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Optimal component-type allocation and replacement time policies for parallel systems having multi-types dependent components

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ABSTRACT

In this work, we discuss some challenging open problems and conjectures recently proposed in the literature for parallel systems with dependent components of multiple types. Specifically, we present necessary conditions for the existence of the unique optimal value which minimizes the mean cost rate for two optimization problems. In the first place, the aim is to find the optimal number of components of each type which minimizes the associated mean cost rate, and secondly, to find the optimal replacement time before system failure. In both cases, we consider copulas to model the dependence structure for components whose lifetimes follow any distribution function. Moreover, in order to illustrate the theoretical results, we provide some numerical studies for specific copulas and marginal distribution functions.

1. Introduction

A parallel system consists of $n \geq 1$ units which fails when all units have failed. This type of system is commonly used in computing systems [1]. For other potential applications see [2]. The study of parallel systems from different perspectives has been considered previously in the literature. Most of those works considered that components' lifetimes were independent (homogeneous or heterogeneous), see e.g. [3–5]. However, in real life, components' lifetimes are often statistically dependent. Clearly, reliability analysis for such a system becomes more complex, since it depends on the joint distribution of the components' lifetimes. The usual framework to model the dependence of a random vector is by using copulas, due to the famous Sklar's Theorem. A number of articles have appeared in recent years in which coherent systems with dependent components' lifetimes are discussed [6–9]. Further, parallel systems are considered the most typical redundancy configuration. More complex redundancy structures have been studied in [10–13], among others.

Preventive maintenance is one of the most popular maintenance strategies in reliability theory, whose purpose is to prevent system failure before it occurs. Age replacement models are essential methods in preventive maintenance of systems. Zhao et al. [2] collected different models to study age replacement times based in cost and availability. Badía et al. [14] analyzed a maintenance policy for a system that can suffer failures of two types: minor and catastrophic. Recently, Zhao et al. [15] studied replacement policies that are collaborative with time of operations, mission durations, minimal repairs and maintenance triggering approaches. Most of the research works on optimal strategies for preventive maintenance have considered systems consisting of the same type of components. Recently, Hashemi et al. [16] investigated coherent systems with multiple types of independent components. They obtained analytical expressions for a cost function and an availability criterion under two preventive maintenance strategies. The case of parallel systems with multiple types of independent components has recently been studied in [17]. Peng et al. [17] derived analytical expressions for cost functions under three maintenance policies. In real situations, units never operate in isolation and can even share workloads, so it is important to take these dependencies into account. For the case of dependent components, Eryilmaz and Ozkut [18] investigated two optimization problems for parallel systems with multiple types of components. Specifically, they provided analytical expressions for two average cost rate functions, one for the optimal number of components and another for the optimal replacement time before system failure. In all these cases, the researchers compute numerically the optimal values for some specific cases, in order to optimize the corresponding objective functions. However, they do not provide any optimal solution valid for the general problem nor conditions that ensure their existence and uniqueness. Existence and uniqueness theorems are essential, since they make it possible to conclude that there exists only one solution which optimizes a given objective function.

In the present work, the systems under consideration have parallel configurations formed by $n(\geq 1)$ dependent components of $K(\geq 1)$ different types, where there exists n_i components of type i, for $i = 1 \ldots, K$, such as $n_1 + \cdots + n_K = n$. This type of system is commonly used in electronics industry. An illustrative example is a silicon

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micro-electro-mechanical system (MEMS) which is composed of multiple non-identical resonators assembled in parallel (see [19,20]). Another real-life example is multi-unit parallel production systems (MuPPSs) which are widely used in manufacturing industries, railway and civil aviation (see [21,22]).

Observe that the number of different types of components satisfies $1 \leq K \leq n$. When $K = 1$, the components are of the same type (identically distributed), whereas, when $K = n$, the components have different distributions (heterogeneous). It is worth mentioning that, when the parallel systems consist of $K = 2$ types of components, its lifetime can be defined through a multiple-outlier model. This kind of models has been studied for instance in [23,24]. When the components are of the same type ($K = 1$, i.e., identically distributed) and independent (i.i.d.), Nakagawa [25] proved two existence and uniqueness theorems, one for the optimal number of units and another for the replacement time by minimizing the associated mean cost rate functions. Here, we provide same kind of results in a more general setting, where the components are of $K(\geq 1)$ different types and they are dependent. Moreover, we delve into optimal solutions of the cost functions provided in [18] and we discuss two conjectures posted in that research article.

Let X_i denote the random lifetime of the n_i components of type i having continuous distribution function F_i with finite mean $\mu_i > 0$, for $i = 1, \dots, K$ and $1 \leq K \leq n$. Let $s = (n_1, \dots, n_K)$ be the component-type allocation vector associated to the lifetimes $X_1, \dots, X_K$. Thus, it is assumed that the random lifetimes of components of the same type are identically distributed, whereas the random lifetimes of components of different type are dependent and heterogeneous. Then, we can obtain the following expression for the distribution function of the system lifetime $X_{n:n} = \max(X_1, \dots, X_K)$ in terms of a distributional copula C and marginal distribution functions $F_1, \dots, F_K$

$$F_{n:n}(t) = C\left(\underbrace{F_1(t), \dots, F_1(t)}_{n_1}, \dots, \underbrace{F_K(t), \dots, F_K(t)}_{n_K}\right) \quad \text{for } t > 0. \tag{1.1}$$

Observe that, for the independent case, C is the product copula ($C = \Pi$), and when $K = 1$, the components are identically distributed, i.e., $F_1 = \cdots = F_K = F$. Further, the mean time to system failure (MTTF) of $X_{n:n}$ is given by

$$\mu_{n:n} = E[X_{n:n}] = \int_0^\infty \bar{F}_{n:n}(t)dt = \int_0^\infty (1 - F_{n:n}(t))dt,$$

where the distribution function is defined in (1.1). Under this setup, we investigate two optimization problems to minimize the mean cost rate function for the optimal component-type allocation vector and the optimal replacement time before system failure. Specifically, we provide conditions under which the existence and uniqueness of an optimal value of replacement time is guaranteed for parallel systems with K types of dependent components. Moreover, we prove that, when $K = 1$, there exists a unique optimal value of n for dependent components having arbitrary marginal distribution functions.

The novelty and contributions of this work can be summarized as follows:

1. Reliability models for parallel systems with dependent components of multiple types are investigated in a general setting (arbitrary marginal distributions and families of copulas).
2. To model the dependency between the components, families of copulas (such as Archimedean and Farlie–Gumbel–Morgenstern (FGM) families) are considered in the reliability models.
3. We provide two existence and uniqueness theorems:
   (a) The first one corresponds to the problem of minimizing the mean cost rate for the optimal number of components, when $K = 1$, for Archimedean copulas and for some FGM copulas. The dependent components have an arbitrary distribution function.
   (b) The second theorem is for the problem of minimizing the mean cost rate for the optimal replacement time before system failure, where the parallel systems are formed by $K \geq 1$ types of dependent components assembled by any copula.
4. Moreover, we provide optimal component-type allocation vector which minimizes the cost mean rate functions for the two optimization problems.

The rest of the manuscript is organized as follows. Firstly, in Section 2, we recall some useful definitions. Section 3 is devoted to investigate the effect of the component-type allocation vector on the distribution function of $X_{n:n}$ by using the majorization theory. We then obtain comparisons among the MTTF's of different parallel systems. In Sections 4 and 5, we examine the two optimization problems and provide some numerical studies to illustrate the new theoretical results. In all numerical examples we use Wolfram Mathematica. First, we define the objective function for each problem, and then, we compute the optimal solution by using the function "NMinimize" predefined in Mathematica. Finally, Section 6 concludes the paper and describes some future works.

For ease of reference, some notations are stated in Table 1.

2. Definitions

In this section, we review some definitions and well-known notions of majorization which will be used later.

Let X and Y be univariate random variables with continuous distribution functions F and G, respectively. Moreover, it is assumed that the random variables are always nonnegative with positive finite mean. The lifetime X will be said to be larger than the lifetime Y in the MTTF order (denoted by $X \geq_{MTTF} Y$) if $E(X) \geq E(Y)$. Analogously, X will be said to be larger than Y in the usual stochastic order (denoted by $X \geq_{ST} Y$), if and only if, $F(t) \leq G(t)$ for all $t \geq 0$. Clearly, if $X \geq_{ST} Y$ then $X \geq_{MTTF} Y$.

Now, let us remember the definition of the class of Archimedean copulas. For a non increasing and continuous function $\phi : [0, \infty) \to [0, 1]$, such that $\phi(0) = 1$, $\phi(\infty) = 0$ and ϕ^{-1} be the right continuous inverse, the copula defined by

$$C(u_1, \dots, u_n) = \phi\left(\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_n)\right), \tag{2.1}$$

with $u_i \in [0, 1]$ for $i = 1, \dots, n$ is called an Archimedean copula with generator ϕ, if $(-1)^k \phi^{[k]}(x) \geq 0$ for $k = 0, \dots, n-2$ and $(-1)^{n-2} \phi^{[n-2]}$ is decreasing and convex. This class of copulas is a rich family of dependence models that includes many well-known copulas such as independence (product) copula, Clayton copula, Frank copula, Gumbel–Hougaard copula, and Ali–Mikhail–Haq (AMH) copula, among others. Another family of copulas, widely used in the literature, is the Farlie–Gumbel–Morgenstern (FGM) family which is defined as follows

$$C(u_1, \dots, u_n) = \prod_{i=1}^n u_i \left(1 + \theta \prod_{i=1}^n (1 - u_i)\right), \tag{2.2}$$

for $\theta \in [-1, 1]$ with $u_i \in [0, 1]$ for $i = 1, \dots, n$. For detailed discussions on copulas and their detailed properties and applications, one may refer to Nelsen [26].

The following definitions introduce the majorization orders we consider in this article. Consider two n-dimensional vectors $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$. Moreover, $a_{1:n} \leq \cdots \leq a_{n:n}$ and $b_{1:n} \leq \cdots \leq b_{n:n}$ denote the increasing arrangements of the components of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively.

Definition 2.1. A vector $\boldsymbol{a}$ is said to be

(i) majorized by another vector $\boldsymbol{b}$, (denoted by $\boldsymbol{a} \overset{m}{\prec} \boldsymbol{b}$), if for each $k = 1, \dots, n-1$, $\sum_{i=1}^k a_{i:n} \geq \sum_{i=1}^k b_{i:n}$ and $\sum_{i=1}^n a_{i:n} = \sum_{i=1}^n b_{i:n}$ hold;

Table 1
Definitions of the used notations.

