Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid

Geometric Methods in Classical Field Theory and Continuous Media

Cédric M. Campos

Tesis doctoral dirigida por Manuel de León Rodríguez y David Martín de Diego

Madrid, Julio 2010

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Métodos Geométricos en Teoría Clásica de Campos y Medios Continuos

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Prólogo

Este trabajo recopila la investigación desarrollada y los resultados obtenidos durante mis cuatro años como becario predoctoral en el Instituto de Ciencias Matemáticas (y anteriormente a su formación, en el Instituto de Matemáticas y Física Funtamental). Su defensa tendrá lugar en la Universidad Autónoma de Madrid con objetivo de obtener el grado de Doctor en Matemáticas. La dirección ha sido llevada a cabo por Manuel de León Rodríguez, Profesor de Investigación y Director del Instituto de Ciencias Matemáticas, y David Martín de Diego, Investigador Científico de la misma institución. En cuanto a tutor he contado con Rafael Orive Illera, Profesor Titular de la Universidad Autónoma de Madrid. De la tarea de lector se ha encargado Marco Castrillón López, Profesor Titular de la Universidad Complutense de Madrid.

La monografía versa sobre teoría clásica de campos de orden superior. El lector podrá encontrar en sus capítulos iniciales una revisión de algunos de los hechos conocidos en mecánica clásica y teoría clásica de campos (de primer orden). En los capítulos finales, se expone la parte original de la memoria con la extensión de estas teorías a campos clásicos de orden superior, centrándose en la problemática de un formalismo canónico hamiltoniano. Algunos ejemplos son propuestos con el fin de facilitar la comprensión y análisis de los resultados obtenidos.

Se ha pretendido dar una organización gradual y un tratamiento unificado de la materia de tal manera que pueda ser usada en posibles desarrollos futuros.

Cédric M. Campos
Madrid, 27 de julio de 2010

Agradecimientos

Hace ya varios años, cuando decidí empezar los estudios de doctorado, ya no con el fin de obtener el grado de doctor, sino más bien con el simple objetivo de continuar lo que había iniciado durante mis estudios de licenciatura, no habría podido imaginar los derroteros que tomaría mi vida ni mi carrera académica. No deja de ser un hecho natural e inherente al devenir del tiempo, pero aún así me sorprende. Después de recorrer este camino, me doy cuenta de que una tesis doctoral supone mucho más que la acumulación de conocimientos y el desarrollo intelectual que se le asocia, lo que de por sí no es poco. Al llegar al final de esta etapa de mi vida, veo como he crecido a nivel personal y profesional, emocional e intelectual. Todo ello no ha sido gratuito, sino gracias a la contribución de las muchas personas con las que me he encontrado a lo largo de este tiempo. Como digo, han sido numerosas y sus aportaciones cuantiosas a la par que diversas, desde señalarme un detalle clave en un problema teórico, a ofrecerme apoyo en una situación comprometida. A todas ellas les estoy enormemente agradecido y, sin lugar a dudas, las tendré presentes siempre que recorra las lineas de estas páginas. Por ello, les debo al menos dejar aquí escrito sus nombres. Quisiera poder dar gracias personalmente a cada una de ellas, recordar cada cara y cada instante, pero temo fallar. De todas formas, voy a enfrentarme a tan ardua tarea.

En primer lugar, quiero agradecer a quiénes han estado conmigo en el día a día, alimentando mi mente con conversaciones científicas, políticas, filosóficas o religiosas, siempre acompañadas de buen humor. Ha sido un placer y un honor compartir mis horas de trabajo con todos ellos, pues he disfrutado enormemente al estar rodeado de un grupo que considero parte de la élite científica española, a la cual desde luego no pertenezco. Sin duda alguna, echaré de menos el reunirnos a comer en la cafetería del CSIC. De entre ellos, quisiera destacar a Manuel y David, mis directores de tesis. Sin ellos, todo esto no sería posible. Me dieron la oportunidad (y no sólo una) de embarcarme en este proyecto, me dieron problemas a los que enfrentarme (en más de un sentido) y medios para resolverlos. Se han convertido en más que mentores para mí. También un espacial agradecimiento a mi compañero de despacho y gran amigo, Ángel. Su rapidez de respuesta nunca dejará de sorprenderme, pese a que desgasta la partícula negativa. Ojalá su fútbol tuviese la misma velocidad. Además, estuvo conmigo en los buenos momentos, en los malos y los peores. Por todo esto, siempre tendrá mi reconocimiento. Tampoco quiero olvidarme de los demás miembros de la plantilla, todos ellos con algo realmente especial. Gracias a Ana (y sus tapers), Ana María, Carolina (enemiga de las impresoras), Dan (por su introducción a la cultural norte-americana y su “fe” en la selección), David (aka RyC One, mi discípulo ubuntero y ex-compañero de habitación), Diego (por los chicles), Emilio (con bici, pero sin perro ni flauta), Fernando (actual compañero de habitación hasta que la Red o una Ramón y Cajal nos separe), José Antonio (único linuxero convencido, pero estancado en Debian), Luis, María (mi amor en el despacho), Mario (gran escalador), Marta (no tan siniestra), Paco, Paz (con su eterna aura primaveral que nos aportaba juventud a todos) y Roberto (artista redomado, pozo de conocimiento, compañero de viajes a Valencia). No me olvido de los que cruzaron el charco y estuvieron aquí, gracias a Poi (por tantas cosas, pero especialmente por los paseos a Malasaña o al centro después de trabajar y las salidas al Yesterday los fines de semana), a Sebastián y Cecilia (por las "argentinadas") y Viviana (por ese humor negro que tanto la hace brillar). Aprovecho este párrafo para agradecer por su trabajo, por un lado, a mis colaboradores Marcelo y Joris y, por otro lado, a Marco, que ha dedicado su tiempo en leer esta tesis, y a Rafa, por acceder a ser mi tutor.

Mi trabajo no sólo se ha visto beneficiado de mi labor en el CSIC, sino también de los congresos a los que he asistido y de las personas con las que allí me he encontrado. Esto ha sido, sin lugar a dudas, gracias a la Red de Geometría, Mecánica y Control y sus miembros. Mi trabajo se basa en el de muchos de ellos y no podían faltar aquí sus nombres. Difícil sería explayarme aquí individualmente y con detalle, pues la virtud de este grupo nace de su unión e intercambio de ideas. Mi agradecimiento a todos ellos por su recibimiento, los buenos consejos y su experiencia, me han sido de gran ayuda. Gracias a Edith, Juan Carlos, David, Diana, Elisa, Miguel C., Narciso, Xavi, María, Eduardo, Jesús, Pepín, Andreu, Javi, Modesto, Silvia, Luis y Miguel.

En ocasiones, las largas horas de trabajo mental se hacían pesadas y me era necesario liberar la energía acumulada (además de mis frustraciones doctorales). Esto lo hacía en los intensos partidos de fútbol sala de los jueves jugados en el pabellón deportivo de la Autónoma. Así que agradezco a Diego el haberme invitado insistentemente a dichos eventos y felicitarle por su más que buen hacer en el terreno de juego. A todos los demás, compañeros y contrincantes, los buenos ratos que he pasado allí. Los golpes han sido duros, pero los goles marcados bien valían la pena. Gracias a Ángel (por cansar al adversario chupándo bola), Dani (subido de Cola-Cao), Eugenio (porterazo), Javi (los dos, a uno por los pases, al otro por la rivalidad), Jorge (siempre tirando de su equipo), Pablo (un maestro), Rafa (siempre decidido), Raúl (los dos), Rubén (generoso), Sasha (aka la Rubia, por el cuerpo a cuerpo), Samuel (lleno de energía), Saúl (por esas patadas) y Wagner. Me atrevo a decir que, pese a mi nula aptitud para este deporte, algo he aprendido y, sin duda, ha sido por ellos.

Además de haberme nutrido intelectualmente en el ámbito laboral, también lo he hecho en mi "propia" casa. Compartir piso tiene, según muchos, numerosos inconvenientes, pero son escasos ante todas sus ventajas, ante la experiencia de convivir y compartir. Pero ello apenas es una pequeña parte de lo que he recibido. Mis compañeros (y allegados) han sido una fuente de riqueza interior, diversidad cultural, confianza, desahogo, risas y diversión. Me siento más que afortunado por haber disfrutado de todo esto día tras día, de las charlas en la cena, las películas en "familia", las salidas en grupo, etc. De todos los compañeros que han pasado por Mariano de Cavia, destaco mi especial afecto y agradecimiento a David (aka zupermercadona), Julie (e Isabella), Mariana y Enrico (y Sara). Tampoco me quiero olvidar de los demás: Marie, Sofie, Tom, Andrew, Chris, Will, Jeff, Mariana (Ángela, Carlos, Diana, Mara y Marcela), Sergio, Viviana, Juanpi y Sofi y Zack (y Virginia).

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Cuatro años son tiempo y, como dicen, *things happen* (ocurren cosas), algunas no muy buenas. Afortunadamente uno encuentra buenos profesionales para lidiar con estas. Por un lado, mi agradecimiento al equipo de fisioterapeutas del Hospital Gregorio Marañón. Gracias a Miriam por recuperar mis manos y a sus compañeros (y demás pacientes) Gema, José Luis, Cristina, María José y Rubén, por hacerme más llevadera la rehabilitación. Por otro lado, gracias a Carmen por enseñarme a andar, por su profesionalidad y ayuda.

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Generamos conocimiento.
Dr. Ángel Castro

Introducción

Una teoría de campos es una teoría física que describe como uno o más campos físicos interactúan con la materia. Un campo físico puede ser entendido como una asignación continua de una magnitud física en cada punto del espacio y el tiempo: por ejemplo, la velocidad de un fluido, el electromagnetismo o incluso la gravedad. Estos son ejemplos de campos macroscópicos o "clásicos" en contraste a los microscópicos o "cuánticos". Nos centraremos en los primeros. En cierto sentido, la teoría clásica de campos es una generalización de la mecánica clásica, en la cual el único campo es la linea temporal.

Desde un punto de vista matemático, los campos clásicos pueden ser descritos como secciones ϕ de un fibrado $\pi : E \to M$. El marco se completa introduciendo una función que abarca la dinámica del sistema físico, la función lagrangiana. Para una teoría clásica de campos, esto es una función $L : J^1\pi \to \mathbb{R}$, donde $J^1\pi$ es el fibrado de jets de orden uno de π. Este fibrado de jets ofrece una descripción geométrica de las derivadas parciales de las coordenadas fibradas de E con respecto a las de M, donde una sección es fijada. Buscamos pues aquellas secciones ϕ del fibrado $\pi : E \to M$, los campos, que extremizan el funcional

$$\mathcal{A}_L(\phi, R) = \int_R L(j^1\phi)\eta,$$

donde η es una forma de volumen prefijada (se da por supuesto que M es orientable), $R \subseteq M$ es una región compacta de M y $j^1\phi$ es la primera prolongación jet de ϕ.

Uno de los resultados más básicos del cálculo variacional es la construcción a partir del funcional anterior de un conjunto de ecuaciones en derivadas parciales, las ecuaciones de Euler-Lagrange

$$\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha} = 0,$$

las cuales deben ser satisfechas por cualquier extremal suave. Más interesante, la propiedad de extremización del problema no depende de la elección particular del sistema de coordenadas (hecho que notó J. L. Lagrange durante sus estudios de mecánica analítica), por tanto debe ser posible escribir las ecuaciones de Euler-Lagrange de forma intrínseca.

La interpretación geométrica de las ecuaciones de Euler-Lagrange se realiza por medio de la así llamada forma de Poincaré-Cartan Ω_L. Esta forma está construida usando la geometría del fibrado de jets y también está relacionada con el trasfondo variacional [97]. Usando esta forma, es posible escribir la ecuaciones de Euler-Lagrange de forma intrínseca. Es más, ϕ satisface las ecuaciones de Euler-Lagrange (es decir, es un punto crítico de la acción $\mathcal{A}_L$) si y sólo si

$$(j^1\phi)^*(i_V\Omega_L) = 0, \quad \text{para todos los vectores tangentes } V \text{ en } TJ^1\pi.$$

Además, esta forma juega un papel importante en la conexión entre las simetrías y las leyes de conservación (see [53]).

Otra manera de describir la evolución de los campos es introduciendo una función dinámica el dual a $J^1\pi$, esto es, una función hamiltoniana $H : J^1\pi^\dagger \to \mathbb{R}$, donde $J^1\pi^\dagger$ es el dual extendido del fibrado de jets de primer orden de π. Entonces, la dinámica del sistema viene descrita gracias a las soluciones de las bien conocidas ecuaciones de Hamilton

$$\frac{\partial H}{\partial u^\alpha} = -\frac{\partial p^i_\alpha}{\partial x^i} \quad \text{y} \quad \frac{\partial H}{\partial p^i_\alpha} = \frac{\partial u^\alpha}{\partial x^i},$$

las cuales son extremales para el principio variacional dado en $J^1\pi^\dagger$ (véanse [31, 61, 115, 134, 147]).

La relación entre estos dos marcos, el formalismo lagrangiano y el hamiltoniano, es descubierto por la transformada de Legendre. Dado un lagrangiano $L : J^1\pi \to \mathbb{R}$, podemos definir el mapa $\mathrm{Leg}_L : J^1\pi \to J^1\pi^\dagger$. Esta función tiene interesantes propiedades como enviar las soluciones de las ecuaciones de Euler-Lagrange a soluciones de las ecuaciones de Hamilton, o bien retrotraer la $(m+1)$-forma de Cartan Ω_H de $J^1\pi^\dagger$ a la $(m+1)$-forma de Poincaré-Cartan $J^1\pi$ (véanse [35, 61, 134, 147]). Es más, cuando L es regular, esto es cuando su "hessiano"

$$\left(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}\right)$$

es regular, la transformada de Legendre Leg_L es un difeomorfismo local en su imagen, la cual es a su vez difeomorfa al dual reducido del fibrado de jets de primer orden, $J^1\pi^\circ$.

En la actualidad, se posee una muy buena comprensión de las teorías de campos de primer orden. Pero muchos lagrangianos que aparecen las teorías de campos son de orden superior (como por ejemplo en elasticidad o gravitación), por tanto es de sumo interés encontrar un marco completamente geométrico también para estas teorías de campos, esto cuando uno considera una función lagrangiana $L : J^k\pi \to \mathbb{R}$, donde $J^k\pi$ es el fibrado de jets de orden k de π. Durante las últimas décadas del pasado siglo, han habido diferentes estudios e intentos para definir de manera global e intrínseca el cálculo variacional de orden superior en varias variables. Los objetivos principales son describir las ecuaciones de Euler-Lagrange asociadas para secciones del fibrado, derivar las formas de Poincaré-Cartan como versión intrínseca las ecuaciones anteriores, y construir transformadas de Legendre adecuadas que nos permitan escribir estas ecuaciones en el marco hamiltoniano (véanse, por ejemplo, [4, 6, 65, 66, 89, 94, 106, 113, 63, 62, 140] para más información).

El marco geométrico estándar de la teoría de campos de orden superior se inicia con la búsqueda de extremales del funcional

$$\mathcal{A}_L(\phi, R) = \int_R L(j^k\phi)\eta,$$

donde como antes η es una forma de volumen prefijada, $R \subseteq M$ es una región compacta y $j^k\phi$ es la prolongación k-jet de ϕ. El calculo variacional establece que los extremales de esta acción integral deben satisfacer las ecuaciones de Euler-Lagrange de orden superior

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u^\alpha_J} = \frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u^\alpha_i} + \frac{\mathrm{d}^2}{\mathrm{d}x^{ij}} \frac{\partial L}{\partial u^\alpha_{1_i+1_j}} - \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^J} \frac{\partial L}{\partial u^\alpha_J} = 0,$$

las cuales son un conjunto de ecuaciones en derivadas parciales en $J^{2k}\pi$. Al igual que en el caso de primer orden, estas ecuaciones no dependen de la elección de coordenadas.

Por tanto, uno puede llegar a preguntarse si existe un objeto canónico que describa geométricamente este conjunto de ecuaciones y sus soluciones. En tal caso, debería de ser una forma de Poincaré-Cartan de orden superior.

La situación está bien establecida para el caso de una variable dependiente (mecánica de orden superior) y para el caso de primer orden [85, 91, 93, 105]. En este último caso, la expresión típica de la forma de poincaré-Cartan asociado en mecánica clásica a un lagrangiano $L : J^1\pi \to \mathbb{R}$ puede ser escrita como $\mathcal{S}^*(dL) + Ldt$, donde $\mathcal{S}^*$ es el adjunto del endomorfismo vertical actuando sobre 1-formas. Con el objetivo de generalizar este concepto a teorías de campos de orden superior, uno necesita definir una aplicación de las 1-formas (la diferencial de L) a m-formas e incorporar de manera global las derivadas de orden superior. Esta es una de las razones para el grado de arbitrariedad en la definición de la forma de Cartan para funciones lagrangianas $L : J^k\pi \to \mathbb{R}$, con $k > 1$ y $\dim M > 1$. En otras palabras, habrán diferentes formas de Cartan definidas a partir de la misma función que definan una formulación intrínseca de las ecuaciones de Euler-Lagrange. La razón principal de este problema es la conmutatividad de las derivadas parciales iteradas. Por tanto, la forma de Cartan es única si (y sólo si) bien k o bien m es igual a uno.

En la literatura, encontramos diferentes aproximaciones para fijar la forma de Cartan en teorías de campos de orden superior. Un trato directo es la aproximación de Aldaya y Azcárraga [4, 6]. Otro punto de vista es el de Arens [8], que consiste en inyectar el fibrado de jets $J^k\pi$ en un fibrado de jets de orden 1 apropiado, con la introducción de numerosas variables dentro de una teoría de multiplicadores de Lagrange. Desde un punto de vista más geométrico, García y Muñoz describen un método global para la construcción de formas de Poincaré-Cartan en el calculo de variaciones de orden superior de espacios fibrados por medio de conexiones (see [88, 89]). En particular, construyen formas de Cartan que dependen de la elección de dos conexiones, una conexión lineal en la base y otra conexión lineal en el fibrado vertical $\mathcal{V}\pi$. Más tarde, Crampin y Saunders [140] proponen el uso de operadores análogos a la estructura casi tangente canónicamente definida en el fibrado tangente de una variedad de configuración dada M para la construcción global de formas de Poincaré-Cartan; este operador depende de la forma de volumen elegida en la base.

En esta monografía, propones un camino alternativo, evitando el uso de estructuras adicionales y trabajando únicamente con objetos intrínsecos del lado lagrangiano y del hamiltoniano. Los resultados pueden encontrase publicados en [24, 25, 26, 27] (para un punto de vista complementario, se sugiere el trabajo de Vitagliano [152]). Con vistas a tratar sistemas lagrangianos singulares, Skinner y Rusk construyen un sistema hamiltoniano en la suma de Whitney $TQ \oplus T^*Q$ de los fibrados tangente y cotangente de una variedad de configuración Q. La ventaja de su acercamiento yace en el hecho de que la condición de segundo orden de la dinámica es satisfecha automáticamente. Esto no ocurre en el lado lagrangiano de la formulación de Gotay y Nester, donde la condition de segundo orden debe de ser considerada tras la implementación del algoritmo de ligaduras (ver [100, 101, 102]), aunque otros formalismos incluyen esta condición de segundo orden desde el principio (ver [34, 36]).

En teorías de campos de orden superior, empezamos con un lagrangiano definido en $J^k\pi$. Consideramos el fibrado $\pi_{W,M} : W \to M$, donde $W = J^k\pi \times_{J^{k-1}\pi} \Lambda_2^m(J^{k-1}\pi)$ es un producto fibrado, el espacio de velocidades y momentos. En W construímos una forma premultisimpléctica haciendo el "pull back" de la forma multisimpléctica canónica de $\Lambda_2^m(J^{k-1}\pi)$, y definimos un formalismo hamiltoniano conveniente gracias al pairing natural canónico y la función lagrangiana dada. Las soluciones de la ecuación de campos son entendidas como secciones integrales de conexiones de Ehresmann en el fibrado $\pi_{W,M} : W \to M$. En este espacio, obtenemos una expresión global, intrínseca y única de una ecuación tipo Cartan para las ecuaciones de Euler-Lagrange para teorías de campos de orden superior. Adicionalmente, obtenemos algoritmo de ligaduras. Nuestro esquema es aplicado a diferentes ejemplos para ilustrar el método.

A parte de la carencia de ambigüedad inherente a nuestra construcción, vale la pena enfatizar que este formalismo se puede extender fácilmente a teorías de campos de orden superior con restricciones o problemas de control óptimo en ecuaciones en derivadas parciales. En este sentido, obtenemos una descripción unificadora y geométrica de ambos tipos de sistemas, con posibles aplicaciones futuras a teorías de reducción por simetrías y la construcción de métodos numéricos que preserven la estructura geométrica (ver [116]). Por tanto, introducimos restricciones en el marco de trabajo, las cuales están representadas geométricamente como una subvariedad $\mathcal{C}$ de $J^k\pi$. En otras palabras, imponemos restricciones en el espacio de secciones donde la acción está definida. El formalismo introducido en [27] es adaptado al caso de teorías de campos restringidas, derivando así un marco intrínseco de las ecuaciones de Euler-Lagrange restringidas. Para la descripción geométrica, inducimos una subvariedad $W_0^{\mathcal{C}}$ de W usando las restricciones dadas por $\mathcal{C}$. Algunos ejemplos son dados para ilustrar la teoría, la cual está recogida en [26].

El Capítulo §1 recopila la notación utilizada a lo largo de la monografía así como el fondo matemático necesario: distribuciones, las diferentes geometrías simplécticas, la estructura del fibrado tangente, etc. También contiene un esquema del algoritmo de Gotay-Nester-Hinds.

El Capítulo §2 es una somera revisión de la mecánica clásica. Describe los principales resultados de la teoría desde el lado lagrangiano y el hamiltoniano.

El Capítulo §3 es una breve introducción a la teoría clásica de campos. Desarrolla la teoría (generalmente sin pruebas) en los diferentes formalismos, el lagrangiano y el hamiltoniano, y los diferentes posibles acercamientos, el variacional y el geométrico. También muestra la relación entre ellas e introduce el formalismo de Skinner y Rusk para teorías de campos.

El Capítulo §4 está dedicado al estudio de la teoría clásica campos clásicos de orden superior. El lector podrá encontrar una primera generalización de los principales objetos geométricos de la teoría de primer orden, señalando las causas de la ambigüedad inherente a la teoría de orden superior. En adelante, el capítulo se centra en la resolución de esta ambigüedad por medio del formalismo de Skinner y Rusk. También introduce restricciones en el esquema. Finalmente, hay una presentación de algunos resultados parciales en la reducción de la arbitrariedad en el espacio de soluciones de la teoría.

Por último, el Capítulo §5 expone un resumen de los principales resultados obtenidos a lo largo de mis estudios, junto con algunas conclusiones y los trabajos futuro que se inician con este tratado.

Introduction

A field theory is a physical theory that describes how one or more physical fields interact with matter. A physical field can be thought of a continuous assignment of a physical quantity at each point of space and time: For instance, the velocity of a fluid, electromagnetism or even gravitation. These are macroscopic or “classical” field examples in contrast to “microscopic” or quantum ones. We will focus on the former. In some sense, classical field theory is a generalization of classical mechanics, in which the only field is the time line.

From the mathematical point of view, classical fields may be described by sections ϕ of a fiber bundle $\pi : E \to M$. The picture is completed by introducing a function that encompasses the dynamics of the physical system, the Lagrangian. For first order field theories, it is a function $L : J^1\pi \to \mathbb{R}$, where $J^1\pi$ is the first-jet bundle of π. This jet bundle gives a geometrical description of the partial derivatives of the fiber coordinates of E with respect to those of M, where a section is fixed. We then look for those sections ϕ of the fiber bundle $\pi : E \to M$, the fields, that extremize the functional

$$\mathcal{A}_L(\phi, R) = \int_R L(j^1\phi)\eta,$$

where η is a fixed volume form (it is assumed that M is orientable and oriented), $R \subseteq M$ is a compact region of M and $j^1\phi$ is the 1st-jet prolongation of ϕ.

The most basic result on variational calculus is the construction from the above functional of a set of partial differential equations, the Euler-Lagrange equations

$$\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha} = 0,$$

which must be satisfied by any smooth extremal. More interesting, the property of extremizing the problem does not depend on the particular chosen coordinate system (fact noted by J. L. Lagrange during his studies of analytical mechanics), therefore it must be able to write the Euler-Lagrange equations in an intrinsic way.

The geometric interpretation of the Euler-Lagrange equations is done by means of the so-called Poincaré-Cartan form Ω_L, which is an $(m+1)$-form $(\dim M = m)$ univocally associated to the Lagrangian. This form is constructed using the geometry of the jet bundle and it is also related with the variational background [97]. Using this form, it is possible to write down the Euler-Lagrange equations in an intrinsic way. Indeed, ϕ satisfies the Euler-Lagrange equations (that is, it is a critical point of the action $\mathcal{A}_L$) if and only if

$$(j^1\phi)^*(i_V\Omega_L) = 0, \quad \text{for all tangent vector } V \text{ in } TJ^1\pi.$$

Moreover, this form plays an important role in the connection between symmetries and conservation laws (see [53]).

Besides, another way to describe the evolution of the fields is by introducing a dynamical function in the dual side of $J^1\pi$, that is, by introducing the Hamiltonian $H : J^1\pi^\dagger \to \mathbb{R}$, where $J^1\pi^\dagger$ is the extended dual first-jet bundle of π. Then, the dynamics of the system is described by means of the solutions of the well known Hamilton's equations

$$\frac{\partial H}{\partial u^\alpha} = -\frac{\partial p^i_\alpha}{\partial x^i} \quad \text{and} \quad \frac{\partial H}{\partial p^i_\alpha} = \frac{\partial u^\alpha}{\partial x^i},$$

which are extremals of a variational principle given in $J^1\pi^\dagger$ (see [31, 61, 115, 134, 147]).

The relation between these two pictures, the Lagrangian and the Hamiltonian formalisms, is unveiled by the Legendre transformation. Given a Lagrangian $L : J^1\pi \to \mathbb{R}$, we may define a mapping $\mathrm{Leg}_L : J^1\pi \to J^1\pi^\dagger$. This function has interesting properties like it maps the solutions of the Euler-Lagrange equation to solutions of the Hamilton's equations or it pulls back the Cartan $(m+1)$-form Ω_H of $J^1\pi^\circ$ into the Poincaré-Cartan $(m+1)$-form of $J^1\pi$ (see [35, 61, 134, 147]). Moreover, when L is regular, that is when its "Hessian"

$$\left(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}\right)$$

is regular, the Legendre map Leg_L is a local diffeomorphism into its image, which is in its turn diffeomorphic to the reduced dual first-jet bundle $J^1\pi^\circ$.

So far, the first-order case of field theories is pretty well understood. But many of the Lagrangians which appear in field theories are of higher order (as for instance in elasticity or gravitation), therefore it is interesting to find a fully geometric setting also for these field theories, that is when one considers a Lagrangian function $L : J^k\pi \to \mathbb{R}$, where $J^k\pi$ is the kth-order jet bundle of π. During the last decades of the past century, there have been different studies and attempts to define in a global and intrinsic way the higher-order calculus of variations in several independent variables. The main objectives are to describe the associated Euler-Lagrange equations for sections of the fiber bundle, to derive Poincaré-Cartan forms for use in intrinsic versions of the above equations, and to construct adequate Legendre maps which permit to write the equations in the Hamiltonian side (see, for instance, [4, 6, 65, 66, 89, 94, 106, 113, 63, 62, 140] for further information).

The standard geometric framework of higher-order field theories starts by looking for the extremals of the functional

$$\mathcal{A}_L(\phi, R) = \int_R L(j^k\phi)\eta,$$

where as before η is a fixed volume form, $R \subseteq M$ is a compact region and $j^k\phi$ is the k-jet prolongation of ϕ. Variational calculus states that the extremizers of this integral action must satisfy the higher-order Euler-Lagrange equations

$$\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u^\alpha_J} = \frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u^\alpha_i} + \frac{\mathrm{d}^2}{\mathrm{d}x^{ij}} \frac{\partial L}{\partial u^\alpha_{ij}} - \cdots + (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^J} \frac{\partial L}{\partial u^\alpha_J} = 0,$$

which is a set of partial differential equations in $J^{2k}\pi$. As in the first order case, these equations do not depend on the chosen coordinates. Thus, one may wonder if there is a canonical object that describes geometrically this set of equations and their solutions. In such a case, it should be a higher-order Poincaré-Cartan form.

The situation is well established for the case of one independent variable (higher order mechanics) and for the case of first order calculus of variations [85, 91, 93, 105]. In this last situation, the typical expression of the Poincaré-Cartan form associated in classical mechanics to a Lagrangian $L : J^1\pi \to \mathbb{R}$ may be written as $\mathcal{S}^*(dL) + Ldt$, where $\mathcal{S}^*$ is the adjoint of the vertical endomorphism acting on 1-forms. In order to generalize this concept to higher order field theories, one needs to define a mapping from 1-forms (the differential of L) to m-forms and to incorporate in a global way the higher order derivatives. This is one of the reasons for the degree of arbitrariness in the definition of Cartan forms for Lagrangian functions $L : J^k\pi \to \mathbb{R}$, with $k > 1$ and $\dim M > 1$. In other words, there will be different Cartan forms which carry out the same function in order to define an intrinsic formulation of Euler-Lagrange equations. The main reason of this problem is the commutativity of repeated partial differentiation. Therefore, the Cartan form is unique if (and only if) either k or m equals one.

In the literature, we find different approaches to fix the Cartan form for higher order field theories. A direct attempt is the approach by Aldaya and Azcárraga [4, 6]. Another point of view is that by Arens [8], which consists of injecting the jet bundle $J^k\pi$ to an appropriate first-order jet bundle by the introduction of a great number of variables into the theory and Lagrange multipliers. From a more geometrical point of view, García and Muñoz described a method of constructing global Poincaré-Cartan forms in the higher order calculus of variations in fibered spaces by means of linear connections (see [88, 89]). In particular they show that the Cartan forms depend on the choice of two connections, a linear connection on the base M and a linear connection on the vertical bundle $\mathcal{V}\pi$. Later, Crampin and Saunders [140] proposed the use of an operator analogous to the almost tangent structure canonically defined on the tangent bundle of a given configuration manifold M for the construction of global Poincaré-Cartan forms; this operator depends on the chosen volume form on the base.

In this monograph, we propose an alternative way, avoiding the use of additional structures, working only with intrinsic objects from both the Lagrangian and Hamiltonian sides. The results main be found published here [24, 25, 26, 27] (for a complementary point of view, see the work by Vitagliano [152]). This formalism is strongly based on the one developed by Skinner and Rusk [142, 143, 144]. In order to deal with singular Lagrangian systems, Skinner and Rusk construct a Hamiltonian system on the Whitney sum $TQ \oplus T^*Q$ of the tangent and cotangent bundles of the configuration manifold Q. The advantage of their approach lies on the fact that the second order condition of the dynamics is automatically satisfied. This does not happen in the Lagrangian side of the Gotay and Nester formulation, where the second-order condition problem has to be considered after the implementation of the constraint algorithm (see [100, 101, 102]), besides other formalisms which include the second-order condition from the very beginning (see [34, 36]).

For higher-order field theories, we start with a Lagrangian function defined on $J^k\pi$. We consider the fibration $\pi_{W,M} : W \to M$, where $W = J^k\pi \times_{J^{k-1}\pi} \Lambda_2^m(J^{k-1}\pi)$ is a fibered product, the velocity-momentum space. On W we construct a premultisymplectic form by pulling back the canonical multisymplectic form of $\Lambda_2^m(J^{k-1}\pi)$, and we define a convenient Hamiltonian from a natural canonical pairing and the given Lagrangian function. The solutions of the field equations are viewed as integral sections of Ehresmann connections in the fibration $\pi_{W,M} : W \to M$. In this space we obtain a global, intrinsic and unique expression for a Cartan type equation for the Euler-Lagrange equations for higher-order field theories. Additionally, we obtain a resultant constraint algorithm. Our scheme is applied to several examples to illustrate our method.

Apart from the lack of ambiguity inherent in our construction, it is worth to emphasize that this formalism is easily extended to the case of higher-order field theories with constraints and optimal control problems for partial differential equations. In this way, we obtain a unified, geometric description of both types of systems, with possible future applications in the theory of symmetry reduction and the construction of numerical methods preserving geometric structure (see [116]). Therefore, we introduce constraints in the picture, which are geometrically defined as a submanifold $\mathcal{C}$ of $J^k\pi$. In other words, we impose the constraints on the space of sections where the action is defined. The formalism introduced in [27] is adapted to the case of constrained field theories, deriving an intrinsic framework of the constrained Euler-Lagrange equations. For the geometrical description, we induce a submanifold $W_0^{\mathcal{C}}$ of W using the constraints given by $\mathcal{C}$. Some examples are given to illustrate the theory, which appears in [26]

Chapter §1 gathers the notation used along the monograph and the basic mathematical background needed: distributions, the different symplectic geometries, the structure of the tangent bundle, etc. There is also a sketch of the Gotay-Nester-Hinds algorithm.

Chapter §2 is a short review of Classical Mechanics. It depicts the main results of the theory from the Lagrangian and the Hamiltonian side.

Chapter §3 is a brief introduction to Classical Field Theory. It develops the theory (generally without proofs) within the different formalisms, Lagrangian and Hamiltonian, and the different approaches, variational and geometrical. It also shows the relation between them and introduces the Skinner-Rusk formalism for field theories.

Chapter §4 is devoted to the study of Higher-Order Classical Field Theory. The reader will may find first a generalization of the main geometric objects of the first order theory, pointing out the causes of the ambiguity inherent to the higher-order theory. Then the chapter focuses on resolution of this ambiguity by means of the Skinner-Rusk formalism. It also introduces constraints in the pictures. Finally there is a presentation of the partial results on the reduction of the arbitrariness in the space of solutions of the theory.

Finally, Chapter §5 exposes a summary of the main results obtained along my studies, together with some conclusions and the future work that starts with this treatise.

Chapter 1

Mathematical background

1.1 Distributions and connections

See [1, 122] for an introduction to the theory of connections.

Definition 1.1. A *distribution* D of dimension m on a manifold P is an assignment to each $p \in P$ of a vector subspace $D(p) \subseteq T_pP$ of dimension m.

1. A distribution D of dimension m is *smooth* if, for each $p_0 \in P$, there exist an open neighborhood U_{p_0} of p_0 and local vector fields $Y_1, \ldots, Y_m \in \mathfrak{X}(U_{p_0})$, such that $Y_1(p), \ldots, Y_m(p)$ span $D(p)$ for every $p \in U_{p_0}$.

2. A submanifold $S \hookrightarrow P$ is said to be an *integral manifold* of a smooth distribution D in TP if $TS = D$ along the points of S. In such a case, D is said to be *integrable*.

3. A smooth distribution D is *involutive* if it is stable under the Lie bracket, that is, if $[D, D] \subseteq D$.

Theorem 1.2 (Frobenius' Theorem)**.** *A smooth distribution D is integrable if and only if it is involutive.*

Definition 1.3. A *connection* Γ in a fiber bundle $\pi_{P,M} : P \to M$ is given by a $\pi_{P,M}$-horizontal distribution H in TP, *i.e.* a distribution H in TP which is complementary to the vertical one $\mathcal{V}\pi_{P,M}$, that is

$$TP = D \oplus \mathcal{V}\pi_{P,M},$$

where $\mathcal{V}\pi_{P,M}(p) = \{V \in T_pP : T_p\pi_{P,M}(V) = 0\}$. This decomposition allow us to define:

1. The *horizontal projector* associated to the connection Γ is the linear map $\mathbf{h} : TP \to D$ defined in the obvious manner.

2. The *horizontal lift* of a tangent vector $X \in TM$ is the unique vector $X^h \in D$ that projects to X, $T\pi_{P,M}(X^h) = X$.

3. If (x^i, y^a) are fibered coordinates on P, then D is locally spanned by the local vector fields
$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} + \Gamma_i^a(x, y)\frac{\partial}{\partial y^a}.$$
The coefficients Γ_i^a are the *Christoffel symbols* of the connection.

Assume that $\pi_{Q,M} : Q \to M$ and $\pi_{P,M} : P \to M$ are two fibrations with the same base manifold M, and that $\Upsilon : Q \to P$ is a surjective submersion (in other words, a fibration as well) preserving the fibrations, say, $\pi_{P,M} \circ \Upsilon = \pi_{Q,M}$ (Diagram 1.1). Let Γ' be a connection in $\pi_{Q,M} : Q \to M$ with horizontal projector $\mathbf{h}'$.

$$\begin{tikzcd} Q \arrow[r, "\Upsilon"] \arrow[dr, "\pi_{Q,M}"'] & P \arrow[d, "\pi_{P,M}"] \\ & M \end{tikzcd}$$

Figure 1.1: Preserved fibration

Definition 1.4. Γ' is said to be *projectable* if the subspaces $(T_q\Upsilon)(D'(q))$ are constant along the fibers of Υ, *i.e.* $(T_{q_1}\Upsilon)(D'(q_1)) = (T_{q_2}\Upsilon)(D'(q_2))$ for every $q_1, q_2 \in \Upsilon^{-1}(p)$, $p \in P$.

If Γ' is projectable, then we define a connection Γ in the fibration $\pi_{P,M} : P \to M$ as follows: The horizontal subspace at $p \in P$ is given by

$$D_p = (T_q\Upsilon)(D'(q)),$$

for an arbitrary q in the fibre of Υ over p. It is routine to prove that D defines a horizontal distribution in the fibration $\pi_{P,M} : P \to M$.

We can choose fibered coordinates (x^i, y^a, z^α) on Q such that (x^i, y^a) are fibered coordinates on P. The Christoffel components of Γ' are obtained by computing the horizontal lift

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} + \Gamma_i^a(x,y,z)\frac{\partial}{\partial y^a} + \Gamma_i^\alpha(x,y,z)\frac{\partial}{\partial z^\alpha}\,.$$

A simple computation shows that Γ' is projectable if and only if the Christoffel components Γ_i^a are constant along the fibres of Υ, say $\Gamma_i^a = \Gamma_i^a(x,y)$. In this case, the horizontal lift of $\partial/\partial x^i$ with respect to Γ is just

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\partial}{\partial x^i} + \Gamma_i^a(x,y)\frac{\partial}{\partial y^a}\,.$$

Conversely, given a connection Γ in the fibration $\pi_{P,M} : P \to M$ and a surjective submersion $\Upsilon : Q \to P$ preserving the fibrations, one can construct connections Γ' in the fibration $\pi_{Q,M} : Q \to M$ which project onto Γ (first, locally and then globally by means of a partition of the unity).

The notion of connection in a fibration admits a useful generalization to submanifolds of the total space. Let $\pi_{P,M} : P \to M$ be a fibration and N a submanifold of P.

Definition 1.5. A *connection in $\pi_{P,M} : P \to M$ along the submanifold N of P* consists of a family of linear mappings

$$\mathbf{h}_p : T_pP \longrightarrow T_pN$$

for all $p \in N$, satisfying the following properties

$$\mathbf{h}_p^2 = \mathbf{h}_p, \quad \ker \mathbf{h}_p = \mathcal{V}_p\,\pi_{P,M},$$

for all $p \in N$. The connection is said to be *smooth (flat)* if the distribution $\operatorname{im}\mathbf{h} \subseteq TN$ is smooth (integrable).

We have the following.

Proposition 1.6. *Let* $\mathbf{h}$ *be a connection in* $\pi_{P,M} : P \to M$ *along a submanifold* N *of* P. *Then:*

1. $\pi_{P,M}(N)$ *is an open subset of* M.
2. $(\pi_{P,M})_{|N} : N \to \pi_{P,M}(N)$ *is a fibration.*
3. *There exists an induced true connection* Γ_N *in the fibration* $(\pi_{PM})_{|N} : N \to \pi_{PM}(N)$ *with the same horizontal subspaces.*
4. Γ_N *is flat if and only if* $\mathbf{h}$ *is flat.*

Proof. See [55, 50]. □

1.2 Multivectors

Definition 1.7. Let P be a n-dimensional differentiable manifold. Sections of $\Lambda^m(TP)$ (with $1 \leq m \leq n$) are called *m-multivector fields* in P. The set of m-multivector fields in P is denoted by $\mathfrak{X}^m(P)$.

Given $X \in \mathfrak{X}^m(P)$, for every $p \in P$, there exists an open neighborhood $U_p \subset P$ and $X_1, \ldots, X_r \in \mathfrak{X}(U_p)$ such that

$$X \underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} X_{i_1} \wedge \ldots \wedge X_{i_m}$$

with $f^{i_1 \ldots i_m} \in \mathcal{C}^\infty(U_p)$ and $m \leq r \leq n$. Of particular interest are those multivector fields whose decomposition may be reduced to a single term.

Definition 1.8. A multivector field $X \in \mathfrak{X}^m(P)$ is *locally decomposable* if, for every $p \in P$, there exists an open neighborhood $U_p \subset P$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that

$$X \underset{U_p}{=} X_1 \wedge \ldots \wedge X_m .$$

The set of locally decomposable m-multivector fields in P is denoted by $\mathfrak{X}^m_d(P)$.

Let $D \subseteq TP$ be an m-dimensional distribution. The sections of $\Lambda^m D$ are locally decomposable m-multivector fields in P.

Definition 1.9. A locally decomposable m-multivector field $X \in \mathfrak{X}^m_d(P)$ and an m-dimensional distribution $D \subseteq TP$ are *associated* whenever X is a section of $\Lambda^m D$.

If $X, X' \in \mathfrak{X}^m_d(P)$ are non-vanishing multivector fields associated with the same distribution D, then there exists a non-vanishing function $f \in \mathcal{C}^\infty(P)$ such that $X' = fX$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in P, whose equivalence classes will be denoted by $\mathcal{D}(X)$.

Theorem 1.10. *There is a bijective correspondence between the set of m-dimensional orientable distributions D in TP and the set of the equivalence classes $\mathcal{D}(X)$ of non-vanishing, locally decomposable m-multivector fields X in P.*

By abuse of notation, $\mathcal{D}(X)$ will also denote the m-dimensional orientable distribution D in TP with whom X is associated.

Definition 1.11. An m-dimensional submanifold $S \hookrightarrow P$ is said to be an *integral manifold* of $X \in \mathfrak{X}^m(P)$ if X spans $\Lambda^m TS$. In such a case, X is said to be *integrable*.

Note that integrable multivector fields are necessarily locally decomposable.

Definition 1.12. A non-vanishing, locally decomposable m-multivector $X \in \mathfrak{X}^m_d(P)$ is *involutive* if its associated distribution $\mathcal{D}(X)$ is involutive.

If a non-vanishing multivector field $X \in \mathfrak{X}^m_d(P)$ is involutive, so is every other in its equivalence class $\mathcal{D}(X)$. Furthermore, by Frobenius' theorem we have the following result.

Corollary 1.13. *A non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive.*

Definition 1.14. Let $\pi_{P,M} : P \to M$ be a fiber bundle with $\dim M = m$. A multivector field $X \in \mathfrak{X}^m(P)$ is said to be *π-transverse* if $\Lambda^m \pi_*(X)$ does not vanish at any point of M, hence M must be orientable.

Proposition 1.15. *If $X \in \mathfrak{X}^m(P)$ is integrable, then X is π-transverse if, and only if, its integral manifolds are sections of $\pi : P \to M$. In this case, if S is an integral manifold of X, then there exists a section $\phi \in \Gamma\pi$ shuch that $S = \mathrm{Im}(\phi)$.*

For more details on multivector fields and their relation with field theories, we refer to [72, 73].

1.3 The geometry of the tangent bundle

Through this section, Q denotes an n-dimensional smooth manifold. Local coordinates in Q are denoted (q^i), and the induced adapted coordinates of TQ and TTQ are denoted (q^i, v^i) and $(q^i, v^i, \dot{q}^i, \dot{v}^i)$, respectively. According to this, vectors $v \in T_qQ$ and $V \in T_v(TQ)$ are respectively of the form

$$v = v^i \frac{\partial}{\partial q^i}\Big|_q \text{ and } V = \dot{q}^i \frac{\partial}{\partial q^i}\Big|_v + \dot{v}^i \frac{\partial}{\partial v^i}\Big|_v.$$

If $\tau_Q : v \in T_qQ \mapsto q \in Q$ denotes the natural projection of TQ onto Q then, given a tangent vector $V \in T_v(TQ)$, we have that $\tau_{TQ}(V) = v$. Besides, we also have the following coordinate expressions (see Diagram 1.2)

$$\tau_Q(q^i, v^i) = (q^i), \;\; \tau_{TQ}(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i) \text{ and } T\tau_Q(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, \dot{q}^i).$$

Definition 1.16. Let $v \in T_qQ$ be a vector tangent to Q at some point $q \in Q$. The *vertical lift* of v at a "point" $w \in T_qQ$ is the tangent vector $v^{\mathrm{v}}_w \in T_w(TQ)$ given by

$$v^{\mathrm{v}}_w(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(w + tv)\big|_{t=0}, \;\; \forall f \in \mathcal{C}^\infty(T_qQ). \tag{1.1}$$

$$\begin{tikzcd} TTQ \arrow[r, "T\tau_Q"] \arrow[d, "\tau_{TQ}"] & TQ \arrow[d, "\tau_Q"] \\ TQ \arrow[r, "\tau_Q"] & Q \end{tikzcd}$$

Figure 1.2: The natural projections

Given a smooth function $f \in \mathcal{C}^\infty(Q)$,

$$\begin{aligned} (T_w\tau_Q)(v_w^{\mathrm{v}})(f) &= v_w^{\mathrm{v}}(f \circ \tau_Q) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \tau_Q)(w + tv)\big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(q)\big|_{t=0} \\ &= 0. \end{aligned}$$

Thus, the vertical lift takes values into the vertical fiber bundle $\mathcal{V}\tau_Q \subset TTQ$. Indeed, for each $w \in T_qQ$, the vertical lift at w,

$$(\cdot)_w^{\mathrm{v}} : T_qQ \longrightarrow \mathcal{V}_w\tau_Q \subset T_wTQ,$$

is a linear isomorphism. It may also be seen as a morphism $X \in \mathfrak{X}(Q) \mapsto X^{\mathrm{v}} \in \mathfrak{X}^{\mathrm{v}}(TQ)$, where $\mathfrak{X}^{\mathrm{v}}(TQ)$ is the module of vector fields over TQ that are vertical with respect to the projection τ_Q. In local coordinates, if $v = v^i \frac{\partial}{\partial q^i}|_q = (q^i, v^i)$ and $w = w^i \frac{\partial}{\partial q^i}|_q = (q^i, w^i)$, then

$$v_w^{\mathrm{v}} = v^i \frac{\partial}{\partial v^i}\bigg|_w = (q^i, w^i, 0, v^i)$$

for the induced adapted local coordinates of TTQ.

Definition 1.17. The *vertical endomorphism* is the linear map $\mathcal{S} : TTQ \longrightarrow TTQ$ that, for any vector $V \in TTQ$, gives the value

$$\mathcal{S}(V) = ((T_v\tau_Q)(V))^{\mathrm{v}}, \tag{1.2}$$

where $v = \tau_{TQ}(V) \in TQ$.

