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Two-grid mixed finite-element approximations to the Navier-Stokes equations based on a Newton-type step

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Abstract

A two-grid scheme to approximate the evolutionary Navier-Stokes equations is introduced and analyzed. A standard mixed finite element approximation is first obtained over a coarse mesh of size H at any positive time $T > 0$. Then, the approximation is postprocessed by means of solving a steady problem based on one step of a Newton iteration over a finer mesh of size $h < H$. The method increases the rate of convergence of the standard Galerkin method in one unit in terms of H and equals the rate of convergence of the standard Galerkin method over the fine mesh h. However, the computational cost is essentially the cost of approaching the Navier-Stokes equations with the plain Galerkin method over the coarse mesh of size H since the cost of solving one single steady problem is negligible compared with the cost of computing the Galerkin approximation over the full time interval $(0, T]$. For the analysis we take into account the loss of regularity at initial time of the solution of the Navier-Stokes equations in the absence of nonlocal compatibility conditions. Some numerical experiments are shown.

Keywords Incompressible Navier–Stokes equations; inf-sup stable finite element methods; static two-grid methods; nonlocal compatibility conditions

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega$ and let us consider the incompressible Navier-Stokes equations

$$\begin{aligned} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \\ \operatorname{div}(u) &= 0 \end{aligned} \qquad (1)$$

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with initial condition $u(\cdot, 0) = u_0$ and homogeneous Dirichlet conditions $u = 0$ on the boundary. In this paper, we study a two-grid postprocessing technique based on Newton iteration. First, a mixed finite-element approximation (u_H, p_H) is computed over a coarse grid of size H. Then, the postprocessed solution $(\tilde{u}, \tilde{p})$ is obtained at any time $T > 0$ as the mixed finite element approximation over a fine mesh $h < H$ to the following steady problem:

$$
\begin{aligned}
-\nu\Delta\tilde{u} + (u_H \cdot \nabla)\tilde{u} + (\tilde{u} \cdot \nabla)u_H + \nabla\tilde{p} + \lambda\tilde{u} &= f - \tfrac{d}{dt}u_H + (u_H \cdot \nabla)u_H + \lambda u_H \\
\mathrm{div}(\tilde{u}) &= 0,
\end{aligned}
\tag{2}
$$

where λ is a positive parameter that is added to have coercivity in the bilinear form associated to problem (2) whose value will be specified later.

In general, two-grid postprocessing techniques improve the rate of convergence of the standard finite element approximations with a slight increase in the computational cost. In [24], [23] several two-level methods are considered to approximate the steady Navier-Stokes equations. They require solving a nonlinear system over a coarse mesh and, depending on the algorithm chosen, one Stokes problem, one linear Oseen problem or one Newton step over the fine mesh. The corresponding algorithms obtain the optimal rate of convergence in the fine mesh for appropriate choices of the coarse mesh diameter H.

Static two-grid or two-grid postprocessing techniques for the evolutionary Navier-Stokes equations have also been studying. In all cases the plain Galerkin method is used to evolve in time and a steady problem is solved at the fine level. More precisely, the problem solved at the fixed step or fine level can be a Stokes problem or an Oseen-type problem both computed with data based on the Galerkin approximation. The Stokes-type postprocess was first developed for spectral methods in [6], [7], [13], [26] and then applied to mixed finite element methods in [2], [3], [11], [12]. The Oseen-postprocess has been studied in [15]. In both cases, the rate of convergence of the postprocessed approximation equals the rate or convergence of the standard finite element approximation over the fine grid and improves in one unit the rate of convergence of the Galerkin method over the coarse grid.

Several authors have also considered two-grid algorithms for the evolutionary Navier-Stokes equations based on evolutionary linearized problems. We refer the reader to [16] for a detailed description of several algorithms of this type that have appeared in the literature. We add to those references two more that have recently appeared. In [19] a two level method based on a linearized time dependent Stokes problem is analyzed considering the case of non-smooth initial data. In [25] Euler-based discretizations of three corrections based on linearized time dependent Stokes, Oseen and Newton problems are considered.

In this paper we concentrate ourselves on the static two-grid approach based on a Newton-type step to the evolutionary Navier-Stokes equations. The reason is twofold, on the one hand the static approaches are computationally cheaper than the dynamical approaches and on the other, although there are three typical linearized methods that can be applied: Stokes, Oseen and Newton, to our knowledge, the last one had not been previously analyzed. We show that with the static two grid method based on Newton-type postprocess we can improve the rate of convergence of the Galerkin approximation in one unit. The inclusion of the *reaction term* $(\tilde{u} \cdot \nabla)u_H$, with the subsequent lost of coercivity of the bilinear form associated to the steady problem, introduces several challenges which are of interest.

As in [11], [12], [15] for the analysis we take into account the loss of regularity suffered by the solutions of the Navier-Stokes equations at the initial time in the absence of nonlocal compatibility conditions. Consequently, for the analysis we do not assume the solution u to have more than second-order partial derivatives bounded in L^2 up to the initial time $t = 0$. Demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations, see [20], [21]. Due to the loss of regularity at $t = 0$, the best error bound that we can obtain is $O(H^5|\log(H)|)$. For this reason we do not analyze higher than cubic finite elements. The same limit in the rate of convergence was found in [21] for standard mixed finite-element approximations and in [11], [15], [16] for two-grid schemes.

The outline of the paper is as follows. We first introduce some preliminaries and notations. In Section 3 we describe the three static linearized approaches to the nonlinear Navier-Stokes equations. The new method is analyzed in Section 5, based on some theoretical results obtained in Section 4. Finally, last section is devoted to show some numerical experiments. We check numerically the rate of convergence of the method predicted by the theory and we compare the static Newton-type approach with the static Stokes and Oseen approaches for different values of the diffusion parameter in order to study the behaviour of the different methods for different values of the Reynolds number.

2 Preliminaries and notations

Along the paper we will denote by $W^{m,q}(\Omega)^d$ the space of Lebesgue integrable functions with m (weak) derivatives in $L^q(\Omega)$. For $q = 2$ we will use the standard notation $H^m(\Omega)^d = W^{m,2}(\Omega)^d$.

For $q \in [1, \infty)$, we will use the following Sobolev embedding formula. There exists a constant C depending on the domain such that

$$\|v\|_{L^{q'}} \le C\|v\|_{W^{s,q}}, \quad \frac{1}{q'} \ge \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad v \in W^{s,q}(\Omega)^d. \tag{3}$$

For $q' = \infty$, (3) holds with $\frac{1}{q} < \frac{s}{d}$. In particular, we will do extensive use of the following cases, which hold simultaneously for both two and three spatial dimensions.

$$\|v\|_{L^{2d}} \le C\|v\|_s, \quad s \ge 1, \quad \|v\|_{L^{2d/(d-1)}} \le C\|v\|_s, \quad s \ge 1/2. \tag{4}$$

We consider the Hilbert spaces:

$$\begin{aligned} \mathcal{H} &= \left\{u \in L^2(\Omega)^d \mid \operatorname{div}(u) = 0,\, u \cdot n_{|\partial\Omega} = 0\right\}, \\ V &= \left\{u \in H_0^1(\Omega)^d \mid \operatorname{div}(u) = 0\right\}, \end{aligned}$$

endowed respectively with the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms. The following inf-sup condition (see [18]) is satisfied

$$\inf_{q\in L^2(\Omega)/\mathbb{R}} \sup_{v\in H_0^1(\Omega)^d} \frac{(q,\nabla\cdot v)}{\|q\|_{L^2(\Omega)/\mathbb{R}}\|v\|_1} \geq \beta \geq 0. \tag{5}$$

For $u \in V$, $v, w \in H_0^1(\Omega)^d$, it also holds

$$((u\cdot\nabla)v, w) = -((u\cdot\nabla)w, v) = -(\nabla\bar{w}\cdot v, u) \tag{6}$$

where $(\nabla\bar{w})_{ij} = \partial_i w_j$.