Notation	Description
n	Total number of components in the parallel system
K	Number of different types of components, $1 \le K \le n$
n_i	Number of components of type i, for $i = 1, \dots, K$
$(n_1, \dots, n_K)$	A component-type allocation vector
X_i or Y_i	Lifetime of component of type i, for $i = 1, \dots, K$
F_i or G_i	Failure time distribution of components of type i
c_{1i}	Acquisition cost of one component of type i
c_2	Additional cost of system failure
$X_{n:n}$ or $Y_{m:m}$	Lifetime of a parallel system
$\mu_{n:n}$ or $\mu_{m:m}$	Mean time to system failure (MTTF) of a parallel system
$F_{n:n}$ or $G_{m:m}$	Failure time distribution of a parallel system
$\overset{m}{\succ}$	Majorization order
$\overset{w}{\succ}$	Weakly supermajorization order
$\succ_w$	Weakly submajorization order
$Z(s, c)$	A mean cost rate function to obtain the optimal component-type allocation vector
$M(s, c, T)$	A mean cost rate function to obtain the optimal replacement time

(ii) weakly supermajorized by another vector $\boldsymbol{b}$, (denoted by $\boldsymbol{a} \overset{w}{\prec} \boldsymbol{b}$), if for each $k = 1, \dots, n$, $\sum_{i=1}^{k} a_{i:n} \ge \sum_{i=1}^{k} b_{i:n}$ holds;
(iii) weakly submajorized by another vector $\boldsymbol{b}$, (denoted by $\boldsymbol{a} \prec_w \boldsymbol{b}$), if for each $k = 1, \dots, n$, $\sum_{i=k}^{n} a_{i:n} \le \sum_{i=k}^{n} b_{i:n}$ holds.

It is well known that the usual majorization order implies both weakly supermajorization and weakly submajorization orders. One may refer to [27] for more details on majorization theory. Throughout the article, the terms increasing and decreasing stand to mean nondecreasing and nonincreasing, respectively. Moreover, the following notation will be used:

$$\mathcal{D}_+ = \left\{ (x_1, \dots, x_n) : x_1 \ge \dots \ge x_n \ge 0 \right\}.$$

3. Comparisons of the systems' MTTF

In this section, we present some useful results to compare the distribution functions of two different parallel systems formed by n dependent components of K different types ($K \le n$). As a direct consequence, the mean time to failure (MTTF) can be also compared.

Firstly, we consider that the dependence structure among the components is defined by the Archimedean family of copulas, and the lifetimes $X_1, \dots, X_K$ are allocated according to the vector $\boldsymbol{s} = (n_1, \dots, n_K)$. Then, combining (1.1) and (2.1), the distribution function of $X_{n:n}$ is given by

$$F_{n:n}(t) = \phi \left(\sum_{i=1}^{K} n_i \phi^{-1}(F_i(t)) \right), \tag{3.1}$$

for $t \ge 0$. In the following result, we give conditions under which the distribution functions are ordered and, therefore, also the system's MTTF.

Proposition 3.1. *Let $X_{n:n}$ and $X_{m:m}$ be two parallel systems with K types of dependent components assembled by the same Archimedean copula and the same heterogeneous lifetimes $X_1, \dots, X_K$ allocated according to the vectors $\boldsymbol{s} = (n_1, \dots, n_K)$ and $\boldsymbol{r} = (m_1, \dots, m_K)$, respectively. Suppose that $\boldsymbol{s}$ and $\boldsymbol{r}$ on $\mathcal{D}_+$ and $F_1 \ge \dots \ge F_K$. If $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$ then $F_{n:n} \ge F_{m:m}$.*

Proof. The distribution function of $X_{n:n}$ is defined in (3.1) and, analogously, the distribution function of $X_{m:m}$ can be defined as

$$F_{m:m}(t) = \phi \left(\sum_{i=1}^{K} m_i \phi^{-1}(F_i(t)) \right),$$

for $t \ge 0$. Let us denote $\psi(\boldsymbol{s}) = \phi \left(\sum_{i=1}^{K} n_i \phi^{-1}(F_i(t)) \right)$ with $\boldsymbol{s} \in \mathcal{D}_+$, then we have

$$\partial_{(i)} \psi(\boldsymbol{s}) = \frac{\partial \psi(\boldsymbol{s})}{\partial n_i} = \phi' \left(\sum_{i=1}^{K} n_i \phi^{-1}(F_i(t)) \right) \phi^{-1}(F_i(t)) \overset{sign}{=} -\phi^{-1}(F_i(t)) \le 0,$$

for $i = 1, 2, \dots, K$, since $\phi' \le 0$. Thus, $\partial_{(i)}\psi$ is decreasing in $i = 1, 2, \dots, K$, if and only if, $\phi^{-1}(F_i(t))$ is increasing in $i = 1, 2, \dots, K$, which holds since $F_1 \ge \dots \ge F_K$ and ϕ^{-1} is decreasing. Observe that $\partial_{(i)}\psi \le 0$, then

$$0 \ge \partial_{(1)}\psi(\boldsymbol{s}) \ge \dots \ge \partial_{(K)}\psi(\boldsymbol{s})$$

for all $\boldsymbol{s} \in \mathcal{D}_+$. Then, from Theorem 3.A.7 in [27], we know that $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$ implies $\psi(\boldsymbol{s}) \ge \psi(\boldsymbol{r})$, that is, $F_{n:n} \ge F_{m:m}$. ■

Remark 3.2. Note that $\partial_{(i)}\psi$ is increasing in $i = 1, 2, \dots, K$, if and only if, $\phi^{-1}(F_i(t))$ is decreasing in $i = 1, 2, \dots, K$, which holds when $F_1 \le \dots \le F_K$. Then

$$-\partial_{(1)}\psi(\boldsymbol{s}) \ge \dots \ge -\partial_{(K)}\psi(\boldsymbol{s}) \ge 0,$$

since $\partial_{(i)}\psi \le 0$ for $i = 1, \dots, K$. Now, again from Theorem 3.A.7 in [27], we know that $\boldsymbol{s} \succ_w \boldsymbol{r}$ implies $-\psi(\boldsymbol{s}) \ge -\psi(\boldsymbol{r})$, and therefore, $F_{n:n} \le F_{m:m}$. Thus, the system's MTTF associated with the allocation vector $\boldsymbol{s}$ is greater than that associated with the vector $\boldsymbol{r}$, that is, $X_{n:n} \ge_{MTTF} X_{m:m}$.

Remark 3.3. It is well known that $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ implies both $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$ and $\boldsymbol{s} \succ_w \boldsymbol{r}$. Then, from Proposition 3.1, $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $F_1 \ge \dots \ge F_K$ imply $F_{n:n} \ge F_{m:m}$, and therefore, $X_{n:n} \le_{MTTF} X_{m:m}$. Whereas, if $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $F_1 \le \dots \le F_K$, then $F_{n:n} \le F_{m:m}$, from Remark 3.2.

As an illustration of Proposition 3.1, let us consider $K = 3$ and $n_1 + n_2 + n_3 = n \le 10$ with $(n_1, n_2, n_3) \in \mathcal{D}_+$. Let us start considering $\boldsymbol{s} = (7, 0, 0)$ and $\boldsymbol{r} = (6, 1, 0)$. Observe that $n_{1:3} = 0 = n_{2:3}$, $n_{3:3} = 7$, $m_{1:3} = 0$, $m_{2:3} = 1$ and $m_{3:3} = 6$. Now, it is easy to check that $n_{1:3} = 0 = m_{1:3}$, $n_{1:3} + n_{2:3} = 0 < 1 = m_{1:3} + m_{2:3}$ and $n_{1:3} + n_{2:3} + n_{3:3} = 7 = m_{1:3} + m_{2:3} + m_{3:3}$. Therefore, from Definition 2.1(i), we get that $(7, 0, 0) \overset{m}{\succ} (6, 1, 0)$. Analogously, by using Definitions 2.1(i) and 2.1(ii), it is easy to verify

$$(6,1,0) \overset{w}{\succ} (7,2,0) \overset{m}{\succ} (7,1,1) \overset{w}{\succ} (8,1,1) \overset{m}{\succ} (5,2,1) \overset{w}{\succ} (6,2,1)$$
$$\overset{m}{\succ} (5,2,2) \overset{w}{\succ} (5,3,2) \overset{m}{\succ} (4,3,3).$$

Then, from Proposition 3.1, we get that $\boldsymbol{s} = (4, 3, 3)$ is the optimal allocation vector which maximizes the system's MTTF for any $F_1 \ge F_2 \ge F_3$ and any Archimedean copula. This means that, once the structure of the system is fixed, that is, n and K are fixed, Proposition 3.1 allows obtaining the best component-type allocation vector within a set, without having to fix previously any distribution function or any copula. The only two restrictions are that the distribution functions of the lifetimes of the components must be ordered in decreasing order and the copula belongs to the family of Archimedean copulas. Moreover, combining Proposition 3.1 and Theorem 4 in [28], the optimal component-type allocation vector is $\boldsymbol{s}^* = (n_1^*, \dots, n_K^*) \in \mathcal{D}_+$ such as $n_1^* + \dots + n_K^* = n$ and

Table 2
Optimal component-type allocation vectors which maximizes the system's MTTF.

n	$s^* = (n_1^*, \dots, n_K^*) \in \mathcal{D}_+$		
	$K = 3$	$K = 4$	$K = 5$
10	(4, 3, 3)	(3, 3, 2, 2)	(2, 2, 2, 2, 2)
30	(10, 10, 10)	(8, 8, 7, 7)	(5, 5, 5, 5, 5)
55	(19, 18, 18)	(14, 14, 14, 13)	(11, 11, 11, 11, 11)
82	(28, 27, 27)	(21, 21, 20, 20)	(17, 17, 16, 16, 16)

Table 3
MTTF values for parallel systems with 3 types of dependent components assembled by Clayton copulas with parameters $\theta \in \{0.5, 2.5, 5\}$ and different component-type allocation vectors.

n_1	n_2	n_3	MTTF(X)		
			$\theta = 0.5$	$\theta = 2.5$	$\theta = 5$
7	0	0	0.8056	0.6590	0.5742
6	1	0	0.8795	0.7250	0.6386
7	2	0	1.0037	0.8198	0.7147
7	1	1	1.3128	1.1297	1.0517
8	1	1	1.3304	1.1389	1.0554
5	2	1	1.3364	1.1475	1.0638
6	2	1	1.3537	1.1563	1.0674
5	2	2	1.6248	1.3801	1.2562
5	3	2	1.6633	1.4021	1.2676
4	3	3	1.8736	1.5660	1.3995

$| n_j^* - n_i^* | \le 1$ for any pair $i \neq j$. In Table 2, for different values of n and K, we compute the optimal component-type allocation vectors which maximizes the system's MTTF.