In adapted coordinates (q^i, v^i) of TQ, the vertical endomorphism has the local expression

$$\mathcal{S} = \mathrm{d}q^i \otimes \frac{\partial}{\partial v^i} \quad \text{or} \quad \mathcal{S}(q^i, v^i, \dot{q}^i, \dot{v}^i) = (q^i, v^i, 0, \dot{q}^i). \tag{1.3}$$

Definition 1.18. The *Liouville* or *dilation vector field* is the vector field Δ over TQ defined by

$$\Delta_v = (v^{\mathrm{v}})_v, \tag{1.4}$$

for any $v \in TQ$.

In adapted coordinates (q^i, v^i) of TQ, Δ is given by

$$\Delta = v^i \frac{\partial}{\partial v^i} = (q^i, v^i, 0, v^i). \tag{1.5}$$

Another way to define the Liouville vector field is as the infinitesimal generator of the 1-parameter group of transformations $\phi_t : v \in TQ \mapsto \mathrm{e}^t v \in TQ$. This definition can easily be translated to any vector bundle.

Definition 1.19. A *second order vector field* or *differential equation* (usually abbreviated *SODE*) is a vector field $X \in \mathfrak{X}(TQ)$ such that $T\tau_Q \circ X = \mathrm{Id}_{TQ}$.

In adapted coordinates (q^i, v^i) of TQ, a SODE is a vector field

$$X = X^i \frac{\partial}{\partial q^i} + Y^i \frac{\partial}{\partial v^i} \quad \text{such that} \quad X^i = v^i.$$

Thus, neither the Liouville vector field nor the vertical lift of a vector field are second order vector fields. Even though, SODEs are characterized by the equation

$$\mathcal{S}(X) = \Delta.$$

Definition 1.20. Given a smooth curve $c : I \longrightarrow Q$, its *(first) lift* to TQ is the smooth curve $c^{(1)} : I \longrightarrow TQ$ such that

$$(c^{(1)}(t_0))(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)\Big|_{t=t_0}.$$

In local adapted coordinates, $c^{(1)} = (c^i, \, \mathrm{d}c^i/\,\mathrm{d}t)$.

Proposition 1.21. *A vector field $X \in \mathfrak{X}(TQ)$ is a SODE if and only if the integral curves of X are lifts of their own projections to Q; that is, if $\tilde{c}$ is an integral curve of X, then*

$$\tilde{c} = (\tau_Q \circ \tilde{c})^{(1)}. \tag{1.6}$$

The curve $c = \tau_Q \circ \tilde{c} : I \longrightarrow Q$ is called a base integral curve *of X or a* solution *of the SODE given by X.*

If $\tilde{c} : I \longrightarrow TQ$ is an integral curve of a SODE $X \in \mathfrak{X}(TQ)$ locally given by $X = (q^i, v^i, v^i, a^i)$ and $c : I \longrightarrow Q$ denotes its base integral curve, then

$$q^i = c^i, \;\; v^i = \frac{\mathrm{d}c^i}{\mathrm{d}t} \text{ and } a^i = \frac{\mathrm{d}^2 c^i}{\mathrm{d}t^2}.$$

Alternatively, the base integral curve c of $\tilde{c}$ satisfies the system of second order differential equations

$$\mathrm{d}^2 c^i/\,\mathrm{d}t^2 = a^i(c^i, \, \mathrm{d}c^i/\,\mathrm{d}t) \qquad (\text{intrinsically} \quad \tilde{c}^{(1)}(t) = X(c^{(1)}(t))).$$

1.4 Symplectic geometry

In some sense, symplectic geometry is complementary to Riemannian geometry. While Riemannian geometry is based on the study of smooth manifolds that carry a non-degenerate symmetric tensor, symplectic geometry covers the study of smooth manifolds that are equipped with a non-degenerate skewsymmetric tensor. Although both have several similarities, by their nature they also show to have strong differences.

Along this section, V and M respectively denote a real vector space and a smooth manifold. They do not necessarily have finite dimension.

Definition 1.22. Let $\omega : V \times V \longrightarrow \mathbb{R}$ be a bilinear map and define the morphism $\omega^\flat : V \longrightarrow V^*$ by

$$\langle \omega^\flat(v) | w \rangle = \omega(v, w).$$

We say that ω is *weakly* (resp. *strongly*) *non-degenerate* whenever $\omega^\flat$ is a monomorphism (resp. an isomorphism).

It turns out that, if V is finite-dimensional, weak and strong non-degeneracy coincide. Thus, in this case, we simply use the term non-degenerate.

Proposition 1.23. *Let V be a finite-dimensional real vector space and let $\omega \in \Lambda^2 V^*$ be a skew-symmetric bilinear map. The following holds,*

1. *ω is non-degenerate if and only if V is even-dimensional ($\dim V = 2n$) and the exterior nth-power ω^n is a volume form on V;*
2. *if ω is non-degenerate, then there exists a basis $(\varepsilon^i)_{i=1}^{2n}$ in V^* such that*

$$(\omega_{ij}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where $\omega = \omega_{ij}\, \varepsilon^i \otimes \varepsilon^j$, 0 is the n-by-n null matrix and I is the n-dimensional identity matrix. Equivalently, $\omega = \sum_{i=1}^n \varepsilon^i \wedge \varepsilon^{n+i}$.

Definition 1.24. A *weak* (resp. *strong*) *symplectic form* on a real vector space V is a weakly (resp. strongly) non-degenerate 2-form ω on V. The pair (V, ω) is called a *weak* (resp. *strong*) *symplectic vector space.*

As before, we avoid the use of the terms weak and strong in the case of finite-dimensional vector spaces.

Example 1.25. Let V be a real vector space of dimension n. Let $(e_i)_{i=1}^n$ be a basis of V and let $(\varepsilon^i)_{i=1}^n$ be its dual counterpart (*i.e.* $\varepsilon^i(e_j) = \delta^i_j$). Then, with some abuse of notation, $\omega = \sum_{i=1}^n \varepsilon^i \wedge e_i$ is a non-degenerate 2-form in $V \times V^*$. Note that ω does not depend on the chosen basis $(e_i)_{i=1}^n$ of V. In fact, ω may be defined intrinsically by the following expression,

$$\omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2).$$

Definition 1.26. Let M be a smooth manifold, a tensor field $\omega \in \Omega^2(M)$ is *weakly* (resp. *strongly*) *non-degenerate* if the bilinear map $\omega_x : T_xM \times T_xM \longrightarrow \mathbb{R}$ is weakly (resp. strongly) non-degenerate, for each $x \in M$.

Proposition 1.27. *Given a tensor field ω over M of type $(0,2)$, let $\omega^\flat : \mathfrak{X}(M) \longrightarrow \Omega(M)$ be the mapping defined by the contraction $\omega^\flat(X) = i_X\omega$. We have that $\omega^\flat$ is $\mathcal{C}^\infty(M)$-linear. Moreover, if ω is weakly (resp. strongly) non-degenerate, then $\omega^\flat$ is injective (resp. bijective).*

Definition 1.28. Let M be a smooth manifold, a *weak* (resp. *strong*) *symplectic form* is a weakly (resp. strongly) non-degenerate 2-form $\omega \in \Omega^2(M)$ which is in addition closed. The pair (M, ω) is called a *weak* (resp. *strong*) *symplectic manifold.*

Theorem 1.29 (Darboux). *Let ω be a 2-form over a finite-dimensional smooth manifold M. Then, (M, ω) is a symplectic manifold if and only if M has even dimension (*$\dim M = 2n$*) and there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ such that ω has locally the form*

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i.$$

Such coordinates are called Darboux *or* canonical coordinates.

Example 1.30 (T^*Q as a symplectic manifold). Let Q be a smooth manifold of dimension n and consider its cotangent bundle T^*Q. We define on T^*Q a 1-form $\Theta \in \Omega(T^*Q)$ by

$$\Theta_\alpha(X) = \alpha((T_\alpha \pi_Q)(X)), \quad X \in T_\alpha(T^*Q), \ \alpha \in T^*Q.$$

The 1-form Θ is known as the *Liouville 1-form*, or also as the *canonical* or *tautological 1-form*. In adapted coordinates (q^i, p_i) of T^*Q, Θ has the local expression

$$\Theta = p_i \, \mathrm{d}q^i.$$

We now define on T^*Q the *canonical 2-form*:

$$\Omega = -\,\mathrm{d}\Theta.$$

From the local expression of Θ, we have that Ω is locally written as

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$$

for the local coordinates (q^i, p_i) of T^*Q. We thus infer that Ω is symplectic and hence it endows T^*Q with a canonical symplectic structure, (T^*Q, Ω).

1.4.1 The Gotay-Nester-Hinds algorithm

By definition, if (M, ω) is a strongly symplectic manifold (posibly of finite dimension), then the equation

$$i_X \omega = \alpha \tag{1.7}$$

has always a unique solution $X \in \mathfrak{X}(M)$, whatever the 1-form $\alpha \in T^*M$ is (Proposition 1.27). In the finite dimensional case and we suppose that $\dim M = 2n$, the solution vector field $X \in \mathfrak{X}(M)$ is locally given by

$$X = \omega^\sharp(\alpha) = (\omega^\flat)^{-1}(\alpha) = \sum_{i,j=1}^{2n} \omega^{ij} \alpha_j \frac{\partial}{\partial x^i}, \tag{1.8}$$

where $(x^1, \ldots, x^{2n})$ are arbitrary local coordinates on M, ω^{ij} is the inverse coeficient matrix of ω, with $\omega = \sum_{1 \le i < j \le 2n} \omega_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j$, and $\alpha = \sum_{j=1}^{2n} \alpha_j \, \mathrm{d}x^j$. If we instead choose Darboux coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ for M and write

$$X = X^i \frac{\partial}{\partial q^i} + X_i \frac{\partial}{\partial p_i} \quad \text{and} \quad \alpha = \alpha_i \, \mathrm{d}q^i + \alpha^i \, \mathrm{d}p_i,$$

then

$$X^i = \alpha^i \quad \text{and} \quad X_i = -\alpha_i. \tag{1.9}$$

This equations will appear again in later sections in slightly different ways.

The aim of the Gotay-Nester-Hinds (GNH) algorithm (see reference [100, 101, 102]) is to study the equation (1.7) whenever the closed 2-form ω is weakly symplectic or degenerate, that is, when ω is *presymplectic*. It manages to circumvent the degeneracy problems that often appear in mechanics, even though it is totally geometric and may be studied appart of any physical meaning. Equation (1.7) could not be solvable for a presymplectic form ω over the whole manifold M, but it could be at some points of M. The objective of the GNH algorithm is to find a submanifold N of M such that equation (1.7) has solutions in N, more precisely, to find the biggest submanifold N of M such that there exists a vector field $X \in \mathfrak{X}(N)$ that satisfies

$$i_{j_* X}\omega|_N = \alpha|_N \tag{1.10}$$

for a prescribed 1-form $\alpha \in \Omega(M)$, where j is the inclusion $j : N \hookrightarrow M$. The manifold N will, of course, depend on the 1-form α.

Remark 1.31. Even though they are quite similar, Equation (1.10) should not be confused with the following one

$$i_X(j^*\omega) = j^*\alpha.$$

While the latter must be satisfied for any vector field Y "over" N, that is

$$(j^*\omega)(X, Y) = (j^*\alpha)(Y), \quad \forall Y \in \mathfrak{X}(N),$$

the former is more restrictive and must be satisfied for any vector field Y "along" N, that is

$$\omega(j_*X, Y) = \alpha(Y), \quad \forall Y \in \mathfrak{X}(j).$$

Given a presymplectic 2-form ω over a manifold M, let $\alpha \in \Omega(M)$ be any 1-form. We start defining the subset M_1 of points x of M such that $\alpha(x)$ is in the range of $\omega^\flat(x)$, that is,

$$M_1 := \left\{x \in M \,:\, \alpha(x) \in \omega^\flat(TM)\right\}.$$

The subset M_1 needs not to be a manifold, fact that is imposed, being $j_1 : M_1 \hookrightarrow M$ the inclusion. The equation (1.7) restricted to M_1,

$$i_X\omega|_{M_1} = \alpha|_{M_1},$$

is solvable, but this does not imply that X is a solution in the sense of equation (1.10). It could be possible that, at some point $x \in M_1$, the vector $X(x)$ dont be tangent to M_1, what will happen when $\alpha(x)$ dont be in the range of $\omega^\flat(x)$ restricted to $j_{1*}(TM_1)$. We are then obliged to define a new "submanifold" $j_2 : M_2 \hookrightarrow M_1$ by

$$M_2 := \left\{x \in M_1 \,:\, \alpha(x) \in \omega^\flat(j_{1*}(TM_1))\right\}.$$

As before, the solutions of the equation (1.7) restricted to M_2,

$$i_X\omega|_{M_2} = \alpha|_{M_2},$$

may not be tangent to M_2, therefore we require that $\alpha|_{M_2}$ be in the range of $\omega^\flat$ restricted to $(j_2 \circ j_1)_*(TM_2)$.

We thus continue this process, defining a chain of further constraint submanifolds

$$\ldots \hookrightarrow M_l \overset{j_l}{\hookrightarrow} M_{l-1} \hookrightarrow \ldots \hookrightarrow M_1 \overset{j_1}{\hookrightarrow} M$$

as follows

$$M_{l+1} := \left\{ x \in M_l \, : \, \alpha(x) \in \omega^\flat((j_1 \circ \cdots \circ j_l)_*(TM_l)) \right\}. \tag{1.11}$$

At each step, we must assume that the constraint set M_l is a smooth manifold (an alternate algorithm for the case when the constraint sets are not smooth submanifolds may be found in [114]). In the end, the algorithm will stop when, for some $k \geq 0$, $M_{k+1} = M_k$. We then take $N := M_k$ and $j := j_k \circ \cdots \circ j_1$. Mainly, two different cases may happen:

– $\dim N = 0$: The Hamiltonian system (M, ω, α) has no dynamics. Furthermore, if $N = \emptyset$, there are no solutions at all and (M, ω, α) does not accurately describe the dynamics of any system. On the contrary, if $N \neq \emptyset$, then N consists of (steady) isolated points.

– $\dim N \neq 0$: (M, ω, α) describes a dynamical system restricted to N and we have completely consistent equations at motion on N of the form

$$(i_X \omega - \alpha)|_N = 0.$$

1.5 Cosymplectic geometry

While symplectic geometry deals with even- dimensional spaces, cosymplectic geometry is the natural extension to study analog structures in odd-dimensional spaces. Through this section, V and M respectively denote a real vector space and a smooth manifold of dimension $2n + 1$.

Definition 1.32. A *cosymplectic vector space* is a triple (V, ω, η) where V is an odd-dimensional real vector space ($\dim V = 2n + 1$), ω is a 2-form on V and η is a 1-form on V such that the exterior product $\omega^n \wedge \eta$ is not null.

Proposition 1.33. *Let V be an odd-dimensional real vector space ($\dim V = 2n + 1$). Given a 2-form ω and a 1-form η on V, define the morphism $\flat : V \longrightarrow V^*$ by*

$$\flat(v) = i_v \omega + \eta(v)\eta.$$

If (V, ω, η) is cosymplectic, then $\flat$ is an isomorphism. In that case, the vector $R = \flat^{-1}(\eta)$ is called the Reeb vector *of the cosymplectic space (V, ω, η).*

Note that the Reeb vector is characterized by the equations

$$i_R \omega = 0, \quad i_R \eta = 1.$$

Example 1.34. Let V be a real vector space of dimension n. Let $(e_i)_{i=1}^n$ be a basis of V and let $(\varepsilon^i)_{i=1}^n$ be its dual counterpart. Define $\omega = \sum_{i=1}^n \varepsilon^i \wedge e_i$, the canonical non-degenerate 2-form in $V \times V^*$ (see example 1.25). Let η be a non-zero covector in $\mathbb{R}$. Then, with some abuse of notation, $(V \times V^* \times \mathbb{R}, \omega, \eta)$ is a cosymplectic vector space.

Definition 1.35. A *cosymplectic manifold* is a triple (M, ω, η) where M is an odd-dimensional smooth manifold ($\dim M = 2n+1$), ω is a closed 2-form on M and η be a closed 1-form on M such that (T_xM, ω_x, η_x) is a cosymplectic vector space for each $x \in M$.

Proposition 1.36. *Let M be an odd-dimensional smooth manifold. Given a 2-form ω and a 1-form η on M, define the map $\flat : \mathfrak{X}(M) \longrightarrow \Omega(M)$ by*

$$\flat(X) = i_X\omega + \eta(X)\eta.$$

If (M, Ω, η) is cosymplectic then $\flat$ is an isomorphism of $\mathcal{C}^\infty(M)$-modules. In that case, $R = \flat^{-1}(\eta)$ is known as the Reeb vector field *of the cosymplectic manifold (M, ω, η).*

Again, note that the Reeb vector field is characterized by the equations

$$i_R\omega = 0, \quad i_R\eta = 1.$$

Proposition 1.37. *Let M be an odd-dimensional smooth manifold ($\dim M = 2n+1$). Given a 2-form ω and a 1-form η on M, the triple (M, Ω, η) is a cosymplectic manifold if and only if there exist local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n, t)$ such that ω and η have locally the expression*

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i, \quad \eta = \mathrm{d}t.$$

Such coordinates are called Darboux *or* canonical coordinates.

Example 1.38 ($T^*Q \times \mathbb{R}$ as a cosymplectic manifold). Let Q be a smooth manifold of dimension n and consider its cotangent bundle T^*Q. Let Ω be the canonical 2-form on T^*Q (see example 1.30) and let η a volume form on $\mathbb{R}$, for instance, $\eta = \mathrm{d}t$. Then, $(T^*Q \times \mathbb{R}, \Omega, \eta)$ is a cosymplectic manifold.

1.6 Multisymplectic geometry

For an introduction to multisymplectic geometry and its use within classical field theory, the reader is refereed to [31, 32]. See also [124, 132].

Through this section, V and M respectively denote a real vector space and a smooth manifold, both of finite dimension.

Definition 1.39. A *multisymplectic k-form* on a real vector space V is a k-form $\omega \in \Lambda^k V^*$ with trival kernel, $\ker \omega = 0$, where the kernel is $\ker \omega := \{v \in V : i_v\omega = 0\}$. The pair (V, ω) is said to be a *multisymplectic vector space*.

A necessary condition to be satisfied by a multisymplectic k-form ω is that $1 < k \leq \dim V$. The non-degeneracy condition $\ker \omega = 0$ is sometimes written as

$$i_v\omega = 0 \Leftrightarrow v = 0.$$

Note also that a multisymplectic 2-form is a symplectic one.

Definition 1.40. Given a k-form $\omega \in \Lambda^k V^*$, we define the mappings

$$\begin{aligned} \omega_j^\flat : \Lambda^j V &\longrightarrow \Lambda^{k-j} V^* \\ \mathrm{v} &\mapsto i_{\mathrm{v}}\omega \end{aligned}$$

for any $1 \leq j \leq k$.

If any of the linear maps $\omega_j^\flat$ is null, then ω must be the zero k-form (and conversely). Thus, $\omega_{k-1}^\flat$ is surjective whenever ω is not zero. If ω is multisymplectic, then $\omega_1^\flat$ is injective.

Example 1.41. Given any real vector space V of dimension n, consider the product $\mathcal{V}_V = V \times \Lambda^k V^*$, with $1 < k \leq n$. We define the $(k+1)$-form Ω_V in $\mathcal{V}_V$ by

$$\Omega_V((v_1,\omega_1),\ldots,(v_{k+1},\omega_{k+1})) := \sum_{i=1}^{k+1}(-1)^i\omega_i(v_1,\ldots,\hat{v}_i,\ldots,v_{k+1})$$

where $(v_i,\omega_i) \in \mathcal{V}_V$ for $i = 1,\ldots,k+1$ and where the hat symbol " ˆ " means that the underlying vector is ommited. If (e_i) is a basis for V, (ε^i) is the corresponding dual basis for V^* and $(e_{i_1\cdots i_k} = e_{i_1} \wedge \cdots \wedge e_{i_k})$ is the basis for $\Lambda^k V$, where $1 \leq i_1 < \cdots < i_k \leq k$, then

$$\Omega_V = \sum_{1\leq i_1<\cdots<i_k\leq k} -e_{i_1\cdots i_k} \wedge \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}.$$

It is easly seen from here that, when $k = 1$, we recover the symplectic 2-form given in the example 1.25.

If E is a proper vector subspace of V, we denote by $\Lambda_r^k V^*$ the collection of k-forms in V that are anihilated when r vectors of E are applied to it,

$$\Lambda_r^k V^* = \left\{\alpha \in \Lambda_r^k V^* \,:\, i_{v_r}\cdots i_{v_1}\alpha = 0,\ \forall v_1,\ldots,v_r \in E\right\}.$$

We then have that $\mathcal{V}_V^r = V \times \Lambda_r^k V^*$ equiped with the restriction of Ω_V to it is a multisymplectic space. Note that, if $E = \{0\}$, we then recover the whole $\mathcal{V}_V$.

Definition 1.42. A *multisymplectic k-form* on a smooth manifold M is a closed k-form $\omega \in \Omega^k(M)$ such that (T_xM,ω_x) is a multisymplectic vector space for each $x \in M$. The pair (M,ω) is said to be a *multisymplectic manifold.*

Example 1.43. Given a smooth manifold of dimension n, consider the fiber bundle $\Lambda^k M$ of k-forms. We define on $\Lambda^k M$ the k-form $\Theta \in \Omega^k(\Lambda^k M)$ by

$$\Theta_\alpha(X_1,\ldots,X_k) = \alpha((T_\alpha\pi_M^k)(X_1),\ldots,(T_\alpha\pi_M^k)(X_k)),\ X_i \in T_\alpha(\Lambda^k M),\ \alpha \in \Lambda^k M,$$

where $\pi_M^k : \Lambda^k M \to M$ is the canonical projection. The k-form Θ is known as the *Liouville k-form*, or also as the *canonical* or *tautological k-form.* In adapted coordinates $(q^i, p_{i_1\cdots i_k})$ of $\Lambda^k M$, Θ has the local expression

$$\Theta = \sum_{1\leq i_1<\cdots<i_k\leq k} p_{i_1\cdots i_k}\, \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_k}.$$

We now define on $\Lambda^k M$ the *canonical* $(k+1)$*-form*:

$$\Omega = -\,\mathrm{d}\Theta.$$

From the local expression of Θ, we have that Ω is locally written as

$$\Omega = \sum_{1\leq i_1<\cdots<i_k\leq k} -\,\mathrm{d}p_{i_1\cdots i_k} \wedge \mathrm{d}q^{i_1} \wedge \cdots \wedge \mathrm{d}q^{i_k}$$

for the local coordinates $(q^i, p_{i_1\cdots i_k})$ of $\Lambda^k M$. We thus infer that Ω is multisymplectic and hence it endows $\Lambda^k M$ with a canonical multisymplectic structure, $(\Lambda^k M, \Omega)$. It is easily seen from here that, when $k = 1$, we recover the canonical symplectic 2-form given in the example 1.30.

If M fibers over a manifold N, $\pi : M \to N$, we denote by $\Lambda^k_r M$ the collection of k-forms over M that are anihilated when r π-vertical vectors are applied to it,

$$\Lambda^k_r M = \left\{ \alpha \in \Lambda^k M \,:\, i_{v_r} \cdots i_{v_1} \alpha = 0,\, \forall v_1, \ldots, v_r \in \mathcal{V}\pi \right\}.$$

We then have that $\Lambda^k_r M$ equiped with the restriction of Ω to it is a multisymplectic space. Nota that, if $N = M$, we then recover the whole $\Lambda^k M$.

Chapter 2

Classical Mechanics

2.1 The Lagrangian formalism for autonomous systems

The Lagrangian formulation of mechanics is set (for simplicity) in a finite dimensional manifold Q (the infinite dimensional case is depicted in [122]), the *configuration space*, whose tangent, TQ, describes the states –position plus velocity– of the system under study. Local coordinates (q^i) on Q induce fiber coordinates (q^i, v^i) on TQ, such that a tangent vector $v \in T_qQ$ at some point $q \in Q$ is written as

$$v = v^i \frac{\partial}{\partial q^i} = v^1 \frac{\partial}{\partial q^1} + \cdots + v^n \frac{\partial}{\partial q^n}.$$

One introduces the *Lagrangian* of the system, a smooth function $L : TQ \longrightarrow \mathbb{R}$, which is in some sense the density cost of a motion in the system. Typically, the Lagrangian is the kinetic energy minus the potential energy of the system,

$$L(q^i, v^i) = \frac{1}{2} m g_{ij} v^i v^j - U(q^i) \quad (L(q^i, v^i) = \frac{1}{2} m g(v_q, v_q) - U(q)),$$

where $g_{ij} = g_{ij}(q)$ is a given metric tensor and m the mass of the particle in motion.

We seek for curves that describe the motion of a particle in our system. It is well known that the trajectories of the system are obtained from a variational procedure. We will thus consider twice differentiable curves $c : [t_0, t_1] \to Q$ joining two fixed points q_0 and q_1 in Q. The set of such curves is

$$\mathcal{C}^2([t_0, t_1], Q, q_0, q_1) = \left\{ c \in \mathcal{C}^2([t_0, t_1], Q) \,:\, c(t_i) = q_i,\ i = 0, 1 \right\},$$

or $\mathcal{C}^2(q_0, q_1)$ for short. Given $c \in \mathcal{C}^2(q_0, q_1)$, denote by $\tilde{c}(t)$ its *lift* to TQ, that is, the curve in TQ that describes the position and velocity of a particle following the original curve (see Definition 1.20). Formally, $\tilde{c}$ is the vector field over c such that

$$\tilde{c}(t)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)(t),$$

for any smooth function $f \in \mathcal{C}^\infty(Q)$. If (q^i, v^i) are adapted coordinates in TQ, then

$$\tilde{c}(t) = (q^i(t), v^i(t)),$$

where one regards $v^i = \mathrm{d}q^i / \mathrm{d}t$ as the velocity of a particle moving along $c(t)$.

If (q^i, v^i) are adapted coordinates on TQ, locally

$$\tilde{c}(t) = (c^i(t), \dot{c}^i(t)),$$

where $c^i(t) = (q^i \circ \tilde{c})(t) = (q^i \circ c)(t)$ and $\dot{c}^i(t) = v^i \circ \tilde{c})(t) = (\mathrm{d}c^i / \mathrm{d}t)(t)$.

2.1.1 Variational approach

Definition 2.1. Given a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$, two fixed points $q_0, q_1 \in Q$ and a fixed time interval $[t_0, t_1]$, the associated *integral action* is the real valued map $\mathcal{A}$ defined on $\mathcal{C}^2([t_0, t_1], Q, q_0, q_1)$ given by

$$\mathcal{A}_L(c) = \int_{t_0}^{t_1} L(\tilde{c}(t))\,\mathrm{d}t = \int_{t_0}^{t_1} L(q^i(t), v^i(t))\,\mathrm{d}t. \tag{2.1}$$

Since we look for a variational approach of the problem, we must describe how the integral action $\mathcal{A}_L$ changes under small perturbations of c and what these perturbations are. One shows that $\mathcal{C}^2(q_0, q_1)$ may be endowed with an infinite-dimensional smooth manifold structure, see [21]. In fact,

$$T_c\mathcal{C}^2(q_0, q_1) = \left\{\delta c \in \mathcal{C}^1([t_0, t_1], TQ) \,:\, \tau_Q \circ \delta c \equiv c,\ \delta c(t_i) = 0,\ i = 0, 1\right\}.$$

Definition 2.2. Let $c \in \mathcal{C}^2([t_0, t_1], Q, q_0, q_1)$, a *variation* of c is a curve c_s in $\mathcal{C}^2(q_0, q_1)$, defined for a small interval $[-\varepsilon, \varepsilon]$, such that $c_0 \equiv c$. An *infinitesimal variation* of c is a vector field δc over c that vanishes at the end points, $\delta c(t_i) = 0$, for $i = 0, 1$.

With this definition, the tangent space $T_c\mathcal{C}^2(q_0, q_1)$ at a curve $c \in \mathcal{C}^2(q_0, q_1)$ is the set of infinitesimal variations δc of c, which are induced by variations c_s of c. More precisely,

$$\delta c(t) = \left.\frac{\mathrm{d}c_s(t)}{\mathrm{d}s}\right|_{s=0},$$

where $t \in [t_0, t_1]$ is fixed.

Definition 2.3. Let $\mathcal{F} : \mathcal{C}^2(q_0, q_1) \longrightarrow \mathbb{R}$ be a functional of class $\mathcal{C}^1$. A *critical point* of $\mathcal{F}$ is a point $c \in \mathcal{C}^2(q_0, q_1)$ such that

$$\left.\frac{\mathrm{d}(\mathcal{F} \circ c_s)}{\mathrm{d}s}\right|_{s=0} = 0,$$

for any variation c_s of c.

Equivalently, one could say that c is a critical point of a functional $\mathcal{F} \in \mathcal{C}^1(\mathcal{C}^2(q_0, q_1))$ if and only if

$$\delta\mathcal{F}(c) \cdot \delta c = 0,$$

for any infinitesimal variation δc of c, which is classically written as

$$\delta\mathcal{F}(c) = 0 \quad \text{or} \quad \frac{\delta\mathcal{F}(c)}{\delta c} = 0.$$

We are now in position to formulate one of the main results in Classical Mechanics, the variational principle of Hamilton, which states that the dynamics of our physical system is determined from the variational problem related to the integral action $\mathcal{A}_L$.

Statement 2.4 (Hamilton's principle)**.** The *motion* of a particle in the Lagrangian system (Q, L) is a critical point of the action functional $\mathcal{A}_L$, that is, a curve $c \in \mathcal{C}^2(q_0, q_1)$ such that $\delta\mathcal{A}_L(c) = 0$.

An easy calculation help us to write this statement in terms of the Lagrangian, obtaining the well known Euler-Lagrange equations.

Theorem 2.5 (The Euler-Lagrange equations)**.** *Consider a given Lagrangian system (Q, L), where $L \in \mathcal{C}^2(TQ)$. A twice differentiable curve $c : [t_0, t_1] \longrightarrow Q$ joining two points $q_0, q_1 \in Q$ is a motion of (Q, L) if and only if the lift $\tilde{c}$ of c to TQ satisfies the differential equations:*

$$\frac{\partial L}{\partial q^i} \circ \tilde{c} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i} \circ \tilde{c}\right) = 0, \tag{2.2}$$

where (q^i, v^i) are adapted coordinates in a neighborhood of $\tilde{c}$.

Proof. Given a curve $c \in \mathcal{C}^2(q_0, q_1)$, let δc be an infinitesimal variation of c tangent to a variation c_s of c. By definition and differentiating under the integral sign, we have that

$$\begin{aligned} \delta\mathcal{A}_L(c) \cdot \delta c &= \left.\frac{\mathrm{d}(\mathcal{A}_L \circ c_s)}{\mathrm{d}s}\right|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{t_0}^{t_1} L(\tilde{c}_s(t))\,\mathrm{d}t\right]\Big|_{s=0} \\ &= \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}s}\left[L(c_s^i(t), \dot{c}_s^i(t))\right]\Big|_{s=0} \mathrm{d}t \\ &= \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^i} \cdot \delta c^i(t) + \frac{\partial L}{\partial v^i} \cdot \delta\dot{c}^i(t)\right] \mathrm{d}t, \end{aligned}$$

provided that ε is small enough such that the image of $\tilde{c}_s : [-\varepsilon, \varepsilon] \times [t_0, t_1] \longrightarrow TQ$ is covered by a single chart with adapted coordinates (q^i, v^i). Integrating by parts the second term and taking into account that δc vanishes at t_0 and t_1, we obtain

$$\delta\mathcal{A}_L(c) \cdot \delta c = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i}\right)\right] \cdot \delta c^i(t)\,\mathrm{d}t.$$

Now, let us suppose that c is a motion of the system. Then, $\delta\mathcal{A}_L(c) \cdot \delta c = 0$ for every infinitesimal variation δc, which holds if and only if

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} = 0.$$

□

2.1.2 Geometric formulation

Definition 2.6. The *Poincaré-Cartan 1-form* is the pull-back of the differential of the Lagrangian function by the vertical endomorphism $\mathcal{S}$, that is, the form

$$\Theta_L = \mathcal{S}^*(\mathrm{d}L). \tag{2.3}$$

The *Poincaré-Cartan 2-form* is then given by

$$\Omega_L = -\,\mathrm{d}\Theta_L. \tag{2.4}$$

In adapted coordinates, the Poincaré-Cartan 1-form reads

$$\Theta_L = \frac{\partial L}{\partial v^i}\,\mathrm{d}q^i \tag{2.5}$$

and the Poincaré-Cartan 2-form

$$\Omega_L = \frac{\partial^2 L}{\partial v^i \partial q^j}\,\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial v^i \partial v^j}\,\mathrm{d}q^i \wedge \mathrm{d}v^j. \tag{2.6}$$

By definition, the Poincaré-Cartan 2-form is exact, and hence closed, but in general needs not to be non-degenerate.

Proposition 2.7. *The Poincaré-Cartan 2-form is non-degenerate if and only if the Lagrangian function is* regular*, that is when the Hessian matrix*

$$\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \tag{2.7}$$

is invertible.

Definition 2.8. The *Lagrangian energy* is the smooth function $E_L \in \mathcal{C}^\infty(TQ)$ defined by

$$E_L = \Delta L - L, \tag{2.8}$$

where Δ denotes the Liouville vector field given in Definition 1.18.

Definition 2.9. Any vector field $X \in \mathfrak{X}(TQ)$ that satisfies the *equation of motion*

$$i_X \Omega_L = \mathrm{d}E_L \tag{2.9}$$

is called a *Lagrangian vector field.*

Theorem 2.10. *If the Lagrangian function L is regular, then there exists a unique vector field $X \in \mathfrak{X}(TQ)$ which is solution of the equation of motion. The Lagrangian vector field X is a second order differential equation and its base integral curves are solutions of the Euler-Lagrange equations* (2.2)*.*

Proof. The existence and uniqueness of a Lagrangian vector field comes out from the fact that Ω_L is non-degenerate when L is regular, hence Ω_L is symplectic and Proposition 1.27 applies. Let X be a generic vector field on TQ whose local expression is

$$X = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{v}^i \frac{\partial}{\partial v^i}$$

for adapted coordinates (q^i, v^i) of TQ and let suppose that X satisfies the equation (2.9). The contraction of the Poincaré-Cartan 2-form Ω_L with X is

$$i_X \Omega_L = \left(\dot{q}^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial v^i \partial q^j} - \dot{v}^j \frac{\partial^2 L}{\partial v^j \partial v^i}\right) \mathrm{d}q^i + \dot{q}^j \frac{\partial^2 L}{\partial v^j \partial v^i}\,\mathrm{d}v^i,$$

and the differential of the Lagrangian energy E_L is

$$\mathrm{d}E_L = \left(v^j \frac{\partial^2 L}{\partial v^j \partial q^i} - \frac{\partial L}{\partial q^i}\right) \mathrm{d}q^i + v^j \frac{\partial^2 L}{\partial v^j \partial v^i}\,\mathrm{d}v^i.$$

Equating coefficients, we have on the one hand that

$$\dot{q}^j \frac{\partial^2 L}{\partial v^j \partial v^i} = v^j \frac{\partial^2 L}{\partial v^j \partial v^i}.$$

Thus, if L is regular, $\dot{q}^j = v^j$, which proves that X is second order. On the other hand, provided that L is regular, we have that

$$\frac{\partial L}{\partial q^i} - v^j \frac{\partial^2 L}{\partial q^j \partial v^i} - \dot{v}^j \frac{\partial^2 L}{\partial v^j \partial v^i} = 0.$$

Let $c : I \longrightarrow Q$ be a base integral curve of the Lagrangian vector field X, then $\dot{q}^i = \dot{c}^i = \mathrm{d}c/\mathrm{d}t$ and $\dot{v}^i = \ddot{c}^i = \mathrm{d}^2 c/\mathrm{d}t^2$. Substituting this in the previous equation and denoting $\tilde{c} = (c^i, \dot{c}^i)$ the lift of c to TQ, we obtain

$$0 = \frac{\partial L}{\partial q^i} \circ \tilde{c} - \frac{\mathrm{d}c^j}{\mathrm{d}t} \frac{\partial^2 L}{\partial q^j \partial v^i} \circ \tilde{c} - \frac{\mathrm{d}\dot{c}^j}{\mathrm{d}t} \frac{\partial^2 L}{\partial v^j \partial v^i} \circ \tilde{c} = \frac{\partial L}{\partial q^i} \circ \tilde{c} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^i} \circ \tilde{c}\right),$$

which are precisely the Euler-Lagrange equations (2.2). □

2.2 The Hamiltoniam formalism for autonomous systems

For a more extended description of the Hamiltonian formalism, please refer to [1, 64, 122].

As for the Lagrangian formalism, the Hamiltonian formulation of mechanics is set in a finite dimensional manifold Q, the *configuration space*, but in contrast, the states –position plus momentum– of the system under study are described by the cotangent bundle T^*Q of Q. Local coordinates (q^i) on Q induce fiber coordinates (q^i, p_i) on T^*Q, such that a 1-form $\alpha \in T_q^*Q$ at some point $q \in Q$ is written as

$$\alpha = p_i \, \mathrm{d}q^i = p_1 \, \mathrm{d}q^1 + \cdots + p_n \, \mathrm{d}q^n.$$

One introduces the *Hamiltonian* of the system, a smooth function $H : T^*Q \longrightarrow \mathbb{R}$, which is in some sense the is the total energy density of the system being described. Typically, the Hamiltonian is the kinetic energy plus the potential energy of the system,

$$H(q^i, p_i) = K(p_i) + U(q^i) \; (= T(p_i) + V(q^i)).$$

Definition 2.11. Given a Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$, the *Hamiltonian vector field* with energy function H is the unique vector field $X \in \mathfrak{X}(M)$ such that

$$i_{X_H}\Omega = \mathrm{d}H,$$

where Ω is the canonical symplectic form of T^*Q.

Theorem 2.12 (Hamilton's equations)**.** *A differentiable curve $c : I \longrightarrow T^*Q$ is an integral curve of X_H if and only if the* Hamilton's equations *hold:*

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \textit{and} \quad \dot{p}_i = \frac{\partial H}{\partial q^i},$$

where $c(t) = (q^i(t), p_i(t))$.

Proposition 2.13. *Given an integral curve $c(t)$ of X_H, we have that $H(c(t))$ is constant.*

Proposition 2.14. *Let $F_t \in \mathrm{Diff}(M)$ be the flow of X_H, then $F_t^*\omega = \omega$, for each t,* i.e. *$\{F_t\}$ is a family of symplectomorphisms.*

2.3 The Legendre transformation

The Legendre transformation is the master key that relates the Lagrangian and the Hamiltonian formalisms. Although the technique is usually used to go from the Lagrangian side to the Hamiltonian one, it can be restated to pass from the latter to the former.

Definition 2.15. Given a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$, the *Legendre transformation* associated to L is the fibered mapping $\mathrm{leg}_L : TQ \longrightarrow T^*Q$ given by

$$\langle \mathrm{leg}_L(v), w\rangle := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(v + \varepsilon \cdot w)\Big|_{\varepsilon=0}.$$

If (q^i, v^i) and (q^i, p_i) denote fiber coordinates on TQ and T^*Q respectively, we then have

$$\mathrm{leg}_L(q^i, v^i) = \left(q^i, p_i = \frac{\partial L}{\partial v^i}\right).$$

Proposition 2.16. *If L is regular, then $\mathrm{leg}_L : TQ \longrightarrow T^*Q$ is a local diffeomorphism.*

Definition 2.17. A Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ is said to be *hyper-regular* whenever leg_L is a global diffeomorphism.

Theorem 2.18. *Given a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$, we have that*

$$\Theta_L = \mathrm{leg}_L^* \Theta \quad \textit{and} \quad \Omega_L = \mathrm{leg}_L^* \Omega.$$

Moreover, if L is hyper-regular and we define the Hamiltonian

$$H := E_L \circ \mathrm{leg}_L^{-1} = v^i \frac{\partial L}{\partial v^i} - L,$$

then the Lagrangian vector field X_L and the Hamiltonian vector field X_H are leg_L-related, i.e. $X_H = (\mathrm{leg}_L)_* X_L$.

2.4 The Tulczyjew's triple

In [147, 148], W. Tulczyjew introduced a purely geometric construction based on a triple of tangent and cotangent bundles in which the theory of classical mechanics fits perfectly. While its extension to higher-order mechanics has been completely achieved (see [30, 45, 48]), there have been some attempts to reproduce it for field theory but with only partial success (for instance, [52, 109]).

Before we give the full picture, we start with two basic definitions.

The *canonical involution* of TTQ is the smooth map $\kappa_Q : TTQ \longrightarrow TTQ$ given by

$$\kappa_M \left(\frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\mathrm{d}}{\mathrm{d}t} \chi(s,t)\Big|_{t=0} \right) \Big|_{s=0} \right) := \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\chi}(s,t)\Big|_{t=0} \right) \Big|_{s=0},$$

where $\chi : \mathbb{R}^2 \longrightarrow Q$ and $\tilde{\chi}(s,t) := \chi(t,s)$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\chi(s,t)|_{t=0} : \mathbb{R} \longrightarrow TQ$.

The *tangent pairing* between TT^*Q and TTQ is the fibered map $\langle \cdot, \cdot \rangle^T : TT^*Q \times_Q TTQ \longrightarrow \mathbb{R}$ given by

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\Big|_{t=0}, \frac{\mathrm{d}}{\mathrm{d}t}\delta(t)\Big|_{t=0} \right\rangle^T := \frac{\mathrm{d}}{\mathrm{d}t} \langle \gamma(t), \delta(t)\rangle^T \Big|_{t=0},$$

where $\gamma : \mathbb{R} \longrightarrow T^*Q$ and $\delta : \mathbb{R} \longrightarrow TQ$ are such that $\pi_Q \circ \gamma \equiv \tau_Q \circ \delta$.

Definition 2.19. The *Tulczyjew's isomorphism* is the map $\alpha : TT^*Q \longrightarrow T^*TQ$ given by

$$\langle \alpha(V), W \rangle := \langle V, \kappa_Q(W) \rangle^T, \quad V \in TT^*Q,\ W \in TTQ.$$

In coordinates,

$$\alpha(q^i, p^i, \dot{q}^i, \dot{p}^i) = (q^i, \dot{q}^i, \dot{p}^i, p^i).$$

Definition 2.20. Define the map $\beta : TT^*Q \longrightarrow T^*T^*Q$ by

$$\beta(V) := i_V \Omega, \quad V \in TT^*Q,$$

where Ω is the canonical symplectic form of T^*Q.

In coordinates,

$$\beta(q^i, p^i, \dot{q}^i, \dot{p}^i) = (q^i, \dot{q}^i, \dot{p}^i, p^i).$$

By means of α and β, TT^*Q may be endowed with two (a priori) different symplectic structures: Let Ω_{TQ} and Ω_{T^*Q} be the canonical symplectic forms of T^*TQ and T^*T^*Q (as cotangent bundles), respectively. Then, both of $\Omega_\alpha = \alpha^*\Omega_{TQ}$ and $\Omega_\beta = \beta^*\Omega_{T^*Q}$ define symplectic structures on TT^*Q which turn out to be the same; more precisely, $\Omega_\alpha = -\Omega_\beta$. Moreover, there is a third canonical symplectic structure on TT^*Q which comes from the complete lift of the canonical symplectic form Ω_Q of Q to TT^*Q, which we denote $\Omega_Q^{(1)}$, and which coincides with the previous ones; more precisely, $\Omega_Q^{(1)} = \Omega_\alpha$. In coordinates,

$$\Theta_\alpha = \alpha^*\Theta_{TQ} = \dot{p}\,\mathrm{d}q + p\,\mathrm{d}\dot{q} \quad \text{and} \quad \Theta_\beta = \beta^*\Theta_{T^*Q} = -\dot{p}\,\mathrm{d}q + \dot{q}\,\mathrm{d}p,$$

where Θ_{TQ} and Θ_{T^*Q} are the Liouville 1-forms on TQ and T^*Q, respectively.

Theorem 2.21. *Given a Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$, consider the associated Hamiltonian vector field $X_H \in \mathfrak{X}(T^*Q)$. The following holds,*

1. *The image of X_H is a Lagrangian submanifold S_{X_H} of (TT^*Q, Ω_β).*

2. *The image of $\mathrm{d}H$ is a Lagrangian submanifold $S_{\mathrm{d}H}$ of (T^*T^*Q, Ω_{T^*Q}).*

3. *The isomorphism β maps one into another,* i.e. $\beta(S_{X_H}) = S_{\mathrm{d}H}$.

Lemma 2.22. *Given a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$, then the image of $\mathrm{d}L$ is a Lagrangian submanifold $S_{\mathrm{d}L}$ of (T^*TQ, Ω_{TQ}).*

Proposition 2.23. *Given an hyper-regular Lagrangian function $L : TQ \longrightarrow \mathbb{R}$, consider the associated Hamiltonian $H = E_L \circ \mathrm{leg}_L^{-1}$. We have that $\alpha^{-1}(S_{\mathrm{d}L}) = S_{X_H} = \beta^{-1}(S_{\mathrm{d}H})$*

$$\begin{tikzcd}
T^*TQ \arrow[dr, "\pi_{TQ}"] & & TT^*Q \arrow[ll, "\alpha"'] \arrow[rr, "\beta"] \arrow[dl, "T\pi_Q"'] \arrow[dr, "\tau_{T^*Q}"] & & T^*T^*Q \arrow[dl, "\pi_{T^*Q}"'] \\
& TQ \arrow[ul, bend left, "\mathrm{d}L"] \arrow[rr, "\mathrm{leg}_L"] & & T^*Q \arrow[ur, bend right, "\mathrm{d}H"'] &
\end{tikzcd}$$

Chapter 3

Classical Field Theory

The main reference for this chapter is the book by Saunders [139], although it does not cover all the sections (references will be provided when necessary). Besides, other basic references are [21, 91, 97, 109, 61, 136].

3.1 Jet bundles

Through this section, (E, π, M) denotes a fiber bundle whose base space M is a smooth manifold of dimension m, and whose fibers have dimension n, thus E is $(m+n)$-dimensional. Adapted coordinate systems in E will be of the form (x^i, u^α), where (x^i) is a local coordinate system in M and (u^α) denotes fiber coordinates.

Definition 3.1. Given a point $x \in M$, two local sections $\phi, \psi \in \Gamma_x\pi$ are *1-equivalent at x* if their value coincide at x, $\phi(x) = \psi(x)$, as well as their tangent maps, $T_x\phi = T_x\psi$. This defines an equivalence relation in $\Gamma_x\pi$. The equivalence class containing ϕ is called the *first order jet of ϕ at x* and is denoted $j^1_x\phi$.

An alternative definition of the previous equivalence relation would be in terms of partial derivatives. Let (x^i, u^α) be a system of adapted local coordinates around $\phi(x)$, ψ will be 1-equivalent to ϕ at x if and only if

$$\phi^\alpha(x) = \psi^\alpha(x) \quad \text{and} \quad \left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_x = \left.\frac{\partial \psi^\alpha}{\partial x^i}\right|_x . \tag{3.1}$$

Definition 3.2. The *first order jet manifold of π*, denoted $J^1\pi$, is the whole collection of first order jets of arbitrary local sections of π, that is,

$$J^1\pi := \left\{ j^1_x\phi : x \in M,\, \phi \in \Gamma_x\pi \right\} .$$

The functions given by

$$\begin{array}{rcl} \pi_1 : J^1\pi & \longrightarrow & M \\ j^1_x\phi & \longmapsto & x \end{array} \tag{3.2}$$

and

$$\begin{array}{rcl} \pi_{1,0} : J^1\pi & \longrightarrow & E \\ j^1_x\phi & \longmapsto & \phi(x) \end{array} \tag{3.3}$$

are called the *source projection* and the *target projection* respectively, and are smooth surjective submersions.

