Automorphisms of Higgs bundle moduli spaces for real groups

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A mi familia.

"Life finds a way." Ian Malcolm

"One of the most amazing things about mathematics is the people who do math aren't usually interested in application, because mathematics itself is truly a beautiful art form. It's structures and patterns, and that's what we love, and that's what we get off on." Danica McKellar

"It is time that we all see gender as a spectrum instead of two sets of opposing ideals. We should stop defining each other by what we are not and start defining ourselves by who we are." Emma Watson

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Realizar una tesis doctoral en matemáticas es un proceso complicado y, muchas veces, solitario. Sin embargo, yo he tenido la suerte de no emprender este camino solo, algo que para mi es esencial. En el ICMAT he coincidido con un variopinto grupo de chavales con los que poco a poco he ido forjando un vínculo de amistad y respeto que durará toda la vida. Hablo por ejemplo de los *No Lemon, no Party* y nuestras eternas comidas y sobremesas: Pablo-*Pepito-Grillo*, Álvaro Co y Álvaro Pi, Irene-*la-infiltrada-del-CBM*, Paco, Charly-*marciano*, también David que es un *No Lemon* más, ... Pero el mayor descubrimiento que he hecho a lo largo de la tesis ha sido Sonja. Una persona pura, sin maldad, llena de vida y con una sonrisa que nunca desaparece de su rostro. Sin duda con ella cualquier cosa es posible. Junto a Miguel y mi bro forman lo que yo llamo: el equipo liante, los cuáles tienen esa habilidad innata de proponerte salir, pongamos un martes "de tranquileo", a tomar una cerveza y terminar cerrando todos los garitos de Madrid. No concibo un doctorado sin vosotros y aunque es posible que el futuro nos lleve por caminos distintos, siempre estaremos unidos por estos años que compartimos juntos. Pero en el ICMAT hay muchas más personas con las que he compartido el día a día y que me han aportado muchísimo tanto a nivel académico como personal. Hablo de las inesperadas y extrañas conversaciones acompañadas de sonoras risotadas que te sacan Ángel y Carlos cada vez que pasas por su despacho o de los peculiares trayectos en autobús con Cristina. También hablo de Eric y Diego, la extraña pareja unida por el azar como acertadamente ha descrito Eric en alguna que otra ocasión. Con vosotros tardé más en arrancar pero echando la vista atrás me doy cuenta de que, entre otras cosas, hemos compartido muchísimos partidos de fútbol, de ping-pong y alguna que otra extraña clase improvisada de flamenco. A Samuel, mi joven padawan, solo tengo una cosa que decirle: Esta tesis NO habría sido posible si llegas a entrar en el ICMAT hace cuatro años. Por último quiero acordarme de mis compañeros de la Autónoma. Allí he encontrado amigos para toda la vida como Serafín y en general un grupo muy sano y alegre lleno de personas a las que admiro y respeto como Leyter, Jose, Carlos, Bea P, Raquel, Bea B, Javi, Felix, Iason, Adri, María, Alessandro, Martí, Sauli, Andrea,... es imposible nombraros a todos porque sois muchísimos pero me alegro mucho de haber coincidido con todos vosotros y de haber estado en la mayoría de vuestras defensas de tesis. Es lo que tiene ser uno de los yogurines.

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Introduction.

Higgs bundles were introduced by Nigel Hitchin around 30 years ago in [38] and they have become an active field of research since then due to the great importance and, sometimes quite unexpected relevance, they gained in such different branches of differential and algebraic geometry or mathematical physics such as surface group representation, gauge theory, hyperkähler geometry or integrable systems among others.

In this paper [38] (and others [39, 40]) Hitchin unveils many amazing results about Higgs bundles and their moduli space. He attacks this study from two different points of view. On the one hand he considered a moduli space of solutions to a set of equations coming from gauge theory, known as Hitchin's equations after him, where the solutions are pairs of objects: a connection A on a principal G-bundle E over a Riemann surface X and a $(1,0)$-form φ on X with values in the (complex) bundle associated to E via the adjoint representation of G. On the other hand, when $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$ a solution to these equations defines a pair (V, φ) where V is a holomorphic rank 2 vector bundle over X and φ is a holomorphic section of $\mathrm{End}(V) \otimes K$, with K the canonical bundle of X. He defined stability notions to such pairs that are consistent with the ones introduced by Mumford for vector bundles. One of the main results given in [38] is the correspondence between rank 2 Higgs bundles with trivial determinant, under these suitable stability conditions, and isomorphism classes of $\mathrm{SL}(2,\mathbb{C})$-local systems on a compact connected Riemann surface. This correspondence can be seen as a generalisation of the classical Narasimhan and Seshadri theorem [48] which establishes a correspondence between polystable vector bundles and irreducible representations of the fundamental group of a compact Riemann surface X in the unitary group $\mathrm{U}(n)$. This result was generalised by Simpson [57] to representations of the fundamental group of X in arbitrary complex reductive Lie groups. As Simpson said this correspondence can also be viewed as a Hodge theorem for non-abelian cohomology and it was the beggining of what is now known as *Nonabelian Hodge theory.*

If we replace $\mathrm{SL}(2,\mathbb{C})$ by a (complex or real) reductive Lie group G this leads to the notion of G-Higgs bundle. Let (G,H,θ,B) be a (complex or real) reductive Lie group where H is a maximal compact subgroup of G and θ is a Lie algebra involution of $\mathfrak{g}$, the lie algebra of G, inducing a decomposition into ± 1-eigenspaces $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$. A G-Higgs bundle over a compact Riemann surface X is then a pair (E,φ) where E is a principal $H^{\mathbb{C}}$-bundle and φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}})\otimes K$ where by $H^{\mathbb{C}}$ we denote the complexification of H and $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to E via the (complex) isotropy representation $\iota^-: H^{\mathbb{C}}\to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$. Observe that when G is complex the complexification $H^{\mathbb{C}}$ of the maximal compact subgroup of G is the group itself and the decomposition into ± 1-eigenspaces is $\mathfrak{g}=\mathfrak{h}\oplus i\mathfrak{h}$ and hence E is a G-bundle and φ is a holomorphic section of the adjoint bundle $E(\mathfrak{g})$ twisted by K.

G-Higgs bundles for some particular real Lie groups are one of the topics of [40] and $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles were, indeed, considered in [38]. The generalisation of the theory of G-Higgs bundle for real reductive Lie groups has been developed further by Bradlow, García-Prada, Gothen, Mundet i Riera, Oliveira and others (see for example [17, 18, 19, 26, 27]). Bradlow, Garcia-Prada, Gothen and Mundet i Riera give in [27] a Hitchin-Kobayashi correspondence for L-twisted Higgs pairs and, in particular, for G-Higgs bundles. Roughly speaking the Hitchin-Kobayashi correspondence says that a Higgs bundle (E,φ) is polystable if and only if there exists a solution to Hitchin's equations and it induces a homeomorphism between the moduli space $\mathcal{M}(G)$ of polystable G-Higgs bundles and the gauge moduli space. This correspondence together with Corlette's generalisation [23] of Donaldson's results [24, 25] extend the correspondence between polystable Higgs bundles and irreducible representation to real reductive Lie groups.

An important tool in the research of Higgs bundles is the study of involutions on $\mathcal{M}(G)$, the moduli space of G-Higgs bundles. The aim of this thesis is to study involutions and higher order automorphisms of different moduli spaces for real groups. Since we want to fix the structure group of the G-Higgs bundle we slightly change the classical notation of these objects denoting them as (G,θ)-Higgs bundles. The first non-trivial involution that one can consider on $\mathcal{M}(G,\theta)$ is defined by the multiplification of the Higgs field by -1:

$$(E,\varphi)\mapsto(E,-\varphi).$$

In fact, we can generalise this involution to obtain an action of $\mathbb{C}^*$ on $\mathcal{M}(G,\theta)$ defined by $(E,\varphi)\mapsto(E,\lambda\varphi)$. The action of $\mathbb{C}^*$ on $\mathcal{M}(G,\theta)$ has been proved to be an important tool in the study of topology aspects of $\mathcal{M}(G,\theta)$ and hence, through the correspondences mentioned above, $\mathcal{R}(G,\theta)$. The study of this action on the moduli space of G-Higgs bundles for complex reductive Lie groups was already carry out by García-Prada and Ramanan in [31] where they give a complete description of its fixed-point. They, indeed, extend their study to automorphism of $\mathcal{M}(G)$ defined by elements of $(H^1(X,Z(G)) \rtimes \mathrm{Out}(G))_n \times \mathbb{C}^*$ and, through nonabelian Hodge correspondence, they consider the action of this set on $\mathcal{R}(G)$, the moduli pace of representations of the fundamental group of X in G. This thesis is devoted, among other problems, to a generalisation of the results obtained by Garcia-Prada and Ramanan to the case of real form of complex semisimple Lie groups. With regards to this, let G be a real form of a connected complex semisimple Lie group $G^{\mathbb{C}}$ with Cartan involution θ. We define $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Int}(G)$, the group of outer automorphisms of G, where $\mathrm{Aut}(G)$ is the group of automorphisms of G and $\mathrm{Int}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ whose elements acts on G via conjugation. Let us denote by $\mathrm{Out}_n(G,\theta)$ the set of elements of order n of the outer automorphism group of G that commute with the Cartan involution and by $H^1(X,Z_\tau)$ the group of isomorphism classes of principal Z_τ-bundles over X, where $Z_\tau = Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota_{\mathbb{C}}^-)$. In the analysis of fixed points there appear certain Higgs bundles over X defined by an automorphism $\eta \in \mathrm{Aut}_n(G,\theta)$. These are $(G,\theta,H^\eta,\zeta_k)$**-Higgs bundles** and $(G,\theta,H_\eta,\zeta_k)$**-Higgs bundles**. They are pairs (E,φ) where E is a holomorphic principal $(H^{\mathbb{C}})^\eta$-bundle (resp. $H_\eta^{\mathbb{C}}$-bundle) and

$$\varphi \in H^0(X, E(\mathfrak{m}_k^{\mathbb{C}}) \otimes K)$$

where $E(\mathfrak{m}_k^{\mathbb{C}})$ is the bundle associated to E via the representation

$$\iota^k : (H^{\mathbb{C}})^\eta \to \mathrm{Gl}(\mathfrak{m}_k^{\mathbb{C}})$$

(resp. $\iota_k : H_\eta^{\mathbb{C}} \to \mathrm{Gl}(\mathfrak{m}_k^{\mathbb{C}})$ and K is the canonical bundle on X. These objects are in fact K-twisted pairs of type ι^k (resp. ι_k) for reductive Lie subgroups of $H^{\mathbb{C}}$. We have, however, two exceptional situations. When $k=0$, from Proposition 1.3.3 we have that $H^\eta \exp(\mathfrak{m}_0) = G^\eta$. In the same fashion from Proposition 1.5.1 we have that $H_\eta \exp(\mathfrak{m}_0) = G_\eta$. Hence a $(G,\theta,H^\eta,\zeta_0)$-Higgs bundle is simply a (G^η,θ)-Higgs bundle and a $(G,\theta,H_\eta,\zeta_0)$-Higgs bundle is just a (G_η,θ)-Higgs bundle in the sense of Definition 2.1.1. When n is even and $l = n/2$ from Proposition 1.3.3 and 1.5.1 we have that a $(G,\theta,H^\eta,\zeta_l)$-Higgs bundle is simply (G^σ,θ)-Higgs bundle and that a $(G,\theta,H_\eta,\zeta_l)$-Higgs bundles is just a (G_σ,θ)-Higgs bundle. These exceptional cases do not require any different stability conditions beyond the classical. Otherwise the stability conditions given in Definition 2.2.6 can be extended to our general situation by replacing $\mathfrak{h}$ with $\mathfrak{h}_0$, and $\mathfrak{m}_s$ and $\mathfrak{m}_s^0$ with their analogue spaces associated to $\mathfrak{m}_k$ as we saw in Remark 2.2.8. Hence we can define $\mathcal{M}(G,\theta,H^\eta,\zeta_k)$, the moduli space of isomorphism classes of polystable

$(G, \theta, H^\eta, \zeta_k)$-Higgs bundles. In the same way we can define $\mathcal{M}(G, \theta, H_\eta, \zeta_k)$, the moduli space of isomorphism classes of polystable $(G, \theta, H_\eta, \zeta_k)$-Higgs bundles. We have the following generalisation of Theorem 6.10 in [31]:

Theorem 0.0.1. *(Theorem 3.6.3) Let $a \in \mathrm{Out}_n(G, \theta)$ and let $\alpha \in H^1(X, Z_\tau)$ such that*

$$\alpha a(\alpha) \cdots a^{n-1}(\alpha) = 1.$$

Let (E, φ) be a (G, θ)-Higgs bundle over X and consider the automorphism

$$\begin{array}{rccc} \iota(a, \alpha, \zeta_k): & \mathcal{M}(G, \theta) & \to & \mathcal{M}(G, \theta) \\ & (E, \varphi) & \mapsto & (a(E) \otimes \alpha, \zeta_k a(\varphi)) \end{array} \tag{0.1}$$

with $\zeta_k = \exp(2\pi i \frac{k}{n})$. Then

(1)
$$\bigsqcup_{\eta' \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_a(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_{\eta'}, \zeta_k) \subset \mathcal{M}(G, \theta)^{\iota(a, \alpha, \zeta_k)},$$

(2)
$$\mathcal{M}(G, \theta)^{\iota(a, \alpha, \zeta_k)}_{simple} \subset \bigsqcup_{\eta' \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_a(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_{\eta'}, \zeta_k)$$

except for $\iota(1, 1, \zeta_0)$, since $\iota(1, 1, \zeta_0) = id$.

In addition, through nonabelian Hodge correspondence for real reductive Lie groups, we also have the following generalisation of Theorem 8.8 in [31].

Theorem 0.0.2. *(Theorem 3.7.8) Let $a \in \mathrm{Out}_n(G, \theta)$ and $\lambda \in \mathcal{R}(Z_\tau) = \mathrm{Hom}(\pi_1(X), Z_\tau)$ such that*

$$\lambda a(\lambda) \cdots a^{n-1}(\lambda) = 1.$$

Consider the automorphism

$$\begin{array}{rccc} \iota(a, \lambda, \pm): & \mathcal{R}(G, \theta) & \to & \mathcal{R}(G, \theta) \\ & \rho & \mapsto & \lambda a^{\pm}(\rho), \end{array}$$

with $a^{\pm}(\rho)$ defined as in Theorem 3.7.5. Then

(1)
$$\bigsqcup_{\eta \in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma) = \lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta, \theta) \subset \mathcal{R}(G, \theta)^{\iota(a, \lambda, +)},$$

(2)

$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,+)}_{irred} \subset \bigsqcup_{\eta\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta,\theta).$$

Notice that $\iota(1,1,+)$ is the identity map. If, in addition, $n=2l$ then

(3)

$$\bigsqcup_{\sigma\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,\lambda,-)},$$

(4)

$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,-)}_{irred} \subset \bigsqcup_{\sigma\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta).$$

The details surrounding Theorem 3.6.3 and 3.7.8 are provided in Chapter 3.

The original correspondence between Higgs bundles and local systems introduced by Hitchin and its applications in the study of fundamental groups has grown in many other directions. One of them is the study of representations of the fundamental group of a punctured Riemann surface in an arbitrary reductive Lie group G with prescribed holonomy around the punctures. This motivates the definition of parabolic Higgs bundles. Parabolic vector bundles were first introduced by Seshadri in [54] and the correspondence for $G=\mathrm{U}(n)$ was proved by Mehta and Seshadri in [46]. The generalisation of this correspondence to arbitrary compact Lie group was carried out by Bhosle and Ramanathan [9], Teleman and Woodward [58] and Balaji and Seshadri [7], assuming appropriate conditions on the holonomy around the punctures. Simpson first considered the situation of non-compactness of the reductive Lie group in [56] where he study the case $G=\mathrm{GL}(n,\mathbb{C})$ and he introduced the study of filtered local systems. A recent paper [15] written by Biquard, García-Prada and Mundet i Riera extends the correspondence to arbitrary real reductive Lie groups. In the last chapter of this thesis we start the study of holomorphic involutions defined on the moduli space of parabolic (G,θ)-Higgs bundles when G is a real form of a complex semisimple Lie group $G^{\mathbb{C}}$ and we try to identify their fixed points.

This thesis is organised as follows.

Chapters 1 and 2 are devoted to a detailed review of the theory of Lie groups, Lie algebras and Higgs bundles, since the techniques and results that appear in this thesis are due above all to the interplay of these three fields. In Chapter 1 we stablish the neccesary background on Lie theory following [37, 44, 50, 33, 31]. We include here some results that we have not found in the literature and are relevant for our Higgs bundle analysis. In Section 1.3 we found our first constribution to this area. Cartan proved that given a semisimple Lie algebra $\mathfrak{g}$ with Cartan involution θ, for any order 2 automorphism η of $\mathfrak{g}$ there exists an inner automorphism φ of $\mathfrak{g}$ such that $\eta' = \varphi\eta\varphi^{-1}$ commutes with θ. In Proposition 1.3.1 we generalised this result to order $n = 2m$ automorphisms of $\mathfrak{g}$ and we extend this result to real connected semisimple Lie groups (see Proposition 1.3.2). These propositions will play an important role in the description of fixed points of the finite order automorphisms studied later. The odd case is not clear, thus throughout this thesis we will consider order n automorphism that commutes with θ, which for n even as we just mentioned is not a restriction. Let $\eta \in \mathrm{Aut}_n(G, \theta)$ and denote also by η its complexification $\eta_{\mathbb{C}} \in \mathrm{Aut}_n(H^{\mathbb{C}})$. In following chapters we introduce Higgs bundles defined by the subgroup

$$H^{\mathbb{C}}_{\eta} = \{h \in H^{\mathbb{C}} \,:\, \eta(h) = c(h)h, \text{ with } c(h) \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)\}$$

hence we devoted Section 1.5 to develop the tools neccesary to work with these Higgs bundles. Some of them were recently studied in [31] and we just generalised them to our situation. A well known fact about G-Higgs bundles when G is real is that a G-Higgs bundle is **simple** if $\mathrm{Aut}(E, \varphi) = Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$. In this sense in Section 1.5 we prove that for any connected real form of a complex semisimple Lie group $G^{\mathbb{C}}$ the subgroup $Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ is equal to $Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})$ and that it is always finite.

In Chapter 2 we take a deep walk through the world of Higgs bundles, moduli spaces and representations. No original ideas or results appear in this Chapter. The content discussed here has been extensively developed over the last 30 years. As we noticed above Higgs bundles for real groups were first introduced by Hitchin [38, 40] and their moduli spaces were systematically studied by Bradlow, García-Prada, Gothen and Mundet i Riera, among others [17, 18, 20, 26, 27]. The aim of this Chapter is to help us to fix some notations and to recall the basic concepts related with the topic of this thesis. In following chapters we introduce principal Higgs bundles defined by automorphims $\eta \in \mathrm{Aut}_n(G, \theta)$. In order to do that we need the notion of **K-twisted Higgs pairs**. We refer the reader to [27] for a complete understanding of this concept. A K-twisted Higgs pair is a pair (E, φ), where E is a holomorphic principal $H^{\mathbb{C}}$-bundle over a compact Riemann surface X and φ is a holomorphic section of $E(\mathbb{V}) \otimes K$, where $E(\mathbb{V})$ is the vector bundle associated to E via the representation $\rho : H^{\mathbb{C}} \to \mathrm{GL}(\mathbb{V})$ and K is the canonical bundle of X. Throughout this thesis we will refer to these objects simply as Higgs pairs. In particular, (G, θ)-Higgs bundles are K-twisted Higgs pair of type ρ when ρ is the isotropy representation. We end this chapter with Section 2.5, where we give a brief introduction to parabolic (G, θ)-Higgs bundles following [15].

The aim of Chapter 3 is to study the action of finite order automorphisms of real semisimple Lie groups G compatible with a fixed Cartan structure θ on the moduli space of (G, θ)-Higgs bundles and to give a complete description of its fixed points subvarieties. Through nonabelian Hodge correspondence we extend this study to representations. The case of semisimple complex Lie groups was accomplished by Garcia-Prada and Ramanan in [31]. At first, we study finite order automorphisms of principal $H^{\mathbb{C}}$-bundles. They give rise to reductions of the structure group of the bundle to the subgroups $(H^{\mathbb{C}})^{\eta}$ and $H^{\mathbb{C}}_{\eta}$ presented in Chapter 1. We also consider finite order automorphisms of $H^{\mathbb{C}}$ twisted by an automorphism of $Z(G^{\mathbb{C}}) \cap Z(H^{\mathbb{C}})$. We then introduce $(G, \theta, H^{\eta}, \zeta_k)$-Higgs bundles and $(G, \theta, H_{\eta}, \zeta_k)$-Higgs bundles mentioned above. The moduli spaces $\mathcal{M}(G, \theta, H^{\eta}, \zeta_k)$ are the ones that will appear in our description of the fixed points of the automorphism

$$\iota(\eta, \zeta_k)(E, \varphi) := (\eta_{\mathbb{C}}(E), \zeta_k \eta_{\mathbb{C}}(\varphi))$$

defined on $\mathcal{M}(G, \theta)$, induced by $\eta \in \mathrm{Aut}_n(G, \theta)$.

We then generalise our study to automorphisms of $\mathcal{M}(G, \theta)$ defined by finite order elements of $H^1(X, Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})) \rtimes \mathrm{Out}(G, \theta) \times \mathbb{C}^*$. To do this we consider an element $(\alpha, a) \in H^1(X, Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})) \rtimes \mathrm{Out}(G, \theta)$ and we define the automorphism

$$\iota(a, \alpha, \zeta_k) : (E, \varphi) \mapsto (\eta_{\mathbb{C}}(E) \otimes \alpha, \zeta_k \eta_{\mathbb{C}}(\varphi))$$

for any $\eta \in \pi^{-1}(a)$, where $\zeta_k = \exp(2\pi i \frac{k}{n})$. The fixed-point subvarieties of this automorphism are described by the moduli spaces $\mathcal{M}(G, \theta, H_{\eta}, \zeta_k)$ defined above. Building upon the theory developed in Chapter 1 and 3 our main results are Theorem 3.5.3 and Theorem 3.6.3. These theorems generalise to real groups some of the main results obtained in [31].

The aforementioned automorphisms induce in a natural way automorphisms on the moduli space of representations through nonabelian Hodge correspondence. In fact, Theorem 3.3.7 gives correspondences between $\mathcal{M}(G, \theta, H^{\eta}, \zeta_0)$ and $\mathcal{M}(G, \theta, H_{\eta}, \zeta_0)$ with $\mathcal{R}(G^{\eta}, \theta)$ and $\mathcal{R}(G_{\eta}, \theta)$ respectively. In addition, if $n = 2l$ then there also exist correspondences between $\mathcal{M}(G, \theta, H^{\eta}, \zeta_l)$ and $\mathcal{M}(G, \theta, H_{\eta}, \zeta_l)$ and $\mathcal{R}(G^{\sigma}, \theta)$ and $\mathcal{R}(G_{\sigma}, \theta)$ respectively, where $\sigma := \theta\eta$. The above theorems can be stated in terms of representation of the fundamental group of X in G. Our main results are summarised in Theorems 3.7.5 and 3.7.8 and generalise Theorems 8.6 and 8.8 in [31].

Chapter 4 is dedicated to ilustrate the main results obtained throughout Chapter 3 through several examples. More precisely, we consider the case of (G, θ)-Higgs bundle for G a connected real form of $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Sp}(n, \mathbb{C})$ and we apply Theorem 3.5.3 to them for the trivial clique $a = 1 \in \mathrm{Out}_2(G, \theta)$.

The objective to which we devote Chapter 5 is to take the first steps in the study of involutions

$$\iota(\sigma, \pm) : (E, \alpha, \{Q_i\}, \varphi, \mathcal{L}) \mapsto (\sigma(E), \sigma(\alpha), \{\sigma(Q_i)\}, \pm\sigma(\varphi), \sigma(\mathcal{L}))$$

defined on the moduli space of parabolic (G, θ)-Higgs bundles. Regarding this, let G be the connected component of the identity of a real form of a complex semisimple Lie group $G^{\mathbb{C}}$. Let $\theta := \tau\mu$ be a fixed Cartan involution where τ is a fixed compact conjugation of $G^{\mathbb{C}}$ and μ is the anti-holomorphic involution defining the real form G. Let X be a compact Riemann surface and $S = \{x_1, \dots, x_r\}$ be a finite set of different points. Let $H = G^\theta$ and $T \subset H$ be a maximal torus. A **parabolic principal bundle** of weight $\alpha = (\alpha_, \dots, \alpha_r) \in \sqrt{-1}\,\overline{\mathcal{A}}$, with $\mathcal{A}$ being an alcove in the Lie algebra of T, is a holomorphic principal bundle E with a choice for any $i \in \{1, \dots, r\}$ of a parabolic structure of weight α_i on x_i. A **parabolic Higgs bundle** (E, φ) **over** (X, S) is a parabolic principal bundle E together with a Higgs field $\varphi \in H^1(X \backslash D, PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D))$ where $PE(\mathfrak{m}^{\mathbb{C}})$ is the sheaf of parabolic sections of $E(\mathfrak{m}^{\mathbb{C}})$. As in Chapter 3 we first study the behaviour of order 2 automorphisms of parabolic principal $H^{\mathbb{C}}$-bundles. They give rise to reductions of the structure group of the bundle to the subgroups $(H^{\mathbb{C}})^\sigma$ of $H^{\mathbb{C}}$ with an appropriate reduction of the parabolic structure of the bundle. This can be done since we consider weights fixed by σ in a σ-invariant alcove. We then introduce $(G, \theta, H^\sigma, \alpha, \pm)$-Higgs bundles and their corresponding moduli spaces. The moduli spaces $\mathcal{M}(G, \theta, H^\sigma, \alpha, \mathcal{L}, \pm)$ are the ones that will appear in our description of the fixed points of the automorphism $\iota(\sigma, \pm)$ induced by $\sigma \in \mathrm{Aut}_2(G, \theta)$.

Introducción.

Los fibrados de Higgs fueron introducidos por Nigel Hitchin hace unos de 30 años en [38] y se ha convertido en un activo campo de investigación desde entonces debido a la gran importancia y, a su vez a la inesperada relevancia, que ganaron en ramas tan diversas de la geometría diferencial y algebraica o de la física matemática como pueden ser la teoría de representaciones de grupos de superficie, las teorías gauges, la geometría Hyperkäler o los sistemas integrables, entre otros.

En este artículo [38] (y en otros [39, 40]) Hitchin desveló muchos resultados asombrosos sobre los fibrados de Higgs y sus espacios de moduli. Atacó el problema desde dos puntos de vista distintos. Por un lado, consideró el espacio de moduli de soluciones a un conjunto de ecuaciones procedentes de las teorías gauges, conocidas después de él como ecuaciones de Hitchin, donde las soluciones son pares de objetos: una conexión A en un G-fibrado principal sobre una superficie de Riemann X y una $(1,0)$-forma φ definida en X con valores en el fibrado (complejo) asociado a E via la representacón adjunta de G. Por otro lado, cuando $G = \mathrm{SU}(2)$ o $\mathrm{SO}(3)$ una solución a estas ecuaciones define un par (V, φ) donde V es un fibrado vectorial holomorfo de rango 2 sobre X y φ es una sección holomorfa de $\mathrm{End}(V) \otimes K$, con K el fibrado canónico sobre X. Hitchin definió nociones de estabilidad para estos pares, las cuáles son consistentes con las introducidas por Mumford para fibrados vectoriales. Uno de los principales resultados obtenidos en [38] es la correspondencia entre fibrados de Higgs de rango 2 con determinante trivial, bajo las nociones de estabilidad anteriormente mencionadas, y las clases de isomorfismos de sistemas locales $\mathrm{SL}(2,\mathbb{C})$ en una superficie de Riemann compacta y conexa. Esta correspondencia se puede entender como una generalización de un resultado clásico, el teorema de Narasimhan y Seshadri [48], que establece una correspondencia entre fibrados vectoriales poliestables y representaciones irreducibles del grupo fundamental de una superficie de Riemann compacta X en el grupo unitario $\mathrm{U}(n)$. Este resultado fue generalizado por Simpson en [57] para representaciones del grupo fundamental de X en un grupo de Lie reductivo complejo cualquiera. Como explica el propio Simpson esta correspondencia se puede ver también como un teorema de Hodge para cohomología no abeliana, dando lugar a lo que ahora llamamos *teoría de Hodge no abeliana*.

Al reemplazar $\mathrm{SL}(2,\mathbb{C})$ por un grupo de Lie reductivo (complejo o real) cualquiera nos topamos con la noción de G-fibrado de Higgs. Sea (G,H,θ,B) un grupo de Lie reductivo (real o complejo) donde H es un subgrupo compacto maximal de G y θ es una involución definida sobre $\mathfrak{g}$, el álgebra de Lie de G, que nos induce una descomposición $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ en ± 1-autoespacios. Un G-fibrado de Higgs sobre una superficie de Riemann compacta X es un par (E,φ) donde E es un $H^{\mathbb{C}}$-fibrado principal y φ es una sección holomorfa de $E(\mathfrak{m}^{\mathbb{C}})\otimes K$, donde por $H^{\mathbb{C}}$ entendemos la complejificación de H y $E(\mathfrak{m}^{\mathbb{C}})$ es el fibrado asociado a E via la representación (compleja) de isotropía $\iota^{-}: H^{\mathbb{C}}\to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$. Obsérvese que cuando G es complejo la complejificación de $H^{\mathbb{C}}$ del subgrupo compacto maximal de G es el propio grupo G y entonces la descomposición dada por θ es $\mathfrak{g}=\mathfrak{h}\oplus i\mathfrak{h}$ y, por tanto, E es un G-fibrado principal holomorfo y φ es una sección holomorfa del fibrado adjunto $E(\mathfrak{g})$ tuisteado por K.

Los G-fibrados de Higgs para ciertos grupos de Lie reales son uno de los temas principales tratados en [40] y, de hecho, los $\mathrm{SL}(2,\mathbb{R})$-fibrados de Higgs ya fueron considerados en [38]. La generalización de la teoría de G-fibrados de Higgs para grupos reales reductivos ha sido profundamente desarrollada por Bradlow, García-Prada, Gothen, Mundet-i-Riera, Oliveira and others (véase por ejemplo [17, 18, 19, 26, 27]). En [27] Bradlow, García-Prada, Gothen y Mundet-i-Riera dan una correspondencia de tipo Hitchin-Kobayashi para pares de Higgs L-tuisteados y, en particular, para G-fibrados de Higgs. *Grosso modo*, la correspondencia de Hitchin-Kobayashi dice que un fibrado de Higgs (E,φ) es poliestable si y solo si existe una solución a las ecuaciones de Hitchin y esto induce un homeomorfismo entre el espacio de moduli $\mathcal{M}(G)$ de clases de isomorfismo de G-fibrados de Higgs poliestables el espacio de moduli gauge. Esta correspondencia junto con la generalizacón de los resultados de Donaldson [24, 25] debida a Corlette [23] extiende la correspondencia entre fibrados de Higgs poliestables y representaciones irreducibles a grupos de Lie reales reductivos.

Una herramienta importante en el estudio de los fibrados de Higgs es el análisis de involuciones definidas en $\mathcal{M}(G)$, el espacio de moduli de G-fibrados de Higgs poliestables. El objetivo principal de esta memoria es el estudio de involuciones y automorfismos de orden superior definidos sobre diferentes espacios de moduli para grupos reales. Como queremos fijar el grupo de estructura del G-fibrado de Higgs cambiamos sutílmente la notación clásica de estos objetos denotándolos como (G,θ)-fibrados de Higgs. La primera involución no trivial que podemos considerar definida en $\mathcal{M}(G,\theta)$ viene dada por la multiplicación del campo de Higgs por -1:

$$(E,\varphi)\mapsto(E,-\varphi).$$

De hecho, podemos generalizar esta involución y definir una acción de $\mathbb{C}^*$ en $\mathcal{M}(G,\theta)$ dada por $(E,\varphi)\mapsto(E,\lambda\varphi)$. Se ha demostrado que la acción de $\mathbb{C}^*$ en $\mathcal{M}(G,\theta)$ es un herramienta importante en el estudio de aspectos topológicos de $\mathcal{M}(G,\theta)$ y, a través de la correspondencia mencionada anteriormente, de $\mathcal{R}(G,\theta)$. El estudio de esta acción en el espacio de moduli de G-fibrados de Higgs para G un grupo reductivo complejo ha sido llevado a cabo por García-Prada y Ramanan en [31] donde dan una descripción completa de sus puntos fijos. De hecho, ellos extienden este estudio al caso de automorfismos de $\mathcal{M}(G)$ definidos por elementos de $(H^1(X,Z(G))\rtimes \mathrm{Out}(G))_n\times\mathbb{C}^*$ y, a tra vés de la correspondencia de Hodge no abeliana, consideran la acción de este conjunto en $\mathcal{R}(G)$, el espacio de moduli de representacines del grupo fundamental de X en G. Esta memoria esta dedicada, entre otros problemas, a generalizar estos resultados para el caso de formas reales de grupos de Lie semisimples complejos. Con respecto a esto, sea G una forma real de un grupo de Lie semisimple conexo complejo $G^{\mathbb{C}}$ con involución de Cartan θ. Definimos $\mathrm{Out}(G):=\mathrm{Aut}(G)/\mathrm{Int}(G)$, el grupo de automorfismos externos de G, donde $\mathrm{Aut}(G)$ es el grupo de automorfismos de G y $\mathrm{Int}(G)$ es un subgrupo normal de $\mathrm{Aut}(G)$ cuyos elementos actúan en G por conjugación. Denotemos por $\mathrm{Out}_n(G,\theta)$ al conjunto de elementos de orden n del grupo de automorfismos externos de G que conmutan con la involución de Cartan y denotemos por $H^1(X,Z_\tau)$ al grupo de cohomología que parametriza las clases de isomorfismos de Z_τ-fibrados principales sobre X, donde $Z_\tau=Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-_{\mathbb{C}})$. En el análisis de puntos fijos aparecen ciertos fibrados de Higgs sobre X definidos por un automorfismo $\eta\in\mathrm{Aut}_n(G,\theta)$. Estos son $(G,\theta,H^\eta,\zeta_k)$-**Higgs bundles** y $(G,\theta,H_\eta,\zeta_k)$-**Higgs bundles**. Son pares (E,φ) donde E es un $(H^{\mathbb{C}})^\eta$-fibrado principal holomorfo (resp. $H^{\mathbb{C}}_\eta$-fibrado principal holomorfo) y

$$\varphi\in H^0(X,E(\mathfrak{m}^{\mathbb{C}}_k)\otimes K)$$

donde $E(\mathfrak{m}^{\mathbb{C}}_k)$ es el fibrado asociado a E via la representación

$$\iota^k:(H^{\mathbb{C}})^\eta\to\mathrm{GL}(\mathfrak{m}^{\mathbb{C}}_k)$$

(resp. $\iota^k:H^{\mathbb{C}}_\eta\to\mathrm{GL}(\mathfrak{m}^{\mathbb{C}}_k)$) y K es el fibrado canónico sobre X. Estos objetos son, en particular, pares de Higgs K-tuisteados de tipo ι^k (resp. ι_k) para subgrupos de Lie reductivos de $H^{\mathbb{C}}$. Tenemos, sin embargo, dos situaciones excepcionales. Cuando $k=0$, de la Proposición 1.3.3 se deduce que $H^\eta\exp(\mathfrak{m}_0)=G^\eta$. De la misma forma, de la Proposición 1.5.1 se tiene que $H_\eta \exp(\mathfrak{m}_0) = G_\eta$. Entonces un $(G, \theta, H^\eta, \zeta_0)$-fibrado de Higgs es simplemente un (G^η, θ)-fibrado de Higgs y un $(G, \theta, H_\eta, \zeta_0)$-fibrado de Higgs es un (G_η, θ)-fibrado de Higgs en el sentido de la definición 2.1.1. Cuando n es par y $l = n/2$ de las Proposiciones 1.3.3y 1.5.1 se deduce que un $(G, \theta, H^\eta, \zeta_l)$-fibrado de Higgs es simplemente un (G^σ, θ)-fibrado de Higgs y que un $(G, \theta, H_\eta, \zeta_l)$-fibrado de Higgs es simplemente un (G_σ, θ)-fibrado de Higgs. Estos casos excepcionales no requieren de ninguna definicin de estabilidad distinta de la clásica. En el resto de los casos las condiciones de estabilidad dadas en la Definición 2.2.6 pueden extenderse a esta situación más general simplemente reemplazando $\mathfrak{h}$ por $\mathfrak{h}_0$, y $\mathfrak{m}_s$ y $\mathfrak{m}_s^0$ por sus correspondientes espacios asociados a $\mathfrak{m}_k$ como vimos en la Observación 2.2.8. Entonces podemos definir $\mathcal{M}(G, \theta, H^\eta, \zeta_k)$, el espacio de moduli de clases de isomorfismos de $(G, \theta, H^\eta, \zeta_k)$-fibrados de Higgs poliestables. De la misma forma, podemos definir $\mathcal{M}(G, \theta, H_\eta, \zeta_k)$, el espacio de moduli de clases de isomorfismos de $(G, \theta, H_\eta, \zeta_k)$-fibrados de Higgs poliestables. Tenemos la siguiente generalización del Teorema 6.10 en [31]:

Theorem 0.0.3. *(Theorem 3.6.3) Sea $a \in \operatorname{Out}_n(G, \theta)$ y sea $\alpha \in H^1(X, Z_\tau)$ tal que*

$$\alpha a(\alpha) \cdots a^{n-1}(\alpha) = 1.$$

Sea (E, φ) un (G, θ)-fibrado de Higgs sobre X y consideremos el automorfismo

$$\begin{array}{rccc} \iota(a, \alpha, \zeta_k): & \mathcal{M}(G, \theta) & \to & \mathcal{M}(G, \theta) \\ & (E, \varphi) & \mapsto & (a(E) \otimes \alpha, \zeta_k a(\varphi)) \end{array} \tag{0.2}$$

con $\zeta_k = \exp(2\pi i \frac{k}{n})$. Entonces

(1)

$$\bigsqcup_{\eta' \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_a(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_{\eta'}, \zeta_k) \subset \mathcal{M}(G, \theta)^{\iota(a, \alpha, \zeta_k)},$$

(2)

$$\mathcal{M}(G, \theta)^{\iota(a, \alpha, \zeta_k)}_{simple} \subset \bigsqcup_{\eta' \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_a(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_{\eta'}, \zeta_k)$$

escepto para $\iota(1, 1, \zeta_0)$, porque $\iota(1, 1, \zeta_0) = id$.

Además, usando la correspondencia de Hodge no abeliana para grupos de Lie reductivos reales, también tenemos la siguiente generalización del Teorema 8.8 en [31].

Theorem 0.0.4. *(Theorem 3.7.8) Sea* $a \in \mathrm{Out}_n(G,\theta)$ *and* $\lambda \in \mathcal{R}(Z_\tau) = \mathrm{Hom}(\pi_1(X), Z_\tau)$ *tal que*

$$\lambda a(\lambda)\cdots a^{n-1}(\lambda) = 1.$$

Consideremos el automorfismo

$$\begin{array}{rccc} \iota(a,\lambda,\pm): & \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ & \rho & \mapsto & \lambda a^{\pm}(\rho), \end{array}$$

con $a^{\pm}(\rho)$ *definido como en el Teorema 3.7.5. Entonces*

(1)

$$\bigsqcup_{\eta\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,\lambda,+)},$$

(2)

$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,+)}_{irred} \subset \bigsqcup_{\eta\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta,\theta).$$

Obsérvese que $\iota(1,1,+)$ *es el automorfismo identidad. Si además,* $n = 2l$ *entonces*

(3)

$$\bigsqcup_{\sigma\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,\lambda,-)},$$

(4)

$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,-)}_{irred} \subset \bigsqcup_{\sigma\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta).$$

Los detalles que rodean a los Teoremas 3.6.3 y 3.7.8 se desarrollan en el Capítulo 3.

La correspondencia original entre fibrados de Higgs y sistemas locales introducida por Hitchin y sus aplicaciones al estudio de representaciones del grupo fundamental ha crecido en muchas otras direcciones. Una de ellas es el estudio de representaciones del grupo fundamental de una superficie de Riemann punteada en un grupo de Lie reductivo cualquiera con holonomía prefijada alrededor de los puntos. Esto motivó la aparición de los fibrados de Higgs parabólicos. Los fibrados vectoriales parabólicos fueron introducidos por Seshadri en [54] y la correspondencia para $G = \mathrm{U}(n)$ fue demostrada por Mehta y Seshadri en [46]. La generalización de esta correspondencia a un grupo de Lie compacto cualquiera fue llevada a cabo por Bhosle y Ramanathan [9], Teleman y Woodward [58] y Balaji y Seshadri en [7], imponiendo unas condiciones determinadas sobre la holonomía alrededor de los puntos. El primero en considerar el caso de grupos de Lie reductivos no compactos fue Simpson en [57], donde estudia el caso para $G = \mathrm{GL}(n, \mathbb{C})$ e introduce los sistemas locales filtrados. Una publicación reciente [**?**] debida a Biquard, García-Prada y Mundet i Riera extiende esta correspondencia a un grupo de Lie reductivo real cualquiera. En el último capítulo de esta memoria empezamos el estudio de involuciones holomorfas definidas en el espacio de moduli de (G, θ)-fibrados de Higgs parabólicos para G una forma real de un grupo simisimple complejo $G^{\mathbb{C}}$ y tratamos de identificar sus puntos fijos.

Esta memoria está organizada de la siguiente manera.

Dedicamos los Capítulos 1 y 2 a dar un profundo repaso a la teoría de los grupos de Lie, las álgebras de Lie y los fibrados de Higgs, ya que las técnicas y los resultados que aparecen a lo largo de esta tesis se deben sobre todo a la interacción de estos tres campos. En el Capítulo 1 se introducen los conocimientos necesarios sobre la teoría de Lie tomando como referencias básicas [37, 44, 50, 33, 31]. Incluímos en este capítulo algunos resultados originales que no hemos encontrado en la literatura y que son relevantes para el estudio de los fibrados de Higgs que realizamos posteriormente. En la Sección 1.3 se encuentra la primera contribución original de la memoria. Cartan probó que dada un álgebra de Lie semisimple $\mathfrak{g}$ junto con una involución de Cartan θ, para cualquier automorfismo de orden 2 η de $\mathfrak{g}$ existe un automorfismo interno φ de $\mathfrak{g}$ tal que $\eta' = \varphi\eta\varphi^{-1}$ conmuta con θ. En la Proposición 1.3.1 generalizamos este hecho para cualquier automorfismo de $\mathfrak{g}$ de orden par y extendemos este resultado al caso de grupos de Lie reales semisimples conexos (véase la Proposición 1.3.2). Estos resultados jugaran un papel importante en el estudio de los puntos fijos de los automorfismos de orden finito planteados en los siguientes capítulos. El caso de orden impar no está claro, por ello a lo largo de esta memoria trabajaremos con automorfismos de orden n que conmuten con θ, que como ya hemos dicho cuando n es par no es una restricción. Sea $\eta \in \mathrm{Aut}_n(G, \theta)$ y denotemos también por η a su complejificación $\eta_{\mathbb{C}} \in \mathrm{Aut}(H^{\mathbb{C}})$. En capítulos posteriores introduciremos fibrados de Higgs definidos por el subgrupo

$$H^{\mathbb{C}}_{\eta} = \{h \in H^{\mathbb{C}} \, : \, \eta(h) = c(h)h, \text{ with } c(h) \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^{-})\},$$

por ello dedicamos la Sección 1.5 a desarrollar las herramientas apropiadas para trabajar con estos objetos. Algunas de esas herramientas han sido recientemente estudiadas en [31] y nosotros simplemente las adaptamos a nuestra situación. Un característica bien conocida sobre los G-fibrados de Higgs de grupo real es que un G-fibrado de Higgs se dice **simple** si $\mathrm{Aut}(E, \varphi) = Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$. Con respecto a esto, en el Capítulo 1.5 se prueba que para cualquier forma real conexa de un grupo de Lie semisimple complejo $G^{\mathbb{C}}$ se tiene que el subgrupo $Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ es igual a $Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})$ y que además este subgrupo es siempre finito.

En el Capítulo 2 damos un largo paseo a través del mundo de los fibrados de Higgs, los espacios de moduli y las representaciones. En este capítulo no se han desarrollado resultados ni ideas propias. El contenido tratado aquí ha sido extensamente desarrollado a lo largo de los últimos 30 aos. Como ya mencionamos anteriormente, los fibrados de Higgs para grupo real fueron introducidos por Hitchin en [38, 39] y sus correspondientes espacios de moduli han sido sistemáticamente estudiados por Bradlow, García-Prada, Gothen y Mundet i Riera, entre otros, [17, 18, 20, 26, 27]. El propósito del Capítulo 2 no es otro que ayudarnos a fijar una notación adecuada y repasar los conceptos básicos relacionados con la temática de esta memoria. En próximos capítulos introduciremos fibrados de Higgs definidos a partir de un automorfismo $\eta \in \mathrm{Aut}_n(G, \theta)$. Para ello necesitaremos la noción de **par de Higgs K-tuisteado** para lo cual remitimos al lector al artículo [27] donde podrá encontrar la información necesaria para lograr un completo entendimiento de estos objetos. Un par de Higgs K-tuisteado es un par (E, φ) donde E es $H^{\mathbb{C}}$-fibrado principal holomorfo sobre una superficie de Riemann X y φ es una sección holomorfa de $E(\mathbb{V}) \otimes K$ donde $E(\mathbb{V})$ es el fibrado vectorial asociado a E via la representación $\rho : H^{\mathbb{C}} \to \mathrm{GL}(\mathbb{V})$ y K el fibrado canónico sobre X. A lo largo de esta tesis nos referiremos a estos objetos simplemente como pares de Higgs. En particular, los (G, θ)-fibrados de Higgs son pares de Higgs K-tusiteados de tipo ρ donde ρ es la representación de isotropía. Terminamos este capítulo con la Sección 2.5 donde introducimos de forma breve los (G, θ)-fibrados de Higgs parabólicos de acuerdo con lo expuesto en [15].