Let $\Pi : L^2(\Omega)^d \longrightarrow \mathcal{H}$ be the $L^2(\Omega)^d$ projection onto $\mathcal{H}$. We denote by A the Stokes operator on Ω:

$$A : \mathcal{D}(A) \subset \mathcal{H} \longrightarrow \mathcal{H}, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap \mathcal{V}.$$

We shall assume that u is a strong solution up to time $t = T$, so that

$$\|u(t)\|_1 \leq M_1, \quad \|u(t)\|_2 \leq M_2, \quad 0 \leq t \leq T, \tag{7}$$

for some constants M_1 and M_2. We shall also assume that there exists a constant $\tilde{M}_2$ such that

$$\|f\|_1 + \|f_t\|_1 + \|f_{tt}\|_1 \leq \tilde{M}_2, \quad 0 \leq t \leq T.$$

Finally, we shall assume that for some $k \geq 2$

$$\sup_{0\leq t\leq T} \left\|\partial_t^{\lfloor k/2\rfloor} f\right\|_{k-1-2\lfloor k/2\rfloor} + \sum_{j=0}^{\lfloor (k-2)/2\rfloor} \sup_{0\leq t\leq T} \left\|\partial_t^j f\right\|_{k-2j-2} < +\infty,$$

so that, according to Theorems 2.4 and 2.5 in [20], there exist positive constants M_k and K_k such that the following bounds hold:

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \leq M_k \tau(t)^{1-k/2}, \tag{8}$$

$$\int_0^t \sigma_{k-3}(s)\big(\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2\big)\,\mathrm{d}s \leq K_k^2, \tag{9}$$

where $\tau(t) = \min(t, 1)$ and $\sigma_n = e^{-\alpha(t-s)}\tau^n(s)$ for some $\alpha > 0$. Observe that for $t \leq T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n. We want to remark that in the error bounds we prove in this paper the final time T is fixed and we are not studying the behavior of the bounds for increasing values of T while, on the contrary, we want to clarify the behavior of the bounds around $t = 0$.

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$ be a family of partitions of suitable domains Ω_h, where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$, and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h. Let $r \geq 2$, we consider the finite-element spaces

$$\begin{aligned} S_{h,r} &= \left\{\chi_h \in \mathcal{C}\left(\overline{\Omega}_h\right) \mid \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0)\right\} \subset H^1(\Omega_h), \\ S_{h,r}^0 &= S_{h,r} \cap H_0^1(\Omega_h), \end{aligned}$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. We shall denote by $(X_{h,r}, Q_{h,r-1})$ the so-called Hood–Taylor element [4, 22], when $r \geq 3$, where

$$X_{h,r} = \left(S_{h,r}^0\right)^d, \quad Q_{h,r-1} = S_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \quad r \geq 3,$$

and the so-called mini-element [8] when $r = 2$, where $Q_{h,1} = S_{h,2} \cap L^2(\Omega_h)/\mathbb{R}$, and $X_{h,2} = (S_{h,2}^0)^d \oplus \mathbb{B}_h$. Here, $\mathbb{B}_h$ is spanned by the bubble functions b_τ, $\tau \in \mathcal{T}_h$, defined by $b_\tau(x) = (d+1)^{d+1}\lambda_1(x)\cdots\lambda_{d+1}(x)$, if $x \in \tau$ and 0 elsewhere, where $\lambda_1(x), \ldots, \lambda_{d+1}(x)$ denote the barycentric coordinates of x. For these elements a uniform inf-sup condition is satisfied (see [4]), that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \geq \beta. \tag{10}$$

The approximate velocity belongs to the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \left\{\chi_h \in H_0^1(\Omega_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1}\right\}.$$

Let $(u_H, p_H) \in (X_{H,r}, Q_{H,r-1})$ be the semi discrete Galerkin approximation to the exact solution (u, p) of the Navier–Stokes equations, that is for $t \in (0, T]$, (u_H, p_H) is the solution of the following problem for all $\phi_H \in X_{H,r}$ and $\psi_H \in Q_{H,r-1}$

$$(\dot{u}_H, \phi_H) + \nu(\nabla u_H, \nabla \phi_H) + b(u_H, u_H, \phi_H) + (\nabla p_H, \phi_H) = (f, \phi_H), \tag{11}$$

$$(\nabla \cdot u_H, \psi_H) = 0, \tag{12}$$

where $b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v, w)$ for any $u, v, w \in H_0^1(\Omega)^d$. The following bounds hold for $2 \leq r \leq 5$, see [11], [20] and [21]

$$\|u(t) - u_H(t)\|_0 + H\|u(t) - u_H(t)\|_1 \leq C\frac{H^r}{t^{(r-2)/2}}, \quad 0 \leq t \leq T, \tag{13}$$

and also,

$$\|p(t) - p_H(t)\|_{L^2/\mathbb{R}} \leq C\frac{H^{r-1}}{t^{(r'-2)/2}}, \quad 0 \leq t \leq T, \tag{14}$$

where $r' = r$ if $r \leq 4$ and $r' = r + 1$ if $r = 5$.

3 The Newton-type problem

Since Navier-Stokes equations are non-linear, it is of interest to study linearized problems related to them. Stokes, Oseen and Newton-type problems provide three ways to linearize the equations. Let $g \in L^2(\Omega)$ and consider the Stokes problem

$$\left.\begin{array}{rcl} -\nu\Delta v + \nabla j &=& g \\ \mathrm{div}(v) &=& 0 \end{array}\right\} \text{ in } \Omega \qquad v = 0, \quad \text{on } \partial\Omega \tag{15}$$

with homogeneous Dirichlet boundary conditions. It is well known (see [21]) that the Stokes projection $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ that approximates problem (15) satisfies the following error bounds for $2 \le m \le r$

$$\begin{aligned} \|v - s_h\|_0 + h\|v - s_h\|_1 &\le Ch^m \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right), \quad (16)\\ \|j - q_h\|_{L^2/\mathbb{R}} &\le Ch^{m-1} \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right). \quad (17)\end{aligned}$$

Let us now consider the following steady Oseen-type problem, which can be obtained by adding a convection term to the Stokes problem $(u \cdot \nabla)v$, where u is a function satisfying $\nabla \cdot u = 0$.

$$\left.\begin{aligned} -\nu\Delta v + (u \cdot \nabla)v + \nabla j &= g \\ \mathrm{div}(v) &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad v = 0. \text{ on } \partial\Omega \tag{18}$$

The mixed finite-element approximation $(w_h, k_h) \in (X_{h,r}, Q_{h,r-1})$ of the Oseen problem (18) satisfies the same bounds as the Stokes projection [15] for $2 \le m \le r$

$$\begin{aligned} \|v - w_h\|_0 + h\|v - w_h\|_1 &\le Ch^m \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right), \quad (19)\\ \|j - k_h\|_{L^2/\mathbb{R}} &\le Ch^{m-1} \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right). \quad (20)\end{aligned}$$

Now, we add to the Oseen problem two new terms. The first one is an extra reaction term $(v \cdot \nabla)u$ which transforms the Oseen problem into a Newton-type problem. Since in general the resulting problem is not coercive, we also add a correction term λv to ensure coercivity. The Newton problem we consider along the paper is then the following:

$$\left.\begin{aligned} -\nu\Delta v + (u \cdot \nabla)v + (v \cdot \nabla)u + \nabla j + \lambda v &= g \\ \mathrm{div}(v) &= 0 \end{aligned}\right\} \text{ in } \Omega \qquad v = 0, \text{ on } \partial\Omega \tag{21}$$

Let us denote by B_u the bilinear form associated to problem (21)

$$B_u(v, w) = \nu(\nabla v, \nabla w) + ((u \cdot \nabla)v, w) + ((v \cdot \nabla)u, w) + \lambda(v, w), \quad v, w \in H_0^1 \tag{22}$$

Continuity of the bilinear form (22) can be derived from

$$\begin{aligned} |B_u(v, w)| &\le \nu\|v\|_1\|w\|_1 + \|u\|_{L^{2d/(d-1)}}\|\nabla v\|_0\|w\|_{L^{2d}} \\ &\quad + \|v\|_{L^{2d/(d-1)}}\|\nabla u\|_0\|w\|_{L^{2d}} + \lambda\|v\|_0\|w\|_0 \\ &\le (\nu + 2C\|u\|_1 + \lambda)\|v\|_1\|w\|_1, \end{aligned} \tag{23}$$

where we have applied Sobolev inequality (4).

