Universidad Autónoma
de Madrid

Extremal problems in spaces of analytic functions

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Thesis submitted to

Universidad Autónoma de Madrid

in partial fulfilment of the requirements for the degree of

DOCTOR IN MATHEMATICS

Supervised by Professor

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Madrid, 2022

A mi familia, especialmente a mis abuelos.

Contents

Agradecimientos

Han pasado muchas cosas desde que empecé mi vida universitaria hace poco más de 9 años y con total sinceridad debo decir que nunca llegué a imaginar que llegaría tan lejos (también en sentido literal, pues estoy escribiendo estas líneas a miles de kilómetros de mi hogar). He trabajado duramente durante aproximadamente 4 años en esta tesis (a veces tocando la frustración con las yemas de los dedos) y pensé que esta parte sería sencilla, pero no podía estar más equivocado. Después de estar unos 30 minutos sufriendo el conocido como *síndrome del folio en blanco*, de lo único que estoy seguro es que no hubiera llegado hasta aquí solo. Muchísima gente ha contribuido, activa o pasivamente, en este trabajo y todos ellos merecen su reconocimiento. No sé si podré hacerles justicia, pero lo intentaré. Pido de forma adelantada disculpas si me olvido de algo. Quien me conoce sabe que puedo ser extraordinariamente despistado y, asumámoslo, es una posibilidad.

Creo que lo mejor es empezar a dar las gracias a todos aquellos que he conocido gracias a las Matemáticas. Probablemente el primero deba ser Francisco García Baladrón, *Paco* para la última generación de sus alumnos, quien logró algo extremadamente difícil. De entre los más de 30 adolescentes que asistíamos a sus clases, al menos uno (sé a ciencia cierta que muchos más, aunque a alguno le cueste reconocerlo) tenía interés por su materia. Quería aprender *de verdad*. Espero que actualmente esté disfrutando de una jubilación más que merecida.

Los siguientes que merecen reconocimiento son, sin lugar a dudas, mis compañeros de carrera en Alicante: Luis, Dani, Dasha, Ramiro, Alejandra, Virginia, Sergio, Martín, Manu, Antonio y Gabri. Me alegro enormemente de haber coincidido con vosotros y del *buen rollo* que tuvimos desde el día 1. Las partidas de *Bang* y las comidas en la cafetería de la Politécnica fueron estrictamente necesarias para superar los ε y las subsucesiones. Por razones de sobra conocidas no hemos podido reunirnos últimamente, pero debemos mantener la tradición de reunirnos (al menos) un par de veces por año.

Por supuesto también debo mencionar a Juan Matías Sepulcre. Su curso de Análisis Complejo me abrió las puertas al mundo de las funciones holomorfas y por el momento aquí sigo. También fue la primera persona que me puso en contacto, aunque fuese superficialmente, con las series de Dirichlet. Espero tener la oportunidad de profundizar este estudio muy pronto.

Mi corta carrera pudo haber acabado por aquel entonces, pero gracias a Dasha (y a Martín, que empezó conmigo) pude encontrar trabajo y con eso permitirme viajar a Madrid para empezar el máster que posteriormente me llevaría al doctorado. Muchas gracias también a todos los compañeros que tuve en Elche. Les deseo lo mejor.

Viniendo de una ciudad pequeña como Ibi, Madrid me parecía (y me sigue pareciendo) inmensa. Gracias a los que serían mis compañeros pude ver sus enormes ventajas y considerarla una segunda casa. Sin lugar a dudas, todo hubiese sido muy distinto sin vosotos.

Fran, tienes una paciencia y tranquilidad inmensas. No cambies. Bea (y por supuesto, su familia) muchas gracias por alquilarme una habitación en vuestro piso... ¡hasta dos veces! Julio, los cafés (y la charla de después) a primera hora de la mañana eran excelentes para empezar el día. Luis, en el fondo eres muy bueno, pero hay que tener paciencia para descubrirlo. Vigila tu espalda cuando tus amigos no lo podamos hacer por ti. Tienes dos excelentes pupilos en Jan y Celia. Florian, no olvido que tengo pendiente hacerte una visita en Bochum. Diego, ojalá nos hubiera dado tiempo a ver a Berto y Andreu en *Nadie sabe nada* (aún estamos a tiempo, aunque nos pilla lejos). Alex, uno de mis hermanos mayores académicos, siempre recordaré las charlas (matemáticas y no matemáticas), los viajes a Alicante y la época en la que compartimos piso. Espero que podamos colaborar más activamente de ahora en adelante. Jaime, Manuel y Javier, ha sido un placer compartir el despacho 413 con vosotros. Y también con Alberto y Marta (¡para ti también se extiende la invitación a *Nadie sabe nada*!). Ha sido una genial coincidencia que de forma completamete inesperada hayamos vuelto a encontrarnos en Noruega. Por supuesto, también me gustaría mencionar a aquellos que, desgraciadamente, no he tenido tanto tiempo de conocer como Bea Pascual, Raquel, Nikita, Ismael, Mikel o Víctor.

Of course, I would like to thank Professor Kristian Seip for his kindness and patience. Even though it was extremely difficult to host a visit during the COVID-19 pandemic, he agreed on supervising me (with the consequent bureaucracy) up to four times. The stay had to be delayed almost a year and a half, but it has been a great opportunity to my education. I want to thank him and Professors Karl-Mikael Perfekt and Ole Fredrik Brevig that they have listened to my ideas and have replied with new suggestions. I also have to thank all the members of Department of Mathematical Sciences at NTNU for their hospitality. Hopefully, I will visit Trondheim again soon.

No puedo olvidarme tampoco de mi director, Dragan Vukotić. Desde que llegué a la Autónoma se mostró dispuesto a enseñarme y he aprendido muchísimo bajo su tutela. Siempre se ha mostrado más que dispuesto a revisar y apoyar mi trabajo, ya sea comprobando las cuentas, sugiriendo nuevas ideas o recomendando mejoras en mi expresión con el inglés (competencia en la cual aún me queda mucho por mejorar). En mis momentos más pesimistas ha sabido animarme, haciéndome ver que tarde o temprano llegarían tiempos mejores. Parece que en casi todas las ocasiones ha tenido razón. Muchas gracias por todo.

Para concluir con los agradecimientos "académicos", me gustaría romper una lanza en favor de la Educación Pública. Me alegra enormemente que el único problema de tipo económico que me he encontrado durante mis estudios ha sido la búsqueda de alojamiento en Madrid (que no es un problema trivial). He tenido la suerte de nacer en un país donde las matrículas nunca han supuesto un problema ni para mi familia ni para mi. Aunque imperfecta (aún queda mucho por mejorar), la Educación Pública ha hecho esto posible y eso es de agradecer. Me siento muy orgulloso de mi formación y sé a ciencia cierta que mi vida sin ella sería muy diferente.

Tengo que dar las gracias además a mi familia, a quienes me gustaría dedicar esta tesis. Aunque me gustaría agradecerselo a todos, por supuesto debo destacar a mis padres, Nuria y Vicente, que siempre me han ofrecido todo lo que tenían y más. También a mis hermanos pequeños, Candela y Marcos. Sabéis que durante los últimos años he estado muy liado y no he pasado por casa tanto como debería, pero me gustaría agradeceros vuestro constante apoyo y cariño. También quiero explícitamente acordarme de mis abuelos ya que, desgraciadamente, no todos ellos han podido ver el final de este camino.

Puesto que tengo la suerte de tener otra familia, también me gustaría dedicarles unas últimas palabras. Antonio, Maribel, Toni y Noelia: muchas gracias por aceptarme como uno más. Esther, pocas cosas puedo añadir aquí que no te haya dicho ya. Has estado conmigo desde el primer día y sabes lo difícil que ha sido todo. Hemos pasado por muchas cosas (como casi cuatro años de distancia intermitente o una pandemia en un apartamento minúsculo) y no me arrepiento de nada. Aún nos quedan muchos retos que superar juntos, pero si es contigo los afrontaré encantado. Eres lo mejor que me ha pasado en la vida y espero compartir el resto de ella contigo. Muchas gracias por soportarme y, sobre todo, por quererme tanto.

Resumen

Tal y como especifica su título, en esta disertación estudiamos una serie de problemas extremales en espacios de funciones analíticas. En algunos de estos problemas somos capaces de proveer una respuesta cerrada, mientras que en otros proporcionamos resultados parciales novedosos. La estructura de esta memoria está planteada de tal forma que cualquier lector con conocimientos elementales en el área (como los que pueda tener otro estudiante de doctorado en una temática afín o un estudiante de máster adelantado) pueda entenderlo con facilidad. Para ello, dividimos esta tesis en cuatro capítulos y dos apéndices.

En el Capítulo 1 puede encontrarse una breve presentación de los espacios de funciones analíticas que van a protagonizar los posteriores capítulos, además de una introducción al concepto de *problema extremal*. Concretamente, se recopilan las definiciones y propiedades elementales de los espacios de Hardy H^p, los espacios de Bergman con peso estándard A^p_α, los espacios de norma mixta $H(p,q,\alpha)$, el espacio de Bloch $\mathscr{B}$ y los espacios de Besov B^q. Es un capítulo fundamentalmente expositivo y en consecuencia únicamente se mostrarán las pruebas especialmente relevantes para el resto de la tesis.

En el Capítulo 2, calculamos el valor preciso de la norma del operador consistente en cancelar un cero de orden arbitrario mediante un factor de Blaschke en el subespacio del espacio de Bergman con peso A^p_α de funciones que se anulan en dicho cero con una multiplicidad mayor o igual de la dada. Más específicamente,

$$\left\| \frac{f}{\varphi_a^k} \right\|_{A^p_\alpha} \leq \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\,\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \|f\|_{A^p_\alpha}, \quad \text{si} \quad f(a) = \ldots = f^{(k-1)}(a) = 0,$$

donde Γ es la función Gamma de Euler.

Como corolario, extendemos una estimación para las derivadas de las funciones pertenecientes a ese mismo subespacio, simplificando su demostración original. El contenido de esta parte de la tesis puede encontrarse en el artículo [73].

A continuación, en el Capítulo 3 estudiamos las normas del operadores de inclusión del espacio de Besov (analítico) B^q en el espacio de Bloch $\mathscr{B}$ y de éste en A^p_α. En el caso de la inclusión de B^q en $\mathscr{B}$ somos capaces de dar una respuesta completa, calculando la norma exacta y describiendo sus funciones extremales. Más concretamente, probamos las siguientes desigualdades:

$$\|f\|_{\mathscr{B}} \le \|f\|_{B^q,1} \le 2^{\frac{q-1}{q}} \|f\|_{B^q,2} \quad \text{y} \quad \|f\|_{\mathscr{B}} \le 2\|f\|_{B^1},$$

donde $\|\cdot\|_{B^q,1}$ y $\|\cdot\|_{B^q,2}$ son las dos normas usuales en B^q, $q > 1$. En el caso de la inclusión del espacio de Bloch en A^p_α podemos estimar asintóticamente el crecimiento de ésta gracias al uso de funciones que se anulan en el origen. En concreto, probamos que existen dos cantidades C_1 y C_2, independientes de p, tal que

$$C_1 p \le \sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\} \le C_2 p, \quad p \ge 1.$$

Además, el uso de las funciones analíticas que se anulan en el origen permite refinar estas estimaciones asintóticas, obteniendo las cotas

$$\begin{aligned} \frac{1}{2e}\frac{1}{\alpha+2} &\le \liminf_{p\to\infty} \frac{\sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\}}{p}, \\ \frac{1}{2e}\frac{1}{\sqrt{(\alpha+1)(\alpha+2)}} &\ge \limsup_{p\to\infty} \frac{\sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\}}{p}. \end{aligned}$$

Este capítulo está basado en la publicación [70].

En el Capítulo 4, el más extenso de esta memoria, estudiamos las inclusiones contractivas entre espacios de norma mixta. Bajo la condición de que $p \ge u$, caracterizamos completamente la contractividad de la inclusión de $H(p,q,a)$ en $H(u,v,b)$. Siendo más específicos, demostramos que el operador inclusión de $H(p,q,a)$ en $H(u,v,b)$, $p \ge u$, tiene norma igual a 1 si, y únicamente si,

$$\text{o bien} \quad q \le v \quad \text{o bien} \quad q > v \;\; \text{y} \;\; aq \le bv.$$

En lugar de estudiar el caso general $p < u$, el cual es considerablemente más complejo que el anterior, se muestran avances parciales novedosos en un problema abierto planteado por O. F. Brevig, J. Ortega-Cerdà, K. Seip y J. Zhao [29] por un lado y

E. H. Lieb y J. P. Solovej [69] por otro (aunque parece que con anterioridad ya era de interés para otros expertos internacionales como M. Pavlović [80, Problema 2.1]), con aplicaciones a otras áreas de las Matemáticas. Para $p > 2$ y $\alpha \geq -1$, probaremos que la norma del operador inclusión de A^2_α en $A^p_{\frac{p}{2}(\alpha+2)-2}$ se mantiene invariante al restringirse al subespacio de funciones con un conjunto finito de coeficientes de Taylor nulos. Esto permite deducir que la norma de dicho operador puede acotarse superiormente por C si, y únicamente si,

$$|f(0)|^p + \frac{p}{2(\alpha+2)} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{\frac{p(\alpha+2)}{2}} \, dA(z) \leq C^p,$$

para toda f tal que $\|f\|_{A^2_\alpha} = 1$. Como corolario, obtenemos la cota uniforme

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \leq \frac{\sqrt[e]{e}}{\sqrt{2}} \|f\|_{A^2_\alpha},$$

que mejora los resultados previamente publicados.

Este capítulo está basado en los trabajos [71] y [72].

Finalmente, el apéndice A sirve como compendio de una serie de desigualdades clásicas y propiedades de las funciones holomorfas que son necesarias para el correcto desarrollo de los capítulos anteriormente descritos. Por su parte, el apéndice B consiste en una brevísima introducción a los espacios de Hardy de series de Dirichlet, los cuales no se abordan directamente en esta tesis pero están fuertemente vinculados a la Sección 4.2.

Abstract

As the title of this dissertation specifies, we study several extremal problems in spaces of analytic functions. In some of these problems we provide a complete answer, whilst in others we deduce new partial results. The structure of this thesis is designed with the aim of being easily comprehensible for any reader with an elementary knowledge in the area (such as a PhD student in a related subject or an advanced Master student). For this purpose, we divide the thesis into four chapters and two appendices.

In Chapter 1 we provide a brief presentation of the spaces of analytic functions which play a leading role in the following chapters, as well as an introduction to the concept of *extremal problem*. Specifically, we collect the definitions and elementary properties of the Hardy spaces H^p, the standard weighted Bergman spaces A^p_α, the mixed norm spaces $H(p,q,\alpha)$, the Bloch space $\mathscr{B}$ and the Besov spaces B^q. It is essentially an expository chapter and thus we only show the proofs particularly relevant for the rest of the thesis.

In Chapter 2, we compute the precise value of the norm of the operator consisting of the cancellation a zero of prescribed order by a suitable Blaschke factor on the corresponding subspace of the weighted Bergman space A^p_α of vanishing functions. More specifically, we prove that

$$\left\| \frac{f}{\varphi_a^k} \right\|_{A^p_\alpha} \leq \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\,\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \|f\|_{A^p_\alpha}, \quad \text{if} \quad f(a) = \ldots = f^{(k-1)}(a) = 0,$$

where Γ is the Euler Gamma function.

As a corollary, we extend an estimate for the derivatives of the functions in the closed subspace defined by the above conditions, simplifying the original proofs. The content of this part of the thesis is based on the article [73].

Then, in Chapter 3 we study the norms of the inclusion operators of the (analytic) Besov space B^q in the Bloch space $\mathscr{B}$ and of the latter in A^p_α. In the case of the inclusion of B^q in $\mathscr{B}$ we are able to give a complete answer, computing the exact norm and describing its extremal functions. More specifically, we prove the following inequalities:

$$\|f\|_{\mathscr{B}} \leq \|f\|_{B^q,1} \leq 2^{\frac{q-1}{q}} \|f\|_{B^q,2} \quad \text{and} \quad \|f\|_{\mathscr{B}} \leq 2\|f\|_{B^1},$$

where $\|\cdot\|_{B^q,1}$ and $\|\cdot\|_{B^q,2}$ are the usual expressions for the B^q norms, $q > 1$. In the case of the inclusion of the Bloch space in A^p_α we can asymptotically estimate the growth of the norm of the inclusion operator by means of the use of vanishing functions at the origin. That is, we show that there exists two quantities C_1 and C_2, independent of p, such that

$$C_1 p \leq \sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\} \leq C_2 p, \quad p \geq 1.$$

In addition, the use of vanishing functions yields the following bounds,

$$\begin{aligned} \frac{1}{2e}\frac{1}{\alpha+2} &\leq \liminf_{p\to\infty} \frac{\sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\}}{p}, \\ \frac{1}{2e}\frac{1}{\sqrt{(\alpha+1)(\alpha+2)}} &\geq \limsup_{p\to\infty} \frac{\sup\left\{\|f\|_{A^p_\alpha} \,:\, \|f\|_{\mathscr{B}} = 1\right\}}{p}. \end{aligned}$$

This chapter is based on the publication [70].

In Chapter 4, the most extensive chapter of this dissertation, we will study the contractive inclusions between mixed norm spaces. Under the condition that $p \geq u$, we completely characterise the contractivity of the inclusion of $H(p,q,a)$ in $H(u,v,b)$. More precisely, we prove that the inclusion operator from $H(p,q,a)$ to $H(u,v,b)$, $p \geq u$, has norm equal to one if, and only if,

$$q \leq v \quad \text{or} \quad q > v \quad \text{and} \quad aq \leq bv.$$

Instead of studying the general case $p < u$, which is way more complex than the previous one, we obtain new partial results in a recent and still open problem posed by O. F. Brevig, J. Ortega-Cerdà, K. Seip and J. Zhao [29] and E. H. Lieb and J. P. Solovej [69] (although it seems to have been of interest to other international experts such as M. Pavlović [80, Problem 2.1] before these publications), with applications to other fields of Mathematics. If $p > 2$ and $\alpha \geq -1$, we prove that the norm of the inclusion operator from A^2_α to $A^p_{\frac{p}{2}(\alpha+2)-2}$ is invariant under the restriction to any subspace of analytic functions with a fixed finite set of vanishing Taylor coefficients. This yields that the norm of the inclusion operator can be bounded from above by a constant C if and only if

$$|f(0)|^p + \frac{p}{2(\alpha+2)} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{\frac{p(\alpha+2)}{2}} \, dA(z) \leq C^p,$$

for every f such that $\|f\|_{A^2_\alpha} = 1$. As a corollary, we deduce the uniform bound

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \leq \frac{\sqrt[e]{e}}{\sqrt{2}} \|f\|_{A^2_\alpha},$$

which improves the previous known results.

This chapter is based on the works [71] and [72].

Finally, the Appendix A will serve as a compendium of some classical inequalities and elementary properties of holomorphic functions that are necessary for the complete development of the chapters described above. Appendix B consists of a brief introduction to Hardy spaces of Dirichlet series, which are not directly treated in this thesis but they are strongly related with the problem studied in Section 4.2.

Notation

Theorems listed by letters instead of numbers are either well-known or have been proved by other authors, so they must not be attributed to the author of this doctoral thesis.

Throughout the document, the following notation will be used frequently:

- The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the natural, integer, real and complex numbers, respectively.

- Given two quantities A and B, depending on other parameters, we will say that they are *comparable* if there exist two absolute constants M and K such that $MA \leq B \leq KA$. In this case, we will write $A \approx B$.

  If only the inequality $B \leq KA$ holds, we will write $B \lesssim A$.

- Given a real number x, $\lceil x \rceil$ will denote the greater integer part of x. That is,

$$\lceil x \rceil := \min\{k \in \mathbb{Z} \,:\, x \leq k\}.$$

- We will write $\log^+ x := \max\{0, \log x\}$.

- Given $z \in \mathbb{C}$, we will use $|z|$, $\operatorname{Re} z$ and $\operatorname{Im} z$ to represent its modulus, real part and imaginary part, respectively. The letters $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} \,:\, |z| = 1\}$ will denote the open unit disc and the unit circle (both centered at the origin) in $\mathbb{C}$.

- Given a set $X \subset \mathbb{C}$ and $c \in \mathbb{C}$, we will write

$$cX := \{cx \,:\, x \in X\}.$$

- Given $X \subset \mathbb{C}$ and a sequence of functions $\{f_n\}_{n\geq 1}$ defined on X, the expression

$$f_n \underset{X}{\rightrightarrows} f,$$

means that $\{f_n\}_{n\geq 1}$ converges uniformly to the function f on X.

- Given two functions f and g, we will use the notation

$$f(z) = \mathcal{O}\big(g(z)\big), \quad z \to z_0,$$

if there exist $\delta, M > 0$ such that $|f(z)| \leq Mg(z)$ if $0 < |z - z_0| < \delta$.

Similarly, we will write

$$f(z) = o\big(g(z)\big), \quad z \to z_0,$$

if $\lim\limits_{z\to z_0} \dfrac{f(z)}{g(z)} = 0$.

- The symbol $\mathbb{C}[z]$ will denote the set of all algebraic polynomials with complex coefficients.

- We will use the notation $\mathscr{H}(\mathbb{D})$ to represent the set of all holomorphic functions in $\mathbb{D}$.

- Given an analytic function f, $\mathscr{Z}(f)$ will denote the zero set of f.

- For any $a \in \mathbb{D}$, the letter φ_a will represent the disc automorphism

$$\varphi_a(z) := \frac{a - z}{1 - \overline{a}z}. \tag{1}$$

- As is usual, we will write $\Gamma(x)$ and $B(x, y)$ to denote the Euler's Gamma and Beta functions, respectively. More specifically, we will use the integral representations

$$\begin{aligned} \Gamma(x) &:= \int_0^\infty t^{x-1}e^{-t}\,dt, \\ B(x,y) &:= \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \end{aligned}$$

if $\operatorname{Re} x, \operatorname{Re} y > 0$.

- For $\beta \geq 1$ and a non-negative integer number n, we will use the notation

$$c_\beta(n) := \binom{n+\beta-1}{n} = \frac{\Gamma(n+\beta)}{\Gamma(\beta)n!}. \tag{2}$$

- If $(V, \|\cdot\|)$ is a normed space and $X \subset V$, then $\overline{X}^{\|\cdot\|}$ will be the closure of X with respect to the norm $\|\cdot\|$.

- Given $p \in (0, \infty)$ and a measure space (X, Σ, μ), we will write $L^p(X, \mu)$ to denote the set of all (equivalence classes of) measurable functions of X which are p-integrable. In the same way, $L^\infty(X, \mu)$ will be the set of all (classes of) functions of X which are essentially bounded.

  If $X \subset \mathbb{C}$ or $X \subset \mathbb{R}$ and μ is the Lebesgue measure of X, we will simply write $L^p(X)$.

  In the case when $X = \mathbb{N}$, Σ is the power set of $\mathbb{N}$ and μ is the counting measure, we will write ℓ^p.

- Unless otherwise indicated, the terms *integrable*, *almost everywhere* or *almost nowhere* must be understood with respect to the Lebesgue measure.

- Given a function (or class of functions) g and a space of functions (or classes of functions) X, we will write the pointwise multiplier operator associated to g as

$$\mathscr{M}_g(f) := gf, \quad \forall f \in X.$$

- Given $\Lambda \subset \mathbb{N} \cup \{0\}$ and $f \in \mathscr{H}(\mathbb{D})$, $P_\Lambda(f)$ will denote the projection on the subspace consisting of those functions spanned by $\{z^n : n \in \Lambda\}$. That is,

$$P_\Lambda(f) = P_\Lambda\left(\sum_{n=0}^{\infty} a_n z^n\right) := \sum_{n\in\Lambda} a_n z^n.$$

Chapter 1

Introduction

As is natural, we start this dissertation with a brief introduction to the spaces of analytic functions which are going to be studied in the following chapters.

At the date of writing this text all of these spaces can be considered as classical, and therefore they have been widely studied. Thus, their elementary properties can be found with detail in several specialised monographs. Consequently, the results in this chapter will be collected mostly without proofs (unless they are relevant for this thesis).

1.1 The Hardy spaces H^p

The first class of spaces of analytic functions that we should introduce is the family of Hardy spaces H^p.