Proposition 3.3. *The first jet manifold of π, $J^1\pi$, may be endowed with a structure of smooth manifold. A system of adapted coordinates (x^i, u^α) on E induces a system of coordinates $(x^i, u^\alpha, u^\alpha_i)$ on $J^1\pi$ such that $x^i(j^1_x\phi) = x^i(x)$, $u^\alpha(j^1_x\phi) = u^\alpha(\phi(x))$ and $u^\alpha_i(j^1_x\phi) = \frac{\partial \phi^\alpha}{\partial x^i}\big|_x$.*

In the induced local coordinates $(x^i, u^\alpha, u^\alpha_i)$, the source and the target projections are written

$$\pi_1(x^i, u^\alpha, u^\alpha_i) = (x^i) \quad \text{and} \quad \pi_{1,0}(x^i, u^\alpha, u^\alpha_i) = (x^i, u^\alpha). \tag{3.4}$$

From here, it is clear that π_1 and $\pi_{1,0}$ are certainly projections (surjective submersions) over M and E, respectively. Therefore, $(J^1\pi, \pi_1, M)$ and $(J^1\pi, \pi_{1,0}, E)$ are fiber bundles. If we consider a change of coordinates $(x^i, u^\alpha) \mapsto (y^j(x^i), v^\beta(x^i, u^\alpha))$ in E, it induces a change of coordinates $(x^i, u^\alpha, u^\alpha_i) \mapsto (y^j(x^i), v^\beta(x^i, u^\alpha), v^\beta_j(x^i, u^\alpha, u^\alpha_i))$ in $J^1\pi$. In this case, the "velocities" transform by the following rule:

$$v^\beta_j = \left(\frac{\partial v^\beta}{\partial x^i} + \frac{\partial v^\beta}{\partial u^\alpha} u^\alpha_i\right) \frac{\partial x^i}{\partial y^j}. \tag{3.5}$$

Note that the change of coordinates is not linear, like in the tangent bundle, but affine.

Proposition 3.4. *The first jet manifold of π, $J^1\pi$, together with the target projection, $\pi_{1,0}$, is an affine bundle over E. The fiber in $J^1\pi$ over a point $u \in E_x$, $J^1_u\pi$, is diffeomorphic to the affine space*

$$\{A \in \mathrm{Lin}(T_xM, T_uE) : T_u\pi \circ A = \mathrm{Id}_{T_xM}\}.$$

The underlying vector bundle has typical fiber

$$\{A \in \mathrm{Lin}(T_xM, T_uE) : T_u\pi \circ A = 0\} = \mathrm{Lin}(T_xM, \mathcal{V}_u\pi).$$

Moreover, the induced coordinate systems $(x^i, u^\alpha, u^\alpha_i)$ are adapted to the affine bundle structure.

Formally, the associated vector bundle to $J^1\pi$ is the bundle over E whose total space is the tensor product $T^*M \otimes_E \mathcal{V}\pi$, that is, the bundle

$$(T^*M \otimes_E \mathcal{V}\pi, (\tau_E|_{\mathcal{V}\pi}) \circ pr_2, E).$$

Let $j^1_x\phi \in J^1\pi$ and consider a typical element $A \in T^*_xM \otimes \mathcal{V}_{\phi(x)}\pi$, the action of A on $j^1_x\phi$ is the 1-jet $j^1_x\psi = j^1_x\phi + A$ such that $\psi(x) = \phi(x)$ and $T_x\psi = T_x\phi + A$. In adapted coordinates,

$$u^\alpha_i(j^1_x\phi + A) = u^\alpha_i(j^1_x\phi) + A^\alpha_i,$$

where

$$A = A^\alpha_i \,\mathrm{d}x^i \otimes \frac{\partial}{\partial u^\alpha}.$$

Despite $(J^1\pi, \pi_{1,0}, E)$ is affine, if we consider a preferred global section and see it as "the zero section", one could thought of $J^1\pi$ as a vector bundle. Obviously, in general, there is no such preferred global section. But, when E is trivial, there it is. Suppose that $E = M \times F$. For each $u \in E$ we define the constant section $\phi_u \in \Gamma\pi$ by

$$\phi_u(x) := (x, pr_2(u)).$$

We then define the zero section $z \in \Gamma\pi_{1,0}$ by

$$z(u) := j^1_x\phi_u = (x^i, u^\alpha, u^\alpha_i = 0).$$

In the particular case where π is the bundle $(\mathbb{R} \times F, pr_1, \mathbb{R})$, $J^1\pi$ turns to be isomorphic to $\mathbb{R} \times TF$.

3.1.1 Prolongations, lifts and contact

Definition 3.5. Let $\phi \in \Gamma\pi$ be a (local) section, its *first prolongation* is the (local) section of $\pi_{1,0}$ given by

$$(j^1\phi)(x) := j^1_x\phi,$$

for every $x \in M$. An arbitrary (local) section σ of π_1 is said to be *holonomic* if it is the first prolongation of a (local) section $\phi \in \Gamma\pi$, that is, if $\sigma = j^1\phi$.

Definition 3.6. Let $f : E \to F$ be a morphism between two fiber bundles (E, π, M) and (F, ρ, N), such that the induced function on the base, $\check{f} : M \to N$, is a diffeomorphism. The *first prolongation of* f is the map $j^1 f : J^1\pi \to J^1\rho$ given by

$$(j^1 f)(j^1_x\phi) := j^1_{\check{f}(x)}\phi_f, \ \forall j^1_x\phi \in J^1\pi,$$

where $\phi_f := f \circ \phi \circ \check{f}^{-1}$.

$$\begin{tikzcd}
J^1\pi \arrow[r, "j^1 f"] \arrow[d, "\pi_{1,0}"] & J^1\rho \arrow[d, "\rho_{1,0}"'] \\
E \arrow[r, "f"] \arrow[d, "\pi"] & F \arrow[d, "\rho"'] \\
M \arrow[r, "\check{f}"] \arrow[u, bend left, "\phi"] \arrow[uu, bend left=60, "j^1\phi"] & N \arrow[u, bend right, "\phi_f"'] \arrow[uu, bend right=60, "j^1\phi_f"']
\end{tikzcd}$$

Note that the first prolongation $j^1 f$ of a morphism f is both, a morphism between $(J^1\pi, \pi_{1,0}, E)$ and $(J^1\rho, \rho_{1,0}, F)$, and a morphism between $(J^1\pi, \pi_1, M)$ and $(J^1\rho, \rho_1, N)$. In each case, the induced functions between the base spaces are f and $\check{f}$, respectively.

If $(x^i, u^\alpha, u^\alpha_i)$ and $(y^j, v^\beta, v^\beta_j)$ denote adapted coordinates in $J^1\pi$ and $J^1\rho$, respectively, then we have

$$y^j(j^1 f) = f^j, \quad v^\beta(j^1 f) = f^\beta \quad \text{and} \quad v^\beta_j(j^1 f) = \left(\frac{\partial f^\beta}{\partial x^i} + u^\alpha_i \frac{\partial f^\beta}{\partial u^\alpha}\right) \cdot \frac{\partial \check{f}^{-i}}{\partial y^j}.$$

The expression between brackets in the last equation is called the *total derivative* of f^β with respect to x^i. We will come back to it later.

Definition 3.7. Let $\phi : M \to E$ be a section of π, $x \in M$ and $u = \phi(x)$. The *vertical differential* of the section ϕ at the point $u \in E$ is the map

$$\begin{array}{rcl} \mathrm{d}^{\mathrm{v}}_u\phi : T_uE & \longrightarrow & \mathcal{V}_u\pi \\ v & \longmapsto & v - T_u(\phi \circ \pi)(v) \end{array}$$

Namely, $\mathrm{d}^{\mathrm{v}}_u\phi := \mathrm{Id}_u - T_u(\phi \circ \pi)$.

Notice that the image of $\mathrm{d}^{\mathrm{v}}_u\phi$ is certainly in $\mathcal{V}_u\,\pi$ since $T_u\pi\circ\mathrm{d}^{\mathrm{v}}_u=0$ and that, in fact, $\mathrm{d}^{\mathrm{v}}_u\phi$ depends only on $j^1_x\phi$. In adapted local coordinates (x^i,u^α) of E,

$$\mathrm{d}^{\mathrm{v}}_u\phi=\left(\mathrm{d}u^\alpha-\frac{\partial\phi^\alpha}{\partial x^i}\,\mathrm{d}x^i\right)\otimes\frac{\partial}{\partial u^\alpha}. \tag{3.6}$$

Definition 3.8. The *canonical structure form* of $J^1\pi$ is the 1-form θ on $J^1\pi$ with values in $\mathcal{V}\,\pi$ defined by

$$\theta_{j^1_x\phi}(V):=(\,\mathrm{d}^{\mathrm{v}}_{\phi(x)}\phi)(T_{j^1_x\phi}\pi_1(V)),\ V\in T_{j^1_x\phi}J^1\pi, \tag{3.7}$$

where ϕ is any representative of $j^1_x\phi\in J^1\pi$. The contraction of the covectors in $\mathcal{V}^*\,\pi$ with θ defines a "distribution" in $T^*J^1\pi$. This distribution is called the *contact module* or the *Cartan codistribution (of order 1)* and it is denoted $\mathcal{C}^1$. Its elements are *contact forms.* The annihilator of $\mathcal{C}^1$ is the *Cartan distribution (of order 1).*

This is the approach taken by Echevaría-Enríquez *et al.* in [71]. In Saunders' terminology (see [139], pages 136–137), θ is one of the elements that conform the "contact structure" of π_1, which is given by a natural decomposition in $\pi^*_{1,0}(\tau_E)$, what is out of our scope.

Note that the expression (3.7) does not depend on the representative ϕ of $j^1_x\phi$, hence it is well defined. In adapted local coordinates $(x^i,u^\alpha,u^\alpha_i)$ of $J^1\pi$,

$$\theta=\left(\,\mathrm{d}u^\alpha-u^\alpha_i\,\mathrm{d}x^i\right)\otimes\frac{\partial}{\partial u^\alpha}. \tag{3.8}$$

In fact, the contact forms $\mathrm{d}u^\alpha-u^\alpha_i\,\mathrm{d}x^i\in\mathcal{C}^1$ are a base of the contact module.

Proposition 3.9. *Let $(x^i,u^\alpha,u^\alpha_i)$ be adapted coordinates on $J^1\pi$, a basis of the Cartan codistribution is given by the* coordinate *or* local contact forms

$$\theta^\alpha=\,\mathrm{d}u^\alpha-u^\alpha_i\,\mathrm{d}x^i. \tag{3.9}$$

Proposition 3.10. *The canonical structure form $\theta\in\Gamma(T^*J^1\pi\otimes_{J^1\pi}\mathcal{V}\,\pi)$ and the contact forms $\omega\in\mathcal{C}^1$ are pulled back to zero by the first prolongation $j^1\phi$ of any section ϕ of π. Moreover, this characterizes the module of contact forms,* i.e.

$$\omega\in\mathcal{C}^1\ \Leftrightarrow\ (j^1\phi)^*\omega=0,\ \forall\phi\in\Gamma\pi. \tag{3.10}$$

A complementary or dual result to the previous one is the following.

Proposition 3.11. *Let $\sigma\in\Gamma\pi_1$ be a (local) section. The following statements are equivalent:*

1. *σ is holonomic.*
2. *σ pulls back to zero any contact form, that is*

$$\sigma^*\omega=0,\ \forall\omega\in\mathcal{C}^1. \tag{3.11}$$

3. *σ pulls back to zero the canonical structure form, that is*

$$\sigma^*\theta=0. \tag{3.12}$$

Notice that the contact forms are $\pi_{1,0}$-basic, which is clear from the coordinate expression (3.9). Though, therefore they may be thought as forms along $\pi_{1,0}$ rather than on $J^1\pi$. In this sense are defined total derivatives.

Definition 3.12. A *total derivative* is a vector field ξ along $\pi_{1,0}$ which is annihilated by the Cartan codistribution (considered now as forms along $\pi_{1,0}$). Given a system of adapted coordinates $(x^i, u^\alpha, u_i^\alpha)$ in $J^1\pi$, the local vector fields defined along $\pi_{1,0}$ by

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} \tag{3.13}$$

are called *coordinate total derivatives.*

It is immediate to check that coordinate total derivatives are total derivatives, in fact they define a basis of such vector fields. Under a change of coordinates, $(x^i, u^\alpha, u_i^\alpha)$ to $(y^j, v^\beta, v_j^\beta)$, a coordinate total derivative transforms linearly by the Jacobian of the underlying change of coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}y^j} = \frac{\partial x^i}{\partial y^j}\frac{\mathrm{d}}{\mathrm{d}x^i}.$$

If $\xi \in \mathfrak{X}(\pi_{1,0})$ is a total derivative with the coordinate representations

$$\xi = \xi^i \frac{\mathrm{d}}{\mathrm{d}x^i} = \xi^j \frac{\mathrm{d}}{\mathrm{d}y^j},$$

where the coefficients ξ^i and ξ^j are functions on $J^1\pi$, then

$$\xi^i = \xi^j \frac{\partial x^i}{\partial y^j}.$$

Definition 3.13. The *total lift* of a vector field $\xi = \xi^i \partial_i$ on M is the unique total derivative that projects on ξ itself, that is, the vector field $\hat{\xi} \in \mathfrak{X}(J^1\pi)$ locally given by

$$\hat{\xi}(j_x^1\phi) = \xi^i(x) \left.\frac{\mathrm{d}}{\mathrm{d}x^i}\right|_{j_x^1\phi}. \tag{3.14}$$

Note that the total lift of the coordinate partial derivatives in M are precisely the coordinate total derivatives.

Now, consider the action of total derivatives on smooth functions over E. If $f \in \mathcal{C}^\infty(E)$, the action of $\mathrm{d}/\mathrm{d}x^i$ on it yields a function $\mathrm{d}f/\mathrm{d}x^i \in \mathcal{C}^\infty(J^1\pi)$. In particular, the action of $\mathrm{d}/\mathrm{d}x^i$ on the coordinate function $u^\alpha \in \mathcal{C}^\infty(E)$, gives as expected

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}x^i} = u_i^\alpha \in \mathcal{C}^\infty(J^1\pi).$$

Another interesting fact is how total derivatives and jets are related. Let $f \in \mathcal{C}^\infty(E)$, $\phi \in \Gamma\pi$ and $\xi \in \mathfrak{X}(M)$, we have

$$\xi(f \circ \phi) = \hat{\xi}(f) \circ j^1\phi, \tag{3.15}$$

and in coordinates

$$\frac{\partial(f \circ \phi)}{\partial x^i} = \frac{\mathrm{d}f}{\mathrm{d}x^i} \circ j^1\phi. \tag{3.16}$$

Finally, note that coordinate total derivatives and ordinary partial derivates do not necesarily conmute:

$$\frac{\partial}{\partial x^i}\frac{\mathrm{d}f}{\mathrm{d}x^j} = \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial f}{\partial x^i}, \quad \frac{\partial}{\partial u^\alpha}\frac{\mathrm{d}f}{\mathrm{d}x^j} = \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial f}{\partial u^\alpha} \quad \text{but} \quad \frac{\partial}{\partial u_i^\alpha}\frac{\mathrm{d}f}{\mathrm{d}x^j} = \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial f}{\partial u_i^\alpha} + \delta_i^j\frac{\partial f}{\partial u^\alpha},$$

where $f \in \mathcal{C}^\infty(E)$.

Definition 3.14. Given a vector field ξ on E, its *first lift* (or *first jet*) is the unique vector field $\xi^{(1)}$ on $J^1\pi$ that is projectable to ξ by $\pi_{1,0}$ and preserves the Cartan codistribution with respect to the Lie derivative, *i.e.* $\mathfrak{L}_{\xi^{(1)}}\omega \in \mathcal{C}^1$ for any $\omega \in \mathcal{C}^1$.

Proposition 3.15. *Let ξ be a vector field on E. If ξ has the local expression*

$$\xi = \xi^i\frac{\partial}{\partial x^i} + \xi^\alpha\frac{\partial}{\partial u^\alpha} \tag{3.17}$$

in adapted coordinates (x^i, u^α) on E, then its first lift $\xi^{(1)}$ has the form

$$\xi^{(1)} = \xi^i\frac{\partial}{\partial x^i} + \xi^\alpha\frac{\partial}{\partial u^\alpha} + \left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}x^i} - u_j^\alpha\frac{\mathrm{d}\xi^j}{\mathrm{d}x^i}\right)\frac{\partial}{\partial u_i^\alpha} \tag{3.18}$$

for the induced coordinates $(x^i, u^\alpha, u_i^\alpha)$ on $J^1\pi$.

Originally, the first lift is defined for π-projectable vector fields on E. The first lift of such vector field ξ is the infinitesimal generator of the first lift of the flow of ξ. Definition 3.14 is a characterization of this property and it is generalized for any kind of vector fields on E (see [71]).

Proposition 3.16. *Let ψ_ε be the flow of a given π-projectable vector field ξ over E. Then, the flow of $\xi^{(1)}$ is the first prolongation of ψ_ε, $j^1\psi_\varepsilon$.*

3.1.2 The vertical endomorphisms

Definition 3.17. Given a 1-jet $j_x^1\phi \in J^1\pi$, let $A \in T_x^*M \otimes \mathcal{V}_{\phi(x)}\,\pi$. The *vertical lift* of A at $j_x^1\phi$ is the tangent vector $A_{j_x^1\phi}^{\mathrm{v}} \in T_{j_x^1\phi}(J^1\pi)$ given by

$$A_{j_x^1\phi}^{\mathrm{v}}(f) = \frac{\mathrm{d}}{\mathrm{d}t}f(j_x^1\phi + tA)\big|_{t=0},\ \ \forall f \in \mathcal{C}^\infty(J_{\phi(x)}^1\pi). \tag{3.19}$$

By the very definition of vertical lift, given a smooth function $f \in \mathcal{C}^\infty(E)$,

$$\begin{aligned}
(T_{j_x^1\phi}\pi_{1,0})(A_{j_x^1\phi}^{\mathrm{v}})(f) &= A_{j_x^1\phi}^{\mathrm{v}}(f \circ \pi_{1,0}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \pi_{1,0})(j_x^1\phi + tA)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}f(\phi(x))\big|_{t=0} \\
&= 0.
\end{aligned}$$

Thus, the vertical lift takes values into the vertical fiber bundle $\mathcal{V}\,\pi_{1,0} \subset TJ^1\pi$. Indeed, it is a morphism of vector bundles over the identity of $J^1\pi$,

$$(\cdot)^{\mathrm{v}} : T^*M \otimes_{J^1\pi} \mathcal{V}\,\pi \longrightarrow \mathcal{V}\,\pi_{1,0}.$$

Note that, this time, the tensor product is taken over $J^1\pi$ and not over E. Note also that for each $j^1_x\phi \in J^1\pi$, the vertical lift at $j^1_x\phi$,

$$(\cdot)^{\mathrm{v}}_{j^1_x\phi} : T^*_x M \otimes \mathcal{V}_{\phi(x)}\,\pi \longrightarrow \mathcal{V}_{j^1_x\phi}\,\pi_{1,0} \subset T_{j^1_x\phi} J^1\pi,$$

is a linear isomorphism. In adapted local coordinates $(x^i, u^\alpha, u^\alpha_i)$, if $A = A^\alpha_i \,\mathrm{d}x^i|_x \otimes \partial/\partial u^\alpha|_{\phi(x)}$, then

$$A^{\mathrm{v}}_{j^1_x\phi} = A^\alpha_i \frac{\partial}{\partial u^\alpha_i}\Big|_{j^1_x\phi} \quad \text{and} \quad (\cdot)^{\mathrm{v}} = \mathrm{d}u^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u^\alpha_i}. \tag{3.20}$$

Definition 3.18. Let $\eta \in \Lambda^m M$ be an arbitrary m-form on M. The *vertical endomorphism* associated to η is the vector valued m-form $\mathcal{S}_\eta : (TJ^1\pi)^m \longrightarrow TJ^1\pi$ that gives

$$\mathcal{S}_\eta(V_1, \ldots, V_m) := \sum_{i=1}^m \left\{\eta^i \otimes [(T_{j^1_x\phi}\pi_{1,0})(V_i) - (T_{j^1_x\phi}(\phi\circ\pi_1))(V_i)]\right\}^{\mathrm{v}}, \tag{3.21}$$

for any m tangent vectors $V_1, \ldots, V_m \in T_{j^1_x\phi}J^1\pi$, and where η^i is the contraction

$$\eta^i := (-1)^{m-i}\eta_x(V_1, \ldots, \widehat{V_i}, \ldots, V_m)$$

with the hatted factor omitted.

Definition 3.19. The *(canonical) vertical endomorphism* $\mathcal{S}$ arises from the natural contraction between the factors in $\mathcal{V}\pi$ of the structure canonical form θ and the factors in $\mathcal{V}^*\pi$ of the vertical lift $(\cdot)^{\mathrm{v}}$; that is

$$\mathcal{S} := \langle \theta, (\cdot)^{\mathrm{v}} \rangle \in \Gamma(T^*J^1\pi \otimes_{J^1\pi} TM \otimes_{J^1\pi} \mathcal{V}\pi_{1,0}). \tag{3.22}$$

In adapted coordinates $(x^i, u^\alpha, u^\alpha_i)$ of $J^1\pi$, the vertical endomorphisms have the local expressions

$$\mathcal{S}_\eta = (\,\mathrm{d}u^\alpha - u^\alpha_j\,\mathrm{d}x^j) \wedge \mathrm{d}^{m-1}x_i \otimes \frac{\partial}{\partial u^\alpha_i} = \theta^\alpha \wedge \mathrm{d}^{m-1}x_i \otimes \frac{\partial}{\partial u^\alpha_i} \tag{3.23}$$

and

$$\mathcal{S} = (\,\mathrm{d}u^\alpha - u^\alpha_j\,\mathrm{d}x^j) \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u^\alpha_i} = \theta^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u^\alpha_i}, \tag{3.24}$$

where $\theta^\alpha = \mathrm{d}u^\alpha - u^\alpha_j\,\mathrm{d}x^j$ are the local contact forms and $\mathrm{d}^{m-1}x_i = i_{\partial/\partial x^i}\,\mathrm{d}^m x$.

3.1.3 Partial Differential Equations

Lemma 3.20. *If N is an open submanifold of M, then $J^1(\pi_N) \simeq \pi_1^{-1}(N)$, where $\pi_N := \pi|_{\pi_1^{-1}(N)}$.*

Definition 3.21. A *first-order differential equation on* π is a closed embedded submanifold $\mathcal{P}$ of the first jet manifold $J^1\pi$. A *solution* of $\mathcal{P}$ is a local section $\phi \in \Gamma_N\pi$, where N is an open submanifold of M, which satisfies $j^1_x\phi \in \mathcal{P}$ for every $x \in N$. A first-order differential equation $\mathcal{P}$ is said to be *integrable at* $z \in \mathcal{P}$ if there is a solution ϕ of $\mathcal{P}$ (around some neighborhood N of $\pi_1(z)$) such that $z = j^1_{\pi_1(z)}\phi$. A first-order differential equation $\mathcal{P}$ is said to be *integrable in a subset* $\mathcal{P}' \subset \mathcal{P}$ if it is integrable at each $z \in \mathcal{P}'$. A first-order differential equation $\mathcal{P}$ is said to be *integrable* if it is integrable at each $z \in \mathcal{P}$.

If l is the codimension of $\mathcal{P}$ ($\dim J^1\pi - \dim \mathcal{P}$), there locally exist submersions $\Psi : J^1\pi \to \mathbb{R}^l$ for whom $\mathcal{P}$ is the zero level set. Written in local coordinates, $\mathcal{P}$ is given by the set of points that satisfy

$$\Psi^\mu(x^i, u^\alpha, u_i^\alpha) = 0, \;\; \mu = 1, \ldots, l.$$

Thus, first-order differential equations are a geometric interpretation of the usual first-order partial differential equations. Under certain conditions, one could solve the previous equation for some of the velocities u_i^α making them to depend on the other variables (then $\pi_{1,0}|_{\mathcal{P}} : \mathcal{P} \to E$ would be a submersion). For simplicity, if $n = 1$ and $1 < l < m$, rearranging conveniently the base variables, the previous equation could be equivalent to the following expression

$$u_{m-l+\mu} = \phi_\mu(x^i, u, u_1, \ldots, u_{m-l}), \;\; \mu = 1, \ldots, l.$$

In the general case, if one projects $\mathcal{P}$ to E by $\pi_{1,0}$, he would obtain a subset $\mathcal{P}^{(0,0)}$ of E, let us assume it is a smooth submanifold, which is not necessarily the whole of E. In such a case, it means that we are dealing with some constraint on the total space E itself. An integral holonomic section $j^1\phi$ of $\mathcal{P}$ will be such that the image of ϕ belongs to $\mathcal{P}^{(0,0)}$ and the image of $j^1\phi$ belongs to $\mathcal{P}^{(1,1)} := J^1\mathcal{P}^{(0,0)} \cap \mathcal{P}$. The submanifold $\mathcal{P}^{(1,1)}$ of $J^1\pi$ introduces new constraints that a solution of $\mathcal{P}$ must satisfy, moreover it represents the first step of the extension to jet bundles of the algorithm to extract the integral part of a differential equation in a tangent bundle, which was presented by Mendella *et al.* in [127] (see also [126, 128]). The general algorithm will be given in Section §4.1.3.

Example 3.22. Given the fiber bundle $pr_1 : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^3$ with global coordinates (x, y, u, v, w), consider the constraint submanifold of $J^1\, pr_1$

$$\begin{aligned}\mathcal{P} = \{(x, y, u, v, w, u_x, u_y, v_x, v_y, w_x, w_y) \in J^1\, pr_1 \;:\; \\ u = 0,\;\; v = w,\;\; u_x = v_y,\;\; u_y = -w_x\}.\end{aligned} \tag{3.25}$$

Then,

$$\mathcal{P}^{(0,0)} = \left\{(x, y, u, v, w) \in \mathbb{R}^2 \times \mathbb{R}^3 \;:\; u = 0,\;\; v = w\right\},$$

and

$$\mathcal{P}^{(1,1)} = J^1\mathcal{P}^{(0,0)} \cap \mathcal{P} = \left\{(x, y, 0, v, v, 0, 0, 0, 0, 0, 0) \in J^1\, pr_1\right\}$$

is the integral part of $\mathcal{P}$. Thus, holonomic integral sections of $\mathcal{P}$ are of the form

$$\phi(x, y) = (x, y, 0, c, c),$$

where c is any real number.

3.1.4 The Dual Jet Bundle

Definition 3.23. The *dual jet bundle* of π, denoted $J^1\pi^\dagger$, is the reunion of the affine maps from $J^1_u\pi$ to $\Lambda^m_{\pi(u)}M$, where u is an arbitrary point of E. Namely,

$$J^1\pi^\dagger := \bigcup_{u\in E} \mathrm{Aff}(J^1_u\pi, \Lambda^m_{\pi(u)}M). \tag{3.26}$$

The functions given by

$$\begin{array}{rcl} \pi_1^\dagger : J^1\pi^\dagger & \longrightarrow & M \\ \omega \in J^1_u\pi^\dagger & \longmapsto & \pi(u) \end{array} \tag{3.27}$$

and

$$\begin{array}{rcl} \pi_{1,0}^\dagger : J^1\pi^\dagger & \longrightarrow & E \\ \omega \in J^1_u\pi^\dagger & \longmapsto & u \end{array} \tag{3.28}$$

where $J^1_u\pi^\dagger = \mathrm{Aff}(J^1_u\pi, \Lambda^m_{\pi(u)}M)$, are called the *dual source projection* and the *dual target projection* respectively.

The duality nature of $J^1\pi^\dagger$ gives rise to a natural pairing between its elements and those of $J^1\pi$. The pairing will be denoted by the usual angular brackets, $\langle\,,\rangle : J^1\pi^\dagger \otimes_E J^1\pi \to \Lambda^m M$.

Proposition 3.24. *The dual jet bundle of π, $J^1\pi^\dagger$, may be endowed with a structure of smooth manifold. A system of adapted coordinates (x^i, u^α) in E induces a system of coordinates $(x^i, u^\alpha, p, p^i_\alpha)$ in $J^1\pi^\dagger$ such that, for any $j^1_x\phi \in J^1\pi$ and any $\omega \in J^1_{\phi(x)}\pi^\dagger$, $x^i(\omega) = x^i(x)$, $u^\alpha(\omega) = u^\alpha(\phi(x))$ and $\langle\omega, j^1_x\phi\rangle = (p + p^i_\alpha u^\alpha_i)\,\mathrm{d}^m x$.*

In the induced local coordinates $(x^i, u^\alpha, p, p^i_\alpha)$, the dual source and the dual target projections are written

$$\pi_1^\dagger(x^i, u^\alpha, p, p^i_\alpha) = (x^i) \quad \text{and} \quad \pi_{1,0}^\dagger(x^i, u^\alpha, p, p^i_\alpha) = (x^i, u^\alpha). \tag{3.29}$$

From here, it is clear that $\pi_1^\dagger$ and $\pi_{1,0}^\dagger$ are certainly projections over M and E respectively. Therefore, $(J^1\pi^\dagger, \pi_1^\dagger, M)$ and $(J^1\pi^\dagger, \pi_{1,0}^\dagger, E)$ are fiber bundles. If we consider a change of coordinates $(x^i, u^\alpha) \mapsto (y^j, v^\beta)$ in E, it induces a change of coordinates $(x^i, u^\alpha, p, p^i_\alpha) \mapsto (y^j, v^\beta, q, q^j_\beta)$ in $J^1\pi^\dagger$. In this case, the "momenta" transform by the following rule:

$$q = \mathrm{Jac}(x(y))\left(p + \frac{\partial u^\alpha}{\partial y^j} p^i_\alpha \frac{\partial y^j}{\partial x^i}\right) \quad \text{and} \quad q^j_\beta = \mathrm{Jac}(x(y))\left(\frac{\partial u^\alpha}{\partial v^\beta} p^i_\alpha \frac{\partial y^j}{\partial x^i}\right), \tag{3.30}$$

where $\mathrm{Jac}(x(y))$ is the Jacobian determinant of the transformation $(y^j) \mapsto (x^i)$. Note that the local volume form and its contraction transforms under the change of coordinates by

$$\mathrm{d}^m y = \mathrm{Jac}(y(x))\,\mathrm{d}^m x \quad \text{and} \quad \mathrm{d}^{m-1} y_j = \mathrm{Jac}(y(x))\frac{\partial x^i}{\partial y^j}\,\mathrm{d}^{m-1} x_i. \tag{3.31}$$

Proposition 3.25. *The dual jet bundle of π, $J^1\pi^\dagger$, together with the dual target projection, $\pi_{1,0}^\dagger$, is a vector bundle over E. Moreover, the induced coordinate systems $(x^i, u^\alpha, p, p^i_\alpha)$ are adapted to the vector bundle structure.*

Definition 3.26. The *reduced dual jet bundle* of π, denoted $J^1\pi^\circ$, is the quotient of $J^1\pi^\dagger$ by constant affine transformations along the fibers of $\pi_{1,0}$. The quotient map will be $\mu : J^1\pi^\dagger \to J^1\pi^\circ$.

Proposition 3.27. *We have that:*

1. *$J^1\pi^\circ$ may be endowed with a structure of smooth manifold;*

2. *$(J^1\pi^\dagger, \mu, J^1\pi^\circ)$ is a smooth vector bundle of rank 1;*

3. *Adapted coordinates* (x^i, u^α) *on* E *induce coordinates* $(x^i, u^\alpha, p^i_\alpha)$ *on* $J^1\pi^\circ$ *such that* $\mu(x^i, u^\alpha, p, p^i_\alpha) = (x^i, u^\alpha, p^i_\alpha)$*, where* $(x^i, u^\alpha, p, p^i_\alpha)$ *are the induced coordinates on* $J^1\pi^\dagger$*.*

The extended and the reduced dual jet bundles of π may also be realized by means of basic and semi-basic forms. Recall that π-basic (resp. π-semi-basic) forms are forms over E annihilated by the contraction with at least one (resp. two) π-vertical vector.

Proposition 3.28. *The extended dual jet bundle,* $J^1\pi^\dagger$*, and the set of* π*-semi-basic* m*-forms over* E*,* $\Lambda^m_2 E$*, with canonical projection* $\Lambda^k\pi_E|_{\Lambda^k_2 E} : \Lambda^k_2 E \to E$ *are isomorphic.*

Proof. Given a semi-basic m-form $\omega \in \Lambda^m_2 E$, let $u = \Lambda^k\pi_E(\omega) \in E$ and consider the function that sends any 1-jet $j^1_x\phi \in J^1_u\pi$ to the pullback of ω by ϕ at x. This does not depend on the representative $\phi \in \Gamma_x\pi$ of $j^1_x\phi$. Moreover, this function is affine with respect to $j^1_x\phi$ thus, this defines a morphism Υ from $\Lambda^m_2 E$ to $J^1\pi^\dagger$ as follows

$$\begin{array}{rccccc} \Upsilon : \Lambda^m_2 E & \longrightarrow & J^1\pi^\dagger & & \\ \omega & \longmapsto & \Upsilon(\omega) : J^1_u\pi & \longrightarrow & \Lambda^m M \\ & & j^1_x\phi & \longmapsto & \phi^*_x\omega \end{array}$$

where $u = \pi^\dagger_{1,0}(\omega)$. It is easy to check that Φ is a smooth isomorphism of vector bundles. $\square$

Semi-basic m-forms $\omega \in \Lambda^m_2 E$ are locally written

$$\omega = \tilde{p}\,\mathrm{d}^m x + \tilde{p}^i_\alpha\,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i.$$

Thus, adapted coordinates (x^i, u^α) on E induce adapted coordinates on $(x^i, u^\alpha, \tilde{p}, \tilde{p}^i_\alpha)$ on $\Lambda^m_2 E$. The isomorphism defined in the previous proof takes then the local expression

$$\Upsilon(x^i, u^\alpha, \tilde{p}, \tilde{p}^i_\alpha) = (x^i, u^\alpha, \tilde{p}, \tilde{p}^i_\alpha) : (x^i, u^\alpha, u^\alpha_i) \in J^1_u\pi \mapsto \tilde{p} + \tilde{p}^i_\alpha u^\alpha_i \in \mathbb{R}.$$

Consider now the set of π-basic forms, $\Lambda^m_1 E$. An arbitrary basic form ω is locally written

$$\omega = \tilde{p}\,\mathrm{d}^m x.$$

Notice that $\Lambda^m_1 E$ coincides with the pullback to E of $\Lambda^m M$ or with the set of constant affine transformations on the fibers of $\pi_{1,0}$.

Corollary 3.29. *The reduced dual jet bundle* $J^1\pi^\circ$ *is canonically isomorphic to the quotient of semi-basic* m*-forms* $\Lambda^m_2 E$ *by the basic* m*-forms* $\Lambda^m_1 E$*, that is* $J^1\pi^\circ \cong \Lambda^m_2 E/\Lambda^m_1 E$*.*

Proof. Let $\Psi : \Lambda^m_2 E \to J^1\pi^\dagger$ be the canonical isomorphism given in Proposition 3.28. Since the set of constant affine transformations on the fibers of $\pi_{1,0}$ coincides with the set of basic m-forms over E, $\mu \circ \Psi$ is constant along the fibers of $\bar{\mu}$ as well as $\Psi \circ \bar{\mu}$ along the fibers of μ. Hence Ψ passes smoothly to the quotient to an isomorphism $\upsilon : \Lambda^m_2 E/\Lambda^m_1 E \to J^1\pi^\circ$. $\square$

While $J^1\pi^\dagger$ is naturally paired with $J^1\pi$, remember that $\Lambda^m_2 E$ has a canonical multisymplectic structure (see Example 1.43). Consider the Liouville m-form Θ on $\Lambda^m_2 E$, which is locally given by the expression

$$\Theta = \tilde{p}\,\mathrm{d}^m x + \tilde{p}^i_\alpha\,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i, \tag{3.32}$$

for adapted coordinates $(x^i, u^\alpha, \tilde{p}, \tilde{p}^i_\alpha)$ on $\Lambda^m_2 E$. Then, the canonical multi-symplectic $(m+1)$-form on $\Lambda^m_2 E$ is

$$\Omega = -\,\mathrm{d}\Theta = -\,\mathrm{d}\tilde{p} \wedge \mathrm{d}^m x - \mathrm{d}\tilde{p}^i_\alpha \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i. \tag{3.33}$$

Thanks to the identification between $J^1\pi^\dagger$ and $\Lambda^m_2 E$ (and their respective quotients), any structure carried by one of them can be translated to the other. In particular, the multi-symplectic form. From now on, no distintion will be made between $J^1\pi^\dagger$ and $\Lambda^m_2 E$ (or between $J^1\pi^\circ$ and $\Lambda^m_2 E/\Lambda^m_1 E$). Although the "dual" notation will be used for sets, coordinates, structures, etc.

3.2 Classical Field Theory

3.2.1 The Lagrangian Formalism

This section is devoted to the first order Lagrangian formalism in jet manifolds. The main ingredients are the following: the Lagrangian density, the Poincaré-Cartan form, the premultisymplectic structure defined from the multimomentum Liouville form and the Legendre transformation. We shall use the same notations as in the previous section.

The variational approach

Definition 3.30. A *Lagrangian density* is a fibered mapping $\mathcal{L} : J^1\pi \to \Lambda^m M$.

Since we assume that M is an oriented manifold, with volume form η, we can write $\mathcal{L} = L\eta$, where $L : J^1\pi \to \mathbb{R}$ is the Lagrangian function. The manifold $J^1\pi$ plays the role of the finite-dimensional configuration space of fields.

Definition 3.31. Given a Lagrangian density $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$, the associated *integral action* is the map $\mathcal{A}_\mathcal{L} : \Gamma\pi \times \mathcal{K} \longrightarrow \mathbb{R}$ given by

$$\mathcal{A}_\mathcal{L}(\phi, R) = \int_R (j^1\phi)^* \mathcal{L}, \tag{3.34}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

Definition 3.32. Let ϕ be a section of π. A *(vertical) variation* of ϕ is a curve $\varepsilon \in I \mapsto \phi_\varepsilon \in \Gamma\pi$ (for some interval $I \subset \mathbb{R}$ containing the 0) such that $\phi_\varepsilon = \varphi_\varepsilon \circ \phi \circ (\check{\varphi}_\varepsilon)^{-1}$, where φ_ε is the flow of a (vertical) π-projectable vector field ξ on E.

Definition 3.33. We say that $\phi \in \Gamma\pi$ is a *critical* or *stationary point* of the Lagrangian action $\mathcal{A}_\mathcal{L}$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\mathcal{A}_\mathcal{L}(\phi_\varepsilon, R_\varepsilon)\right]\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_{R_\varepsilon} (j^1\phi_\varepsilon)^* \mathcal{L}\right]\Bigg|_{\varepsilon=0} = 0, \tag{3.35}$$

for any variation ϕ_ε of ϕ whose associated vector field vanishes outside of $\pi^{-1}(R)$.

Lemma 3.34. *Let $\phi_\varepsilon = \varphi_\varepsilon \circ \phi \circ (\check{\varphi}_\varepsilon)^{-1}$ be a variation of a section $\phi \in \Gamma\pi$. If ξ denotes the infinitesimal generator of φ_ε, then*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[((j^1\phi_\varepsilon) \circ \check{\varphi}_\varepsilon)^*_x \omega\right]\Big|_{\varepsilon=0} = (j^1\phi)^*_x(\mathfrak{L}_{\xi^{(1)}}\omega), \tag{3.36}$$

for any differential form $\omega \in \Omega(J^1\pi)$.

Proof. From Proposition 3.16, we have that $\xi^{(1)}$ is the infinitesimal generator of $j^1\varphi_\varepsilon$. We then obtain by a direct computation,

$$\begin{aligned}(j^1\phi)_x^*(\mathfrak{L}_{\xi^{(1)}}\omega) &= (j^1\phi)_x^*\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^1\varphi_\varepsilon)^*\omega\right]\Big|_{\varepsilon=0}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^1\varphi_\varepsilon\circ j^1\phi)_x^*\omega\right]\Big|_{\varepsilon=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^1\phi_\varepsilon\circ j^1\check{\varphi}_\varepsilon)_x^*\omega\right]\Big|_{\varepsilon=0}.\end{aligned}$$

□

Theorem 3.35 (The Euler-Lagrange equations)**.** *Given a fiber section $\phi\in\Gamma\pi$, let us consider an infinitesimal variation ϕ_ε of it such that the support R of the associated vector field $\xi\in\mathfrak{X}(E)$ is contained in a coordinate chart (x^i) of M. We then have that the variation of the Lagrangian action $\mathcal{A}_\mathcal{L}$ at ϕ is given by*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_\mathcal{L}(\phi_\varepsilon,R)\Big|_{\varepsilon=0} &= \int_R (j^2\phi)^*\left[(\xi^\alpha-u_i^\alpha\xi^i)\left(\frac{\partial L}{\partial u^\alpha}-\frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha}\right)\right]\mathrm{d}^m x\\ &\quad -\int_{\partial R}(j^1\phi)^*\left[\xi^i L+(\xi^\alpha-u_i^\alpha\xi^i)\frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^{m-1}x_i,\end{aligned}\tag{3.37}$$

where $(x^i,u^\alpha,u_i^\alpha,u_{ij}^\alpha)$ are adapted coordinates on $J^2\pi$. Moreover, ϕ is a critical point of the Lagrangian action $\mathcal{A}_\mathcal{L}$ if and only if it satisfies the Euler-Lagrange equations

$$(j^2\phi)^*\left(\frac{\partial L}{\partial u^\alpha}-\frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha}\right)=0 \tag{3.38}$$

on the interior of M, plus the boundary conditions

$$(j^1\phi)^*L=(j^1\phi)^*\frac{\partial L}{\partial u_i^\alpha}=0, \tag{3.39}$$

on the boundary ∂M of M.

Proof. Let us denote by ξ the vector field associated to the variation ϕ_ε. By Proposition 3.34 and Cartan's formula $\mathfrak{L}=\mathrm{d}\circ i+i\circ\mathrm{d}$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_\mathcal{L}(\phi_\varepsilon,R_\varepsilon)\Big|_{\varepsilon=0} &= \int_R \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^1\phi_\varepsilon\circ\check{\varphi}_\varepsilon)^*\mathcal{L}\right]\Big|_{\varepsilon=0}\\ &= \int_R (j^1\phi)^*(\mathfrak{L}_{\xi^{(1)}}\mathcal{L})\\ &= \int_R (j^1\phi)^*\,\mathrm{d}(i_{\xi^{(1)}}\mathcal{L})+\int_R (j^1\phi)^* i_{\xi^{(1)}}\,\mathrm{d}\mathcal{L}\\ &= \int_{\partial R}(j^1\phi)^* i_{\xi^{(1)}}\mathcal{L}+\int_R (j^1\phi)^*(\xi^{(1)}(L)\,\mathrm{d}^m x-\mathrm{d}L\wedge i_{\xi^{(1)}}\,\mathrm{d}^m x).\end{aligned}$$

So as to develop the last three terms, we shall use the coordinate expression of $\xi^{(1)}$ given in Proposition 3.15. Therefore, the first boundary integral is

$$\int_{\partial R}(j^1\phi)^* i_{\xi^{(1)}}\mathcal{L}=\int_{\partial R}(j^1\phi)^*\left(L\xi^i\,\mathrm{d}^{m-1}x_i\right).$$

For the second term, taking into account Equation (3.16) and using integration by parts, we obtain

$$\begin{aligned}
&\int_R (j^1\phi)^*[\xi^{(1)}(L)\,\mathrm{d}^m x] = \\
&\quad = \int_R (j^1\phi)^* \left[\xi^i \frac{\partial L}{\partial x^i} + \xi^\alpha \frac{\partial L}{\partial u^\alpha} + \left(\frac{\mathrm{d}\xi^\alpha}{\mathrm{d}x^i} - u_j^\alpha \frac{\mathrm{d}\xi^j}{\mathrm{d}x^i}\right)\frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^m x \\
&\quad = \int_R (j^2\phi)^* \left[\xi^i \frac{\partial L}{\partial x^i} + \xi^\alpha \frac{\partial L}{\partial u^\alpha} + \frac{\mathrm{d}}{\mathrm{d}x^i}\left(\xi^\alpha - u_j^\alpha \xi^j\right)\frac{\partial L}{\partial u_i^\alpha} + u_{ji}^\alpha \xi^j \frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^m x \\
&\quad = \int_R (j^2\phi)^* \left[\xi^i \frac{\partial L}{\partial x^i} + \xi^\alpha \frac{\partial L}{\partial u^\alpha} - \left(\xi^\alpha - u_j^\alpha \xi^j\right)\frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha} + u_{ji}^\alpha \xi^j \frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^m x \\
&\qquad + \int_{\partial R} (j^1\phi)^* \left[\left(\xi^\alpha - u_j^\alpha \xi^j\right)\frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^{m-1}x_i.
\end{aligned}$$

And the third term is

$$\begin{aligned}
&\int_R (j^1\phi)^*(\mathrm{d}L \wedge i_{\xi^{(1)}}\,\mathrm{d}^m x) = \\
&\quad = \int_R (j^1\phi)^* \left[\left(\frac{\partial L}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial L}{\partial u^\alpha}\,\mathrm{d}u^\alpha + \frac{\partial L}{\partial u_i^\alpha}\,\mathrm{d}u_i^\alpha\right)\wedge \xi^j\,\mathrm{d}^{m-1}x_j\right] \\
&\quad = \int_R (j^2\phi)^* \left[\xi^i\left(\frac{\partial L}{\partial x^i} + u_i^\alpha \frac{\partial L}{\partial u^\alpha} + u_{ji}^\alpha \frac{\partial L}{\partial u_j^\alpha}\right)\right]\mathrm{d}^m x.
\end{aligned}$$

Adding the three developments that we have computed, some terms cancel out and, rearranging properly the remaining ones, we obtain the first statement of the theorem.

If we now suppose that R is contained in the interior of M, as ξ is null outside of R, so it is $\xi^{(1)}$ outside of R and, by smoothness, on its boundary ∂R. Thus, if ϕ is a critical point of $\mathcal{A}_\mathcal{L}$, we then must have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_\mathcal{L}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = \int_R (j^2\phi)^* \left[\left(\xi^\alpha - u_i^\alpha \xi^i\right)\left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha}\right)\right]\mathrm{d}^m x = 0,$$

for any vertical field ξ whose compact support is contained in $\pi^{-1}(R)$. We thus infer that ϕ shall satisfy the higher-order Euler-Lagrange equations (3.38) on the interior of M.

