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BOUNDEDNESS OF DIFFERENTIAL TRANSFORMS FOR ONE-SIDED FRACTIONAL POISSON-TYPE OPERATOR SEQUENCE

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ABSTRACT. Let $\mathcal{P}_\tau f$ be given by

$$\mathcal{P}_\tau f(t) = \frac{1}{4\ \Gamma(\alpha)} \int_0^{+\infty} \frac{\tau^2\ e^{-\tau^2/(4s)}}{s^{1+}} f(t-s)ds,\ \tau>0,\ t\in\mathbb{R},\ 0<\alpha<1.$$

It is known that the function $U\ \ (t,\tau) = \mathcal{P}_\tau f(t)$ is a classical solution to the extension problem

$$-D_{\text{left}}U\ \ + \frac{1-2\alpha}{\tau} U_\tau\ + U_{\tau\tau} = 0, \quad \text{in } \mathbb{R}\times(0,\infty)$$

and

$$\lim_{\tau\to 0^+} \mathcal{P}_\tau f(t) = f(t), \quad a.e. \text{ and in } L^p(\mathbb{R},w)\text{-norm}, w\in A_p^-.$$

In this paper, we analyze the convergence speed of a series related with $\mathcal{P}_\tau f$ by discussing the behavior of the family of operators

$$T_N f(t) = \sum_{j=N_1}^{N_2} v_j(\mathcal{P}_{a_{j+1}} f(t) - \mathcal{P}_{a_j} f(t)),\ N=(N_1,N_2)\in\mathbb{Z}^2 \quad \text{with} \quad N_1<N_2,$$

where $\{v_j\}_{j\in\mathbb{Z}}$ is a bounded number sequence, and $\{a_j\}_{j\in\mathbb{Z}}$ is a ρ-lacunary sequence of positive numbers, that is, $1<\rho\le a_{j+1}/a_j$, for all $j\in\mathbb{Z}$. We shall show the boundedness of the maximal operator

$$T^* f(t) = \sup_N |T_N f(t)|, \quad t\in\mathbb{R},$$

in the one-sided weighted Lebesgue spaces $L^p(\mathbb{R},\omega)(\omega\in A_p^-)$, $1<p<\infty$. As a consequence we infer the existence of the limit, in norm and almost everywhere, of the family $T_N f$ for functions in $L^p(\mathbb{R},\omega)$. Results for $L^1(\mathbb{R},\omega)(\omega\in A_1^-)$, $L^\infty(\mathbb{R})$ and $BMO(\mathbb{R})$ are also obtained.

It is also shown that the local size of T^*f, for functions f having local support, is the same with the order of a singular integral. Moreover, if $\{v_j\}_{j\in\mathbb{Z}}\in\ell^p(\mathbb{Z})$, we get an intermediate size between the local size of singular integrals and Hardy-Littlewood maximal operator.

1. INTRODUCTION

Let $\mathcal{P}_\tau^\alpha f$ be given by

$$\mathcal{P}_\tau^\alpha f(t) = \frac{1}{4^\alpha\Gamma(\alpha)} \int_0^{+\infty} \frac{\tau^{2\alpha} e^{-\tau^2/(4s)}}{s^{1+\alpha}} f(t-s)ds,\ \tau>0,\ t\in\mathbb{R},\ 0<\alpha<1. \tag{1.1}$$

This is a fractional Poisson-type operator on the line, which can be found in [3]. It is known that the Poisson-type operator appeared when solving the extension problem, see [5, 12, 13]. In [3], the authors showed that $\mathcal{P}_\tau^\alpha$ is a classical solution to a version of extension problem for the given initial data f in a weighted space $L^p(w)$, where w satisfies the one-sided A_p condition. Moreover, in this extension problem, they proved that the fractional derivatives on the line are Dirichlet-to-Neumann

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operators. Precisely, it is shown that for functions $f \in L^p(\mathbb{R}, w), w \in A_p^-, 1 < p < \infty$, the function $U^\alpha(t,\tau) = \mathcal{P}_\tau^\alpha f(t)$ is a classical solution to the extension problem

$$\begin{cases} -D_{\text{left}} U^\alpha + \dfrac{1-2\alpha}{\tau} U_\tau^\alpha + U_{\tau\tau}^\alpha = 0, & \text{in } \mathbb{R}\times(0,\infty), \\ \lim\limits_{\tau\to 0^+} \mathcal{P}_\tau^\alpha f(t) = f(t), & a.e. \text{ and in } L^p(\mathbb{R},w)\text{-norm.} \end{cases}$$

Moreover, for $c_\alpha := \dfrac{4^{\alpha-1/2}\Gamma(\alpha)}{\Gamma(1-\alpha)} > 0$,

$$-c_\alpha \lim_{\tau\to 0^+} \tau^{1-2\alpha} U_\tau^\alpha(t,\tau) = (D_{\text{left}})^\alpha f(t), \quad \text{in the distributional sense.}$$

In the above formulas,

$$D_{\text{left}} f(t) = \lim_{s\to 0^-} \frac{f(t)-f(t-s)}{s} \quad \text{and} \quad (D_{\text{left}})^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(t-s)-f(t)}{s^{\alpha+1}} ds.$$

By A_p^- we denote the class of lateral weights introduced by E.Sawyer [11], see (2.2) and (2.3).

The purpose of this note is to give some extra information about the convergence of the family $\{\mathcal{P}_\tau^\alpha f\}_{\tau>0}$. In order to do this, we shall discuss the behavior of the series

$$\sum_{j\in\mathbb{Z}} v_j (\mathcal{P}_{a_{j+1}}^\alpha f(t) - \mathcal{P}_{a_j}^\alpha f(t)),$$

where $\{v_j\}_{j\in\mathbb{Z}}$ is a sequence of bounded numbers and $\{a_j\}_{j\in\mathbb{Z}}$ is a ρ-lacunary sequence of positive numbers, that is, $1 < \rho \le a_{j+1}/a_j$, for all $j \in \mathbb{Z}$. This way to analyze convergence of sequences was considered by Jones and Rosemblatt for ergodic averages(see [7]), and latter by Bernardis et al. for differential transforms(see [2]).

For each $N \in \mathbb{Z}^2$, $N = (N_1, N_2)$ with $N_1 < N_2$, we define the sum

$$T_N^\alpha f(t) = \sum_{j=N_1}^{N_2} v_j (\mathcal{P}_{a_{j+1}}^\alpha f(t) - \mathcal{P}_{a_j}^\alpha f(t)). \tag{1.2}$$

We shall consider the maximal operator

$$T_\alpha^* f(t) = \sup_N |T_N^\alpha f(t)|, \quad t \in \mathbb{R}.$$

Along the paper, we shall denoteT^* to be T_α^* for simply. (1.3)

The supremum are taken over all $N = (N_1, N_2) \in \mathbb{Z}^2$ with $N_1 < N_2$.

In order to prove the results, we shall use the vector-valued Calderón-Zygmund theory in an essential way. In the proof of the maximal operator T^*, we shall use a kind of Cotlar's lemma that in some sense is parallel to the classical Cotlar's inequality used to control the maximal operator of the truncations in the Calderón-Zygmund theory. Looking to the first set of our results, the reader could have the impression that the operator T^* is of the same size of the maximal operator $\mathcal{M}^-$. In this line of thought we present a series of results contained in Theorem 1.3 and Theorem 1.4 in which it is shown that the size of T^* acting over functions of compact support is in fact of the order of a singular integral. At this point we want to observe the analogy of our operators with martingale transforms. On the other hand if we consider the sequence of Rademacher functions $\{r_j\}_{j\in\mathbb{Z}}$, by Kintchine's inequality we have

$$\Big(\sum_{j\in\mathbb{Z}} |\mathcal{P}_{a_{j+1}}^\alpha f(\cdot) - \mathcal{P}_{a_j}^\alpha f(\cdot)|^2\Big)^{1/2}_{L^p(\mathbb{R})} \le K_p \Big\| \sum_{j\in\mathbb{Z}} r_j(\cdot)(\mathcal{P}_{a_{j+1}}^\alpha f(\cdot) - \mathcal{P}_{a_j}^\alpha f(\cdot)) \Big\|_{L^p(\ )\ L^p(\mathbb{R})}.$$

In other words, as a by product of our results we get the boundedness of the operator

$$\Big(\sum_{j\in\mathbb{Z}} |\mathcal{P}_{a_{j+1}}^\alpha f(\cdot) - \mathcal{P}_{a_j}^\alpha f(\cdot)|^2\Big)^{1/2}$$

in the same spaces that we get for operator T^*. Finally, in Theorem 1.4 it is also shown that if we assume the sequence $\{v_j\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, then the local behavior of T^* is approaching to the maximal operator as $p \to 1^+$. Now we present our main results.

Theorem 1.1. *Let $0 < \alpha < 1$, $\{v_j\}_{j\in\mathbb{Z}}$ a sequence of bounded numbers and $\{a_j\}_{j\in\mathbb{Z}}$ a ρ-lacunary sequence of positive numbers. Let T^* be defined in (1.3).*

(a) For any $1 < p < \infty$ and $\omega \in A_p^-$, there exists a constant C depending on p, ρ, α, ω and $\|v\|_{l^\infty(\mathbb{Z})}$ such that

$$\|T^* f\|_{L^p(\mathbb{R},\omega)} \leq C \|f\|_{L^p(\mathbb{R},\omega)},$$

for all functions $f \in L^p(\mathbb{R},\omega)$.

(b) For any $\omega \in A_1^-$, there exists a constant C depending on ρ, α, ω and $\|v\|_{l^\infty(\mathbb{Z})}$ such that

$$\omega\left(\{-\infty < t < +\infty : |T^* f(t)| > \lambda\}\right) \leq C\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R},\omega)}, \quad \lambda > 0,$$

for all functions $f \in L^1(\mathbb{R},\omega)$.

(c) Given $f \in L^\infty(\mathbb{R})$, then either $T^* f(t) = \infty$ for all $t \in \mathbb{R}$, or $T^* f(t) < \infty$ for a.e. $t \in \mathbb{R}$. And in this later case, there exists a constant C depending on ρ, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that

$$\|T^* f\|_{BMO(\mathbb{R})} \leq C \|f\|_{L^\infty(\mathbb{R})}.$$

(d) Given $f \in BMO(\mathbb{R})$, then either $T^* f(t) = \infty$ for all $t \in \mathbb{R}$, or $T^* f(t) < \infty$ for a.e. $t \in \mathbb{R}$. And in this later case, there exists a constant C depending on ρ, α and $\|v\|_{l^\infty(\mathbb{Z})}$ such that

$$\|T^* f\|_{BMO(\mathbb{R})} \leq C \|f\|_{BMO(\mathbb{R})}. \tag{1.4}$$

We have denoted by $L^p(\mathbb{R},\omega), 1 \leq p < \infty$, the Lebesgue space of measurable functions satisfying

$$\int_{\mathbb{R}} |f(t)|^p \omega(t) dt < \infty,$$

and $L^\infty(\mathbb{R})$ the space of measurable functions such that $\operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t)| < \infty$. Both of them are with the obvious norms. Also, we define $BMO(\mathbb{R})$ as the space of measurable functions such that for any interval B,

$$\frac{1}{|B|}\int_B f(t) - f_B \, dt \leq C < \infty,$$

and $\|f\|_{BMO(\mathbb{R})} = \sup_B \frac{1}{|B|}\int_B f(t) - f_B \, dt$, where $f_B = \frac{1}{|B|}\int_B f(t)dt$. For more details, see [6].