Next, let us show a particular case for $K = 3$ and $n \le 10$. If we assume that the dependence among X_1, X_2 and X_3 is modeled by a Clayton family of copulas with generator $\phi(t) = (\theta t + 1)^{-1/\theta}$, where $\theta > 0$ is the dependence parameter of the copula function, then, from (3.1), we have

$$F_{n:n}(t) = \left(n_1 F_1^{-\theta}(t) + n_2 F_2^{-\theta}(t) + n_3 F_3^{-\theta}(t) + 1 - n\right)^{-1/\theta}. \quad (3.2)$$

Let us suppose that the marginal failure time distributions of the components are exponentials with hazard rates 1, 2 and 3, i.e. $F_1(t) = 1 - e^{-3t}$, $F_2(t) = 1 - e^{-2t}$ and $F_3(t) = 1 - e^{-t}$ for $t \ge 0$. It is easy to verify that $F_1 \ge F_2 \ge F_3$. Then, we can write

$$MTTF(X) = \int_0^\infty \left(1 - \left(n_1(1 - e^{-3t})^{-\theta} + n_2(1 - e^{-2t})^{-\theta} + n_3(1 - e^{-t})^{-\theta} + 1 - n\right)^{-1/\theta}\right) dt.$$

In Table 3, we compute the system's MTTF for three different values of the dependence parameter, say $\theta \in \{0.5, 2.5, 5\}$. Clearly, the greatest MTTF corresponds with the vector $\boldsymbol{s} = (4, 3, 3)$ for any value of θ, which is according to Proposition 3.1 and Table 2. Also notice that in each row the MTTF's are in decreasing order. It is worth mentioned that the Clayton copula given by

$$C_\theta(u_1, \dots, u_n) = (u_1^{-\theta} + \cdots + u_n^{-\theta} - n + 1)^{-1/\theta}, \quad \text{for } u_i \in [0, 1], \quad (3.3)$$

for $i = 1, \dots, n$, is positively ordered, that is, $C_{\theta_1} \le C_{\theta_2}$ whenever $\theta_1 \le \theta_2$ (see [26]). In general, if we fix the allocation vector $(n_1, \dots, n_K)$ and C_{ϕ_i} is an Archimedean copula for $i = 1, 2$ such as $C_{\phi_1} \le C_{\phi_2}$ then $F_{n:n}^{(\phi_1)} \le F_{n:n}^{(\phi_2)}$. In other words, the MTTF of the first system (with associated copula C_{ϕ_1}) is greater than that of the second system. Another Archimedean copulas positively ordered are the Ali–Mikhail–Haq and the Gumbel–Hougaard families of copulas (see Exercise 2.32 and Example 4.12, respectively, in [26]). We state these results in the following proposition whose proof can be found in Appendix A.

Proposition 3.4. *Let $X_{n:n}$ and $Y_{m:m}$ be two parallel systems with K types of dependent components assembled by Archimedean copulas C_X, C_Y and heterogeneous lifetimes $X_1, \dots, X_K$, $Y_1, \dots, Y_K$ allocated according to the vectors $\boldsymbol{s} = (n_1, \dots, n_K)$ and $\boldsymbol{r} = (m_1, \dots, m_K)$ on $\mathcal{D}_+$, respectively. Suppose that $C_X \ge C_Y$, $F_1 \ge \cdots \ge F_K$, $G_1 \ge \cdots \ge G_K$ and $F_i \ge G_i$ for $i = 1, \dots, K$. If $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$ then $F_{n:n} \ge G_{m:m}$.*

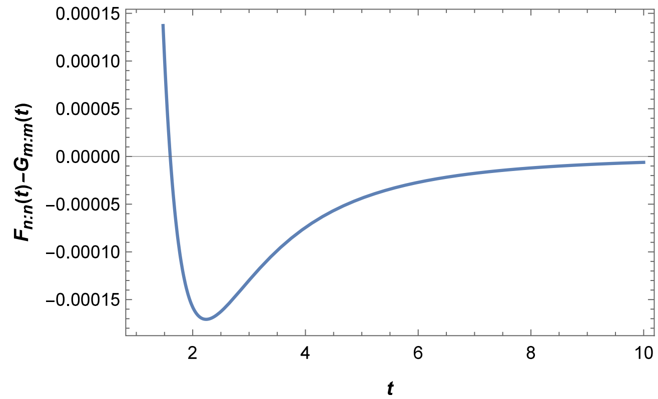

Fig. 1. Plots of $F_{n:n} - G_{m:m}$ for the two 2-out-of-3 systems considered in Counterexample 3.5.

It is worth noting that Proposition 3.4 may not hold for more complex systems like κ-out-of-n systems as we show in the following counterexample.

Counterexample 3.5. *Let us consider two 2-out-of-3 systems with $n = 3$ components of $K = 2$ different types. Assume that $F_1(t) = 1 - (1 + 2.5t)^{-3.5}$, $F_2(t) = 1 - (1 + 2t)^{-3.5}$, $G_1(t) = 1 - (1 + 2.5t)^{-3}$ and $G_2(t) = 1 - (1 + 2t)^{-3}$. Observe that $F_1 \ge F_2$, $G_1 \ge G_2$ and $F_i \ge G_i$ for $i = 1, 2$. On the other hand, we suppose that the dependence between the lifetimes of the components is modeled by a Gumbel–Hougaard copula, which belongs to the family of Archimedean copulas, given by*

$$C_\theta(u_1, u_2, u_3) = \exp\left(-\left((-\log u_1)^\theta + (-\log u_2)^\theta + (-\log u_3)^\theta\right)^{1/\theta}\right),$$

for $\theta \in [1, \infty)$ and $u_i \in [0, 1]$ for $i = 1, 2, 3$. We assume $C_X = C_{\theta_1}$ and $C_Y = C_{\theta_2}$ with $\theta_1 = 15$ and $\theta_2 = 2$. Therefore, $C_X \ge C_Y$ since the Gumbel–Hougaard copula is positively ordered. Finally, we consider $\boldsymbol{s} = (3, 0)$ and $\boldsymbol{r} = (2, 1)$. Then, it is easy to check that $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$. Therefore, all the conditions on Proposition 3.4 are satisfied. However, from Fig. 1, it is evident that the distribution functions $F_{n:n}$ and $G_{m:m}$ cross each other (they are not ordered), since $F_{n:n} - G_{m:m}$ changes its sign from positive to negative. This means that Proposition 3.4 cannot be extended in a straightforward manner to more complex systems such as κ-out-of-n systems.

Next, we derive a result for parallel systems with $K = 2$ types of dependent components assembled by any copula (see Appendix A for its proof).

Proposition 3.6. *Let $X_{n:n}$ and $Y_{m:m}$ be two parallel systems with $K = 2$ types of dependent components assembled by copulas C_X and C_Y and heterogeneous lifetimes X_1, X_2 and Y_1, Y_2 allocated according to the vectors $\boldsymbol{s} = (n_1, n_2)$ and $\boldsymbol{r} = (m_1, m_2)$, respectively. Suppose that $C_X \ge C_Y$, $F_1 \ge F_2$, $G_1 \ge G_2$ and $F_i \ge G_i$ for $i = 1, 2$. If $n_1 + n_2 = m_1 + m_2$ and $n_2 \le m_2$ then $F_{n:n} \ge G_{m:m}$.*

Observe that $F_1 \ge F_2$ means that components of type 1 are worse than components of type 2. And analogously, $F_i \ge G_i$, for $i = 1, 2$, means that the components of the first system are worse than those of the second system. Therefore, Proposition 3.6 indicates that the parallel systems more reliable and efficient (in the sense of having a higher reliability function) is the one with the greatest number of best components. This is an intuitive conclusion. However, this conclusion may not hold for more complex systems as for instance 2-out-of-3 systems. Note that all the conditions on Proposition 3.6 are satisfied in Counterexample 3.5, since $n_1 + n_2 = 3 = m_1 + m_2$ and $n_2 = 0 < 1 = m_2$. Nevertheless, as before, from Fig. 1, we get that the

Table 4
MTTF values for parallel systems with 2 types of dependent components assembled by a FGM copula with parameter $\theta \in \{-0.5, 1\}$ and distribution functions F_1, F_2 and G_1, G_2, respectively, and different component-type allocation vectors.

n_1	n_2	$\theta = -0.5$		$\theta = 1$	
		MTTF(X)	MTTF(Y)	MTTF(X)	MTTF(Y)
5	0	0.420162	0.570933	0.419955	0.570635
4	1	0.522798	0.724102	0.522582	0.723820
3	2	0.614696	0.853072	0.614461	0.852784
2	3	0.697418	0.963203	0.697152	0.962880
1	4	0.772264	1.058470	0.771945	1.058060
0	5	0.840324	1.141870	0.839911	1.141270

distribution functions $F_{n:n}$ and $G_{m:m}$ are not ordered. This means that Proposition 3.6 may not hold for κ-out-of-n systems.

In order to show a practical utility of Proposition 3.6, let us consider parallel systems with $K = 2$ types of components and $n_1 + n_2 = 5$. We suppose that the dependence among X_1, X_2 and Y_1, Y_2 is modeled by the FGM copula given by (2.2). Then, from (1.1), the distribution function is

$$F_{n:n}(t) = \prod_{i=1}^{K} F_i^{n_i}(t) \left(1 + \theta \prod_{i=1}^{K} (1 - F_i(t))^{n_i} \right), \tag{3.4}$$

with $\theta \in [-1, 1]$. We assume that the distribution functions of the corresponding lifetimes are $F_1(t) = \left(1 - e^{-4t}\right)^{0.5}$, $F_2(t) = \left(1 - e^{-2t}\right)^{0.5}$, $G_1(t) = 1 - e^{-4t}$ and $G_2(t) = 1 - e^{-2t}$ for $t \geq 0$. It is easy to verify that $F_1 \geq F_2$, $G_1 \geq G_2$ and $F_i \geq G_i$ for $i = 1, 2$, but F_2 and G_1 are not ordered since they intersect at the point $t = 0.2406$ (obtained from Wolfram Mathematica). Under this assumptions, the MTTF's are

$$MTTF(X) = \int_0^{\infty} \left(1 - \left(1 - e^{-4t}\right)^{0.5\, n_1} \left(1 - e^{-2t}\right)^{0.5\, n_2} \left(1 + \theta \left(1 - \left(1 - e^{-4t}\right)^{0.5}\right)^{n_1} \left(1 - \left(1 - e^{-2t}\right)^{0.5}\right)^{n_2}\right)\right) dt$$

and

$$MTTF(Y) = \int_0^{\infty} \left(1 - \left(1 - e^{-4t}\right)^{n_1} \left(1 - e^{-2t}\right)^{n_2} \left(1 + \theta e^{-(4n_1 + 2n_2)t}\right)\right) dt.$$

Next, we compute the MTTF's for these two parallel systems and present the results in Table 4. For each value of θ, observe that the MTTF's are ordered in increasing order in each row and the greatest value corresponds to the component-type allocation vector $s = (0, 5)$ and distribution functions G_1 and G_2, which is according to Proposition 3.6. Also notice that, for each parallel system, the MTTF's are ordered in decreasing order with respect to the dependence parameter θ. This fact is due to the FGM family is positively ordered, i.e., if $\theta_1 \leq \theta_2$ then $C_{\theta_1} \leq C_{\theta_2}$ (see Exercise 3.22 in [26]). Hence, the greatest value of the MTTF's will correspond to $\theta = -1$.