Finally, if R has common boundary with M and ϕ is a critical point of $\mathcal{A}_\mathcal{L}$, from the above results, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_\mathcal{L}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = -\int_{\partial R\cap\partial M} (j^1\phi)^* \left[\xi^i L + \left(\xi^\alpha - u_i^\alpha \xi^i\right)\frac{\partial L}{\partial u_i^\alpha}\right]\mathrm{d}^{m-1}x_i = 0.$$

Since this is true for any vector field ξ whose compact support is contained in $\pi^{-1}(R)$, then the boundary conditions (3.39) follows. □

Remark 3.36. In the definition 3.33 of critical point of the Lagrangian action $\mathcal{A}_\mathcal{L}$, we have considered the widest range of variations, with the consequent decrement of the set of possible solutions. There, two different requirements on the variations could have been made, deriving in a broader set of solutions. First, we could have imposed verticality to the variations, resulting in a substantial simplification of the proof, and we would still have obtained the Euler-Lagrange equations (3.38) but, this time, without the restriction $(j^1\phi)^*L = 0$ on ∂M. The same set of solution would have been obtained with verticality only along the boundary ∂M. Secondly, we could have imposed null variations along ∂M, which would have implied no restrictions of the solutions of the Euler-Lagrange equations along ∂M, neither $(j^1\phi)^*L = 0$ nor $(j^1\phi)^*\partial L/\partial u_i^\alpha = 0$.

If we have avoided these assumptions and followed this more general procedure is to stress out the strong relation with the geometric structure of jet bundles, in particular with the so-called Poincaré-Cartan form, which will appear clear in the next section.

The geometric approach

Definition 3.37. The *Poincaré-Cartan m-form* associated with the Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$ is defined by

$$\Theta_{\mathcal{L}} := \mathcal{L} + \langle \mathcal{S},\, \mathrm{d}\mathcal{L} \rangle\,, \tag{3.40}$$

where $\mathcal{S}$ is the canonical vertical endomorphism of $J^1\pi$ and $\langle \mathcal{S},\, \mathrm{d}\mathcal{L} \rangle$ is the contraction between the factors in $\mathcal{V}\pi_1$ of $\mathcal{S}$ and those in $T^*J^1\pi$ of $\mathrm{d}\mathcal{L}$. The *Poincaré-Cartan* $(m+1)$-*form* associated with $\mathcal{L}$ is defined by

$$\Omega_{\mathcal{L}} := -\,\mathrm{d}\Theta_{\mathcal{L}}. \tag{3.41}$$

In local coordinates, if $\mathcal{L} = L\,\mathrm{d}^m x$, we get:

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \left(L - u_i^\alpha \frac{\partial L}{\partial u_i^\alpha} \right) \mathrm{d}^m x + \frac{\partial L}{\partial u_i^\alpha} du^\alpha \wedge \mathrm{d}^{m-1}x_i && (3.42)\\
&= \mathcal{L} + \frac{\partial L}{\partial u_i^\alpha}\,\theta^\alpha \wedge \mathrm{d}^{m-1}x_i,\\
\Omega_{\mathcal{L}} &= -\,\mathrm{d}\left(L - u_i^\alpha \frac{\partial L}{\partial u_i^\alpha} \right) \wedge \mathrm{d}^m x - \mathrm{d}\left(\frac{\partial L}{\partial u_i^\alpha} \right) \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i && (3.43)\\
&= -\theta^\alpha \wedge \left(\frac{\partial L}{\partial u^\alpha}\,\mathrm{d}^m x - \mathrm{d}\left(\frac{\partial L}{\partial u_i^\alpha} \right) \wedge \mathrm{d}^{m-1}x_i \right).
\end{aligned}$$

If we conveniently denote $\hat{p}_\alpha^i := \frac{\partial L}{\partial u_i^\alpha}$ and $\hat{p} := L - \hat{p}_\alpha^i u_i^\alpha$, then the local expression of the Poincaré-Cartan forms are now

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \hat{p}\,\mathrm{d}^m x + \hat{p}_\alpha^i\,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i\\
\Omega_{\mathcal{L}} &= -\,\mathrm{d}\hat{p} \wedge \mathrm{d}^m x - \mathrm{d}\hat{p}_\alpha^i \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i
\end{aligned}$$

which are formally the expression of the Liouville forms of $J^1\pi^\dagger$ (compare with equations (3.32) and (3.33)). This is not a mere coincidence but, as we will see, an evidence of the strong relation between the Lagrangian and the Hamiltonian formalisms.

Remark 3.38. Instead of using the canonical vertical endomorphism $\mathcal{S}$ in the definition 3.37 of the Poincaré-Cartan m-form Θ_L, we could have used the vertical endomorphism $\mathcal{S}_\eta$ associated to a volume form η on M. If we define

$$\Theta_{\mathcal{L}} := \mathcal{L} + \langle \mathcal{S}_\eta,\, \mathrm{d}L \rangle\,, \tag{3.44}$$

where $\mathcal{L} = L\eta$ and $\langle \mathcal{S}_\eta,\, \mathrm{d}L \rangle$ is the contraction between the factors in $TJ^1\pi$ of $\mathcal{S}_\eta$ and those in $T^*J^1\pi$ of $\mathrm{d}L$, it turns out that this definition does not depend on the chosen volume form η and coincides with the previous definition.

Remark 3.39. The Poincaré-Cartan m-form Θ_L is also called *DeDonder* form by Binz, Śniatycki and Fisher [21], since this was the name used by Cartan (who attributed its construction to DeDonder) to distinguish it from the Poincaré-Cartan form in Mechanics. In the quoted book by Binz *et al.* the reader can find interesting historical remarks concerning Field theories.

Proposition 3.40. *The Poincaré-Cartan forms satisfy the following properties:*

1. *The Poincaré-Cartan m-form $\Theta_{\mathcal{L}}$ is $\pi_{1,0}$-semi-basic,* i.e. *it is annihilated by any $\pi_{1,0}$-vertical vector $X \in \mathcal{V}\pi_{1,0}$,*
$$i_X \Theta_{\mathcal{L}} = 0. \tag{3.45}$$
2. *The Poincaré-Cartan m-form $\Theta_{\mathcal{L}}$ is annihilated by any pair of π_1-vertical vectors $X, Y \in \mathcal{V}\pi_1$,*
$$i_X i_Y \Theta_{\mathcal{L}} = 0. \tag{3.46}$$
3. *The Poincaré-Cartan $(m+1)$-form $\Omega_{\mathcal{L}}$ is annihilated by any pair of $\pi_{1,0}$-vertical vectors $X, Y \in \mathcal{V}\pi_{1,0}$,*
$$i_X i_Y i_Z \Omega_{\mathcal{L}} = 0. \tag{3.47}$$
4. *The Poincaré-Cartan $(m+1)$-form $\Omega_{\mathcal{L}}$ is annihilated by any triple of π_1-vertical vectors $X, Y, Z \in \mathcal{V}\pi_1$,*
$$i_X i_Y i_Z \Omega_{\mathcal{L}} = 0. \tag{3.48}$$
5. *Let ξ be a vector field on E, we then have that*
$$(j^1\phi)^* \mathfrak{L}_{\xi^{(1)}} \mathcal{L} = (j^1\phi)^* \mathfrak{L}_{\xi^{(1)}} \Theta_{\mathcal{L}}. \tag{3.49}$$

Theorem 3.41. *A section $\phi \in \Gamma\pi$ is a critical point of the Lagrangian action $\mathcal{A}_{\mathcal{L}}$ if and only if its first prolongation satisfies*

$$(j^1\phi)^*(i_\xi \Omega_{\mathcal{L}}) = 0, \tag{3.50}$$

for any vector field $\xi \in J^1\pi$.

Lemma 3.42. *Given a section $\sigma \in \Gamma\pi_{1,0}$ and a vector field $\xi \in \mathfrak{X}(J^1\pi)$ tangent to $\operatorname{im}\sigma$, we have that*
$$\sigma^*(i_\xi \Omega_{\mathcal{L}}) = 0.$$

Proof. Along the image of σ, ξ shall have the form $\xi = T\sigma(v)$ for some vector field $v \in \mathfrak{X}(M)$. Then,
$$\sigma^*(i_\xi \Omega_{\mathcal{L}}) = \sigma^*(i_{T\sigma(v)} \Omega_{\mathcal{L}}) = i_v \sigma^*(\Omega_{\mathcal{L}}) = 0,$$
since $\sigma^*(\Omega_{\mathcal{L}})$ is an $(m+1)$-form on M which has dimension m. □

Lemma 3.43. *Given a section $\phi \in \Gamma\pi$ and a $\pi_{1,0}$-vertical vector field $\xi \in \mathfrak{X}(J^1\pi)$, we have that*
$$(j^1\phi)^*(i_\xi \Omega_{\mathcal{L}}) = 0.$$

Proof. Let $\xi = \xi_i^\alpha \partial/\partial u_i^\alpha$ be the local expression of the $\pi_{1,0}$-vertical vector field ξ. Then, thanks to the local expression (3.43) of $\Omega_{\mathcal{L}}$, we get

$$i_\xi \Omega_{\mathcal{L}} = -\xi_i^\alpha \frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta} (\theta^\beta \wedge \mathrm{d}^{m-1} x_j),$$

which is annihilated by $j^1\phi$ since θ^β is contact. □

Proof of Theorem 3.41. Let ϕ_ε be a vertical variation with compact support $R \subset M$ of a section $\phi \in \Gamma\pi$ and let ξ be its infinitesimal generator. By Proposition 3.40 and Cartan's formula, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{A}_{\mathcal{L}}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} &= \int_R (j^1\phi)^* (\mathfrak{L}_{\xi^{(1)}} \mathcal{L}) \\ &= \int_R (j^1\phi)^* (\mathfrak{L}_{\xi^{(1)}} \Theta_{\mathcal{L}}) \\ &= -\int_R (j^1\phi)^* (i_{\xi^{(1)}} \Omega_{\mathcal{L}}) + \int_{\partial R} (j^1\phi)^* (i_{\xi^{(1)}} \Theta_{\mathcal{L}}). \end{aligned}$$

Thus, using the Euler-Lagrange equations, Theorem 3.35, and the local expression (3.42) of the Poincaré-Cartan form $\Theta_{\mathcal{L}}$, we deduce that

$$\int_R (j^1\phi)^* (i_{\xi^{(1)}} \Omega_{\mathcal{L}}) = -\int_R (j^2\phi)^* \left[(\xi^\alpha - u_i^\alpha \xi^i) \left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u_i^\alpha} \right) \right] \mathrm{d}^m x.$$

Therefore, ϕ satisfies the Euler-Lagrange equations (3.38) if and only if

$$(j^1\phi)^* (i_{\xi^{(1)}} \Omega_{\mathcal{L}}) = 0,$$

for any compactly supported π-projectable vector field $\xi \in \mathfrak{X}(E)$. Using partitions of the unity, we may generalize this to any π-projectable vector field $\xi \in \mathfrak{X}(E)$.

Finally, any vector field $\xi \in \mathfrak{X}(J^1\pi)$ may be split into the sum of a vector field on $J^1\pi$ tangent to the image of $j^1\phi$, the first lift to $J^1\pi$ of a vector field on E and a $\pi_{1,0}$-projectable vector field on $J^1\pi$. The assertion of the theorem follows from the previous lemmas. □

Definition 3.44. The *DeDonder equation* is the following equation in terms of sections σ of $\pi_1 : J^1\pi \to M$:

$$\sigma^*(i_\xi \Omega_{\mathcal{L}}) = 0, \ \forall \xi \in \mathfrak{X}(J^1\pi). \tag{3.51}$$

Using Lemma 3.42, we have that a section $\sigma \in \Gamma\pi_1$ still satisfies the DeDonder equation if we only consider π_1-vertical vector fields $\xi \in \mathfrak{X}(J^1\pi)$. Taking this into account, an easy computation using the local expression (3.43) of $\Omega_{\mathcal{L}}$ shows that the DeDonder equation is locally written

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u_i^\alpha} - \frac{\partial \sigma^\beta}{\partial x^i} \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} - \frac{\partial \sigma_j^\beta}{\partial x^i} \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} + \left(\frac{\partial \sigma^\beta}{\partial x^i} - u_i^\beta \right) \frac{\partial^2 L}{\partial u_i^\beta \partial u^\alpha} = 0, \tag{3.52}$$

$$\left(\frac{\partial \sigma^\beta}{\partial x^j} - u_j^\beta \right) \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} = 0. \tag{3.53}$$

Definition 3.45. A Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$ is *regular* whenever its Hessian with respect to the velocities

$$\left(\frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i}\right)$$

is non-degenerate.

Proposition 3.46. *Let $\mathcal{L} : J^1\pi \to \Lambda^m M$ be a regular Lagrangian density. A section $\sigma \in \Gamma\pi_1$ satisfies the DeDonder equation if and only if σ is holonomic,* i.e. *$\sigma = j^1\phi$ for some $\phi \in \Gamma\pi$, and ϕ satisfies the Euler-Lagrange equations.*

Proposition 3.47. *The Poincaré-Cartan $(m+1)$-form $\Omega_{\mathcal{L}}$ is multisymplectic, whenever $m > 1$, and cosymplectic (together with the volume form η), whenever $m = 1$, if and only if the Lagrangian density $\mathcal{L}$ is regular.*

Proof. Provided that $m > 1$, let $(x^i, u^\alpha, u^\alpha_i)$ be adapted coordinates on $J^1\pi$. A straightforward computation shows that

$$\begin{aligned}
i_{\frac{\partial}{\partial x^j}}\Omega_{\mathcal{L}} &= (\dots) - \frac{\partial^2 L}{\partial u^\gamma_k \partial u^\alpha_i}\, \mathrm{d}u^\gamma_k \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-2}x_{ij} \\
i_{\frac{\partial}{\partial u^\beta}}\Omega_{\mathcal{L}} &= (\dots) + \frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i} \mathrm{d}u^\alpha_i \wedge \mathrm{d}^{m-1}x_j \\
i_{\frac{\partial}{\partial u^\beta_j}}\Omega_{\mathcal{L}} &= (\dots) - \frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i} \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i
\end{aligned}$$

where the indicated terms are the only ones with the corresponding m-form. Thus, assume that $v \in TJ^1\pi$ is such that $i_\xi\Omega_{\mathcal{L}} = 0$ and it is locally written in the given coordinates

$$\xi = \xi^j \frac{\partial}{\partial x^j} + \xi^\beta \frac{\partial}{\partial u^\beta} + \xi^\beta_j \frac{\partial}{\partial u^\beta_j}.$$

If $\mathcal{L}$ is regular, then all coefficients of ξ must be zero and $\Omega_{\mathcal{L}}$ is multisymplectic. Reciprocally, if $\Omega_{\mathcal{L}}$ is multisymplectic, then the “Hessian” of L has trivial kernel, *i.e.* $\mathcal{L}$ is regular.

For the case $m = 1$, consider coordinates (t, q^α, v^α) on $J^1\pi$. In these coordinates, after Equation (3.43), the Poincaré-Cartan $(m+1)$-form has the form

$$\Omega_{\mathcal{L}} = -\,\mathrm{d}\left(\frac{\partial L}{\partial v^\alpha}\right) \wedge \mathrm{d}q^\alpha + v^\alpha\, \mathrm{d}\left(\frac{\partial L}{\partial v^\alpha}\right) \wedge \mathrm{d}t - \frac{\partial L}{\partial q^\alpha}\, \mathrm{d}q^\alpha \wedge \mathrm{d}t.$$

A straightforward computation shows

$$\Omega^n_{\mathcal{L}} \wedge \mathrm{d}t = \det\left(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}\right) \mathrm{d}q^1 \wedge \mathrm{d}v^1 \wedge \dots \wedge \mathrm{d}q^n \wedge \mathrm{d}v^n \wedge \mathrm{d}t,$$

which is a volume form if and only if $\mathcal{L}$ is regular. □

Connections and multivector fields

Let Γ be a a connection in the fibration $\pi_1 : J^1\pi \to M$ with horizontal projector $\mathbf{h}$. So $\mathbf{h}$ is locally expressed as follows:

$$\mathbf{h} = \mathrm{d}x^i \otimes \left(\frac{\partial}{\partial x^i} + A_i^\alpha \frac{\partial}{\partial u^\alpha} + A_{ji}^\alpha \frac{\partial}{\partial u_j^\alpha} \right). \tag{3.54}$$

Univocally associated to this connection, there is a class of locally decomposable multivector fields $\mathcal{D}(X) \subset \mathfrak{X}_d^m(J^1\pi)$ locally expressed as follows (see Section §1.2):

$$X = f \bigwedge_{i=1}^{m} X_i = f \bigwedge_{i=1}^{m} \left(\frac{\partial}{\partial x^i} + A_i^\alpha \frac{\partial}{\partial u^\alpha} + A_{ji}^\alpha \frac{\partial}{\partial u_j^\alpha} \right). \tag{3.55}$$

Proposition 3.48. *Consider the* dynamical equations

$$i_{\mathbf{h}}\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}} \tag{3.56}$$

in terms of horizontal projectors $\mathbf{h}$ *of connections* Γ *in the fibration* $\pi_1 : J^1\pi \to M$*, and*

$$i_X\Omega_{\mathcal{L}} = 0 \tag{3.57}$$

in terms of locally decomposable m*-multivector fields* $X \in \mathfrak{X}_d^m(J^1\pi)$*. We have that both equation are locally written*

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u_i^\alpha} - A_i^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} - A_{ji}^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} + \left(A_i^\beta - u_i^\beta \right) \frac{\partial^2 L}{\partial u_i^\beta \partial u^\alpha} = 0, \tag{3.58}$$

$$\left(A_j^\beta - u_j^\beta \right) \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} = 0, \tag{3.59}$$

where $(x^i, u^\alpha, u_i^\alpha)$ *are adapted coordinates on* $J^1\pi$ *and the* A*'s are the coefficients of* $\mathbf{h}$ *and* X *given in* (3.54) *and* (3.55)*, respectively. It turns out that, if* $\mathbf{h}$ *and* X *are associated, then* $\mathbf{h}$ *satisfies* (3.56) *if and only if* X *satisfies* (3.57)*.*

Moreover, if Γ *and/or* X *are integrable, then they satisfy the previous equations if and only if its integral sections* $\sigma \in \Gamma\pi_1$ *satisfy the DeDonder equation.*

Proof. Using the local expression (3.43), we obtain on the one hand

$$i_{\mathbf{h}}\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}} + \left[\left(A_j^\beta - u_j^\beta \right) \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} \right] \mathrm{d}u_i^\alpha \wedge \mathrm{d}^m x$$
$$+ \left[\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u_i^\alpha} - A_i^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} - A_{ji}^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} + \left(A_j^\beta - u_j^\beta \right) \frac{\partial^2 L}{\partial u_j^\beta \partial u^\alpha} \right] \mathrm{d}u^\alpha \wedge \mathrm{d}^m x.$$

On the other hand, a cumbersome computation[1] yields

$$\begin{aligned} (-1)^m i_X \Omega_{\mathcal{L}} \ = \ & \underbrace{\left[\left(\frac{\partial \hat{p}_\alpha^i}{\partial x^i} - \frac{\partial \hat{p}}{\partial u^\alpha} \right) + A_j^\beta \left(\frac{\partial \hat{p}_\alpha^j}{\partial u^\beta} - \frac{\partial \hat{p}_\beta^j}{\partial u^\alpha} \right) + A_{ji}^\beta \frac{\partial \hat{p}_\alpha^i}{\partial u_j^\beta} \right]}_{\lambda_\alpha} \mathrm{d}u^\alpha \\ & - \underbrace{\left[\frac{\partial \hat{p}}{\partial u_i^\alpha} + A_j^\beta \frac{\partial \hat{p}_\beta^j}{\partial u_i^\alpha} \right]}_{\lambda_\alpha^i} \mathrm{d}u_i^\alpha - \left(A_k^\alpha \lambda_\alpha - A_{ik}^\alpha \lambda_\alpha^i \right] \mathrm{d}x^k, \end{aligned}$$

where $\hat{p}_\alpha^i = \frac{\partial L}{\partial u_i^\alpha}$ and $\hat{p} = L - u_i^\alpha \hat{p}_\alpha^i$, and where we have assumed that $f = 1$.

We deduce form here that, if $\mathbf{h}$ or X satisfy the corresponding equations (3.56) and (3.57), then their coefficients must satisfy equations (3.58) and (3.59). The first assertion of the theorem is now clear.

For the second statement, suppose that $\mathbf{h}$ and/or X are integrable and let $\sigma \in \Gamma\pi_1$ be an integral section of any of them. Then, we have that $\frac{\partial \sigma^\alpha}{\partial x^i} = A_i^\alpha$ and $\frac{\partial \sigma_j^\alpha}{\partial x^i} = A_{ji}^\alpha$, what yields to the local expressions (3.52) and (3.52) for σ of the DeDonder equation (3.51). □

Notice that σ being an integral section of $\mathbf{h}$ or X does not mean that σ is holonomic, which is the case when $\mathcal{L}$ is regular as Proposition 3.46 assures.

Proposition 3.49. *Let $\mathcal{L} : J^1\pi \to \Lambda^m M$ be a regular Lagrangian density. Then there exists a semi-holonomic connection Γ in $\pi_1 : J^1\pi \to M$ satisfying*

$$i_{\mathbf{h}}\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}}, \tag{3.60}$$

where $\mathbf{h}$ is the horizontal projector of Γ. Such a connection Γ will be called an Euler-Lagrange connection *for $\mathcal{L}$.*

Proof. Given a locally finite open covering $\{U_\lambda^1\}_{\lambda\in\Lambda}$ be of $J^1\pi$ with fibered coordinates, let $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a partition of the unity subordinate to $\{U_\lambda^1\}_{\lambda\in\Lambda}$. For each $\lambda \in \Lambda$, we define a horizontal projector $\mathbf{h}_\lambda$ on U_λ^1 as follows: Assuming that $\mathbf{h}_\lambda$ must be described as in the local expression (3.54), we take $A_j^\alpha = u_j^\alpha$ and we determine A_{ij}^α by means of the equation (3.58).

Denote by $\mathbf{v}_\lambda = \mathrm{Id}_{TJ^1\pi} - \mathbf{h}_\lambda$ the vertical projector and extend it by zero

$$\tilde{\mathbf{v}}_\lambda(u) = \begin{cases} \alpha_\lambda(u)\mathbf{v}_\lambda(u) & \text{if } u \in \mathrm{supp}(\alpha_\lambda) \\ 0 & \text{if } u \notin \mathrm{supp}(\alpha_\lambda) \end{cases}$$

for any $u \in J^1\pi$. Now, we put

$$\mathbf{v}(u) = \sum_{\lambda\in\Lambda} \tilde{\mathbf{v}}_\lambda(u).$$

A direct computation shows that

$$\mathrm{im}(\tilde{\mathbf{v}}_\lambda(u)) = \begin{cases} \mathrm{im}(\mathbf{v}_\lambda(u)) & \text{if } u \in \mathrm{supp}(\alpha_\lambda) \\ 0 & \text{if } u \notin \mathrm{supp}(\alpha_\lambda) \end{cases}$$

[1] Here, we have used the formulae

$$\begin{aligned} (-1)^m i_X(\alpha \wedge \mathrm{d}^m x) &= \alpha - \alpha(X_j)\,\mathrm{d}x^j, \\ (-1)^m i_X(\beta \wedge \alpha^i \wedge \mathrm{d}^{m-1}x_i) &= \beta(X_j)\alpha^j(X_i)\,\mathrm{d}x^i - \beta(X_j)\alpha^i(X_i)\,\mathrm{d}x^j - \beta(X_j)\alpha^j + \alpha^j(X_j)\beta, \end{aligned}$$

where α, α^i and β are 1-forms and where X is the m-vector $X = \bigwedge_{i=1}^m X_i$.

from which one deduces that $\mathrm{im}(\mathbf{v}(u)) \subseteq \sum_{\lambda\in\Lambda} \mathrm{im}(\mathbf{v}_\lambda(u)) \subseteq \mathcal{V}_u\,\pi_1$. Furthermore, we have

$$\begin{aligned}
\mathbf{v}(u)|_{\mathcal{V}_u\,\pi_1} &= \sum_{\lambda\in\Lambda} \tilde{\mathbf{v}}_\lambda(u)|_{\mathcal{V}_u\,\pi_1} \\
&= \sum_{\lambda\in\Lambda} \alpha_\lambda(u)\mathbf{v}_\lambda(u)|_{\mathcal{V}_u\,\pi_1} \\
&= \sum_{\lambda\in\Lambda} \alpha_\lambda(u)\,\mathrm{Id}_{TJ^1\pi}\,|_{\mathcal{V}_u\,\pi_1} \\
&= \mathrm{Id}_{TJ^1\pi}\,|_{\mathcal{V}_u\,\pi_1}.
\end{aligned}$$

So we deduce that $\mathbf{v}$ is a globally well defined vertical projector over $J^1\pi$, thus it induces a semi-holonomic connection Γ in $\pi_1 : J^1\pi \to M$ (by construction) satisfying (3.60). □

Corollary 3.50. *Let $\mathcal{L} : J^1\pi \to \Lambda^m M$ be a regular Lagrangian density. Then there exists a semi-holonomic multivector field $X \in \mathfrak{X}^m_d(J^1\pi)$ satisfying*

$$i_X\Omega_{\mathcal{L}} = 0. \tag{3.61}$$

Such a connection multivector field X will be called an Euler-Lagrange multivector *for $\mathcal{L}$.*

Remark 3.51. In order to discuss the uniqueness, suppose that Γ_1 and Γ_2 are two solutions of (3.60). If we denote by T the tensor field $T = \mathbf{h}_1 - \mathbf{h}_2$, difference of the two horizontal projectors then, using that Γ_1 and Γ_2 are semi-holonomic, we deduce that T is locally given by $T = T^\alpha_{ij}\dfrac{\partial}{\partial u^\alpha_i} \otimes dx^j$. Moreover, $i_T\,\Omega_{\mathcal{L}} = 0$ implies that

$$T^\alpha_{ij}\frac{\partial^2 L}{\partial u^\alpha_i \partial u^\beta_j} = 0\ ,$$

for all $\beta \in \{1, \ldots, n\}$. Since we have n equations and nm^2 unknowns, the solutions at each point form a vector space of dimension $nm^2 - n$, taking into account the regularity assumption on $\mathcal{L}$. Therefore, the solutions of 3.60 are given by $\mathbf{h} + T$, where $\mathbf{h}$ is the horizontal projector of a particular solution, T is a tensor field of type (1,1) on $J^1\pi$ such that it takes values in the vertical bundle $V\pi_{1,0}$, it vanishes when it is applied to π_1-vertical vector fields and $i_T\,\Omega_{\mathcal{L}} = 0$. In fact, the space $\mathcal{T}$ of all tensor fields T is a $C^\infty(J^1\pi)$-module with local dimension $n(m^2-1)$. If $\dim M = 1$, then there exists a unique solution $\Gamma_{\mathcal{L}}$ of (3.60).

3.2.2 The Hamiltonian Formalism

See [31, 68, 69, 73, 75, 77, 91, 92, 115, 133, 134, 136, 137].

Definition 3.52. A *Hamiltonian section* is a section $h : J^1\pi^\circ \to J^1\pi^\dagger$ of $\mu : J^1\pi^\dagger \to J^1\pi^\circ$.

Definition 3.53. A *Hamiltonian density* is a smooth function $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ such that $i_\xi\,\mathrm{d}\mathcal{H} = i_\xi\Omega$ for any μ-vertical vector field $\xi \in \mathcal{V}\,\mu$.

In coordinates $(x^i, u^\alpha, p, p^i_\alpha)$, a Hamiltonian section $h \in \Gamma\mu$ is locally described by

$$h(x^i, u^\alpha, p^i_\alpha) = (x^i, u^\alpha, p = -H(x^i, u^\alpha, p^i_\alpha), p^i_\alpha), \tag{3.62}$$

where the smooth function H, which is locally defined, is called the *Hamiltonian function.*

Given a Hamiltonian density $\mathcal{H}$, let $\xi \in \mathcal{V}\mu$ be a μ-vertical vector field. In local coordinates $(x^i, u^\alpha, p, p^i_\alpha)$, ξ shall has the form $\xi = \xi_0 \partial/\partial p$, for some locally defined function ξ_0 on $J^1\pi^\dagger$. In order to satisfy the definition, we must have

$$i_\xi \,\mathrm{d}\mathcal{H} = i_\xi(\,\mathrm{d}\bar{H} \wedge \mathrm{d}^m x) = \xi_0 \frac{\partial \bar{H}}{\partial p}\,\mathrm{d}^m x = -\xi_0 \,\mathrm{d}^m x = i_\xi \Omega,$$

where $\mathcal{H} = \bar{H}\eta$. Since this shall be true for any $\xi \in \mathcal{V}\mu$, we have that Hamiltonian density $\mathcal{H}$ is in turn locally described by

$$\bar{H}(x^i, u^\alpha, p, p^i_\alpha) = p + H(x^i, u^\alpha, p^i_\alpha). \tag{3.63}$$

The smooth function H coincides with the previous Hamilton function in the following sense.

Proposition 3.54. *The space of Hamiltonian sections and the family of Hamiltonian densities are in bijective correspondence. In fact, a Hamiltonian section h and a Hamiltonian density $\mathcal{H}$ are univocally related by the condition* $\operatorname{im} h = \mathcal{H}^{-1}(0)$*. In this case, we say that they are* associated.

By means of this relation, we may relate also section of π_1° and $\pi_1^\dagger$.

Corollary 3.55. *Let $h : J^1\pi^\circ \to J^1\pi^\dagger$ be a Hamiltonian section associated with a Hamiltonian density $\mathcal{H} \in \Omega^m(\pi_1^\dagger)$. A section $\sigma \in \Gamma\pi_1^\circ$, defines a section $\bar\sigma = h \circ \sigma \in \Gamma\pi_1^\dagger$ such that $\bar\sigma^*\mathcal{H} = 0$. Reciprocally, a section $\bar\sigma \in \Gamma\pi_1^\dagger$ with $\bar\sigma^*\mathcal{H} = 0$ defines a section $\sigma = \mu \circ \bar\sigma \in \Gamma\pi_1^\circ$ such that $\bar\sigma = h \circ \sigma$. In both cases, we say that σ and $\bar\sigma$ are* associated.

Definition 3.56. Let $h \in \Gamma\mu$ be a Hamiltonian section. The *Cartan m-form* associated to h is defined by

$$\Theta_h := h^*\Theta. \tag{3.64}$$

The *Cartan $(m+1)$-form* associated to h is defined by

$$\Omega_h := -\,\mathrm{d}\Theta_h = h^*\Omega. \tag{3.65}$$

It is worth to recall that the Liouville form and the canonical one are locally given by

$$\Theta = p\,\mathrm{d}^m x + p^i_\alpha \,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i, \tag{3.66}$$
$$\Omega = -\,\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^i_\alpha \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i; \tag{3.67}$$

in adapted local coordinates $(x^i, u^\alpha, p, p^i_\alpha)$ on $J^1\pi^\dagger$. Thus, in the corresponding induced coordinates $(x^i, u^\alpha, p^i_\alpha)$ on $J^1\pi^\circ$, we have that the Cartan forms are given by

$$\Theta_h = -H\,\mathrm{d}^m x + p^i_\alpha \,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i, \tag{3.68}$$
$$\Omega_h = \mathrm{d}H \wedge \mathrm{d}^m x - \mathrm{d}p^i_\alpha \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i. \tag{3.69}$$

Proposition 3.57. *Let $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ be a Hamiltonian density associated with a Hamiltonian section $h \in \Gamma\mu$. We have that*

$$\Theta = \mu^*\Theta_h + \mathcal{H} \quad \text{and} \quad \Omega = \mu^*\Omega_h - \mathrm{d}\mathcal{H}. \tag{3.70}$$

Proof. The second equation follows from the first, which is immediate from the previous local expressions. □

Definition 3.58. Let $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ be a Hamiltonian density. The *Cartan m-form* associated to $\mathcal{H}$ is defined by

$$\Theta_{\mathcal{H}} := \Theta - \mathcal{H}. \tag{3.71}$$

The *Cartan $(m+1)$-form* associated to $\mathcal{H}$ is defined by

$$\Omega_{\mathcal{H}} := -\mathrm{d}\Theta_{\mathcal{H}} = \Omega + \mathrm{d}\mathcal{H}. \tag{3.72}$$

The Hamilton equations

Definition 3.59. Given a Hamiltonian section $h \in \Gamma$, the associated *(reduced) Hamiltonian action* is the map $\mathcal{A}_h : \Gamma\pi_1^\circ \times \mathcal{K} \to \mathbb{R}$ given by

$$\mathcal{A}_h(\sigma, R) = \int_R \sigma^*\Theta_h, \tag{3.73}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

Let $h : J^1\pi^\circ \to J^1\pi^\dagger$ be a Hamiltonian section associated with a Hamiltonian density $\mathcal{H} \in \Omega^m(\pi_1^\dagger)$. Given a section $\sigma : M \to J^1\pi^\circ$ of $\pi_1^\dagger : J^1\pi^\dagger \to M$, the composition $\bar\sigma = h\circ\sigma : M \to J^1\pi^\dagger$ defines a section of $\pi_1^\dagger : J^1\pi^\dagger \to M$. Note that, in general, a section $\bar\sigma \in \Gamma\pi_1^\dagger$ does not define a section $\sigma \in \Gamma\pi_1^\circ$ such that $\bar\sigma = h \circ \sigma$, which is only true when $\bar\sigma^*\mathcal{H} = 0$ (from Proposition 3.54). Besides, for this particular section $\sigma \in \Gamma\pi_1^\circ$, we have that

$$\sigma^*\Theta_h = \bar\sigma^*(\mu^*\Theta_h) = \bar\sigma^*\Theta_{\mathcal{H}}.$$

Thus, the extremals of $\mathcal{A}_h$ coincide through h with the extremals restricted to $\bar\sigma^*\mathcal{H} = 0$ of the following integral action.

Definition 3.60. Given a Hamiltonian density $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$, the associated *(extended) Hamiltonian action* is the map $\mathcal{A}_{\mathcal{H}} : \Gamma\pi_1^\dagger \times \mathcal{K} \to \mathbb{R}$ given by

$$\mathcal{A}_{\mathcal{H}}(\sigma, R) = \int_R \sigma^*\Theta_{\mathcal{H}}, \tag{3.74}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

Theorem 3.61 (Hamilton's equations)**.** *Let $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ be a Hamiltonian density associated with a Hamiltonian section $h \in \Gamma\mu$. Critical points of each integral action are characterized by the* Hamilton equations *plus boundary conditions, which are*

$$\sigma^*(i_\xi\Omega_h) = 0 \quad \text{and} \quad \sigma^*(i_\xi\Theta_h) =_{\partial M} 0, \quad \forall\xi \in \mathfrak{X}(J^1\pi^\circ), \tag{3.75}$$

for a critical point $\sigma \in \Gamma\pi_1^\circ$ of $\mathcal{A}_h$ and

$$\bar\sigma^*(i_\xi\Omega_{\mathcal{H}}) = 0 \quad \text{and} \quad \bar\sigma^*(i_\xi\Theta_{\mathcal{H}}) =_{\partial M} 0, \quad \forall\xi \in \mathfrak{X}(J^1\pi^\dagger), \tag{3.76}$$

for a critical point $\bar{\sigma} \in \Gamma\pi_1^\dagger$ *of* $\mathcal{A}_\mathcal{H}$. *Moreover, in both cases, the Hamilton equations have the "common" local expression*

$$\frac{\partial H}{\partial u^\alpha} = -\frac{\partial \sigma_\alpha^i}{\partial x^i} \quad \text{and} \quad \frac{\partial H}{\partial p_\alpha^i} = \frac{\partial \sigma^\alpha}{\partial x^i}, \tag{3.77}$$

and the boundary conditions are

$$\sigma^* H =_{\partial M} 0 \quad \text{and} \quad \sigma_\alpha^i =_{\partial M} 0, \tag{3.78}$$

where $(x^i, u^\alpha, p, p_\alpha^i)$ *and* $(x^i, u^\alpha, p_\alpha^i)$ *are adapted coordinates on* $J^1\pi^\dagger$ *and* $J^1\pi^\circ$ *respectively,* $\sigma = (x^i, \sigma^\alpha, \sigma_\alpha^i)$ *and* $\bar{\sigma} = (x^i, \sigma^\alpha, \sigma_0, \sigma_\alpha^i)$.

Proof. We begin by determining the variation of the reduced Hamiltonian action $\mathcal{A}_h$. Given a section σ of $\pi_1^\circ : J^1\pi^\circ \to M$, let R be a compact region of M and $\sigma_\varepsilon = \varphi_\varepsilon \circ \sigma \circ (\check{\varphi}_\varepsilon)^{-1}$ a variation of σ where the infinitesimal generator $\xi \in \mathfrak{X}(J^1\pi^\circ)$ of φ_ε has its support contained in $(\pi_1^\circ)^{-1}(R)$. Applying a result similar to Lemma 3.34 and Cartan's formula, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\mathcal{A}_h(\sigma_\varepsilon, R_\varepsilon)\right]\Big|_{\varepsilon=0} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_R (\varphi_\varepsilon \circ \sigma)^* \Theta_h\right]\Bigg|_{\varepsilon=0} \\ &= \int_R \sigma^*(\mathfrak{L}_\xi \Theta_h) \\ &= -\int_R \sigma^*(i_\xi \Omega_h) + \int_{\partial R} \sigma^*(i_\xi \Theta_h). \end{aligned}$$

Thus, σ is a critical point of $\mathcal{A}_h$ if an only if

$$\int_R \sigma^*(i_\xi \Omega_h) - \int_{\partial R} \sigma^*(i_\xi \Theta_h) = 0$$

for any compact region $R \subseteq M$ and any vector field $\xi \in \mathfrak{X}(J^1\pi^\circ)$ whose support is contained in $(\pi_1^\circ)^{-1}(R)$.

Now, assume that σ is a critical point of $\mathcal{A}_h$. If R is a compact region contained in the interior of M, then any vector field $\xi \in \mathfrak{X}(J^1\pi^\circ)$ whose support is contained in $(\pi_1^\circ)^{-1}(R)$ must be null along the fibers over the boundary of R. Indeed, for such R and σ, we have

$$\int_R \sigma^*(i_\xi \Omega_h) = 0.$$

Varying R and ξ, and using partitions of the unity, we deduce that

$$\sigma^*(i_\xi \Omega_h) = 0$$

for every vector field $\xi \in \mathfrak{X}(J^1\pi^\circ)$.

In a similar way, we deduce the boundary condition

$$\sigma^*(i_\xi \Theta_h) =_{\partial M} 0$$

for every vector field $\xi \in \mathfrak{X}(J^1\pi^\circ)$.

Finally, let us compute the local expression of $\sigma^*(i_\xi\Omega_h)$. Given adapted coordinates $(x^i, u^\alpha, p^i_\alpha)$ on $J^1\pi^\circ$, if we denote $\sigma = (x^i, \sigma^\alpha, \sigma^i_\alpha)$, we then have

$$\begin{aligned}\sigma^*(i_\xi\Omega_h) &= \sigma^*\Bigg[\left(\xi^\alpha\frac{\partial H}{\partial u^\alpha} + \xi^i_\alpha\frac{\partial H}{\partial p^i_\alpha}\right)\mathrm{d}^m x - \left(\frac{\partial H}{\partial u^\alpha}\,\mathrm{d}u^\alpha + \frac{\partial H}{\partial p^i_\alpha}\,\mathrm{d}p^i_\alpha\right)\wedge\xi^j\,\mathrm{d}^{m-1}x_j \\ &\qquad - \xi^i_\alpha\,\mathrm{d}u^\alpha\wedge\mathrm{d}^{m-1}x_i + \xi^\alpha\,\mathrm{d}p^i_\alpha\wedge\mathrm{d}^{m-1}x_i - \mathrm{d}p^i_\alpha\wedge\mathrm{d}u^\alpha\wedge\xi^j\,\mathrm{d}^{m-2}x_{ij}\Bigg] \\ &= \Bigg[-\xi^j\left(\frac{\partial\sigma^\alpha}{\partial x^j}\frac{\partial H}{\partial u^\alpha} + \frac{\partial\sigma^i_\alpha}{\partial x^j}\frac{\partial H}{\partial p^i_\alpha} + \frac{\partial\sigma^i_\alpha}{\partial x^i}\frac{\partial\sigma^\alpha}{\partial x^j} - \frac{\partial\sigma^j_\alpha}{\partial x^i}\frac{\partial\sigma^i_\alpha}{\partial x^j}\right) \\ &\qquad + \xi^\alpha\left(\frac{\partial H}{\partial u^\alpha} + \frac{\partial\sigma^i_\alpha}{\partial x^i}\right) + \xi^i_\alpha\left(\frac{\partial H}{\partial p^i_\alpha} - \frac{\partial\sigma^\alpha}{\partial x^i}\right)\Bigg]\,\mathrm{d}^m x,\end{aligned}$$

where we have used the relation

$$\mathrm{d}x^k\wedge\mathrm{d}^{m-2}x_{ij} = \delta^k_j\,\mathrm{d}^{m-1}x_i - \delta^k_i\,\mathrm{d}^{m-1}x_j.$$

Provided σ is a critical point of $\mathcal{A}_h$, since $\sigma^*(i_\xi\Omega_h)$ must be null for any $\xi\in\mathfrak{X}(J^1\pi^\circ)$, we therefore shall have that

$$\frac{\partial H}{\partial u^\alpha} = -\frac{\partial\sigma^i_\alpha}{\partial x^i} \quad\text{and}\quad \frac{\partial H}{\partial p^i_\alpha} = \frac{\partial\sigma^\alpha}{\partial x^i},$$

which are precisely the local expression of the Hamilton equations.

For the boundary condition $\sigma^*(i_\xi\Theta_h) = 0$ over ∂M, we proceed in the same way, and we get that locally

$$\sigma^*(i_\xi\Theta_h) = \sigma^*(-H\xi^i - p^i_\alpha u^\alpha_j\xi^j - p^j_\alpha u^\alpha_j\xi^i - p^i_\alpha\xi^\alpha)\,\mathrm{d}^{m-1}x_i = 0,$$

which implies that

$$\sigma^* H = \sigma^i_\alpha = 0,$$

along ∂M.

The proof of the theorem for the case of the extended Hamiltonian action $\mathcal{A}_\mathcal{H}$ is completely analogous. □

Remark 3.62. Even though the proof seems to be valid only for the multidimensional case ($m > 1$), because of the fact that the $(m-2)$-form $\mathrm{d}^{m-2}x_{ij}$ appears explicitly in the development of $\sigma^*(i_\xi\Omega_h)$, it remains valid when $m = 1$. In fact, in this case, the terms with $\mathrm{d}^{m-2}x_{ij}$ would disappear and $\mathrm{d}^{m-1}x_i = 1$.

Remark 3.63. It follows from the derivation of the local expression of the Hamilton's equations and the boundary conditions that the considerations made in Remark 3.36 are still valid here. If we had restricted the variations to π°_1-vertical or $\pi^\dagger_1$-vertical ones over the whole of M or only over ∂M, then only the boundary condition $\sigma^* H = 0$ along ∂M would have remained. Moreover, if we had considered null variations at the border ∂M, then any boundary condition would had remained and we would be free to fix them.

The main difference between the reduced and the extended formalism is that, in the extended one, there are a wider number of critical sections since there are no restrictions on the component $\sigma_0 = p\circ\bar\sigma$ of a critical section $\bar\sigma$. Nonetheless, critical sections of the reduced Hamiltonian action $\mathcal{A}_h$ "are always" critical sections of the extended Hamiltonian action $\mathcal{A}_\mathcal{H}$.

Corollary 3.64. *A section $\sigma \in \Gamma\pi_1^\circ$ is a critical point of the reduced Hamiltonian action $\mathcal{A}_h$ if and only if the associated section $\bar\sigma = h \circ \sigma \in \Gamma\pi_1^\dagger$ is a critical point of the extended Hamiltonian action $\mathcal{A}_{\mathcal{H}}$ (and vice versa).*

Proof. The proof is trivial using the coordinate expression (3.77) of the Hamilton equations (3.75) and (3.76), or taking into account the relation (3.70) between the Cartan forms and that $\Theta_{\mathcal{H}}$ and $\Omega_{\mathcal{H}}$ are both μ-basic. □

Let Γ be a a connection in the fibration $\pi_1^\dagger : J^1\pi^\dagger \to M$ with horizontal projector $\mathbf{h}$. So $\mathbf{h}$ is locally expressed as follows:

$$\mathbf{h} = \mathrm{d}x^j \otimes \left(\frac{\partial}{\partial x^j} + A_j^\alpha \frac{\partial}{\partial u^\alpha} + B_{\alpha j}^{\ i} \frac{\partial}{\partial p_\alpha^i} + C_j \frac{\partial}{\partial p} \right). \tag{3.79}$$

Univocally associated to this connection, there is a class of locally decomposable multivector fields $\mathcal{D}(X) \subset \mathfrak{X}_d^m(J^1\pi^\dagger)$ locally expressed as follows (see Section §1.2):

$$X = f \bigwedge_{j=1}^m X_j = f \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + A_j^\alpha \frac{\partial}{\partial u^\alpha} + B_{\alpha j}^{\ i} \frac{\partial}{\partial p_\alpha^i} + C_j \frac{\partial}{\partial p} \right). \tag{3.80}$$

Let $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ be a Hamiltonian density and $h : J^1\pi^\circ \to J^1\pi^\dagger$ the associated Hamiltonian section. A connection Γ in $\pi_1^\circ : J^1\pi^\circ \to M$ (and any associated multivector field $X \in \mathfrak{X}_d^m(J^1\pi^\circ)$) induces a connection $\bar\Gamma$ in $\pi_1^\dagger : J^1\pi^\dagger \to M$ (and associated multivector fields $\bar X \in \mathfrak{X}_d^m(J^1\pi^\dagger)$) along $\mathrm{im}(h)$ such that

$$C_j = -\frac{\partial H}{\partial x^j} - A_j^\alpha \frac{\partial H}{\partial u^\alpha} - B_{\alpha j}^{\ i} \frac{\partial H}{\partial p_\alpha^i}.$$

Proposition 3.65. *Let $\mathcal{H} : J^1\pi^\dagger \to \Lambda^m M$ be a Hamiltonian density and $h : J^1\pi^\circ \to J^1\pi^\dagger$ the associated Hamiltonian section.*

1. *The* (extended) dynamical equations

$$i_{\bar{\mathbf{h}}}\Omega_{\mathcal{H}} = (m-1)\Omega_{\mathcal{H}} \quad \text{and} \quad i_{\bar X}\Omega_{\mathcal{H}} = 0 \tag{3.81}$$

   in terms of the horizontal projectors $\bar{\mathbf{h}}$ of connections $\bar\Gamma$ in $\pi_1^\dagger : J^1\pi^\dagger \to M$ and locally decomposable multivector fields $\bar X \in \mathfrak{X}_d^m(J^1\pi^\dagger)$ are equivalent whenever $\bar{\mathbf{h}}$ and $\bar X$ are associated. Moreover, the integral sections $\bar\sigma$ of solutions $\bar{\mathbf{h}}$ or $\bar X$ of the extended dynamical equations (3.81) are solutions of the extended Hamilton equation (3.76).

2. *The* (reduced) dynamical equations

$$i_{\mathbf{h}}\Omega_h = (m-1)\Omega_h \quad \text{and} \quad i_X\Omega_h = 0 \tag{3.82}$$

   in terms of the horizontal projectors $\mathbf{h}$ of connections Γ in $\pi_1^\circ : J^1\pi^\circ \to M$ and locally decomposable multivector fields $X \in \mathfrak{X}_d^m(J^1\pi^\circ)$ are equivalent whenever $\mathbf{h}$ and X are associated. Moreover, the integral sections σ of solutions $\mathbf{h}$ or X of the reduced dynamical equations (3.82) are solutions of the reduced Hamilton equation (3.75).