The proof of the last theorem contains three steps:

(A) We prove the following uniform boundedness of the family of operators T_N^α: from $L^p(\mathbb{R},\omega)$ into $L^p(\mathbb{R},\omega)$, $1 < p < \infty$, from $L^1(\mathbb{R},\omega)$ into weak-$L^1(\mathbb{R},\omega)$, from $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$, and from $BMO(\mathbb{R})$ into $BMO(\mathbb{R})$, see Theorem 2.6.

(B) The following pointwise Cotlar's type inequality

$$\sup_{-M\leq N_1<N_2\leq M} |T^\alpha_{N_1,N_2} f(t)| \leq C\left\{\mathcal{M}^-(T^\alpha_{-M,M} f)(t) + \mathcal{M}_q^- f(t)\right.,$$

see Theorem 3.2.

(C) The boundedness of $\mathcal{M}^-$ and the uniform boundedness of $T^\alpha_{-M,M}$ in $L^p(\mathbb{R},\omega)$ show the boundedness of the maximal operator T^* in $L^p(\mathbb{R},\omega)$. The use of the vector-valued Calderón-Zygmund theory allows us to get all of the statements in Theorem 1.1.

The last theorem has the following consequence.

Theorem 1.2. *(a) If $1 < p < \infty$ and $\omega \in A_p^-$, then $T_N^\alpha f$ converges a.e. and in $L^p(\mathbb{R},\omega)$ norms for all $f \in L^p(\mathbb{R},\omega)$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$.*

(b) If $p = 1$ and $\omega \in A_1^-$, then $T_N^\alpha f$ converges a.e. and in measure for all $f \in L^1(\mathbb{R},\omega)$ as $N = (N_1, N_2)$ tends to $(-\infty, +\infty)$.

The dichotomy results announced in Theorem 1.1, parts (c) and (d), about $L^\infty(\mathbb{R})$ and $BMO(\mathbb{R})$ are motivated, in part, by the existence of a bounded function f such that $T^*f(t)=\infty$ as the following theorem shows.

Theorem 1.3. *There exist bounded sequence $\{v_j\}_{j\in\mathbb{Z}}$, ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$ and $f\in L^\infty(\mathbb{R})$ such that $T^*f(t)=\infty$ for all $t\in\mathbb{R}$.*

This last theorem also says that the operator T^* is essentially bigger than the operator $\mathcal{P}^*f(t)=\sup_\tau \mathcal{P}_\tau^{1/2}f(t)$ which is bounded in $L^p(\mathbb{R},\omega), 1<p<\infty$, and in $L^\infty(\mathbb{R})$, see [3].

On the other hand, if $f=\chi_{(0,1)}$ and $\mathcal{H}$ is the Hilbert transform, it is easy to see that $\frac{1}{r}\int_r^0 \mathcal{H}(f)(x)dx \sim \log\frac{e}{r}$ as $r\to 0^+$. In general, this is the growth of a singular integral applied to a bounded function at the origin. The following theorem shows that the growth of the function T^*f for bounded function f at the origin is of the same order of a singular integral operator.

Theorem 1.4. *(a) Let $\{v_j\}_{j\in\mathbb{Z}}\in l^p(\mathbb{Z})$ for some $1\le p\le\infty$. For every $f\in L^\infty(\mathbb{R})$ with support in the unit ball $B=B(0,1)$, for any ball $B_r\subset B$ with $2r<1$, there exists a constant $C>0$ such that*

$$\frac{1}{|B_r|}\int_{B_r}|T^*f(t)|\,dt\le C\left(\log\frac{2}{r}\right)^{1/p'}\|v\|_{l^p(\mathbb{Z})}\,\|f\|_{L^\infty(\mathbb{R})}.$$

(b) When $1<p<\infty$, for any $\varepsilon>0$, there exist a ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$, a sequence $\{v_j\}_{j\in\mathbb{Z}}\in\ell^p(\mathbb{Z})$ and a function $f\in L^\infty(\mathbb{R})$ with support in the unit ball $B=B(0,1)$, satisfying the following statement: for any ball $B_r\subset B$ with $2r<1$, there exists a constant $C>0$ such that

$$\frac{1}{|B_r|}\int_{B_r}|T^*f(t)|\,dt\ge C\left(\log\frac{2}{r}\right)^{1/(p-\varepsilon)'}\|v\|_{l^p(\mathbb{Z})}\,\|f\|_{L^\infty(\mathbb{R})}.$$

(c) When $p=\infty$, there exist a ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$, a sequence $\{v_j\}_{j\in\mathbb{Z}}\in l^\infty(\mathbb{Z})$ and $f\in L^\infty(\mathbb{R})$ with support in the unit ball $B=B(0,1)$, satisfying the following statements: for any ball $B_r\subset B$ with $2r<1$, there exists a constant $C>0$ such that

$$\frac{1}{|B_r|}\int_{B_r}|T^*f(t)|\,dt\ge C\left(\log\frac{2}{r}\right)\|v\|_{l^\infty(\mathbb{Z})}\,\|f\|_{L^\infty(\mathbb{R})}.$$

In the statements above, $p'=\dfrac{p}{p-1}$, and if $p=1$, $p'=\infty$.

Some related results about the local behavior of variation operators can be found in [4]. One dimensional results about the variation of some convolutions operators can be found in [8].

The organization of the paper is as follows. In Section 2, we will get the kernel estimates to see that the kernel K_N^α is a vector-valued Caldrón-Zygmund kernel, and then we can get the uniform boundedness of T_N^α, i.e. Theorem 2.6. And with a Cotlar's inequality, we can get the proof of Theorem 1.1 in Section 3. In Section 4, we will give the proof of Theorem 1.3 and Theorem 1.4.

Throughout this paper, the symbol C in an inequality always denotes a constant which may depend on some indices, but never on the functions f in consideration.

2. Uniform L^p boundedness of the operators T_N^α

We shall need the following lemma.

Lemma 2.1. *Let $0<\alpha<1$. Then for any complex number z_0 with $Re z_0>0$ and $|\arg z_0|\le\pi/4$, we have*

$$\int_0^\infty e^{-z_0u}e^{-\frac{z_0}{u}}\frac{du}{u^\alpha}=z_0^{1-\alpha}\int_0^\infty\frac{e^{-r}e^{-z_0^2/r}}{r^{2-\alpha}}dr.$$

Proof. Let $\varphi_0 = \arg z_0$. Assume that $0 \le \varphi_0 \le \pi/4$. The case $-\pi/4 \le \varphi_0 \le 0$ is completely analogous. Define the ray in the complex plane

$$\mathrm{ray}_{\varphi_0} := \{z = re^{i\varphi_0} : 0 < r < \infty\}.$$

And then let $\mathcal{C}$ denote the sector in the real part of the complex plane, with $0 \le \arg z \le \varphi_0$ but truncated at $c_\varepsilon : |z| = \varepsilon$ and $C_R : |z| = \varepsilon$. In fact, the boundary of $\mathcal{C}$ consists four parts: C_ε, C_R, ray_{φ_0} and positive half part of the real line.

Let us consider the complex function

$$F(u) = \frac{e^{-z_0/u}e^{-uz_0}}{u^\alpha},$$

which is holomorphic function when $u \neq 0$. Thus, by the Cauchy theorem, we have $\iint_{\mathcal{C}} F(u)du = 0$. We first calculate

$$\int_{C_\varepsilon} F(u)du = \iint_0^{\varphi_0} \frac{e^{-z_0/(e^{i\theta}\varepsilon)}e^{-z_0\varepsilon e^{i\theta}}}{\varepsilon^\alpha e^{i\alpha\theta}} i\varepsilon e^{i\theta}d\theta = \iint_0^{\varphi_0} \frac{e^{-|z_0|e^{i(\varphi_0-\theta)}/\varepsilon}e^{-|z_0|\varepsilon e^{i(\varphi_0+\theta)}}}{\varepsilon^\alpha e^{i\alpha\theta}} i\varepsilon e^{i\theta}d\theta \ .$$

Since $\varphi_0 < \pi/4$, $\varphi_0 - \theta < \pi/2$ and $\varphi_0 + \theta < \pi/2$. Hence

$$\int_{C_\varepsilon} F(u)du \le \varepsilon^{1-\alpha} \iint_0^{\varphi_0} e^{-|z_0|\cos(\varphi_0-\theta)/\varepsilon}e^{-|z_0|\varepsilon\cos(\varphi_0+\theta)}d\theta \to 0,$$

as $\varepsilon \to 0$. Similarly, along the curve C_R, we have

$$\int_{C_\varepsilon} F(u)du \le \iint_0^{\varphi_0} e^{-|z_0|\cos(\varphi_0-\theta)/R}e^{-|z_0|R\cos(\varphi_0+\theta)}R^{1-\alpha}d\theta.$$

If $\varphi_0 < \pi/4$,

$$\int_{C_R} F(u)du \le e^{-C_{z_0}R^2}R^{1-\alpha} \iint_0^{\varphi_0} e^{-\cos(\varphi_0-\theta)}d\theta \to 0,$$

as $R \to \infty$. But for the case $\varphi_0 = \pi/4$, $\varphi_0 + \theta$ can be $\pi/2$, then we can not take the limit as above. However, we have

$$\begin{aligned}\int_{C_R} F(u)du &\le \iint_0^{\frac{\pi}{4}} e^{-|z_0|\cos(\frac{\pi}{4}-\theta)/R}e^{-|z_0|R\cos(\frac{\pi}{4}+\theta)}R^{1-\alpha}d\theta \\ &\le \int_0^{\frac{\pi}{4}} e^{-|z_0|R\cos(\frac{\pi}{4}+\theta)}R^{1-\alpha}d\theta \le \iint_0^{\frac{\pi}{4}} e^{-|z_0|R\sin(\frac{\pi}{4}-\theta)}R^{1-\alpha}d\theta \\ &\le \int_0^{\frac{\pi}{4}} e^{-|z_0|R\sin\omega}R^{1-\alpha}d\omega \le \iint_0^{\frac{\pi}{4}} e^{-|z_0|\frac{2}{\pi}R\omega}R^{1-\alpha}d\omega,\end{aligned}$$

where we have changed variable $\omega = \pi/4 - \theta$ and used the inequality $2\omega/\pi \le \sin\omega$. Thus we have

$$\int_{C_R} F(u)du \le \frac{\pi}{2|z_0|}R^{-\alpha} \iint_0^{\infty} e^{-u}du \le CR^{-\alpha} \to 0, \quad R \to \infty.$$

Therefore, we conclude that $\iint_{C_R} F(u)du = 0$ for $|\arg z| \le \pi/4$.

At last, by the Cauchy theorem, we then get

$$\int_0^\infty F(u)du = \int_{\mathrm{ray}_{\varphi_0}} F(u)du.$$

Taking $u = sz_0$, we have

$$\int_0^\infty F(u)du = \iint_{\mathrm{Ray}_{\varphi_0}} \frac{e^{-1/s}e^{-sz_0^2}}{s^\alpha z_0^\alpha} z_0 ds = z_0^{1-\alpha}\int_0^\infty \frac{e^{-r}e^{-z_0^2/r}}{r^{2-\alpha}}dr.$$

Then this lemma is completely proved. □

Remark 2.2. *Notice that the integral*

$$\frac{y^{2s}}{4^s\Gamma(s)}\int_0^\infty e^{-y^2/(4\tau)}e^{-\tau(i\rho+\lambda)}\,\frac{d\tau}{\tau^{1+s}},\quad \rho\in\mathbb{R},\ \lambda\geq 0,\ 0<s<1.$$

is absolutely convergent.