Coercivity comes from the antisymmetry of the convection term $((u\cdot\nabla)v, v) = 0$ (6) and the following bound

$$|((v \cdot \nabla)u, v)| \le \|\nabla u\|_\infty \|v\|_0^2 \tag{24}$$

that is

$$B_u(v,v) \geq \nu\|\nabla v\|_0^2 + (\lambda - \|\nabla u\|_\infty)\|v\|_0^2. \tag{25}$$

Observe that due to the reaction term $((v\cdot\nabla)u, v)$, Newton-type problem becomes non coercive, so it is necessary to include the correction term λv in (21). Now, assuming

$$\lambda - \|\nabla u\|_\infty \geq 0 \tag{26}$$

we get that $B_u(v,w)$ is coercive being ν the constant of coercivity.

Let us observe that the norm $\|\nabla u\|_\infty$ is bounded. By (3) and (8) we get

$$\|\nabla u\|_\infty \leq C\|\nabla u\|_{3/2+\alpha} \leq C\|u\|_{5/2+\alpha} \leq M_3 t^{-1/2}.$$

Since we solve the Newton-type problem at a fixed time $T > 0$ we can bound $\|\nabla u\|_\infty \leq CM_3T^{-1/2}$. Let us also observe that since $\|\nabla u\|_\infty$ is not known in practice we can replace in (26) $\|\nabla u\|_\infty$ by $\|\nabla u_H\|_\infty$ to have a computable algorithm.

Remark 1 We want to remark that that assumption $u \in L^\infty(0,T;W^{1,\infty}(\Omega))$ is also required in other related references. See for example [5] where the analysis of the continuous interior penalty finite element method for the time-dependent Navier-Stokes equation is considered, [1] where the local projection finite element stabilization for the time-dependent incompressible Navier-Stokes problem is analyzed and [9] where the authors prove error bounds for stabilized finite element methods for the Oberbeck-Boussinesq model.

The bilinear form $B_u(v,w)$ is thus continuous and coercive on the whole space H_0^1. In particular, this is also true for the divergence free space V. Then, by the Lax-Milgram theorem and the aid of the continuous inf-sup condition (5), there exists a unique solution (v,j) of the problem (21). Regularity conditions can be obtained from the general theory of elliptic problems [17]:

$$\|v\|_2 + \|j\|_{H^1(\Omega)/\mathbb{R}} \leq C\|g\|_0.$$

Now, by (6), the dual problem of (21) is

$$\left.\begin{array}{rcl} -\nu\Delta v - (u\cdot\nabla)v - \nabla\bar{v}\cdot u + \lambda v + \nabla j &=& g, \\ \mathrm{div}(v) &=& 0, \end{array}\right\} \text{ in } \Omega \qquad v = 0, \quad \text{on } \partial\Omega \tag{27}$$

Let $D_u(v,w)$ be the bilinear form associated to the problem (27)

$$D_u(v,w) = \nu(\nabla v, \nabla w) - ((u\cdot\nabla)v, w) - (\nabla\bar{v}\cdot u, w) + \lambda(v,w)$$

that satisfies $D_u(v,w) = B_u(w,v)$. The bilinear form D_u is also continuous and coercive so existence, uniqueness and regularity conditions of problem (27) can be obtained arguing exactly as before. In particular the following regularity condition for the dual problem holds true:

$$\|v\|_2 + \|j\|_{H^1(\Omega)/\mathbb{R}} \leq C\|g\|_0. \tag{28}$$

In the following lemma we get error bounds for the mixed finite-element approximation to problem (21) analogous to bounds (16)-(17) and (19)-(20) for the Stokes (15) and the Oseen (18) problems respectively.

Lemma 1 *Let u be the velocity of the Navier-Stokes equations (1) and let (v, j) be the solution of the linearized problem (21). Consider the discrete variational problem for all $\phi_h \in X_{h,r}$ and $\psi_h \in Q_{h,r-1}$*

$$\begin{aligned} \nu(\nabla v_h, \nabla\phi_h) + ((u\cdot\nabla)v_h, \phi_h) + ((v_h\cdot\nabla)u, \phi_h) + \lambda(v_h, \phi_h) \\ +(\nabla j_h, \phi_h) = (g, \phi_h), \\ (\nabla\cdot v_h, \psi_h) = 0. \end{aligned} \quad (29)$$

Then, there exists a unique solution $(v_h, j_h) \in (X_{h,r}, Q_{h,r-1})$ of (29) which satisfies the following bounds for $2 \le m \le r$

$$\|v - v_h\|_0 + h\|v - v_h\|_1 \le Ch^m \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right), \quad (30)$$

$$\|j - j_h\|_{L^2/\mathbb{R}} \le Ch^{m-1} \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right). \quad (31)$$

Proof Let us observe that the bilinear form $B_u(v, w)$ is also coercive and continuous in the discrete divergence free subspace $V_{h,r}$. Applying Lax-Milgram Theorem and the discrete inf-sup condition (10), the existence and uniqueness of the discrete solution of (29) is obtained.

Now, we observe that problem (21) can be rewritten as an Oseen-type problem with right-hand side $g - v\cdot\nabla u - \lambda v$ as follows

$$\begin{aligned} -\nu\Delta v + (u\cdot\nabla)v + \nabla j &= g - v\cdot\nabla u - \lambda v, \\ \text{div}(v) &= 0. \end{aligned} \quad (32)$$

Let $(w_h, k_h) \in (X_{h,r}, Q_{h,r-1})$ be the finite-element approximation of (32) defined for all $(\phi_h, \psi_h) \in (X_{h,r}, Q_{h,r-1})$ by

$$\begin{aligned} \nu(\nabla w_h, \nabla\phi_h) + ((u\cdot\nabla)w_h, \phi_h) + (\nabla k_h, \phi_h) = (g, \phi_h) - ((v\cdot\nabla)u, \phi_h) - \lambda(v, \phi_h) \\ (\nabla\cdot w_h, \psi) = 0. \end{aligned} \quad (33)$$

We decompose the error in two parts

$$v - v_h = (v - w_h) + (w_h - v_h). \quad (34)$$

The first term on the right-hand side above is bounded in (19). For the second term $e_h = w_h - v_h$ we subtract (29) from (33) and project onto the divergence free space to get

$$\begin{aligned} \nu(\nabla e_h, \nabla\phi_h) + ((u\cdot\nabla)e_h, \phi_h) + ((e_h\cdot\nabla)u, \phi_h) + \lambda(e_h, \phi_h) \\ = (((w_h - v)\cdot\nabla)u, \phi_h) + \lambda(w_h - v, \phi_h), \quad \forall\phi_h \in V_{h,r}. \end{aligned} \quad (35)$$

Taking $\phi_h = e_h$ and using the coercivity of the bilinear form B_u we get a first estimation of $\|e_h\|_1$

$$\nu\|e_h\|_1^2 \le \|w_h - v\|_0\|\nabla u\|_\infty\|e_h\|_0 + \lambda\|w_h - v\|_0\|e_h\|_0, \quad (36)$$

from which

$$\|e_h\|_1 \le C\|w_h - v\|_0 \le h^m \left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right), \quad (37)$$

where in the last inequality we have applied (19).