El propósito del Capítulo 3 es estudiar la acción de automorfismos de orden finito de grupos de Lie reales semisimples G compatibles con una estructura de Cartan fija θ en el espacio de moduli de (G, θ)-fibrados de Higgs y dar una descripción completa de sus subvariedades de puntos fijos. A través de la correspondencia de Hodge no abeliana extendemos estos resultados al estudio de representaciones. El caso de grupos de Lie semisimples complejos fue acometido por García-Prada y Ramanan en [31]. Primero analizamos automorfismos de orden finito de $H^{\mathbb{C}}$-fibrados principales. Estos dan lugar a reducciones del grupo de estructura del fibrado a subgrupos $(H^{\mathbb{C}})^{\eta}$ y $H^{\mathbb{C}}_{\eta}$ presentados en el Capítulo 1. También se consideran automorfismos de orden finito de $H^{\mathbb{C}}$ tuisteado por un automorfismos de $Z(G^{\mathbb{C}}) \cap Z(H^{\mathbb{C}})$. Introduciremos luego los $(G, \theta, H^{\eta}, \zeta_k)$-fibrados de Higgs y los $(G, \theta, H_\eta, \zeta_k)$-fibrados de Higgs mencionados anteriormente. Los espacios de moduli $\mathcal{M}(G, \theta, H^\eta, \zeta_k)$ son los que aparecerán en la descripción de los puntos fijos del automorfismo

$$iota(\eta, \zeta_k)(E, \varphi) := (\eta_{\mathbb{C}}(E), \zeta_k \eta_{\mathbb{C}}(\varphi))$$

definido en $\mathcal{M}(G, \theta)$, inducido por $\eta \in \mathrm{Aut}_n(G, \theta)$.

Luego generalizamos este estudio a automorfismos de $\mathcal{M}(G, \theta)$ definidos por elementos de orden finito de $H^1(X, Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})) \rtimes \mathrm{Out}(G, \theta) \times \mathbb{C}^*$. Para ello dado $(\alpha, a) \in H^1(X, Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})) \rtimes \mathrm{Out}(G, \theta) \times \mathbb{C}^*$ se define el automorfismo

$$\iota(a, \alpha, \zeta_k) : (E, \varphi) \mapsto (\eta_{\mathbb{C}}(E) \otimes \alpha, \zeta_k \eta_{\mathbb{C}}(\varphi))$$

para algún $\eta \in \pi^{-1}(a)$, donde $\zeta_k = \exp(2\pi i \frac{k}{n})$. Las subvariedades de puntos fijos de este automorfismo se describen a través de los espacios de moduli $\mathcal{M}(G, \theta, H_\eta, \zeta_k)$ definidos anteriormente. Basándonos en los resultados obtenidos y desarrollados en el Capítulo 1 y 3 nuestros resultados principales son los dados en los Teoremas 3.5.3 y 3.6.3. Estos teoremas generalizan algunos de los resultados obtenidos en [31] para el caso de grupos de Lie reales.

los automorfismos anteriormente mencionados inducen de forma natural automorfismos en el espacio de moduli de representaciones a través de la teoría de Hodge no abeliana. De hecho, el Teorema 3.3.7 da una correspondencia entre $\mathcal{M}(G, \theta, H^\eta, \zeta_0)$ y $\mathcal{M}(G, \theta, H_\eta, \zeta_0)$ y $\mathcal{R}(G^\eta, \theta)$ y $\mathcal{R}(G_\eta, \theta)$ respectivamente. Si además, $n = 2l$ entonces entonces también existen correspondencias entre $\mathcal{M}(G, \theta, H^\eta, \zeta_l)$ y $\mathcal{M}(G, \theta, H_\eta, \zeta_l)$ y $\mathcal{R}(G^\sigma, \theta)$ y $\mathcal{R}(G_\sigma, \theta)$ respectivamente, donde $\sigma := \theta\eta$. Los teoremas anteriormente mencionados pueden traducirse al lenguaje de representaciones del grupo fundamental de X en G. Nuestros resultados principales se resumen en los Teoremas 3.7.5 y 3.7.8 y generalizan los Teoremas 8.6 y 8.8 en [31].

El Capítulo 4 está dedicado a ilustrar, a través de diversos ejemplos, los resultados principales obtenidos a lo alrgo del Capítulo 3. Más concretamente, se considera el caso de (G, θ)-fibrados de Higgs para G una forma real conexa de $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(n, \mathbb{C})$ y $\mathrm{Sp}(n, \mathbb{C})$ y aplicamos el Teorema 3.7.5 para estos objetos para el clique trivial $a = 1 \in \mathrm{Out}_2(G, \theta)$.

El objetivo al que se consagra el Capítulo 5 es a dar los primeros pasos en el estudio de involuciones

$$\iota(\sigma, \pm) : (E, \alpha, \{Q_i\}, \varphi, \mathcal{L}) \mapsto (\sigma(E), \sigma(\alpha), \{\sigma(Q_i)\}, \pm\sigma(\varphi), \sigma(\mathcal{L}))$$

definidas en el espacio de moduli de (G, θ)-fibrados de higgs parabólicos. Con respecto a esto, sea G la componente conexa en la identidad de una forma real de un grupo de Lie semisimple complejo G^{CC}. Sea $\theta := \mu\tau$ una involución de Cartan prefijada donde τ es una conjugación compacta prefijada de $G^{\mathbb{C}}$ y μ es la involución anti-holomorfa que define la forma real G. Sea X una superficie de Riemann y $s = \{x_1, \ldots, x - r\}$ un conjunto finito de puntos distintos en X. Sea $H = G^\theta$ y $T \subset H$ un toro maximal. Un **fibrado parabólico principal** de peso $\alpha = (\alpha_1, \ldots, \alpha_r) \in \sqrt{-1}\,\overline{\mathcal{A}}$ con $\overline{\mathcal{A}}$ siendo una alcoba en el álgebra de Lie de T, es un fibrado principal holomorfo E con una elección para cada $i \in \{frm[o]--, \ldots, r\}$ de una estructura parabólica de peso α_i en x_i. Un **fibrado de Higgs parabólico** (E, φ) **sobre** (X, S) es un fibrado principal parabólico E junto con un campo de Higgs $\varphi \in H^0(X \backslash S, PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(S))$, donde $PE(\mathfrak{m}^{\mathbb{C}})$ es el haz de secciones parabólicas de $E(\mathfrak{m}^{\mathbb{C}})$. Como en el Capítulo 3 primero estudiamos el comportamiento de automofismos de orden 2 definidos sobre $H^{\mathbb{C}}$-fibrados parabólicos principales. Esto dará lugar a reducciones del grupo de estructura del fibrado a los subgrupos $(H^{\mathbb{C}})^\sigma$ de $H^{\mathbb{C}}$ que llevara asociado una reducción de la estructura parabólica del fibrado. Esto es posible porque estamos considerando pesos fijos bajo la involución σ en una alcoba σ-invariante. Luego introducimos la noción de $(G, \theta, H^\sigma, \alpha, \pm)$-fibrado de Higgs e introducimos sus correspondientes espacios de moduli. los espacios de moduli $\mathcal{M}(G, \theta, H^\sigma, \alpha, \mathcal{L}, \pm)$ serán los que aparecan en nuestra descripción de puntos fijos del automorfismo $\iota(\sigma, \pm)$ inducido por $\sigma \in \mathrm{Aut}_2(G, \theta)$.

Contents

Chapter 1

Background on Lie theory.

We start by recalling some facts about Lie theory that will be neccesary later. We follow [37, 44, 50, 33].

1.1 Complex Lie algebras and real forms

Definition 1.1.1. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{g}_{\mathbb{R}}$ be its underlying real Lie algebra. A **real form of** $\mathfrak{g}$ is real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}_{\mathbb{R}}$ such that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. In this situation we can naturally identify $\mathfrak{g}$ with $\mathfrak{g}_0 \otimes \mathbb{C}$. In fact there exists a **conjugation**, i.e. an antilinear homomorphism, σ of $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ onto itself defined by

$$\sigma(X + iY) = X - iY, \text{ for all } X, Y \in \mathfrak{g}_0.$$

Remark 1.1.2. Coversely any conjugation $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = \mathrm{Id}$ defines the real form

$$\mathfrak{g}^\sigma = \{X \in \mathfrak{g} \text{ such that } \sigma(X) = X\}.$$

Thus there exists a bijection between real forms and conjugations of $\mathfrak{g}$.

Example 1.1.3. Let $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) \otimes \mathbb{C} = \mathfrak{sl}(n, \mathbb{C})$ be its complexification. Since

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sl}(n, \mathbb{R}) \oplus i\, \mathfrak{sl}(n, \mathbb{R}),$$

then $\mathfrak{sl}(n, \mathbb{R})$ is a real form of $\mathfrak{sl}(n, \mathbb{C})$. Moreover, if we consider the involution $\sigma : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C})$ given by $A \mapsto \overline{A}$, then $\mathfrak{sl}(n, \mathbb{C})^\sigma = \mathfrak{sl}(n, \mathbb{R})$. In general, any real Lie algebra is a real form of its complexification.

Definition 1.1.4. Let $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ be two real forms of $\mathfrak{g}$. If σ and σ' are the corresponding conjugations for $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ respectively, then we say that the real forms are **isomorphic** if and only if there exists an automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ such that the following diagram

$$\begin{array}{ccccc} & & \phi & & \\ & \mathfrak{g} & \longrightarrow & \mathfrak{g} & \\ \sigma & \downarrow & & \downarrow & \sigma' \\ & \mathfrak{g} & \longrightarrow & \mathfrak{g} & \\ & & \phi & & \end{array}$$

commutes. Furthermore, we will say that they are **inner isomorphic** if and only if $\phi \in \mathrm{Int}(\mathfrak{g})$ where $\mathrm{Int}(\mathfrak{g})$ denotes the **group of inner automorphisms of** $\mathfrak{g}$, i.e. the automorphisms of $\mathfrak{g}$ that are of the form $\exp(\mathrm{ad}_X)$ with

$$\begin{array}{cccc} \mathrm{ad}_X : & \mathfrak{g} & \to & \mathfrak{g} \\ & Y & \mapsto & [X, Y] \end{array}$$

the adjoint representation of $\mathfrak{g}$, where we denote the Lie bracket by $[.,.]$ and $X \in \mathfrak{g}$.

If we denote the set of conjugation of $\mathfrak{g}$ by $\mathrm{Conj}(\mathfrak{g})$ then the set of isomorphism classes of real forms of $\mathfrak{g}$ is in bijection with $\mathrm{Conj}(\mathfrak{g})/\sim_c$ where the equivalence relation $\sim_c$ for σ and $\sigma' \in \mathrm{Conj}(\mathfrak{g})$ is defined by

$$\sigma \sim_c \sigma' \text{ if there exists a } \phi \in \mathrm{Aut}(\mathfrak{g}) \text{ such that } \sigma' = \phi\sigma\phi^{-1}. \tag{1.1}$$

In order to replace conjugations of $\mathfrak{g}$ by $\mathbb{C}$-linear involutions we need to remind the notion of semisimple Lie algebra.

Definition 1.1.5. A Lie algebra $\mathfrak{g}$ is called **simple** if it has no nontrivial ideals and **semisimple** if it is a direct sum of simple Lie algebras.

For any Lie algebra $\mathfrak{g}$ we define the **Killing form** $\mathrm{B} : \mathfrak{g} \times \mathfrak{g} \to K$ as

$$\mathrm{B}(X, Y) := \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

Observe that the Killing form is symmetric. In addition $\mathrm{B}([X, Y], Z) = \mathrm{B}(X, [Y, Z])$ for any $X, Y, Z \in \mathfrak{g}$. We have the following characterization of semisimple Lie algebras.

Theorem 1.1.6 (Cartan's second criterion)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non-degenerate.*

Definition 1.1.7. Let $\mathfrak{g}$ be a real semisimple lie algebra. A **Cartan involution** of $\mathfrak{g}$ is an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\mathrm{B}_\theta(X, Y) := -\mathrm{B}(X, \theta(Y))$ is a positive definite bilinear form.

A Cartan involution θ of $\mathfrak{g}$ defines a decomposition of the Lie algebra in ± 1-eigenspaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{1.2}$$

satisfying $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Thus $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and any subalgebra of $\mathfrak{m}$ is commutative. In the literature this decomposition is known as a **Cartan decomposition** of $\mathfrak{g}$. One special feature of a Cartan decomposition is that the Killing form B is negative definite on $\mathfrak{h}$ and positive-definite on $\mathfrak{m}$. This shows that the subalgebra $\mathfrak{h}$ is a maximal compact subalgebra of $\mathfrak{g}$ for any cartan decomposition. The subspace $\mathfrak{m}$ coincides with the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form B and is called **Cartan subspace** in $\mathfrak{g}$.

Conversely a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satisfying the above conditions determines a Cartan involution θ on $\mathfrak{g}$ defined as follows.

$$\theta(x) := \begin{cases} +x & \text{if } x \in \mathfrak{h}, \\ -x & \text{if } x \in \mathfrak{m}. \end{cases}$$

Example 1.1.8. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ be a real semisimple Lie algebra and $\mathfrak{h} = \mathfrak{so}(n)$. Let $\mathfrak{m} = \mathrm{Sym}(n, \mathbb{R})$ be the space of all real symmetric matrices of order n with zero trace. One can see that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a Cartan decomposition and that it induces the involution $\theta(X) = -X^t$, for all $X \in \mathfrak{g}$. Conversely consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ and the Cartan involution $\theta(X) = -X^t$. It follows that $\mathfrak{h} = \mathfrak{g}^\theta = \mathfrak{so}(n)$ and $\mathfrak{m} = \mathrm{Sym}(n, \mathbb{R})$.

Example 1.1.9. As we see below an important aspect about semisimple complex Lie algebras is the existence of a unique (up to conjugation) compact real form $\mathfrak{u}$ of $\mathfrak{g}$. The decomposition

$$\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$$

is a Cartan decomposition of $\mathfrak{g}$. Here $\theta = \tau$ is the real structure corresponding to the real form $\mathfrak{u}$. This is the case for example of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with $\theta(X) = (\overline{X}^t)^{-1}$ and $\mathfrak{u} = \mathfrak{g}^\theta = \mathfrak{su}(n)$.

It is a well known fact (see [33] for further details) that for any semisimple complex Lie algebra $\mathfrak{g}$ and for any $\sigma \in \mathrm{Conj}(\mathfrak{g})$ we can find an antilinear involution τ defining a compact real form $\mathfrak{u} = \mathfrak{g}^\tau \subset \mathfrak{g}$ such that τ commutes with σ. Hence we have a map

$$\begin{array}{rcl} \mathrm{Conj}(\mathfrak{g}) & \to & \mathrm{Aut}(\mathfrak{g}) \\ \sigma & \mapsto & \eta := \tau\sigma. \end{array}$$

In the literature τ is known as a **compact conjugation**. The resulting involution η is $\mathbb{C}$-linear and depends on the choice of the compact conjugation.

Let us consider the equivalence relation $\sim$ in $\mathrm{Conj}(\mathfrak{g})$ defined by

$$\sigma \sim \sigma' \text{ if there exists a } \phi \in \mathrm{Int}(\mathfrak{g}) \text{ such that } \sigma' = \phi\sigma\phi^{-1}, \tag{1.3}$$

where as we noticed at the beginning of this section $\mathrm{Int}(\mathfrak{g})$ is the normal subgroup of the linear algebraic group $\mathrm{Aut}(\mathfrak{g})$ of all automorphism of $\mathfrak{g}$, generated by all elements of the form $\exp(\mathrm{ad}x)$ where ad is the adjoint representation of $\mathfrak{g}$ and $x \in \mathfrak{g}$. The group $\mathrm{Int}(\mathfrak{g})$ is known as the **group of inner automorphism of $\mathfrak{g}$**. One has the following.

Proposition 1.1.10. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathrm{Aut}_2(\mathfrak{g})$ be the set of $\mathbb{C}$-linear involutions of $\mathfrak{g}$. Consider the equivalence relations $\sim_c$ and $\sim$ in $\mathrm{Conj}(\mathfrak{g})$ described in (1.1) and (1.3) and defines in an analogue way the same relations $\sim$ and $\sim_c$ in $\mathrm{Aut}(\mathfrak{g})$. We have the following bijective correspondences:*

1. $\mathrm{Conj}(\mathfrak{g})/\sim_c \longleftrightarrow \mathrm{Aut}_2(\mathfrak{g})/\sim_c$,
2. $\mathrm{Conj}(\mathfrak{g})/\sim \longleftrightarrow \mathrm{Aut}_2(\mathfrak{g})/\sim$.

As a consequence of Proposition 1.1.10 we obtain a bijective correspondence between the set of isomorphism classes of real forms of $\mathfrak{g}$ and $\mathrm{Aut}_2(\mathfrak{g})/\sim_c$.

1.2 Automorphism of complex Lie groups and real forms

For this section, we follow [31]. See also [3].

Definition 1.2.1. Let G be a complex Lie group and $G_{\mathbb{R}}$ be its underlying real Lie group. A **real form** of G is a real Lie subgroup $G_0 \subset G_{\mathbb{R}}$ such that G_0 is the fixed point set of a **conjugation or antiholomorphic involution** σ of G.

We make the following observations.

1. According to our definition of real form for a complex Lie group G with Lie algebra $\mathfrak{g}$, there may exist a real Lie subgroup G_0 such that its Lie algebra is a real form of $\mathfrak{g}$ without G_0 being a real form of G.

   Example 1.2.2. $G_0 = \mathrm{SO}_0(p,q)$, the connected component of the identity of $\mathrm{SO}(p,q)$, is not a real form of $G = \mathrm{SO}(n,\mathbb{C})$ with $n = p+q$ since the fixed point set of the conjugation $\sigma(X) = \mathrm{I}_{p,q}(X^t)^{-1}\mathrm{I}_{p,q}$ where

   $$\mathrm{I}_{p,q} = \begin{pmatrix} -\mathrm{I}_p & 0 \\ 0 & \mathrm{I}_q \end{pmatrix}$$

   is $G^\sigma = \mathrm{SO}(p,q)$ and there is no other possible conjugation having $\mathrm{SO}_0(p,q)$ as fixed point subgroup of G.

2. Any real form G_0 of a connected, simply connected Lie group G is connected.

A Lie group is said to be **semisimple** if its Lie algebra is semisimple. Let G be a complex semisimple Lie group. It is well known that a compact real form U of G always exists and it is unique up to conjugation. We can then define a **compact conjugation** of G to be the antiholomorphic involution $\tau : G \to G$ such that $G^\tau = U$.

Example 1.2.3. Let $G = \mathrm{SL}(n,\mathbb{C})$, its compact real form is $\mathrm{SU}(n)$. In fact, $\mathrm{SL}^\tau(n,\mathbb{C}) = \mathrm{SU}(n)$ for the conjugation $\tau : X \mapsto (\overline{X}^t)^{-1}$.

There exists, however, a more general concept than that of semisimple Lie groups that will appear throughout this thesis. Following Knapp [44] a **reductive Lie group** is a tuple (G, H, θ, B) where

1. G is a Lie group with **reductive Lie algebra** $\mathfrak{g}$. By a reductive Lie algebra we mean a Lie algebra $\mathfrak{g}$ whose adjoint representation is completely reducible.

2. H is a maximal compact subgroup of G.

3. θ is a Lie algebra involution of $\mathfrak{g}$ and it induces a decomposition into ± 1-eigenspaces
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$
where the $+1$-eigenspace $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is the vector subspace defined by the eigenvalue $\lambda = -1$.

4. B is a θ, $\mathrm{Ad}(G)$-invariant, non-degenerate bilinear form with respect to $\mathfrak{h} \perp_B \mathfrak{m}$, where B is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$.

5. The map
$$\begin{array}{rcl} H \times \mathfrak{m} & \rightarrow & G \\ (h, X) & \mapsto & h \exp(X) \end{array}$$
is a diffeomorphism.

Remark 1.2.4. Let G be a Lie group with reductive Lie algebra $\mathfrak{g}$. The triple (H, θ, B) that makes G a reductive Lie group is known as **Cartan data for G**.

A **reductive Lie subgroup** (G', H', θ', B') of a reductive Lie group (G, H, θ, B) is Lie subgroup $G' \subset G$ such that the Cartan data (H', θ', B') is obtained as intersection and restriction of (H, θ, B) to G'. For instance, $\mathrm{O}(n, \mathbb{C})$ is a reductive Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ with $H = \mathrm{O}(n)$, $\theta(X) = (X^t)^{-1}$ and B the Killing form.

Remark 1.2.5. When G is a connected semisimple Lie group all the Cartan data is reduced to choosing a maximal compact subgroup H of G. In fact, one can take B to be the killing form since it is negative definite on $\mathfrak{h}$, the Lie algebra of H and set $\mathfrak{m}$ to be the orthogonal complement to $\mathfrak{h}$ with respect to B. Finally, we define θ as
$$\theta(X) = \begin{cases} X & \text{if } X \in \mathfrak{h}, \\ -X & \text{if } X \in \mathfrak{m}. \end{cases}$$

Let us now denote by $\mathrm{Aut}(G)$ the group of all holomorphic automorphisms of G. We define $\mathrm{Int}(G)$ to be the normal subgroup of $\mathrm{Aut}(G)$ given by **inner automorphism** of G, i.e. by automorphism of the form
$$\mathrm{Int}(g)(h) := ghg^{-1} \text{ for } h, g \in G.$$
The quotient
$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Int}(G) \tag{1.4}$$

is known as the subgroup of **outer holomorphic automorphisms of** G and we hence have an extension

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

We define for Lie groups the equivalence relations $\sim_c$ and $\sim$ as in the case of Lie algebras replacing $\mathrm{Int}(\mathfrak{g})$ by $\mathrm{Int}(G)$. Consider the map

$$\begin{array}{rcl} \mathrm{Conj}(G) & \to & \mathrm{Aut}_2(G) \\ \sigma & \mapsto & \theta := \sigma\tau. \end{array}$$

where τ is the compact conjugation of G. We then have a similar proposition to Proposition 1.1.10.

Proposition 1.2.6. *Let G be a complex semisimple Lie group and $\mathrm{Aut}_2(G)$ be the set of holomorphic involutions of G. Consider the equivalence relations $\sim_c$ and $\sim$. We have the following bijective correspondences:*

1. $\mathrm{Conj}(G)/\sim_c \longleftrightarrow \mathrm{Aut}_2(G)/\sim_c$,
2. $\mathrm{Conj}(G)/\sim \longleftrightarrow \mathrm{Aut}_2(G)/\sim$.

We also have the following.

Proposition 1.2.7. *Let G be a complex semisimple Lie group.*

1. *Let $\sigma' \sim \sigma$ with $\sigma, \sigma' \in \mathrm{Conj}(G)$ i.e $\sigma' = \mathrm{Int}(g)\sigma\mathrm{Int}(g)^{-1}$ for any $g \in G$. Then $G^{\sigma'} = \mathrm{Int}(g)G^{\sigma}$.*
2. *Let $\eta' \sim \eta$ with $\eta, \eta' \in \mathrm{Aut}_2(G)$ i.e $\eta' = \mathrm{Int}(g)\eta\mathrm{Int}(g)^{-1}$ for any $g \in G$. Then $G^{\eta'} = \mathrm{Int}(g)G^{\eta}$.*

As a consequence of the previous propositions we have that the set of isomorphic real forms of G is in bijection with $\mathrm{Aut}_2(G)/\sim_c$.

We can rephrase the first correspondence of Proposition 1.2.6 in terms of the **Galois cohomology** of the Galois group of the field extension $\mathbb{R} \subset \mathbb{C}$, on $\mathrm{Aut}(G)$. We first recall some facts about **non-abelian cohomology**. We follow the approach in ([53], Chapter 3).

Let A be a group acted on by another group Γ, meaning that every $\gamma \in \Gamma$ defines an automorphism of A also denoted by γ such that $\gamma(ab) = \gamma(a)\gamma(b)$ for any $a, b \in A$. We first define $Z^1(\Gamma, A)$ the **set of 1-cocycles of Γ in** A, where a **1-cocycle** of Γ in A is a map of $\gamma \in \Gamma \mapsto a_\gamma \in A$ such that for any $\gamma_1, \gamma_2 \in \Gamma$,

$$a_{\gamma_1\gamma_2} = a_{\gamma_1}\gamma_1(a_{\gamma_2}).$$

We say that the cocycles a_{γ_1} and a_{γ_2} are **cohomologous** if and only

$$a_{\gamma_2} = b^{-1}a_{\gamma_1}\gamma_1(b), \tag{1.5}$$

for any $b \in A$. One can verify that this is an equivalence relation $\sim$ in $Z^1(\Gamma, A)$ and hence we can define $H^1(\Gamma, A)$, **the first cohomology set of Γ in** A, as the quotient of $Z^1(\Gamma, A)$ by this relation.

In our situation, let $\sigma \in \mathrm{Aut}_2(G)$ and consider the group $\Gamma = \{1, \sigma\}$. This group is isomorphic to $\mathbb{Z}/2$. A **1-cocycle** is then given by a map $\sigma \mapsto a_\sigma$ from $\mathbb{Z}/2$ to $\mathrm{Aut}(G)$ such that $a_\sigma\sigma(a_\sigma) = 1$. Now let a'_σ be another 1-cocycle, we say that a_σ and a'_σ are cohomologous if and only if $a_\sigma = f^{-1}a'_\sigma\sigma(f)$ for any $f \in \mathrm{Aut}(G)$.

We have the following (see [31]).

Proposition 1.2.8. *Let $\sigma \in \mathrm{Aut}(G)$. Let us define the set*

$$S_\sigma := \{s \in G \,:\, s\sigma(s) \in Z(G)\},$$

where $Z(G)$ is the centre of G. Then

(1) $Z(G)$ acts on S_σ by multiplication.

(2) G acts on the right on S_σ by

$$s \cdot g := g^{-1}s\sigma(g), \text{ with } g \in G \text{ and } s \in S_\sigma.$$

(3) Let $H^1_\sigma(\mathbb{Z}/2, \mathrm{Ad}(G))$ be the cohomology set defined by the action of $\mathbb{Z}/2$ in $\mathrm{Ad}(G)$ given by $\sigma \in \mathrm{Aut}_2(G)$. There is a bijection

$$H^1_\sigma(\mathbb{Z}/2, \mathrm{Ad}(G)) \leftrightarrow S_\sigma/(Z(G) \times G).$$

Let $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Int}(G)$ be the group of **outer automorphisms** of G. We make the following observation.

Remark 1.2.9. Let $\pi : \mathrm{Aut}_2(G) \to \mathrm{Out}_2(G)$ be the natural projection. Observe that for any $\sigma, \sigma' \in \mathrm{Aut}_2(G)$ such that $\pi(\sigma) = a = \pi(\sigma')$ we have that $H^1_\sigma(\mathbb{Z}/2, \mathrm{Ad}(G))$ and $H^1_\sigma(\mathbb{Z}/2, \mathrm{Ad}(G))$ are in bijection. It makes sense then to denote this cohomology set by $H^1_a(\mathbb{Z}/2, \mathrm{Ad}(G))$.

For a more detailed account of these notions we refer the reader to [31].

1.3 Finite order automorphisms of semisimple real Lie groups equipped with a Cartan involution.

We now focus our attention to the study of automorphisms of a real semisimple Lie group. The approach will be similar to the one made in the complex case. We generalise to real Lie groups some of the results obtained by García-Prada and Ramanan [31] for complex semisimple Lie groups.

Let G be a real semisimple Lie group (not necessarily connected) with lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a given Cartan decomposition. Let θ be its corresponding involution and let H be the analytic subgroup of G with Lie algebra $\mathfrak{h}$. Then there exists a Lie group involution of G with differential θ. This involution of G, that we will also denote by θ, is called **global Cartan involution** of G or simply **Cartan involution** of G. In addition, the subgroup of G fixed by θ is a maximal compact subgroup of G and coincides with H. Cartan proved that any real semisimple Lie algebra has a Cartan involution and that it is unique up to conjugation. Observe then that H is also unique up to conjugation. In addition recall that for any semisimple Lie group G with cartan involution θ there exists a surjective diffeomorphism $H \times \mathfrak{m} \to G$ given by $(h, X) \mapsto h\exp(X)$ known as **global Cartan decomposition**.

As in the complex situation consider the linear algebraic group $\mathrm{Aut}(G)$ of all automorphisms of G. The normal subgroup of $\mathrm{Aut}(G)$ is also known as the group $\mathrm{Int}(G)$ of **inner automorphisms** of G. Recall that the quotient

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Int}(G)$$

is called the **group of outer automorphisms of** G. Then we have an extension

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1. \tag{1.6}$$

Now, let us consider $\mathrm{Aut}(G,\theta)$ the **group of all automorphisms of** G **commuting with** θ. In this case the group $\mathrm{Int}(G,\theta)$ of **inner automorphisms of** G **that commute with** θ is the normal subgroup of $\mathrm{Aut}(G,\theta)$ and coincide with $\mathrm{Int}(H)$.

The quotient

$$\mathrm{Out}(G,\theta) := \mathrm{Aut}(G,\theta)/\mathrm{Int}(G,\theta)$$

is then the group of **outer automorphisms of** G **that commute with** θ and as above we have an extension

$$1 \to \mathrm{Int}(G,\theta) \to \mathrm{Aut}(G,\theta) \to \mathrm{Out}(G,\theta) \to 1. \tag{1.7}$$

If we denote by $\mathrm{Aut}(\mathfrak{g})$, $\mathrm{Int}(\mathfrak{g})$ and $\mathrm{Out}(\mathfrak{g})$ the group of all automorphisms (respectively inner and outer automorphisms) of $\mathfrak{g}$ then we have the analogous extensions to (1.6) and (1.7):

$$1 \to \mathrm{Int}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Out}(\mathfrak{g}) \to 1.$$

$$1 \to \mathrm{Int}(\mathfrak{g},\theta) \to \mathrm{Aut}(\mathfrak{g},\theta) \to \mathrm{Out}(\mathfrak{g},\theta) \to 1.$$

If G is connected then $\mathrm{Int}(\mathfrak{g})$ is cannonically isomorphic to $\mathrm{Ad}(G)$, where Ad is the adjoint representation of G in $\mathfrak{g}$. In addition, if $\widetilde{G}$ is the universal cover of G then $\mathrm{Int}(\widetilde{G}) \cong \mathrm{Ad}(\widetilde{G}) = \mathrm{Ad}(G) \cong \mathrm{Int}(G)$.

Notice that the Cartan decomposition (1.2) allow us to define the restriction of the adjoint representation of G to H giving rise to two representations: The **adjoint representation**

$$\mathrm{Ad}_H : H \to \mathrm{GL}(\mathfrak{h})$$

of H on $\mathfrak{h}$ and the **isotropy representation**

$$\iota : H \to \mathrm{GL}(\mathfrak{m})$$

of H on $\mathfrak{m}$ (see [37] for further details). In addition let $H^{\mathbb{C}}$ be the complexification of H and let $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ be the complexifications of $\mathfrak{h}$ and $\mathfrak{m}$ respectively. We then define the following representations.

$$\iota^+ : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{h}^{\mathbb{C}})$$

and

$$\iota^- : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}}).$$

Notice that $\iota^+_{|_H} = \mathrm{Ad}_H$, that is ι^+ is the complexification of Ad_H. In the same way $\iota^-_{|_H} = \iota$, that is ι^- is the complexification of ι. We also refer to both as the adjoint and the isotropy representations.

Let $\mathrm{Aut}_n(G)$, $\mathrm{Aut}_n(G,\theta)$, $\mathrm{Out}_n(G)$ and $\mathrm{Out}_n(G,\theta)$ be the sets of elements of order n in $\mathrm{Aut}(G)$, $\mathrm{Aut}(G,\theta)$, $\mathrm{Out}(G)$ and $\mathrm{Out}(G,\theta)$, respectively. It is well known (see Chapter VI of [44] for further details) that for any $\eta \in \mathrm{Aut}_2(G)$ there exists an inner automorphism φ of G such that $\eta' = \varphi\eta\varphi^{-1}$ commutes with the Cartan involution θ. Hence it seems natural to ask whether can we extend this result to higher order automorphisms. With the following two propositions we give an answer to this matter.

Proposition 1.3.1. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ. Then for any $\eta \in \mathrm{Aut}_n(\mathfrak{g})$ with $n = 2m$, there exists $\varphi \in \mathrm{Int}(G)$ such that $\eta' := \varphi\eta\varphi^{-1}$ commutes with θ.*

Proof. Let $\eta \in \mathrm{Aut}_n(\mathfrak{g})$ with $n = 2m$. Let us define $\omega = \eta^m\theta$, observe that $\eta^m \in \mathrm{Aut}_2(\mathfrak{g})$. Hence ω is an automorphism of $\mathfrak{g}$ and by Proposition 1.96 in [44] it leaves $B_{\mathfrak{g}}$ invariant. Since θ is a Cartan involution B_θ is an inner product for $\mathfrak{g}$. For any $X, Y \in \mathfrak{g}$ we have

$$\mathrm{B}(\omega X, \theta Y) = \mathrm{B}(X, \omega^{-1}\theta Y) = \mathrm{B}(X, \omega^{-1}\theta Y) = \mathrm{B}(X, \theta\omega Y)$$

and then for any $X, Y \in \mathfrak{g}$

$$\mathrm{B}_\theta(\omega X, Y) = \mathrm{B}_\theta(X, \omega Y).$$

Thus ω is symmetric and its square $\rho = \omega^2$ is positive definite. Write ρ^r for the positive-definite r^{th} power of ρ, $-\infty < \rho < \infty$. By Lemma 6.15 in [44] ρ^r is in $\mathrm{Int}(\mathfrak{g})$. Also note that

$$\rho\theta = \omega^2\theta = \eta^m\theta\eta^m\theta\theta = \theta\theta\eta^m\theta\eta^m = \theta\omega^{-2} = \theta\rho^{-1}.$$

Now in terms of a basis that diagonalizes ρ it is easy to see that $\rho_{ii}^r\theta_{ij} = \theta_{ij}\rho_{jj}^{-r}$ and therefore $\rho^r\theta = \theta\rho^{-r}$. Let $\varphi = \rho^{1/4}$. Then

$$(\varphi\theta\varphi^{-1})\eta^{m+1} = \rho^{1/4}\theta\rho^{-1/4}\eta^{m+1} = \rho^{1/2}\theta\eta^m\eta = \rho^{1/2}\omega^{-1}\eta = \eta\rho^{1/2}\omega^{-1} =$$

$$= \eta\rho^{1/2}\theta\eta^m = \eta\rho^{1/4}\theta\rho^{-1/4}\eta^m = \eta(\varphi\theta\varphi^{-1})\eta^m.$$

From this we get $(\varphi\theta\varphi^{-1})\eta = \eta(\varphi\theta\varphi^{-1})$ or equivalently $(\varphi\eta\varphi^{-1})\theta = \theta(\varphi\eta\varphi^{-1})$. □

Proposition 1.3.2. *Let G be a real connected semisimple Lie group with Cartan involution θ. Let η be an order n automorphism on G with n even, then there exists $\varphi \in \mathrm{Int}(G)$ such that $\eta' := \varphi\eta\varphi^{-1}$ commutes with θ.*

Proof. Let $\widetilde{G}$ be the unique simple connected semisimple Lie group with semisimple Lie algebra $\mathfrak{g}$. Let θ be its Cartan involution and $\eta \in \mathrm{Aut}_n(\mathfrak{g}, \theta)$. If we denote by $\widetilde{\theta}$ and $\widetilde{\eta}$ the corresponding involutions on $\widetilde{G}$, then $\widetilde{\theta}$ is a Cartan involution on $\widetilde{G}$ (see Theorem 6.31, [44]) and there exists $\widetilde{\varphi}$ an inner automorphism of $\widetilde{G}$ such that $\widetilde{\eta}' := \widetilde{\varphi}\widetilde{\eta}\widetilde{\varphi}^{-1}$ commutes with $\widetilde{\theta}$.

Now, let G be any real semisimple Lie group with semisimple Lie algebra $\mathfrak{g}$ and let $\overline{\theta}$ be the Cartan involution on G such that $d\overline{\theta} = \theta$ (see Theorem 6.31, [44]). Let $\overline{\eta} \in \mathrm{Aut}_n(G)$ then there exists $\widetilde{\eta}$ an order n automorphism of $\widetilde{G}$ such that $\pi(\widetilde{\eta}) = \overline{\eta}$, where $\pi : \widetilde{G} \to G$ is the covering map. Let $\widetilde{\varphi}$ be the inner automorphism of $\widetilde{G}$ such that $\widetilde{\eta}'$ commutes with $\widetilde{\theta}$. Since $\mathrm{Int}(\widetilde{G})$ is isomorphic to $\mathrm{Int}(G)$ we associate to each $\widetilde{\varphi} \in \mathrm{Int}(\widetilde{G})$ an inner automorphism $\overline{\varphi}$ of G. Thus we define $\overline{\eta}' := \overline{\varphi}\,\overline{\eta}\,\overline{\varphi}^{-1}$. Clearly, $\widetilde{\eta}'$ is the preimage of $\overline{\eta}'$ and since π is a group homomorphism $\overline{\eta}'$ c ommutes with $\overline{\theta}$. □

The analogous results for the odd case are not clear. Hence when needed we will consider order n automorphisms that commute with θ, which for n even, thanks to Proposition 1.3.2, is not a restriction. An order n automorphism η gives rise to a finite decomposition

$$\mathfrak{g} = \bigoplus_{k=0}^{n-1} \mathfrak{g}_k, \tag{1.8}$$

where $\mathfrak{g}_k$ is the eigenspace for the eigenvalue $\zeta_k := \exp(2\pi i \frac{k}{n})$ of the automorphism of $\mathfrak{g}$ defined by η. If, indeed, η commutes with θ, as we are assuming, then these finite order automorphisms leave invariant $\mathfrak{h}$ and $\mathfrak{m}$. Hence we can also decompose $\mathfrak{h}$ and $\mathfrak{m}$ as

$$\mathfrak{h} = \bigoplus_{k=0}^{n-1} \mathfrak{h}_k \quad \text{and} \quad \mathfrak{m} = \bigoplus_{k=0}^{n-1} \mathfrak{m}_k.$$

Observe that the above decompositions satisfy

$$[\mathfrak{h}_k, \mathfrak{h}_l] \subset \mathfrak{h}_{k+l}, \quad [\mathfrak{h}_k, \mathfrak{m}_l] \subset \mathfrak{m}_{k+l}, \quad [\mathfrak{m}_k, \mathfrak{m}_l] \subset \mathfrak{h}_{k+l}. \tag{1.9}$$

Clearly $\mathfrak{h}_0$ is the Lie algebra of $H^\eta := (G^\theta)^\eta$. Thus the restriction of the isotropy representation ι_- to H^η gives rise to representations

$$\iota^k : H^\eta \to \mathrm{GL}(\mathfrak{m}_k).$$

By abuse of notation we will denote their complexifications

$$\iota^k : (H^{\mathbb{C}})^\eta \to \mathrm{GL}(\mathfrak{m}_k^{\mathbb{C}}) \tag{1.10}$$

in the same way. It would be interesting to find all the **symmetric pairs** $(\mathfrak{h}_0, \mathfrak{m}_k)$. A symmetric pair in the sense of Helgason [37] is a pair $(\mathfrak{h}_0, \mathfrak{m}_k)$ such that

$$[\mathfrak{h}_0, \mathfrak{h}_0] \subset \mathfrak{h}_0, \quad [\mathfrak{h}_0, \mathfrak{m}_k] \subset \mathfrak{m}_k, \quad [\mathfrak{m}_k, \mathfrak{m}_k] \subset \mathfrak{h}_0. \tag{1.11}$$

Hence $(\mathfrak{h}_0, \mathfrak{m}_0)$ is always a symmetric pair. Furthermore if n is odd there is no other symmetric pair. If n is even, then $(\mathfrak{h}_0, \mathfrak{m}_l)$ with $l = n/2$ is the only symmetric pair besides $(\mathfrak{h}_0, \mathfrak{m}_0)$. We prove the following.

Proposition 1.3.3. *Let $\sigma := \theta\eta$ with $\eta \in \mathrm{Aut}_n(G, \theta)$. Let $H^\eta = H_0$ be the Lie subgroup of H whose lie algebra is $\mathfrak{h}_0$. Then there exists a diffeomorphism*

$$\begin{array}{rcl} H_0 \times \mathfrak{m}_0 & \to & G^\eta \\ (h, X) & \mapsto & h \exp(X). \end{array}$$

If $n = 2l$ there exists also a diffeomorphism

$$\begin{array}{rcl} H_0 \times \mathfrak{m}_l & \to & G^\sigma \\ (h, X) & \mapsto & h \exp(X). \end{array}$$

Proof. For any $a \in G$ there exist $h \in H$ and $X \in \mathfrak{m}$ such that $a = h \exp(X)$. In particular let us take any element $h \in H_0$ and any $X \in \mathfrak{m}_0$ or $\mathfrak{m}_l$ and consider $g = h \exp(X)$. Since for any finite automorphism γ of G we have $\gamma \circ \exp = \exp \circ \gamma$ (here we are denoting $d\gamma$ also by γ). Then $\gamma(g) = \gamma(h \exp(X)) = \gamma(h) \exp(\gamma(X))$. On the one hand if $X \in \mathfrak{m}_0$ then $\eta(X) = X$, since $\eta(h) = h$ for any $h \in H_0$ we thus have $\eta(g) = \eta(h) \exp(\eta(X)) = g$, i.e. $g \in G^\eta$. On the other hand, if $X \in \mathfrak{m}_l$ then $\sigma(g) = \sigma(h) \exp(\sigma(X))$. As $\sigma(h) = \theta(\eta(h)) = h$ for any $h \in H_0$ and $\sigma(X) = \theta(\eta(X)) = X$ for any $X \in \mathfrak{m}_l$ then $\sigma(g) = g$ and hence $g \in G^\sigma$. □

We define the following equivalence relations for η, $\eta' \in \mathrm{Aut}_n(G)$ and $\sigma, \sigma' \in \mathrm{Aut}_n(G,\theta)$:

$$\eta \sim \eta' \text{ if there is } \varphi \in \mathrm{Int}(G) \text{ such that } \eta' = \varphi\eta\varphi^{-1},$$

$$\sigma \sim_\theta \sigma' \text{ if there is } \varphi \in \mathrm{Int}(H) \text{ such that } \sigma' = \varphi\sigma\varphi^{-1}.$$

Let $[\eta]_\theta$ denote the image of η under the natural map $\mathrm{Aut}(G,\theta) \to \mathrm{Aut}(G,\theta)/\sim_\theta$. Let η, $\eta' \in \mathrm{Aut}_n(G,\theta)$ and consider the natural map

$$\pi : \mathrm{Aut}_n(G,\theta) \to \mathrm{Out}_n(G,\theta).$$

We say that η and η' are inner equivalent if there is $\varphi \in \mathrm{Int}(H)$ such that $\eta' = \varphi\eta$ or equivalently, $\pi(\eta) = \pi(\eta')$.

Proposition 1.3.4. *Let G be a semisimple real Lie group. The map $\pi : \mathrm{Aut}_n(G,\theta) \to \mathrm{Out}_n(G,\theta)$ descends to define a surjective map*

Proof. Let η_1, $\eta_2 \in \mathrm{Aut}_n(G,\theta)$ such that $[\eta_1]_\theta = [\eta_2]_\theta$. This means that $\eta_2 = \mathrm{Int}(h)\eta_1\mathrm{Int}(h)^{-1}$ for some $h \in H$. We define $\tilde{h} := h\eta_1(h^{-1})$. One easily checks that $\mathrm{Int}(\tilde{h}) \in \mathrm{Int}(H)$ and that $\eta_2 = \mathrm{Int}(\tilde{h})\eta_1$, i.e. η_1 and η_2 are inner equivalent. $\square$

Consider the map $\mathcal{cl} : \mathrm{Aut}_n(G,\theta)/\sim_\theta \to \mathrm{Out}_n(G,\theta)$ defined in Proposition 1.3.4. Following [31] we refer to the image of $[\eta]_\theta \in \mathrm{Aut}_n(G,\theta)/\sim_\theta$ by this map as the **clique** of $[\eta]_\theta$. Clearly $\mathcal{cl}^{-1}(1) = \mathrm{Int}_n(H)/\sim_\theta$.

Proposition 1.3.5. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$. Consider the set*

$$S_\eta^{n,\theta} := \{s \in G^\theta \ : s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s) = z \in Z(G)\}$$

Then

1. *$Z(G)$ acts on $S_\eta^{n,\theta}$ by multiplication.*
2. *H acts on $S_\eta^{n,\theta}$ by*

$$h * s := h^{-1}s\eta(h) \text{ for all } h \in S_\eta^{n,\theta}.$$

3. Let $\pi : \mathrm{Aut}_n(G,\theta) \to \mathrm{Out}_n(G,\theta)$ *be the natural projection. Let* $a \in \mathrm{Out}_n(G,\theta)$ *and* $\eta \in \pi^{-1}(a)$*. Then the map* $\psi : S_\eta^{n,\theta} \to \pi^{-1}(a)$ *defined by* $s \mapsto \mathrm{Int}(s)\eta$ *gives a bijection*

$$S_\eta^{n,\theta}/(Z(G)\times H) \longleftrightarrow \mathfrak{cl}_n^{-1}(a),$$

where $\mathfrak{cl}_n : \mathrm{Aut}_n(G,\theta)/\sim_\theta \to \mathrm{Out}_n(G,\theta)$ *is the map induced by* π*. According to [31], we will refer to it as the* n-***clique*** *map.*

4. In particular, let $\eta = id \in \mathrm{Aut}_n(G,\theta)$ *then* $S_{id}^{n,\theta} = \{s \in G^\theta \ : s^n \in Z(G)\}$ *and the map* $s \mapsto \mathrm{Int}(s)$ *defines a bijection*

$$S_{id}^{n,\theta}/(Z(G)\times H) \leftrightarrow \mathrm{Int}_n(G,\theta)/\sim_\theta .$$

Proof. This proof is similar to that of Proposition 2.7 in [31].

(1) Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and let $s \in S_\eta^{n,\theta}$. Let $z \in Z(G)$, since $Z(G) \subset H$ and η^k leaves $Z(G)$ invariant for all $k = 1,\dots,n-1$, we have that

$$zs\eta(zs) = zs\eta(zs)\eta^2(zs)\cdots\eta^{n-1}(zs) =$$
$$= z\eta(z)\eta^2(z)\cdots\eta^{n-1}(z)s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s) \in Z(G)$$

and $zs \in H$ hence $zs \in S_\eta^{n,\theta}$.

