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A UNIVERSAL HÖLDER ESTIMATE UP TO DIMENSION 4 FOR STABLE SOLUTIONS TO HALF-LAPLACIAN SEMILINEAR EQUATIONS

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ABSTRACT. We study stable solutions to the equation $(-\Delta)^{1/2}u = f(u)$, posed in a bounded domain of $\mathbb{R}^n$. For nonnegative convex nonlinearities, we prove that stable solutions are smooth in dimensions $n \leq 4$. This result, which was known only for $n = 1$, follows from a new interior Hölder estimate that is completely independent of the nonlinearity f.

A main ingredient in our proof is a new geometric form of the stability condition. It is still unknown for other fractions of the Laplacian and, surprisingly, it requires convexity of the nonlinearity. From it, we deduce higher order Sobolev estimates that allow us to extend the techniques developed by Cabré, Figalli, Ros-Oton, and Serra for the Laplacian. In this way we obtain, besides the Hölder bound for $n \leq 4$, a universal $H^{1/2}$ estimate in all dimensions.

Our L^∞ bound is expected to hold for $n \leq 8$, but this has been settled only in the radial case or when $f(u) = \lambda e^u$. For other fractions of the Laplacian, the expected optimal dimension for boundedness of stable solutions has been reached only when $f(u) = \lambda e^u$, even in the radial case.

CONTENTS

1. Introduction and results

The regularity of stable solutions to semilinear equations $-\Delta u = f(u)$ has been a long-standing problem in elliptic PDEs since the 1970s. Important efforts have been devoted to investigate the optimal dimension up to which stable solutions are bounded. This problem has been recently solved by Cabré, Figalli, Ros-Oton, and Serra [3], by proving that stable solutions are regular in dimensions $n \le 9$ for all nonnegative nonlinearities f. The result is optimal since there exist examples of singular H^1 stable solutions in dimensions $n \ge 10$. For further details, see [3, 10] and the references therein.

The goal of this paper is to study the same question in the fractional setting. We consider the equation

$$(-\Delta)^s u = f(u) \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ and $(-\Delta)^s$ is the fractional Laplacian,

$$(-\Delta)^s w(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{w(x) - w(z)}{|x - z|^{n+2s}} \, dz \quad s \in (0, 1),$$

with $c_{n,s}$ being a positive normalizing constant (see [4]). In this case, the few known results (mainly those contained in the three papers [21, 19, 23] described below) reach the expected optimal dimension for boundedness of stable solutions only when $f(u) = \lambda e^u$, even in the radial case. In Figure 1 such dimension, which was found by Ros-Oton in [19] and it is given by condition (1.11) below, is compared with the available results.

Note that (1.1) is the Euler-Lagrange equation of the functional

$$E(w) := \frac{c_{n,s}}{4} \int\!\!\int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|w(x) - w(z)|^2}{|x - z|^{n+2s}} \, dx\,dz - \int_\Omega F(w) \, dx \tag{1.2}$$

where $F(t) := \int_0^t f(\theta) \, d\theta$ and $\Omega^c := \mathbb{R}^n \setminus \Omega$. A solution $u: \mathbb{R}^n \to \mathbb{R}$ to (1.1) —or critical point of E— is said to be *stable* if the second variation of E at u is nonnegative, i.e., $\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0} E(u + \varepsilon\xi) \ge 0$ for all $\xi \in H^s(\mathbb{R}^n)$ with compact support in Ω. This is equivalent to requiring that

$$\int_\Omega f'(u)\xi^2 \, dx \le [\xi]^2_{H^s(\mathbb{R}^n)} \quad \text{for all } \xi \in H^s(\mathbb{R}^n) \text{ with compact support in } \Omega. \tag{1.3}$$

Recall that for $s \in (0, 1)$ and $U \subset \mathbb{R}^n$, we define $H^s(U) := \{w \in L^2(U) : [w]_{H^s(U)} < +\infty\}$, where

$$[w]^2_{H^s(U)} := \frac{c_{n,s}}{2} \int_U \int_U \frac{|w(x) - w(z)|^2}{|x - z|^{n+2s}} \, dx\,dz.$$

Notice that stability is considered among functions which agree with u outside Ω, and therefore *local minimizers* of the energy (i.e., minimizers under small perturbations which do not change the exterior values of u) are stable solutions.

Our interest lies in nonnegative nonlinearities that grow superlinearly at $+\infty$. In this case, it is easy to see that the energy (1.2) is unbounded below and hence admits no absolute minimizer. However, as we will see in Section 1.1, there are important instances in which nonconstant stable solutions exist.

In this article we study the problem for the half-Laplacian ($s = 1/2$), a case which is of special interest in view of its applications. Indeed, $(-\Delta)^{1/2}$ and related first order integro-differential operators appear in the modeling of important physical phenomena, such as the Peierls-Nabarro model for crystal dislocations or the Benjamin-Ono equation in fluid dynamics. This occurs since the half-Laplacian is the Dirichlet to Neumann map associated to the harmonic extension in the half-space.

An important illustration of the key role played by the dimension in our problem is given by the function $u(x) = \log|x|^{-1}$ ($x \in \mathbb{R}^n$), which solves $(-\Delta)^{1/2}u = \lambda_0 e^u$ in B_1 for some constant $\lambda_0 > 0$ and belongs to $H^{1/2}(B_1) \cap L^1_{1/2}(\mathbb{R}^n)$ —see (1.4) below for this last space. Using the fractional Hardy inequality one can check that, in dimensions $n \geq 9$, u is a stable solution.

On the other hand, for the equation $(-\Delta)^{1/2}u = f(u)$ in $\Omega \subset \mathbb{R}^n$, with $f \geq 0$ and under the Dirichlet condition $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, it is known that stable solutions are bounded in Ω

- when $n = 1$, if f is convex (Ros-Oton and Serra [21]);
- when $n \leq 4$, if $ff''/(f')^2$ has a limit at infinity (Ros-Oton and Serra [21]);
- when $n \leq 8$, if $f(u) = \lambda e^u$ and Ω is symmetric and convex with respect to all the coordinate directions (Ros-Oton [19]);
- when $2 \leq n \leq 8$, if $\Omega = B_1$ (Sanz-Perela [23]).

In view of these results, it is natural to conjecture that stable solutions to $(-\Delta)^{1/2}u = f(u)$ in $\Omega \subset \mathbb{R}^n$ are always bounded in Ω whenever $n \leq 8$, not only for $f(u) = \lambda e^u$ in symmetric domains but for a wider class of nonlinearities and domains. Here the dimension $n = 8$ would be optimal, by the explicit stable solution for $n \geq 9$ exhibited above. In this paper we make progress towards the solution of this conjecture by extending some of the techniques developed for the local case in [3] by Cabré, Figalli, Ros-Oton, and Serra.

The following is our main result. It provides a universal interior Hölder estimate for stable solutions u to (1.1) in dimensions $n \leq 4$, as well as an $H^{1/2}$ bound in every dimension. Both estimates give a control in terms of a very weak norm of u, namely the quantity $\|u\|_{L^1_{1/2}(\mathbb{R}^n)}$. Here and through the paper, for $s \in (0,1)$ we denote by $L^1_s(\mathbb{R}^n)$ the space of measurable functions for which the norm

$$\|w\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|w(x)|}{(1+|x|^2)^{\frac{n+2s}{2}}}\, dx \tag{1.4}$$

is finite.

Theorem 1.1. *Let $n \geq 1$ and $u \in C^2(B_1) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2}u = f(u)$ in $B_1 \subset \mathbb{R}^n$, where f is a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$.*
Then,

$$[u]_{H^{1/2}(B_{1/2})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \tag{1.5}$$

for some dimensional constant C. In addition,

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \qquad \text{if } 1 \leq n \leq 4, \tag{1.6}$$

for some dimensional constants $\alpha > 0$ and C.

The most remarkable feature of this result is that estimates (1.5) and (1.6) do not depend on the nonlinearity f at all, which is only assumed to be nonnegative and convex. This is a notorious difference with all the available results mentioned above, in which the L^∞ estimates depended on the particular nonlinearity f appearing in the equation. This feature will allow us to establish a Liouville result for entire stable solutions, Corollary 1.5 below.

In addition, Theorem 1.1 is the first result in which no prescribed exterior Dirichlet condition is assumed. This will be of great importance in order to include the case $n = 1$ (by looking at the solution as defined in $\mathbb{R}^2$ after adding an artificial variable) since our basic inequality towards the Hölder estimate (1.6) —which is (1.16) below— requires[1] $n \geq 2$.

The Hölder interior estimate of Theorem 1.1 for $n \leq 4$ can be combined with the moving planes method to obtain an $L^\infty(\Omega)$ bound for stable solutions to the Dirichlet problem in convex domains Ω under zero exterior data. Indeed, as proved in [21], one can start the moving planes argument at points on the boundary of a convex domain Ω, obtaining an L^∞ estimate in a neighborhood of $\partial\Omega$. Thus, one concludes the following result.

Corollary 1.2. *Let $1 \leq n \leq 4$ and let $\Omega \subset \mathbb{R}^n$ be any bounded convex C^1 domain. Let $u \in L^\infty(\Omega) \cap H^{1/2}(\mathbb{R}^n)$ be a stable solution to*

$$\begin{cases} (-\Delta)^{1/2} u = f(u) & in \ \Omega, \\ u = 0 & in \ \mathbb{R}^n \setminus \Omega, \end{cases}$$

where f is a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$.
 Then,

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^1(\Omega)}$$

for some constant C depending only on Ω.

Also in convex domains Ω, an $H^s(\mathbb{R}^n)$ estimate for bounded stable solutions to (1.1) vanishing outside Ω is known to hold in every dimension and for all $s \in (0,1)$. This was proved by Ros-Oton and Serra in [21] using the Pohozaev identity for the fractional Laplacian combined with some regularity estimates near $\partial\Omega$. Instead, our interior $H^{1/2}$ bound (1.5) does not assume any particular exterior data —and also holds for all dimensions.

Although Theorem 1.1 and Corollary 1.2 are stated as a priori estimates for C^2 or L^∞ solutions, they also hold for a bigger class of stable solutions. This is discussed in Remark 2.1 below, where we comment also on the different notions of solution to our semilinear equation.

As a main ingredient in the proof of Theorem 1.1, we establish a new geometric[2] form of the stability condition, expressed through the harmonic extension[3] v of u in $\mathbb{R}^{n+1}_+$. It is stated in (1.8) below, and it is a fractional analogue of a well-known inequality of Sternberg and

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[1] More generally, for $s \in (0,1]$ it requires $n > 2s$; see Proposition 2.2 below for $s \in (0,1)$ and [3] for $s = 1$.
[2] Although we do not use it in this article, it is worth noticing that the quantity $\mathcal{A}$ defined in (1.7) controls a geometric quantity: the second fundamental form of the level sets of v. Indeed, in the set $\{|\nabla v| > 0\}$ it holds $\mathcal{A}^2 = |\nabla_T|\nabla v||^2 + |B|^2|\nabla v|^2$, where $|B|^2 = |B(x)|^2$ denotes the square of the second fundamental form of the level set of v passing through $x \in \mathbb{R}^{n+1}_+$, and ∇_T denotes the tangential gradient along such level set; see Lemma 2.1 in [26] for a detailed proof.
[3] Recall that the equation $(-\Delta)^{1/2} u = f(u)$ is equivalent to $\partial_\nu v = f(v)$ on $\partial\mathbb{R}^{n+1}_+$.

Zumbrun [26, 27] for stable solutions to $-\Delta u = f(u)$. Surprisingly (when comparing it with the proof in the local case), to establish (1.8) we need to further assume that f is convex.

Theorem 1.3. *Let $n \geq 1$ and $u \in C^2(\Omega) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2}u = f(u)$ in a domain $\Omega \subset \mathbb{R}^n$, where f is a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$ and define*

$$\mathcal{A} := \begin{cases} \left(\left(\sum_{i,j=1}^{n+1} v_{ij}^2 - \sum_{j=1}^{n+1} \left(\sum_{i=1}^{n+1} v_{ij} \frac{v_i}{|\nabla v|} \right)^2 \right) \right)^{1/2} & \text{if } |\nabla v| > 0, \\ 0 & \text{if } |\nabla v| = 0. \end{cases} \tag{1.7}$$

Then,

$$\int_{\mathbb{R}^{n+1}_+} \mathcal{A}^2 \eta^2 \, dx \leq \iint_{\mathbb{R}^{n+1}_+} |\nabla v|^2 |\nabla \eta|^2 \, dx \tag{1.8}$$

for every Lipschitz function η with compact support in $\overline{\Omega} \times [0, +\infty)$.

An analogue of Theorem 1.3 for fractional powers $s \neq 1/2$ is not known at the moment. In its proof we strongly use that $s = 1/2$; see Remark 3.1 for details. We believe that, if available, it would allow extending Theorem 1.1, in certain dimensions, to all powers $s \in (0, 1)$.

Prior to our work, the inequality of Sternberg and Zumbrun had been extended to the fractional setting by Sire and Valdinoci [25], but in a weaker form than ours. Their inequality involves only horizontal derivatives of v: they obtained a similar quantity to $\mathcal{A}$ in (1.7), but where the indices run only from 1 to n, and not to $n+1$. As a consequence, no one has used it successfully for proving boundedness of stable solutions in bounded domains —although it is the main ingredient in a simple proof of the fractional De Giorgi conjecture in $\mathbb{R}^2$; see [25].

Instead, from our new geometric form of the stability condition we will obtain an L^2 estimate for all second derivatives of v, including those involving the y variable. This bound will be crucial in the proof of Theorem 1.1; see Section 1.3 for more details.

Before going into the main ideas in our proofs, let us introduce an important class of stable solutions and the known results on their regularity.

1.1. **Extremal solutions. Known results for $s \in (0, 1)$.** Given a power $s \in (0, 1)$, consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.9}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $\lambda > 0$ is a real parameter, and

$$f \in C^1([0, \infty)), \ f \text{ is nondecreasing, } f(0) > 0, \text{ and } \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.10}$$

It is known (see [21]) that problem (1.9) admits an increasing family of minimal[4] stable solutions $\{u_\lambda \, : \, 0 < \lambda < \lambda^\star\}$, with $u_\lambda > 0$ being bounded in Ω, up to a certain finite extremal

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[4]Here, *minimal* means that u_λ is smaller than any other solution (and, as a byproduct, smaller than any supersolution).

parameter $\lambda^\star$. For $\lambda > \lambda^\star$ there is no solution to (1.9), even in the L^1-weak sense.[5] The pointwise limit of $\{u_\lambda\}$, as $\lambda \nearrow \lambda^\star$, is a stable L^1-weak solution of (1.9) for $\lambda = \lambda^\star$. Such solution, denoted by $\bar{u}$, is called the *extremal solution* of (1.9). A fundamental question is to determine whether it is bounded or not.

The main result of our paper can be used (through Corollary 1.2) to establish the boundedness of the extremal solution in dimensions $n \leq 4$ for $s = 1/2$ when Ω is a convex domain and f is a convex nonlinearity.

Corollary 1.4. *For some $\gamma > 0$, let f be a $C^{1,\gamma}$ convex function satisfying* (1.10). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex C^1 domain and let $\bar{u}$ be the extremal solution to* (1.9) *for $s = 1/2$.*
  If $1 \leq n \leq 4$, then $\bar{u} \in L^\infty(\Omega)$.

The result is established by applying the estimate of Corollary 1.2, for $\lambda < \lambda^\star$, to the bounded solutions u_λ. Then, since it is easy to prove that $\bar{u} \in L^1(\Omega)$, the estimates for u_λ are uniform in λ and one can take the limit $\lambda \to \lambda^\star$ to deduce the boundedness of $\bar{u}$.

Besides Corollary 1.4, the available results on the boundedness of the extremal solution for the fractional problem (1.9) are those contained in [21], [19], and [23]. We briefly describe them next. See Figure 1 for a schematic representation of the dimensions achieved in each result.

In [21], Ros-Oton and Serra showed that if f is convex then $\bar{u}$ is bounded whenever $n < 4s$, extending to the nonlocal setting the arguments of Nedev [18]. The authors also established the boundedness of the extremal solution in dimensions $n < 10s$ under the more restrictive assumption of f being C^2 and $ff''/(f')^2$ having a limit at infinity; this is an extension of the arguments of Crandall and Rabinowitz [6]. The estimates for stable solutions in [21] depend on f, in contrast with ours, and do not provide any estimate for $n = 1$ if s is small. The results from [21] have been recently extended to the case of systems of two equations by Fazly [12].