Inserting (37) in (34) and applying (19) again we complete the proof of the bound for the H^1 norm in (30)

$$\|v - v_h\|_1 \quad \le \quad Ch^{m-1}\left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right). \tag{38}$$

In view of (37) we have proved a super-convergent result in the H^1 norm between the Galerkin approximation to the velocity of problem (21) and the Oseen approximation to the velocity of the same problem defined in (33).

For the pressure bound, we decompose

$$j - j_h = (j - k_h) + (k_h - j_h) \tag{39}$$

and subtract again (29) from (33) to obtain

$$\begin{aligned}\nu(\nabla e_h, \nabla\phi_h) + ((u\cdot\nabla)e_h, \phi_h) + (\nabla(k_h - j_h), \phi_h) = (((v_h - v)\cdot\nabla)u, \phi_h)\\ +\lambda(v_h - v, \phi_h) \quad \forall\phi_h \in X_{h,r}.\end{aligned}$$

Applying the inf-sup condition (10) and (4) it follows

$$\beta\|k_h - j_h\|_{L^2/\mathbb{R}} \quad \le \quad \nu\|e_h\|_1 + C\|u\|_{1/2}\|e_h\|_1 + C\|u\|_1\|v_h - v\|_1 + \lambda\|v_h - v\|_1,$$

so that applying (37) and (38) the proof of (31) is finished.

To bound the zero norm we use duality arguments. Let us observe that

$$\|e_h\|_0 = \sup_{\varphi\in L^2(\Omega)^d\ \varphi\neq 0} \frac{|(e_h, \varphi)|}{\|\varphi\|_0}$$

so that for each fixed $\varphi \in L^2(\Omega)^d$, we introduce the dual problem of (21).

$$\left.\begin{array}{rcl} -\nu\Delta\alpha - (u\cdot\nabla)\alpha - \nabla\bar{\alpha}\cdot u + \nabla\gamma + \lambda\alpha &=& \varphi,\\ \mathrm{div}(\alpha) &=& 0, \end{array}\right\} \text{ in } \Omega \\ \alpha = 0, \quad \text{on } \partial\Omega. \tag{40}$$

Let (α_h, γ_h) be the Stokes projection of the solution (α, γ) of (40). This approximation satisfies, by (16), (17) and (28), the following bounds

$$\begin{aligned}\|\alpha - \alpha_h\|_0 + h\|\alpha - \alpha_h\|_1 \le Ch^2(\|\alpha\|_2 + \|\gamma\|_{H^1/\mathbb{R}}) \le Ch^2\|\varphi\|_0,\\ \|\gamma - \gamma_h\|_{L^2/\mathbb{R}} \le Ch(\|\alpha\|_2 + \|\gamma\|_{H^1/\mathbb{R}}) \le Ch\|\varphi\|_0.\end{aligned} \tag{41}$$

Integrating by parts, we get

$$\begin{aligned}(\varphi, e_h) &= \nu(\nabla e_h, \nabla\alpha) + ((u\cdot\nabla)e_h, \alpha) + ((e_h\cdot\nabla)u, \alpha) + \lambda(\alpha, e_h) - (\gamma, \nabla\cdot e_h)\\ &= \nu(\nabla e_h, \nabla(\alpha - \alpha_h)) + ((u\cdot\nabla)e_h, \alpha - \alpha_h) + ((e_h\cdot\nabla)u, \alpha - \alpha_h) \quad (42)\\ &\quad + \lambda(e_h, \alpha - \alpha_h) + (\gamma_h - \gamma, \nabla\cdot e_h) + \nu(\nabla e_h, \nabla\alpha_h) + ((u\cdot\nabla)e_h, \alpha_h)\\ &\quad + ((u\cdot\nabla)e_h, \alpha_h) + \lambda(e_h, \alpha_h).\end{aligned}$$

Now the terms depending on $\alpha - \alpha_h$ are easily bounded applying (4) and (41)

$$\begin{aligned}(\varphi, e_h) &\leq \nu\|e_h\|_1\|\alpha-\alpha_h\|_1 + \|u\|_{1/2}\|e_h\|_1\|\alpha-\alpha_h\|_1 \\ &\quad +\|u\|_1\|e_h\|_1\|\alpha-\alpha_h\|_1 + \lambda\|e_h\|_1\|\alpha-\alpha_h\|_1 + \|\gamma-\gamma_h\|_{L^2/\mathbb{R}}\|e_h\|_1 \\ &\quad +\nu(\nabla e_h, \nabla\alpha_h) + ((u\cdot\nabla)e_h, \alpha_h) + ((e_h\cdot\nabla)u, \alpha_h) + \lambda(e_h, \alpha_h) \\ &\leq Ch\|e_h\|_1\|\varphi\|_0 + \nu(\nabla e_h, \nabla\alpha_h) + ((u\cdot\nabla)e_h, \alpha_h) \\ &\quad +((e_h\cdot\nabla)u, \alpha_h) + \lambda(e_h, \alpha_h). \end{aligned} \tag{43}$$

For the other terms we take $\phi_h = \alpha_h$ in (35) and apply (4) to get

$$\begin{aligned}&\nu(\nabla e_h, \nabla\alpha_h) + ((u\cdot\nabla)e_h, \alpha_h) + ((e_h\cdot\nabla)u, \alpha_h) + \lambda(e_h, \alpha_h) \\ &\quad = (((w_h - v)\cdot\nabla)u, \alpha_h) + \lambda(w_h - v, \alpha_h) \\ &\quad \leq C\|\nabla u\|_{L^{2d}}\|w_h - v\|_0\|\alpha_h\|_{L^{2d/(d-1)}} + \lambda\|w_h - v\|_0\|\alpha_h\|_0 \\ &\quad \leq C\|u\|_2\|w_h - v\|_0\|\alpha_h\|_1 + \lambda\|w_h - v\|_0\|\alpha_h\|_1 \\ &\quad \leq C\|w_h - v\|_0\|\varphi\|_0, \end{aligned} \tag{44}$$

where the constant C depends on M_2 (see (7)) and the last inequality is due to

$$\|\alpha_h\|_1 \leq \|\alpha_h - \alpha\|_1 + \|\alpha\|_1 \leq Ch\|\alpha\|_2 + \|\alpha\|_2 \leq C\|\varphi\|_0. \tag{45}$$

Inserting (44) in (43) and applying (19) we obtain the following bound

$$\begin{aligned}\|e_h\|_0 &\leq Ch\|e_h\|_1 + C\|w_h - v\|_0 \\ &\leq Ch^m\left(\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}\right).\end{aligned}$$

Applying triangle inequality and (19) again we conclude the proof. □

4 Some auxiliary results

We now state some results that will be used to get the rate of convergence of the new two-grid method. Let us denote by B_H the bilinear form defined by

$$B_H(v, w) = \nu(\nabla v, \nabla w) + ((u_H\cdot\nabla)v, w) + ((v\cdot\nabla)u_H, w) + \lambda(v, w), \quad v, w \in H_0^1,$$

where u_H is the mixed finite-element approximation to u defined in (11)-(12). Continuity of B_H can be proved in the following way

$$B_H(v, w) = B_u(v, w) - (B_u - B_H)(v, w) \leq C(\nu + 2\|u\|_1 + 2\|u_H - u\|_1 + \lambda)\|v\|_1\|w\|_1.$$