2.1. **Uniform L^2-boundedness.**

It is known that, see [3], the Fourier transform of $\mathcal{P}_\tau^\alpha f$ is

$$\widehat{\mathcal{P}_\tau^\alpha f}(\rho)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-r}e^{-i\rho\tau^2/4r}\hat{f}(\rho)\frac{dr}{r^{1-\alpha}}.$$

By $\widehat{f}(\rho)$ we denote the Fourier transform of the function f, that is,

$$\widehat{f}(\rho)=\frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} f(x)e^{-ix\rho}\,dx,\quad \rho\in\mathbb{R}.$$

Theorem 2.3. *There is a constant C, depending on α and $\|v\|_{l^\infty(\mathbb{Z})}$, such that*

$$\sup_N \|T_N^\alpha f\|_{L^2(\mathbb{R})}\leq C\|f\|_{L^2(\mathbb{R})}.$$

Proof. Let $f\in L^2(\mathbb{R})$. Using the Plancherel theorem, we have

$$\|T_N^\alpha f\|_{L^2(\mathbb{R})}=\Big\|\sum_{j=N_1}^{N_2} v_j(\mathcal{P}_{a_{j+1}}^\alpha f-\mathcal{P}_{a_j}^\alpha f)\Big\|_{L^2(\mathbb{R})}\leq C\,\|v\|_{l^\infty(\mathbb{Z})}\Big\|\sum_{j=-\infty}^{\infty}\int_{a_j}^{a_{j+1}}\partial_\tau\widehat{\mathcal{P}_\tau^\alpha f}\,d\tau\Big\|_{L^2(\mathbb{R})}.$$

Observe that,

$$\partial_\tau\widehat{\mathcal{P}_\tau^\alpha f}(\rho)=C\partial_\tau\int_0^\infty e^{-r}e^{-\frac{\tau^2}{4r}(i\rho)}\widehat{f}(\rho)\,\frac{dr}{r^{1-\alpha}}=C\int_0^\infty e^{-r}\tau(i\rho)e^{-\frac{\tau^2}{4r}(i\rho)}\widehat{f}(\rho)\,\frac{dr}{r^{2-\alpha}}.$$

Note that the Fourier transform above is well defined, see Remark 2.2. Then we deduce that

$$\|T_N^\alpha f\|_{L^2(\mathbb{R})}\leq C\ \Big\|\widehat{f}(\rho)\int_0^\infty\int_0^\infty e^{-r}\tau(i\rho)e^{-\frac{\tau^2}{4r}(i\rho)}\,\frac{dr}{r^{2-\alpha}}\,d\tau\Big\|_{L^2(\mathbb{R})}.$$

Changing variable $z_0=\tau\sqrt{i\rho}$, by using Lemma 2.1 , we have

$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-r}\tau(i\rho)e^{-\frac{\tau^2}{4r}(i\rho)}\,\frac{dr}{r^{2-\alpha}}\,d\tau&=\int_0^\infty\int_0^\infty e^{-r}z_0e^{-\frac{z_0^2}{4r}}\,\frac{dr}{r^{2-\alpha}}\,dz_0\\&=2^{1-\alpha}\int_0^\infty z_0^\alpha\int_0^\infty e^{-\frac{z_0}{2u}}e^{-\frac{z_0}{2}u}\frac{du}{u^\alpha}\,dz_0.\end{aligned}$$

Since $|\arg z_0|=\pi/4$, we have $|e^{-z_0/(2u)}|\leq e^{-c|z_0|/u}$ and $|e^{-z_0u/2}|\leq e^{-c|z_0|u}$, where $c=\sqrt{2}/4$. Then

$$\begin{aligned}&\int_0^\infty z_0^\alpha\int_0^\infty e^{-z_0/u}e^{-z_0u}\frac{du}{u^\alpha}dz_0\ \leq\int_0^\infty|z_0|^\alpha\int_0^\infty e^{-c|z_0|/u}e^{-c|z_0|u}\frac{du}{u^\alpha}dz_0\\&\leq\int_0^\infty|z_0|^{2\alpha-1}\int_0^\infty e^{-c|z_0|^2/v}e^{-cv}\frac{dv}{v^\alpha}dz_0=\int_0^\infty|\sqrt{i\rho}|^{2\alpha}\,\tau^{2\alpha-1}\int_0^\infty e^{-c(|\sqrt{i\rho}|\tau)^2/v}e^{-cv}\frac{dv}{v^\alpha}d\tau\\&=\int_0^\infty\int_0^\infty(|\sqrt{i\rho}|\tau)^{2\alpha-1}e^{-c(m\tau)^2/v}d(|\sqrt{i\rho}|\tau)e^{-cv}\frac{dv}{v^\alpha}\\&=\int_0^\infty\int_0^\infty\tau^{2\alpha-1}e^{-c\tau^2/v}d\tau e^{-cv}\frac{dv}{v^\alpha}\leq C\int_0^\infty e^{-cv}dv\leq C.\end{aligned}$$

Then the proof of the theorem is complete. □

2.2. **Uniform L^p-boundedness.**

Let us come back to the definition of the operators T_N^α, see (1.2). By using the formula (1.1), we have

$$\begin{aligned} T_N^\alpha f(t) &= \sum_{j=N_1}^{N_2} \Big(v_j (\mathcal{P}_{a_{j+1}}^\alpha f(t) - \mathcal{P}_{a_j}^\alpha f(t)) \\ &= \frac{1}{4^\alpha \Gamma(\alpha)} \sum_{j=N_1}^{N_2} \Big(v_j \iint_0^{+\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} f(t-s)\, ds \\ &= \int_0^{+\infty} K_N^\alpha(s) f(t-s)\, ds = \int_{-\infty}^t K_N^\alpha(t-s) f(s)\, ds, \end{aligned}$$

where

$$K_N^\alpha(s) = \frac{1}{4^\alpha \Gamma(\alpha)} \sum_{j=N_1}^{N_2} v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}}. \tag{2.1}$$

The kernel $K_N^\alpha(s)$ is supported in $(0, +\infty)$. Our study of T_N^α will be related to the one-sided Calderón-Zygmund operators. In particular, we shall look for Lebesgue estimates with absolute continuous measures $w(x)dx$, where w is a weight in any of the classes $A_p^\pm$ defined by E. Sawyer, see [11]. This classes were introduced in relation with the boundedness of the one-sided Hardy-Littlewood maximal operator $\mathcal{M}^-$ defined by

$$\mathcal{M}^- f(t) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \iint_{-\varepsilon}^0 |f(t+s)|\, ds.$$

We recall the results that we shall use related with weights for $\mathcal{M}^-$:

(1) The operator $\mathcal{M}^-$ is of weak type $(1,1)$ with respect to the measure $\omega(t)dt$ if and only if $\omega \in A_1^-$, i.e., there exists C such that

$$\mathcal{M}^+ \omega \le C\omega \quad a.e., \tag{2.2}$$

where $\mathcal{M}^+$ is the right-sided Hardy Littlewood maximal operator defined as

$$\mathcal{M}^+ f(t) = \sup_{\varepsilon>0} \frac{1}{\varepsilon} \iint_0^{\varepsilon} |f(t+s)|\, ds.$$

(2) The operator $\mathcal{M}^-$ is bounded in $L^p(\omega)$, $1 < p < \infty$, if and only if $\omega \in A_p^-$, i.e., if there exists C such that for any three points $a < b < c$

$$\iint_a^b \omega^{1-p'} \Big)^{\frac{1}{p'}} \Big(\iint_b^c \omega \Big)^{\frac{1}{p}} \le C(c-a), \tag{2.3}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

For more details about the one-sided weights, see [1, 2, 11].

Theorem 2.4. *Let K_N^α be the kernel defined in (2.1). For any $s \neq 0$, there exists constant C depending on α and $\|v\|_{l^\infty(\mathbb{Z})}$ (but not on N) such that*

i) $|K_N^\alpha(s)| \le \dfrac{C}{s}$,

ii) $|\partial_s K_N^\alpha(s)| \le \dfrac{C}{s^2}$.

The proof of Theorem 2.4 involves an estimate we will repeat several times, so we formulate it in the following remark.

Remark 2.5. *Along the paper, we shall use frequently the estimate* $x^A e^{-x/B} \leq Ce^{-x/B'}$ *with* $x, A, B, B', C > 0$.

Proof of Theorem 2.4. For *i*), we have

$$|K_N^\alpha(s)| \leq C \sum_{j=-\infty}^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} = C \sum_{j=-\infty}^{\infty} a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)} \ \frac{1}{s^{1+\alpha}}.$$

Observe that, by Remark 2.5,

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)} = \sum_{j=-\infty}^{\infty} \int_{a_j}^{a_{j+1}} \partial_u \left(u^{2\alpha} e^{-u^2/(4s)} \right) du \\
&\leq \int_0^\infty (2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s}) e^{-u^2/(4s)} \ du \leq C \int_0^\infty (u^{2\alpha-1} + \frac{u^{2\alpha+1}}{2s}) e^{-u^2/(4s)} \ du \\
&\leq C\sqrt{s} \Big(\int_0^\infty (\sqrt{s})^{2\alpha-1} \left(\frac{u}{\sqrt{s}} \right)^{2\alpha-1} e^{-\frac{1}{4}(u/\sqrt{s})^2} d\frac{u}{\sqrt{s}} \\
&\quad + s^{\alpha-1/2} \int_0^\infty \left(\frac{u}{\sqrt{s}} \right)^{2\alpha+1} e^{-\frac{1}{4}(u/\sqrt{s})^2} d\frac{u}{\sqrt{s}} \Big) \\
&\leq C s^\alpha.
\end{aligned}$$

Then $|K_N^\alpha(s)| \leq \dfrac{C}{s}$. This proves *i).*

For *ii)*, we can write

$$\begin{aligned}
K_N^\alpha(s) &= C \sum_{j=N_1}^{N_2} v_j \left(a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)} \right) \\
&= C \sum_{j=N_1}^{N_2} \frac{1}{s^{1+\alpha}} v_j \int_{a_j}^{a_{j+1}} \left(2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s} \right) e^{-\frac{u^2}{4s}} du.
\end{aligned}$$

The partial derivative $\partial_s K_N^\alpha(s)$ consists two parts. The first part is

$$\begin{aligned}
I &= C \sum_{j=N_1}^{N_2} \frac{1}{s^{1+\alpha}} v_j \int_{a_j}^{a_{j+1}} \left(\frac{u^{2\alpha+1}}{2s^2} + \left(2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s} \right) \frac{u^2}{4s^2} \right) e^{-\frac{u^2}{4s}} du \\
&= C \sum_{j=N_1}^{N_2} \frac{1}{s^{1+\alpha}} v_j \int_{a_j}^{a_{j+1}} \left(\frac{(\alpha+1)u^{2\alpha+1}}{2s^2} - \frac{u^{2\alpha+3}}{8s^3} \right) e^{-\frac{u^2}{4s}} du.
\end{aligned}$$

And the second part is

$$\begin{aligned}
II &= C \sum_{j=N_1}^{N_2} \partial_s \left(\frac{1}{s^{1+\alpha}} \right) v_j \int_{a_j}^{a_{j+1}} \left(2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s} \right) e^{-\frac{u^2}{4s}} du \\
&= C \sum_{j=N_1}^{N_2} \left(-\frac{1+\alpha}{s^{2+\alpha}} \right) v_j \int_{a_j}^{a_{j+1}} \left(2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s} \right) e^{-\frac{u^2}{4s}} du.
\end{aligned}$$

Then by using Remark 2.5 again, we have

$$|I| \leq C \frac{1}{s^{1+\alpha}} \int_0^\infty \frac{(\alpha+1)u^{2\alpha+1}}{2s^2} - \frac{u^{2\alpha+3}}{8s^3} \ e^{-\frac{u^2}{4s}} du \leq C \frac{1}{s^{1+\alpha}} s^{\alpha-1} \leq \frac{C}{s^2},$$

and

$$|II| \leq C \ \frac{1+\alpha}{s^{2+\alpha}} \int_0^\infty (2\alpha u^{2\alpha-1} - \frac{u^{2\alpha+1}}{2s}) e^{-\frac{u^2}{4s}} du \ \leq C \frac{s^\alpha}{s^{2+\alpha}} \leq \frac{C}{s^2}.$$

Combining the estimates I and II, we have

$$|\partial_s K_N^\alpha(s)| \leq \frac{C}{s^2}.$$

All the estimates above are true uniform for N. The proof of the Theorem 2.4 is complete. □

From Theorems 2.3, 2.4, and standard Calderón-Zygmund theory, we can get the uniform estimate in $L^p(\mathbb{R}, w)$ $(1 < p < \infty,\ w \in A_p)$ of the operators T_N^α. Here, A_p denotes the classical Muckenhoupt A_p weights, see [10]. However, to the one-side nature of the kernel, we can apply Theorem 2.1 in [1] to get the uniform boundedness in $L^p(\mathbb{R}, w)$ of the operators T_N^α with $w \in A_p^-$ in the following.