(2) Let $h \in H$ and $s \in S_\eta^{n,\theta}$ then let $h * s := h^{-1}s\eta(h)$. Since $s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s) \in Z(G)$ we have

$$(h{*}s)\eta(h{*}s)\cdots\eta^{n-1}(h{*}s) = h^{-1}s\eta(h)\eta(h^{-1})\eta(s)\eta^2(h)\cdots\eta^{n-1}(h^{-1})\eta^{n-1}(s)h$$
$$= s\eta(s)\cdots\eta^{n-1}(s) \in Z(G),$$

hence $h * s \in S_\eta^\theta$ if and only if $h * s \in H$ but this is clear since

$$\theta(h*s) = \theta(h^{-1})\theta(s)\theta(\eta(h)) = h^{-1}s\eta(h) = h*s.$$

(3) First let us check that for any $h \in H$ then $\mathrm{Int}(h)\eta \in \mathrm{Aut}_n(G,\theta)$ if and only if $h \in S_\eta^{n,\theta}$. This is true since, as we can easily compute, $(\mathrm{Int}(h)\eta)^n = \mathrm{Id}$ is equivalent to

$$h\eta(h)\eta^2(h)\cdots\eta^{n-1}(h)g\eta^{n-1}(h)^{-1}\cdots\eta^2(h)^{-1}\eta(h)^{-1}h^{-1} = g,$$

for all $g \in G$ meaning that

$$h\eta(h)\eta^2(h)\cdots\eta^{n-1}(h) \in Z(G).$$

Then we have that $s \to \mathrm{Int}(s)\eta$ defines a surjective map $\psi : S_\eta^{n,\theta} \to \pi^{-1}(a)$. Since $\mathrm{Int}(s) = \mathrm{Int}(zs)$ for all $z \in Z(G)$, ψ descends to a map on $S_\eta^{n,\theta}/Z(G)$. On the other hand let $s' = h * s$ with $s \in S_\eta^{n,\theta}$ and $h \in H$. Then

$$\psi(s') = \mathrm{Int}(s')\eta = \mathrm{Int}(h^{-1})\mathrm{Int}(s)\mathrm{Int}(\eta(h))\eta.$$

However since $\mathrm{Int}(\eta(h))\eta(g) = \eta(\mathrm{Int}(h)g)$ for every $g \in G$, meaning that $\mathrm{Int}(\eta(h))\eta = \eta\mathrm{Int}(h)$, we have that $\psi(s') = \mathrm{Int}(h^{-1})\psi(s)\mathrm{Int}(h)$ and hence $\psi(s') \sim_\theta \psi(s)$.

(4) Follows from (3) and the fact that $\mathcal{cl}^{-1}(1) = \mathrm{Int}_n(H)/\sim_\theta$.

□

Proposition 1.3.5 gives an interpretation of $\mathcal{cl}^{-1}(a)$ in terms of non-abelian cohomology.

Proposition 1.3.6. *Let $H^1_\eta(\mathbb{Z}/n, H/Z(G))$ be the cohomology set defined by the action of $\mathbb{Z}/n$ in $H/Z(G)$ given by $\eta \in \mathrm{Aut}_n(G,\theta)$. Then there is a bijection*

$$H^1_\eta(\mathbb{Z}/n,\, H/Z(G)) \longleftrightarrow S_\eta^{n,\theta}/(Z(G) \times H). \tag{1.12}$$

Proof. We define $\Gamma := \{Id, \eta, \eta^2, \dots, \eta^{n-1}\} \simeq \mathbb{Z}/n$ with η a generator of Γ. We can consider the natural action of $\mathbb{Z}/n$ on $H/Z(G)$ given by the action of Id and η^i on H for $i = 1, \dots, n-1$. Let $s \in S_\eta^{n,\theta}$ and let $\widetilde{s}$ the image of s in $H/Z(G)$. Hence a 1-cocycle g is just given by an element of $H/Z(G)$, $\widetilde{s} := g_\eta$ defining a bijection between $S_\eta^{n,\theta}$ and $Z^1(\mathbb{Z}/n, H/Z(G))$. The cocycle condition allows us to determine the elements g_{η^i} by the recursive formula

$$g_{\eta^i} = g_\eta \eta(g_{\eta^{i-1}}).$$

Then $1 = g_{\eta^n} = \widetilde{s}\eta(\widetilde{s})\dots\eta^{n-1}(\widetilde{s})$. Let g' be another cocycle and let $\widetilde{s}' := g'_\eta$. They are cohomologous if there exist $\widetilde{h} \in H/Z(G)$ satisfying

$$\widetilde{s}' = h^{-1}\widetilde{s}\eta(h).$$

Since this is by definition the action of $H/Z(G)$ in $S_\eta^{n,\theta}$, this defines the cohomology set $H^1_\eta(\mathbb{Z}/n, H/Z(G))$. □

As a result of Proposition 1.3.5 and 1.3.6 it is clear that if η, $\eta' \in \mathrm{Aut}_n(G,\theta)$ are such that $\pi(\eta) = \pi(\eta') = a$ then there is a bijection

$$H^1_\eta(\mathbb{Z}/n, H/Z(G)) \longleftrightarrow H^1_{\eta'}(\mathbb{Z}/n, H/Z(G)).$$

Thence this cohomology set could be denoted by $H^1_a(\mathbb{Z}/n, H/Z(G))$. We have prooved the following.

Proposition 1.3.7. *Let $a \in \mathrm{Out}_n(G,\theta)$. There is a bijection*

$$\mathfrak{cl}_n^{-1}(a) \longleftrightarrow H^1_a(\mathbb{Z}/n, H/Z(G)),$$

and hence a bijection

$$\mathrm{Aut}_n(G,\theta)/\sim_\theta \longleftrightarrow \bigcup_{a\in \mathrm{Out}_n(G,\theta)} H^1_a(\mathbb{Z}/n, H/Z(G)).$$

Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and consider now the set

$$S^{n,\tau}_\eta := \{s \in H^{\mathbb{C}} \ : \ s\eta(s)\cdots\eta^{n-1}(s) = z \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)\}. \tag{1.13}$$

We then have similar results to the ones in Proposition 1.3.5.

Proposition 1.3.8. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$. Then*

1. *$Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ acts on $S^{n,\tau}_\eta$ by multiplication.*
2. *$H^{\mathbb{C}}$ acts on $S^{n,\tau}_\eta$ by*

$$h * s := h^{-1}s\eta(s) \text{ for all } h \in S^{n,\tau}_\eta.$$

Proof. (1) Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and let $s \in S^{n,\tau}_\eta$. Let $z \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$. Since $Z_\tau \subset H^{\mathbb{C}}$ and η^k leaves $Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ invariant for all $k = 1, \ldots, n-1$, we have that $z \in S^{n,\tau}_\eta$. Thus $z\eta(z)\cdots\eta^{n-1}(z) \in Z_\tau$. Since $s \in S^{n,\tau}_\eta$, then $z\eta(z)\eta^2(z)\cdots\eta^{n-1}(z)s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s) \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$. Hence $zs \in S^{n,\tau}_\eta$.

(2) Let $h \in H^{\mathbb{C}}$ and $s \in S^{n,\tau}_\eta$ and let us define $h * s := h^{-1}s\eta(h)$. Since $s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s) \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ we have

$$\begin{aligned}(h*s)\eta(h*s)\cdots\eta^{n-1}(h*s) &= h^{-1}s\eta(h)\cdots\eta^{n-1}(h^{-1})\eta^{n-1}(s)h \\ &= h^{-1}s\eta(s)\eta^2(s)\cdots\eta^{n-1}(s)h \\ &= s\eta(s)\cdots\eta^{n-1}(s) \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-).\end{aligned}$$

Hence $h * s \in S_\eta^{n,\tau}$ if $h * s \in H^{\mathbb{C}}$ but this is clear since s and h are in $H^{\mathbb{C}}$.

□

Lemma 1.3.9. *There is a bijection*

$$S_\eta^{n,\tau}/((Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)) \times H^{\mathbb{C}}) \leftrightarrow H^1_\eta(\mathbb{Z}/n, H^{\mathbb{C}}/(Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-))).$$

Proof. Close to the reasoning used in the proof of Proposition 1.3.6. □

Remark 1.3.10. Finally notice that much of the results studied throughout Sections 1.2 and 1.3 can be generalised to reductive Lie groups. See, for instance, [44, 49].

1.4 Weyl Alcove of a compact Lie group and conjugacy classes.

In this section we follow [21].

Definition 1.4.1. A **Cartan subgroup** of a compact connected Lie group H is a maximal connected Abelian subgroup, i.e a maximal torus of H. Its Lie algebra is a Cartan subalgebra.

For disconnected compact Lie groups there are several inequivalent definitions of a Cartan subgroup(see[21], 177 for further details). Let H be a compact connected Lie group with Lie algebra $\mathfrak{h}$ and let $\langle \cdot, \cdot \rangle$ be an inner product H-invariant on $\mathfrak{h}$. Fix a maximal torus T in H, i.e. a Cartan subgroup of H and let $\mathfrak{t} \subset \mathfrak{h}$ be its Lie (Cartan) subalgebra.

Definition 1.4.2. The nontrivial weights (see Chapter II, [21]) of the adjoint representation $\mathrm{Ad} : H \times \mathfrak{h} \to \mathfrak{h}$ are called the **roots** of H. We differentiate between types of roots:

- The **global** roots of H are the global weights $\vartheta : T \to \mathrm{U}(1)$.
- Their corresponding linear forms $\Theta : \mathfrak{t} \to i\mathbb{R}$ are known as the **infinitesimal** roots of H.

- Their complexification $\theta : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ are the **complex** roots of H and the linear forms $\alpha = \Theta/2\pi i$ are the **real** roots. We denote by $R\langle\mathbb{C}\rangle$ the set of complex roots and by $R\langle\mathbb{R}\rangle = R$ the set of real roots.

Definition 1.4.3. A subset S of a root system, i.e. a set of non-zero roots in $\mathfrak{t}$ satisfying certain properties (again we refer to [21], p.197), is called a **system of simple roots** if S is linearly independent in $\mathfrak{t}$ and every root $\beta \in R$ may be written as

$$\beta = \sum_{\alpha \in S} m_\alpha \cdot \alpha$$

where m_α are integers such that either all $m_\alpha \geq 0$ (**positive roots**) or all $m_\alpha \leq 0$ (**negative roots**).

Fix a system of real simple roots and denote by R_+ the set of positive roots. The family of affine hyperplanes in $\mathfrak{t}$

$$\mathcal{H}_{\alpha n} = \alpha^{-1}\{n\}, \quad \alpha \in R_+, \quad n \in \mathbb{Z}$$

together with the union $\mathfrak{t}_s = \bigcup_{\alpha,n} L_{\alpha n}$ is called the **Stiefel diagram of** H. This diagram is the inverse image of the set of singular points of T under the exponential map $\exp : \mathfrak{t} \to T$.

The set $\mathfrak{t} - \mathfrak{t}_s$ of regular points in $\mathfrak{t}$ decomposes into convex connected components called **alcoves** or **chambers of the Stiefel diagram**. Alcoves are open by definition. Let $k = \text{rank}(H)$ and $\overline{\mathcal{A}}$ denote the closure of $\mathcal{A}$. A **wall** of an alcove $\mathcal{A}$ is one of the subsets of $\overline{\mathcal{A}} \cap \mathcal{H}_{\alpha n}$ of $\mathfrak{t}$ with dimension $k - 1$.

We next define $\text{Conj}(H)$ to be the space of **conjugacy classes** of H. That is, $\text{Conj}(H)$ is the orbit space of H under the action of H via

$$\begin{array}{rcl} H \times H & \to & H \\ (h_1, h_2) & \mapsto & h_1 h_2 h_1^{-1}. \end{array}$$

If H is connected every element of H is conjugate to an element of T. Thus every element of H lies in a Cartan subgroup. Let $N(T)$ be the normaliser of T. We define the **Weyl group** of H to be $W := N(T)/T$. Then there exists a canonical homeomorphism

$$\begin{array}{rcl} T/W & \to & \text{Conj}(H) \\ [\,t\,]_W & \mapsto & [\,t\,]_H, \end{array}$$

taking the orbit of t under the action of the Weyl group to the conjugacy class of t.

Now, consider the W-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{t}$. This defines an isomorphism $b:\mathfrak{t}\to\mathfrak{t}^*$. Recall that we call **co-roots** to the elements of $\mathfrak{t}$ defined by

$$\alpha^* = 2b^{-1}(\alpha)/\langle\alpha,\alpha\rangle.$$

The **co-character lattice** $\Lambda_{cochar}\subset\mathfrak{t}$ is defined as the kernel of the exponential map $\exp:\mathfrak{t}\to T$. The co-roots define a character lattice $\Lambda_{coroot}\subset\Lambda_{cochar}\subset\mathfrak{t}$. We have that $\pi_1(H)=\Lambda_{cochar}/\Lambda_{coroot}$. In particular, if H is simply connected then $\Lambda_{coroot}=\Lambda_{cochar}$. We define the **affine Weyl group** to be $W_{\text{aff}}:=\Lambda_{cochar}\rtimes W$. Then there exists homeomorphisms

$$\text{Conj}(H)\simeq T/W\simeq\mathfrak{t}/W_{\text{aff}}.$$

Finally, let us define

$$\sqrt{-1}\mathcal{A}' := \{\alpha\in\sqrt{-1}\,\overline{\mathcal{A}}\text{ s.t. }\text{Spec}(\text{ad}(\alpha))\subset(-1,1)\}$$

and

$$\sqrt{-1}W\mathcal{A}' := \bigcup_{w\in W}\sqrt{-1}w\mathcal{A}'.$$

The following Propositions (see apendix A of [15]) will play an important role in the definition of parabolic Higgs bundle given in Section 2.5.

Proposition 1.4.4. *Let $\mathcal{A}\subset\mathfrak{t}$ be an alcove of H such that 0 is in the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$. Then:*

1. *If $\alpha\in\sqrt{-1\mathcal{A}}$ then* $\text{Spec}(\text{ad}(\alpha))\subset(-1,1)$.
2. *We have $\sqrt{-1}\mathcal{A}\subset\sqrt{-1}\mathcal{A}'$.*
3. *We have $\sqrt{-1}W\mathcal{A}'=\{\alpha\in\sqrt{-1}\,\mathfrak{t}$ s.t.* $\text{Spec}(\text{ad}(\alpha))\subset(-1,1)\}$
4. *For any $\alpha\in\sqrt{-1}\,\overline{\mathcal{A}}$ there exist $k\in\mathbb{Z}$ and $\lambda\in\dfrac{\sqrt{-1}}{2\pi}\Lambda_{cochar}$ such that*

$$k\alpha+\lambda\in\sqrt{-1}W\mathcal{A}'.$$

Proposition 1.4.5. *Let $H^{\mathbb{C}}$ be the complexification of a connected compact Lie group H and let $T^{\mathbb{C}}$ be the complexification of a cartan subgroup of H. Then every element of $H^{\mathbb{C}}$ is conjugate to an element of $T^{\mathbb{C}}$, which can be written as* $\exp(\alpha)\exp(s)$ *with $\alpha\in\overline{\mathcal{A}}, s\sqrt{-1}\mathfrak{t}$ and $[\alpha,s]=0$.*

Recall that in Section 1.3 we define the following equivalence relation:

$\sigma \sim \sigma'$ if there is $\varphi \in \mathrm{Int}(H)$ such that $\sigma' = \varphi\sigma\varphi^{-1}$ for any $\sigma, \sigma' \in \mathrm{Aut}_2(H)$.

Consider the natural projection $\pi : \mathrm{Aut}_2(H) \to \mathrm{Out}_2(H)$ and let σ, $\sigma' \in \mathrm{Aut}_2(H)$. Let us denote by $[\sigma]$ the image of σ under the natural map

$$\mathrm{Aut}_2(H) \to \mathrm{Aut}_2(H)/\sim .$$

Recall that we have the surjective map

$$\begin{array}{rccc} \mathcal{cl}: & \mathrm{Aut}_2(H)/\sim & \longrightarrow & \mathrm{Out}_2(H) \\ & [\sigma] & \mapsto & a. \end{array}$$

Notice that $\sigma \in \mathrm{Aut}_2(H)$ acts in a natural way on $\mathrm{Conj}(H)$, the set of conjugacy classes of H. This action descends to an action of $\mathrm{Out}_2(H)$ on $\mathrm{Conj}(H)$ since $\mathrm{Int}(H)$ acts trivially on $\mathrm{Conj}(H)$. In the same way, let $a := \mathcal{cl}(\sigma)$ for some $\sigma \in \mathrm{Aut}_2(H)$. Let $\sigma' \in \mathcal{cl}(a)^{-1}$. The action defined by σ coincide with σ' on $\mathrm{Conj}(H)$ since $\mathrm{Int}(H)$ acts trivially on $\mathrm{Conj}(H)$ and hence

$$\sigma'([x]) = (\mathrm{int}(h)\sigma)(\mathrm{int}(h^{-1})[x]) = (\mathrm{int}(h)\sigma)([x]) = \mathrm{int}(h)(\sigma([x])) = \sigma([x]).$$

Let $\sigma \in \mathrm{Aut}_2(H)$, let H^σ be the subgroup of elements of H fixed by σ. Let us denote by $\mathrm{Conj}(H^\sigma)$ the set of conjugacy classes of the subgroup H^σ. There exists a natural map (no necessarily injective)

$$\begin{array}{rccc} f_\sigma: & \mathrm{Conj}(H^\sigma) & \to & \mathrm{Conj}(H) \\ & [x]_{H^\sigma} & \mapsto & [x]_H. \end{array} \tag{1.14}$$

Let us denote by $\overline{\mathrm{Conj}}(H^\sigma)$ the image of $\mathrm{Conj}(H^\sigma)$ in $\mathrm{Conj}(H)$ via f_σ. We can state the following.

Theorem 1.4.6. *Let H be a real compact connected semisimple Lie group and $a \in \mathrm{Out}_2(H)$. Then*

$$\bigcup_{[\sigma]\in \mathcal{cl}^{-1}(a)} \overline{\mathrm{Conj}}(H^\sigma) \subset \mathrm{Conj}(H)^a.$$

Proof. Let $a \in \mathrm{Out}_2(H)$ and $\sigma \in \pi^{-1}(a)$. Let us denote by $\widetilde{[x]}_H$ the image of $[x]_{H^\sigma}$ in $\mathrm{Conj}(H)$. Let $y \in \widetilde{[x]}_H$, this means that there exists $h \in H$ such that $y = hxh^{-1}$. Hence

$$\sigma(y) = \sigma(h)\sigma(x)\sigma(h)^{-1} = \sigma(h)x\sigma(h)^{-1},$$

since $x \in H^\sigma$. This means that $\sigma(y) \in \widetilde{[x]}_H$ for any $y \in \widetilde{[x]}_H$. Then $\sigma(\widetilde{[x]}_H) = \widetilde{[x]}_H$. Finally, consider $\sigma' \in \mathcal{cl}^{-1}(a)$, that is $\sigma' = \varphi\sigma\varphi^{-1}$ where $\varphi = \mathrm{int}(\overline{h})$ with $\overline{h} \in H$. Then,

$$\sigma'(\widetilde{[x]}_H) = (\varphi\sigma\varphi^{-1})(\widetilde{[x]}_H) = \overline{h}\sigma(\overline{h}^{-1})\sigma(\widetilde{[x]}_H)\sigma(\overline{h})\overline{h}^{-1} = \sigma(\widetilde{[x]}_H),$$

as we wished. □

1.5 The subgroup $H_\eta^{\mathbb{C}}$.

Let G be a connected real semisimple Lie group with Cartan involution θ. As usual let $\mathfrak{g}$ be its Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let H be the maximal compact subgroup of G whose Lie algebra is $\mathfrak{h}$ and let $\eta \in \mathrm{Aut}_n(G, \theta)$. We consider now the subgroup of H defined by

$$H_\eta := \left\{ h \in H \;:\; \eta(h) = c(h)h, \text{ with } c(h) \in Z(H) \cap \mathrm{Ker}(\iota). \right\} \tag{1.15}$$

Observe that if G is connected and semisimple then the kernel of the adjoint representation of G is $Z(G)$ and $Z(G) \subset Z(H) \subset H$. Since H is also connected, we have

$$Z(H) \cap \mathrm{Ker}(\iota) = \{h \in H \text{ s.t. } x = x_{\mathfrak{h}} + x_{\mathfrak{m}} = \mathrm{Ad}_h(x_{\mathfrak{h}}) + \iota_h(x_{\mathfrak{m}}) \text{ for any } x \in \mathfrak{g}\}.$$

Hence $Z(H) \cap \mathrm{Ker}(\iota) = \mathrm{Ker}((\mathrm{Ad}_G)_{|H}) = Z(G)_{|H} = Z(G)$. It makes sense then to define

$$G_\eta := \left\{ g \in G \;:\; \eta(g) = c(g)g, \text{ with } c(g) \in Z(H) \cap \mathrm{Ker}(\iota) = Z(G). \right\}$$

Similar to Proposition 1.3.3 we have the following

Proposition 1.5.1. *Let $\sigma := \theta\eta$ with $\eta \in \mathrm{Aut}_n(G,\theta)$. Let H_η be the Lie subgroup of H whose lie algebra is $\mathfrak{h}_0$. Then there exists a diffeomorphism*

$$\begin{array}{rcl} H_\eta \times \mathfrak{m}_0 & \to & G_\eta \\ (h,X) & \mapsto & h\exp(X). \end{array}$$

If $n = 2k$ there exists also a diffeomorphism

$$\begin{array}{rcl} H_\eta \times \mathfrak{m}_k & \to & G_\sigma \\ (h,X) & \mapsto & h\exp(X). \end{array}$$

For reasons that will be apparent in the following chapters we need to study the complex situation. Let us denote by $H_\eta^{\mathbb{C}}$ the complexification of (1.15). Since H is compact and connected, $H^{\mathbb{C}} = H\exp(i\mathfrak{h})$ and $\mathrm{Ker}(\mathrm{Ad}_{|_H}) = Z(H)$ then $Z(H)\cap \mathrm{Ker}(\iota) = Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)$ and we have

$$H_\eta^{\mathbb{C}} = \{h \in H^{\mathbb{C}} \;:\; \eta_{\mathbb{C}}(h) = c(h)h, \text{ with } c(h) \in Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)\}, \tag{1.16}$$

where $\eta_{\mathbb{C}}$ is the complexification of $\eta_{|_H}$. From now on we will denote $\eta_{\mathbb{C}}$ also by η and $Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)$ by Z_τ. We have the following.

Proposition 1.5.2. *If G is a connected real form of a semisimple complex Lie group $G^{\mathbb{C}}$ then $Z_\tau := Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota_{\mathbb{C}}^-) = Z(H^{\mathbb{C}})\cap Z(G^{\mathbb{C}})$.*

Proof. Let $h \in Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota_{\mathbb{C}}^-)$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}}\oplus\mathfrak{m}^{\mathbb{C}}$ the Cartan decomposition of the Lie algebra of $G^{\mathbb{C}}$. Then for any $X \in \mathfrak{g}^{\mathbb{C}}$ we write $X = X_1 + X_2$ with $X_1 \in \mathfrak{h}^{\mathbb{C}}$ and $X_2 \in \mathfrak{m}^{\mathbb{C}}$. Recall that $\mathrm{Ker}(\iota_{\mathbb{C}}^{\pm}) = \{h \in H^{\mathbb{C}} : \iota_{\mathbb{C}}^{\pm}(h) = id\}$. Since the kernel of the adjoint representation of any connected Lie group A to its Lie algebra $\mathfrak{a}$ is equal to the center $Z(A)$ of A, we have that $\mathrm{Ad}(h)(X) = \iota^+(h)(X_1)+\iota^-(h)(X_2) = X_1 + X_2 = X$. Meaning that $\mathrm{Ad}(h) = id$ for any $X \in \mathfrak{g}^{\mathbb{C}}$ then $h \in Z(G^{\mathbb{C}})$. □

From now on G will be a connected real form of a semisimple complex Lie group $G^{\mathbb{C}}$ and hence $Z_\tau = Z(G^{\mathbb{C}})\cap Z(H^{\mathbb{C}})$.

Proposition 1.5.3. *The subgroup Z_τ is finite.*

Proof. On the one hand it is clear that if G is not a Lie group of **Hermitian type** then Z_τ is finite since $\mathfrak{z}(\mathfrak{h}) = 0$. On the other hand suppose that G is a lie group of Hermitian type, meaning that $M = G/H$ is a symmetric space of Hermitian type. These groups are characterized by the fact that the center $\mathfrak{z}(\mathfrak{h})$ of the Lie algebra $\mathfrak{h}$ of H is isomorphic to $\mathbb{C}$ (see [37]). Moreover, any symmetric space of hermitian type is simply connected, hence M is a product

$$M = M_1 \times \cdots \times M_r, \tag{1.17}$$

where the factors M_i are irreducible Hermitian symmetric spaces $M_i = G_i/H_i$, with G_i simple. The irreducible Hermitian symmetric spaces of compact (and non-compact) type are classified in Table 1.1 (for further details see [37]).

$G^{\mathbb{C}}$	Compact type	Non-compact type
$\mathrm{SL}(p+q,\mathbb{C})$	$\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$
$\mathrm{SO}(2n,\mathbb{C})$	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$
$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$
$\mathrm{SO}(n+2,\mathbb{C})$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$	$\mathrm{SO}(n,2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$
$E_6^{\mathbb{C}}$	$E_6/(\mathrm{SO}(10)\times\mathrm{SO}(2))$	$E_6/(\mathrm{SO}(10)\times\mathrm{SO}(2))$
$E_7^{\mathbb{C}}$	$E_7/(E_6\times\mathrm{SO}(6))$	$E_7/(E_6\times\mathrm{SO}(6))$

Table 1.1: Irreducible Hermitian symmetric spaces $M = G/H$ of compact and non-compact type.

From Table 1.1 one can verify that for any semisimple Lie group defining an irreducible Hermitian symmetric space, Z_τ is finite since $Z(G^{\mathbb{C}})$ is. Hence from (1.17) we realize that for any connected semisimple real form G of Hermitian type Z_τ is finite. □

Since Z_τ is finite the Lie algebra $\mathfrak{h}_0$ of H^η, is also the Lie algebra of H_η. Hence if we take the decomposition of $\mathfrak{g}$ into the direct sum of ζ_k-eigenspaces $\mathfrak{g}_k$ given by η, as we did in the previous section, it makes sense to consider the restriction of the isotropy representation $\iota_- : H \to \mathrm{GL}(\mathfrak{m})$ to H_η. This gives rise to representations

$$\iota_\eta^k : H_\eta \to \mathrm{GL}(\mathfrak{m}_k).$$

We denote their complexifications

$$\iota_\eta^k : H_\eta^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_k^{\mathbb{C}}) \tag{1.18}$$

in the same fashion. Notice that the automorphism $\eta_{\mathbb{C}}$ descends to an automorphism of $\mathrm{Ad}(H^{\mathbb{C}}) = H^{\mathbb{C}}/Z(H^{\mathbb{C}})$ since it sends $Z(H^{\mathbb{C}})$ in $Z(H^{\mathbb{C}})$. We can make then the following observation.

Proposition 1.5.4. $\mathrm{Ad}(H^{\mathbb{C}})^\eta = \mathrm{Ad}(H^{\mathbb{C}}_\eta) := H^{\mathbb{C}}_\eta/Z(H^{\mathbb{C}}_\eta)$, *where we denote also by* η *the involution defined on* $\mathrm{Ad}(H^{\mathbb{C}})$.

Now, $(H^{\mathbb{C}})^\eta$ is a normal subgroup of $H^{\mathbb{C}}_\eta$ then we have the exact sequence

$$1 \to (H^{\mathbb{C}})^\eta \to H^{\mathbb{C}}_\eta \to \Gamma_\eta := H^{\mathbb{C}}_\eta/(H^{\mathbb{C}})^\eta \to 1. \tag{1.19}$$

The following Proposition generalises to real groups Proposition 2.16 in [31].

Proposition 1.5.5. (a) *The map* $c : H^{\mathbb{C}}_\eta \to Z_\tau$ *defined by the element* $c(h)$ *appearing in (1.15) is a homomorphism. Moreover* $\mathrm{Ker}\, c = (H^{\mathbb{C}})^\eta$ *and hence* c *induces an injective homomorphism* $\tilde{c} : \Gamma_\eta \to Z_\tau$.

(b) *The action of* η *on* $H^{\mathbb{C}}$ *restricts to an action on* $H^{\mathbb{C}}_\eta$*. hence* c *is* η*-equivariant with respect to this action and the natural action of* η *in* Z_τ*. In fact, this action descends to* Γ_η *and hence* $\tilde{c}$ *is* η*-equivariant.*

(c) *For every* $h \in H^{\mathbb{C}}_\eta$*, we have*

$$c(h)\eta(c(h))\cdots\eta^{n-1}((c(h)) = 1,$$

then the image of $\tilde{c}$ *in* Z_τ *is in*

$$Z_a := \{z \in Z_\tau \ : \ za(z)\cdots a^{n-1}(z) = 1\}$$

with $a = \pi(\eta)$*, where* π *is the projection* $\mathrm{Aut}_n(G,\theta) \to \mathrm{Out}_n(G,\theta)$.

Proof. It is easy to see that c is a homomorphism. In fact, for any $g, h \in H^{\mathbb{C}}_\eta$,

$$c(gh) = \eta(gh)(gh)^{-1} = \eta(g)\eta(h)h^{-1}g^{-1} = \eta(g)c(h)g^{-1} = \eta(g)g^{-1}c(h) = c(g)c(h).$$

Moreover, $\mathrm{Ker}\, c = \{h \in H^{\mathbb{C}}_\eta \ : \ c(h) = e\} = (H^{\mathbb{C}})^\eta$, and hence c descends to an injective homomorphism $\tilde{c}$ from Γ_η to Z_τ proving (a).

To prove (b) we note that for each $h \in H^{\mathbb{C}}_\eta$ we have $\eta(\eta(h)) = \eta(c(h))\eta(h)$. Since by definition $\eta(\eta(h)) = c(\eta(h))\eta(h) = c(\eta(h))c(h)h$, we conclude that $c(\eta(h)) = \eta(c(h))$ and hence $\eta(c(h)) \in Z_\tau$. Since $\eta((H^{\mathbb{C}})^\eta) = (H^{\mathbb{C}})^\eta$ then (b) follows.

Now, since

$$h = \eta^n(h) = \eta^{n-1}((c(h))\cdots\eta(c(h))c(h)h,$$

then $\eta^{n-1}((c(h))\cdots\eta(c(h))c(h) = 1$. By straightforward computation we have (c).

$\square$

Proposition 1.5.6. *Let* η *and* η' *be in* $\mathrm{Aut}_n(G,\theta)$ *such that* $\eta \sim_\theta \eta'$ *meaning that* $\eta' = \mathrm{Int}(h)\eta\mathrm{Int}(h)^{-1}$ *for any* $h \in H^{\mathbb{C}}$*.*

(a) The map $x \mapsto \mathrm{Int}(h)x$ *defines an isomorphism* $f_h : H^{\mathbb{C}}_\eta \to H^{\mathbb{C}}_{\eta'}$*, which induces an isomorphism* $\tilde{f}_h : \Gamma_\eta \to \Gamma_{\eta'}$*.*

(b) Let $c : H^{\mathbb{C}}_\eta \to Z_\tau$ *and* $c' : H^{\mathbb{C}}_{\eta'} \to Z_\tau$ *be the homomorphisms corresponding to* η *and* η' *and* $\tilde{c}$ *and* $\tilde{c'}$ *the induced homomorphisms defined in the previous proposition. Then* $c = c'f_h$ *and* $\tilde{c} = \tilde{c'}\tilde{f}_h$*.*

Proof. (a) follows from (b) in Proposition 1.5.5. Notice that for any $x \in H^{\mathbb{C}}$ we define $x' = f_h(x) = \mathrm{Int}(h)x$ and by straightforward computation we get $c'(x')x' = c(\mathrm{Int}(h)^{-1}(x'))x'$ proving (b). $\square$

1.6 Involutions.

Let G be a real semisimple Lie group and θ a fixed Cartan involution on G. Let us consider $\mathrm{Aut}_2(G,\theta)$ the group of all involutions of G that commute with θ. We define $\mathrm{Int}(G,\theta)$ and $\mathrm{Out}_2(G,\theta)$ as Section 1.3. Let $\mathfrak{g}$ be the Lie algebra of G and $\theta \in \mathrm{Aut}_2(\mathfrak{g})$ be its corresponding Cartan involution on $\mathfrak{g}$ defining a Cartan decomposition of $\mathfrak{g}$ in ±1-eigenspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (recall that we are denoting $d\theta$ also by θ).

As we notice at the beginning of Section 1.3 for any $\eta \in \mathrm{Aut}_2(\mathfrak{g})$ there exists an inner automorphism φ of $\mathfrak{g}$ such that $\eta' = \varphi\eta\varphi^{-1}$ commutes with the Cartan involution θ. Then we have the following Corollary to Proposition 1.3.2.

Corollary 1.6.1. *Let* G *be a real semisimple Lie group,* θ *a Cartan involution and* η *another involution on* G*. Then there exists* $\varphi \in \mathrm{Int}(G)$ *such that* $\eta' := \varphi\eta\varphi^{-1}$ *commutes with* θ*.*

Let us consider involutions $\eta \in \mathrm{Aut}_2(G,\theta)$. These involutions give rise to a fine decomposition. Since η commutes with θ, it leaves invariant $\mathfrak{h}$ and $\mathfrak{m}$, we thus can further decompose $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{h}_+ \oplus \mathfrak{h}_-) \oplus (\mathfrak{m}_+ \oplus \mathfrak{m}_-), \tag{1.20}$$

where $\mathfrak{h}_+$ and $\mathfrak{h}_-$ are the (± 1)-eigenspaces of $\mathfrak{h}$ defined by η and $\mathfrak{m}_+$, $\mathfrak{m}_-$ are the (± 1)-eigenspaces of $\mathfrak{m}$ defined by η. Note that $\mathfrak{h}_+$ is a Lie subalgebra of $\mathfrak{g}$, hence we will define H_+ to be the Lie subgroup of H whose Lie algebra is $\mathfrak{h}_+$.

Proposition 1.6.2. *The decomposition (1.20) satisfies the following relations:*

$$[\mathfrak{h}_+, \mathfrak{h}_+] \subset \mathfrak{h}_+, \;\; [\mathfrak{h}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+, \;\; [\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{h}_+,$$

$$[\mathfrak{h}_+, \mathfrak{m}_-] \subset \mathfrak{m}_-, \;\; [\mathfrak{m}_-, \mathfrak{m}_-] \subset \mathfrak{h}_+.$$

Proof. Straightforward computation from the fact that the Lie brackets and the involutions η and θ commute. □

We have the following special case of Proposition 1.3.3.

Proposition 1.6.3. *Let $\sigma := \theta\eta$. Then there exist two diffeomorphisms*

$$\begin{array}{rcl} H_+ \times \mathfrak{m}_+ & \to & G^\eta \\ (h\,, X_{\mathfrak{m}_+}) & \mapsto & h \exp X_{\mathfrak{m}_+} \end{array}$$

and

$$\begin{array}{rcl} H_+ \times \mathfrak{m}_- & \to & G^\eta \\ (h\,, X_{\mathfrak{m}_-}) & \mapsto & h \exp X_{\mathfrak{m}_-} \end{array}$$

Now let us define $\sim$ and $\sim_\theta$ as in Section 1.3. By Corollary 1.6.1 we have a biyection

$$\mathrm{Aut}_2(G)/\sim \;\longleftrightarrow\; \mathrm{Aut}_2(G,\theta)/\sim_\theta\;. \tag{1.21}$$

As a particular case of Propositions 1.3.5 and 1.3.6 we have the following result. In this Proposition we extend to real groups the results obtained by O. García-Prada and S. Ramanan for complex Lie groups in [31] and are rewrite here for completness.

Proposition 1.6.4. *Let G be a real connected semisimple Lie group with a fixed Cartan involution θ. Let $\eta \in \mathrm{Aut}_2(G,\theta)$ then*

(1) *The projection* $\pi : \mathrm{Aut}_2(G,\theta) \to Out_2(G,\theta)$ *descends to define a surjective map*

$$\mathcal{cl} : \mathrm{Aut}_2(G,\theta)/\sim_\theta \longrightarrow \mathrm{Out}_2(G,\theta).$$

(2) *Let* $\eta \in \mathrm{Aut}_2(G,\theta)$. *Consider the set*

$$S_\eta^\theta := \{s \in G^\theta \;:\; s\eta(s) \in Z(G)\}.$$

Then the map $\psi : S_\eta^\theta \to \pi^{-1}(a)$ *defined by* $s \to \mathrm{Int}(s)\eta$ *gives a bijection*

$$S_\eta^\theta/(Z(G)\times H) \longleftrightarrow \mathcal{cl}^{-1}(a),$$

where $\mathcal{cl} : \mathrm{Aut}_2(G,\theta)/\sim_\theta \to \mathrm{Out}_2(G,\theta)$ *is the map induced by* π *and the action of* H *and* $Z(G)$ *on* S_η^θ *is defined in Proposition 1.3.5. In particular, there is a bijection*

$$S_{id}^\theta/(Z(G)\times H) \longleftrightarrow \mathrm{Int}_2(G,\theta)/\sim_\theta .$$

(3) *Let* $H_\eta^1(\mathbb{Z}/2, H/Z(G))$ *the cohomology set defined by the action of* $\mathbb{Z}/2$ *in* $H/Z(G)$ *given by* $\eta \in \mathrm{Aut}_2(G,\theta)$. *Then, there is a bijection*

$$H_\eta^1(\mathbb{Z}/2, H/Z(G)) \longleftrightarrow S_\eta^\theta/(Z(G)\times H).$$

It is clear that if η, $\eta' \in Aut_2(G,\theta)$ *are such that* $\pi(\eta) = \pi(\eta') = a$ *then this cohomology set could be denoted by* $H_a^1(\mathbb{Z}/2, H/Z(G))$.

(4) *There is a bijection*

$$\mathcal{cl}^{-1}(a) \longleftrightarrow H_a^1(\mathbb{Z}/2, H/Z(G))$$

and hence a bijection

$$\mathrm{Aut}_2(G,\theta)/\sim_\theta \longleftrightarrow \bigcup_{a\in \mathrm{Out}_2(G,\theta)} H_a^1(\mathbb{Z}/2, H/Z(G)).$$

We now particularise to $n = 2$ the set $S_\eta^{\tau,n}$ defined in (1.13):

$$S_\eta^\tau := \{s \in H^{\mathbb{C}} \;:\; s\eta(s) \in Z_\tau\}. \tag{1.22}$$

Then we have the following.

Lemma 1.6.5. *There is a bijection*

$$S_\eta^\tau/(Z_\tau \times H^{\mathbb{C}}) \longleftrightarrow H_\eta^1(\mathbb{Z}/2, H^{\mathbb{C}}/Z_\tau),$$

where Z_τ acts on S_η^τ by multiplication and the action of $H^{\mathbb{C}}$ on S_η^τ is defined in Proposition 1.3.8.

Proof. It follows from Lemma 1.3.9 for $n = 2$. □

Finally, we have an equivalence result to Proposition 1.5.5 in Section 1.5 for this particular case.

Proposition 1.6.6. (a) *The map $c : H_\eta^{\mathbb{C}} \to Z_\tau$ defined by the element $c(h)$ appearing in the Definition (1.15) is a homomorphism. Moreover we have that $\operatorname{Ker} c = (H^{\mathbb{C}})^\eta$ and hence c induces an injective homomorphism $\tilde{c} : \Gamma_\eta \to Z_\tau$.*

(b) *The action of η on $H^{\mathbb{C}}$ restricts to an action on $H_\eta^{\mathbb{C}}$. hence c is η-equivariant with respect to this action and the natural action of η in Z_τ. In fact, this action descends to Γ_η and hence $\tilde{c}$ is η-equivariant.*

(c) *$\eta(c(h)) = c(h^{-1})$, then the image of $\tilde{c}$ in Z_τ is in*

$$Z_a := \{z \in Z_\tau \ : \ a(z) = z^{-1}\}$$

and contains

$$\{a(z)z^{-1} \ : \ z \in Z_\tau\}$$

with $a = \pi(\eta)$, where π is the projection $\operatorname{Aut}_2(G,\theta) \to \operatorname{Out}_2(G,\theta)$.

1.7 Complex Lie groups regarded as real forms

In this thesis we generalise the results obtained in [31]. In order to do that realise that a complex Lie group can be seen as a real form of another complex Lie group. Let G be a complex semisimple Lie group. If we consider the antiholomorphic involution

$$\begin{array}{rccc} \sigma : & (G,J)\times(G,-J) & \to & (G,J)\times(G,-J) \\ & (g_1,g_2) & \mapsto & (g_2,g_1), \end{array}$$

where J and $-J$ are complex structures on G, then $G = (G\times G)^\sigma$, meaning that G is a real form of the complex semisimple Lie group $G \times G$.

In this situation we can make the following observations:

1. Let τ be the compact conjugation for G and let $\mathfrak{g}$ be its Lie algebra. As usual the compact conjugation induces a decomposition of the Lie algebra in $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of the maximal compact subgroup $H = G^{\tau}$ of G. In this particular case $\tau := \sigma \circ (\tau \times \tau)$, where $(\tau \times \tau)$ is a compact conjugation defined on $G \times G$.
2. $Z_{\tau} = Z(G)$.
3. $\mathrm{Int}_n(G, \tau) = \mathrm{Int}_n(H)$ and $\mathrm{Int}_n(H^{\mathbb{C}}) = \mathrm{Int}_n(G)$.

Chapter 2

Review on Higgs bundles.

The content discussed in this chapter has been extensively developed over the past years by many authors in some different contexts. Higgs bundle for real groups were first considered by Hitchin [38, 40] and further studied by Bradlow, García-Prada, Gothen and Mundet i Riera [17, 18, 20, 26, 27] and others. Despite this, we give a brief review to the theory of G-Higgs bundles for G a real reductive Lie group in order to fix notations and recall some important concepts related with the topic of this thesis. We will mainly follow the approach given in [27, 49].

2.1 G-Higgs-bundles for real Lie groups.

Let G be a reductive Lie group and let (H, θ, B) be its Cartan data (see Remark 1.2.4). As usual, let $\mathfrak{h}$ be the Lie algebra of H and let $\mathfrak{g}$ be the Lie algebra of G with θ-eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

This decomposition complexifies to

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}},$$

where $\mathfrak{h}^{\mathbb{C}}$ is the Lie algebra of $H^{\mathbb{C}}$. Recall that the group H acts linearly on $\mathfrak{m}$ through the Adjoint representation. We extend this action to a linear holomorphic action

$$\iota^{-} : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$$

of $H^{\mathbb{C}}$ on $\mathfrak{m}^{\mathbb{C}}$ known also as the isotropy representation.

Definition 2.1.1. A (G,θ)**-Higgs bundle over a compact Riemann surface** X is a pair (E,φ) consisting of a principal holomorphic $H^{\mathbb{C}}$-bundle E over X and a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}})\otimes K$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to E via the isotropy representation ι^- and $K=T^*X$ is the **holomorphic cotangent bundle** or **canonical bundle** of X.

Two (G,θ)-Higgs bundles (E,φ) and (F,ψ) are isomorphic if there is an isomorphism $f:E\to F$ such that the induced isomorphism $f_{\iota^-}:E(\mathfrak{m}^{\mathbb{C}})\to F(\mathfrak{m}^{\mathbb{C}})$ sends φ to ψ.

Remark 2.1.2. In the literature it is common to find (G,θ)-Higgs bundles referred simply as G-Higgs bundles. We choose this more precise term to emphasise the role of θ since we are interested in fixing the structure group of the holomorphic bundle E. Observe also that when G is compact $\mathfrak{m}=0$ and hence a (G,θ)-Higgs bundle is simply a holomorphic principal $G^{\mathbb{C}}$-bundle.

Remark 2.1.3. At the end of Chapter 1 we noticed that a complex reductive Lie group G can be seen as a real form of the complex semisimple lie group $(G,J)\times(G,-J)$, for any complex structure J, with respect to the antiholomorphic involution

$$\sigma:(g_1,g_2)\mapsto(g_2,g_1).$$

Hence let us consider G a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and let X be a compact Riemann surface. Let

$$\mathrm{Ad}:G\to \mathrm{GL}(\mathfrak{g}),$$

be the adjoint representation of G in $\mathfrak{g}$. If we fix τ a compact conjugation of G then $\theta:=\sigma\circ(\tau\times\tau)=\tau$. Let H be the corresponding maximal compact subgroup of G such that $G^\tau=H$. Its complexification $H^{\mathbb{C}}$ is G since G is a complex semisimple Lie group and H is its compact real form. Hence, in this situation, a (G,θ)-Higgs bundle is just a pair (E,φ) where E is a holomorphic G-bundle and $\varphi\in H^0(X,E(\mathfrak{g})\otimes K)$ since $\mathfrak{g}\oplus\mathfrak{g}=\mathfrak{g}^{\mathbb{C}}=\mathfrak{h}^{\mathbb{C}}\oplus\mathfrak{m}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}=\mathfrak{g}$. This means that such a pair (E,φ) is a G**-Higgs bundle** in the classical sense for G a complex semisimple Lie group. These objects were introduced in 1987 by Hitchin [38] for $G=\mathrm{SL}(n,\mathbb{C})$.

For reasons that will become apparent throughout the following sections we need a generalisation of (G,θ)-Higgs bundles. Let

$$\rho:H^{\mathbb{C}}\to\mathrm{GL}(\mathbb{V})$$

be a complex representation of a complex redutive Lie group $H^{\mathbb{C}}$ on a hermitian vector space $\mathbb{V}$. This defines a holomorphic action of $H^{\mathbb{C}}$ on $\mathbb{V}$.

Definition 2.1.4. A **K-twisted Higgs pair of type** ρ is a pair (E, φ), where E is a holomorphic principal $H^{\mathbb{C}}$-bundle over a compact riemann surface X and φ is a holomorphic section of $E(\mathbb{V}) \otimes K$, where $E(\mathbb{V})$ is the vector bundle associated to E via the representation ρ and K is the canonical bundle of X. Throughout this thesis we will refer to these objects simply as Higgs pairs. Notice that (G, θ)-Higgs bundles are K-twisted Higgs pair of type ρ when ρ is the isotropy representation.