Moreover, for H small enough B_H, is also coercive. Using the same decomposition as before and the coercivity of the bilinear form B_u we get

$$\begin{aligned}B_H(v, v) = B_u(v, v) - (B_u - B_H)(v, v) &\geq B_u(v, v) - 2\|u_H - u\|_1\|v\|_1^2 \\ &\geq \left(\nu - 2C\frac{H^{r-1}}{t^{(r-2)/2}}\right)\|v\|_1^2.\end{aligned}$$

Then, the coercivity is attained when u_H is a good enough approximation to u, i.e., when H is small enough. More precisely:

Remark 2 Let us observe that for $t > 0$ and $H < (t^{(r-2)/2}\nu/2C)^{1/(r-1)}$ the bilinear form (46) is coercive. Taking for example $H < (t^{(r-2)/2}\nu/(4C))^{1/(r-1)}$ we get

$$|B_H(v,v)| \geq \frac{\nu}{2}\|v\|_1^2, \quad \forall v \in H_0^1. \tag{46}$$

We want to remark that we are not studying the convection dominating regime in which $\nu \to 0$. Some works concerning this case are for example, [5], [1] where the continuous interior penalty method and the local projection stabilization method for the Navier-Stokes equations are considered and [10] where a plain Galerkin method with grad-div stabilization for the Navier-Stokes equations is analyzed. In all these papers error bounds with constants independent on the viscosity parameter ν can be found.

Altogether, we can state that the method can be applied whenever condition (26), that assures coercivity of the continuous problem, is satisfied, for any strictly positive time $t > 0$, for any strictly positive value of the viscosity parameter ν and whenever H is small enough, $H < (t^{(r-2)/2}\nu/(4C))^{1/(r-1)}$. Under these assumptions the bilinear form B_H is coercive which assures the stability of the method.

The following two lemmas establish some bounds for the temporal derivative of the Galerkin error and for the dual norm or the Galerkin error. Their proofs can be found in [14, Lemma 4] for the case $r = 2$, [11, Lemma 5.1] for $r = 3, 4$ and in [11, p. 226], respectively.

Lemma 2 *Let (u, p) be the solution of* (1) *and let u_H be the mixed finite-element approximation to u defined in (11)-(12). Let A the Stokes operator defined by (7). Then, there exists a positive constant C such that*

$$\|u_t(t) - \dot{u}_H(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} H^r \left|\log(H)\right|^{r'}, \ t \in (0,T], \ r = 2,3,4, \tag{47}$$

$$\|A^{-1}\Pi\left(u_t(t) - \dot{u}_H(t)\right)\|_0 \leq \frac{C}{t^{(r-1)/2}} H^{r+1} \left|\log(H)\right|, \ t \in (0,T], \ r = 3,4, \tag{48}$$

where $r' = 2$ when $r = 2$ and $r' = 1$ otherwise.

Lemma 3 *Let (u, p) be the solution of (1) and let u_H be the mixed finite-element approximation to u defined in (11)-(12). Then, there exists a positive constant C such that*

$$\|u(t) - u_H(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} H^{r+1} \left|\log(H)\right|, \ t \in (0,T], \quad r = 3,4. \tag{49}$$

5 The Newton-type two-grid method

The postprocessing technique we propose is a two-level or two-grid method. In the first level, we choose a coarse mesh of size H and compute the mixed finite-element approximation (u_H, p_H) to (u,p) defined by (11)-(12). In the second level, the discrete velocity and pressure $(u_H(t), p_H(t))$ are postprocessed by solving the following linear Newton-type problem: find $(\tilde u_h(t), \tilde p_h(t)) \in (X_{h,r}, Q_{h,r-1})$, $h < H$, satisfying for all $\phi_h \in X_{h,r}$ and $\psi_h \in Q_{h,r-1}$

$$\begin{aligned}\nu(\nabla \tilde u_h(t), \nabla\phi_h) + ((u_H(t)\cdot\nabla)\tilde u_h(t),\phi_h) + ((\tilde u_h(t)\cdot\nabla)u_H(t),\phi_h) + \lambda(\tilde u_h(t),\phi_h)\\ +(\nabla \tilde p_h(t),\phi_h) = (f(t) - \dot u_H(t),\phi_h) + ((u_H(t)\cdot\nabla)u_H(t),\phi_h) + \lambda(u_H,\phi_h) \quad (50)\\ (\nabla\cdot \tilde u_h(t),\psi_h) = 0.\end{aligned}$$

Equations (50) can also be solved over a higher order mixed finite-element space over the same grid. For simplicity in the exposition we will only consider the case in which we refine the mesh at the postprocessing step.

We now state the main result of the paper that bounds the error of the post-processed approximation.

Theorem 1 *Let (u,p) be the solution of (1) and $(\tilde u_h, \tilde p_h)$ be the solution of (50). Then, for h and H small enough the following bounds hold for $t \in (0,T]$, $m = 0,1$:*

$$\|u(t) - \tilde u_h(t)\|_1 \le Ch + \frac{C}{t^{1/2}} H^2 |\log(H)|^2, \ r = 2, \quad (51)$$

$$\|u(t) - \tilde u_h(t)\|_m \le \frac{C}{t^{(r-2)/2}} h^{r-m} + \frac{C}{t^{(r-1)/2}} H^{r+1-m} |\log(H)|, \ r = 3,4. \quad (52)$$

$$\|p(t) - \tilde p_h(t)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{(r-2)/2}} h^{r-1} + \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, \ r = 2,3,4, \quad (53)$$

where $r' = 2$ for $r = 2$ and $r' = 1$ otherwise.

Proof Let us consider the auxiliary problem (21) with $g = f - u_t + (u\cdot\nabla)u + \lambda u$. Clearly the solution (u,p) of the Navier-Stokes equations (1) is also the unique solution of this problem. We will denote by (v_h, j_h) its mixed finite-element approximation. Let us decompose

$$u - \tilde u_h = (u - v_h) + (v_h - \tilde u_h) \quad (54)$$

and denote by $e_h = v_h - \tilde u_h$. To bound the first term we apply Lemma 1. For the second, we subtract (50) from (29) projecting onto the discrete divergence free space $V_{h,r}$. Then,

$$\begin{aligned}\nu(\nabla e_h, \nabla\phi_h) + ((u_H\cdot\nabla)e_h,\phi_h) + ((e_h\cdot\nabla)u_H,\phi_h) + \lambda(e_h,\phi_h) =\\ (\dot u_H - u_t,\phi_h) + (((u_H - u)\cdot\nabla)v_h,\phi_h) + ((v_h\cdot\nabla)(u_H - u),\phi_h) + ((u\cdot\nabla)u,\phi_h)\\ -((u_H\cdot\nabla)u_H,\phi_h) + \lambda(u - u_H,\phi_h), \quad \forall\phi_h \in V_{h,r}.\end{aligned}$$

Adding and subtracting to the right-hand side the following terms

$$(((u_H - u)\cdot\nabla)u,\phi_h) + ((u\cdot\nabla)(u_H - u),\phi_h)$$

and reordering terms we get

$$\begin{aligned}
\nu(\nabla e_h, \nabla\phi_h) + ((u_H\cdot\nabla)e_h,\phi_h) + ((e_h\cdot\nabla)u_H,\phi_h) + \lambda(e_h,\phi_h) = \\
(\dot{u}_H - u_t,\phi_h) + (((u_H-u)\cdot\nabla)(v_h-u),\phi_h) + (((v_h-u)\cdot\nabla)(u_H-u),\phi_h) \\
+(((u_H-u)\cdot\nabla)(u-u_H),\phi_h) + \lambda(u-u_H,\phi_h).
\end{aligned}$$