Theorem 2.6. *Let T_N^α be the family of operators defined in (1.2), we have the following statements.*

(a) For any $1 < p < \infty$ and $\omega \in A_p^-$, there exists a constant C depending on p, α, $\|v\|_{l^\infty(\mathbb{Z})}$ and ω(not on N) such that

$$\|T_N^\alpha f\|_{L^p(\mathbb{R},\omega)} \leq C \|f\|_{L^p(\mathbb{R},\omega)},$$

for all functions $f \in L^p(\mathbb{R}, \omega)$.

(b) For any $\omega \in A_1^-$, there exists a constant C depending on α, $\|v\|_{l^\infty(\mathbb{Z})}$ and ω(not on N) such that

$$\omega\left(\{t \in \mathbb{R} : |T_N^\alpha f(t)| > \lambda\}\right) \leq C\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R},\omega)}, \quad \lambda > 0,$$

for all functions $f \in L^1(\mathbb{R}, \omega)$.

(c) There exists a constant C depending on α and $\|v\|_{l^\infty(\mathbb{Z})}$(not on N) such that

$$\|T_N^\alpha f\|_{BMO(\mathbb{R})} \leq C \|f\|_{L^\infty(\mathbb{R})},$$

for all functions $f \in L^\infty(\mathbb{R})$.

(d) There exists a constant C depending on α and $\|v\|_{l^\infty(\mathbb{Z})}$(not on N) such that

$$\|T_N^\alpha f\|_{BMO(\mathbb{R})} \leq C\|f\|_{BMO(\mathbb{R})},$$

for all functions $f \in BMO(\mathbb{R})$).

The constants C appeared above all are independent with N.

As we have said before the proof of (a) and (b) in the theorem above is obtained by using Theorem 2.1 in [1]. On the other hand the proof of (c) and (d) are standard in the Calderón-Zygmund theory and it can be found in [9].

3. Boundedness of the maximal operator T^*

In this section, we will give the proof of Theorem 1.1 related to the boundedness of the maximal operator T^*. The next proposition, parallel to Proposition 3.2 in [2], shows that, without lost of generality, we may assume that

$$1 < \rho \leq \frac{a_{j+1}}{a_j} \leq \rho^2, \quad j \in \mathbb{Z}. \tag{3.1}$$

Proposition 3.1. *Given a ρ-lacunary sequence $\{a_j\}_{j\in\mathbb{Z}}$ and a multiplying sequence $\{v_j\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$, we can define a ρ-lacunary sequence $\{\eta_j\}_{j\in\mathbb{Z}}$ and $\{\omega_j\}_{j\in\mathbb{Z}} \in l^\infty(\mathbb{Z})$ verifying the following properties:*

(i) $1 < \rho \leq \eta_{j+1}/\eta_j \leq \rho^2$, $\quad \|\omega_j\|_{l^\infty(\mathbb{Z})} = \|v_j\|_{l^\infty(\mathbb{Z})}$.

(ii) For all $N = (N_1, N_2)$ there exists $N' = (N_1', N_2')$ with $T_N^\alpha = \tilde{T}_{N'}^\alpha$, where $\tilde{T}_{N'}^\alpha$ is the operator defined in (1.2) for the new sequences $\{\eta_j\}_{j\in\mathbb{Z}}$ and $\{\omega_j\}_{j\in\mathbb{Z}}$.

Proof. We follow closely the ideas in the proof of Proposition 3.2 in [2]. We include it at here for completeness.

Let $\eta_0 = a_0$, and let us construct η_j for positive j as follows (the argument for negative j is analogous). If $\rho^2 \geq a_1/a_0 \geq \rho$, define $\eta_1 = a_1$. In the opposite case where $a_1/a_0 > \rho^2$, let $\eta_1 = \rho a_0$. It verifies $\rho^2 \geq \eta_1/\eta_0 = \rho \geq \rho$. Further, $a_1/\eta_1 \geq \rho^2 a_0/\rho a_0 = \rho$. Again, if $a_1/\eta_1 \leq \rho^2$, then $\eta_2 = a_1$. If this is not the case, define $\eta_2 = \rho^2 a_0 \leq a_1$. By the same calculations as before, η_0, η_1, η_2 are part of a lacunary sequence satisfying (3.1). To continue the sequence, either $\eta_3 = a_1$ (if $a_1/\eta_2 \leq \rho^2$) or

$\eta_2 = \rho^3\eta_0$ (if $a_1/\eta_2 > \rho^2$). Since $\rho > 1$, this process ends at some j_0 such that $\eta_{j_0} = a_1$. The rest of the elements η_j are built in the same way, as the original a_k plus the necessary terms put in between two consecutive a_j to get (3.1).

Let $J(j) = \{k : a_{j-1} < \eta_k \leq a_j\}$, and $\omega_k = v_j$ if $k \in J(j)$. Then

$$v_j(\mathcal{P}^\alpha_{a_{j+1}} f(t) - \mathcal{P}^\alpha_{a_j} f(t)) = \sum_{k\in J(j)} \omega_k(\mathcal{P}^\alpha_{a_{k+1}} f(t) - \mathcal{P}^\alpha_{a_k} f(t)).$$

If $M = (M_1, M_2)$ is the number such that $\eta_{M_2+1} = a_{N_2+1}$ and $\eta_{M_1} = a_{N_1}$, then we get

$$T^\alpha_N f(t) = \sum_{j=N_1}^{N_2} v_j(\mathcal{P}^\alpha_{a_{j+1}} f(t) - \mathcal{P}^\alpha_{a_j} f(t)) = \sum_{k=M_1}^{M_2} \omega_k(\mathcal{P}^\alpha_{\eta_{k+1}} f(t) - \mathcal{P}^\alpha_{\eta_k} f(t)) = \tilde{T}^\alpha_M f(t),$$

where $\tilde{T}^\alpha_M$ is the operator defined in (1.2) related with sequences $\{\eta_k\}_{k\in\mathbb{Z}}$, $\{\omega_k\}_{k\in\mathbb{Z}}$, α and $M = (M_1, M_2)$. □

It follows from this proposition that it is enough to prove all the results of this article in the case of a ρ-lacunary sequence satisfying (3.1). For this reason, in the rest of the article we assume that $\{a_j\}_{j\in\mathbb{Z}}$ satisfies (3.1) without saying it explicitly.

In order to prove Theorem 1.1, we need a Cotlar's type inequality to control the operator T^* by some one-sided Hardy-Littlewood maximal operators.

For any $M \in \mathbb{Z}^+$, let

$$T^*_M f(t) = \sup_{-M\leq N_1<N_2\leq M} |T^\alpha_N f(t)|, \quad -\infty < t < +\infty.$$

Theorem 3.2. *For each $q \in (1, +\infty)$, there exists a constant C depending on q, $\|v\|_{l^\infty(\mathbb{Z})}$, α and ρ such that for every $M \in \mathbb{Z}^+$,*

$$T^*_M f(t) \leq C\left\{\mathcal{M}^-(T^\alpha_{-M,M} f)(t) + \mathcal{M}^-_q f(t) \quad , \quad -\infty < t < +\infty,\right.$$

where

$$\mathcal{M}^-_q f(t) = \sup_{\varepsilon>0}\left(\frac{1}{\varepsilon}\int_{-\varepsilon}^{0} |f(t+s)|^q\, ds\right)^{\frac{1}{q}}.$$

Proof. Since the operators T^α_N are given by convolutions, they are invariant under translations, and therefore it is enough to prove the theorem for $t = 0$. Observe that, for $N = (N_1, N_2)$,

$$T^\alpha_N f(t) = T^\alpha_{N_1,M} f(t) - T^\alpha_{N_2+1,M} f(t),$$

with $-M \leq N_1 < N_2 \leq M$. Then, it suffices to estimate $T^\alpha_{m,M} f(0)$ for $|m| \leq M$ with constants independent of m and M. Let us split f as

$$\begin{aligned} f(t) &= f(t)\chi_{(-a^2_{m+1},0]}(t) + f(t)\chi_{(-\infty,-a^2_{m+1}]}(t) + f(t)\chi_{(0,+\infty)} \\ &=: f_1(t) + f_2(t) + f_3(t), \end{aligned}$$

for $-\infty < t < +\infty$.

First, notice that $T^\alpha_{m,M} f_3(0) = 0$. Then, we have

$$\begin{aligned} T^\alpha_{m,M} f(0) \;&\leq\; T^\alpha_{m,M} f_1(0) \;+\; T^\alpha_{m,M} f_2(0) \\ &=: I + II. \end{aligned}$$

For I, by the mean value theorem, we have

$$\begin{aligned}
I = \ & T^{\alpha}_{m,M} f_1(0) \\
= \ & C_\alpha \int_0^{+\infty} \sum_{j=m}^{M} \left(v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} f_1(-s) ds \right. \\
\leq \ & C_\alpha \left\| v \right\|_{l^\infty(\mathbb{Z})} \sum_{j=m}^{M} \left(\frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} + a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \left| f_1(-s) \right| ds \right. \\
\leq \ & C_{\alpha,v} \int_0^{+\infty} \sum_{j=m}^{M} \left(\frac{1}{a_{j+1}^2} + \frac{1}{a_j^2} \right) \left| f_1(-s) \right| ds \\
\leq \ & C_{\alpha,v} (\rho^4 + 1) \int_0^{+\infty} \sum_{j=m}^{M} \left(\frac{1}{a_{j+1}^2} \left| f_1(-s) \right| ds \quad (\text{since} \quad \rho \leq \frac{a_{j+1}}{a_j} \leq \rho^2) \right. \\
\leq \ & C_{\alpha,v,\rho} \frac{1}{a_{m+1}^2} \int_0^{+\infty} \sum_{j=m}^{M} \left(\frac{a_{m+1}^2}{a_j^2} \left| f_1(-s) \right| ds \right. \\
\leq \ & C_{\alpha,v,\rho} \frac{1}{a_{m+1}^2} \int_0^{+\infty} \left(\rho^4 + \sum_{j=m}^{M} \frac{1}{\rho^{2(j-m)}} \right) \left| f_1(-s) \right| ds \\
\leq \ & C_{\alpha,v,\rho} \frac{1}{a_{m+1}^2} \int_0^{+\infty} (\rho^2 - 1) \rho^{4\alpha} \left(\rho^4 + \sum_{j=0}^{+\infty} \left(\frac{1}{\rho^{2j}} \right) \left| f_1(-s) \right| ds \right. \\
\leq \ & C_{\alpha,v,\rho} \frac{1}{a_{m+1}^2} \int_{-a_{m+1}^2}^{0} \left| f(s) \right| \ ds \\
\leq \ & C_{\alpha,\rho,v} \mathcal{M}_q^- f(0).
\end{aligned}$$