If $p > 0$, let $M_p(r, f)$ be the integral mean of order p of the analytic function f on the circle of radius r centred at the origin,

$$\begin{aligned} M_p(r,f) &:= \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^p \, dt \right)^{\frac{1}{p}}, \quad p < \infty, \\ M_\infty(r,f) &:= \max_{|z|=r} \{ |f(z)| \}. \end{aligned}$$

We define *the Hardy space H^p* as the set of all analytic functions in $\mathbb{D}$ whose integral means M_p are uniformly bounded:

$$H^p := \left\{ f \in \mathscr{H}(\mathbb{D}) \, : \, \sup_{r \in [0,1)} \{ M_p(r,f) \} < \infty \right\}.$$

Historically, we can set the origin of these spaces in 1915, with the publication of the so-called Hardy convexity theorem, although the first explicit definition is due to F. Riesz in 1923 [86], who set the notation H^p in honour of G. H. Hardy.

Theorem A (Hardy's convexity theorem, [50])**.** *Let $f \in \mathscr{H}(\mathbb{D})$ and $0 < p \leq \infty$. Then*

(a) $M_p(r, f)$ is an increasing function of r. In addition, if f is not constant then $M_p(r, f)$ is strictly increasing.

(b) The function $\log M_p(r, f)$ is a convex function of $\log r$.

Observe that, if $p = \infty$, Hardy's convexity theorem is just maximum modulus principle and Hadamard's three-circle theorem which were known before its publication. Moreover, using G. H. Hardy's own words [50], his interest in these properties of the means M_p had its origin in conversations with H. Bohr and E. Landau, who guessed that the result should be true for $p = 1$. During the first half of the twentieth century many other famous analysts contributed to the further development of the theory.

An alternative proof which is based on the subharmonicity of $|f|^p$ to G. H. Hardy's demonstration can be found in later works of F. Riesz [85]. This idea is explained in the monograph [38] (Theorem 1.6, page 9).

Due to the monotonicity of $M_p(r, f)$ with respect to the radius r, we can equip H^p with the following norm

$$\|f\|_{H^p} := \lim_{r \to 1^-} M_p(r, f).$$

It is immediate that, due to Hölder's inequality, $\|f\|_{H^p}$ is an increasing function of p, and in particular the inclusion $H^p \subset H^q$ holds if $p > q$.

It is easy to check that the mapping $\|\cdot\|_{H^p}$ is indeed a norm if $p \geq 1$, whilst $d(f, g) := \|f - g\|_{H^p}^p$ is distance-invariant under translations if $0 < p < 1$. In any case, we can endow H^p with a topology thanks to $\|\cdot\|_{H^p}$.

Two topics of interest in the theory of H^p spaces are the description of the zero sets of their functions (which are completely understood) and the existence of non-tangential limits almost everywhere in $\mathbb{T}$.

The first work about the zero sets of H^p functions is due to, again, F. Riesz [86]. Using Jensen's formula, he proved that the sequence of zeros $\{z_n\}_{n \geq 1}$ of a non-identically zero function $f \in H^p$ must verify the *Blaschke's condition*

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty,$$

named after W. Blaschke [20], who showed that, under this condition, the infinite product

$$B(z) := z^m \prod_{z_n \neq 0} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z},$$

converges uniformly on compact sets of $\mathbb{D}$ and defines an analytic function in $\mathbb{D}$ whose zero set (counting multiplicities) is exactly the sequence $\{z_n\}_{n\geq 1}$. Moreover, $|B| < 1$ in $\mathbb{D}$ and $|B(z)| = 1$ for almost every $z \in \mathbb{T}$. An analytic function satisfying these two last properties is known as an *inner function*, a term coined by A. Beurling in his seminal work [19].

Indeed, these *Blaschke products* (finite or not) lead to a natural factorization for H^p functions.

Theorem B (Riesz's factorization theorem, [86]). *Let $f \in H^p$ non-identically zero. Then, there exists a Blaschke product B and a non-vanishing function $g \in H^p$ such that $f = Bg$ in $\mathbb{D}$.*

A proof of Riesz's factorization theorem can be found in [38] (Theorem 2.5, page 20), [49] (Theorem 2.3, page 53) or [66] (page 68).

With respect to the existence of non-tangential limits and their properties, the starting point of this theory is due to P. Fatou [44], who proved that every bounded harmonic (and, in particular, analytic) function has non-tangential limits almost everywhere in $\mathbb{T}$ (this result justifies the existence of such limits for the Blaschke products previously introduced) and that these limits cannot vanish on an arc of $\mathbb{T}$ unless f is identically zero.

Later, F. and M. Riesz [87] extended this result to H^1 functions. In addition, they showed that if the boundary values vanish at a set of positive measure (not necessarily an arc) then $f \equiv 0$.

Afterwards, G. Szegő [98] showed that any non-identically zero function $f \in H^2$ verifies that $\log|f|$ is integrable over $\mathbb{T}$.

Finally, in the same paper where he proved his factorization theorem, F. Riesz [86] generalised these results for every H^p, $p > 0$. For the sake of clarity, we collect this result below.

Theorem C (F. Riesz, [86])**.** *For every $p > 0$ and $f \in H^p$, the non-tangential limit*

$$f(e^{it}) := \lim_{z \to e^{it}, z \in S_K(t)} f(z),$$

where $S_K(t) = \{z \in \mathbb{D} : |e^{it} - z| < K(1 - |z|)\}$, $K > 1$, exists for almost every $t \in [0, 2\pi]$, $f \in L^p(\mathbb{T})$ and

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) - f(e^{it}) \right|^p \, dt = 0.$$

In addition, if $f \not\equiv 0$ then $\log |f| \in L^1(\mathbb{T})$.

Remark. *The region $S_K(t)$ is known as a Stolz angle. In fact, it can be substituted by any region $\Omega \subset \mathbb{D}$ such that $e^{it} \in \overline{\Omega}$ and any path $\gamma : (0,1) \to \Omega$ with $\lim_{r \to 1^-} \gamma(r) = e^{it}$ is not tangent to $\mathbb{T}$.*

As a particular case, if we choose as a path the radius that joins the origin with the point e^{it}, the limit $\lim_{r \to 1^-} f(re^{it})$ is called "radial limit" instead of "non-tangential limit".

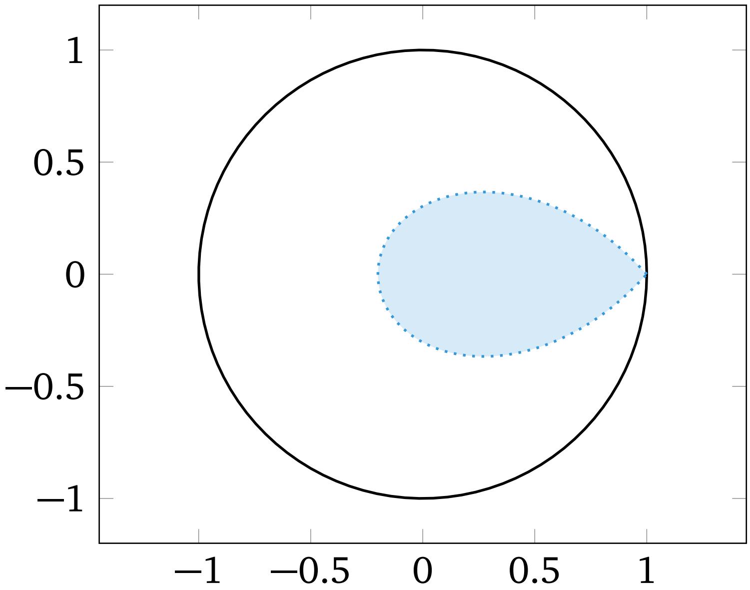

Figure 1: The Stolz angle $S_{\frac{3}{2}}(0)$ corresponds to the region enclosed inside the blue curve.

A proof of Theorem C can be found in [66] (pages 70 and 71).

All these properties of H^p spaces can be deduced from a different point of view using the *Nevanlinna class N*

$$N := \left\{ f \in \mathscr{H}(\mathbb{D}) : \sup_{r \in [0,1)} \left\{ \int_0^{2\pi} \log^+ |f(re^{it})| \, dt \right\} < \infty \right\}.$$

Since $\log^+ x \le \frac{x}{e}$ for all $x > 0$, it is clear that $H^p \subset N$ for all $p > 0$. See [38] (Sections 2.1, 2.2 and 2.3) or [49] (Section 5 in Chapter 2) to consult the proofs of these results using the class N.

A first consequence of Theorem C is that

$$\begin{aligned} \|f\|_{H^p} &= \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p\, dt\right)^{\frac{1}{p}}, \quad p<\infty, \\ \|f\|_{H^\infty} &= \operatorname*{ess\,sup}_{|z|=1}\{|f(z)|\}. \end{aligned}$$

As a particular case, we can write $\|f\|_{H^2}$ in terms of the Taylor coefficients of f. That is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, due to Parseval's identity

$$M_2(r,f) = \sqrt{\sum_{n=0}^{\infty} |a_n|^2 r^{2n}}, \quad \forall r \in [0,1),$$

and therefore $\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2$ as an application of the monotone convergence theorem. It is a straightforward computation to check that this norm comes from the inner product

$$\langle f, g\rangle_{H^2} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\, dt = \sum_{n=0}^{\infty} a_n \overline{b_n}, \quad \forall f, g \in H^2,$$

where $\{a_n\}_{n\ge 0}$ and $\{b_n\}_{n\ge 0}$ are the sequences of Taylor coefficients of f and g, respectively. In other words, the space H^2 will be a Hilbert space (once we prove that it is closed).

This interpretation of H^2, jointly with Riesz's factorization (Theorem B), provides us with a powerful tool to prove results in H^p (typically inequalities). This technique consists of the following three steps:

1. First, we prove the result for the H^2 case.

2. Then, we extend the result for non-vanishing H^p functions.

3. Finally, we complete the proof for arbitrary H^p functions by factoring out Blaschke products.

An important example of the potential of this tool is the deduction of the sharp pointwise bounds for H^p functions. Due to the relevance of this result in this dissertation, we will include the proof.

Theorem D (Pointwise bounds in H^p, [38] (page 144)). *Let $p > 0$ and $\zeta \in \mathbb{D}$. Then*

$$|f(\zeta)| \le \frac{\|f\|_{H^p}}{(1-|\zeta|^2)^{\frac{1}{p}}}, \quad \forall f \in H^p,$$

and equality is attained if and only if f is a constant multiple of

$$f_\zeta(z) = \frac{1}{(1-\overline{\zeta}z)^{\frac{2}{p}}}, \quad \forall z \in \mathbb{D}.$$

Proof.

1. Let $f \in H^2$. Due to the Cauchy-Schwarz inequality and the representation of $\|f\|_{H^2}$ in terms of the Taylor coefficients of f,

$$|f(\zeta)| = \left|\sum_{n=0}^{\infty} a_n \zeta^n\right| \le \sqrt{\sum_{n=0}^{\infty} |a_n|^2}\sqrt{\sum_{n=0}^{\infty} |\zeta|^{2n}} = \frac{\|f\|_{H^2}}{\sqrt{1-|\zeta|^2}}.$$

   Moreover, equality is possible if, and only if, $a_n = c\overline{\zeta}^n$, for some $c \in \mathbb{C}$ and every $n \ge 0$, so $f(z) = c(1-\overline{\zeta}z)^{-1}$.

2. Suppose that $f \in H^p$ does not vanish in $\mathbb{D}$. Then the function $g(z) := f(z)^{\frac{p}{2}}$ belongs to H^2 and, in addition to this, $\|f\|_{H^p}^p = \|g\|_{H^2}^2$. Consequently,

$$|f(\zeta)| = |g(\zeta)|^{\frac{2}{p}} \le \left(\frac{\|g\|_{H^2}}{\sqrt{1-|\zeta|^2}}\right)^{\frac{2}{p}} = \frac{\|f\|_{H^p}}{(1-|\zeta|^2)^{\frac{1}{p}}}.$$

3. Finally, take $f \in H^p$. Due to Riesz's factorization theorem, there exist a Blaschke product B and a non-vanishing function $g \in H^p$ such that $f = Bg$. Since B is an inner function,

$$|f(\zeta)| = |B(\zeta)g(\zeta)| \le |g(\zeta)| \le \frac{\|g\|_{H^p}}{(1-|\zeta|^2)^{\frac{1}{p}}} = \frac{\|f\|_{H^p}}{(1-|\zeta|^2)^{\frac{1}{p}}}.$$

   Equality is attained if and only if $|B(\zeta)| = 1$ and then the open mapping theorem yields that B is constant and $g(z) = (1-\overline{\zeta}z)^{\frac{-2}{p}}$.

□

Theorem D is a prime example of an *extremal problem*, that is, the precise computation of the supremum of some quantity (typically the modulus or the real part of some functional or the norm of an operator) under some constraints, as well as the determination (or not) of the *extremal functions*, *i.e.*, functions for which the supremum is attained. These problems are the main topic of this doctoral thesis. The specific results will be developed in the following chapters.

Thanks to its reproducing kernel properties for $p = 2$ and in honour of some of the analysts who pointed out its use, the function

$$k_\zeta(z) := \frac{1}{1 - \overline{\zeta} z}, \quad \forall z, \zeta \in \mathbb{D},$$

is called Szegő (or Riesz) kernel.

Once the pointwise estimates in H^p are known, a standard argument of normal families (using Montel's theorem) shows that $(H^p, \|\cdot\|_{H^p})$ is a Banach space if $p \geq 1$, whilst (H^p, d) is a complete metric space and invariant under translations if $0 < p < 1$.

With respect to the role of the polynomials, it can be proven that they are dense in H^p for any $0 < p < \infty$ [38, Theorem 3.3].

If $p \geq 1$, we can recover $f \in H^p$ from its boundary values and the *Poisson kernel*

$$P(r,t) := \operatorname{Re}\left\{\frac{1 + re^{it}}{1 - re^{it}}\right\} = \frac{1 - r^2}{1 - 2r\cos t + r^2}, \quad r \in [0,1), t \in [0, 2\pi].$$

That is, if $f \in H^p$ for some $1 \leq p$, then f can be written as the *Poisson-Stieltjes integral* of its boundary values (F. and M. Riesz, [87])

$$f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t - \theta) f(e^{i\theta})\, d\theta.$$

In other words, we can identify in a unique manner any function in H^p, $1 \leq p$, with the class of functions in $L^p(\mathbb{T})$ corresponding to its boundary values.

If $p \geq 1$, the *Fourier coefficients* of f with respect to the exponential system $\{e^{ikt}\}_{k \in \mathbb{Z}}$,

$$\tilde{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it}) e^{-ikt}\, dt, \quad \forall k \in \mathbb{Z},$$

are well defined. Due to the strong convergence to the boundary values (Theorem C) and the fact that f is analytic, we can characterise H^p, $p \geq 1$, as follows

$$H^p = \{f \in L^p(\mathbb{T}) \,:\, \tilde{f}(k) = 0, \forall k < 0\}. \tag{1.1}$$

For more information about H^p spaces, see the monographs [38], [49], [63], [66] or [80].

1.2 The weighted Bergman spaces

Another very natural class of spaces of analytic functions is that of the Bergman spaces, named after S. Bergman [18].

If $p \in (0,\infty)$ and $\alpha \in (-1,\infty)$, we define the *standard weighted Bergman space* A^p_α as follows:

$$A^p_\alpha := \left\{ f \in \mathscr{H}(\mathbb{D}) \, : \, \int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z) < \infty \right\},$$

where $dA(z) := \frac{r}{\pi}\, dr\, dt$ is the normalised Lebesgue measure in $\mathbb{D}$. In other words, $A^p_\alpha = \mathscr{H}(\mathbb{D}) \cap L^p(\mathbb{D}, \mu_\alpha)$ with

$$d\mu_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha \, dA(z).$$

If $\alpha = 0$, we will simply call *Bergman spaces* the family of spaces $A^p := A^p_0$.

More generally, given a positive function $\omega \in L^1([0,1])$, we can define the *weighted Bergman space* A^p_ω as the set of all analytic functions f such that

$$\|f\|^p_{A^p_\omega} := \int_{\mathbb{D}} |f(z)|^p \omega(|z|) \, dA(z) = \int_0^1 M_p^p(r,f) 2r\omega(r)\, dr.$$

Without loss of generality, we assume that $\displaystyle\int_0^1 2r\omega(r)\, dr = 1$, so $\|1\|_{A^p_\omega} = 1$.

In a similar way to L^p spaces, the expression

$$\|f\|_{A^p_\omega} := \left(\int_{\mathbb{D}} |f(z)|^p \omega(|z|)\, dA(z) \right)^{\frac{1}{p}},$$

defines a norm if $p \geq 1$, whilst $d(f,g) := \|f-g\|^p_{A^p_\omega}$ is a distance-invariant metric under translations if $0 < p < 1$. Due to the normalization imposed on ω and Hölder's inequality, it is clear that $\|f\|_{A^q_\omega} \leq \|f\|_{A^p_\omega}$ whenever $p > q$.

Since $|f|^p$ is a subharmonic function if $f \in \mathscr{H}(\mathbb{D})$ and $p > 0$, we can check that point evaluations are bounded in A^p_ω. Indeed, sub-mean value property and the monotonicity of M_p yield

$$|f(\zeta)|^p \leq \frac{1}{\rho^2} \int_{|z-\zeta|<\rho} |f|^p \, dA \leq \frac{8|\zeta|}{1-|\zeta|} M_p^p(|\zeta|+\rho, f), \quad \frac{1-|\zeta|}{2} < \rho < 1-|\zeta|,$$

if $\frac{1}{2} < |\zeta| < 1$ (it is enough to prove the boundedness in this case due to maximum modulus principle). Thus, because ω is a positive weight, we deduce the pointwise estimate

$$|f(\zeta)| \leq \left(\frac{4|\zeta|}{(1-|\zeta|)\int_{(1+|\zeta|)/2}^{1} r\omega(r)\,dr} \right)^{\frac{1}{p}} \|f\|_{A^p_\omega}.$$

In the case of A^p_α spaces the argument can be refined to get a sharp estimate.

Theorem E (Pointwise estimates in A^p_α, [76, 99])**.** *If $\zeta \in \mathbb{D}$ and $f \in A^p_\alpha$, then*

$$|f(\zeta)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|\zeta|^2)^{\frac{2+\alpha}{p}}}, \tag{1.2}$$

and equality is attained if and only if f is a multiple of

$$f_\zeta(z) := \frac{1}{(1-\overline{\zeta}z)^{\frac{2(2+\alpha)}{p}}}.$$

Using the dominated convergence theorem and characteristic functions of the disc tangent to the unit circle and centred at z, it can actually be proven that

$$|f(z)| = o\left(\frac{1}{(1-|z|^2)^{\frac{2(2+\alpha)}{p}}} \right), \quad |z| \to 1^-,$$

if $f \in A^p_\alpha$.

Again, Montel's theorem yields that A^p_ω is a closed subspace of $L^p(\mathbb{D}, \omega(|z|)\,dA(z))$. That is, A^p_ω is a Banach space if $p \geq 1$, whereas if $0 < p < 1$ it is a complete metric space whose metric is invariant under translations.

If $p = 2$, the spaces A^2_ω are actually *reproducing kernel Hilbert spaces* (*i.e.*, Hilbert spaces where point evaluations are continuous functionals) whose scalar products can be written in two equivalent ways:

$$\langle f, g \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(|z|)\,dA(z) = \sum_{n=0}^{\infty} \int_0^1 2r^{2n+1}\omega(r)\,dr\, a_n\overline{b_n}.$$

In the special case of standard weights,

$$\langle f, g \rangle_{A^2_\alpha} = \int_{\mathbb{D}} f(z)\overline{g(z)}\,d\mu_\alpha(z) = \sum_{n=0}^{\infty} \frac{a_n\overline{b_n}}{c_{\alpha+2}(n)},$$

where $\{c_{\alpha+2}(n)\}_{n\geq 0}$ are the binomial coefficients introduced in (2) of the Notation chapter.

The *reproducing kernel* for A^2_α, that is, the function $K_\zeta \in A^2_\alpha$ such that $f(\zeta) = \langle f, K_\zeta \rangle_{A^2_\alpha}$ for every $f \in A^2_\alpha$, can be explicitly computed in this case

$$K_\zeta(z) = \frac{1}{(1-\overline{\zeta}z)^{\alpha+2}}, \quad \forall z, \zeta \in \mathbb{D},$$

and it is known as the *Bergman kernel*.

The reader should be aware that some of the properties of H^p spaces previously discussed cannot be extended to A^p_ω, even for the standard weights.

With respect to the existence of non-tangential limits, we can find functions whose properties are diametrically opposed to H^p spaces. Due to the works of R. E. A. C. Paley and A. Zygmund [77, 78, 104], we know that there exists a sequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ with $\inf\left\{\frac{n_{k+1}}{n_k} : k \geq 1\right\} > 1$ and $\{\varepsilon_k\}_{k\geq 1} \subset \{-1, 1\}$ such that the *lacunary series* (or *Hadamard gap series*)

$$f(z) = \sum_{k=1}^{\infty} \varepsilon_k z^{n_k}, \tag{1.3}$$

belongs to A^p_α but it has radial limits almost nowhere in $\mathbb{T}$. See [38, Theorem A.5] or [43, Section 3.2] for further details.

The characterisation of the *zero sets* of A^p_α functions is still an open problem, although we know enough to show that there are notable differences with H^p.

More precisely, we say that a sequence $\{z_n\}_{n\geq 1} \subset \mathbb{D}$ is a zero set of A^p_α if there exists $f \in A^p_\alpha$ such that f does not vanish in $\mathbb{D} \setminus \{z_n\}_{n\geq 1}$ and the multiplicity of the zero $\zeta \in \{z_n\}_{n\geq 1}$ is equal to the number of indices n such that $\zeta = z_n$. For H^p spaces, we know that their zero-sets coincide with the set of Blaschke sequences, independently of the value $p > 0$.

As a part of his doctoral thesis and its related works, C. Horowitz [61, 62] proved that, if $q < p$, there exist A^q zero sets which are not A^p zero sets. Thus, there is a dependence with respect to the exponent of the space. Due to Jensen's formula it can be deduced that

$$\sum_{n=1}^{\infty} (1-|z_n|)\left(\log \frac{1}{1-|z_n|}\right)^{-1-\varepsilon} < \infty, \quad \forall \varepsilon > 0, \tag{1.4}$$

if $\{z_n\}_{n\geq 1}$ is an A^p zero set, and therefore it is "almost" a Blaschke sequence. However, the property (1.4) cannot be extended to $\varepsilon = 0$ (see [43], Corollary 3 in Chapter 4).

The arguments given by C. Horowitz can be adapted to other values of α, because all A^p_α have similar growth estimates (Theorem E).

However, the necessary condition (1.4) cannot be sufficient. As a consequence of a more general result by H. S. Shapiro and A. L. Shields [92], if a non-identically zero function $f \in A^p_\alpha$ verifies that its zero set is included in a radius $\{re^{it} \ : \ 0 \ \leq \ r \ < \ 1\}$, then it is necessarily a Blaschke sequence. Thus, an hypothetical geometric characterization of A^p_α zero sets cannot depend exclusively on the moduli of the roots.

There is no analogous result to Reisz factorization in A^p_α. That is, given an A^p_α zero set $\{z_n\}_{n\geq 1}$ there does not exist $G \in A^p_\alpha$ such that for every $f \in A^p_\alpha$ which vanishes in $\{z_n\}_{n\geq 1}$ we have that $\mathscr{M}_{G^{-1}} f \in A^p_\alpha$ and moreover

$$\|\mathscr{M}_{G^{-1}} f\|_{A^p_\alpha} = \|f\|_{A^p_\alpha}.$$

Indeed, A. Aleman, P. L. Duren, M. J. Martín and D. Vukotić [4] proved that, among all the spaces of analytic functions introduced in this dissertation, the H^p spaces are the only ones with isometric zero divisors.

However, for the unweighted case ($\alpha = 0$) it is known that there exists a *contractive divisor*, in other words, a function G whose associated operator $\mathscr{M}_{G^{-1}}$ has norm equal to 1.

The origin of this theory is due to H. Hedenmalm [53] for A^2, and was extended to other values of p by P. L. Duren, D. Khavinson, H. S. Shapiro and C. Sundberg [40, 41, 42].

Although the terminology might suggest the opposite, we must stress that these divisors (which are unique whenever $p \geq 1$, since A^p_α is strictly convex in this case) are not characterised by the contractivity of $\mathscr{M}_{G^{-1}}$, but they are extremal functions for the extremal problem

$$\sup\left\{\operatorname{Re} f^{(k)}(0) \ : \ \|f\|_{A^p_\alpha} \leq 1, f \text{ vanishes at } \{z_n\}_{n\geq 1}\right\}, \tag{1.5}$$

where $k = k(\{z_n\}_{n\geq 1})$ is the smallest integer such that $f^{(k)}(0) \neq 0$ for some $f \in A^p_\alpha$ whose zero set includes $\{z_n\}_{n\geq 1}$.