Taking $\phi_h = e_h$ and applying (46) we get for H small enough

$$\begin{aligned}
\frac{\nu}{2}\|e_h\|_1^2 \quad\leq\quad & |(\dot{u}_H - u_t, e_h)| + |(((u_H-u)\cdot\nabla)(v_h-u),e_h)| \\
& +|(((v_h-u)\cdot\nabla)(u_H-u),e_h)| + |(((u_H-u)\cdot\nabla)(u-u_H),e_h)| \\
& +|\lambda(u-u_H,e_h)|. \qquad (55)
\end{aligned}$$

We will bound each term on the right-hand side of (55). To this end, we write

$$\begin{aligned}
|(\dot{u}_H - u_t, e_h)| &\leq \|\dot{u}_H - u_t\|_{-1}\|e_h\|_1, \\
|(((u_H-u)\cdot\nabla)(v_h-u),e_h)| &\leq C\|u_H-u\|_1\|v_h-u\|_1\|e_h\|_1, \\
|(((v_h-u)\cdot\nabla)(u_H-u),e_h)| &\leq C\|v_h-u\|_1\|u_H-u\|_1\|e_h\|_1, \\
|(((u_H-u)\cdot\nabla)(u-u_H),e_h)| &\leq C\|u-u_H\|_1^2\|e_h\|_1, \\
|\lambda(u-u_H,e_h)| &\leq \lambda\|u-u_H\|_0\|e_h\|_0 \leq \lambda\|u-u_H\|_0\|e_h\|_1.
\end{aligned}$$

And then we get

$$\begin{aligned}
\frac{\nu}{2}\|e_h\|_1 \quad\leq\quad & \|u_t - \dot{u}_H\|_{-1} + 2C\|v_h-u\|_1\|u_H-u\|_1 \\
& +C\|u-u_H\|_1^2 + \lambda\|u-u_H\|_0. \qquad (56)
\end{aligned}$$

To bound the first term above we apply (47)

$$\|u_t - \dot{u}_H\|_{-1} \leq \frac{C}{t^{(r-1)/2}} H^r \left|\log(H)\right|^{r'}, \qquad (57)$$

where $r' = 2$ for $r = 2$ and $r' = 1$ for $r = 3, 4$. For the other terms, we apply (30) with $m = 2$ together with (13)

$$\begin{aligned}
\|u - v_h\|_1\|u_H - u\|_1 &\leq \frac{C}{t^{(r-2)/2}} hH^{r-1}, \\
\|u - u_H\|_1^2 &\leq C\frac{H^{r-1}}{t^{(r-2)/2}}\frac{H^{\min(2,r-1)}}{t^{1/2}} = C\frac{H^{r+\min(1,r-2)}}{t^{(r-1)/2}}, \qquad (58) \\
\|u - u_H\|_0 &\leq \frac{C}{t^{(r-2)/2}} H^r.
\end{aligned}$$

Inserting now (57) and (58) in (56) and keeping only the biggest terms in the error bound we get

$$\|e_h\|_1 \quad\leq\quad \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, \ r = 2, 3, 4, \qquad (59)$$

where $r' = 2$ for $r = 2$ and $r' = 1$ for $r = 3, 4$. Applying triangle inequality together with (30) we conclude

$$\|u(t) - \tilde{u}_h(t)\|_1 \leq \frac{C}{t^{(r-2)/2}} h^{r-1} + \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}.$$

For the pressure bound, we decompose

$$p - \tilde{p}_h = (p - j_h) + (j_h - \tilde{p}_h). \tag{60}$$

Let us denote by $r_h = j_h - \tilde{p}_h$. We subtract again (50) from (29) to get

$$\begin{aligned}\nu(\nabla e_h, \nabla\phi_h) + ((u_H \cdot \nabla)e_h, \phi_h) + ((e_h \cdot \nabla)u_H, \phi_h) + \lambda(e_h, \phi_h) + (\nabla r_h, \phi_h) = \\ (\dot{u}_H - u_t, \phi_h) + (((u_H - u) \cdot \nabla)(v_h - u), \phi_h) + (((v_h - u) \cdot \nabla)(u_H - u), \phi_h) \\ +(((u_H - u) \cdot \nabla)(u - u_H), \phi_h) + \lambda(u - u_H, \phi_h),\end{aligned}$$

for all $\phi_h \in X_{h,r}$. Using the continuity of B_H and applying the inf-sup condition (10) we get

$$\begin{aligned}\beta\|r_h\|_{L^2/(R)} \quad &\leq \quad C\|e_h\|_1 + \|u_t - \dot{u}_H\|_{-1} + 2C\|v_h - u\|_1\|u_H - u\|_1 \\ &\quad +C\|u - u_H\|_1^2 + \lambda\|u - u_H\|_0.\end{aligned}$$

Applying (59) to bound $\|e_h\|_1$ together with (57) and (58) we reach

$$\beta\|r_h\|_{L^2/(R)} \leq \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}.$$

To conclude the error bound for the pressure we apply decomposition (60) and (31).

To bound the L^2 norm of the error in the velocity we argue by duality exactly as in the proof of Lemma 1. We write

$$\|e_h\|_0 = \sup_{\varphi \in L^2(\Omega)^d \ \varphi \neq 0} \frac{|(e_h, \varphi)|}{\|\varphi\|_0},$$

and consider the following dual problem for each $\varphi \in L^2(\Omega)^d$

$$\left.\begin{aligned}-\nu\Delta\alpha - (u \cdot \nabla)\alpha - \nabla\bar{\alpha} \cdot u + \nabla\gamma + \lambda\alpha &= \varphi, \\ \mathrm{div}(\alpha) &= 0,\end{aligned}\right\} \text{ in } \Omega$$
$$\alpha = 0, \quad \text{on } \partial\Omega.$$

As in Lemma 1 we denote by (α_h, γ_h) the Stokes projection of this dual problem. Then, we observe that (43) holds. Using (55) instead of (35) in (43) we reach

$$\begin{aligned}(e_h, \varphi) \quad &\leq \quad Ch\|e_h\|_1\|\varphi\|_0 + \nu(\nabla e_h, \nabla\alpha_h) + ((u \cdot \nabla)e_h, \alpha_h) + ((e_h \cdot \nabla)u, \alpha_h) \\ &\quad +\lambda(e_h, \alpha_h) \\ &\leq \quad Ch\|e_h\|_1\|\varphi\|_0 + (\dot{u}_H - u_t, \alpha_h) + (((u_H - u) \cdot \nabla)(\tilde{u}_h - u), \alpha_h) \\ &\quad +(((\tilde{u}_h - u) \cdot \nabla)(u_H - u), \alpha_h) \\ &\quad +(((u_H - u) \cdot \nabla)(u - u_H), \alpha_h) + \lambda(u - u_H, \alpha_h).\end{aligned} \tag{61}$$

To get the error bound in L^2 it is necessary to improve some of the bounds used in the proof of the H^1 norm. For the temporal Galerkin error, we apply (47), (48) and (41) to obtain

$$\begin{aligned} |(\dot{u}_H - u_t, \alpha_h)| &= |(\dot{u}_H - u_t, \alpha_h - \alpha) + (\dot{u}_H - u_t, \alpha)| \\ &\leq C\|u_t - \dot{u}_H\|_{-1}\|\alpha_h - \alpha\|_1 + C\|A^{-1}\Pi(\dot{u} - u_t)\|_0\|A\alpha\|_0 \\ &\leq C\|u_t - \dot{u}_H\|_{-1}h\|\varphi\|_0 + C\|A^{-1}\Pi(\dot{u} - u_t)\|_0\|\varphi\|_0 \\ &\leq \frac{C}{t^{(r-1)/2}}H^{r+1}\,|\log(H)|\,\|\varphi\|_0. \end{aligned} \tag{62}$$