For part II,

$$\begin{aligned}
II = \ & T^{\alpha}_{m,M} f_2(0) \ = \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{m,M} f_2(0) \ du \\
\leq \ & \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{-M,M} f(u) \ du + \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{-M,M} f_1(u) \ du \\
& + \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{m,M} f_2(u) - T^{\alpha}_{m,M} f_2(0) \ du \\
& + \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{-M,m-1} f_2(u) \ du \\
=: \ & A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

(If $m = -M$, we understand that $A_4 = 0$.) It is clear that

$$A_1 \leq \mathcal{M}^- (T^{\alpha}_{-M,M} f)(0).$$

For A_2, by the uniform boundedness of T^{α}_N in Theorem 2.6, we get

$$\begin{aligned}
A_2 \leq \ & \frac{1}{a_m^2} \int_{-a_m^2}^{0} \ T^{\alpha}_{-M,M} f_1(u) \ ^q \, du \Bigg)^{1/q} \leq C \left(\frac{1}{a_m^2} \int_{\mathbb{R}} \left| f_1(u) \right|^q du \right)^{1/q} \\
& = C \ \ \frac{1}{a_m^2} \int_{-a_m^2}^{0} \left| f(u) \right|^q du \Bigg)^{1/q} \leq C \mathcal{M}_q^- f(0).
\end{aligned}$$

For the third term A_3, with $-a_m^2 \le u \le 0$, by the mean value theorem and Theorem 2.4, we have

$$\begin{aligned}
T^\alpha_{m,M} f_2(u) - T^\alpha_{m,M} f_2(0) \ &= \int_{-\infty}^{u} K^\alpha_{m,M}(u-s) f_2(s) ds - \int_{-\infty}^{0} K^\alpha_{m,M}(-s) f_2(s) ds \\
&\le \int_{-\infty}^{u} K^\alpha_{m,M}(u-s) - K^\alpha_{m,M}(-s) \ |f_2(s)|\, ds + \int_{u}^{0} K^\alpha_{m,M}(-s) f_2(s) ds \\
&= \iint_{-\infty}^{-a_{m+1}^2} K^\alpha_{m,M}(u-s) - K^\alpha_{m,M}(-s) \ |f(s)|\, ds \\
&= \sum_{j=m+1}^{+\infty} \left(\iint_{-a_{j+1}^2}^{-a_j^2} K^\alpha_{m,M}(u-s) - K^\alpha_{m,M}(-s) \ |f(s)|\, ds \right. \\
&= \sum_{j=m+1}^{+\infty} \left(\iint_{-a_{j+1}^2}^{-a_j^2} \partial_t K^\alpha_{m,M}(t)\big|_{t=\xi_j} \ |u|\, |f(s)|\, ds \quad (a_j^2 - a_m^2 \le \xi_j \le a_{j+1}^2) \right. \\
&\le C \sum_{j=m+1}^{+\infty} \left(\iint_{-a_{j+1}^2}^{-a_j^2} \frac{|u|}{|\xi_j|^2} |f(s)|\, ds \le C \sum_{j=m+1}^{+\infty} \left(\frac{a_m^2}{\left(a_j^2 - a_m^2\right)^2} \int_{-a_{j+1}^2}^{0} |f(s)|\, ds \right. \right. \\
&\le C \sum_{j=m+1}^{+\infty} \left(\frac{a_m^2}{a_j^2} \cdot \frac{\rho^4}{(\rho^4-1) a_{j+1}^2} \iint_{-a_{j+1}^2}^{0} |f(s)|\, ds \right. \\
&\le C \sum_{j=m+1}^{+\infty} \left(\frac{1}{\rho^{2(j-m)}} \ \mathcal{M}^- f(0) \right. \\
&\le C \mathcal{M}^-_q f(0).
\end{aligned}$$

Then,

$$A_3 = \frac{1}{a_m^2} \iint_{-a_m^2}^{0} T^\alpha_{m,M} f_2(u) - T^\alpha_{m,M} f_2(0) \ du \le C \mathcal{M}^-_q f(0).$$

For the latest one, A_4, we have

$$A_4 = \frac{1}{a_m^2} \iint_{-a_m^2}^{0} T^\alpha_{-M,m-1} f_2(u) \ du \le \frac{1}{a_m^2} \int_{-a_m^2}^{0} \int_{-\infty}^{-a_{m+1}^2} K^\alpha_{-M,m-1}(u-s) f_2(s) \ ds du.$$

Then, we consider the inner integral appeared in the above inequalities first. Since $-a_m^2 \le u \le 0$, $-\infty < s \le -a_{m+1}^2$ and the sequence $\{a_j\}_{j\in\mathbb{Z}}$ is ρ-lacunary sequence, we have $|u-s| \sim |s|$. From this and by the mean value theorem, we get

$$\begin{aligned}
&\iint_{-\infty}^{-a_{m+1}^2} K^\alpha_{-M,m-1}(u-s) f_2(s) \ ds \\
&= \sum_{k=m+1}^{+\infty} \iint_{-a_{k+1}^2}^{-a_k^2} \sum_{j=-M}^{m-1} \left(v_j \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4(u-s))} - a_j^{2\alpha} e^{-a_j^2/(4(u-s))}}{(u-s)^{1+\alpha}} f(s) \ ds \right. \\
&\le \sum_{k=m+1}^{+\infty} \left(\iint_{-a_{k+1}^2}^{-a_k^2} \sum_{j=-M}^{m-1} v_j \frac{(a_{j+1}-a_j) \xi_j^{2\alpha-1} e^{-\xi_j^2/(4(u-s))}}{(u-s)^{1+\alpha}} f(s) \ ds \quad (a_j \le \xi_j \le a_{j+1}) \right. \\
&\le C \|v\|_{l^\infty(\mathbb{Z})} \sum_{k=m+1}^{+\infty} \left(\iint_{-a_{k+1}^2}^{-a_k^2} \sum_{j=-M}^{m-1} \left(\frac{\rho^{4\alpha}(\rho^2-1) a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} \ |f(s)|\, ds \right. \right. \\
&\le C_{\rho,v,\alpha} \sum_{k=m+1}^{+\infty} \left(\frac{1}{a_k^2} \iint_{-a_{k+1}^2}^{-a_k^2} \sum_{j=-M}^{m-1} \frac{a_j^{2\alpha}}{a_k^{2\alpha}} |f(s)|\, ds \right.
\end{aligned}$$

$$
\begin{aligned}
&\leq C_{\rho,v,\alpha} \sum_{k=m+1}^{+\infty} \left(\frac{1}{a_{k+1}^2} \iint_{a_{k+1}^2}^{-a_k^2} \sum_{j=-M}^{m-1} \rho^{-2\alpha(k-j)} \, |f(s)| \, ds \right. \\
&\leq C_{\rho,v,\alpha} \sum_{k=m+1}^{+\infty} \left(\frac{\rho^{-2\alpha(k-m+1)}}{a_{k+1}^2} \int_{-a_{k+1}^2}^{-a_k^2} |f(s)| \, ds \right. \\
&\leq C_{\rho,v,\alpha} \sum_{k=m+1}^{+\infty} \left(\frac{1}{\rho^{2\alpha(k-m+1)}} \frac{1}{a_{k+1}^2} \iint_{a_{k+1}^2}^{0} |f(s)| \, ds \right. \\
&\leq C_{\rho,v,\alpha} \sum_{k=m+1}^{+\infty} \left(\frac{1}{\rho^{2\alpha(k-m+1)}} \mathcal{M}^- f(0) \right. \\
&\leq C_{\rho,v,\alpha} \mathcal{M}_q^- f(0).
\end{aligned}
$$

Hence,

$$A_4 \leq C \mathcal{M}_q^- f(0).$$

Combining the estimates above for A_1, A_2, A_3 and A_4, we get

$$II \leq \mathcal{M}^-(T^\alpha_{-M,M} f)(0) + C \mathcal{M}_q^- f(0).$$

And, then we have

$$T^\alpha_{m,M} f(0) \ \leq C \left(\mathcal{M}^-(T^\alpha_{-M,M} f)(0) + \mathcal{M}_q^- f(0) \right) \left(\right.$$

As the constants C appeared above all depend on $\|v\|_{l^\infty(\mathbb{Z})}$, ρ and α, not on m, M, we complete the proof. □

Now we can start the proof of Theorem 1.1.

Proof of Theorem 1.1. For each $\omega \in A_p^-$, choose $1 < q < p < \infty$ such that $\omega \in A_{p/q}^-$. Then, it is well known that the maximal operators $\mathcal{M}^-$ and $\mathcal{M}_q^-$ are bounded in $L^p(\mathbb{R}, \omega)$. On the other hand, by Theorem 2.6, the operators T_N^α are uniformly bounded in $L^p(\mathbb{R}, \omega)$ with $\omega \in A_p^-$. Hence

$$
\begin{aligned}
\|T_M^* f\|_{L^p(\omega)} &\leq C \left(\ \mathcal{M}^-(T^\alpha_{-M,M} f) \ _{L^p(\omega)} + \ \mathcal{M}_q^- f \ _{L^p(\omega)} \right) \left(\right. \\
&\leq C \left(\ T^\alpha_{-M,M} f \ _{L^p(\omega)} + \|f\|_{L^p(\omega)} \right) \left(\leq C \, \|f\|_{L^p(\omega)} \right. .
\end{aligned}
$$

Note that the constants C appeared above do not depend on M. Consequently, letting M increase to infinity, we get the proof of the L^p boundedness of T^*. This completes the proof of part (a) of the theorem.

In order to prove (b), we consider the $\ell^\infty(\mathbb{Z}^2)$-valued operator $\mathcal{T} f(t) = \{T_N^\alpha f(t)\}_{N \in \mathbb{Z}^2}$. Since $\|\mathcal{T} f(t)\|_{\ell^\infty(\mathbb{Z}^2))} = T^* f(t)$, by using (a) we know that the operator $\mathcal{T}$ is bounded from $L^p(\mathbb{R}, \omega)$ into $L^p_{\ell^\infty(\mathbb{Z}^2)}(\mathbb{R}, \omega)$, for every $1 < p < \infty$ and $\omega \in A_p^-$. The kernel of the operator $\mathcal{T}$ is given by $\mathcal{K}^\alpha(t) = \{K_N^\alpha(t)\}_{N \in \mathbb{Z}^2}$. By Theorem 2.4 and the vector valued version of Theorem 2.1 in [1], we get that the operator $\mathcal{T}$ is bounded from $L^1(\mathbb{R}, \omega)$ into weak- $L^1_{\ell^\infty(\mathbb{Z}^2)}(\mathbb{R}, \omega)$ for $\omega \in A_1^-$. Hence, as $\|\mathcal{T} f(t)\|_{\ell^\infty(\mathbb{Z}^2)} = T^* f(t)$, we get the proof of (b).

