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REMARKS ON ORNSTEIN'S NON-INEQUALITY IN $\mathbb{R}^{2\times 2}$

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Abstract

We give a very concise proof of Ornstein's L^1 non-inequality for first- and second-order operators in two dimensions. The proof just needs a two-dimensional laminate supported on three points.

Given two linear constant-coefficient homogeneous kth order differential operators $\mathcal{P}_1$, $\mathcal{P}_2$ and a number $1 \leq p \leq \infty$, consider the inequality

$$\|\mathcal{P}_1\varphi\|_{L^p(\mathbb{R}^n)} \leq C_p\|\mathcal{P}_2\varphi\|_{L^p(\mathbb{R}^n)}, \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m), \tag{0.1}$$

where $0 < C_p$ is some constant. When does such an estimate hold?

The case $1 < p < \infty$ is very classical: if we take $\mathcal{P}_1 = \mathrm{D}^k$ to be the kth order gradient then (0.1) holds if and only if $\mathcal{P}_2$ is an elliptic operator (in the sense that it has injective symbol); this is a classical result which goes back to the work of Calderón and Zygmund [**4**]. We refer the reader to [**13**] for a generalization of (0.1) which allows for operators with non-trivial kernels.

At the end-points $p = 1$ or $p = \infty$, (0.1) never holds, except in trivial situations: this was proved, respectively, by Ornstein [**23**] and Mityagin [**20**], but see also [**9**] for the $p = \infty$ case. In some circumstances one can deduce the result for $p = \infty$ from the one for $p = 1$, see for instance [**3**, **19**] for the case $\mathcal{P}_2 = \operatorname{div}$, and in fact the result for $p = 1$ is much more difficult. Similar results also hold in the anisotropic setting, see [**14**, **24**].

The failure of (0.1) when $p = 1$ can also be deduced from the Kirchheim–Kristensen convexity theorem [**15**, **16**]. Besides providing a concise proof of Ornstein's result, their theorem also has applications to the regularity of hessians of rank-one convex functions and to the characterization of gradient Young measures [**17**, **18**]. Both the failure of (0.1) when $p = 1$, $\mathcal{P}_1 = \mathrm{D}, \mathcal{P}_2 = \mathcal{E}$, as well as the existence of rank-one convex functions on three-dimensional spaces with irregular hessians, were proved by the first author and collaborators in [**5**, **6**] through constructions with unbounded laminates (the so-called staircase laminates) introduced in [**10**, **11**]. See also [**1**] and [**12**, **22**] for related problems for $p > 1$. Such laminates can be used to provide a fairly explicit deformation showing

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Ornstein's non-inequality, but their construction is somewhat complicated as it includes an infinite process.

The purpose of this note is to give an alternative proof of Ornstein's result when $n = m = 2$ and $k = 1$. These assumptions encompass the case where $\mathcal{P}_1 = \mathrm{D}$ and $\mathcal{P}_2$ is any of the operators

$$(\mathrm{div}, \mathrm{curl}), \qquad \mathcal{E}u \equiv \frac{1}{2}(\mathrm{D}u + (\mathrm{D}u)^{\mathrm{T}}), \qquad \bar{\partial}u \equiv \mathcal{E}u - \frac{\mathrm{div}\, u}{2}\mathrm{Id}_2,$$

which appear respectively in electromagnetism, linearized elasticity and complex analysis. Our strategy is similar to the one of Kirchheim–Kristensen [15, 16] and in fact we prove a particular case of their convexity theorem. However, our approach is less elaborate, as the singular value decomposition reduces the problem to building a laminate on the diagonal matrices and, since the integrand of interest is 1-homogeneous, the laminate is very simple. An interesting question that we do not address here is whether, at least for integrands with symmetries, there exists a Kirchheim–Kristensen theory for $p > 1$.

For a bounded open set $\Omega \subset \mathbb{R}^n$ and a function $f \colon \mathbb{R}^{m\times n} \to \mathbb{R}$, we recall that:

(a) f is *quasiconvex* at $A \in \mathbb{R}^{m\times n}$ if $0 \leq \int_\Omega f(A + \mathrm{D}\varphi) - f(A)\,\mathrm{d}x$ for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^m)$;
(b) f is *rank-one convex* if $t \mapsto f(A + tX)$ is convex for all $A, X \in \mathbb{R}^{m\times n}$ with $\mathrm{rank}\, X = 1$;
(c) f is *positively 1-homogeneous* if $f(tA) = tf(A)$ for all $t > 0$ and all $A \in \mathbb{R}^{m\times n}$;
(d) f is *1-homogeneous* if $f(tA) = |t| f(A)$ for all $t \in \mathbb{R}$ and all $A \in \mathbb{R}^{m\times n}$.

It is well known that the definition in (a) is independent of Ω and that (a) $\Rightarrow$ (b), see [7], although in general the converse is not true [27]. In [16], the following theorem was proved:

THEOREM 0.1 *(Kirchheim–Kristensen) Let $f \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be positively 1-homogeneous and rank-one convex. Then f is convex at all matrices X with* $\mathrm{rank}\, X \leq 1$.