Proof. We prove only the extended case, since the reduced one is completely analogous. Under the given assumptions, we have on the one hand that

$$i_{\bar{\mathbf{h}}}\Omega_{\mathcal{H}} = (m-1)\Omega_{\mathcal{H}} + (\,\mathrm{d}H + B_{\alpha i}^{\ i}\,\mathrm{d}u^\alpha - A_i^\alpha\,\mathrm{d}p_\alpha^i) \wedge \mathrm{d}^m x.$$

On the other hand, we have that

$$\begin{aligned}(-1)^m i_{\bar{X}}\Omega_{\mathcal{H}} &= \left(\frac{\partial H}{\partial u^\alpha} + B_{\alpha i}^{\ i}\right)\mathrm{d}u^\alpha + \left(\frac{\partial H}{\partial p_\alpha^i} - A_i^\alpha\right)\mathrm{d}p_\alpha^i \\ &\quad + \left(A_i^\alpha B_{\alpha j}^{\ i} - A_j^\alpha B_{\alpha i}^{\ i} - A_j^\alpha\frac{\partial H}{\partial u^\alpha} + B_{\alpha j}^{\ i}\frac{\partial H}{\partial p_\alpha^i}\right)\mathrm{d}x^j.\end{aligned}$$

Therefore, the dynamical equations (3.81) are written in terms of the coefficients of $\bar{\mathbf{h}}$ and $\bar{X}$

$$\frac{\partial H}{\partial u^\alpha} = -B_{\alpha i}^{\ i} \quad\text{and}\quad \frac{\partial H}{\partial p_\alpha^i} = A_i^\alpha.$$

We deduce from here that the dynamical equations (3.81) are equivalent whenever $\bar{\mathbf{h}}$ and $\bar{X}$ are associated.

If $\bar{\sigma}$ is an integral section of $\bar{\mathbf{h}}$ or $\bar{X}$, then $B_{\alpha j}^{\ i} = \partial\sigma_\alpha^i/\partial x^j$ and $A_i^\alpha = \partial\sigma^\alpha/\partial x^i$, and we recover the local Hamilton equations (3.77). □

3.2.3 The Legendre transformation

Definition 3.66. Given a Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$, the *extended Legendre transformation* is the map $\mathrm{Leg}_{\mathcal{L}} : J^1\pi \to J^1\pi^\dagger$ defined as follows: let $j_x^1\phi \in J^1\pi$, for any m tangent vectors $\xi_1, \ldots, \xi_m \in T_{\phi(x)}E$, then $\mathrm{Leg}_{\mathcal{L}}(j_x^1\phi)$ gives

$$\mathrm{Leg}_{\mathcal{L}}(j_x^1\phi)(\xi_1, \ldots, \xi_m) := (\Theta_{\mathcal{L}})_{j_x^1\phi}(\bar{\xi}_1, \ldots, \bar{\xi}_m), \tag{3.83}$$

where $\bar{\xi}_i$ is any tangent vector to $J^1\pi$ at $j_x^1\phi$ that projects to ξ_i.

The *reduced Legendre transformation* is the map $\mathrm{leg}_{\mathcal{L}} : J^1\pi \to J^1\pi^\circ$ defined by $\mathrm{leg}_{\mathcal{L}} := \mu \circ \mathrm{Leg}_{\mathcal{L}}$.

Recall that the Poincaré-Cartan form $\Theta_{\mathcal{L}}$ is $\pi_{1,0}$-basic and π_1-semi-basic (see Proposition 3.40). We thus have that, in one hand, the Legendre transformation does not depend on the chosen vectors $\bar{\xi}_1, \ldots, \bar{\xi}_m$ and, in the other hand, the image of $\mathrm{Leg}_{\mathcal{L}}$ are π-semi-basic m-forms over E. Henceforth, the Legendre transformation $\mathrm{Leg}_{\mathcal{L}}$ is well defined and gives values in $J^1\pi^\dagger$. Furthermore, from the definition, both Legendre transformations are clearly morphisms of fiber bundles over the identity of E, which is also clear from their local expressions

$$\mathrm{Leg}_{\mathcal{L}}(x^i, u^\alpha, u_i^\alpha) = \left(x^i, u^\alpha, p = L - u_i^\alpha\frac{\partial L}{\partial u_i^\alpha}, p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha}\right), \tag{3.84}$$

$$\mathrm{leg}_{\mathcal{L}}(x^i, u^\alpha, u_i^\alpha) = \left(x^i, u^\alpha, p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha}\right). \tag{3.85}$$

Proposition 3.67. *Let $\mathcal{L} : J^1\pi \to \Lambda^m M$ be a Lagrangian density. The following statement are equivalent:*

1. *$\mathcal{L}$ is regular;*

2. $\mathrm{Leg}_{\mathcal{L}} : J^1\pi \to J^1\pi^\dagger$ *is an immersion;*

3. $\mathrm{leg}_{\mathcal{L}} : J^1\pi \to J^1\pi^\circ$ *is a local diffeomorphism.*

Definition 3.68. A Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$ is *hiper-regular* whenever $\mathrm{leg}_{\mathcal{L}}$ is a global diffeomorphism.

In such a case, we have that $J^1\pi$, $\mathrm{im}(\mathrm{Leg}_{\mathcal{L}})$ and $J^1\pi^\circ$ are diffeomorphic. Moreover, $h := \mathrm{Leg}_{\mathcal{L}} \circ \mathrm{leg}_{\mathcal{L}}^{-1}$ is a Hamiltonian section and $\mathrm{im}(\mathrm{Leg}_{\mathcal{L}})$ is the 0-level set of the Hamiltonian density associated to h.

Proposition 3.69. *Let $\mathcal{L} : J^1\pi \to \Lambda^m M$ be any Lagrangian density. Then, we have*

$$\mathrm{Leg}_{\mathcal{L}}^* \Theta = \Theta_{\mathcal{L}} \quad \textit{and} \quad \mathrm{Leg}_{\mathcal{L}}^* \Omega = \Omega_{\mathcal{L}}. \tag{3.86}$$

Furthermore, if $\mathcal{L}$ is hiper-regular, we may define the Hamiltonian section $h := \mathrm{Leg}_{\mathcal{L}} \circ \mathrm{leg}_{\mathcal{L}}^{-1}$ and consider the Hamiltonian density $\mathcal{H}$ associated to h. We then have

$$\mathrm{Leg}_{\mathcal{L}}^* \Theta_{\mathcal{H}} = \Theta_{\mathcal{L}} \quad \textit{and} \quad \mathrm{Leg}_{\mathcal{L}}^* \Omega_{\mathcal{H}} = \Omega_{\mathcal{L}}, \tag{3.87}$$

$$\mathrm{leg}_{\mathcal{L}}^* \Theta_h = \Theta_{\mathcal{L}} \quad \textit{and} \quad \mathrm{leg}_{\mathcal{L}}^* \Omega_h = \Omega_{\mathcal{L}}. \tag{3.88}$$

Proof. The first equation derives easily from the local expressions (3.42) of $\Theta_{\mathcal{L}}$, (3.66) of Θ and (3.84) of $\mathrm{Leg}_{\mathcal{L}}$. The others follows directly. □

Theorem 3.70 (The equivalence theorem)**.** *Given a hiper-regular Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$, let $h := \mathrm{Leg}_{\mathcal{L}} \circ \mathrm{leg}_{\mathcal{L}}^{-1}$ be the induced Hamiltonian section and $\mathcal{H}$ the Hamiltonian density associated to h. If a section $\sigma_1 \in \Gamma\pi_1$ satisfies the DeDonder equation* (3.51),

$$\sigma_1^*(i_\xi \Omega_{\mathcal{L}}) = 0, \quad \forall \xi \in \mathfrak{X}(J^1\pi),$$

then the sections $\sigma_2 = \mathrm{leg}_{\mathcal{L}} \circ \sigma_1 \in \Gamma\pi_1^\circ$ and $\bar{\sigma}_2 = \mathrm{Leg}_{\mathcal{L}} \circ \sigma_1 \in \Gamma\pi_1^\dagger$ satisfies the corresponding Hamilton equations (3.75) *and* (3.76),

$$\sigma_2^*(i_\xi \Omega_h) = 0, \quad \forall \xi \in \mathfrak{X}(J^1\pi^\circ),$$

and

$$\bar{\sigma}_2^*(i_\xi \Omega_{\mathcal{H}}) = 0, \quad \forall \xi \in \mathfrak{X}(J^1\pi^\dagger).$$

Reciprocally, if $\sigma_2 \in \Gamma\pi_1^\circ$ (resp. $\bar{\sigma}_2 \in \Gamma\pi_1^\dagger$ with $\bar{\sigma}_2^*\mathcal{H} = 0$) satisfy the corresponding Hamilton equation, then $\sigma_1 = \mathrm{leg}_{\mathcal{L}}^{-1} \circ \sigma_2 \in \Gamma\pi_1$ (resp. $\sigma_1 = \mathrm{leg}_{\mathcal{L}}^{-1} \circ \mu \circ \bar{\sigma}_2 \in \Gamma\pi_1$) satisfies the DeDonder equation.

Remark 3.71. Observe that the Lagrangian boundary conditions (3.39) are transformed by the Legendre map to the Hamiltonian boundary conditions (3.78). Therefore, the variations considered within the theory must be correspond properly in the Lagrangian and the Hamiltonian side as stated in Remark 3.36 and Remark 3.63.

3.2.4 The Skinner and Rusk formalism

What follows may be found in here [70, 50].

Definition 3.72. The fibered product

$$W := J^1\pi \times_E J^1\pi^\dagger \quad (\text{resp. } W^\circ := J^1\pi \times_E J^1\pi^\circ) \tag{3.89}$$

is called the *mixed space of velocities and extended (*resp. *reduced) momenta*. The canonical projections are denoted $pr_1 : W \to J^1\pi$ and $pr_2 : W \to J^1\pi^\dagger$ (resp. with abuse of notation $pr_1 : W^\circ \to J^1\pi$ and $pr_2 : W^\circ \to J^1\pi^\circ$). The projections as a fiber bundle over E and M are $\pi_{W,E} = \pi_{1,0} \circ pr_1$ and $\pi_{W,E} = \pi_1 \circ pr_1$ (resp. $\pi_{W^\circ,E} = \pi_{1,0} \circ pr_1$ and $\pi_{W^\circ,E} = \pi_1 \circ pr_1$). We still denote the canonical projection by $\mu : W \to W^\circ$.

We deduce from Propositions 3.3 and 3.24 that adapted coordinates (x^i, u^α) in E induce adapted coordinates $(x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha)$ on W, where (u^α_i) and (p, p^i_α) are fibered coordinates on $J^1\pi \to E$ and $J^1\pi^\dagger \to E$, respectively. Accordingly, we have adapted coordinates $(x^i, u^\alpha, u^\alpha_i, p^i_\alpha)$ on W°.

$$\begin{tikzcd}
 & W \arrow[r, "\mu"] \arrow[dl, "pr_1"'] \arrow[dd, "\pi_{W,E}"] \arrow[dr, "pr_2"] & W^\circ \arrow[dr, "pr_2"] & \\
J^1\pi \arrow[dr, "\pi_{1,0}"] & & J^1\pi^\dagger \arrow[r, "\mu"] \arrow[dl] & J^1\pi^\circ \arrow[dll] \\
 & E \arrow[d, "\pi"] & & \\
 & M & &
\end{tikzcd}$$

The Liouville form Θ and the canonical multisymplectic form Ω of $J^1\pi^\dagger$ are pulled back to W by pr_2, which we continue denoting by the same letters. It should be noticed that (W, Ω) is no longer multisymplectic, but pre-multisymplectic. We also have to our disposal the natural pairing natural pairing $\langle\, , \rangle : J^1\pi^\dagger \times_E J^1\pi$. Therefore, we have the fibered map

$$\begin{tikzcd}
W = J^1\pi \times_E J^1\pi^\dagger \arrow[r, "\Phi"] \arrow[d, "\pi_{W,E}"'] & \Lambda^m M \arrow[d] \\
E \arrow[r, "\pi"] & M
\end{tikzcd}$$

where $\Phi := \langle pr_2, pr_1 \rangle$. If we realize $J^1\phi^\dagger$ as the space of semi-basic m-forms over E (see 3.28), then Φ takes the form

$$\Phi(w) = \phi^*_x(\omega), \tag{3.90}$$

where $w = (j^1_x\phi, \omega) \in W$ and $\phi^\dagger_{1,0}(\omega) = \phi(x)$. In the local coordinates $(x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha)$ of W, the "internal" pairing Φ is given by

$$\Phi(x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha) = (p + p^i_\alpha u^\alpha_i)\, \mathrm{d}^m x. \tag{3.91}$$

Observe that we have used the map in the proof of the identification between $J^1\pi^\dagger$ and $\Lambda^m_2 E$.

Together with the pairing Φ and the pre-multisymplectic $(m+1)$-form Ω, we introduce a Lagrangian density to define a Hamiltonian density on W and, let us say, the corresponding Cartan forms.

Definition 3.73. Assume that $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$ is a Lagrangian density.

1. The *Hamiltonian density* (or the *generated energy density* following [151]) associated to $\mathcal{L}$ on W is the map $\mathcal{H} : W \to \Lambda^m M$ defined by
$$\mathcal{H} = \Phi - \mathcal{L} \circ pr_1 . \tag{3.92}$$
2. The *Hamiltonian section* associated to $\mathcal{L}$ on W is the unique section $h : W^\circ \to W$ of $\mu : W \to W^\circ$ whose image coincides with the 0-level set of $\mathcal{H}$, *i.e.* such that $\operatorname{im} h = \mathcal{H}^{-1}(0)$.
3. The *Hamiltonian submanifold* of W, let say W_0, is identified with the 0-level set of the associated Hamiltonian density $\mathcal{H}$ or the image of the associated Hamiltonian section h, that is,
$$W_0 := \{w \in W \ : \ \mathcal{H}(w) = 0\} = \operatorname{im}(h). \tag{3.93}$$

In fibered coordinates $(u_i^\alpha, p, p_\alpha^i)$ of W and (u_i^α, p_α^i) of W°, the Hamiltonian density, section and submanifold are respectively given by

$$\begin{aligned}
\mathcal{H}(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i) &= (p + p_\alpha^i u_i^\alpha - L)\, \mathrm{d}^m x; && (3.94)\\
h(x^i, u^\alpha, u_i^\alpha, p_\alpha^i) &= (x^i, u^\alpha, u_i^\alpha, p = L - p_\alpha^i u_i^\alpha, p_\alpha^i); && (3.95)\\
W_0 &= \left\{(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i) \in W \ : \ p = L - p_\alpha^i u_i^\alpha\right\}. && (3.96)
\end{aligned}$$

From here, we may observe that $\mathcal{H}$ corresponds precisely to a Hamiltonian density in the sense of Definition 3.53: For any vertical μ-vector field ξ, we do have $i_\xi\Omega = i_\xi\, \mathrm{d}\mathcal{H}$. Even if it is obvious, it is worth to note that W° and W_0 are diffeomorphic, being h the diffeomorphism between them.

Definition 3.74. Given a Lagrangian density $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$. Let $\mathcal{H}$ be the associated Hamiltonian density and $h \in \Gamma\mu$ the associated Hamiltonian section.

1. The *Cartan m-form* and *$(m+1)$-form* associated to $\mathcal{H}$ are
$$\Theta_\mathcal{H} := \Theta - \mathcal{H} \quad \text{and} \quad \Omega_\mathcal{H} := -\,\mathrm{d}\Theta_\mathcal{H} = \Omega -\, \mathrm{d}\mathcal{H}. \tag{3.97}$$
2. The *Cartan m-form* and *$(m+1)$-form* associated to h are
$$\Theta_h := h^*\Theta \quad \text{and} \quad \Omega_h := -\,\mathrm{d}\Theta_h = h^*\Omega. \tag{3.98}$$

Following Proposition 3.57, one could check that
$$\Theta_\mathcal{H} = \mu^*\Theta_h \quad \text{and} \quad \Omega_\mathcal{H} = \mu^*\Omega_h. \tag{3.99}$$

In fibered coordinates $(u_i^\alpha, p, p_\alpha^i)$ of W and (u_i^α, p_α^i) of W°, the Cartan forms are given by

$$\Theta_\mathcal{H} =(L - p_\alpha^i u_i^\alpha)\, \mathrm{d}^m x + p_\alpha^i\, \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i \tag{3.100}$$
$$\Omega_\mathcal{H} = \left(\frac{\partial L}{\partial u^\alpha}\, \mathrm{d}u^\alpha + \frac{\partial L}{\partial u_i^\alpha}\, \mathrm{d}u_i^\alpha - u_i^\alpha\, \mathrm{d}p_\alpha^i + p_\alpha^i\, \mathrm{d}u_i^\alpha\right) \wedge \mathrm{d}^m x -\, \mathrm{d}p_\alpha^i \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i, \tag{3.101}$$

where $\mathcal{L} = L\,\mathrm{d}^m x$. Hence Θ_h and Ω_h have formally the same developments.

As we have already stated, while $(J^1\pi^\dagger, \Omega)$ was multisymplectic, (W, Ω) is only pre-multisymplectic and so are $(W, \Omega_{\mathcal{H}})$, $(W_0, \Omega_{\mathcal{H}}|_{TW_0})$ and (W°, Ω_h).

We are now in position to introduce the equation that establishes the field dynamics within the Skinner-Rusk formalism. As in the previous sections §3.2.1 and §3.2.2, we could do it by means of horizontal projectors of a given connection or using the associated multivector field. In this case, we will restrict to the method of horizontal projectors, but the reader may check that the same equations will follow considering multivector fields.

Definition 3.75. The *dynamical equation* is the following equation in terms of horizontal projectors $\mathbf{h}$ of the corresponding connections Γ in $\pi_{W,M} : W \to M$:

$$i_{\mathbf{h}}\Omega_{\mathcal{H}} = (m-1)\Omega_{\mathcal{H}}. \tag{3.102}$$

As we are going to see, the previous equation is only solvable in a subset W_1' of W. If we require that W_1' be a smooth submanifold of W and that the solutions be horizontal projectors of connections along W_1', we will end up with further restrictions on the projectors and, whenever $\mathcal{L}$ is not regular, with further constraints on the manifold along which the connections are defined. This chain of consequences is known as the Gotay-Nester-Hinds algorithm, although it was originally defined for classical mechanics. The submanifold W_1' is called the *first constraint manifold* and it is obtained at the first step of the algorithm. The *final constraint manifold* W_f' along which the solutions lie is obtained as a "fix point" and final step of the algorithm.

Theorem 3.76. *The solutions of the dynamical equation* (3.102) *restricted to W_0 are, in the best case, horizontal projectors of connections along a submanifold W_f of W_0. In particular, if $\mathbf{h}$ is such a solution, which is assumed to be written in the form*

$$\mathbf{h} = \mathrm{d}x^j \otimes \left(\frac{\partial}{\partial x^j} + A^\alpha{}_j \frac{\partial}{\partial u^\alpha} + A^\alpha_{ij} \frac{\partial}{\partial u^\alpha_i} + B_{\alpha j}^{\ i} \frac{\partial}{\partial p^i_\alpha} + C_j \frac{\partial}{\partial p} \right), \tag{3.103}$$

then it must satisfy the equation of holonomy

$$A^\alpha{}_i = u^\alpha_i, \tag{3.104}$$

the equations of dynamics

$$B_{\alpha j}^{\ j} = \frac{\partial L}{\partial u^\alpha}, \tag{3.105}$$

$$p^i_\alpha = \frac{\partial L}{\partial u^\alpha_i}, \tag{3.106}$$

plus the equations of tangency

$$B_{\alpha j}^{\ i} = \frac{\partial^2 L}{\partial x^j \partial u^\alpha_i} + u^\beta \frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_i} + A^\beta_{kj} \frac{\partial^2 L}{\partial u^\beta_k \partial u^\alpha_i}, \tag{3.107}$$

$$C_j = \frac{\partial L}{\partial x^j} + u^\alpha_j \frac{\partial L}{\partial u^\alpha} - B_{\alpha j}^{\ i} u^\alpha_i. \tag{3.108}$$

The submanifold W_f is contained in the submanifold of W_0 defined by

$$W_1 := \left\{ (x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha) \in W \ : \ p = L - p^i_\alpha u^\alpha_i,\ p^i_\alpha = \frac{\partial L}{\partial u^\alpha_i} \right\}, \tag{3.109}$$

and coincides with it when $\mathcal{L}$ is regular.

Proof. We begin by defining W_1 as the subset of W where point-wise solutions of the dynamical equation (3.102) exist, that is

$$W_1' := \{w \in W \ / \ \exists \mathbf{h}_w : T_wW \longrightarrow T_wW \text{ linear such that } \mathbf{h}_w^2 = \mathbf{h}_w, \\ \ker \mathbf{h}_w = \mathcal{V}_w\,\pi_{W,M},\ i_{\mathbf{h}_w}\Omega_{\mathcal{H}}(w) = (m-1)\Omega_{\mathcal{H}}(w)\}.$$

For a given point $w \in W$, we fix a chart around it with coordinates $(x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha)$ and consider an arbitrary horizontal projector $\mathbf{h}_w$ in T_WW. Then, $\mathbf{h}_w$ must certainly have the form (3.103). We therefore compute

$$i_{\mathbf{h}_w}\Omega_{\mathcal{H}} - (m-1)\Omega_{\mathcal{H}} = \\ \left[\left(B_{\alpha i}^{\ i} - \frac{\partial L}{\partial u^\alpha}\right)\mathrm{d}u^\alpha + \left(p^i_\alpha - \frac{\partial L}{\partial u^\alpha_i}\right)\mathrm{d}u^\alpha_i + (u^\alpha_i - A^\alpha_{\ i})\,\mathrm{d}p^i_\alpha\right]\mathrm{d}^m x,$$

equating to zero, we deduce that, in order to be a solution of the dynamical equation (3.102), $\mathbf{h}_w$ must be defined over a point w that satisfies Equation (3.106) and its coefficients the equations (3.104) and (3.105).

By a reasoning in terms of partitions of the unity similar to the one given in the proof of Proposition 3.49, we obtain a horizontal projector $\mathbf{h} : T_{W_1'}W \to T_{W_1'}W$ defined over W_1' which satisfies the dynamical equation (3.102). We now restrict $\mathbf{h}$ to be defined over $W_1 := W_0 \cap W_1'$, hence obtaining a horizontal projector $\mathbf{h} : T_{W_1}W \to T_{W_1}W$ defined over W_1 which satisfies the dynamical equation (3.102). But we still have to ensure that $\mathbf{h}$ is a horizontal projector along W_1, that is $\mathbf{h}$ takes values in TW_1: Therefore, we impose the tangency condition $\mathbf{h}_w(T_wW) \subset T_wW_1$, $\forall w \in W_1$. This latter condition is equivalent to having

$$\mathbf{h}\left(\frac{\partial}{\partial x^j}\right)\left(p^i_\alpha - \frac{\partial L}{\partial u^\alpha_i}\right) = 0 \quad \text{and} \quad \mathbf{h}\left(\frac{\partial}{\partial x^j}\right)\left(p^i_\alpha - \frac{\partial L}{\partial u^\alpha_i}\right) = 0,$$

which in turn is equivalent (using the previous relations) to equations (3.107) and (3.108).

By combining the first equation of dynamics (3.105) with the first equation of tangency (3.107), we get

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u^\alpha_i} - u^\beta_i \frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_i} - A^\beta_{ji}\frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i} = 0.$$

If $\mathcal{L}$ is regular, nothing else can be stated than that W_f coincides with W_1 and that $\mathbf{h}$ defines a connection along it. Otherwise, depending on the non-regularity of $\mathcal{L}$, the last equation could derive restrictions in W_0 to obtain the first constraint manifold (see remark 3.77 below). Nevertheless, it is contained in the submanifold W_1 given above. □

Remark 3.77. It shall be said that Theorem 3.76 remains true when the dynamical equation (3.102) is considered on the whole of W (instead of restricted to W_0), but then W_1 should be changed by W_1', so the tangency condition (3.108) is no longer available.

We may note in the Theorem's proof that, while the coefficients $A^\alpha_{\ i}$ and C_j of $\mathbf{h}$ are completely determined (equations (3.104) and (3.108)), the coefficients $B_{\alpha j}^{\ i}$ are overdetermined (equations (3.105) and (3.107)), what gives an extra restriction on the coefficients A^α_{ij} for each α:

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u^\alpha_i} - u^\beta_i \frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_i} - A^\beta_{ji}\frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i} = 0. \tag{3.110}$$

Although, the latter coefficients cannot be completely determined in general.

When the base manifold M has dimension $m = 1$, further restrictions could be obtained on the manifold along which $\mathbf{h}$ is defined, depending on if the Lagrangian density is regular or not. In this case, W_1 would include these restrictions and they will give further tangency conditions to determine the coefficients of $\mathbf{h}$. Assume that $M = 1$ and let $(t, q^\alpha, v^\alpha, p, p_\alpha)$ denote adapted coordinates on W. By Theorem 3.76, a solution $\mathbf{h}$ of the dynamical equation (3.102) would satisfy in particular the equations

$$B_\alpha = \frac{\partial L}{\partial q^\alpha} \quad \text{and} \quad B_\alpha = \frac{\partial^2 L}{\partial t \partial v^\alpha} + v^\beta \frac{\partial^2 L}{\partial q^\beta \partial v^\alpha} + A^\beta \frac{\partial^2 L}{\partial v^\beta \partial v^\alpha},$$

where

$$\mathbf{h} = \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial q^\alpha} + A^\alpha \frac{\partial}{\partial v^\alpha} + B_\alpha \frac{\partial}{\partial p^i_\alpha} + C \frac{\partial}{\partial p} \right).$$

Therefore, if $\mathcal{L}$ is not regular and we consider an element (V^α) in the kernel of $\frac{\partial^2 L}{\partial v^\beta \partial v^\alpha}$, then

$$\left(\frac{\partial L}{\partial q^\alpha} - \frac{\partial^2 L}{\partial t \partial v^\alpha} - v^\beta \frac{\partial^2 L}{\partial q^\beta \partial v^\alpha} \right) V^\alpha = 0,$$

which is a new restriction that determines the submanifold where $\mathbf{h}$ is defined.

Analogously, if the base manifold is multidimensional ($m > 1$) then, a possible way to obtain constraints derived from Equation (3.110) is to find an non-trivial element V^α such that

$$\frac{\partial^2 L}{\partial u^\beta_j \partial u^\alpha_i} V^\alpha = 0, \quad \forall i, j, \beta,$$

but this is no an easy task. The new constraint would then be

$$\left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u^\alpha_i} - u^\beta_i \frac{\partial^2 L}{\partial u^\beta \partial u^\alpha_i} \right) V^\alpha = 0.$$

Unfortunately, this method cannot be used in the general case ($m > 1$ and $n > 1$).

However, in both cases ($m = 1$ or $m > 1$), a remarkable fact is that the "semi-holonomy" of $\mathbf{h}$ yields immediately (Equation (3.104)) whether the Lagrangian density is regular or not, which differs from the Lagrangian formalism (see Proposition 3.49 or Corollary 3.50). Taking this into account, there is a clear analogy between Equation (3.110) and the equations derived in the proof of Proposition 3.48.

Example 3.78. Consider the fiber bundle $pr_1 : \mathbb{R}^4 \to \mathbb{R}^2$ with global adapted coordinates (x, y, u, v) and base volume form $\mathrm{d}x \wedge \mathrm{d}y$. We consider the Lagrangian function $L : J^1 pr_1 \to \mathbb{R}$

$$L = uv + (u_x + v_x)(u_y + v_y).$$

In this case, Equation (3.110) reads

$$\begin{aligned} v - A^u_{yx} - A^v_{yx} - A^u_{xy} - A^v_{xy} &= 0, \\ u - A^u_{yx} - A^v_{yx} - A^u_{xy} - A^v_{xy} &= 0. \end{aligned}$$

From where we deduce that $u = v$, hence the first constraint submanifold is

$$W_1 = \{ w \in W \ : \ p_0 = uv - p^x q^y, \ p^x = u_y + v_y = q^x, \ p^y = u_x + v_x = q^y, \ u = v \}.$$

Requiring that **h** be defined along, we obtain the second and final constraint submanifold

$$W_f = W_2 = \{w \in W_1 \;:\; u_x = u_y\}$$

with the corresponding tangency conditions on **h**.

Proposition 3.79. *Let Ω_1 denote the pullback of $\Omega_{\mathcal{H}}$ to W_1 by the natural inclusion $i : W_1 \hookrightarrow W$, that is $\Omega_1 = i^*(\Omega_{\mathcal{H}})$. Suppose that $\dim M > 1$ (resp. $\dim M = 1$). The $(m+1)$-form Ω_1 is multisymplectic (resp. cosymplectic together with η) if and only if $\mathcal{L}$ is regular.*

Proof. First of all, assume that $m > 1$ and let us make some considerations. By definition, Ω_1 is multisymplectic whenever Ω_1 has trivial kernel, that is,

$$\text{if } v \in TW_1,\; i_v\Omega_1 = 0 \iff v = 0\ .$$

This is equivalent to say that

$$\text{if } v \in i_*(TW_1),\; i_v\Omega_{\mathcal{H}}|_{i_*(TW_1)} = 0 \iff v = 0\ .$$

Let $v \in TW$ be a tangent vector whose coefficients in an adapted basis are given by

$$v = \gamma^i \frac{\partial}{\partial x^i} + A^\alpha \frac{\partial}{\partial u^\alpha} + A_i^\alpha \frac{\partial}{\partial u_i^\alpha} + B_\alpha^i \frac{\partial}{\partial p_\alpha^i} + C \frac{\partial}{\partial p}.$$

Using the local expression (3.101), we may compute the contraction of $\Omega_{\mathcal{H}}$ by v,

$$\begin{aligned} i_v\Omega_{\mathcal{H}} = & - B_\alpha^i \,\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + A^\alpha \,\mathrm{d}p_\alpha^i \wedge \mathrm{d}^{m-1}x_i - \gamma^j \,\mathrm{d}p_\alpha^i \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-2}x_{ij} \\ & + \left(A_i^\alpha p_\alpha^i + B_\alpha^i u_i^\alpha - A^\alpha \frac{\partial L}{\partial u^\alpha} - A_i^\alpha \frac{\partial L}{\partial u_i^\alpha} \right) \mathrm{d}^m x \\ & - \gamma^j \left(p_\alpha^i \,\mathrm{d}u_i^\alpha + u_i^\alpha \,\mathrm{d}p_\alpha^i - \frac{\partial L}{\partial u^\alpha} \mathrm{d}u^\alpha - \frac{\partial L}{\partial u_i^\alpha} \mathrm{d}u_i^\alpha \right) \wedge \mathrm{d}^{m-1}x_j. \end{aligned} \tag{3.111}$$

In addition to this, let us consider a vector $v \in TW$ tangent to W_1, that is $v \in i_*(TW_1)$, we then have that

$$\mathrm{d}\left(p_\alpha^i - \frac{\partial L}{\partial u_i^\alpha} \right)(v) = 0 \quad \text{and} \quad \mathrm{d}\left(p + p_\alpha^i u_i^\alpha - L \right)(v) = 0,$$

which leads us to the following relations for the coefficients of v:

$$B_\alpha^i \;=\; \gamma^j \frac{\partial^2 L}{\partial x^j \partial u_i^\alpha} + A^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} + A_j^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} \tag{3.112}$$

$$C \;=\; \gamma^j \frac{\partial L}{\partial x^j} + A^\alpha \frac{\partial L}{\partial u^\alpha} - B_\alpha^i u_i^\alpha. \tag{3.113}$$

It is important to note that, even though the coefficient A_i^α explicitly appears in the previous equations (3.111) and (3.112), for such a vector $v \in i_*(TW_1)$, the terms associated to these A_i^α cancel out in the development of $i_v\Omega_{\mathcal{H}}$, Equation (3.111). Thus, a tangent vector $v \in i_*(TW_1)$ is in the kernel of $\Omega_{\mathcal{H}}$ if and only if its coefficients satisfy the following relations

$$\gamma^j = 0,\; A_i^\alpha = 0,\; A_j^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} = 0,\; B_\alpha^i = 0,\; C = 0.$$

These considerations being made, the assertion is now clear for the multidimensional case.

Now, let suppose $m = 1$ and consider coordinates $(t, q^\alpha, v^\alpha, p, p_\alpha)$ on W, which induce coordinates (t, q^α, v^α) on W_1. In these coordinates, the Cartan $(m+1)$-form is written

$$\Omega_{\mathcal{H}} = -\,\mathrm{d}p_\alpha \wedge \mathrm{d}q^\alpha + v^\alpha\,\mathrm{d}v^\alpha \wedge \mathrm{d}t + p_\alpha\,\mathrm{d}v^\alpha \wedge \mathrm{d}t -\,\mathrm{d}L \wedge \mathrm{d}t$$

and its pull back to W_1

$$\Omega_1 = -\,\mathrm{d}\left(\frac{\partial L}{\partial v^\alpha}\right) \wedge \mathrm{d}q^\alpha + v^\alpha\,\mathrm{d}\left(\frac{\partial L}{\partial v^\alpha}\right) \wedge \mathrm{d}t - \frac{\partial L}{\partial q^\alpha}\,\mathrm{d}q^\alpha \wedge \mathrm{d}t.$$

A straightforward computation shows that

$$\Omega_1^n \wedge \mathrm{d}t = \det\left(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}\right)\mathrm{d}q^1 \wedge \mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}q^n \wedge \mathrm{d}v^n \wedge \mathrm{d}t,$$

which is a volume form if and only if $\mathcal{L}$ is regular. □

Corollary 3.80. *Under the same assumptions, we have: $(J^1\pi, \Omega_{\mathcal{L}})$, $(J^1\pi^0, \Omega_h)$ and (W_1, Ω_1) are (globally) locally multisymplecticomorphic (resp. cosymplecticomorphic together with η when $m = 1$) if and only if $\mathcal{L}$ is (hyper)regular. Indeed, $W_1 = graph(\mathrm{Leg}_{\mathcal{L}})$ and the corresponding multisymplecticomorphisms (resp. cosymplecticomorphisms) are*

$$\begin{tikzcd} & W_1 \arrow[dl, "pr_1|_{W_1}"'] \arrow[dr, "\mu\circ pr_2|_{W_1} = \mathrm{leg}_{\mathcal{L}} \circ pr_1|_{W_1}"] & \\ J^1\pi \arrow[rr, "\mathrm{leg}_{\mathcal{L}}"'] & & J^1\pi^0 \end{tikzcd} \tag{3.114}$$

In the following proposition, W_f denotes the final constraint submanifold, which coincides with W_1 whenever $\mathcal{L}$ is regular.

Proposition 3.81. *Let σ be a section of $\pi_{W_f,M} : W_f \longrightarrow M$ and denote $\bar{\sigma} = i \circ \sigma$ and $\phi = \pi_{W_f,E} \circ \sigma$, where $i : W_f \hookrightarrow W$ is the canonical inclusion. If $\bar{\sigma}$ is an integral section of* **h**, *then the Lagrangian part $\sigma_1 = pr_1 \circ \sigma$ of σ is* holonomic, i.e. $\sigma_1 = j^1\phi$, *and satisfies the Euler-Lagrange equations:*

$$j^2\phi^*\left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_j}\right) = 0. \tag{3.115}$$

Proof. If $\sigma = (x^i, \sigma^\alpha, \sigma^\alpha_i, \sigma_0, \sigma^i_\alpha)$ is an integral section of **h**, then

$$\frac{\partial \sigma^\alpha}{\partial x^j} = A^\alpha{}_j,\ \frac{\partial \sigma^\alpha_i}{\partial x^j} = A^\alpha_{ij},\ \frac{\partial \sigma^i_\alpha}{\partial x^j} = B_{\alpha j}^{\ i} \text{ and } \frac{\partial \sigma_0}{\partial x^j} = C_j,$$

where the A's, B's and C's are the coefficients given in (3.103). From Equation (3.104), we have that σ_1 is holonomic, since $\sigma^\alpha_i = \partial\sigma^\alpha/\partial x^i$. On the other hand, using the equations (3.105) and (3.106), we obtain:

$$\begin{aligned} 0 &= \frac{\partial L}{\partial u^\alpha} \circ j^1\phi - \frac{\partial \sigma^j_\alpha}{\partial x^j}; \\ \sigma^i_\alpha &= \frac{\partial L}{\partial u^\alpha_i} \circ j^1\phi. \end{aligned}$$

We then have

$$0 = (j^1\phi)^* \frac{\partial L}{\partial u^\alpha} - (j^2\phi)^* \left(\frac{\mathrm{d}}{\mathrm{d}x^j} \frac{\partial L}{\partial u^\alpha_j} \right),$$

which is precisely the Euler-Lagrange equations. □

$$\begin{tikzcd}
 & & W = J^1\pi \times_E J^1\pi^\dagger \arrow[ddll, "pr_2"'] \arrow[ddrr, "pr_1"] & & \\
 & W_1' \arrow[ur] & & & \\
J^1\pi \arrow[ddrr] \arrow[dddrr, "\pi_1"'] & & W_1 \arrow[ul] \arrow[ll] \arrow[dd] \arrow[drr] & & J^1\pi^\dagger \arrow[d] \\
 & & & & J^1\pi^\circ \arrow[dll] \arrow[ddll, "\pi_1^\circ"] \\
 & & E \arrow[d, "\pi"'] & & \\
 & & X & &
\end{tikzcd}$$

Definition 3.82. Let $\mathcal{H}$ be the Hamiltonian density associated to a given Lagrangian density $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$. The associated *(extended) Hamiltonian action* is the map $\mathcal{A}_\mathcal{H} : \Gamma\pi_{W,M} \times \mathcal{K} \to \mathbb{R}$ given by

$$\mathcal{A}_\mathcal{H}(\sigma, R) := \int_R \sigma^*(\Theta_\mathcal{H}), \tag{3.116}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

It is called Hamilton-Pontryagin principle for field theories in [151].

Theorem 3.83. *A section $\sigma : M \to W$ of $\pi_{W,M} : W \to M$ is a critical point of the Hamiltonian action $\mathcal{A}_\mathcal{H}$ if and only if it satisfies the local equations*

$$\sigma_i^\alpha = \frac{\partial \sigma^\alpha}{\partial x^i}, \quad \frac{\partial \sigma^i_\alpha}{\partial x^i} = \frac{\partial L}{\partial u^\alpha}, \quad \textit{and} \quad \sigma^i_\alpha = \frac{\partial L}{\partial u^\alpha_i} \tag{3.117}$$

on M, and

$$L(x^i, \sigma^\alpha, \sigma^\alpha_i) = 0 \quad \textit{and} \quad \sigma^i_\alpha = \frac{\partial L}{\partial u^\alpha_i}\Big|_{\sigma(x)} = 0 \tag{3.118}$$

on the boundary ∂M of M, where $(x^i, u^\alpha, u^\alpha_i, p, p^i_\alpha)$ denotes adapted coordinates on W and $\sigma = (x^i, \sigma^\alpha, \sigma^\alpha_i, \sigma_0, \sigma^i_\alpha)$.

Proof. As usual, given a section $\sigma \in \Gamma\pi_1^\dagger$ and a compact region $R \subseteq M$, let $\sigma_\varepsilon = \varphi_\varepsilon \circ \sigma \circ (\check{\varphi}_\varepsilon)^{-1}$ be a variation of σ such that the infinitesimal generator ξ of φ_ε vanishes outside of $(\pi_1^\dagger)^{-1}(R)$. The variation of the Hamiltonian action $\mathcal{A}_\mathcal{H}$ is then given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\mathcal{A}_\mathcal{H}(\sigma_\varepsilon, R_\varepsilon)]\Big|_{\varepsilon=0} &= \int_R \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[(\varphi_\varepsilon \circ \sigma)^*\Theta_\mathcal{H}]\Big|_{\varepsilon=0} \\
&= \int_R \sigma^*(\mathfrak{L}_\xi \Theta_\mathcal{H}) \\
&= -\int_R \sigma^*(i_\xi \Omega_\mathcal{H}) + \int_{\partial R \cap \partial M} \sigma^*(i_\xi \Theta_\mathcal{H}).
\end{aligned}$$

We deduce from here that σ is a critical point of $\mathcal{A}_{\mathcal{H}}$ if and only if

$$\sigma^*(i_\xi \Omega_{\mathcal{H}}) = 0 \quad \text{and} \quad \sigma^*(i_\xi \Theta_{\mathcal{H}}) \underset{\partial M}{=} 0, \quad \forall \xi \in \mathfrak{X}(W).$$

Using local coordinates $(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i)$ on W and denoting $\sigma = (x^i, \sigma^\alpha, \sigma_i^\alpha, \sigma_0, \sigma_\alpha^i)$, we compute on the one hand

$$\begin{aligned}\sigma^*(i_\xi \Omega_{\mathcal{H}}) = \Bigg[& \xi^\alpha \left(\frac{\partial \sigma_\alpha^i}{\partial x^i} - \frac{\partial L}{\partial u^\alpha} \right) + \xi_i^\alpha \left(\sigma_\alpha^i - \frac{\partial L}{\partial u_i^\alpha} \right) + \xi_\alpha^i \left(\sigma_i^\alpha - \frac{\partial \sigma^\alpha}{\partial x^i} \right) + \\ & + \xi^j \left(\frac{\partial \sigma_\alpha^i}{\partial x^j} \frac{\partial \sigma^\alpha}{\partial x^i} - \frac{\partial \sigma_\alpha^i}{\partial x^i} \frac{\partial \sigma^\alpha}{\partial x^j} - \sigma_\alpha^i \frac{\partial \sigma_i^\alpha}{\partial x^j} - \sigma_i^\alpha \frac{\partial \sigma_\alpha^i}{\partial x^j} + \frac{\partial L}{\partial u^\alpha} \frac{\partial \sigma^\alpha}{\partial x^j} + \frac{\partial L}{\partial u_i^\alpha} \frac{\partial \sigma_i^\alpha}{\partial x^j} \right) \Bigg] \mathrm{d}^m x,\end{aligned}$$

and on the other hand

$$\sigma^*(i_\xi \Theta_{\mathcal{H}}) = \left[\xi^j (L - \sigma_\alpha^i \sigma_i^\alpha) + \xi^\alpha \sigma_\alpha^j + \xi^j \sigma_\alpha^i \frac{\partial \sigma^\alpha}{\partial x^j} - \xi^i \sigma_\alpha^j \frac{\partial \sigma^\alpha}{\partial x^j} \right] \mathrm{d}^{m-1} x_j.$$

From here, we conclude that, in order to be a critical point of $\mathcal{A}_{\mathcal{H}}$, σ must satisfy the equations (3.117) and (3.118). □

Note that equations in (3.117) are equivalent to equations (3.104–3.106) when we consider an integral section of a solution $\mathbf{h}$ of the dynamical equation (3.102). They also correspond to the Euler-Lagrange equations (3.38) (combine the second and the third one), to the Hamilton's equations (3.77) (define $H = u_i^\alpha p_\alpha^i - L$ and consider the first two equations) and the Legendre transform (3.84) (take the third equation). In the same way, the boundary conditions (3.118) are equivalent to those the Lagrangian side, Equation (3.39), and those of the Hamiltonian side, Equation (3.78) (see remarks 3.36 and 3.63).

Definition 3.84. Let $\mathcal{L} : J^1\pi \longrightarrow \Lambda^m M$ be a Lagrangian density. The associated *(extended) Hamiltonian-Pontryagin action* is the map $\mathcal{A}_{\mathcal{L}} : \Gamma\pi_{W,M} \times \mathcal{K} \to \mathbb{R}$ given by

$$\mathcal{A}_{\mathcal{L}}(\sigma, R) := \int_R \left(\mathcal{L} \circ \sigma_1 + \left\langle \sigma_1^\dagger, j^1\sigma_0 \right\rangle - \left\langle \sigma_1^\dagger, \sigma_1 \right\rangle \right) \tag{3.119}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

In fact, the Hamiltonian-Pontryagin action 3.84 coincides with the Hamiltonian action 3.82 as stated by the next result.

Theorem 3.85. *A section $\sigma : M \to W$ of $\pi_{W,M} : W \to M$ is a critical point of the Hamiltonian-Pontryagin action $\mathcal{A}_{\mathcal{L}}$ if and only if it satisfies the local equations*

$$\sigma_i^\alpha = \frac{\partial \sigma^\alpha}{\partial x^i}, \quad \frac{\partial \sigma_\alpha^i}{\partial x^i} = \frac{\partial L}{\partial u^\alpha}, \quad \textit{and} \quad \sigma_\alpha^i = \frac{\partial L}{\partial u_i^\alpha} \tag{3.120}$$

on M, and

$$\sigma_\alpha^i = \left. \frac{\partial L}{\partial u_i^\alpha} \right|_{\sigma(x)} = 0 \tag{3.121}$$

on the boundary ∂M of M, where $(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i)$ denotes adapted coordinates on W and $\sigma = (x^i, \sigma^\alpha, \sigma_i^\alpha, \sigma_0, \sigma_\alpha^i)$.

Proof. Given a section $\sigma \in \Gamma\pi_1^\dagger$ and a compact region $R \subseteq M$, we have that the variation of the Hamiltonian-Pontryagin action $\mathcal{A}_\mathcal{L}$ with respect to a variation $\delta\sigma$ of σ is given by

$$\begin{aligned}
\left.\frac{\delta \mathcal{A}_\mathcal{L}}{\delta\sigma}\right|_{(\sigma,R)} \cdot \delta\sigma &= \int_R \frac{\delta}{\delta\sigma}\left[L(x^i,\sigma^\alpha,\sigma_i^\alpha) + \sigma_\alpha^i\left(\frac{\partial\sigma^\alpha}{\partial x^i} - \sigma_i^\alpha\right)\right]\Bigg|_\sigma \cdot \delta\sigma \,\mathrm{d}^m x \\
&= \int_R \left[\frac{\partial L}{\partial u^\alpha}\delta\sigma^\alpha + \frac{\partial L}{\partial u_i^\alpha}\delta\sigma_i^\alpha + \delta\sigma_\alpha^i\left(\frac{\partial\sigma^\alpha}{\partial x^i} - \sigma_i^\alpha\right) + \sigma_\alpha^i\left(\frac{\partial}{\partial x^i}\delta\sigma^\alpha - \delta\sigma_i^\alpha\right)\right]\mathrm{d}^m x \\
&= \int_R \left[\left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial\sigma_\alpha^i}{\partial x^i}\right)\delta\sigma^\alpha + \left(\frac{\partial L}{\partial u_i^\alpha} - \sigma_\alpha^i\right)\delta\sigma_i^\alpha + \left(\frac{\partial\sigma^\alpha}{\partial x^i} - \sigma_i^\alpha\right)\delta\sigma_\alpha^i\right]\mathrm{d}^m x \\
&\quad + \int_{\partial R} \sigma_\alpha^i\delta\sigma^\alpha \,\mathrm{d}^{m-1}x_i.
\end{aligned}$$

where $(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i)$ denotes adapted coordinates on W and $\sigma = (x^i, \sigma^\alpha, \sigma_i^\alpha, \sigma_0, \sigma_\alpha^i)$. We thus deduce that σ is a critical point of $\mathcal{A}_\mathcal{L}$, *i.e.* $\delta\mathcal{A}_\mathcal{L}/\delta\sigma = 0$, if and only if the relations (3.120) and (3.121) are satisfied. □

Here, the boundary conditions (3.121) differ from the boundary conditions (3.118), since in the proof we have considered vertical variations.