For (c), we shall prove that if $f \in L^\infty(\mathbb{R})$ and there exists $t_0 \in \mathbb{R}$ such that $T^* f(t_0) < \infty$, then $T^* f(t) < \infty$ for *a.e.* $t \in \mathbb{R}$. Given $t \neq t_0$. Set $f_1 = f \chi_{(t_0 - 4|t_0 - t|, \ t_0 + 4|t_0 - t|)}$ and $f_2 = f - f_1$. Note that T^* is L^p-bounded for any $1 < p < \infty$. Then $T^* f_1(t) < \infty$, because $f_1 \in L^p(\mathbb{R})$, for any $1 < p < \infty$.

On the other hand, as the kernel K_N is supported in $\mathbb{R}^+$, we have

$$\begin{aligned}
&T_N^\alpha f_2(t) - T_N^\alpha f_2(t_0)\\
&= \iint_{-\infty}^{t} K_N^\alpha(t-s) f_2(s) ds - \iint_{-\infty}^{t_0} K_N^\alpha(t_0-s) f_2(s) ds\\
&= \iint_{-\infty}^{t_0-4|t_0-t|} \left(K_N^\alpha(t-s) - K_N^\alpha(t_0-s)\right) f_2(s) ds\\
&\le \iint_{-\infty}^{t_0-4|t_0-t|} |\partial_s K_N^\alpha(\xi(s))|\, |t-t_0|\, |f_2(s)|\, ds \quad (t-s \le \xi(s) \le t_0-s)\\
&\le C \iint_{-\infty}^{t_0-4|t_0-t|} \frac{|t-t_0|}{(t-s)^2} |f_2(s)|\, ds\\
&\le C \|f\|_{L^\infty(\mathbb{R})} < \infty.
\end{aligned}$$

Hence

$$\|T_N^\alpha f_2(t) - T_N^\alpha f_2(t_0)\|_{l^\infty(\mathbb{Z}^2)} \le C \|f\|_{L^\infty(\mathbb{R})}$$

and therefore $T^* f(t) = \|T_N^\alpha f(t)\|_{l^\infty(\mathbb{Z}^2)} \le C < \infty$. For the $L^\infty - BMO$ boundedness, we will prove it later.

(*d*) Let t_0 be one point in $\mathbb{R}$ such that $T^* f(t_0) < \infty$. Set $I = [t_0 - 4|t_0-t|,\ t_0 + 4|t_0-t|]$ with $t \neq t_0$. And we decompose f to be

$$f = (f-f_I)\chi_I + (f-f_I)\chi_{I^c} + f_I =: f_1 + f_2 + f_3.$$

Note that T^* is L^p-bounded for any $1 < p < \infty$. Then $T^* f_1(t) < \infty$, because $f_1 \in L^p(\mathbb{R})$, for any $1 < p < \infty$. And $T_N^\alpha f_3 = 0$, since $\mathcal{P}_{a_j}^\alpha f_3 = f_3$ for any $j \in \mathbb{Z}$. On the other hand, as the kernel K_N is supported in $\mathbb{R}^+$, we have

$$\begin{aligned}
&T_N^\alpha f_2(t) - T_N^\alpha f_2(t_0)\\
&= \int_{-\infty}^{t} K_N^\alpha(t-s) f_2(s) ds - \int_{-\infty}^{t_0} K_N^\alpha(t_0-s) f_2(s) ds\\
&= \iint_{-\infty}^{t_0-4|t_0-t|} \left(K_N^\alpha(t-s) - K_N^\alpha(t_0-s)\right) f_2(s) ds\\
&\le \iint_{-\infty}^{t_0-4|t_0-t|} |\partial_s K_N^\alpha(\xi(s))|\, |t-t_0|\, |f_2(s)|\, ds \quad (t-s \le \xi(s) \le t_0-s)\\
&\le C \iint_{-\infty}^{t_0-4|t_0-t|} \frac{|t-t_0|}{(t-s)^2} |f_2(s)|\, ds\\
&\le C \sum_{k=2}^{+\infty} \left(|t-t_0| \iint_{t_0-2^{k+1}|t_0-t|}^{t_0-2^k|t_0-t|} \frac{|f(s)-f_I|}{|t-s|^2} ds\right.\\
&\le C \sum_{k=2}^{+\infty} \left(\frac{|t-t_0|}{(2^{k+1}|t-t_0|)^2} \iint_{t_0-2^{k+1}|t_0-t|}^{t_0-2^k|t_0-t|} |f(s)-f_I| ds\right.\\
&\le C \sum_{k=2}^{+\infty} \left(\frac{|t-t_0|}{(2^{k+1}|t-t_0|)^2} \iint_{t_0-2^{k+1}|t_0-t|}^{t_0+2^{k+1}|t_0-t|} |f(s)-f_I| ds\right.\\
&= C \sum_{k=2}^{+\infty} \left(2^{-(k+1)} \frac{1}{2^{k+1}|t-t_0|} \int_{I_{k+1}} |f(s)-f_I| ds\right.\\
&\le C \sum_{k=2}^{+\infty} 2^{-(k+1)} \frac{1}{2^{k+1}|t-t_0|} \iint_{I_{k+1}} \left(f(s) - f_{I_{k+1}} + \sum_{l=2}^{k} f_{I_{l+1}} - f_{I_l}\right) \left(ds\right.
\end{aligned}$$

$$\leq C\sum_{k=2}^{+\infty}2^{-(k+1)}\frac{1}{2^{k+1}|t-t_0|}\int_{I_{k+1}}\left(f(s)-f_{I_{k+1}}\right.\ +2k\left\|f\right\|_{BMO(\mathbb{R})}\Big)\,ds$$
$$\leq C\sum_{k=2}^{+\infty}2^{-(k+1)}(1+2k)\left\|f\right\|_{BMO(\mathbb{R})}$$
$$\leq C\left\|f\right\|_{BMO(\mathbb{R})},$$

where $I_{k+1}=[t_0-2^{k+1}|t_0-t|,t_0+2^{k+1}|t_0-t|]$ for any $k\in\mathbb{N}$. Hence

$$\|T_N^\alpha f_2(t)-T_N^\alpha f_2(t_0)\|_{l^\infty(\mathbb{Z}^2)}\leq C\left\|f\right\|_{BMO(\mathbb{R})}$$

and therefore $T^*f(t)=\|T_N^\alpha f(t)\|_{l^\infty(\mathbb{Z}^2)}\leq C<\infty$.

Now, we shall prove the estimate (1.4) for functions such that $T^*f(t)<\infty$ *a.e.* For any $h>0$ and t_0 such that $T^*f(t_0)<\infty$, consider the integral $I=(t_0,t_0+h)$ and $f_I=\frac{1}{h}\int_I f(t)dt$. We have $T^*f_I(t)=0$. Let $f(t)=f_1(t)+f_2(t)+f_I$, where $f_1(t)=(f(t)-f_I)\chi_{(t_0-4h,t_0+4h)}(t)$ and $f_2(t)=(f(t)-f_I)\chi_{(-\infty,t_0-4h)}(t)+(f(t)-f_I)\chi_{(t_0+4h,+\infty)}(t)$. Then,

$$\frac{1}{h}\int_{t_0}^{t_0+h}|T^*f(t)-(T^*f)_I|\,dt=\frac{1}{h}\int_{t_0}^{t_0+h}\ \frac{1}{h}\int_{t_0}^{t_0+h}(T^*f(t)-T^*f(s))\,ds\ \ dt$$
$$\leq\frac{1}{h^2}\int_{t_0}^{t_0+h}\int_{t_0}^{t_0+h}|T^*f(t)-T^*f(s)|\,dsdt$$
$$=\frac{1}{h^2}\int_{t_0}^{t_0+h}\int_{t_0}^{t_0+h}\|T_N^\alpha f(t)\|_{l^\infty(\mathbb{Z}^2)}-\|T_N^\alpha f(s)\|_{l^\infty(\mathbb{Z}^2)}\ \ dsdt$$
$$\leq\frac{1}{h^2}\int_{t_0}^{t_0+h}\int_{t_0}^{t_0+h}\|T_N^\alpha f(t)-T_N^\alpha f(s)\|_{l^\infty(\mathbb{Z}^2)}dsdt$$
$$\leq\frac{1}{h^2}\int_{t_0}^{t_0+h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_1(t)-T_N^\alpha f_1(s)\|_{l^\infty(\mathbb{Z}^2)}dsdt$$
$$+\frac{1}{h^2}\int_{t_0}^{t_0+h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_2(t)-T_N^\alpha f_2(s)\|_{l^\infty(\mathbb{Z}^2)}dsdt$$
$$=:A+B.$$

The Hölder inequality and L^2-boundedness of T^* imply that

$$A\leq\frac{1}{h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_1(t)\|_{l^\infty(\mathbb{Z}^2)}dt+\frac{1}{h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_1(s)\|_{l^\infty(\mathbb{Z}^2)}ds$$
$$\leq\ \left(\frac{1}{h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_1(t)\|_{l^\infty(\mathbb{Z}^2)}^2dt\right)^{1/2}+\ \left(\frac{1}{h}\int_{t_0}^{t_0+h}\|T_N^\alpha f_1(s)\|_{l^\infty(\mathbb{Z}^2)}^2ds\right)^{1/2}$$
$$\leq C\frac{1}{h^{1/2}}\left\|f_1\right\|_{L^2(\mathbb{R})}\leq C\left\|f\right\|_{BMO(\mathbb{R})}.$$

For B, since $t_0\leq t,s\leq t_0+h$ and the support of f_2 is $(-\infty,t_0-4h)\bigcup(t_0+4h,+\infty)$, we have

$$T_N^\alpha f_2(t)-T_N^\alpha f_2(s)$$
$$=\int_{-\infty}^{t}K_N^\alpha(t-u)f_2(u)du-\int_{-\infty}^{t_0}K_N^\alpha(s-u)f_2(u)du$$
$$=\int_{-\infty}^{t_0-4h}\left(K_N^\alpha(t-u)-K_N^\alpha(s-u)\right)f_2(u)du$$
$$\leq\int_{-\infty}^{t_0-4h}|\partial_uK_N^\alpha(\xi(u))|\,|t-s|\,|f_2(u)|\,du\quad(t-u\leq\xi(u)\leq t_0-u)$$

$$
\begin{aligned}
&\le C\int_{-\infty}^{t_0-4h}\frac{|t-s|}{(t-u)^2}\,|f_2(u)|\,du\\
&\le C\sum_{k=2}^{+\infty}\int_{t_0-2^{k+1}h}^{t_0-2^kh}\frac{h\,|f(u)-f_I|}{|t-u|^2}du\\
&\le C\sum_{k=2}^{+\infty}\frac{h}{(2^{k+1}h)^2}\int_{t_0-2^{k+1}h}^{t_0-2^kh}|f(u)-f_I|du\\
&\le C\sum_{k=2}^{+\infty}\frac{h}{(2^{k+1}h)^2}\int_{t_0-2^{k+1}h}^{t_0+2^{k+1}h}|f(u)-f_I|du\\
&= C\sum_{k=2}^{+\infty}2^{-(k+1)}\frac{1}{2^{k+1}h}\iint_{I_{k+1}}|f(u)-f_I|du\\
&\le C\sum_{k=2}^{+\infty}2^{-(k+1)}\frac{1}{2^{k+1}h}\int_{I_{k+1}}\Big(f(u)-f_{I_{k+1}}+\sum_{l=2}^{k}f_{I_{l+1}}-f_{I_l}\Big)du\\
&\le C\sum_{k=2}^{+\infty}2^{-(k+1)}\frac{1}{2^{k+1}h}\iint_{I_{k+1}}\Big(f(u)-f_{I_{k+1}}+2k\,\|f\|_{BMO(\mathbb{R})}\Big)du\\
&\le C\sum_{k=2}^{+\infty}2^{-(k+1)}(1+2k)\,\|f\|_{BMO(\mathbb{R})}\\
&\le C\,\|f\|_{BMO(\mathbb{R})},
\end{aligned}
$$

where I_{k+1} denotes the interval $[t_0-2^{k+1}h, t_0+2^{k+1}h]$. Hence, we have $B\le C\,\|f\|_{BMO(\mathbb{R})}$. Then by the arbitrary of t_0 and $h>0$, we proved

$$\|T^*f\|_{BMO(\mathbb{R})}\le C\,\|f\|_{BMO(\mathbb{R})}.$$

For the second part of (c), we can deduce it from the BMO-boundedness of T^* and the inclusion of $L^\infty(\mathbb{R})\subset BMO(\mathbb{R})$. This completes the proof of Theorem 1.1. □

Now we shall prove Theorem 1.2.