Actually, the weight is important in order to get the contractivity, because not every A^p_α has this property. In the Hilbert space framework, it is known that the contractivity is lost in A^2_α if $\alpha > 1$ (H. Hedenmalm and K. Zhu [58]) whilst it is preserved if $-1 < \alpha \leq 1$ (H. Hedenmalm, $\alpha = 1$, [54], S. M. Shimorin [93, 94]).

Despite these differences, the Hardy spaces H^p are, in some sense, the limit of the weighted Bergman spaces A^p_α. That is, if $f \in H^p$ [103] then

$$\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}. \tag{1.6}$$

For further information about the vast theory of Bergman spaces, we refer the reader to the monographs [43] and [55].

1.3 The mixed norm spaces $H(p,q,a)$

A family of spaces which generalises standard weighted Bergman spaces are the *mixed norm spaces*. Given $0 < p,q \le \infty$ and $a > 0$, we define the space $H(p,q,a)$ as

$$H(p,q,a) := \left\{ f \in \mathscr{H}(\mathbb{D}) \,:\, \|f\|_{p,q,a} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{p,q,a} &:= \left(aq \int_0^1 2r(1-r^2)^{aq-1} M_p^q(r,f)\, dr \right)^{\frac{1}{q}}, \quad 0 < q < \infty, \\ \|f\|_{p,\infty,a} &:= \sup_{r\in[0,1)} \left\{ (1-r^2)^a M_p(r,f) \right\}. \end{aligned}$$

With this notation, it follows that

$$A^p_\alpha = H\left(p,p,\frac{\alpha+1}{p}\right),$$

and, in addition to this, $\|f\|_{A^p_\alpha} = \|f\|_{p,p,\frac{\alpha+1}{p}}$ for every analytic function f.

The spaces $H(p,q,a)$ also generalise the so-called *growth spaces* or *Korenblum spaces*

$$\mathscr{A}^{-a} := \left\{ f \in \mathscr{H}(\mathbb{D}) \,:\, \sup_{z\in\mathbb{D}} \left\{ |f(z)|(1-|z|^2)^a \right\} < \infty \right\} = H(\infty,\infty,a), \quad a > 0,$$

which are closely related to the space A^p_α.

Remark. *The Hardy space H^p cannot be identified with any mixed norm space, because it should correspond to $H(p,\infty,0)$, but the third parameter must be positive. However, it is the limit of mixed norm spaces in the sense of* (1.6).

Point evaluations are uniformly bounded in compact sets of $\mathbb{D}$ (see for example [12, Proposition 4.9] or [63, Proposition 7.1.1]). By the same arguments as in the previous sections, $H(p,q,a)$ is a complete space. If $p,q \ge 1$, then the mapping $\|\cdot\|_{p,q,a}$ is actually a norm and therefore $(H(p,q,a), \|\cdot\|_{p,q,a})$ is a Banach space.

Although the integrals from the definition of these spaces appeared in the work of G. H. Hardy and J. E. Littlewood [51], the $H(p,q,a)$ spaces were introduced as such by T. M. Flett [45, 46]. The results about $H(p,q,a)$ are really scattered in the literature. More information about mixed norm spaces can be found in Chapter 7 of [63].

1.4 The Bloch space $\mathscr{B}$

Another way of defining spaces of analytic functions consists of imposing conditions on the derivatives, typically such as asking for their membership in other classical space of analytic functions. However, the outcome of this method is not necessarily a "new" space, that is, a set of functions which is not included in other classification. For example, as a consequence of a remarkable work of G. H. Hardy and J. E. Littlewood [51] (see also [38, Theorem 5.5]) it follows that

$$f \in H(p,\infty,a) \quad \Longleftrightarrow \quad f' \in H(p,\infty,a+1),$$

if $p, a > 0$, and therefore the space $\{f \in \mathscr{H}(\mathbb{D}) \,:\, f' \in H(p,\infty,a+1)\}$ will be, once it is endowed with a topology, isomorphic to $H(p,\infty,a)$.

Likewise, the *Littlewood-Paley identity*:

$$\|f\|_{H^2}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|^2}\, dA(z), \quad \forall f \in \mathscr{H}(\mathbb{D}), \tag{1.7}$$

is well known. As a consequence, $f \in H^2$ if and only if $f' \in A^2_\omega$, where $\omega(r) = -2\log r$. See [81] for further information about how to characterise spaces of analytic functions by means of conditions on the derivatives.

One of the most important spaces that can be defined by this method is the *Bloch space* $\mathscr{B}$, namely, the set of all holomorphic functions in $\mathbb{D}$ such that its *Bloch seminorm*,

$$\rho_{\mathscr{B}}(f) := \sup_{z\in\mathbb{D}}\{|f'(z)|(1-|z|^2)\},$$

is finite. In other words, $f \in \mathscr{B}$ if and only if $f' \in H(\infty,\infty,1)$.

The origin of this class of functions is a theorem by A. Bloch [22], which states that there exists an absolute constant δ such that for every function f analytic in a domain $\Omega \supset \overline{\mathbb{D}}$ with the normalisation $f(0) = 0$ and $f'(0) = 1$ verifies that there exists an open set $\Omega' \subset \Omega$ where f is univalent (*i.e.*, analytic and one-to-one) in Ω' and, furthermore, $f(\Omega')$ contains an euclidean disc of radius δ. The supremum of these quantities δ is called the *Bloch constant*, and it is represented with the letter B in the specialised bibliography.

E. Landau [68] proved that the regularity of f could be weakened to $f \in \mathscr{H}(\mathbb{D})$, and proposed the use of the Bloch functions (term coined by Ch. Pommerenke in 1970 [82]) in order to estimating B. Moreover, he considered the following related problem

$$L := \inf\left\{R_f \,:\, f(0) = 0, f'(0) = 1\right\},$$

where R_f is the radius of the largest disc included in $f(\mathbb{D})$, $f \in \mathcal{H}(\mathbb{D})$. This value L is known as the *Landau constant*.

On the date of this dissertation, the explicit values of B and L are unknown but they can be estimated with an error reasonably small.

On the one hand, it is known that

$$0.433\ldots = \frac{\sqrt{3}}{4} \leq B \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)} = 0.471\ldots$$

The bound $\frac{\sqrt{3}}{4}$ was deduced by L. V. Ahlfors [2], and it is known that it is not sharp (M. Heins, [59]). Later it was explicitly improved by M. Bonk [28] (and H. Chen and P. M. Gauthier [33] afterwards) but, since the difference is minimal, for the sake of clarity we show the bound deduced by L. V. Ahlfors in this dissertation.

The upper bound for B is due to L. V. Ahlfors and H. Grunsky [3], and it is conjectured to be sharp (see the first section in Chapter 12 of [35], for example).

On the other hand, the Landau constant L can be estimated as follows:

$$\frac{1}{2} \leq L \leq \sqrt[3]{4}\pi \frac{\Gamma^3\left(\frac{1}{3}\right)}{\Gamma^4\left(\frac{1}{4}\right)} = 0.554\ldots$$

These lower and upper bounds were obtained by L. V. Ahlfors [2] and E. Landau [68], respectively. Observe that, in particular, $B < L$.

Turning to the properties of $\mathcal{B}$, it is elementary to check that $\rho_{\mathcal{B}}$ is indeed a seminorm and, moreover, $\rho_{\mathcal{B}}(f) = 0$ if and only if f is constant. The easiest way to equip $\mathcal{B}$ with a norm is the following:

$$\|f\|_{\mathcal{B}} := |f(0)| + \rho_{\mathcal{B}}(f).$$

It follows from the definition of $\rho_{\mathcal{B}}$ that point evaluations are locally bounded in $\mathcal{B}$.

Theorem F (Pointwise bounds in $\mathcal{B}$, [7, 55])**.** *If $f \in \mathcal{H}(\mathbb{D})$, then the following sharp inequality holds*

$$|f(\zeta) - f(0)| \leq \frac{1}{2} \log\left(\frac{1+|\zeta|}{1-|\zeta|}\right) \rho_{\mathcal{B}}(f). \tag{1.8}$$

Observe that Theorem F yields that $\mathcal{B} \subset A^p_\alpha$ for any $p > 0$ and $\alpha > -1$, and furthermore $H^p \not\subset \mathcal{B}$, $0 < p < \infty$, ($f(z) = (1-z)^{-\gamma} \in H^p$ if $0 < \gamma < \frac{1}{p}$). On the other hand, a lacunary series belongs to $\mathcal{B}$ if, and only if, its Taylor coefficients are bounded (see [7, 82] or Theorem 7.6.4 from [63]), so the function introduced in (1.3) is included in the Bloch space and has radial limits almost nowhere. Thus, $\mathscr{B} \not\subset H^p$ for every $p > 0$. The inclusion $\mathscr{B} \subset A^p_\alpha$ will be covered in detail later.

It is well-known that $\mathscr{B}$ is a Banach space when equipped with the norm $\|\cdot\|_{\mathscr{B}}$ (see Proposition 2 in Chapter 2 of [43], for example).

Just like H^∞, $\mathscr{B}$ is not separable [7] but it contains a closed space which is separable. We define the *little Bloch space* B_0 as the closure of the polynomials with respect to the topology induced by $\|\cdot\|_{\mathscr{B}}$,

$$\mathscr{B}_0 := \overline{\mathbb{C}[z]}^{\|\cdot\|_{\mathscr{B}}}.$$

Equivalently, $\mathscr{B}_0$ can be characterised by the following geometric condition [7]:

$$f \in \mathscr{B}_0 \quad \Longleftrightarrow \quad \lim_{|z|\to 1^-} \left\{ |f'(z)|(1-|z|^2) \right\} = 0.$$

Two fundamental references to understand the Bloch space $\mathscr{B}$ are the seminal papers of Ch. Pommerenke [82] and J. M. Anderson, J. Clunie and Ch. Pommerenke [7].

The Bloch space is closely linked with univalent functions. That is, for any $f \in \mathscr{B}$ there exist a univalent function g and $a \in \mathbb{C}$ such that $f = a \log g'$ [82]. Thus, some sections about the Bloch space can be found in books devoted to univalent functions (like [83, 84]).

$\mathscr{B}$ and $\mathscr{B}_0$ take an important place in the duality of A^p_α (*e.g.*, $\mathscr{B}$ can be identified with the dual of A^1 and, in turn, the dual of $\mathscr{B}_0$ is isomorphic to A^1) and in the study of the *Bergman projection*

$$P_\alpha(f)(z) := \langle f, K_z \rangle_{A^2_\alpha} = \int_{\mathbb{D}} \frac{f(\zeta)}{(1-z\overline{\zeta})^{\alpha+2}} \, d\mu_\alpha(\zeta), \quad \forall z \in \mathbb{D}.$$

For these reasons, $\mathscr{B}$ can be found in several monographs dedicated to A^p_α spaces, like [43], [55], [63] o [102].

1.5 The analytic Besov spaces B^q

Finally, if $q \in (1, \infty)$ the (analytic) Besov space B^q is defined as

$$\begin{aligned} \rho_q(f) &:= \left((q-1) \int_{\mathbb{D}} (1-|z|^2)^{q-2} |f'(z)|^q \, dA(z) \right)^{\frac{1}{q}}, \\ B^q &:= \left\{ f \in \mathscr{H}(\mathbb{D}) : \rho_q(f) < \infty \right\}. \end{aligned}$$

That is, $f \in B^q$ if, and only if, $f' \in A^q_{q-2}$.

Similarly to $\mathscr{B}$, ρ_q is a (complete) seminorm in B^q. The space B^q is a Banach space when equipped with one of the following norms:

$$\begin{aligned} \|f\|_{B^q,1} &:= |f(0)| + \rho_q(f), \\ \|f\|_{B^q,2} &:= \left(|f(0)|^q + \rho_q(f)^q\right)^{\frac{1}{q}}, \end{aligned}$$

although this second expression is used when emphasizing that B^q is a *Dirichlet type* space (see [34], for example). They are called this way because the particular case B^2 is known as the *Dirichlet space* and it is often denoted by $\mathscr{D}$. $(\mathscr{D}, \|\cdot\|_{B^2,2})$ is a Hilbert space and, like H^2 and A^2_α, the norm $\|f\|_{B^2,2}$ can be computed in terms of the Taylor coefficients of f

$$\|f\|_{\mathscr{D}} := \|f\|_{B^2,2} = \sqrt{|a_0|^2 + \sum_{n=1}^{\infty} n|a_n|^2}.$$

The norms $\|\cdot\|_{B^q,1}$ and $\|\cdot\|_{B^q,2}$ actually are equivalent because of the following straightforward estimates

$$\|f\|_{B^q,2} \leq \|f\|_{B^q,1} \leq 2^{\frac{q-1}{q}} \|f\|_{B^q,2}, \quad \forall f \in B^q.$$

Since $M_1(r, f')$ is an increasing function of r, $M_1(r, f')(1-r^2)^{-1}$ is integrable if and only if f is constant, and thus B^1 requires a different definition. B^1 is defined as the set of all functions $f \in \mathscr{H}(\mathbb{D})$ that can be represented as

$$f(z) = \sum_{k=1}^{\infty} b_k \varphi_{a_k}(z), \tag{1.9}$$

where $\{b_k\}_{k\geq 1} \in \ell^1$, $\{a_k\}_{k\geq 1} \subset \mathbb{D}$ and φ_a is the disc automorphism introduced in (1).

It can be proven that $(B^1, \|\cdot\|_{B^1})$, endowed with the norm

$$\|f\|_{B^1} := \inf\left\{ \|\{c_n\}_{n\geq 1}\|_{\ell^1} \;:\; f(z) = \sum_{n=1}^{\infty} c_n \varphi_{a_n}(z) \right\},$$

is a Banach space. Alternatively, $f \in B^1$ if and only if $f'' \in A^1$ [9] and, in addition to this,

$$\|f''\|_{A^1} \approx \|f - f(0) - f'(0)z\|_{B^1}.$$

From this point of view, the connection of B^1 with the other Besov spaces is more clear.

One of the main reasons for which the B^q spaces have been studied is that they are *(strictly) conformally invariant*. That is, $\rho_q(f) = \rho_q(f \circ \varphi)$ for any $f \in B^q$ and ever disc automorphism φ. Indeed, J. Arazy, S. D. Fisher and J. Peetre [9] proved that B^1 is minimal among conformally invariant spaces that satisfy some basic conditions.

For further bibliography about Besov spaces, check Section 5.3 of [102].

Chapter 2

Extremal problems for vanishing functions in Bergman spaces

Probably one of the most frequently studied operators acting on spaces of analytic functions is the *shift operator*, which can be written as

$$\mathscr{M}_z(f)(z) = zf(z), \quad \forall z \in \mathbb{D}.$$

It is trivial that $\mathscr{M}_z$ is a linear operator. Furthermore, if X is one of the spaces introduced in Chapter 1, then $\mathscr{M}_z X \subset X$ and thus it is continuous due to closed graph theorem.

The structure of (closed) invariant subspaces of this operator is of great interest in Functional Analysis. For H^p spaces, it was described in a seminal paper by A. Beurling for $p = 2$ [19, Theorem IV] and generalized by T. P. Srinivasan and J.-K. Wang [95]. However, on Bergman spaces the situation is much more complicated, as was shown by C. Apostol, H. Bercovici, C. Foiaș and C. Pearcy [8]. A partial but important analogue of Beurling's theorem for the Bergman space was proved in 1996 by A. Aleman, S. Richter and C. Sundberg [5]. We refer the reader to [57] for an alternative exposition of the one given in [8].

A relevant property used in [57] is that, if M is a closed subspace, then $\mathscr{M}_z M$ is also closed if and only if $\mathscr{M}_z$ is bounded from below (here again closed graph theorem plays an important role), a fact that is true for A^p_α and it is widely known by the experts in the area.

However, during the elaboration of this doctoral thesis we were not able to find in the bibliography the explicit computation of the smallest constant $C = C(p, \alpha)$ such that

$$\|f\|_{A^p_\alpha} \leq C \|\mathscr{M}_z f\|_{A^p_\alpha}, \quad \forall f \in A^p_\alpha,$$

or equivalently

$$\|\mathscr{M}_{z^{-1}} f\|_{A^p_\alpha} \leq C \|f\|_{A^p_\alpha}, \quad \forall f \in A^p_\alpha, f(0) = 0, \tag{2.1}$$

where

$$\mathscr{M}_{z^{-1}}(f)(z) = \frac{f(z)}{z}, \quad \forall z \in \mathbb{D}.$$

is the restriction (to the subspace of functions that vanish at the origin) of the *backward shift operator*, which maps f to $\frac{f(z)-f(0)}{z}$.

This is the starting point of this chapter. Our main objective consists of solving the extremal problem (2.1), as well as computing the precise norm of some similar operators in Bergman-type spaces, deducing as corollaries new proofs for results previously known.

The content of this chapter is based on the publication [73].

2.1 A sharp estimate for dividing out a single Blaschke factor

Let $\omega \in L^1([0,1])$ be positive and $p > 0$, and consider the corresponding A^p_ω space introduced in Section 1.2.

We are not just going to compute the norm of the backward shift in A^p_ω, but we are going to extend it to the family of operators by which we cancel a zero of arbitrary order at the origin.

Theorem 1. *If $k \geq 1$, then*

$$\|\mathscr{M}_{z^{-k}} f\|_{A^p_\omega} \leq \left(\frac{\int_0^1 r\omega(r)\, dr}{\int_0^1 r^{kp+1}\omega(r)\, dr} \right)^{\frac{1}{p}} \|f\|_{A^p_\omega},$$

whenever f has a zero of multiplicity at least k at the origin. Moreover, equality holds if and only if $f(z) = cz^k$ for some $c \in \mathbb{C}$.

Proof. Suppose that $f \in A^p_\omega$ has a zero of order at least k at $z = 0$. Then, there exists $g \in \mathscr{H}(\mathbb{D})$ such that

$$f(z) = z^k g(z).$$

That is, $g = \mathscr{M}_{z^{-k}} f$. It is easy to check that $g \in A^p_\omega$, because for any $\rho \in (0,1)$

$$\int_{\mathbb{D}\setminus\rho\mathbb{D}} |g(z)|^p \omega(|z|)\, dA(z) \leq \frac{1}{\rho^{kd}} \int_{\mathbb{D}\setminus\rho\mathbb{D}} |f(z)|^p \omega(|z|)\, dA(z) < \infty,$$

whilst the integral over the disc $\rho\mathbb{D}$ is clearly bounded.

Take the probability measure $d\sigma(r) := \frac{r\omega(r)\,dr}{\int_0^1 x\omega(x)\,dx}$. Since $g \in \mathscr{H}(\mathbb{D})$, we know that $M_p(r,g)$ is an increasing function of r and therefore we can use Chebyshev's inequality (see Appendix A.3) to get the desired bound

$$\begin{aligned} \|f\|^p_{A^p_\omega} &= 2\left(\int_0^1 r\omega(r)\,dr\right)\left(\int_0^1 r^{kp} M_p^p(r,g)\,d\sigma(r)\right) \\ &\geq 2\left(\int_0^1 r\omega(r)\,dr\right)\left(\int_0^1 r^{kp}\,d\sigma(r)\right)\left(\int_0^1 M_p^p(r,g)\,d\sigma(r)\right) \\ &= \frac{\int_0^1 r^{kp+1}\omega(r)\,dr}{\int_0^1 r\omega(r)\,dr}\|g\|^p_{A^p_\omega}. \end{aligned}$$

Since r^{kp} is a strictly increasing function, equality is attained if and only if $M_p(r,g)$ is constant, but this is possible only if g is constant. Hence, equality holds if and only if f is a monomial of degree k. $\square$

If ω is a standard weight (*i.e.*, if $\|\cdot\|_{A^p_\omega} = c\|\cdot\|_{A^p_\alpha}$ for some $\alpha > -1$ and $c > 0$) the structure of the weight allows us to "move" the zero to any arbitrary point $a \in \mathbb{D}$ using operators based on conformal mappings. More specifically, we will show that we can substitute z^k with the *Blaschke factor* $\varphi_a^k(z)$, $a \in \mathbb{D}$, without changing the corresponding operator norm.

Theorem 2. *Let $p > 0$, $\alpha > -1$ and $a \in \mathbb{D}$. Then, for every $k \geq 1$*

$$\left\|\mathscr{M}_{\varphi_a^{-k}} f\right\|_{A^p_\alpha} \leq \left(\frac{\Gamma\left(\frac{kp}{2}+\alpha+2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)}\right)^{\frac{1}{p}} \|f\|_{A^p_\alpha},$$

whenever $f(a) = \ldots = f^{(k-1)}(a) = 0$.

In addition, equality is attained if and only if

$$f(z) = c(1-|a|^2)^{\frac{\alpha+2}{p}} \frac{(a-z)^k}{(1-\overline{a}z)^{k+\frac{2(\alpha+2)}{p}}},$$

for some constant c.

Proof. If $a = 0$ and $f(0) = \ldots = f^{(k-1)}(0) = 0$, the estimate

$$\|\mathscr{M}_{z^{-k}} f\|_{A^p_\alpha} \leq \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \|f\|_{A^p_\alpha},$$

is consequence of the general case covered in Theorem 1.

If $a \neq 0$, the automorphism φ_a is an *involution* (that is, its inverse function coincides with itself), $\varphi_a(0) = a$ and, in addition to this, the operator

$$T_a f(z) := \left(\varphi_a'(z)\right)^{\frac{\alpha+2}{p}} f\left(\varphi_a(z)\right), \tag{2.2}$$

preserves the norm in A^p_α.

That is, using the change of variables $z = \varphi_a(\zeta)$, since the identities $\varphi_a' \circ \varphi_a = \frac{1}{\varphi_a'}$ and $1 - |\varphi_a(\zeta)|^2 = |\varphi_a'(\zeta)|\,(1-|\zeta|^2)$ hold in $\mathbb{D}$, we have

$$\begin{aligned}
\|T_a f\|^p_{A^p_\alpha} &= \int_{\mathbb{D}} |\varphi_a'(z)|^{\alpha+2} \left| f\left(\varphi_a(z)\right)\right|^p d\mu_\alpha(z) \\
&= (\alpha+1) \int_{\mathbb{D}} (1-|\varphi(\zeta)|^2)^\alpha \left|\varphi_a'\left(\varphi_a(\zeta)\right)\right|^{\alpha+2} |f(\zeta)|^p |\varphi_a'(\zeta)|^2 \, dA(\zeta) \\
&= \int_{\mathbb{D}} |f(\zeta)|^p \, d\mu_\alpha(\zeta) = \|f\|^p_{A^p_\alpha}.
\end{aligned}$$

It is easy to check that $T_a \mathscr{M}_{\varphi_a^{-k}} = \mathscr{M}_{z^{-k}} T_a$. Indeed,

$$\begin{aligned}
T_a \mathscr{M}_{\varphi_a^{-k}} f(z) &= \left(\varphi_a'(z)\right)^{\frac{\alpha+2}{p}} \mathscr{M}_{\varphi_a^{-k}} f\left(\varphi_a(z)\right) \\
&= \left(\varphi_a'(z)\right)^{\frac{\alpha+2}{p}} \frac{f\left(\varphi_a(z)\right)}{z^k} \\
&= \mathscr{M}_{z^{-k}} T_a f(z).
\end{aligned}$$

Finally, if $f \in A^p_\alpha$ has a zero of order at least k in a, we have

$$\begin{aligned}
\left\| \mathcal{M}_{\varphi_a^{-k}} f \right\|_{A^p_\alpha} &= \left\| T_a \mathcal{M}_{\varphi_a^{-k}} f \right\|_{A^p_\alpha} \\
&= \| \mathcal{M}_{z^{-k}} T_a f \|_{A^p_\alpha} \\
&\leq \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \| T_a f \|_{A^p_\alpha} \\
&= \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \| f \|_{A^p_\alpha} .
\end{aligned}$$

Theorem 1 yields that equality is possible if and only if there exists $c \in \mathbb{C}$ such that

$$T_a f(\zeta) = c\zeta^k,$$

and since φ_a is a disc automorphism,

$$f\big(\varphi_a(\zeta)\big) = c\zeta^k \big(\varphi_a'(\zeta)\big)^{-\frac{\alpha+2}{p}} \quad \Longrightarrow \quad f(z) = c\varphi_a^k(z)\big(\varphi_a'(z)\big)^{\frac{\alpha+2}{p}},$$

from where we deduce the expression for the extremal functions. □

Remark. *The operators introduced in* (2.2) *have not been chosen on a mere whim, but they are quite natural in the theory of weighted Bergman spaces* A^p_α.