Remark 2.1.5. There even exists a more general notion of that of K-twisted Higgs pair, where a general line bundle L play the role of the canonical bundle K in Definition 2.1.4. Such a pair is known as a **L-twisted Higgs pair**. We refer the reader to [27] for a complete understading of these concepts.

2.2 Parabolic subgroups and stability.

Let G be a real reductive Lie group with Cartan data (H, θ, B). In order to define the moduli space of (G, θ)-Higgs bundles we need the notions of (semi, poly)stability. Before defining stability we recall some basic facts about parabolic subgroups.

Definition 2.2.1. Given $s \in i\mathfrak{h}$ we define:

$$P_s = \{h \in H^{\mathbb{C}} \,:\, e^{ts}he^{-ts} \text{ is bounded as } t \to \infty\}$$

$$\mathfrak{p}_s = \{x \in \mathfrak{h}^{\mathbb{C}} \,:\, \mathrm{Ad}(e^{ts})(x) \text{ is bounded as } t \to \infty\}$$

$$L_s = \{h \in H^{\mathbb{C}} \,:\, \lim_{t\to\infty} e^{ts}he^{-ts} = h\}$$

$$\mathfrak{l}_s = \{x \in \mathfrak{h}^{\mathbb{C}} \,:\, \lim_{t\to\infty} \mathrm{Ad}(e^{ts})(x) = 0\}.$$

Notice that $\mathfrak{p}_s$ and $\mathfrak{l}_s$ are Lie subalgebras of $\mathfrak{h}^{\mathbb{C}}$ and P_s and L_s are connected Lie subgroups of $H^{\mathbb{C}}$. We call P_s and $\mathfrak{p}_s$ (respectively L_s and $\mathfrak{l}_s$) the **parabolic** (respectively **Levi) subgroup and subalgebra** associated to s.

In the same fashion we define:

$$\mathfrak{m}_s = \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \lim_{t\to 0} \iota^-(e^{ts})x \text{ exists}\},$$
$$\mathfrak{m}_s^0 = \{x \in \mathfrak{m}^{\mathbb{C}} \ : \ \lim_{t\to \infty} \iota^-(e^{ts})x = 0\}.$$

Recall that a **character** of a complex Lie algebra $\mathfrak{h}^{\mathbb{C}}$ is a complex linear map $\mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ which factors through $\mathfrak{h}^{\mathbb{C}}/[\mathfrak{h}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}]$. Let $\mathfrak{z}$ be the centre of $\mathfrak{h}$ and let us denote by $\mathfrak{z}^{\mathbb{C}}$ the centre of $\mathfrak{h}^{\mathbb{C}}$. For a parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{h}^{\mathbb{C}}$ let $\mathfrak{l}$ be a corresponding Levi subalgebra with centre $\mathfrak{z}_{\mathfrak{l}}$. Hence one has that $(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^* \cong \mathfrak{z}_{\mathfrak{l}}^*$, then a character χ of $\mathfrak{p}$ comes from an element of $\mathfrak{z}_{\mathfrak{l}}^*$. Any $s \in \mathfrak{z}_{\mathfrak{l}}$ yields a character χ_s of $\mathfrak{p}_s$ since $B(s, [\mathfrak{p}_s, \mathfrak{p}_s]) = 0$. Conversely, if we use the Killing form, from $\chi \in \mathfrak{z}_{\mathfrak{l}}^*$ we get an element $s_\chi \in \mathfrak{z}_{\mathfrak{l}}$. The character χ is said to be **antidominant** if $\mathfrak{p} \subset \mathfrak{p}_{s_\chi}$ and **strictly antidominant** if $\mathfrak{p} = \mathfrak{p}_{s_\chi}$. Note that any $s \in i\mathfrak{h}$ defines clearly a strictly antidominant character $\mathfrak{p}_s$. Let P be a parabolic subgroup. Given a character $\chi : P \to \mathbb{C}^*$ of P, denote by χ_* the corresponding character of $\mathfrak{p}$. We say that χ is (strictly) antidominant if χ_* is.

Let $\chi : P \to \mathbb{C}^*$ be an antidominant character of P and let $\sigma \in H^0(X, E/P)$ a reduction of the structure group of E to P. Denote by E_σ the corresponding principal bundle.

Definition 2.2.2. The **degree**, $\deg(E)(\sigma, \chi)$, of E with respect to σ and χ, is the degree of the line bundle obtained by extending the structure group of E_σ through χ, i.e.

$$\deg(E)(\sigma, \chi) = \deg(E_\sigma \times_\chi \mathbb{C}^*). \tag{2.1}$$

Remark 2.2.3. Given an element $s \in i\mathfrak{h}$ and a reduction of the structure group of E to P_s we can also define $\deg(E)(\sigma, s)$, even though χ_s may not lift to a character of P_s.

To do this define

$$H_s := H \cap L_s \quad \text{and} \quad \mathfrak{h}_s := \mathfrak{h} \cap \mathfrak{l}_s.$$

Then H_s is a maximal compact subgroup of L_s. Since the inclusions $H_s \subset L_s$ and $L_s \subset P_s$ are homotopy equivalences, given a reduction σ of the structure group of E to P_s one can further restrict the structure group to H_s in a unique way up to homotopy.

Definition 2.2.4. Let E be a $H^{\mathbb{C}}$ principal bundle and let E'_σ be the resulting reduced principal H_s-bundle. Consider a connection A on E'_σ and let F_A be its curvature. Then $\chi_s(F_A)$ is a 2-form on X with values in $i\mathbb{R}$ and

$$\deg(E)(\sigma, s) := \frac{i}{2\pi}\int_X \chi_s(F_A). \tag{2.2}$$

Remark 2.2.5. If χ_s can be lifted to a character $\widetilde{\chi}_s$ of P_s then (2.2) coincides with $\deg(E)(\sigma, \widetilde{\chi}_s)$ defined in (2.1).

We can now define the stability of a (G,θ)-Higgs bundle.

Definition 2.2.6. Let $\alpha \in i\mathfrak{z}(\mathfrak{h})$. A (G,θ)-Higgs bundle (E,φ) over X is:

α-semistable if $\deg(E)(\sigma, s) - B(\alpha, s) \geq 0$, for any $s \in i\mathfrak{h}$ and any reduction of structure group σ of E to P_s such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$.

α-stable if $\deg(E)(\sigma, s) - B(\alpha, s) > 0$, for any $s \in i\mathfrak{h} \setminus \mathrm{Ker}(d\iota)$ and any reduction of structure group σ of E to P_s such that $\varphi \in H^0(X, E_\sigma(\mathfrak{m}_s) \otimes K)$.

α-polystable if it is α-semistable and anytime $\deg(E)(\sigma, s) - B(\alpha, s) = 0$ for any $s \in i\mathfrak{h}$ and any reduction of structure group σ of E to P_s, there exists a reduction σ_{L_s} of E_σ to the corresponding Levi subgroup L_s of P_s such that φ takes values in $H^0(X, E_{\sigma_{L_s} f}(\mathfrak{m}_s^0) \otimes K)$.

We can then define $\mathcal{M}^\alpha(G,\theta)$ as the **moduli space of isomorphism classes of α-polystable (G,θ)-Higgs bundle**. Let us now fix $d = c(E) \in \pi_1(H^{\mathbb{C}})$ a certain characteristic class of E. This class give the topological classification of $H^{\mathbb{C}}$-bundles E over X. Hence we obtain a subset $\mathcal{M}_d^\alpha(G,\theta) \subset \mathcal{M}^\alpha(G,\theta)$ of isomorphism classes of α-polystable (G,θ)-Higgs bundles with $c(E) = d$.

Remark 2.2.7. If $\mathfrak{z}(\mathfrak{h}) = 0$ then Definition 2.2.6 becomes easier. We just replace the left hand of the inequalities by

$$\deg(E)(\sigma, s)) \geq (> \text{ or } =)\, 0.$$

This is not necessarily the case even if G is semisimple since if the symmetric space $M = G/H$ is of hermitian type then $\mathfrak{z}(\mathfrak{h}) \neq 0$ (see Proposition 1.2.42 in [49]). However, when we relate (G,θ)-Higgs bundles to representations of the universal central extension of the fundamental group of X in G we ask α to lie also in the center, $\mathfrak{z}(\mathfrak{g})$, of $\mathfrak{g}$. Since in our analysis we will be concerned with (G,θ)-Higgs bundles for G semisimple, then $\mathfrak{z}(\mathfrak{g})=0$ and hence $\alpha=0$. In this case we relate (G,θ)-Higgs bundles to representations of the fundamental group of X in G. We are interested in studying these relations and hence we will take α to be zero. In this case, we refer to α-(semi,poly)stability as 0-**(semi,poly)stability** or just **(semi,poly)stability**. We can then define $\mathcal{M}(G,\theta)$ the **moduli space of polystable (G,θ)-Higgs bundle**. Then when we talk about stability we will be referring to Definition 2.2.6 for $\alpha=0$.

The notions of stability extend also to Higgs pairs.

Remark 2.2.8. Let $H^{\mathbb{C}}$ be a complex reductive Lie group with lie algebra $\mathfrak{h}^{\mathbb{C}}$ and let $\mathbb{V}$ be a hermitian vector space acted on by $H^{\mathbb{C}}$ via a representation ρ. Consider (E,φ) a Higgs pair of type ρ over X and let $\alpha\in i\mathfrak{z}(\mathfrak{h}^{\mathbb{C}})_{|\mathbb{R}}\subset\mathfrak{z}(\mathfrak{h}^{\mathbb{C}})$ and $s\in i\mathfrak{h}$. Let us define

$$\mathbb{V}_s^- := \{v\in\mathbb{V}\text{ s.t. } \rho(e^{ts})v \text{ is bounded as } t\to\infty\}$$

and

$$\mathbb{V}_s^0 := \{v\in\mathbb{V}\text{ s.t. } \rho(e^{ts})v=v \text{ for any } t\in\mathbb{R}\}.$$

They are complex vector subspaces invariant under the action of a parabolic (respect. Levi) subgroup of $H^{\mathbb{C}}$. Hence, for any holomorphic sections σ and σ_L of $E(H^{\mathbb{C}}/P)$ and $E_\sigma(P/L)$ respectively, we define $E(\mathbb{V})^-_{\sigma,\chi}$ and $E(\mathbb{V})^-_{\sigma_L,\chi}$ to be the vector bundles associated to E and E_σ via the action of the Parabolic (respectively Levi) subgroup on $\mathbb{V}_s^-$ and $\mathbb{V}_s^0$. Hence the notions of (semi,poly)stability for Higgs pairs are the same as those of Definition 2.2.6 but replacing $\mathfrak{m}_s$ and $\mathfrak{m}_s^0$ by $\mathbb{V}_s$ and $\mathbb{V}_s^0$ respectively.

Remark 2.2.9. We hence define the **moduli space of isomorphism classes of α-polystable Higgs pairs** and the same observations made for (G,θ)-Higgs bundles apply here.

Finally, let (E,φ) be a (G,θ)-Higgs bundle notice that

$$Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)\subset\mathrm{Aut}(E,\varphi),$$

since $Z(H^{\mathbb{C}})\subset\mathrm{Aut}(E)$ and the induced action of $Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)$ in $E(\mathfrak{m}^{\mathbb{C}})$ preserves the Higgs field and hence $Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-)$ is a central subgroup of $\mathrm{Aut}(E,\varphi)$. We have the following.

Proposition 2.2.10. *A (G,θ)-Higgs bundle (E,φ) is said to be **simple** if*

$$\mathrm{Aut}(E,\varphi)=Z(H^{\mathbb{C}})\cap\mathrm{Ker}(\iota^-).$$

From now on, unless otherwise indicated, G will be a connected semisimple real Lie group with a fixed Cartan involution θ.

Remark 2.2.11. Recall that if G is a complex semisimple Lie group a G-Higgs bundle (E,φ) is said to be simple if $\mathrm{Aut}(E,\varphi)=Z(G)$. If, in addition, (E,φ) is stable then it is a smooth point in the moduli space. This is not necessarily the case when G is a real semisimple Lie group. In this situation the smoothness of (E,φ) is given by the vanishing of an obstruction class in $\mathbb{H}^2(C^\bullet(E,\varphi))$, where $\mathbb{H}^2(C^\bullet(E,\varphi))$ is the second hypercohomology group of the deformation complex of (E,φ) (see [14] for more details). In particular, $\mathbb{H}^2(C^\bullet(E,\varphi))=0$ is a sufficient condition for the smoothness of (E,φ). The vanishing of an obstruction class in $\mathbb{H}^2(C^\bullet(E,\varphi))$ is also needed in the complex situation, but in that case

$$\mathbb{H}^0(C^\bullet(E,\varphi))=0$$

and by Serre duality for hypercohomology

$$\mathbb{H}^2(C^\bullet(E,\varphi))\cong\mathbb{H}^0(C^\bullet(E,\varphi))^*=0.$$

2.3 The Hitchin equations and the Hitchin-Kobayashi correspondence.

Let G be a reductive (not necessarily connected, see [29]) Lie group with Cartan data (H,θ,B). Let us consider $H^{\mathbb{C}}$ the complexification of the maximal compact subgroup H of G fixed by θ and let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ be the complexifications of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Recall that $\mathfrak{g}^{\mathbb{C}}=\mathfrak{h}^{\mathbb{C}}\oplus\mathfrak{m}^{\mathbb{C}}$ is the complex version of the Cartan decomposition (1.2) and it is given by the $\mathbb{C}$-linear Cartan involution θ. Let us denote by $\widetilde{\tau}$ the compact conjugation of $\mathfrak{g}^{\mathbb{C}}$. Observe that the $+1$-eigenspace of $\widetilde{\tau}$ is a maximal compact subalgebra of $\mathfrak{g}^{\mathbb{C}}$ whose intersections with $\mathfrak{h}^{\mathbb{C}}$ is $\mathfrak{h}$. Let E be a principal $H^{\mathbb{C}}$-bundle over a compact riemann surface X, we extend $\widetilde{\tau}$ to

$$\tau:\Omega^1(X,E(\mathfrak{m}^{\mathbb{C}}))\to\Omega^1(X,E(\mathfrak{m}^{\mathbb{C}}))$$

by combining complex conjugation on 1-form and $\widetilde{\tau}$. It locally looks like

$$\tau(\omega\otimes\varphi):=\overline{\omega}\otimes\tau(\varphi),$$

for any complex 1-form ω and any section φ of $E(\mathfrak{m}^{\mathbb{C}})$. In the same way we extend $\widetilde{\mu}$. Let us denote by J the complex structure defined on the tangent bundle TX of X. There is an isomorphism

$$\begin{array}{rcl} \Omega^1(E(\mathfrak{m})) & \to & \Omega^{1,0}(X, E(\mathfrak{m}^{\mathbb{C}})) \\ \varphi & \mapsto & \dfrac{\varphi - iJ\varphi}{2}, \end{array} \tag{2.3}$$

and the inverse given by

$$\psi = \varphi - \tau(\varphi).$$

Let $H \subset G$ be a maximal compact subgroup of G. Recall that a **reduction of structure group** of a principal G-bundle $E \to X$ to H is a global section of the bundle

$$E/H \simeq E \times_G (G/H).$$

On the other hand, H is also a maximal compact subgroup of $H^{\mathbb{C}}$. Recall that a **metric** in a principal $H^{\mathbb{C}}$-bundle E over X is a reduction of structure group to $H \subset H^{\mathbb{C}}$. We say that a metric h is **harmonic** if it is a critical point of the energy functional.

Given a principal $H^{\mathbb{C}}$-bundle E and a reduction of structure group $E_H \subset E$, there is a unique connection A (known as Chern connection) on E_H compatible with the metric reduction and the holomorphic structure on E. This means that the $(0,1)$-part of A induces the holomorphic structure on E.

Let (E, φ) be a (G, θ)-Higgs bundle over a compact Riemann surface X. We will denote by $(\mathbb{E}, \varphi)$ the C^{∞}-objects underlying the Higgs pair. Observe that the Higgs field is a $(1,0)$-form, $\varphi \in \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}}))$. Consider a reduction h of the structure group of the smooth $H^{\mathbb{C}}$-bundle E to H and consider

$$\tau_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$$

the isomorphism induced by τ.

Theorem 2.3.1. *Let $\alpha \in \mathfrak{z}(\mathfrak{h})$. A (G, θ)-Higgs bundle (E, φ) is α-polystable if and only if there exists a reduction h of the structure group of E to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = -i\alpha\omega, \tag{2.4}$$

where F_h is the curvature of the Chern connection on E corresponding to h and ω is a volume form of X.

Remark 2.3.2. We can generalise Theorem 2.3.1 to Higgs pairs (an in general to L-twisted pairs) in a natural way just by replacing τ_h by the convenient map defined by

$$\rho^* : E_h(\mathfrak{u}(B))^* \to E_h(\mathfrak{h})^*,$$

the map induced from the action of $\mathfrak{h}$ on B (See [27] for further details).

Remark 2.3.3. As we noticed in Remark 2.2.7, in order to relate Higgs bundles with representation of the fundamental group of X in G later on we will require α to be zero. Hence we rewrite Theorem 2.3.1 in an easier way.

Theorem 2.3.4. *A (G,θ)-Higgs bundle (E,φ) is 0-polystable (or just polystable) if and only if there exists a reduction h of the structure group of E to H satisfying the Hitchin equation*

$$F_h - [\varphi, \tau_h(\varphi)] = 0. \tag{2.5}$$

Theorem 2.3.4 was first proved by Hitchin [38] for $G = SL(2,\mathbb{C})$ and generalise by Simpson [55, 57] for any semisimple complex Lie group. The proof of Theorem 2.3.1 for an arbitrary real reductive Lie group when (E,φ) is stable was accomplished in [28] and the general polystable situation is a particular case of Theorem 2.24 in [27].

From now on we will assume that $\alpha = 0$. In terms of connections, let us fix $\mathbb{E}$ the C^∞-principal $H^{\mathbb{C}}$-bundle with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. Consider pairs (A,φ) consisting on a H-connection A and $\varphi \in \Omega^{1,0}(X, \mathbb{E}_H(\mathfrak{m}^{\mathbb{C}}))$ such that φ is holomorphic for the corresponding holomorphic structure $\overline{\partial}_A$ defined by A. Consider $\mathcal{H}$ the gauge group of $\mathbb{E}_H$, this group acts on the space of solutions to **Hitchin's equations** (2.6) for such a pair (A,φ):

$$\begin{array}{l} F_A - [\varphi, \tau_h(\varphi)] = 0 \\ \overline{\partial}_A \varphi = 0, \end{array} \tag{2.6}$$

where the second equation just means that φ is holomorphic with respect to the Dolbeaut operator as we noticed above. The moduli space of solutions to Hitchin's equation (A,φ) on $\mathbb{E}_H$ is

$$\mathcal{M}_d^{gauge}(G,\theta) := \{(A,\varphi) \text{ verifying } (2.6)\}/\mathcal{H}$$

Now, let us consider $\mathcal{M}_d(G,\theta)$ the moduli space of polystable (G,θ)-Higgs bundle of topological type d. This means that the objects of $\mathcal{M}_d(G,\theta)$ are the (G,θ)-Higgs bundles whose underlying C^∞-bundle $\mathbb{E}$ is the bundle obatined from $\mathbb{E}_H$ by extension of the structure group to $H^{\mathbb{C}}$. The following is a consequence of Theorem 2.3.4.

Theorem 2.3.5. *There is a homeomorphism*

$$\mathcal{M}_d(G,\theta) \cong \mathcal{M}_d^{gauge}(G,\theta).$$

The best way to understand this homeomorphism is to think about the moduli space $\mathcal{M}_d(G,\theta)$ as pairs $(\overline{\partial}_E,\varphi)$, where $\overline{\partial}_E$ is a $\overline{\partial}$-operator defining a holomorphic structure on the C^∞ principal $H^{\mathbb{C}}$-bundle $\mathbb{E}$ obtained from $\mathbb{E}_H$ by extension of the structure group $H \hookrightarrow H^{\mathbb{C}}$, and $\varphi \in \Omega^{1,0}(X,\mathbb{E}(\mathfrak{m}^{\mathbb{C}}))$ such that $\overline{\partial}_E\varphi = 0$. These pairs are in one-to-one correspondence with (G,θ)-Higgs bundles (E,φ) since E is the holomorphic $H^{\mathbb{C}}$-bundle defined by the pair $(\mathbb{E},\overline{\partial}_E)$ and $\overline{\partial}_E\varphi = 0$ is equivalent to $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. According to this reinterpretation, the moduli space $\mathcal{M}_d(G,\theta)$ of polystable (G,θ)-Higgs bundle (E,φ) can be identified with the orbit space

$$\{(\overline{\partial}_E,\varphi) \text{ such that } (\overline{\partial}_E,\varphi) \text{ defines a polystable } (G,\theta)\text{-Higgs bundle}\}/\mathcal{H}^{\mathbb{C}},$$

where $\mathcal{H}^{\mathbb{C}}$, the complexification of $\mathcal{H}$, is the gauge group of $\mathbb{E}$. Since there is a natural bijection between Chern connections on $\mathbb{E}_H$ and holomorphic structures on $\mathbb{E}$, obtained by indentifying the Chern-connection with its $(0,1)$-part, the correspondence given in Theorem 2.3.5 can be translate in the following way:

In the $\mathcal{H}^{\mathbb{C}}$-orbit of a polystable (G,θ)-Higgs bundle $(\overline{\partial}_{E'},\varphi')$ we can find another (G,θ)-Higgs bundle $(\overline{\partial}_E,\varphi)$ whose corresponding pair (d_A,φ) is unique up to $\mathcal{H}$-gauge transformations, and satisfies the Hitchin equation (2.5).

2.4 The moduli space of representations

Let G be a real semisimple Lie group with Cartan involution θ. An important aspect of (G,θ)-Higgs bundles is its relation with representation of the fundamental group of X in G. In this section we will explain this relation.

Let X be a closed oriented surface of genus g. By a **representation** of the fundamental group of X to G we mean a homomorphism $\rho : \pi_1(X) \to G$. Let us denote by $\mathrm{Hom}(\pi_1(X),G)$ the set of all these homomorphisms. It is a real analytic variety, which is algebraic if G is algebraic.

Definition 2.4.1. Let $\rho : \pi_1(X) \to G$ be a representation. We will say that ρ is a **reductive representation** if ρ composed with the adjoint representation of G in the Lie algebra of $\mathfrak{g}$ decomposes as a sum of irreducible representations. If G is algebraic this condition is equivalent to the Zariski closure of the image of $\pi_1(X)$ in G being reductive group. We will denote by $\mathrm{Hom}^+(\pi_1(X),G)$ the subset of all reductive representations.

Remark 2.4.2. If G is compact every representation is reductive.

Let $g \in G$ and $\rho \in \mathrm{Hom}(\pi_1(X), G)$, the group G acts on $\mathrm{Hom}(\pi_1(X), G)$ by the relation

$$(g \cdot \rho)(\gamma) := g\rho(\gamma)g^{-1},$$

for $\gamma \in \pi_1(X)$. If we restricts to $\mathrm{Hom}^+(\pi_1(X), G)$ the orbit space is Haussdorf. Hence we have the following definition.

Definition 2.4.3. The **moduli space of representations** of $\pi_1(X)$ in G to be the orbit space

$$\mathcal{R}(G, \theta) := \mathrm{Hom}^+(\pi_1(X), G)/G.$$

It has a structure of a real analytic variety (see for example [32]) which is algebraic if G is algebraic and is a complex variety if G is complex.

Let $\mathrm{Hom}(\pi_1(X), G)/G$ be the set of equivalence classes of representations. We can associate to each representation $\rho \in \mathrm{Hom}(\pi_1(X), G)$ a flat (G, θ)-bundle

$$E_\rho := E \times_\rho \widetilde{X},$$

where $\widetilde{X}$ is the universal cover of X and $\pi_1(X)$ acts on G via ρ. This relation gives an identification between $\mathrm{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat (G, θ)-bundles, which is parametrized by the (nonabelian) cohomology set $H^1(X, G)$. Hence the characteristic class of E_ρ give us a way to define a topological invariant $c(\rho) := c(E_\rho) \in \pi_1(G) = \pi_1(H)$ to a representation ρ.

For a fixed $d \in \pi_1(G)$, we define

$$\mathcal{R}_d(G, \theta) := \{[\rho]_G \in \mathcal{R}(G, \theta) \text{ s.t. } c(\rho) := c(E_\rho) = d\},$$

the moduli space of reductive representation with topological invariant d. We give now the following non-abelian Hodge theorem for representations of the fundamental group of a closed Riemann surface in a semisimple connected real Lie group G.

Theorem 2.4.4. *Let G be a semisimple connected real lie group . There exists a homeomorphism*

$$\mathcal{R}_d(G, \theta) \cong \mathcal{M}_d(G, \theta).$$

Under this homeomorphism the irreducible representations of $\pi_1(X)$ in G are in correspondence with the stable and simple (G, θ)-Higgs bundles.

Remark 2.4.5. The result is also true for G a real form of a complex reductive but not semisimple Lie group $G^{\mathbb{C}}$ but if we require the topological class of the $G^{\mathbb{C}}$-Higgs bundle (E,φ) (given by an element $d \in \pi_1(G^{\mathbb{C}})$) to be trivial. However if we do not impose the above condition on the topological class of (E,φ) there is a similar correspondence involving representations of the universal central extension of the fundamental group.

One possible approach to understand the non-abelian Hodge correspondence given by Theorem 2.4.4 is the following. With the same notation used in Section 2.3, let $\mathbb{E} \to X$ be a C^∞-principal $H^{\mathbb{C}}$-bundle over X with fixed topological class $d \in \pi_1(G) = \pi_1(H)$. As we noticed in Section 2.3, from the Cartan decomposition (1.2), we get that every G-connection D on $\mathbb{E}$ decompose uniquely as a sum of the covariant derivative d_A of an H-connection A on $\mathbb{E}_H$ and $\psi \in \Omega^1(X, \mathbb{E}_H(\mathfrak{m}))$. Let F_A be the curvature of the H-connection A, the flatness of D is given by the equations

$$\begin{aligned} &F_A + \tfrac{1}{2}[\psi,\psi] = 0 \\ &d_A\psi = 0. \end{aligned} \tag{2.7}$$

Consider now, the set of equations

$$\begin{aligned} &F_A + \tfrac{1}{2}[\psi,\psi] = 0 \\ &d_A\psi = 0 \\ &d_A^*\psi = 0. \end{aligned} \tag{2.8}$$

These equations are motivated by the following Proposition.

Proposition 2.4.6. *Let h be a metrics in a flat principal $H^{\mathbb{C}}$-bundle $E \to X$ and let (d_A,ψ) be a pair defined by the flat G-connection D on $\mathbb{E}$ as above. Then h is harmonic if and only if*

$$d_A^*\psi = 0.$$

Observe that the equations (2.8) are invariant under the action of the gauge group $\mathcal{H}$ of $\mathbb{E}_H$. The following result was independently proved by Donaldson [24] (for $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle) and Corlette [23] (for arbitrary complex groups and dimensions).

Theorem 2.4.7. *There is a homeomorphism*

$$\{\text{Reductive } G\text{-connections } D \text{ s.t. } F_D = 0\}/\mathcal{G} \cong \{(d_A,\psi) \text{ satisfying } 2.8\}/\mathcal{H}.$$

As we noticed above the first two equations in (2.8) imply that the connection $D = d_A + \psi$ is flat, hence Theorem 2.4.7 means that in each $\mathcal{G}$-orbit of a reductive flat G-connection D we can find a flat G-connection $D' = g(D) = d_{A'} + \psi'$ such that the last condition in (2.8) is satisfied, i.e such that h is harmonic. Finally notice that there exists a homeomorphism

$$\{(d_A, \varphi) \text{ satisfying } 2.6\}/\mathcal{H} \cong \{(d_A, \psi) \text{ satisfying } (2.8)\}/\mathcal{H},$$

given by the correspondence

$$(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau_h(\varphi)),$$

induced by the isomorphism (2.3).

2.5 Parabolic (G, θ)-Higgs bundles

Let G be the connected component at the identity of a real form of a complex semisimple Lie group $G^{\mathbb{C}}$. Let $\theta := \tau\mu$ be a fixed Cartan involution, where τ is a compact conjugation of $G^{\mathbb{C}}$ and μ is the anti-holomorphic involution defining the real form G. Let H be its corresponding maximal compact subgroup of G such that $G^{\theta} = H$ and denote by $H^{\mathbb{C}}$ its complexification. Consider $\sigma^{\mathbb{C}} \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with θ, μ and τ and $\sigma \in \mathrm{Aut}_2(G, \theta)$ its decomplexification or realification.

Let X be a compact connected Riemann surface and $S = \{x_1, \dots, x_r\}$ a finite set of different points of X. Let D be the divisor $x_1 + \cdots + x_r$. Let $T \subset H \subset H^{\mathbb{C}}$ be a maximal torus and $\mathfrak{g}, \mathfrak{h}, \mathfrak{t}$ their respective Lie algebras. Let $\mathcal{A} \subset \mathfrak{t}$ be an alcove containing $0 \in \mathfrak{t}$. Let $\alpha_i \in \sqrt{-1}\,\bar{\mathcal{A}}$. A **parabolic structure** of weight α_i on a principal $H^{\mathbb{C}}$-bundle E over a point x_i is a choice of a subgroup

$$Q_i \subset E(H^{\mathbb{C}})_{x_i} := \{\phi : E_{x_i} \longrightarrow H^{\mathbb{C}} \text{ s.t. } \phi(\xi h) = h^{-1}\phi(\xi)h\}$$

such that in some trivialization $\xi \in E_{x_i}$, we have

$$P_{\alpha_i} = \{\phi(\xi),\ \phi \in Q_i\}. \tag{2.9}$$

Let $\alpha = (\alpha_1, \dots, \alpha_r)$ a r-tuple with elements in $\sqrt{-1}\,\bar{\mathcal{A}}$. A **parabolic principal $H^{\mathbb{C}}$-bundle** of weight α is a holomorphic principal $H^{\mathbb{C}}$-bundle E with a choice for any $i \in \{1, \dots, r\}$ of a parabolic structure of weight α_i on x_i.

Let μ be the set of eigenvalues of $\mathrm{ad}(\alpha_i)$ acting on $\mathfrak{m}^{\mathbb{C}}$ and let $\xi \in E$ be a trivialization near x_i compatible with the parabolic structure Q_i such that

$$E(\mathfrak{m}^{\mathbb{C}}) = \bigoplus_{\mu} \mathfrak{m}^{\mathbb{C}}_{\mu}.$$

A section $\varphi \in E(\mathfrak{m}^{\mathbb{C}})$ is in the sheaf $PE(\mathfrak{m}^{\mathbb{C}})$ of **parabolic sections** of $E(\mathfrak{m}^{\mathbb{C}})$ (resp. $NE(\mathfrak{m}^{\mathbb{C}})$ of **strictly parabolic sections**) if φ is holomorhic on $X - D$ and it is meromorphic on x_i and if $a-1 < \mu \leq a$ (resp. $a-1 \leq \mu < a$) for some integer a then $\varphi_\mu = O(z^\alpha)$.

Definition 2.5.1. A **parabolic (strictly parabolic)** (G,θ)**-Higgs bundle** (E,φ) **over** (X,D) is a parabolic principal $H^{\mathbb{C}}$-bundle such that $\varphi \in PE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$ (resp. $\varphi \in NE(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$).

We now define the **residue** $\mathrm{Res}_{x_i}\varphi$ **of** φ **at the points** x_i. Take α_i in $\sqrt{-1}\mathcal{A}$. We can identify the fibre of $PE(\mathfrak{m}^{\mathbb{C}})$ at x_i with $\mathfrak{m}^{\mathbb{C}}$. Hence we can project the residue of φ to the space $\widetilde{\mathfrak{m}}^0_i \subset E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to

$$\widetilde{\mathfrak{m}}^0_{\alpha_i} := \mathrm{Ker}_{\mathfrak{m}^{\mathbb{C}}}\{\mathrm{Ad}(\exp 2\pi\sqrt{-1}\alpha_i) - 1\}$$

and we denote this projection by $\mathrm{Gr}\,\mathrm{Res}_{x_i}\varphi$.

Remark 2.5.2. If we take the weights α_i in $\sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$, where

$$\mathcal{A}'_{\mathfrak{g}} := \{\alpha \in \overline{\mathcal{A}} \text{ such that the eigenvalues } \lambda \text{ of } \mathrm{ad}(\alpha) \text{ on } \mathfrak{g} \text{ satisfy } |\lambda| < 1\},$$

the Higgs field φ is a meromorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \times K$ with a simple pole at $x_i \in D$ and then

$$\mathrm{Res}_{x_i}\varphi \in \mathfrak{m}_i,$$

where $\mathfrak{m}_i$ is the subset of $E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to

$$\mathfrak{m}_{\alpha_i} := \{v \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota^-(e^{t\alpha})v \text{ is bounded as } t \to \infty\}$$

and its projection $\mathrm{Gr}\,\mathrm{Res}_{x_i}\varphi$ is in $\mathfrak{m}^0_i \subset \mathfrak{m}_i \subset E(\mathfrak{m}^{\mathbb{C}})_{x_i}$ corresponding to

$$\mathfrak{m}^0_{\alpha_i} := \{v \in \mathfrak{m}^{\mathbb{C}} \ : \ \iota^-(e^{t\alpha})v = v \text{ for every } t\},$$

where $\iota^- : H^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ is the complexification of the isotropy representation of H to $\mathfrak{m}$.

2.5.1 Stability of parabolic (G,θ)-Higgs bundles and Hitchin-Kobayashi correspondence

Let denote by $\mathfrak{z}$ the centre of $\mathfrak{h}$. In general, (semi, poly)stability of parabolic (G,θ)-Higgs bundles depends on an element β lying in $\imath\mathfrak{z}$. One also requires β to lie in $\mathfrak{z}(\mathfrak{g})$, the centre of $\mathfrak{g}$, in order to relate (G,θ)-Higgs bundles with representation of the fundamental group of X in G. Since we are dealing with semisimple Lie groups the centre of G is trivial and hence β is zero. However, we will develop this theory for a general element in $\imath\mathfrak{z}$ in order to cover the case of reductive Lie groups.

Let $s \in i\mathfrak{h}$. Let $P_s \subset H^{\mathbb{C}}$ and L_s be the corresponding parabolic and Levi subgroups of $H^{\mathbb{C}}$ and $\mathfrak{p}_s$ and $\mathfrak{l}_s$ be their Lie subalgebras. Let χ_s be the corresponding antidominant character of $\mathfrak{p}_s$. We define

$$\mathfrak{m}_s = \{v \in \mathfrak{m}^{\mathbb{C}} \,:\, \iota^-(e^{ts})v \text{ is bounded as } t \to \infty\},$$
$$\mathfrak{m}_s^0 = \{v \in \mathfrak{m}^{\mathbb{C}} \,:\, \iota^-(e^{ts})v = v \text{ for every t}\}.$$

Observe that if G is complex then $\mathfrak{m}_s = \mathfrak{p}_s$ and $\mathfrak{m}_s^0 = \mathfrak{l}_s$. In [15] Biquard, García-Prada and Ignasi define the **parabolic degree** as follows:

$$\operatorname{pardeg}_\alpha(E)(\sigma,\chi) := \deg(E)(\sigma,\chi) - \sum_i \deg((Q_i,\alpha_i),(E_\sigma(P)_{x_i},\chi)).$$

That is, the parabolic degree consists of the sum of the ususal degree and the relative degree (see Appendix B in [15]) of two simpleces on $\partial_\infty(G/H)$ where each simplex is determined by a parabolic subgroup Q_i and an element $\alpha_i \in i\mathfrak{h}$.

Definition 2.5.3. Let $\beta \in i\mathfrak{z}(\mathfrak{h})$. A parabolic (G,θ)-Higgs bundle (E,φ) over (X,D) is:

β**-semistable** if

$$\operatorname{pardeg}(E)(\sigma,\chi_s) - \langle\beta,s\rangle \geq 0,$$

for any $s \in i\mathfrak{h}$ and any holomorphic reduction of structure group σ of E to P_s such that $\varphi_{|X\backslash D} \in H^0(X\backslash D, E_\sigma(\mathfrak{m}_s)\otimes K)$.

β**-stable** if

$$\operatorname{pardeg}(E)(\sigma,\chi_s) - \langle\beta,s\rangle > 0,$$

for any $s \in i\mathfrak{h}\setminus\operatorname{Ker}(d\iota)$ and any reduction of structure group σ of E to P_s such that $\varphi_{|X\backslash D} \in H^0(X\backslash D, E_\sigma(\mathfrak{m}_s)\otimes K)$.

β-polystable if it is semistable and anytime

$$\operatorname{pardeg}(E)(\sigma, \chi_s) = 0$$

for any $s \in i\mathfrak{h}$ and any reduction of structure group σ of E to P_s, then there is a meromorphically equivalent parabolic Higgs bundle (E', φ) and a holomorphic reduction σ_{L_s} of E_σ to the corresponding Levi subgroup L_s of P_s such that

1. $\varphi_{|X\setminus D}$ takes values in $H^0(X\setminus D, E_{\sigma_{L_s}}(\mathfrak{m}_s^0) \otimes K)$,
2. the bundle E_{L_s} has a parabolic stucture at the points of the divisor D compatible with that of E' in the sense that the parabolic bundle E' is induced from E_{L_s} through $L_s \hookrightarrow H^{\mathbb{C}}$. In particular, this means that the parabolic weights of E lie in $\mathfrak{l}_s$.

We will denote by $\mathcal{M}_\beta(G, \theta, \alpha) := \mathcal{M}_\beta(X, D, G, \theta, \alpha)$ the **moduli space of isomorphism classes of β-polystable parabolic (G, θ)-Higgs bundles (E, φ) on (X, D) with parabolic weight $\alpha = (\alpha_1, \ldots, \alpha_r)$.** The moduli space of polystable parabolic (G, θ)-Higgs bundle will be simply denoted by $\mathcal{M}(G, \theta, \alpha)$.

Now, let L_i be the Levi subgroup of Q_i and $\widetilde{L}_i$ the subgroup corresponding to

$$\widetilde{L}_{\alpha_i} := \operatorname{Stab}_{H^{\mathbb{C}}}(\exp 2\pi\sqrt{-1}\alpha_i).$$

Notice that if $\alpha_i \in \sqrt{-1}\mathcal{A}'_{\mathfrak{g}}$ then $\widetilde{L}_i = L_i$ and $\widetilde{\mathfrak{m}}_i^0 = \mathfrak{m}_i^0$. We denote by $\widetilde{\mathfrak{m}}_i^0/\widetilde{L}_i$ the set of $\widetilde{L}_i$-orbits. We define a map

$$\varrho : \mathcal{M}_\beta(G, \theta, \alpha) \to \prod_i (\widetilde{\mathfrak{m}}_i^0/\widetilde{L}_i)$$

by taking the $\widetilde{L}_i$-orbit of $\operatorname{Gr}\operatorname{Res}_{x_i}\varphi \in \widetilde{\mathfrak{m}}_i^0$. Let $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_r)$ with fixed orbits $\mathcal{L}_i \in \widetilde{\mathfrak{m}}_i^0/\widetilde{L}_i$, we define the moduli space $\mathcal{M}_\beta(G, \theta, \alpha, \mathcal{L}) := \varrho^{-1}(\mathcal{L})$. This is a subspace of $\mathcal{M}_\beta(G, \theta, \alpha)$.

Finally we give a brief description of the Hitchin-Kobayashi correspondence given in [15]. Let G be a real form of a complex semisimple Lie group with Cartan involution θ, X be a compact Riemann surface and let D be a divisor as above. Choose a smoth 2-form ω on $X \setminus D$ and assume that $\int \omega = 2\pi$. Let (E, φ) be a parabolic (G, θ)-Higgs bundle on (X, D) and let $\beta \in \sqrt{-1}\zeta(\mathfrak{h})$ as in Definition 2.5.1. We are looking for a metric $h \in \Gamma(X \setminus D, E(H\ H^{\mathbb{C}}))$, satisfying the β-Hermite-Einstein equation

$$R_h - [\varphi, \tau_h(\varphi)] + \sqrt{-1}\beta\omega = 0, \tag{2.10}$$

where R_h is the curvature of the connection A_h compatible with the holomorphic structure of E and the metric h and τ_h as always is the conjugation on $\Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})$ defined by combining the metric h and the conjugation on X from $(1,0)$-forms to $(0,1)$-forms.

Let us first recall the notion of α-adapted metric.

Definition 2.5.4. Let h be a metric for E away from the divisor D. We say that h is an α_i**-adapted metric** if for any parabolic point x_i the following holds. Choose a local holomorphic trivialisation ξ_i of E near x_i compatible with the parabolic structure and choose a local holomorphic coordinate z. Then there is some meromorphic gauge transformation $h \in PE(H^{\mathbb{C}})$, in the sheaf of parabolic sections of $E(H^{\mathbb{C}})$, near x_i such that in the trivialisation $g(\xi_i)$ one has

$$h = h_0 \cdot |z|^{-\alpha_i} e^c,$$

where h_0 is the standard constant metric. $\mathrm{Ad}(|z|^{-\alpha_i})c = o(\log|z|)$, $\mathrm{Ad}(|z|^{-\alpha_i})dc \in L^2$ and $\mathrm{Ad}(|z|^{-\alpha_i})F_h \in L^1$ and F_h is the curvature of the unique connection A_h compatible with the holomorphic structure of E and the metric h.

We construct a singular α-adapted metric h_0 which gives an approximate solution to the equations. In order to do that we decompose

$$\mathrm{GrRes}_{x_i}\varphi = s_i + Y_i$$

into its semisimple part and its nilpotent part. We complete Y_i into a Konstant-rallis $\mathfrak{sl}_2$-triple (see Appendix A.3 in [15]) (H_i, X_i, Y_i) and then the model for the metric is:

$$h_0 = |z|^{-2\alpha_i}(-\ln|z|)^{H_i}.$$

Hence we have the following (see [15] for further details).

Theorem 2.5.5. *Let (E, φ) be a parabolic (G, θ)-Higgs bundle, equipped with an initial metric h_0. Then (E, φ) is β-polystable if and only if it admits a metric h satisfying the Hermite-Einstein equation (2.10), α_i-adapted to h_0 at each puncture x_i and quasi-isometric to h_0.*

Chapter 3

Holomorphic automorphisms of Higgs bundle moduli spaces.

Throughout this chapter, G is the connected component of the identity of a real form of a complex semisimple Lie group $G^{\mathbb{C}}$ with fixed Cartan involution θ, and H is its maximal compact subgroup, i.e. the subgroup of points of G fixed by θ. The corresponding Lie algebras will be denoted by gothic letters. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be the corresponding Cartan decomposition of $\mathfrak{g}$ induced by θ. This decomposition complexifies to

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}.$$

For Lie theoretical facts and notation regarding semisimple real or complex Lie groups we refer to Chapter 1.

3.1 Automorphisms of principal bundles.

In this section we follow the approach taken in [31] with slightly different nuances due to the fact that we are studying real semisimple Lie groups rather than complex semisimple Lie groups.

Let E be a principal $H^{\mathbb{C}}$-bundle and A an order n automorphism. We define the morphism $f_A : E \to H^{\mathbb{C}}$ by the formula

$$A(\xi) = \xi f_A(\xi) \text{ for every } \xi \in E. \tag{3.1}$$

We have the following two lemmas.

Lemma 3.1.1. *The map f_A is equivariant for the right action of $H^{\mathbb{C}}$ on E and the (right) adjoint action of $H^{\mathbb{C}}$ on itself, namely $x \mapsto h^{-1}xh$ for $x, h \in H^{\mathbb{C}}$.*

Proof. Let $h \in H^{\mathbb{C}}$, $\xi \in E$. On the one hand $A(\xi h) = (\xi h) f_A(\xi h)$. On the other hand, $A(\xi h) = A(\xi)h = \xi f_A(\xi)h$, hence $h f_A(\xi h) = f_A(\xi)h$. □

Lemma 3.1.2. *If A_1, $A_2 \in \mathrm{Aut}(E)$, then $f_{A_1A_2} = f_{A_1} f_{A_2}$.*

Proof. Let $\xi \in E$. Then $\xi f_{A_1A_2}(\xi) = (A_1A_2)(\xi) = A_1(A_2\xi) = A_1(\xi f_{A_2}(\xi)) = A_1(\xi) f_{A_2}(\xi) = \xi f_{A_1}(\xi) f_{A_2}(\xi)$. □

As a consequence of the previous assertions we have.

Proposition 3.1.3. *Let A be an automorphism of E such that $A^n = z \in Z_\tau$, where $Z_\tau = Z(H^{\mathbb{C}}) \cap Z(G^{\mathbb{C}})$ (see Section 1.5 for further details). Then:*

(1) f_A maps E onto a single orbit $S(E)$ of the set

$$S_n = \{s \in H^{\mathbb{C}} \, : \, s^n \in Z_\tau)\}$$

under the action of $H^{\mathbb{C}}$ by inner automorphisms.

(2) Every element $s \in S(E)$ defines a reduction of structure group of E to $Z_{H^{\mathbb{C}}}(s)$, the centralizer of s of $H^{\mathbb{C}}$.

Proof. First observe that $f_{A^n} = (f_A)^n$ is the constant map $\xi \in E \mapsto z \in Z_\tau$ by Lemma 3.1.2. Let $s := f_A(\xi)$ then f_A maps E into S_n. In addition, f_A defines a morphism $\widetilde{f_A} : E \to H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ given by the GIT quotient for the right action $x \mapsto h^{-1}xh$ of $H^{\mathbb{C}}$ on itself. By Lemma 3.1.1 $\widetilde{f_A}$ is constant on the fibres of E then it descends to a constant morphism $\widetilde{\widetilde{f_A}} : X = E/H^{\mathbb{C}} \to H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ since X is projective and $H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ is affine. Now, $s := f_A(\xi) \in S_n$ for all $\xi \in E$ and hence s is a semi-simple element of $H^{\mathbb{C}}$. Then s is a stable point for the GIT quotient and hence $s = f_A(\xi)$ lies on a single orbit $S(E) = S_n/H^{\mathbb{C}}$ (for the given action of $H^{\mathbb{C}}$ on S_n) for every $\xi \in E$.

