Journal of Functional Analysis 280 (2021) 108946

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Weighted conformal invariance of Banach spaces of analytic functions

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A R T I C L E I N F O

Article history:
Received 5 August 2020
Accepted 28 January 2021
Available online 2 February 2021
Communicated by K. Seip

MSC:
46E15
47B33

Keywords:
Banach space
Integration operator
Weighted composition operator
Weighted conformal invariance

A B S T R A C T

We consider Banach spaces of analytic functions in the unit disc which satisfy a weighted conformal invariance property, that is, for a fixed $\alpha > 0$ and every conformal automorphism φ of the disc, $f \to f \circ \varphi(\varphi')^\alpha$ defines a bounded linear operator on the space in question, and the family of all such operators is uniformly bounded in operator norm. Many common examples of Banach spaces of analytic functions like Korenblum growth classes, Hardy spaces, standard weighted Bergman and certain Besov spaces satisfy this condition. The aim of the paper is to develop a general approach to the study of such spaces based on this property alone. We consider polynomial approximation, duality and complex interpolation, we identify the largest and the smallest as well as the "unique" Hilbert space satisfying this property for a given $\alpha > 0$. We investigate the weighted conformal invariance of the space of derivatives, or anti-derivatives with the induced norm, and arrive at the surprising conclusion that they depend entirely on the properties of the (modified) Cesàro operator acting on the original space. Finally, we prove that this last result implies a John-Nirenberg type estimate for analytic functions g with the property that the integration operator

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https://doi.org/10.1016/j.jfa.2021.108946

$f \to \int_0^z f(t)g'(t)dt$ is bounded on a Banach space satisfying the weighted conformal invariance property.

1. Introduction

Conformal invariance plays a crucial role in the theory of Banach spaces of analytic functions on the unit disc $\mathbb{D}$. In particular, it turns out to be a powerful tool in understanding analytic functions with bounded mean oscillation on the boundary [8], or Bloch functions [22]. These ideas led to the rich theory of the so-called Q_p-spaces (see Xiao's book [21]) and their natural generalization, the Q_K-spaces introduced by Essén and Wulan (see [11], [12]). All of the spaces mentioned here can be defined following a common pattern, that is, using a conformally invariant seminorm. More precisely, let $Aut(\mathbb{D})$ be the group of conformal automorphisms of the unit disc in the complex plane, i.e. linear fractional maps of the form

$$\varphi(z) = \lambda \frac{z+a}{1+\bar{a}z},\ z \in \mathbb{D}, \quad a \in \mathbb{D},\ |\lambda| = 1,$$

mapping the unit disc onto itself. Following the ideas in [4], let X be a Banach space of analytic functions in $\mathbb{D}$ which contains the constants and is invariant under the operators of composition with $\varphi \in Aut(\mathbb{D})$, and set

$$\mathcal{M}_0(X) = \{f \in X :\ \|f\|_0 = \sup_{\varphi \in Aut(\mathbb{D})} \|f \circ \varphi - f \circ \varphi(0)\|_X < \infty\}, \tag{1}$$

and $\|f\|_{\mathcal{M}_0(X)} = |f(0)| + \|f\|_0$. Then it turns out that $BMOA = \mathcal{M}_0(H^2)$, the Bloch space satisfies $B = \mathcal{M}_0(A_0^2)$, $Q_p = \mathcal{M}_0(D^{2,p})$, $p \in (0,1)$, where H^2 is the Hardy space, A_0^2 the Bergman space and $D^{2,p}$ denotes the standard weighted Dirichlet space (see Section 2 for the definitions of these spaces). Finally, Q_K is constructed in the same way starting with the weighted Dirichlet space with weight K. There are a number of interesting results concerning such *Möbius invariant spaces*. For example, Rubel and Timoney [17] showed that the Bloch space is the largest space defined this way and Arazy, Fisher and Peetre [7] proved that the smallest is the Besov space B^1 consisting of analytic functions in $\mathbb{D}$ whose second derivative is integrable with respect to area measure. Arazy and Fisher [6] proved that, up to equivalence of norms, the unweighted Dirichlet space is the only Hilbert space which occurs this way.

The present paper continues the investigation along this line, but using the *weighted conformal invariance* instead, that is, invariance under the weighted composition $f \to f \circ \varphi(\varphi')^\alpha$, where $\alpha > 0$ is fixed. We should point out from the beginning that weighted conformal invariance is completely different from the property described above. This type of condition is related merely to the growth than to the oscillation of functions. In fact, according to Remark 1 c), Section 2, the spaces mentioned above cannot satisfy our conditions for weighted conformal invariance.

One motivation for our study is the fact that most common examples of Banach spaces of analytic functions in $\mathbb{D}$ like Korenblum growth classes, standard weighted Bergman and Besov spaces and Hardy spaces satisfy this property for a fixed $\alpha > 0$, and it turns out that this type of conformal invariance is responsible for a number of their common properties. For example, another important source of inspiration for the present work are the recent results in [5] where it is shown that for spaces satisfying the weighted conformal invariance condition for some $\alpha \in (0, 1)$ the usual Hilbert matrix (acting on the sequence of Taylor coefficients) induces a bounded linear operator whose spectrum is completely determined by α.

In order to give a precise definition of the objects involved in this paper, let $Hol(\Omega)$ denote the locally convex space of analytic functions in the open set $\Omega \subset \mathbb{C}$.

We consider Banach spaces X consisting of analytic functions in the unit disc $\mathbb{D}$ with the following properties:

1) X is continuously contained in $Hol(\mathbb{D})$.
2) X contains $Hol(\rho\mathbb{D})$, for all $\rho > 1$.
3) There exist constants $\alpha = \alpha(X),\ K = K(X) > 0$, such that for every $\varphi \in Aut(\mathbb{D})$, the linear map defined by $W^\alpha_\varphi f = f \circ \varphi(\varphi')^\alpha$, is bounded on X and satisfies $\|W^\alpha_\varphi\| \le K$.

Throughout in what follows, a space X with the properties 1)-3) will be called a *conformally invariant of index* $\alpha = \alpha(X)$. The aim of this paper is to investigate examples, methods of construction, as well as to establish some of the basic properties of such spaces and of some operators acting on them. The paper is organized as follows.

Section 2 contains a list of natural spaces which fulfil the axioms 1)-3).

Section 3 begins by emphasizing some natural objects related to such spaces, like the pointwise multipliers or their weak products (projective tensor products) and their relation to weighted conformal invariance. Other interesting objects related to these spaces are two abelian groups of operators emerging from 3), namely the group of composition with rotations $\{R_t :\ t \in [0, 2\pi)\}$ with $R_t f(z) = f(e^{it}z)$, the representation on $\mathcal{B}(X)$ of the hyperbolic group $\{W^\alpha_{\psi_a} : a \in (-1, 1)\}$, where $\psi_a(z) = \frac{z+a}{1+az},\ a \in (-1, 1)$, together with the semigroup of dilations defined for $r \in [0, 1]$ by $D_r f(z) = f(rz)$. The boundedness of $D_r,\ r \in [0, 1]$ follows directly from 2). In general, none of these groups is strongly continuous on the spaces in question, while the semigroup is not necessarily strongly continuous at $r = 1$.

Assuming only the uniform boundedness of $\{R_t :\ t \in [0, 2\pi)\}$ in $\mathcal{B}(X)$ instead of 3), we arrive at the interesting result (Theorem 1) that the density of the polynomials in X is equivalent to any of the following statements:

a) The strong continuity of $t \to R_t,\ t \in [0, 2\pi)$,

b) The strong continuity of $r \to D_r$ at $r = 1$ (from the left).

A sufficient condition for this is that point evaluations are dense in the dual of X (Theorem 2). At its turn, this result implies that polynomials are dense in X whenever the space is reflexive.

When X is conformally invariant of index $\alpha > 0$, and polynomials are dense in X, the full group $\{W_\varphi^\alpha : \varphi \in Aut(\mathbb{D})\}$ becomes strongly continuous with respect to the relative topology of $Hol(\mathbb{D})$ on $Aut(\mathbb{D})$. Moreover, we can represent the dual of X as a conformally invariant space of the same index. This is achieved using the pairing induced by the Hilbert space H_α determined by the reproducing kernel $k^\alpha(z, w) = (1 - \overline{w}z)^{-2\alpha}$ (Theorem 3).

In Section 4 we determine the largest and smallest conformally invariant Banach space of a given index $\alpha > 0$, which extends the results in [17] and [7] to this context. The largest space is the Korenblum growth class $\mathcal{A}^{-\alpha}$ while the smallest is either a weighted Bergman or a weighted Besov space. However, the main result of the section is the appropriate version of the Arazy-Fisher theorem [6] in this context. In Theorem 5 we prove that, up to equivalence of norms, the Hilbert space H_α defined above is the unique conformally invariant Hilbert space of index $\alpha > 0$. It is easy to see that H_α is a weighted Bergman space when $\alpha > \frac{1}{2}$, $H_{\frac{1}{2}}$ is the Hardy space H^2, while for $\alpha < \frac{1}{2}$, H_α is a weighted Dirichlet (Besov) space. A related result has been proved in [15], showing that under certain assumptions the unique Hilbert spaces that have a unitary weighted composition operator are these H_α and such operators are our W_φ^α in 3). Considering unitary operators immediately implies an identity for the reproducing kernel of the space which becomes a powerful tool in that proof. Without this assumption the approach is considerably more involved and is somewhat related to the idea in [6] where the key step is the amenability of the hyperbolic group. In our proof this property is only partly used, since our argument is essentially based on asymptotic estimates of $\|W_{\psi_a}^\alpha \zeta^n\|$ when $a \to 1^-$.

Section 5 contains two applications of the previous results. We focus first on the analogue of (1), i.e. for a given Banach space X satisfying 1) and 2) we consider the subspace $\mathcal{M}_\alpha(X)$ consisting of $f \in X$ with $W_\varphi^\alpha f \in X$, $\varphi \in Aut(\mathbb{D})$,

$$\|f\|_{\mathcal{M}_\alpha} = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|_X < \infty.$$

It turns out that $\mathcal{M}_\alpha(X)$ has a tractable structure in the case when the original space X is itself conformally invariant of index $\beta > 0$. In this case, $\mathcal{M}_\alpha(X)$ is either trivial, equal to X, or it is a space of pointwise multipliers. On the other hand, if X is not conformally invariant, we show by an example that $\mathcal{M}_\alpha(X)$ may have a very complicated structure which differs from the examples presented in Section 2. The second topic is complex interpolation. We only consider the pair given by the largest and smallest conformally invariant spaces of a given index $\alpha > 0$ and use the classical idea of E.M. Stein [19] to show that this chain of spaces consists of weighted Bergman and for $\alpha < 1$, weighted Besov spaces. Surprisingly enough the Hardy spaces are excluded from the chains.

Section 6 is devoted to three types of operators acting on conformally invariant Banach spaces: differentiation, taking the anti-derivative and general integration operators of the form $f \to T_g f$, where $T_g f(z) = \int_0^z f(t)g'(t)dt$. Here the symbol $g \in Hol(\mathbb{D})$ is fixed. In the first two cases we consider the ranges of these operators with the induced norm. The common intuition based on the so-called Littlewood-Paley identity (estimate) in weighted Berman spaces, or H^2, suggests that if X is conformally invariant of index $\alpha > 0$, then:
(i) The space of derivatives $D(X) = \{f' : f \in X\}$ with the induced norm is conformally invariant of index $\alpha + 1$,
(ii) When $\alpha > 1$, the space of anti-derivatives $A(X) = \{f : f' \in X\}$ with the induced norm is conformally invariant of index $\alpha - 1$.

The remarkable fact revealed by Theorem 7 which may be seen as the main result of this paper, is that under the assumption the polynomials are dense in X, both assertions above depend entirely on the properties of the linear map which acts on the sequence of Taylor coefficients by taking the Cesàro means, or more precisely the modified version

$$\mathcal{C}\left(\sum_{n\geq 0} f_n \zeta^n\right) = \sum_{n\geq 0} \zeta^{n+1}\left(\frac{1}{n+1}\sum_{k=0}^{n} f_k\right) \Leftrightarrow \mathcal{C}f(z) = \int_0^z \frac{f(t)}{1-t}dt.$$

It turns out that (i) holds true if and only if $\mathcal{C} \in \mathcal{B}(X)$. Moreover, in this case (ii) holds true if and only if $I_X - \mathcal{C}$ is invertible on X.