If $p \neq 2$ *(that is, in the non-Hilbert space framework), F. Forelli [47] and W. Rudin [90] proved that the unique surjective isometries of* H^p *(we recall that, in the sense of* (1.6)*,* H^p *can be understood as* A^p_{-1}*) are given by* (2.2)*, whilst C. J. Kolaski [64, 65] extended the result to* A^p_α.

2.2 A sharp estimate for the derivative

As an application of Theorem 2, we can estimate precisely the k-th derivative at the point a of a function $f \in A^p_\alpha$ such that $f^{(j)}(a) = 0$ for every $j \in \{0, \dots, k-1\}$. For $p \geq 1$ and $\alpha = 0$, these estimates were deduced independently by K. Yu. Osipenko and M. I. Stessin [76] and D. Vukotić [100]. The proof given below considerably simplifies both proofs, and also extends the result for every A^p_α even for $p < 1$.

Corollary 3. *Let* $p > 0$, $\alpha > -1$ *and* $a \in \mathbb{D}$. *If* $f \in A^p_\alpha$ *has a zero of order at least* k *in* a, *then*

$$\left| f^{(k)}(a) \right| \leq k! \left(\frac{\Gamma\left(\frac{kp}{2} + \alpha + 2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)} \right)^{\frac{1}{p}} \frac{\| f \|_{A^p_\alpha}}{(1-|a|^2)^{k+\frac{\alpha+2}{p}}},$$

and this inequality is sharp.

Proof. First, since $f(a) = \ldots = f^{(k-1)}(a) = 0$, it is elementary to check

$$\lim_{z\to a} \frac{f(z)}{(z-a)^k} = \frac{f^{(k)}(a)}{k!}.$$

Due to Theorem 2 and the pointwise estimates for A^p_α (Theorem E), we have

$$\begin{aligned}\left|\frac{f(z)}{\varphi_a^k(z)}\right| &\leq \frac{\left\|\mathscr{M}_{\varphi_a^k} f\right\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}}} \\ &\leq \left(\frac{\Gamma\left(\frac{kp}{2}+\alpha+2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)}\right)^{\frac{1}{p}} \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}, \quad \forall z \in \mathbb{D}.\end{aligned}$$

Finally, taking limits as $z \to a$ we deduce

$$\frac{|f^{(k)}(a)|(1-|a|^2)^k}{k!} \leq \left(\frac{\Gamma\left(\frac{kp}{2}+\alpha+2\right)}{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)}\right)^{\frac{1}{p}} \frac{\|f\|_{A^p_\alpha}}{(1-|a|^2)^{\frac{\alpha+2}{p}}}.$$

A straightforward computation shows that the upper bound is attained for

$$f(z) = c\varphi_a^k(z)\left(\varphi_a'(z)\right)^{\frac{\alpha+2}{p}}, \quad \forall z \in \mathbb{D},$$

and therefore the estimate is sharp. □

Remark. *Theorem 2 yields that the function*

$$(-1)^k \left(\frac{\Gamma(\alpha+2)\Gamma\left(\frac{kp}{2}+1\right)}{\Gamma\left(\frac{kp}{2}+\alpha+2\right)}\right)^{\frac{1}{p}} \varphi_a^k(z),$$

*(and consequently any of its rotations) is a contractive divisor in A^p_α for the class of functions that vanish on the finite set $z_1 = \ldots = z_k = a$. However, this does not mean that this function is **the** contractive divisor defined as the extremal function of* (1.5). *Corollary 3 yields that the two concepts coincide if $a = 0$, but in general this is not true.*

Chapter 3

Inclusions between some spaces of different type

The inclusions between the spaces introduced in Chapter 1 have been widely studied, so they are completely understood. Since all these spaces are complete, normal families and closed graph theorem (Theorem 2.15, [91]) yield that the inclusion $X \subset Y$ implies that the inclusion operator $\iota(f) := f$ is continuous and hence bounded. That is, there exists a constant $M > 0$ such that

$$\|f\|_Y \leq M\|f\|_X, \quad \forall f \in X.$$

In general, the *norm of the inclusion*

$$\|\iota\| := \sup\{\|f\|_Y \,:\, \|f\|_X = 1\},$$

is not known. The computation or estimate (as precise as possible) of this norm are the main objectives in this chapter.

Another topic covered in this chapter is the extremal functions. When possible, we try to determine their other properties such as their uniqueness, integral conditions they must satisfy or, in the best scenario, an explicit expression for extremal functions.

Clearly $\|\iota\|$ depends on the choice of $\|\cdot\|_X$ and $\|\cdot\|_Y$. We restrict ourselves to the norms (or distances) introduced in Chapter 1.

The content of this chapter is based on the publication [70].

3.1 The inclusion of B^q in $\mathscr{B}$

There are many ways of proving that $B^q \subset \mathscr{B}$. One way is to show that B^q is (strictly) conformally invariant (*i.e.*, $\rho_q(f) = \rho_q(f \circ \varphi)$ for every $f \in B^q$ and φ a disc automorphism) and using the fact that $\mathscr{B}$ is (under some conditions) maximal in this class of spaces [88], as a counterpart of B^1, which is minimal [9]. However, it is much easier to prove that $B^q \subset \mathscr{B}$ via the pointwise bounds of A^p_α functions (Theorem E). If $f \in B^q$ for some $q > 1$, then

$$|f'(z)|(1-|z|^2) \leq \|f'\|_{A^q_{q-2}}, \quad \forall z \in \mathbb{D}, \tag{3.1}$$

and therefore $f \in \mathscr{B}$.

If $f \in B^1$ then f is indeed a bounded analytic function. The well-known Schwarz-Pick lemma yields that $H^\infty \subset \mathscr{B}$. Thus $B^1 \subset \mathscr{B}$.

At this point, we are able to compute the norm of the inclusion $B^q \subset \mathscr{B}$ for every $q \geq 1$.

Proposition 4.

(a) Let $q > 1$. Then

$$\|f\|_{\mathscr{B}} \leq \|f\|_{B^q,1} \quad \textit{and} \quad \|f\|_{\mathscr{B}} \leq 2^{\frac{q-1}{q}} \|f\|_{B^q,2}.$$

In addition, if the equality is attained in any of the above inequalities, then f must be of either of the following forms

$$\begin{aligned} f_0(z) &:= \alpha z + \beta, \\ f_\zeta(z) &:= \gamma \frac{1-|\zeta|^2}{\overline{\zeta}} \frac{1}{1-\overline{\zeta} z} + \delta, \quad 0 < |\zeta| < 1, \end{aligned}$$

for some suitable constants α, β, γ and δ.

(b) The following sharp estimate holds

$$\|f\|_{\mathscr{B}} \leq 2\|f\|_{B^1}, \quad \forall f \in B^1,$$

but equality is not attained for any non-identically zero function.

Proof.

(a) If $f \in B^q$, then (3.1) implies

$$\|f\|_{\mathscr{B}} = |f(0)| + \rho_{\mathscr{B}}(f) \le |f(0)| + \|f'\|_{A^q_{q-2}} = \|f\|_{B^q,1} \le 2^{\frac{q-1}{q}}\|f\|_{B^q,2}.$$

Note that if either $\|f\|_{\mathscr{B}} = \|f\|_{B^p,1}$ or $\|f\|_{\mathscr{B}} = 2^{\frac{q-1}{q}}\|f\|_{B^q,2}$, then necessarily $\rho_{\mathscr{B}}(f) = \|f'\|_{A^q_{q-2}}$. But polynomials are dense in B^q and the inclusion operator is continuous, so indeed $B^q \subset \mathscr{B}_0$. Thus, there exists $\zeta \in \mathbb{D}$ such that

$$|f'(\zeta)|(1-|\zeta|^2) = \rho_{\mathscr{B}}(f) = \|f'\|_{A^q_{q-2}},$$

and therefore we deduce that $f'(z) = cK_\zeta(z) = c(1-\overline{\zeta}z)^{-2}$ for some constant c, from where we get the expression for f.

(b) Let $f \in B^1$ such that $\|f\|_{B^1} = 1$. If $\{b_k\}_{k\ge1}$ is an admissible sequence of coefficients for f in the formula (1.9), then it is clear that

$$|f(z)| \le \sum_{k=1}^{\infty}|b_k|, \quad \forall z \in \mathbb{D},$$

so $\|f\|_{H^\infty} \le \|f\|_{B^1}$. Thus, we can apply the Schwarz-Pick lemma to the function $\frac{f}{\|f\|_\infty}$ in order to obtain $|f'(z)|(1-|z|^2) \le \|f\|_{H^\infty} - \frac{|f(z)|^2}{\|f\|_{H^\infty}}$ for all $z \in \mathbb{D}$. Summing up,

$$\|f\|_{\mathscr{B}} = |f(0)| + \rho_{\mathscr{B}}(f) \le 2\|f\|_{H^\infty} \le 2.$$

This inequality is sharp, because if we test with the sequence of automorphisms $\{\varphi_{1-k^{-1}}\}_{k\ge1}$ it is immediate to check that $\|\varphi_{1-k^{-1}}\|_{B^1} = 1$ and $\|\varphi_{1-k^{-1}}\|_{\mathscr{B}} = 2-k^{-1}$ for every $k \ge 1$. However, if equality holds for a non-identically zero function f (whose B^1 norm is assumed to be 1), then in particular $\|f\|_{\mathscr{B}} = 2\|f\|_{H^\infty} = 2$. This leads to

$$|f(0)| = \rho_{\mathscr{B}}(f) = \|f\|_{H^\infty} = 1,$$

which is in a clear contradiction with the open mapping theorem. Therefore, there are no extremal functions for this inclusion.

□

Remark.

1. *It can be checked that* $\|f_\zeta\|_{\mathscr{B}} = \|f_\zeta\|_{B^q,1}$, $\zeta \in \mathbb{D}$, *for any choice of* α, β, γ *and* δ. *On the other hand* $\|f_\zeta\|_{\mathscr{B}} = 2^{\frac{q-1}{q}}\|f_\zeta\|_{B^q,2}$, *whenever* $|\alpha| = |\beta| = 2^{-\frac{1}{q}}\|f_0\|_{B^q,2}$, *if* $\zeta = 0$, *or* $|\gamma| = \left|\gamma\frac{1-|\zeta|^2}{\zeta} + \delta\right| = 2^{-\frac{1}{q}}\|f_\zeta\|_{B^q,2}$ *otherwise.*

2. *The estimate* $\|f\|_{\mathscr{B}} \leq 2\|f\|_{H^\infty}$ *is widely known and can be found in many works (like the outstanding paper of J. M. Anderson, J. Clunie and Ch. Pommerenke [7]), but the non-existence of extremal functions, although rather straightforward, is not explicitly stated in the literature.*

3.2 The inclusion of $\mathscr{B}$ in A^p_α

Again, the easiest way of proving the inclusion of the Bloch space in every standard weighted Bergman space A^p_α is through the pointwise estimates in $\mathscr{B}$. In other words, if $f \in \mathscr{B}$ then its growth is at most logarithmic (Theorem F) and therefore it is p-integrable with respect to the measure μ_α for every $p > 0$ and $\alpha > -1$. Given this growth estimate, we can deduce the following properties of the inclusion operator of $\mathscr{B}$ in A^p_α.

Proposition 5. *For every* $\alpha > -1$ *and* $p > 0$ *the following statements are true:*

(a) *The inclusion operator of* $\mathscr{B}$ *in* A^p_α *is a compact operator.*

(b) *There exists* $f \in \mathscr{B}$ *such that*

$$\frac{\|f\|_{A^p_\alpha}}{\|f\|_{\mathscr{B}}} = \sup_{\|g\|_{\mathscr{B}}\leq 1}\left\{\|g\|_{A^p_\alpha}\right\}.$$

Proof.

(a) Let $\{f_n\}_{n\geq 1} \subset \mathscr{B}$ a sequence whose Bloch norms are bounded by a constant M. As a consequence of Theorem F, we have that this sequence is uniformly bounded on compact sets of $\mathbb{D}$, and therefore Montel's theorem yields that there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ and an analytic function f such that

$$f_{n_k} \underset{K}{\rightrightarrows} f, \quad \forall K \Subset \mathbb{D}.$$

On the other hand, due to Theorem F

$$|f_{n_k}(z)|^p \leq \left(1+\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right)^p M^p, \quad \forall z \in \mathbb{D},$$

and the dominated convergence theorem yields that

$$\lim_{k\to\infty} \|f_{n_k}\|_{A^p_\alpha} = \lim_{k\to\infty}\left(\int_{\mathbb{D}} |f_{n_k}(z)|^p \, d\mu_\alpha(z)\right)^{\frac{1}{p}} = \left(\int_{\mathbb{D}} |f(z)|^p \, d\mu_\alpha(z)\right)^{\frac{1}{p}} = \|f\|_{A^p_\alpha}.$$

Finally, since $\{f_{n_k}\}_{k\geq 1}$ converges to f μ_α-almost everywhere and $\left\{\|f_{n_k}\|_{A^p_\alpha}\right\}_{k\geq 1}$ converges to $\|f\|_{A^p_\alpha}$, it follows from Riesz's lemma (see Lemma 1 in Chapter 2 of [38] or [75]) that

$$\lim_{k\to\infty} f_{n_k} = f \text{ in } A^p_\alpha.$$

(b) The proof is similar to the one of part (a).

$\square$

At this point, for the sake of simplicity we will use the notation

$$C_\alpha(p) := \max_{\|f\|_{\mathcal{B}}\leq 1} \left\{\|f\|_{A^p_\alpha}\right\}.$$

Given $f \in \mathcal{H}(\mathbb{D})$, due to an elementary property of Measure Theory,

$$\lim_{p\to\infty} \|f\|_{A^p_\alpha} = \operatorname*{ess\,sup}_{z\in\mathbb{D}}\{|f(z)|\} = \|f\|_{H^\infty},$$

and, since $H^\infty \subsetneq \mathcal{B}$, it follows that

$$\lim_{p\to\infty} C_\alpha(p) = \infty.$$

Thus, a first question to consider is: in the absence of the precise value of $C_\alpha(p)$, what is its order of growth as p tends to infinity?

We recall the following result about the growth of $M_p^p(r,f)$ which can be deduced from Hölder's inequality and the Cauchy-Riemann equations.

Theorem G. *If $p \geq 1$, then*

$$\frac{d}{dr}M_p^p(r,f) \leq pM_p^{p-1}(r,f)M_p(r,f'), \quad \forall f \in \mathcal{H}(\mathbb{D}).$$

Proof. See [6, Lemma 1] or page 82 in [38]. $\square$

Theorem 6. *There exists $K_\alpha > 0$ such that*

$$K_\alpha \le \frac{C_\alpha(p)}{p} \le \max\left\{1, \frac{B\left(\frac{1}{2}, \alpha+1\right)}{2}\right\}, \quad \forall p \ge 1. \tag{3.2}$$

In other words, $C_\alpha(p) \approx p$.

Proof. First, we prove the upper bound. Take $f \in \mathscr{B}$ and assume $f(0) = 0$. Using integration by parts, the growth estimates in $\mathscr{B}$ (Theorem F) and Theorem G, we deduce

$$\begin{aligned} \|f\|_{A^p_\alpha}^p &= (\alpha+1)\int_0^1 2r(1-r^2)^\alpha M_p^p(r,f)\,dr \\ &= \int_0^1 (1-r^2)^{\alpha+1}\frac{d}{dr}M_p^p(r,f)\,dr \\ &\le p\int_0^1 (1-r^2)^{\alpha+1}M_p^{p-1}(r,f)M_p(r,f')\,dr \\ &\le p\int_0^1 (1-r^2)^{\alpha}M_p^{p-1}(r,f)\,dr\|f\|_{\mathscr{B}}. \end{aligned}$$

Writing $d\sigma(r) := \frac{2(1-r^2)^\alpha dr}{B(\frac{1}{2},\alpha+1)}$, the straightforward change of variable $x = r^2$ yields that σ is a probability measure in the interval $(0,1)$. We can apply Chebyshev's inequality (Theorem O) and Hölder's inequality to get

$$\begin{aligned} \|f\|_{A^p_\alpha}^p &\le \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}p\int_0^1 M_p^{p-1}(r,f)\,d\sigma(r)\,\|f\|_{\mathscr{B}} \\ &\le \frac{B^2\left(\frac{1}{2},\alpha+1\right)}{4}p(\alpha+1)\int_0^1 2rM_p^{p-1}(r,f)\,d\sigma(r)\,\|f\|_{\mathscr{B}} \\ &\le \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}p\|f\|_{A^p_\alpha}^{p-1}\|f\|_{\mathscr{B}}, \end{aligned}$$

and therefore $\|f\|_{A^p_\alpha} \le \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}p\|f\|_{\mathscr{B}}$ whenever $f(0) = 0$.

In general, if $f \in \mathscr{B}$

$$\begin{aligned} \|f\|_{A^p_\alpha} &\le |f(0)| + \|f - f(0)\|_{A^p_\alpha} \\ &\le |f(0)| + \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}p\|f-f(0)\|_{\mathscr{B}} \\ &= |f(0)| + \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}p\,\rho_{\mathscr{B}}(f) \qquad (3.3) \\ &\le \max\left\{1, \frac{B\left(\frac{1}{2},\alpha+1\right)}{2}\right\}p\|f\|_{\mathscr{B}}. \end{aligned}$$

In order to prove the reverse inequality, consider the test function $f(z) = \frac{1}{2}\log\frac{1}{1-z}$. Observe that $f(0) = 0$ and, in addition to this,

$$|f'(z)|(1-|z|^2) = \frac{1}{2}\frac{1-|z|^2}{|1-z|} \leq \frac{1+|z|}{2},$$

and hence $\|f\|_{\mathscr{B}} = 1$.

Using polar coordinates centred at 1, we have

$$\begin{aligned}
\|f\|^p_{A^p_\alpha} &= \frac{\alpha+1}{2^p}\int_{\mathbb{D}}(1-|z|^2)^\alpha \left|\log\left(\frac{1}{1-z}\right)\right|^p dA(z)\\
&\geq \frac{\alpha+1}{2^p}\int_{\mathbb{D}}(1-|z|^2)^\alpha \left|\log\left|\frac{1}{1-z}\right|\right|^p dA(z)\\
&= \frac{1}{\pi 2^p}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{2|\cos t|} r^{\alpha+1}(2|\cos t|-r)^\alpha \left|\log\frac{1}{r}\right|^p dr\,dt\\
&= \frac{1}{\pi 2^{p-1}}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2|\cos t|} r^{\alpha+1}(2|\cos t|-r)^\alpha \left|\log\frac{1}{r}\right|^p dr\,dt\\
&\geq \frac{1}{\pi 2^{p-1}}\int_{\frac{2\pi}{3}}^{\pi}\int_0^{1} r^{\alpha+1}(2|\cos t|-r)^\alpha \left|\log\frac{1}{r}\right|^p dr\,dt\\
&\geq \frac{M_\alpha}{2^{p-1}}\int_0^1 r^{\alpha+1}\log^p\frac{1}{r}\,dr\\
&\underbrace{=}_{r=e^{-u}} \frac{M_\alpha}{2^{p-1}}\int_0^\infty u^p e^{-(\alpha+2)u}\,du\\
&= \frac{M_\alpha}{2^{p-1}(\alpha+2)^{p+1}}\int_0^\infty x^p e^{-x}\,dx = \frac{M_\alpha}{2^{p-1}(\alpha+2)^{p+1}}\Gamma(p+1),
\end{aligned}$$

where

$$M_\alpha := \frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\min\{(2|\cos(t)|-1)^\alpha, (2|\cos(t)|)^\alpha\}\,dt.$$

We have, thus, proved that

$$\frac{C_\alpha(p)}{p} \geq \frac{1}{2(\alpha+2)}\left(\frac{2M_\alpha}{\alpha+2}\right)^{\frac{1}{p}}\frac{\Gamma^{\frac{1}{p}}(p+1)}{p}, \quad \forall p \geq 1, \tag{3.4}$$

and the existence of such a constant K_α follows from Stirling's formula for Euler's Γ function. $\square$

Corollary 7. *Let $\alpha \geq 0$. If $p \leq \frac{2}{B\left(\frac{1}{2},\alpha+1\right)}$, then*

$$\|f\|_{A^p_\alpha} \leq \|f\|_{\mathscr{B}}, \quad \forall f \in \mathscr{B},$$

and equality is attained if and only if f is constant.

In particular, $C_\alpha(p) = 1$ for all such values of p.

Proof. Observe that, if $1 \leq p \leq \frac{2}{B\left(\frac{1}{2},\alpha+1\right)}$, by the arguments given in the proof of Theorem 6 (see (3.3)) it follows that

$$\|f\|_{A^p_\alpha} \leq |f(0)| + \frac{B\left(\frac{1}{2},\alpha+1\right)}{2} p\rho_{\mathscr{B}}(f) \leq \|f\|_{\mathscr{B}}.$$

Moreover, the equality is possible if and only if the seminorm $\rho_{\mathscr{B}}$ is attained at every point. That is,

$$|f'(z)| = \frac{\rho_{\mathscr{B}}(f)}{1-|z|^2}, \quad \forall z \in \mathbb{D}.$$

Since f' is an analytic function whose modulus is a radial function, then f' must be a monomial (see Theorem M in Appendix A), and this is possible if and only if $\rho_{\mathscr{B}}(f) = 0$.

On the other hand, if $p < 1$, since $\alpha \geq 0$ we have that $1 \leq \frac{2}{B\left(\frac{1}{2},\alpha+1\right)}$ and due to Hölder's inequality

$$\|f\|_{A^p_\alpha} \leq \|f\|_{A^1_\alpha} \leq \|f\|_{\mathscr{B}}.$$

□

In other words, for any $\alpha \geq 0$ we know the precise value of $C_\alpha(p)$ if p belongs to $\left(0, \frac{2}{B\left(\frac{1}{2},\alpha+1\right)}\right]$. In the particular case of $\alpha = 0$, that is, in the case of the (unweighted) Bergman spaces A^p, it turns out to be the interval $(0,1]$. However, we are going to show that the identity $C_\alpha(p) = 1$ can be extended to other values of p.

Theorem 8. *Let $\alpha \geq 0$. Then, for every $f \in \mathscr{B}$ the following inequality holds*

$$\|f\|_{A^2_\alpha} \leq \|f\|_{\mathscr{B}}. \tag{3.5}$$

In addition, equality is attained of and only if f is a constant function.

If $\alpha = 0$, it seems that (3.5) could be deduced in [7] using the expression for $\|f\|_{A^2}$ in terms of the Taylor coefficients of f, but in this dissertation we are going to give an alternative proof. For this purpose, we will make use of the so-called *Hardy-Stein identity*, which can be understood as a refinement of Theorem G

$$\frac{d}{dr} M_p^p(r,f) = \frac{p^2}{2r} \int_{r\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \, dA(z), \quad \forall r \in (0,1). \tag{3.6}$$

A proof of this identity can be found in Section A.2 in the Appendix of this thesis.

Proof of Theorem 8. Take $f \in \mathscr{B}$. Then, using integration by parts and the Hardy-Stein identity (3.6) we deduce

$$\begin{aligned}
\|f\|_{A^2_\alpha}^2 &= (\alpha+1) \int_0^1 2r(1-r^2)^{\alpha+1} M_2^2(r,f)\, dr \\
&= |f(0)|^2 + \int_0^1 (1-r^2)^{\alpha+1} \frac{d}{dr} M_2^2(r,f)\, dr \\
&= |f(0)|^2 + 2\int_0^1 \frac{(1-r^2)^{\alpha+1}}{r} \int_{r\mathbb{D}} |f'(z)|^2 \, dA(z)\, dr \\
&\le |f(0)|^2 + 2\int_0^1 \frac{(1-r^2)^{\alpha+1}}{r} \int_{r\mathbb{D}} \frac{1}{(1-|z|^2)^2} \, dA(z)\, dr\, \rho_{\mathscr{B}}(f)^2 \\
&= |f(0)|^2 + \frac{\rho_{\mathscr{B}}(f)^2}{\alpha+1} \le (|f(0)| + \rho_{\mathscr{B}}(f))^2 = \|f\|_{\mathscr{B}}^2.
\end{aligned}$$

It is clear that equality is only possible if $\rho_{\mathscr{B}}(f)$ is attained at every point in $\mathbb{D}$, and therefore f is constant. $\square$

Due to the expression for $\|f\|_{\mathscr{B}}$, it is natural to focus on the subspace of $\mathscr{B}$ consisting of all functions which vanish at the origin. For the sake of clarity, we will write

$$\tilde{C}_\alpha(p) = \max_{\|f\|_{\mathscr{B}} \le 1, f(0)=0} \left\{ \|f\|_{A^p_\alpha} \right\}. \tag{3.7}$$

Note that writing the maximum in (3.7) is justified since, similarly to Proposition 5, this restricted inclusion operator is also compact. Moreover, observe that $\tilde{C}_\alpha$ is a strictly increasing function of p because of the existence of extremal functions and the equality conditions in Hölder's inequality. Using the same idea, it is easy to check that $\tilde{C}_\alpha$ (and also C_α) is a convex function, and therefore it is continuous.