(2) follows from the fact that for any $s \in S(E)$ and ξ, ξ' on a fibre of $E \to X$, they belong to $S := f_A^{-1}(s)$ if and only if $\xi' = \xi h$ for some $h \in H^{\mathbb{C}}$ and $f_A(\xi) = s = f_A(\xi') = f_A(\xi h) = h^{-1} f_A(\xi) h$. This means that h is in the centralizer $Z_{H^{\mathbb{C}}}(s)$ of $f_A(\xi) = s$ of $H^{\mathbb{C}}$. Thus $Z_{H^{\mathbb{C}}}(s)$ acts upon transitively on fibres on the set S. It gives a reduction of the structure group of E to $Z_{H^{\mathbb{C}}}(s)$. □

Remark 3.1.4. Let $\eta = \mathrm{Int}(s)$. Since $Z_{H^{\mathbb{C}}}(s) = (H^{\mathbb{C}})^{\eta}$ we can rewrite (2) in Proposition 3.1.3 in terms of $(H^{\mathbb{C}})^{\eta}$.

Let us now consider $\eta \in \mathrm{Aut}_n(G, \theta)$. There exists an isomorphism between the total spaces of $\eta(E) := E \times_{\eta} H^{\mathbb{C}}$, **the bundle associated to E via** η, and the total space of a principal $H^{\mathbb{C}}$-bundle $E \to X$ whose $H^{\mathbb{C}}$-action is given by

$$\xi \cdot h := \xi \eta(h).$$

The previous isomorphism is equivalent to a bijection $A : E \to E$ such that $A(\xi h) = A(\xi)\eta(h)$ for all $\xi \in E$, and for all $h \in H^{\mathbb{C}}$. We will refer to this as an **η-twisted automorphism** and we will denote by $\mathrm{Aut}_{\eta}(E)$ the set of η-twisted automorphisms.

Remark 3.1.5. The aim of this section is to introduce some tools and neccesary results on automorphisms of principal bundles to apply them to the study of fixed points subset of the moduli space of polystable (G, θ)-Higgs bundles (E, φ) under the action of $\mathrm{Out}_n(G, \theta)$. In fact, in following sections we will require (E, φ) to be simple. This indirectly impose an additional condition to A. Recall that a (G, θ)-Higgs bundle (E, φ) is simple if and only if $\mathrm{Aut}(E, \varphi) = Z_{\tau}$ (see Proposition 2.2.10). Hence since $A^n \in \mathrm{Aut}(E, \varphi)$ we need A^n to be in Z_{τ}.

Let $A \in \mathrm{Aut}_{\eta}(E)$. We have the following.

Lemma 3.1.6. *Let $\eta_1, \eta_2 \in \mathrm{Aut}(G, \theta)$ and $A_i \in \mathrm{Aut}_{\eta_i}(E)$ for $i = 1, 2$. Let $f_{A_i} : E \to H^{\mathbb{C}}$ be the morphism given by (3.1). Then*

(1) f_{A_i} is $H^{\mathbb{C}}$-equivariant for the right action of $H^{\mathbb{C}}$ on E and the right action of $H^{\mathbb{C}}$ on itself given by

$$h \mapsto \widetilde{h}^{-1} h \eta_i(\widetilde{h}) \text{ for all } h, \widetilde{h} \in H^{\mathbb{C}}.$$

(2) $A_1 A_2 \in \mathrm{Aut}_{\eta_1 \eta_2}(E)$.

(3) $f_{A_1A_2} = f_{A_1} \cdot \eta_1(f_{A_2})$.

Proof. (1) On the one hand we have $A_i(\xi h) = A_i(\xi)\eta_i(h) = \xi f_{A_i}(\xi)\eta_i(h) = \xi h h^{-1} f_{A_i}(\xi)\eta_i(h)$ for all $\xi \in E$ and $h \in H^{\mathbb{C}}$. On the other hand we have $A_i(\xi h) = (\xi h) f_{A_i}(\xi h)$ for all $\xi \in E$ and $h \in H^{\mathbb{C}}$. Thus $f_{A_i}(\xi h) = h^{-1} f_{A_i}(\xi)\eta_i(h)$ for all $\xi \in E$ and $h \in H^{\mathbb{C}}$.

(2) For any $\xi \in E$ and $h \in H^{\mathbb{C}}$ we have

$$A_1(A_2(\xi h)) = A_1(A_2(\xi)\eta_2(h)) = A_1(A_2(\xi))\eta_1(\eta_2(h)),$$

then $A_1A_2 \in \mathrm{Aut}_{\eta_1\eta_2}(E)$.

(3) First observe that $f_{A_1A_2} = f_{A_1} \cdot \eta_1(f_{A_2})$ means that $f_{A_1A_2}(\xi) = f_{A_1}(\xi)\eta_1(f_{A_2}(\xi))$ for every $\xi \in E$. Hence let $\xi \in E$, the proof of (3) is given by the following straightforward computation:

$$\xi f_{A_1A_2}(\xi) = (A_1A_2)(\xi) = A_1(\xi f_{A_2}(\xi)) = A_1(\xi)\eta_1(f_{A_2}(\xi)) = \xi f_{A_1}(\xi)\eta_1(f_{A_2}(\xi)).$$

□

Consider the set

$$S_\eta^{n,\tau} := \{s \in H^{\mathbb{C}} \ : \ s\eta(s)\cdots\eta^{n-1}(s) = z \in Z_\tau\},$$

defined in Chapter 1. Here is the twisted version of Proposition 3.1.3.

Proposition 3.1.7. *Let E be a $H^{\mathbb{C}}$-bundle over a compact Riemann surface X. Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and $A \in \mathrm{Aut}_\eta(E)$ such that $A^n = z \in Z_\tau \subset \mathrm{Aut}(E)$. Then:*

(1) The morphism f_A defined in (3.1) maps E onto a single orbit $S(E)$ of the set $S_\eta^{n,\tau}$ under the action of $H^{\mathbb{C}}$ defined as in Proposition 1.3.8 .

(2) Every element $s \in S(E)$ defines a reduction of the structure group of E to $(H^{\mathbb{C}})^{\eta'}$, where $\eta' = \mathrm{Int}(s)\eta$, and $(H^{\mathbb{C}})^{\eta'}$ is the subgroup of fixed points under η'.

Proof. We proceed in the same way we did in the proof of Proposition 3.1.3. First observe that $f_{A^n} = f_A \cdot \eta(f_A)\cdots\eta^{n-1}(f_A)$ is the constant map

$$\begin{array}{rccc} f_z: & E & \to & Z_\tau \\ & \xi & \mapsto & z \end{array} \tag{3.2}$$

by Lemma 3.1.6. Thus $s := f_A(\xi)$ belongs to $S_\eta^{n,\tau}$. In addition, f_A defines a morphism $\widetilde{f_A} : E \to H^{\mathbb{C}} /\!\!/_\eta H^{\mathbb{C}}$ given by the GIT quotient for the right η-twisted action $x \mapsto h^{-1}x\eta(h)$ of $H^{\mathbb{C}}$ on itself for all $x, h \in H^{\mathbb{C}}$. By Lemma 3.1.6 $\widetilde{f_A}$ is constant on the fibres of E then it descends to a constant morphism $\widetilde{\widetilde{f_A}} : X = E/H^{\mathbb{C}} \to H^{\mathbb{C}} /\!\!/_\eta H^{\mathbb{C}}$ since X is projective and $H^{\mathbb{C}} /\!\!/_\eta H^{\mathbb{C}}$ is affine. Now, using the same argument put forward by García-Prada and Ramanan in [31] we claim that $s := f_A(\xi) \in S_\eta^{n,\tau}$ is a stable point for the GIT quotient $H^{\mathbb{C}} /\!\!/_\eta H^{\mathbb{C}}$ and hence $f_A(\xi)$ lie on a single orbit for the η-twisted action of $H^{\mathbb{C}}$ on $S_\eta^{n,\tau}$ for every $\xi \in E$, proving (1).

(2) follows from the fact that for any $s \in S(E)$ and ξ, ξ' on a fibre of $E \to X$, they belong to $S := f_A^{-1}(s)$ if and only if $\xi' = \xi h$ for some $h \in H^{\mathbb{C}}$ and $f_A(\xi) = s = f_A(\xi') = f_A(\xi h) = h^{-1} f_A(\xi)\eta(h)$. This means that h is in $(H^{\mathbb{C}})^{\eta'}$ for $\eta' = \mathrm{Int}(s)\eta$. Thus $(H^{\mathbb{C}})^{\eta'}$ acts upon transitively on fibres on the set S. It gives a reduction of the structure group of E to $(H^{\mathbb{C}})^{\eta'}$. □

3.1.1 Involutions on principal bundles.

Let us consider $\eta \in \mathrm{Aut}_2(G, \theta)$. As noticed above, there exists an isomorphism between the total spaces of E and $\eta(E)$, the bundle associated to E via η, and this is equivalent to having an η-twisted automorphism $A : E \to E$ such that $A(\xi h) = A(\xi)\eta(h)$ for all $\xi \in E$, and for all $h \in H^{\mathbb{C}}$. Observe that $A^2 \in \mathrm{Aut}(E, \varphi)$ since $\eta^2 = id$. Hence $A^2 \in Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota_{\mathbb{C}}^-)$ (see Proposition 2.2.10). Consider the set

$$S_\eta^\tau := \{s \in H^{\mathbb{C}} \,:\, s\eta(s) \in Z_\tau\}, \tag{3.3}$$

defined in Section 1.6. Taking into account Lemma 3.1.6 for $n = 2$, we have the following special case of Proposition 3.1.7.

Proposition 3.1.8. *Let E be a $H^{\mathbb{C}}$-bundle over a compact Riemann surface X. Let $\eta \in \mathrm{Aut}_2(G, \theta)$ and $A \in \mathrm{Aut}_\eta(E)$ such that $A^2 = z \in Z_\tau \subset \mathrm{Aut}(E)$. Then:*

(1) The morphism f_A defined in (3.1) maps E onto a single orbit $S(E)$ of the set S_η^τ under the action of $H^{\mathbb{C}}$ defined as in Proposition 1.3.5.

(2) Every element $s \in S(E)$ defines a reduction of the structure group of E to

$(H^{\mathbb{C}})^{\eta'}$, *where* $\eta' = \mathrm{Int}(s)\eta$, *and* $(H^{\mathbb{C}})^{\eta'}$ *is the subgroup of fixed points under* η'.

Proof. Straightforward from Proposition 3.1.7 for $n = 2$. □

3.2 (η, Z_τ)-twisted automorphisms.

As in [31], consider automorphisms of $H^{\mathbb{C}}$-principal bundles that involve the subgroup Z_τ.

Definition 3.2.1. Let $\eta \in \mathrm{Aut}_n(G, \theta)$. We define (η, Z_τ)**-twisted automorphisms** as morphisms $A : E \to E$ such that A is fibre-preserving and for each $h \in H^{\mathbb{C}}$, $\xi \in E$ we have that $A(\xi h) = A(\xi) z \eta(h)$, where $z \in Z_\tau$ depends on A, η, ξ and h. We will denote the set of (η, Z_τ)-twisted automorphisms of E by $\mathrm{Aut}_\eta^{Z_\tau}(E)$.

Let us consider the fibre product $E \times_X \alpha$ of the principal $H^{\mathbb{C}}$ bundle E and a principal Z_τ-bundle α over X. This is a principal $(H^{\mathbb{C}} \times Z_\tau)$-bundle. Following [31] we will write the action of Z_τ on α on the left. Using the multiplication, we can extend the structure group of $E \times_X \alpha$ obtaining a principal $H^{\mathbb{C}}$-bundle denoted by $E \otimes \alpha$. This bundle is defined as the quotient of $E \times_X \alpha$ by the action of Z_τ given by $z *_{Z_\tau} (\xi, a) = (\xi z^{-1}, za)$ for any $z \in Z_\tau$. We will also denote the image of (ξ, a) in $E \otimes \alpha$ by $\xi \otimes a$.

Remark 3.2.2. Let $\eta \in \mathrm{Aut}_n(G, \theta)$. An isomorphism between E and $\eta(E) \otimes \alpha$ can be identified with a (η, Z_τ)-twisted automorphism.

Let $A \in \mathrm{Aut}_\eta^{Z_\tau}(E)$. We easily adapt the proof of Lemma 3.1.6 to get the following.

Lemma 3.2.3. *Let* $\eta_1, \eta_2 \in \mathrm{Aut}(G, \theta)$ *and* $A_i \in \mathrm{Aut}_{\eta_i}^{Z_\tau}(E)$ *for* $i = 1, 2$. *Let* $f_{A_i} : E \to H^{\mathbb{C}}$ *be the morphism given by the formula (3.1). Then*

(1) Let $A \in \mathrm{Aut}_\eta^{Z_\tau}(E)$ *for any* $\eta \in \mathrm{Aut}(G, \theta)$. *Then* f_A *is* $H^{\mathbb{C}}$*-equivariant for the right action of* $H^{\mathbb{C}}$ *on* E *and the right action of* $H^{\mathbb{C}}$ *on itself given by*

$$h \mapsto z(A, \eta, h, \xi)\widetilde{h}^{-1} h \eta(\widetilde{h}) \text{ for all } h, \widetilde{h} \in H^{\mathbb{C}}, \xi \in E,$$

where $z(A,\eta,h,\xi)$ depends on A,η,h and ξ. From now we will refer to $z(A,\eta,h,\xi)$ simply as z.

(2) $A_1A_2 \in \mathrm{Aut}^{Z_\tau}_{\eta_1\eta_2}(E)$.

(3) $f_{A_1A_2}(\xi) = zf_{A_1}(\xi)\cdot\eta_1(f_{A_2}(\xi))$.

Finally we have the (η, Z_τ)-twisted version of Proposition 3.1.7.

Proposition 3.2.4. *Let E be a principal $H^{\mathbb{C}}$-bundle over a compact Riemann surface X. Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and $A \in \mathrm{Aut}^{Z_\tau}_\eta(E)$ such that $A^n = f$, for a function $f : E \to Z_\tau$. Then:*

(1) The morphism f_A defined as in (3.1) maps E onto a single orbit $S(E)$ of the set $S^{n,\tau}_\eta$ under the action of $Z_\tau \times H^{\mathbb{C}}$, where Z_τ acts in $S^{n,\tau}_\eta$ by multiplication and $H^{\mathbb{C}}$ acts in $S^{n,\tau}_\eta$ as in Proposition 1.3.5 .

(2) Every element $s \in S(E)$ defines a reduction of the structure group of E to $(H^{\mathbb{C}})_{\eta'}$, where $\eta' = \mathrm{Int}(s)\eta$, and $(H^{\mathbb{C}})_{\eta'}$ is defined as in Section 1.5.

Proof. We will follow step by step the proof of Proposition 3.1.7. By (3) in Lemma 3.2.3 we have

$$s := f(\xi) = f_{A^n}(\xi) = zf_A(\xi)\eta(f_A(\xi))\cdots\eta^{n-1}(f_A(\xi))$$

for all $\xi \in E$, where z depends on ξ, A and η. Since $s \in Z_\tau$ then $s \in S^{n,\tau}_\eta$. In addition, f_A defines a morphism $\widetilde{f_A} : E \to H^{\mathbb{C}}/\!\!/_\eta\, (Z_\tau \times H^{\mathbb{C}})$ given by the GIT quotient for the action given by $x \mapsto zh^{-1}x\eta(h)$ of $Z_\tau \times H^{\mathbb{C}}$ on $H^{\mathbb{C}}$ for all $x, h \in H^{\mathbb{C}}$ and $z \in Z_\tau$. By Lemma 3.2.3 f_A is constant on the fibres, hence it descends to a constant morphism $\widetilde{f_A} : X = E/H^{\mathbb{C}} \to H^{\mathbb{C}}/\!\!/_\eta\, (Z_\tau \times H^{\mathbb{C}})$ since X is projective and $H^{\mathbb{C}}/\!\!/_\eta\, (Z_\tau \times H^{\mathbb{C}})$ is affine. For the rest of the proof we follow (1) in Proposition 3.1.7.

To prove (2) let we define $S := f_A^{-1}(s)$ for $s \in S(E)$. For any ξ, ξ' on a fibre of $E \to X$, they belong to $S := f_A^{-1}(s)$ if and only if $\xi' = \xi h$ for some $h \in H^{\mathbb{C}}$ and $f_A(\xi) = s = f_A(\xi') = f_A(\xi h) = zh^{-1}f_A(\xi)\eta(h)$. This means that h is in $(H^{\mathbb{C}})_{\eta'}$ for $\eta' = \mathrm{Int}(s)\eta$. Thus $(H^{\mathbb{C}})_{\eta'}$ acts upon transitively on fibres on the set S. It gives a reduction of the structure group of E to $(H^{\mathbb{C}})_{\eta'}$. □

3.3 Higgs bundles defined by η.

In this section we introduce certain Higgs bundles defined by an automorphism $\eta \in \mathrm{Aut}_n(G,\theta)$ where as usual G is a connected real form of a complex semisimple Lie group and θ is its Cartan involution. They will play an important role in the study of the automorphisms of $\mathcal{M}(G,\theta)$ considered in Section 3.5.

Recall that the Lie algebra of H^η and H_η coincide (we refer to Sections 1.3 and 1.5 for further details) and that the decomposition of $\mathfrak{g}$ defined by η in (1.8) allow us to define the representations

$$\iota^k : (H^{\mathbb{C}})^\eta \to \mathrm{GL}(\mathfrak{m}_k^{\mathbb{C}}) \tag{3.4}$$

and

$$\iota_k : H_\eta^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_k^{\mathbb{C}}). \tag{3.5}$$

Then we have the following definitions.

Definition 3.3.1. Let $\zeta_k := \exp(2\pi i \frac{k}{n})$ for $0 \leq k \leq n-1$. A $(G,\theta,H^\eta,\zeta_k)$**-Higgs bundle over a compact Riemann surface** X is a pair (E,φ) where E is a principal $(H^{\mathbb{C}})^\eta$-bundle over X and φ is a section of $E(\mathfrak{m}_k^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}_k^{\mathbb{C}})$ is the bundle associated to E via the representation (1.10) defined above and K is the canonical bundle on X.

Definition 3.3.2. Let $\zeta_k := \exp(2\pi i \frac{k}{n})$ for $0 \leq k \leq n-1$. A $(G,\theta,H_\eta,\zeta_k)$**-Higgs bundles over a compact Riemann surface** X is a pair (E,φ) where E is a principal $H_\eta^{\mathbb{C}}$-bundle over X and φ is a section of $E(\mathfrak{m}_k^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}_k^{\mathbb{C}})$ is the bundle associated to E via the representation (1.18) defined above and K is the canonical bundle on X.

Remark 3.3.3. These objets are, in fact, K-twisted Higgs pair for a reductive Lie group of type ι^k (resp., ι_k). However, we have two special situation. On the one hand for $k=0$, from Proposition 1.3.3 we have that $H^\eta \exp(\mathfrak{m}_0) = G^\eta$. In the same fashion from Proposition 1.5.1 we have that $H_\eta \exp(\mathfrak{m}_0) = G_\eta$. Hence a $(G,\theta,H^\eta,\zeta_0)$-Higgs bundle is simply a (G^η,θ)-Higgs bundle and a $(G,\theta,H_\eta,\zeta_0)$-Higgs bundle is just a (G_η,θ)-Higgs bundle in the sense of Definition 2.1.1. On the other hand, for $p=n/2$ with n even, from Proposition 1.3.3 and 1.5.1 we have that a $(G,\theta,H^\eta,\zeta_p)$-Higgs bundle is simply (G^σ,θ)-Higgs bundle and that a $(G,\theta,H_\eta,\zeta_p)$-Higgs bundles is just a (G_σ,θ)-Higgs bundle. In these particular cases the notions of (semi)stability and polystability are hence defined in Definition 2.2.6. Otherwise the stability conditions given in Definition 2.2.6 can be extended to our general situation by replacing $\mathfrak{h}$ with $\mathfrak{h}_0$, and $\mathfrak{m}_s$ and $\mathfrak{m}_s^0$

with their analogue spaces associated to $\mathfrak{m}_k$ as we saw in Remark 2.2.8. Hence we can define $\mathcal{M}^\alpha(G,\theta,H^\eta,\zeta_k)$ the moduli space of isomorphism classes of polystable $(G,\theta,H^\eta,\zeta_k)$-Higgs bundles. In the same way we can define $\mathcal{M}(G,\theta,H_\eta,\zeta_k)$ the moduli space of isomorphism classes of polystable $(G,\theta,H_\eta,\zeta_k)$-Higgs bundles.

Remark 3.3.4. Notice that the subgroups H^η and H_η (defined in Chapter 1) complexify to $(H^{\mathbb{C}})^\eta$ and $H_\eta^{\mathbb{C}}$, respectively. This is why we decide to denote these objects as $(G,\theta,H^\eta,\zeta_k)$ and $(G,\theta,H_\eta,\zeta_k)$-Higgs bundles despite the fact that the structure groups of these bundles are $(H^{\mathbb{C}})^\eta$ and $H_\eta^{\mathbb{C}}$ respectively. We apologise since we know that it could result confusing, however we did not find a simpler way to encode all the meaningful information refering to these objects while emphasising the duality between them.

As we pointed out in Remark 2.3.2 since a $(G,\theta,H^\eta,\zeta_k)$-Higgs bundle (E,φ) is a Higgs pair of type $\rho:(H^{eta})^{\mathbb{C}}\to \mathrm{GL}(\mathfrak{m}_k^{\mathbb{C}})$ there exist Hitchin equations attached to these objects. With the same notation used in Theorem 2.3.4 we have the following.

Theorem 3.3.5. *Let (E,φ) be a $(G,\theta,H^\eta,\zeta_k)$-Higgs bundle. Then (E,φ) is polystable if and only if there exists a reduction h of the structure group of E to H^η satisfying the Hitchin equation*

$$F_h-[\varphi,\tau_h(\varphi)]=0,$$

where F_h is the curvature of the Chern connection on E with respect to h.

Proof. We refer the reader to [27, Theorem 2.24], where Gacia-Prada, Gothen and Mundet i Riera prooved Theorem 3.3.5 for a more general setting where (E,φ) is an L-twisted pair. Observe that the cases $k=0,1$ are covered by Theorem 2.3.4 since we are dealing with (G^η,θ)-Higgs bundles and (G^σ,θ)-Higgs bundle, respectively. □

Similar equations to the one appearing in Theorem 3.3.5 also exists and are attached to $(G,\theta,H_\eta,\zeta_k)$-Higgs bundles.

Theorem 3.3.6. *Let (E,φ) be a $(G,\theta,H_\eta,\zeta_k)$-Higgs bundle. Then (E,φ) is polystable if and only if there exists a reduction h of the structure group of E to H_η satisfying the Hitchin equation*

$$F_h-[\varphi,\tau_h(\varphi)]=0, \tag{3.6}$$

where F_h is the curvature of the Chern connection on E with respect to h.

3.3.1 Representations of the fundamental group

Let G be the connected component at the identity of a real form of a complex semisimple Lie group G and let θ be its Cartan involution. Recall that we impose the stability parameter α to lie in the center of $\mathfrak{g}$ and hence to be zero. Then we force the topological class d of (E, φ) to be zero. With the notation introduced in this section, we have the following result whose proof is similar to Theorem 2.4.4.

Theorem 3.3.7. *Let $\eta \in \mathrm{Aut}_n(G, \theta)$ with and consider $\sigma := \theta\eta$. There exist the following homeomorphisms*

1. $\mathcal{M}(G, \theta, H^\eta, \zeta_0) \cong \mathcal{R}(G^\eta, \theta)$.
2. $\mathcal{M}(G, \theta, H_\eta, \zeta_0) \cong \mathcal{R}(G_\eta, \theta)$.

In addition, if $n = 2l$ then there also exist the following homeomorphisms.

1. $\mathcal{M}(G, \theta, H^\eta, \zeta_l) \cong \mathcal{R}(G^\eta, \sigma)$.
2. $\mathcal{M}(G, \theta, H_\eta, \zeta_l) \cong \mathcal{R}(G_\eta, \sigma)$.

Under these homeomorphisms the irreducible representation are in correspondence with the stable and simple objects.

Remark 3.3.8. Notice that G^η and G_η are not necessarily semisimple or connected. Also observe that in the order 2 case, $(G, \theta, H^\eta, \zeta_k)$- Higgs bundles (respectively $(G, \theta, H_\eta, \zeta_k)$- Higgs bundles) always have an interpretation in terms of representations of the fundamental group of X since there is no other Higgs pairs besides (G^η, θ) or (G^σ, θ)-Higgs bundles (respectively, (G_θ, θ) or (G_σ, θ)-Higgs bundles).

3.3.2 Higgs bundles defined by involutions.

We now consider the case of $\eta \in \mathrm{Aut}_2(G, \theta)$, where as usual G is the connected component at the identity of a real form of a complex semisimple Lie group and θ

is its Cartan involution. These will appear in the study of involutions of $\mathcal{M}(G,\theta)$ considered in following sections.

As we saw in Chapter 1, since η commutes with θ, we can further decompose $\mathfrak{g}$ as direct sum of (± 1)-eigenspaces of $\mathfrak{h}$ and $\mathfrak{m}$ defined by η. Thus the complexification of the isotropy representation of H to $\mathfrak{m}$ restricts to two representations

$$(\iota^-)^+ : H_+^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_+^{\mathbb{C}}) \text{ and } (\iota^-)^- : H_+^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_-^{\mathbb{C}}).$$

Remark 3.3.9. In order not to confuse the reader with more unneccesary notation and to be consistent with the one introduced in Section 3.3, we change the notation used in Section 1.6 in the following way. We will denote the subgroup $H_+^{\mathbb{C}}$ by $(H^{\mathbb{C}})^\eta$ and its Lie algebra by $\mathfrak{h}_0^{\mathbb{C}}$. Now since the case $n=2$ is endowed with a special duality we decide to keep the notation used to name the subspaces $\mathfrak{m}_+$ and $\mathfrak{m}_-$. Thus we use $\mathfrak{m}_+^{\mathbb{C}}$ to refer to $\mathfrak{m}_0^{\mathbb{C}}$ and $\mathfrak{m}_-^{\mathbb{C}}$ to refer to $\mathfrak{m}_1^{\mathbb{C}}$. Accordingly, we refer to Higgs bundles defined by $\eta \in \mathrm{Aut}_2(G,\theta)$ as $(G,\theta,H^\eta,\pm)$-Higgs bundle in the sense of Definition 3.3.1 for $n=2$. That is, a $(G,\theta,H^\eta,\pm)$-Higgs bundle over X is a pair (E,φ) where E is a principal $(H^{\mathbb{C}})^\eta$-bundle and φ is a section of $E(\mathfrak{m}_k^{\mathbb{C}}) \otimes K$, where K is the canonical bundle on X and $E(\mathfrak{m}_k^{\mathbb{C}})$ is the bundl e associated to E via the representations $(\iota^-)^\pm : (H^{\mathbb{C}})^\eta \to \mathfrak{m}_\pm^{\mathbb{C}}$.

As we notice in Section 3.3 when $n=2p$ is even, $(G,\theta,H^\eta,\zeta_0)$-Higgs bundles are just (G^η,θ)-Higgs bundles and $(G,\theta,H^\eta,\zeta_p)$-Higgs bundles are just (G^σ,θ)-Higgs bundles with $\sigma := \theta\eta$. In our case $p=1$, then $(G,\theta,H^\eta,+)$-Higgs bundle are simply (G^η,θ)-Higgs bundles and $(G,\theta,H^\eta,-)$-Higgs bundle are simply (G^σ,θ)-Higgs bundles and there is no more other options. We will use interchangeably the $(G,\theta,H^\eta,\pm)$-Higgs bundles notation or the (G^η,θ) or (G^σ,θ) Higgs bundles notation.

Now, from Section 1.5, we have the short exact sequence

$$1 \to (H^{\mathbb{C}})^\eta \to H_\eta^{\mathbb{C}} \to \Gamma_\eta \to 1. \tag{3.7}$$

From Proposition 1.5.4 and from the representations $(\iota^-)^\pm$ introduced above, we can also consider the restrictions of the complexification of the isotropy representation of H to $H_\eta^{\mathbb{C}}$ in order to get two more representations

$$\iota_\pm^- : H_\eta^{\mathbb{C}} \to \mathrm{Gl}(\mathfrak{m}_\pm^{\mathbb{C}}).$$

Then, as we noticed for $(G,\theta,H^\eta,\pm)$-Higgs bundle, by a $(G,\theta,H_\eta,\pm)$-Higgs bundles we mean a pair (E,φ) as defined in Definition 3.3.2 for $n=2$. That is, a pair

(E, φ) where E is a principal $H_\eta^{\mathbb{C}}$-bundle and φ is a section of $E(\mathfrak{m}_\pm^{\mathbb{C}}) \otimes K$, where $E(\mathfrak{m}_\pm^{\mathbb{C}})$ is the bundle associated to E via the representations $(\iota^-)_\pm : H_\eta^{\mathbb{C}} \to \mathfrak{m}_\pm^{\mathbb{C}}$ and K is the canonical bundle on X

From Proposition 1.5.1, it is clear that $(H_\eta^{\mathbb{C}}, +)$-Higgs bundles are (G_η, θ)-Higgs bundles and $(H_\eta^{\mathbb{C}}, -)$-Higgs bundles are (G_σ, θ)-Higgs bundles and there is no more other choices. We will use interchangeably these notations.

3.4 Extending $(G, \theta, H^\eta, \zeta_k)$-Higgs bundles and $(G, \theta, H_\eta, \zeta_k)$-Higgs bundles to (G, θ)-Higgs bundles.

Let (E, φ) be a $(G, \theta, H^\eta, \zeta_k)$-Higgs bundle defined as in the previous section. Denote by $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ the principal $H^{\mathbb{C}}$-bundle obtained by extending the structure group of E to $H^{\mathbb{C}}$. Since $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = \oplus_k E(\mathfrak{m}_k^{\mathbb{C}})$ we define $\varphi_{H^{\mathbb{C}}}$ to be the section of $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$ associated to the Higgs field φ by taking 0 in the $E(\mathfrak{m}_l^{\mathbb{C}})$ component for $l \neq k$. We called the resulting (G, θ)-Higgs bundle $(E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}), \varphi_{H^{\mathbb{C}}})$ **the extension of** (E, φ). In the same manner, a **reduction of** a (G, θ)-Higgs bundle (E, φ) to a $(G, \theta, H^\eta, \zeta_k)$-Higgs bundle $(E_{(H^{\mathbb{C}})^\eta}, \varphi_k)$ is given by a reduction of the structure group of E to an $(H^{\mathbb{C}})^\eta$-bundle $E_{(H^{\mathbb{C}})^\eta}$ and by a Higgs field φ_k that takes values in $E_{(H^{\mathbb{C}})^\eta}(\mathfrak{m}_k^{\mathbb{C}}) \otimes K$. We consider an analogue construction for $(G, \theta, H_\eta, \zeta_k)$-Higgs bundles.

We can now state the following.

Proposition 3.4.1. *(1) Let (E, φ) be a polystable $(G, \theta, H^\eta, \zeta_k)$-Higgs bundle. Then the corresponding (G, θ)-Higss bundle $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ is also polystable. Hence the correspondence $(E, \varphi) \mapsto (E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ defines a map*

$$\mathcal{M}(G, \theta, H^\eta, \zeta_k) \to \mathcal{M}(G, \theta).$$

(2) Let (E, φ) be a (G, θ)-Higgs bundle and consider $(E_{(H^{\mathbb{C}})^\eta}, \varphi_k)$ its reduction to a $((H^{\mathbb{C}})^\eta, \zeta_k)$-Higgs bundle. If (E, φ) is (semi,poly)stable then $(E_{(H^{\mathbb{C}})^\eta}, \varphi_k)$ is also (semi,poly)stable.

(3) Let η, $\eta' \in \mathrm{Aut}_n(G, \theta)$ such that $\eta' = \mathrm{Int}(h)\eta\mathrm{Int}(h)$, with $h \in H$. Then $\mathcal{M}((G, \theta, H^\eta, \zeta_k)$ and $\mathcal{M}(G, \theta, H^{\eta'}, \zeta_k)$ are canonically isomorphic and since

Int(h) *acts trivially in* $\mathcal{M}(G,\theta)$ *we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}(G,\theta,H^{\eta},\zeta_k) & \to & \mathcal{M}(G,\theta) \\ \downarrow & \nearrow & \\ \mathcal{M}(G,\theta,H^{\eta'},\zeta_k). & & \end{array}$$

Proof. From Theorem 3.3.5 we have that if (E,φ) is polystable then there exists a reduction ρ of structure group from $(H^{\mathbb{C}})^{\eta}$ to H^{η}.Then ρ defines a reduction of structure group of $E_{H^{\mathbb{C}}}$ to H since $E((H^{\mathbb{C}})^{\eta}/H^{\eta}) \subset E_{H^{\mathbb{C}}}(H^{\mathbb{C}}/H)$. This reduction satisfies the Hitchin equations (2.5) given in Theorem 2.3.4. Hence by Theorem 2.3.4 $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ is polystable proving (1).

To prove (2) let us first suppose that $(E_{(H^{\mathbb{C}})^{\eta}},\varphi_k)$ is not semistable. On the one hand this means that there exists an $s \in i\mathfrak{h}$ defining a parabolic subgroup P_s of $(H^{\mathbb{C}})^{\eta}$ and a reduction μ of the structure group of $E_{(H^{\mathbb{C}})^{\eta}}$ to P_s such that $\deg(E_{(H^{\mathbb{C}})^{\eta}})(s,\mu) < 0$. On the other hand, $s \in i\mathfrak{h}$ also defines a parabolic subgroup $\widetilde{P}_s$ of $H^{\mathbb{C}}$ and the reduction μ defines a reduction $\widetilde{\mu}$ of E to $\widetilde{P}_s$ such that $\deg(E)(s,\widetilde{\mu}) = \deg(E_{(H^{\mathbb{C}})^{\eta}})(s,\mu)$ (the method to see this is described in [?] and [27]). If (E,φ) is semistable then $\deg(E_{(H^{\mathbb{C}})^{\eta}})(s,\mu) = \deg(E)(s,\widetilde{\mu}) \geq 0$. Using the same argument we prove stability and polystability.

(3) Define the principal $H^{\mathbb{C}}$-bundle $\eta(E) := E \times_{\eta_{\mathbb{C}}} H^{\mathbb{C}}$. Observe that if we associate to the Higgs field φ a section $\eta(\varphi)$ of $\eta(E)(\mathfrak{m}^{\mathbb{C}})$, then we get another (G,θ)-Higgs bundles $(\eta(E),\eta(\varphi))$. As we will see later, If η is an inner automorphism of H then (E,φ) and $(\eta(E),\eta(\varphi))$ turn out to be isomorphic, meaning that Int(H) acts trivially on the moduli space $\mathcal{M}(G,\theta)$ of isomorphism classes of polystable (G,θ)-Higgs bundles . □

We rephrase Proposition 3.4.1 to get the analogous results for the case of $(G,\theta,H_{\eta},\zeta_k)$-Higgs bundle.

Proposition 3.4.2. *(1) Let (E,φ) be a polystable $(G,\theta,H_{\eta},\zeta_k)$-Higgs bundle. Then the corresponding (G,θ)-Higss bundle $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ is also polystable. Hence the map $(E,\varphi) \mapsto (E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ defines a map $\mathcal{M}(G,\theta,H_{\eta},\zeta_k) \to \mathcal{M}(G,\theta)$.*

(2) Let (E,φ) a (G,θ)-Higgs bundle and consider $(E_{H^{\mathbb{C}}_{\eta}},\varphi_k)$ its reduction to a $(G,\theta,H_{\eta},\zeta_k)$-Higgs bundle. If (E,φ) is (semi,poly)stable then $(E_{H^{\mathbb{C}}_{\eta}},\varphi_k)$ is also (semi,poly)stable.

(3) Let η, $\eta' \in \mathrm{Aut}_n(G,\theta)$ such that $\eta' = \mathrm{Int}(h)\eta\mathrm{Int}(h)$, with $h \in H$. Then $\mathcal{M}(G,\theta,H_\eta,\zeta_k)$ and $\mathcal{M}(G,\theta,H_{\eta'},\zeta_k)$ are canonically isomorphic and since $\mathrm{Int}(h)$ acts trivially in $\mathcal{M}(G)$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(G,\theta,H_\eta,\zeta_k) & \rightarrow & \mathcal{M}(G,\theta) \\ \downarrow & \nearrow & \\ \mathcal{M}(G,\theta,H_{\eta'},\zeta_k). & & \end{array}$$

Proof. The proof is analogous to that of Proposition 3.4.1. □

3.5 Finite order outer automorphisms of Higgs bundle moduli spaces.

Let $\mathrm{Aut}(G,\theta)$, $\mathrm{Int}(G,\theta)$ and $\mathrm{Out}(G,\theta)$ be as defined in Chapter 1. Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and let (E,φ) be a (G,θ)-Higgs bundle over X. We define

$$\iota(\eta,\zeta_k)(E,\varphi) := (\eta_{\mathbb{C}}(E), \zeta_k\eta(\varphi)), \tag{3.8}$$

where $(\eta_{\mathbb{C}}(E), \zeta_k\eta(\varphi))$ is the (G,θ)-Higgs bundle defined by taking the associated principal bundle $\eta_{\mathbb{C}}(E) := E \times_{\eta_{\mathbb{C}}} H^{\mathbb{C}}$ to E via η and associating to the section φ of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$ a section $\eta_{\mathbb{C}}(\varphi)$ of $\eta_{\mathbb{C}}(E)(\mathfrak{m}^{\mathbb{C}}) \times K$. One can check that if $\eta \in \mathrm{Int}(G,\theta)$ then the Higgs bundle $(\eta_{\mathbb{C}}(E), \eta_{\mathbb{C}}(\varphi))$ is isomorphic to (E,φ). Hence the group $\mathrm{Out}_n(G,\theta)$ acts on the set of isomorphism classes of (G,θ)-Higgs bundles. Since (semi, poly)stability is preserved by the action of $\mathrm{Aut}_n(G,\theta)$ then $\mathrm{Out}_n(G,\theta)$ acts on the moduli space of (G,θ)-Higgs bundles $\mathcal{M}(G,\theta)$. From now we will denote $\eta_{\mathbb{C}}(E)$ by $\eta(E)$.

We also have an action of $\mathbb{C}^*$ on the set of (G,θ)-Higgs bundles. For any $\lambda \in \mathbb{C}^*$ and any (G,θ)-Higgs bundle (E,φ) over X we define

$$\lambda(E,\varphi) := (E, \lambda\varphi).$$

Observe that stability, semistability and polystability are preserved by this action. Hence this defines an action of $\mathbb{C}^*$ on $\mathcal{M}(G,\theta)$. In particular if we take $\lambda = -1$ this becomes an involution on $\mathcal{M}(G,\theta)$. In Section 3.6.1 we will develop this case.

Proposition 3.5.1. *Let (E,φ) be a (G^η,ζ_k)-Higgs bundle with $\theta\in \mathrm{Aut}_n(G,\theta)$ and let $(E_{H^\mathbb{C}},\varphi_{H^\mathbb{C}})$ be the corresponding extension to a (G,θ)-Higgs bundle described in Section 3.4. Then $(E_{H^\mathbb{C}},\varphi_{H^\mathbb{C}})$ is isomorphic to $(\eta(E_{H^\mathbb{C}}),\zeta_k\eta(\varphi_{H^\mathbb{C}}))$.*

Proof. We obtain $E_{H^\mathbb{C}}$ by extending the structure group of E to $H^\mathbb{C}$. Notice that $\eta^\mathbb{C}_{|H^\eta}=\mathrm{Id}$, then there is a canonical isomorphism of $E_{H^\mathbb{C}}$ with $\eta(E_{H^\mathbb{C}})=E_{H^\mathbb{C}}\times_\eta H^\mathbb{C}$. If φ takes values in the ζ_k-eigenspace of $\mathrm{ad}(\theta)$, it gives rise to a Higgs field on $E_{H^\mathbb{C}}$ on which $\mathrm{ad}(\theta)$ acts as ζ_k. □

Proposition 3.5.2. *Let (E,φ) be a simple (G,θ)-Higgs bundle isomorphic to $(E,\zeta_k\varphi)$ and $\eta\in\mathrm{Aut}_n(G,\theta)$. Then, except for $k=0$ and $\eta\in\mathrm{Int}(H)$ we have the following.*

(1) The structure group of E can be reduced to $(H^\mathbb{C})^{\eta'}$ with $\eta'=\mathrm{Int}(s)\eta$ and $s\in S^{n,\tau}_\eta$, with $S^{n,\tau}_\eta$ as defined in (1.13). In addition, s is unique up to the action of $H^\mathbb{C}$ and Z_τ as defined in (1) and (2) of Proposition 1.3.8 .

(2) If the Higgs field $\varphi\neq 0$, (E,φ) reduces to a $(G,\theta,H^\eta,\zeta_k)$-Higgs bundle, where $\eta'=\mathrm{Int}(s)\eta$.

Proof. Let $\eta\in\mathrm{Aut}_n(G,\theta)$ and let A be an isomorphism between E and $\eta(E)$ such that $\iota^-(A)(\varphi)=\eta(\varphi)$, where $\iota^-(A)(\varphi)$ is the automorphism of $E(\mathfrak{m}^\mathbb{C})\otimes K$ induced by A. As we explain in Section 3.1 we can think A as an η-twisted automorphism of E. Since η is of order n, A^n is an isomorphism of (E,φ) and since (E,φ) is simple $A^n=z\in Z_\tau$. Now, in Proposition 3.1.7 we proved that the map f_A defined in (3.1) maps E onto a unique orbit in $S^{n,\tau}_\eta$ under the action of $H^\mathbb{C}$ defined in Proposition 1.13. Moreover if we choose another isomorphism $A':E\to\eta(E)$ such that $\iota^-(A')(\varphi)=\eta(\varphi)$, then $A'A^{-1}=z'$ for $z\in Z_\tau$ meaning that $A'=z'A$ with $z'\in Z_\tau$ then if f_A defines an element $s\in S^{n,\tau}_\eta$, $f_{A'}$ defines an element $z's$, therefore the orbit defined by $f_{A'}$ is gi ven by the action Z_τ by multiplication on the orbit defined by f_A. Hence we obtain a single $(Z_\tau\times H^\mathbb{C})$-orbit in $S^{n,\tau}_\eta$ and by Proposition 3.1.7 this defines a reduction of structure group of E to $(H^\mathbb{C})^{\eta'}$ with $\eta'=\mathrm{Int}(s)\eta$, proving (1).

To prove (2), let $\eta'=\mathrm{Int}(s)\eta$ for some $s\in S^{n,\tau}_\eta/(Z_\tau\times H^\mathbb{C})$. Then we have a reduction of the structure group of E to $(H^\mathbb{C})^{\eta'}$. Let denote the reduced bundle by $E_{(H^\mathbb{C})^{\eta'}}$. The adjoint bundle decompose in

$$E(\mathfrak{m}^\mathbb{C})=E_{(H^\mathbb{C})^{\eta'}}(\mathfrak{m}^\mathbb{C}_0)\oplus E_{(H^\mathbb{C})^{\eta'}}(\mathfrak{m}^\mathbb{C}_p),$$

where $\mathfrak{m}^{\mathbb{C}}_p$ and $\mathfrak{m}^{\mathbb{C}}_0$ are the (ζ_k)-eigenspaces for $k = 0, p$ of $\mathfrak{m}^{\mathbb{C}}$ with respect to η'. Clearly, $\iota^-(A)(\varphi) = \zeta_k\eta(\varphi)$ is equivalent to $\varphi \in H^0(X, E_{(H^{\mathbb{C}})^{\eta'}}(\mathfrak{m}^{\mathbb{C}}_{\zeta_k}) \otimes K)$ for $k = 0, p$. Finally, since Z_τ is finite then $Z_\tau \subset H$. Using semidirect product it is only a matter of computation to prove that $s \in H$. □

The following theorem generalises to real groups one of the main results obtained in [31].

Theorem 3.5.3. *Let $a \in \mathrm{Out}_n(G, \theta)$. Consider the automorphism*

$$\begin{array}{rccc} \iota(a, \zeta_k): & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E,\varphi) & \mapsto & (a(E), \zeta_k a(\varphi)) \end{array}$$

where $\zeta_k = \exp 2\pi\frac{k}{n}$. Then

(1)
$$\bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^\eta,\zeta_k) \subset \mathcal{M}(G,\theta)^{\iota(a,\zeta_k)},$$

(2)
$$\mathcal{M}(G,\theta)^{\iota(a,\zeta_k)}_{simple} \subset \bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^\eta,\zeta_k).$$

Notice that $\iota(1,1)$ is the identity map.

Proof. Let $\eta \in Aut_n(G,\theta)$ and let (E,φ) be a (G,θ)-Higgs bundle. Recall that $\mathrm{Int}_n(G,\theta)$ acts trivially on $\mathcal{M}(G,\theta)$ and there exists a bijection between $S^{n,\tau}_\eta$ and $H^1_\eta(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)$.Let $\eta \in \mathrm{Out}_n(G,\theta)$ and $(E,\varphi) \in \mathcal{M}(G,\theta,H^\eta,\zeta_k)$. By (1) in Proposition 3.4.1 the image of (E,φ) in $\mathcal{M}(G,\theta)$ is given by an extension to a $(G,\theta,H^\eta,\zeta_k)$-Higgs bundle $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$. As we proved in Proposition 3.5.1 $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ is isomorphic to $(\eta(E_{H^{\mathbb{C}}}),\zeta_k\eta(\varphi_{H^{\mathbb{C}}}))$ hence $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}}) \in \mathcal{M}(G,\theta)^{\iota(\eta,\zeta_k)}$. Finally by (3) in Proposition 3.4.1 we get that if $\eta,\eta' \in \mathrm{Aut}_n(G,\theta)$ such that $\eta' \sim_\theta \eta$ then $\mathcal{M}(G,\theta,H^\eta,\zeta_k) \sim \mathcal{M}(G,\theta,H^{\eta'},\zeta_k)$ and their images coincide in $\mathcal{M}(G,\theta)$.

On the other hand, let $(E,\varphi) \in \mathcal{M}(G,\theta)$ a simple (G,θ)-Higgs bundle such that $(E,\varphi) \simeq (\eta(E),\eta(\varphi))$. The result follows from Proposition 3.5.2 combined with (2) and (3) of Proposition 3.4.1 and Lemma 1.3.9. □

3.6 Automorphisms defined by elements of order n in $H^1(X, Z_\tau) \rtimes \mathrm{Out}(G, \theta)$.

Let (E, φ) be a principal (G, θ)-Higgs bundle over a compact Riemann surface X. Consider the group $H^1(X, Z_\tau)$, of isomorphism classes of principal Z_τ-bundles over X. Let $\alpha \in H^1(X, Z_\tau)$ be a principal Z_τ-bundle. The fibre product $E \times_X \alpha$ has the structure of a principal $(H^{\mathbb{C}} \times Z_\tau)$-bundle as we explain at the begining of Section 3.2.