Chapter 4

Higher Order Classical Field Theory

In this chapter, we will find the main original contributions of this memory. For it, we will first extend the notions of jets to an arbitrary order, that is, higher-order jets. We will find, as an important result, an unambiguous and intrinsic formalism for the higher-order calculus of variations. The case of constrained calculus will be also analyzed. The main results appear in [24, 25, 26, 27] and in a forthcoming paper. As a basic reference in what follows, the reader is refereed to the book by Saunders [139].

Through this section, (E,π,M) denotes a fiber bundle whose base space M is a smooth manifold of dimension m, and whose fibers have dimension n, thus E is $(m+n)$-dimensional. Adapted coordinate systems in E will be of the form (x^i,u^α), where (x^i) is a local coordinate system in M and (u^α) denotes fiber coordinates.

Lower case Latin (resp. Greek) letters will usually denote indexes that range between 1 and m (resp. 1 and n). Capital Latin letters will usually denote multi-indexes whose length ranges between 0 and k (see Appendix §A). In particular and if nothing else it is stated, I and J will usually denote multi-indexes whose length goes from 0 to $k-1$ and 0 to k, respectively; and K (and sometimes R) will denote multi-indexes whose length is equal to k. The Einstein notation for repeated indexes and multi-indexes is understood but, for clarity, in some cases the summation for multi-indexes will be indicated.

4.1 Higher Order Jet bundles

Definition 4.1. Given a point $x\in M$, two local sections $\phi,\psi\in\Gamma_x\pi$ are *k-equivalent at x* if their value coincide at x, as well as their partial derivatives up to order k

$$\phi(x)=\psi(x) \text{ and } \left.\frac{\partial^k\phi^\alpha}{\partial x^{i_1}\cdots\partial x^{i_k}}\right|_x=\left.\frac{\partial^k\psi^\alpha}{\partial x^{i_1}\cdots\partial x^{i_k}}\right|_x,$$

for all $1\le\alpha\le n$, $1\le i_j\le m$, $1\le j\le k$. This defines an equivalence relation in $\Gamma_x\pi$. The equivalence class containing ϕ is called the *kth jet of ϕ at x* and is denoted $j^k_x\phi$.

The notion of k-equivalency is independent of the chosen coordinate system (adapted or not), thus so is the equivalence relation that it defines (see [61, 64, 139], for more details).

Definition 4.2. The *kth jet manifold of π*, denoted $J^k\pi$, is the whole collection of kth jets of arbitrary local sections of π, that is,

$$J^k\pi:=\left\{j^k_x\phi : x\in M,\ \phi\in\Gamma_x\pi\right\}.$$

The functions given by

$$\begin{array}{rcl} \pi_k : J^k\pi & \longrightarrow & M \\ j^k_x\phi & \longmapsto & x \end{array} \tag{4.1}$$

and

$$\begin{array}{rcl} \pi_{k,0} : J^k\pi & \longrightarrow & E \\ j^k_x\phi & \longmapsto & \phi(x) \end{array} \tag{4.2}$$

are called the *kth source projection* and the *kth target projection* respectively.

From the definitions, it is trivial to see that $j^0_x\phi = \phi(x)$, $J^0\pi = E$, $\pi_0 = \pi$ and $\pi_{0,0} = \mathrm{Id}_E$.

Proposition 4.3. *The kth jet manifold of π, $J^k\pi$, may be endowed with a structure of smooth manifold. A system of adapted coordinates (x^i, u^α) on E induces a system of coordinates (x^i, u^α_I) (with $0 \leq |I| \leq k$) on $J^1\pi$ such that*

$$x^i(j^k_x\phi) = x^i(x) \quad \textit{and} \quad u^\alpha_I(j^k_x\phi) = \left.\frac{\partial^{|I|}\phi^\alpha}{\partial x^I}\right|_x.$$

In the induced local coordinates (x^i, u^α_I), the source and the target projections are written

$$\pi_k(x^i, u^\alpha_I) = (x^i) \quad \text{and} \quad \pi_{k,0}(x^i, u^\alpha_I) = (x^i, u^\alpha). \tag{4.3}$$

From here, it is clear that π_k and $\pi_{k,0}$ are certainly projections (surjective submersions) over M and E, respectively. Therefore, $(J^k\pi, \pi_k, M)$ and $(J^k\pi, \pi_{k,0}, E)$ are fiber bundles.

If we consider a change of coordinates $(x^i, u^\alpha) \mapsto (y^j, v^\beta)$ in E, it induces a change of coordinates $(x^i, u^\alpha_I) \mapsto (y^j, v^\beta_J)$ in $J^1\pi$. In this case, the "velocities" transform by the following rule:

$$\begin{aligned} v^\beta_{J+1_j} &= \left(\frac{\partial v^\beta_J}{\partial x^i} + u^\alpha_{I+1_i}\frac{\partial v^\beta_J}{\partial u^\alpha_I}\right)\frac{\partial x^i}{\partial y^j} \\ &= \frac{\partial v^\beta_J}{\partial x^i}\frac{\partial x^i}{\partial y^j} + u^\alpha_{I'}\sum_{I+1_i=I'}\frac{\partial v^\beta_J}{\partial u^\alpha_I}\frac{\partial x^i}{\partial y^j}, \end{aligned} \tag{4.4}$$

from where we deduce that coordinates of a particular order depend only on coordinates of equal or lower order, that is

$$v^\beta_J = v^\beta_J(x^i, u^\alpha_I) \ : \ |I| \leq |J|.$$

Even more, the changes have a polynomial expansion and it is affine from order to order (*cf.* [139]).

Proposition 4.4. *For each $0 \leq l \leq k$, define the map*

$$\begin{array}{rcl} \pi_{k,l} : J^k\pi & \longrightarrow & J^l\pi \\ j^k_x\phi & \longmapsto & j^l_x\phi. \end{array} \tag{4.5}$$

We have that $(J^k\pi, \pi_{k,l}, J^l\pi)$ are smooth fiber bundles to which the induced coordinates (x^i, u^α_I) are adapted. Moreover, for the particular case $l = k-1$, $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$ is an affine bundle, being its associated vector bundle

$$\pi^*_{k-1}(S^kT^*M) \otimes_{J^{k-1}\pi} \pi^*_{k-1,0}(\mathcal{V}\pi),$$

where S^kT^*M is the space of symmetric covariant tensors of order k over M and $\mathcal{V}\pi$ the vertical bundle of π.

$$
\begin{tikzcd}
J^k\pi \arrow[r, "\pi_{k,k-1}"] \arrow[ddddr, "\pi_k"] & \cdots \arrow[r, "\pi_{3,2}"] & J^2\pi \arrow[r, "\pi_{2,1}"] \arrow[ddr, "\pi_2"] & J^1\pi \arrow[r, "\pi_{1,0}"] \arrow[dr, "\pi_1"] & E \arrow[d, "\pi"] \\
 & & & & M \arrow[u, bend right, "\phi"'] \arrow[llllu, bend left, "j^k\phi"]
\end{tikzcd}
$$

Figure 4.1: Chain of jets

In the induced local coordinates (x^i, u^α_I) of $J^k\pi$, with $0 \le |I| \le k$, and (x^i, u^α_J) of $J^l\pi$, with $0 \le |J| \le l \le k$, we have the obvious local expression

$$\pi_{k,l}(x^i, u^\alpha_I) = (x^i, u^\alpha_J).$$

4.1.1 Prolongations, lifts and contact

Definition 4.5. Let $\phi \in \Gamma\pi$ be a (local) section, its *kth prolongation* is the (local) section of $\pi_{k,0}$ given by

$$(j^k\phi)(x) := j^k_x\phi,$$

for every $x \in M$. An arbitrary (local) section σ of π_k is said to be *holonomic* if it is the kth prolongation of a (local) section $\phi \in \Gamma\pi$, that is, if $\sigma = j^k\phi$.

Definition 4.6. Let $f : E \to F$ be a bundle morphism between two fiber bundles (E, π, M) and (F, ρ, N), such that the induced function on the base, $\check{f} : M \to N$, is a diffeomorphism. The *kth prolongation of* f is the map $j^k f : J^k\pi \to J^k\rho$ given by

$$(j^k f)(j^k_x\phi) := j^k_{\check{f}(x)}\phi_f, \ \forall j^k_x\phi \in J^k\pi,$$

where $\phi_f := f \circ \phi \circ \check{f}^{-1} \in \Gamma\rho$.

$$
\begin{tikzcd}
J^k\pi \arrow[r, "j^k f"] \arrow[d, "\pi_{k,0}"] & J^k\rho \arrow[d, "\rho_{k,0}"'] \\
E \arrow[r, "f"] \arrow[d, "\pi"] & F \arrow[d, "\rho"'] \\
M \arrow[r, "\check{f}"] \arrow[u, bend left, "\phi"] \arrow[uu, bend left=60, "j^k\phi"] & N \arrow[u, bend right, "\phi_f"'] \arrow[uu, bend right=60, "j^k\phi_f"']
\end{tikzcd}
$$

Figure 4.2: The kth prolongation of a morphism

Note that the kth prolongation $j^k f$ of a morphism f is not only a morphism between $(J^k\pi, \pi_{k,0}, E)$ and $(J^k\rho, \rho_{k,0}, F)$, and a morphism between $(J^k\pi, \pi_k, M)$ and $(J^k\rho, \rho_k, N)$, but also a morphism between the intermediate lth jet bundles $(J^k\pi, \pi_{k,l}, J^l\pi)$ and $(J^k\rho,$

$\rho_{k,l}, J^l\rho)$, for $0 < l < k$. In each case, the induced functions between the corresponding base spaces are f, $\check{f}$ and $j^l f$, respectively.

If (x^i, u_I^α) and $(y^j, v^\beta, v_J^\beta)$ denote adapted coordinates in $J^k\pi$ and $J^k\rho$, respectively, then we have

$$f^\beta_{J+1_j} = v^\beta_{J+1_j} \circ j^k f = \left(\frac{\partial f^\beta_J}{\partial x^i} + u^\alpha_{I+1_i}\frac{\partial f^\beta_J}{\partial u^\alpha_I}\right)\cdot\frac{\partial \check{f}^{-i}}{\partial y^j}.$$

The expression between brackets is called the *total derivative* of f_J^β with respect to x^i. We will come back to it later.

Definition 4.7. Let $\phi : M \to E$ be a section of π, $x \in M$ and $u = j_x^{k-1}\phi$. The *vertical differential* of the section ϕ at the point $u \in J^{k-1}\pi$ is the map

$$\begin{aligned} \mathrm{d}^{\mathrm{v}}_u\phi : T_u J^{k-1}\pi &\longrightarrow \mathcal{V}_u\,\pi_{k-1} \\ v &\longmapsto v - T_u(j^{k-1}\phi\circ\pi_{k-1})(v) \end{aligned}$$

Namely, $\mathrm{d}^{\mathrm{v}}_u\phi := \mathrm{Id}_u - T_u(j^{k-1}\phi\circ\pi_{k-1})$.

Notice that the image of $\mathrm{d}^{\mathrm{v}}_u\phi$ is certainly in $\mathcal{V}_u\,\pi_{k-1}$ since $T_u\pi_{k-1}\circ\mathrm{d}^{\mathrm{v}}_u\phi = 0$ and that, in fact, $\mathrm{d}^{\mathrm{v}}_u\phi$ depends only on $j^k_x\phi$. In adapted local coordinates (x^i, u^α_I) of $J^{k-1}\pi$,

$$\mathrm{d}^{\mathrm{v}}_u\phi = \left(\mathrm{d}u^\alpha_I - \frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}}\,\mathrm{d}x^i\right)\otimes\frac{\partial}{\partial u^\alpha_I}. \tag{4.6}$$

Definition 4.8. The *canonical structure form* of $J^k\pi$ is the 1-form θ on $J^k\pi$ with values in $\mathcal{V}\,\pi_{k-1}$ defined by

$$\theta_{j^k_x\phi}(V) := (\,\mathrm{d}^{\mathrm{v}}_{j^{k-1}_x\phi}\phi)(T_{j^k_x\phi}\pi_{k,k-1}(V)),\ V\in T_{j^k_x\phi}J^k\pi, \tag{4.7}$$

where ϕ is any representative of $j^k_x\phi \in J^k\pi$. The contraction of the covectors in $\mathcal{V}^*\,\pi_{k-1}$ with θ defines a "distribution" in $T^*J^k\pi$. This distribution is called the *contact module* or the *Cartan codistribution (of order k)* and it is denoted $\mathcal{C}^k$. Its elements are *contact forms.* The annihilator of $\mathcal{C}^k$ is the *Cartan distribution (of order k).*

Note that the expression (4.7) does not depend on the representative ϕ of $j^k_x\phi$, hence it is well defined. In adapted local coordinates $(x^i, u^\alpha_I, u^\alpha_K)$ of $J^k\pi$, where $0\le|I|\le k-1$ and $|K| = k$,

$$\theta = \left(\mathrm{d}u^\alpha_I - u^\alpha_{I+1_i}\,\mathrm{d}x^i\right)\otimes\frac{\partial}{\partial u^\alpha_I}. \tag{4.8}$$

In fact, the contact forms $\mathrm{d}u^\alpha_I - u^\alpha_{I+1_i}\,\mathrm{d}x^i \in \mathcal{C}^k$ are a base of the contact module.

Proposition 4.9. *Let $(x^i, u^\alpha_I, u^\alpha_K)$ be adapted coordinates on $J^k\pi$, where $0\le|I|\le k-1$ and $|K| = k$, a basis of the Cartan codistribution is given by the* coordinate contact forms

$$\theta^\alpha_I = \mathrm{d}u^\alpha_I - u^\alpha_{I+1_i}\,\mathrm{d}x^i. \tag{4.9}$$

Proposition 4.10. *The canonical structure form $\theta\in\Gamma(T^*J^k\pi\otimes_{J^k\pi}\mathcal{V}\,\pi)$ and the contact forms $\omega\in\mathcal{C}^k$ are pulled back to zero by the kth prolongation $j^k\phi$ of any section ϕ of π. Moreover, this characterizes the module of contact forms,* i.e.

$$\omega\in\mathcal{C}^k \;\Leftrightarrow\; (j^k\phi)^*\omega = 0,\ \forall\phi\in\Gamma\pi. \tag{4.10}$$

Proof. Let $\omega \in \Omega(J^k\pi)$ be an arbitrary form. We can write ω as the linear combination

$$\omega = \omega_i \,\mathrm{d}x^i + \omega_\alpha^J \,\mathrm{d}u_J^\alpha,\ 0 \leq |J| \leq k,$$

where the ω's are unknown functions on $J^k\pi$. Given any section ϕ of π, we have that

$$(j^k\phi)^*\omega = \left(\omega_i \circ j^k\phi + (\omega_\alpha^I \circ j^k\phi) \cdot \frac{\partial^{|I|+1}\phi^\alpha}{\partial x^{I+1_i}} + (\omega_\alpha^K \circ j^k\phi) \cdot \frac{\partial^{k+1}\phi^\alpha}{\partial x^{K+1_i}}\right) \mathrm{d}x^i = 0.$$

Since two k-equivalent sections at a point $x \in M$ coincide on their partial derivatives at x up to order k, we deduce that

$$\omega_\alpha^K = 0 \quad \text{and} \quad \omega_i + \omega_\alpha^I u_{I+1_i}^\alpha = 0.$$

Substituting ω_i and ω_α^K in the initial expression of ω, we obtain

$$\omega = -\omega_\alpha^I u_{I+1_i}^\alpha \,\mathrm{d}x^i + \omega_\alpha^J \,\mathrm{d}u_J^\alpha = \omega_\alpha^I (\,\mathrm{d}u_I^\alpha - u_{I+1_i}^\alpha \,\mathrm{d}x^i) = \omega_\alpha^J \theta_I^\alpha,$$

which proofs the sufficiency by Proposition 4.9.

The necessity is immediate. □

A complementary or dual result to the previous one is the following.

Proposition 4.11. *Let $\sigma \in \Gamma\pi_k$ be a (local) section. The following statements are equivalent:*

1. *σ is holonomic.*

2. *σ pulls back to zero any contact form, that is*

$$\sigma^*\omega = 0,\ \forall \omega \in \mathcal{C}^k. \tag{4.11}$$

Notice that the contact forms are $\pi_{k,k-1}$-basic, which is clear from the coordinate expression (4.9). Though, therefore they may be thought as forms along $\pi_{k,k-1}$ rather than on $J^k\pi$. In this sense are defined total derivatives.

Definition 4.12. A *total derivative* is a vector field ξ along $\pi_{k,k-1}$ which is annihilated by the Cartan codistribution (as forms along $\pi_{k,k-1}$). Given a system of adapted coordinates $(x^i, u^\alpha, u_I^\alpha, u_K^\alpha)$ in $J^k\pi$, where $0 \leq |I| \leq k-1$ and $|K| = k$, the local vector fields defined along $\pi_{1,0}$ by

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + u_{I+1_i}^\alpha \frac{\partial}{\partial u^\alpha} \tag{4.12}$$

are called *coordinate total derivatives.*

It is immediate to check that coordinate total derivatives are total derivatives, in fact they define a basis of such vector fields. Under a change of coordinates, (x^i, u^α) to (y^j, v^β), a coordinate total derivative transforms linearly by the Jacobian of the underlying change of coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}y^j} = \frac{\partial x^i}{\partial y^j} \frac{\mathrm{d}}{\mathrm{d}x^i}.$$

If $\xi \in \mathfrak{X}(\pi_{k,k-1})$ has the different coordinate representations

$$\xi = \xi^i \frac{\mathrm{d}}{\mathrm{d}x^i} = \xi^j \frac{\mathrm{d}}{\mathrm{d}y^j},$$

where the coefficients ξ^i and ξ^j are functions on $J^k\pi$. Then,

$$\xi^i = \xi^j \frac{\partial x^i}{\partial y^j}.$$

Definition 4.13. The *total lift* of a vector field $\xi = \xi^i \partial_i$ on M is the unique total derivative that projects on ξ itself, that is, the vector field $\hat{\xi}^k$ along $\pi_{k,k-1}$ locally given by

$$\hat{\xi}^k(j^k_x\phi) = \xi^i(x) \frac{\mathrm{d}}{\mathrm{d}x^i}\bigg|_{j^k_x\phi}.$$

Note that the total lift of the coordinate partial derivatives in M are precisely the coordinate total derivatives.

Now, consider the action of total derivatives on smooth functions over $J^{k-1}\pi$. If $f \in \mathcal{C}^\infty(J^{k-1}\pi)$, the action of $\mathrm{d}/\mathrm{d}x^i$ on it yields a function $\mathrm{d}f/\mathrm{d}x^i \in \mathcal{C}^\infty(J^k\pi)$. In particular, the action of $\mathrm{d}/\mathrm{d}x^i$ on the coordinate function $u^\alpha \in \mathcal{C}^\infty(E)$, gives as expected

$$\frac{\mathrm{d}u^\alpha_I}{\mathrm{d}x^i} = u^\alpha_{I+1_i} \in \mathcal{C}^\infty(J^k\pi), \quad \forall\, 0 \leq |I| \leq k-1.$$

Another interesting fact is how total derivatives and jets are related. Let $f \in \mathcal{C}^\infty(J^l\pi)$, $l < k$, $\phi \in \Gamma\pi$ and $\xi \in \mathfrak{X}(M)$, we have

$$\xi(f \circ j^l\phi) = \hat{\xi}^k(f) \circ j^{l+1}\phi,$$

in coordinates

$$\frac{\partial(f \circ \phi)}{\partial x^i} = \frac{\mathrm{d}f}{\mathrm{d}x^i} \circ j^k\phi. \tag{4.13}$$

Finally, note that coordinate total derivatives and ordinary partial derivates do not necesarilly conmute:

$$\frac{\partial}{\partial x^i}\frac{\mathrm{d}f}{\mathrm{d}x^j} = \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial f}{\partial x^i}, \quad \frac{\partial}{\partial u^\alpha}\frac{\mathrm{d}f}{\mathrm{d}x^j} = \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial f}{\partial u^\alpha} \quad \text{but} \quad \frac{\partial}{\partial u^\alpha_J}\frac{\mathrm{d}f}{\mathrm{d}x^i} = \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial f}{\partial u^\alpha_J} + \delta^J_{I+1_i}\frac{\partial f}{\partial u^\alpha_I},$$

where $f \in \mathcal{C}^\infty(E)$. Nevertheless, coordinate total derivatives do commute, what allow us to use the multi-index notation with iterated coordinate total derivatives.

Proposition 4.14. *Let $f \in \mathcal{C}^\infty(J^l\pi)$, then $\frac{\mathrm{d}f}{\mathrm{d}x^i} \in \mathcal{C}^\infty(J^{l+1}\pi)$ and $\frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\mathrm{d}f}{\mathrm{d}x^i} \in \mathcal{C}^\infty(J^{l+2}\pi)$. Moreover, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\mathrm{d}f}{\mathrm{d}x^i} = \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\mathrm{d}f}{\mathrm{d}x^j}.$$

Definition 4.15. Given a vector field ξ on E, its *kth lift* (or *kth jet*) is the unique vector field $\xi^{(k)}$ on $J^k\pi$ that is projectable to ξ by $\pi_{k,0}$ and preserves the Cartan codistribution with respect to the Lie derivative.

Proposition 4.16. *Let ξ be a vector field on E. If ξ has the local expression*

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial u^\alpha} \tag{4.14}$$

in adapted coordinates (x^i, u^α) on E, then its kth-lift $\xi^{(k)}$ has the form

$$\xi^{(k)} = \xi^i \frac{\partial}{\partial x^i} + \xi^\alpha_J \frac{\partial}{\partial u^\alpha_J} \tag{4.15}$$

for the induced coordinates (x^i, u^α_J) on $J^k\pi$, where

$$\xi^\alpha_0 = \xi^\alpha \quad \textit{and} \quad \xi^\alpha_{I+1_i} = \frac{\mathrm{d}\xi^\alpha_I}{\mathrm{d}x^i} - u^\alpha_{I+1_j}\frac{\mathrm{d}\xi^j}{\mathrm{d}x^i}. \tag{4.16}$$

In particular, if ξ is vertical with respect to π, then $\xi^\alpha_J = \mathrm{d}^{|J|}\xi^\alpha / \mathrm{d}x^J$.

Proof. Since $\xi^{(k)}$ is $\pi_{k,0}$-projectable to ξ, it must have the form

$$\xi^{(k)} = \xi^i \frac{\partial}{\partial x^i} + \xi^\alpha_J \frac{\partial}{\partial u^\alpha_J}$$

where $\xi^\alpha_0 = \xi^\alpha$ and where the remaining components ξ^α_J, with $|J| = 1, \ldots, k$, still have to be determined.

Note that the preserving condition is equivalent to require that the Lie derivatives by $\xi^{(k)}$ of the elements of any fixed base of the Cartan codistribution $\mathcal{C}^k$ are still contact forms. Thus, consider the base $\{\theta^\alpha_I\}$ given in Proposition (4.9) and let us compute the Lie derivative of its elements by $\xi^{(k)}$. Using the Cartan's formula $\mathfrak{L} = \mathrm{d} \circ i + i \circ \mathrm{d}$, we obtain

$$\begin{aligned}
\mathfrak{L}_{\xi^{(k)}}\theta^\alpha_I &= \mathfrak{L}_{\xi^{(k)}}(\,\mathrm{d}u^\alpha_I - u^\alpha_{I+1_i}\,\mathrm{d}x^i) \\
&= \frac{\partial \xi^\alpha_I}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial \xi^\alpha_I}{\partial u^\beta_J}\,\mathrm{d}u^\beta_J - u^\alpha_{I+1_i}\frac{\partial \xi^i}{\partial x^j}\,\mathrm{d}x^j - u^\alpha_{I+1_i}\frac{\partial \xi^i}{\partial u^\beta}\,\mathrm{d}u^\beta - \xi^\alpha_{I+1_i}\,\mathrm{d}x^i
\end{aligned}$$

Adding and subtracting properly some terms, we have

$$\begin{aligned}
\mathfrak{L}_{\xi^{(k)}}\theta^\alpha_I &= \frac{\partial \xi^\alpha_I}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial \xi^\alpha_I}{\partial u^\beta_{\tilde I}}\theta^\beta_{\tilde I} + \frac{\partial \xi^\alpha_I}{\partial u^\beta_{\tilde I}} u^\beta_{\tilde I+1_i}\,\mathrm{d}x^i + \frac{\partial \xi^\alpha_I}{\partial u^\beta_K}\,\mathrm{d}u^\beta_K \\
&\quad - u^\alpha_{I+1_i}\frac{\partial \xi^i}{\partial x^j}\,\mathrm{d}x^j - u^\alpha_{I+1_i}\frac{\partial \xi^i}{\partial u^\beta}\theta^\beta - u^\alpha_{I+1_i}\frac{\partial \xi^i}{\partial u^\beta}u^\beta_j\,\mathrm{d}x^j - \xi^\alpha_{I+1_i}\,\mathrm{d}x^i.
\end{aligned}$$

As $\mathfrak{L}_{\xi^{(k)}}\theta^\alpha_I$ is required to be contact,

$$\xi^\alpha_{I+1_i} = \frac{\partial \xi^\alpha_I}{\partial x^i} + u^\beta_{\tilde I+1_i}\frac{\partial \xi^\alpha_I}{\partial u^\beta_{\tilde I}} - u^\alpha_{I+1_j}\left(\frac{\partial \xi^j}{\partial x^i} + u^\beta_i \frac{\partial \xi^j}{\partial u^\beta}\right) \quad \text{and} \quad \frac{\partial \xi^\alpha_I}{\partial u^\beta_K} = 0.$$

From the first equation we deduce that ξ^α_J depends only on u^α_I's with $|I| \leq |J|$, which agrees with the second one. Rewriting the former in terms of the coordinate total derivatives (4.12), we finally obtain

$$\xi^\alpha_{I+1_i} = \frac{\mathrm{d}\xi^\alpha_I}{\mathrm{d}x^i} - u^\alpha_{I+1_j}\frac{\mathrm{d}\xi^j}{\mathrm{d}x^i}.$$

The final statement is clear from here. $\square$

Corollary 4.17. *Under the same assumptions, we have that the components of the kth lift $\xi^{(k)} \in \mathfrak{X}(J^k\pi)$ of a vector field $\xi \in \mathfrak{X}(E)$ are explicitly given by*

$$\xi_J^\alpha = \frac{\mathrm{d}^{|J|}\xi^\alpha}{\mathrm{d}x^J} - \sum_{\substack{I_u+I_\xi=J \\ |I_u|,|I_\xi|\neq 0}} \frac{J!}{I_u!I_\xi!} u^\alpha_{I_u+1_l} \frac{\mathrm{d}^{|I_\xi|}\xi^l}{\mathrm{d}x^{I_\xi}} - u_l^\alpha \frac{\mathrm{d}^{|J|}\xi^l}{\mathrm{d}x^J}. \tag{4.17}$$

Proof. We proceed by induction on the length $|J|$ of a multi-index $J \in \mathbb{N}^m$. For $J = 1_j$, with $1 \leq j \leq m$, we obtain

$$\xi_j^\alpha = \frac{\mathrm{d}\xi^\alpha}{\mathrm{d}x^j} - u_l^\alpha \frac{\mathrm{d}\xi^l}{\mathrm{d}x^j},$$

which agrees with the recursive formula (4.15) (and also with (3.18)). Let us assume that the theorem is true for multi-indexes up to length $k-1 \geq 1$ and consider a multi-index $K \in \mathbb{N}^m$ of length k. For any decomposition $K = J + 1_j$, where $J \in \mathbb{N}^m$ and $1 \leq j \leq m$, we have

$$\begin{aligned} \xi^\alpha_{J+1_j} &= \frac{\mathrm{d}\xi_J^\alpha}{\mathrm{d}x^j} - u^\alpha_{J+1_l}\frac{\mathrm{d}\xi^l}{\mathrm{d}x^j} \\ &= \frac{\mathrm{d}^{|J|+1}\xi^\alpha}{\mathrm{d}x^{J+1_j}} - \sum_{\substack{I_u+I_\xi=J \\ |I_u|,|I_\xi|\neq 0}} \frac{J!}{I_u!I_\xi!}\left(u^\alpha_{I_u+1_j+1_l}\frac{\mathrm{d}^{|I_\xi|}\xi^l}{\mathrm{d}x^{I_\xi}} + u^\alpha_{I_u+1_l}\frac{\mathrm{d}^{|I_\xi|+1}\xi^l}{\mathrm{d}x^{I_\xi+1_j}}\right) \\ &\quad - u^\alpha_{1_j+1_l}\frac{\mathrm{d}^{|J|}\xi^l}{\mathrm{d}x^J} - u_l^\alpha\frac{\mathrm{d}^{|J|+1}\xi^l}{\mathrm{d}x^{J+1_j}} - u^\alpha_{J+1_l}\frac{\mathrm{d}\xi^l}{\mathrm{d}x^j}, \end{aligned}$$

where we have used the formula (4.15) and the induction hypothesis. We multiply each member of the equality by $K(j)/|K|$ and sum over all the decompositions of the type $K = J + 1_j$, what gives us thanks to Lemma A.4

$$\begin{aligned} \xi_K^\alpha &= \frac{\mathrm{d}^{|K|}\xi^\alpha}{\mathrm{d}x^K} - \sum_{J+1_j=K}\frac{K(j)}{|K|}\sum_{\substack{I_u+I_\xi=J \\ |I_u|,|I_\xi|\neq 0}} \frac{J!}{I_u!I_\xi!}\left(u^\alpha_{I_u+1_j+1_l}\frac{\mathrm{d}^{|I_\xi|}\xi^l}{\mathrm{d}x^{I_\xi}} + u^\alpha_{I_u+1_l}\frac{\mathrm{d}^{|I_\xi|+1}\xi^l}{\mathrm{d}x^{I_\xi+1_j}}\right) \\ &\quad - \sum_{J+1_j=K}\frac{K(j)}{|K|}\left(u^\alpha_{1_j+1_l}\frac{\mathrm{d}^{|J|}\xi^l}{\mathrm{d}x^J} + u^\alpha_{J+1_l}\frac{\mathrm{d}\xi^l}{\mathrm{d}x^j}\right) - u_l^\alpha\frac{\mathrm{d}^{|K|}\xi^l}{\mathrm{d}x^K} \\ &= \frac{\mathrm{d}^{|K|}\xi^\alpha}{\mathrm{d}x^K} - \sum_{\substack{I_u+I_\xi+1_j=K \\ |I_u|,|I_\xi|\neq 0}} \frac{K(j)}{|K|}\frac{(I_u+I_\xi)!}{I_u!I_\xi!}\left(u^\alpha_{I_u+1_j+1_l}\frac{\mathrm{d}^{|I_\xi|}\xi^l}{\mathrm{d}x^{I_\xi}} + u^\alpha_{I_u+1_l}\frac{\mathrm{d}^{|I_\xi|+1}\xi^l}{\mathrm{d}x^{I_\xi+1_j}}\right) \\ &\quad - \sum_{J+1_j=K}\frac{K(j)}{|K|}\left(u^\alpha_{1_j+1_l}\frac{\mathrm{d}^{|J|}\xi^l}{\mathrm{d}x^J} + u^\alpha_{J+1_l}\frac{\mathrm{d}\xi^l}{\mathrm{d}x^j}\right) - u_l^\alpha\frac{\mathrm{d}^{|K|}\xi^l}{\mathrm{d}x^K}. \end{aligned}$$

We now need to rearrange properly the middle terms. For the first one, we substitute

$I_u + 1_j$ by J_u and I_ξ by J_ξ, obtaining

$$\sum_{\substack{I_u+I_\xi+1_j=K\\|I_u|,|I_\xi|\neq 0}} \frac{K(j)}{|K|}\frac{(I_u+I_\xi)!}{I_u!I_\xi!}u^\alpha_{I_u+1_j+1_l}\frac{\mathrm{d}^{|I_\xi|}\xi^l}{\mathrm{d}x^{I_\xi}} =$$

$$= \sum_{\substack{J_u+J_\xi=K\\|I_u|\geq 2,|J_\xi|\neq 0}} \sum_{I_u+1_j=J_u} \frac{J_u(j)}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}}$$
$$= \sum_{\substack{J_u+J_\xi=K\\|I_u|\geq 2,|J_\xi|\neq 0}} \frac{|J_u|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}},$$

where we have use the fact that $K(j)(I_u + I_\xi)! = K!$ and $J_u(j)I_u! = J_u!$ (Lemma A.1) and again the identity (A.7). For the third middle term, we substitute $I_u + 1_j$ by J_u and I_ξ by J_ξ, obtaining

$$\sum_{J+1_j=K} \frac{K(j)}{|K|}u^\alpha_{1_j+1_l}\frac{\mathrm{d}^{|J|}\xi^l}{\mathrm{d}x^J} = \sum_{\substack{J_u+J_\xi=K\\|I_u|=1,|J_\xi|\neq 0}} \frac{|J_u|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}}$$

where we have use the fact that $K(j)J_\xi! = K!$ and $|J_u| = J_u! = 1$. The second and forth middle terms are rearranged accordingly. We thus arrive to

$$\begin{aligned}
\xi^\alpha_K &= \frac{\mathrm{d}^{|K|}\xi^\alpha}{\mathrm{d}x^K} - u^\alpha_l\frac{\mathrm{d}^{|K|}\xi^l}{\mathrm{d}x^K}\\
&\quad - \sum_{\substack{J_u+J_\xi=K\\|I_u|\geq 2,|J_\xi|\neq 0}} \frac{|J_u|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}} - \sum_{\substack{J_u+J_\xi=K\\|I_u|=1,|J_\xi|\neq 0}} \frac{|J_u|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}}\\
&\quad - \sum_{\substack{J_u+J_\xi=K\\|I_u|\neq 0,|J_\xi|\geq 2}} \frac{|J_\xi|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}} - \sum_{\substack{J_u+J_\xi=K\\|I_u|\neq 0,|J_\xi|=1}} \frac{|J_\xi|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}}\\
&= \frac{\mathrm{d}^{|K|}\xi^\alpha}{\mathrm{d}x^K} - \sum_{\substack{J_u+J_\xi=K\\|J_u|,|J_\xi|\neq 0}} \frac{|J_u|+|J_\xi|}{|K|}\frac{K!}{J_u!J_\xi!}u^\alpha_{J_u+1_l}\frac{\mathrm{d}^{|J_\xi|}\xi^l}{\mathrm{d}x^{J_\xi}} - u^\alpha_l\frac{\mathrm{d}^{|K|}\xi^l}{\mathrm{d}x^K},
\end{aligned}$$

which is the desired formula since $|J_u| + |J_\xi| = |K|$. □

Originally, the kth lift is defined for π-projectable vector fields on E. The kth lift of such vector field ξ is the infinitesimal generator of the kth lift of the flow of ξ. Definition 4.15 is a characterization of this property and it is generalized for any kind of vector fields on E (see [71]).

Proposition 4.18. *Let ψ_ε be the flow of a given π-projectable vector field ξ over E. Then, the flow of $\xi^{(k)}$ is the kth prolongation of ψ_ε, $j^k\psi_\varepsilon$.*

4.1.2 On the definition of vertical endomorphisms

We are going to face one of the first problems in order to define a canonical geometric Lagrangian formalism. There is no natural extension of the notions of vertical endomorphism for first order theories (see Section §3.1.2). However, an alternative approach was developed by Saunders in [138].

Definition 4.19. Given a k-jet $j^k_x\phi \in J^k\pi$, let $A \in S^kT^*_xM \otimes_{j^k_x\phi} \mathcal{V}_{\phi(x)}\pi$. The *vertical lift* of A at $j^k_x\phi$ is the tangent vector $A^{\mathrm{v}}_{j^k_x\phi} \in T_{j^k_x\phi}(J^k\pi)$ given by

$$A^{\mathrm{v}}_{j^k_x\phi}(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(j^k_x\phi + tA)\big|_{t=0}, \ \forall f \in \mathcal{C}^\infty(J^k_{j^k_x\phi}\pi). \tag{4.18}$$

By the very definition of vertical lift, given a smooth function $f \in \mathcal{C}^\infty(J^{k-1}\pi)$,

$$\begin{aligned}
(T_{j^k_x\phi}\pi_{k,k-1})(A^{\mathrm{v}}_{j^k_x\phi})(f) &= A^{\mathrm{v}}_{j^k_x\phi}(f\circ\pi_{k,k-1}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\pi_{k,k-1})(j^k_x\phi + tA)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(\phi(x))\big|_{t=0} \\
&= 0.
\end{aligned}$$

Thus, the vertical lift takes values into the vertical fiber bundle $\mathcal{V}\pi_{k,k-1} \subset TJ^k\pi$. Indeed, it is a morphism of vector bundles over the identity of $J^k\pi$,

$$(\cdot)^{\mathrm{v}} : S^kT^*M \otimes_{J^k\pi} \mathcal{V}\pi \longrightarrow \mathcal{V}\pi_{k,k-1}.$$

Note that, this time, the tensor product is taken over $J^k\pi$ and not over E. Note also that for each $j^k_x\phi \in J^k\pi$, the vertical lift at $j^k_x\phi$,

$$(\cdot)^{\mathrm{v}}_{j^k_x\phi} : S^kT^*_xM \otimes \mathcal{V}_{\phi(x)}\pi \longrightarrow \mathcal{V}_{j^k_x\phi}\pi_{k,k-1} \subset T_{j^k_x\phi}J^k\pi,$$

is a linear isomorphism. In adapted local coordinates (x^i, u^α_J), if $A = A^\alpha_K\,\mathrm{d}x^K|_x \otimes \partial/\partial u^\alpha|_{\phi(x)}$, where $\mathrm{d}x^K$ is the symmetric tensor product of the local 1-forms $\mathrm{d}x^{i_1}, \dots,$ $\mathrm{d}x^{i_k}$ with $K = 1_{i_1} + \cdots + 1_{i_k}$, then

$$A^{\mathrm{v}}_{j^k_x\phi} = A^\alpha_K \frac{\partial}{\partial u^\alpha_K}\Big|_{j^k_x\phi} \quad \text{and} \quad (\cdot)^{\mathrm{v}} = \mathrm{d}u^\alpha \otimes \frac{\delta}{\delta x^K} \otimes \frac{\partial}{\partial u^\alpha_K}, \tag{4.19}$$

where $\delta/\delta x^K$ is the dual counterpart of $\mathrm{d}x^K$.

Now, we would like to use this vertical lift in order to generalize the definitions of the vertical endomorphisms of first order, definitions 3.18 and 3.19. Nevertheless, the ideas that are behind these definitions seem to not work for this one. In the case of the volume dependent vertical endomorphism 3.18, one would like to define a skewsymmetric map $\mathcal{S}_\eta : (TJ^k\pi)^m \to TJ^k\pi$ using the volume form η and the vertical lift $(\cdot)^{\mathrm{v}}$, but there is no chance to obtain an element in the domain of $(\cdot)^{\mathrm{v}}$ from a tangent vector in $TJ^k\pi$. In the case of the canonical vertical endomorphism 3.19, we look for a map $\mathcal{S} : T^*M \otimes TJ^k\pi \to \mathcal{V}\pi_{k,0}$, but the contraction of the canonical form η with the vertical lift $(\cdot)^{\mathrm{v}}$ simply does not give what one would expect.

Therefore, we are forced to try to generalize these objects my means of their local descriptions, equations (3.23) and (3.24). For instance, the obvious formula for a canonical vertical endomorphism for higher-order jet bundles would be

$$\mathcal{S}^k := \sum_{|I|=0}^{k-1} \theta_I^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u_{I+1_i}^\alpha}. \tag{4.20}$$

Unfortunately, this local definition does not behave as expected under a change of coordinates. Let (x^i, u_I^α) and (y^j, v_J^β) denote two systems of adapted coordinates in $J^k\pi$ then, following the transformation rules (4.4), we have

$$\theta_I^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u_{I+1_i}^\alpha} = \sum_{|I|+1\le|\tilde{J}|\le k} \frac{\partial u_I^\alpha}{\partial v_J^\beta} \cdot \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial v_{\tilde{J}}^{\tilde{\beta}}}{\partial u_{I+1_i}^\alpha} \cdot \theta_J^\beta \otimes \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial v_{\tilde{J}}^{\tilde{\beta}}},$$

where we have omitted some of the summation symbols for clarity. For the second order case, after further computations, this is translated to

$$\begin{aligned}
\theta^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u^\alpha} + \theta_{i'}^\alpha \otimes \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial u_{i'i}^\alpha} = \\
= \theta^\beta \otimes \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial v_j^\beta} + \theta_{j'}^\beta \otimes \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial v_{j'j}^\beta} \\
+ \left(\frac{\partial^2 u^\alpha}{\partial y^{j'} \partial v^\beta} + v_{j'}^{\beta'} \frac{\partial^2 u^\alpha}{\partial v^{\beta'} \partial v^\beta} \right) \cdot \frac{\partial v^{\tilde{\beta}}}{\partial u^\alpha} \theta_{j'}^\beta \otimes \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial v_{j'j}^{\tilde{\beta}}}.
\end{aligned}$$

Obviously, this is not invariant under a change of coordinates.

4.1.3 Partial differential equations

Lemma 4.20. *If N is an open submanifold of M, then $J^k(\pi_N) \simeq \pi_k^{-1}(N)$.*

Definition 4.21. A *differential equation on* π is a closed embedded submanifold $\mathcal{P}$ of the jet manifold $J^k\pi$. The *order* of $\mathcal{P}$ is the largest natural number r satisfying $\pi_{r,r-1}^{-1}(\pi_{k,r-1}) \neq \pi_{k,r}\mathcal{P}$. A *solution* of $\mathcal{P}$ is a local section $\phi \in \Gamma_N\pi$, where N is an open submanifold of M, which satisfies $j_x^k\phi \in \mathcal{P}$ for every $x \in N$. A differential equation $\mathcal{P}$ is said to be *integrable at* $z \in \mathcal{P}$ if there is a solution ϕ of $\mathcal{P}$ (around some neighborhood N of $\pi_k(z)$) such that $z = j_{\pi_k(z)}^k\phi$. A first-order differential equation $\mathcal{P}$ is said to be *integrable in a subset* $\mathcal{P}' \subset \mathcal{P}$ if it is integrable at each $z \in \mathcal{S}$. A first-order differential equation $\mathcal{P}$ is said to be *integrable* if it is integrable at each $z \in \mathcal{P}$.

If l is the codimension of $\mathcal{P}$ ($\dim J^k\pi - \dim \mathcal{P}$), there locally exist submersions $\Psi : J^k\pi \to \mathbb{R}^l$ for whom $\mathcal{P}$ is the zero level set. Written in local coordinates, $\mathcal{P}$ is given by the set of points that satisfy

$$\Psi^\mu(x^i, u_J^\alpha) = 0, \;\; \mu = 1, \ldots, l.$$

Thus, differential equations are a geometric interpretation of the usual kth-order partial differential equations. Under certain conditions (for instance, if $\pi_{k,k-1}|_{\mathcal{P}} : \mathcal{P} \to J^{k-1}\pi$ is a surjective submersion), one could solve the previous equation for some of the highest-order velocities u^α_K making them to depend on the other variables. For simplicity, if $n = 1$ and we fix l multi-indexes K of length k, which we denote with a hat $\hat{K}$, the previous equation could be equivalent to the following expression

$$u_{\hat{K}} = \phi_{\hat{K}}(x^i, u_I, u_{\check{K}}),$$

where the multi-index with check accent, $\check{K}$, is a multi-index of length k complementary to those of $\hat{K}$. For instance, in the equation

$$u_{xy} = u_y \cdot u_{xx} + u_x \cdot u_{yy}$$

defined on $J^2\pi$ where $\pi = pr_1 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ with global coordinates (x, y, u), the "hat" multi-index would be $xy = 1_x + 1_y$ and the "check" ones $xx = 1_x + 1_x$ and $yy = 1_y + 1_y$.

In what follows, constrained coordinates will be denoted generically with a hat accent " $\hat{}$ ", while free coordinates will be denoted with a check accent " $\check{}$ ", *i.e.* $u^{\hat{\alpha}}_{\hat{K}}$ and $u^{\check{\alpha}}_{\check{K}}$. Note that in general, $\hat{\alpha}$ and $\check{\alpha}$ or $\hat{K}$ and $\check{K}$ may coincide, what do not are the pairs $(\hat{\alpha}, \hat{K})$ and $(\check{\alpha}, \check{K})$.

Remark 4.22. As our ultimate goal is to characterize holonomic jet sections that belong to $\mathcal{P}$, one could look for a submanifold $\mathcal{P}'$ of $\mathcal{P}$ consisting of the image of such sections. The submanifold $\mathcal{P}'$ is given by the constraint functions of $\mathcal{P}$ plus their consequences up to order k, that is, Ψ^μ, $\frac{\mathrm{d}\Psi^\mu}{\mathrm{d}x^i}$, $\frac{\mathrm{d}^2\Psi^\mu}{\mathrm{d}x^{ij}}$, etc. Geometrically, $\mathcal{P}'$ is obtained as the output of the following recursive process:

$$\mathcal{P}^{(s,r)} := \begin{cases} \mathcal{P}, & s = 0, \quad r = k; \\ \pi_{k,0}(\mathcal{P}^{(s-1,k)}), & s > 0, \quad r = 0; \\ J^1\mathcal{P}^{(s,r-1)} \cap \pi_{k,r}(\mathcal{P}^{(s-1,k)}), & s > 0, \quad 0 < r \leq k; \end{cases} \tag{4.21}$$

which stops when, for some step $s \geq 0$, $\mathcal{P}^{(s+1,k)} = \mathcal{P}^{(s,k)}$. This algorithm is a generalization to jet bundles of the method given in [127] by Mendella *et al.* to extract the integral part of a differential equation in a tangent bundle. The reader is also refereed to the alternative approach by Gasqui [90].

For instance, if one considers the null divergence restriction $u_x + v_y = 0$ in the 2nd-order jet manifold of $pr_1 : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, then the resulting manifold $\mathcal{P}^{(2,2)} = \mathcal{P}^{(1,2)}$ is given by the restrictions $u_x + v_y = 0$, $u_{xt} + v_{yt} = 0$, $u_{xx} + v_{xy} = 0$ and $u_{xy} + v_{yy} = 0$ (see Example 4.61). Note that in this particular example, the original equation is in essence a 1st-order differential equation while, after the recurrence algorithm, it 2nd-order one since it has been considered in $J^2\,pr_1$.

4.1.4 Iterated jet bundles

In Section §3.1, we already saw that $J^1\pi$ is a fiber bundle over M. We thus may consider the first jet bundle $J^1\pi_1$ of the first source projection $\pi_1 : J^1\pi \to M$. Bearing this idea in mind, we could even consider arbitrary iteration of jet bundles of any order. Of greater importance are the jet bundles which are the first jet of a $(k-1)$th source projections: $J^1\pi_{k-1}$.

Definition 4.23. Let $k, l \geq 0$, the (l, k)*-iterated jet bundle of* π is the lth jet bundle $J^l\pi_k$ of the kth source projection $\pi_k : J^k\pi \to M$.