4. Optimal number of components

Let c_{1i} be the acquisition cost of one component of type i, for $i = 1, 2, \ldots, K$. Denote by c_2 the additional cost of the system which is replaced at failure. All costs are positive real-values. Then, according to Eryilmaz and Ozkut [18], the mean cost rate for a parallel system that consists of multiple types of possibly dependent components is given by

$$Z(s, c) = \frac{n_1 c_{11} + \cdots + n_K c_{1K} + c_2}{\mu_{n:n}} \tag{4.1}$$

where $s = (n_1, \ldots, n_K)$ with $n_1 + \cdots + n_K \leq n_0$, $c = (c_{11}, \ldots, c_{1K})$ and $\mu_{n:n} = E[X_{n:n}]$. One of our goals in this section is to provide optimal component-type allocation vectors that minimize the cost function given in (4.1). To do this, firstly, let c and $c^* = \left(c_{11}^*, \ldots, c_{1K}^*\right)$ be two cost vectors. Observe that, if $\mu_{n:n} \leq \mu_{m:m}$, from (4.1), it is evident that if c and c^* verify

$$n_1 c_{11} + \cdots + n_K c_{1K} \geq m_1 c_{11}^* + \cdots + m_K c_{1K}^*, \tag{4.2}$$

then

$$Z(s, c) \geq Z(r, c^*),$$

where $r = (m_1, \ldots, m_K)$. For instance, let us consider two component-type allocation vectors $s = (4, 1)$, $r = (3, 2)$, and let $c = (8, 1)$ and $c^* = (6, 5)$ be two cost vectors. It is easy to check that $s \overset{m}{\succ} r$ and $c \overset{w}{\succ} c^*$ but $c \not\succ_w c^*$. A simple computation shows that, in this case, (4.2) holds. However, the condition $c \overset{w}{\succ} c^*$ not always implies (4.2). Thus, if we consider the same two component-type allocation vectors than before and let $\tilde{c} = (6.5, 1)$ and $c^* = (6, 5)$ the two cost vectors, again $\tilde{c} \overset{w}{\succ} c^*$ but $\tilde{c} \not\succ_w c^*$. Note that, now, $n_1 \tilde{c}_{11} + n_2 \tilde{c}_{12} \leq m_1 c_{11}^* + m_2 c_{12}^*$. In the following result, we prove (see Appendix A), for the family of the Archimedean copulas, that the sufficient condition which implies (4.2), is $c \succ_w c^*$.

Proposition 4.1. *Let $X_{n:n}$ and $X_{m:m}$ be two parallel systems with K types of dependent components assembled by an Archimedean copula and heterogeneous lifetimes $X_1, \ldots, X_K$ allocated according to the vectors $s = (n_1, \ldots, n_K)$ and $r = (m_1, \ldots, m_K)$, respectively. Suppose that $s, r, c, c^* \in \mathcal{D}_+$ and $F_1 \geq \cdots \geq F_K$. Then $s \overset{m}{\succ} r$ and $c \succ_w c^*$ imply*

$$Z(s, c) \geq Z(r, c^*).$$

Remark 4.2. Specifically, for a common cost vector $c \in \mathcal{D}_+$, if $s \overset{m}{\succ} r$ then $\left(n_1 c_{11}, \ldots, n_K c_{1K}\right) \succ_w \left(m_1 c_{11}, \ldots, m_K c_{1K}\right)$, and therefore $Z(s, c) \geq Z(r, c)$.

Remark 4.3. Note that Proposition 4.1 can be generalized to parallel systems with different Archimedean copulas and different lifetimes random samples under the same assumptions than those in Proposition 3.4.

Remark 4.4. It is worth mentioning that, from Proposition 3.H.3.c in [27], we know that $s \succ_w r$ and $c \succ_w c^*$ imply $sc \succ_w rc^*$, that is, condition (4.2) holds. However, from Remark 3.2, we get that $\mu_{n:n} \geq \mu_{m:m}$ whenever $F_1 \leq \cdots \leq F_K$ and $s \succ_w r$. Therefore, in this case, we cannot conclude that $Z(s, c) \geq Z(r, c^*)$ nor $Z(s, c) \leq Z(r, c^*)$. Hence, this problem needs further research.

As an application of Proposition 4.1, let us consider parallel systems with $K = 4$ types of components where $n_1 + n_2 + n_3 + n_4 = 15$ and $s = (n_1, n_2, n_3, n_4) \in \mathcal{D}_+$. Let X_1, X_2, X_3, X_4 be a set of heterogeneous lifetimes whose dependence structure is defined by a Gumbel–Hougaard copula with generator $\phi(t) = e^{-t^{1/\theta}}$ for $\theta > 1$. For $\theta = 1$ the Gumbel–Hougaard copula models independence. Then, from (3.1), we get

$$F_{n:n}(t) = \exp\left(-\left(n_1(-\log F_1(t))^{\theta} + n_2(-\log F_2(t))^{\theta} + n_3(-\log F_3(t))^{\theta} + n_4(-\log F_4(t))^{\theta}\right)^{1/\theta}\right), \tag{4.3}$$

for $t \geq 0$. In addition, we assume that the lifetimes of the components have exponential distributions with hazard rates $\lambda_i \in \{2.4, 1.6, 1.2, 0.8\}$ for $i = 1, \ldots, 4$, respectively. Then, it is easy to verify that $F_1 \geq F_2 \geq F_3 \geq F_4$. By considering the presented copula and distribution functions, from (4.3), the MTTF is

$$MTTF(X) = \int_0^{\infty} \left(1 - \exp\left(-\left(\sum_{i=1}^{4} n_i \left(-\log(1 - e^{-\lambda_i t})\right)^{\theta}\right)^{1/\theta}\right)\right) dt.$$

Let us take two cost vectors $c = (2.5, 1.8, 1, 1)$ and $c^* = (2, 1.5, 1.5, 1.2)$. It is easy to verify that $c \succ_w c^*$. Next, we study the effects of the parameter θ and of the component-type allocation vectors (n_1, n_2, n_3, n_4) on the mean cost rates. The results are presented in Table 5.

In Table 5, when θ decreases (the components get less dependent), the mean cost rate decreases. And it also decreases when the component-type allocation vector is smaller in the sense of the usual majorization order. Observe that, in this case, the optimal component-type allocation vector is $(4, 4, 4, 3)$ which minimizes the mean cost rate.

Table 5
Mean cost rates for parallel systems with 4 types of dependent components assembled by Gumbel–Hougaard copulas with dependence parameters $\theta \in \{1.3, 2.5, 5\}$ and exponential distribution functions such as $F_1 \geq F_2 \geq F_3 \geq F_4$, two cost vectors $c = (2.5, 1.8, 1, 1)$ and $c^* = (2, 1.5, 1.5, 1.2)$ and different component-type allocation vectors $s = (n_1, n_2, n_3, n_4)$.

n_1	n_2	n_3	n_4	$\theta = 5$		$\theta = 2.5$		$\theta = 1.3$	
				$Z(\mathbf{s},\mathbf{c})$	$Z(\mathbf{s},\mathbf{c}^*)$	$Z(\mathbf{s},\mathbf{c})$	$Z(\mathbf{s},\mathbf{c}^*)$	$Z(\mathbf{s},\mathbf{c})$	$Z(\mathbf{s},\mathbf{c}^*)$
15	0	0	0	65.1061	52.0849	49.4439	39.5552	33.0723	26.4578
12	2	1	0	39.6081	32.6252	34.1433	28.1238	24.898	20.5085
11	2	1	1	26.0102	21.7668	23.9167	20.0149	18.9641	15.8702
9	2	2	2	21.79	19.1115	19.0497	16.7081	14.5637	12.7734
6	4	3	2	19.597	17.9399	16.8914	15.4631	12.6172	11.5503
5	5	3	2	19.0811	17.569	16.4091	15.1088	12.2041	11.2369
5	4	3	3	17.7237	16.6203	14.9493	14.0186	11.0056	10.3204
4	4	4	3	16.6523	16.2394	13.9629	13.6167	10.1874	9.93479

This is consistent with Proposition 4.1 since

$$(15,0,0,0) \overset{m}{\succ} (12,2,1,0) \overset{m}{\succ} (11,2,1,1) \overset{m}{\succ} (9,2,2,2) \overset{m}{\succ} (6,4,3,2) \overset{m}{\succ} (5,5,3,2) \overset{m}{\succ} (5,4,3,3) \overset{m}{\succ} (4,4,4,3).$$

From Proposition 4.1, it is evident that we should focus on the following set of all component-type allocation vectors

$$\mathcal{A}_n = \{(n_1,\dots,n_K) \in \mathcal{D}_+ : n_1 + \cdots + n_k = n\}.$$

Now, from Proposition 4.1 and Theorem 4 in [28], we get the following result for the family of Archimedean copulas.

Theorem 4.5. *Under the assumptions of Proposition 4.1, the optimal component-type allocation vector which minimizes the cost mean rate $Z(s,c)$ defined in (4.1) is $s^* = (n_1^*,\dots,n_K^*) \in \mathcal{A}_n$ such as $\left|n_j^* - n_i^*\right| \leq 1$ for any pair $i \neq j$.*

It is worth mentioning that we can apply all the above results to the case of parallel systems with heterogeneous and independent components, since the product copula is a particular case of the Archimedean family of copulas. Specifically, we obtain the product copula when $\phi(t) = e^{-t}$. In addition, by using the results of this section, we can also compare a parallel system which components are dependent with another one formed by independent components. In particular, from Theorem 4.6.2 and Corollary 4.6.3 in [26], we know that if an Archimedean copula C_θ can be extended to a n-dimensional copula, then $\Pi \leq C_\theta$. This is the case of the Frank family for $\theta > 0$, Gumbel–Hougaard family for $\theta > 1$, the Ali–Mikhail–Haq family for $\theta > 0$ and Clayton copula for $\theta > 0$, among others.

For the case $K = 2$, we obtain the optimal component-type allocation vector for any copula in the following result. The proof is straightforward combining Proposition 3.6, Theorem 4 in [28] and Proposition 3.H.3.c in [27].

