1) \leq \mathrm{mult}_X(\pi(\xi_1))$ *for all $\xi_1 \in X_1$.*

[2] A ring (resp. scheme) is said to be *strictly equidimensional* if it is equidimensional and it does not have embedded primes (resp. components).

Remark B.0.17. It is worth noting that Dade's proof is a quite involved one, that makes use of several results due to Nagata [27], Chevalley [19, §13.1], etc. For further information we refer to [34, Remark 6.13].

Corollary B.0.18. *Let S be a regular excellent ring, and fix an element $f \in S$. Then the order of f at points of* $\mathrm{Spec}(S)$ *is an upper semi-continuous function, i.e., the set*

$$\{\xi \in \mathrm{Spec}(S) \mid \nu_\xi(f) \geq n\}$$

is closed in $\mathrm{Spec}(S)$ *for each $n \in \mathbb{N}$.*

Proof. Set $H = \mathrm{Spec}(S/\langle f \rangle)$. By Proposition A.0.14, we have that $\mathrm{mult}_H(\xi) = \nu_\xi(f)$ for all $\xi \in H$. Thus the claim follows from part i) of Dade's theorem. □

Appendix C

Étale topology

In this appendix we review the notions of étale morphism and étale neighborhood. For a detailed introduction to this topic we refer to [30].

Definition C.0.1. Let B be an A-algebra. B is said to be *étale* over A (resp. *quasi-étale*) if the following conditions hold:

i) B is of finite type over A (resp. essentially of finite type over A).

ii) For any A-algebra C, any ideal $J \subset C$ with $J^2 = 0$, and any morphism of A-algebras $B \to C/J$, there exists a unique homomorphism $B \to C$ so that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C/J \\ A \arrow[u] \arrow[r] & C. \arrow[u] \end{tikzcd}$$

Lemma C.0.2 ([4, Proposition VI.4.7, p. 116])**.** *Let B be an étale A-algebra. Then:*

i) For any multiplicative subset $S \subset B$, the ring $S^{-1}B$ is quasi-étale over A.

ii) For any A-algebra A', the ring $B \otimes_A A'$ is étale over A'.

iii) If a B-algebra B' is étale over B, then B' is étale over A.

Theorem C.0.3 ([30, Theorem V.2, p. 55])**.** *Let B be an algebra of finite type over A. Then B is étale over A if and only if the morphism $A \to B$ is flat and $\Omega^1_{B/A} = 0$.*

Corollary C.0.4. *Let B be an algebra of finite type over A. For a prime ideal $\mathfrak{q} \subset B$, set $\mathfrak{p} = A \cap \mathfrak{q}$ and assume that the following conditions hold:*

i) $B_{\mathfrak{q}}$ *is flat over* $A_{\mathfrak{p}}$.

ii) $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$.

iii) $B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ *is a finite separable extension of* $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.

Then there exists an element $f \in B \setminus \mathfrak{q}$ so that B_f is étale over A.

Proof. By [4, Theorem V.5.5, p. 101], condition i) implies that B is flat over A locally at $\mathfrak{q}$. On the other hand, ii) and iii) yield $(\Omega^1_{B/A})_{\mathfrak{q}} = 0$ by [4, Proposition VI.3.3, p. 112]. Since B is of finite type over A, we have that $\Omega^1_{B/A}$ is a finite B-module, and hence $\Omega^1_{B/A}$ is 0 locally at $\mathfrak{q}$. Thus Theorem C.0.3 implies that B is étale over A locally at $\mathfrak{q}$. □

Lemma C.0.5. *Let k be an arbitrary ring, A a k-algebra, and B an étale A-algebra. Let M be a B-module. Then, for any k-linear derivative $\delta : A \to M$, there exists a unique k-linear derivative of B into M, say $\delta' : B \to M$, so that the following diagram commutes:*

$$\begin{tikzcd} B \arrow[r, "\delta'"] & M. \\ A \arrow[u] \arrow[ur, "\delta"'] & \end{tikzcd}$$

In other words, δ can be extended to a unique k-linear derivation of B into M.