Theorem 7 has an interesting application regarding the integration operators defined above. There is a vast literature on the subject (see for example, [1]). Even in this generality, the boundedness of T_g can be characterized in terms of g (see Proposition 6). Here we are concerned with an idea of Pomerenke [16] who used the resolvent of such operators to derive the well-known John-Nirenberg inequality for $BMOA$ functions. We show that a similar inequality holds in the general context as well. More precisely, we prove that if T_g is bounded on the conformally invariant Banach space X then there exists $\delta > 0$ such that $\{\exp[\lambda(g \circ \varphi - g \circ \varphi(0))] : |\lambda| \leq \delta, \varphi \in Aut(\mathbb{D})\}$ is a bounded subset of X.

2. Examples

The purpose of this section is to list a number of examples of conformally invariant Banach spaces in the unit disc. Before doing so, we make some remarks which follow directly from the axioms and will be used frequently in what follows.

Remark 1.

a) From 2) we have by the closed graph theorem that for every $\rho > 1$, the inclusion map from $Hol(\rho\mathbb{D})$ into X is continuous.
b) The bounded operators considered in 3) are invertible, $(W_\varphi^\alpha)^{-1} = W_{\varphi^{-1}}^\alpha$.

c) The number $\alpha(X)$ in 3) is unique. Indeed, if $W_\varphi^\alpha, W_\varphi^\beta$, $\varphi \in Aut(\mathbb{D})$ are uniformly bounded and, say, $\alpha < \beta$, then by 2), $W_\varphi^\alpha W_{\varphi^{-1}}^\beta 1 = (\varphi')^{\alpha-\beta}$ is uniformly bounded in X which leads to a contradiction since the values at the origin of these functions are unbounded when $\varphi \in Aut(\mathbb{D})$. The same argument shows that the spaces $\mathcal{M}_0(X)$ defined by (1) are not conformally invariant of index $\alpha > 0$ unless they are trivial.

We now turn to the examples.

Example 1. In many cases the operators defined in 3) are isometries on the spaces in question and this property follows by a straightforward change of variable. Such examples are the usual Hardy spaces H^p, $p \geq 1$, with $\alpha(H^p) = \frac{1}{p}$, the Korenblum growth classes

$$\mathcal{A}^{-\gamma} = \{f \in Hol(\mathbb{D}) \,:\, \|f\|_{\mathcal{A}^{-\gamma}} = \sup_{|z|<1} (1-|z|^2)^\gamma |f(z)| < \infty\},$$

or their “little oh” version

$$\mathcal{A}_0^{-\gamma} = \{f \in \mathcal{A}^{-\gamma} \,:\, \lim_{|z|\to 1} (1-|z|^2)^\gamma |f(z)| = 0\},$$

with $\alpha(\mathcal{A}^{-\gamma}) = \gamma > 0$. The same holds for the standard weighted Bergman spaces

$$A_\beta^p = \left\{ f \in Hol(\mathbb{D}) \,:\, \|f\|_{A_\beta^p}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta dA(z) < \infty \right\},$$

with $\alpha(A_\beta^p) = \frac{2+\beta}{p}$. Here A denotes the normalized area measure on $\mathbb{D}$.

Of course, any Banach space X satisfying 1)-3) can be endowed with the equivalent norm

$$\|f\|_\alpha = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|,$$

which makes these operators isometric.

Example 2. The Banach space

$$\mathcal{A}^{\log} = \{f \in Hol(\mathbb{D}) \,:\, \|f\|_{\mathcal{A}^{\log}} = \sup_{|z|<1} \frac{1}{\log \frac{2}{1-|z|^2}} |f(z)| < \infty\},$$

satisfies 1) and 2), but fails to satisfy 3) for any $\alpha > 0$. Indeed, for $\varphi \in Aut(\mathbb{D})$ and $\psi = \varphi^{-1}$,

$$\|W_\varphi^\alpha f\|_{\mathcal{A}^{\log}} = \sup_{|z|<1} \frac{1}{\log \frac{2}{1-|z|^2}} |f(\varphi(z))||\varphi'(z)|^\alpha$$

$$= \sup_{|w|<1} \frac{1}{\log \frac{2}{1-|w|^2}} |f(w)| \frac{\log \frac{2}{1-|w|^2}}{\log \frac{2}{1-|\psi(w)|^2}} |\psi'(w)|^{-\alpha},$$

and for fixed $w \in \mathbb{D}$,

$$\sup_{\psi \in Aut(\mathbb{D})} \frac{\log \frac{2}{1-|w|^2}}{\log \frac{2}{1-|\psi(w)|^2}} |\psi'(w)|^{-\alpha} = \infty,$$

which implies that $\sup_{\varphi \in Aut(\mathbb{D})} \|f \circ \varphi (\varphi')^\alpha\|_{\mathcal{A}^{\log}} = \infty$, for all nonzero $f \in \mathcal{A}^{\log}$.

Example 3. The standard weighted Besov spaces $B^{p,\beta},\ p \geq 1, \beta > -1$ consist of analytic functions in $\mathbb{D}$ whose derivative belongs to A^p_β and are normed by

$$\|f\|_{B^{p,\beta}} = |f(0)| + \|f'\|_{A^p_\beta}.$$

When $p = 2$ $B^{2,\beta}$ is also called a *weighted Dirichlet space* and is denoted by $D^{2,\beta}$. They satisfy 1)-3) if $p < \beta + 2$ and in this case $\alpha(B^{p,\beta}) = \frac{\beta+2}{p} - 1$. The assertion will follow from a more general result, Theorem 7 below. The condition $p < \beta + 2$, is essential here. For example, $B^{2,0} = D^{2,0}$ does not satisfy 3) for any $\alpha > 0$.

Example 4. Let $\beta > -1, 0 < \gamma \leq 1,\ \beta - \gamma + 2 > 0$, $p \geq 1$, and consider the space $\mathcal{Q}_{p,\beta,\gamma}$, consisting of analytic functions f in $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} = \sup_{\substack{h \in (0,1) \\ t \in [0,2\pi]}} h^{-\gamma} \int_{S_h(t)} |f(z)|^p (1-|z|^2)^\beta dA(z) < \infty,$$

where S_h is the usual Carleson box $S_h(t) = \{re^{is} :\ 0 \leq 1-r \leq h,\ |t-s| \leq h\}$. The "little oh" version $\mathcal{Q}^0_{p,\beta,\gamma}$ consists of analytic functions f in $\mathbb{D}$ such that

$$\lim_{h \to 0} h^{-\gamma} \sup_{t \in [0,2\pi]} \int_{S_h(t)} |f(z)|^p (1-|z|^2)^\beta dA(z) = 0,$$

and is a closed subspace of $\mathcal{Q}_{p,\beta,\gamma}$. Then $\mathcal{Q}_{p,\beta,\gamma},\ \mathcal{Q}^0_{p,\beta,\gamma}$ satisfy 1)-3) with $\alpha = \frac{\beta-\gamma+2}{p}$.

Indeed, by a standard estimate (see [13, p. 239, Lemma 3.3] and its proof) we have

$$\|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} \sim \sup_{a \in \mathbb{D}} (1-|a|^2)^\gamma \int_{\mathbb{D}} \frac{|f(z)|^p}{|1-\overline{a}z|^{2\gamma}} (1-|z|^2)^\beta dA(z).$$

Moreover, for every $\varphi \in Aut(\mathbb{D})$ we have

$$\frac{1}{|1-\overline{a}z|^{2\gamma}} = \frac{|\varphi'(a)|^\gamma |\varphi'(z)|^\gamma}{|1-\overline{\varphi(a)}\varphi(z)|^{2\gamma}}.$$

Thus, with α as above we can use the fact that $W_\varphi^{\alpha+\frac{\gamma}{p}}$ is an isometry on A^p_β to obtain

$$\begin{aligned}\|W_\varphi^\alpha f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} &\sim \sup_{a\in\mathbb{D}}(1-|a|^2)^\gamma|\varphi'(a)|^\gamma\int_{\mathbb{D}}\frac{|W_\varphi^{\alpha+\frac{\gamma}{p}}f(z)|^p}{|1-\overline{\varphi(a)}\varphi(z)|^{2\gamma}}(1-|z|^2)^\beta dA(z)\\ &=\sup_{a\in\mathbb{D}}(1-|\varphi(a)|^2)^\gamma\int_{\mathbb{D}}\frac{|f(z)|^p}{|1-\overline{\varphi(a)}z|^{2\gamma}}(1-|z|^2)^\beta dA(z)\\ &\sim \|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}}.\end{aligned}$$

The condition $0<\gamma\le 1$ only ensures that $\mathcal{Q}_{p,\beta,\gamma}$ is not a growth class. For $0<\beta\le 1$, the spaces $\mathcal{Q}_{2,\beta,\beta}$ consist of derivatives of functions in the standard Q_β-spaces [21]. In particular, $\mathcal{Q}_{2,1,1}$ consists of derivatives of $BMOA$-functions. The norms are equivalent to the original ones modulo constants.

The above example can be somewhat refined as we shall see in the next section.

3. Basic properties

3.1. Standard objects emerging from the definition

Multipliers and weak products. Let us recall first two standard notions regarding Banach spaces satisfying 1).

Definition 1. Given two Banach spaces X, Y with the property 1), the space $Mult(X, Y)$ consists of analytic functions u in $\mathbb{D}$ with $uX \subset Y$ with the norm

$$\|u\|_{Mult(X,Y)} = \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|uf\|_Y.$$

The functions in $Mult(X, Y)$ are usually called *pointwise multipliers from X into Y*. By the closed graph theorem each $u \in Mult(X, Y)$ defines a bounded multiplication operator $M_u : X \to Y$, $M_u f = uf$, and $\|u\|_{Mult(X,Y)}$ equals the operator norm of M_u. In particular, it follows that $Mult(X, Y)$ is a Banach space.

Definition 2. Given two Banach spaces X, Y with the property 1), their weak product $X \odot Y$ consists of analytic functions f in $\mathbb{D}$ which can be represented in the form

$$f = \sum_{n\ge1} g_n h_n, \quad g_n \in X,\ h_n \in Y,\ \sum_{n\ge1}\|g_n\|_X\|h_n\|_Y < \infty. \tag{2}$$

The norm of $f \in X \odot Y$ is defined by

$$\|f\|_{X\odot Y} = \inf \sum_{n\geq 1} \|g_n\|_X \|h_n\|_Y,$$

where the infimum is taken over all representations of f in the form (2).

This can be identified with the projective tensor product $X\hat{\otimes}Y$, and thus it is a Banach space [18] which obviously satisfies 1) and if X or Y satisfies 2) then $X \odot Y$ also satisfies it. Some elementary properties of the spaces defined above are given by the following result.

Proposition 1. *Let X, Y be conformally invariant Banach spaces of indices α, respectively β.*

(i) If $\alpha > \beta$, then $Mult(X,Y) = \{0\}$.
(ii) If $\alpha < \beta$ and $Mult(X,Y)$ satisfies 2), then it is conformally invariant of index $\beta - \alpha$.
(iii) $X \odot Y$ is conformally invariant of index $\alpha + \beta$.

Proof. (i) If $u \in Mult(X,Y)$ we have

$$\sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\beta u W_{\varphi^{-1}}^\alpha 1\|_Y < \infty,$$

hence by 1),

$$\sup_{\varphi\in Aut(\mathbb{D})} |W_\varphi^\beta u W_{\varphi^{-1}}^\alpha 1(0)| < \infty.$$

In particular, for the choice $\varphi(z) = \frac{z+a}{1+\bar{a}z}$, $a \in \mathbb{D}$, we get

$$\sup_{a\in\mathbb{D}} |u(a)|(1-|a|^2)^{\beta-\alpha} < \infty.$$

Then the maximum principle implies that $u = 0$.

(ii) A similar computation gives for $f \in X$

$$W_\varphi^\beta uf = W_\varphi^{\beta-\alpha} u W_\varphi^\alpha f.$$

Since W_φ^α is invertible on X, it follows that $W_\varphi^{\beta-\alpha} uX \subset Y$.

(iii) Is immediate from Remark 1. $\square$

Remark 2.

1) If X is conformally invariant of index $\alpha > 0$, $Mult(X) = Mult(X,X)$ is invariant under composition with conformal automorphisms. Indeed, if the multiplication operator M_u is bounded on X, then $W_\varphi^\alpha M_u W_{\varphi^{-1}}^\alpha = M_{u\circ\varphi}$.

2) The spaces $Mult(B^{p,\beta}, A^p_\beta)$, $1 \le p < \beta+2$, $\frac{\beta+2}{p} - 1 = \frac{\gamma+2}{p}$, are of particular interest. They consist of analytic functions f with the property that $|f|^p(1-|z|^2)^\beta dA$ is a Carleson measure for $B^{p,\beta}$. Such spaces are not completely understood in this generality. If $f \in Mult(B^{p,\beta}, A^p_\beta)$, it follows that its primitive belongs to $Q'_{p,\beta,\beta+2-p}$. However, it is known [2] that when $p = 2$, $0 < \beta < 1$, this inclusion is strict.