Let us observe that (62) is valid for $r \geq 3$. Next two terms in (61) are bounded applying (52), (13) with $m = 2$ and (45)

$$\begin{aligned} &|(((u_H - u)\cdot\nabla)(\tilde{u}_h - u), \alpha_h)| + |(((u_H - u)\cdot\nabla)(u - u_H), \alpha_h)| \\ &\quad \leq C\|u_H - u\|_1\|\tilde{u}_h - u\|_1\|\alpha_h\|_1 \\ &\quad \leq \frac{CH^2}{t^{(r-1)/2}}\left(t^{1/2}h^{r-1} + H^r|\log(H)|\right)\|\varphi\|_0. \end{aligned} \tag{63}$$

For the fifth term we apply (58) and (45) to get

$$|(((u_H - u)\cdot\nabla)(u - u_H), \alpha_h)| \quad \leq \quad C\frac{H^{r+1}}{t^{(r-1)/2}}\|\varphi_h\|_0. \tag{64}$$

Finally, for the last term in (62) we use (49) and (45) to obtain

$$|(u - u_H, \alpha_h)| \leq \frac{C}{t^{(r-1)/2}}H^{r+1}\,|\log(H)|\,\|\varphi_h\|_0. \tag{65}$$

Inserting (62), (63), (64) and (65) in (61) and keeping only the biggest terms in the error bound we get

$$\|e_h\|_0 \leq Ch\|e_h\|_1 + \frac{C}{t^{(r-1)/2}}H^{r+1}\,|\log(H)|\,.$$

Applying (59), decomposition (54) and (30) we finally reach (52) for $m = 0$. □

Remark 3 We want to observe that projecting onto the discrete divergence free space $V_{h,r}$ in the proof of Theorem 1 simplifies the task of getting the error bounds for the velocity. This technique can be applied whenever inf-sup stable mixed finite elements are considered.

6 Numerical experiments

In this section, we carry out several experiments for studying both the order of convergence and the comparative behaviour of the three postprocessed procedures: Stokes, Oseen and Newton. The experiments are computed in the unit square $\Omega = [0,1] \times [0,1]$, using the mini-element over a regular triangulation based on the set of nodes $(i/N, j/N)$ $i, j = 1, 2 \ldots N$, where $N = 1/H$ is the Galerkin spatial resolution. In the time integration we use a semi-implicit trapezoidal rule, where spatial derivatives are treated implicitly. The size of the time step is chosen so that temporal errors are negligible compared to spatial errors. Once the mini-element is obtained, the bubble part is removed and the errors of the linear part are computed. It has been reported in [15] that the linear part is a better approximation to the solution than the complete linear-bubble vector velocity. The bubble part is considered only for stability reasons. In this experiment, we take the following functions

$$\begin{aligned} u^1(x,y,t) &= \pi t \sin^2(\pi x)\sin(2\pi y), \\ u^2(x,y,t) &= -\pi t \sin^2(\pi y)\sin(2\pi x), \\ p(x,y,t) &= 5tx^2y. \end{aligned} \tag{66}$$

and calculate the forcing term $f(x,t)$ so that (66) is the solution of the Navier Stokes equations for different values of the diffusion parameter ranging from $\nu = 1$ to $\nu = 0.001$. The Galerkin approximations are obtained integrating up to time $T = 1$ in the coarse mesh of size $H = 1/N$. Then, there are post-processed at the fixed final time over a finer mesh of size $h < H$ small enough to retain the asymptotic behavior of the rate of convergence. Computational cost of the post-processed step is typically of the order of a single time step over the fine mesh h. In this experiment, the sizes of the coarse meshes are given by $N = 40$, $N = 50$, $N = 70$, $N = 85$ and for the fine meshes $n = 175$, $n = 223$, $n = 273$ and $n = 353$, respectively. Although for the mini-element, optimal values for the fine mesh in the sense of Theorem 1 are those obtained taking $h = H^2$, in practice the rate of convergence of the methods can be reached taking exponents less than 2. This is the reason why we chose the above values of n that correspond to the smallest values for which increasing them we do not achieve smaller errors.

Figure 1 shows the L^2 velocity errors of the first component of the velocity for the Galerkin approximation and the three post-processed procedures: Stokes, Oseen and Newton, with respect to the above mentioned values for the coarse and fine meshes. Each of the pictures correspond to a different value of the diffusion parameter, ranging from $\nu = 1$, $\nu = 0.1$, $\nu = 0.01$ to $\nu = 0.001$ from left to right and up to down. For the Newton-type algorithm we have carried out computations with different values of the parameter λ. In this experiment, for all values of λ we chose in our computations, including $\lambda = 0$, the approximations were always computable, i.e., problem (50) had a unique solution. Then, although, theoretically, λ should be chosen greater or equal to $\|\nabla u\|_\infty$ according to (26) (or in practice $\lambda \geq \|\nabla u_H\|\infty$ since u is in general unknown) in the experiments shown in this section we have chosen the values of λ that produce better errors for the method. These values depend on the diffusion parameter. In all the pictures, we have chosen $\lambda = 0.1, 0.5, 1.2$ and 2 for $\nu = 1, 0.1, 0.01$ and 0.001 respectively.

It can be observed in Figure 1 that, as expected, in general the post-processed procedures do not increase the L^2 rate of convergence of the Galerkin method, although there is a slight increase for $\nu = 1$. This is a particular case due to

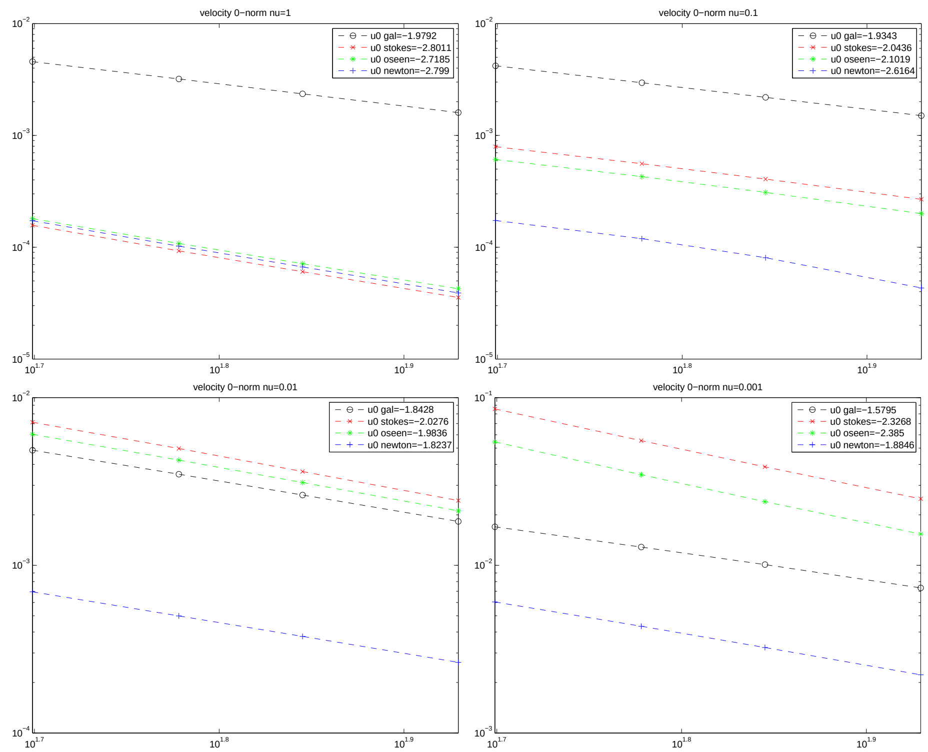

Figure 1: L^2 errors for the first velocity component with $\nu = 1$, $\nu = 0.1$, $\nu = 0.01$ and $\nu = 0.001$

the use of linear elements (case $r = 2$) . For $\nu = 1$ all the postprocessed methods produce the same errors and much smaller than the Galerkin errors. The comparison between the methods changes with the value of ν. As ν decreases the difference between the errors of the new method and the other two postprocessed methods increases being for $\nu = 0.001$ the new procedure the only one producing much smaller errors than the Galerkin method. As expected, all the methods deteriorate when decreasing the value of ν due to the loss of coercivity but the new method is the one behaving better.