Proposition 4.6. *Let $X_{n:n}$ be a parallel system with $K = 2$ types of dependent components assembled by a copula C and heterogeneous lifetimes X_1, X_2 allocated according to the vector $s = (n_1, n_2)$. Suppose that $c, c^* \in \mathcal{D}_+$ such as $c \succ_w c^*$, $F_1 \geq F_2$ and $s \in \mathcal{A}_n$. Then, the optimal component-type allocation vector which minimizes the cost mean rate $Z(s,c)$ defined in (4.1) is $s^* = (n_1^*, n_2^*) \in \mathcal{A}_n$ such as $\left|n_1^* - n_2^*\right| \leq 1$.*

Finally, it is evident that we can also use the results presented in this section to parallel systems with a single type of components. In this case, $K = 1$, i.e, the components are dependent and identically distributed, so $F_1 = \cdots = F_K = F$. Eryilmaz and Ozkut [18] showed that, for dependent and identically distributed components having exponential distributions with unit mean assembled by a Clayton copula, there exists a unique optimal value of n which minimizes

$$Z(n,c) = \frac{nc_1 + c_2}{\mu_{n:n}}, \tag{4.4}$$

for $c = (c_1, c_2)$. Next, we extend the result proved in [18] to the case where the components have arbitrary marginal lifetime distributions assembled by any copula belongs to the Archimedean family or to the FGM copulas with $\theta \in [0,1]$, but first we need to prove the following lemma whose proof can be found in Appendix A.

Table 6
Optimal number of components for parallel systems with components assembled by Ali–Mikhail–Haq copulas with different values of θ and Weibull distribution functions with shape parameter α.

θ	α					
	1		2		10	
	n^*	$Z(n^*,\mathbf{c})$	n^*	$Z(n^*,\mathbf{c})$	n^*	$Z(n^*,\mathbf{c})$
−1	6	6.53014	3	10.0098	1	11.3483
−0.75	6	6.53026	3	10.0245	1	11.4411
−0.5	6	6.53037	3	10.0401	1	11.5525
0	6	6.53061	3	10.0746	1	11.8671
0.5	6	6.53086	4	10.0907	2	12.1968
0.75	6	6.53098	4	10.0950	2	12.2596
0.95	6	6.53108	4	10.0987	2	12.3408

Lemma 4.7. *Let $X_1,\dots,X_n$ be a set of dependent random variables having a common distribution function F assembled by the Archimedean family of copulas or by the FGM copula with $\theta \in [0,1]$. Then, $F_{n-1:n-1} - F_{n:n}$ is decreasing in n for any fixed $t \geq 0$.*

Remark 4.8. It is worth mentioning that $F_{n-1:n-1} - F_{n:n}$ decreasing in n implies that $\mu_{n:n} - \mu_{n-1:n-1}$ is decreasing in n, i.e., $\mu_{n:n} - \mu_{n-1:n-1} > \mu_{n+1:n+1} - \mu_{n:n}$, and therefore, we have that $2\mu_{n:n} > \mu_{n+1:n+1} + \mu_{n-1:n-1}$ holds for any Archimedean copula and for the FGM copula with $\theta \in [0,1]$. However, when the dependence parameter θ of the FGM copula satisfies $-1 \leq \theta < 0$, then the function p defined in (A.3) can be non-monotonic (see Fig. 2) and therefore, $F_{n-1:n-1} - F_{n:n}$ can be not decreasing in n.

On the other hand, from Proposition 2.1(i) in [24], we know that $F_{n-1:n-1} \geq F_{n:n}$. Therefore, $\mu_{n:n} \geq \mu_{n-1:n-1}$, that is, $\mu_{n:n} - \mu_{n-1:n-1} \geq 0$ for both Archimedean and FGM copulas with any dependence parameter. Moreover, observe that the functions ℓ and p defined in (A.1) and (A.3), respectively, go to zero as $n \to \infty$ since $u \in [0,1]$. Hence, $\mu_{n:n} - \mu_{n-1:n-1}$ goes to zero as $n \to \infty$.

Theorem 4.9. *For any copula belongs to the family of the Archimedean copulas and for the FGM copulas with $\theta \in [0,1]$, there exists a unique optimal value of n, say n^*, which minimizes the mean cost rate $Z(n,c)$ defined in (4.4), where $c = (c_1, c_2)$, for any distribution function F.*

As a useful application of Theorem 4.9, we determine the optimal number of components for parallel systems consisting of single type of dependent components. We compute optimal n^* for $c_2/c_1 = 10$ when $F(t) = 1 - e^{-t^\alpha}$ with $\alpha > 0$ and the dependence structure is defined by a copula C_θ where θ is the dependence parameter. Observe that, when $\alpha = 1$, the components have exponential distributions with hazard rate 1. Under the abovementioned assumptions, the objective function is given by

$$Z(n,c) = \frac{nc_1 + c_2}{\int_0^1 \left(\alpha(1-u)(-\log(1-u))^{1-1/\alpha}\right)^{-1}\left(1 - C_\theta(u)\right) du}. \tag{4.5}$$

Table 6 presents the results for the Ali–Mikhail–Haq family, which belongs to the family of the Archimedean copulas, such that $C_\theta(u) = u^n\left(1 - \theta(1-u)^n\right)^{-1}$ for $u \in [0,1]$ and $\theta \in [-1,1)$. When $\theta = 0$, then $C_\theta = \Pi$, i.e., the components are independent. Also notice that this family of copulas is positively quadrant dependent (PQD) for $\theta \geq 0$ and negatively quadrant dependent (NQD) for $\theta \leq 0$.

In Table 6, the effects of parameters θ and α on the mean cost rate have been investigated. Thus, when θ increases, the optimal point

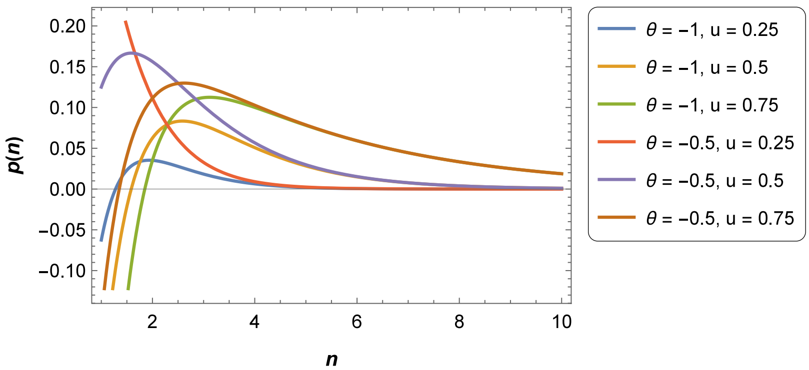

Fig. 2. Plot of the function p defined in (A.3) for $n \geq 1$ and FGM copulas with different values of $\theta \in [-1, 0)$ and different values of $u \in [0, 1]$.

which minimizes (4.5) increases, and when α increases, the optimal point decreases.

Eryilmaz and Ozkut [18] established the following conjecture: "In general, if the components in a parallel system consisting of single type of dependent components are positively dependent (PQD), then $n_I^* \geq n_D^*$ where n_I^* and n_D^* represent the optimal values which minimize (4.4) for the independent and dependent cases, respectively." However, for the Ali–Mikhail–Haq family of copulas having a PQD property ($\theta \geq 0$), as it is clear from Table 6, the optimal number of components could be smaller under independence ($\theta = 0$). Hence, the conjecture established in [18] is not true in general.

5. Optimal replacement time

Assume that the system is replaced at time T or at failure, whichever occurs first. Thus, a cost $n_1 c_{11} + \cdots + n_K c_{1K}$ is suffered for a non-failed system that is replaced at time T and a cost $n_1 c_{11} + \cdots + n_K c_{1K} + c_2$ is suffered for a failed system. Then, the mean cost rate for a parallel system, $X_{n:n}$, that consists of n dependent components of K types is given by

$$M(\boldsymbol{s}, \boldsymbol{c}, T) = \frac{P(X_{n:n} > T)\sum_{i=1}^K n_i c_{1i} + P(X_{n:n} \leq T)(\sum_{i=1}^K n_i c_{1i} + c_2)}{E[\min(X_{n:n}, T)]} = \frac{\bar{F}_{n:n}(T)\sum_{i=1}^K n_i c_{1i} + F_{n:n}(T)(\sum_{i=1}^K n_i c_{1i} + c_2)}{E[\min(X_{n:n}, T)]},$$

where $\bar{F}_{n:n} = 1 - F_{n:n}$ is the reliability function of the parallel system $X_{n:n}$, $\boldsymbol{s} = (n_1, \ldots, n_k)$ and $\boldsymbol{c} = (c_{11}, \ldots, c_{1K})$. Then, it is easy to verify that the above function can be rewritten as follows

$$M(\boldsymbol{s}, \boldsymbol{c}, T) = \frac{n_1 c_{11} + \cdots + n_K c_{1K} + c_2 F_{n:n}(T)}{\mu_{n:n}(T)}, \tag{5.1}$$

where $\mu_{n:n}(T) = E[\min(X_{n:n}, T)] = \int_0^T (1 - F_{n:n}(t))dt$. Note that Eq. (5.1) is equal to Equation (34) in Eryilmaz and Ozkut [18], where the study of conditions on the existence of a unique T that minimizes (5.1) was left as future work. In this section, we solve this problem, but first we study the effects of $\boldsymbol{s}$ and $\boldsymbol{c}$ on the mean cost rate function. Specifically, in the following result, we show that the function $M(\boldsymbol{s}, \boldsymbol{c}, T)$ decreases when $\boldsymbol{s}$ and $\boldsymbol{c}$ also decrease in the majorization sense for a fixed time T (see Appendix A for its proof).

Proposition 5.1. *Let $X_{n:n}$ and $X_{m:m}$ be two parallel systems with K types of dependent components assembled by an Archimedean copula and heterogeneous lifetimes $X_1, \ldots, X_K$ allocated according to the vectors $\boldsymbol{s} = (n_1, \ldots, n_K)$ and $\boldsymbol{r} = (m_1, \ldots, m_K)$, respectively. Suppose that $\boldsymbol{s}, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{c}^* \in \mathcal{D}_+$ and $F_1 \geq \cdots \geq F_K$. Then $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $\boldsymbol{c} \succ_w \boldsymbol{c}^*$ imply*

$$M(\boldsymbol{s}, \boldsymbol{c}, T) \geq M(\boldsymbol{r}, \boldsymbol{c}^*, T).$$

Let us see an application of the above result. We assume that the parallel systems have $K = 3$ types of dependent components with exponential or Weibull lifetime distributions such as $F_1(t) = 1 - e^{-5t}$, $F_2(t) = 1 - e^{-2t}$ and $F_3(t) = 1 - e^{-t^{1.2}}$ for $t \geq 0$. These three distribution functions verify $F_1 \geq F_2 \geq F_3$. We suppose that the dependent components are assembled by the Ali–Mikhail–Haq family of copulas defined by

$$C_\theta(u_1, \ldots, u_n) = \frac{u_1 \cdots u_n}{1 - \theta(1 - u_1)\cdots(1 - u_n)}, \quad \text{for } \theta \in [-1, 1),$$

and $u_i \in [0, 1]$ for $i = 1, \ldots, n$. Then, from (1.1), we have

$$F_{n:n}(t) = \frac{F_1^{n_1}(t) F_2^{n_2}(t) F_3^{n_3}(t)}{1 - \theta(1 - F_1(t))^{n_1}(1 - F_2(t))^{n_2}(1 - F_3(t))^{n_3}}.$$

Let us consider parallel systems with 36 components of 3 types. In particular, we take $\boldsymbol{s} = (15, 11, 10)$, $\boldsymbol{r} = (13, 12, 11)$, and let $\boldsymbol{c} = (8, 3, 1)$ and $\boldsymbol{c}^* = (6, 4, 1)$ be two cost vectors. It is easy to verify that $\boldsymbol{s}, \boldsymbol{r}, \boldsymbol{c}, \boldsymbol{c}^* \in \mathcal{D}_+$, $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $\boldsymbol{c} \succ_w \boldsymbol{c}^*$. In Fig. 3, we plot the functions $M(\boldsymbol{s}, \boldsymbol{c}, T)$ as a function of T when $\theta = 0.5$ and $c_2 = 15$. As it can be seen in Fig. 3 the mean cost rate functions are ordered which is according to Proposition 5.1.