One-parameter abelian operator groups. The group $Aut(\mathbb{D})$ contains several abelian one-parameter subgroups. The generic examples are the group of rotations $\{\varphi_t : \varphi_t(z) = e^{it}z,\ t \in [0, 2\pi)\}$, and the hyperbolic group $\{\psi_a : \psi_a(z) = \frac{z+a}{1+az}, a \in (-1, 1)\}$. Of course when X is conformally invariant of index $\alpha > 0$, the corresponding operators $\{W^\alpha_{\varphi_t} : t \in [0, 2\pi)\}$, $\{W^\alpha_{\psi_a} : a \in (-1, 1)\}$ form one-parameter abelian groups of operators, but in general, these groups fail to be strongly continuous. Another important object related to approximations is the semigroup of dilations $\{D_r : r \in [0, 1]\}$ defined by

$$D_r f(z) = f(rz). \tag{3}$$

Sometimes we shall write $D_r f = f_r$. By 2) and the closed graph theorem it follows that each D_r, $r > 0$ is bounded on X and the semigroup is strongly continuous on $[0, 1)$. The question of main interest is whether it is strongly continuous from the left at $r = 1$. Again, this property fails to hold in full generality.

An example for the above assertions is the space $X = \mathcal{A}^{-1}$. If $f(z) = (z-i)^{-1}$, it follows easily that the functions $t \to W^\alpha_{\varphi_t} f$, $a \to W^\alpha_{\psi_a} f, r \to D_r f$ are not norm-continuous in $\mathcal{A}^{-1}$ on $[0, 2\pi)$, $(-1, 1)$, or $[0, 1]$.

Under the additional assumption that polynomials are dense in the space, the whole group $\{W^\alpha_\varphi : \varphi \in Aut(\mathbb{D})\}$, as well as the above semigroup become strongly continuous. For the semigroup $\{D_r : r \in [0, 1]\}$ the assertion is immediate from 2), while for the group it is proved below.

Proposition 2. *Assume that X is conformally invariant of index $\alpha > 0$, and that polynomials are dense in X. If (φ_n) is a sequence in $Aut(\mathbb{D})$ which converges uniformly on compact subsets of $\mathbb{D}$ to $\varphi \in Aut(\mathbb{D})$, then $W^\alpha_{\varphi_n}$ converges strongly to W^α_φ.*

Proof. From $\varphi_n(0) \to \varphi(0)$, $\varphi'_n(0) \to \varphi'(0)$ it follows that (φ_n) converges to φ uniformly in $\rho\mathbb{D}$, for some $\rho > 1$. Then for every polynomial f, $W^\alpha_{\varphi_n} f \to W^\alpha_\varphi f$ uniformly in $\rho\mathbb{D}$, hence $W^\alpha_{\varphi_n} f \to W^\alpha_\varphi f$ in X by 2). Since polynomials are dense in X and the operator norms $\|W^\alpha_{\varphi_n}\|_{\mathcal{B}(X)}$ are bounded above, the result follows. □

If polynomials are dense in X, the one-parameter abelian groups considered above have densely defined, closed infinitesimal generators. They are given by

$$A_\alpha f = \frac{d}{dt} W^\alpha_{\varphi_t} f|_{t=0}, \quad A_\alpha f(z) = izf'(z) + i\alpha f(z), \tag{4}$$

$$\mathcal{D}_\alpha f = \frac{d}{da} W^\alpha_{\psi_a} f|_{a=0}, \quad \mathcal{D}_\alpha f(z) = (1-z^2)f'(z) - 2\alpha z f(z). \tag{5}$$

The infinitesimal generator of $\{D_r : r \in [0,1]\}$ is $-iA_0$. All of these unbounded operators are considered on their maximal domain of definition. $\mathcal{D}_{\frac{1}{2}}$ plays a crucial role in the description of the spectrum of the Hilbert matrix on conformally invariant spaces of index $\alpha \in (0,1)$ obtained in [5].

3.2. Polynomial approximation

This is a central question regarding Banach spaces of analytic functions in $\mathbb{D}$, and in many cases it is addressed with help of the dilation semigroup. In the framework considered here, this is intimately related to the rotation group given by $R_t = e^{-i\alpha t} W^\alpha_{\varphi_t}$, $\varphi_t(z) = e^{it}z, t \in [0, 2\pi]$. The results below (partially related to the work of A.E. Taylor [20]) hold for any Banach space X which satisfies 1), 2) and the weaker condition

3') $R_t \in \mathcal{B}(X)$, and $\|R_t\|$ is uniformly bounded in $t \in [0, 2\pi]$.

Recall from the previous paragraph that by 2) the semigroup of dilations $\{D_r : r \in [0,1]\}$ defined by (3) is contained in $\mathcal{B}(X)$ and is strongly continuous on $[0, 1)$.

Theorem 1. *Let X satisfy 1), 2) and 3'). The following are equivalent:*

(i) $t \to R_t$ is strongly continuous in $[0, 2\pi]$,
(ii) $r \to D_r$ is strongly continuous from the left at $r = 1$,
(iii) Polynomials are dense in X.

Proof. (i) $\Rightarrow$ (ii). For $r \in (0,1)$, let $P_r(e^{it}) = \frac{1-r^2}{|e^{it}-r|^2}$ be the Poisson kernel at $r \in (0,1)$. Then for $f \in X$, $t \to P_r(e^{it})R_t f$ is a continuous X-valued function on $[-\pi, \pi]$, and its Bochner integral satisfies for all $z \in \mathbb{D}$,

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R_t f \, dt \right)(z) = f(rz) = D_r f(z),$$

i.e.

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R_t f \, dt = D_r f.$$

Then from the standard estimates for such integrals we obtain for every $\delta > 0$,

$$\|D_r f - f\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) \|R_t f - f\| dt$$

$$\leq \sup_{|t|<\delta} \|R_t f - f\| + \frac{\left(1 + \sup_{t\in[0,2\pi]} \|R_t\|\right) \|f\|}{2\pi} \int_{|t|>\delta} P_r(e^{it}) dt.$$

Given $\varepsilon > 0$ we choose $\delta > 0$ such that $\sup_{|t|<\delta} \|R_t f - f\| < \varepsilon$ and let $r \to 1^-$ in the above inequality to obtain

$$\limsup_{r\to 1^-} \|D_r f - f\| \leq \varepsilon,$$

i.e. $D_r f \to f$ in X. (ii) $\Rightarrow$ (iii). From 2) it follows immediately that for fixed $r \in (0, 1)$, $f_r = D_r f$ can be approximated by polynomials in X, which gives (iii). (iii) $\Rightarrow$ (i). Again by 2) we conclude that $t \to R_t f$ is strongly continuous in $[0, 2\pi]$, whenever f is a polynomial, hence by (iii) this holds true for any $f \in X$. $\square$

There is an important sufficient condition for the density of polynomials in such spaces. The result is interesting in its own right as well as its applications. Throughout in what follows we shall denote by X' the dual of the Banach space X and by T' the transpose of $T \in \mathcal{B}(X),\ T'l(f) = l(Tf),\ f \in X, l \in X'$.

Theorem 2. *Let X satisfy 1), 2) and 3'). If the linear span of point evaluations $l_w(f) = f(w),\ f \in X, w \in \mathbb{D}$, is dense in X', then polynomials are dense in X.*

Proof. Note first that for fixed $w \in \mathbb{D}$, we have $R'_t l_w = l_{e^{it}w}$, and that $t \to R'_t l_w$ is continuous on $[-\pi, \pi]$. Indeed, using 1) we have for $f = \sum_{n\geq 0} f_n \zeta^n$,

$$|f(e^{it}w) - f(e^{is}w)| \leq |t - s| \sum_{n\geq 0} n|w|^n |f_n| \leq |t - s| c_w \|f\|,$$

with $c_w > 0$ independent of f. In other words, $\|R'_t l_w - R'_s l_w\|_{X'} \leq c_w |t - s|$, which proves the claim. Now 3') together with the density of $span\{l_w :\ w \in \mathbb{D}\}$ in X' imply that $t \to R'_t$ is strongly continuous on $[-\pi, \pi]$. Thus the Bochner integral

$$T_r l = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R'_t l dt,$$

defines a bounded linear operator on X' which obviously satisfies

$$T_r l_w = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R'_t l_w dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) l_{e^{it}w} dt = l_{rw} = D'_r l_w.$$

Thus, $\{D'_r : r \in [0,1]\}$ is bounded in $\mathcal{B}(X')$, and $D'_r l_w \to l_w$, $r \to 1^-$, for any fixed $w \in \mathbb{D}$. This proves that $D_r f \to f$, $r \to 1^-$, weakly in X and using again 2) we conclude that polynomials are weakly dense in X. The result follows. □

In general, the density of polynomials in X does not imply that the linear span of point evaluations is dense in X'. The weighted Bergman spaces A^1_β, $\beta > -1$, provide such examples. A direct application of Theorem 2 is as follows.

Corollary 1. *Assume that X satisfies 1), 2) and 3'). If X is reflexive then polynomials are dense in X.*

Proof. If $\Lambda \in X''$ annihilates all point evaluations, from reflexivity we have $\Lambda(l) = l(f)$, for some $f \in X$, and since $\Lambda(l_w) = f(w) = 0$, $w \in \mathbb{D}$, it follows that $f = 0$, hence $\Lambda = 0$. Thus the linear span of point evaluations is dense in X' and Theorem 2 gives the desired result. □

3.3. Duality

It is a well-known fact that if X is a Banach space of analytic functions in $\mathbb{D}$ containing the polynomials as a dense subset, then its dual X' can be represented as a Banach space of analytic functions as well. We are interested in a representation which preserves conformal invariance of index α which can be achieved with a suitable pairing. Given $\alpha > 0$, let H_α denote the Hilbert space with reproducing kernel

$$k^\alpha(z,w) = (1 - \overline{w}z)^{-2\alpha}, \quad z, w \in \mathbb{D}.$$

When $\alpha > \frac{1}{2}$ we have $H_\alpha = A^2_{2\alpha-2}$, $H_{\frac{1}{2}} = H^2$, and when $\alpha < \frac{1}{2}$, we have $H_\alpha = D^{2,2\alpha}$. Let $\langle \cdot, \cdot \rangle_\alpha$ be the scalar product induced by the kernel k^α. The reason for choosing this kernel (pairing) is the obvious identity

$$(1 - \overline{w}z)^{-2\alpha} = \overline{\varphi'^{\,\alpha}(w)}\, \varphi'^{\,\alpha}(z)(1 - \overline{\varphi(w)}\varphi(z))^{-2\alpha}, \quad z, w \in \mathbb{D},\ \varphi \in Aut(\mathbb{D}), \tag{6}$$

which says that W^α_φ is unitary on H_α. Note that

$$k^\alpha(z,w) = 1 + \sum_{n \geq 1} \frac{2\alpha \cdots (2\alpha + n - 1)}{n!} \overline{w}^n z^n, \tag{7}$$

and that for fixed $w \in \mathbb{D}$, the series on the right converges in any Banach space X which satisfies 1) and 2). Indeed, 2) implies that $Hol(\rho\mathbb{D})$ is continuously contained in X, $\rho > 1$, in particular, if $\zeta(z) = z$, $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} = 1$ (here we have used also 1)). Consequently, we obtain that if $l \in X'$, the function

$$Ul(w) = l\left(k^\alpha(\cdot, \overline{w})\right) = l\left((1 - w\zeta)^{-2\alpha}\right), \quad w \in \mathbb{D},$$

is analytic in $\mathbb{D}$. In fact, from (7) we have

$$Ul(w) = l(1) + \sum_{n\geq 1} \frac{2\alpha\cdots(2\alpha+n-1)}{n!} w^n l(\zeta^n), \tag{8}$$

and from $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} = 1$ we see that the series converges uniformly on each compact subset of $\mathbb{D}$.

This gives a linear map $U : X' \to Hol(\mathbb{D})$. We shall denote by X'_α its range, $X'_\alpha = UX'$.

Theorem 3. *Let X be conformally invariant of index $\alpha > 0$, and assume that polynomials are dense in X. Then with respect to the norm $\|Ul\| = \|l\|$, X'_α becomes a Banach space of analytic functions which is conformally invariant of index α. Moreover, every $l \in X'$ can be represented in the form*

$$l(f) = \lim_{r\to 1^-} \langle f_r, g_r\rangle_\alpha, \quad f \in X,$$

with $g = Ul \in X'_\alpha$.