Proof of Theorem 1.2. First, we shall see that if φ is a test function, then $T_N^\alpha\varphi(t)$ converges for all $t\in\mathbb{R}$. In order to prove this, it is enough to see that for any (L,M) with $0<L<M$, the series

$$A=\sum_{j=L}^{M}v_j(\mathcal{P}^\alpha_{a_{j+1}}\varphi(t)-\mathcal{P}^\alpha_{a_j}\varphi(t)) \text{ and } B=\sum_{j=-M}^{-L}v_j(\mathcal{P}^\alpha_{a_{j+1}}\varphi(t)-\mathcal{P}^\alpha_{a_j}\varphi(t))$$

converge to zero, when $L,M\to+\infty$. By the mean value theorem, following the arguments in the proof of Theorem 3.2, we have

$$
\begin{aligned}
|A|&\le C_\alpha\,\|v\|_{l^\infty(\mathbb{Z})}\int_0^\infty\sum_{j=L}^{M}\frac{\xi_j^{2\alpha-1}e^{-\xi_j^2/(4s)}(a_{j+1}-a_j)}{s^{1+\alpha}}\,|\varphi(t-s)|ds,\quad(\exists\ a_j\le\xi_j\le a_{j+1})\\
&\le C_{\alpha,v}\int_0^{+\infty}\rho^{4\alpha}(\rho^2-1)\sum_{j=L}^{M}\frac{a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}|\varphi(t-s)|ds,\quad(\text{since}\quad\rho\le\frac{a_{j+1}}{a_j}\le\rho^2)\\
&\le C_{\alpha,v,\rho}\int_0^{+\infty}\sum_{j=L}^{M}\frac{C}{a_j^2}|\varphi(t-s)|ds\\
&\le C_{\alpha,v,\rho}\left(\frac{1}{a_L^2}\sum_{j=L}^{M}\frac{a_L^2}{a_j^2}\right)\int_0^{+\infty}|\varphi(t-s)|ds
\end{aligned}
$$

$$\leq C_{\alpha,v,\rho}\frac{\rho^2}{\rho^2-1}\|\varphi\|_{L^1(\mathbb{R})}\frac{1}{a_L^2}\longrightarrow 0,\quad \text{as } L,M\to+\infty.$$

On the other hand, as the integral of the kernels are zero, we can write

$$\begin{aligned}B&=C_\alpha\int_0^{+\infty}\sum_{j=-M}^{-L}\left(v_j\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}(\varphi(t-s)-\varphi(t))ds\right.\\&=C_\alpha\left\{\int_0^1+\int_1^\infty\right\}\left(\sum_{j=-M}^{-L}\left(v_j\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}(\varphi(t-s)-\varphi(t))ds\right.\right.\\&=:B_1+B_2.\end{aligned}$$

Proceeding as in the case A, and by using the fact that φ is a test function, we have

$$\begin{aligned}|B_1|&=C_\alpha\int_0^1\sum_{j=-M}^{-L}\left(v_j\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{1+\alpha}}(\varphi(t-s)-\varphi(t))ds\right.\\&\leq C_\alpha\|\varphi'\|_{L^\infty(\mathbb{R})}\int_0^1\sum_{j=-M}^{-L}\left(v_j\frac{a_{j+1}^{2\alpha}e^{-a_{j+1}^2/(4s)}-a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^{\alpha}}ds\right.\\&\leq C_{\alpha,\varphi}\|v\|_{l^\infty(\mathbb{Z})}\int_0^1\rho^{4\alpha}(\rho^2-1)\sum_{j=-M}^{-L}\left(\frac{a_j^{2\alpha}e^{-a_j^2/(4s)}}{s^\alpha}ds\right.\\&\leq C_{\alpha,\varphi,v,\rho}a_{-L}^{2\alpha}\sum_{j=-M}^{-L}\left(\frac{a_j^{2\alpha}}{a_{-L}^{2\alpha}}\int_0^1\frac{1}{s^\alpha}ds\right.\\&\leq C_{\alpha,\varphi,v,\rho}\frac{\rho^{2\alpha}}{\rho^{2\alpha}-1}a_{-L}^{2\alpha}\longrightarrow 0,\quad\text{as}\quad L,M\to+\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}|B_2|&\leq C_{\alpha,\rho}\|v\|_{l^\infty(\mathbb{Z})}\|\varphi\|_{L^\infty(\mathbb{R})}\int_1^\infty\sum_{j=-M}^{-L}\left(\frac{a_j^{2\alpha}}{s^{1+\alpha}}ds\leq C_{\alpha,v,\varphi,\rho}\sum_{j=-M}^{-L}\left(a_j^{2\alpha}\int_1^\infty\frac{1}{s^{1+\alpha}}ds\right.\right.\\&\leq C_{\alpha,v,\varphi,\rho}a_{-L}^{2\alpha}\sum_{j=-M}^{-L}\left(\frac{a_j^{2\alpha}}{a_{-L}^{2\alpha}}\leq C_{\alpha,\varphi,v,\rho}\frac{\rho^{2\alpha}}{\rho^{2\alpha}-1}a_{-L}^{2\alpha}\longrightarrow 0,\quad\text{as}\quad L,M\to+\infty.\right.\right.\end{aligned}$$

As the set of test functions is dense in $L^p(\mathbb{R})$, by Theorem 1.1 we get the *a.e.* convergence for any function in $L^p(\mathbb{R})$. Analogously, since $L^p(\mathbb{R})\cap L^p(\mathbb{R},\omega)$ is dense in $L^p(\mathbb{R},\omega)$, we get the *a.e.* convergence for functions in $L^p(\mathbb{R},\omega)$ with $1\leq p<\infty$. By using the dominated convergence theorem, we can prove the convergence in $L^p(\mathbb{R},\omega)$-norm for $1<p<\infty$, and also in measure. □

4. Proofs of Theorems 1.3 and 1.4

In this section, we will give the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Let f be the function defined by

$$f(s)=\sum_{k\in\mathbb{Z}}\left((-1)^k\chi_{(-a^{2k+1},-a^{2k}]}(s),\right.$$

where $a>1$ is a real number that we shall fix it later. It is easy to see that

$$f(a^{2j}s)=(-1)^jf(s).\tag{4.1}$$

Let $a_j=a^j$. Then

$$\mathcal{P}_{a_j}^\alpha f(t)=\frac{1}{4^\alpha\Gamma(\alpha)}\int_0^{+\infty}\frac{a^{2\alpha j}e^{-a^{2j}/(4s)}}{s^{1+\alpha}}f(t-s)ds=\frac{1}{4^\alpha\Gamma(\alpha)}\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(t-a^{2j}u)\frac{du}{u}.$$

So

$$\mathcal{P}^{\alpha}_{a_j} f(0) = \frac{1}{4^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-a^{2j}u)\frac{du}{u} = (-1)^j \frac{1}{4^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u}.$$

We observe that

$$\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}}\ f(-u)\ \frac{du}{u} \le \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} \frac{du}{u} = 4^{\alpha}\Gamma(\alpha) < \infty.$$

Hence

$$\lim_{R\to+\infty} \int_R^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+} \int_0^{\varepsilon} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = 0.$$

On the other hand, $\lim\limits_{a\to+\infty} \int_1^a \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = \lim\limits_{a\to+\infty} \int_1^a \frac{e^{-1/(4u)}}{u^{\alpha}} \frac{du}{u} = C > 0$. Hence we can choose $a > 1$ big enough such that

$$\begin{aligned}\int_1^a \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = \int_1^a \frac{e^{-1/(4u)}}{u^{\alpha}} \frac{du}{u} &> \int_0^{1/a} \frac{e^{-1/(4u)}}{u^{\alpha}} \frac{du}{u} + \int_{a^2}^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} \frac{du}{u} \\ &> \int_0^{1/a} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} + \int_{a^2}^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u}\ .\end{aligned}$$

In other words, with the $a > 1$ fixed above, there exists constant $C_1 > 0$ such that

$$\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = C_1. \tag{4.2}$$

Hence

$$\mathcal{P}^{\alpha}_{a_j} f(0) - \mathcal{P}^{\alpha}_{a_{j+1}} f(0)\ = \frac{2C_1}{4^{\alpha}\Gamma(\alpha)} > 0.$$

Therefore we have

$$\sum_{j\in\mathbb{Z}} \mathcal{P}^{\alpha}_{a_{j+1}} f(0) - \mathcal{P}^{\alpha}_{a_j} f(0)\ = \infty.$$

By using (4.1) and changing variable we get

$$\mathcal{P}^{\alpha}_{a_j} f(t) = \frac{1}{4^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(t - a^{2j}u)\frac{du}{u} = (-1)^j \frac{1}{4^{\alpha}\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u}.$$

Then

$$\begin{aligned}&\mathcal{P}^{\alpha}_{a_{j+1}} f(t) - \mathcal{P}^{\alpha}_{a_j} f(t) \\ &= \frac{(-1)^{j+1}}{4^{\alpha}\Gamma(\alpha)} \Big\{ \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2(j+1)}} - u\right)\frac{du}{u} + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u} \Big\}.\end{aligned} \tag{4.3}$$

By the dominated convergence theorem, we know that

$$\lim_{h\to 0} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(h-u)\frac{du}{u} = \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = C_1 > 0,$$

where C_1 is the constant appeared in (4.2). So, there exists $0 < \eta_0 < 1$, such that, for $|h| < \eta_0$,

$$\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(h-u)\frac{du}{u} \ge \frac{1}{2} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f(-u)\frac{du}{u} = \frac{C_1}{2}.$$

Then, for each $t \in \mathbb{R}$, we can choose $j \in \mathbb{Z}$ such that $\dfrac{|t|}{a^j} < \eta_0$ (there are infinite j satisfying this condition), and we have

$$\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2(j+1)}} - u\right)\frac{du}{u} + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u} \ge C_1 > 0.$$

Choosing $v_j = (-1)^{j+1},\ j \in \mathbb{Z}$, by (4.3) we have, for any $t \in \mathbb{R}$,

$$\begin{aligned} T^* f(t) &\geq \sum_{|\frac{t}{a^j}|<\eta_0} (-1)^{j+1} \left(\mathcal{P}^{\alpha}_{a_{j+1}} f(t) - \mathcal{P}^{\alpha}_{a_j} f(t) \right) \\ &= \frac{1}{4^{\alpha}\Gamma(\alpha)} \sum_{|\frac{t}{a^j}|<\eta_0} \left(\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2(j+1)}} - u \right) \frac{du}{u} + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^{\alpha}} f\left(\frac{t}{a^{2j}} - u \right) \frac{du}{u} \right) \\ &= \infty. \end{aligned}$$

We complete the proof of Theorem 1.3. □

Also, we will give the proof of Theorem 1.4 which gives a local growth characterization of the operator T^* with $f \in L^{\infty}(\mathbb{R}^n)$.