Shortly after [21], Ros-Oton [19] obtained an optimal result in the case of the exponential nonlinearity $f(u) = e^u$, extending the arguments of [7] commented below, by showing that $\bar{u}$ is bounded whenever[6]

$$\frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})} < \frac{\Gamma(\frac{n}{2})\Gamma(1+s)}{\Gamma(\frac{n-2s}{2})} \tag{1.11}$$

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[5]We say that $u \in L^1(\Omega)$ is a stable L^1-weak solution of (1.9) if $f(u)\operatorname{dist}(\cdot, \partial\Omega)^s \in L^1(\Omega)$ and

$$\int_\Omega u\,(-\Delta)^s \zeta \, dx' = \int_\Omega \lambda f(u)\zeta \, dx'$$

for all ζ such that ζ and $(-\Delta)^s\zeta$ are bounded in Ω and $\zeta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.
[6]Condition (1.11) is equivalent to

$$|x'|^{\frac{2s+n}{2}}(-\Delta)^s|x'|^{\frac{2s-n}{2}} < \lim_{\varepsilon \to 0} \frac{2s}{\varepsilon}|x'|^{n-\varepsilon}(-\Delta)^s|x'|^{2s-n+\varepsilon} \quad \text{for } x' \in \mathbb{R}^n \setminus \{0\},$$

which is indeed the expression arising in [19]. Note that the two quantities in this inequality are constants independent of $|x'|$.

and   is symmetric and convex with respect to all the coordinate directions. In Figure 1 below we represent graphically the dimensions given by (1.11). In particular, we see that (1.11) holds in dimensions $n \leq 7$ for all $s \in (0, 1)$, and additionally in dimension $n = 8$ for some fractions s which include $s = 1/2$.
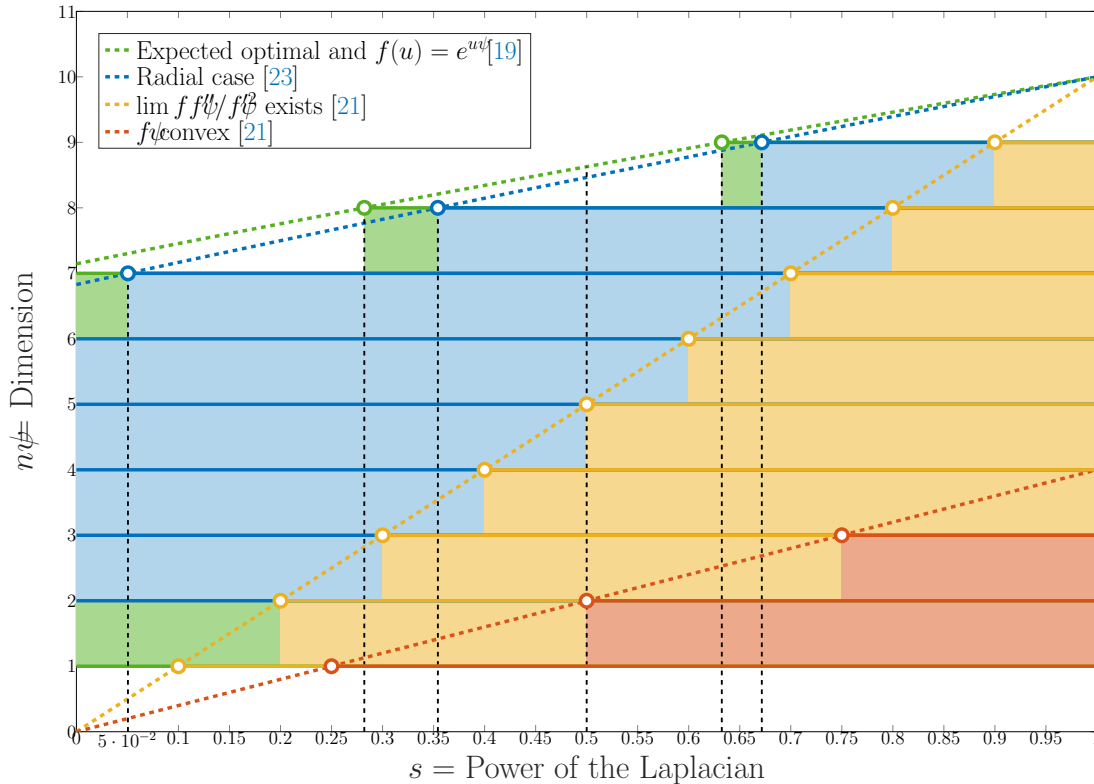

FIGURE 1. Graphical representation of the known results on the boundedness of the extremal solution. For each color, the horizontal solid lines represent the rank of s for which u is bounded in that dimension. The dashed colored lines represent the threshold values (n, s) found in each paper: in green, the values given by inequality (1.11); in blue, $n = 2(s + 2 + \sqrt{2(s+1)})$; in yellow, $n = 10s$; in red, $n = 4s$. These lines correspond, in descending order, to the four cases listed in the legend. The black dashed vertical lines help to locate some limiting values of s.

Condition (1.11) is optimal since, when (1.11) fails, $\log |x|^{-2s}$ is a singular stable solution to $(-\Delta)^s u = \lambda e^u$ in all of $\mathbb{R}^n$ for some $\lambda > 0$ (see Remark 2.2 of [19]).

One expects (1.11) to provide the optimal range for the values of n and s in which stable solutions are bounded also for all nonlinearities f (or for a large class of them). Nevertheless, this is not known even in the radial case,[7] as explained next.

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[7] For radially symmetric solutions, the local equation $-\Delta u = f(u)$ becomes a second order ODE. Instead, although $(-\Delta)^s u = f(u)$ can be written as a one-dimensional equation in the radial case, it still involves an integro-differential operator which is much more involved than an ODE.

The problem in a ball is studied in a recent paper by Sanz-Perela [23]. It is proved that in dimensions $2 \leq n < 2(s + 2 + \sqrt{2(s+1)})$ the extremal solution to (1.9), with $\Omega = B_1$, is bounded. This extends the results of Cabré and Capella [2] for the local case, and of Capella, Dávila, Dupaigne, and Sire [5] for another nonlocal operator. The condition on the dimension in this radial case (in blue in Figure 1) is slightly worse than the expected optimal one (1.11) (in green), but it is a significant improvement of $n < 10s$ (in yellow). In Section 1.3 we will explain why [23] and the current work do not reach the expected optimal dimensions.

A further question of interest is the partial regularity of singular stable solutions. In another very recent work, Hyder and Yang [16] have established that the dimension of the singular set of a stable solution to $(-\Delta)^s u = e^u$ in Ω is at most $n - 10s$.

Condition (1.11) in the case $s = 1/2$ appeared in the literature for the first time in the paper [7] by Dávila, Dupaigne, and Montenegro. They dealt with a boundary reaction problem which is related to, but different from, our equation $(-\Delta)^{1/2} u = \partial_\nu v = f(v)$ on $\partial \mathbb{R}^{n+1}_+$ (where v is the harmonic extension of u in the half-space). Besides the case $f(u) = \lambda e^u$, [7] also studied power-like nonlinearities, for which a more precise condition on the dimension, now depending on the power p, was found. It corresponds to (1.12) below for $s = 1/2$.

1.2. Entire solutions.

A related issue to our problem is the classification of entire solutions to $(-\Delta)^s u = f(u)$ in all of $\mathbb{R}^n$ which are stable or, more generally, have finite Morse index. This problem has been treated very recently in a series of works, in which the condition (1.11) appears again.

For the power case $f(u) = |u|^{p-1}u$, Dávila, Dupaigne, and Wei [8] give a complete classification of finite Morse index solutions. Namely, they prove that when either $1 < p < (n + 2s)/(n - 2s)$, or $p > (n + 2s)/(n - 2s)$ and

$$\frac{\Gamma^2(\frac{n+2s}{4})}{\Gamma^2(\frac{n-2s}{4})} < p\frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})}, \tag{1.12}$$

every finite Morse index entire solution is identically zero. Note that taking the limit $p \to +\infty$ in (1.12), one obtains precisely the expected optimal condition (1.11).

In the case $f(u) = e^u$, Hyder and Yang [15] have recently proved that no finite Morse index entire solution exists if (1.11) holds —Duong and Nguyen [9] had previously reached the condition $n < 10s$.

Our main result allows to establish a Liouville theorem for stable solutions in dimensions $n \leq 4$ for general nonnegative convex nonlinearities. We follow the ideas of Dupaigne and Farina [11] for the local case, which consist of applying a universal Hölder estimate to the blow-downs of a stable solution.

Corollary 1.5. *Let $u \in C^2(\mathbb{R}^n) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2} u = f(u)$ in $\mathbb{R}^n$, where f is a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Assume that*

$$u(x') \geq -c_0 \log(2 + |x'|) \quad \text{for } x' \in \mathbb{R}^n, \tag{1.13}$$

where $c_0 > 0$ is a constant.

If $1 \leq n \leq 4$, then u must be constant.

1.3. **Outline of the proofs.** Here we summarize the main ideas in the proofs of Theorems 1.1 and 1.3. We use the extension problem for the half-Laplacian as a main tool. For this, let us settle our notation.

Through all the paper we will denote points in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, +\infty)$ by $x = (x', y)$. Recall that the half-Laplacian of a regular enough function $u: \mathbb{R}^n \to \mathbb{R}$ can be computed through its harmonic extension in $\mathbb{R}^{n+1}_+$. Indeed, if v solves $\Delta v = 0$ in $\mathbb{R}^{n+1}_+$ and $v(\cdot, 0) = u$, then $-v_y(\cdot, 0) = (-\Delta)^{1/2} u$.

We will denote balls in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ by B_R and $\mathcal{B}_R$ respectively, while

$$\mathcal{B}_R^+ := \mathcal{B}_R \cap \mathbb{R}^{n+1}_+ = \{x = (x', y) \in \mathbb{R}^{n+1}_+ \; : \; |(x', y)| < R\}$$

will be half-balls in $\mathbb{R}^{n+1}_+$. If $x_0' \in \mathbb{R}^n$, $\mathcal{B}_R^+(x_0') := (x_0', 0) + \mathcal{B}_R^+$. In addition, we will use the notation

$$r := |x| \quad \text{and} \quad v_r := \frac{x}{r} \cdot \nabla v(x), \quad \text{for } x \in \mathbb{R}^{n+1}_+.$$

The general strategy to obtain estimates for stable solutions consists of choosing an appropriate test function in the stability condition (1.3), which for $s = 1/2$ can be written, using the extension problem, as

$$\int f'(u)\xi^2 \, dx' \leq \iint_{\mathbb{R}^{n+1}_+} |\nabla \xi|^2 \, dx \tag{1.14}$$

for every $\xi \in H^1(\mathbb{R}^{n+1}_+)$ such that its trace in $\mathbb{R}^n$ has compact support in ; see Section 2. The choices of test function taken in each of the previously mentioned papers [21, 19, 23] were the following (up to a cut-off): $\xi = h(u)$ for some h depending on f in [21], $\xi = |x'|^{-a}$ for some $a > 0$ in [19], and $\xi = |x'|^{-b}(x' \cdot \nabla_{x'} v)$ for some $b > 0$ in [23]. Instead, we will make use of two new test functions, (1.15) and (1.17) below.

Our proof of Theorem 1.1 follows the main lines of that in [3] for the local case, but confronts a delicate issue in a compactness argument (which remains open for $s \neq 1/2$). The main arguments in the proof can be summarized in four steps:

• **Step 1.** A key point is to choose

$$\xi = r^{1+(1-n)/2} v_r \zeta = |x|^{-(n-1)/2}(x \cdot \nabla v)\zeta \tag{1.15}$$

as test function in the stability condition (1.14), where v is the harmonic extension of u and ζ is a smooth cut-off function. This leads to the following bound (see Proposition 2.2):

$$\int_{\mathcal{B}_{1/2}^+} r^{1-n} v_r^2 \, dx \leq C \iint_{\mathcal{B}_{3/4}^+ \setminus \mathcal{B}_{1/2}^+} |\nabla v|^2 \, dx \quad \text{if } 2 \leq n \leq 4. \tag{1.16}$$

The crucial fact here is that, after this choice of ξ, the nonlinearity f no longer appears in the estimates.

The condition $2 \leq n \leq 4$ (which for general s reads $2s < n < 10s$ in Proposition 2.2 below) is not the expected optimal range of dimensions. The reason why our proof does not reach such range is that, although we believe that the choice $\xi|_{\{y=0\}} = |x'|^{-(n-1)/2}(x' \cdot \nabla_{x'} u)\zeta$ should be optimal, the election of its extension (1.15) is not. The same issue occurs in the radial case treated in [23], where still another different extension of the test function $\xi|_{\{y=0\}}$ is chosen.

● **Step 2.** To relate the left and right-hand sides of (1.16), we prove that, under a doubling condition on $|\nabla v|^2 \, dx$, the quantities v_r and $\nabla v \psi$ are comparable, in L^2, in an annulus (see Lemma 4.1). This combined with (1.16) will lead to the geometric-decay bound

$$\iint_{\mathcal{B}_R^+} r^{1-n} v_\psi^2 \, dx \leq CR^{2\alpha} \|\nabla v\|_{L^2(\mathcal{B}_{3/4}^+)}^2 \quad \text{if } 2 \leq n \leq 4,$$

for all small enough radii $R\psi$ and for some $\alpha > 0$.

In order to establish the comparability of v_r and $\nabla v \psi$ in annuli we proceed as in [3], using a compactness argument. To carry it on, a control on higher order Sobolev norms of $v \psi$ is needed, and for this it is crucial to use the new stability condition in geometric form, (1.8), given in Theorem 1.3.

To establish Theorem 1.3 we choose

$$\xi \psi = |\nabla v| \eta \psi \tag{1.17}$$

in the stability condition. At some point along the proof, to show the sign of a certain term, we use crucially the convexity and nonnegativeness of f.

● **Step 3.** Combining the decay of the weighted radial derivative and a Morrey-type estimate from [1] (Lemma 6.1), we obtain the Hölder bound

$$[u]_{C \ (\overline{B}_{1/2})} \leq C \|\nabla v\|_{L^2(\mathcal{B}_{3/4}^+)} \quad \text{if } 2 \leq n \leq 4.$$

At this point, it will only remain to control the L^2 norm of $\nabla v \psi$ by the $L_{1/2}^1(\mathbb{R}^n)$ norm of u. This is done in Section 5. For this, the key point is to realize that, thanks to Theorem 1.3, we have a control of the L^2 norm of $D^2 v \psi$ in terms of a lower order norm of v. Using this and two interpolation results from [1] (proved in Appendix B below), we control the H^1 norm of $v \psi$ by its L^1 norm and, in turn, by the $L_{1/2}^1$ norm of u.

This will also lead, through a trace inequality, to the $H^{1/2}$ bound (5.2) for $u \psi$ in all dimensions.

● **Step 4.** Finally, to obtain the result in dimension $n \psi = 1$, we just need to consider the solution as defined in $\mathbb{R}^2$ after the addition of an artificial variable, and apply the result for $n = 2$. In Section 7 we establish the necessary results to carry out this process.

1.4. **Plan of the article.** In Section 2 we establish the key estimate (1.16) of Step 1, which we carry on in the general setting of the equation $(-\Delta)^s u = f(u)$ with $s \in (0, 1)$. In Section 3 we establish the new geometric stability inequality given by Theorem 1.3, while Section 4 is devoted to Step 2 described above. Then, in Section 5 we establish the $H^{1/2}$ estimate of Step 3, leading to the proofs of Theorem 1.1 and Corollaries 1.2, 1.4 and 1.5 in Section 6. Finally, in Section 7 we present some results related to adding an artificial variable.

In the appendices we collect some results that are used through the paper, and we recall the proofs of the two interpolation inequalities from [1] needed in Section 5.

2. A KEY WEIGHTED RADIAL DERIVATIVE ESTIMATE

The goal of this section is to prove estimate (1.16). Although in this paper we only consider the case $s = 1/2$, for future reference all the computations in this section are done for the equation $(-\Delta)^s u = f(u)$ in B_1, with $s \in (0, 1)$.

Along this section we will assume that $u \in C^2(B_1) \cap L^1_s(\mathbb{R}^n)$ and that it solves (pointwise) the equation $(-\Delta)^s u = f(u)$ in B_1, with $f \in C^{1,\gamma}$ for some $\gamma > 0$. In the following remark we comment on these assumptions and discuss some extensions of our results to less regular classes of stable solutions.

Remark 2.1. We can consider three classes of stable solutions to (1.1) in terms of their regularity: L^1-weak solutions —defined in footnote 5—, energy solutions —critical points of the functional $E(\cdot)$ in (1.2)—, and pointwise (or classical) solutions. As mentioned in Remark 2.1 of [21] (see [24] for nonzero values in $\mathbb{R}^n \setminus$), these three notions of solution coincide whenever the solution is bounded and f sufficiently regular.[8] Thus, our main result holds for bounded L^1-weak or energy solutions.

When u is not assumed a priori to be bounded, the situation is more delicate. On the one hand, our result leads to the interior Hölder regularity of energy stable solutions or, more generally, L^1-weak stable solutions which belong to the energy space associated to $E(\cdot)$. Indeed, it can be proved that any stable solution u in these classes can be approximated by bounded stable solutions u_k (at least when f is increasing and convex and $s \geq 1/2$; see [24]). Then, the estimates for u_k pass on the limiting function u.