The study of conditions on the existence of a unique T that minimizes (5.1) was left as an open problem in [18]. We give an answer to this problem in the following result, whose proof can be found in Appendix A. First, let us define the function

$$\eta_i\Big(\underbrace{u_1, \ldots, u_1}_{n_1}, \ldots, \underbrace{u_K, \ldots, u_K}_{n_K}\Big) = \frac{(1 - u_i)\,\partial_{(i)} C\Big(\overbrace{u_1, \ldots, u_1}^{n_1}, \ldots, \overbrace{u_K, \ldots, u_K}^{n_K}\Big)}{1 - C\Big(\underbrace{u_1, \ldots, u_1}_{n_1}, \ldots, \underbrace{u_K, \ldots, u_K}_{n_K}\Big)}, \tag{5.2}$$

with $u_i \in [0, 1]$ for $i = 1, \ldots, K$, where C is the distributional copula associated to $X_{n:n}$ and

$$\partial_{(i)} C(u_1, \ldots, u_K) = \frac{\partial C(u_1, \ldots, u_K)}{\partial u_i},$$

for $i = 1, \ldots, K$. If the system consists of a single type of dependent components ($K = 1$), then

$$\eta(u) = \frac{(1 - u)C'(u)}{1 - C(u)}, \tag{5.3}$$

with $u \in [0, 1]$.

Theorem 5.2. *Let $X_{n:n}$ be a parallel system with K types of dependent components assembled by a copula C and heterogeneous lifetimes $X_1, \ldots, X_K$ allocated according to the vectors $\boldsymbol{s} = (n_1, \ldots, n_K)$. If X_i is IFR for $i = 1, \ldots, K$ and the function η_i is increasing in $(0, 1)^K$, then there exists an optimal value of T, say T^*, which minimizes the mean cost rate $M(\boldsymbol{s}, \boldsymbol{c}, T)$ defined in (5.1).*

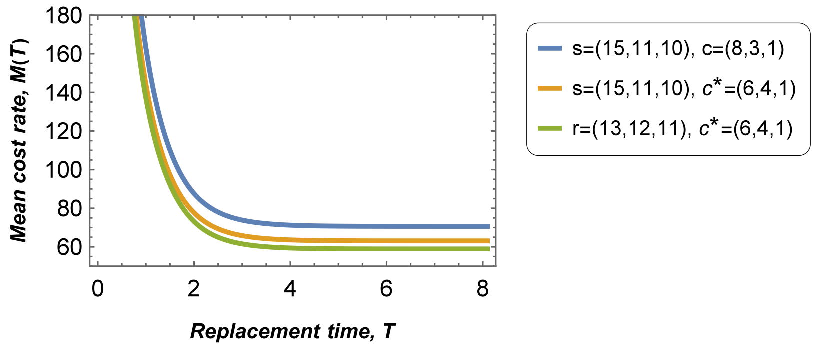

Fig. 3. Plot of the mean cost rate functions $M(s, c, T)$ for $s = (15, 11, 10)$, $r = (13, 12, 11)$, $c = (8, 3, 1)$ and $c^* = (6, 4, 1)$, and an AMH copula with dependence parameter $\theta = 0.5$.

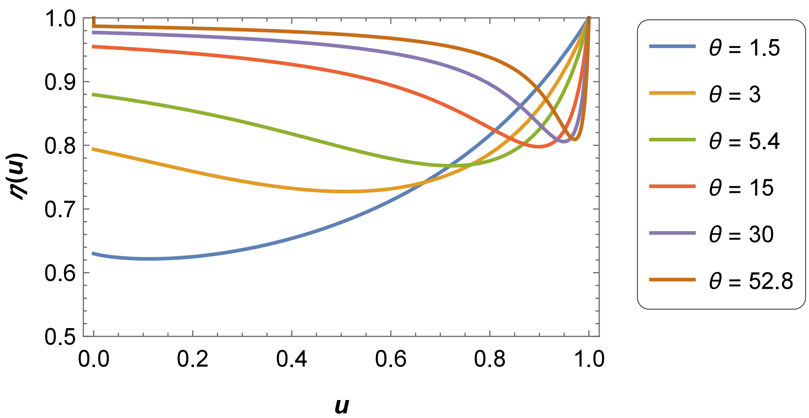

Fig. 4. Plots of the function η defined in (5.5) for $u \in (0, 1)$, $n = 2$ and Clayton copulas with dependence parameter $\theta \in \{1.5, 3, 5.4, 15, 30, 52.8\}$.

Remark 5.3. If the hazard rate of X_i and η_i are both strictly increasing to infinity for $i = 1, \ldots, K$, then T^* is unique and finite.

It is worth mentioning that Weibull distributions have hazard rate functions strictly increasing when the shape parameter is greater than 1. If the shape parameter is 1, then the distribution is exponential. Generalized Gamma distributions with shape parameters α and ν are IFR when $\alpha \geq 1$ and $\alpha\nu \geq 1$. Another type of IFR distributions are the Gompertz–Makeham distributions. In general, if the probability density function is log-concave, then the reliability function is also log-concave, i.e., it is IFR. For more details about IFR distributions, see [29].

By the symmetry of the Archimedean copulas, from (5.2) and Theorem 5.2, it is enough to study the monotonicity of

$$\eta_1\left(u_1, \ldots, u_K\right) = \frac{n_1\left(1-u_1\right)\phi'(\sum_{i=1}^K n_i \phi^{-1}(u_i))}{\phi'(\phi^{-1}(u_1))\left(1-\phi(\sum_{i=1}^K n_i \phi^{-1}(u_i))\right)} \tag{5.4}$$

in $(0,1)^K$. For example, for parallel systems with $K = 1$ type of components assembled by a Clayton copula with generator $\phi(t) = (\theta t + 1)^{-1/\theta}$ with $\theta > 0$, we obtain $\phi^{-1}(x) = \theta^{-1}(x^{-\theta} - 1)$ for $x \in [0, 1]$ and from (5.4), we get

$$\eta(u) = \frac{n(1-u)u^{-\theta-1}\left(nu^{-\theta}-n+1\right)^{-\frac{1}{\theta}-1}}{1-\left(nu^{-\theta}-n+1\right)^{-1/\theta}}, \tag{5.5}$$

with $u \in (0, 1)$. In Fig. 4, we plot the function η defined in (5.5) when $n = 2$ for different values of the dependence parameter θ. As it can be seen in Fig. 4, the function η is not increasing (nor decreasing) in $u \in (0, 1)$ for $\theta \in \{1.5, 3, 5.4, 15, 30, 52.8\}$. Therefore, the IFR class is not always preserved under the formation of parallel systems with dependent components (e.g. for exponential distributions). Then, from Theorem 5.2, we know that, in some cases, the optimal T^* may be infinity, i.e., a unit is replaced only at failure.

For parallel systems consisting of single type of dependent components ($K = 1$), note that Theorem 5.2 can be applied to any distributional copula verifying that the function η defined in (5.3) is increasing in $u \in (0, 1)$. For this case, we prove that the Gumbel–Hougaard family of copulas for $\theta \geq 1$ and the Clayton family of copulas for $0 < \theta \leq 1$ verify such condition (see Appendix B). For the FGM copulas, we show in Fig. 5 that this family also verify that the function η defined in (5.3) is increasing for different values of the dependence parameter θ and n. Therefore, Theorem 5.2 can be applied for these families of copulas and for any IFR marginal distribution function F. In particular, let us assume that the component lifetimes follow Weibull distributions with shape parameter $\alpha > 1$, that is, $F(t) = 1 - e^{-t^\alpha}$ for $t \geq 0$. Further, we suppose that the dependence structure is defined by Gumbel–Hougaard copulas with parameter $\theta \geq 1$. We compute the optimal replacement times (T^*) for different values of c_2/c_1, θ and n. Tables 7 and 8 present the results for $\alpha = 1.2$ and $\alpha = 2.5$, respectively. It is worth mentioning that, for the values of c_2/c_1, α, θ and n considered in Tables 7 and 8, the conditions of Remark 5.3 are satisfied, so the existence and uniqueness of T^* are guaranteed.

According to Table 7 ($\alpha = 1.2$), when $c_2/c_1 = 10$, $n = 10$ and $\theta = 6.5$, the system should be replaced at time 6.5421 and the corresponding mean cost rate value is 17.4714. Moreover, note that the Gumbel–Hougaard copula with $\theta = 6.5$ (more dependent components) has the highest T^* values in all cases, except when $c_2/c_1 = 20$ and $n = 5$, in which case the highest value of T^* corresponds to the independent copula ($\theta = 1$). That is, for all cases except one, the more dependence among the components the later replacement time. However, the same conclusion cannot be obtained from Table 8 ($\alpha = 2.5$), since this table shows that when the dependence parameter θ increases, T^* decreases in all cases. That is, the more dependence among the components the earlier replacement time. Therefore, we can conclude that the marginal distribution functions affect the optimal replacement time. In

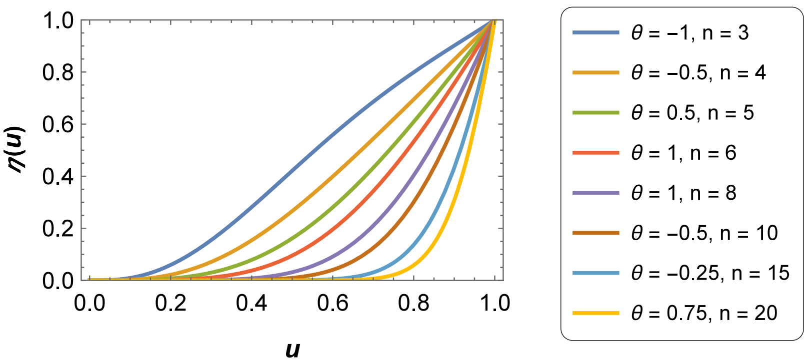

Fig. 5. Plots of the function η defined in (5.3) for the FGM copula with $\theta \in \{-1, -0.5, -0.25, 0.5, 0.75, 1\}$ and $n \in \{3, 4, 5, 6, 8, 10, 15, 20\}$.