It is trivial that $\tilde{C}_\alpha(p) \leq C_\alpha(p)$ for every $p > 0$. On the other hand, if $p \geq 1$ and $\tilde{C}_\alpha(p) \geq 1$, triangle inequality yields that

$$\|f\|_{A^p_\alpha} \leq |f(0)| + \|f - f(0)\|_{A^p_\alpha} \leq |f(0)| + \tilde{C}_\alpha(p)\rho_{\mathscr{B}}(f) \leq \tilde{C}_\alpha(p)\|f\|_{\mathscr{B}},$$

if $f \in \mathscr{B}$. Summing up,

$$\tilde{C}_\alpha(p) \geq 1 \quad \Longleftrightarrow \quad C_\alpha(p) = \tilde{C}_\alpha(p), \tag{3.8}$$

whenever $p \geq 1$.

Similarly, we can check that $\tilde{C}_\alpha(p) < 1$ implies that $C_\alpha(p) = 1$. That is,

$$C_\alpha(p) = \max\left\{1, \tilde{C}_\alpha(p)\right\}, \tag{3.9}$$

for every $p \geq 1$.

Since $C_\alpha(p)$ is completely characterised in terms of $\tilde{C}_\alpha(p)$, this allows us to extend the results of Corollary 7 and Theorem 8 to further values of p.

Theorem 9. *Let $\alpha \geq 0$. There exists $p_\alpha > 2\max\left\{1, B^{-1}\left(\frac{1}{2}, \alpha+1\right)\right\}$ such that*

$$\|f\|_{A^p_\alpha} \leq \|f\|_{\mathscr{B}}, \quad \forall f \in B,$$

for every $p \in (0, p_\alpha]$.

Proof. Due to Corollary 7 and Theorem 8,

$$\|f\|_{A_\alpha^{\frac{2}{B(1/2,\alpha+1)}}} \leq \|f\|_{\mathscr{B}} \quad \text{and} \quad \|f\|_{A^2_\alpha} \leq \|f\|_{\mathscr{B}},$$

for every $f \in \mathscr{B}$, but the equality is possible if and only if f is constant. Thus,

$$\tilde{C}_\alpha\left(\frac{2}{B\left(\frac{1}{2}, \alpha+1\right)}\right), \tilde{C}_\alpha(2) < 1.$$

Since $C_\alpha(p) \approx p$ (Theorem 6) and (3.9), we have that $\tilde{C}_\alpha$ is not bounded, and then there exists a unique p_α such that $\tilde{C}_\alpha(p_\alpha) = 1$.

Finally, the monotonicity of $\tilde{C}_\alpha$ and the identity (3.9) yield that

$$\|f\|_{A^p_\alpha} \leq \|f\|_{\mathscr{B}}, \quad \forall f \in \mathscr{B} \text{ and } \forall p \in (0, p_\alpha].$$

$\square$

Remark. *If $\alpha = 0$, we can show that, in fact, that $p_0 \in \left(2, \frac{25}{4}\right)$.*

That is, using the test function $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ is easy to check that $f(0) = 0$, $\|f\|_{\mathscr{B}} = 1$ and, after some suitable changes of variables,

$$\begin{aligned}
\|f\|_{A^p}^p &= \frac{1}{2^p}\int_{\mathbb{D}}\left|\log\frac{1+z}{1-z}\right|^p dA(z) \\
&= \frac{1}{2^{p-2}}\int_{\{\operatorname{Re}\zeta>0\}}|\log\zeta|^p\,\frac{1}{|\zeta+1|^4}\,dA(\zeta) \\
&= \frac{1}{2^{p-2}}\int_{\{-\frac{\pi}{2}<\operatorname{Im}\xi<\frac{\pi}{2}\}}\frac{|\xi|^p}{|e^{\xi}+1|^4}|e^{\xi}|^2\,dA(\xi) \\
&= \frac{1}{\pi 2^{p-4}}\int_0^{\infty}\int_0^{\frac{\pi}{2}}\frac{(x^2+y^2)^{\frac{p}{2}}}{(e^x+2\cos(y)+e^{-x})^2}\,dy\,dx,
\end{aligned}$$

and then, after some numerical estimates using Wolfram Alpha, $\tilde{C}\left(\frac{25}{4}\right) \geq \|f\|_{A^{\frac{25}{4}}} > 1$.

We will now show that, asymptotically, the upper bound in Theorem 6 can be improved significantly. To do this, we are going to study how we can improve the upper bounds for $\tilde{C}_\alpha(2n)$, being n a positive integer, making use of the expression for $\|\cdot\|_{A^2_\alpha}$ in terms of the Taylor coefficients, a fact that we have not used until now.

Lemma 10. *If $\alpha > -1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{H}(\mathbb{D})$, then*

$$(\alpha+1)(\alpha+2)\sum_{n=1}^{\infty}\frac{n}{n+\alpha+2}\frac{|a_n|^2}{c_{\alpha+2}(n)} = \int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^2\,d\mu_\alpha(z).$$

In addition, if $f(0) = f'(0) = \ldots = f^{(k-1)}(0) = 0$ for some $k \geq 1$, then

$$\|f\|_{A^2_\alpha}^2 \leq \frac{k+\alpha+2}{(\alpha+1)(\alpha+2)k}\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^2\,d\mu_\alpha(z).$$

Proof. It is a consequence of Parseval's identity

$$M_2^2(r, f') = \sum_{n=1}^{\infty} n^2|a_n|^2 r^{2(n-1)}, \quad \forall r \in (0,1),$$

the monotone convergence theorem and that the fact that the sequence $\left\{\frac{n}{n+\alpha+2}\right\}_{n\geq 1}$ is increasing. □

Theorem 11. *For any $\alpha > -1$ and $n \geq 2$*

$$\tilde{C}_\alpha(2n) \leq \frac{1}{\sqrt{(\alpha+1)(\alpha+2)}} \left(\frac{(\alpha+1)(\alpha+2)\Gamma(n+\alpha+3)n!}{\Gamma(\alpha+4)} \tilde{C}_\alpha^2(2) \right)^{\frac{1}{2n}}.$$

Proof. We provide a proof by induction. Observe that, if $\|f\|_{\mathscr{B}} = 1$ and $f(0)$, then the first two Taylor coefficients of f^2 are 0. Using Lemma 10, we deduce that

$$\begin{aligned} \|f\|_{A_\alpha^4}^4 = \|f^2\|_{A_\alpha^2}^2 &\leq \frac{\alpha+4}{2(\alpha+1)(\alpha+2)} \int_{\mathbb{D}} \left|(f^2)'(z)\right|^2 (1-|z|^2)^2 \, d\mu_\alpha(z) \\ &= \frac{2(\alpha+4)}{(\alpha+1)(\alpha+2)} \int_{\mathbb{D}} |f(z)|^2 |f'(z)|^2 (1-|z|^2)^2 \, d\mu_\alpha(z) \\ &\leq \frac{2(\alpha+4)}{(\alpha+1)(\alpha+2)} \|f\|_{A_\alpha^2}^2 \leq \frac{2(\alpha+4)}{(\alpha+1)(\alpha+2)} \tilde{C}_\alpha^2(2), \end{aligned}$$

and consequently we have proved the result for $n = 2$.

Now, assume that

$$\tilde{C}_\alpha(2n) \leq \frac{1}{\sqrt{(\alpha+1)(\alpha+2)}} \left(\frac{(\alpha+1)(\alpha+2)\Gamma(n+\alpha+3)n!}{\Gamma(\alpha+4)} \tilde{C}_\alpha^2(2) \right)^{\frac{1}{2n}},$$

for some $n \geq 2$.

Again, the function f^{n+1} verifies that $f^{n+1}(0) = \ldots = (f^{n+1})^{(n)}(0) = 0$ and then

$$\begin{aligned} \|f\|_{A_\alpha^{2(n+1)}}^{2(n+1)} = \|f^{n+1}\|_{A_\alpha^2}^2 &\leq \frac{n+\alpha+3}{(\alpha+1)(\alpha+2)(n+1)} \int_{\mathbb{D}} \left|(f^{n+1})'(z)\right|^2 (1-|z|^2)^2 \, d\mu_\alpha(z) \\ &= \frac{(n+\alpha+3)(n+1)}{(\alpha+1)(\alpha+2)} \int_{\mathbb{D}} |f(z)|^{2n} |f'(z)|^2 (1-|z|^2)^2 \, d\mu_\alpha(z) \\ &\leq \frac{\Gamma(n+\alpha+4)(n+1)!}{(\alpha+1)^n(\alpha+2)^n \Gamma(\alpha+4)} \tilde{C}_\alpha^2(2), \end{aligned}$$

which proves the theorem. $\square$

Theorem 12. *For any $\alpha > -1$*

$$\begin{aligned} \liminf_{p\to\infty} \frac{C_\alpha(p)}{p} &\geq \frac{1}{2e} \frac{1}{\alpha+2} \\ \limsup_{p\to\infty} \frac{C_\alpha(p)}{p} &\leq \frac{1}{2e} \frac{1}{\sqrt{(\alpha+1)(\alpha+2)}} \end{aligned}$$

Proof. The first bound comes from (3.4) and Stirling's formula.

On the other hand, if p is large enough, we have

$$\begin{aligned} \frac{C(p)}{p} &= \frac{\tilde{C}_\alpha(p)}{p} \\ &\leq \frac{2\left\lceil \frac{p}{2} \right\rceil}{p} \frac{\tilde{C}_\alpha\left(2\left\lceil \frac{p}{2} \right\rceil\right)}{2\left\lceil \frac{p}{2} \right\rceil}. \end{aligned}$$

Finally, we deduce the upper bound for the limit superior using Stirling's formula. □

Note that, since both estimates are asymptotically equivalent when $\alpha \to \infty$, it seems that the quantity $C_\alpha(p)p^{-1}$ might be convergent when $p \to \infty$.

Furthermore, due to the fact that $\mathscr{B}$ is not included in any H^p, which can be understood as A^p_{-1} in the sense of (1.6), the accurate asymptotic approximation should blow up when α tends to -1.

Given these considerations, we conjecture the following result.

Conjecture 1. *For any $\alpha > -1$,*

$$\lim_{p\to\infty} \frac{C_\alpha(p)}{p/\left(2e\sqrt{(\alpha+1)(\alpha+2)}\right)} = 1.$$

We end the chapter by deducing an integral condition that the extremal functions must satisfy.

Theorem 13. *Let $\alpha > -1$ and $p > 1$. If $f \in \mathscr{B}$ with $f(0) = 0$ is extremal, then*

$$\int_{\mathbb{D}} |f(z)|^p z \, d\mu_\alpha(z) = \frac{p}{2(\alpha+2)} \overline{f'(0)} \int_{\mathbb{D}} |f(z)|^{p-2} f(z) \, d\mu_\alpha(z).$$

Proof. Take $f \in \mathscr{B}$ such that $f(0) = 0$ and $\|f\|_{A^p_\alpha} = \tilde{C}_\alpha(p)\|f\|_{\mathscr{B}}$. For any $a \in \mathbb{D}$, consider the function

$$g_a(z) := f\left(\varphi_a(z)\right) - f(a),$$

where φ_a is the disc automorphism introduced in 1.

Note that $g_a(0) = 0$ and, since $\mathscr{B}$ is (strictly) conformally invariant,

$$\rho_{\mathscr{B}}(g_a) = \rho_{\mathscr{B}}(f \circ \varphi_a) = \rho_{\mathscr{B}}(f),$$

and thus $\|g_a\|_{A^p_\alpha} \leq \|f\|_{A^p_\alpha}$ for every $a \in \mathbb{D}$.

Using the change of variables $z = \varphi_a(\zeta)$, we get

$$\|g_a\|_{A^p_\alpha}^p = \int_{\mathbb{D}} \left|f(\zeta) - f(a)\right|^p |\varphi_a'(\zeta)|^{\alpha+2}\, d\mu_\alpha(\zeta).$$

Observe that

$$|\varphi_a'(\zeta)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}\zeta|^4} = 1 + 4\,\mathrm{Re}\{\overline{a}\zeta\} + o(|a|),$$

and from here

$$|\varphi_a'(\zeta)|^{\alpha+2} = 1 + 2(\alpha+2)\,\mathrm{Re}\{\overline{a}\zeta\} + o(|a|).$$

Consider the following partition of $\mathbb{D}$

$$\mathbb{D}_a^+ := \{\zeta \in \mathbb{D} : |f(\zeta)| > |f(a)|\} \quad \text{and} \quad \mathbb{D}_a^- = \mathbb{D} \setminus \mathbb{D}_a^+.$$

- If $\zeta \in \mathbb{D}_a^-$, since $p > 1$ and $f(a) = 0$,

  $$|f(\zeta) - f(a)|^p \le 2^p |f(a)|^p = o(|a|), \quad a \to 0,$$

  and therefore the dominated convergence theorem yields that

  $$\begin{aligned} \int_{\mathbb{D}_a^-} |f(\zeta) - f(a)|^p |\varphi_a'(\zeta)|^{\alpha+2}\, d\mu_\alpha(\zeta) &= o(|a|), \\ \int_{\mathbb{D}_a^-} |f(\zeta)|^p\, d\mu_\alpha(\zeta) &= o(|a|), \end{aligned}$$

  as $a \to 0$.

- If $\zeta \in \mathbb{D}_a^+$, then

  $$\left(f(\zeta) - f(a)\right)^{\frac{p}{2}} = f(\zeta)^{\frac{p}{2}} - \frac{p}{2} f(\zeta)^{\frac{p}{2}-1} f'(0)a + o(|a|), \quad a \to 0,$$

  and consequently

  $$|f(\zeta) - f(a)|^p = |f(\zeta)|^p - p|f(\zeta)|^{p-2}\,\mathrm{Re}\{f(\zeta)\overline{f'(0)a}\} + o(|a|), \quad a \to 0.$$

  Then, due to the dominated convergence theorem, the integral of $|f(\zeta) - f(0)|^p |\varphi_a'(\zeta)|^{\alpha+2}$ with respect to the measure μ_α on $\mathbb{D}_a^+$ is equal to

  $$\int_{\mathbb{D}_a^+} |f(\zeta)|^{p-2}\,\mathrm{Re}\left\{\left(2(\alpha+2)|f(\zeta)|^2\zeta - pf(\zeta)\overline{f'(0)}\right)\overline{a}\right\}\, d\mu_\alpha(\zeta)$$
  $$+ \int_{\mathbb{D}_a^+} |f(\zeta)|^p\, d\mu_\alpha(\zeta) + o(|a|), \quad a \to 0.$$

In other words, writing $a = re^{it}$ with $r > 0$ and $t \in [0, 2\pi)$, we have proved that

$$\operatorname{Re}\left\{ e^{-it} \int_{\mathbb{D}_a^+} \left[2(\alpha+2)|f(\zeta)|^p \zeta - p\overline{f'(0)}|f(\zeta)|^{p-2} f(\zeta) \right] d\mu_\alpha(\zeta) \right\} \leq \frac{o(r)}{r}, \quad r \to 0^+.$$

Finally, applying one last time the dominated convergence theorem, it follows that

$$\int_{\mathbb{D}\setminus\mathscr{Z}(f)} |f(\zeta)|^p \zeta \, d\mu_\alpha(\zeta) = \frac{p}{2(\alpha+2)} \overline{f'(0)} \int_{\mathbb{D}\setminus\mathscr{Z}(f)} |f(\zeta)|^{p-2} f(\zeta) \, d\mu_\alpha(\zeta),$$

but, since f is not identically zero, $\mu_\alpha(\mathscr{Z}(f)) = 0$ and therefore

$$\int_{\mathbb{D}} |f(\zeta)|^p \zeta \, d\mu_\alpha(\zeta) = \frac{p}{2(\alpha+2)} \overline{f'(0)} \int_{\mathbb{D}} |f(\zeta)|^{p-2} f(\zeta) \, d\mu_\alpha(\zeta).$$

□

Corollary 14. *If $f \in \mathscr{B}$, $f(0) = 0$, satisfies that $\|f\|_{A^2_\alpha} = \tilde{C}_\alpha(2)\|f\|_{\mathscr{B}}$, then f is orthogonal to $\mathscr{M}_z f$ in A^2_α.*

Chapter 4

Contractive inclusions between mixed norm spaces

As was pointed out in Chapter 1, the mixed norm spaces $H(p,q,a)$ have not been as extensively studied as other families of spaces of analytic functions. As a consequence of this, the papers devoted to $H(p,q,a)$ are really scattered in the literature.

There are many partial results that cover various special cases of inclusions between two spaces $H(p,q,a)$ and $H(u,v,b)$ (or the characterisation of the pointwise multipliers, a related problem). For example, we mention the works of G. H. Hardy and J. E. Littlewood [51], T. M. Flett [45, 46], P. Ahern and M. Jevtić [1], Ó. Blasco [21], S. M. Buckley, P. Koskela and D. Vukotić [30] or K. L. Avetisyan [13, 14].

In 2015, I. Arévalo [10, 11] put together all possible cases and provided a unified proof for all the possible inclusions.

Theorem H (Inclusion Theorem, [10, 11]). *Let $0 < p, q, u, v \leq \infty$ and $a, b > 0$.*

(a) If $p \geq u$, then

$$H(p,q,a) \subset H(u,v,b) \quad \Longleftrightarrow \quad \begin{cases} a < b \\ a = b \quad \text{and} \quad q \leq v. \end{cases}$$

(b) If $p < u$, then

$$H(p,q,a) \subset H(u,v,a) \quad \Longleftrightarrow \quad \begin{cases} a + \frac{1}{p} < b + \frac{1}{u} \\ a + \frac{1}{p} = b + \frac{1}{u} \quad \text{and} \quad q \leq v. \end{cases}$$

Since computing the precise norm of the corresponding inclusion operator for any possible choice of p, q, u, v, a and b is a really ambitious problem, in this chapter we are going to focus on the following question: Under which conditions on p, q, u, v, a and b does the inequality

$$\|f\|_{u,v,b} \leq \|f\|_{p,q,a},$$

hold for any $f \in H(p,q,a)$? That is, when is the inclusion $H(p,q,a) \subset H(u,v,b)$ *contractive*?

The possible strategy to deal with this problem depends considerably on the relation between p and u. Thus, we will study the cases $p \geq u$ and $p < u$ separately.

The content of this chapter is based on the works [71] and [72].

4.1 The case $p \geq u$

We are going to provide a complete answer for this case. To this end, we will discuss all the possible subcases, ending the section with the characterisation of the contractivity of the inclusion $H(p,q,a) \subset H(u,v,b)$ if $p \geq u$.

Proposition 15. *Assume that $p \geq u$, $\infty > q \geq v$ and $aq \leq bv$. Then*

$$\|f\|_{u,v,b} \leq \|f\|_{p,q,a}, \quad \forall f \in H(p,q,b),$$

and equality is attained if and only if f is constant (unless the trivial case $p = u$, $q = v$ and $a = b$).

Proof. Take $f \in H(p,q,a)$. Since v is finite, we can use the change of variables $r(s) = \sqrt{1 - s^{\frac{1}{bv}}}$, $0 < s < 1$, to write $\|f\|_{u,v,b}$ as follows

$$\begin{aligned} \|f\|_{u,v,b} &= \left(bv \int_0^1 2r(1-r^2)^{bv-1} M_u^v(r,f)\, dr \right)^{\frac{1}{v}} \\ &= \left(\int_0^1 M_u^v\left(\sqrt{1-s^{\frac{1}{bv}}}, f\right) ds \right)^{\frac{1}{v}}. \end{aligned}$$

Because of $p \geq u$ and $q \geq v$, applying Hölder's inequality twice we deduce

$$\|f\|_{u,v,b} \leq \left(\int_0^1 M_p^q\left(\sqrt{1-s^{\frac{1}{bv}}}, f\right) \right)^{\frac{1}{q}}.$$

Observe that, since $aq \leq bv$, then

$$1 - s^{\frac{1}{bv}} \leq 1 - s^{\frac{1}{aq}}, \quad \forall s \in (0,1),$$

and finally, because of the monotonicity of M_p

$$\begin{aligned} \|f\|_{u,v,b} &\leq \left(\int_0^1 M_p^q \left(\sqrt{1 - s^{\frac{1}{bv}}}, f \right) ds \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^1 M_p^q \left(\sqrt{1 - s^{\frac{1}{aq}}}, f \right) ds \right)^{\frac{1}{q}} = \|f\|_{p,q,a}. \end{aligned} \tag{4.1}$$

Note that equality is attained in any of the three inequalities if and only if f is constant (unless $H(p,q,a) = H(u,v,b)$). □

The proof of the next sufficient condition for $H(p,q,a) \subset H(u,v,b)$ to be contractive needs the following lemma.

Lemma 16. *Assume $p \geq u$ and $q \leq v < \infty$. If $f \in H(p,q,a)$, then*

$$\Phi(r) := \left(aq \int_r^1 2\rho(1-\rho^2)^{aq-1} M_p^q(\rho, f)\, d\rho \right)^{\frac{v}{q}} - av \int_r^1 2\rho(1-\rho^2)^{av-1} M_u^v(\rho, f)\, d\rho,$$

is a decreasing function in the interval $(0,1)$. *Moreover, if f is not constant then Φ is strictly decreasing (unless $p = u$ and $q = v$).*

Proof. If $f \in H(p,q,a)$, note that Φ is well-defined for every $r \in (0,1)$. Due to the fundamental theorem of calculus, we deduce that $\Phi'(r) \leq 0$ if and only if

$$\Psi(r) := M_u^v(r,f) - \left(aq \int_r^1 2\rho(1-\rho^2)^{aq-1} M_p^q(\rho, f)\, d\rho \right)^{\frac{v}{q}-1} (1-r^2)^{a(q-v)} M_p^q(r,f),$$

is non-positive in $(0,1)$.

Since $M_p(r,f)$ is an increasing function of the radius, we get

$$\Psi(r) \leq M_u^v(r,f) - M_p^v(r,f) \leq 0.$$

Observe that if f is not constant then $\Psi(r) < 0$ for any $r \in (0,1)$ and therefore Φ is strictly decreasing. □

Proposition 17. *If $p \geq u$, $q \leq v$ and $a \leq b$, then*

$$\|f\|_{u,v,b} \leq \|f\|_{p,q,a}.$$

Proof. We will split the proof into three cases.

- **Case 1:** If $q = v = \infty$, then

  $$(1-r^2)^b M_u(r,f) \leq (1-r^2)^b M_p(r,f) \leq (1-r^2)^{b-a} \|f\|_{p,\infty,a}, \quad \forall r \in [0,1),$$

  and hence $\|f\|_{u,\infty,b} \leq \|f\|_{p,\infty,a}$.

  Equality is attained whenever $|f(0)| = \|f\|_{p,\infty,a}$.

- **Case 2:** If $q < v = \infty$, for every $r \in [0,1)$ we have

  $$\begin{aligned}(1-r^2)^{bq} M_u^q(r,f) &\leq bq \int_r^1 2\rho(1-\rho^2)^{bq-1} M_u^q(\rho,f)\, d\rho \\ &\leq \|f\|_{u,q,b}^q \leq \|f\|_{p,q,a}^q,\end{aligned}$$

  due to Proposition 15. Thus, equality is possible if and only if f is constant.

- **Case 3:** If $q \leq v < \infty$, using Lemma 16 we deduce

  $$0 = \Phi(1) \leq \Phi(0) = \|f\|_{p,q,b}^v - \|f\|_{u,v,b}^v,$$

  and again due to Proposition 15

  $$\|f\|_{u,q,b} \leq \|f\|_{p,q,b} \leq \|f\|_{p,q,a}.$$

  and equality is attained if and only if f is constant (unless $p = u$, $q = v$ and $a = b$).