On the other hand, as in the local case $s = 1$, our main result does not hold for general L^1-weak solutions, as there exist unbounded L^1-weak solutions which satisfy the stability hypothesis in the dimensions given by Theorem 1.1; see [24]. The issue here is that these unbounded L^1-weak solutions are not in the energy space associated to $E(\cdot)$.

To state the main result of this section, we need to introduce the extension problem for all the fractional powers $s \in (0, 1)$ of the Laplacian. Recall that if v solves

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v = u & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases}$$

where $a := 1 - 2s$, then

$$\frac{\partial v}{\partial \nu^a} := -\lim_{y \downarrow 0} y^a v_y = \frac{1}{d_s}(-\Delta)^s u \tag{2.1}$$

for a positive constant d_s which depends only on s, and such that $d_{1/2} = 1$. The function v, which is smooth in $\mathbb{R}^{n+1}_+$, is called the *s-harmonic extension* of u and can be expressed in terms of u through convolution with a Poisson kernel —see (A.1) in our first appendix. Along the paper we will always denote by v the s-harmonic extension of u.

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[8]Indeed, if u is an L^1-weak solution to $(-\Delta)^s u = f(u)$ in B_1 which is bounded in B_1 (note that in the paper [21] L^1-weak solutions are called *weak solutions*), and $f \in C^{1,\gamma}$ for some $\gamma > \max\{0, 1 - 2s\}$, then by considering the convolution of u with a standard mollifier and using regularity results for the fractional Laplacian it follows readily that $u \in C^2(B_1)$ —see Corollaries 2.3 and 2.5 in [20]; note that here we do not use the stability of u.

Proposition 2.2. *Let $n \geq 1$, $s \in (0,1)$, and let $u \in C^2(B_1) \cap L^1_s(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^s u = f(u)$ in $B_1 \subset \mathbb{R}^n$, where f is a $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the s-harmonic extension of u.*

Then,

$$\int_{\mathcal{B}^+_{1/2}} y^{1-2s} r^{2s-n} v_r^2 \, \mathrm{d}x \leq C \iint_{\mathcal{B}^+_{3/4} \setminus \mathcal{B}^+_{1/2}} y^{1-2s} |\nabla v|^2 \, \mathrm{d}x \quad \text{if } n \in (2s, 10s),$$

for some constant C depending only on n and s.

This result will follow from making a particular choice of the test function in the stability condition, once this condition is written in the extended space $\mathbb{R}^{n+1}_+$. Namely, definition (1.3) is equivalent to requiring

$$\int f'(u)\xi^2 \, \mathrm{d}x \leq d_s \iint_{\mathbb{R}^{n+1}_+} y^a |\nabla\xi|^2 \, \mathrm{d}x \tag{2.2}$$

for every $\xi \in H^1(\mathbb{R}^{n+1}_+, y^a)$ whose trace in $\mathbb{R}^n$ has compact support in Ω. This follows easily from the fact that the constant d_s in (2.2), which also appears in the previous relation (2.1), is the optimal constant in the trace inequality $[\xi(\cdot,0)]^2_{H^s(\mathbb{R}^n)} \leq d_s [\xi]^2_{H^1(\mathbb{R}^{n+1}_+, y^a)}$; see [13, Section 5].

The following lemma is the first step towards Proposition 2.2. It is proved by taking $\xi = (x \cdot \nabla v)\eta = r v_r \eta$ in the stability condition (2.2). The crucial point here is that, after this particular choice, the nonlinearity f no longer appears in the inequality.

Lemma 2.3. *Let $n \geq 1$, $s \in (0,1)$, and let $u \in C^2(B_1) \cap L^1_s(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^s u = f(u)$ in $B_1 \subset \mathbb{R}^n$, where f is a $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the s-harmonic extension of u.*

Then, for every $\eta \in \mathrm{Lip}(\overline{\mathbb{R}^{n+1}_+})$ with compact support in $\mathcal{B}^+_1 \cup B_1$ it holds

$$s \int_{\mathcal{B}^+_1} y^a \left\{ (n - 2s)\eta^2 + r(\eta^2)_r \right\} |\nabla v|^2 \, \mathrm{d}x - 2s \int_{\mathcal{B}^+_1} y^a r v_r \nabla v \cdot \nabla(\eta^2) \, \mathrm{d}x$$
$$\leq \iint_{\mathcal{B}^+_1} y^a r^2 v_r^2 |\nabla \eta|^2 \, \mathrm{d}x. \tag{2.3}$$

Proof. Throughout the proof we will use the notation $v_i = \partial_i v = \partial_{x_i} v$, $v_y = v_{n+1}$, and $x_{n+1} = y$.

The key idea is to take $\xi = \mathbf{c}\eta$ in the stability condition (2.2), with $\mathbf{c} = x \cdot \nabla v = r v_r$, and use that $\mathbf{c}$ satisfies an appropriate equation for the linearized operator $d_s \partial_{\nu^a} - f'(u)$ (see (2.7) below). To carry out this program we need to prove some regularity for $\mathbf{c}$. In what follows, we will use that, for every $R < 1$, the functions v, $y^a \partial_y v$, $\nabla_{x'} v$, and $y^a \partial_y \nabla_{x'} v$ are continuous in $\overline{\mathcal{B}^+_R}$, that $D^2_{x'} v$ is bounded in $\mathcal{B}^+_R$, and that $v \in H^1(\mathcal{B}^+_R, y^a)$. All these statements are proved in Lemma A.3.

Regarding $\mathbf{c}$, we next show two properties of this function. First, note that $\mathbf{c}$ is continuous up to $\{y = 0\}$ and $\mathbf{c}(\cdot, 0) = x' \cdot \nabla_{x'} u$ in $B_R \subset \{y = 0\}$ for $R < 1$. Indeed, $\mathbf{c} = x' \cdot \nabla_{x'} v + y v_y$

and, since $\nabla_{x'} v$ and $y^a v_y$ are continuous up to $\{y = 0\}$ in $\mathcal{B}_R^+$, the claim follows using that $|y v_y| = |y^a v_y y^{1-a}| \le C y^{2s} \to 0$ as $y \downarrow 0$.

Second, we show that $\mathbf{c} \in H^1(\mathcal{B}_R^+, y^a)$ for all $R < 1$, and as a consequence $\xi = \mathbf{c}\eta \in H^1(\mathcal{B}_R^+, y^a)$ and thus it is an admissible test function for the stability condition (2.2) —since η is C^1 and has compact support in $\mathcal{B}_1^+ \cup B_1$. To see that $\mathbf{c} \in H^1(\mathcal{B}_R^+, y^a)$, recall that $v \in H^1(\mathcal{B}_R^+, y^a)$ for $R < 1$, and thus $\mathbf{c} \in L^2(\mathcal{B}_R^+, y^a)$. Hence, we only need to check that $\nabla \mathbf{c} \in L^2(\mathcal{B}_R^+, y^a)$. For $i = 1, \ldots, n$, we have

$$\mathbf{c}_{x_i} = v_{x_i} + x' \cdot \nabla_{x'} v_{x_i} + y v_{x_i y}.$$

Clearly the first two terms belong to $L^2(\mathcal{B}_R^+, y^a)$ since $\nabla_{x'} v$ and $D_{x'}^2 v$ are bounded in $\mathcal{B}_R^+$ and y^a is integrable near 0. For the last term we use that $y^a v_{x_i y}$ is continuous up to $\{y = 0\}$ to obtain that $|v_{x_i y}| \le C y^{-a}$ for some constant C, and thus $y^a |y v_{x_i y}|^2 \le C y^{2-a} = C y^{1+2s}$, which yields the desired integrability. Finally, for the vertical derivative of $\mathbf{c}$ we use the equation $\Delta v + a v_y / y = 0$ to obtain

$$\mathbf{c}_y = x' \cdot \nabla_{x'} v_y + v_y + y v_{yy} = x' \cdot \nabla_{x'} v_y + (1-a) v_y - y \Delta_{x'} v, \tag{2.4}$$

and using similar arguments as the previous ones, we see that all the terms belong to $L^2(\mathcal{B}_R^+, y^a)$.

Note also that, since $y^a v_y$, $y^a \nabla_{x'} v_y$, and $y^{2-2s} \Delta_{x'} v$ are continuous up to $\{y = 0\}$ (in $\mathcal{B}_R^+$), (2.4) leads to the flux $y^a \mathbf{c}_y$ being continuous and well defined up to $B_R \subset \{y = 0\}$.

We can now take $\xi = \mathbf{c}\eta$ in the stability condition (2.2) to get

$$\int_{B_1} f'(u) \mathbf{c}^2 \eta^2 \, dx' \le d_s \iint_{\mathcal{B}_1^+} y^a \nabla \mathbf{c} \cdot \nabla (\mathbf{c}\eta^2) \, dx + d_s \iint_{\mathcal{B}_1^+} y^a \mathbf{c}^2 |\nabla \eta|^2 \, dx. \tag{2.5}$$

We claim that

$$d_s \int_{\mathcal{B}_1^+} y^a \nabla \mathbf{c} \cdot \nabla (\mathbf{c}\eta^2) \, dx = \iint_{B_1} (f'(u)\mathbf{c} + 2s f(u)) \, \mathbf{c}\eta^2 \, dx'. \tag{2.6}$$

To see this, we first show that

$$\operatorname{div}(y^a \nabla \mathbf{c}) = 0 \quad \text{in } \mathbb{R}_+^{n+1} \quad \text{and} \quad d_s \frac{\partial \mathbf{c}}{\partial \nu^a} - f'(u)\mathbf{c} = 2s f(u) \quad \text{in } B_1. \tag{2.7}$$

The first identity is obtained after a simple computation using that $\Delta v + a v_y / y = 0$:

$$y^{-a} \operatorname{div}(y^a \nabla \mathbf{c}) = \Delta(x \cdot \nabla v) + \frac{a}{y}(x \cdot \nabla v)_y$$

$$= x \cdot \nabla \left(-a \frac{v_y}{y} \right) + 2\Delta v + \frac{a}{y}(v_y + x \cdot \nabla v_y)$$

$$= -a x \cdot \nabla \left(\frac{v_y}{y} \right) + \Delta v + \frac{a}{y} x \cdot \nabla v_y = a y \frac{v_y}{y^2} + \Delta v = 0.$$

To show the second one, recall that from (2.4), for $y > 0$ we have

$$-y^a \mathbf{c}_y = x' \cdot (-y^a \partial_y \nabla_{x'} v) + 2s(-y^a v_y) + y^{2-2s} \Delta_{x'} v.$$

Now, the desired identity follows after taking the limit $y \downarrow 0$ in the above expression, by using Lemma A.3 (b), that $d_s \partial_{\nu^a} v = f(u)$ in B_1, and that $\Delta_{x'} v$ is bounded —and thus $|y^{2-2s} \Delta_{x'} v| \leq C y^{2-2s} \to 0$ as $y \downarrow 0$. We have established (2.7).

To prove (2.6), we integrate by parts in $\mathcal{B}_R^+ \cap \{y > \delta\}$ (a set where $\mathbf{c}$ is smooth) and use the first identity in (2.7) to obtain

$$d_s \int_{\mathcal{B}_R^+ \cap \{y > \delta\}} y^a \nabla \mathbf{c} \cdot \nabla(\mathbf{c}\eta^2)\, dx = -d_s \iint_{\mathcal{B}_R} \delta^a \mathbf{c}_y(x',\delta)\mathbf{c}(x',\delta)\eta^2(x',\delta)\, dx'.$$

Taking the limit $\delta \to 0$ and using that $\mathbf{c}$ and $y^a \mathbf{c}_y$ are continuous up to $\{y = 0\}$ (as proved before), from (2.7) we conclude (2.6).

Combining (2.6) with (2.5) we get

$$-2s \int_{B_1} \frac{f(u)}{d_s} \mathbf{c}\eta^2\, dx' \leq \iint_{\mathcal{B}_1^+} y^a \mathbf{c}^2 |\nabla \eta|^2\, dx.$$

Next, we rewrite the left-hand side of this inequality after an integration by parts (now using that $d_s \partial_{\nu^a} v = f(u)$ in B_1 and that v is s-harmonic), obtaining

$$-2s \iint_{\mathcal{B}_1^+} y^a (\nabla v \cdot \nabla \mathbf{c})\eta^2\, dx - 2s \int_{\mathcal{B}_1^+} y^a \mathbf{c}\nabla v \cdot \nabla(\eta^2)\, dx \leq \iint_{\mathcal{B}_1^+} y^a \mathbf{c}^2 |\nabla \eta|^2\, dx. \tag{2.8}$$

We treat the first integral in this inequality by using the same idea as in the proof of the Pohozaev identity, to get rid of second order derivatives of v. Note first that

$$\nabla v \cdot \nabla \mathbf{c} = \sum_{j=1}^{n+1} v_j \partial_j \sum_{i=1}^{n+1} x_i v_i = \sum_{j=1}^{n+1} v_j^2 + \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} x_i v_j v_{ij} = |\nabla v|^2 + x \cdot \frac{\nabla |\nabla v|^2}{2}. \tag{2.9}$$

We now claim that

$$-\iint_{\mathcal{B}_1^+} y^a \left(x \cdot \nabla |\nabla v|^2\right) \eta^2\, dx = (n + 2 - 2s) \int_{\mathcal{B}_1^+} y^a |\nabla v|^2 \eta^2\, dx + \int_{\mathcal{B}_1^+} y^a |\nabla v|^2 r(\eta^2)_r\, dx. \tag{2.10}$$

To see this, we proceed as before (integrating by parts on $\mathcal{B}_1^+ \cap \{y > \delta\}$, where all the functions are smooth) to obtain

$$-\int_{\mathcal{B}_1^+ \cap \{y > \delta\}} y^a \left(x \cdot \nabla |\nabla v|^2\right) \eta^2\, dx = \int_{\mathcal{B}_1^+ \cap \{y > \delta\}} \mathrm{div}(y^a x)|\nabla v|^2 \eta^2\, dx$$

$$+ \iint_{\mathcal{B}_1^+ \cap \{y > \delta\}} y^a |\nabla v|^2 r(\eta^2)_r\, dx$$

$$+ \iint_{\mathcal{B}_1} \delta^{a+1}\left(|\nabla_{x'} v(x',\delta)|^2 + |v_y(x',\delta)|^2\right)\eta^2(x',\delta)\, dx'.$$

Now, using that $\nabla_{x'} v$ and $y^a v_y$ are continuous up to $\{y = 0\}$ (locally in B_1, but recall that η has compact support in B_1), we have

$$\delta^{a+1}\left(|\nabla_{x'} v(x',\delta)|^2 + |v_y(x',\delta)|^2\right) \leq C\delta^{2-2s} + C'\delta^{1-a}|\delta^a v_y(x',\delta)|^2 \leq C(\delta^{2-2s} + \delta^{2s}),$$

and thus (2.10) is obtained after letting $\delta \to 0$ and using that $\mathrm{div}(y^a x) = (n + 1 + a)y^a = (n + 2 - 2s)y^a$.

From (2.9) and (2.10) we get

$$-\iint_{\mathcal{B}_1^+} y^a(\nabla v \cdot \nabla \mathbf{c})\eta^2 \,\mathrm{d}x = -\int_{\mathcal{B}_1^+} y^a|\nabla v|^2\eta^2 \,\mathrm{d}x - \frac{1}{2}\iint_{\mathcal{B}_1^+} y^a\left(x \cdot \nabla|\nabla v|^2\right)\eta^2 \,\mathrm{d}x$$

$$= \frac{n-2s}{2}\int_{\mathcal{B}_1^+} y^a|\nabla v|^2\eta^2 \,\mathrm{d}x + \frac{1}{2}\iint_{\mathcal{B}_1^+} y^a|\nabla v|^2 r(\eta^2)_r \,\mathrm{d}x.$$

Combining this with (2.8) we conclude the proof. □

Remark 2.4. The second identity in (2.7) can be established directly "downstairs", that is, without using the extension. Indeed, since $\mathbf{c}$ is the s-harmonic extension of $x' \cdot \nabla_{x'}u$, it holds $d_s\partial_{\nu^a}\mathbf{c} = (-\Delta)^s(x' \cdot \nabla_{x'}u)$. Then, the second expression in (2.7) follows from the identity

$$(-\Delta)^s(x' \cdot \nabla_{x'}w) = x' \cdot \nabla_{x'}\{(-\Delta)^s w\} + 2s(-\Delta)^s w,$$

(which holds for all functions w) by taking $w = u$. The identity can be proved easily using the definition of the fractional Laplacian, or using that $x' \cdot \nabla_{x'}w(x') = \frac{\mathrm{d}}{\mathrm{d}h}|_{h=1}w(hx')$ and the scaling properties of $(-\Delta)^s$.