Proof of Theorem 1.4. First, we prove the theorem in the case $1 < p < \infty$. Since $2r < 1$, we know that $B \backslash B_{2r} \neq \emptyset$. Let $f(t) = f_1(t) + f_2(t)$, where $f_1(t) = f(t)\chi_{B_{2r}}(t)$ and $f_2(t) = f(t)\chi_{B\backslash B_{2r}}(t)$. Then

$$|T^* f(t)| \leq |T^* f_1(t)| + |T^* f_2(t)| .$$

By Theorem 1.1,

$$\frac{1}{|B_r|} \int_{B_r} |T^* f_1(t)|\, dt \leq \left(\frac{1}{|B_r|} \int_{B_r} |T^* f_1(t)|^2\, dt \right)^{1/2} \leq C \left(\frac{1}{|B_r|} \int_{\mathbb{R}} |f_1(t)|^2\, dt \right)^{1/2} \leq C \|f\|_{L^{\infty}(\mathbb{R})} .$$

We also know that, for any $j \in \mathbb{Z}$,

$$\int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}}\, ds \leq \int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} + a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}}\, ds = 2 \cdot 4^{\alpha}\Gamma(\alpha). \tag{4.4}$$

Then, by Hölder's inequality, (4.4) and Fubini's Theorem, for $1 < p < \infty$ and any $N = (N_1, N_2)$, we have

$$\begin{aligned} &\sum_{j=N_1}^{N_2} v_j \left(\mathcal{P}^{\alpha}_{a_{j+1}} f_2(t) - \mathcal{P}^{\alpha}_{a_j} f_2(t) \right) \\ &\leq C \sum_{j=N_1}^{N_2} v_j \int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s))}}{s^{1+\alpha}} f_2(t-s)\, ds \\ &\leq C \|v\|_{l^p(\mathbb{Z})} \left(\sum_{j=N_1}^{N_2} \left(\int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} |f_2(t-s)|\, ds \right)^{p'} \right)^{1/p'} \\ &\leq C \|v\|_{l^p(\mathbb{Z})} \left(\sum_{j=N_1}^{N_2} \left\{ \int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} |f_2(t-s)|^{p'}\, ds \right\} \right. \\ &\qquad \left. \times \left\{ \int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}}\, ds \right\}^{p'/p} \right)^{1/p'} \\ &\leq C \|v\|_{l^p(\mathbb{Z})} \left(\sum_{j=N_1}^{N_2} \left(\int_0^{\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} |f_2(t-s)|^{p'}\, ds \right) \right)^{1/p'} \\ &\leq C \|v\|_{l^p(\mathbb{Z})} \left(\int_0^{\infty} \sum_{j=-\infty}^{+\infty} \frac{a_{j+1}^{2\alpha} e^{-a_{j+1}^2/(4s)} - a_j^{2\alpha} e^{-a_j^2/(4s)}}{s^{1+\alpha}} |f_2(t-s)|^{p'}\, ds \right)^{1/p'} \end{aligned}$$

$$\leq C\,\|v\|_{l^p(\mathbb{Z})}\left(\int_0^\infty \frac{1}{|s|}\,|f_2(t-s)|^{p'}\;ds\right)^{1/p'}$$
$$\leq C\,\|v\|_{l^p(\mathbb{Z})}\int_{\mathbb{R}}\frac{1}{|t-s|}\,|f_2(s)|^{p'}\;ds.$$

For $s\in B\backslash B_{2r}$ and $t\in B_r$, we have $r\leq|t-s|\leq 2$. Then, we get

$$\frac{1}{|B_r|}\int_{B_r}|T^*f_2(t)|\,dt \leq C\frac{1}{|B_r|}\int_{B_r}\left(\int_{\mathbb{R}}\frac{1}{|t-s|}\,|f_2(s)|^{p'}\;ds\right)^{1/p'}dt$$
$$\leq C\frac{\|f\|_{L^\infty(\mathbb{R})}}{|B_r|}\int_{B_r}\int_{r\leq|t-s|\leq 2}\frac{1}{|t-s|}\;ds\Bigg)^{1/p'}dt$$
$$\sim\left(\log\frac{2}{r}\right)^{1/p'}\|f\|_{L^\infty(\mathbb{R})}.$$

Hence,

$$\frac{1}{|B_r|}\int_{B_r}|T^*f(t)|\,dt\leq C\left(1+\left(\log\frac{2}{r}\right)^{1/p'}\right)\|f\|_{L^\infty(\mathbb{R})}\leq C\left(\log\frac{2}{r}\right)^{1/p'}\|f\|_{L^\infty(\mathbb{R})}.$$

For the case $p=1$ and $p=\infty$, the proof is similar and easier. Then we get the proof of (a).

For (b), when $1<p<\infty$, for any $0<\varepsilon<p-1$, let

$$f(t)=\sum_{k=-\infty}^{0}(-1)^k\chi_{(-a^{2k},-a^{2k-1}]}(t)\ \text{ and }\ a_j=a^j,$$

with $a>1$ being fixed later. Then, the support of f is contained in $[-1,0)$, and $\{a_j\}_{j\in\mathbb{Z}}$ is a ρ-lacunary sequence with $\rho=a>1$. We observe that

$$\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}\ \leq\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}=4^\alpha\Gamma(\alpha)<\infty.$$

Hence

$$\lim_{R\to+\infty}\int_R^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=0\quad\text{ and }\quad\lim_{\varepsilon\to0}\int_0^{\varepsilon}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=0.$$

Also there exists a constant $C>0$ such that $\lim_{a\to+\infty}\int_{a^{-1}}^1\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=\lim_{a\to+\infty}\int_{a^{-1}}^1\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}=C$. So we can choose $a>1$ big enough such that

$$\int_{a^{-1}}^1\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=\int_{a^{-1}}^1\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}\geq 10\quad\left(\int_0^{1/a^2}\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}+\int_{a-1}^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}\right)$$
$$>10\quad\left(\int_0^{1/a^2}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}\ +\ \int_{a-1}^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}\ \right).$$

Therefore, there exists a constant $C_1>0$ such that

$$\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=C_1>0 \tag{4.5}$$

and

$$0<\int_0^{1/a^2}\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}+\int_{a-1}^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u}\leq\frac{C_1}{9}. \tag{4.6}$$

On the other hand, by the dominated convergence theorem, we have

$$\lim_{h\to0}\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(h-u)\frac{du}{u}=\int_0^{+\infty}\frac{e^{-1/(4u)}}{u^\alpha}f(-u)\frac{du}{u}=C_1>0,$$

where C_1 is the constant appeared in (4.5). So, there exists $0 < \eta_0 < 1$, such that, for $|h| < \eta_0$,

$$\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} f(h-u)\frac{du}{u} \geq \frac{1}{2}\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} f(-u)\frac{du}{u} = \frac{C_1}{2}. \tag{4.7}$$

It can be checked that

$$f(a^{2j}t) = (-1)^j f(t) + (-1)^j \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}(t)$$

when $j \leq 0$. We will always assume $j \leq 0$ in the following. By changing variable,

$$\begin{aligned}
\mathcal{P}^\alpha_{a_j} f(t) &= \frac{1}{4^\alpha\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} f(t-a^{2j}u)\frac{du}{u} \\
&= \frac{(-1)^j}{4^\alpha\Gamma(\alpha)} \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} \left\{ f\left(\frac{t}{a^{2j}} - u\right) + \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j}} - u\right) \right\} \frac{du}{u}.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathcal{P}^\alpha_{a_{j+1}} f(t) - \mathcal{P}^\alpha_{a_j} f(t) \\
&= \frac{(-1)^{j+1}}{4^\alpha\Gamma(\alpha)} \Big\{ \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} f\left(\frac{t}{a^{2(j+1)}} - u\right)\frac{du}{u} + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} f\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u} \\
&\quad + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j+2}} - u\right)\frac{du}{u} \\
&\quad + \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u} \Big\}.
\end{aligned} \tag{4.8}$$

For given η_0 as above, let $2r < 1$ such that $r < \eta_0^2$ and $r \sim a^{2J_0}\eta_0$ for a certain negative integer J_0. If $J_0 \leq j \leq 0$, we have $\dfrac{r}{a^{2j}} < \eta_0$. And, for any $-r \leq t \leq r$ we have

$$-1 \cdot \chi_{[a-1,+\infty)}(u) \leq \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j+2}} - u\right) \leq \chi_{[a-1,+\infty)}(u)$$

and

$$-1 \cdot \chi_{[a-1,+\infty)}(u) \leq \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j}} - u\right) \leq \chi_{[a-1,+\infty)}(u).$$

Hence, for the third and fourth integrals in (4.8), by (4.6) we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} \sum_{k=1}^{-j-1} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j+2}} - u\right)\frac{du}{u} + \\
&\qquad \int_0^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha} \sum_{k=1}^{-j} (-1)^k \chi_{(-a^{2k},-a^{2k-1}]}\left(\frac{t}{a^{2j}} - u\right)\frac{du}{u} \\
&\qquad\qquad \geq (-2)\int_{-1}^{+\infty} \frac{e^{-1/(4u)}}{u^\alpha}\frac{du}{u} \geq -\frac{2C_1}{9}.
\end{aligned} \tag{4.9}$$

So, for any $t \in [-r, r]$ and $J_0 \leq j \leq 0$, combining (4.8), (4.7) and (4.9), we have

$$\mathcal{P}^\alpha_{a_{j+1}} f(t) - \mathcal{P}^\alpha_{a_j} f(t) \geq C_\alpha \cdot \left(C_1 - \frac{2C_1}{9}\right) = C \cdot C_1 > 0.$$

We choose the sequence $\{v_j\}_{j\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ given by $v_j = (-1)^{j+1}(-j)^{-\frac{1}{p-\varepsilon}}$, then for $N = (J_0, 0)$, we have

$$\frac{1}{2r}\int_{[-r,r]} |T^* f(t)|\, dt \geq \frac{1}{2r}\iint_{[-r,r]} |T_N^\alpha f(t)|\, dt \geq \frac{1}{4^\alpha\Gamma(\alpha)}\frac{1}{2r}\iint_{[-r,r]} \sum_{j=J_0}^{0} \left(C\cdot C_1\cdot(-j)^{-\frac{1}{p-\varepsilon}}\right) dt$$

$$\geq C_{p,\varepsilon,\alpha}\cdot C_1\cdot(-J_0)^{\frac{1}{(p-\varepsilon)'}} \sim \left(\log\frac{2}{r}\right)^{\frac{1}{(p-\varepsilon)'}}.$$

For (c), let $v_j = (-1)^{j+1}$, $a_j = a^j$ with $a > 1$ and $0 < \eta_0 < 1$ fixed in the proof of (b). Consider the same function f as in (b). Then, $\|v\|_{l^\infty(\mathbb{Z})} = 1$ and $\|f\|_{L^\infty(\mathbb{R})} = 1$. By the same argument as in (b), with $N = (J_0, 0)$ and $0 < \alpha < 1$, we have

$$\frac{1}{2r}\int_{[-r,r]} |T^* f(t)|\, dt \geq \frac{1}{2r}\iint_{[-r,r]} |T_N^\alpha f(t)|\, dt \geq \frac{1}{4^\alpha\Gamma(\alpha)}\frac{1}{2r}\iint_{[-r,r]} \sum_{j=J_0}^{0} C_1 dt \geq \frac{C_1}{4^\alpha\Gamma(\alpha)}\cdot(-J_0) \sim \log\frac{2}{r}.$$

□

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