Recall that using the multiplication $m : H^{\mathbb{C}} \times Z_\tau \to H^{\mathbb{C}}$ we can extend the structure group of $E \times_X \alpha$ obtaining a principal $H^{\mathbb{C}}$-bundle $E \otimes \alpha := (E \times_X \alpha) \times_m H^{\mathbb{C}}$.

Since $Z_\tau = Z(H^{\mathbb{C}}) \cap \mathrm{Ker}(\iota^-)$ then $E(\mathfrak{m}^{\mathbb{C}}) = (E \otimes \alpha)(\mathfrak{m}^{\mathbb{C}})$. Hence the action of $H^1(X, Z_\tau)$ on the moduli space of (G, θ)-Higgs bundles $\mathcal{M}(G, \theta)$ is defined by

$$\alpha \cdot (E, \varphi) := (E \otimes \alpha, \varphi).$$

Observe that $\mathrm{Int}(H)$ acts trivially on Z_τ. This defines an action of $\mathrm{Out}_n(G, \theta)$ on Z_τ and hence an action of $\mathrm{Out}_n(G, \theta)$ on $H^1(X, Z_\tau)$.

Let us consider the semidirect product $H^1(X, Z_\tau) \rtimes \mathrm{Out}(G, \theta)$ defined by

$$(\beta, \eta_2) \cdot (\alpha, \eta_1) := (\beta \otimes \eta_2(\alpha), \eta_2\eta_1).$$

Notice that $H^1(X, Z_\tau) \rtimes \mathrm{Out}(G, \theta)$ acts on $\mathcal{M}(G, \theta)$ in the following way: for any (E, φ) polystable (G, θ)-Higgs bundle and for any $(\alpha, \eta) \in H^1(X, Z_\tau) \rtimes \mathrm{Out}(G, \theta)$

$$(\alpha, \eta) \cdot (E, \varphi) := (\eta(E) \otimes \alpha, \eta(\varphi))$$

Then let $(\alpha, \eta) \in (H^1(X, Z_\tau) \rtimes \mathrm{Out}(G, \theta))_n$, this means that $\eta \in \mathrm{Out}_n(G, \theta)$ and $\eta(\alpha) = \alpha^{-1}$. We can define involutions

$$\begin{array}{rccl} \iota(\alpha, \eta, \zeta_k) : & \mathcal{M}(G, \theta) & \to & \mathcal{M}(G, \theta) \\ & (E, \varphi) & \mapsto & (\eta(E) \otimes \alpha, \zeta_k \eta(\varphi)), \end{array}$$

with $\zeta_k = \exp(2\pi i \frac{k}{n})$.

Recall from Chapter 1 that given $\eta \in \mathrm{Aut}_n(G,\theta)$ we are able to define subgroups $(H^{\mathbb{C}})^\eta$, $H^{\mathbb{C}}_\eta$ of $H^{\mathbb{C}}$ and give the exact sequence (1.19). Let $\underline{H^{\mathbb{C}}_\eta}$ be the sheaf of continuous functions with values in $H^{\mathbb{C}}_\eta$. Consider $H^1(X, \underline{H^{\mathbb{C}}_\eta})$ the set of isomorphism classes of principal $H^{\mathbb{C}}_\eta$-bundles over X. Define the map

$$\gamma_\eta : H^1(X, \underline{H^{\mathbb{C}}_\eta}) \to H^1(X, \Gamma_\eta) \tag{3.9}$$

induced by the exact sequence (1.19), where by Γ_η we mean the sheaf of locally constant functions with values in Γ_η. This map associates to every $H^{\mathbb{C}}_\eta$-bundle an invariant $\gamma \in H^1(X, \Gamma_\eta)$. If we fix it, we can define the moduli space $\mathcal{M}_\gamma(G, \theta, H_\eta, \zeta_k)$ of polystable Higgs bundles (E, φ) such that $\gamma_\eta(E) = \gamma$. Notice that if for any Higgs bundles (E, φ) we have that $\gamma(E) = e \in H^1(X, \Gamma_\eta)$, the identity element, then the structure group of E reduces to $(H^{\mathbb{C}})^\eta$. We have the following result analogue to the one obtained in [31].

Proposition 3.6.1. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and let*

$$c_\eta : H^1(X, \Gamma_\eta) \to H^1(X, Z_\tau) \tag{3.10}$$

be the map induced by the homomorphism $\widetilde{c} : \Gamma_\eta \to Z_\tau$ defined in Proposition 1.5.5. We have:

(1) The map c_η is injective.

(2) Let $\gamma_\eta : H^1(X, \underline{H^{\mathbb{C}}_\eta}) \to H^1(X, \Gamma_\eta)$ be the map (3.9). Let (E, φ) be a $(G, \theta, H_\eta, \zeta_k)$-Higgs bundle with $\gamma_\eta(E) = \gamma$ and let $\alpha := c_\eta(\gamma)$. Let $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ be the extension of (E, φ) to a (G, θ)-Higgs bundle. Then $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ is isomorphic to $(\eta(E_{H^{\mathbb{C}}}), \zeta_k \eta(\varphi_{H^{\mathbb{C}}}))$.

Proof. Since Γ_η and Z_τ are finite, $H^1(X, \Gamma_\eta)$ and $H^1(X, Z_\tau)$ parameterize principal Γ_η-flat and Z_τ bundles respectively. Then (see for example [47, Lemma 1] for further details)

$$H^1(X, \Gamma_\eta) = \mathrm{Hom}(\pi_1(X), \Gamma_\eta)$$

and

$$H^1(X, Z_\tau) = \mathrm{Hom}(\pi_1(X), Z_\tau).$$

Hence the map (3.10) rewrite as

$$c_\eta : \mathrm{Hom}(\pi_1(X), \Gamma_\eta) \to \mathrm{Hom}(\pi_1(X), Z_\tau)$$

and (1) follows.

To prove (2) let (E,φ) be a $(G,\theta,H_\eta,\zeta_k)$-Higgs bundle. On the one hand recall that we can also extend (E,φ) to get a (G,θ)-Higgs bundle $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$. On the other hand using the exact sequence (1.19) and the homomorphism $\widetilde{c}$ we can also extend the structure group of (E,φ) to get a principal Z_τ-bundle α. Consider $\iota : H^{\mathbb{C}}_\eta \to H^{\mathbb{C}}$, the inclusion of $H^{\mathbb{C}}_\eta$ in $H^{\mathbb{C}}$. By extension of the structure group of E through $\eta \circ \iota$ we obtain $\eta(E_{H^{\mathbb{C}}})$. Let now x be in $H^{\mathbb{C}}_\eta$, then $\eta(x) = xx^{-1}\eta(x) = x(\widetilde{c}\circ f)(x)$ where $f : H^{\mathbb{C}}_\eta \to \Gamma_\eta$ is the natural surjection. Recall that $E_{H^{\mathbb{C}}} \otimes \alpha$ is obtained by extension of structure group using the multiplication map $m : H^{\mathbb{C}} \times Z_\tau \to H^{\mathbb{C}}$. Hence $\eta(E_{H^{\mathbb{C}}}) \otimes \alpha$ is obtained by e xtension of the structure group of E via $h \to (\eta(h), \eta(h)^{-1}h) \to h$ proving (2). □

The following generalise to real groups Proposition 6.8 in [31].

Proposition 3.6.2. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and $\alpha \in H^1(X,Z_\tau)$ such that*

$$\alpha\eta(\alpha)\cdots\eta^{n-1}(\alpha) = 1,$$

and let (E,φ) be a simple (G,θ)-Higgs bundle isomorphic to $(\eta(E)\otimes\alpha,\zeta_k\eta(\varphi))$. We have the following.

(1) The structure group of E can be reduced to $H^{\mathbb{C}}_{\eta'}$ with $\eta' = \mathrm{Int}(s)\eta$ and $s \in S(E)$ defined in Proposition 3.2.4 for n. In addition, if $\gamma_\eta(E) = \gamma$ where γ_η is the map defined in (3.9) and $\gamma \in H^1(X,\Gamma_{\eta'})$ then $c_{\eta'}(\gamma) = \alpha$, where $c_{\eta'}$ is the map defined in Proposition 3.6.1.

(2) The Higgs field φ takes values in the ζ_k-eigenspace of the automorphism of $\mathfrak{m}^{\mathbb{C}}$ defined by η'. We conclude that (E,φ) reduces to a $(G,\theta,H_{\eta'},\zeta_k)$-Higgs bundle.

(3) For any $\eta \in \mathrm{Int}_n(G,\theta)$ and $\alpha = 1$, (E,φ) is always isomorphic to $(\eta(E),\eta(\varphi))$.

Proof. Let E be a principal $H^{\mathbb{C}}$-bundle over a compact Riemann surface X. Recall that we denote by $\xi \otimes a$ the image of $(\xi,a) \in E \times_X \alpha$ in $E \otimes \alpha$. Also recall that the action of $H^{\mathbb{C}}$ on $E \times_X \alpha$ given by

$$(\xi,a)h := (\xi h, a)$$

goes down to an action of $H^{\mathbb{C}}$ on $E \otimes \alpha$,

$$(\xi \otimes a)\cdot h = \xi h \otimes a$$

that makes it a principal $H^{\mathbb{C}}$-bundle. Notice that the bundle $\eta(E)\otimes\alpha$ is just the quotient of $E\times\alpha$ under the action of Z_τ given by

$$(\xi, a) * z := (\xi\eta(z), z^{-1}a),$$

for any $z\in Z_\tau$. The action of $H^{\mathbb{C}}$ on $E\otimes\alpha$ given above induces an action of $H^{\mathbb{C}}$ on $\eta(E)\otimes\alpha$,

$$(\xi\otimes a)\cdot h = \xi\eta(h)\otimes a$$

making it a principal $H^{\mathbb{C}}$-bundle.

We assume that there is an isomorphism between the (G,θ)-Higgs bundle (E,φ) and $(\eta(E)\otimes\alpha, \zeta_k\eta(\varphi))$. It gives an isomorphism A of principal bundles between E and $\eta(E)\otimes\alpha$. This isomorphism induces an isomorphism

$$\overline{A}_k : \eta^k(E)\to\eta^{k+1}(E)\otimes\eta^k(\alpha)$$

and hence an isomorphism

$$\widetilde{A}_k : \eta^k(E)\otimes\beta\to\eta^{k+1}(E)\otimes\beta\otimes\eta^k(\alpha)$$

with $\beta\in H^1(X,Z_\tau)$. Hence we get an isomorphism

$$B := \widetilde{A}_{n-1}\circ\cdots\circ\widetilde{A}_1 : \eta(E)\otimes\alpha\to E\otimes\alpha\otimes\eta(\alpha)\otimes\cdots\otimes\eta^{n-1}(\alpha) = E$$

since $\alpha\eta(\alpha)\cdots\eta^{n-1}(\alpha)=1$. Then composing B with A we get an automorphism of E and hence an automorphism of (E,φ), which is given by an element $z\in Z_\tau$. In fact, one can check that $B\circ A$ is a (η, Z_τ)-twisted automorphism as defined in Section 3.2. From this one can see that $s\eta(s)\cdots\eta^{n-1}(s)\in Z_\tau$, i.e. $s\in S_\eta^{n,\tau}$, where $S_\eta^{n,\tau}$ is defined in (1.13). From (2) in Proposition 3.2.4 $s\in S(E)$ defines a reduction of the structure group of E to $(H^{\mathbb{C}})_{\eta'}$, where $\eta'=\mathrm{Int}(s)\eta$, and

$$(H^{\mathbb{C}})_{\eta'} = \{h\in H^{\mathbb{C}} : \eta(h) = hc(h), \text{ with } c(h)\in Z_\tau\}.$$

Recall from Section 3.1 that given a (η, Z_τ)-twisted automorphism A there exist a morphism $f_A : E\to H^{\mathbb{C}}$ defined by $A(\xi)=\xi f_A(\xi)$ such that

$$f_A(\xi h) = h^{-1}zf_A(\xi)\eta(h),$$

for $z=z(A,\eta,\xi,h)\in Z_\tau$ and it descends to a morphism $f_A : E\to H^{\mathbb{C}}/Z_\tau$ such that

$$f_A(\xi h) = h^{-1}f_A(\xi)\eta(h).$$

Now, consider the isomorphism $A : E \to \eta(E) \otimes \alpha$ as above, for every $\xi \in E$ there exists $a \in \alpha$ such that $A(\xi) = \xi h s \eta(h)^{-1} \otimes a$ for some $h \in H^{\mathbb{C}}$. Let

$$E' = \{\xi \in E : A(\xi) = \xi s \otimes a \text{ for some } a \in \alpha\} \subset E.$$

One can see that E' provides the reduction of structure group of E to $H^{\mathbb{C}}_{\eta'}$. Let Z'_τ be the image of $H^{\mathbb{C}}_{\eta'}$ in Z_τ given by the homomorphism

$$\begin{array}{rccl} \widetilde{c}: & H^{\mathbb{C}}_{\eta'} & \to & Z_\tau \\ & h & \mapsto & h\eta'(h)^{-1} = c(h)^{-1}. \end{array}$$

If $\xi \in E'$ then there exists a unique a_ξ such that $A(\xi) = \xi s \otimes a_\xi$. Notice that

$$A(\xi h) = (\xi s \otimes a_\xi) \cdot h = \xi s \eta'(h) \otimes a_\xi = \xi h (h^{-1} s \eta'(h)) \otimes a_{\xi h} = \xi h s \otimes a_{\xi h} c(h)$$

Then $a_{\xi h} = a_\xi \widetilde{c}(h)$ for all $\xi \in E'$ and $h \in H^{\mathbb{C}}_{\eta'}$. From this we get that $a_{E'}$ is a principal bundle which provides a reduction of the structure group of α to Z'_τ proving (1).

Statement (2) is trivial. □

We generalise now the main result obtained by O. García-Prada and S. Ramanan in [31].

Theorem 3.6.3. *Let $a \in \mathrm{Out}_n(G, \theta)$ and let $\alpha \in H^1(X, Z_\tau)$ such that*

$$\alpha a(\alpha) \cdots a^{n-1}(\alpha) = 1.$$

Consider the automorphism

$$\begin{array}{rccl} \iota(a, \alpha, \zeta_k): & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E, \varphi) & \mapsto & (a(E) \otimes \alpha, \zeta_k a(\varphi)). \end{array} \tag{3.11}$$

Then

(1)
$$\bigcup_{[\eta] \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_\eta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_\eta, \zeta_k) \subset \mathcal{M}(G,\theta)^{\iota(a,\alpha,\zeta_k)},$$

(2)
$$\mathcal{M}(G,\theta)^{\iota(a,\alpha,\zeta_k)}_{simple} \subset \bigcup_{[\eta] \in H^1_a(\mathbb{Z}/n,\, H^{\mathbb{C}}/Z_\tau),\, c_\eta(\gamma) = \alpha} \widetilde{\mathcal{M}}_\gamma(G, \theta, H_\eta, \zeta_k)$$

except for $\iota(1, 1, \zeta_0)$, since $\iota(1, 1, \zeta_0) = \mathrm{Id}$.

Remark 3.6.4. Theorem 3.6.3 is a generalisation of Theorem 3.5.3 where we considered α to be the neutral element in $H^1(X, Z_\tau)$.

Proof. Let $\eta \in \mathrm{Out}_n(G,\theta)$, $\alpha \in H^1(X, Z_\tau)$ such that $\eta(\alpha) = \alpha^{-1}$ and let (E,φ) be a (G,θ)-Higgs bundle. Note that $(E,\varphi) \simeq (\eta(E) \otimes \alpha, \zeta_k \eta(\varphi))$ is equivalent to $(E,\varphi) \simeq (\widetilde{\eta}(E) \otimes \alpha, \zeta_k \widetilde{\eta}(\varphi))$ for any $\widetilde{\eta} \in \mathrm{Aut}_n(G,\theta)$ such that $\pi(\widetilde{\eta}) = \eta$ with $\pi : \mathrm{Aut}_n(G,\theta) \to \mathrm{Out}_n(G,\theta)$ the natural projection. Hence let $\eta \in \mathrm{Aut}_n(G,\theta)$ and $(E,\varphi) \in \mathcal{M}_\gamma(G,\theta,H_\eta,\zeta_k)$. By (1) in Proposition 3.4.2 the extended (G,θ)-Higgs bundle $(E_{H^\mathbb{C}}, \varphi_{H^\mathbb{C}})$ give us the image of (E,φ) in $\mathcal{M}(G,\theta)$. Recall that from Proposition 3.6.1 $(E_{H^\mathbb{C}}, \varphi_{H^\mathbb{C}})$ is isomorphic to $(\eta(E_{H^\mathbb{C}}) \otimes \alpha, \zeta_k \eta(\varphi_{H^\mathbb{C}}))$ with $\alpha = c_\eta(\gamma)$, then $(E_{H^\mathbb{C}}, \varphi_{H^\mathbb{C}}) \in \mathcal{M}(G,\theta)^{\iota(\eta,\alpha,\zeta_k)}$. Finally let $\eta' \in \mathrm{Aut}_n(G,\theta)$ such that $\eta' \sim_\theta \eta$, then by (3) in Proposition 3.4.2 $\mathcal{M}(H^\mathbb{C}_\eta,\theta)$ and $\mathcal{M}(H^\mathbb{C}_{\eta'},\theta)$ are isomorphic and their images coincide in $\mathcal{M}(G,\theta)$.

(2) is a consequence of Proposition 3.6.2 combined with (2) and (3) in Proposition 3.4.2 and Lemma 1.3.9. □

3.6.1 Involutions of Higgs bundle moduli spaces.

Now we will focus our results on the case of involutions. As usual let G be a connected real semisimple Lie group with Cartan involution θ and let $\mathfrak{g}$ be its Lie algebra. Recall that in this case $\mathfrak{g}$ decomposes as direct sum of ± 1-eigenspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Consider $\eta \in \mathrm{Out}_2(G,\theta)$. As we explained in the previous Section $\mathrm{Out}_2(G,\theta)$ acts on $\mathcal{M}(G,\theta)$, the moduli space of (g,θ)-Higgs bundles, and allows us to define what we called $(G,\theta,H^\eta,\pm)$-Higgs bundles and $(G,\theta,H_\eta,\pm)$-Higgs bundles. Recall from Subsection 3.3.2 that we denote by $\mathcal{M}(G,\theta,H^\eta,\pm)$ and $\mathcal{M}(G,\theta,H_\eta,\pm)$ the moduli spaces of polystable $(G,\theta,H^\eta,\pm)$-Higgs bundles and $(G,\theta,H_\eta,\pm)$-Higgs bundles respectively. In this situation notice that

$$\mathcal{M}((H^\mathbb{C})^\eta,+) = \mathcal{M}(G^\eta,\theta),$$

$$\mathcal{M}((H^\mathbb{C})^\eta,-) = \mathcal{M}(G^\sigma,\theta),$$

$$\mathcal{M}(H^\mathbb{C}_\eta,+) = \mathcal{M}(G_\eta,\theta),$$

$$\mathcal{M}(H^\mathbb{C}_\eta,-) = \mathcal{M}(G_\sigma,\theta),$$

where $\sigma = \eta\theta$. Theorem 3.5.3 specialise to the following.

Theorem 3.6.5. *Let* $a \in \mathrm{Out}_2(G,\theta)$. *Consider the involutions*

$$\begin{array}{rccc} \iota(\eta,\pm): & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E,\varphi) & \mapsto & (a(E), \pm a(\varphi)). \end{array}$$

Then

(1)

$$\bigcup_{[\eta]\in H^1_a(\mathbb{Z}/2, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^\eta,\pm) \subset \mathcal{M}(G,\theta)^{\iota(a,\pm)},$$

(2)

$$\mathcal{M}(G,\theta)^{\iota(a,\pm)}_{simple} \subset \bigcup_{[\eta]\in H^1_a(\mathbb{Z}/2, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^\eta,\pm).$$

Remark 3.6.6. When η is the trivial element we only have the involution

$$\begin{array}{rccc} \iota: & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E,\varphi) & \mapsto & (E,-\varphi). \end{array}$$

since $\iota(1,+)$ is the identity map. We have the following.

Theorem 3.6.7. *Let* $a \in \mathrm{Out}_2(G,\theta)$ *be the trivial element. Consider the involution*

$$\begin{array}{rccc} \iota: & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E,\varphi) & \mapsto & (E,-\varphi). \end{array}$$

Then

(1)

$$\bigcup_{[\eta]\in \mathrm{Int}_2(H)/\sim_\theta} \widetilde{\mathcal{M}}(G,\theta,\eta,-) \subset \mathcal{M}(G,\theta)^{\iota},$$

(2)

$$\mathcal{M}(G,\theta)^{\iota}_{simple} \subset \bigcup_{[\eta]\in \mathrm{Int}_2(H)/\sim_\theta} \widetilde{\mathcal{M}}(G,\theta,\eta,-).$$

Theorem 3.6.3 specialise to the following.

Theorem 3.6.8. *Let* $a \in \mathrm{Out}_2(G,\theta)$ *and let* $\alpha \in H^1(X,Z_\tau)$ *such that* $a(\alpha) = \alpha^{-1}$. *Consider the involutions*

$$\begin{array}{rccc} \iota(a,\alpha,\pm): & \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ & (E,\varphi) & \mapsto & (a(E)\otimes\alpha, \pm a(\varphi)). \end{array} \tag{3.12}$$

Then

(1)
$$\bigcup_{[\eta]\in H^1_a(\mathbb{Z}/2,\, H^{\mathbb{C}}/Z_\tau),\, c_\eta(\gamma)=\alpha} \widetilde{\mathcal{M}}_\gamma(G,\theta,H_\eta,\pm) \subset \mathcal{M}(G)^{\iota(a,\alpha,\pm)},$$

(2)
$$\mathcal{M}(G)^{\iota(a,\alpha,\pm)}_{simple} \subset \bigcup_{[\eta]\in H^1_a(\mathbb{Z}/2,\, H^{\mathbb{C}}/Z_\tau),\, c_\eta(\gamma)=\alpha} \widetilde{\mathcal{M}}_\gamma(G,\theta,H_\eta,\pm)$$

except for $\iota(1,1,+)$, since $\iota(1,1,+)=id$.

3.7 The action of $H^1(X,Z_\tau)\rtimes \mathrm{Out}(G,\theta)$ on the moduli space of representations

The automorphisms studied in Sections 3.5 and 3.6 induce in a natural way automorphisms on the moduli space of representations through the homeomorphism between $\mathcal{R}(G,\theta)$ and $\mathcal{M}(G,\theta)$ given by Theorem 2.4.4. In this section we give a complete description of this automorphisms and study their fixed points subsets.

Let G be the connected component at the identity of a real form of a complex semisimple Lie group $G^{\mathbb{C}}$ and let θ be its Cartan involution. Let μ be the conjugation defining the real form of $G^{\mathbb{C}}$ and let $\tau := \theta\mu$ be a fixed compact conjugation of $G^{\mathbb{C}}$. We denote in the same way their corresponding complexifications and involutions at the Lie algebra level. Let $H = G^\theta$ be the maximal compact subgroup of G defined by θ and denote its complexification by $H^{\mathbb{C}}$. Hence τ and μ coincide on $H^{\mathbb{C}}$. In addition, If $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ is the complexification of the Cartan decomposition of $\mathfrak{g}$ then $\tau=\mu$ on $\mathfrak{h}^{\mathbb{C}}$ and $\tau=-\mu$ on $\mathfrak{m}^{\mathbb{C}}$.

Let us consider (E,φ) a polystable (G,θ)-Higgs bundle over a compact Riemann surface X. From Theorem 2.3.4 there exists a metric h in E such that it satisfies the Hitchin equation and in that case

$$D = d_h + \varphi - \tau_h(\varphi)$$

defines a flat G-connection on the bundle E, where τ_h is defined in Section 2.3 from the compact conjugation τ of $G^{\mathbb{C}}$, h and the complex conjugation on $(1,0)$-forms on X and d_h is an H-connection compatible with h and the holomorphic structure $\overline{\partial}_E$ of E. In the same way, we extend θ and μ to

$$\theta_h : \Omega^{1,0}(E(\mathfrak{g}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{g}^{\mathbb{C}})),$$

and

$$\mu_h : \Omega^{1,0}(E(\mathfrak{g}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{g}^{\mathbb{C}})).$$

In particular, since $\theta \in \mathrm{Aut}_2(\mathfrak{g}^{\mathbb{C}})$ and μ and θ commutes, they extend to

$$\theta_h : \Omega^{1,0}(E(\mathfrak{h}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{h}^{\mathbb{C}})), \quad \theta_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}))$$

and

$$\mu_h : \Omega^{1,0}(E(\mathfrak{h}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{h}^{\mathbb{C}}) \quad \mu_h : \Omega^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to \Omega^{0,1}(E(\mathfrak{m}^{\mathbb{C}}).$$

We have the following proposition similar to Proposition 8.4 in [31].

Proposition 3.7.1. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and $\sigma_h := \theta_h\eta_h$ where η_h is similarly constructed to τ_h and let us denote their differentials also by η_h and σ_h. Then*

1. *The flat G-connection corresponding to $(\eta(E), \zeta_0\eta(\varphi))$ is given by $\eta_h(D)$.*
2. *If $n = 2l$ the flat G-connection corresponding to $(\eta(E), \zeta_l\eta(\varphi))$ is given by $\sigma_h(D)$*

Proof. Let (E,φ) be a (G,θ)-Higgs bundle over X. Recall that $\tau = \mu$ on $\mathfrak{h}^{\mathbb{C}}$. In fact, τ and μ extends to τ_h and μ_h (see Section 2.3 for further details) and one can check that $\tau_h(\varphi) = -\mu_h(\varphi)$. Hence we can rewrite d_h as $d_h = \overline{\partial}_E + \tau_h(\overline{\partial}_E)$ (or $d_h = \overline{\partial}_E - \mu_h(\overline{\partial}_E)$). From this we have that

$$\eta_h(D) = \eta_h(\overline{\partial}_E) + \eta_h(\tau_h(\overline{\partial}_E)) + \eta_h(\varphi) - \eta_h(\tau_h(\varphi)).$$

Since $\zeta_0 = 1$, $\eta\tau = \tau\eta$ and $\eta_h(\overline{\partial}_E) = \overline{\partial}_{\eta_h(E)}$ we have that

$$\eta_h(D) = \overline{\partial}_{\eta_h(E)} + \tau_h(\overline{\partial}_{\eta_h(E)}) + \zeta_0\eta_h(\varphi) + \tau_h(\zeta_0\eta_h(\varphi))$$

and hence (1) is proved. To prove (2) suppose that $= 2l$. Then, since $\tau_h(\overline{\partial}_E) = \mu_h(\overline{\partial}_E)$ and $\tau_h(\varphi) = -\mu_h(\varphi)$

$$\begin{aligned}
\sigma_h(D) &= \sigma_h(\overline{\partial}_E) + \sigma_h\tau_h(\overline{\partial}_E) + \sigma_h(\varphi) - \sigma_h\tau_h(\varphi) \\
&= \sigma_h(\overline{\partial}_E) + \sigma_h\mu_h(\overline{\partial}_E) + \sigma_h(\varphi) + \sigma_h\mu_h(\varphi) \\
&= \sigma_h(\overline{\partial}_E) + \eta_h\tau_h(\overline{\partial}_E) + \sigma_h(\varphi) + \eta_h\tau_h(\varphi).
\end{aligned}$$

Now, since $\zeta_l = -1$, $\theta_h(\overline{\partial}_E) = \overline{\partial}_E$ and $\theta_h(\varphi) = \zeta_l\varphi$ we have

$$\begin{aligned}
\sigma_h(D) &= \eta_h\theta_h(\overline{\partial}_E) + \tau_h\eta_h(\overline{\partial}_E) + \eta_h\theta_h(\varphi) + \eta_h\tau_h(\varphi) \\
&= \eta_h(\overline{\partial}_E) + \tau_h\eta_h(\overline{\partial}_E) + \eta_h(\zeta_l\varphi) + \tau_h\eta_h(\varphi) \\
&= \overline{\partial}_{\eta_h(E)} + \tau_h(\overline{\partial}_{\eta_h(E)}) + \zeta_l\eta_h(\varphi) - \tau_h(\zeta_l\eta_h(\varphi))
\end{aligned}$$

and hence $\sigma_h(D)$ is the flat G-connection corresponding to $(\eta(E), \zeta_l\eta(\varphi))$. □

As a corollary of Proposition 3.7.1 join with Theorem 2.4.4 and 2.4.7 we have the following.

Proposition 3.7.2. *Let $\eta \in \mathrm{Aut}_n(G,\theta)$ and let $\sigma := \eta\theta$. Let (E,φ) be a polystable (G,θ)-Higgs bundle and let us denote its corresponding reductive representation by $\rho \in \mathcal{R}(G,\theta)$. Then:*

1. *The n-automorphism of $\mathcal{M}(G,\theta)$ given by*

$$\begin{array}{rcl} \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ (E,\varphi) & \mapsto & (\eta(E),\zeta_0\eta(\varphi)) \end{array}$$

corresponds with the automorphism

$$\begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \eta(\rho). \end{array}$$

In addition, if $n = 2l$ then,

(2) The n-automorphism of $\mathcal{M}(G,\theta)$ given by

$$\begin{array}{rcl} \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ (E,\varphi) & \mapsto & (\eta(E),\zeta_l\eta(\varphi)) \end{array}$$

corresponds with the automorphism

$$\begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \sigma(\rho). \end{array}$$

Remark 3.7.3. If we consider just arbitrary automorphisms of $\mathcal{M}(G,\theta)$ defined by elements in $\mathrm{Aut}_2(G,\theta)$ then there only exist involutions

$$\begin{array}{rcl} \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ (E,\varphi) & \mapsto & (\eta(E),\pm\eta(\varphi)) \end{array}$$

and hence we get involutions

$$\begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \eta(\rho) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \sigma(\rho). \end{array}$$

Remark 3.7.4. Recall that by a slight abuse of notation we are denoting $\eta_{\mathbb{C}}$ and $\sigma_{\mathbb{C}}$ just as η and σ. Then the automorphisms $\rho \mapsto \eta(\rho)$ and $\rho \mapsto \sigma(\rho)$ defined in Proposition 3.7.2 only depends on the cohomology classes of $[\eta_{\mathbb{C}}] \in H^1_\eta(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)$ and $[\sigma_{\mathbb{C}}] \in H^1_\eta(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)$, respectively.

Let $a \in \mathrm{Out}_n(G,\theta)$ such that $\eta \in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)$ and $\sigma := \eta\theta \in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)$. For any $\rho \in \mathcal{R}(G,\theta)$ we define

$$a^+(\rho) := \theta(\rho) \text{ and if } n = 2l,\ a^-(\rho) := \sigma(\rho).$$

Let us denote by $\mathcal{R}_{irred}(G,\theta) \subset \mathcal{R}(G,\theta)$ the subset of irreducible representations of $\pi_1(X)$ in G. Recall that under the homeomorphism described in Theorem 2.4.4 irreducible representations of $\pi_1(X)$ in G are in correspondence with the stable and simple (G,θ)-Higgs bundles.

Now, from Theorem 3.3.7 we have similar results to the one obtained in Propositions 3.4.1 and 3.4.2 for the moduli space of representations $\mathcal{R}(G^\eta,\theta), \mathcal{R}(G_\eta,\theta)$, $\mathcal{R}(G^\sigma,\theta)$ and $\mathcal{R}(G_\sigma,\theta)$. If we denote by $\widetilde{\mathcal{R}}(G^\eta,\theta), \widetilde{\mathcal{R}}(G_\eta,\theta)$, $\widetilde{\mathcal{R}}(G^\sigma,\theta)$ and $\widetilde{\mathcal{R}}(G_\sigma,\theta)$, respectively, their images in $\mathcal{R}(G,\theta)$ we have the following generalisation of Theorem 8.6 in [31].

Theorem 3.7.5. *Let $a \in \mathrm{Out}_n(G,\theta)$. Consider the automorphism*

$$\begin{array}{rccc} \iota(a,\pm): & \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ & \rho & \mapsto & a^\pm(\rho) \end{array}$$

Then

(1)
$$\bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{R}}(G^\eta,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,+)},$$

(2)
$$\mathcal{R}(G,\theta)^{\iota(a,+)}_{irred} \subset \bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{R}}(G^\eta,\theta).$$

Notice that $\iota(1,+)$ is the identity map. If, in addition, $n = 2l$ then

(3)
$$\bigsqcup_{[\sigma]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{R}}(G^\sigma,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,-)},$$

(4)

$$\mathcal{R}(G,\theta)^{\iota(a,-)}_{irred} \subset \bigcup_{[\sigma]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{R}}(G^\sigma,\theta).$$

with $\sigma = \tau\eta$.

Proof. Theorem 3.7.5 is a consequence of Theorem 3.3.7 and Theorem 3.5.3. □

In order to study the fixed points of automorphism of $\mathcal{R}(G,\theta)$ determined by automorphism of $\mathcal{M}(G.\theta)$ defined by elements of order n in $H^1(X,Z) \rtimes \mathrm{Out}(G,\theta)$ first notice that Proposition 3.7.2 can be extended to our situation in the following way.

Proposition 3.7.6. *Let* $\eta \in \mathrm{Aut}_n(G,\theta)$ *and let* $\sigma := \eta\theta$. *Let* $\alpha \in H^1(X,Z_\tau)$ *be a flat* Z_τ*-bundle over* X *such that* $\alpha\eta(\alpha)\cdots\eta^{n-1}(\alpha) = 1$ *and denote by* λ *its corresponding representation. Observe that this is equivalent* $\lambda\eta(\lambda)\cdots\eta^{n-1}(\lambda) = \lambda\sigma(\lambda)\cdots\sigma^{n-1}(\lambda) = 1$. *Let* (E,φ) *be a polystable* (G,θ)*-Higgs bundle and let us denote its corresponding reductive representation by* $\rho \in \mathcal{R}(G,\theta)$. *Then:*

1. *The* n*-automorphism of* $\mathcal{M}(G,\theta)$ *given by*

$$\begin{array}{rcl} \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ (E,\varphi) & \mapsto & (\eta(E)\otimes\alpha, \zeta_0\eta(\varphi)) \end{array}$$

corresponds with the automorphism

$$\begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \lambda\eta(\rho). \end{array} \tag{3.13}$$

In addition, if $n = 2l$ *then,*

(2) The n*-automorphism of* $\mathcal{M}(G,\theta)$ *given by*

$$\begin{array}{rcl} \mathcal{M}(G,\theta) & \to & \mathcal{M}(G,\theta) \\ (E,\varphi) & \mapsto & (\eta(E)\otimes\alpha, \zeta_l\eta(\varphi)) \end{array}$$

corresponds with the automorphism

$$\begin{array}{rcl} \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ \rho & \mapsto & \lambda\sigma(\rho). \end{array} \tag{3.14}$$

Remark 3.7.7. The automorphisms (3.13) and (3.14) are well defined since $Z_\tau \subset H$ is compact and hence every representation $\lambda : \pi_1(X) \to Z_\tau$ is reductive. In addition, a representation from $\pi_1(X)$ to $Z_\tau \times G$ is a pair (λ, ρ) consisting on two representations $\lambda : \pi_1(X) \to Z_\tau$ and $\rho : \pi_1(X) \to G$. This representation extends to a representation from $\pi_1(X)$ to G in the following way.

$$\begin{array}{ccc} Z_\tau \times G & \xrightarrow{m} & G \\ {\scriptstyle (\lambda,\rho)} \uparrow & \nearrow {\scriptstyle \lambda\cdot\rho} & \\ \pi_1(X), & & \end{array}$$

where $m : Z_\tau \times G \to G$ is the standar multiplication on G. Notice that by $\lambda\eta(\rho)$ and $\lambda\sigma(\rho)$ we mean representations

$$\begin{array}{rccc} \lambda\eta(\rho): & \pi_1(X) & \to & G \\ & \gamma & \mapsto & \lambda(\gamma)\eta(\rho)(\gamma) \end{array}$$

and

$$\begin{array}{rccc} \lambda\sigma(\rho): & \pi_1(X) & \to & G \\ & \gamma & \mapsto & \lambda(\gamma)\sigma(\rho)(\gamma). \end{array}$$

Recall the exact sequence (1.19) given in Section 1.5. Analogously, for $\sigma = \eta\theta$ we define $\Gamma_\sigma := H^{\mathbb{C}}_\sigma / (H^{\mathbb{C}})^\sigma$ and then we have the exact sequence

$$1 \to (H^{\mathbb{C}})^\sigma \to H^{\mathbb{C}}_\sigma \to \Gamma_\sigma \to 1. \tag{3.15}$$

Observe that since $H^{\mathbb{C}} = (G^{\mathbb{C}})^\theta$ then $\Gamma_\eta = \Gamma_\sigma$. Using a similar construction to the one employed to define the map (3.9), from (1.19) and (3.15) we define maps

$$\widetilde{\gamma}_\eta : \mathcal{R}(G_\eta, \theta) \to \mathcal{R}(\Gamma_\eta, \theta) \quad \text{and} \quad \widetilde{\gamma}_\sigma : \mathcal{R}(G_\sigma, \theta) \to \mathcal{R}(\Gamma_\sigma, \theta),$$

where $\mathcal{R}(\Gamma_\eta, \theta) = \mathrm{Hom}(\pi_1(X), \Gamma_\eta)$ and $\mathcal{R}(\Gamma_\sigma, \theta) = \mathrm{Hom}(\pi_1(X), \Gamma_\sigma)$. These maps assign to every representation $\rho \in \mathcal{R}(G_\eta, \theta)$ (resp. $\rho \in \mathcal{R}(G_\sigma, \theta)$) an invariant $\widetilde{\gamma}_\eta(\rho) \in \mathcal{R}(G_\eta, \theta)$ (resp. $\widetilde{\gamma}_\eta(\rho) \in \mathcal{R}(G_\sigma, \theta)$). We denote by $\mathcal{R}_\gamma(G_\eta, \theta)$ and $\mathcal{R}_\gamma(G_\sigma, \theta)$ the moduli space of reductive representations ρ from $\pi_1(X)$ to G_η (resp. to G_σ) such that $\gamma_\eta(\rho) = \gamma$ (resp. $\gamma_\sigma(\rho) = \gamma$). We will denote their images in $\mathcal{R}(G, \theta)$ by $\widetilde{\mathcal{R}}_\gamma(G_\eta, \theta)$ and $\widetilde{\mathcal{R}}_\gamma(G_\sigma, \theta)$. We also have injective homomorphisms

$$\widetilde{c}_\eta : \mathcal{R}(\Gamma_\eta, \theta) \to \mathcal{R}(Z_\tau) \quad \text{and} \quad \widetilde{c}_\sigma : \mathcal{R}(\Gamma_\sigma, \theta) \to \mathcal{R}(Z_\tau),$$

where $\mathcal{R}(Z_\tau) = \mathrm{Hom}(\pi_1(X), Z_\tau)$. This follows from Proposition 1.5.5 and 3.6.1 and from the fact that $\Gamma_\sigma = \Gamma_\eta$.

We give now a generalisation of Theorem 8.8 in [31].

Theorem 3.7.8. *Let $a \in \mathrm{Out}_n(G,\theta)$ and $\lambda \in \mathcal{R}(Z_\tau)$ such that*

$$\lambda a(\lambda) \cdots a^{n-1}(\lambda) = 1.$$

Consider the automorphism

$$\begin{array}{rccc} \iota(a,\lambda,\pm): & \mathcal{R}(G,\theta) & \to & \mathcal{R}(G,\theta) \\ & \rho & \mapsto & \lambda a^{\pm}(\rho), \end{array}$$

with $a^{\pm}(\rho)$ defined as in Theorem 3.7.5. Then

(1)
$$\bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,\lambda,+)},$$

(2)
$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,+)}_{irred} \subset \bigcup_{[\eta]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\eta(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\eta,\theta).$$

Notice that $\iota(1,1,+)$ is the identity map. If, in addition, $n = 2l$ then

(3)
$$\bigcup_{[\sigma]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta) \subset \mathcal{R}(G,\theta)^{\iota(a,\lambda,-)},$$

(4)
$$\mathcal{R}(G,\theta)^{\iota(a,\lambda,-)}_{irred} \subset \bigcup_{[\sigma]\in H^1_a(\mathbb{Z}/n, H^{\mathbb{C}}/Z_\tau),\, \widetilde{c}_\sigma(\gamma)=\lambda} \widetilde{\mathcal{R}}_\gamma(G_\sigma,\theta).$$

Proof. Theorem 3.7.8 is a consequence of Theorem 3.3.7 and Theorem 3.6.3. □

Chapter 4

Examples.

4.1 Higgs bundles and stability.

Let X be a compact Riemann surface. A **Higgs bundle over a** X is a pair (V, ϕ) where $V \to X$ is a holomorphic vector bundle and the Higgs field is a K-twisted endomorphism $\phi : V \to V \times K$, where K is the canonical line bundle of X, i.e. $\phi \in H^0(X, \mathrm{End}(V) \otimes K)$. These objects were introduced by Hitchin [38] when studying solutions of the self-dual Yang-Mills equations. Let then (V, ϕ) be a Higgs bundle, the **slope** of V is defined as

$$\mu(V) := \deg V / \mathrm{rank}\, V,$$

where $\deg V$ is the **degree of** V, i.e. the integral of the first Chern class $c_1(V)$.

Definition 4.1.1. A subbundle $W \subseteq E$ is φ-invariant if and only if $\phi(W) \subseteq W \otimes K$. We say that

- A Higgs bundle (V, φ) is **stable** if $\mu(W) < \mu(V)$ for any proper φ-invariant subbundle $W \subset V$.
- A Higgs bundle (V, φ) is **semistable** if $\mu(W) \leq \mu(V)$ for any φ-invariant subbundle $W \subseteq V$.
- A Higgs bundle (V, φ) is **polystable** if

  $$(V, \phi) = (W_1 \oplus \cdots \oplus W_r\, ,\, \phi_1 \oplus \cdots \oplus \phi_r)$$

  where each (W_i, ϕ_i) is a stable Higgs bundle of slope $\mu(W_i) = \mu(V)$.

Now, let G be a complex reductive Lie group and denote by U its maximal compact subgroup. Recall that a G-Higgs bundle over X in the classical sense is a pair consisting on a holomorphic principal G-bundle $E \to X$ and a holomorphic section $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$, where $E(\mathfrak{g})$ is the bundle associated to E via the adjoint representation and K is the canonical bundle. Recall that the stability conditions for these objects are similar to the ones given in Definition 2.2.6 where, in this case, $\alpha \in \mathfrak{z}(\mathfrak{u})$ and we replace $H^{\mathbb{C}}$ by G and $\mathfrak{m}^{\mathbb{C}}$ by $\mathfrak{g}^{\mathbb{C}}$. When $G = \mathrm{GL}(n, \mathbb{C})$ the standard representation of $\rho : G \to \mathbb{C}^n$ give a one-to-one correspondence between vector bundles V of rank n and principal $\mathrm{GL}(n, \mathbb{C})$-bundles E. In fact, $E(\mathfrak{gl}(n, \mathbb{C})) = \mathrm{End}(V)$. Then $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles recover the notion of Higgs bundles in the classical sense. When $H^{\mathbb{C}}$ is a closed Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ the standard representation ρ restricts to a standard representation $\rho' : H^{\mathbb{C}} \to \mathbb{C}^n$. Hence this gives rise to Higgs bundles with an extra structure coming from the group $H^{\mathbb{C}}$. It is sometimes preferable to work with Higgs bundles associated to the standard representation rather than with G-Higgs bundles.

We now recall some concrete examples of G-higgs bundles for the complex semisimple classical Lie groups.

4.1.1 $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles.

Let $G = \mathrm{SL}(n, \mathbb{C})$ and $H = \mathrm{SU}(n)$ be a maximal compact subgroup of G. Observe that its complexification $H^{\mathbb{C}}$ is again G. A $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle is a pair (E, φ) where E is a holomorphic principal $\mathrm{SL}(n, \mathbb{C})$-bundle and φ is a holomorphic section of $E(\mathfrak{sl}(n, \mathbb{C})) \otimes K$. If we consider the standard representation ρ of $\mathrm{SL}(n, \mathbb{C})$ on $\mathbb{C}^n$ then we can associate to E a holomorphic rank n vector bundle $V = E \times_\rho \mathbb{C}^n$ with trivial determinant and $E(\mathfrak{sl}(n, \mathbb{C})) = \mathrm{End}_0(V)$ where by $\mathrm{End}_0(V)$ we mean the bundle of traceless endomorphism of V. Thus $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles are in one-to-one correspondence with classical Higgs bundles (V, φ) where V is a rank n holomorphic vector bundle with trivial determinant and the Higgs field is a traceless homeomorphism $\varphi : V \to V \otimes K$, that is φ is a section of $H^0(X, \mathrm{End}(V) \otimes K)$ and $\mathrm{tr}(\varphi) = 0$.

Remark 4.1.2. The center of $\mathfrak{su}(n)$ is trivial since $Z(\mathrm{SU}(n)) = Z(\mathrm{SL}(n, \mathbb{C})$ is finite. Hence the only possible value for which stability of $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles is defined is $\alpha = 0$.

Following [27] the (semi,poly)stability conditions for $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles are equivalent to the following.

Definition 4.1.3. A $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle (V, φ) is semistable if for any φ-invariant subbundle $V' \subset V$ we have that $\deg(V') \leq 0$. Furthermore (V, φ) is stable if for any nonzero strict φ-invariant subbundle $V' \subsetneq V$ we have that $\deg(V') < 0$.

Definition 4.1.4. A $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle (V, φ) is polystable if it is semistable and for each φ-invariant subbundle $V' \subset V$ such that $\deg(V') = 0$ there is another φ-invariant subbundle $V'' \subset V$ such that $V = V' \oplus V''$. The moduli space of isomorphism classes of polystable $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles was introduced by Hitchin in [38].