From the inequality of Lemma 2.3 and choosing $\eta = r^{(2s-n)/2}\zeta$ (properly regularized near the origin), with ζ being a cut-off function, we can establish Proposition 2.2.

Proof of Proposition 2.2. For $\varepsilon \in (0, 1/2)$, we take the Lipschitz function

$$\eta = \begin{cases} \varepsilon^{-\alpha/2} & \text{in } \mathcal{B}_\varepsilon^+, \\ r^{-\alpha/2}\zeta & \text{in } \mathcal{B}_1^+ \setminus \mathcal{B}_\varepsilon^+, \end{cases}$$

where $0 \le \alpha \le n - 2s$ is to be chosen later and $\zeta = \zeta(r)$ is a cut-off function satisfying $\zeta = 1$ for $r \le 1/2$ and $\zeta = 0$ for $r \ge 3/4$. We use this choice of η in (2.3). To let $\varepsilon \to 0$ in the resulting inequality, we use the dominated convergence theorem taking into account that $y^a|\nabla v|^2 = y^a(|\nabla_{x'}v|^2 + v_y^2) \le C(y^{1-2s} + y^{2s-1})$ in $\mathcal{B}_{3/4}^+$ —by Lemma A.3 (b)— and that $\alpha \ge 2s - n$. We conclude that $I_1 + I_2 \le I_3$, where

$$I_1 := s\iint_{\mathcal{B}_1^+} y^a\left\{(n - 2s)r\bar{r}^\alpha\zeta^2 + r(r\bar{r}^\alpha\zeta^2)_r\right\}|\nabla v|^2 \,\mathrm{d}x,$$

$$I_2 := -2s\iint_{\mathcal{B}_1^+} y^a r v_r \nabla v \cdot \nabla(r\bar{r}^\alpha\zeta^2) \,\mathrm{d}x,$$

and

$$I_3 := \iint_{\mathcal{B}_1^+} y^a r^2 v_r^2|\nabla(r\bar{r}^{\alpha/2}\zeta)|^2 \,\mathrm{d}x.$$

Now, we compute each of the previous integrals separately, using that $\zeta \equiv 1$ in $\mathcal{B}^+_{1/2}$ —and thus $\nabla\zeta \equiv 0$ in $\mathcal{B}^+_{1/2}$. First, note that

$$I_1 = s(n - 2s - \gamma)\iint_{\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha |\nabla v|^2 \, dx$$

$$+ s\iint_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha |\nabla v|^2\big((n - 2s - \gamma)\zeta^2 + r(\zeta^2)_r\big)\, dx.$$

For I_2 we have

$$I_2 = 2\gamma s\int_{\mathcal{B}^+_{1/2}} y^a r^{-\alpha} v_r^2 \, dx + 2s\iint_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}} y^a r^{-\alpha}\big(\gamma v_r^2\zeta^2 - r v_r\nabla v\cdot\nabla(\zeta^2)\big)\, dx.$$

Last, since $|\nabla(r^{\alpha/2}\zeta)|^2 = (\gamma^2/4)r^{\alpha-2}\zeta^2 + r^{-\alpha}|\nabla\zeta|^2 - \gamma r^{-\alpha-1}\zeta\zeta_r$, we get

$$I_3 = \frac{\gamma^2}{4}\int_{\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha v_r^2 \, dx + \iint_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha v_r^2 \left(\frac{\gamma^2}{4}\zeta^2 + r^2|\nabla\zeta|^2 - \gamma r\zeta\zeta_r\right)\, dx.$$

Next, we estimate the three previous integrals over $\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}$, noticing that the integrands can be bounded by $Cy^a|\nabla v|^2$, with C depending only on n, s, and γ. Hence, from the inequality $I_1 + I_2 \leq I_3$ we get

$$s(n - 2s - \gamma)\iint_{\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha |\nabla v|^2 \, dx + \gamma\left(2s - \frac{\gamma}{4}\right)\int_{\mathcal{B}^+_{1/2}} y^a r^{\gamma}\overline{r}^\alpha v_r^2 \, dx \leq C\int_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}} y^a|\nabla v|^2 \, dx.$$

Note that $\gamma(2s - \gamma/4) > 0$ whenever $\gamma \in (0, 8s)$. Thus, if we take $\gamma = n - 2s$, since the proposition assumes $n \in (2s, 10s)$ we finally obtain the desired estimate

$$\int_{\mathcal{B}^+_{1/2}} y^a r^{2s-n} v_r^2 \, dx \leq C\iint_{\mathcal{B}^+_{3/4}\setminus\mathcal{B}^+_{1/2}} y^a|\nabla v|^2 \, dx$$

with a constant C depending only on n and s. $\qquad\square$

3. A NEW GEOMETRIC FORM OF THE STABILITY CONDITION

In this section we establish Theorem 1.3, which gives a new geometric form of the stability condition. As a corollary, we will obtain an L^2 estimate for the full Hessian of the harmonic extension v of a stable solution u. This last result, stated in Corollary 3.2 below, will be crucial in the following sections.

Proof of Theorem 1.3. We take $\xi = \mathbf{c}\eta$ in the stability condition (1.14), where η is a Lipschitz function with compact support in $\mathbb{R}^n \times [0, +\infty)$ (as in the statement of the theorem) and $\mathbf{c}$ is a function (to be chosen later) in $C^1_{\mathrm{loc}}(\mathbb{R}^n \times [0, +\infty))$, smooth in $\mathbb{R}^{n+1}_+$, and such that $\mathbf{c}\,\Delta\mathbf{c} \in C_{\mathrm{loc}}(\mathbb{R}^n \times [0, +\infty))$. Notice that, as a consequence, $\xi \in H^1(\mathbb{R}^n \times [0, +\infty))$. We obtain

$$\int_{\mathbb{R}^n} f'(u)\mathbf{c}^2\eta^2 \, dx \leq \iint_{\mathbb{R}^{n+1}_+} \nabla\mathbf{c}\cdot\nabla(\mathbf{c}\eta^2) \, dx + \iint_{\mathbb{R}^{n+1}_+} \mathbf{c}^2|\nabla\eta|^2 \, dx.$$

We now integrate by parts the first term on the right-hand side. For the integration by parts to be fully justified, we perform it in $\{y > \delta\}$ (where $\mathbf{c}$ is smooth) and then we let $\delta \to 0$, using dominated convergence and the continuity of $\mathbf{c}\,\mathbf{c}_y$ and $\mathbf{c}\,\Delta\mathbf{c}$. We get

$$\int \left(\mathbf{c}\,\mathbf{c}_y + f'(u)\mathbf{c}^2\right)\eta^2\,\mathrm{d}x' \leq \int_{\mathbb{R}^{n+1}_+} \left(\mathbf{c}^2|\nabla\eta|^2 - \mathbf{c}\,\Delta\mathbf{c}\,\eta^2\right)\,\mathrm{d}x. \tag{3.1}$$

We wish to take $\mathbf{c}$ to be $|\nabla v|$ in the previous inequality. Nevertheless, since this function may not be C^1 (if ∇v vanishes somewhere), we consider instead the smooth function

$$\mathbf{c}_\varepsilon := \left(|\nabla v|^2 + \varepsilon^2\right)^{1/2},$$

for $\varepsilon > 0$, which is locally C^1 in $\overline{\Omega} \times [0, +\infty)$ since $v \in C^2_{\mathrm{loc}}(\overline{\Omega} \times [0, +\infty))$ by Lemma A.3 (c). Moreover, an easy computation using that v is harmonic shows that

$$\mathbf{c}_\varepsilon \Delta\mathbf{c}_\varepsilon = \sum_{i,j=1}^{n+1} v_{ij}^2 - \sum_{j=1}^{n+1}\left(\sum_{i=1}^{n+1} v_{ij}\frac{v_i}{\mathbf{c}_\varepsilon}\right)^2 \geq \mathcal{A}^2 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{3.2}$$

In particular, $\mathbf{c}_\varepsilon \Delta\mathbf{c}_\varepsilon \in C_{\mathrm{loc}}(\overline{\Omega} \times [0, +\infty))$ —since $v \in C^2_{\mathrm{loc}}(\overline{\Omega} \times [0, +\infty))$. As a consequence, we can choose $\mathbf{c}$ in (3.1) to be $\mathbf{c}_\varepsilon$ and conclude, taking into account (3.2),

$$\int_{\mathbb{R}^{n+1}_+} \mathcal{A}^2\eta^2\,\mathrm{d}x + \iint \left(\mathbf{c}_\varepsilon \partial_y\mathbf{c}_\varepsilon + f'(u)\mathbf{c}_\varepsilon^2\right)\eta^2\,\mathrm{d}x' \leq \int_{\mathbb{R}^{n+1}_+} \mathbf{c}_\varepsilon^2|\nabla\eta|^2\,\mathrm{d}x.$$

Next, we claim that

$$\mathbf{c}_\varepsilon \partial_y\mathbf{c}_\varepsilon + f'(u)\mathbf{c}_\varepsilon^2 \geq \varepsilon^2 f'(u) \quad \text{in } \Omega \subset \{y = 0\}. \tag{3.3}$$

Once this is proved, the result will follow by letting $\varepsilon \to 0$.

To establish (3.3), we first compute, for $y > 0$,

$$\mathbf{c}_\varepsilon \partial_y\mathbf{c}_\varepsilon = \sum_{i=1}^{n+1} v_i\,v_{iy} = \nabla_{x'}v \cdot \nabla_{x'}v_y + v_y\,v_{yy}. \tag{3.4}$$

Now, on the one hand note that $-\partial_y\nabla_{x'}v = f(u)\nabla_{x'}u$ in Ω by Lemma A.3 (b). Multiplying this identity by $\nabla_{x'}v$ we get

$$\nabla_{x'}v \cdot \nabla_{x'}v_y = -f'(u)|\nabla_{x'}u|^2 \quad \text{in } \Omega.$$

On the other hand, since $v_{yy} = -\Delta_{x'}v$ and $\Delta_{x'}v$ is continuous up to $\{y = 0\}$, we see that

$$\lim_{y\downarrow 0} v_{yy} = -\Delta_{x'}u = (-\Delta_{x'})^{1/2}(-\Delta_{x'})^{1/2}u = (-\Delta_{x'})^{1/2}f(u) \quad \text{in } \Omega.$$

Using the previous identities together with $-v_y = f(u)$ in Ω, (3.4) leads to

$$\mathbf{c}_\varepsilon \partial_y\mathbf{c}_\varepsilon = -f'(u)|\nabla_{x'}u|^2 - f(u)(-\Delta_{x'})^{1/2}f(u) \quad \text{in } \Omega.$$

As a consequence, noticing that $f'(u)\mathbf{c}_\varepsilon^2 = f'(u)|\nabla_{x'}u|^2 + f'(u)f(u)^2 + \varepsilon^2 f'(u)$ in Ω, we have

$$\mathbf{c}_\varepsilon \partial_y\mathbf{c}_\varepsilon + f'(u)\mathbf{c}_\varepsilon^2 = -f(u)(-\Delta_{x'})^{1/2}f(u) + f'(u)f(u)^2 + \varepsilon^2 f'(u)$$
$$= f(u)\left\{f'(u)(-\Delta_{x'})^{1/2}u - (-\Delta_{x'})^{1/2}f(u)\right\} + \varepsilon^2 f'(u).$$

Finally, using that $f(t_1) - f(t_2) \leq f'(t_1)(t_1 - t_2)$ for all $t_1, t_2 \in \mathbb{R}$ (since f is convex), we see that $(-\Delta_{x'})^{1/2} f(u) \leq f'(u)(-\Delta_{x'})^{1/2} u$. Hence, since f is nonnegative, the claim (3.3) follows. $\square$

Remark 3.1. In the previous proof, we have used strongly that $s = 1/2$, both in the choice of the test function $\xi = |\nabla v| \eta$ and when using that $-\Delta = (-\Delta)^{1/2} \circ (-\Delta)^{1/2}$. It is not clear to us how to extend the previous arguments to other powers $s \in (0,1)$. On the one hand, $|\nabla v|$ is not an appropriate test function anymore, at least when $s < 1/2$ (indeed, since v_y behaves as $-y^{2s-1} f(v)$ near $\{y = 0\}$, $|\nabla v|$ may be singular at almost all points in $\{y = 0\}$). On the other hand, one would say that the analogue decomposition of $-\Delta$ for powers $s \neq 1/2$ should be $-\Delta = (-\Delta)^s \circ (-\Delta)^{1-s}$, but its is not clear which information on $(-\Delta)^{1-s}$ is to be used.

From Theorem 1.3 we can now obtain an H^1 estimate for the gradient of the harmonic extension of a stable solution. The estimate is independent of the nonlinearity f, a fact that will be crucial in the following sections. Note that the norms are now computed on the intersection of $\mathbb{R}_+^{n+1}$ with balls not necessarily centered at $\partial \mathbb{R}_+^{n+1}$, since this will be useful later on.

Corollary 3.2. *Let $n \geq 1$ and let $u \in C^2(B_1) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2} u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the harmonic extension of u in $\mathbb{R}_+^{n+1}$.*

Then, for every $0 < R_1 < R_2 \leq 1$ and every $x_0 \in \mathcal{B}_1$ such that $\mathcal{B}_{R_2}(x_0) \subset \mathcal{B}_1$,

$$\|\nabla v\|_{L^2(\mathcal{B}_{R_1}(x_0) \cap \{y>0\})} + (R_2 - R_1) \left\| D^2 v \right\|_{L^2(\mathcal{B}_{R_1}(x_0) \cap \{y>0\})} \leq C \|\nabla v\|_{L^2(\mathcal{B}_{R_2}(x_0) \cap \{y>0\})}$$

for some positive dimensional constant C.

Proof. The bound for the L^2 norm of ∇v in $\mathcal{B}_{R_1}(x_0) \cap \{y > 0\}$ is trivial; thus we only need to prove an L^2 estimate for $D^2 v$. To carry this out, choose a cut-off function η with compact support in $\mathcal{B}_{R_2}(x_0)$ such that $\eta \equiv 1$ in $\mathcal{B}_{R_1}(x_0)$ —hence its gradient is supported in $\mathcal{B}_{R_2}(x_0) \setminus \mathcal{B}_{R_1}(x_0)$ and, in this set, $|\nabla \eta| \leq C/(R_2 - R_1)$ for some dimensional constant C. Now, since $\mathcal{B}_{R_2}(x_0) \subset \mathcal{B}_1$, we can use Theorem 1.3 with this choice of η to obtain

$$\int_{\mathcal{B}_{R_1}(x_0) \cap \{y>0\}} \mathcal{A}^2 \, dx \leq \frac{C}{(R_2 - R_1)^2} \iint_{(\mathcal{B}_{R_2}(x_0) \setminus \mathcal{B}_{R_1}(x_0)) \cap \{y>0\}} |\nabla v|^2 \, dx.$$

The corollary will follow once we show that the square of every element of $D^2 v$ can be controlled pointwise by $C\mathcal{A}^2$ for some dimensional constant C.

Let us prove this last assertion. At every point $x \in \mathbb{R}_+^{n+1} \cap \{\nabla v \neq 0\}$ we consider the normal unitary vector to the level set of v passing through x,

$$\nu = \frac{\nabla v(x)}{|\nabla v(x)|},$$

and any orthonormal basis $\{\tau_1, \ldots, \tau_n, \nu\}$ of $\mathbb{R}^{n+1}$. Expressing $D^2 v(x)$ in this basis, we see that

$$\mathcal{A}^2(x) = \sum_{i,j=1}^{n} (D^2 v(x)[\tau_i, \tau_j])^2 + \sum_{i=1}^{n} (D^2 v(x)[\tau_i, \nu])^2;$$

recall that $\mathcal{A}^2$ is defined in (1.7). Now, by the symmetry of D^2v, it is clear that $\mathcal{A}^2(x)$ controls the square of every entry in $D^2v(x)$ except for $D^2v(x)[\nu, \nu]$. However, since v is harmonic, we have that

$$D^2v(x)[\nu, \nu] = -\sum_{i=1}^{n} D^2v(x)[\tau_i, \tau_i],$$

and thus we have the desired control at every point $x \in \mathbb{R}^{n+1}_+ \cap \{\nabla v \neq 0\}$.

Finally, by noticing that $D^2v = 0$ a.e. in $\mathbb{R}^{n+1}_+ \cap \{\nabla v = 0\}$ since ∇v is Lipschitz (see for instance [17, Theorem 6.19]), we conclude the proof. $\qquad \square$

4. DECAY OF THE WEIGHTED RADIAL DERIVATIVE

In this section, we establish the geometric decay in R of a weighted L^2 norm of v_r in $\mathcal{B}^+_R$, where v is the harmonic extension of a stable solution. The main ingredient is the following key lemma. It establishes that, under a doubling assumption on $|\nabla v|^2 \, dx$, the full gradient of v is controlled, in L^2, by the radial derivative of v in an annulus. To prove this result we use, as in [3], a compactness argument combined with the nonnegativeness of the nonlinearity f and the nonexistence of nonconstant 0-homogeneous superharmonic functions.