Figure 2 shows the H^1 norm of the error for the first component of the velocity. Considering this norm, and in agreement with the theory, it is remarkable the increment of the rate of convergence of the new postprocessed method in one unit in terms of H (same increment can be observed for the other two postprocessed methods). Contrary to the situation shown in Figure 1 for the L^2 errors, in the H^1 norm the three postprocessed methods produce always smaller errors than the Galerkin method, being again Newton-type the best post-processing procedure. The Stokes method reveals itself to be the worst postprocess in all cases. Since the Galerkin errors lie on a perfect straight line we can deduce from the figure the Galerkin errors for other values of N different

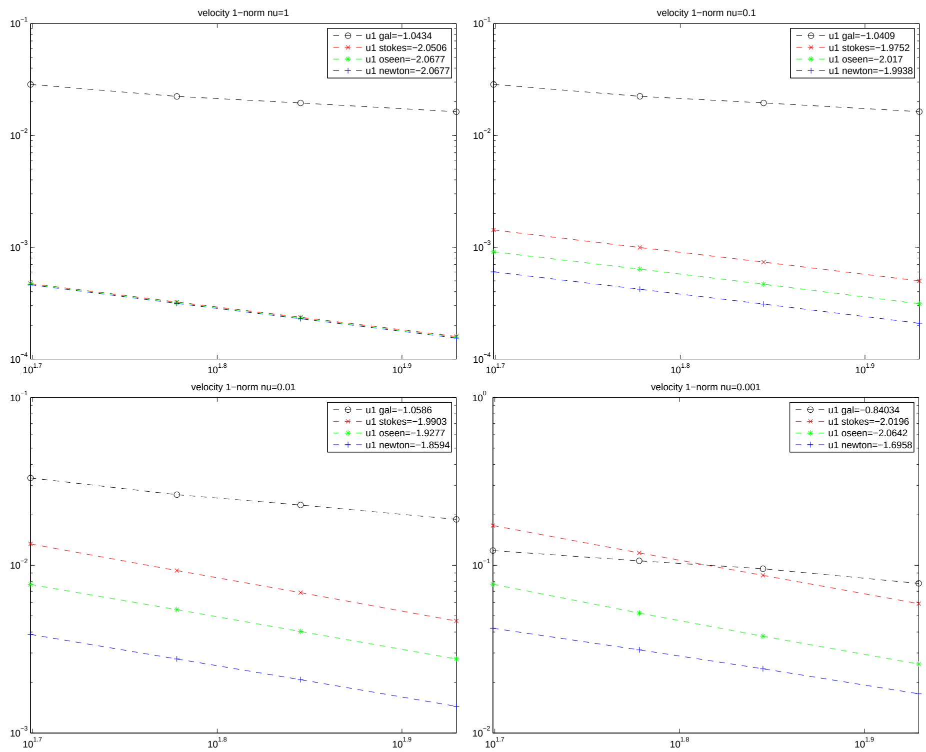

Figure 2: H^1 errors for the first velocity component with $\nu = 1$, $\nu = 0.1$, $\nu = 0.01$ and $\nu = 0.001$

from those plotted. In particular, for example, for $\nu = 0.01$ the Galerkin error for $n = 175$ is around 0.008 while the error for $n = 353$ is 0.004. On the other hand, the Newton-type postprocess gives for $(N, n) = (40, 175)$ an error of size 0.004 while for the last plotted value $(N, n) = (85, 353)$ the error is around 0.001. This means that although in the picture we have compared the postprocessed errors with the coarse-mesh Galerkin errors if we compare the postprocessed errors with the fine-mesh Galerkin errors the postprocessed procedure gives still better errors than the Galerkin method. However, the computational cost of the postprocessed method is essentially the same as the Galerkin method over the coarse mesh. This means that those errors on the same vertical line are achieved with almost the same computational CPU time. Then, from the picture we can deduce that not only the new method produces smaller errors but is also considerably more efficient.

Finally, Figure 3 shows the L^2 errors for the pressure. We can observe that the postprocessed pressures of the Newton-type method have a rate of convergence of order 2, according to Theorem 1. As in Figure 1, Newton-type postprocess is the best one, Stokes is never worse than Galerkin while Oseen is worse than the Galerkin method for $\nu = 0.001$.

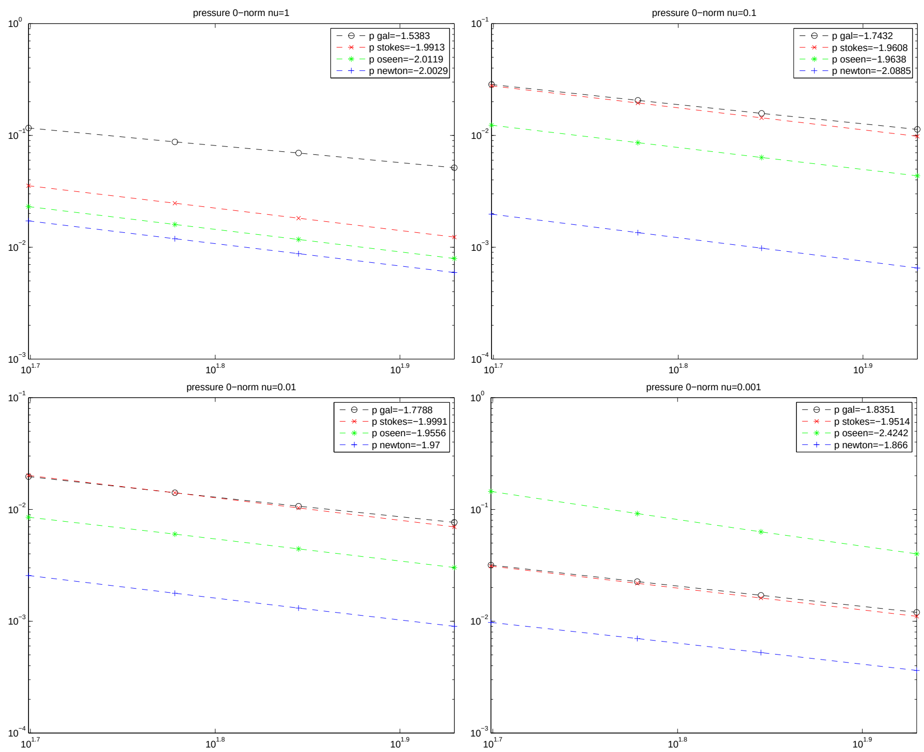

Figure 3: L^2 errors for the pressure with $\nu = 1$, $\nu = 0.1$, $\nu = 0.01$ and $\nu = 0.001$

In summary, the new method we analyze is the one giving better errors for all values of the Reynolds number computed in the experiment. Our results are in agreement with the numerical examples of [25] where, as mentioned in the introduction, the three two-grid methods are compared but with a dynamical (time-dependent) implementation of the linearized problem over the fine mesh. We want to remark that the static two-grid approach we analyze has the advantage of being computationally more efficient since a single steady problem is solved at the selected time in which we need to get an approximation over a finer mesh.

Finally, we want to mention that the increased accuracy of the new Newton-type postprocessing compared with the plain Galerkin method can be applied for the interesting task of getting a posteriori error estimations for the Galerkin method as was already studied in [14] for the Stokes postprocessing. We want to explore the behavior of a similar a posteriori error estimation but based on the Newton-type postprocessing in a future work.

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