4.1.2 $\mathrm{SO}(n, \mathbb{C})$-Higgs bundles

Let $G = \mathrm{SO}(n, \mathbb{C})$ and $H = \mathrm{SO}(n)$ be a maximal compact subgroup of G. A $\mathrm{SO}(n, \mathbb{C})$-Higgs bundles is a par (E, φ) where E is a holomorphic principal $\mathrm{SO}(n, \mathbb{C})$-bundle and φ is a holomorphic section of $E(\mathfrak{so}(n, \mathbb{C})) \otimes K$. With the standard representation $\rho : \mathrm{SO}(n, \mathbb{C}) \to \mathbb{C}^n$ we can associate to E a holomorphic rank n vector bundle $V = E \times_\rho \mathbb{C}^n$ with trivial determinant together with a non-degenerate symmetric quadratic form Q (see for example [5]). In this situation,

$$E(\mathfrak{so}(n, \mathbb{C})) = \{f \in \mathrm{End}_0(V) \text{ s.t. } Q(f\cdot, \cdot) = -Q(\cdot, f\cdot)\}.$$

Thus, $\mathrm{SO}(n, \mathbb{C})$-Higgs bundles are in one-to-one correspondence with triples (V, Q, φ) where V is a rank n holomorphic vector bundle with trivial determinant, Q is a non-degenerate symmetric quadratic form and the Higgs field $\varphi : V \to V \otimes K$ is a traceless homeomorphism such that it is skew-symmetric with resp. to the quadratic form Q.

Remark 4.1.5. The center of $\mathfrak{so}(n)$ is trivial since $Z(\mathrm{SO}(n))$ is finite. Hence the only possible value for which stability of $\mathrm{SO}(n, \mathbb{C})$-Higgs bundle is defined is $\alpha = 0$.

Acording to [5] a $\mathrm{SO}(n, \mathbb{C})$-higgs bundle (V, Q, φ) is semistable if for any isotropic φ-invariant subbundle $V' \subset V$ (where by isotropic subbundle we mean a subbundle V' such that $V' \subset V^{\perp_Q}$) we have that $\deg(V) \leq 0$. Moreover, (V, Q, φ) is stable if for any nonzero strict φ-invariant isotropic subbundle $V' \subsetneq V$ we have that $\deg(V') < 0$. Finally, a $\mathrm{SO}(n, \mathbb{C})$-higgs bundle (V, Q, φ) is polystable if it is semistable and for each φ-invariant subbundle $V' \subset V$ such that $\deg(V') = 0$ there is another φ-invariant subbundle $V'' \subset V$ such that $V = V' \oplus V''$.

4.1.3 Sp($n,\mathbb{C}$)-Higgs bundles.

Let $G = \mathrm{Sp}(2n,\mathbb{C})$ and $H = \mathrm{Sp}(2n)$ be a maximal compact subgroup of G. A $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundles is a par (E,φ) where E is a holomorphic principal $\mathrm{Sp}(2n,\mathbb{C})$-bundle and φ is a holomorphic section of $E(\mathfrak{sp}(2n,\mathbb{C}))\otimes K$. With the standard representation $\rho : \mathrm{Sp}(2n,\mathbb{C}) \to \mathbb{C}^{2n}$ we can associate to E a holomorphic rank $2n$ symplectic vector bundle $(V = E\times_\rho \mathbb{C}^{2n}, \Omega)$ over X where Ω is a holomorphic section of $V^*\wedge V^*$ such that the restriction to each fiber is a symplectic form ω, i.e. for any $x \in X, (E_x, \Omega_x = \omega)$ is a symplectic vector space. In this situation,

$$E(\mathfrak{sp}(2n,\mathbb{C})) = \{f \in \mathrm{End}(V) \text{ s.t. } \Omega(f\cdot,\cdot) = -\Omega(\cdot, f\cdot)\}.$$

Hence a $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundles is a triple (V,Ω,φ) where (V,Ω) is a holomorphic rank $2n$ symplectic vector bundle and the Higgs field $\varphi : V \to V\otimes K$ is a homeomorphism such that it is skew-symmetric with resp. to Ω.

Remark 4.1.6. The center of $\mathfrak{sp}(2n,\mathbb{C})$ is trivial since $Z(\mathrm{Sp}(2n,\mathbb{C}))$ is finite. Hence the only possible value for which stability of $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundle is defined is $\alpha = 0$.

According to [27] the classical notions of (semi,poly)stability conditions for $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundle are equivalent to the following definitions.

Definition 4.1.7. A $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundle (V,Ω,φ) is semistable if for any φ-invariant isotropic subbundle $V' \subset V$ we have that $\deg(V') \leq 0$. Furthermore (V,φ) is stable if for any nonzero strict isotropy φ-invariant subbundle $V' \subsetneq V$ we have that $\deg(V') < 0$.

Definition 4.1.8. A $\mathrm{Sp}(2n,\mathbb{C})$-Higgs bundle (V,Ω,φ) is polystable if it is semistable and for any strict isotropy (resp. coisotropic) nonzero φ-invariant subbundle $V' \subset V$ such that $\deg(V') = 0$ there is a coisotropic (resp. isotropic) φ-invariant subbundle $V'' \subset V$ such that $V = V' \oplus V''$.

4.2 (G,θ)-Higgs bundles for the classical groups and stability

4.2.1 Real forms of $\mathrm{SL}(n,\mathbb{C})$

It is a well known fact that the real forms of $\mathrm{SL}(n,\mathbb{C})$, besides the compact form $\mathrm{SU}(n)$, are $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SU}(p,q)$ and $\mathrm{SU}^*(2m)$. We first consider the case of $(\mathrm{SL}(n,\mathbb{R}),\theta)$-Higgs bundles where $\theta(X)=(X^t)^{-1}$. The Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of $\mathrm{SL}(n,\mathbb{R})$ consists of all $n\times n$ matrices with vanishing trace. Its cartan decomposition is

$$\mathfrak{sl}(n,\mathbb{R})=\mathfrak{h}\oplus\mathfrak{m}, \tag{4.1}$$

where $\mathfrak{h}=\mathfrak{so}(n,\mathbb{R})$ is the lie algebra of the maximal compact subgroup

$$\mathrm{SO}(n,\mathbb{R})=\{A\in\mathrm{SL}(n,\mathbb{R})\ s.t.\ AA^t=\mathrm{Id}\}$$

of $\mathrm{SL}(n,\mathbb{R})$ and

$$\mathfrak{m}=\{A\in\mathfrak{sl}(n,\mathbb{R})\ s.t.\ A=A^t\}$$

i.e. $\mathfrak{m}$ is the vector space of traceless symmetric matrices. We will denote it by $\mathrm{Sym}_n(\mathbb{R})$. Decomposition (4.1) complexify to

$$\mathfrak{sl}(n,\mathbb{C})=\mathfrak{so}(n,\mathbb{C})\oplus\mathrm{Sym}_n(\mathbb{C}).$$

Applying Definition 2.1.1 to this particular case we have the following.

Definition 4.2.1. A $(\mathrm{SL}(n,\mathbb{R}),\theta)$**-Higgs bundle** is a pair (E,φ) where E is principal $\mathrm{SO}(n,\mathbb{C})$-bundle and $\varphi\in H^0(X,E(\mathrm{Sym}_n(\mathbb{C}))\otimes K)$.

As we already noticed along this chapter it is sometimes useful to see (G,θ)-Higgs bundles for the classical groups as $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles with an additional structure. In this sense, there is a 1-to-1 correspondence between $\mathrm{SL}(n,\mathbb{R})$-Higgs bundles and tuples (V,Q,φ) defined as follows. Using the standard representation ρ_s of $\mathrm{SO}(n,\mathbb{C})$ in $\mathbb{C}^n$, we can associate to the principal $\mathrm{SO}(n,\mathbb{C})$-bundle E a pair $(\mathbb{E},Q)$, where $\mathbb{E}$ is a rank n holomorphic vector bundle

$$\mathbb{E}=E\times_{\rho_s}\mathbb{C}^n$$

with trivial determinant and Q is a non-degenerate symmetric quadratic form $Q:\mathbb{E}\times\mathbb{E}\to\mathbb{C}$. This of course induces an isomorphism $Q:V\xrightarrow{\sim}V^*$. In this situation the Higgs field is a section $\varphi\in H^0(X,\mathrm{End}(V)\otimes K)$ which satisfies

1. Tr(φ)=0
2. $Q(\varphi\xi_1, \xi_2) = Q(\xi_1, \varphi\xi_2)$.

Recall that from the vector bundles point of view stability is defined in terms of slopes (see Section 4.1). Notice that the stability conditions for the principal and vector bundle points of view coincide in a natural way.

We consider now the case where $G = \mathrm{SU}(p,q)$ is the group of $n \times n$ indefinite unitary matrices of signature (p,q) with determinant 1. Let $\theta_{p,q}(X) = \mathrm{I}_{p,q} X \mathrm{I}_{p,q}$ be a fixed Cartan involution of G. The Lie algebra $\mathfrak{su}(p,q)$ of G is

$$\left\{ \begin{pmatrix} Z_1 & Z_2 \\ \overline{Z}_2^t & Z_3 \end{pmatrix} \begin{array}{l} \text{such that } \; Z_1, Z_3 \text{ skew-hermitian, of order } p \text{ and } q, \text{ respectively,} \\ \mathrm{tr}(Z_1) + \mathrm{Tr}(Z_3) = 0 \text{ and } Z_2 \text{ arbitrary.} \end{array} \right\}$$

At the Lie algebra level $\theta_{p,q}$ induces a Cartan decomposition

$$\mathfrak{su}(p,q) = \mathfrak{s}(\mathfrak{u}(p) \times \mathfrak{u}(q)) \oplus \left\{ \begin{pmatrix} 0 & Z_2 \\ \overline{Z}_2^t & 0 \end{pmatrix} \text{ such that } \; Z_2 \text{ arbitrary} \right\},$$

which complexify to

$$\mathfrak{sl}(p+q, \mathbb{C}) = \mathfrak{s}(\mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C})) \oplus \mathfrak{m}^{\mathbb{C}},$$

where $\mathfrak{m}^{\mathbb{C}}$ is the set of all off-diagonal elements of $\mathfrak{sl}(p+q,\mathbb{C})$. Observe that $\mathfrak{s}(\mathfrak{u}(p) \times \mathfrak{u}(q))$ is the Lie algebra of H =S(U(p) × U(q)), the maximal compact subgroup of SU(p,q) determined by θ. Its complexification is

$$H^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})),$$

whose corresponding Lie algebra is $\mathfrak{s}(\mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C}))$. We hence have the following.

Definition 4.2.2. A $(\mathrm{SU}(p,q), \theta_{p,q})$-**Higgs bundle** is a pair (E, φ) where E is principal S(GL$(p,\mathbb{C})$ × GL$(q,\mathbb{C})$)-bundle and the Higgs field φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$.

Now in the same way we did for $(\mathrm{SL}(n,\mathbb{R}), \theta)$-Higgs bundles, we can see $(\mathrm{SU}(p,q), \theta)$-Higgs bundle as GL$(n,\mathbb{C})$-Higgs bundles with aditional structures. More precisely, using the standard representation ρ_s of S(GL$(p,\mathbb{C})$ × GL$(q,\mathbb{C})$)

in $\mathbb{C}^n$, we can associate to the principal $\mathrm{S}(\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C}))$-bundle a vector bundle $\mathbb{E}=E\times_\rho\mathbb{C}^n$, this is equivalent to saying that $\mathbb{E}=V_p\oplus V_q$ where V_p and V_q are holomorphic vector bundles with $\mathrm{Rank}(V_p)=p$, $\mathrm{Rank}(V_q)=q$ and such that $\det V_p\otimes\det V_q=\mathcal{O}$. In this sense, the associated bundle $E(\mathfrak{m}^{\mathbb{C}})$ can be expressed in terms of V_p and V_q as follows:

$$E(\mathfrak{m}^{\mathbb{C}})=\mathrm{Hom}(V_q,V_p)\oplus\mathrm{Hom}(V_p,V_q)$$

and hence the Higgs field is given by

$$\varphi=\begin{pmatrix}0&\beta\\ \gamma&0\end{pmatrix},$$

where the non-zero components in the Higgs field are

$$\beta:V_q\to V_p\otimes K$$

and

$$\gamma:V_p\to V_q\otimes K.$$

From the vector bundles point of view stability is defined in terms of slopes. In this regard we say that a pair $(E=V_p\oplus V_q,\varphi)$ as above is semistable if $\mu(E')\leq\mu(E)$ for all φ-invariant subbundles E' of the form $E'=V'\oplus V''$ where $V'\subset V_p$ and $V''\subset V_q$ such that

$$\beta:V''\to V'\otimes K$$

and

$$\gamma:V'\to V''\otimes K.$$

Furthermore, $(E=V_p\oplus V_q,\varphi)$ is stable if for all strict φ-invariant subbundles E' of the form $E'=V'\oplus V''$ with $V'\subset V_p$ and $V''\subset V_q$ such that

$$\beta:V''\to V'\otimes K$$

and

$$\gamma:V'\to V''\otimes K$$

is $\mu(E')<\mu(E)$. Finally we say that $(E=V_p\oplus V_q,\varphi)$ is polystable if it is a direct sum of $\mathrm{SU}(p,q)$-Higgs bundles all of the same slope.

Finally, let $n=2m$ and consider $G=\mathrm{SU}^*(2m)$ the group of matrices in $\mathrm{SL}(n=2m,\mathbb{C})$ which commutes with the transformation $\psi:\mathbb{C}^{2m}\to\mathbb{C}^{2m}$ given by

$$(z_1,\ldots,z_{2n})\mapsto(\overline{z}_{n+1},\ldots,\overline{z}_{2n},-\overline{z}_1,\ldots,-\overline{z}_n).$$

Let $\theta(X) = -\mathrm{J}_m(X^t)^{-1}\mathrm{J}_m$ be a fixed Cartan involution of G. The Lie algebra $\mathfrak{su}^*(2m)$ of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \text{ s.t. } \; Z_1, Z_2 \text{ are complex matrices and } \mathrm{Tr}(Z_1) + \mathrm{Tr}(\overline{Z}_1) = 0 \right\}.$$

At the Lie algebra level θ induces a Cartan decomposition

$$\mathfrak{su}^*(2m) = \mathfrak{sp}(m) \oplus \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \text{ s.t. } \; Z_1 = \overline{Z}_1^t, \, Z_2 \in \mathfrak{so}(m, \mathbb{C}) \right\},$$

which complexify to

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sp}(m, \mathbb{C}) \oplus \left\{ A \in \mathfrak{sl}(2m, \mathbb{C}) \text{ s.t. } \mathrm{J}_m A^t \mathrm{J}_m = -A \right\}.$$

Observe that $\mathfrak{sp}(m)$ is the Lie algebra of $H = \mathrm{Sp}(m)$, the maximal compact subgroup of $\mathrm{SU}^*(2m)$ determined by θ. Its complexification, $H^{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$, is the group of $2m \times 2m$ complex symplectic matrices whose Lie algebra is $\mathfrak{sp}(m, \mathbb{C})$. We hence have the following.

Definition 4.2.3. A $(\mathrm{SU}^*(2m), \theta)$**-Higgs bundle** is a pair (E, φ) where E is principal $\mathrm{Sp}(m, \mathbb{C})$-bundle over X and the Higgs field φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ A \in \mathfrak{sl}(2m, \mathbb{C}) \text{ s.t. } \mathrm{J}_m A^t \mathrm{J}_m = -A \right\}$$

Now, as we noticed above, it is convenient to see $(\mathrm{SU}^*(2m), \theta)$-Higgs bundle as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles with aditional structures. More precisely, If we denote by $\mathbb{V}$ the standard complex representation of $\mathrm{Sp}(m, \mathbb{C})$ and ω denotes the standard symplectic form on $\mathbb{V}$ then

$$\mathfrak{m}^{\mathbb{C}} = \{\xi \in \mathrm{End}(\mathbb{V}) \text{ s.t. } \omega(\xi \cdot, \cdot) = \omega(\cdot, \xi \cdot)\}.$$

Hence given a rank $2m$ holomorphic vector bundle $\mathbb{V}$ endowed with a symplectic form ω, we denoted by $S^2\mathbb{V}$ the bundle of endomorphisms ξ of $\mathbb{V}$ which are symmetric with respect to ω.

Knowing this, a $(\mathrm{SU}^*(2m), \theta)$-Higgs bundle over X can be seen as a triple $(\mathbb{V}, \omega, \varphi)$ such that $\mathbb{V}$ is a holomorphic $2m$-rank vector bundle, ω is a symplectic form defined on $\mathbb{V}$ and the Higgs field $\varphi \in \mathrm{H}^0(S^2\mathbb{V} \otimes K)$ is a K-twisted endomorphism $\mathbb{V} \to \mathbb{W} \otimes K$ symmetric with respect to ω.

4.2.2 Real forms of $\mathrm{SO}(n, \mathbb{C})$

It is a well known fact that the real forms of $\mathrm{SO}(n, \mathbb{C})$, besides the compact form $\mathrm{SO}(n)$, are $\mathrm{SO}(p,q)$ and $\mathrm{SO}^*(2m)$. We first consider the case of $(\mathrm{SO}(p,q), \theta)$-Higgs bundles where $\theta(X) = \mathrm{I}_{p,q} X \mathrm{I}_{p,q}$. The Lie algebra $\mathfrak{so}(p,q)$ of $\mathrm{SO}(p,q)$ consists of all $p+q \times +q$ skew-symmetric matrices with vanishing trace. Its cartan decomposition is

$$\mathfrak{so}(p,q) = \mathfrak{h} \oplus \mathfrak{m}, \tag{4.2}$$

where $\mathfrak{h} = \mathfrak{so}(p) \times \mathfrak{so}(q)$ is the lie algebra of the maximal compact subgroup

$$\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q))$$

of $\mathrm{SO}(p,q)$ and

$$\mathfrak{m} = \{ \begin{pmatrix} 0 & X_2 \\ X_2^t & 0 \end{pmatrix} \in \mathfrak{so}(p,q) \text{ s.t. } X_2 \text{ is a } (p \times q)\text{-real matrix} \}.$$

Decomposition (4.2) complexify to

$$\mathfrak{so}(p+q, \mathbb{C}) = \mathfrak{so}(p, \mathbb{C}) \oplus \mathfrak{so}(q, \mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}},$$

where

$$\mathfrak{m}^{\mathbb{C}} = \{ \begin{pmatrix} 0 & X_2 \\ X_2^t & 0 \end{pmatrix} \in \mathfrak{so}(p+q, \mathbb{C}) \text{ s.t. } X_2 \text{ is a } (p \times q)\text{-complex matrix} \}.$$

Applying Definition 2.1.1 to this particular case we have the following.

Definition 4.2.4. A $(\mathrm{SO}(p,q), \theta)$**-Higgs bundle** is a pair (E, φ) where E is principal $\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$-bundle and $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$.

As we know it is sometimes useful to see (G, θ)-Higgs bundles for the classical groups as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles with an additional structure. In this sense, there is a 1-to-1 correspondence between $(\mathrm{SO}(p,q), \theta)$-Higgs bundles and tuples $(V_p, V_q, Q_{V_p}, Q_{W_q}, \varphi)$ defined as follows. The complex vector space V associated to the standard representation of $\mathrm{S}(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}))$ is decomposed in $V = V_p \oplus V_q$, where V_p and V_q are complex vector spaces of dimension p and q with orthogonal structures Q_{V_p} and Q_{V_q} respectively. The Higgs field φ is a holomorphic section in $H^0(X, \mathrm{Hom}(V_q, V_p) \oplus \mathrm{Hom}(V_p, V_q) \otimes K)$ given by

$$\varphi = \begin{pmatrix} 0 & \nu \\ -\nu^T & 0 \end{pmatrix},$$

where ν^T is the orthogonal transpose of ν, that is $\nu^T = q_{V_q}^{-1}\nu^t q_{V_p}^{-1}$, with q_{V_p} and q_{V_q} the isomorphisms between V_p and V_q and their corresponding dual spaces induced by the orthogonal structures (see [5] for more details).

Now, let $n = 2m$ and consider $G = \mathrm{SO}^*(2m)$ the group of matrices in $\mathrm{SO}(n = 2m, \mathbb{C})$ fixed under the involution $X \mapsto J_m(\overline{X}^t)^{-1}J_m^{-1}$. Let $\theta(X) = \mathrm{J}_m X \mathrm{J}_m^{-1}$ be a fixed Cartan involution of G. The Lie algebra $\mathfrak{so}^*(2m)$ of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \text{ s.t. } Z_1, Z_2 \text{ are complex matrices and } Z_1^t + \overline{Z}_1 = 0 \text{ and } Z_2^t - \overline{Z}_2 = 0 \right\}.$$

At the Lie algebra level θ induces a Cartan decomposition

$$\mathfrak{so}^*(2m) = \mathfrak{u}(m) \oplus i \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \text{ s.t. } X_1, X_2 \in \mathfrak{so}(m) \right\}.$$

which complexify to

$$\mathfrak{so}(2m, \mathbb{C}) = \mathfrak{gl}(m, \mathbb{C}) \oplus \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \text{ s.t. } X_1, X_2 \in \mathfrak{so}(m, \mathbb{C}) \right\}$$

Observe that $\mathfrak{u}(m)$ is the Lie algebra of $H = \mathrm{U}(m)$, the maximal compact subgroup of $\mathrm{SO}^*(2m)$ determined by θ. We hence have the following.

Definition 4.2.5. A $(\mathrm{SO}^*(2m), \theta)$**-Higgs bundle** is a pair (E, φ) where E is principal $\mathrm{GL}(m, \mathbb{C})$-bundle over X and the Higgs field φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \text{ s.t. } X_1, X_2 \in \mathfrak{so}(m, \mathbb{C}) \right\}.$$

It is useful to see $(\mathrm{SO}^*(2m), \theta)$-Higgs bundle as $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles with aditional structures. The vector bundle $\mathbb{E}$ associated to $H^{\mathbb{C}}$ via the standard representation has an orthogonal and symplectic structure J, since $J^t = J^{-1}$ and $J^2 = -\mathrm{Id}$. This induces a decomposition in $\pm i$-eigenspaces $\mathbb{E} = V \oplus V^*$, where V is a rank $2m$ vector bundle.

Knowing this, a $(\mathrm{SO}^*(2m), \theta)$-Higgs bundle (E, φ) over X can be seen as a pair (V, φ) where $E = V \oplus V^*$ and the Higgs field φ is of the form $\varphi(v_1, v_2) = (\beta(v_2), \gamma(v_1))$ where $\beta : V^* \to V \otimes K$ and $\gamma : V \to V^* \otimes K$ are skew-symmetric. That is,

$$\varphi = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \text{ s.t. } \beta = -\beta^t \text{ and } \gamma = -\gamma^t \right\}.$$

4.2.3 Real forms of $\mathrm{Sp}(n, \mathbb{C})$

It is a well known fact that the real forms of $\mathrm{Sp}(n, \mathbb{C})$, besides the compact form $\mathrm{Sp}(n)$, are $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{Sp}(2p, 2q)$. Consider the case of $(\mathrm{Sp}(n, \mathbb{R}), \theta)$-Higgs bundles where $\theta(X) = (X^t)^{-1}$. The Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ of $\mathrm{Sp}(n, \mathbb{R})$ consists of all $2n \times 2n$-real symplectic matrices. Its cartan decomposition is

$$\mathfrak{sp}(n, \mathbb{R}) = \mathfrak{h} \oplus \mathfrak{m}, \tag{4.3}$$

where $\mathfrak{h} = \mathfrak{u}(n)$ is the lie algebra of the maximal compact subgroup $\mathrm{U}(n)$ of $\mathrm{Sp}(n, \mathbb{R})$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & -X_1 \end{pmatrix} \text{ s.t. } X_1, X_2 \text{ are real matrices with } X_1 = X_1^t \text{ and } X_2 = X_2^t \right\}.$$

Decomposition (4.3) complexify to

$$\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}},$$

where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & -X_1 \end{pmatrix} \text{ s.t. } X_1, X_2 \text{ are real matrices with } X_1 = X_1^t \text{ and } X_2 = X_2^t \right\}.$$

Applying Definition 2.1.1 to this particular case we have the following.

Definition 4.2.6. A $(\mathrm{Sp}(n, \mathbb{R}), \theta)$-**Higgs bundle** is a pair (E, φ) where E is principal $\mathrm{GL}(n, \mathbb{C})$-bundle and $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$.

In this case $(\mathrm{Sp}(n, \mathbb{R}), \theta)$-Higgs bundles are in one-to-one correspondence with tuples (V, β, γ) in the following sense. Let $\mathbb{E}$ be the vector bundle associated to $H^{\mathbb{C}}$ via the standard representation. It has an orthogonal and symplectic structure J that induces a decomposition in $\pm i$-eigenspaces $\mathbb{E} = V \oplus V^*$, where V is a rank n holomorphic vector bundle. The Higgs field φ is of the form $\varphi(v_1, v_2) = (\beta(v_2), \gamma(v_1))$ where $\beta : V^* \to V \otimes K$ and $\gamma : V \to V^* \otimes K$ are symmetric. That is,

$$\varphi = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \text{ s.t. } \beta = \beta^t \text{ and } \gamma = \gamma^t \right\}.$$

See [26] for a betetr understanding of this objects.

Now, $G = \mathrm{Sp}(2p, 2q)$ be the group of matrices in $\mathrm{Sp}(2(p+q), \mathbb{C})$ fixed under the involution $X \mapsto K_{p,q}(\overline{X}^t)^{-1}K_{p,q}$, where

$$K_{p,q} = \begin{pmatrix} -\mathrm{I}_p & 0 & 0 & 0 \\ 0 & \mathrm{I}_q & 0 & 0 \\ 0 & 0 & -\mathrm{I}_p & 0 \\ 0 & 0 & 0 & \mathrm{I}_q \end{pmatrix}. \tag{4.4}$$

Let $\theta(X) = K_{p,q}XK_{p,q}$ be a fixed Cartan involution of G. The Lie algebra $\mathfrak{sp}(2p, 2q)$ of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ \overline{Z}_{12}^t & Z_{22} & Z_{14}^t & Z_{24} \\ -\overline{Z}_{13} & \overline{Z}_{14} & \overline{Z}_{11} & -\overline{Z}_{12} \\ \overline{Z}_{14}^t & -\overline{Z}_{24} & -Z_{12}^t & \overline{Z}_{22} \end{pmatrix} \begin{array}{c} \text{s.t. } Z_{ij} \text{ are complex matrices with} \\ \\ Z_{11}^t + \overline{Z}_{11} = 0, \quad Z_{22}^t + \overline{Z}_{22} = 0, \\ Z_{13} = Z_{13}^t, \qquad Z_{24} = Z_{24}^t. \end{array} \right\}$$

At the Lie algebra level θ induces a Cartan decomposition

$$\mathfrak{sp}(2p, 2q) = \mathfrak{h} \oplus \mathfrak{m}$$

where

$$\mathfrak{h} = \left\{ \begin{pmatrix} Z_{11} & 0 & Z_{13} & 0 \\ 0 & Z_{22} & 0 & Z_{24} \\ -\overline{Z}_{13} & 0 & \overline{Z}_{11} & 0 \\ 0 & -\overline{Z}_{24} & 0 & \overline{Z}_{22} \end{pmatrix} \begin{array}{c} \text{s.t. } Z_{ij} \text{ are complex matrices with} \\ \\ Z_{11} \in \mathfrak{u}(p), \quad Z_{22} \in \mathfrak{u}(q), \\ Z_{13} = Z_{13}^t, \quad Z_{24} = Z_{24}^t. \end{array} \right\}.$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & Z_{12} & 0 & Z_{14} \\ Z_{12}^t & 0 & Z_{14}^t & 0 \\ 0 & \overline{Z}_{14} & 0 & -\overline{Z}_{12} \\ \overline{Z}_{14}^t & 0 & -Z_{12}^t & 0 \end{pmatrix} \text{for } Z_{ij} \text{ arbitrary complex matrices.} \right\}$$

Observe that $\mathfrak{h} \simeq \mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$ via the map

$$\begin{pmatrix} Z_{11} & 0 & Z_{13} & 0 \\ 0 & Z_{22} & 0 & Z_{24} \\ -\overline{Z}_{13} & 0 & \overline{Z}_{11} & 0 \\ 0 & -\overline{Z}_{24} & 0 & \overline{Z}_{22} \end{pmatrix} \to \left\{ \begin{pmatrix} Z_{11} & Z_{13} \\ -\overline{Z}_{13} & \overline{Z}_{11} \end{pmatrix}, \begin{pmatrix} Z_{22} & Z_{24} \\ -\overline{Z}_{24} & \overline{Z}_{22} \end{pmatrix} \right\}. \tag{4.5}$$

This decomposition complexify to

$$\mathfrak{sp}(2(p+q),\mathbb{C}) = \mathfrak{sp}(p,\mathbb{C}) \oplus \mathfrak{sp}(q,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}$$

where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ \begin{pmatrix} 0 & Z_{12} & 0 & Z_{14} \\ Z_{13}^t & 0 & Z_{14}^t & 0 \\ 0 & Z_{32} & 0 & -Z_{13}^t \\ Z_{32}^t & 0 & -Z_{12}^t & 0 \end{pmatrix} \text{ for } Z_{ij} \text{ arbitrary complex matrices.} \right\}$$

Observe that $\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$ is the Lie algebra of $H = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$, the maximal compact subgroup of $\mathrm{Sp}(2p, 2q)$ determined by θ. We hence have the following.

Definition 4.2.7. A $(\mathrm{Sp}(2p, 2q), \theta)$**-Higgs bundle** is a pair (E, φ) where E is principal $\mathrm{Sp}(p,\mathbb{C}) \times \mathrm{Sp}(q,\mathbb{C})$-bundle over X and the Higgs field φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$.

In this case $(\mathrm{Sp}(2p, 2q), \theta)$-Higgs bundles are in one-to-one correspondence with tuples $(V_{2p}, V_{2q}, \beta, \gamma)$ in the following sense. Let $\mathbb{E}$ be the vector bundle associated to $H^{\mathbb{C}}$ via the standard representation. It may be written as $\mathbb{E} = V_{2p} \oplus V_{2q}$, where V_{2p} and V_{2q} are rank $2p$ and $2q$ holomorphic vector bundle respectively with symplectic structures. The Higgs field φ is of the form

$$\varphi = \left\{ \begin{pmatrix} 0 & -\gamma^T \\ \gamma & 0 \end{pmatrix} \text{ s.t. } \gamma : V_{2p} \to V_{2q} \otimes K \text{ and } -\gamma^T : V_{2q} \to V_{2p} \otimes K \right\},$$

where by γ^T is the symplectic transpose of the map γ defined via the symplectic isomorphisms $q_V 2p : V_{2p} \to V_{2p}^*$ and $q_V 2q : V_{2q} \to V_{2q}^*$ and the dual action γ^t of γ.

4.3 Involutions of the moduli space of $\mathrm{SL}(n, \mathbb{R})$-Higgs bundles.

Let $G = \mathrm{SL}(n, \mathbb{R})$ with $n \geq 2$ and let $\theta(X) := (X^t)^{-1}$ be its fixed Cartan involution. It is our burden not being able to determine the group of outer automorphisms of a real form of a semisimple complex Lie group. This implies that we can not give the entire description of its fixed points subvarieties as we theoretically did in Theorem 3.5.3. Anyway we can give partial examples to illustrate the main results of Section 3 since we can always consider the trivial clique $a = 1 \in \mathrm{Out}_2(G, \theta)$. Consider the cohomology set $H^1_a(\mathbb{Z}/2\mathbb{Z}, \mathrm{SO}(n,\mathbb{C})/Z_\tau)$ for the trivial clique $a = 1$ where in this case

$$Z_\tau = Z(\mathrm{SO}(n,\mathbb{C})) \cap Z(\mathrm{SL}(n,\mathbb{C})) = Z(\mathrm{SO}(n,\mathbb{C})).$$

Hence this cohomology set parametrise the conjugacy classes of real forms of $\mathrm{SO}(n,\mathbb{C})$ that belong to the trivial clique (see [3] for further details about these cohomology sets). Recall that the conjugacy classes of the real forms of $\mathrm{SO}(n,\mathbb{C})$ are the compact real form $\tau(X) = \overline{X}$, $\gamma_{p,q}(X) = \mathrm{I}_{p,q}\overline{X}\mathrm{I}_{p,q}$ and if $n = 2m$ $\gamma_*(X) = \mathrm{J}_m\overline{X}\mathrm{J}_m^{-1}$, where

$$\begin{pmatrix} 0 & -\mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix}.$$

The conjugacy classes of the real forms of $\mathrm{SO}(n,\mathbb{C})$ corresponding to the trivial clique are $\gamma_{2p,n-2p}$ and γ_* if $n = 2(2m+1) = 4m+2$. The corresponding elements in $\mathrm{Aut}_2(\mathrm{SO}(n,\mathbb{C}))$ are

$$\eta_{2p,2n-2p}(X) = (\gamma_{2p,2n-2p} \circ \tau)(X) = \mathrm{I}_{2p,2n-2p}X\mathrm{I}_{2p,2n-2p}$$

and

$$\eta_*(X) = (\gamma_* \circ \tau)(X) = \mathrm{J}_m X \mathrm{J}_m^{-1}.$$

By abuse of notation we will use the same terminology to denote their correspondigng Lie algebra involution. Recall that the maximal compact subgroup H of $\mathrm{SL}(n,\mathbb{R})$ is $\mathrm{SO}(n)$ and its complexification $\mathrm{SO}(n,\mathbb{C})$ is the structure group of a $(\mathrm{SL}(n,\mathbb{R}),\theta)$-Higgs bundle (E,φ). Also recall that the Cartan decomposition

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{Sym}_n(\mathbb{R})$$

complexifies to

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{so}(n,\mathbb{C}) \oplus \mathrm{Sym}_n(\mathbb{C}).$$

One has that

$$(H^{\eta_{2p,n-2p}}, \mathrm{Sym}_n^{\eta_{2p,n-2p}}(\mathbb{R})) = (\mathrm{S}(\mathrm{O}(2p,\mathbb{R})\times\mathrm{O}(n-2p,\mathbb{R})), \mathrm{Sym}_{2p}(\mathbb{R})\oplus\mathrm{Sym}_{n-2p}(\mathbb{R})) \tag{4.6}$$

which complexifies to

$$((H^{\mathbb{C}})^{\eta^{\mathbb{C}}_{2p,n-2p}}, \mathrm{Sym}_n(\mathbb{C})^{\eta^{\mathbb{C}}_{2p,n-2p}}) = (\mathrm{S}(\mathrm{O}(2p,\mathbb{C})\times\mathrm{O}(n-2p,\mathbb{C})), \mathrm{Sym}_{2p}(\mathbb{C})\oplus\mathrm{Sym}_{n-2p}(\mathbb{C})). \tag{4.7}$$

Now, let us define

$$\sigma_{2p,n-2p}(X) := (\eta_{2p,n-2p} \circ \theta)(X) = \mathrm{I}_{2p,n-2p}(X^t)^{-1}\mathrm{I}_{2p,n-2p}.$$

Observe that $\sigma_{2p,n-2p}(X) = \eta_{2p,n-2p}(X)$ on H since $\theta(h) = h$ for all $h \in H$. One hence has that

$$(H^{\sigma_{2p,n-2p}}, \mathrm{Sym}_n^{\sigma_{2p,n-2p}}(\mathbb{R})) = (\mathrm{S}(\mathrm{O}(2p) \times \mathrm{O}(n-2p)), \qquad (4.8)$$
$$\mathfrak{m} = \{\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \text{ for any real matrix } B\})$$

which complexifies to

$$((H^{\mathbb{C}})^{\sigma^{\mathbb{C}}_{2p,n-2p}}, \mathrm{Sym}_n(\mathbb{C})^{\sigma^{\mathbb{C}}_{2p,n-2p}}) = (\mathrm{S}(\mathrm{O}(2p,\mathbb{C}) \times \mathrm{O}(n-2p,\mathbb{C})), \qquad (4.9)$$
$$\{\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} \text{ for any complex matrix } B\}).$$

Remark 4.3.1. Notice that (4.6) and (4.8) are symmetric pairs in the sense of Helgason ([37]) and we hence have the following.

1. $G^{\eta_{2p,n-2p}} = \mathrm{S}(\mathrm{GL}(2p,\mathbb{R}) \times \mathrm{GL}(n-2p,\mathbb{R}))$ decomposes in
$$\mathrm{S}(\mathrm{O}(2p) \times \mathrm{O}(n-2p)) \exp(\mathrm{Sym}_n(\mathbb{R})).$$
2. The global Cartan decomposition of $G^{\sigma_{2p,n-2p}} = \mathrm{SO}(2p, n-2p)$ is
$$\mathrm{S}(\mathrm{O}(2p) \times \mathrm{O}(n-2p)) \exp(\mathfrak{m}).$$
3. $G^{\eta_{2p,n-2p}}$ complexifies to $\mathrm{S}(\mathrm{GL}(2p,\mathbb{C}) \times \mathrm{GL}(n-2p,\mathbb{C}))$ and $G^{\sigma_{2p,n-2p}}$ complexifies to $\mathrm{SO}(n,\mathbb{C})$ and their global Cartan decompositions are given by the symmetrics pairs (4.7) and (4.9) respectively.

Let now $n = 2m$, on the one hand notice that there is a bijection

$$A + iB \longleftrightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad (4.10)$$

between complex invertible matrices $X = A + iB$ of rank m and real invertible matrices of rank $2m$ and that this bijection do not preserve determinants.

On the other hand one has that

$$\mathrm{J}_m X \mathrm{J}_m^{-1} = X$$

if and only if

$$X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

with A and B real invertible matrices of rank m. From (4.10) we have the following.

$$(H^{\eta_*}, \mathrm{Sym}_n^{\eta_*}(\mathbb{R})) = (\mathrm{U}(m), \mathrm{Sym}_m(\mathbb{C})) \tag{4.11}$$

which complexifies to

$$((H^{\mathbb{C}})^{\eta_*^{\mathbb{C}}}, \mathrm{Sym}_n(\mathbb{C})^{\eta_*^{\mathbb{C}}}) = (\mathrm{GL}(m,\mathbb{C}), \mathrm{Sym}_m(\mathbb{C}) \oplus \mathrm{Sym}_m(\mathbb{C})). \tag{4.12}$$

In turn we define

$$\sigma_*(X) := (\eta_* \circ \theta)(X) = \mathrm{J}_m (X^t)^{-1} \mathrm{J}_m.$$

Observe that $\sigma_*(X) = \eta_*(X)$ on H since $\theta(h) = h$ for all $h \in H$. One hence has that

$$(H^{\sigma_*}, \mathrm{Sym}_n^{\sigma_*}(\mathbb{R})) = (\mathrm{U}(m), \{\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \text{ s.t. } A \in \mathfrak{so}(m) \text{ and } B \in \mathrm{Sym}_m(\mathbb{R})\}) \tag{4.13}$$

which complexifies to

$$((H^{\mathbb{C}})^{\sigma_*^{\mathbb{C}}}, \mathrm{Sym}_n(\mathbb{C})^{\sigma_*^{\mathbb{C}}}) = (\mathrm{GL}(m,\mathbb{C}), \mathfrak{so}(m,\mathbb{C}) \oplus \mathrm{Sym}_m(\mathbb{C})). \tag{4.14}$$

From (4.10) we have that

$$\mathrm{SL}(n,\mathbb{R})^{\eta_*} = \mathrm{GL}(m,\mathbb{C}).$$

Notice that

$$\mathrm{SL}(n,\mathbb{R})^{\sigma_*} = \mathrm{Sp}(m,\mathbb{R})$$

where by $\mathrm{Sp}(m,\mathbb{R})$ we mean the group of linear transformations of $\mathbb{R}^{2m}$ over $\mathbb{R}$ that preserve the symplectic form.

Remark 4.3.2. (4.11) and (4.13) are also symmetric pairs in the sense of Helgason ([37]). We then have the following.

1. The global Cartan decomposition of $\mathrm{GL}(m,\mathbb{C})$) is

$$\mathrm{U}(m)\exp(\mathrm{Sym}_m(\mathbb{C})).$$

2. The global Cartan decomposition of $\mathrm{Sp}(m,\mathbb{R})$ is
$$\mathrm{U}(m)\exp(\mathfrak{so}(m)\oplus \mathrm{Sym}_m(\mathbb{R})).$$
3. G^{η_*}, G^{σ_*} complexify to $\mathrm{GL}(m,\mathbb{C})\times\mathrm{GL}(m,\mathbb{C})$ and $\mathrm{Sp}(m,\mathbb{C})$ respectively, and their global Cartan decompositions are given by the symmetrics pairs (4.12) and (4.14).

Hence from Theorem 3.5.3 the fixed point subsets for the involution
$$\begin{array}{rccc}\iota(a,\pm): & \mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta) & \to & \mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta)\\ & (E,\varphi) & \mapsto & (a(E),\pm a(\varphi))\end{array}$$
are described by $\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_{p,q}},\pm)$ and $\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_*},\pm)$. In particular, from Remark 4.3.1 and 4.3.2,
$$\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_{2p,2n-2p}},+)=\mathcal{M}(\mathrm{SL}(n,\mathbb{R})^{\eta_{2p,2n-2p}},\theta),$$
$$\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_{2p,2n-2p}},-)=\mathcal{M}(\mathrm{SL}(n,\mathbb{R})^{\sigma_{2p,2n-2p}},\theta),$$
$$\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_*},+)=\mathcal{M}(\mathrm{SL}(n,\mathbb{R})^{\eta_*},\theta)$$
and
$$\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_*},-)=\mathcal{M}(\mathrm{SL}(n,\mathbb{R})^{\sigma_*},\theta).$$

From Theorem 3.7.5 we have homeomorphisms
$$\begin{array}{rcl}\mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_{2p,2n-2p}},+) & \simeq & \mathcal{R}(\mathrm{S}(\mathrm{GL}(2p,\mathbb{R})\times\mathrm{GL}(n-2p,\mathbb{R}))),\\ \mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_{2p,2n-2p}},-) & \simeq & \mathcal{R}(\mathrm{SO}(2p,n-2p)),\\ \mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_*},+) & \simeq & \mathcal{R}(\mathrm{GL}(m,\mathbb{C})),\\ \mathcal{M}(\mathrm{SL}(n,\mathbb{R}),\theta,\mathrm{SO}(n)^{\eta_*},-) & \simeq & \mathcal{R}(\mathrm{Sp}(m,\mathbb{R})).\end{array}$$

4.4 Involutions of the moduli space of $(\mathrm{SU}^*(2m),\theta)$-Higgs bundles.

Let $n=2m$ and consider $G=\mathrm{SU}^*(2m)$ the group of matrices in $\mathrm{SL}(n=2m,\mathbb{C})$ which commutes with the transformation $\psi:\mathbb{C}^{2m}\to\mathbb{C}^{2m}$ given by
$$(z_1,\ldots,z_{2n})\mapsto(\overline{z}_{n+1},\ldots,\overline{z}_{2n},-\overline{z}_1,\ldots,-\overline{z}_n).$$

Let $\theta(X) = -\mathrm{J}_m (X^t)^{-1}\mathrm{J}_m$ be a fixed Cartan involution of G. The Lie algebra $\mathfrak{su}^*(2m)$ of G is

$$\mathfrak{g} = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \text{ s.t. } \ Z_1, Z_2 \text{ are complex matrices and } \mathrm{Tr}(Z_1) + \mathrm{Tr}(\overline{Z}_1) = 0 \right\}.$$

At the Lie algebra level θ induces a Cartan decomposition

$$\mathfrak{su}^*(2m) = \mathfrak{sp}(m) \oplus \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \text{ s.t. } \ Z_1 = \overline{Z}_1^t,\ Z_2 \in \mathfrak{so}(m, \mathbb{C}) \right\},$$

which complexifies to

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{sp}(m, \mathbb{C}) \oplus \left\{ A \in \mathfrak{sl}(2m, \mathbb{C}) \text{ s.t. } \mathrm{J}_m A^t \mathrm{J}_m = -A \right\}.$$

Observe that $\mathfrak{sp}(m)$ is the Lie algebra of $H = \mathrm{Sp}(m)$, the maximal compact subgroup of $\mathrm{SU}^*(2m)$ determined by θ. Recall that a $(\mathrm{SU}^*(2m), \theta)$-Higgs bundle is a pair (E, φ) where E is principal $\mathrm{Sp}(m, \mathbb{C})$-bundle over X and the Higgs field φ is a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, where

$$\mathfrak{m}^{\mathbb{C}} = \left\{ A \in \mathfrak{sl}(2m, \mathbb{C}) \text{ s.t. } \mathrm{J}_m A^t \mathrm{J}_m = -A \right\}.$$

As in Section 4.3 consider the trivial clique $a = 1 \in \mathrm{Out}_2(\mathrm{SU}^*(2m), \theta)$. Consider the cohomology set $H^1_a(\mathbb{Z}/2\mathbb{Z}, \mathrm{Sp}(m, \mathbb{C})/Z_\tau)$ for the trivial clique $a = 1$ where in this case

$$Z_\tau = Z(\mathrm{Sp}(m, \mathbb{C})) \cap Z(\mathrm{SL}(2m, \mathbb{C})) = Z(\mathrm{Sp}(m, \mathbb{C})).$$

Hence this cohomology set parametrise the conjugacy classes of real forms of $\mathrm{Sp}(m, \mathbb{C})$ that belong to the trivial clique. Recall that the conjugacy classes of the real forms of $\mathrm{Sp}(m, \mathbb{C})$ are the compact real form

$$\tau(X) = (\overline{X}^t)^{-1}, \quad \gamma_{p,q}(X) = K_{p,q}(\overline{X}^t)^{-1}K_{p,q} \text{ and } \overline{\gamma}(X) = \overline{X},$$

where

$$K_{p,q} = \begin{pmatrix} -\mathrm{I}_p & 0 & 0 & 0 \\ 0 & \mathrm{I}_q & 0 & 0 \\ 0 & 0 & -\mathrm{I}_p & 0 \\ 0 & 0 & 0 & -\mathrm{I}_q \end{pmatrix}.$$

The conjugacy classes of the real forms of $\mathrm{Sp}(m, \mathbb{C})$ corresponding to the trivial clique are $\sigma_{p,q}$ for all p, q such that $p + q = m$. The corresponding elements in $\mathrm{Aut}_2(\mathrm{Sp}(m, \mathbb{C}))$ are

$$\eta_{p,q}(X) = (\gamma_{p,q} \circ \tau)(X) = K_{p,q} X K_{p,q}.$$

Recall that the maximal compact subgroup H of $\mathrm{SU}^*(2m)$ is $\mathrm{Sp}(m)$ and its complexification $\mathrm{Sp}(m,\mathbb{C})$ is the structure group of a $(\mathrm{SU}^*(2m),\theta)$-Higgs bundle (E,φ). Also recall that the Cartan decomposition

$$\mathfrak{su}^*(2m) = \mathfrak{sp}(m) \oplus \{A \in \mathfrak{sl}(2m,\mathbb{R}) \text{ s.t. } -J_m A^t J_m = A\}$$

complexifies to

$$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sp}(m,\mathbb{C}) \oplus \{A \in \mathfrak{sl}(2m,\mathbb{C}) \text{ s.t. } -J_m A^t J_m = A\}.$$

Let us denote $\{A \in \mathfrak{sl}(2m,\mathbb{R}) \text{ s.t. } -J_m A^t J_m = A\}$ by $\mathfrak{m}$. One has that

$$(\mathrm{Sp}(m)^{\eta_{p,q}}, \mathfrak{m}^{\eta_{p,q}}) = (\mathrm{Sp}(p)\times\mathrm{Sp}(q), \{A \in \mathrm{sl}(2m,\mathbb{R}) \text{ s.t. } A \text{ is of the form } P\}) \tag{4.15}$$

where

$$P = \begin{pmatrix} Z_{11} & 0 & Z_{13} & 0 \\ 0 & Z_{22} & 0 & Z_{24} \\ Z_{31} & 0 & -Z_{11}^t & 0 \\ 0 & Z_{42} & 0 & -Z_{22}^t \end{pmatrix},$$

with $Z_{13} = Z_{13}^t$, $Z_{24} = Z_{24}^t$, $Z_{31} = Z_{31}^t$, $Z_{42} = Z_{42}^t$.