Lemma 4.1. *Let $n \geq 1$ and let $u \in C^2(B_2) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2}u = f(u)$ in $B_2 \subset \mathbb{R}^n$, with f a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$. Assume that*

$$\int_{\mathcal{B}^+_1} |\nabla v|^2 \, dx \geq \delta \iint_{\mathcal{B}^+_2} |\nabla v|^2 \, dx$$

for some constant $\delta > 0$.

Then,

$$\int_{\mathcal{B}^+_{3/2}} |\nabla v|^2 \, dx \leq C_\delta \iint_{\mathcal{B}^+_{3/2} \setminus \mathcal{B}^+_1} |x \cdot \nabla v|^2 \, dx$$

for some constant C_δ depending only on n and δ.

Proof. By contradiction, assume the result to be false. Then, there exists a sequence of stable solutions u_k in B_2 such that their harmonic extensions v_k satisfy

$$\iint_{\mathcal{B}^+_1} |\nabla v_k|^2 \, dx \geq \delta \iint_{\mathcal{B}^+_2} |\nabla v_k|^2 \, dx, \tag{4.1}$$

$$\iint_{\mathcal{B}^+_{3/2}} |\nabla v_k|^2 \, dx = 1, \tag{4.2}$$

and

$$\iint_{\mathcal{B}^+_{3/2} \setminus \mathcal{B}^+_1} |x \cdot \nabla v_k|^2 \, dx \to 0 \quad \text{as } k \to \infty.$$

Note that each $u_k = v_k(\cdot, 0)$ solves a different equation, $(-\Delta)^{1/2}u_k = f_k(u_k)$ in B_2, for some convex nonlinearities $f_k \geq 0$. To ensure (4.2), we use that the class of solutions considered in the lemma is invariant under multiplication by constants.

The crucial point in the following arguments is that the estimate from Corollary 3.2 does not depend on the nonlinearity f_k. Thus, we will be able to use it to obtain uniform estimates in k. The details go as follows.

Using Corollary 3.2 (rescaled to hold in B_2 and with $x_0 = 0$ and $R_1 = 3/2$ and $R_2 = 2$ after the rescaling), the doubling assumption (4.1), and (4.2), we get

$$\|\nabla v_k\|_{H^1(\mathcal{B}^+_{3/2})}^2 \leq C \|\nabla v_k\|_{L^2(\mathcal{B}^+_2)}^2 \leq \frac{C}{\delta} \|\nabla v_k\|_{L^2(\mathcal{B}^+_1)}^2 \leq \frac{C}{\delta}$$

where C is a positive dimensional constant. Therefore, by Rellich's compactness theorem, up to removing a constant from v_k, a subsequence of v_k converges in $H^1(\mathcal{B}^+_{3/2})$ to some function v_∞. Moreover,

$$\fint\!\!\!\!\!\fint_{\mathcal{B}^+_{3/2}} |\nabla v_\infty|^2 \, \mathrm{d}x = 1 \tag{4.3}$$

and

$$\fint\!\!\!\!\!\fint_{\mathcal{B}^+_{3/2}\setminus\mathcal{B}^+_1} |x \cdot \nabla v_\infty|^2 \, \mathrm{d}x = 0.$$

This yields that v_∞ is a 0-homogeneous function in $\mathcal{B}^+_{3/2} \setminus \mathcal{B}^+_1$.

Notice that each v_k is harmonic in $\mathcal{B}^+_{3/2}$. In addition, v_k has nonnegative flux on $B_{3/2} \subset \{y = 0\}$ in the weak H^1 sense, i.e.,

$$\fint\!\!\!\!\!\fint_{\mathcal{B}^+_{3/2}} \nabla v_k \cdot \nabla \varphi \, \mathrm{d}x \geq 0$$

for every nonnegative smooth function φ with compact support in $\mathcal{B}^+_{3/2} \cup B_{3/2}$. By the H^1 convergence the same two properties hold for v_∞. Therefore, by considering the even reflection of v_∞ across $\partial\mathbb{R}^{n+1}_+$ we obtain a function, still denoted by v_∞, which is 0-homogeneous in $\mathcal{B}_{3/2} \setminus \mathcal{B}_1$ and weakly superharmonic in $\mathcal{B}_{3/2}$.

As a consequence of the mean value property for superharmonic functions, v_∞ is bounded below in $\partial\mathcal{B}_{5/4}$ and thus, by 0-homogeneity, also in $\mathcal{B}_{3/2} \setminus \mathcal{B}_1$. Moreover, $\inf_{\mathcal{B}_{3/2}\setminus\mathcal{B}_1} v_\infty = \inf_{\mathcal{B}_{1/8}(x_0)} v_\infty$ for some $x_0 \in \partial\mathcal{B}_{5/4}$. Therefore, the strong maximum principle ([14, Theorem 8.19]) yields that v_∞ is constant in $\mathcal{B}_{3/2} \setminus \mathcal{B}_1$, say $v_\infty \equiv c_0$.

Finally, since $v_{\infty|\partial\mathcal{B}_1} = c_0$, by superharmonicity it follows that $v_\infty \geq c_0$ in $\mathcal{B}_{3/2}$. But since $v_\infty \equiv c_0$ in $\mathcal{B}_{3/2} \setminus \mathcal{B}_1$, the strong maximum principle for superharmonic functions yields that $v_\infty \equiv c_0$ in $\mathcal{B}_{3/2}$. This contradicts (4.3) and concludes the proof. $\square$

Thanks to the control of the gradient in terms of the radial derivative (under a doubling assumption), given by the previous lemma, we can establish the estimate that will lead to Hölder continuity. Note that here we need to assume $2 \leq n \leq 4$.

Proposition 4.2. *Let $u \in C^2(B_1) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2}u = f(u)$ in $B_1 \subset \mathbb{R}^n$, with f a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the harmonic extension of u in $\mathbb{R}^{n+1}_+$.*

If $2 \le n \le 4$, then

$$\int_{\mathcal{B}_R^+} r^{1-n} v_r^2 \, dx \le C R^{2\alpha} \iint_{\mathcal{B}_{3/4}^+} |\nabla v|^2 \, dx \quad \text{for } R \le 1/2,$$

where   and C are positive dimensional constants.

Proof. For $j \ge 0$, define

$$a_j := 2^{-j(1-n)} \int_{\mathcal{B}_{1/2^{j+1}}^+} |\nabla v|^2 \, dx \quad \text{and} \quad b_j := \iint_{\mathcal{B}_{1/2^{j+1}}^+} r^{1-n} v_r^2 \, dx.$$

By Proposition 2.2 (used with $s = 1/2$) there exists a dimensional constant C for which $b_0 \le C \|\nabla v\|_{L^2(\mathcal{B}_{3/4}^+)}^2$, and thus

$$a_0 \le M \quad \text{and} \quad b_0 \le M, \quad \text{where} \quad M = \max\{1, C\} \iint_{\mathcal{B}_{3/4}^+} |\nabla v|^2 \, dx.$$

Clearly, $b_j \le b_{j-1}$ for $j \ge 1$. Moreover, by applying Proposition 2.2 to $v(\cdot/2^j)$, it follows[9] that

$$a_j + b_j \le L_1 a_{j-1} \quad \text{for } j \ge 1, \tag{4.4}$$

for some positive dimensional constant L_1. Furthermore, by applying Lemma 4.1 (with $\delta = 2^{-n}$) to $v(\cdot/2^{j+1})$ with $j \ge 1$, we get that if

$$2^{-j(1-n)} \int_{\mathcal{B}_{1/2^{j+1}}^+} |\nabla v|^2 \, dx \ge 2^{-n} 2^{-j(1-n)} \iint_{\mathcal{B}_{1/2^j}^+} |\nabla v|^2 \, dx = \frac{1}{2} \left(2^{-(j-1)(1-n)} \iint_{\mathcal{B}_{1/2^j}^+} |\nabla v|^2 \, dx \right)$$

then

$$2^{-j(1-n)} \int_{\mathcal{B}_{1/2^{j+1}}^+} |\nabla v|^2 \, dx \le 2^{-j(1-n)} \iint_{\mathcal{B}_{3/2^{j+2}}^+} |\nabla v|^2 \, dx$$

$$\le C \int_{\mathcal{B}_{3/2^{j+2}}^+ \setminus \mathcal{B}_{1/2^{j+1}}^+} r^{1-n} v_r^2 \, dx \le C \iint_{\mathcal{B}_{1/2^j}^+ \setminus \mathcal{B}_{1/2^{j+1}}^+} r^{1-n} v_r^2 \, dx$$

for some dimensional constant C. That is, if $j \ge 1$ and $a_j \ge (1/2)a_{j-1}$, then $a_j \le L_2(b_{j-1}-b_j)$ for some other dimensional constant L_2. By (4.4), this leads to $b_j \le 2L_1 L_2(b_{j-1}-b_j)$ provided that $a_j \ge (1/2)a_{j-1}$.

Summarizing, there exist a dimensional constant L such that

$$b_j \le b_{j-1} \quad \text{and} \quad a_j + b_j \le L a_{j-1} \quad \text{for } j \ge 1,$$

and

$$\text{if} \quad a_j \ge \frac{1}{2} a_{j-1} \quad \text{then} \quad b_j \le L(b_{j-1} - b_j) \quad \text{for } j \ge 1.$$

---

[9]It is useful to note that a_j and b_j are adimensional quantities, and that Proposition 2.2 can be written equivalently having the weight r^{2s-n} inside the integral of the right-hand side.

Hence, from Lemma 3.2 of [3] if follows that there exist two dimensional constants $\theta \in (0,1)$ and $C_0 > 0$ for which

$$b_j \leq C_0 \theta^j M \quad \text{for } j \geq 0. \tag{4.5}$$

Taking α such that $(1/2)^{2\alpha} = \theta$ we conclude the proof. $\qquad\square$

Remark 4.3. Although the decay of the weighted radial derivative given by Proposition 4.2 will suffice to establish our main result, we note that, with the current tools, the same decay can be proved for the full gradient. That is,

$$\iint_{\mathcal{B}_R^+} r^{1-n}|\nabla v|^2 \, \mathrm{d}x \leq CR^{2\alpha} M \quad \text{for } R \leq 1/2, $$

where M is defined as in the previous proof. For this, one first proves by induction that there exists a dimensional constant $C_\star > 0$ such that

$$d_k := \iint_{\mathcal{B}_{1/2^{k-1}}^+ \setminus \mathcal{B}_{1/2^k}^+} r^{1-n}|\nabla v|^2 \, \mathrm{d}x \leq C_\star \theta^k M \quad \text{for } k \geq 2. \tag{4.6}$$

From this, one concludes that

$$\iint_{\mathcal{B}_{1/2^j}^+} r^{1-n}|\nabla v|^2 \, \mathrm{d}x = \sum_{k=j+1}^\infty \iint_{\mathcal{B}_{1/2^{k-1}}^+ \setminus \mathcal{B}_{1/2^k}^+} r^{1-n}|\nabla v|^2 \, \mathrm{d}x \leq C_\star \sum_{k=j+1}^\infty \theta^k M = C\theta^j M$$

for some dimensional constant C.

To show (4.6), in each induction step we have the following dichotomy: either $d_{k+1} \leq \theta d_k$ (and then (4.6) follows readily from the induction hypothesis $d_k \leq C_\star \theta^k M$), or $d_{k+1} > \theta d_k$. In this second case, a suitable doubling assumption for $|\nabla v|^2 \, \mathrm{d}x$ is at our disposal. This allows us to use Lemma 4.1 (after an appropriate scaling) and hence control the L^2 norm of the gradient in the annulus by the L^2 norm of the radial derivative. From this and the decay given by (4.5), we deduce (4.6).

5. H^1 CONTROL OF THE HARMONIC EXTENSION

In this section we establish a further ingredient towards the proof of Theorem 1.1. In view of our last result, Proposition 4.2, we must control the H^1 norm of the harmonic extension v of a stable solution u by the $L^1_{1/2}$ norm of u in $\mathbb{R}^n$. This is the content of the next proposition.

Proposition 5.1. *Let $n \geq 1$ and let $u \in C^2(B_1) \cap L^1_{1/2}(\mathbb{R}^n)$ be a stable solution to $(-\Delta)^{1/2}u = f(u)$ in $B_1 \subset \mathbb{R}^n$, where f is a nonnegative convex $C^{1,\gamma}$ function for some $\gamma > 0$. Let v be the harmonic extension of u in $\mathbb{R}_+^{n+1}$.*

Then,

$$\|\nabla v\|_{L^2(\mathcal{B}_{1/2}^+)} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \tag{5.1}$$

for some dimensional constant C.

As a consequence,

$$[u]_{H^{1/2}(B_{1/2})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \tag{5.2}$$

for some other dimensional constant C.

Proof. The main idea of the proof is, as in [1], to use the interpolation results of Appendix B combined with the L^2 estimate for D^2v from Corollary 3.2. Since some error terms will appear in the right-hand side of the estimates, we will use an abstract lemma of L. Simon to absorb them into the left-hand side. In order to use this lemma, we establish our estimates in generic balls $\mathcal{B}_R(x_0) \subset \mathcal{B}_1$ (not necessarily half-balls centered at points $x_0 = (x_0', 0) \in \{y = 0\}$) after intersecting them with $\mathbb{R}^{n+1}_+$. Through the proof we will use the letter C to denote a dimensional constant which may change in each appearance.

Given $x_0 \in \mathcal{B}_1 \subset \mathbb{R}^{n+1}$ and $R \in (0, 1]$ such that $\mathcal{B}_R(x_0) \subset \mathcal{B}_1$, we cover the set $\mathcal{B}_{R/2}(x_0) \cap \{y > 0\}$ (up to a set of measure zero) with a family of disjoint open cubes $Q_j \subset \mathbb{R}^{n+1}_+$ of side-length $l_n R$, for some dimensional number l_n small enough such that $Q_j \subset \mathcal{B}_{3R/4}(x_0)$.

Now, given $\varepsilon \in (0, 1)$, in each cube we use the interpolation results of Propositions B.1 and B.2 (with $\mathbb{R}^n$ replaced by $\mathbb{R}^{n+1}$), properly rescaled to hold in the cubes Q_j, and taking $\tilde{\varepsilon} = \varepsilon^{3/2}$ in Proposition B.2. Note that v is C^2 in each $\overline{Q_j}$ by Lemma A.3 (c). We obtain

$$R^{2-(n+1)} \int_{Q_j} |\nabla v|^2 \, dx \le CR^{3-(n+1)} \varepsilon \iint_{Q_j} |\nabla v||D^2v| \, dx + CR^{2-(n+1)}\varepsilon \iint_{Q_j} |\nabla v|^2 \, dx$$

$$+ CR^{-2(n+1)}\varepsilon^{-2-3(n+1)/2} \left(\iint_{Q_j} |v| \, dx \right)^2$$

for every $\varepsilon \in (0, 1)$. Multiplying the above inequality by $R^{2(n+1)}$ and using $2R^3|\nabla v||D^2v| \le R^2|\nabla v|^2 + R^4|D^2v|^2$, we get

$$R^{n+3} \int_{Q_j} |\nabla v|^2 \, dx \le CR^{n+5}\varepsilon \int_{Q_j} |D^2v|^2 \, dx + CR^{n+3}\varepsilon \iint_{Q_j} |\nabla v|^2 \, dx$$

$$+ C\varepsilon^{-2-3(n+1)/2} \left(\iint_{Q_j} |v| \, dx \right)^2.$$

Adding up all these inequalities (using that the disjoint cubes Q_j cover $\mathcal{B}_{R/2}(x_0) \cap \{y > 0\}$ and are contained in $\mathcal{B}_{3R/4}(x_0) \cap \{y > 0\}$), we obtain

$$R^{n+3} \int_{\mathcal{B}_{R/2}(x_0)\cap\{y>0\}} |\nabla v|^2 \, dx \le CR^{n+5}\varepsilon \iint_{\mathcal{B}_{3R/4}(x_0)\cap\{y>0\}} |D^2v|^2 \, dx$$

$$+ CR^{n+3}\varepsilon \iint_{\mathcal{B}_{3R/4}(x_0)\cap\{y>0\}} |\nabla v|^2 \, dx$$

$$+ C\varepsilon^{-2-3(n+1)/2} \left(\iint_{\mathcal{B}_{3R/4}(x_0)\cap\{y>0\}} |v| \, dx \right)^2.$$

Next, we combine this information with Corollary 3.2 (used with $R_1 = 3R/4$ and $R_2 = R$), which gives that

$$R^2 \int_{\mathcal{B}_{3R/4}(x_0)\cap\{y>0\}} |D^2v|^2 \, dx \le C \iint_{\mathcal{B}_R(x_0)\cap\{y>0\}} |\nabla v|^2 \, dx.$$

We deduce that

$$R^{n+3} \int_{\mathcal{B}_{R/2}(x_0) \cap \{y>0\}} |\nabla v|^2 \, \mathrm{d}x \leq C R^{n+3} \varepsilon \iint_{\mathcal{B}_R(x_0) \cap \{y>0\}} |\nabla v|^2 \, \mathrm{d}x$$
$$+ C \varepsilon^{-2-3(n+1)/2} \left(\iint_{\mathcal{B}_1^+} |v| \, \mathrm{d}x \right)^2 . \tag{5.3}$$

Applying now an abstract result of L. Simon —that we use as stated in [3, Lemma A.4] taking $= n+3$ and $\sigma(\mathcal{B}_R(x_0)) := \|\nabla v\|^2_{L^2(\mathcal{B}_R(x_0) \cap \{y>0\})}$, after choosing ε dimensionally small in (5.3)— we get

$$\int_{\mathcal{B}_{1/2}^+} |\nabla v|^2 \, \mathrm{d}x \leq C \left(\iint_{\mathcal{B}_1^+} |v| \, \mathrm{d}x \right)^2 . \tag{5.4}$$

Finally, to bound the L^1 norm of v in terms of its trace we use Lemma A.1, which gives

$$\iint_{\mathcal{B}_1^+} |v| \, \mathrm{d}x \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \tag{5.5}$$

for some dimensional constant C. This concludes the proof of (5.1).