Proof. If polynomials are dense in X then U is injective, since by (8) $Ul = 0$, implies that $l(\zeta^n) = 0,\ n \geq 0$, i.e. $l = 0$. Then $\|Ul\| = \|l\|$ defines a norm on X'_α which becomes isometrically isomorphic to X', in particular, it is a Banach space. The fact that X'_α satisfies 1) follows also directly from (8). To verify 2), let $\rho > 1$, let $g \in Hol(\rho^2\mathbb{D})$, and set $g^*(z) = \overline{g}(\overline{z})$. Then the dilation $g^*_\rho \in H_\alpha$, and by 1), $f \to f_{\frac{1}{\rho}}$ defines a bounded linear map from X into H_α. Thus

$$l(f) = \langle f_{\frac{1}{\rho}}, g^*_\rho\rangle_\alpha, \quad f \in X$$

defines an element $l \in X'$, and a direct calculation gives $Ul(w) = g(w)$. To see 3) we use the identity (6) in the form

$$k^\alpha(z,\overline{w}) = \overline{\varphi'^{\,\alpha}(\overline{w})}\,\varphi'^{\,\alpha}(z)(1 - \overline{\varphi(\overline{w})}\varphi(z))^{-2\alpha}, \quad z, w \in \mathbb{D},\ \varphi \in Aut(\mathbb{D}).$$

If $z = \varphi^{-1}(\lambda)$, from $\varphi'(z)(\varphi^{-1})'(\lambda) = 1$ and the above equality, we get

$$W^\alpha_{\varphi^{-1}} k^\alpha(\cdot,\overline{w})(\lambda) = (\varphi^*)'^{\,\alpha}(w)(1 - \varphi^*(w)\lambda)^{-2\alpha}, \quad \lambda, w \in \mathbb{D},$$

where, as before, $\varphi^*(w) = \overline{\varphi(\overline{w})}$. This leads to

$$W^\alpha_{\varphi^*} Ul = U(W^\alpha_{\varphi^{-1}})'l,$$

and the result follows from the fact that X satisfies 3). Finally, (8) together with another direct computation gives for $f = \sum_{n\geq 0} f_n\zeta^n$, $r \in (0,1)$,

$$\langle f_r, (Ul)_r \rangle_\alpha = \sum_{n\geq 0} r^{2n} f_n l(\zeta^n) = l(f_{r^2}).$$

Thus, by Theorem 1,

$$\lim_{r\to 1^-} \langle f_r, (Ul)_r \rangle_\alpha = l(f),$$

and the result follows. □

4. The largest and the smallest space. The Hilbert space case

In this section we show that there is a largest and a smallest Banach space of analytic functions in $\mathbb{D}$, conformally invariant of a given index $\alpha > 0$, and that amongst such spaces there exists a unique Hilbert space.

4.1. Largest and smallest space

If X is conformally invariant of index $\alpha > 0$, it contains the constants, hence by 3)

$$\sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\alpha 1\| = \sup_{\varphi\in Aut(\mathbb{D})} \|(\varphi')^\alpha\| < \infty. \tag{9}$$

Thus, a good candidate for the smallest space with this property is

$$X_\alpha^{min} = \left\{ f \in Hol(\mathbb{D}) : \ f = \sum_j a_j(\varphi_j')^\alpha, \ \varphi_j \in Aut(\mathbb{D}), \ a_j \in \mathbb{C}, \ \sum_j |a_j| < \infty \right\}, \tag{10}$$

with the norm

$$\|f\| = \inf \left\{ \sum_j |a_j| : \ f = \sum_j a_j(\varphi_j')^\alpha \right\}.$$

This space obviously satisfies 1) and 3) and by (9), it is continuously contained in any conformally invariant space of index α. It turns out that X_α^{min} can be identified either with a weighted Bergman space, or a weighted Besov space.

Lemma 1. *If $\alpha > 1$, then $X_\alpha^{min} = A^1_{\alpha-2}$, and if $\alpha \leq 1$, $X_\alpha^{min} = B^{1,\alpha-1}$. In all cases the norms are equivalent.*

Proof. A standard estimate (see [14], Chapter I) shows that $\{(\varphi')^\alpha : \ \varphi \in Aut(\mathbb{D})\}$ is bounded in $A^1_{\alpha-2}$, when $\alpha > 1$, and in $B^{1,\alpha-1}$, when $\alpha \leq 1$. Thus X_α^{min} is continuously contained in the space indicated in the statement. The reverse (continuous) inclusion follows directly from the general atomic decomposition theorem proved in [10]. □

With the lemma in hand we can prove the main result of the paragraph.

Theorem 4. *If X is conformally invariant of index $\alpha > 0$, then X is continuously contained in $X_\alpha^{max} = \mathcal{A}^{-\alpha}$, and X_α^{min} is continuously contained in X.*

Proof. We have already seen at the beginning of the paragraph that X_α^{min} is continuously contained in X. For the remaining part, let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $a, z \in \mathbb{D}$, and use 1) and 3) to conclude that there exists $K_1 > 0$, such that for all $f \in X$,

$$\sup_{a\in\mathbb{D}} |W_{\varphi_a}^\alpha f(0)| = \sup_{a\in\mathbb{D}} (1-|a|^2)^\alpha |f(a)| \le K_1\|f\|,$$

which completes the proof. □

4.2. The Hilbert space case

We shall prove that the only Hilbert space which is conformally invariant of index $\alpha > 0$ is the space H_α introduced in §3.3, i.e. $H_\alpha = A^2_{2\alpha-2}$ when $\alpha > \frac{1}{2}$, $H_{\frac{1}{2}} = H^2$, and $H_\alpha = D^{2,2\alpha}$ when $\alpha < \frac{1}{2}$.

We begin with a useful observation derived from the results in §3.2.

Lemma 2. *If X is a conformally invariant Hilbert space of index $\alpha > 0$, then there exists a scalar product on X which induces an equivalent norm and has the property that monomials form an orthogonal basis in X.*

Proof. X is reflexive, hence polynomials are dense in X, by Corollary 1. Consequently, by Theorem 1 (i), the group $\{R_t : \ t \in [0, 2\pi]\}$ is strongly continuous. Set

$$\|f\|_1^2 = \int_0^{2\pi} \|R_t f\|^2 dt.$$

Clearly, $\|\cdot\|_1$ is equivalent to the original norm. The induced scalar product is

$$\langle f, g\rangle_1 = \int_0^{2\pi} \langle R_t f, R_t g\rangle dt,$$

hence for $n \neq m$,

$$\langle \zeta^n, \zeta^m\rangle_1 = \int_0^{2\pi} e^{i(n-m)t} \langle \zeta^n, \zeta^m\rangle dt = 0,$$

which completes the proof. □

Using the equivalent norm given by Lemma 2, it follows that X consists of all analytic functions $f = \sum_{n\geq 0} f_n\zeta^n$ in $\mathbb{D}$ with

$$\|f\|^2 = \sum_{n\geq 0} |f_n|^2 v_n < \infty, \tag{11}$$

where $v_n = \|\zeta^n\|^2 > 0$. From (7) we have that H_α consists of all analytic functions $f = \sum_{n\geq 0} f_n\zeta^n$ in $\mathbb{D}$ with

$$\|f\|^2_{H_\alpha} = \sum_{n\geq 0} |f_n|^2 v_{n,\alpha} < \infty,$$

where

$$v_{0,\alpha} = 1, \quad v_{n,\alpha} = \frac{n!}{2\alpha\cdots(2\alpha+n-1)}, \quad n \geq 1. \tag{12}$$

Here is a simple observation regarding the weights $v_n,\ n \geq 0$.

Lemma 3. *There exists $c > 0$ such that*

$$\sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2} \leq c(n+1)^{2\alpha},$$

for all $n \geq 0$.

Proof. With the rotationally invariant norm considered above, the estimate

$$\sup_{\varphi\in Aut(\mathbb{D})} \|(\varphi')^\alpha\| = \sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\alpha 1\| \leq \sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\alpha\| v_0^{\frac{1}{2}} = c_1,$$

translates to

$$\sup_{a\in(0,1)} (1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k} \leq c_1^2.$$

Then for $a^2 = 1 - \frac{1}{n+1}$ we have

$$\frac{1}{(n+1)^{2\alpha}} \sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2} \leq c_2 \sup_{a\in(0,1)} (1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k} \leq c_1^2 c_2. \quad \square$$

For $a \in (0,1)$, let $\psi_a(z) = \frac{z+a}{1+az},\ z \in \mathbb{D}$. We shall use some identities and estimates for the scalar products

$$C_{n,k,\alpha}(a) = \langle W_{\psi_a}^\alpha \zeta^n, \zeta^k\rangle_{H_\alpha} = v_{k,\alpha} \frac{[\psi_a^n(\psi_a')^\alpha]^{(k)}(0)}{k!}. \tag{13}$$

Lemma 4. *(i)* $C_{n,k,\alpha}(a) \in \mathbb{R},\ k, n \geq 0, a \in (0,1)$, *and for fixed* $n \geq 0,\ a \in (0,1)$, $|C_{n,k,\alpha}(a)|, |\frac{d}{da}C_{n,k,\alpha}(a)| = o(b^k),\ k \to \infty$ *for any* $b \in (a,1)$, *while for fixed* $k, n \geq 0$, $\lim_{a\to 1^-} C_{n,k,\alpha}(a) = 0$.
(ii) If $1 \leq n \leq k,\ a \in (0,1)$, *we have*

$$C_{n,k+1,\alpha}(a) = -aC_{n,k,\alpha}(a) + \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha}C_{n-1,k,\alpha+\frac{1}{2}}(a).$$

(iii) For $k, n \geq 1$ $a \in (0,1)$,

$$C_{n,k,\alpha}(a) - \frac{k+1+2\alpha}{k+1}C_{n,k+2,\alpha}(a) = -\frac{1-a^2}{k+1}\frac{d}{da}C_{n,k+1,\alpha}(a).$$

(iv) Consequently, for $1 \leq n \leq k,\ a \in (0,1)$

$$C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a) \leq \frac{n^2(1-a^2)}{4\alpha^2 a^2}C^2_{n-1,k,\alpha+\frac{1}{2}}(a) + \frac{1}{a(k+1)}\frac{d}{da}[(1-a^2)C^2_{n,k+1,\alpha}(a)] + \frac{2}{k+1}C^2_{n,k+1,\alpha}(a) - \frac{4\alpha}{a(k+1)}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a).$$

Proof. (i) follows directly from the definition together with the fact that $W_{\psi_a}\zeta^n$ converges weakly to 0 in H_α when $a \to 1^-$. (ii) Since $\psi_a(z) = a + \frac{(1-a^2)z}{1+az}$, it follows for any $f \in H_\alpha$,

$$\langle \psi_a^n(\psi_a')^\alpha, f\rangle_{H_\alpha} = \sum_{j=0}^{n}\binom{n}{j}a^{n-j}(1-a^2)^{j+\alpha}\left\langle \frac{\zeta^j}{(1+a\zeta)^{j+2\alpha}}, f\right\rangle_{H_\alpha}.$$

Furthermore,

$$\left\langle \frac{1}{(1+a\zeta)^{2\alpha}}, f\right\rangle_{H_\alpha} = \overline{f}(-a),$$

and if $j \geq 1$,

$$\begin{aligned}\left\langle \frac{\zeta^j}{(1+a\zeta)^{j+2\alpha}}, f\right\rangle_{H_\alpha} &= \frac{(-1)^j}{2\alpha\cdots(2\alpha+j-1)}\frac{d^j}{da^j}\left\langle \frac{1}{(1+a\zeta)^{2\alpha}}, f\right\rangle_{H_\alpha}\\ &= \frac{1}{2\alpha\cdots(2\alpha+j-1)}\overline{f^{(j)}}(-a).\end{aligned}$$

Thus for $f = \zeta^k$, since $k \geq n$, we obtain

$$C_{n,k,\alpha}(a) = \sum_{j=0}^{n}\binom{n}{j}(-1)^{k-j}a^{n+k-2j}(1-a^2)^{j+\alpha}\prod_{0\leq l<j}\frac{k-l}{2\alpha+l},$$

where, as usual we set the product over the empty set to be 1, i.e. the first term in the above sum is $(-1)^k a^{n+k}(1-a^2)^\alpha$. This implies

$$\begin{aligned}&C_{n,k+1,\alpha}(a) = -aC_{n,k,\alpha}(a)\\&+\sum_{j=1}^{n}\binom{n}{j}(-1)^{k+1-j}a^{n+k+1-2j}(1-a^2)^{j+\alpha}\left(\prod_{0\le l<j}\frac{k+1-l}{2\alpha+l}-\prod_{0\le l<j}\frac{k-l}{2\alpha+l}\right).\end{aligned}$$