(4.15) complexifies to

$$(\mathrm{Sp}(m,\mathbb{C})^{\eta_{p,q}^{\mathbb{C}}}, \mathfrak{m}_{\mathbb{C}}^{\eta_{p,q}^{\mathbb{C}}}) = (\mathrm{Sp}(p,\mathbb{C})\times\mathrm{Sp}(q,\mathbb{C}), \{A \in \mathrm{sl}(2m,\mathbb{C}) \text{ s.t. } A \text{ is of the form } P\}). \tag{4.16}$$

Let us now define $\sigma_{p,q}(X) := (\eta_{p,q}\circ\theta)(X) = \widetilde{K}_{p,q}(X^t)^{-1}\widetilde{K}_{p,q}$ with

$$\widetilde{K}_{p,q} = \begin{pmatrix} 0 & 0 & -\mathrm{I}_p & 0 \\ 0 & 0 & 0 & \mathrm{I}_q \\ \mathrm{I}_p & 0 & 0 & 0 \\ 0 & -\mathrm{I}_q & 0 & 0 \end{pmatrix}.$$

Observe that $\sigma_{p,q} = \eta_{p,q}$ on H since $\theta(h) = h$ for all $h \in H$. One hence has that

$$(\mathrm{Sp}(m)^{\sigma_{p,q}}, \mathfrak{m}^{\sigma_{p,q}}) = (\mathrm{Sp}(p)\times\mathrm{Sp}(q), \{A \in \mathrm{sl}(2m,\mathbb{R}) \text{ s.t. } A \text{ is of the form } Q\}), \tag{4.17}$$

where

$$Q = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ \overline{Z}_{31} & -\overline{Z}_{32} & Z_{11}^t & Z_{21}^t \\ \overline{Z}_{32}^t & \overline{Z}_{42} & Z_{12}^t & Z_{22}^t \end{pmatrix},$$

with $Z_{13}=-Z_{13}^t$, $Z_{24}=-Z_{24}^t$, $Z_{31}=-Z_{31}^t$, $Z_{42}=-Z_{42}^t$.

(4.17) complexifies to

$$(\mathrm{Sp}(m,\mathbb{C})^{\eta_{p,q}^{\mathbb{C}}},\mathfrak{m}_{\mathbb{C}}^{\eta_{p,q}^{\mathbb{C}}})=(\mathrm{Sp}(p,\mathbb{C})\times\mathrm{Sp}(q,\mathbb{C}),\{A\in \mathrm{sl}(2m,\mathbb{C}) \text{ s.t. } A \text{ is of the form } Q\}). \tag{4.18}$$

Remark 4.4.1. Notice that (4.15) and (4.17) are symmetric pairs in the sense of Helgason ([37]) and we hence have the following.

1. $G^{\eta_{p,q}}$ decomposes in
$$(\mathrm{Sp}(p)\times\mathrm{Sp}(q))\exp\mathfrak{m}^{\eta_{p,q}}.$$
2. The global Cartan decomposition of $G^{\sigma_{p,q}}$ is
$$(\mathrm{Sp}(p)\times\mathrm{Sp}(q))\exp\mathfrak{m}^{\sigma_{p,q}}.$$
3. $G^{\eta_{p,q}}$ complexifies to the subgroup of $\mathrm{SL}(n,\mathbb{C})$ given by matrices of the form
$$\begin{pmatrix} Z_{11} & 0 & Z_{13} & 0\\ 0 & Z_{22} & 0 & Z_{24}\\ Z_{31} & 0 & Z_{33} & 0\\ 0 & Z_{42} & 0 & Z_{44}\end{pmatrix}$$
and $G^{\sigma_{p,q}}$ complexifies to complexifies to the subgroup of $\mathrm{SL}(n,\mathbb{C})$ whose matrices are of the form
$$\begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14}\\ Z_{21} & Z_{22} & Z_{23} & Z_{24}\\ \overline{Z}_{31} & -\overline{Z}_{32} & -\overline{Z}_{33} & \overline{Z}_{34}\\ -\overline{Z}_{41} & \overline{Z}_{42} & \overline{Z}_{43} & -\overline{Z}_{44}\end{pmatrix}.$$
Their global Cartan decompositions are given by the symmetrics pairs (4.16) and (4.18) respectively.

Hence from Theorem 3.5.3 the fixed point subsets for the involution

$$\begin{array}{rccc}\iota(a,\pm): & \mathcal{M}(\mathrm{SU}^*(2m),\theta) & \to & \mathcal{M}(\mathrm{SU}^*(2m),\theta)\\ & (E,\varphi) & \mapsto & (a(E),\pm a(\varphi))\end{array}$$

are described by $\mathcal{M}(\mathrm{SU}^*(2m), \theta, \mathrm{Sp}(m)^{\eta_{p,q}}, \pm)$ for any p, q such that $p + q = m$. In particular, from Remark 4.4.1,

$$\mathcal{M}(\mathrm{SU}^*(2m), \theta, \mathrm{Sp}(m)^{\eta_{p,q}}, +) = \mathcal{M}((\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \exp \mathfrak{m}^{\eta_{p,q}}, \theta)$$

and

$$\mathcal{M}(\mathrm{SU}^*(2m), \theta, \mathrm{Sp}(m)^{\eta_{p,q}}, -) = \mathcal{M}((\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \exp \mathfrak{m}^{\sigma_{p,q}}, \theta).$$

From Theorem 3.7.5 we have homeomorphisms

$$\begin{array}{rcl} \mathcal{M}(\mathrm{SU}^*(2m), \theta, \mathrm{Sp}(m)^{\eta_{p,q}}, +) & \simeq & \mathcal{R}((\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \exp \mathfrak{m}^{\eta_{p,q}}), \\ \mathcal{M}(\mathrm{SU}^*(2m), \theta, \mathrm{Sp}(m)^{\eta_{p,q}}, -) & \simeq & \mathcal{R}((\mathrm{Sp}(p) \times \mathrm{Sp}(q)) \exp \mathfrak{m}^{\sigma_{p,q}}). \end{array}$$

Chapter 5

Holomorphic involutions of parabolic (G,θ)-Higgs bundles for real groups

5.1 Order two automorphisms of parabolic principal $H^{\mathbb{C}}$-bundles

Let $H^{\mathbb{C}}$ be a complex reductive Lie group with maximal compact subgroup H and maximal torus T and denote by $\mathfrak{h}^{\mathbb{C}}$, $\mathfrak{h}$ and $\mathfrak{t}$ their corresponding Lie algebras. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a r-tuple with elements in $\sqrt{-1}\mathcal{A} \subset \mathfrak{t}$, X be a compact Riemann surface and $\{x_1 \ldots, x_r\}$ be a finite set of differents points of X. Recall that a parabolic principal $H^{\mathbb{C}}$-bundle (E, Q) over $(X, D = x_1 + \cdots + x_r)$ of weights α is a holomorphic principal $H^{\mathbb{C}}$-bundle with a choice, for any i, of a parabolic structure of weight α_i on x_i. We refer the reader to Section 2.5 for more details. Let us denote by $\mathsf{PBun}(H^{\mathbb{C}}, X, D)$ the set of parabolic principal $H^{\mathbb{C}}$-bundle over (X, D). Let us consider $\sigma \in \mathrm{Aut}_2(H)$ and denote its complexification and its differential also by σ. Notice that given σ there always exists a maximal torus $T \subset H$ and an alcove $\mathcal{A} \subset \mathfrak{t}$ such that $0 \in \overline{\mathcal{A}}$ invariant under σ. This is a consequence of Theorem 1 p. 329 joint with Proposition 2 p. 326 in [16]. Hence σ acts on the set of weights α sending a weight $\alpha_i \in \alpha$ to a weight $\sigma(\alpha_i) \in \alpha$.

Now, recall from Section 2.5 that attached to any parabolic principal bundle

E we have a parabolic structure $Q_i \subset E(H^{\mathbb{C}})_{x_i}$ of weight α_i over x_i. Let us define

$$\sigma(Q_i) := \{\ E_{x_i} \xrightarrow{\ \phi\ } H^{\mathbb{C}} \xrightarrow{\ \sigma\ } H^{\mathbb{C}} \ \text{ s.t. } \phi \in Q_i\}.$$

We wonder if $\sigma(Q_i)$ is the parabolic structure of weight $\sigma(\alpha_i)$ over x_i attached to $\sigma(E) := E \times_\sigma H^{\mathbb{C}}$. To see that this is in fact the case, notice that a principal $H^{\mathbb{C}}$-bundle $\sigma(E)$ is isomorphic to the $H^{\mathbb{C}}$-bundle whose total space is E and the $H^{\mathbb{C}}$-action is

$$\xi \cdot h := \xi\sigma(h), \text{ for any } \xi \in E, h \in H^{\mathbb{C}}.$$

This isomorphism is given by

$$\begin{array}{ccc} E & \longleftrightarrow & a(E) \\ \xi & \mapsto & [\xi, e] \\ \xi a(h) & \leftarrow & [\xi, h] = [\xi a(h), e]. \end{array} \tag{5.1}$$

and we say that $\sigma(E)$ is the bundle associated to E via σ. Under this isomorphism

$$Q_i \subset E(H^{\mathbb{C}})_{x_i} := \{\phi : E_{x_i} \longrightarrow H^{\mathbb{C}} \text{ s.t. } \phi(\xi h) = h^{-1}\phi(\xi)h\}$$

transform to

$$\begin{aligned} Q_i' \subset \sigma(E(H^{\mathbb{C}}))_{x_i} &:= \{\widetilde{\phi} : E_{x_i} \longrightarrow H^{\mathbb{C}} \text{ s.t. } \widetilde{\phi}(\xi h) = \sigma(h)^{-1}\widetilde{\phi}(\xi)\sigma(h)\} \\ &= \{\ E_{x_i} \xrightarrow{\ \phi\ } H^{\mathbb{C}} \xrightarrow{\ \sigma\ } H^{\mathbb{C}} \ \text{ s.t. } \phi \in Q_i\} = \sigma(Q_i). \end{aligned}$$

Since Q_i is a parabolic structure on E, for some trivialization $\xi \in E_{x_i}$, we have that

$$P_{\alpha_i} = \{\phi(\xi),\ \phi \in Q_i\}.$$

One can check that $\sigma(P_{\alpha_i}) = P_{\sigma(\alpha_i)}$ and hence we have that

$$P_{\sigma(\alpha_i)} = \sigma(P_{\alpha_i}) = \sigma(\{\phi(\xi),\ \phi \in Q_i\}) = \{(\sigma\circ\phi)(\xi),\ \phi \in Q_i\} = \{\phi(\xi),\ \phi \in \sigma(Q_i)\}.$$

Observe that for some trivialisation $\xi' \in \sigma(E)_{x_i}$ there exists $h \in H^{\mathbb{C}}$ and $\xi \in E_{x_i}$ such that $\xi' = \xi\sigma(h)$ and then $\widetilde{\phi}(\xi') = h^{-1}\widetilde{\phi}(\xi)h$. Hence we conclude that $\sigma(Q_i)$ is a parabolic structure on $\sigma(E)$ of weight $\sigma(\alpha_i)$ over x_i. We can then define an involutive map

$$\begin{array}{rrcl} \iota(\sigma) : & \mathsf{PBun}(H^{\mathbb{C}}, X, D) & \rightarrow & \mathsf{PBun}(H^{\mathbb{C}}, X, D) \\ & (E, \alpha = (\alpha_1, \ldots, \alpha_r), \{Q_i\}) & \mapsto & (\sigma(E), \sigma(\alpha) = (\sigma(\alpha_1), \ldots, \sigma(\alpha_r)), \{\sigma(Q_i)\}), \end{array} \tag{5.2}$$

where $\sigma(E)$, $\sigma(\alpha_i)$ and $\sigma(Q_i)$ are defined as above for any $i = 1, \ldots, r$. Observe that this map becomes an involution of $\mathsf{PBun}(H^{\mathbb{C}}X, D, \alpha)$, the set of parabolic principal $H^{\mathbb{C}}$-bundle over (X, D) of weights $\alpha = (\alpha_1, \ldots, \alpha_r)$ if, in addition, we ask σ to fix the weights. In that case we have that $\sigma(P_{\alpha_i}) = P_{\sigma(\alpha_i)} = P_{\alpha_i}$ and hence for some trivialisation $\xi \in E(H^{\mathbb{C}})_{x_i}$,

$$\{\phi(\xi),\ \phi \in Q_i\} = P_{\alpha_i} = P_{\sigma(\alpha_i)} = \sigma(P_{\alpha_i}) = \sigma(\{\phi(\xi),\ \phi \in Q_i\}) = \{\phi(\xi),\ \phi \in \sigma(Q_i)\}.$$

Thus $\sigma(Q_i) = Q_i$ for all $i = 1, \ldots, r$.

As we already noticed above (5.1) is an isomorphism between $\sigma(E) := E \times_\sigma H^{\mathbb{C}}$ the bundle associated to E via σ and the principal bundle whose total space is E and the action of $H^{\mathbb{C}}$ is

$$\xi \cdot h := \xi a(h), \text{ for any } \xi \in E, h \in H^{\mathbb{C}}.$$

Recall that the isomorphism (5.1) is equivalent to a σ-twisted automorphism $A \in \mathrm{Aut}_\sigma(E)$, that is, a bijection $A : E \to E$ such that $A(\xi h) = A(\xi)\sigma(h)$ for all $\xi \in E$ and $h \in H^{\mathbb{C}}$.

Let G be a real form of a complex semisimple Lie group $G^{\mathbb{C}}$ and let θ be its Cartan involution. As usual, let H be the maximal compact subrgoup of G defined by θ. Consider $\sigma \in \mathrm{Aut}_2(G, \theta)$, we have an analogue to Proposition 3.1.8 (see Section 3.1).

Proposition 5.1.1. *Let E be a parabolic principal $H^{\mathbb{C}}$-bundle over (X, D) of weights $\alpha = (\alpha_1, \ldots, \alpha_r)$. Let $\sigma \in \mathrm{Aut}_2(G, \theta)$ and $A \in \mathrm{Aut}_\sigma(E)$ such that*

$$A^2 = z \in Z_\tau = Z(G^{\mathbb{C}}) \cap Z(H^{\mathbb{C}})$$

and $\sigma(\alpha_i) = \alpha_i$ for all $i = 1, \ldots, r$. Then:

(1) The morphism f_A defined in (3.1) maps E onto a single orbit $S(E)$ of the set S^τ_σ under the action of $H^{\mathbb{C}}$ defined as in Proposition 1.3.5.

(2) Every element $s \in S(E)$ defines a reduction of the structure group of E to $(H^{\mathbb{C}})^{\sigma'}$, where $\sigma' = \mathrm{Int}(s)\sigma$ and by $(H^{\mathbb{C}})^{\sigma'}$ we mean the subgroup of fixed points of $H^{\mathbb{C}}$ under σ'.

Proof. (1) is straightforward from Proposition 3.1.7 for $n = 2$. To prove (2) notice that the reasoning is analogous to the one done in Proposition 3.1.7 for $n = 2$

and we only need to explain what happens with the parabolic structure when reducing the structure group of the bundle.

In this sense, let us first consider $\sigma \in \mathrm{Aut}_2(H)$ such that $\overline{\mathcal{A}}$ is a σ-invariant alcove and the weights $\alpha = (\alpha_1, \ldots, \alpha_r)$ are fixed by σ then define $\mathfrak{t}^\sigma := \mathfrak{t} \cap \mathfrak{h}^\sigma$ to be corresponding Cartan subalgebra of H^σ and $\overline{\mathcal{A}}^\sigma \subset \mathfrak{t}^\sigma$ the corresponding alcove. Then

$$\alpha_i \in \sqrt{-1}\,\overline{\mathcal{A}}^\sigma \subset \sqrt{-1}\mathfrak{t}^\sigma$$

for all $i = 1, \ldots, r$. Recall that a parabolic subgroup $P_{\alpha_i} \subset H^{\mathbb{C}}$ defined by the weight $\alpha_i \subset \sqrt{-1}\,\overline{\mathcal{A}} \subset \sqrt{-1}\mathfrak{h}$ is characterized by

$$P_{\alpha_i} = \{h \in H^{\mathbb{C}} \text{ s.t. } e^{t\alpha_i} h e^{-t\alpha - i} \text{ is bounded as } t \to \infty\}.$$

Hence

$$P'_{\alpha_i} = P_{\alpha_i} \cap (H^{\mathbb{C}})^\sigma = \{h \in (H^{\mathbb{C}})^\sigma \text{ s.t. } e^{t\alpha_i} h e^{-t\alpha_i} \text{ is bounded as } t \to \infty\}$$

is a parabolic subgroup of $(H^{\mathbb{C}})^\sigma$ defined by α_i. It follows that the parabolic structures over α_i corresponding to P'_{α_i} for all i are

$$\begin{aligned} Q'_i &= \{\phi \in E'((H^{\mathbb{C}})^\sigma)_{x_i} \text{ s.t. } \phi(\xi) \in P'_{\alpha_i} \text{ for some trivialisation } \xi\} \\ &= \{\phi \in E'((H^{\mathbb{C}})^\sigma)_{x_i} \text{ s.t. } \phi(\xi) \in P_{\alpha_i} \cap (H^{\mathbb{C}})^\sigma \text{ for some trivialisation } \xi\} \end{aligned}$$

Meaning that Q'_i is a parabolic substructure of Q_i.

Now consider $\sigma' = \mathrm{Int}(s)\sigma$ in that we know that there exists a σ'-invariant Cartan subalgebra $\mathfrak{t}'$ and alcove $\mathcal{A}'$. In fact, we know that $\mathfrak{t}' = h\mathfrak{t}h^{-1}$ for some $h \in H$. In the same way we define the corresponding σ'-invariant alcove $\mathcal{A}' := h\mathcal{A}h^{-1}$. Observe that we get new weights

$$\alpha'_i := h\alpha_i h^{-1} \in \sqrt{-1}\,\overline{\mathcal{A}'}$$

for all $i = 1, \ldots, r$ and they are fixed under σ'. Hence $P'_{\alpha'_i} = hP_{\alpha_i}h^{-1}$ and we define the corresponding parabolic structure

$$Q'_i := \{\phi \in E((H^{\mathbb{C}}))_{x_i} \text{ s.t. } \phi(\xi) \in P'_{\alpha'_i} = hP_{\alpha_i}h^{-1} \text{ for some trivialisation } \xi\}$$

In this new setting when we reduce the structure group of E to $(H^{\mathbb{C}})^{\sigma'}$ define $\mathfrak{t}'^{\sigma'}$ and $\mathcal{A}'^{\sigma'}$ as above and we get to an equivalent conclusion. □

5.2 Parabolic Higgs bundles defined by involutions.

Let G be the connected component at the identity of a real form of a complex semisimple Lie group $G^{\mathbb{C}}$. Let $\theta := \tau\mu$ be a fixed Cartan involution, where τ is a compact conjugation of $G^{\mathbb{C}}$ and μ is the anti-holomorphic involution defining the real form G. Let H be the corresponding maximal compact subgroup of G such that $G^{\theta} = H$ and denote by $H^{\mathbb{C}}$ its complexification. Consider $\sigma^{\mathbb{C}} \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with θ, μ and τ and $\sigma \in \mathrm{Aut}_2(G, \theta)$ its realification.

As we saw in Chapter 1, since σ commutes with θ, we can further decompose $\mathfrak{g}$ as direct sum of (± 1)-eigenspaces of $\mathfrak{h}$ and $\mathfrak{m}$ defined by σ. Thus the complexification of the isotropy representation of H to $\mathfrak{m}$ restricts to two representations

$$\iota_+^- : (H^{\sigma})^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_+^{\mathbb{C}}) \text{ and } \iota_-^- : (H^{\sigma})^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}_-^{\mathbb{C}}).$$

Definition 5.2.1. A **parabolic $(G, \theta, H^{\sigma}, \alpha, \pm)$-Higgs bundle over a compact Riemann surface (X, D) of weights $\alpha = (\alpha_1, \dots, \alpha_r)$** is a pair (E, φ) where E is a parabolic principal $(H^{\mathbb{C}})^{\sigma}$-bundle over (X, D) of weights α and φ is a section of $PE(\mathfrak{m}_{\pm}^{\mathbb{C}}) \otimes K(D)$, where $PE(\mathfrak{m}_{\pm}^{\mathbb{C}})$ is the sheaf of parabolic sections of $E(\mathfrak{m}_{\pm}^{\mathbb{C}})$, the bundle associated to E via the representations defined above.

Remark 5.2.2. Notice that parabolic $(G, \theta, H^{\sigma}, \alpha, +)$-Higgs bundles are simply parabolic (G^{σ}, θ)-Higgs bundles in the sense of Definition 2.5.1. Moreover, from Proposition 1.3.3 if we define $\gamma := \theta\sigma$ then parabolic $(G, \theta, H^{\sigma}, \alpha, -)$-Higgs bundles are just parabolic (G^{γ}, θ)-Higgs bundles. Hence the notions of β-(semi,poly)stability and the Hitchin-Kobayashi correspondence for this objects are the ones introduced in Section 2.5.1. Hence we denote by $\mathcal{M}_{\beta}(G, \theta, H^{\sigma}, \alpha, \pm)$ or just $\mathcal{M}_{\beta}(G^{\sigma}, \theta, \alpha)$ or $\mathcal{M}_{\beta}(G^{\gamma}, \theta, \alpha)$ the **moduli space of isomorphism classes of β-polystable parabolic $((G, \theta, H^{\sigma}, \alpha\pm)$-Higgs bundles (E, φ) on (X, D) with parabolic weight $\alpha = (\alpha_1, \dots, \alpha_r)$.** The moduli space of polystable parabolic $(G, \theta, H^{\sigma}, \alpha, \pm)$-Higgs bundle will be simply denoted by $\mathcal{M}(G, \theta, H^{\sigma}, \alpha, \pm)$. We will use interchangeably these notations.

Remark 5.2.3. Recall that there is a map (see Section 2.5.1)

$$\varrho : \mathcal{M}(G, \theta, H^{\sigma}, \alpha, \pm) \to \prod_i ((\widetilde{\mathfrak{m}}_{\pm})_i^0 / \widetilde{L}_i^{\sigma}),$$

where $(\widetilde{\mathfrak{m}}_{\pm})_i^0 \subset E(\mathfrak{m}_{\pm}^{\mathbb{C}})_{x_i}$ is the space corresponding to

$$(\widetilde{\mathfrak{m}}_{\pm})_{\alpha_i}^0 := \mathrm{Ker}_{\mathfrak{m}_{\pm}^{\mathbb{C}}}(\mathrm{Ad}(\exp 2\pi\sqrt{-1}\alpha_i) - 1)$$

and $\widetilde{L}_i^\sigma$ is the subgroup corresponding to

$$\widetilde{L}_{\alpha_i} := \mathrm{Stab}_{(H^{\mathbb{C}})^\sigma}((\exp 2\pi\sqrt{-1}\alpha_i) - 1).$$

Let $\mathcal{L} := (\mathcal{L}_1, \dots \mathcal{L}_r)$ with fixed orbits $\mathcal{L}_i \in (\widetilde{\mathfrak{m}}_\pm)_i^0/\widetilde{L}_i^\sigma$ hence we define the moduli space $\mathcal{M}(G, \theta, H^\sigma, \alpha, \mathcal{L}, \pm) := \varrho^{-1}(\mathcal{L})$.

5.2.1 Extending parabolic Higgs bundles defined by involutions to parabolic (G, θ)-Higgs bundles.

Let (E, φ) be a parabolic $(G, \theta, H^\sigma, \alpha, \pm)$-Higgs bundle over (X, D) defined as in the Section 5.2 where σ is as in the previous section and $\alpha = (\alpha_1, \dots, \alpha_r)$. Denote by $E_{H^{\mathbb{C}}}$ the principal parabolic $H^{\mathbb{C}}$-bundle of weights $\alpha = (\alpha_1, \dots, \alpha_r)$ obtained by extending the structure group of E to $H^{\mathbb{C}}$. With this extension the parabolic structure attached to E extends, in a natural way, to a parabolic structure attached to $E_{H^{\mathbb{C}}}$. Since

$$PE_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = PE(\mathfrak{m}_+^{\mathbb{C}}) \oplus PE(\mathfrak{m}_-^{\mathbb{C}})$$

we define $\varphi_{H^{\mathbb{C}}}$ to be the section of $PE_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$ associated to the Higgs field φ by taking 0 in the $PE(\mathfrak{m}_\mp^{\mathbb{C}})$ component. Then the pair $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ is a parabolic (G, θ)-Higgs bundle over (X, D) of weights $\alpha = (\alpha_1, \dots \alpha_r)$. We call this pair **extension of** (E, φ). In the same manner, a **reduction of** a parabolic (G, θ)-Higgs bundle (E, φ) over (X, D) of weights $\alpha = (\alpha_1, \dots, \alpha_r)$ to a parabolic $(G, \theta, H^\sigma, \alpha, \pm)$-Higgs bundle $(E_{(H^{\mathbb{C}})^\sigma}, \varphi_\pm)$ over (X, D) of weights $\alpha = (\alpha_1, \dots, \alpha_r)$ is given by a reduction of structure group of E to a parabolic $(H^{\mathbb{C}})^\sigma$-bundle $E_{(H^{\mathbb{C}})^\sigma}$ as defined in Section 5.1 and by a Higgs field $\varphi_\pm$ that takes values in $PE_{(H^{\mathbb{C}})^\sigma}(\mathfrak{m}_\pm^{\mathbb{C}}) \otimes K(D)$.

Proposition 5.2.4. *(1) Let (E, φ) be a parabolic polystable $(G, \theta, H^\sigma, \alpha, \pm)$-Higgs bundle over (X, D) of weights $\alpha = (\alpha_1, \dots, \alpha_r)$. Then the corresponding parabolic (G, θ)-Higgs bundle $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ over (X, D) of weights $\alpha = (\alpha_1, \dots, \alpha_r)$ is also polystable. Hence the correspondence $(E, \varphi) \mapsto (E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ defines a map*

$$\mathcal{M}(G, \theta, H^\sigma, \alpha, \mathcal{L}, \pm) \to \mathcal{M}(G, \theta, \alpha, \mathcal{L}, \pm)$$

(2) Let (E, φ) be a parabolic (G, θ, α)-Higgs bundle and consider $(E_{(H^{\mathbb{C}})^\sigma}, \varphi_\pm)$ its reduction to a parabolic $(G, \theta, H^\sigma, \alpha, \pm)$-Higgs bundle. If (E, φ) is (semi,poly)stable then $(E_{(H^{\mathbb{C}})^\sigma}, \varphi_\pm)$ is also (semi,poly)stable.

Proof. First observe that when we extend (E, φ) to a parabolic (G, θ)-Higgs bundle $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ as we mentioned above , the model metric h_0 corresponding to (E, φ) extends to a quasi-isometric local model metric h'_0 corresponding to $(E_{H^{\mathbb{C}}}, \varphi_{H^{\mathbb{C}}})$ since $\varphi_{H^{\mathbb{C}}}$ is just the section of $PE_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K(D)$ associated to the Higgs field φ by taking 0 in the $PE(\mathfrak{m}^{\mathbb{C}}_{\mp})$ component and hence $\mathrm{GrRes}_{x_i}\varphi_{H^{\mathbb{C}}} = \mathrm{GrRes}_{x_i}\varphi$. Then the proof of (1) is analogous to the one given in Proposition 3.4.1.

The reasoning to proof (2) is analogous to the one done in (2) in Proposition 3.4.1. □

5.3 Involutive map of the moduli spaces of parabolic (G, θ)-Higgs bundles with a single point in the divisor

Let G be the connected component at the identity of a real form of a semisimple Lie group $G^{\mathbb{C}}$ and let θ be its Cartan involution. As we noticed in Chapter 1 $\tau := \theta\mu$ is a compact conjugation of $G^{\mathbb{C}}$, where μ is the involution defining G. Let us consider $\sigma \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that

$$\sigma\theta = \theta\sigma,$$

$$\sigma\tau = \tau\sigma,$$

$$\sigma\mu = \mu\sigma.$$

By abuse of notation we denote $d\sigma \in \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$, $\sigma_{|\mathbb{R}} \in \mathrm{Aut}_2(G, \theta)$ and $d\sigma_{|\mathbb{R}} \in \mathrm{Aut}_2(\mathfrak{g}, d\theta)$ also by σ. Consider

$$\pi : \mathrm{Aut}_2(G, \theta) \to \mathrm{Out}_2(G, \theta)$$

the natural projection of $\mathrm{Aut}_2(G, \theta)$ on $\mathrm{Out}_2(G, \theta)$ introduced in Section 1.3. For any (G, θ)-Higgs bundle (E, φ), we define,

$$\iota(\sigma, \pm)(E, \varphi) := (\sigma(E), \pm\sigma(\varphi)), \tag{5.3}$$

where $\sigma(E) := E \times_\sigma H^{\mathbb{C}}$ is the principal $H^{\mathbb{C}}$-bundle associated to E via σ and $\sigma(\varphi)$ is its corresponding Higgs field. This is well define since $\sigma\theta = \theta\sigma$. One can check that if $\sigma \in \mathrm{Int}_2(G, \theta)$ then the Higgs bundle $(\sigma(E), \pm\sigma(\varphi))$ is isomorphic to (E, φ). Hence the group $\mathrm{Out}_2(G, \theta)$ acts on the set of isomorphism classes of (G, θ)-Higgs bundles.

Now, let (E, φ) be a parabolic (G, θ)-Higgs bundle over (X, x) with parabolic weight α on x and parabolic structure Q. We define an equivalent map to (5.3) for the case of parabolic bundles.

$$\iota(\sigma, \pm): \quad (E, \alpha, Q, \varphi, \mathcal{L}) \quad \mapsto \quad (\sigma(E), \sigma(\alpha), \sigma(Q), \pm\sigma(\varphi), \sigma(\mathcal{L})). \tag{5.4}$$

Notice that (5.4) is well defined. To see that recall that in Section 5.1 we proved that $\iota(\sigma) : (E, \alpha, Q) \mapsto (\sigma(E), \sigma(\alpha), \sigma(Q))$ is well defined. Hence we only need to check what happens with the residue of φ at $x \in X$. For that we will use the following remark.

Remark 5.3.1. Consider $\overline{L}_\alpha \subset \overline{P}_\alpha \subset G^{\mathbb{C}}$ the parabolic and Levi subgroup of $G^{\mathbb{C}}$ of weight α as defined above and $L_\alpha \subset P_\alpha \subset H^{\mathbb{C}}$. Then

$$\overline{\mathfrak{p}}_\alpha = \mathfrak{p}_\alpha \oplus \mathfrak{m}_\alpha \subset \mathfrak{g}^{\mathbb{C}},$$

where $\overline{\mathfrak{p}}_\alpha$ and $\mathfrak{p}_\alpha$ are the Lie algebras of $\overline{P}_\alpha$ and P_α and $\mathfrak{m}_\alpha \subset \mathfrak{m}^{\mathbb{C}}$ is defined in Section 2.5.

Recall that σ sends $\overline{P}_\alpha$ to $\overline{P}_{\sigma(\alpha)}$, then $\mathrm{d}\sigma$ sends $\overline{\mathfrak{p}}_\alpha$ to $\overline{\mathfrak{p}}_{\mathrm{d}\sigma(\alpha)}$. In particular, $\mathrm{d}\sigma$ sends $\mathfrak{p}_\alpha$ to $\mathfrak{p}_{\mathrm{d}\sigma(\alpha)}$. From this fact and Remark 5.3.1 we get that $\mathrm{d}\sigma$ sends $\mathfrak{m}_\alpha$ to $\mathfrak{m}_{\mathrm{d}\sigma(\alpha)}$. Reasoning in the same way we have that σ and $\mathrm{d}\sigma$ send $\widetilde{L}_\alpha$ and $\widetilde{\mathfrak{m}}^0_\alpha$ to $\widetilde{L}_{\sigma(\alpha)}$ and $\widetilde{\mathfrak{m}}^0_{\mathrm{d}\sigma(\alpha)}$ respectively. Hence $\sigma(\mathcal{L}) \in \widetilde{\mathfrak{m}}^0_{\mathrm{d}\sigma(\alpha)}/\widetilde{L}_{\sigma(\alpha)}$ and thus $\iota(\sigma, \pm)$ is well defined.

Remark 5.3.2. Notice that $\iota(\sigma, \pm)$ preserves the relative degree and pardegree since by Proposition A.2.1 in [22] σ is an isometry with respect to Tits distance (see [22, 35, 42] for further details about Tits distance). It also works well with the parabolic subgroups of $H^{\mathbb{C}}$ appearing in Definition 2.5.3 hence it preserves (semi,poly)stability.

5.4 Involutive map of the moduli spaces of parabolic (G, θ)-Higgs bundles with r point in the divisor

The involutive map studied in Section 5.3 can be defined in a more general setting where the divisor D consists of more than one point. Let (E, φ) be a (G, θ)-Higgs bundle over (X, D) with weights $\alpha = \{\alpha_1, \dots, \alpha_r\}$ where $D = x_1 + \cdots + x_r$ and consider $\sigma \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that

$$\sigma\theta = \theta\sigma,$$
$$\sigma\tau = \tau\sigma,$$
$$\sigma\mu = \mu\sigma.$$

Consider the involutive map $\iota(\sigma, \pm)$ that sends

$$(E, \alpha, \{Q_i\}, \varphi, \{\mathcal{L}_1, \dots, \mathcal{L}_r\}) \ \mapsto \ (\sigma(E), \sigma(\alpha), \{\sigma(Q_i)\}, \pm\sigma(\varphi), \{\sigma(\mathcal{L}_1), \dots \sigma(\mathcal{L}_r)\}). \tag{5.5}$$

Notice that (5.5) is also well defined since as we noticed in Section 5.3 we only need to check what happens with the residue of φ at $x_i \in X$ for $i = 1, \dots, r$. This is a straightforward generalisation of what we did in Section 5.3 since σ sends $\overline{P}_{\alpha_i}$ and $\widetilde{L}_{\alpha_i}$ to $\overline{P}_{\sigma(\alpha_i)}$ and $\widetilde{L}_{\sigma(\alpha_i)}$ respectively. Hence

$$\sigma(\mathcal{L}_i) \in \widetilde{\mathfrak{m}}^0_{\mathrm{d}\sigma(\alpha_i)}/\widetilde{L}_{\sigma(\alpha_i)}$$

for $i = 1, \dots, r$ and $\iota(\sigma, \pm)$ is well-defined. Finally observe that by Remark 5.3.2 this map preserves (semi,poly)stability.

5.5 Involutions of parabolic (G, θ)-Higgs bundle Moduli spaces

We are interested in the study of order two holomorphic automorphisms of the moduli space of isomorphism classes of polystable parabolic (G, θ)-Higgs bundle of parabolic weights $\alpha = \{\alpha_1, \dots, \alpha_r\}$ over $(X, D = x_1 + \cdots + x_r)$. Once we have seen that $\iota(\sigma, \pm)$ is indeed a map between moduli spaces of parabolic (G, θ)-higgs bundles, we fix some parameters in the moduli space.

Let us consider $\mathcal{M}(G, \theta, \alpha, \mathcal{L})$ the moduli space of parabolic (G, θ)-Higgs bundles over $(X, D = x_1 + \cdots + x_r)$ of fixed parabolic weights $\alpha = \{\alpha_1, \dots, \alpha_r\}$, parabolic structure $\{Q_i\}$ and fixed orbits $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_r)$. Let us consider the map

$$\begin{array}{rccc} \iota(\sigma, \pm): & \mathcal{M}(G, \theta, \alpha, \mathcal{L}) & \to & \mathcal{M}(G, \theta, \sigma(\alpha), \sigma(\mathcal{L})) \\ & (E, \alpha, \{Q_i\}, \varphi, \mathcal{L}) & \mapsto & (\sigma(E), \sigma(\alpha), \{\sigma(Q_i)\}, \pm\sigma(\varphi), \sigma(\mathcal{L})). \end{array}$$

studied in the previous sections and restict us to the case of involutions of $\mathcal{M}(G,\theta,\alpha,\mathcal{L})$. This is well defined if and only if

$$\sigma(\alpha_i)=\alpha_i \tag{5.6}$$

and

$$\sigma(\mathcal{L}_i)=\mathcal{L}_i \tag{5.7}$$

for all $i=1,\dots,r$.

The following proposition is similar to Proposition 3.5.1.

Proposition 5.5.1. *Let (E,φ) be a parabolic $(G,\theta,H^\sigma,\pm)$-Higgs bundle of weights $\alpha=(\alpha_1,\dots,\alpha_r)$ over (X,D) and let $\sigma\in\mathrm{Aut}(G^{\mathbb{C}})$ such that it commutes with θ,μ and τ and such that it verifies Conditions (5.6) and (5.7). Let $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}}$ be the corresponding extension to a (G,θ)-Higgs bundle described at the end of Section 5.2. Then $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ is isomorphic to $(\sigma(E_{H^{\mathbb{C}}}),\pm\sigma(\varphi_{H^{\mathbb{C}}}))$.*

Proposition 5.5.2. *Let (E,φ) be a simple parabolic (G,θ)-Higgs bundle of weights $\alpha=(\alpha_1,\dots,\alpha_r)$ over (X,D) isomorphic to $(\sigma(E),\pm\sigma(\varphi))$ with $\sigma\in\mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with θ,μ and τ and such that it verifies Condition (5.6). Then,*

(1) The structure group of E can be reduced to $(H^{\mathbb{C}})^{\sigma'}$ with $\sigma'=\mathrm{Int}(s)\sigma$ and $s\in S_\sigma^\tau$, with S_σ^τ as defined in (1.13) for $n=2$. In addition, s is unique up to the action of $H^{\mathbb{C}}$ and Z_τ as defined in (1) and (2) of Proposition 1.3.8 for $n=2$.

(2) If the Higgs field $\varphi\neq 0$, (E,φ) reduces to a parabolic $(G,\theta,H^{\sigma'},\alpha','\pm)$-Higgs bundle of weights $\alpha=(\alpha'_1,\dots,\alpha'_r)$ over (X,D), where $\sigma'=\mathrm{Int}(s)\sigma$ and it verifies Condition (5.6) for α'.

Proof. Let $\sigma\in\mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with θ,μ and τ and such that it verifies Conditions (5.6) and (5.7) and let A be an isomorphism between E and $\sigma(E)$ such that $\iota^-(A)(\varphi)=\sigma(\varphi)$, where $\iota^-(A)(\varphi)$ in this case is the automorphism of $PE(\mathfrak{m}^{\mathbb{C}})\otimes K(D)$ induced by A. As we explain in Sections 3.1 and 5.1 we can think A as an σ-twisted automorphism of E. Since σ is of order 2 and verifies Condition (5.6), A^2 is an isomorphism of parabolic (G,θ)-Higgs bundles (E,φ) and since (E,φ) is simple $A^2=z\in Z_\tau$. Now, in Proposition 5.1.1 we proved that the map f_A defined in (3.1) maps E onto a unique orbit in S_σ^τ under the action of $H^{\mathbb{C}}$ defined in Proposition 1.13. Moreover if we choose another isomorphism

$A' : E \to \sigma(E)$ such that $\iota^-(A')(\varphi) = \sigma(\varphi)$, then $A'A^{-1} = z'$ for $z \in Z_\tau$ meaning that $A' = z'A$ with $z' \in Z_\tau$ then if f_A defines an element $s \in S_\eta^{n,\tau}$, $f_{A'}$ defines an element $z's$, therefore the orbit defined by $f_{A'}$ is given by the action Z_τ by multiplication on the orbit defined by f_A. Hence we obtain a single $(Z_\tau \times H^{\mathbb{C}})$-orbit in S_σ^τ and by Proposition 5.1.1 this defines a reduction of structure group of E to $(H^{\mathbb{C}})^{\sigma'}$ with $\sigma' = \mathrm{Int}(s)\sigma$. Recall that there exist weights $\alpha_i' := h\alpha_i h^{-1}$ for some $h \in H$ such that $\sigma'(\alpha_i') = \alpha_i'$ for every $i = 1, \ldots, r$, proving (1).

To prove (2), let $\sigma' = \mathrm{Int}(s)\sigma$ for some $s \in S_\sigma^\tau/(Z_\tau \times H^{\mathbb{C}})$. Then we have a reduction of the structure group of E to $(H^{\mathbb{C}})^{\sigma'}$. Let denote the reduced bundle by $E_{(H^{\mathbb{C}})^{\sigma'}}$. The adjoint bundle decompose in

$$PE(\mathfrak{m}^{\mathbb{C}}) = PE_{(H^{\mathbb{C}})^{\sigma'}}(\mathfrak{m}_-^{\mathbb{C}}) \oplus PE_{(H^{\mathbb{C}})^{\sigma'}}(\mathfrak{m}_+^{\mathbb{C}}),$$

where $\mathfrak{m}_+^{\mathbb{C}}$ and $\mathfrak{m}_-^{\mathbb{C}}$ are the $(\pm)$-eigenspaces of $\mathfrak{m}^{\mathbb{C}}$ with respect to σ'. Clearly, $\iota^-(A)(\varphi) = \pm\sigma(\varphi)$ is equivalent to $\varphi \in H^0(X, E_{(H^{\mathbb{C}})^{\sigma'}}(\mathfrak{m}_\pm^{\mathbb{C}}) \otimes K)$. Finally, since Z_τ is finite then $Z_\tau \subset H$. Using semidirect product it is only a matter of computation to prove that $s \in H$. □

We have the following.

Theorem 5.5.3. *Let $\sigma \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with τ,μ and θ and such that it verifies Conditions (5.6) and (5.7). Consider the involution*

$$\begin{array}{rccl} \iota(\sigma,\pm): & \mathcal{M}(G,\theta,\alpha,\mathcal{L}) & \to & \mathcal{M}(G,\theta,\alpha,\mathcal{L}) \\ & (E,\alpha,\{Q_i\},\varphi,\mathcal{L}) & \mapsto & (\sigma(E),\sigma(\alpha)=\alpha,\{\sigma(Q_i)=Q_i\},\pm\sigma(\varphi),\sigma(\mathcal{L})=\mathcal{L}). \end{array}$$

Then

(1)

$$\bigcup_{\sigma' \in H^1_\sigma(\mathbb{Z}/2\mathbb{Z}, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^{\sigma'},\alpha',\mathcal{L}',\pm) \subset \mathcal{M}(G,\theta,\alpha,\mathcal{L})^{\iota(\sigma,\pm)},$$

(2)

$$\mathcal{M}(G,\theta)^{\iota(\sigma,\pm)}_{simple} \subset \bigcup_{\sigma' \in H^1_\sigma(\mathbb{Z}/2\mathbb{Z}, H^{\mathbb{C}}/Z_\tau)} \widetilde{\mathcal{M}}(G,\theta,H^{\sigma'},\alpha',\mathcal{L}',\pm).$$

Notice that $\iota(1,1)$ is the identity map.

Proof. Let $\sigma \in \mathrm{Aut}_2(G^{\mathbb{C}})$ such that it commutes with τ,μ and θ and such that it verifies Conditions (5.6) and (5.7). Recall that there exists a bijection between S^{τ}_{σ} and $H^1_{\sigma}(\mathbb{Z}/2, H^{\mathbb{C}}/Z_{\tau})$. Let $\sigma' \in H^1_{\sigma}(\mathbb{Z}/2, H^{\mathbb{C}}/Z_{\tau})$ and consider $(E,\varphi) \in \mathcal{M}(G,\theta,H^{\sigma'},\alpha',\mathcal{L}',\pm)$. By (1) in Proposition 5.2.4 the image of (E,φ) in $\mathcal{M}(G,\theta,\alpha,\mathcal{L})$ is given by an extension to a parabolic (G,θ)-Higgs bundle $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ of weights $\alpha = (\alpha_1,\dots,\alpha_r)$. Note that when we extend (E,φ) to a parabolic (G,θ)-Higgs bundle $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ we can choose appropiate weights $\alpha = (\alpha_1,\dots,\alpha_r)$ in order to verify Conditions (5.6) and (5.7). As we proved in Proposition 5.5.1 $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}})$ is isomorphic to $(\sigma(E_{H^{\mathbb{C}}}),\pm\sigma(\varphi_{H^{\mathbb{C}}}))$ hence $(E_{H^{\mathbb{C}}},\varphi_{H^{\mathbb{C}}}) \in \mathcal{M}(G,\theta,\alpha,\mathcal{L})^{\iota(\sigma,\pm)}$. On the other hand, let $(E,\varphi) \in \mathcal{M}(G,\theta,\alpha,\mathcal{L})$ be a parabolic simple (G,θ)-Higgs bundle of weights $\alpha = (\alpha_1,\dots,\alpha_r)$ over (X,D) such that $(E,\varphi) \simeq (\sigma(E),\pm\sigma(\varphi))$. The result follows from Proposition 5.5.2 combined with (2) of Proposition 5.2.4 and Lemma 1.3.9. $\square$

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