To establish the second estimate of the proposition, we take a cut-off function ζ such that $\zeta \equiv 1$ in $\mathcal{B}_{1/8}^+$ and $\zeta \equiv 0$ in $\mathbb{R}_+^{n+1} \setminus \mathcal{B}_{1/4}^+$. Now, by the well-known trace inequality $[w(\cdot, 0)]_{H^{1/2}(\mathbb{R}^n)} \leq [w]_{H^1(\mathbb{R}_+^{n+1})}$ used with $w = v\zeta$, we obtain

$$[u]^2_{H^{1/2}(B_{1/8})} \leq [u\zeta]^2_{H^{1/2}(\mathbb{R}^n)} \leq \iint_{\mathcal{B}_{1/4}^+} |\nabla (v\zeta)|^2 \, \mathrm{d}x \leq C \left(\int_{\mathcal{B}_{1/4}^+} |\nabla v|^2 + \iint_{\mathcal{B}_{1/4}^+} |v|^2 \right)$$

Using Proposition B.2 as before (in a finite collection of disjoint cubes Q_j covering $\mathcal{B}_{1/4}^+$ but all contained in $\mathcal{B}_{1/2}^+$, with $\tilde{\varepsilon} = 1/2$, and then adding up the resulting inequalities), we get that

$$\int_{\mathcal{B}_{1/4}^+} |v|^2 \leq C \iint_{\mathcal{B}_{1/2}^+} |\nabla v|^2 + C \left(\iint_{\mathcal{B}_{1/2}^+} |v| \right)^2 ,$$

and therefore

$$[u]^2_{H^{1/2}(B_{1/8})} \leq C \int_{\mathcal{B}_{1/2}^+} |\nabla v|^2 + C \left(\iint_{\mathcal{B}_{1/2}^+} |v| \right)^2 .$$

To conclude, we use the already proved estimate (5.1) together with (5.5) to obtain

$$[u]_{H^{1/2}(B_{1/8})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} .$$

Finally, to replace $B_{1/8}$ by $B_{1/2}$ in the above inequality and thus conclude (5.2), we use a standard covering and scaling argument, noticing that all the estimates are independent of the nonlinearity f and hence can be applied to rescaled stable solutions. □

6. Proof of the main results

In this last section we finally establish Theorem 1.1 as well as Corollaries 1.2 and 1.4. In the case $n = 1$ we borrow the results from next section.

To establish the Hölder bound of Theorem 1.1, we will use a Morrey-type estimate from [1] adapted to radial derivatives. It is stated in Lemma 6.1 below. Notice that in the current boundary setting, to obtain Hölder regularity on $\mathbb{R}^n = \partial \mathbb{R}^{n+1}_+$ it will suffice to have a geometric-decay bound for radial derivatives with respect to points on $\mathbb{R}^n = \partial \mathbb{R}^{n+1}_+$. That is, we do not need control on the full gradient, neither on radial derivatives with respect to points in $\mathbb{R}^{n+1}_+$.

Here and through the section, given a function w defined in $\mathbb{R}^{n+1}_+$ and a point $z' \in \mathbb{R}^n$, we will denote the radius with respect to z' by

$$r_{z'} = r_{z'}(x) := |x - (z', 0)|, \quad x \in \mathbb{R}^{n+1}_+,$$

and the radial derivative of w with respect to z' by

$$w_{r_{z'}}(x) := \frac{x - (z', 0)}{|x - (z', 0)|} \cdot \nabla w(x), \quad x \in \mathbb{R}^{n+1}_+.$$

Lemma 6.1 ([1]). *Let $z' \in \mathbb{R}^n$, $d > 0$, and w be a C^1 function in $\overline{\mathcal{B}^+_d(z')} \subset \overline{\mathbb{R}^{n+1}_+}$. Assume that, for some positive constants C_1,*

$$\fint_{\mathcal{B}^+_R(z')} |w_{r_{z'}}| \, dx \leq C_1 R^{n+\alpha} \quad \text{for all } R \leq d. \tag{6.1}$$

Let S be any measurable subset of $\mathcal{B}^+_d(z')$ and $w_S := \frac{1}{|S|} \int_S w \, dx$. Then,

$$|w(z', 0) - w_S| \leq C C_1 \frac{d^{n+1}}{|S|} d^\alpha$$

for some constant C depending only on n and α.

Proof. The proof is that of Lemma C.1 in [1] (which, in turn, follows that of Chapter 7 in [14]). Given $x \in S$, we have

$$w(x) - w(z', 0) = \int_0^{|x-(z',0)|} \frac{x - (z', 0)}{|x - (z', 0)|} \cdot \nabla w \left((z', 0) + \rho \frac{x - (z', 0)}{|x - (z', 0)|} \right) \, d\rho$$

$$= \int_0^{|x-(z',0)|} w_{r_{z'}} \left((z', 0) + \rho \frac{x - (z', 0)}{|x - (z', 0)|} \right) \, d\rho.$$

Averaging in $x \in S$, taking absolute values, and using spherical coordinates centered at $(z', 0)$ (i.e., $x = (z', 0) + \tilde{\rho}\omega$, with $\tilde{\rho} = |x - (z', 0)| = r_{z'}$ and $\omega \in \mathbb{S}^n_+$, where $\mathbb{S}^n_+ := \mathbb{S}^n \cap \{y > 0\}$), we

get

$$
\begin{aligned}
|w(z',0) - w_S| &\leq \frac{1}{|S|} \int_S \mathrm{d}x \fint_0^{|x-(z',0)|} \mathrm{d}\rho\, w_{r_{z'}}\left((z',0) + \rho\frac{x-(z',0)}{|x-(z',0)|}\right) \\
&\leq \frac{1}{|S|} \int_{\mathcal{B}_d^+(z')} \mathrm{d}x \fint_0^d \mathrm{d}\rho\; w_{r_{z'}}\left((z',0) + \rho\frac{x-(z',0)}{|x-(z',0)|}\right) \\
&= \frac{1}{|S|} \fint_0^d \mathrm{d}\tilde\rho\, \tilde\rho^n \int_{\mathbb{S}_+^n} \mathrm{d}\omega \fint_0^d \mathrm{d}\rho\; w_{r_{z'}}\left((z',0) + \rho\omega\right) \\
&= \frac{d^{n+1}}{(n+1)|S|} \int_{\mathbb{S}_+^n} \mathrm{d}\omega \fint_0^d \mathrm{d}\rho\; w_{r_{z'}}\left((z',0) + \rho\omega\right) \\
&= \frac{d^{n+1}}{(n+1)|S|} \fint_{\mathcal{B}_d^+(z')} r^{-n}|w_{r_{z'}}|\,\mathrm{d}x.
\end{aligned}
$$

Let us now bound this last integral. For this, set

$$
\varphi(t) := \fint_{\mathcal{B}_t^+(z')} |w_{r_{z'}}|\,\mathrm{d}x
$$

and use that

$$
\varphi'(t) = \fint_{\partial \mathcal{B}_t(z') \cap \{y>0\}} |w_{r_{z'}}|\,\mathrm{d}\mathcal{H}^n.
$$

Integrating by parts and using (6.1) we get

$$
\begin{aligned}
\int_{\mathcal{B}_d^+(z')} r^{-n}|w_{r_{z'}}|\,\mathrm{d}x &= \fint_0^d \mathrm{d}t \fint_{\partial \mathcal{B}_t(z')\cap\{y>0\}} \mathrm{d}\mathcal{H}^n\, t^{-n}|w_{r_{z'}}| = \fint_0^d t^{-n}\varphi'(t)\,\mathrm{d}t \\
&= d^{-n}\varphi(d) + n\fint_0^d t^{-n-1}\varphi(t)\,\mathrm{d}t \\
&\leq C_1 d^{-n} d^{n+\alpha} + nC_1 \fint_0^d t^{-n-1} t^{n+\alpha}\,\mathrm{d}t \leq CC_1 d^\alpha,
\end{aligned}
$$

establishing the result. $\qquad\square$

Once we have the previous result at hand, we can proceed now with the proof of our main theorem, which provides the Hölder estimate for stable solutions in dimensions $n \leq 4$.

Proof of Theorem 1.1. The $H^{1/2}$ estimate (1.5) has been already proved in Proposition 5.1. From now on, we assume that $2 \leq n \leq 4$, and notice that once the Hölder estimate (1.6) is established in dimension $n = 2$, then by the results of next section (Lemmas 7.1 and 7.2), it will hold as well for stable solutions $u: \mathbb{R} \to \mathbb{R}$ in dimension $n = 1$. For this, we simply define $w(x_1, x_2) := u(x_1)$ and apply estimate (1.6) to w.

Now, since $2 \leq n \leq 4$, by Proposition 4.2 we know that

$$
\int_{\mathcal{B}_R^+} |v_r|\,\mathrm{d}x \leq \left(\fint_{\mathcal{B}_R^+} r^{n-1}\,\mathrm{d}x\right)^{1/2} \left(\fint_{\mathcal{B}_R^+} r^{-n}|v_r|^2\,\mathrm{d}x\right)^{1/2} \leq CR^{n+\alpha}\|\nabla v\|_{L^2(\mathcal{B}_{3/4}^+)}
$$

for $R \leq 1/2$, where α and C are some positive dimensional constants. Replacing the origin by any $z' \in \overline{B}_{1/8} \subset \mathbb{R}^n$ in the above inequality (since the equation solved by u is invariant under translations in $\mathbb{R}^n$), we get that

$$\fint\kern-0.6em\fint_{\mathcal{B}_R^+(z')} |v_{r_{z'}}| \, dx \leq C R^{n+\alpha} \|\nabla v\|_{L^2(\mathcal{B}_{3/4}^+(z'))} \leq C R^{n+\alpha} \|\nabla v\|_{L^2(\mathcal{B}_{7/8}^+)} \quad \text{for all } R \leq 1/2.$$

Using Proposition 5.1 to control $\|\nabla v\|_{L^2(\mathcal{B}_{7/8}^+)}$ (after a covering and scaling argument) by $\|u\|_{L^1_{1/2}(\mathbb{R}^n)}$, we conclude

$$\fint\kern-0.6em\fint_{\mathcal{B}_R^+(z')} |v_{r_{z'}}| \, dx \leq C_0 R^{n+\alpha} \|u\|_{L^1_{1/2}(\mathbb{R}^n)} \quad \text{for all } z' \in \overline{B}_{1/8} \text{ and } R \leq 1/2, \qquad (6.2)$$

for some other dimensional constant C_0.

From these geometric-decay bounds for radial derivatives, we now obtain a Hölder estimate for u in $\overline{B}_{1/8}$. For this, given z' and $\tilde{z}'$ in $\overline{B}_{1/8}$, we set

$$d := |z' - \tilde{z}'| \leq 1/4 \quad \text{and} \quad S := \mathcal{B}_d^+(z') \cap \mathcal{B}_d^+(\tilde{z}').$$

Note that $|S| = c(n) d^{n+1}$ for some dimensional constant $c(n)$. Hence, using Lemma 6.1 in $\mathcal{B}_d^+(z')$ and $\mathcal{B}_d^+(\tilde{z}')$ —in both cases with $w = v$, $C_1 = C_0 \|u\|_{L^1_{1/2}(\mathbb{R}^n)}$, and with d and S as defined above—, we obtain

$$|u(z') - u(\tilde{z}')| \leq |v(z',0) - v_S| + |v_S - v(\tilde{z}',0)| \leq C C_1 d^\alpha = C C_0 \|u\|_{L^1_{1/2}(\mathbb{R}^n)} |z' - \tilde{z}'|^\alpha$$

for some constant C depending only on n and α. Hence,

$$[u]_{C^\alpha(\overline{B}_{1/8})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)}.$$

Now, we bound the L^∞ norm of u as follows. For x' and z' in $B_{1/8}$, by the previous estimate we have

$$|u(x')| \leq |u(x') - u(z')| + |u(z')| \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)} + |u(z')|.$$

Integrating with respect to z' in $B_{1/8}$ we deduce

$$\|u\|_{L^\infty(B_{1/8})} \leq C \left(\|u\|_{L^1_{1/2}(\mathbb{R}^n)} + \|u\|_{L^1(B_{1/8})} \right) \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)}.$$

We conclude that

$$\|u\|_{C^\alpha(\overline{B}_{1/8})} \leq C \|u\|_{L^1_{1/2}(\mathbb{R}^n)}.$$

Finally, we prove the claimed estimate in $\overline{B}_{1/2}$ by using a standard covering and scaling argument. $\square$

Once the main theorem is proved, we can establish Corollary 1.2.

Proof of Corollary 1.2. We may assume $u \not\equiv 0$, and thus $u > 0$ in Ω by the maximum principle. Now, on the one hand, since Ω is convex, by Proposition 1.8 of [21], there exist two positive constants δ and C, both depending only on Ω, such that

$$\|u\|_{L^\infty(\Omega \setminus K_\delta)} \leq C \|u\|_{L^1(\Omega)},$$

where $K_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \delta\}$.

On the other hand, by regularity theory (see Remark 2.1), $u \in C^2(\Omega)$. Thus, we can use Theorem 1.1, that together with a covering and scaling argument leads to

$$\|u\|_{L^\infty(K_\delta)} \leq C \|u\|_{L^1(\Omega)},$$

for some constant C depending only on Ω (to control the $L^1_{1/2}$ norm of u, recall that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$).

Combining both estimates we conclude the desired result. $\qquad\square$

With Corollary 1.2 at hand, we can establish our boundedness result for the extremal solution.

Proof of Corollary 1.4. For $\lambda < \lambda^\star$, let u_λ be the minimal solution to (1.9) with $s = 1/2$. Since $u_\lambda \in L^\infty(\Omega) \cap H^{1/2}(\mathbb{R}^n)$ and it is a stable solution, from Corollary 1.2 we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C \|u_\lambda\|_{L^1(\Omega)}.$$

But now, since $u^\star \in L^1(\Omega)$ and $0 \leq u_\lambda \leq u^\star$, the right-hand side of the previous inequality is bounded independently of λ. As a consequence, we deduce our result by letting $\lambda \to \lambda^\star$. $\quad\square$

We conclude the section by establishing our Liouville result.

Proof of Corollary 1.5. First, note that if $\tilde{u}$ is a stable solution to $(-\Delta)^{1/2}\tilde{u} = f(\tilde{u})$ in B_1, and $\tilde{v}$ is its harmonic extension in $\mathbb{R}^{n+1}_+$, then

$$[\tilde{u}]_{C^\gamma(\overline{B}_{1/2})} \leq C \|\tilde{v}\|_{L^1(\mathcal{B}^+_1)} \qquad \text{if } 1 \leq n \leq 4, \tag{6.3}$$

where $\gamma > 0$ and C are dimensional constants. To see this, it is enough to carry out the proof of Theorem 1.1, but using estimate (5.4) instead of (5.1) to control the H^1 seminorm of $\tilde{v}$ in (6.2).

Our proof now follows that of [11]. Given u as in the statement of the corollary, for each $R > 2$ define $u_R(x) := u(Rx)$. Since the harmonic extension of u_R in $\mathbb{R}^{n+1}_+$ is $v_R(x) := v(Rx)$, by applying the estimate (6.3) to these rescaled functions (which are stable solutions to $(-\Delta)^{1/2}u_R = Rf(u_R)$ in B_1) we obtain

$$|u(x) - u(z)| \leq C R^{-\alpha}|x - z|^\alpha \fint_{\mathcal{B}^+_R} |v| \, dx \quad \text{for } x \text{ and } z \text{ in } B_{R/2}.$$

Note that here we use crucially that the estimate (6.3) does not depend on the nonlinearity.