Now with the above convention it is easy to verify that for $j \geq 1$, we have

$$\begin{aligned}&\prod_{0\le l<j}\frac{k+1-l}{2\alpha+l}-\prod_{0\le l<j}\frac{k-l}{2\alpha+l}=\left(\prod_{0\le l<j}\frac{1}{2\alpha+l}\prod_{0\le l<j-1}(k-l)\right)((k+1)-(k-j+1))\\&=\frac{j}{2\alpha}\prod_{0\le l<j-1}\frac{k-l}{2\alpha+1+l}.\end{aligned}$$

This leads to

$$\begin{aligned}&C_{n,k+1,\alpha}(a) = -aC_{n,k,\alpha}(a)\\&+\frac{n(1-a^2)^{\frac12}}{2\alpha}\sum_{j=1}^{n}\binom{n-1}{j-1}(-1)^{k-j+1}a^{n-1+k-2(j-1)}(1-a^2)^{j-1+\alpha+\frac12}\prod_{0\le l<j-1}\frac{k-l}{2\alpha+1+l}\\&=-aC_{n,k,\alpha}(a)+\frac{n(1-a^2)^{\frac12}}{2\alpha}C_{n-1,k,\alpha+\frac12}(a),\end{aligned}$$

which proves the identity in the statement.
(iii) Recall that $\{W^\alpha_{\psi_a} : a \in (-1,1)\}$ is a unitary group on H_α with infinitesimal generator $\mathcal{D}_\alpha$ given by (5). Then $i\mathcal{D}_\alpha$ is selfadjoint on H_α, i.e. $\mathcal{D}^*_\alpha = -\mathcal{D}_\alpha$ on this space. Moreover, taking into account the parametrization of the group, or by a direct calculation, it follows that

$$\mathcal{D}_\alpha W^\alpha_{\psi_a} f = W^\alpha_{\psi_a}\mathcal{D}_\alpha f = (1-a^2)\frac{d}{da}W^\alpha_{\psi_a} f,$$

whenever f is in the domain of $\mathcal{D}_\alpha$, in particular, when f is a polynomial. Since $\mathcal{D}_\alpha\zeta^{k+1} = (k+1)\zeta^k - (k+1+2\alpha)\zeta^{k+2},\ k \geq 1$, we obtain

$$\begin{aligned}&C_{n,k,\alpha}(a)-\frac{k+1+2\alpha}{k+1}C_{n,k+2,\alpha}(a)=\frac{1}{k+1}\langle W^\alpha_{\psi_a}\zeta^n,\mathcal{D}_\alpha\zeta^{k+1}\rangle_{H_\alpha}\\&=-\frac{1}{k+1}\langle\mathcal{D}_\alpha W^\alpha_{\psi_a}\zeta^n,\zeta^{k+1}\rangle_{H_\alpha}=-\frac{1-a^2}{k+1}\langle\frac{d}{da}W^\alpha_{\psi_a}\zeta^n,\zeta^{k+1}\rangle_{H_\alpha}\\&=-\frac{1-a^2}{k+1}\frac{d}{da}C_{n,k+1,\alpha}(a).\end{aligned}$$

(iv) An elementary computation yields

$$
\begin{aligned}
C_{n,k,\alpha}^2(a) - C_{n,k+2,\alpha}^2(a) &= 2C_{n,k,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) - (C_{n,k,\alpha}(a) \\
&- C_{n,k+2,\alpha}(a))^2 = 2(C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a))(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) \\
&- \frac{2}{a}C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) - (C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a))^2 \\
&\leq (C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a))^2 - \frac{2}{a}C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)).
\end{aligned}
\tag{14}
$$

By (ii) we have $C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a) = \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha a}C_{n-1,k,\alpha+\frac{1}{2}}(a)$, and by (iii),

$$
\begin{aligned}
&C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) \\
&\quad = C_{n,k+1,\alpha}(a)\left(C_{n,k,\alpha}(a) - \frac{k+1+2\alpha}{k+1}C_{n,k+2,\alpha}(a)\right) \\
&\qquad + \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a) \\
&\quad = -\frac{1-a^2}{k+1}C_{n,k+1,\alpha}(a)\frac{d}{da}C_{n,k+1,\alpha}(a) + \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a) \\
&\quad = -\frac{1}{2(k+1)}\frac{d}{da}[(1-a^2)C_{n,k+1,\alpha}^2(a)] - \frac{a}{k+1}C_{n,k+1,\alpha}^2(a) \\
&\qquad + \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a).
\end{aligned}
$$

By replacing these in the last line of (14), we obtain the desired inequality. $\square$

We can now turn to our result about conformally invariant Hilbert spaces.

Theorem 5. *Let X be a Hilbert space which is conformally invariant of index $\alpha > 0$. Then $X = H_\alpha$ and the corresponding norms are equivalent.*

Proof. Without loss of generality, we can assume that X is equipped with the norm given by Lemma 2. As above, let $v_n = \|\zeta^n\|^2$, and let $v_{n,\alpha}$ be given by (12). It will be sufficient to show that there exists $\eta > 0$, depending only on X, such that

$$
v_n \leq \eta v_{n,\alpha}, \quad n \geq 0. \tag{15}
$$

Indeed, a straightforward argument shows that X' consists of all analytic functions $f = \sum_{n\geq 0} f_n\zeta^n$ in $\mathbb{D}$ with

$$
\|f\|_{X'}^2 = \sum_{n\geq 0} |f_n|^2 \frac{v_{n,\alpha}^2}{v_n} < \infty.
$$

If (15) holds for any space X as in the statement, it holds for X' as well, which implies $\frac{v_{n\alpha}^2}{v_n} \leq \eta' v_{n\alpha}$ for all n, that is,

$$\frac{1}{\eta'} v_{n\alpha} \leq v_n \leq \eta v_{n\alpha}, \quad n \geq 0,$$

and the result follows.

To verify the claim (15), we consider the disc-automorphisms $\psi_a(z) = \frac{z+a}{1+az}$, $z \in \mathbb{D}$, $a \in (0,1)$, and use the conformal invariance of X to conclude that

$$v_n = \|\zeta^n\|^2 \leq c_1 \|W_{\psi_a}^\alpha \zeta^n\|^2 = c_1 \sum_{k=0}^\infty C_{n,k,\alpha}^2(a) \frac{v_k}{v_{k,\alpha}^2}, \quad a \in (0,1), \tag{16}$$

where $C_{n,k,\alpha}(a)$ are given by (13). For $n \geq 2$, $a \in (0,1)$, let

$$S_n(a) = \sum_{k=n}^\infty (C_{n,k,\alpha}^2(a) - C_{n,k+2,\alpha}^2(a)) \sum_{j=n}^k \frac{v_j}{v_{j,\alpha}^2}.$$

By Lemma 3 and Lemma 4 (i), for fixed $a \in (0,1)$ we can interchange the order of summation and obtain,

$$\begin{aligned} S_n(a) &= \sum_{j=n}^\infty \frac{v_j}{v_{j,\alpha}^2} \sum_{k \geq j} (C_{n,k,\alpha}^2(a) - C_{n,k+2,\alpha}^2(a)) \\ &= \sum_{j=n}^\infty \frac{v_j}{v_{j,\alpha}^2} (C_{n,j,\alpha}^2(a) + C_{n,j+1,\alpha}^2(a)) \\ &= \sum_{j=0}^\infty \frac{v_j}{v_{j,\alpha}^2} (C_{n,j,\alpha}^2(a) + C_{n,j+1,\alpha}^2(a)) - \sum_{j=0}^{n-1} \frac{v_j}{v_{j,\alpha}^2} (C_{n,j,\alpha}^2(a) + C_{n,j+1,\alpha}^2(a)). \end{aligned}$$

Thus by another application of Lemma 4 (i) we conclude that

$$v_n = \|\zeta^n\|^2 \leq c_1 \sum_{k=0}^\infty C_{n,k,\alpha}^2(a) \frac{v_k}{v_{k,\alpha}^2} \leq c_1 S_n(a) + o(1), \quad a \to 1^-. \tag{17}$$

In order to estimate $S_n(a)$ when $a \to 1^-$, for $k \geq n$, let

$$\begin{aligned} B_{n,k,\alpha}(a) = {} & \frac{n^2(1-a^2)}{4\alpha^2 a^2} C_{n-1,k,\alpha+\frac{1}{2}}^2(a) + \frac{1}{a(k+1)} \frac{d}{da} [(1-a^2) C_{n,k+1,\alpha}^2(a)] \\ & + \frac{2}{k+1} C_{n,k+1,\alpha}^2(a) - \frac{4\alpha}{a(k+1)} C_{n,k+1,\alpha}(a) C_{n,k+2,\alpha}(a), \end{aligned}$$

and use the inequality in Lemma 4 (iv) to obtain

$$S_n(a) \le \sum_{k=n}^{\infty} B_{n,k,\alpha}(a) \sum_{j=n}^{k} \frac{v_j}{v_{j,\alpha}^2}. \tag{18}$$

We want to use the estimate in Lemma 3, but at this stage it cannot be applied since the numbers $B_{n,k,\alpha}(a)$ might be negative. However, it turns out that their (weighted) averages can be controlled. If $a \in (0,1)$, then

$$\begin{aligned}\frac{1}{a-a^2}\int_{a^2}^{a} sB_{n,k,\alpha}(s)ds &= \frac{n^2}{4\alpha^2(a-a^2)}\int_{a^2}^{a} \frac{(1-s^2)}{s} C^2_{n-1,k,\alpha+\frac{1}{2}}(s)ds \\ &+ \frac{1}{(k+1)(a-a^2)}\int_{a^2}^{a} \frac{d}{ds}[(1-s^2)C^2_{n,k+1,\alpha}(s)]ds \\ &+ \frac{2}{(k+1)(a-a^2)}\int_{a^2}^{a} sC^2_{n,k+1,\alpha}(s)ds \\ &- \frac{4\alpha}{(k+1)(a-a^2)}\int_{a^2}^{a} C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)ds.\end{aligned}$$

Note that

$$\int_{a^2}^{a} \frac{d}{ds}[(1-s^2)C^2_{n,k+1,\alpha}(s)]ds \le (1-a^2)C^2_{n,k+1,\alpha}(a),$$

and

$$-\int_{a^2}^{a} C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)ds \le \int_{a^2}^{a} |C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)|ds.$$

From (18) we infer that

$$\frac{1}{a-a^2}\int_{a^2}^{a} sS_n(s)ds \le \sum_{k=n}^{\infty}\sum_{j=n}^{k} \frac{v_j}{v_{j,\alpha}^2}\frac{1}{a-a^2}\int_{a^2}^{a} sB_{n,k,\alpha}(s)ds,$$

where the interchange of the sum and the integral is justified by Lemma 4 (i) and Lemma 3. Now use the above estimates in order to replace $\frac{1}{a-a^2}\int_{a^2}^{a} sB_{n,k,\alpha}(s)ds$ by a sum of four nonnegative terms, and then apply Lemma 3 to arrive at

$$\frac{1}{a-a^2}\int_{a^2}^{a} sS_n(s)ds \le c\frac{1}{a-a^2}\int_{a^2}^{a} \left(S_n^{I}(s) + S_n^{II}(s) + S_n^{III}(s)\right) ds + cS_n^{IV}(a), \tag{19}$$

where c is the constant in Lemma 3, and

$$S_n^I(s) = \frac{n^2(1-s^2)}{4\alpha^2 s}\sum_{k=0}^{\infty}(k+1)^{2\alpha}C^2_{n-1,k,\alpha+\frac{1}{2}}(s),$$

$$S_n^{II}(s) = 2s\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(s),$$

$$S_n^{III}(s) = 4\alpha\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}|C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)|,$$

$$S_n^{IV}(a) = \frac{1-a^2}{(a-a^2)}\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(a).$$