If (x^i, u^α) are adapted coordinates on E and (x^i, u^α_I), $0 \le |I| \le k$, are the corresponding induced local coordinates on $J^k\pi$, adapted coordinates on the iterated jet bundle $J^l\pi_k$ will have the form

$$(x^i, u^\alpha_{I;J}), \text{ where } 0 \le |I| \le k,\ 0 \le |J| \le l,$$

such that, for any local section $\psi : M \to J^k\pi$ of π_k,

$$u^\alpha_{I;J}(j^l_x\psi) = \left.\frac{\partial^{|J|}\psi^\alpha_I}{\partial x^J}\right|_x,$$

being $\psi^\alpha_I = u^\alpha_I \circ \psi$.

Let $\phi : M \to E$ be a local section of π around $x \in M$, then its kth prolongation $j^k\phi$ is a local section of π_k and, hence, its lth jet at x, $j^l_x(j^k\phi)$, is and element of the iterated jet bundle $J^l\pi_k$. Besides, $j^{k+l}_x\phi$ is an element of the higher order jet bundle $J^{k+l}\pi$. In fact, $J^{k+l}\pi$ is naturally embedded in $J^l\pi_k$.

Definition 4.24. The map $i_{l,k} : J^{k+l}\pi \to J^l\pi_k$ is defined by

$$i_{l,k}(j^{k+l}_x\phi) = j^l_x(j^k\phi).$$

The elements in the image of $i_{l,k}$ are called *holonomic*.

Do not confuse this concept with the one given in Definition 4.5, even though they are related. A holonomic iterated jet $j^l_x\sigma \in J^l\pi_k$ (in the sense of 4.24), is the jet of a holonomic jet $\sigma = j^k\phi$ (in the sense of 4.5) or, the iterated jet of a fixed section $j^l_x\sigma = j^l_x(j^k\phi)$.

In adapted coordinates $(x^i, u^\alpha_{I;J})$ on $J^l\pi_k$ and (x^i, u^α_K) on $J^{k+l}\pi$, where $0 \le |I| \le k$, $0 \le |J| \le l$ and $0 \le |K| \le k + l$,

$$u^\alpha_{I;J}(i_{l,k}(j^{k+l}_x\phi)) = u^\alpha_{I+J}.$$

It follows that $J^{k+l}\pi$ may be seen as the submanifold of holonomic jets of $J^l\pi_k$ given by the coordinate expression

$$J^{k+l}\pi = \left\{ j^l_x\psi \in J^l\pi_k \ :\ u^\alpha_{I_1;J_1}(j^l_x\psi) = u^\alpha_{I_2;J_2}(j^l_x\psi) \text{ whenever } I_1 + J_1 = I_2 + J_2 \right\}.$$

$$\begin{tikzcd}
 & J^l\pi_k \arrow[dl, "\pi_{k+l,l}"'] \arrow[d, "i_{l,k}"] \arrow[dr, "\pi_{k+l,k}"] & \\
J^l\pi \arrow[dr, "\pi_l"'] & J^l\pi_k \arrow[l, "j^l\pi_{k,0}"'] \arrow[r, "(\pi_k)_{l,0}"] \arrow[d, "(\pi_k)_l"] & J^k\pi \arrow[dl, "\pi_k"] \\
 & M &
\end{tikzcd}$$

Figure 4.3: Iterated jets

Do not confuse this notion of holonomy with the one given in Definition 4.1.1. The former refers to holonomy as an iterated jet, the latter as a jet section by itself.

In contrast to the elements of $i_{l,k}(J^{k+l}\pi)$, that are called holonomic, the elements of $J^l\pi_k$ are sometimes refereed as *non-holonomic jets*, even though the holonomic jets belong to it. But there are still a set of particular interest between them when $l = 1$ (see [139]).

Two different maps may be defined from $J^1\pi_k$ to $J^1\pi_{k-1}$. First, the composition of the target projection $(\pi_k)_{1,0} : J^1\pi_k \to J^k\pi$ with the natural embedding $i_{1,k-1} : J^k\pi \hookrightarrow J^1\pi_{k-1}$. And secondly, the first prolongation $j^1\pi_{k,k-1}$ of $\pi_{k,k-1} : J^k\pi \to J^{k-1}\pi$ as a morphism over the identity on M. Finally, we recall that the vector bundle associated to the affine bundle $(\pi_{k-1})_{1,0} : J^1\pi_{k-1} \to J^{k-1}\pi$ is

$$(\tau_{J^{k-1}\pi}|_{\mathcal{V}\pi_{k-1}}) \circ pr_2 : T^*M \otimes_{J^{k-1}\pi} \mathcal{V}\pi_{k-1} \longrightarrow J^{k-1}\pi.$$

Definition 4.25. The *k-jet Spencer operator* is the map

$$D_k : J^1\pi_k \longrightarrow T^*M \otimes_{J^{k-1}\pi} \mathcal{V}\pi_{k-1}$$

such that $D_k(j^1_x\psi)$ is the unique element of $T^*M \otimes_{J^{k-1}\pi} \mathcal{V}\pi_{k-1}$ whose affine action on $J^1\pi_{k-1}$ maps $(i_{1,k-1} \circ (\pi_k)_{1,0})(j^1_x\psi)$ to $(j^1\pi_{k,k-1})(j^1_x\psi)$

In local coordinates, the k-jet Spencer operator has the expression

$$D_k(x^i, u^\alpha_J, u^\alpha_{I;i}) = \sum_{|I|=0}^{k-1} (u^\alpha_{I;i} - u^\alpha_{I+1_i})\, \mathrm{d}x^i \otimes \frac{\partial}{\partial u^\alpha_I}. \tag{4.22}$$

Definition 4.26. The semi-holonomic $(k+1)$-jet manifold $\hat{J}^{k+1}\pi$ is the submanifold $D_k^{-1}(0)$ of $J^1\pi_k$.

From the local expression of the k-jet Spencer operator, it follows that

$$\hat{J}^{k+1}\pi = \left\{ j^1_x\psi \in J^1\pi_k \ : \ u^\alpha_{I;i}(j^1_x\psi) = u^\alpha_{I+1_i}(j^1_x\psi) \text{ when } 0 \leq |I| \leq k \right\}.$$

We now have the inclusions $i_{i,k}(J^{k+1}) \subset \hat{J}^{k+1}\pi \subset J^1\pi_k$. In terms of coordinates, we may say that the semi-holonomic manifold $\hat{J}^{k+1}\pi$ is the collection of elements of $J^1\pi_k$ whose coordinates are symmetric with respect to the multi-indexes up to order k, whereas the holonomic manifold $J^{k+1}\pi$ is the collection of elements of $\hat{J}^{k+1}\pi$ whose coordinates are in addition symmetric with respect to the multi-indexes of order $k+1$. We may take $(x^i, u^\alpha_J, u^\alpha_{K;i})$ as coordinates of $\hat{J}^{k+1}\pi$, where $0 \leq |J| \leq k$ and $|K| = k$.

4.1.5 The kth Dual Jet Bundle

There are mainly two possible choices to define the dual space of $J^k\pi$. For our purposes, one of them is not valid, while the other will introduce some problems in the formulation of dynamics. In spite of it all, we shall show the reason of this election.

Recall that $\pi_{k,k-1} : J^k\pi \to J^{k-1}\pi$ is an affine bundle (Proposition 4.4). Thus, we may consider its affine dual

$$\mathcal{A} := \bigcup_{u \in J^{k-1}\pi} \mathrm{Aff}(J^k_u\pi, \Lambda^m_{\pi_{k-1}(u)}M).$$

Note that, in this case, the affine nature of this space takes only in consideration the highest order component of $J^k\pi$, what is clearer when one considers local coordinates. Let (x^i, u^α_J) denote adapted coordinates on $J^k\pi$, where $0 \leq |J| \leq k$, then they induce coordinates $(x^i, u^\alpha_I, p^k, p^K_\alpha)$ in $\mathcal{A}$, where $0 \leq |I| \leq k-1$ and $|K| = k$. Note that there is only one coordinate p^k, with little k. The pairing will then be

$$p^k + p^K_\alpha u^\alpha_K.$$

Roughly speaking, this space has the nice property of having as many momenta (plus one) as highest order velocities has $J^k\pi$. Nonetheless, the lack of taking care of the lower order velocities is too important to neglect it.

A workaround could be to consider fiber products of this space for each "level" $J^l\pi$ from $l = 1$ to $l = k$. Then we would have a space whose coordinates will take the form $(x^i, u^\alpha_I, p^1, \dots, p^k, p^J_\alpha)$, where $0 \leq |I| \leq k-1$ and $1 \leq |J| \leq k$. The problem now is that there are many affine components p^l, which would give a lack of unicity when the Hamiltonian formalism would be introduced. Moreover, there is not a canonically define pairing since there are pairings defined at each level but not globally.

The alternative to all of this is to consider the iterated jet $J^1\pi_{k-1}$ and its dual space as affine bundle over $J^{k-1}\pi$. As already seen, $J^k\pi$ is affinely embedded into $J^1\pi_{k-1}$, thus it makes sense to restrict the elements of $J^1(\pi_{k-1})^\dagger$ to $J^k\pi$.

Definition 4.27. The *kth dual jet bundle* of π, denoted $J^k\pi^\dagger$, is the reunion of the affine maps from $J^1_u\pi_{k-1}$ to $\Lambda^m_{\pi_{k-1}(u)}M$, where u is an arbitrary point of $J^{k-1}\pi$. Namely,

$$J^k\pi^\dagger := J^1(\pi_{k-1})^\dagger = \bigcup_{u\in J^{k-1}\pi} \mathrm{Aff}(J^1_u\pi_{k-1}, \Lambda^m_{\pi_{k-1}(u)}M). \tag{4.23}$$

The functions given by

$$\begin{array}{rcl} \pi^\dagger_k : J^k\pi^\dagger & \longrightarrow & M \\ \omega \in J^k_u\pi^\dagger & \longmapsto & \pi_{k-1}(u) \end{array} \tag{4.24}$$

and

$$\begin{array}{rcl} \pi^\dagger_{k,0} : J^k\pi^\dagger & \longrightarrow & E \\ \omega \in J^k_u\pi^\dagger & \longmapsto & \pi_{k-1,0}(u) \end{array} \tag{4.25}$$

where $J^k_u\pi^\dagger = \mathrm{Aff}(J^k_u\pi, \Lambda^m_{\pi(u)}M)$, are called the *kth dual source projection* and the *kth dual target projection* respectively. Finally, we denote $\pi^\dagger_{k,k-1}$ the map

$$\begin{array}{rcl} \pi^\dagger_{k,k-1} : J^k\pi^\dagger & \longrightarrow & J^{k-1}\pi \\ \omega \in J^k_u\pi^\dagger & \longmapsto & u \end{array} \tag{4.26}$$

and $\pi^\dagger_{k,l} := \pi^\dagger_{k,k-1} \circ \pi_{k-1,l}$, for $0 \leq l \leq k-1$.

The duality nature of $J^k\pi^\dagger$ gives rise to a natural pairing between its elements and those of $J^k\pi$. The pairing will be denoted by the usual angular brackets, $\langle\, ,\rangle : J^k\pi^\dagger \otimes_{J^{k-1}\pi} J^k\pi \to \Lambda^m M$.

Proposition 4.28. *The kth dual jet bundle of π, $J^k\pi^\dagger$, may be endowed with a structure of smooth manifold. A system of adapted coordinates (x^i, u^α) in E induces a system of coordinates $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ in $J^k\pi^\dagger$, where $0 \leq |I| \leq k-1$, such that, for any $j^k_x\phi \in J^k\pi$ and any $\omega \in J^k_{j^{k-1}_x\phi}\pi^\dagger$, $x^i(\omega) = x^i(x)$, $u^\alpha_I(\omega) = u^\alpha_I(j^{k-1}_x\phi)$ and $\langle\omega, j^k_x\phi\rangle = (p + p^{Ii}_\alpha u^\alpha_{I+1_i})\,\mathrm{d}^m x$.*

In the induced local coordinates $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$, the kth dual source and target projections are respectively written

$$\pi^\dagger_k(x^i, u^\alpha_I, p, p^{Ii}_\alpha) = (x^i) \quad \text{and} \quad \pi^\dagger_{k,0}(x^i, u^\alpha_I, p, p^{Ii}_\alpha) = (x^i, u^\alpha), \tag{4.27}$$

and for the intermediate projections

$$\pi^\dagger_{k,l}(x^i, u^\alpha_I, p, p^{Ii}_\alpha) = (x^i, u^\alpha_J), \tag{4.28}$$

where $1 \leq |J| \leq l \leq k-1$. From here, it is clear that $\pi^\dagger_k$ and $\pi^\dagger_{1,0}$ are certainly projections over M and E respectively. Therefore, $(J^k\pi^\dagger, \pi^\dagger_k, M)$ and $(J^k\pi^\dagger, \pi^\dagger_{k,0}, E)$ are fiber bundles. If we consider a change of coordinates $(x^i, u^\alpha) \mapsto (y^j, v^\beta)$ in E, it induces a change of coordinates $(x^i, u^\alpha_I, p, p^{Ii}_\alpha) \mapsto (y^j, v^\beta_I, q, q^{Jj}_\beta)$ in $J^k\pi^\dagger$. In this case, the "momenta" transform by the following rule.

Proposition 4.29. *Let (x^i, u^α) and (y^j, v^β) be adapted coordinates on E and let $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ and $(y^j, v^\beta_J, q, q^{Jj}_\beta)$ be the corresponding induced coordinates on the space of semi-basic forms $\Lambda^m_2 J^{k-1}\pi$, where $0 \leq |I|, |J| \leq k$. We have that the fiber coordinates (with respect to $J^{k-1}\pi$) transform according to the following rule:*

$$p = \mathrm{Jac}(y(x)) \left(\frac{\partial v^\beta_J}{\partial x^i} q^{Jj}_\beta \frac{\partial x^i}{\partial y^j} + q \right), \tag{4.29}$$

$$p^{Ii}_\alpha = \mathrm{Jac}(y(x)) \left(\frac{\partial v^\beta_J}{\partial u^\alpha_I} q^{Jj}_\beta \frac{\partial x^i}{\partial y^j} \right). \tag{4.30}$$

Proof. First of all, recall that $J^k\pi^\dagger = J^1(\pi_{k-1})^\dagger$ is canonically isomorphic to the space of semi-basic form $\omega \in \Lambda^m_2 J^k\pi$ (Proposition 3.28). We only have to write an arbitrary semi-basic form $\omega \in \Lambda^m_2 J^k\pi$ in the two different systems of coordinates and use the transformation rules (3.31) to get:

$$\begin{aligned}
\omega &= p\,\mathrm{d}^m x + p^{Ii}_\alpha\,\mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i \\
&= q\,\mathrm{d}^m y + q^{Jj}_\beta\,\mathrm{d}v^\beta_J \wedge \mathrm{d}^{m-1}y_j \\
&= \mathrm{Jac}(y(x)) \left[q\,\mathrm{d}^m x + q^{Jj}_\beta \frac{\partial x^i}{\partial y^j} \left(\frac{\partial v^\beta_J}{\partial x^k}\,\mathrm{d}x^k + \frac{\partial v^\beta_J}{\partial u^\alpha_I}\,\mathrm{d}u^\alpha_I \right) \wedge \mathrm{d}^{m-1}x_i \right] \\
&= \mathrm{Jac}(y(x)) \left[\left(q + \frac{\partial v^\beta_J}{\partial x^i} q^{Jj}_\beta \frac{\partial x^i}{\partial y^j} \right) \mathrm{d}^m x + \left(\frac{\partial v^\beta_J}{\partial u^\alpha_I} q^{Jj}_\beta \frac{\partial x^i}{\partial y^j} \right) \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i \right].
\end{aligned}$$

If we now compare the coefficients of the first and last expressions, we obtain the desired result. □

While the kth jet bundle $J^k\pi$ projects over the lower order jet bundles (Diagram 4.1), the kth dual jet bundle is "embedded" into the $(k+1)$th dual jet bundle by means of the pullback of the affine projection $\pi_{k+1,k}$ (Diagram 4.4).

Proposition 4.30. *The kth dual jet bundle of π, $J^k\pi^\dagger$, together with the kth dual projection, $\pi^\dagger_{k,k-1}$, is a vector bundle over $J^{k-1}\pi$. Moreover, the induced coordinate systems $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ are adapted to the vector bundle structure.*

$$\begin{tikzcd}
J^{k+1}\pi^\dagger \arrow[dr] & J^k\pi^\dagger \arrow[l, hook', "\pi^*_{k+1,k}"'] \arrow[dr] & \cdots \arrow[l, hook', "\pi^*_{k,k-1}"'] \arrow[dr] & J^2\pi^\dagger \arrow[l, hook', "\pi^*_{3,2}"'] \arrow[dr] & J^1\pi^\dagger \arrow[l, hook', "\pi^*_{2,1}"'] \arrow[dr] & & \\
J^{k+1}\pi \arrow[r, "\pi_{k+1,k}"] & J^k\pi \arrow[r, "\pi_{k,k-1}"] & \cdots \arrow[r, "\pi_{3,2}"] & J^2\pi \arrow[r, "\pi_{2,1}"] & J^1\pi \arrow[r, "\pi_{1,0}"] & E \arrow[r, "\pi"] & M
\end{tikzcd}$$

Figure 4.4: Chain of dual jets

Definition 4.31. The *reduced kth dual jet bundle* of π is $J^k\pi^\circ := J^1(\pi_{k-1})^\circ$ (see Definition 3.26), which is isomorphic to $\Lambda^m_2 J^{k-1}\pi/\Lambda^m_1 J^{k-1}\pi$ (Corollary 3.29).

Proposition 4.32. *We have that:*

1. *$J^k\pi^\circ$ may be endowed with a structure of smooth manifold;*

2. *$(J^k\pi^\dagger, \mu, J^k\pi^\circ)$ is a smooth vector bundle of rank 1;*

3. *adapted coordinates (x^i, u^α) on E induce coordinates $(x^i, u^\alpha_I, p^{Ii}_\alpha)$ on $J^k\pi^\circ$ such that $\mu(x^i, u^\alpha_I, p, p^{Ii}_\alpha) = (x^i, u^\alpha_I, p^{Ii}_\alpha)$, where $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ are the induced coordinates on $J^k\pi^\dagger$.*

Before we end this section, we summarize the important geometrical ingredients that he dual jet bundle $J^k\pi^\dagger$ posses. In first place, it has a canonical multisymplectic structure which is carried form the realization of $J^k\pi^\dagger$ as a semi-basic forms $\Lambda^m_2 J^{k-1}\pi$ over $J^{k-1}\pi$ (see equations (3.32) and (3.33) and Definition 4.27). Moreover, its elements are naturally paired with those of $J^k\pi$ (see Proposition 4.28). We recall that a form Ω is multi-symplectic if it is closed and if its contraction with a single tangent vector is injective, that is, $i_V\Omega = 0$ if and only if $V = 0$.

Definition 4.33. The *Liouville* or *tautological m-form* on $J^k\pi^\dagger$ is the form given by

$$\Theta_\omega(V_1, \ldots, V_m) = ((\pi^\dagger_{k,k-1})^*\omega)(V_1, \ldots, V_m),\ \omega \in J^k\pi^\dagger,\ V_1, \ldots, V_m \in T_\omega J^k\pi^\dagger, \tag{4.31}$$

where $\pi^\dagger_{k,k-1}$ is the natural projection from $J^k\pi^\dagger$ to $J^{k-1}\pi$. The *Liouville* or *canonical multi-symplectic $(m+1)$-form* on $J^k\pi^\dagger$ is

$$\Omega = -\,\mathrm{d}\Theta. \tag{4.32}$$

Definition 4.34. The natural pairing between $J^k\pi$ and its dual $J^k\pi^\dagger$ is the fibered map $\Phi : J^k\pi \times_{J^{k-1}\pi} J^k\pi^\dagger \to \Lambda^m M$ given by

$$\Phi(j^k_x\phi, \omega) = (j^{k-1}\phi)^*_{j^{k-1}_x\phi}\omega. \tag{4.33}$$

Let $(x^i, u^\alpha_I, u^\alpha_K)$ and $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ denote adapted coordinates on $J^k\pi$ and $J^k\pi^\dagger$ respectively, where $|I| = 0, \ldots, k-1$ and $|K| = k$. Then, the tautological form and the canonical one are locally written

$$\Theta = p\,\mathrm{d}^m x + p^{Ii}_\alpha\,\mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i \quad \text{and} \quad \Omega = -\,\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i, \tag{4.34}$$

and the fibered pairing between the elements of $J^k\pi$ and $J^k\pi^\dagger$ is locally written

$$\Phi(x^i, u^\alpha_I, u^\alpha_K, p, p^{Ii}_\alpha) = (p + p^{Ii}_\alpha u^\alpha_{I+1_i})\,\mathrm{d}^m x. \tag{4.35}$$

4.2 Higher Order Classical Field Theory

4.2.1 Variational Calculus

The dynamics in classical field theory is specified giving a Lagrangian density: A Lagrangian density is a mapping $\mathcal{L} : J^k\pi \to \Lambda^m M$. Fixed a volume form η on M, there is a smooth function $L : J^k\pi \to \mathbb{R}$ such that $\mathcal{L} = L\eta$.

Definition 4.35. Given a Lagrangian density $\mathcal{L} : J^k\pi \longrightarrow \Lambda^m M$, the associated *integral action* is the map $\mathcal{A}_\mathcal{L} : \Gamma\pi \times \mathcal{K} \longrightarrow \mathbb{R}$ given by

$$\mathcal{A}_\mathcal{L}(\phi, R) = \int_R (j^k\phi)^*\mathcal{L}, \tag{4.36}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

Definition 4.36. Let ϕ be a section of π. A *(vertical) variation* of ϕ is a curve $\varepsilon \in I \mapsto \phi_\varepsilon \in \Gamma\pi$ (for some interval $I \subset \mathbb{R}$) such that $\phi_\varepsilon = \varphi_\varepsilon \circ \phi \circ \check{\varphi}_\varepsilon^{-1}$, where φ_ε is the flow of a (vertical) π-projectable vector field ξ on E.

When ξ is vertical, then its flow φ_ε is an automorphism of fiber bundles over the identity for each $\varepsilon \in I$.

Definition 4.37. We say that $\phi \in \Gamma\pi$ is a *critical* or *stationary point* of the Lagrangian action $\mathcal{A}_\mathcal{L}$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\mathcal{A}_\mathcal{L}(\phi_\varepsilon, R)\right]\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_R (j^k\phi_\varepsilon)^*\mathcal{L}\right]\Big|_{\varepsilon=0} = 0, \tag{4.37}$$

for any vertical variation ϕ_ε of ϕ whose associated vertical field vanishes outside of $\pi^{-1}(R)$.

Lemma 4.38. *Let $\phi_\varepsilon = \varphi_\varepsilon \circ \phi \circ \check{\varphi}_\varepsilon^{-1}$ be a variation of a section $\phi \in \Gamma\pi$. If ξ denotes the infinitesimal generator of φ_ε, then*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^k(\varphi_\varepsilon \circ \phi)_x^*\omega\right]\Big|_{\varepsilon=0} = (j^k\phi)_x^*(\mathfrak{L}_{\xi^{(k)}}\omega), \tag{4.38}$$

for any differential form $\omega \in \Omega(J^k\pi)$.

Proof. From Proposition 4.18, we have that $\xi^{(k)}$ is the infinitesimal generator of $j^k\varphi_\varepsilon$. We then obtain by a direct computation,

$$(j^k\phi)_x^*(\mathfrak{L}_{\xi^{(k)}}\omega) = (j^k\phi)_x^*\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^k\varphi_\varepsilon)^*\omega\right]\Big|_{\varepsilon=0}\right) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[(j^k\varphi_\varepsilon \circ j^k\phi)_x^*\omega\right]\Big|_{\varepsilon=0}.$$

□

The following lemma will show to be useful in the variational derivation of the higher-order Euler-Lagrange equations.

Lemma 4.39 (Higher-order integration by parts). *Let $R \subset M$ be a smooth compact region and let $f, g : R \longrightarrow \mathbb{R}$ be two smooth functions. Given any multi-index $J \in \mathbb{N}^m$, we have that*

$$\int_R \frac{\partial^{|J|} f}{\partial x^J} g \, \mathrm{d}^m x = (-1)^{|J|} \int_R f \frac{\partial^{|J|} g}{\partial x^J} \, \mathrm{d}^m x + \sum_{I_f + I_g + 1_i = J} \lambda(I_f, I_g, J) \int_{\partial R} \frac{\partial^{|I_f|} f}{\partial x^{I_f}} \frac{\partial^{|I_g|} g}{\partial x^{I_g}} \, \mathrm{d}^{m-1} x_i, \tag{4.39}$$

where λ is given by the expression

$$\lambda(I_f, I_g, J) := (-1)^{|I_g|} \cdot \frac{|I_f|! \cdot |I_g|!}{|J|!} \cdot \frac{J!}{I_f! \cdot I_g!}. \tag{4.40}$$

Proof. In this proof, we will use the shorthand notation $f_J = \frac{\partial^{|J|} f}{\partial x^J}$.

We proceed by induction on the length l of the multi-index J. The case $l = |J| = 0$ is a trivial identity and the case $l = |J| = 1$ is the well known formula of integration by parts

$$\int_R f_i g \, \mathrm{d}^m x = \int_{\partial R} f g \, \mathrm{d}^{m-1} x_i - \int_R f g_i \, \mathrm{d}^m x.$$

Thus, let us suppose that the result is true for any multi-index $J \in \mathbb{N}^m$ up to length $l > 1$, in order to show that it is also true for any multi-index $K \in \mathbb{N}^m$ of length $l + 1$. Let J and $1 \leq j \leq m$ such that $J + 1_j = K$. We then have,

$$\begin{aligned} \int_R f_{J+1_j} g \, \mathrm{d}^m x &= -\int_R f_J g_j \, \mathrm{d}^m x + \int_{\partial R} f_J g \, \mathrm{d}^{m-1} x_j \\ &= (-1)^{l+1} \int_R f g_{J+1_j} \, \mathrm{d}^m x + \int_{\partial R} f_J g \, \mathrm{d}^{m-1} x_j \\ &\quad - \sum_{I_f + I_{g_j} + 1_i = J} \lambda(I_f, I_{g_j}, i) \int_{\partial R} f_{I_f} g_{I_{g_j} + 1_j} \, \mathrm{d}^{m-1} x_i, \end{aligned}$$

where we have used the first-order integration formula in first place, to then apply the induction hypothesis. We now multiply each member by $(J(j) + 1)/(l + 1)$ and sum over $J + 1_j = K$. Using the multi-index identity (A.7), we have

$$\begin{aligned} \int_R f_K g \, \mathrm{d}^m x &= (-1)^{l+1} \int_R f g_K \, \mathrm{d}^m x + \sum_{J+1_j = K} \frac{J(j) + 1}{l + 1} \int_{\partial R} f_J g \, \mathrm{d}^{m-1} x_j \\ &\quad - \sum_{J+1_j = K} \frac{J(j) + 1}{l + 1} \sum_{I_f + I_{g_j} + 1_i = J} \lambda(I_f, I_{g_j}, i) \int_{\partial R} f_{I_f} g_{I_{g_j} + 1_j} \, \mathrm{d}^{m-1} x_i. \end{aligned}$$

It only remain to rearrange properly the last two terms to express them in the stated form. Clearly,

$$\begin{aligned} &\sum_{J+1_j = K} \frac{J(j) + 1}{l + 1} \int_{\partial R} f_J g \, \mathrm{d}^{m-1} x_j = \\ &\qquad = \sum_{\substack{I_f + I_g + 1_i = K \\ |I_g| = 0}} (-1)^{|I_g|} \cdot \frac{|I_f|! \cdot |I_g|!}{|K|!} \cdot \frac{K!}{I_f! \cdot I_g!} \int_{\partial R} \frac{\partial^{|I_f|} f}{\partial x^{I_f}} \frac{\partial^{|I_g|} g}{\partial x^{I_g}} \, \mathrm{d}^{m-1} x_i. \end{aligned}$$

The last term is a little bit more tricky,

$$
\begin{aligned}
\sum_{J+1_j=K} \frac{J(j)+1}{l+1} \sum_{I_f+I_{g_j}+1_i=J} (-1)^{|I_{g_j}|+1} \cdot \frac{|I_f|! \cdot |I_{g_j}|!}{|J|!} \cdot \frac{J!}{I_f! \cdot I_{g_j}!} \int_{\partial R} f_{I_f} g_{I_{g_j}+1_j} \, \mathrm{d}^{m-1}x_i = \\
= \sum_{I_f+I_{g_j}+1_i+1_j=K} (-1)^{|I_{g_j}|+1} \cdot \frac{|I_f|! \cdot |I_{g_j}|!}{|K|!} \cdot \frac{K!}{I_f! \cdot I_{g_j}!} \int_{\partial R} f_{I_f} g_{I_{g_j}+1_j} \, \mathrm{d}^{m-1}x_i \\
= \sum_{\substack{I_f+I_g+1_i=K \\ |I_g|\geq 1}} (-1)^{|I_g|} \sum_{I_{g_j}+1_j=I_g} \frac{I_g(j)}{|I_g|} \cdot \frac{|I_f|! \cdot |I_g|!}{|K|!} \cdot \frac{K!}{I_f! \cdot I_g!} \int_{\partial R} f_{I_f} g_{I_g} \, \mathrm{d}^{m-1}x_i \\
= \sum_{\substack{I_f+I_g+1_i=K \\ |I_g|\geq 1}} (-1)^{|I_g|} \cdot \frac{|I_f|! \cdot |I_g|!}{|K|!} \cdot \frac{K!}{I_f! \cdot I_g!} \int_{\partial R} f_{I_f} g_{I_g} \, \mathrm{d}^{m-1}x_i
\end{aligned}
$$

where we have used the identity (A.7) again. The result is now clear. □

In the following version of the higher-order Euler-Lagrange equations, we restrict ourselves to vertical variations for simplicity, although it is possible to use also non-vertical variation like in Theorem 3.35.

Theorem 4.40 (The higher-order Euler-Lagrange equations)**.** *Given a fiber section $\phi \in \Gamma\pi$, let us consider an infinitesimal variation $\phi_\varepsilon = \varphi_\varepsilon \circ \phi$ of it such that the support R of the associated vertical vector field ξ is contained in a coordinate chart (x^i). We then have that the variation of the Lagrangian action $\mathcal{A}_\mathcal{L}$ at ϕ is given by*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{A}_\mathcal{L}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = \sum_{|J|=0}^{k} \Bigg[(-1)^{|J|} \int_R (j^{2k}\phi)^* \left(\xi^\alpha \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^m x \\
+ \sum_{I_\xi+I_L+1_i=J} \lambda(I_\xi, I_L, J) \int_{\partial R} (j^{2k}\phi)^* \left(\xi^\alpha_{I_\xi} \frac{\mathrm{d}^{|I_L|}}{\mathrm{d}x^{I_L}} \frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^{m-1}x_i \Bigg],
\end{aligned}
\tag{4.41}
$$

where $R_\varepsilon = \check{\varphi}_\varepsilon(R)$. Moreover, ϕ is a critical point of the Lagrangian action $\mathcal{A}_\mathcal{L}$ if and only if it satisfies the higher-order Euler-Lagrange equations

$$
(j^{2k}\phi)^* \left(\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u_J^\alpha} \right) = 0 \tag{4.42}
$$

on the interior of M, plus the boundary conditions

$$
\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I} \frac{\partial L}{\partial u_J^\alpha} = 0, \;\; 0 \leq |I| < |J| \leq k, \tag{4.43}
$$

on the boundary ∂M of M.

Proof. Let us denote by ξ the vertical field associated to the variation ϕ_ε. By Proposition 4.38, Cartan's formula $\mathfrak{L} = \mathrm{d} \circ i + i \circ \mathrm{d}$ and Proposition 4.18, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_{\mathcal{L}}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} &= \int_R (j^k\phi)^*(\mathfrak{L}_{\xi^{(k)}}\mathcal{L}) \\
&= \int_R (j^k\phi)^* \,\mathrm{d}(i_{\xi^{(k)}}\mathcal{L}) + \int_R (j^k\phi)^* i_{\xi^{(k)}}\,\mathrm{d}\mathcal{L} \\
&= \int_{\partial R} (j^k\phi)^* i_{\xi^{(k)}}\mathcal{L} + \int_R (j^k\phi)^*(\xi^{(k)}(L)\,\mathrm{d}^m x - \mathrm{d}L \wedge i_{\xi^{(k)}}\,\mathrm{d}^m x) \\
&= \int_R (j^k\phi)^* \left(\sum_{|J|=0}^{k} \frac{\mathrm{d}^{|J|}\xi^\alpha}{\mathrm{d}x^J}\frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^m x
\end{aligned}$$

If we now apply the higher-order integration by parts (4.39) and we take into account that Equation (4.13), we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_{\mathcal{L}}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = \sum_{|J|=0}^{k} \Bigg[&(-1)^{|J|} \int_R (j^{2k}\phi)^* \left(\xi^\alpha \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J}\frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^m x \\
&+ \sum_{I_\xi + I_L + 1_i = J} \lambda(I_\xi, I_L, J) \int_{\partial R} (j^{2k}\phi)^* \left(\frac{\mathrm{d}^{|I_\xi|}\xi^\alpha}{\mathrm{d}x^{I_\xi}} \frac{\mathrm{d}^{|I_L|}}{\mathrm{d}x^{I_L}}\frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^{m-1}x_i \Bigg],
\end{aligned}$$

which is the first statement of our theorem.

If we now suppose that R is contained in the interior of M, as ξ is null outside of R, so it is $\xi^{(k)}$ outside of R and, by smoothness, on its boundary ∂R. Thus, if ϕ is a critical point of $\mathcal{A}_{\mathcal{L}}$, we then must have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_{\mathcal{L}}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = \int_R (j^{2k}\phi)^* \left(\xi^\alpha \sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J}\frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^m x = 0,$$

for any vertical field ξ whose compact support is contained in $\pi^{-1}(R)$. We thus infer that ϕ shall satisfy the higher-order Euler-Lagrange equations (4.42) on the interior of M.

Finally, if R has common boundary with M, we then have that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}_{\mathcal{L}}(\phi_\varepsilon, R)\Big|_{\varepsilon=0} = \sum_{|J|=0}^{k} \sum_{I_\xi + I_L + 1_i = J} \lambda(I_\xi, I_L, J) \int_{\partial R \cap \partial M} (j^{2k}\phi)^* \left(\xi^\alpha_{I_\xi} \frac{\mathrm{d}^{|I_L|}}{\mathrm{d}x^{I_L}}\frac{\partial L}{\partial u_J^\alpha} \right) \mathrm{d}^{m-1}x_i = 0. \tag{4.44}$$

As this is true for any vertical field ξ whose compact support is contained in $\pi^{-1}(R)$, we deduce the boundary conditions (4.43). □

4.2.2 Variational Calculus with Constraints

We consider a constraint submanifold $i : \mathcal{C} \hookrightarrow J^k\pi$ of codimension l, which is locally annihilated by l functionally independent constraint functions Ψ^μ, where $1 \le \mu \le l$. The constraint submanifold $\mathcal{C}$ is supposed to fiber over the whole of M. Here one could use the algorithm given in 4.22 so as to use the consequence up to order k given by the constraint submanifold $\mathcal{C}$.

We now look for extremals of the Lagrangian action (4.36) restricted to those sections $\phi \in \Gamma\pi$ whose k-jet takes values in $\mathcal{C}$ (see [83, 121]). We will use the Lagrange multiplier theorem that follows.

Theorem 4.41 (Abraham, Marsden & Ratiu [2])**.** *Let $\mathcal{M}$ be a smooth manifold, $f : \mathcal{M} \longrightarrow \mathbb{R}$ be $\mathcal{C}^r$, $r \geq 1$, $\mathcal{F}$ a Banach space, $g : \mathcal{M} \longrightarrow \mathcal{F}$ a smooth submersion and $\mathcal{N} = g^{-1}(0)$. A point $\phi \in N$ is a critical point of $f|_N$ if and only if there exists $\lambda \in \mathcal{F}^*$, called a* Lagrange multiplier*, such that ϕ is a critical point of $f - \langle\lambda, g\rangle$.*

In order to apply the Lagrange multiplier theorem, we need to define constraints as the 0-level set of some function g. We configure therefore the following setting: choose the smooth manifold $\mathcal{M}$ to be the space of local sections $\Gamma_R\pi = \{\phi : R \subset M \to E : \pi \circ \phi = \mathrm{Id}_M\}$, for some compact region $R \subset M$. The Banach space $\mathcal{F}$ is the set of smooth functions $\mathcal{C}^\infty(R, \mathbb{R}^l)$, provided with the L^2-norm. The constraint function Ψ induces a constraint function on the space of local sections $\Gamma_R\pi$ by mapping each section ϕ to the evaluation of its k-lift by the constraint, that is,

$$g : \phi \in \Gamma_R\pi \mapsto \Psi \circ j^k\phi \in \mathcal{C}^\infty(R, \mathbb{R}^l).$$

Note that the 0-level set $\mathcal{N} = g^{-1}(0)$ is the set of sections whose k-lift takes values in the constraint manifold $\mathcal{C}$ (over R).

We therefore obtain that a section $\phi : M \longrightarrow E$ is a regular critical point of the integral action $\mathcal{A}_\mathcal{L}$ restricted to $\mathcal{C}$ if and only if there exists a Lagrange multiplier $\lambda \in (\mathcal{C}^\infty(R, \mathbb{R}^l))^*$ such that ϕ is a critical point of $\mathcal{A}_\mathcal{L} - \langle\lambda, g\rangle$. A priori, we cannot assure that the pairing $\langle\lambda, g(\phi)\rangle$ has an integral expression of the type $\int_R \lambda_\mu \Psi^\mu \circ j^k\phi \, \mathrm{d}^m x$ for some functions $\lambda_\mu : R \longrightarrow \mathbb{R}$. Henceforth, we shall suppose that it is the case.

Remark 4.42. In Theorem 4.41 appears some regularity conditions that exclude the so-called *abnormal solutions*. In general, given a critical point $\phi \in \mathcal{N} = g^{-1}(0)$ of $f_{|\mathcal{N}}$, the classical Lagrange multiplier theorem claims that there exists a nonzero element $(\lambda_0, \lambda) \in \mathbb{R} \times \mathcal{F}^*$ such that ϕ is a critical point of

$$\lambda_0 f - \langle\lambda, g\rangle \,. \tag{4.45}$$

Under the submersivity condition on g, that is ϕ is a *regular critical point*, it is possible to guarantee that $\lambda_0 \neq 0$ and dividing by λ_0 in (4.45) we obtain the characterization of critical points given in Theorem 4.41. The critical points ϕ with vanishing Lagrange multiplier, that is, $\lambda_0 = 0$ are called *abnormal critical points*.

In the sequel we will only study the regular critical points, but our developments are easily adapted for the case of abnormality (adding the Lagrange multiplier λ_0 and studying separately both cases, $\lambda_0 = 0$ and $\lambda_0 = 1$).

Proposition 4.43 (Constrained higher-order Euler-Lagrange equations)**.** *Let $\phi \in \Gamma\pi$ be a critical point of the Lagrangian action $\mathcal{A}_\mathcal{L}$ given in* (4.36) *restricted to those sections of π whose kth lift take values in the constraint submanifold $\mathcal{C} \subset J^k\pi$. If the associated Lagrange multiplier λ is regular enough, then there must exist l smooth functions λ_μ :*

$R \subset M \to \mathbb{R}$ *that satisfy together with* ϕ *the* constrained higher-order Euler-Lagrange equations

$$(j^{2k}\phi)^* \left(\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \left(\frac{\partial L}{\partial u_J^\alpha} - \lambda_\mu \frac{\partial \Psi^\mu}{\partial u_J^\alpha} \right) \right) = 0. \tag{4.46}$$

Proof. The proof is a direct application of Theorem 4.40 and Theorem 4.41. □

4.2.3 The Skinner-Rusk formalism

The generalization of the Skinner-Rusk formalism to higher order classical field theories will take place in the fibered product

$$W_0 = J^k\pi \times_{J^{k-1}\pi} \Lambda_2^m(J^{k-1}\pi). \tag{4.47}$$

The results of this section constitute the main developments of our paper [27]. The first order case is covered in [50, 70]; see also [143, 144] for the original treatment by Skinner and Rusk. The projection on the i-th factor will be denoted pr_i (with $i = 1, 2$) and the projection as fiber bundle over $J^{k-1}\pi$ will be $\pi_{W_0, J^{k-1}\pi} = \pi_{k,k-1} \circ pr_1$ (see Diagram 4.5). On W_0, adapted coordinate systems are of the form $(x^i, u_I^\alpha, u_K^\alpha, p_\alpha^{I,i}, p)$, where $|I| = 0, \ldots, k-1$ and $|K| = k$.

$$\begin{tikzcd}
 & W_0 \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] \arrow[dd, "\pi_{W_0,J^{k-1}\pi}"] & \\
J^k\pi \arrow[dr, "\pi_{k,k-1}"] & & \Lambda_2^m(J^{k-1}\pi) \arrow[dl] \\
 & J^{k-1}\pi \arrow[d, "\pi_{k-1}"] & \\
 & M &
\end{tikzcd}$$

Figure 4.5: The Skinner-Rusk framework

Assume that $L : J^k\pi \longrightarrow \mathbb{R}$ is a Lagrangian function. Together with the pairing Φ (Proposition 4.28), we use this Lagrangian L to define a dynamical function H (corresponding to the Hamiltonian) on W_0:

$$H = \Phi - L \circ pr_1 \,. \tag{4.48}$$

Consider the canonical multisymplectic $(m+1)$-form Ω on $\Lambda_2^m(J^{k-1}\pi)$ (Equation (4.32)), whose pullback to W_0 shall be denoted also by Ω. We define on W_0 the $(m+1)$-form

$$\Omega_H = \Omega + \mathrm{d}H \wedge \eta. \tag{4.49}$$

In adapted coordinates

$$H = p_\alpha^{I,i} u_{I+1_i}^\alpha + p - L(x^i, u_I^\alpha, u_K^\alpha) \tag{4.50}$$

$$\Omega_H = -\,\mathrm{d}p_\alpha^{I,i} \wedge \mathrm{d}u_I^\alpha \wedge \mathrm{d}^{m-1}x_i + \left(p_\alpha^{I,i}\,\mathrm{d}u_{I+1_i}^\alpha + u_{I+1_i}^\alpha\,\mathrm{d}p_\alpha^{I,i} - \frac{\partial L}{\partial u_J^\alpha}\,\mathrm{d}u_J^\alpha \right) \wedge \mathrm{d}^m x \tag{4.51}$$

where $|I| = 0, \ldots, k-1$ and $|J| = 0, \ldots, k$.

The dynamical equation

We search for an Ehresmann connection Γ in the fiber bundle $\pi_{W_0,M} : W_0 \longrightarrow M$ whose horizontal projector be a solution of the *dynamical equation* (see Section §1.1):

$$i_{\mathbf{h}}\Omega_H = (m-1)\Omega_H. \tag{4.52}$$

We will show that such a solution does not exist on the whole W_0. Thus, we need to restrict to the space on where such a solution exists, that is on

$$\begin{aligned} W_1 \quad = \quad & \{w \in W_0 \ / \ \exists \mathbf{h}_w : T_wW_0 \longrightarrow T_wW_0 \text{ linear such that } \mathbf{h}_w^2 = \mathbf{h}_w, \\ & \ker \mathbf{h}_w = (V\pi_{W_0,M})_w,\ i_{\mathbf{h}_w}\Omega_H(w) = (m-1)\Omega_H(w)\}. \end{aligned} \tag{4.53}$$

Remark 4.44. Equation (4.52) is a generalization of equations that usually appear in first order field theories. In this particular case, from a given Lagrangian function $L : J^1\pi \to \mathbb{R}$ we may construct a unique $(m+1)$-form Ω_L (the Poincaré-Cartan $(m+1)$-form). Hence, we have a geometrical characterization of the Euler-Lagrange equations for L as follows. Let Γ be an Ehresmann connection in $\pi_{1,0} : J^1\pi \to M$, with horizontal projector $\mathbf{h}$. Consider the equation

$$i_{\mathbf{h}}\Omega_L = (n-1)\Omega_L. \tag{4.54}$$

If $\mathbf{h}$ has locally the from

$$\mathbf{h}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + A_i^\alpha \frac{\partial}{\partial u^\alpha} + A_{ji}^\alpha \frac{\partial}{\partial u_j^\alpha},$$

then a direct computation shows that equation (4.54) holds if and only if

$$(A_i^\alpha - u_i^\alpha)\left(\frac{\partial^2 L}{\partial u_i^\alpha \partial u_j^\beta}\right) = 0, \tag{4.55}$$

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u_i^\alpha} - A_i^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} - A_{ji}^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} + (A_j^\beta - u_j^\beta)\frac{\partial^2 L}{\partial u^\alpha \partial u_j^\beta} = 0. \tag{4.56}$$

(see [54]). If the lagrangian L is regular, then Eq. (4.55) implies that $A_i^\alpha = u_i^\alpha$ and therefore Eq. (4.56) becomes

$$\frac{\partial L}{\partial u^\alpha} - \frac{\partial^2 L}{\partial x^i \partial u_i^\alpha} - A_i^\beta \frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} - A_{ji}^\beta \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha} = 0. \tag{4.57}$$

Now, if $\sigma(x^i) = (x^i, \sigma^\alpha(x), \sigma_i^\alpha(x))$ is an integral section of $\mathbf{h}$ we would have

$$u_i^\alpha = \frac{\partial \sigma^\alpha}{\partial x^i} \quad \text{and} \quad A_{ij}^\alpha = \frac{\partial \sigma_i^\alpha}{\partial x^j},$$

which proves that Eq. (4.57) is nothing but the Euler-Lagrange equations for L.

We may think Equation (4.52) as a generalization of equation 4.54 giving the Euler-Lagrange equations for higher-order field theories in a univocal way, as we will see.