Proof. Let $B * M$ denote the B-algebra formed by the module $B \oplus M$, endowed with the product

$$(b, m) \cdot (b', m') = (bb', bm' + b'm).$$

Note that, by construction, each derivation from B into M, say $\varepsilon : B \to M$, induces morphism of a B-algebras from B into $B * M$. Namely, ε induces the map $\mathrm{id}_B \oplus \varepsilon$. In fact, this relation induces a one-to-one correspondence between $\mathrm{Der}(B, M)$ and $\mathrm{Hom}_B(B, B * M)$. Similarly, we have a natural morphism of A-algebras associated to the derivation $\delta : A \to M$, say $\varphi : A \to B * M$, given by $\varphi(a) = (a, \delta(a))$.

Next consider the map $\pi : B * M \to B$ given by $(b, m) \mapsto b$, whose kernel is the ideal $J = 0 \oplus M$. Since B is étale over A, and $J^2 = 0$, the identity on B can be lifted to a unique morphism $\varphi' : B \to B * M$ which makes the following diagram commutative:

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}_B"] \arrow[dr, "\varphi'"] & B \\ A \arrow[u] \arrow[r, "\varphi"] & B * M. \arrow[u, "\pi"'] \end{tikzcd}$$

Set $\varphi' = \mathrm{id}_B \oplus \delta'$, with $\delta' \in \mathrm{Hom}_k(B, M)$. Then one readily checks that $\delta' : B \to M$ is a derivative that extends δ. Moreover, since φ' is unique, it follows that δ' is the unique derivative with this property. $\square$

Definition C.0.6. A morphism of schemes $Y \to X$ is said to be *étale* if it is of finite type, Y is flat over X, and $\Omega^1_{Y/X} = 0$.

Remark C.0.7. The previous definition is equivalent to require that $Y \to X$ is locally given by affine morphisms of the form $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, where B is étale over A.

Theorem C.0.8 ([4, Theorem V.5.1, p. 99])**.** *Étale morphisms are open.*

Lemma C.0.9 ([10, §32.1, p. 188])**.** *Étale morphisms preserve multiplicity, i.e., if $f : Y \to X$ is an étale morphism, then* $\mathrm{mult}_Y(\zeta) = \mathrm{mult}_X(f(\zeta))$ *for all* $\zeta \in Y$.

Remark C.0.10. Let $Y \to X$ be an étale morphism. Fix a point $\zeta \in Y$, and let ξ denote its image in X. Using the characterization of Corollary C.0.4, one can check that

$$\mathrm{Gr}_{\mathfrak{M}_\zeta}(\mathcal{O}_{Y,\zeta}) \simeq \mathrm{Gr}_{\mathfrak{M}_\xi}(\mathcal{O}_{X,\xi}) \otimes_{k(\xi)} k(\zeta),$$

where $k(\zeta)$ is separably algebraic over $k(\xi)$. This implies that $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{Y,\zeta}$ have the same Hilbert-Samuel function and, in particular, that they have the same multiplicity.

Definition C.0.11. Let X be a scheme and fix a point $\xi \in X$. A morphism of schemes $Y \to X$ is said to be an *étale neighborhood of X at ξ* if it is étale and Y contains a point ζ mapping to ξ. Sometimes the morphism $Y \to X$ is omitted and one simply says that Y is an étale neighborhood of X at ξ.

The name of étale topology comes from the following observation: given two étale neighborhoods Y and Y' of X at ξ, there is a commutative diagram

$$\begin{tikzcd} & Y \times_X Y' \arrow[dl] \arrow[dr] \arrow[dd] & \\ Y \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

where all the morphisms are étale (see Lemma C.0.2). Thus $Y \times_X Y'$ can be regarded as an étale neighborhood of X at ξ which dominates Y and Y'.

Example C.0.12. An open immersion of schemes is étale. Moreover, if U_1 and U_2 are two (Zariski) open neighborhoods of X at ξ, then $U_1 \times_X U_2 = U_1 \cap U_2$.

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