Now, we need to bound the average of $|v|$ in $\mathcal{B}^+_R$. To do it, we claim first that there exists a positive constant c_1 for which

$$v(x) \geq -c_0(\log|x| + c_1) \quad \text{if } |x| \geq 1, \tag{6.4}$$

where c_0 is the constant appearing in the lower bound (1.13) for u. Assuming this claim to be true, we obtain

$$|v(x)| \leq |v(x) + c_0(\log|x| + c_1)| + |c_0(\log|x| + c_1)| = v(x) + 2c_0(\log|x| + c_1).$$

Moreover, since v is harmonic and has nonnegative flux on B_1, the mean value property

$$\fint_{\mathcal{B}_R^+} v \, dx \leq v(0)$$

holds.[10]

As a consequence, using the previous two bounds we get

$$\fint_{\mathcal{B}_R^+} |v| \, dx \leq \fint_{\mathcal{B}_R^+} v \, dx + 2c_0 \left(\log R + c_1\right) \leq v(0) + 2c_0 \left(\log R + c_1\right)$$

and thus

$$|u(x') - u(z')| \leq CR^{-\alpha} |x' - z'|^\alpha \big(v(0) + 2c_0 \left(\log R + c_1\right)\big).$$

Letting $R \to +\infty$ we deduce that u is constant.

It remains to establish (6.4). We use the Poisson kernel (A.1) to express v in terms of u, together with the lower bound (1.13). For $x = (x', y) \in \mathbb{R}_+^{n+1}$ with $|x| \geq 1$, we see that

$$v(x) \geq -c_0 \, p_{n,1/2} \iint_{\mathbb{R}^n} \frac{y \log(2 + |z'|)}{(|x' - z'|^2 + y^2)^{\frac{n+1}{2}}} \, dz'$$

$$\geq -c_0 \, p_{n,1/2} \iint_{\mathbb{R}^n} \frac{y \log(|x|(2 + |z'|/|x|))}{(|x' - z'|^2 + y^2)^{\frac{n+1}{2}}} \, dz'$$

$$= -c_0 \left(\log |x| + p_{n,1/2} \iint_{\mathbb{R}^n} \frac{y \log(2 + |z'|/|x|)}{(|x' - z'|^2 + y^2)^{\frac{n+1}{2}}} \, dz' \right)$$

$$= -c_0 \left(\log |x| + p_{n,1/2} \frac{y}{|x|} \iint_{\mathbb{R}^n} \frac{\log(2 + |\tilde{z}'|)}{(|x'/|x| - \tilde{z}'|^2 + (y/|x|)^2)^{\frac{n+1}{2}}} \, d\tilde{z}' \right)$$

$$=: -c_0 \left(\log |x| + p_{n,1/2} \, \varphi(x/|x|) \right).$$

Finally, let us show that the function $\varphi : \mathbb{S}_+^n \to \mathbb{R}$ defined above is bounded. To do this, note first that $\varphi = \varphi(x/|x|)$ does not depend on $x'/|x|$ and can be written as

$$\tilde{y} \iint_{\mathbb{R}^n} \frac{\log(2 + |\tilde{z}'|)}{(|e_1 - \tilde{z}'|^2 + \tilde{y}^2)^{\frac{n+1}{2}}} \, d\tilde{z}'$$

where $\tilde{y} := y/|x| \in (0, 1]$. Since $|\tilde{y}| \leq 1$, it suffices to control this quantity when integrating only in $\{|e_1 - \tilde{z}'| < 1\}$. But we have

$$\tilde{y} \iint_{\mathcal{B}_1(e_1)} \frac{\log(2 + |\tilde{z}'|)}{(|e_1 - \tilde{z}'|^2 + \tilde{y}^2)^{\frac{n+1}{2}}} \, d\tilde{z}' \leq C \int_0^1 \frac{\tilde{y} \rho^{n-1}}{(\rho^2 + \tilde{y}^2)^{\frac{n+1}{2}}} \, d\rho \leq C \int_0^{+\infty} \frac{t^{n-1}}{(t^2 + 1)^{\frac{n+1}{2}}} \, dt < +\infty.$$

This establishes the claim (6.4) and concludes the proof. $\qquad\square$

---

[10]This follows from the same usual proof of the mean value property for superharmonic functions in full balls, by adapting it to half-balls. Or, alternatively, by applying it to the even reflexion of v across $\{y = 0\}$ (which is superharmonic in the weak H^1 sense).

7. Adding artificial variables

In this section we present the two results that allowed us to establish the Hölder estimate of Theorem 1.1 in dimension $n = 1$ from the estimate in higher dimensions, by adding artificial variables. In this section, points in $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ will be denoted by $x = (x', x_m)$.

The first result concerns the control of the L^1_s norm when adding artificial variables.

Lemma 7.1. *Let $m \geq 2$ and let $u : \mathbb{R}^{m-1} \to \mathbb{R}$ belong to $L^1_s(\mathbb{R}^{m-1})$. Define $w : \mathbb{R}^m \to \mathbb{R}$ by*

$$w(x', x_m) := u(x').$$

Then,

$$\|w\|_{L^1_s(\mathbb{R}^m)} \leq C \|u\|_{L^1_s(\mathbb{R}^{m-1})}$$

for some constant C depending only on m and s.

Proof. Let $Q^m = (-1, 1)^m \subset \mathbb{R}^m$. We have

$$\|w\|_{L^1_s(\mathbb{R}^m)} = \int_{Q^m} \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}} \, dx + \iint_{\mathbb{R}^m \setminus Q^m} \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}} \, dx.$$

The first integral is bounded easily using that $Q^m = Q^{m-1} \times (-1, 1)$:

$$\int_{Q^m} \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}} \, dx \leq 2 \iint_{Q^{m-1}} |u(x')| \, dx' \leq C \|u\|_{L^1_s(\mathbb{R}^{m-1})}.$$

To estimate the second one, we split it further:

$$\iint_{\mathbb{R}^m \setminus Q^m} \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}} \, dx = I_1 + I_2,$$

where

$$I_1 := \int_{-1}^{1} dx_m \int_{\mathbb{R}^{m-1} \setminus Q^{m-1}} dx' \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}}$$

and

$$I_2 := 2 \iint_{1}^{+\infty} dx_m \iint_{\mathbb{R}^{m-1}} dx' \frac{|u(x')|}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}}.$$

Now, on the one hand

$$I_1 \leq 2 \iint_{\mathbb{R}^{m-1} \setminus Q^{m-1}} \frac{|u(x')|}{(1 + |x'|^2)^{\frac{m-1+2s}{2}}} \, dx' \leq C \|u\|_{L^1_s(\mathbb{R}^{m-1})}.$$

On the other hand,

$$I_2 = 2 \int_{\mathbb{R}^{m-1}} dx' |u(x')| \int_{1}^{+\infty} \frac{dx_m}{(1 + |x'|^2 + x_m^2)^{\frac{m+2s}{2}}}.$$

and note that, for $\lambda > 0$,

$$\int_1^{+\infty} \frac{\mathrm{d}x_m}{(\lambda^2 + x_m^2)^{\frac{m+2s}{2}}} = \frac{1}{\lambda^{m+2s}} \int_1^{+\infty} \frac{\mathrm{d}x_m}{(1 + (x_m/\lambda)^2)^{\frac{m+2s}{2}}}$$

$$\leq \frac{1}{\lambda^{m-1+2s}} \int_0^{+\infty} \frac{\mathrm{d}t}{(1 + t^2)^{\frac{m+2s}{2}}} \leq \frac{C}{\lambda^{m-1+2s}}.$$

This yields $I_2 \leq C \left\| u \right\|_{L_s^1(\mathbb{R}^{m-1})}$ (taking $\lambda = \sqrt{1 + |x'|^2}$), which concludes the proof. $\qquad\square$

Now we prove that stability is preserved after the addition of artificial variables. Here we denote by B_1^m the unit ball in $\mathbb{R}^m$.

Lemma 7.2. *Let $m \geq 2$ and let $u : \mathbb{R}^{m-1} \to \mathbb{R}$, with $u \in C^2(B_1^{m-1}) \cap L_s^1(\mathbb{R}^{m-1})$, be a stable solution of $(-\Delta)^s u = f(u)$ in B_1^{m-1}. Define $w : \mathbb{R}^m \to \mathbb{R}$ by*

$$w(x', x_m) := u(x').$$

Then, $w \in C^2(B_1^m) \cap L_s^1(\mathbb{R}^m)$ and it is a stable solution of $(-\Delta)^s w = f(w)$ in B_1^m.

Proof. First, that $w \in L_s^1(\mathbb{R}^m)$ follows from Lemma 7.1, and a straightforward computation shows that w solves $(-\Delta)^s w = f(w)$ in B_1^m (see for instance [22, Lemma 2.1]). Now, let us check that w is stable. We will show that the stability inequality (2.2) holds for all C^∞ functions $\xi = \xi(x, y)$ with compact support in $\mathbb{R}_+^{m+1} \cup B_1^m$. Then, the result will follow by density.

Given such a function ξ, we define

$$\overline{\xi}^2(x', y) := \int_{\mathbb{R}} \xi^2(x', x_m, y) \, \mathrm{d}x_m.$$

We have

$$\int_{B_1^m} f'(w(x))\xi^2(x, 0) \, \mathrm{d}x = \int_{B_1^{m-1}} \mathrm{d}x' f'(u(x')) \int_{\mathbb{R}} \mathrm{d}x_m \, \xi^2(x', x_m, 0)$$

$$= \int_{B_1^{m-1}} f'(u(x'))\overline{\xi}^2(x', 0) \, \mathrm{d}x',$$

and using the stability of u in B_1^{m-1}, we deduce

$$\int_{B_1^m} f'(w)\xi^2 \, \mathrm{d}x \leq d_s \int_0^{+\infty} \mathrm{d}y \, y^{1-2s} \int_{\mathbb{R}^{m-1}} \mathrm{d}x' |\nabla_{(x', y)}\overline{\xi}(x', y)|^2. \tag{7.1}$$

Now, for $(x', y) \in \{\overline{\xi} \neq 0\}$ and $i = 1, \ldots, m-1$ we have

$$\overline{\xi}_{x_i}(x', y) = \overline{\xi}(x', y)^{-1} \int_{\mathbb{R}} \xi(x', x_m, y)\xi_{x_i}(x', x_m, y) \, \mathrm{d}x_m,$$

and the same holds for the derivative with respect to the extension variable y. Using the Cauchy-Schwarz inequality we see that

$$\overline{\xi}_{x_i}^2(x', y) \leq \int_{\mathbb{R}} \xi_{x_i}^2(x', x_m, y) \, \mathrm{d}x_m,$$

and similarly for $\overline{\xi}_y^2$. Hence

$$|\nabla_{(x',y)}\overline{\xi}(x',y)|^2 \le \fint_{\mathbb{R}} |\nabla\xi(x',x_m,y)|^2 \, dx_m.$$

Using this in (7.1), and the fact that since $\overline{\xi}$ is Lipschitz, $\nabla\overline{\xi} = 0$ a.e. in $\{\overline{\xi} = 0\}$ (see for instance [17, Theorem 6.19]), we obtain that

$$\int_{B_1^m} f'(w)\xi^2 \, dx \le d_s \fint_0^{+\infty} dy\, y^{1-2s} \fint_{\mathbb{R}^m} dx\, |\nabla\xi(x,y)|^2.$$

This establishes the stability of w. $\qquad\square$

APPENDIX APPENDIX A. AUXILIARY REGULARITY LEMMATA

In this section we collect some auxiliary results used along the paper. Recall that $a := 1-2s$ and that the s-harmonic extension of a function $u : \mathbb{R}^n \to \mathbb{R}$ is iven by

$$v(x',y) = \fint_{\mathbb{R}^n} u(z')P_s(x'-z',y)\, dz', \quad \text{with} \quad P_s(\bar{x}',y) = p_{n,s}\frac{y^{2s}}{(|\bar{x}'|^2+y^2)^{\frac{n+2s}{2}}}. \quad \text{(A.1)}$$

Here, $p_{n,s}$ is a normalizing positive constant, depending only on n and s, which makes $P_s(\cdot,y)$ integrate 1 in $\mathbb{R}^n$ for every $y > 0$.

We first present a result stating that the weighted L^p norm of the s-harmonic extension of a function can be controlled in terms of L^p and L_s^1 norms of its trace. It is a simple application of Young's convolution inequality.

Lemma A.1. *Let $n \ge 1$, $s \in (0,1)$, $p \in [1,+\infty)$, $R > 0$, and $u \in L^p(B_{2R}) \cap L_s^1(\mathbb{R}^n)$. Let $a = 1-2s$ and let v be the s-harmonic extension of u.*
   Then,

$$\|v\|_{L^p(\mathcal{B}_R^+,y^a)} \le C_R\left(\|u\|_{L^p(B_{2R})} + \|u\|_{L_s^1(\mathbb{R}^n)}\right)$$

for some constant C_R depending only on n, s, p, and R.

Proof. First, note that

$$\int_{\mathcal{B}_R^+} y^a|v|^p \, dx \le \fint_0^R dy\, y^a \int_{B_R} dx' \left|\fint_{\mathbb{R}^n} u(z')P_s(x'-z',y)\, dz'\right|^p$$

$$\le 2^p \fint_0^R dy\, y^a \int_{B_R} dx' \left|\fint_{B_{2R}} u(z')P_s(x'-z',y)\, dz'\right|^p$$

$$+ 2^p \fint_0^R dy\, y^a \int_{B_R} dx' \left|\fint_{\mathbb{R}^n\setminus B_{2R}} u(z')P_s(x'-z',y)\, dz'\right|^p.$$

Now, on the one hand, defining $\widetilde u := u\chi_{B_{2R}}$, for $y > 0$ we have

$$\int_{B_R} dx' \left(\iint_{\mathcal{B}_{2R}} u(z')P_s(x'-z',y)\,dz'\right)^p = \iint_{\mathcal{B}_R} |\widetilde u * P_s(\cdot,y)|^p(x')\,dx'$$
$$\leq \|\widetilde u * P_s(\cdot,y)\|_{L^p(\mathbb{R}^n)}^p$$
$$\leq \|\widetilde u\|_{L^p(\mathbb{R}^n)}^p \|P_s(\cdot,y)\|_{L^1(\mathbb{R}^n)}^p = \|u\|_{L^p(B_{2R})}^p,$$

where we have used Young's convolution inequality. On the other hand, we claim that

$$P_s(x'-z',y) \leq \overline{C}_R \frac{y^{2s}}{(1+|z'|^2)^{\frac{n+2s}{2}}} \quad \text{if } x' \in B_R \text{ and } z' \in \mathbb{R}^n \setminus B_{2R}, \tag{A.2}$$

where the constant $\overline{C}_R$ depends only on n, s, and R. Indeed, since $|x'| \leq |z'|/2$, we have that

$$1 + |z'|^2 \leq \frac{1+4R^2}{R^2} \frac{|z'|^2}{4} \leq \frac{1+4R^2}{R^2}|x'-z'|^2 \leq \frac{1+4R^2}{R^2}(|x'-z'|^2 + y^2).$$

Therefore

$$\int_{B_R} dx' \left(\iint_{\mathbb{R}^n \setminus B_{2R}} u(z')P_s(x'-z',y)\,dz'\right)^p \leq C_R y^{2sp} \|u\|_{L^1_s(\mathbb{R}^n)}^p,$$

with C_R depending only on n, s, p, and R.

The desired estimate follows from the fact that y^a is integrable in $(0,R)$. $\qquad\square$

We next establish an L^∞ estimate for the s-harmonic extension of a function, as well as for its first and second horizontal derivatives (i.e., its derivatives in $x' \in \mathbb{R}^n$). The result will be used to prove Lemma A.3 below.

Lemma A.2. *Let $n \geq 1$, $s \in (0,1)$, $R > 0$, and $u \in C^2(\overline{B}_{4R}) \cap L^1_s(\mathbb{R}^n)$. Let v be the s-harmonic extension of u.*

Then,

$$\|v\|_{L^\infty(\mathcal{B}_R^+)} + \|\nabla_{x'} v\|_{L^\infty(\mathcal{B}_R^+)} + \|D^2_{x'}v\|_{L^\infty(\mathcal{B}_R^+)} \leq C_R \left(\|u\|_{C^2(\overline{B}_{4R})} + \|u\|_{L^1_s(\mathbb{R}^n)}\right)$$

for a constant C_R depending only on n, s, and R.

Proof. Let $\zeta: \mathbb{R}^n \to [0,1]$ be a smooth cut-off function such that $\zeta \equiv 1$ in B_{2R} and $\zeta \equiv 0$ in $\mathbb{R}^n \setminus B_{4R}$. Then, writing $u = \zeta u + (1-\zeta)u$, for $(x',y) \in \mathcal{B}_R^+$ we have

$$v(x',y) = \int_{\mathbb{R}^n} \zeta(x'-z')u(x'-z')P_s(z',y)\,dz' + \iint_{\mathbb{R}^n} (1-\zeta(z'))u(z')P_s(x'-z',y)\,dz'. \tag{A.3}$$

Now, the first term is estimated by $\|u\|_{L^\infty(B_{4R})}$ using simply that ζ has compact support in B_{4R} and that P_s is positive and integrates 1, while for the second one we note that $1-\zeta$ is zero in B_{2R} and thus we can use (A.2) to estimate the integral by $C_R \|u\|_{L^1_s(\mathbb{R}^n)}$.