$\square$

Propositions 15 and 17 might give the wrong sensation that if $p \geq u$ the inclusion $H(p,q,a) \subset H(u,v,b)$, if it takes place, is always contractive. However, this is absolutely false. That is, if we fix p, q, u, v and a positive, under the conditions $p \geq u$ and $q > v$, due to Theorem H we have that $H(p,q,a) \not\subset H(u,v,a)$ and, however, $H(p,q,a) \subset H(u,v,b)$ for every $b < a$. Thus, there exists $f \in H(p,q,a)$ such that $\|f\|_{p,q,a} = 1$ and, in addition to this,

$$\lim_{b \to a^+} \|f\|_{u,b,v} = \infty.$$

In other words, there exists $\delta > 0$ such that the inclusion $H(p,q,a) \subset H(u,v,b)$ is not contractive if $b \in (a, a+\delta)$.

We are going to show that, under the complementary hypothesis of Propositions 15 and 17, contractivity is not possible.

Proposition 18. *Suppose $p \geq u$, $q > v$ and $a < b$. If $aq > bv$, then the inclusion $H(p,q,a) \subset H(u,v,b)$ is not contractive.*

Proof. The idea of the proof is simple. It is enough to show that there exists $f \in H(p,q,a)$ such that

$$\|f\|_{p,q,a} < \|f\|_{u,v,b}.$$

Since the finiteness (or not) of q is important, we separate the argument into two cases.

- **Case 1:** If $q = \infty$, consider the function $f(z) = (1+z^2)^a$. For any $r \in [0,1)$,

$$(1-r^2)^a M_p(r,f) \leq (1-r^2)^a M_\infty(r,f) = (1-r^4)^a \leq 1,$$

and therefore $\|f\|_{p,\infty,a} = 1$. On the other hand, since v is finite

$$\|f\|_{u,v,b}^v = bv \int_0^1 2r(1-r^2)^{bv-1} M_u^v(r,f)\, dr > |f(0)|^v = 1.$$

- **Case 2:** Assume $q < \infty$. Without loss of generality, we can suppose that $p = \infty$ and $H(u,v,b)$ is actually is a standard weighted Bergman space. If not, Hölder's inequality yields that

$$\|f\|_{p,q,a} \leq \|f\|_{\infty,q,a} \quad \text{and} \quad \|f\|_{A^{\min\{u,v\}}_{bv-1}} \leq \|f\|_{u,v,b},$$

for every holomorphic function f.

Let $w = \min\{u,v\}$ and $\gamma \in \left(0, \frac{a}{2}\right)$. Consider the sequence of functions

$$f_n(z) = \frac{1}{(1-z^{2n})^{2\gamma}}, \quad \forall n \geq 1.$$

On the one hand,

$$\begin{aligned} \|f_n\|_{\infty,q,a} &= aq \int_0^1 2r(1-r^2)^{aq-1} M_\infty^q(r,f_n)\, dr \\ &= aq \int_0^1 2r(1-r^2)^{aq-1} \frac{1}{(1-r^{2n})^{2\gamma q}}\, dr \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(aq+1)(nk)!}{\Gamma(nk+aq+1)} c_{2\gamma q}(k) \\ &= 1 + \frac{\Gamma(aq+1)}{\Gamma(2\gamma q)} \sum_{k=1}^{\infty} \frac{\Gamma(k+2\gamma q)}{k!} \frac{(nk)!}{\Gamma(nk+aq+1)}. \end{aligned}$$

As a consequence of the *Euler-Gauss formula* [39, Section 9.6]

$$\Gamma(x) = \lim_{m\to\infty} \frac{m! m^x}{x(x+1)\dots(x+m)}, \quad \forall x > 0, \tag{4.2}$$

and the multiplicative property $\Gamma(x+1) = x\Gamma(x)$, we have

$$\lim_{m\to\infty} \frac{m! m^{aq}}{\Gamma(m+aq+1)} = 1.$$

Thus, there exists $m_1 > 0$ such that

$$\frac{(nk)!(nk)^{aq}}{\Gamma(nk+aq+1)} \leq \frac{3}{2}, \quad \forall k \geq 1,$$

if $n \geq m_1$. In particular, there exists $C_1 = C_1(q, a, \gamma) > 0$ such that

$$\|f_n\|_{\infty,q,a}^q \leq 1 + \frac{C_1}{n^{aq}}, \quad \forall n \geq m_1.$$

On the other hand, similarly to the computation of $\|f\|_{\infty,q,a}$,

$$\begin{aligned}
\|f_n\|_{A^w_{bv-1}}^w &= bv \int_{\mathbb{D}} (1-|z|^2)^{bv-1} \frac{1}{|1-z^{2n}|^{2\gamma w}}\, dA(z) \\
&= bv \int_0^1 2r(1-r^2)^{bv-1} \left(\sum_{k=0}^{\infty} c^2_{\gamma\omega}(k) r^{4nk} \right) dr \\
&= 1 + \frac{\Gamma(bv+1)}{\Gamma^2(\gamma w)} \sum_{k=1}^{\infty} \frac{\Gamma^2(k+\gamma w)}{k!^2} \frac{(2nk)!}{\Gamma(2nk+bv+1)}
\end{aligned}$$

Due to (4.2), there exists $m_2 > 0$ such that

$$\frac{1}{2} \leq \frac{(2nk)!(2nk)^{bv}}{\Gamma(2nk+bv+1)}, \quad \forall k \geq 1,$$

if $n \geq m_2$. That is, there is a constant $C_2 = C_2(w, v, b, \gamma) > 0$ such that

$$\|f_n\|_{A^w_{bv-1}}^w \geq 1 + \frac{C_2}{n^{bv}}, \quad \forall n \geq m_2.$$

Finally, if $n \geq \max\{m_1, m_2\}$,

$$n^{bv} \left(\|f_n\|_{A^w_{bv-1}}^q - \|f_n\|_{\infty,q,a}^q \right) \geq n^{bv} \left(\|f_n\|_{A^w_{bv-1}}^w - \|f_n\|_{\infty,q,a}^q \right) \geq C_2 - \frac{C_1}{n^{aq-bv}}.$$

In other words, there exists n large enough such that $\|f_n\|_{\infty,q,a} < \|f_n\|_{A^w_{bv-1}}$.

$\square$

We are going to see that, under certain conditions, we can find other examples among the inner functions. First, we are going to need the following lemma.

Lemma 19.

(a) If $\alpha > -1$, $k \geq 1$ and $p > 0$

$$\lim_{x\to 1^-}(1-x)^{1+\alpha}\sum_{n=0}^{\infty}\frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)}c_{\alpha+1}(n)x^n = 1.$$

(b) If $\alpha > 0$, $k \geq 1$ and $p > 0$

$$\lim_{x\to 1^-}(1-x)^{\alpha}\sum_{n=0}^{\infty}\frac{(n+\alpha+1)(n+\alpha)n\Gamma\left(n+\alpha+3\right)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+2)!\Gamma\left(n+\frac{kp}{2}+\alpha+4\right)}c_{\alpha}(n)x^n = 1.$$

Proof.

(a) As a consequence of the Euler-Gauss formula,

$$\lim_{n\to\infty}\frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)} = 1.$$

That is, for all $\varepsilon > 0$ there exists N such that

$$1-\varepsilon \leq \frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)} \leq 1+\varepsilon,$$

if $n \geq N$. Then,

$$1-\varepsilon \leq \lim_{x\to 1^-}(1-x)^{\alpha+1}\sum_{n=N}^{\infty}\frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)}c_{\alpha+1}(n)x^n \leq 1+\varepsilon,$$

and because of the independence of N with respect to x,

$$1-\varepsilon \leq \lim_{x\to 1^-}(1-x)^{\alpha+1}\sum_{n=0}^{\infty}\frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)}c_{\alpha+1}(n)x^n \leq 1+\varepsilon.$$

(b) The proof is analogous to the part (a), since

$$\lim_{n\to\infty}\frac{(n+\alpha+2)(n+\alpha+1)n\Gamma\left(n+\alpha+3\right)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+2)!\Gamma\left(n+\frac{kp}{2}+\alpha+4\right)} = 1.$$

□

Proposition 20. *Assume that $u \le p < \infty$, $v < q < \infty$ and $aq > bv$. If $vb \le 1$ or $\frac{v}{bv-1} > \frac{q}{aq-1}$, and k is an integer such that ku and kv are greater than or equal to 2, then there exists a disc automorphism φ such that*

$$\|\varphi^k\|_{p,q,a} < \|\varphi^k\|_{u,v,b}.$$

Proof. Without loss of generality, we may assume that both spaces are weighted Bergman spaces (that is, $p = q$ and $u = v$). In order to simplify the notation, we will write $\alpha = aq - 1$ and $\beta = bv - 1$. We are going to find $a \in \mathbb{D}$ such that

$$\|\varphi_a^k\|_{A^p_\alpha} < \|\varphi_a^k\|_{A^u_\beta},$$

under the conditions $p > u$, $\alpha > \beta$ and

$$\beta \le 0 \quad \text{or} \quad \frac{u}{\beta} > \frac{p}{\alpha}.$$

Using the change of variables $z = \varphi_a(\zeta)$, we have

$$\begin{aligned}
\|\varphi_a^k\|^p_{A^p_\alpha} &= (\alpha+1)\int_{\mathbb{D}} |\varphi_a(z)|^{kp}(1-|z|^2)^\alpha \, dA(z) \\
&= (\alpha+1)\int_{\mathbb{D}} |\zeta|^{kp}(1-|\zeta|^2)^\alpha \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{\alpha+2} dA(\zeta) \\
&= (1-|a|^2)^{\alpha+2}\Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{kp}{2}+1\right)}{\Gamma\left(n+\frac{kp}{2}+\alpha+2\right)} c^2_{\alpha+2}(n)|a|^{2n}.
\end{aligned}$$

The dominated convergence theorem ($|\varphi_a| < 1$ in $\mathbb{D}$ for all $a \in \mathbb{D}$) yields that

$$\lim_{|a|\to 1^-} \{\|\varphi_a^k\|_{A^p_\alpha}\} = 1.$$

Assume that $a \in (0,1)$ and consider the auxiliary function

$$G(a) := \|\varphi^k_{\sqrt{a}}\|^p_{A^u_\beta} - \|\varphi^k_{\sqrt{a}}\|^p_{A^p_\alpha}.$$

Since $\lim\limits_{a\to 1^-} G(a) = 0$, it is enough to find some $\delta = \delta(p,u,\alpha,\beta) > 0$ such that G is decreasing in $(\delta, 1)$.

It is immediate that

$$G'(a) = \frac{p}{u}\|\varphi^k_{\sqrt{a}}\|^{u\left(\frac{p}{u}-1\right)}_{A^u_\beta} \frac{d}{da}\|\varphi^k_{\sqrt{a}}\|^u_{A^u_\beta} - \frac{d}{da}\|\varphi^k_{\sqrt{a}}\|^p_{A^p_\alpha}.$$

Because of the expression for $\|\varphi^k_{\sqrt{a}}\|^p_{A^p_\alpha}$ using power series, we deduce

$$\begin{aligned}
\frac{d}{da}\|\varphi^k_{\sqrt{a}}\|^p_{A^p_\alpha} &= \frac{d}{da}\left((1-a)^{\alpha+2}\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{kp}{2}+1\right)}{\Gamma\left(n+\frac{kp}{2}+\alpha+2\right)}c^2_{\alpha+2}(n)a^n\right)\\
&= \frac{kp}{2}(1-a)^{\alpha+1}\sum_{n=0}^{\infty}\frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)}c_{\alpha+1}(n)a^n.
\end{aligned}$$

Observe that, in particular, $\|\varphi^k_{\sqrt{a}}\|^p_{A^p_\alpha}$ is an increasing function of a for any p and α. In addition, the statement (a) of Lemma 19 implies that $\lim\limits_{a\to 1^-} G'(a)=0$.

Using the notation,

$$\begin{aligned}
A_{n,p,k,\alpha} &:= \frac{(n+\alpha+1)\Gamma(n+\alpha+3)\Gamma\left(n+\frac{kp}{2}+1\right)}{(n+1)!\Gamma\left(n+\frac{kp}{2}+\alpha+3\right)}c_{\alpha+1}(n),\\
B_{n,p,k,\alpha} &:= \frac{\Gamma(n+\alpha+3)\Gamma(n+\alpha+2)\Gamma\left(n+\frac{kp}{2}+1\right)}{\Gamma(\alpha+1)(n+2)!n!\Gamma\left(n+\frac{kp}{2}+\alpha+4\right)}\left(n\left((\alpha+2)\frac{kp}{2}-\alpha\right)\right.\\
&\quad \left.+\left(\alpha(\alpha+3)+4\right)\frac{kp}{2}-\alpha(\alpha+3)\right),
\end{aligned}$$

we have

$$\frac{d}{da}\left((1-a)^{\alpha+1}\sum_{n=0}^{\infty}A_{n,p,k,\alpha}a^n\right)=(1-a)^{\alpha}\sum_{n=0}^{\infty}B_{n,p,k,\alpha}a^n,$$

and then

$$\begin{aligned}
\frac{2}{kp}G''(a) &= \left(\frac{p}{u}-1\right)\|\varphi^k_{\sqrt{a}}\|^{u\left(\frac{p}{u}-2\right)}_{A^u_\beta}\frac{d}{da}\|\varphi^k_{\sqrt{a}}\|^u_{A^u_\beta}(1-a)^{\beta+1}\sum_{n=0}^{\infty}A_{n,u,k,\beta}a^n\\
&+ \|\varphi^k_{\sqrt{a}}\|^{u\left(\frac{p}{u}-1\right)}_{A^u_\beta}(1-a)^{\beta}\sum_{n=0}^{\infty}B_{n,u,k,\beta}a^n-(1-a)^{\alpha}\sum_{n=0}^{\infty}B_{n,p,k,\alpha}a^n.
\end{aligned}$$

Note that $B_{n,p,k,\alpha}, B_{n,u,k,\beta}\ge 0$ due to the hypotheses $ku\ge 2$ and $p>u$. At this point, we analyse the different cases that we can encounter.

- **Case 1:** If $-1<\beta<\alpha\le 0$, Euler-Gauss formula implies

$$\lim_{n\to\infty} nB_{n,p,k,\alpha}=\frac{(\alpha+2)\frac{kp}{2}-\alpha}{\Gamma(\alpha+1)}\lim_{n\to\infty}n^{\alpha},$$

  and thus that sequence is convergent due to the assumptions on α. Then, there exist constants $K,C>0$ such that

$$G''(a)>(1-a)^{\alpha}\left(K(1-a)^{\beta-\alpha}-C\log\frac{1}{1-a}\right),$$

and then there exists $\delta > 0$ such that G'' is positive in $(\delta, 1)$. This shows that G' is negative in $(\delta, 1)$ and therefore G is decreasing in the same interval. In other words,

$$\|\varphi^k_{\sqrt{a}}\|_{A^p_\alpha} < \|\varphi^k_{\sqrt{a}}\|_{A^u_\beta}, \quad \forall a \in (\delta, 1).$$

- **Case 2:** If $-1 < \beta < 0 < \alpha$, using the fact that the inclusion $A^p \subset A^p_\alpha$ is contractive, we can apply the previous case to find a disc automorphism such that

$$\|\varphi^k\|_{A^p_\alpha} \leq \|\varphi^k\|_{A^p} < \|\varphi^k\|_{A^u_\beta}.$$

- **Case 3:** If $\beta = 0$, part (b) of Lemma 19 yields that there exists $K, C > 0$ such that

$$G''(a) > K \log \frac{1}{1-a} - C,$$

and then $G'' > 0$ in some interval $(\delta, 1)$, from where we prove the statement of the proposition.

- **Case 4:** Finally, if $0 < \beta$, it follows from part (b) of Lemma 19 that

$$\frac{2}{kp} \lim_{a \to 1^-} G''(a) = k\left(\frac{u}{\beta} - \frac{p}{\alpha}\right) > 0,$$

showing that G'' must be positive in some $(\delta, 1)$, and then the statement of the proposition holds.

$\square$

Summing up, we can collect all the results in the following theorem.

Theorem 21. *Assume $p \geq u$ and $a \leq b$. Then the inclusion $H(p,q,a) \subset H(u,v,b)$ is contractive if, and only if,*

$$q \leq v \quad \text{or} \quad q > v \text{ and } aq \leq vb.$$

Corollary 22. *Suppose $p \geq q$ and $A^p_\alpha \subset A^q_\beta$. Then the corresponding inclusion operator is contractive if and only if $\alpha \leq \beta$.*

4.2 An open problem for $p < u$

Observe that a key property used throughout Section 4.1 is Hölder's inequality. That is, the inequality

$$M_u(r, f) \leq M_p(r, f),$$

if $u \leq p$, plays a vital role.

However, since $p < u$ in the present section, the techniques used previously cannot be employed to obtain sharp estimates. This results in a much more complicated problem than the one studied in Section 4.1, and therefore we need to develop new tools in order to deduce analogous results. A unified result like Theorem 21 seems to be extremely unlikely, so new *ad hoc* strategies have to be developed for each kind of inclusion.

This section is devoted to the study of a specific open question, which is of relevance in view of its possible applications to other fields of Mathematics, which we briefly review later.

Unfortunately we are not able to solve it at the time of writing this dissertation, but we are going to provide some partial answers.

4.2.1 Statement of the conjecture

The starting point of the problem is the so-called *Carleman's inequality*, which we state below.

Theorem I (Carleman's inequality [32]). *For every $p > 0$,*

$$\|f\|_{A^{2p}} \leq \|f\|_{H^p}, \tag{4.3}$$

and equality is possible if and only if $f(z) = c(1 - \overline{\zeta} z)^{-\frac{2}{p}}$ for some $c \in \mathbb{C}$ and $\zeta \in \mathbb{D}$.

That is, the inclusion operator from H^p to A^{2p} is contractive. Regarding the history of this problem, G. H. Hardy and J. E. Littlewood proved the inclusion of H^p in a range of mixed norm spaces (for example, see [51, Theorem 31] or [38, Theorem 5.11]), including $A^q_{\frac{q}{p}-2}$ for $p < q$. However, it seems that T. Carleman [32] was the first author who deduced (4.3) (although his proof was not complete since Riesz's factorization had not been published yet) and its connection with the classical isoperimetric inequality. See [74], [97, Theorem 19.9] or [101] for later references.

In 1987, J. Burbea extended Carleman's inequality to a larger class of standard weighted Bergman spaces.

Theorem J (Burbea [31]). *Let $p > 0$ and k a positive integer. Then, for any $f \in H^p$ we have*

$$\|f\|_{A^{kp}_{k-2}} \leq \|f\|_{H^p}. \tag{4.4}$$

In addition, equality is possible if and only if $f(z) = c(1-\overline{\zeta}z)^{-\frac{2}{p}}$ for some $c \in \mathbb{C}$ and $\zeta \in \mathbb{D}$.

Heavily inspired by an application of such inequalities to Hardy spaces of Dirichlet series by H. Helson [60], in 2018 O. F. Brevig, J. Ortega-Cerdà, K. Seip and J. Zhao [29] conjectured that (4.4) should hold for any $k \geq 1$, not necessarily an integer. In fact, it seems that this question has been of interest to experts even before, but it was in [29] where an overwhelming evidence in favour of this conjecture was shown.

After a suitable normalization using Riesz's factorization, their conjecture can be stated as follows.

Conjecture A (Brevig, Ortega-Cerdà, Seip, Zhao [29]). *If $p > 2$, then*

$$\|f\|_{A^p_{\frac{p}{2}-2}} \leq \|f\|_{H^2}, \quad \forall f \in H^2,$$

and equality is attained if and only if f is a constant multiple of the Szegő kernel

$$k_\zeta(z) = \frac{1}{1-\overline{\zeta}z}.$$

Remark. *The original statement of Conjecture A in [29] is*

$$\sqrt{\sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{2/q}(n)}} = \|f\|_{A^2_{\frac{2}{q}-2}} \leq \|f\|_{H^q}, \quad \forall f \in H^q. \tag{4.5}$$

if $\{a_n\}_{n\geq 0}$ are the Taylor coefficients of f and $0 < q \leq 2$.

Recently, A. Kulikov [67] proved that there exists $\delta = \delta(q) > 0$ such that

$$\sqrt{|a_0|^2 + \frac{q}{2}|a_1|^2 + \delta \sum_{n=2}^{\infty} \frac{|a_n|^2}{(n+1)^{\frac{2}{q}-1}}} \leq \|f\|_{H^q}.$$

This implies that (4.5) *holds, at least, for linear polynomials (fact that was previously obtained by A. Bondarenko, W. Heap and K. Seip [27]) and that the asymptotic for the remainder is correct.*

A brief description of Helson's inequality and the application of (4.5) to Hardy spaces of Dirichlet spaces can be found in Appendix B.

With a view to applications in Mathematical Physics, E. H. Lieb and J. P. Solovej stated that a similar inequality to Conjecture A should exist for any A^2_α (see [69]).

Conjecture B (Lieb, Solovej). *Let $\alpha > -1$. If $p > 2$, then*

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \le \|f\|_{A^2_\alpha}, \quad \forall f \in A^2_\alpha.$$

In addition to this, equality is possible if and only if f is a constant multiple of the Bergman kernel

$$K_\zeta(z) = \frac{1}{(1-\overline{\zeta}z)^{\alpha+2}}.$$

Remark. *In relation to other spaces present in this dissertation, observe that, if true, Conjecture B would imply that*

$$\|f\|_{B^{p,2}} \le \|f\|_{\mathscr{D}}, \quad \forall f \in \mathscr{D},$$

if $p > 2$.

Recalling that H^2 can be understood as A^2_{-1}, we will provide a unified strategy to study Conjectures A and B. Indeed, (1.6) yields that the inclusion of H^2 in $A^p_{\frac{p}{2}-2}$ is contractive if Conjecture B is true for any $\alpha > -1$. The converse result does not seem to be straightforward, but there appears to be a strong feeling among the experts that a possible solution of Conjecture A could lead to an answer to Conjecture B.

4.2.2 Vanishing functions and reformulation of the problem

Theorem 23. *Let $\Lambda \subset \mathbb{N} \cup \{0\}$ be a finite set. Then, for every $\alpha \ge -1$ and $p > 2$, the following identity holds*

$$\sup_{\|f\|_{A^2_\alpha} \le 1} \left\{ \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \right\} = \sup \left\{ \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \ : \ \|f\|_{A^2_\alpha} \le 1, f^{(j)}(0) = 0 \textit{ for all } j \in \Lambda \right\}.$$

Proof. Let M_Λ be the restricted supremum on the right, and take $f \in A^2_\alpha$ such that $\|f\|_{A^2_\alpha} \le 1$. Consider $\{r_n\}_{n\ge1} \subset (0,1)$ any sequence convergent to 1.

We recall that, after a suitable choice of the parameters, the operators introduced in (2.2)

$$T_a(f)(z) := \left(\varphi_a'(z)\right)^{\frac{\alpha+2}{2}} f\left(\varphi_a(z)\right), \quad a \in \mathbb{D}, \tag{4.6}$$

are isometries in both A^2_α and $A^p_{\frac{p}{2}(\alpha+2)-2}$. Therefore, the sequence of functions

$$f_n := T_{r_n}(f), \quad n \geq 1,$$

verifies that $\|f_n\|_{A^2_\alpha} = \|f\|_{A^2_\alpha}$ and $\|f_n\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} = \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}}$ for all n.

Then, using triangle inequality and the fact that $\|f_n - P_\Lambda(f_n)\|_{A^2_\alpha} \leq \|f_n\|_{A^2_\alpha} \leq 1$,

$$\begin{aligned} \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} &= \|f_n\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \\ &\leq \|P_\Lambda(f_n)\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} + \|f_n - P_\Lambda(f_n)\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \\ &\leq \sum_{j\in\Lambda} \frac{|f_n^{(j)}(0)|}{j!} \|z^j\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} + M_\Lambda. \end{aligned}$$

Since Λ is finite, it is enough to prove that $f_n^{(j)}(0) \to 0$ when $n \to \infty$ in order to see that M_Λ is an admissible bound for the unrestricted supremum.