Table 7
Optimal replacement times for parallel systems consisting of single type of dependent components assembled by Gumbel–Hougaard copulas with dependence parameters $\theta \in \{1, 2, 6.5\}$ and Weibull distribution functions with $\alpha = 1.2$.

c_2/c_1	n	θ					
		1		2		6.5	
		T^*	$M(T^*)$	T^*	$M(T^*)$	T^*	$M(T^*)$
10	2	0.6280	6.6357	0.6843	9.2914	0.9501	11.5235
	3	0.8556	5.8931	0.8543	9.1702	1.3019	12.2966
	5	1.2347	5.9275	1.1747	9.7619	2.1502	13.8165
	8	1.6541	6.6457	1.6115	11.0243	4.2405	16.0096
	10	1.8763	7.2049	1.9009	11.8874	6.5421	17.4714
	25	3.0464	11.3015	7.4074	17.9105	20.0209	28.4514
20	2	0.4194	9.2395	0.3958	14.3581	0.4885	19.3734
	3	0.6233	7.5732	0.5163	13.3743	0.6405	20.1481
	5	0.9597	7.1533	0.7293	13.4797	0.9349	21.9020
	8	1.3264	7.7454	0.9886	14.6815	1.3972	24.4006
	10	1.5157	8.3015	1.1366	15.6174	1.7474	25.9503
	25	2.3896	12.6974	2.0359	22.3999	8.2260	36.5804

Table 8
Optimal replacement times for parallel systems consisting of single type of dependent components assembled by Gumbel–Hougaard copulas with dependence parameters $\theta \in \{1, 2, 6.5\}$ and Weibull distribution functions with $\alpha = 2.5$.

c_2/c_1	n	θ					
		1		2		6.5	
		T^*	$M(T^*)$	T^*	$M(T^*)$	T^*	$M(T^*)$
10	2	0.5925	4.3660	0.5116	5.6206	0.4662	6.8640
	3	0.7528	4.8026	0.6224	6.5453	0.5525	8.5115
	5	0.9496	6.0277	0.7680	8.4189	0.6764	11.4100
	8	1.1202	7.9589	0.9065	11.0975	0.8088	15.2348
	10	1.1973	9.2284	0.9738	12.8016	0.8795	17.5823
	25	1.4950	18.1775	1.2708	24.3874	1.2541	32.9706
20	2	0.5041	5.0730	0.4133	6.8733	0.3599	8.7923
	3	0.6655	5.3657	0.5164	7.7647	0.4303	10.7520
	5	0.8648	6.5329	0.6526	9.7059	0.5303	14.1854
	8	1.0368	8.4813	0.7811	12.5524	0.6351	18.6848
	10	1.1140	9.7786	0.8427	14.3767	0.6897	21.4275
	25	1.4041	19.0021	1.1003	26.8365	0.9594	39.0186

fact, from Theorem 5.2, under dependence, we know that the optimal replacement time can be infinity (a unit is replaced only at failure) when the marginal distribution functions are not IFR, as for example holds for exponential distributions.

On the other hand, Eryilmaz and Ozkut [18] established the following conjecture: "In general, if the components in a parallel system consisting of single ($K = 1$) type of dependent components are positively dependent (PQD), then $T_I^* \leq T_D^*$ where T_I^* and T_D^* represent the optimal values which minimize (5.1) for the independent and dependent cases, respectively." However, for the Gumbel–Hougaard family of copulas, which is PQD (see [26]), the optimal replacement time could be larger under independence ($\theta = 1$), as it is shown in Table 8. Hence, the conjecture established in [18] is not true in general.

Finally, from these two tables, we observe that when c_2/c_1 increases, T^* decreases for fixed n and θ, which means that if costs increase, then the system should be replaced earlier. And when n increases, T^* increases for fixed c_2/c_1 and θ, i.e., the higher the number of system components the later the replacement time.

6. Conclusions and future work

In this work, we have study in depth the two optimization problems as well as the conjectures posted in [18]. Furthermore, we have solved an open problem posed by Eryilmaz and Ozkut [18] related to the existence and uniqueness of optimal replacement time for parallel systems with multiple types of dependent components. The dependence between the components has been modeled by different families of copulas. We have also reported numerical studies in order to show the applicability of the new results. Furthermore, based on them, we have investigated the effects of the dependence and of the marginal distribution functions. We can conclude that, in general, the optimal values depend on both the copula and the marginal distributions. The same conclusion follows from Theorem 5.2, since it states that an unique optimal replacement time exists whenever the marginal distribution functions are IFR and a function that depends on the distributional copula is increasing. Therefore, when simulations are carried out, it is important to consider distributions which have the IFR property, such as Weibull, Generalized Gamma, Gompertz–Makeham distributions, etc.

It is clear that real data plays an essential role in evaluating preventive maintenance strategies for the correct operation of systems. However, obtaining information about lifetimes of real systems results very difficult. Moreover, according to Jobge and Scarf [30], in practical situations, a lot of data is needed to fit models with dependence. As a consequence, most researchers combine numerical analysis and simulations to study the performance of their maintenance strategies by using different computational algorithms. One may refer to [30] for a recent review on maintenance modeling and optimization. Nevertheless, before using any computational algorithm, it is important to know that the optimal solution to an optimization problem exists and it is unique.

Besides, the theoretical results proved in this manuscript can be applied to systems with any number of components ($n \geq 1$), arbitrary marginal distributions and different families of copulas, even some results hold for any copula. Therefore, the setting considered in this new manuscript is very wide. Thus, practitioners may use our results to real data sets (if available). In the case of having a real data set of system lifetimes, the empirical joint distribution function associated with the data could be obtained, the IFR property could be tested (if necessary), and finally, the optimal solution could be computed.

To finish, a number of recommendations for future research are given. Firstly, it would be interesting to study if Theorem 4.5 also holds for other families of copulas different than the Archimedean family. Secondly, future research could continue to explore conditions on the generator ϕ of the Archimedean copula under which the function η defined in (5.4) is increasing when $K \geq 2$. Another possibility would be to investigate the two optimization problems studied in this manuscript for κ-out-of-n systems consisting of multiple types of dependent components. These are some open problems that we expect to report in future.

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CRediT authorship contribution statement

Nuria Torrado: Conceptualization, Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

Here, we provide the proofs of the theorems and propositions presented in the previous sections.

Proof of Proposition 3.4. On the one hand, let $X_{m:m}$ be a parallel system with dependent components assembled by an Archimedean copula C_X and lifetimes $X_1, \dots, X_K$ allocated according the vector $\boldsymbol{r}$ on D_+. Then, from Proposition 3.1, we know that $\boldsymbol{s} \overset{w}{\succ} \boldsymbol{r}$ implies $F_{n:n} \geq G_{m:m}$ since $F_1 \geq \cdots \geq F_K$. On the other hand, let $X^*_{m:m}$ be a parallel system with dependent components assembled by an Archimedean copula C_Y and lifetimes $X_1, \dots, X_K$ allocated according the vector $\boldsymbol{r}$. Then, from (1.1), we get

$$C_X\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \dots, \underbrace{F_K(t), \dots, F_K(t)}_{m_K}\Big)$$
$$\geq C_Y\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \dots, \underbrace{F_K(t), \dots, F_K(t)}_{m_K}\Big),$$

since $C_X \geq C_Y$. Finally, we have

$$C_Y\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \dots, \underbrace{F_K(t), \dots, F_K(t)}_{m_K}\Big)$$
$$\geq C_Y\Big(\underbrace{G_1(t), \dots, G_1(t)}_{m_1}, \dots, \underbrace{G_K(t), \dots, G_K(t)}_{m_K}\Big)$$

whenever $F_i \geq G_i$ for $i = 1, \dots, K$, since C_Y is an increasing function. This complete the proof. ■

Proof of Proposition 3.6. On the one hand, let us consider a parallel system with dependent components assembled by the copula C_X and lifetimes X_1, X_2 allocated according to the vector $\boldsymbol{r}$. Then, from Proposition 2.3(i) in [24], we know that

$$C_X\Big(\underbrace{F_1(t), \dots, F_1(t)}_{n_1}, \underbrace{F_2(t), \dots, F_2(t)}_{n_2}\Big) \geq C_X\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \underbrace{F_2(t), \dots, F_2(t)}_{m_2}\Big),$$

when $F_1 \geq F_2$ and $n_2 \leq m_2$. On the other hand, let $X^*_{n:n}$ be a parallel system with dependent components assembled by the copula C_Y and lifetimes X_1, X_2 allocated according the vector $\boldsymbol{r}$. Then, we get

$$C_X\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \underbrace{F_2(t), \dots, F_2(t)}_{m_2}\Big) \geq C_Y\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \underbrace{F_2(t), \dots, F_2(t)}_{m_2}\Big),$$

since $C_X \geq C_Y$. Finally, we have

$$C_Y\Big(\underbrace{F_1(t), \dots, F_1(t)}_{m_1}, \underbrace{F_2(t), \dots, F_2(t)}_{m_2}\Big) \geq C_Y\Big(\underbrace{G_1(t), \dots, G_1(t)}_{m_1}, \underbrace{G_2(t), \dots, G_2(t)}_{m_2}\Big)$$

whenever $F_i \geq G_i$ for $i = 1, 2$, since C_Y is an increasing function. This complete the proof. ■

Proof of Proposition 4.1. From Proposition 3.1, we know that $F_{n:n} \geq F_{m:m}$ and therefore $X_{n:n} \leq_{MTTF} X_{m:m}$, that is, $\mu_{n:n} \underset{m}{\leq} \mu_{m:m}$ where $\mu_{m:m} = E[X_{m:m}] = \int_0^\infty \bar{F}_{m:m}(t)dt$, since $\boldsymbol{s} \underset{w}{\succ} \boldsymbol{r}$ implies $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$. Moreover, from Proposition 3.H.3.c in [27], if $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $\boldsymbol{c} \succ_w \boldsymbol{c}^*$, then

$$\left(n_1 c_{11}, \dots, n_K c_{1K}\right) \succ_w \left(m_1 c^*_{11}, \dots, m_K c^*_{1K}\right).$$

Therefore, the inequality (4.2) holds. Then, from (4.1), we get $Z(\boldsymbol{s}, \boldsymbol{c}) \geq Z(\boldsymbol{r}, \boldsymbol{c}^*)$. ■

Proof of Lemma 4.7. First, we consider the Archimedean family of copulas. Then, from (3.1), we get

$$F_{n-1:n-1}(t) - F_{n:n}(t) = \phi((n-1)\phi^{-1}(F(t))) - \phi(n\phi^{-1}(F(t))),$$

for any generator ϕ. Let us denote

$$\ell(u, n) = \phi((n-1)\phi^{-1}(u)) - \phi(n\phi^{-1}(u)). \tag{A.1}$$

Then, $F_{n-1:n-1}(t) - F_{n:n}(t) = \ell(F(t), n)$. Taking the partial derivative of $\ell(u, n)$ with respect to n, we obtain

$$\frac{\partial \ell(u, n)}{\partial n} = \phi^{-1}(u)\Big(\phi'((n-1)\phi^{-1}(u)) - \phi'(n\phi^{-1}(u))\Big).$$

Clearly, $(n-1)\phi^{-1}(u) < n\phi^{-1}(u)$. Now, because ϕ is convex, we have $\phi'((n-1)\phi^{-1}(u)) < \phi'(n\phi^{-1}(u))$ which implies that $\partial\ell(u, n)/\partial n < 0$ for any $n \geq 1$, i.e., ℓ is strictly decreasing in n. Therefore,

$$F_{n-2:n-2}(t) - F_{n-1:n-1}(t) = \ell(F(t), n-1) > \ell(F(t), n) = F_{n-1:n-1}(t) - F_{n:n}(t),$$

for any fixed $t \geq 0$. In other words, $F_{n-1:n-1} - F_{n:n}$ is decreasing in n for any Archimedean copula.