Last, to estimate the horizontal derivatives of v we proceed similarly, now differentiating (A.3) and using, for the second integral (where $|x'-z'| \geq |z'| - |x'| \geq R$), that, for $i,j = 1,\dots,n$,

$$|\partial_i P_s(x'-z',y)| + |\partial_i\partial_j P_s(x'-z',y)| \leq C_R P_s(x'-z',y) \quad \text{if } |x'-z'| \geq R$$

for some constant C_R depending only on n, s, and R. This last bound can be easily obtained from the explicit expression of the Poisson kernel P_s just noticing that $|x - z| \geq R$ yields

$$\frac{|x_i - z_i|}{|x - z|^2 + y^2} + \frac{1}{|x - z|^2 + y^2} \leq \frac{1}{|x - z|} + \frac{1}{|x - z'|^2} \leq \frac{1}{R} + \frac{1}{R^2}.$$

$\square$

In the following lemma we collect some regularity results for the s-harmonic extension of a solution to a fractional semilinear equation in B_1. These are the ingredients that we have referred to in our computations through the paper. The main issue here is that $u \in L_s^1(\mathbb{R}^n)$ is the only control that we have of u outside B_1. Instead, under more restrictive assumptions on the solution —for instance in [4] for entire solutions in $L^\infty(\mathbb{R}^n)$—, most of these results are known.

Lemma A.3. *Let $n \geq 1$, $s \in (0,1)$, and let $u \in C^2(B_1) \cap L_s^1(\mathbb{R}^n)$ be a solution to $(-\Delta)^s u = f(u)$ in $B_1 \subset \mathbb{R}^n$, where f is a $C^{1,\gamma}$ function for some $\gamma > 0$. Let $a = 1 - 2s$ and let v be the s-harmonic extension of u.*

Then, for every $R < 1$,

(a) $v \in H^1(\mathcal{B}_R^+, y^a)$.
(b) The functions v, $y^a \partial_y v$, $\nabla_{x'} v$, and $y^a \partial_y \nabla_{x'} v$ are continuous in $\overline{\mathcal{B}_R^+}$. Moreover,

$$- d_s \lim_{y \downarrow 0} y^a \partial_y \nabla_{x'} v = f'(u) \nabla_{x'} u \quad \text{in } B_R, \tag{A.4}$$

where d_s is the constant appearing in (2.1).
(c) $D_{x'}^2 v$ is bounded in $\mathcal{B}_R^+$. For $s = 1/2$ we additionally[11] have that $v \in C^2(\overline{\mathcal{B}_R^+})$.

Proof. Since $u \in L_s^1(\mathbb{R}^n)$, the function v given by (A.1) is well-defined and smooth in $\mathbb{R}_+^{n+1}$, and satisfies $\mathrm{div}(y^a \nabla v) = 0$ in $\mathbb{R}_+^{n+1}$. Therefore, the main issue is to prove that the regularity results stated in the lemma hold up to $\{y = 0\}$.

First, to show that v is continuous up to $\{y = 0\}$ we simply use its expression as a convolution with P_s. Indeed, from (A.3) we see that the first integral in that expression is the convolution of an approximation of the identity with a compactly supported function which is continuous in B_1 (and thus the result is continuous in B_1 as $y \downarrow 0$), while the second integral can be bounded by $C_R y^{2s} \|u\|_{L_s^1(\mathbb{R}^n)}$ thanks to (A.2).

Next, it is well known and easy to verify —using (A.1)— that $y^a v_y$ is continuous in $\overline{\mathcal{B}_R^+}$, and that

$$d_s \frac{\partial v}{\partial \nu^a} := -d_s \lim_{y \downarrow 0} y^a v_y = (-\Delta)^s u = f(u) \quad \text{in } B_R. \tag{A.5}$$

That $v \in H^1(\mathcal{B}_R^+, y^a)$ follows readily using that y^a and y^{-a} are integrable. Indeed, on the one hand v and its horizontal derivatives can be controlled in $L^2(\mathcal{B}_R^+, y^a)$ using Lemma A.2 (appropriately rescaled and after a covering argument). On the other hand, to control the

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[11]Note that, for $s < 1/2$, in general v is not even C^1 up to $\{y = 0\}$. Indeed, since $y^a v_y$ is continuous up to the boundary, v_y will generally blow up at $\{y = 0\}$.

vertical derivative (in y) we use that $y^a v_y$ is continuous up to $\{y = 0\}$, and thus $y^a v_y^2 \leq C y^{-a}$, from which the desired integrability follows.

Let us now establish the regularity of v_{x_i} for $i = 1, \ldots, n$. Note that we cannot use the previous results applied to v_{x_i} instead of v, since we do not have any control of v_{x_i} outside B_1. To carry out the proof, we consider the weak equation solved by v, i.e.,

$$d_s \int_{\mathcal{B}_1^+} y^a \nabla v \cdot \nabla \varphi \, dx = \iint_{B_1} f(u) \varphi \, dx' \tag{A.6}$$

for every $\varphi \in C^1(\overline{\mathcal{B}_1^+})$ with compact support in $\mathcal{B}_1^+ \cup B_1$. Note that (A.6) follows readily from the equation of v after integrating by parts in $\mathcal{B}_1^+ \cap \{y > \delta\}$ (where v is smooth) and letting $\delta \to 0$, using (A.5). Now, for $h > 0$, we consider the difference quotient

$$D_h^i v(x) := \frac{v(x + h e_i) - v(x)}{h}.$$

From (A.6), we get that for every $x_0' \in B_R$ and $\rho > 0$ such that $B_{2\rho}(x_0') \subset B_1$, the function $D_h^i v$ (with $h < \rho$) solves weakly

$$\begin{cases} \operatorname{div}(y^a \nabla D_h^i v) = 0 & \text{in } \mathcal{B}_\rho^+(x_0'), \\ d_s \dfrac{\partial D_h^i v}{\partial \nu^a} = D_h^i[f(v)] & \text{in } B_\rho(x_0'). \end{cases}$$

Now, using Lemma 4.5 of [4], it follows that

$$\|D_h^i v\|_{C^\gamma(\overline{\mathcal{B}_{\rho/2}^+}(x_0'))} + \|y^a \partial_y D_h^i v\|_{C^\gamma(\overline{\mathcal{B}_{\rho/2}^+}(x_0'))} \leq C_h \tag{A.7}$$

for some γ depending on n, s, and ρ, and a constant C_h depending on n, s, ρ, $\|D_h^i v\|_{L^\infty(\mathcal{B}_\rho^+(x_0'))}$, and $\|D_h^i[f(u)]\|_{C^\gamma(\overline{B}_\rho(x_0'))}$. To establish the regularity for v_{x_i}, we need to show that C_h can be controlled uniformly as $h \to 0$. On the one hand, $\|D_h^i v\|_{L^\infty(\mathcal{B}_\rho^+(x_0'))}$ is controlled by $\|v_{x_i}\|_{L^\infty(\mathcal{B}_\rho^+(x_0'))}$, which is finite thanks to Lemma A.2 (applied after appropriate translations and rescalings). On the other hand, $\|D_h^i[f(u)]\|_{C^\gamma(\overline{B}_\rho(x_0'))}$ can be controlled by the norm $\|f'(u) u_{x_i}\|_{C^\gamma(\overline{B}_\rho(x_0'))}$, which is also bounded by hypothesis. Thus, taking the limit $h \to 0$ in (A.7), we establish the Hölder regularity for v_{x_i} and $y^a \partial_y v_{x_i}$ in $\overline{\mathcal{B}_{\rho/2}^+}(x_0')$. Since x_0' can be taken arbitrarily in B_R, we get the desired result. Note that once we have proved that $y^a \partial_y v_{x_i}$ is continuous up to $\{y = 0\}$, the identity (A.4) follows readily.

Finally, Lemma A.2 established the boundedness of $D_x^2 v$. To conclude the proof, let us show that $v \in C^2(\overline{\mathcal{B}_R^+})$ when $s = 1/2$. For this, take $\zeta : \mathbb{R}^n \to \mathbb{R}$ a cut-off function with compact support in $B_{R+2\rho}$ for some $\rho < (1 - R)/2$, and such that $\zeta \equiv 1$ in $B_{R+\rho}$. Define $w(x, y) := (P_{1/2}(\cdot, y) * (u\zeta))(x)$, which is harmonic in $\mathbb{R}_+^{n+1}$ and satisfies $w(\cdot, 0) = u\zeta$ in $\mathbb{R}^n$, and set $\bar{v} := v - w$. Now, on the one hand, since $\bar{v}$ is harmonic in $\mathcal{B}_{R+\rho}^+$ and $\bar{v} = 0$ on $B_{R+\rho}$, by standard estimates for harmonic functions (taking for instance the odd extension across $\{y = 0\}$), we have that $\bar{v} \in C^2(\overline{\mathcal{B}_R^+})$. On the other hand, since $u\zeta \in C^2(\mathbb{R}^n)$ and has compact support, by the previous results (applied to $u\zeta$ and its derivatives) it follows readily

that w, w_y, w_{x_i}, w_{x_iy}, and $w_{x_ix_j}$ are continuous in $\overline{\mathcal{B}_R^+}$, for every $i,j = 1,\ldots,n$. Using that w is harmonic, we deduce that w_{yy} is also continuous in $\overline{\mathcal{B}_R^+}$, concluding the proof. $\qquad\square$

Appendix Appendix B. Two simple interpolation inequalities in cubes

The following is an interpolation inequality from [1], where it was conceived in order to give quantitative proofs of the Hölder regularity result of [3] for stable solutions to the local equation $-\Delta u = f(u)$. For completeness, we present its proof next. Note that, in contrast with the interpolation inequality of [14, Theorem 7.28], which requires an extension theorem in $\mathbb{R}^n$, establishing (B.1) in a cube of $\mathbb{R}^n$ is immediate once it is proved in dimension one.

Proposition B.1 ([1]). *Let $Q = (0,1)^n \subset \mathbb{R}^n$, $p \geq 1$, and $w \in C^2(\overline{Q})$.*
Then, for every $\varepsilon \in (0,1)$,

$$\int_Q |\nabla w|^p \,\mathrm{d}x \leq n^{p/2+1}\, p\, \varepsilon \fint_Q |\nabla w|^{p-1}\, |D^2 w|\, \mathrm{d}x + n^{p/2+1}\left(\frac{18}{\varepsilon}\right)^p\left(\fint_Q |w|^p\,\mathrm{d}x\right). \tag{B.1}$$

Proof. We follow [1]. We start proving (B.1) in dimension $n = 1$. First, we claim that given $\delta \in (0,1)$ and $w \in C^2([0,\delta])$, it holds

$$\int_0^\delta |w'|^p\,\mathrm{d}x \leq p\,\delta \fint_0^\delta |w'|^{p-1}\,|w''|\,\mathrm{d}x + 9^p\delta^{-p}\int_0^\delta |w|^p\,\mathrm{d}x. \tag{B.2}$$

To show this, take $x_0 \in [0,\delta]$ such that $|w'(x_0)| = \min_{[0,\delta]}|w'|$. Then, for $0 < y < \frac{\delta}{3} < \frac{2\delta}{3} < z < \delta$, since $(w(z) - w(y))/(z - y)$ is equal to w' at some point, it follows that $|w'(x_0)| \leq 3\delta^{-1}(|w(y)| + |w(z)|)$. Integrating this inequality first in y and later in z, we get that $|w'(x_0)| \leq 9\delta^{-2}\int_0^\delta |w|\,\mathrm{d}x$, and taking powers we obtain

$$|w'(x_0)|^p \leq 9^p\delta^{-p-1}\int_0^\delta |w|^p\,\mathrm{d}x \quad \text{for } p \in [1,\infty). \tag{B.3}$$

Now, for $x \in (0,\delta)$, we integrate $(|w'|^p)'$ in the interval with end points x and x_0, to get

$$|w'(x)|^p \leq p\int_0^\delta |w'|^{p-1}\,|w''|\,\mathrm{d}x + |w'(x_0)|^p.$$

Combining this inequality with (B.3) and integrating in $x \in (0,\delta)$, we conclude (B.2).

Once the previous claim (on integrals in $[0,\delta]$) is proved, we can establish (B.1) in dimension one. Let $w \in C^2([0,1])$. Now, for any given integer $k > 1$ we divide $(0,1)$ into k disjoint intervals of length $\delta = 1/k \in (0,1)$. Since (B.2) did not require any specific boundary values of w, we can use the inequality in each of these intervals of length $1/k$ and then add them all up to obtain

$$\int_0^1 |w'|^p\,\mathrm{d}x \leq \frac{p}{k}\int_0^1 |w'|^{p-1}\,|w''|\,\mathrm{d}x + (9k)^p\int_0^1 |w|^p\,\mathrm{d}x. \tag{B.4}$$

Now, since $0 < \varepsilon < 1$, in (B.4) we can choose $k \in \mathbb{Z}$ such that $1 < \frac{1}{\varepsilon} \leq k < \frac{2}{\varepsilon}$, establishing the result in dimension one.

Finally, we show the result for $w \in C^2([0,1]^n)$, with $n \geq 2$. We will denote $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Using the result in dimension 1 for every x', we get

$$\fint_Q |w_{x_1}|^p \, dx = \int_{(0,1)^{n-1}} dx' \fint_0^1 dx_1 \, |w_{x_1}(x)|^p$$

$$\leq p\,\varepsilon \int_{(0,1)^{n-1}} dx' \fint_0^1 dx_1 \, |w_{x_1}(x)|^{p-1} \, |w_{x_1 x_1}(x)|$$

$$+ (18\varepsilon^{-1})^p \int_{(0,1)^{n-1}} dx' \fint_0^1 dx_1 \, |w(x)|^p$$

$$= p\,\varepsilon \fint_Q |w_{x_1}(x)|^{p-1} \, |w_{x_1 x_1}(x)| \, dx + (18\varepsilon^{-1})^p \fint_Q |w(x)|^p \, dx.$$

Since the same inequality holds for the partial derivatives with respect to each variable x_i instead of x_1, adding up all the inequalities and using that $|\nabla v|^p \leq n^{p/2}(|v_{x_1}|^p + \ldots + |v_{x_n}|^p)$, we obtain (B.1). $\qquad\square$

The following second interpolation inequality is well known and follows immediately from Poincaré's inequality. It allows us to replace, when $p = 2$, the L^2 norm of w in (B.1) by the square of its L^1 norm (at the price of adding a reabsorbable small factor of the H^1 norm).

Proposition B.2. *Let $Q = (0,1)^n \subset \mathbb{R}^n$, $p \geq 1$, and $w \in C^2(\overline{Q})$.*
Then, for every $\tilde\varepsilon \in (0,1)$,

$$\int_Q |w|^p \, dx \leq C\,\tilde\varepsilon^p \fint_Q |\nabla w|^p \, dx + C\,\tilde\varepsilon^{-n(p-1)} \left(\fint_Q |w| \, dx \right)^p \tag{B.5}$$

for some constant C depending only on n and p.

Proof. From Poincaré's inequality for functions w in a cube Q_δ of side-length δ, which reads $\|w - w_{Q_\delta}\|_{L^p(Q_\delta)} \leq C\delta \|\nabla w\|_{L^p(Q_\delta)}$ (where w_Q is the average of w in Q), we obtain

$$\|w\|_{L^p(Q_\delta)} \leq C\delta \|\nabla w\|_{L^p(Q_\delta)} + |Q_\delta|^{1/p} |w_{Q_\delta}| = C\delta \|\nabla w\|_{L^p(Q_\delta)} + \delta^{-n(p-1)/p} \fint_{Q_\delta} |w| \, dx.$$

Now, for any given integer $k > 1$ we divide Q into k^n disjoint cubes Q_j of side-length $\delta = 1/k$. Using the previous inequality in each cube Q_j we have

$$\int_{Q_j} |w|^p \, dx \leq \frac{C}{k^p} \fint_{Q_j} |\nabla w|^p \, dx + C k^{n(p-1)} \left(\fint_{Q_j} |w| \, dx \right)^p.$$

Adding up all these inequalities and noticing that $\sum_j \left(\fint_{Q_j} |w| \, dx \right)^p \leq \left(\sum_j \fint_{Q_j} |w| \, dx \right)^p = \left(\fint_Q |w| \, dx \right)^p$, the desired estimate (B.5) follows after choosing an integer k such that $1 < \tilde\varepsilon^{-1} \leq k \leq 2\tilde\varepsilon^{-1}$. $\qquad\square$

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