From the obvious estimate $(k+1)^{2\alpha} \le c_2 \min\{v^{-1}_{k,\alpha+\frac{1}{2}}, (k+1)v^{-1}_{k+1,\alpha}, (k+1)v^{-1}_{k+2,\alpha}\}$, valid for some fixed constant $c_2 > 0$ and all integers $k \ge 0$, we conclude that

$$\begin{aligned} S_n^I(s) &\le c_2\frac{n^2(1-s^2)}{4\alpha^2 s}\sum_{k=0}^{\infty}v^{-1}_{k,\alpha+\frac{1}{2}}C^2_{n-1,k,\alpha+\frac{1}{2}}(s) = c_2\frac{n^2(1-s^2)}{4\alpha^2 s}\|W^{\alpha+\frac{1}{2}}_{\psi_a}\zeta^{n-1}\|^2_{H_{\alpha+\frac{1}{2}}} \\ &= c_2\frac{n^2(1-s^2)}{4\alpha^2 s}v_{n-1,\alpha+\frac{1}{2}}, \end{aligned}$$

and similarly,

$$\begin{aligned} &S_n^{II}(s) \le 2c_2\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha} = 2c_2 v_{n,\alpha}, \quad S_n^{III}(s) \le c_2 4\alpha\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha} = c_2 4\alpha v_{n,\alpha}, \\ &S_n^{IV}(a) \le c_2\frac{1+a}{a}\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha} = c_2\frac{1+a}{a}v_{n,\alpha} \end{aligned}$$

where in the estimate of $S_n^{III}(s)$ we have used the Cauchy-Schwartz inequality. In particular, $\lim_{a\to 1^-}\frac{1}{a-a^2}\int_{a^2}^a S_n^I(s)\,ds = 0$, and there exists $c_3 > 0$ such that

$$\frac{1}{a-a^2}\int_{a^2}^{a} S_n^{II}(s)\,ds,\ \frac{1}{a-a^2}\int_{a^2}^{a} S_n^{III}(s)\,ds,\ S_n^{IV}(a) \le c_3 v_{n,\alpha},$$

for all $n \ge 2$ and all $a \in (\frac{1}{2}, 1)$. Thus from (17)

$$v_n = \|\zeta^n\|^2 \le 3c_1\frac{1}{a-a^2}\int_{a^2}^{a} sS_n(s)ds + o(1) \le c_4 v_{n,\alpha} + o(1), \quad a \to 1^-,$$

and the claim (15) follows by letting $a \to 1^-$. This completes the proof of the theorem. $\square$

5. Applications

5.1. Conformally invariant subspaces

Following the idea in [4] we attempt to construct a conformally invariant space starting with an arbitrary Banach space X of analytic functions in $\mathbb{D}$ which satisfies 1) and 2). We simply set for $\alpha > 0$,

$$\mathcal{M}_\alpha(X) = \{f \in X: \ (\varphi')^\alpha f \circ \varphi \in X, \ \varphi \in Aut(\mathbb{D}), \ \|f\|_{\mathcal{M}_\alpha} = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|_X < \infty\}. \tag{20}$$

Clearly, $\mathcal{M}_\alpha(X) \subset X$, with equality if and only if X is conformally invariant of index α. Moreover, $\mathcal{M}_\alpha(X)$ is a Banach space satisfying 1) and 3), but it is not clear whether it satisfies the condition 2).

Proposition 3. *Let X be a Banach space satisfying 1) and 2). Then $\mathcal{M}_\alpha(X)$ is conformally invariant of index α if and only if $\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\|_X < \infty$. Moreover, if $\mathcal{A}^{-\alpha} \subset X$ then $\mathcal{M}_\alpha(X) = \mathcal{A}^{-\alpha}$*

Proof. This is a direct application of Theorem 4. The condition $\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\|_X < \infty$, is equivalent to $X_\alpha^{min} \subset \mathcal{M}_\alpha(X)$. If it holds, then $\mathcal{M}_\alpha(X)$ satisfies 2) because X_α^{min} does. Conversely, if $\mathcal{M}_\alpha(X)$ satisfies 2), it is conformally invariant of index α, hence it must contain X_α^{min}. If $\mathcal{A}^{-\alpha} \subset X$, the inclusion is continuous by the closed graph theorem, hence $\mathcal{A}^{-\alpha} \subset \mathcal{M}_\alpha(X)$. Thus, $\mathcal{M}_\alpha(X)$ is conformally invariant of index α, and by Theorem 4 we have $\mathcal{A}^{-\alpha} = \mathcal{M}_\alpha(X)$. $\square$

Some interesting examples occur this way.

Example 5. If $p \geq 1$, and $\alpha < \frac{1}{p}$, a direct computation based on a change of variable reveals that $\mathcal{M}_\alpha(H^p)$ consists of $f \in H^p$ with

$$\sup_{a \in \mathbb{D}} (1 - |a|^2)^{1-\alpha p} \int_0^{2\pi} \frac{|f(e^{it})|^p}{|e^{it} - a|^{2-2\alpha p}} dt < \infty.$$

If arcs on the unit circle are denoted by I, and their length by $|I|$, the above condition is equivalent to

$$\sup_I |I|^{\alpha p - 1} \int_I |f(e^{it})|^p dt < \infty.$$

In a similar way it follows for $p \geq 1$, $\beta > -1$, and $\frac{\beta+1}{p} \leq \alpha < \frac{\beta+2}{p}$, that $\mathcal{M}_\alpha(A_\beta^p) = \mathcal{Q}_{p,\beta,\beta+2-\alpha p}$.

Let us also note that for $\gamma > 0$, we have $\mathcal{M}_\alpha(\mathcal{A}^{-\gamma}) = \{0\}$, when $\alpha > \gamma$, $\mathcal{M}_\gamma(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma}$, and $\mathcal{M}_\alpha(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\alpha}$, when $0 < \alpha < \gamma$. All these examples are actually consequences of a general fact which is proved below.

Proposition 4. *Let X be conformally invariant of index $\gamma > 0$. Then $\mathcal{M}_\alpha(X) = \{0\}$, when $\alpha > \gamma$, $\mathcal{M}_\gamma(X) = X$, and when $0 < \alpha < \gamma$, $\mathcal{M}_\alpha(X) = Mult(X^{min}_{\gamma-\alpha}, X)$.*

Proof. Let $\alpha > \gamma$, and let $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$, $a, z \in \mathbb{D}$. If $f \in M_\alpha(X)$ and $z \in K$, with K a compact subset of $\mathbb{D}$, by 1)

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{\gamma-\alpha}|f(z)| &\lesssim \sup_{a\in\mathbb{D}}|(\varphi_a'(z))^{\gamma-\alpha}f(z)| = \sup_{a\in\mathbb{D}}|W^\gamma_{\varphi_a}W^\alpha_{\varphi_a}f(z)| \\ &\lesssim \sup_{a\in\mathbb{D}}\|W^\gamma_{\varphi_a}W^\alpha_{\varphi_a}f\|_X \lesssim \sup_{a\in\mathbb{D}}\|W^\alpha_{\varphi_a}f\|_X \le \|f\|_{\mathcal{M}_\alpha(X)}.\end{aligned}$$

Hence, $f = 0$. If $0 < \alpha < \gamma$, note that rotations are bounded in X and for f analytic in $\mathbb{D}$ we have

$$W^\gamma_{\varphi_a}(f(\varphi_a')^{\gamma-\alpha}) = W^\alpha_{\varphi_a}f.$$

If $f \in Mult(X^{min}_{\gamma-\alpha}, X)$, then the left hand side is bounded in X, uniformly in $\varphi \in Aut(\mathbb{D})$, which gives $f \in \mathcal{M}_\alpha(X)$. Conversely, if $f \in \mathcal{M}_\alpha(X)$, then the right hand side is bounded in X, uniformly in $\varphi \in Aut(\mathbb{D})$, and by 3), the same holds for $f(\varphi')^{\gamma-\alpha}$, which gives $f \in Mult(X^{min}_{\gamma-\alpha}, X)$, by (10). $\square$

The situation is more complicated in the case when X is not conformally invariant. We close the paragraph with an example of this type.

Example 6. Let

$$A^1_{\log^{-2}} = \left\{ f \in Hol(\mathbb{D}) \,:\, \|f\|_{A^1_{\log^{-2}}} = \int_{\mathbb{D}} |f(z)| \frac{1}{\log^2 \frac{2}{1-|z|^2}} dA(z) < \infty \right\}.$$

Then:
a) $A^1_{\log^{-2}}$ is not conformally invariant of any index $\alpha > 0$,
b) $\mathcal{M}_\alpha(A^1_{\log^{-2}}) = 0$, when $\alpha > 2$, and $\mathcal{M}_\alpha(A^1_{\log^{-2}}) = \mathcal{A}^{-\alpha}$ when $0 < \alpha \le 1$,
c) If $1 < \alpha \le 2$ we have for every $\varepsilon > 0$,

$$\begin{aligned} &Mult(X^{min}_{2-\alpha}, A^1_0) \subset \mathcal{M}_\alpha(A^1_{\log^{-2}}) \subset Mult(X^{min}_{2-\alpha+\varepsilon}, A^1_\varepsilon) && 1 < \alpha < 2, \\ &A^1_0 \subset \mathcal{M}_\alpha(A^1_{\log^{-2}}) \subset Mult(X^{min}_\varepsilon, A^1_\varepsilon) && \alpha = 2. \end{aligned}$$

It is easily seen that the multiplier spaces which appear in c) are strictly contained in the corresponding growth class $\mathcal{A}^{-\alpha}$. We were not able to relate the formal definition of

$\mathcal{M}_\alpha(A^1_{\log^{-2}})$ to standard objects in this area, for example those considered in Section 2. It remains a challenging question to do so. For this reason, we have appealed to the obvious fact that there exists $C > 0$, and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that for all $g \in Hol(\mathbb{D})$, we have

$$\|g\|_{A^1_\varepsilon} \le C_\varepsilon \|g\|_{A^1_{\log^{-2}}} \text{ and } \|g\|_{A^1_{\log^{-2}}} \le C\|g\|_{A^1_0}. \tag{21}$$

To sketch a proof of these assertions, note that for $0 < \alpha \le 1$, we have $\mathcal{A}^{-\alpha} \subset A^1_{\log^{-2}}$, so that Proposition 3 gives the second part of b). Since A^1_ε is conformally invariant of index $2 + \varepsilon$ (see Example 1), the other part of b) follows by (21) together with Proposition 4 by choosing $\varepsilon < \alpha - 2$. c) follows by the same argument. Finally, note that by (21) and Proposition 4, c) implies a).

5.2. Interpolation

Interpolating between conformally invariant Banach spaces of analytic functions is certainly meaningful and might lead to interesting examples. In view of 1), any pair of such spaces is *compatible* in the sense of interpolation theory (see [9]), but, in general, describing the intermediate spaces is a difficult task. We shall consider the extreme case, that is, we are going to apply the complex interpolation method to the couple $(X^{max}_\alpha, X^{min}_\alpha)$, $\alpha > 0$. Our result is essentially based on the following lemma which is actually a well known result. Given a positive measurable function v on $\mathbb{D}$ we denote by $L^p(v) = L^p(vdA)$, $1 \le p < \infty$, and $L^\infty(v) = v^{-1}L^\infty(dA)$.

Lemma 5. *Let $\gamma > -1$, $\delta > 0$, $p \in [1, \infty)$. For $\beta > \max\{\gamma, \delta\}$, $f \in L^1((1 - |\zeta|^2)^\beta)$, and $z \in \mathbb{D}$, define*

$$P_\beta f(z) = (\beta + 1)\int\limits_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^{\beta+2}}(1 - |w|^2)^\beta dA(w), \quad Q_\beta f(z) = P_\beta(f(1 - |\zeta|^2))(z).$$

Then P_β is a bounded projection from $L^\infty((1-|\zeta|^2)^\delta)$ onto $\mathcal{A}^{-\delta}$, and from $L^p\left((1 - |\zeta|^2)^\gamma\right)$ onto A^p_γ. Moreover, Q_β extends to a continuous bijection from $\mathcal{A}^{-\delta-1}$ onto $\mathcal{A}^{-\delta}$, from A^p_γ onto $A^p_{\gamma-p}$, when $\gamma > p - 1$, and from A^p_γ onto $B^{p,\gamma}$, when $\gamma \le p - 1$.

Proof. The proof of the assertions regarding P_β can be found, for example, in the first chapter of [14]. Using also the equality

$$(Q_\beta f)' = (\beta + 1)P_{\beta+1}\overline{\zeta}f,$$

we conclude from the first part that Q_β is a bounded linear operator between the spaces in the statement. Its injectivity is obvious, since if f belongs to the spaces in the statement, and $Q_\beta f = 0$, then all Taylor coefficients of f must vanish. To see the surjectivity, note that from the above equality it follows by a straightforward calculation that whenever $u\zeta^{-1} \in L^\infty((1-|\zeta|^2)^{\delta+1})$, or $u\zeta^{-1} \in L^p\left((1-|\zeta|^2)^\gamma\right)$, we have

$$(\beta+1)P_{\beta+1}u = (Q_\beta\overline{\zeta}^{-1}u)' = (\beta+1)P_{\beta+1}\overline{\zeta}P_{\beta+1}\overline{\zeta}^{-1}u = (Q_\beta P_{\beta+1}\overline{\zeta}^{-1}u)'$$

Therefore, if $f \in \mathcal{A}^{-\delta-1}$, or $f \in A^p_\gamma$ with $f(0) = 0$, we have

$$(\beta+1)f = (Q_\beta P_{\beta+1}\overline{\zeta}^{-1}f)'.$$

Moreover, since $Q_\beta\zeta^n = c_{n\beta}\zeta^n$, with $c_{n\beta} \neq 0$, the range of Q_β contains all polynomials of first degree. Thus, the range of Q_β contains all anti-derivatives of functions in $\mathcal{A}^{-\delta-1}$, respectively in A^p_γ. Then the surjectivity of Q_β follows from standard results (see again [14]). $\square$

For a compatible pair of Banach spaces (X, Y) we shall denote by $[X, Y]_\theta,\ \theta \in (0,1)$, the corresponding complex interpolation space.