Secondly, for the FGM family of copulas, from (3.4), we have

$$\begin{aligned} F_{n-1:n-1}(t) - F_{n:n}(t) &= F^{n-1}(t)\big(1 + \theta(1-F(t))^{n-1}\big) - F^n(t)\big(1 + \theta(1-F(t))^n\big) \\ &= F^{n-1}(t)\big(1 + \theta(1-F(t))^{n-1} - F(t)\big(1 + \theta(1-F(t))^n\big)\big) \\ &= F^{n-1}(t)\big(1 - F(t) + \theta(1-F(t))^{n-1}\left(1 - F(t)(1-F(t))\right)\big), \end{aligned} \tag{A.2}$$

for $\theta \in [-1, 1]$. Let us define

$$p(u, n) = u^{n-1}\left(1 - u + \theta(1-u)^{n-1}(1 - u(1-u))\right), \quad \text{for } \theta \in [-1, 1], \tag{A.3}$$

and $u \in [0, 1]$. Taking the partial derivative of $p(u, n)$ with respect to n, we obtain

$$\frac{\partial p(u, n)}{\partial n} \overset{sign}{=} \theta(1-u)^{n-1}(1 - u(1-u))(\log(1-u) + \log(u)) + (1-u)\log(u) < 0,$$

whenever $\theta \in [0,1]$, i.e., the function p is strictly decreasing in n for $\theta \in [0,1]$. The rest of the proof follows as before and hence it is omitted. ■

Proof of Theorem 4.9. Following the idea in [25], we need to find a number n such as $Z(n+1,\boldsymbol{c}) \geq Z(n,\boldsymbol{c})$ and $Z(n,\boldsymbol{c}) < Z(n-1,\boldsymbol{c})$ for a fixed cost vector $\boldsymbol{c} = (c_1, c_2)$. Observe that these two conditions are equivalent to

$$L(n) \geq \frac{c_2}{c_1} \quad \text{and} \quad L(n-1) < \frac{c_2}{c_1}, \tag{A.4}$$

whenever $\mu_{n:n} > 0$ for all n and where

$$L(n) = \frac{\mu_{n:n}}{\mu_{n+1:n+1} - \mu_{n:n}} - n,$$

for $n \geq 1$. Now, we get

$$\begin{aligned} L(n+1) - L(n) &= \frac{\mu_{n+1:n+1}}{\mu_{n+2:n+2} - \mu_{n+1:n+1}} - \frac{\mu_{n:n}}{\mu_{n+1:n+1} - \mu_{n:n}} - 1 \\ &= \frac{\mu_{n+1:n+1}}{\mu_{n+2:n+2} - \mu_{n+1:n+1}} - \frac{\mu_{n:n}}{\mu_{n+1:n+1} - \mu_{n:n}} \\ &\quad - \frac{\mu_{n+1:n+1} - \mu_{n:n}}{\mu_{n+1:n+1} - \mu_{n:n}} \\ &= \mu_{n+1:n+1} \left(\frac{1}{\mu_{n+2:n+2} - \mu_{n+1:n+1}} - \frac{1}{\mu_{n+1:n+1} - \mu_{n:n}} \right). \end{aligned}$$

Then, from Lemma 4.7, we know that $\mu_{n+1:n+1} - \mu_{n:n}$ is decreasing in n (see Remark 4.8) and therefore, $L(n+1) - L(n) > 0$, that is, the function L is strictly increasing in n. Now, since

$$\mu_{n:n} - \mu = (\mu_{n:n} - \mu_{n-1:n-1}) + (\mu_{n-1:n-1} - \mu_{n-2:n-2}) + \cdots + (\mu_{3:3} - \mu_{2:2}) + (\mu_{2:2} - \mu)$$

then

$$\mu_{n:n} - \mu > (n-1)(\mu_{n:n} - \mu_{n-1:n-1}) > (n-1)(\mu_{n+1:n+1} - \mu_{n:n})$$

which is equivalent to $n(\mu_{n:n} - \mu_{n+1:n+1}) > \mu - \mu_{n+1:n+1}$, and therefore,

$$L(n) \geq \frac{\mu}{\mu_{n+1:n+1} - \mu_{n:n}} - 1, \quad \text{for } n \geq 1.$$

Finally, from Remark 4.8, we know that

$$\lim_{n \to \infty} (\mu_{n+1:n+1} - \mu_{n:n}) = 0.$$

Therefore, $L(n)$ is strictly increasing to infinity, and then, there exists a finite and unique n^* which satisfies (A.4), and it minimizes $Z(n,\boldsymbol{c})$. ■

Proof of Proposition 5.1. From Proposition 3.1, we know that $F_{n:n} \geq F_{m:m}$ and therefore $\mu_{n:n}(T) \leq \mu_{m:m}(T)$ for any T fixed. Moreover, as we proved in Proposition 4.1, the inequality in (4.2) holds whenever $\boldsymbol{s} \overset{m}{\succ} \boldsymbol{r}$ and $\boldsymbol{c} \succ_w \boldsymbol{c}^*$. Therefore, from (5.1), we have $M(\boldsymbol{s},\boldsymbol{c},T) \geq M(\boldsymbol{r},\boldsymbol{c}^*,T)$. ■

Proof of Theorem 5.2. Differentiating $M(\boldsymbol{s},\boldsymbol{c},T)$ with respect to T, and setting it equal to zero, we get

$$c_2 f_{n:n}(T)\mu_{n:n}(T) - \left(n_1 c_{11} + \cdots + n_K c_{1K} + c_2 F_{n:n}(T)\right)\bar{F}_{n:n}(T) = 0, \tag{A.5}$$

where $f_{n:n}$ is the probability density function of the parallel system $X_{n:n}$. Then, (A.5) is equivalent to

$$\frac{f_{n:n}(T)\mu_{n:n}(T) - F_{n:n}(T)\bar{F}_{n:n}(T)}{\bar{F}_{n:n}(T)} = \frac{n_1 c_{11} + \cdots + n_K c_{1K}}{c_2}$$

if and only if

$$h_{n:n}(T)\mu_{n:n}(T) - F_{n:n}(T) = \frac{n_1 c_{11} + \cdots + n_K c_{1K}}{c_2},$$

where $h_{n:n}$ is the hazard rate function of $X_{n:n}$. Now, from Theorem 3.2 in [31], we know that there exists a value of T, say T^*, which minimizes (5.1) if $h_{n:n}$ is an increasing function, that is, if $X_{n:n}$ is increasing failure rate (IFR).

Now, observe that the function defined in (5.2) can be rewritten as follows

$$\eta_i(u_1,\ldots,u_K) = \frac{(1-u_i)\,\partial_{(i)}\mathbf{H}(1-u_1,\ldots,1-u_K)}{\mathbf{H}(1-u_1,\ldots,1-u_K)},$$

where $\mathbf{H}(1-u_1,\ldots,1-u_K) = 1 - C(u_1,\ldots,u_K)$ is a *generalized domination function* such as $\bar{F}_{n:n} = \mathbf{H}(1-F_1,\ldots,1-F_K)$. Then, from Proposition 2.5(i) in [32], we know that $X_{n:n}$ is IFR if η_i is increasing in $(0,1)^K$ and X_i is IFR for $i = 1,\ldots,K$. ■

Appendix B

Firstly, we consider the Gumbel–Hougaard family of copulas with

$$C_\theta(u) = \exp(-(n(-\log u)^\theta)^{1/\theta}) = u^{n^{1/\theta}},$$

for $u \in [0,1]$ and $\theta \geq 1$, from (5.3), we get

$$\eta(u) = \frac{n^{1/\theta}(1-u)}{u} \cdot \frac{u^{n^{1/\theta}}}{1-u^{n^{1/\theta}}} = \frac{n^{1/\theta}(1-u)}{u} \cdot \left(\frac{1}{1-u^{n^{1/\theta}}} - 1 \right).$$

Taking the derivative of η, we obtain

$$\eta'(u) \overset{sign}{=} u^{n^{1/\theta}} + (1-u)n^{1/\theta} - 1 := s_1(u).$$

Observe that $s_1(0) = n^{1/\theta} - 1 > 0$ for $n > 1$ and $s_1(1) = 0$. Now, $s_1'(u) \overset{sign}{=} u^{n^{1/\theta}-1} - 1 < 0$, i.e., the function s_1 is decreasing and therefore $s_1(u) > 0$ for $u \in [0,1]$. In other words, η is strictly increasing for $u \in (0,1)$ and $\theta \geq 1$, which implies that $h_{n:n}(T)$ is strictly increasing.

Secondly, for the Clayton family of copulas, differentiating the function defined in (5.5) with respect to u, we have

$$\begin{aligned} \eta'(u) &\overset{sign}{=} n - nu\left(nu^{-\theta} - n + 1\right)^{1/\theta} + u^\theta(n-1)(1+\theta(1-u)) \\ &\quad \times \left(\left(nu^{-\theta} - n + 1\right)^{1/\theta} - 1\right) := s_2(u). \end{aligned}$$

Note that $s_2(0) = n$ and $s_2(1) = 0$. Taking the derivative of s_2, we get

$$s_2' \overset{sign}{=} n(\theta-1)\left(nu^{-\theta} - n + 1\right)^{1/\theta} - n\theta - \theta u^\theta(n-1)\left(\left(nu^{-\theta} - n + 1\right)^{1/\theta} - 1\right) < 0,$$

when $\theta \leq 1$. Then, s_2 is a decreasing function and therefore $s_2 > 0$. Hence, η is strictly increasing for $u \in (0,1)$ and $0 < \theta \leq 1$, which implies that $h_{n:n}(T)$ is strictly increasing.

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