The reader may find other results concerning positively *1*-homogeneous rank-one convex functions in [8, 21, 25]. In the planar case, there is a particularly simple proof of Theorem 0.1 for 1-homogeneous functions:

LEMMA 0.2 *Let $f \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ be 1-homogeneous and rank-one convex. Then $f \geq 0$ and moreover, as $f(0) = 0$, f is convex at zero.*

Proof. For $A \in \mathbb{R}^{2\times 2}$ the singular value decomposition yields $Q, R \in \mathrm{O}(2)$ and $\Lambda \in \mathbb{R}^{2\times 2}_{\mathrm{diag}}$ such that $A = Q\Lambda R$; moreover, the entries of Λ are non-negative. Let us write $(x, y) \equiv diag(x, y)$. If $0 \neq A$ then, by homogeneity, we can assume that $\Lambda = (1, y)$, where $y \geq 0$. The measure

$$\nu = \frac{1}{2}\delta_{Q(2,-2y)R} + \frac{1}{3}\delta_{Q(1,y)R} + \frac{1}{6}\delta_{Q(-2,-2y)R}$$

is a laminate with barycentre $Q(1,-y)R$. Indeed, we have the splittings

$$(1, -y) \to \frac{1}{3}(1, y) + \frac{2}{3}(1, -2y) \to \frac{1}{3}(1, y) + \frac{1}{6}(-2, -2y) + \frac{1}{2}(2, -2y)$$

and the map $A \mapsto QAR$ is rank-preserving. Since f is rank-one convex and 1-homogeneous,

$$\begin{aligned} f(Q(1,-y)R) &\leq \frac{1}{2}f(Q(2,-2y)R) + \frac{1}{3}f(Q(1,y)R) + \frac{1}{6}f(Q(-2,-2y)R) \\ &= f(Q(1,-y)R) + \frac{1}{3}f(A) + \frac{1}{3}f(-A). \end{aligned}$$

Hence $0 \leq f(A) + f(-A) = 2f(A)$ and the proof is finished. □

REMARK 0.3 An identical proof gives the same conclusion if $f\colon \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$; in this case one takes $R = Q^{\mathrm{T}}$, since symmetric matrices are diagonalizable by orthogonal matrices.

From Lemma 0.2, we get a two-dimensional version of Ornstein's non-inequality:

THEOREM 0.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $\mathcal{P}_i$ be first-order differential operators, $i = 1, 2$, acting on $\varphi \in C_c^\infty(\Omega, \mathbb{R}^2)$ by $\mathcal{P}_i\varphi = P_i(D\varphi)$, where $P_i \in Lin(\mathbb{R}^{2\times 2}, \mathbb{R}^{d_i})$.*
Suppose that there is a constant C such that

$$\|\mathcal{P}_1\varphi\|_{L^1} \leq C\|\mathcal{P}_2\varphi\|_{L^1} \qquad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^2). \tag{0.2}$$

Then there is $T \in Lin(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ such that $P_1 = T \circ P_2$. Moreover, the same conclusion is true if we require that (0.2) holds only for those φ of the form $\varphi = \nabla\phi$ for some $\phi \in C_c^\infty(\Omega, \mathbb{R})$.

Proof. Consider the function $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $f(A) = C\|P_2A\| - \|P_1A\|$. Its quasiconvex envelope $f^{\mathrm{qc}}\colon \mathbb{R}^{2\times 2} \to [-\infty, \infty)$ is given by the Dacorogna formula

$$f^{\mathrm{qc}}(A) = \inf_{\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)} \int_{\mathbb{R}^2} f(A + \mathrm{D}\varphi)\,\mathrm{d}x;$$

it is easily checked that f^{qc} is 1-homogeneous, since the same holds for f. Note that (0.2) is equivalent to $f^{\mathrm{qc}}(0) \geq 0$; thus $f^{\mathrm{qc}} > -\infty$ everywhere and hence f^{qc} is rank-one convex. Applying Lemma 0.2 we see that $0 \leq f^{\mathrm{qc}} \leq f$ and so we must have $\ker P_2 \subseteq \ker P_1$. Take $T = P_1P_2^\dagger$, where $P_2^\dagger$ is the *Moore–Penrose inverse*, defined by

$$P_2^\dagger \equiv \left(P_2|_{(\ker P_2)^\perp}\right)^{-1} \mathrm{Proj}_{\mathrm{im}\, P_2}.$$

Since $P_2^\dagger P_2$ is the orthogonal projection onto $(\ker P_2)^\perp$, the conclusion follows.

The last part is identical, except that we replace Lemma 0.2 with Remark 0.3: if (0.2) holds for all potential vector fields then $(f|_{\mathbb{R}^{2\times 2}_{\mathrm{sym}}})^{\mathrm{qc}}(0) \geq 0$, see [2, 26] for quasiconvexity on $\mathbb{R}^{n\times n}_{\mathrm{sym}}$. □

In particular, from the second part of Theorem 0.4 we recover [23, Part 1]:

COROLLARY 0.5 *Given a bounded open set $\Omega \subset \mathbb{R}^2$, there is no constant C such that*

$$\int_\Omega |\partial_{x_1x_2}\phi(x)|\,dx \leq C\int_\Omega |\partial_{x_1x_1}\phi(x)| + |\partial_{x_2x_2}\phi(x)|\,dx \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

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