Theorem 6. *For $0 < \theta < 1$ if $\alpha > \theta$, then $[X^{max}_\alpha, X^{min}_\alpha]_\theta = A^{\frac{1}{\theta}}_{\frac{\alpha}{\theta}-2}$, and if $\alpha \leq \theta$, then $[X^{max}_\alpha, X^{min}_\alpha]_\theta = B^{\frac{1}{\theta},\frac{\alpha+1}{\theta}-2}$.*

Proof. We want to find $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta,\ \theta \in (0,1)$. By the Stein interpolation theorem [19], if $\theta \in (0,1)$

$$\begin{aligned}[L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta &= L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{1-\theta}{\theta}(\alpha+1)+\alpha-1}) \\ &= L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2}).\end{aligned}$$

Since $\mathcal{A}^{-\alpha-1} \subset L^\infty((1-|\zeta|^2)^{\alpha+1}),\ A^1_{\alpha-1} \subset L^1\left((1-|\zeta|^2)^{\alpha-1}\right)$, it follows by definition that for $\theta \in (0,1)$

$$[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta \subset [L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta = L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2}).$$

Moreover, since $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta \subset \mathcal{A}^{-\alpha-1}$, it consists of analytic functions in $\mathbb{D}$, hence $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta \subset A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$. On the other hand, by standard interpolation theory (see Theorem 4.1.2 in [9]) and Lemma 5, for $\beta > \max\{\alpha+1, \frac{\alpha+1}{\theta}-2\}$ we have that

$$P_\beta : [L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta = L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2}) \to [\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta,$$

is bounded for all $\theta \in (0,1)$. Since P_β is onto, we obtain hence $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta \supset A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$. The fact that the norms on these two spaces are equivalent follows by the closed graph theorem.

Finally another application of Lemma 5 and Theorem 4.1.2 in [9] shows that Q_β is an invertible linear operator from $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta$ onto $[X^{max}_\alpha, X^{min}_\alpha]_\theta$ for all $\theta \in (0,1)$, hence the result follows from Lemma 5. $\square$

6. Derivatives, anti-derivatives and integration operators

6.1. Spaces of derivatives and anti-derivatives

Let X be a conformally invariant space of index $\alpha > 0$. We are interested in the spaces

$$D(X) = \{f' : \ f \in X\}, \quad A(X) = \{f \in Hol(\mathbb{D}) : \ f' \in X\}.$$

They are endowed with the norms

$$\|g\|_{D(X)} = \|G\|_X, \ G(z) = \int_0^z g(w)dw, \quad \|g\|_{A(X)} = |g(0)| + \|g'\|_X.$$

The norm on $D(X)$ is actually equivalent to the standard one $\|g\|_{D(X),1} = \inf_{c\in\mathbb{C}} \|G + c\|_X$, where g, G are related as above. The following observation is entirely based on standard estimates and we omit its proof.

Proposition 5. *$D(X), A(X)$ satisfy 1), 2) and $D(A(X)) = A(D(X)) = X$. Moreover,*
(i) $X^{min}_{\alpha+1} \subset D(X) \subset X^{max}_{\alpha+1} = \mathcal{A}^{-\alpha-1}$, and the inclusions are continuous.
(ii) For $\alpha > 1$, $X^{min}_{\alpha-1} \subset A(X) \subset X^{max}_{\alpha-1} = \mathcal{A}^{-\alpha+1}$, and the inclusions are continuous.

The purpose is to investigate the conformal invariance of these spaces. For the standard examples we have (see for example [14])

$$D(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma-1}, \quad D(A^p_\beta) = A^p_{\beta+p}, \ p \geq 1, \beta > -1,$$

and

$$A(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma+1}, \ \gamma > 1, \quad A(A^p_\beta) = A^p_{\beta-p}, \ p \geq 1, \beta > p - 1,$$

or $A(A^p_\beta) = B^{p,\beta}, \ p \geq 1, \beta \leq p - 1$. In other words (assuming for the moment the assertions in Example 3), if X is any of the spaces listed above and $\alpha > 0$ is its index of conformal invariance, then $D(X)$ is conformally invariant of index $\alpha + 1$ and when $\alpha > 1$, $A(X)$ is conformally invariant of index $\alpha - 1$.

It turns out that the result continues to hold for many other conformally invariant spaces. Surprisingly enough, this property is closely related to the behaviour on the spaces in question of the modified Cesàro $\mathcal{C}$, operator, formally defined by

$$\mathcal{C}f(z) = \int_0^z \frac{f(w)}{1-w}dw, \quad f \in Hol(\mathbb{D}).$$

Our result is as follows.

Theorem 7. *Let X be a conformally invariant space of index $\alpha > 0$, such that polynomials are dense in X.*
(i) $D(X)$ is conformally invariant of index $\beta > 0$, if and only if $\beta = \alpha + 1$ and $\mathcal{C} \in \mathcal{B}(X)$.
(ii) Assume that $\mathcal{C} \in \mathcal{B}(X)$. Then $A(X)$ is conformally invariant of index $\beta > 0$, if and only if $\alpha > 1$, $\beta = \alpha - 1$, and $I_X - \mathcal{C}$ is invertible.

Note that part (ii) implies the assertions in Example 3. Indeed, it is known that $\mathcal{C}$ is bounded on A^p_β, $p \geq 1, \beta > -1$, and its resolvent set consists of points $\lambda \in \mathbb{C} \setminus \{0\}$, such that $(1-\zeta)^{-\frac{1}{\lambda}} \in A^p_\beta$ (see [3], Theorem 5.2). In particular, $I - \mathcal{C}$ is invertible on A^p_β if and only if $\beta + 2 > p$. In this case, by part (ii) of the above theorem $B^{p,\beta}$ is conformally invariant with index $\frac{\beta+2}{p} - 1$.

Our argument involves two families of linear operators formally defined for $a \in \mathbb{D}$ by

$$\mathcal{C}_a f(z) = \int_0^z \frac{f(w)}{1-aw} dw, \quad T_a f(z) = \frac{1}{1-az} \int_0^z f(w) dw, \quad z \in \mathbb{D}, f \in Hol(\mathbb{D}). \tag{22}$$

Their relation to the modified Cesàro operator $\mathcal{C}$ is explained in the next two lemmas.

Lemma 6. *Let $\sigma \in \{0, 1\}$ and let X be conformally invariant of index $\alpha > \sigma$, such that polynomials are dense in X. For $f \in Hol(\mathbb{D})$ and $a \in \mathbb{D}$ let*

$$T^\sigma f(z) = (1-z)^{-\sigma} \int_0^z \frac{f(w)}{(1-w)^{1-\sigma}} dw, \quad T^\sigma_a f(z) = (1-az)^{-\sigma} \int_0^z \frac{f(w)}{(1-aw)^{1-\sigma}} dw.$$

Then the following are equivalent:

i) $T^\sigma \in \mathcal{B}(X)$.
ii) $T^\sigma_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ and $\sup_{a \in \mathbb{D}} \|T^\sigma_a\|_{\mathcal{B}(X)} < \infty$.
iii) There exists $\delta \in (0, 1)$ such that $T^\sigma_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ with $\delta \leq |a| < 1$ and $\sup_{\delta \leq |a| < 1} \|T^\sigma_a\|_{\mathcal{B}(X)} < \infty$.

Proof. i) $\Rightarrow$ ii) For every $f \in Hol(\mathbb{D})$ and $a \in \mathbb{D}$ we have

$$T^\sigma_a f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a(e^{it}) e^{-it} R_t T^\sigma R_{-t} f(z) dt,$$

where $P_a(e^{it}) = \frac{1-|a|^2}{|a-e^{it}|^2}$ is the Poisson kernel at a. If $T^\sigma \in \mathcal{B}(X)$, then, by Theorem 1, $t \to P_a(e^{it}) R_t T^\sigma R_{-t}$ is strongly continuous on $[-\pi, \pi]$, hence for $f \in X$ the right hand side becomes a Bochner integral. Thus

$$T_a^\sigma f = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})e^{-it}R_t T^\sigma R_{-t} f dt, \quad f \in X,\, a \in \mathbb{D},$$

and

$$\|T_a^\sigma f\| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})\|R_t T^\sigma R_{-t} f\| dt \leq (\sup_{t\in[-\pi,\pi]} \|R_t\|_{\mathcal{B}(X)})^2 \|T^\sigma\|_{\mathcal{B}(X)}\|f\|_X,$$

for all $f \in X$ and $a \in \mathbb{D}$.
ii) $\Rightarrow$ iii) It is trivial.
iii) $\Rightarrow$ i) Assume that there exists $\delta \in (0,1)$ such that $\{T_a^\sigma :\ \delta \leq |a| < 1\}$ is bounded in $\mathcal{B}(X)$. The boundedness of T^σ will follow directly from the closed graph theorem once we show that $T^\sigma f \in X$, whenever $f \in X$. To this end, we verify that $\mathcal{D}_{T^\sigma} = \{f \in X :\ T^\sigma f \in X\}$, is both closed and dense in X.

To prove that $\mathcal{D}_{T^\sigma}$ is closed, we use the equality

$$(T^\sigma f)_r(z) = (1-rz)^{-\sigma}\int_0^1 \frac{rzf(srz)}{(1-srz)^{1-\sigma}} ds = rT_r^\sigma f_r(z). \tag{23}$$

Let (f_n) be a sequence in $\mathcal{D}_{T^\sigma}$, $f \in X$ with $\|f_n - f\| \to 0$. Given $\varepsilon > 0$, choose $n_\varepsilon \geq 1$, such that $\|f_n - f_m\| < \varepsilon$, $n, m > n_\varepsilon$. For such m, n use Theorem 1 to find $r = r(m,n) \in (\delta, 1)$, with

$$\|T^\sigma f_n - (T^\sigma f_n)_r\| < \varepsilon, \quad \|T^\sigma f_m - (T^\sigma f_m)_r\| < \varepsilon.$$

By another application of Theorem 1, there exists $c > 0$, independent of m, n, r such that $\|(f_n)_r - (f_m)_r\| \leq c\|f_n - f_m\| < c\varepsilon$. Thus by (23)

$$\begin{aligned}\|T^\sigma f_n - T^\sigma f_m\| &< 2\varepsilon + \|(T^\sigma f_n)_r - (T^\sigma f_m)_r\| = 2\varepsilon + r\|T_r^\sigma (f_n)_r - T_r^\sigma (f_m)_r\| \\ &\leq 2\varepsilon + c\sup_{\rho\in(\delta,1)} \|T_\rho^\sigma\|_{\mathcal{B}(X)}\|f_n - f_m\| < (2 + c\sup_{\rho\in(\delta,1)} \|T_\rho^\sigma\|_{\mathcal{B}(X)})\varepsilon,\end{aligned}$$

i.e. $(T^\sigma f_n)$ is Cauchy in X. Since $T^\sigma f_n(z) \to T^\sigma f(z)$, $z \in \mathbb{D}$, we obtain that $f \in \mathcal{D}_{T^\sigma}$, that is, $\mathcal{D}_{T^\sigma}$ is closed.

To verify that $\mathcal{D}_{T^\sigma}$ is dense in X, set

$$\mathcal{D}_0 = \{f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) :\ f(1) = 0\}, \quad \mathcal{D}_1 = \left\{f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) :\ \int_0^1 f(w)dw = 0\right\},$$

and observe that if $f \in \mathcal{D}_\sigma$, then $T^\sigma f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) \subset X$, i.e. $f \in \mathcal{D}_{T^\sigma}$. We claim that $\mathcal{D}_\sigma$ is a dense subspace of X. Indeed, if $l \in \mathcal{D}_\sigma^\perp$, and $g \in \cup_{\rho>1} Hol(\rho\mathbb{D})$, then if $\sigma = 0$,

$$l(g) = l(g(1) + g - g(1)) = l(1)g(1),$$

and similarly, if $\sigma = 1$,

$$l(g) = l\left(\int_0^1 g(w)dw + g - \int_0^1 g(w)dw\right) = l(1)\int_0^1 g(w)dw.$$

If $l(1) \neq 0$ we see in both cases that the restriction of l to the bounded set $\{(\varphi')^\alpha : \varphi \in Aut(\mathbb{D})\} \subset X$ is unbounded, which is a contradiction. Hence $l(1) = 0$ which implies that $l = 0$ since it annihilates all polynomials.