Using Cauchy's integral formula, we have

$$\begin{aligned} \frac{|f_n^{(j)}(0)|}{j!} &= \left| \frac{1}{2\pi i} \int_{|z|=\frac{1}{2}} \frac{f_n(z)}{z^{j+1}} dz \right| \\ &\leq \frac{2^{j-1}}{\pi} \left(\frac{4}{3}\right)^{\frac{\alpha+2}{2}} \int_0^{2\pi} \left(1 - \left|\varphi_{r_n}(2^{-1}e^{it})\right|^2\right)^{\frac{\alpha+2}{2}} \left|f\left(\varphi_{r_n}(2^{-1}e^{it})\right)\right| dt. \end{aligned}$$

The function f belongs to A^2_α, and consequently we know that

$$\lim_{|z|\to 1^-} \left\{ |f(z)|(1-|z|^2)^{\frac{\alpha+2}{2}} \right\} = 0.$$

Also, it is an elementary computation to see that the interval $\left[\frac{r_n - 2^{-1}}{1-2^{-1}r_n}, \frac{r_n+2^{-1}}{1+2^{-1}r_n}\right]$ is a diameter of the circle $\varphi_{r_n}(2^{-1}\mathbb{T})$. Then, it is immediate that $f_n^{(j)}(0)$ must converge to 0. $\square$

Remark. *A frequent choice for Λ will be the discrete interval $\{0, \ldots, n\}$, $n \geq 0$. In other words, we can restrict our attention to A^2_α functions with a zero of order arbitrary large at the origin. If $\alpha = -1$ (that is, in the H^2 setting) this strategy is completely against the Riesz's factorization procedure described in Chapter 1. In other words, we are going to use non-standard techniques in order to deduce an equivalent formulation of the conjecture.*

The other key idea of this section is that we will not use the standard expression for $\|\cdot\|_{A^p_{\frac{p}{2}(\alpha+2)-2}}$. Instead, we are going to employ an alternative expression for this norm deduced from the Hardy-Stein identity.

Indeed, using integration by parts, the Hardy-Stein identity, Fubini's theorem and an appropriate changes of variables, we obtain that the $A^p_{\frac{p}{2}(\alpha+2)-2}$ norm of $f \in A^2_\alpha$ can be computed as follows.

$$\begin{aligned}
\|f\|^p_{A^p_{\frac{p(\alpha+2)}{2}-2}} &= \left(\frac{p(\alpha+2)}{2}-1\right)\int_0^1 2r(1-r^2)^{\frac{p(\alpha+2)}{2}-2} M_p^p(r,f)\,dr \\
&= |f(0)|^p + \int_0^1 (1-r^2)^{\frac{p(\alpha+2)}{2}-1}\frac{d}{dr}M_p^p(r,f)\,dr \\
&= |f(0)|^p + \frac{p^2}{2}\int_0^1 \frac{(1-r^2)^{\frac{p(\alpha+2)}{2}-1}}{r}\int_{r\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}\,dA(z)\,dr \\
&= |f(0)|^p + \frac{p^2}{2}\int_{\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}\int_{|z|}^1 \frac{(1-r^2)^{\frac{p(\alpha+2)}{2}-1}}{r}\,dr\,dA(z) \\
&= |f(0)|^p + \frac{p^2}{4}\int_{\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}\int_{|z|^2}^1 \frac{(1-x)^{\frac{p(\alpha+2)}{2}-1}}{x}\,dx\,dA(z) \\
&= |f(0)|^p + \frac{p^2}{4}\int_{\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}}\int_0^1 \frac{s^{\frac{p(\alpha+2)}{2}-1}}{1-(1-|z|^2)s}\,ds\,dA(z).
\end{aligned}$$

This expression for the norm $\|\cdot\|_{A^p_{\frac{p}{2}(\alpha+2)-2}}$ and Theorem 23 are enough to prove the next equivalence.

Theorem 24. *Let $\alpha \ge -1$, $p > 2$ and $C \ge 1$. The following statements are equivalent:*

(a) For any $f \in A^2_\alpha$,

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \le C\|f\|_{A^2_\alpha}.$$

(b) For every $f \in A^2_\alpha$ such that $\|f\|_{A^2_\alpha} = 1$, the estimate

$$|f(0)|^p + \frac{p}{2}\int_{\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}}\int_0^1 \frac{s^{\alpha+1}}{1-(1-|z|^2)s}\,ds\,dA(z) \le C^p,$$

holds.

(c) For every $f \in A^2_\alpha$ such that $\|f\|_{A^2_\alpha} = 1$, the estimate

$$\frac{p}{2}\int_{\mathbb{D}} |f'(z)|^2|f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}}\int_0^1 \frac{s^{\alpha+1}}{1-(1-|z|^2)s}\,ds\,dA(z) \le C^p,$$

holds.

(d) *There exists* $M > 0$ *such that for every* $f \in A^2_\alpha$ *with* $\|f\|_{A^2_\alpha} = 1$ *and* $f^{(j)}(0) = 0$*,* $0 \le j \le M$*, the estimate*

$$\frac{p}{2}\int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}} \int_0^1 \frac{s^{\alpha+1}}{1-(1-|z|^2)s}\, ds\, dA(z) \le C^p,$$

holds.

Proof. If $z \in \mathbb{D}\setminus\{0\}$, integration by parts yields that

$$\begin{aligned} H(p,z) &:= \frac{p}{2}\int_0^1 \frac{s^{\frac{p(\alpha+2)}{2}-1}}{1-(1-|z|^2)s}\, ds \\ &= \frac{1}{\alpha+2}\left(\frac{1}{|z|^2} - (1-|z|^2)\int_0^1 \frac{s^{\frac{p(\alpha+2)}{2}}}{\left(1-(1-|z|^2)s\right)^2} ds\right). \end{aligned} \tag{4.7}$$

In particular, $H(p,z)$ is an increasing function of p. Then,

$$|f(0)|^p + \frac{p}{2}\int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}} \int_0^1 \frac{s^{\alpha+1}}{1-(1-|z|^2)s}\, ds\, dA(z) \le \|f\|^p_{A^p_{\frac{p}{2}(\alpha+2)-2}},$$

for every $f \in A^2_\alpha$. Therefore, (a) implies (b).

It is trivial that (b) implies (c), and that (c) implies (d).

Finally, assume that (d) is true. Due to Theorem 23, it is enough to prove that for any $\varepsilon > 0$ there exists a sufficiently large m such that

$$\|f\|^p_{A^p_{\frac{p}{2}(\alpha+2)-2}} \le C^p + \varepsilon,$$

if $\|f\|_{A^2_\alpha} = 1$ and $f^{(j)}(0) = 0$ for every $0 \le j \le m$.

Observe that (4.7) yields that $H(p,z)$ is a concave function of p, and therefore its graph lies below the tangent line at any point. In particular, we have

$$H(p,z) \le H(2,z) + (p-2)\frac{\partial H}{\partial p}(2,z), \quad \forall z \in \mathbb{D}\setminus\{0\},$$

for any $p > 2$. We are going to estimate the integrals

$$\begin{aligned} I_1 &:= \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}} \frac{\partial H}{\partial p}(2,z)\, dA(z), \\ I_2 &:= \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}} H(2,z)\, dA(z). \end{aligned}$$

Using the geometric series and the monotone convergence theorem, we obtain

$$H(p,z) = \frac{p}{2}\int_0^1 \sum_{n=0}^{\infty} s^{\frac{p(\alpha+2)}{2}+n-1}(1-|z|^2)^n\, ds = \frac{p}{2}\sum_{n=0}^{\infty}\frac{(1-|z|^2)^n}{n+\frac{p(\alpha+2)}{2}},$$

and, similarly,

$$\frac{\partial H}{\partial p}(2,z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n(1-|z|^2)^n}{(n+\alpha+2)^2}. \tag{4.8}$$

Using the pointwise estimates of A^2_α (Theorem E), (4.8), Parseval's identity and the monotone convergence theorem, we have

$$\begin{aligned} I_1 &\leq \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{\alpha+2}\frac{\partial H}{\partial p}(2,z)\, dA(z) \\ &= \frac{1}{2}\sum_{k=m+1}^{\infty}|a_k|^2k^2\int_0^1 2r^{2(k-1)+1}(1-r^2)^{\alpha+2}\frac{\partial H}{\partial p}(2,r)\, dr \\ &= \frac{1}{2}\sum_{k=m+1}^{\infty}|a_k|^2k^2\sum_{n=1}^{\infty}\frac{n}{(n+\alpha+2)^2}B(k,n+\alpha+3), \end{aligned}$$

where $\{a_k\}_{k\geq m+1}$ is the sequence of Taylor coefficients of f. Then, since $\|f\|_{A^2_\alpha}=1$, we deduce

$$I_1 \leq \frac{1}{2\Gamma(\alpha+2)}\sup_{k\geq m+1}\left\{\frac{\Gamma(k+\alpha+2)k}{(k-1)!}\sum_{n=1}^{\infty}\frac{n}{(n+\alpha+2)^2}B(k,n+\alpha+3)\right\}.$$

Observe that the last bound for I_1 is uniform in f. We are going to check that

$$\lim_{k\to\infty}\frac{\Gamma(k+\alpha+2)k}{(k-1)!}\sum_{n=1}^{\infty}\frac{n}{(n+\alpha+2)^2}B(k,n+\alpha+3) = 0.$$

Consider the probability measure $d\sigma(x) := (\alpha+4)(1-x)^{\alpha+3}\,dx$ in $(0,1)$. Then, Chebyshev's inequality (Theorem O) yields that

$$\begin{aligned} B(k,n+\alpha+3) &= \frac{1}{\alpha+4}\int_0^1 x^{k-1}(1-x)^{n-1}\, d\sigma(x) \\ &\leq \frac{1}{\alpha+4}\int_0^1 x^{k-1}\, d\sigma(x)\int_0^1(1-x)^{n-1}\, d\sigma(x) \\ &= \frac{\alpha+4}{n+\alpha+3}B(k,\alpha+4) \\ &= \frac{\Gamma(\alpha+5)(k-1)!}{(n+\alpha+3)\Gamma(k+\alpha+4)}, \quad \forall n\geq 1. \end{aligned}$$

Thus,

$$\frac{\Gamma(k+\alpha+2)k}{(k-1)!}\sum_{n=1}^{\infty}\frac{n}{(n+\alpha+2)^2}B(k,n+\alpha+3)\lesssim\frac{k}{(k+\alpha+3)(k+\alpha+2)}.$$

In other words, we have proved that for every $\varepsilon>0$ there exists m large enough such that

$$I_1\leq\frac{2}{p(p-2)}\varepsilon,$$

if $f(0)=\ldots=f^{(m)}(0)=0$. Since the assumption (d) is true, without loss of generality we can assume that $m>M$ and then

$$\frac{p}{2}I_2\leq C^p.$$

Summarising, we have shown that

$$\|f\|^p_{A^p_{\frac{p}{2}(\alpha+2)-2}}\leq\frac{p}{2}I_2+\frac{p(p-2)}{2}I_1\leq C^p+\varepsilon,$$

if $\|f\|_{A^2_\alpha}=1$ and its first m Taylor coefficients are zero. Thus, the inclusion operator from A^2_α to $A^p_{\frac{p}{2}(\alpha+2)-2}$ has norm less than or equal to C if (d) is true. □

Remark. *Roughly speaking, Theorem 24 could be understood as follows: we can subtract from the integral kernel*

$$\int_0^1\frac{s^{\frac{p(\alpha+1)}{2}-1}}{1-(1-|z|^2)s}\,ds,$$

any quantity with a decay of at least $(1-|z|^2)$*,* $|z|\to1^-$*, and the resulting maximisation problem is essentially equivalent.*

Following this idea, another alternative quantity to maximise is

$$\frac{p}{2(\alpha+2)}\int_{\mathbb{D}}|f'(z)|^2|f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}}\,dA(z). \tag{4.9}$$

However, expression

$$\frac{p}{2}\int_{\mathbb{D}}|f'(z)|^2|f(z)|^{p-2}(1-|z|^2)^{\frac{p(\alpha+2)}{2}}\int_0^1\frac{s^{\alpha+1}}{1-(1-|z|^2)s}\,ds\,dA(z),$$

seems to be more suitable because the measure

$$d\sigma_f(z):=|f'(z)|^2(1-|z|^2)^{\alpha+2}\int_0^1\frac{s^{\alpha+1}}{1-(1-|z|^2)s}\,ds\,dA(z),$$

is a probability measure in $\mathbb{D}$ *if* $\|f\|_{A^2_\alpha}=1$ *and* $f(0)=0$*.*

As a sample application of Theorem 24 to a possible solution to the conjecture, we are going to give an alternative proof for $p = 2k$, case that has already been solved by several authors like O. F. Brevig, J. Ortega-Cerdà, K. Seip and J. Zhao [29], E. H. Lieb and J. P. Solovej [69] or D. Békollè, J. Gonessa and B. F. Sehba for $\alpha = 0$ [17].

Theorem K (Extended Carleman's inequality). *If $\alpha \geq -1$ and k is a positive integer, then*

$$\|f\|_{A^{2k}_{k(\alpha+2)-2}} \leq \|f\|_{A^2_\alpha}, \quad \forall f \in A^2_\alpha.$$

Proof. We will provide a proof by induction. If $k = 1$ the result is obvious (the two spaces are the same).

Assume that the inclusion $A^2_\alpha \subset A^{2k}_{k(\alpha+2)-2}$ is contractive for some k, and take f with $\|f\|_{A^2_\alpha} = 1$. We are going to see that the expression (4.9),

$$I_{k+1} := \frac{k+1}{\alpha+2} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{2k} (1-|z|^2)^{(k+1)(\alpha+2)}\, dA(z)$$

can be bounded by 1.

Let $g = f^{k+1}$. If $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ are the sequences of Taylor coefficients of f, f^k and f^{k+1}, respectively, then

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}.$$

Therefore, using the chain rule and Parseval's identity, we have

$$\begin{aligned} I_{k+1} &= \frac{1}{(k+1)(\alpha+2)} \int_{\mathbb{D}} |g'(z)|^2 (1-|z|^2)^{(k+1)(\alpha+2)}\, dA(z) \\ &= \frac{1}{(k+1)(\alpha+2)} \sum_{n=1}^{\infty} n^2 \int_0^1 x^{n-1}(1-x)^{(k+1)(\alpha+2)}\, dx |c_n|^2 \\ &= \frac{1}{(k+1)(\alpha+2)} \sum_{n=1}^{\infty} n^2 B\big(n, (k+1)(\alpha+2)+1\big) |c_n|^2. \end{aligned}$$

On the other hand, due to the Cauchy-Schwarz inequality,

$$\begin{aligned} |c_n|^2 &\leq \left(\sum_{j=0}^{n} |a_j||b_n - j| \right)^2 \\ &\leq \left(\sum_{j=0}^{n} c_{\alpha+2}(j) c_{k(\alpha+2)}(n-j) \right) \left(\sum_{j=0}^{n} \frac{|a_j|^2}{c_{\alpha+2}(j)} \frac{|b_{n-j}|^2}{c_{k(\alpha+2)}(n-j)} \right). \end{aligned}$$

Using the fact that $\{c_\beta(n)\}_{n\geq 0}$ is the sequence of Taylor coefficients of $(1-z)^{-\beta}$ (or, alternatively, the properties of Γ, B and binomial theorem), it is clear that

$$\sum_{j=0}^{n} c_{\alpha+2}(j)c_{k(\alpha+2)}(n-j) = c_{(k+1)(\alpha+2)}(n), \quad \forall n \geq 0,$$

and therefore

$$\frac{n^2 B\big(n,(k+1)(\alpha+2)+1\big)}{(k+1)(\alpha+2)} c_{(k+1)(\alpha+2)}(n) = \frac{n}{n+(k+1)(\alpha+2)}, \quad \forall n \geq 1. \tag{4.10}$$

Summarising,

$$\begin{aligned} I_{k+1} &\leq \sum_{n=1}^{\infty} \frac{n}{n+(k+1)(\alpha+2)} \sum_{j=0}^{n} \frac{|a_j|^2}{c_{\alpha+2}(j)} \frac{|b_{n-j}|^2}{c_{k(\alpha+2)}(n-j)} \\ &\leq \|f\|^2_{A^2_\alpha} \|f^k\|^2_{A^2_{k(\alpha+2)-2}} \\ &= \|f\|^{2k}_{A^{2k}_{k(\alpha+2)-2}}, \end{aligned}$$

and then the inclusion of A^2_α in $A^{2(k+1)}_{(k+1)(\alpha+2)-2}$ is contractive due to the induction hypothesis. $\square$

Finally, we are going to derive a new necessary and sufficient condition for the contractivity.

Lemma 25. *Let $\alpha \geq -1$ and $p > 2$. If f verifies that $\|f\|_{A^2_\alpha} = 1$ and*

$$|f(0)| = \sup_{z\in\mathbb{D}} \left\{ |f(z)|(1-|z|^2)^{\frac{\alpha+2}{2}} \right\},$$

then

$$F_p(f) := |f(0)|^p + \frac{p}{2} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{\frac{p(\alpha+2)}{2}} \int_0^1 \frac{s^{\alpha+1}}{1-(1-|z|^2)s}\, ds\, dA(z) \leq 1,$$

and equality is possible if and only if f is constant.

Proof. By the conditions on f, we have

$$F_p(f) \leq |f(0)|^{p-2} \left(1 + \frac{p}{2}(1-|f(0)|^2)\right).$$

If $p > 2$, a direct computation shows that the function $x^{p-2}\left(x^2 + \frac{p}{2}(1-x^2)\right)$ is an strictly increasing function of x in $[0,1]$. In other words, $F_f(p) \leq 1$.

On the other hand, equality is possible only if $|f(0)| = 1 = \|f\|_{A^2_\alpha}$, and thus f must be constant. $\square$

Theorem 26. *Let $p > 2$ and $\alpha \geq -1$. The following statements are equivalent:*

(a) *The inclusion $A^2_\alpha \subset A^p_{\frac{p}{2}(\alpha+2)-2}$ is contractive.*

(b) *For any f such that $\|f\|_{A^2_\alpha} = 1$, there exists g with $\|g\|_{A^2_\alpha} = 1$ and*

$$|g(0)| = \sup_{z\in\mathbb{D}} \left\{ |g(z)|(1-|z|^2)^{\frac{\alpha+2}{2}} \right\},$$

such that

$$F_p(f) \leq F_p(g).$$

Proof. If the inclusion is contractive, then Theorem 24 yields that

$$F_p(f) \leq 1, \quad \text{if} \quad \|f\|_{A^2_\alpha} = 1,$$

an so the constant function 1 is a possible choice for g.

On the other hand, if we can find such a function g for each normalised $f \in A^2_\alpha$, then Lemma 25 implies that

$$F_p(f) \leq F_p(g) \leq 1,$$

and Theorem 24 yields that the inclusion of A^2_α in $A^p_{\frac{p}{2}(\alpha+2)-2}$ is contractive. □

Remark. *Unfortunately, the quantity $F_p(f)$ is not invariant under the isometries T_a introduced in* (4.6). *A straightforward counterexample is given by the normalised reproducing kernels. That is, if $p > 2$, $\zeta \in \mathbb{D}\setminus\{0\}$ and $\tilde{K}_\zeta = \frac{K_\zeta}{\|K_\zeta\|_{A^2_\alpha}}$, we see that*

$$F_p(\tilde{K}_\zeta) < \|\tilde{K}_\zeta\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} = 1 = F_p\big(T_\zeta(\tilde{K}_\zeta)\big),$$

since $T_\zeta(\tilde{K}_\zeta)$ is constant.

4.2.3 Explicit bounds

Finally, we will find explicit values for C in order to get

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \leq C\|f\|_{A^2_\alpha}, \quad \forall f \in A^2_\alpha,$$

if p is not an even integer.

Corollary 27. *If $\alpha \geq -1$ and $2(k-1) < p \leq 2k$ for some integer $k \geq 2$, then*

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \leq \left(\frac{p}{2}\right)^{\frac{1}{p}} \frac{1}{(k-1)^{\frac{k}{p}-\frac{1}{2}} k^{\frac{1}{2}-\frac{k-1}{p}}} \|f\|_{A^2_\alpha}, \quad \forall f \in A^2_\alpha.$$

Proof. Take $f \in A^2_\alpha$ such that $\|f\|_{A^2_\alpha} = 1$ and $f(0) = \ldots = f^{(m)}(0) = 0$. We will prove that

$$I_p := \frac{p}{2(\alpha+2)} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{p-2} (1-|z|^2)^{\frac{p(\alpha+2)}{2}} dA(z) \le \frac{p}{2} \frac{1}{(k-1)^{k-\frac{p}{2}} k^{\frac{p}{2}-k+1}} + o(1), \quad m \to \infty.$$

Note that, because of (4.10) and Theorem K,

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{2(k-2)} (1-|z|^2)^{(k-1)(\alpha+2)} \, dA(z) &\le \frac{\alpha+2}{k-1} + o(1), \\ \int_{\mathbb{D}} |f'(z)|^2 |f(z)|^{2(k-1)} (1-|z|^2)^{k(\alpha+2)} \, dA(z) &\le \frac{\alpha+2}{k} + o(1), \end{aligned}$$

if $m \to \infty$ uniformly in f.

Since $p \in \big(2(k-1), 2k\big]$, then $p^{-1} = \theta\big(2(k-1)\big)^{-1} + (1-\theta)(2k)^{-1}$ for some $\theta \in [0,1)$. It is elementary to see that

$$\theta = (k-1)\left(\frac{2k}{p} - 1\right) \quad \text{and} \quad 1-\theta = k\left(1 - \frac{2(k-1)}{p}\right),$$

and therefore Hölder's inequality yields

$$I_p \le \frac{p}{2(\alpha+2)} \left(\frac{\alpha+2}{k-1} + o(1)\right)^{k-\frac{p}{2}} \left(\frac{\alpha+2}{k} + o(1)\right)^{\frac{p}{2}-k+1} \le \frac{p}{2} \frac{1}{(k-1)^{k-\frac{p}{2}} k^{\frac{p}{2}-k+1}} + o(1),$$

when $m \to \infty$. $\square$

For the sake of clarity, $C(p)$ will denote the bound obtained in Corollary 27.

$$C(p) := \left(\frac{p}{2}\right)^{\frac{1}{p}} \frac{1}{(k-1)^{\frac{k}{p}-\frac{1}{2}} k^{\frac{1}{2}-\frac{k-1}{p}}}, \quad 2(k-1) < p \le 2k.$$

Because of $C(2k) = 1$ and that $C(p) \ge 1$ for all p, a straightforward computation shows that the maximum of C in $\big(2(k-1), 2k\big]$ is attained at $p = 2e\frac{(k-1)^k}{k^{k-1}}$. Note that this point belongs to $\big(2(k-1), 2k\big]$ by the classical estimates

$$\left(1 + \frac{1}{k-1}\right)^{k-1} < e < \left(1 + \frac{1}{k-1}\right)^{k}, \quad \forall k \ge 2.$$

Therefore

$$C(p) \le \sqrt{\frac{k-1}{k}} e^{\frac{k^{k-1}}{2e(k-1)^k}}, \quad \forall p \in \big(2(k-1), 2k\big].$$

Lemma 28. *The function*

$$\sqrt{\frac{x-1}{x}e^{\frac{x^{x-1}}{2e(x-1)^x}}}, \quad x>1,$$

is decreasing.

Proof. We are going to show that

$$I(x) := \log\left(1-\frac{1}{x}\right)+\frac{1}{e}\left(1+\frac{1}{x-1}\right)^{x-1}\frac{1}{x-1},$$

is decreasing in $(1,\infty)$.

It is a direct computation to show that

$$I'(x) = \frac{1}{x(x-1)}\left[1+\frac{1}{e}\left(1+\frac{1}{x-1}\right)^{x-1}\left(x\log\left(1+\frac{1}{x-1}\right)-2-\frac{1}{x-1}\right)\right],$$

and consequently it is enough to see that

$$J(x) := \left(1+\frac{1}{x-1}\right)^{x-1}\left(x\log\left(1+\frac{1}{x-1}\right)-2-\frac{1}{x-1}\right), \quad x>1,$$

is increasing.

Similarly,

$$J'(x) = \left(1+\frac{1}{x-1}\right)^{x-1}\left(x\log^2\left(1+\frac{1}{x-1}\right)-\frac{2x-1}{x-1}\log\left(1+\frac{1}{x-1}\right)+\frac{x^2-x+1}{x(x-1)^2}\right).$$

Observe that the discriminant of the quadratic trinomial

$$P_x(y) := x^2(x-1)^2y^2-(2x-1)x(x-1)y+x^2-x+1,$$

is $-3x^2(x-1)^2$. Thus, $P_x(y)\geq 0$ and then J is increasing. $\square$

From Lemma 28 we deduce the uniform bound

$$\|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} \leq \frac{\sqrt[e]{e}}{\sqrt{2}}\|f\|_{A^2_\alpha} = 1.021\ldots\|f\|_{A^2_\alpha}, \quad \forall f\in A^2_\alpha,$$

for all $p>2$.