Finally, to see that the graph of T^σ is closed, assume $\|f_n - f\| \to 0,\ \|T^\sigma f_n - g\| \to 0$, with $f_n, f, g \in X$. Then $T^\sigma f_n(z) \to T^\sigma f(z),\ z \in \mathbb{D}$, i.e. $T^\sigma f = g$. $\square$

Lemma 7. *Let X be conformally invariant of index $\alpha > 1$, such that polynomials are dense in X, and assume that $\mathcal{C} \in \mathcal{B}(X)$. Then $I_X - \mathcal{C}$ is invertible if and only if $T_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ and $\sup_{a\in\mathbb{D}} \|T_a\|_{\mathcal{B}(X)} < \infty$.*

Proof. With the notation in Lemma 6 we have $T_a = T_a^1,\ a \in \mathbb{D}$. Using integration by parts we obtain the identity

$$T^1\mathcal{C}f = T^1 f - \mathcal{C}f = \mathcal{C}T^1 f, \quad f \in Hol(\mathbb{D}),$$

or equivalently,

$$(I_{Hol(\mathbb{D})} + T^1)(I_{Hol(\mathbb{D})} - \mathcal{C}) = (I_{Hol(\mathbb{D})} - \mathcal{C})(I_{Hol(\mathbb{D})} + T^1) = I_{Hol(\mathbb{D})}. \tag{24}$$

If $I_X - \mathcal{C}$ is invertible, then by (24) it follows that $(I_X - \mathcal{C})^{-1} = I_X + T^1$. In particular, $T^1 \in \mathcal{B}(X)$, and by Lemma 6, $\{T_a : a \in \mathbb{D}\}$ is bounded in $\mathcal{B}(X)$. Conversely, if $\{T_a : a \in \mathbb{D}\}$ is bounded in $\mathcal{B}(X)$, by Lemma 6 we have that $T^1 \in \mathcal{B}(X)$, and by (24) we obtain $I_X + T^1 = (I_X - \mathcal{C})^{-1}$. $\square$

Proof of Theorem 7. (i) Assume that $D(X)$ is conformally invariant of index $\beta > 0$. Then

$$\sup_{\varphi\in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{D(X)} < \infty.$$

By Proposition 5 $D(X)$ is continuously contained in $\mathcal{A}^{-\alpha-1}$, hence

$$\sup_{\varphi\in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{\mathcal{A}^{-\alpha-1}} < \infty,$$

which implies that $\beta \leq \alpha + 1$. On the other hand,

$$\sup_{\varphi\in Aut(\mathbb{D})} \|((\varphi')^\alpha)'\|_{D(X)} < \infty,$$

and $D(X)$ is continuously contained in $\mathcal{A}^{-\beta}$, hence

$$\sup_{\varphi\in Aut(\mathbb{D})} \|((\varphi')^{\alpha})'\|_{\mathcal{A}^{-\beta}} < \infty,$$

which implies $\alpha + 1 \le \beta$, i.e. $\beta = \alpha + 1$.

For $\varphi = \lambda\varphi_a \in Aut(\mathbb{D})$, with $\varphi_a = \frac{\overline{a}-z}{1-az}$ and $f \in X$ we have

$$(W^{\alpha}_{\varphi} f)' = W^{\alpha+1}_{\varphi} f' + ((\varphi')^{\alpha})' f \circ \varphi = W^{\alpha+1}_{\varphi} f' + \frac{((\varphi')^{\alpha})'}{(\varphi')^{\alpha}} W^{\alpha}_{\varphi} f,$$

which can be rewritten as

$$W^{\alpha+1}_{\varphi} f' = (W^{\alpha}_{\varphi} f)' - 2a\alpha(C_a W^{\alpha}_{\varphi} f)'. \tag{25}$$

Now replace $f \in X$ by $W^{\alpha}_{\varphi^{-1}} f$ to obtain

$$W^{\alpha+1}_{\varphi}(W^{\alpha}_{\varphi^{-1}} f)' = f' - 2a\alpha(\mathcal{C}_a f)'.$$

Since $D(X)$ is conformally invariant of index $\alpha + 1$, the left hand side stays bounded in $D(X)$ when $\varphi \in Aut(\mathbb{D})$ and $f \in X$ with $\|f\|_X \le 1$. This implies that, setting $\delta \in (0, 1)$, $\|(\mathcal{C}_a f)'\|_{D(X)}$ stays bounded when φ, f are as above and $\delta \le |a| < 1$, i.e. $\sup_{\delta\le a<1} \|\mathcal{C}_a\|_{\mathcal{B}(X)} < \infty$. By Lemma 6 with $\sigma = 0$ we obtain $\mathcal{C} \in \mathcal{B}(X)$. Conversely, if $\mathcal{C}$ is bounded on X, use again Lemma 6 with $\sigma = 0$, to conclude that the second term on the right hand side of (25) stays bounded in $D(X)$ when $\varphi \in Aut(\mathbb{D})$ and $f \in X$, $\|f\| \le 1$, which implies that $\|W^{\alpha+1}_{\varphi}\|_{\mathcal{B}(D(X))}$ stays bounded when $\varphi \in Aut(\mathbb{D})$.

(ii) Assume that $A(X)$ is conformally invariant of index $\beta > 0$ and that $\mathcal{C} \in \mathcal{B}(X)$. If $\alpha \le 1$, by direct integration we see that $A(X)$ is continuously contained in the growth class $\mathcal{A}^{\log}$ from Example 2. But then we can easily verify that

$$\sup_{\varphi\in Aut(\mathbb{D})} \|(\varphi')^{\beta}\|_{A(X)} = \infty,$$

which is a contradiction. Thus, $\alpha > 1$. The proof of the equality $\beta = \alpha - 1$ is identical to the corresponding argument in the proof of (i) and will be omitted. The remaining part of the proof is similar to the above as well. For $f \in A(X)$, $\varphi = \lambda\varphi_a \in Aut(\mathbb{D})$, with $\varphi_a = \frac{\overline{a}-z}{1-az}$ and $\varphi^{-1} = \mu\varphi_b$, write

$$(W^{\alpha-1}_{\varphi} f)' = W^{\alpha}_{\varphi} f' + ((\varphi')^{\alpha-1})'(f \circ \varphi - f(0)) + f(0)((\varphi')^{\alpha-1})'.$$

A direct calculation gives

$$W^{\alpha}_{\varphi^{-1}}((\varphi')^{\alpha-1})'(f \circ \varphi - f(0)) = -2b(\alpha - 1)\frac{f - f(0)}{1 - b\zeta} = -2b(\alpha - 1)T_b f',$$

hence the above equality becomes

$$(W_\varphi^{\alpha-1} f)' = W_\varphi^\alpha f' - 2b(\alpha-1) W_\varphi^\alpha T_b f' + f(0)((\varphi')^{\alpha-1})'. \tag{26}$$

Let $f(0) = 0$ and apply $W_{\varphi-1}^\alpha$ on both sides to obtain

$$W_{\varphi-1}^\alpha (W_\varphi^{\alpha-1} f)' = f' - 2b(\alpha-1) T_b f'.$$

By assumption, the left hand side stays bounded in X when $\varphi \in Aut(\mathbb{D})$, and $\|f\|_{A(X)} \le 1$, $f(0) = 0$, and so does the first term on the right. Since the condition on f is equivalent to $\|f'\|_X \le 1$ it follows that setting $\delta \in (0,1)$, $T_b \in \mathcal{B}(X)$, $\delta \le |b| < 1$, and $\sup_{\delta \le |b| < 1} \|T_b\|_{\mathcal{B}(X)} < \infty$. Thus by Lemma 6 with $\sigma = 1$ and Lemma 7, $I_X - \mathcal{C}$ is invertible on X. To see the converse, use first Proposition 5 (ii) to conclude that $|W_\varphi^{\alpha-1} f(0)|$ and $|f(0)| \|((\varphi')^{\alpha-1})'\|_X$ stay bounded when $\varphi \in Aut(\mathbb{D})$ and $\|f\|_{A(X)} \le 1$. If $\mathcal{C} \in \mathcal{B}(X)$, and $I_X - \mathcal{C}$ is invertible, then by Lemma 7 we have $\sup_{a \in \mathbb{D}} \|T_a\|_{\mathcal{B}(X)} < \infty$ and we conclude that the right hand side of (26) stays bounded in X when $\varphi \in Aut(\mathbb{D})$ and $\|f\|_{A(X)} \le 1$. Thus, $W_\varphi^{\alpha-1} \in \mathcal{B}(A(X))$ and $\sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^{\alpha-1}\|_{\mathcal{B}(A(X))} < \infty$, which completes the proof. □

6.2. Integration operators

We shall apply our results to investigate a class of integration operators containing the modified Cesàro operator from the previous paragraph. These operators are formally defined by

$$T_g f(z) = \int_0^z f(w) g'(w) dw, \quad f \in Hol(\mathbb{D}),$$

where the *symbol* $g \in Hol(\mathbb{D})$, is fixed. There is a vast literature on the subject concerning boundedness and compactness of such operators, and more recently, even their spectral properties (see for example, [1] [3] and the references therein). Here we shall only discuss boundedness in the general context of conformal invariant Banach spaces of analytic functions. We start with a simple result whose statement is self-explanatory. However, in many cases it turns out to be an important observation related to the characterization of the symbols g which generate bounded operators T_g. We shall use the notations in the previous paragraph for arbitrary Banach spaces satisfying 1) and 2).

Proposition 6. *Let X, Y be Banach spaces satisfying 1) and 2). Then $T_g : X \to Y$ is bounded if and only if $g' \in Mult(X, D(Y))$, and the norms $\|T_g\|_{\mathcal{B}(X,Y)}$, $\|g'\|_{Mult(X,D(Y))}$ are equivalent.*

Proof. By the closed graph theorem $T_g \in \mathcal{B}(X, Y)$ if and only if $T_g f \in Y$ whenever $f \in X$, or equivalently, $g'f \in D(Y)$ whenever $f \in X$. The remaining part is also straightforward. □

Remark 3. If $X, D(Y)$ are conformally invariant of indices $\alpha > 0$, respectively $\beta > \alpha$ and $T_g \in \mathcal{B}(X, Y)$, then by Proposition 1 it follows that the integration operators generated by $g_\varphi = \int_0^z W_\varphi^{\beta-\alpha} g'(w)dw,\ \varphi \in Aut(\mathbb{D})$ are uniformly bounded in $\mathcal{B}(X, Y)$.

We shall be concerned with the case when $X = Y$. The following result provides a nice necessary condition for boundedness of such operators.

Corollary 2. *Let X be conformally invariant of index $\alpha > 0$ such that polynomials are dense in X. Assume also that $\mathcal{C} \in \mathcal{B}(X)$. If $T_g \in \mathcal{B}(X)$, then there exist $c, \delta > 0$ such that for all $\lambda \in \mathbb{C},\ |\lambda| \leq \delta$, and all $\varphi \in Aut(\mathbb{D})$, $\exp(\lambda(g \circ \varphi - g \circ \varphi(0))) \in X$, with*

$$\| \exp(\lambda(g \circ \varphi - g \circ \varphi(0)))\|_X \leq c.$$

Proof. By Theorem 7, $D(X)$ is conformally invariant of index $\alpha + 1$ Then Remark 3 applies and we obtain that the family $\{T_{g_\varphi} :\ \varphi \in Aut(\mathbb{D})\}$ is bounded in $\mathcal{B}(X)$, where

$$T_{g_\varphi} f(z) = \int_0^z f(w)(g \circ \varphi)'(w)dw, \quad z \in \mathbb{D},\ f \in X.$$

Choose $\delta > 0$, such that

$$\delta \|T_{g_\varphi}\|_{\mathcal{B}(X)} < \frac{1}{2}, \quad \varphi \in Aut(\mathbb{D}).$$

Differentiating and solving an ordinary linear differential equation of first order we obtain that the unique solution f_λ of

$$f - \lambda T_{g_\lambda} f = 1,$$

is given by $f_\lambda = \exp(\lambda(g \circ \varphi - g \circ \varphi(0)))$. Now for $|\lambda| < \delta$, $I_X - \lambda T_{g_\varphi}$ is invertible with $\|(I_X - \lambda T_{g_\varphi})^{-1}\|_{\mathcal{B}(X)} < 2$, so that,

$$\|f_\lambda\|_X = \|(I_X - \lambda T_{g_\varphi})^{-1} 1\|_X < 2\|1\|_X,$$

and the result follows. □

The idea of exponentiating via the resolvents of T_g is due to Pomerenke [16]. When $X = H^2$, one can use it to prove the John-Nirenberg inequality for BMO (see [13]). In the general context considered here, necessary condition for boundedness of T_g provided by Corollary 2 is probably not sufficient.

Acknowledgment

The second author was supported by the research project PID2019-106870GB-I00 from MICINN, Spain.

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