The best known constant so far was due to O. F. Brevig, J. Ortega-Cerdà, K. Seip and J. Zhao [29]. They proved that

$$\|f\|_{A^p_{\frac{p}{2}-2}} \leq \sqrt{\frac{2}{e\log 2}}\|f\|_{H^2} = 1.03\ldots\|f\|_{H^2}, \quad \forall f\in H^2,$$

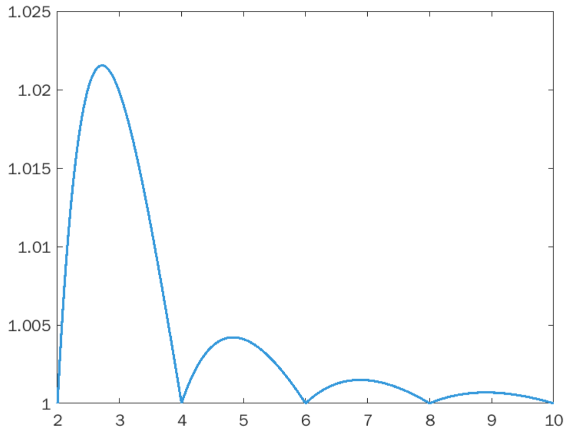

Figure 2: Plot of the function $C(p)$ in $(2,10)$ using *Matlab*.

if $p \geq 4$. Thus, Corollary 27 represents a slight improvement (both in the value of the uniform bound and in the range of admissible p) with respect to the previously published works.

Another consequence of Corollary 27 is that H^2 is the limit of $A^p_{\frac{p}{2}-2}$ in the sense of (1.6).

Corollary 29. *If $f \in H^2$,*

$$\lim_{p\to 2^+} \|f\|_{A^p_{\frac{p}{2}-2}} = \|f\|_{H^2}.$$

Proof. Without loss of generality, assume $2 < p < 4$. Due to the monotonicity of $M_p(r,f)$, we can apply Hölder's inequality and Chebyshev's inequality to deduce

$$\begin{aligned}
\|f\|^p_{A^p_{\frac{p}{2}-2}} &= \left(\frac{p}{2}-1\right)\int_0^1 2r(1-r^2)^{\frac{p}{2}-2} M_p^p(r,f)\,dr \\
&\geq \left(\frac{p}{2}-1\right)\int_0^1 2r(1-r^2)^{\frac{p}{2}-2} M_2^p(r,f)\,dr \\
&\geq \left(\frac{p}{2}-1\right)\int_0^1 2r(1-r^2)^{\frac{p}{2}-2} M_2^2(r,f)\,dr \int_0^1 2rM_2^{p-2}(r,f)\,dr.
\end{aligned}$$

That is, Corollary 27 (for $\alpha=-1$) implies

$$\|f\|^2_{A^2_{\frac{p}{2}-2}} \int_0^1 2rM_2^{p-2}(r,f)\,dr \leq \|f\|^p_{A^p_{\frac{p}{2}-2}} \leq \frac{p}{2^{\frac{p}{2}}}\|f\|^p_{H^2},$$

and the convergence follows from (1.6) and the dominated convergence theorem. $\square$

Because of

$$\lim_{p\to 2^+} \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}} = \|f\|_{A^2_\alpha}, \quad \forall f \in A^2_\alpha,$$

another open (and harder) problem is the following: Given $f \in A^2_\alpha$, is it true that its $A^p_{\frac{p}{2}(\alpha+2)-2}$ norm is a monotonic function of p?

The best result so far on this question is due to F. Bayart, O. F. Brevig, A. Haimi, J. Ortega-Cerdà and K.-M. Perfekt [16].

Theorem L (Bayart, Brevig, Haimi, Ortega-Cerdà, Perfekt [16]). *Let* $0 < p$ *and* $\beta > \frac{\sqrt{17}-7}{4}$. *Then*

$$\|f\|_{A^{\frac{p(\beta+3)}{\beta+2}}_{\beta+1}} \leq \|f\|_{A^p_\beta}, \quad \forall f \in A^p_\beta$$

and equality holds if and only if

$$f(z) = \frac{c}{(1-\overline{\zeta}z)^{\frac{2(\beta+2)}{p}}},$$

for some $c \in \mathbb{C}$ *and* $\zeta \in \mathbb{D}$.

Choosing $\beta = \frac{p}{2}(\alpha+2)-2$, we have

$$\|f\|_{A^{p+\frac{2}{\alpha+2}}_{\frac{p+2/(\alpha+2)}{2}(\alpha+2)-2}} \leq \|f\|_{A^p_{\frac{p}{2}(\alpha+2)-2}}.$$

In other words, the monotonicity is true whenever we take “steps of length $\frac{2}{\alpha+2}$”.

Conclusiones

El estudio llevado a cabo en esta tesis ha dado lugar a cuatro trabajos, los cuales detallamos a continuación:

- A. Llinares, *Norm of inclusions between some spaces of analytic functions*, Banach J. Math. Anal. **16** (2022), no. 1, Artículo No. 9, 14 pp.
  DOI: 10.1007/s43037-021-00161-7.

- A. Llinares y D. Vukotić, *Contractive inequalities for mixed norm spaces and the Beta function*. J. Math. Anal. Appl. **509** (2022), no. 1, Artículo No. 125938, 9 pp.
  DOI: 10.1016/j.jmaa.2021.125938.

- A. Llinares y D. Vukotić, *Extremal problems for vanishing functions in Bergman spaces*, Proc. Amer. Math. Soc. (Aceptado).
  DOI: 10.1090/proc/15797.

- A. Llinares, *On a conjecture about contractive inequalities for weighted Bergman spaces*. Prepublicación (2021) (Enviado a publicar).
  arXiv: 2112.09962

Muchos de los problemas tratados han sido completamente resueltos, como el cálculo de las normas del operador dado por la división por una potencia de un automorfismo del disco (para aquellas funciones con ceros en dicho punto) y la inclusión de B^q en $\mathscr{B}$; además de la caracterización de la contractividad de la inclusión de $H(p,q,a)$ en $H(u,v,b)$ siempre que $p \geq u$. No obstante, otros problemas como el cálculo de la norma precisa de la inclusión de $\mathscr{B}$ en A^p_α o la contractividad del operador inclusión de A^2_α en $A^p_{\frac{p}{2}(\alpha+2)-2}$ siguen estando abiertos, debido a su complejidad.

Es de especial relevancia este último problema, dado el interés que éste suscita por sus posibles aplicaciones en distintas áreas. La amplia evidencia en favor de la contractividad de la inclusión entre estos dos espacios, reforzada por los contenidos de esta última prepublicación, anima a seguir trabajando en busca de una solución completa.

Conclusions

The work carried out in this thesis has resulted in four publications, which we list below:

- A. Llinares, *Norm of inclusions between some spaces of analytic functions*, Banach J. Math. Anal. **16** (2022), no. 1, Paper No. 9, 14 pp.
  DOI: 10.1007/s43037-021-00161-7.

- A. Llinares and D. Vukotić, *Contractive inequalities for mixed norm spaces and the Beta function*. J. Math. Anal. Appl. **509** (2022), no. 1, Paper No. 125938, 9 pp.
  DOI: 10.1016/j.jmaa.2021.125938.

- A. Llinares and D. Vukotić, *Extremal problems for vanishing functions in Bergman spaces*, Proc. Amer. Math. Soc. (Accepted).
  DOI: 10.1090/proc/15797.

- A. Llinares, *On a conjecture about contractive inequalities for weighted Bergman spaces*. Preprint (2021) (Submitted for publication).
  arXiv: 2112.09962

Some of the problems treated in this thesis have been completely solved. For example, we have computed the precise value of the norm of the operator of dividing out by a power of a disc automorphism (for functions that vanish in the disc), or the inclusion operator of B^q in $\mathscr{B}$. In addition, we have characterised the contractivity of the inclusion of $H(p,q,a)$ in $H(u,v,b)$ whenever $p \geq u$. However, other problems such as the precise norm of the inclusion of $\mathscr{B}$ in A^p_α or the contractivity of the inclusion operator of A^2_α in $A^p_{\frac{p}{2}(\alpha+2)-2}$ are still open due to their complexity.

The latter problem is of particular relevance, given the interest in its applications to different areas. The strong evidence in favour of the contractivity between these two spaces, reinforced by the contents of this last preprint, encourages further work towards a complete solution.

Appendices

Appendix A

Elementary properties of analytic functions and classical inequalities

In this appendix we review some elementary properties of holomorphic functions which are needed in this thesis. Some of them are widely known but, for the sake of completeness, proofs will be provided. We also prove a classical inequality due to Chebyshev.

A.1 Analytic functions with radial moduli

A key property which follows from maximum modulus principle is that the only analytic functions whose moduli are *radial functions* are the monomials.

The following hypothesis can be extended to further kinds of domains but, since this dissertation is devoted to extremal functions in spaces of analytic functions on the unit disc, we will focus on $\mathbb{D}$.

Theorem M. *Let $f \in \mathscr{H}(\mathbb{D})$ such that*

$$|f(z)| = \left|f(|z|)\right|, \quad \forall z \in \mathbb{D}.$$

Then, there exist $c \in \mathbb{C}$ and $n \geq 0$ such that

$$f(z) = cz^n.$$

Proof. Assume first that f does not vanish in $\mathbb{D}$. Hence $g = \frac{1}{f}$ is also analytic, and due to maximum modulus principle it follows that, for each $r \in (0,1)$, the infimum and the supremum of $|f|$ on the disc $r\mathbb{D}$ must be attained on $r\mathbb{T}$. Since $|f|$ is a radial function, the two quantities must coincide, and therefore $|f|$ is constant on $r\mathbb{D}$. Finally, the open mapping theorem implies that f is constant.

On the other hand, suppose that f has at least one zero $\zeta \in \mathbb{D}$. If $\zeta \neq 0$, then f vanishes on the circle $|\zeta|\mathbb{T}$ and consequently, due to the identity principle for holomorphic functions, f is identically zero. Without loss of generality, assume that $\zeta = 0$ is the unique zero of f and let n be the (finite) multiplicity of this root. Thus, there exists $h \in \mathscr{H}(\mathbb{D})$ such that $h(0) \neq 0$ and

$$f(z) = z^n h(z), \quad \forall z \in \mathbb{D}.$$

Note that $|h|$ is a radial function and does not vanish in $\mathbb{D}$ (since f is not identically zero), so h must be constant. That is,

$$f(z) = h(0) z^n.$$

□

A.2 The Hardy-Stein identity

Another elementary property, which is fundamental in this doctoral thesis, is the Hardy-Stein identity, or simply Hardy's identity in some books, named after G. H. Hardy (it appeared in his remarkable work [50]) and P. Stein (who used it in order to produce an alternative proof for M. Riesz's theorem regarding the harmonic conjugates of functions in the *harmonic Hardy space*, h^p, for $1 < p$ [96]). For further historical information, see [79].

Theorem N (Hardy-Stein identity). *If $p > 0$ and $f \in \mathscr{H}(\mathbb{D})$, then*

$$\frac{d}{dr} M_p^p(r,f) = \frac{p^2}{2r} \int_{r\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \, dA(z), \quad \forall r \in (0,1).$$

The idea of the proof that we show in this text can be found in Section 8.2 of [84].

Proof. Consider the *Wirtinger derivatives*

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If $g \in \mathscr{C}^2(\mathbb{D})$, the Laplacian of g, Δg, can be rewritten in terms of the Wirtinger derivatives

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} =: \Delta g = 4\frac{\partial^2 g}{\partial z \partial \overline{z}}.$$

On the other hand, if we compute Δg using polar coordinates, we deduce

$$r^2 \Delta g = \left(r\frac{\partial}{\partial r}\right)^2 g + \frac{\partial^2}{\partial t^2} g.$$

In particular, if $p > 0$ and $f \in \mathscr{H}(\mathbb{D})$, then taking $g(z) := |f(z)|^p$ we have

$$p^2|z|^2|f(z)|^{p-2}|f'(z)|^2 = \left(r\frac{\partial}{\partial r}\right)^2 |f(z)|^p + \frac{\partial^2}{\partial t^2}|f(z)|^p,$$

whenever z is not a zero of f.

Integrating with respect to t, due to the dominated convergence theorem, we obtain

$$r\frac{d}{dr}\left(r\frac{d}{dr}M_p^p(r,f)\right) = \frac{p^2 r^2}{2\pi}\int_0^{2\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2\, dt,$$

for any $r \in (0,1)$.

Finally, integrating with respect to r, we have

$$\begin{aligned} r\frac{d}{dr}M_p^p(r,f) &= \frac{p^2}{2}\int_0^r \rho \int_0^{2\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2\, dt\, d\rho \\ &= \frac{p^2}{2}\int_{r\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\, dA(z), \end{aligned}$$

from where we deduce Hardy-Stein identity $\square$

A.3 Chebyshev's inequality

The next inequality, which was attributed to P. L. Chebyshev by C. Hermite, can be found (for discrete measures) in the classical book of G. H. Hardy, J. E. Littlewood and G. Pólya [52]. We include a proof of the general case for the sake of completeness. The proof follows the old idea from a paper of F. Franklin [48].

Theorem O (Chebyshev's inequality)**.** *Let σ a probability measure on the interval (a,b). Suppose that $f, g \in L^1\big((a,b),\sigma\big)$ verify that fg is also integrable with respect to σ and*

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0$$

for almost every (x,y) with respect to the product measure induced by σ on the square $(a,b)^2 := (a,b) \times (a,b)$. Then

$$\int_a^b f(x)g(x)\,d\sigma(x) \geq \int_a^b f(x)\,d\sigma(x) \int_a^b g(x)\,d\sigma(x).$$

Proof. Let $F(x,y) := \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)$. Since f, g and fg are integrable with respect to σ, due to Fubini's theorem

$$0 \leq \frac{1}{2}\int_{(a,b)^2} F(x,y)d\sigma(x,y) = \int_a^b f(x)g(x)d\sigma(x) - \int_a^b f(x)d\sigma(x)\int_a^b g(x)d\sigma(x),$$

proving the inequality. □

Remark. *From the proof it can be deduced that equality holds if and only if $F(x,y)=0$ almost everywhere (with respect to the product measure induced by σ on $(a,b)^2$).*

In particular, if f or g is a strictly monotonic function, equality is attained if and only if the other function is constant.

On the other hand, if f and g have different monotonicity, then

$$\int_a^b f(x)g(x)\,d\sigma(x) \leq \int_a^b f(x)\,d\sigma(x) \int_a^b g(x)\,d\sigma(x).$$

Appendix B

Hardy spaces of Dirichlet series

This appendix is a brief introduction to the inspiration of Conjecture A, posed by O. F. Brevig, J. Ortega-Cerdà, K. Seip and J.Zhao [29]. In addition to this reference, this appendix is based on the foundational papers of this family of spaces of analytic functions, due to H. Hedenmalm, P. Lindqvist and K. Seip [56] and F. Bayart [15], and H. Helson's paper [60].

B.1 Definition

We say that a series is a *(standard) Dirichlet series* if it is of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{B.1}$$

where $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ and $s = \sigma + it$. If $\{a_n\}_{n\geq 1}$ only has a finite quantity of non-zero elements (that is, if it is a finite sum), we say that (B.1) is a *Dirichlet polynomial*.

Following the idea of H. Bohr [23] of treating the quantities $z_j = p_j^{-s}$, where $\{p_j\}_{j\geq 1}$ is the sequence of prime numbers, as independent complex variables we can identify (B.1) (at least formaly) with the power series

$$\sum_{\alpha \in \mathcal{N}} a_\alpha \prod_{j=1}^{\infty} z_j^{\alpha_j}, \tag{B.2}$$

where the set $\mathcal{N}$, also known as the *narrow cone*, is the set of all sequences of non-negative integers which are eventually equal to 0. This association between Dirichlet series and power series in infinitely many variables is what we call the *Bohr lift*. This dual expression for Dirichlet series yields two equivalent ways of defining the Hardy spaces of Dirichlet series.

On the one hand, we recall that the classical H^p space, $p \geq 1$, can be identified using its boundary values as explained in (1.1). That is,

$$H^p = \{f \in L^p(\mathbb{T}) \,:\, \tilde{f}(k) = 0, \forall k < 0\}.$$

Similarly, for any positive integer d and any $p \geq 1$ we can define the d-dimensional Hardy space $H^p\left(\mathbb{D}^d\right)$ as

$$H^p\left(\mathbb{D}^d\right) := \left\{f \in L^p\left(\mathbb{T}^d\right) \,:\, \tilde{f}(k_1, \ldots, k_d) = 0 \text{ if } k_j < 0 \text{ for some } j\right\},$$

where here $\tilde{f}(k_1, \ldots, k_d)$ is the Fourier coefficient of f corresponding to the element of the basis $z_1^{k_1} \ldots z_d^{k_d}$.

In the limit case, we define the *Hardy space of Dirichlet series* $\mathscr{H}^p$, $p \geq 1$, as the set

$$\mathscr{H}^p := \left\{f \in L^p\left(\mathbb{T}^\infty\right) \,:\, \tilde{f}(\alpha) = 0 \text{ if } \alpha \notin \mathscr{N}\right\}.$$

Observe that, since the Fourier coefficients induce continuous functionals, both spaces are Banach spaces when equipped with the corresponding L^p norms.

$H^p\left(\mathbb{D}^d\right)$ functions extend to $\mathbb{D}^d$ using the convolution with a suitable Poisson kernel, whilst $\mathscr{H}^p$ can be extended to $\mathbb{D}^\infty \cap \ell^2$ due to the pointwise bound

$$|f(z)| \leq \left(\prod_{j=1}^{\infty} \frac{1}{1-|z_j|^2}\right)^{\frac{1}{p}} \|f\|_{\mathscr{H}^p}, \quad p < \infty. \tag{B.3}$$

On the other hand, $\mathscr{H}^p$ can be defined as the closure of the Dirichlet polynomials under the *Besicovitch norm*,

$$\left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\sum_{n=1}^{N} \frac{a_n}{n^{it}}\right|^p dt\right)^{\frac{1}{p}}, \tag{B.4}$$

which is actually a norm if $p \geq 1$. Indeed, the Bohr lift establishes an isometric isomorphism between the additive language of power series in the narrow cone and the multiplicative structure of Dirichlet series, so we will simply write $\|\cdot\|_{\mathscr{H}^p}$ to denote both $L^p(\mathbb{T}^\infty)$ norm and Besicovitch norm. Another advantage of (B.4) is that we can define $\mathscr{H}^p$ even for $0 < p < 1$.

Similarly to the standard Hardy space H^2, it turns out that $\mathscr{H}^2$ is also a Hilbert space and the norm of a function in it can be computed in terms of the coefficients of the Dirichlet series

$$\left\|\sum_{n=1}^{\infty}\frac{a_n}{n^s}\right\|_{\mathscr{H}^2}=\sum_{n=1}^{\infty}|a_n|^2.$$

It is easy to see that the Cauchy-Schwarz inequality implies that any Dirichlet series belonging to $\mathscr{H}^2$ is convergent in the right half-plane $\mathbb{C}_{\frac{1}{2}} := \left\{\sigma + it \,:\, \sigma > \frac{1}{2}\right\}$, and due to Bohr's theorem [24] it is in fact uniformly convergent in any half-plane contained in $\mathbb{C}_{\frac{1}{2}}$, so it is an analytic function there. Indeed, the same can be said for a general $\mathscr{H}^p$, $p \geq 1$, thanks to (B.3).

One last word regarding Hardy spaces of Dirichlet series. The space $\mathscr{H}^\infty$ can be defined in a similar way, but we do not cover this case since it is not strictly needed in order to explain the motivation of the problem treated in Section 4.2. For further information, *cf.* [56].

Now that we have introduced the $\mathscr{H}^p$ spaces, we can formulate Helson's inequality and its possible generalizations if Conjecture A is true.

B.2 Helson's inequality

In 2006, H. Helson [60] published the natural generalization of Carleman's inequality to Hardy spaces of Dirichlet series.

Theorem P (Helson's Inequality). *For any $N \geq 1$ and $a_1, \ldots, a_N \in \mathbb{C}$, we have*

$$\sqrt{\sum_{n=1}^{N}\frac{|a_n|^2}{d(n)}} \leq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\sum_{n=1}^{N}\frac{a_n}{n^{it}}\right| dt, \tag{B.5}$$

where $d(n)$ is the number of divisors of n.

In other words, if $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathscr{H}^1$, then

$$\sqrt{\sum_{n=1}^{\infty}\frac{|a_n|^2}{d(n)}} \leq \|f\|_{\mathscr{H}^1}. \tag{B.6}$$

Observe that, because H^1 is a subspace of $\mathscr{H}^1$ (their Fourier coefficients are supported only in the first variable), (B.6) is actually a refinement of Carleman's inequality.

Since Helson's argument is very elegant and it is strongly related with the content of Section 4.2, we reproduce his proof here.

Proof. We use the definition of $\mathscr{H}^1$ as a subspace of $L^1(\mathbb{T}^\infty)$. Observe that Bohr's lift yields $a_n = \tilde{f}(\alpha)$, where α is the sequence of powers of the prime decomposition of n, and thus $\alpha \in \mathscr{N}$. Then, (B.5) is equivalent to

$$\sqrt{\sum_{\alpha\in\mathscr{N}} \frac{\left|\tilde{f}(\alpha)\right|^2}{\prod_{j\geq 1}(\alpha_j+1)}} \leq \|f\|_{\mathscr{H}^1}.$$

Since trigonometric polynomials (the additive counterpart of Dirichlet polynomials) are dense in $\mathscr{H}^1$, it is enough to prove it for a polynomial

$$p(z_1,\dots,z_d) = \sum_{\alpha} \tilde{p}(\alpha_1,\dots,\alpha_d) z_1^{\alpha_1}\dots z_d^{\alpha_d},$$

where, of course, this sum is finite.

Let $\{T_j\}_{j\geq 1}$ be the family of continuous linear operators uniquely determined by

$$T_j(z^\alpha) := \frac{1}{\sqrt{\alpha_j+1}} z^\alpha, \quad \forall \alpha \in \mathscr{N}.$$

Observe that this notation allows us, once again, rewriting (B.5) for the polynomial p as

$$\|T_1\dots T_d p\|_{L^2(\mathbb{T}^d)} \leq \|p\|_{L^1(\mathbb{T}^d)}. \tag{B.7}$$

For a fixed $(t_2,\dots,t_d) \in (0,2\pi)^{d-1}$, applying Carleman's inequality to the first variable we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \left|T_1\dots T_d p(e^{it_1},\dots,e^{it_d})\right|^2 dt_1 \leq \left(\frac{1}{2\pi}\int_0^{2\pi} \left|T_2\dots T_d p(e^{it_1},\dots,e^{it_d})\right| dt_1\right)^2,$$

and thus

$$\|T_1\dots T_d p\|_{L^2(\mathbb{T}^d)} \leq \left(\frac{1}{(2\pi)^{d-1}}\int_{(0,2\pi)^{d-1}} \left(\frac{1}{2\pi}\int_0^{2\pi} \left|T_2\dots T_d p(e^{it_1},\dots,e^{it_d})\right| dt_1\right)^2 dt_2\dots dt_d\right)^{\frac{1}{2}}.$$

Then, the integral version of Minkowski's inequality yields

$$\|T_1\dots T_d p\|_{L^2(\mathbb{T}^d)} \leq \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{(2\pi)^{d-1}}\int_{(0,2\pi)^{d-1}} \left|T_2\dots T_d p(e^{it_1},\dots,e^{it_d})\right|^2 dt_2\dots dt_d\right)^{\frac{1}{2}} dt_1.$$

Finally, we can repeat the same idea $d-1$ times in order to get (B.7). □

Observe that the contractivity in Carleman's inequality plays a vital role in the argument given by H. Helson. Any bound strictly larger than 1 for the norm of the inclusion operator from H^1 to A^2 results in an upper bound which blows up when d tends to infinity. In other words, there is no inequality at all if we use a bound different from 1.

Indeed, we could replace Carleman's inequality by another appropriate contractive inclusion of Hardy spaces in order to deduce new Helson-type inequalities in $\mathscr{H}^p$. For example, the inequality

$$\sqrt{\sum_{n=1}^{\infty} \frac{|a_n|^2}{d_{2/q}(n)}} \leq \|f\|_{\mathscr{H}^q}, \quad 0 < q \leq 2, \tag{B.8}$$

where

$$\begin{aligned} d_{2/q}(1) &:= 1, \\ d_{2/q}(n) &:= \prod_{\alpha_j \neq 0} \binom{\alpha_j + 2/q - 1}{\alpha_j}, \quad n \geq 2, \end{aligned}$$

would hold if Conjecture A were true.

Since $\{d_\alpha(n)\}_{n\geq 1}$, $\alpha > 1$, is the sequence of the coefficients of $\zeta^\alpha(s)$ as a Dirichlet series (where $\zeta(s)$ is the Riemann Zeta function) [25], then (B.8) would have applications in the study of the pseudomoments of $\zeta(s)$. For further references, see [26, 27].

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