In a local chart $(x^i, u_J^\alpha, p_\alpha^{I,i}, p)$ of W_0, a horizontal projector $\mathbf{h}$ must have the expression:

$$\mathbf{h} = \left(\frac{\partial}{\partial x^j} + A_{Jj}^\alpha \frac{\partial}{\partial u_J^\alpha} + B_{\alpha j}^{Ii} \frac{\partial}{\partial p_\alpha^{I,i}} + C_j \frac{\partial}{\partial p}\right) \otimes \mathrm{d}x^j, \tag{4.58}$$

where $|I| = 0, \dots, k-1$ and $|J| = 0, \dots, k$. We then obtain that

$$
\begin{aligned}
i_{\mathbf{h}}\Omega_H - (m-1)\Omega_H &= \left(B^{Ii}_{\alpha i}\, \mathrm{d}u^\alpha_I - A^\alpha_{Ii}\, \mathrm{d}p^{I,i}_\alpha + p^{I,i}_\alpha\, \mathrm{d}u^\alpha_{I+1_i} + u^\alpha_{I+1_i}\, \mathrm{d}p^{I,i}_\alpha - \frac{\partial L}{\partial u^\alpha_J}\, \mathrm{d}u^\alpha_J \right) \wedge \mathrm{d}^m x \\
&= \left[\left(B^{\ i}_{\alpha i} - \frac{\partial L}{\partial u^\alpha} \right) \mathrm{d}u^\alpha + \sum_{|I'|=1}^{k-1} \left(B^{I'i}_{\alpha i} - \frac{\partial L}{\partial u^\alpha_{I'}} \right) \mathrm{d}u^\alpha_{I'} + \sum_{|I|=0}^{k-2} p^{I,i}_\alpha\, \mathrm{d}u^\alpha_{I+1_i} \right. \\
&\quad - \sum_{|K|=k} \frac{\partial L}{\partial u^\alpha_K}\, \mathrm{d}u^\alpha_K + \sum_{|I|=k-1} p^{I,i}_\alpha\, \mathrm{d}u^\alpha_{I+1_i} \\
&\quad \left. + \sum_{|I|=0}^{k-1} (u^\alpha_{I+1_i} - A^\alpha_{Ii})\, \mathrm{d}p^{I,i}_\alpha \right] \wedge \mathrm{d}^m x.
\end{aligned}
$$

Equating this to zero and using Lemma A.5, we have that

$$
\begin{aligned}
A^\alpha_{Ii} &= u^\alpha_{I+1_i}, \ |I| = 0, \dots, k-1, \ i = 1, \dots, m; && (4.59) \\
B^{\ j}_{\alpha j} &= \frac{\partial L}{\partial u^\alpha}; && (4.60) \\
p^{I,i}_\alpha &= \frac{I(i)+1}{|I|+1} \left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} - B^{I+1_i j}_{\alpha j} + Q^{Ii}_\alpha \right), \ |I| = 0, \dots, k-2, \ i = 1, \dots, m; && (4.61) \\
p^{I,i}_\alpha &= \frac{I(i)+1}{|I|+1} \left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} + Q^{Ii}_\alpha \right), \ |I| = k-1, \ i = 1, \dots, m; && (4.62)
\end{aligned}
$$

where the Q's are arbitrary functions such that

$$
\sum_{I+1_i=J} \frac{I(i)+1}{|I|+1} Q^{Ii}_\alpha = 0, \ \text{with } |J| = 1, \dots, k. \tag{4.63}
$$

Remark 4.45. The ambiguity in the definition of the Legendre transform, and therefore of the Cartan form, becomes apparent in the equations (4.61) and (4.62), as noted by Crampin and Saunders (see [140]). There are too many momentum variables to be related univocally with the velocity counterpart. To fix this, a choice of arbitrary functions Q satisfying (4.63) must be done. The choice may be encoded as an additional geometric structure, like a connection.

Applying (4.63) to (4.61) and (4.62), and using the identity (A.7), we finally obtain the equations

$$
\begin{aligned}
A^\alpha_{Ii} &= u^\alpha_{I+1_i}, \ \text{with } |I| = 0, \dots, k-1, \ i = 1, \dots, m; && (4.64) \\
0 &= \frac{\partial L}{\partial u^\alpha} - B^{\ j}_{\alpha j}; && (4.65) \\
\sum_{I+1_i=J} p^{I,i}_\alpha &= \frac{\partial L}{\partial u^\alpha_J} - B^{Jj}_{\alpha j}, \ \text{with } |J| = 1, \dots, k-1; && (4.66) \\
\sum_{I+1_i=K} p^{I,i}_\alpha &= \frac{\partial L}{\partial u^\alpha_K}, \ \text{with } |K| = k. && (4.67)
\end{aligned}
$$

Notice that equation (4.67) is the constraint that defines the space W_1; and that (4.64), (4.65) an d(4.66) are conditions on coefficients of the horizontal projectors $\mathbf{h}$.

Note also that, for the time being, the A's with greatest order index and the C's remain undetermined, as well as the most part of the B's. From the definition of W_1, we know that for each point $w \in W_1$ there exists a horizontal projector $\mathbf{h}_w : T_wW_0 \longrightarrow T_wW_0$ satisfying equation (4.52). However, we cannot ensure that such $\mathbf{h}_w$, for each $w \in W_1$, will take values in T_wW_1. Therefore, we impose the natural regularizing condition $\mathbf{h}_w(T_wW_0) \subset T_wW_1$, $\forall w \in W_1$. This latter condition is equivalent to having

$$\mathbf{h}\left(\frac{\partial}{\partial x^j}\right)\left(\sum_{I+1_i=K} p_\alpha^{I,i} - \frac{\partial L}{\partial u_K^\alpha}\right) = 0,$$

which in turn is equivalent (using (4.58) and (4.64)) to

$$\sum_{I+1_i=K} B_{\alpha j}^{Ii} = \frac{\partial^2 L}{\partial x^j \partial u_K^\alpha} + \sum_{|I|=0}^{k-1} u_{I+1_j}^\beta \frac{\partial^2 L}{\partial u_I^\beta \partial u_K^\alpha} + \sum_{|J|=k} A_{Jj}^\beta \frac{\partial^2 L}{\partial u_J^\beta \partial u_K^\alpha}, \tag{4.68}$$

with $|K| = k$. Thus, if the matrix of second order partial derivatives of L with respect to the "velocities" of highest order

$$\left(\frac{\partial^2 L}{\partial u_J^\beta \partial u_K^\alpha}\right) \tag{4.69}$$

is non-degenerate, then the highest order A's are completely determined in terms of the highest order B's. In the sequel, we will say that the Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$ is *regular* if, for any system of adapted coordinates the matrix, (4.69) is non-degenerate.

Up to now, no meaning has been assigned to the coordinate p. Consider the submanifold W_2 of W_1 defined by the restriction $H = 0$. In other words, W_2 is locally characterized by the equation

$$p = L - p_\alpha^{I,i} u_{I+1_i}^\alpha.$$

As before, we cannot ensure that a solution $\mathbf{h}$ of the dynamical equation (4.52) takes values in TW_2. We thus impose to $\mathbf{h}$ the regularizing condition $\mathbf{h}_w(T_wW_0) \subset T_wW_2$, $\forall w \in W_2$, or equivalently $\mathbf{h}(\partial/\partial x^j)(H) = 0$. Therefore, the coefficients of the linear mapping $\mathbf{h}$ are governed by the equations (4.64), (4.65), (4.66), (4.68) and in addition

$$C_j = \frac{\partial L}{\partial x^j} + A_{Jj}^\alpha \frac{\partial L}{\partial u_J^\alpha} - A_{I+1_ij}^\alpha p_\alpha^{I,i} - B_{\alpha j}^{Ii} u_{I+1_i}^\alpha. \tag{4.70}$$

Note that, thanks to the Lemma A.5 and Equation (4.67), the terms with A's with multi-index of length k cancel out, and the A's with lower multi-index are already determined. So, in some sense, the C's depend only on the B's.

Description of the solutions

The relations (4.66) (with $|J| = k - 1$) and (4.68) can be seen as a system of linear equations with respect to the B's. When $k = 1$, equation (4.65) should be considered instead of equation (4.66). In the following, we are going to suppose that $n = 1$, since the dimension of the fibres is irrelevant for our purposes and we may ignore it. The number of B's with order $k - 1$ (with multi-index length $k - 1$) is given by

$$\binom{m-1+k-1}{m-1} \cdot m^2$$

and the number of equations with such B's is

$$\binom{m-1+k}{m-1}\cdot m+\binom{m-1+k-1}{m-1}.$$

An easy computation shows that the system is overdetermined if and only if $k=1$ or $m=1$ (examples 4.51 and 4.52), and completely determined when $k=m=2$. In all other cases the system is underdetermined, but it still has maximal rank.

Proposition 4.46. *Suppose that $k\geq 2$ and $m\geq 2$. Then, the system of linear equations with respect to the B's*

$$\sum_{j=1}^{m} B_j^{Jj} = \frac{\partial L}{\partial u_J}-\sum_{I+1_i=J} p^{I,i}; \tag{4.71}$$

$$\sum_{I+1_i=K} B_j^{Ii} = \frac{\partial^2 L}{\partial x^j\partial u_K}+\sum_{|I|=0}^{k-1} u_{I+1_j}\frac{\partial^2 L}{\partial u_I\partial u_K}+\sum_{|J|=k} A_{Jj}\frac{\partial^2 L}{\partial u_J\partial u_K}; \tag{4.72}$$

where $|J|=k-1$, $j=1,\ldots,m$ and $|K|=k$, has maximal rank.

Proof. In a first step, we are going to describe how to write the matrix of coefficients. Then, we will select the proper columns of this matrix to obtain a new square matrix of maximal size. We finally shall prove that this matrix has maximal rank.

The matrix of coefficients will be a rectangular matrix formed by 1's and 0's. The columns will be indexed by the indexes of the B's, and the rows by the indexes of the first partial derivatives that appear in the equations (4.71) and (4.72). As B_j^{Ii} has three indexes, the columns of the matrix of coefficients will organized in a superior level by the index i, in a middle level by the index j and in an inferior level by the multi-index I. The rows will be organized at the top by the index J for the first equation, (4.71), and at the bottom by the index j and then by the multi-index K for the second equation, (4.72).

As the matrix of coefficients has more columns than rows, we shall build a second matrix that has as many columns and rows as the matrix of coefficients has rows. To do that, we select a column of the matrix of coefficients for each row index using the following *algorithm* (for the sake of simplicity):

```
ForEach (j,K)
   Define G={(I,i):I+1_i=K}
   If Cardinal(G)=1
      Select the column (i,j,I)
   ElseIf
      Select a column (i,j,I) such that (I,i) is in G and i\neq j
   EndIf
EndFor
ForEach J
   If J(1)=k-1
      Select the column (m,m,J)
   ElseIf
      Select the column (1,1,J)
   EndIf
EndFor
```

Now, this matrix being defined and since it is full of 0's and has only few 1's, we are going to develop its determinant by rows and columns. Notice that the columns selected at line 6 have only one 1 each, thus we can cross out the rows an columns related to these 1's. Now the rows at the bottom part of the remaining matrix (related to the second equations) have only one 1 each, thus we can also cross out the rows an columns related to these 1's. Now, the remaining matrix has the property of having only one 1 per column and row (there must be at least one 1 per row and column, and no two 1's may be at the same row or column), thus its determinant is not zero and the matrix of coefficients has maximal rank. □

Example 4.47. If we consider the "simple" case of third order ($k = 3$) with two independent variables ($m = 2$), then we will obtain a system of 11 equations with 12 unknowns. The matrix of coefficients will take the form

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

where the columns are labeled in order by: $(1, 1, 1_1 + 1_1)$, $(1, 1, 1_1 + 1_2)$, $(1, 1, 1_2 + 1_2)$, $(1, 2, 1_1 + 1_1)$, $(1, 2, 1_1 + 1_2)$, $(1, 2, 1_2 + 1_2)$, $(2, 1, 1_1 + 1_1)$, $(2, 1, 1_1 + 1_2)$, $(2, 1, 1_2 + 1_2)$, $(2, 2, 1_1+1_1)$, $(2, 2, 1_1+1_2)$, $(2, 2, 1_2+1_2)$; and where the rows are ordered by: 1_1+1_1, 1_1+1_2, 1_2+1_2, $(1, 1_1+1_1+1_1)$, $(1, 1_1+1_1+1_2)$, $(1, 1_1+1_2+1_2)$, $(1, 1_2+1_2+1_2)$, $(2, 1_1+1_1+1_1)$, $(2, 1_1 + 1_1 + 1_2)$, $(2, 1_1 + 1_2 + 1_2)$, $(2, 1_2 + 1_2 + 1_2)$. The previous algorithm would select all the columns but the eleventh (which corresponds to the label $(2, 2, 1_1 + 1_2)$) in the following order: $(1, 1, 1_1 + 1_1)$, $(2, 1, 1_1 + 1_1)$, $(2, 1, 1_1 + 1_2)$, $(2, 1, 1_2 + 1_2)$, $(1, 2, 1_1 + 1_1)$, $(1, 2, 1_1 + 1_2)$, $(1, 2, 1_2 + 1_2)$, $(2, 2, 1_2 + 1_2)$, $(2, 2, 1_1 + 1_1)$, $(1, 1, 1_1 + 1_2)$, $(1, 1, 1_2 + 1_2)$. Note that the resulting matrix is regular.

The problem get worst with a little increment of the order or the number of independent variables. For instance the case $k = 5$ and $m = 6$ gives a system of 1.638 equations and 4.536 unknowns.

Another way to interpret the tangency condition (4.68) is the following one: Let us suppose we are dealing with a first order Lagrangian (example 4.51, equation (4.89)). One could apply the theory of connections to the Lagrangian setting and the Hamiltonian one as separate frameworks. We know that they must be related by means of the Legendre transform and so are the horizontal projectors induced by these connections. Thus, equation (4.89) is nothing else than the relation between the coefficients of these horizontal projectors.

The reduced mixed space W_2

In section §4.2.3 we reduced the space W_1 to W_2 by considering the constraint $H = 0$, which is a way of interpreting the coordinate p as the Hamiltonian function. But W_2 is not a mere instrument to get rid off the coordinate p or the coefficients C_j. As the premultisymplectic form Ω_H, it encodes the dynamics of the system and, when L is regular, it is a multisymplectic space. Indeed, when $k = 1$, W_2 is diffeomorphic to $J^1\pi$ (*cf.* de León *et al.* [50]), which is not true for higher order cases.

Proposition 4.48. *Let $W_2 = \{w \in W_1 \ :\ H(w) = 0\}$ and define the $(m+1)$-form Ω_2 as the pullback of Ω_H to W_2 by the natural inclusion $i : W_2 \hookrightarrow W_0$, that is $\Omega_2 = i^*(\Omega_H)$. Suppose that* $\dim M > 1$*, then, the $(m+1)$-form Ω_2 is multisymplectic if and only if L is regular.*

Proof. First of all, let us make some considerations. By definition, Ω_2 is multisymplectic whenever Ω_2 has trivial kernel, that is,

$$\text{if } v \in TW_2, i_v\Omega_2 = 0 \iff v = 0\ .$$

This is equivalent to say that

$$\text{if } v \in i_*(TW_2), i_v\Omega_H|_{i_*(TW_2)} = 0 \iff v = 0\ .$$

Let $v \in TW_0$ be a tangent vector whose coefficients in an adapted basis are given by

$$v = \lambda^i\frac{\partial}{\partial x^i} + A^\alpha_J\frac{\partial}{\partial u^\alpha_J} + B^{Ii}_\alpha\frac{\partial}{\partial p^{Ii}_\alpha} + C\frac{\partial}{\partial p}.$$

Using the expression (4.51), we may compute the contraction of Ω_H by v,

$$\begin{aligned} i_v\Omega_H \;=\; & -B^{Ii}_\alpha\,\mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + A^\alpha_I\,\mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}^{m-1}x_i - \lambda^j\,\mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-2}x_{ij} \\ & + \left(A^\alpha_{I+1_i}p^{Ii}_\alpha + B^{Ii}_\alpha u^\alpha_{I+1_i} - A^\alpha_J\tfrac{\partial L}{\partial u^\alpha_J}\right)\mathrm{d}^m x \\ & -\lambda^j\left(p^{Ii}_\alpha\,\mathrm{d}u^\alpha_{I+1_i} + u^\alpha_{I+1_i}\,\mathrm{d}p^{Ii}_\alpha - \tfrac{\partial L}{\partial u^\alpha_J}\,\mathrm{d}u^\alpha_J\right) \wedge \mathrm{d}^{m-1}x_j. \end{aligned} \tag{4.73}$$

On the other hand, if we now suppose that v is tangent to W_2 in W_0, that is $v \in i_*(TW_2)$, we then have that

$$\mathrm{d}\left(\sum_{I+1_i=K} p^{Ii}_\alpha - \frac{\partial L}{\partial u^\alpha_K}\right)(v) = 0 \quad\text{and}\quad \mathrm{d}H(v) = 0, \tag{4.74}$$

which leads us to the following relations for the coefficients of v,

$$\sum_{I+1_i=K} B^{Ii}_\alpha = \lambda^i\frac{\partial^2 L}{\partial x^i\partial u^\alpha_K} + A^\beta_J\frac{\partial^2 L}{\partial u^\beta_J\partial u^\alpha_K} \text{ and} \tag{4.75}$$

$$A^\alpha_{I+1_i}p^{Ii}_\alpha + B^{Ii}_\alpha u^\alpha_{I+1_i} + C - \lambda^i\tfrac{\partial L}{\partial x^i} - A^\alpha_J\tfrac{\partial L}{\partial u^\alpha_J} = 0. \tag{4.76}$$

It is important to note that thanks to Lemma A.2 and the equation (4.67) which defines W_1 (and hence W_2), the terms in (4.73) and (4.76) involving A's with multi-index of length k cancel each other out.

These considerations being made, suppose that Ω_2 is multisymplectic and, by *reductio ad absurdum*, suppose in addition that L is not regular, which means that the matrix

$$\left(\frac{\partial^2 L}{\partial u^\beta_{K'} \partial u^\alpha_K}\right)$$

has non-trivial kernel. Let $v \in TW_0$ be a tangent vector such that all its coefficients are null except the A's of highest order which are in such a way they are mapped to zero by the "hessian" of L. Such a vector v fulfills the restrictions (4.75) and (4.76), thus it must be tangent to W_2 in W_0, $v \in i_*(TW_2)$. But, as $i_v\Omega_H$ has no A's of highest order, it must be zero, $i_v\Omega_H = 0$, which is a contradiction.

Conversely, let us suppose that L is regular, then equation (4.67) defines implicitly the coordinates u^α_K as functions of the other coordinates. That is, locally there exist functions $f^\alpha_K(x^i, u^\alpha_I, p^{I,i}_\alpha)$ such that $u^\alpha_K = f^\alpha_K$ on $i(W_2)$. Furthermore, given a system of adapted coordinates $(x^i, u^\alpha_I, u^\alpha_K, p^{I,i}_\alpha, p)$ on W_0, $(x^i, u^\alpha_I, p^{I,i}_\alpha)$ defines a coordinate system on W_0 and the inclusion is given by:

$$(x^i, u^\alpha_I, p^{I,i}_\alpha) \in W_2 \hookrightarrow (x^i, u^\alpha_I, f^\alpha_K, p^{I,i}_\alpha, L - \sum_{|I|=0}^{k-2} p^{I,i}_\alpha u^\alpha_{I+1_i} - \sum_{|I|=k-1} p^{I,i}_\alpha f^\alpha_{I+1_i}) \in W_0.$$

From equation (4.51), we can compute an explicit expression of the $(m+1)$-form Ω_2 in this coordinate system,

$$\begin{aligned}
\Omega_2 \;=\; & -\sum_{|I|=0}^{k-1} \mathrm{d}p^{I,i}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i \\
& + \left[\sum_{|I|=0}^{k-2} \left(p^{I,i}_\alpha \,\mathrm{d}u^\alpha_{I+1_i} + u^\alpha_{I+1_i}\,\mathrm{d}p^{I,i}_\alpha\right) - \sum_{|I|=0}^{k-1} \frac{\partial L}{\partial u^\alpha_I}\,\mathrm{d}u^\alpha_I\right] \wedge \mathrm{d}^m x \\
& + \left[\sum_{|I|=k-1} \left(p^{I,i}_\alpha \,\mathrm{d}f^\alpha_{I+1_i} + f^\alpha_{I+1_i}\,\mathrm{d}p^{I,i}_\alpha\right) - \sum_{|K|=k}\sum_{I+1_i=K} p^{I,i}_\alpha\,\mathrm{d}f^\alpha_K\right] \wedge \mathrm{d}^m x,
\end{aligned}$$

where we have used equation (4.67) in the last term. Note that, by Lemma A.2, the first and last terms of the last bracket cancel each other out. Now,

$$\begin{aligned}
i_{\partial/\partial x^j}\Omega_2 &= \mathrm{d}p^{I,i}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-2}x_{ij} - [\ldots] \wedge \mathrm{d}^{m-1}x_j \\
i_{\partial/\partial u^\alpha_I}\Omega_2 &= \mathrm{d}p^{I,i}_\alpha \wedge \mathrm{d}^{m-1}x_i + \left(\sum_{J+1_j=I} p^{Jj}_\alpha - \frac{\partial L}{\partial u^\alpha_I}\right)\mathrm{d}^m x \\
i_{\partial/\partial p^{I,i}_\alpha}\Omega_2 &= \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + u^\alpha_{I+1_i}\,\mathrm{d}^m x,
\end{aligned}$$

where $0 \le |I| \le k-1$. We deduce from here that the kernel of Ω_2 is trivial, $\ker\Omega_2 = \{0\}$, and Ω_2 is multisymplectic. □

Note 4.49. In the particular case when $\dim M = 1$, the Lagrangian function $L : J^k\pi \longrightarrow \mathbb{R}$ is regular if and only if the pair $(\Omega_2, \tau^*_{W_2,M}dt)$ is a *cosymplectic structure* on W_2. We recall that a cosymplectic structure on a manifold N of odd dimension $2\bar{n}+1$ is a pair which consists of a closed 2-form Ω and a closed 1-form η such that $\eta \wedge \Omega^{\bar{n}}$ is a volume form.

We remark that, if the Lagrangian L is regular or (from Proposition 4.46) if $k, m > 1$, then there locally exist solutions $\mathbf{h}$ of the dynamical equations (4.52) on W_2 that give rise to connections Γ in the fibration $\pi_{W_0 M} : W_0 \longrightarrow M$ along the submanifold W_2 (see Section §1.1). In such a case, a global solution is obtained using partitions of the unity, and we obtain by restriction a connection $\bar{\Gamma}$, with horizontal projector $\bar{\mathbf{h}}$, in the fibre bundle $\pi_{W_2 M} : W_2 \longrightarrow M$, which is a solution of equation (4.52) when it is restricted to W_2 (in fact, we have a family of such solutions).

In some cases, but only when $\dim M = 1$ or $k = 1$, it would be necessary to consider a subset W_3 defined in order to satisfy the tangency conditions (4.68) and (4.70):

$$\begin{aligned} W_3 \;\; = \;\; & \{ w \in W_2 \;/\; \exists \mathbf{h}_w : T_w W_0 \longrightarrow T_w W_2 \text{ linear such that } \mathbf{h}_w^2 = \mathbf{h}_w, \\ & \ker \mathbf{h}_w = (V\pi_{W_0,M})_w, \; i_{\mathbf{h}_w} \Omega_H(w) = (m-1)\Omega_H(w) \}. \end{aligned}$$

We will assume that W_3 is a submanifold of W_2. If $\mathbf{h}_w(T_w W_0)$ is not contained in $T_w W_3$, we go to the third step, and so on. At the end, and if the system has solutions, we will find a final constraint submanifold W_f, fibered over M (or over some open subset of M) and a connection Γ_f in this fibration such that Γ_f is a solution of equation (4.52) restricted to W_f.

In any case, one obtains the Euler-Lagrange equations. In the following result, W_f denotes the final constraint manifold, which is W_2 when $k, m > 1$, and $\mathbf{h}$ the horizontal projector of a connection in $\pi_{W_2,M} : W_f \longrightarrow M$ along W_f, which is solution of the dynamical equation.

Proposition 4.50. *Let $\bar{\sigma}$ be a section of $\pi_{W_f,M} : W_f \longrightarrow M$ and denote $\sigma = i \circ \bar{\sigma}$, where $i : W_f \hookrightarrow W_0$ is the canonical inclusion. If $\bar{\sigma}$ is an integral section of $\mathbf{h}$, then $\bar{\sigma}$ is* holonomic*, in the sense that*

$$pr_1 \circ \sigma = j^k(\pi_{W_f,E} \circ \bar{\sigma}), \tag{4.77}$$

and satisfies the higher-order Euler-Lagrange equations:

$$j^{2k}(\pi_{W_f,E} \circ \bar{\sigma})^* \left(\sum_{|J|=0}^{k} (-1)^{|J|} \frac{\mathrm{d}^{|J|}}{\mathrm{d}x^J} \frac{\partial L}{\partial u_J^\alpha} \right) = 0. \tag{4.78}$$

Proof. If $\sigma = (x^i, \sigma_J^\alpha, \sigma_\alpha^{I,i}, \tilde{\sigma})$ is an integral section of $\mathbf{h}$, then

$$\frac{\partial \sigma_J^\alpha}{\partial x^j} = A_{Jj}^\alpha, \; \frac{\partial \sigma_\beta^{Ii}}{\partial x^j} = B_{\beta j}^{Ii} \text{ and } \frac{\partial \tilde{\sigma}}{\partial x^j} = C_j,$$

where the A's, B's and C's are the coefficients given in (4.58). From equation (4.64), we have that σ is holonomic, in the sense that $\sigma_{I+1_i}^\alpha = \partial \sigma_I^\alpha / \partial x^i$. On the other hand, using the equations (4.65), (4.66) and (4.67), we obtain the relations (where $\phi = pr_1 \circ \sigma$):

$$0 \;\; = \;\; \frac{\partial L}{\partial u^\alpha} \circ \phi - \frac{\partial \sigma_\alpha^{\;j}}{\partial x^j}; \tag{4.79}$$

$$\sum_{I+1_i=J} \sigma_\alpha^{I,i} \;\; = \;\; \frac{\partial L}{\partial u_J^\alpha} \circ \phi - \frac{\partial \sigma_\alpha^{Jj}}{\partial x^j}, \text{ with } |J| = 1, \ldots, k-1; \tag{4.80}$$

$$\sum_{I+1_i=K} \sigma_\alpha^{I,i} \;\; = \;\; \frac{\partial L}{\partial u_K^\alpha} \circ \phi, \text{ with } |K| = k. \tag{4.81}$$

From the equations (4.79) and (4.80) for $|J|=1$ we get

$$\begin{aligned} 0 &= \frac{\partial L}{\partial u^\alpha}\circ\phi - \frac{\partial \sigma_\alpha{}^j}{\partial x^j} \\ &= (j^0\phi)^*\frac{\partial L}{\partial u^\alpha} - \sum_{|I|=1}(j^1\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) + \sum_{|I|=1}\sum_i \frac{\partial^{|I|}}{\partial x^I}\frac{\partial \sigma_\alpha^{Ii}}{\partial x^i}. \end{aligned}$$

Applying now Lemma A.2 on the last term and repeating this process until $|I|=k-1$ we reach

$$\begin{aligned} 0 &= (j^0\phi)^*\frac{\partial L}{\partial u^\alpha} - \sum_{|I|=1}(j^1\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) + \sum_{|J|=2}\frac{\partial^{|J|}}{\partial x^J}\sum_{I+1_i=J}\sigma_\alpha^{Ii} \\ &= (j^0\phi)^*\frac{\partial L}{\partial u^\alpha} - \sum_{|I|=1}(j^1\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) + \sum_{|I|=2}(j^2\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) - \sum_{|I|=2}\sum_i \frac{\partial^{|I|}}{\partial x^I}\frac{\partial \sigma_\alpha^{Ii}}{\partial x^i} \\ &= \sum_{|I|=0}^{k-1}(-1)^{|I|}(j^{|I|}\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) - (-1)^{k-1}\sum_{|I|=k-1}\sum_i \frac{\partial^{|I|}}{\partial x^I}\frac{\partial \sigma_\alpha^{Ii}}{\partial x^i} \\ &= \sum_{|I|=0}^{k-1}(-1)^{|I|}(j^{|I|}\phi)^*\left(\frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I}\frac{\partial L}{\partial u_I^\alpha}\right) - (-1)^{k-1}\sum_{|K|=k}\frac{\partial^{|K|}}{\partial x^K}\sum_{I+1_i=K}\sigma_\alpha^{Ii}, \end{aligned}$$

where by abuse of notation $j^l\phi = j^{k+l}(\pi_{W_f,E}\circ\bar\sigma)$. Finally, it only rest to use equation (4.81) to prove the desired result. □

Examples

First, we are going to study the particular cases when $k=1$ and $m=1$, which correspond to the First Order Classical Field Theory and to the Higher Order Mechanical Systems, respectively. Theoretic results for these cases are very well known [15, 50, 70, 103] and we are only going to recover these results from our general setting. In addition, these particular cases will clarify the general procedure.

Example 4.51 (First order Lagrangians ($k=1$)). Let us suppose that $k=1$, which corresponds to the case of first order Lagrangians. In that case the velocity-momentum space is $W_0 = J^1\pi\otimes_E\Lambda_2^m E$, with adapted coordinates $(x^i, u^\alpha, u_i^\alpha, p, p_\alpha^i)$. The premultisymplectic $(m+1)$-form would be

$$\Omega_H = -\,\mathrm{d}p_\alpha^i\wedge\mathrm{d}u^\alpha\wedge\mathrm{d}^{m-1}x_i + \left(p_\alpha^i\,\mathrm{d}u_i^\alpha + u_i^\alpha\,\mathrm{d}p_\alpha^i - \frac{\partial L}{\partial u^\alpha}\mathrm{d}u^\alpha - \frac{\partial L}{\partial u_i^\alpha}\mathrm{d}u_i^\alpha\right)\wedge\mathrm{d}^m x, \quad (4.82)$$

and horizontal projectors on TW_0 would have locally the form:

$$\mathbf{h} = \left(\frac{\partial}{\partial x^j} + A_j^\alpha\frac{\partial}{\partial u^\alpha} + A_{ij}^\alpha\frac{\partial}{\partial u_i^\alpha} + B_{\alpha j}{}^i\frac{\partial}{\partial p_\alpha^i} + C_j\frac{\partial}{\partial p}\right)\otimes\mathrm{d}x^j. \quad (4.83)$$

Solutions of the dynamical equation would satisfy the relations

$$\sum_{j=1}^m B_{\alpha j}{}^j = \frac{\partial L}{\partial u^\alpha}; \quad (4.84)$$

$$p_\alpha^i = \frac{\partial L}{\partial u_i^\alpha},\ \text{for } i=1,\dots,m; \quad (4.85)$$

$$A_i^\alpha = u_i^\alpha,\ \text{for } i=1,\dots,m; \quad (4.86)$$

from which we deduce the Euler-Lagrange equations

$$j^2(\pi_{W_2,M}\circ\sigma)^*\left(\frac{\partial L}{\partial u^\alpha}-\sum_{i=1}^m\frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u_i^\alpha}\right)=0, \tag{4.87}$$

where W_2 is defined by

$$W_2=\left\{(x^i,u^\alpha,u_i^\alpha,p,p_\alpha^i)\in W_1\ :\ p_\alpha^i=\frac{\partial L}{\partial u_i^\alpha},\ p=L-\sum_{i=1}^m p^iu_i\right\}. \tag{4.88}$$

We then obtain the tangency conditions:

$$B_{\alpha j}^{\ i} = \frac{\partial^2 L}{\partial x^j\partial u_i^\alpha}+u_j^\beta\frac{\partial^2 L}{\partial u^\beta\partial u_i^\alpha}+\sum_{l=1}^m A_{lj}^\beta\frac{\partial^2 L}{\partial u_l^\beta\partial u_i^\alpha}, \tag{4.89}$$

$$C_j = \frac{\partial L}{\partial x^j}+u_j^\alpha\frac{\partial L}{\partial u^\alpha}-B_{\alpha j}^{\ i}u_i^\alpha. \tag{4.90}$$

Note that (4.89) is the relation that would appear between the coefficients of a Lagrangian and a Hamiltonian setting through the Legendre transform. For simplicity, suppose that $n=1$ and ignore the α's and β's that appear above. Consider the linear system of equations with respect to the B's formed by equations (4.84) and (4.89). This system is overdetermined since it has m^2+1 equations and only m^2 variables (B_j^i).

Example 4.52 (Higher order mechanical systems ($m=1$)). Let us suppose that $m=1$, which corresponds to the case of mechanical systems. In that case the velocity-momentum space is $W_0=J^k\pi\times_{J^{k-1}\pi}\Lambda_2^m(J^{k-1}\pi)$. Since here a multi-index J is of the form (l) with $1\leq l\leq k$, we change the usual notation for coordinates to

$$u_J^\alpha\longrightarrow u_{|J|}^\alpha\quad\text{and}\quad p_\alpha^{I,1}\longrightarrow p_\alpha^{|I|+1},$$

and we adapt the remaining objects to this notation. So adapted coordinates on W_0 are of the form $(x,u^\alpha,u_l^\alpha,p,p_\alpha^l)$, where $l=1,\dots,k$. The premultisymplectic $(m+1)$-form would be

$$\Omega_H=-\sum_{l=0}^{k-1}\mathrm{d}p_\alpha^{l+1}\wedge\mathrm{d}u_l^\alpha+\sum_{l=1}^k\left(p_\alpha^l\,\mathrm{d}u_l^\alpha+u_l^\alpha\,\mathrm{d}p_\alpha^l\right)\wedge\mathrm{d}x-\sum_{l=0}^k\frac{\partial L}{\partial u_l^\alpha}\mathrm{d}u_l^\alpha\wedge\mathrm{d}x, \tag{4.91}$$

and horizontal projectors on TW_0 would have locally the form:

$$\mathbf{h}=\left(\frac{\partial}{\partial x}+\sum_{l=0}^k A_l^\alpha\frac{\partial}{\partial u_l^\alpha}+\sum_{l=1}^k B_\alpha^l\frac{\partial}{\partial p_\alpha^l}+C\frac{\partial}{\partial p}\right)\otimes\mathrm{d}x. \tag{4.92}$$

Solutions of the dynamical equation would satisfy the relations

$$B_\alpha^1 = \frac{\partial L}{\partial u^\alpha}; \tag{4.93}$$

$$p_\alpha^l = \frac{\partial L}{\partial u_l^\alpha}-B_\alpha^{l+1},\ \text{for } l=1,\dots,k-1; \tag{4.94}$$

$$p_\alpha^k = \frac{\partial L}{\partial u_k^\alpha}; \tag{4.95}$$

$$A_l^\alpha = u_{l+1}^\alpha,\ \text{for } l=0,\dots,k-1. \tag{4.96}$$

which we use to get the Euler-Lagrange equations

$$j^{2k}(\pi_{W_2,M}\circ\sigma)^*\left(\sum_{l=0}^{k}(-1)^l\frac{\mathrm{d}^l}{\mathrm{d}x^l}\frac{\partial L}{\partial u_l^\alpha}\right)=0, \tag{4.97}$$

where W_2 is defined by

$$W_2=\left\{(x^i,u^\alpha,u_l^\alpha,p,p_\alpha^l)\in W_1\ :\ p_\alpha^k=\frac{\partial L}{\partial u_k^\alpha},\ p=L-\sum_{l=1}^{k}p_\alpha^l u_l^\alpha\right\}. \tag{4.98}$$

We then obtain the tangency conditions:

$$B_\alpha^k \;=\; \frac{\partial^2 L}{\partial x\partial u_k^\alpha}+\sum_{l=0}^{k-1}u_{l+1}^\beta\frac{\partial^2 L}{\partial u_l^\beta\partial u_k^\alpha}+A_{k'}^\beta\frac{\partial^2 L}{\partial u_{k'}^\beta\partial u_k^\alpha}=0; \tag{4.99}$$

$$C \;=\; \frac{\partial L}{\partial x}+\sum_{l=0}^{k-1}u_{l+1}^\alpha\frac{\partial L}{\partial u_l^\alpha}+A_k^\alpha\frac{\partial L}{\partial u_k^\alpha}-\sum_{l=1}^{k}\left(A_l^\alpha p_\alpha^l+B_{\alpha j}^l u_l^\alpha\right). \tag{4.100}$$

Note that, thanks to equation (4.95), the terms in (4.100) with coefficient A_k cancel out. Now, for simplicity, suppose that $n=1$ and ignore the α's and β's that appear above. Consider the linear system of equations with respect to the B's formed by equations (4.94) (with $l=k-1$) and (4.99). This system is overdetermined since it has 2 equations and only one variable (B^k).

Example 4.53 (The loaded and clamped plate). Let us set $M=\mathbb{R}^2$ and $E=\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3$, and consider the Lagrangian

$$L(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})=\frac{1}{2}(u_{xx}^2+2u_{xy}^2+u_{yy}^2-2qu),$$

where $q=q(x,y)$ is the normal load on the plate. Given a regular region R of the plane, we look for the extremizers of the functional $I(u)=\int_R L$ such that $u=\partial u/\partial n=0$ on the border ∂R, where n is the normal exterior vector. The Euler-Lagange equation associated to the problem is

$$u_{xxxx}+2u_{xxyy}+u_{yyyy}=q. \tag{4.101}$$

Written in the multi-index notation, the Lagrangian has the form

$$L(j^2\phi)=\frac{1}{2}(u_{(2,0)}^2+2u_{(1,1)}^2+u_{(0,2)}^2-2qu)$$

and the Euler-Lagrange equation reads

$$u_{(4,0)}+2u_{(2,2)}+u_{(0,4)}=q.$$

The velocity-momentum space is $W_0=J^2\pi\times_{J^1\pi}\Lambda_2^2(J^1\pi)$, with adapted coordinates $(x,y,u_x,\ u_y,u_{xx},u_{xy},u_{yy},p,p^x,p^y,p^{yy},p^{xy},p^{yx},p^{yy})$. It is straightforward to write down the premultisymplectic 3-form and a general horizontal projector on TW_0, so we are not going to do it here. Even so, the coefficients of solutions of the dynamical equation would satisfy the relations

$$B_x{}^{,x} + B_y{}^{,y} = -2q \qquad \begin{array}{rcl} -p^x &=& B_x^{x,x} + B_y^{x,y} \\ -p^y &=& B_x^{y,x} + B_y^{y,y} \end{array} \qquad \begin{array}{rcl} p^{xx} &=& u_{xx} \\ p^{xy} + p^{yx} &=& 2u_{xy} \\ p^{yy} &=& u_{yy} \end{array} \tag{4.102}$$

where the latter ones are the equations that define W_1. The tangency condition on W_1 gives us the relations

$$\begin{array}{rcl} B_x^{x,x} &=& A_{xx,x} \\ B_x^{x,y} + B_x^{y,x} &=& 2A_{xy,x} \\ B_x^{y,y} &=& A_{yy,x} \end{array} \qquad \begin{array}{rcl} B_y^{x,x} &=& A_{xx,y} \\ B_y^{x,y} + B_y^{y,x} &=& 2A_{xy,y} \\ B_y^{y,y} &=& A_{yy,y} \end{array} \tag{4.103}$$

from where we can see that the Lagrangian is "regular", since

$$\left(\frac{\partial^2 L}{\partial u_K \partial u_{K'}} \right)_{|K|=|K'|=2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.104}$$

Finally, we remark that the middle equations of (4.102) and (4.103) form a 8×8 linear system of equations on the B's, which is completely determined.

Example 4.54 (The Camassa-Holm equation). In 1993, Camassa and Holm introduced the following completely integrable bi-Hamiltonian equation (see [23]):

$$v_t - v_{yyt} = -3vv_y + 2v_y v_{yy} + vv_{yyy}, \tag{4.105}$$

which is used to model the breaking waves in shallow waters as the Korteweg–de Vries equation. But, as the former is of higher order, we are going to use it as example.

The CH equation (4.105) is expressed in terms of the Eulerian or spatial velocity field $u(y,t)$, and it is the Euler-Poincaré equation of the reduced Lagrangian

$$l(v) = \frac{1}{2} \int \left(v^2 + v_y^2 \right) \,\mathrm{d}y. \tag{4.106}$$

To give a multisymplectic approach to the problem, as Kouranbaeva and Shkoller did (see [112]), we must express the CH equation (4.105) in Lagrangian terms. Thus, we shall use the Lagrangian variable $u(x,t)$ that arises as the solution of

$$\frac{\partial u(x,t)}{\partial t} = v(u(x,t),t). \tag{4.107}$$

The independent variables (x,t) are coordinates for the base space $M = S^1 \times \mathbb{R}$, and the dependent variable $u(x,t)$ is a fiber coordinate for the total space $E = S^1 \times \mathbb{R} \times \mathbb{R} = S^1 \times \mathbb{R}^2$. The Lagrangian action is now written as

$$L(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = \frac{1}{2}(u_x u_t^2 + u_x^{-1} u_{xt}^2) \tag{4.108}$$

The coefficients of a horizontal projector which is solution of the dynamical equation must satisfy

$$\begin{array}{rcl} {B_x}^{,x} + {B_t}^{,t} & = & 0 \\ p^x & = & 1/2(u_t^2 - (u_{xt}/u_x)^2) - (B_x^{x,x} + B_t^{x,t}) \\ p^t & = & u_x u_t - (B_x^{t,x} + B_t^{t,t}) \\ p^{xx} & = & 0 \\ p^{xt} + p^{tx} & = & u_{xt}/u_x \\ p^{tt} & = & 0 \end{array} \tag{4.109}$$

where the last three are the equations that define W_1. The tangency condition on W_1 gives us the relations

$$\begin{array}{rcl} B_x^{x,x} & = & 0 \\ B_x^{x,t} + B_x^{t,x} & = & -u_x^{-1} u_{xx} u_{xt} + A_{xt,x} u_x^{-1} \\ B_x^{t,t} & = & 0 \\ B_t^{x,x} & = & 0 \\ B_t^{x,t} + B_t^{t,x} & = & -(u_{xt}/u_x)^2 + A_{xt,t} u_x^{-1} \\ B_t^{t,t} & = & 0 \end{array} \tag{4.110}$$

from where we can see that the Lagrangian is clearly "singular", since

$$\left(\frac{\partial^2 L}{\partial u_K \partial u_{K'}} \right)_{|K|=|K'|=2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u_x^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.111}$$

Again, we may form a completely determined system of linear equations on the B's with the corresponding relations of (4.102) and the equations (4.110).

Example 4.55 (First order Lagrangian as second order). For the sake of simplicity, let suppose that $n = 1$. Given a first order Lagrangian $L : J^1\pi \longrightarrow \mathbb{R}$, extend it to a second order Lagrangian, $\bar{L} = L \circ \pi_{2,1}$. Consider the first and second order velocity-momenta mixed spaces $W_0^1 = J^1\pi \times_E \Lambda_2^m E$ and $W_0^2 = J^2\pi \times_{J^1\pi} \Lambda_2^m J^1\pi$, with adapted coordinates (x^i, u, u_i, p, p^i) and $(x^i, u, u_i, u_K, p, p^i, p^{ij})$ (with $|K| = 2$), respectively. Let $\pi_0^{2,1} : W_0^2 \longrightarrow W_0^1$ be the natural projection (Diagram 4.6).

$$\begin{tikzcd}
W_0^2 \arrow[r, "\pi_0^{2,1}"] \arrow[d] \arrow[dr] & W_0^1 \arrow[d] \arrow[dr] & \\
J^2\pi \arrow[r, "\pi_{2,1}"] \arrow[dr, "\bar{L}"] & J^1\pi \arrow[r] \arrow[d, "L"] \arrow[dr, "\pi_1"] & E \arrow[d, "\pi"] \\
 & \mathbb{R} & M
\end{tikzcd}$$

Figure 4.6: The 1st and 2nd order Lagrangian settings

We are going to apply the theory we have developed here to the systems given by each Lagrangian. Consider the premultisymplectic forms Ω_H and $\Omega_{\bar{H}}$, where H and $\bar{H}$ are the corresponding dynamical functions (equations (4.48) and (4.49)). Let $\mathbf{h}$ and $\bar{\mathbf{h}}$

denote solutions of the respective dynamical equations on (W_0^1, Ω_H) and $(W_0^2, \Omega_{\bar{H}})$. They would locally have the form

$$\begin{aligned}
\mathbf{h} &= \left(\frac{\partial}{\partial x^j} + A_j \frac{\partial}{\partial u} + A_{ij} \frac{\partial}{\partial u_i} + B_j^i \frac{\partial}{\partial p^i} + C_j \frac{\partial}{\partial p}\right) \otimes \mathrm{d}x^j, \\
\bar{\mathbf{h}} &= \left(\frac{\partial}{\partial x^j} + \bar{A}_j \frac{\partial}{\partial u} + \bar{A}_{ij} \frac{\partial}{\partial u_i} + \bar{A}_{Kj} \frac{\partial}{\partial u_K} + \bar{B}_j^i \frac{\partial}{\partial p^i} + \bar{B}_j^{ki} \frac{\partial}{\partial p^{ki}} + \bar{C}_j \frac{\partial}{\partial p}\right) \otimes \mathrm{d}x^j,
\end{aligned}$$

where $|K| = 2$. We then obtain the relations

$$B_j^j = \frac{\partial L}{\partial u}, \tag{4.112}$$

$$p^i = \frac{\partial L}{\partial u_i}, \tag{4.113}$$

$$A_i = u_i, \tag{4.114}$$

for $(W_0^1, \Omega_H, \mathbf{h})$; and

$$\bar{B}_j^j = \frac{\partial L}{\partial u}, \tag{4.115}$$

$$p^i = \frac{\partial L}{\partial u_i} - \bar{B}_j^{ij}, \tag{4.116}$$

$$p^{ij} + p^{ji} = (1_i + 1_j)! \cdot \frac{\partial \bar{L}}{\partial u_{1_i + 1_j}} = 0, \tag{4.117}$$

$$\bar{A}_i = u_i, \tag{4.118}$$

$$\bar{A}_{ij} = u_{1_i+1_j}, \tag{4.119}$$

for $(W_0^2, \Omega_{\bar{H}}, \bar{\mathbf{h}})$. Equations (4.113) and (4.117), together with $H = 0$ and $\bar{H} = 0$, define the corresponding submanifolds W_2^1 and W_2^2 of W_0^1 and W_0^2.

We notice that, even though $\bar{L}$ is in some sense the same Lagrangian than L, a solution of the dynamical equation on W_0^1 may be easily determined, while in W_0^2 the space of solutions has grown (there are more coefficients to be determined). We thus infer from here, that a solution $\bar{\mathbf{h}}$ of the dynamical equation in W_0^2 must satisfy an extra condition. Since $p = L - p^i u_i + 0$ in W_2^2, the projection $\pi_0^{2,1}$ maps W_2^2 to W_2^1. We therefore impose to a solution $\bar{\mathbf{h}}$ of the dynamical equation along W_2^2 to be in addition projectable to a solution $\mathbf{h}$ of the dynamical equation along W_2^1. In such a case, we would have that

$$\bar{B}_j^{ij} = 0 \tag{4.120}$$

which implies that the following equation

$$p^i = \frac{\partial L}{\partial u_i} \tag{4.121}$$

is now a restriction in W_2^2. So, by tangency condition, we get

$$\bar{B}_j^i = \frac{\partial^2 L}{\partial x^j \partial u_i} + u_j \frac{\partial^2 L}{\partial u \partial u_i} + u_{1_k+1_j} \frac{\partial^2 L}{\partial u_k \partial u_i} + 0 = \frac{\mathrm{d}}{\mathrm{d}x^j} \frac{\partial L}{\partial u_i}. \tag{4.122}$$

Combining this with equation (4.112), we finally obtain

$$\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u_j} = 0, \tag{4.123}$$

which is the Euler-Lagrange equation.

It is worth to remark here that, at this time, the Euler-Lagrange equation has not been deduced by the process shown in the proof of Proposition 4.50, but directly from the projectability condition, although the previous Euler-Lagrange equation may be recovered from any of the two settings.

4.2.4 Constraints within the Skinner-Rusk Formalism

As in the previous section, we begin by considering a constraint submanifold $i : \mathcal{C} \hookrightarrow J^k\pi$ of codimension l, which is locally annihilated by l functionally independent constraint functions Ψ^μ, where $1 \leq \mu \leq l$. The constraint submanifold $\mathcal{C}$ is supposed to fiber over the whole of M and it is not necessarily generated from a previous constraint submanifold by the process shown in Remark 4.22. We define in the restricted velocity-momentum space $W_0 = \{w \in W : \mathcal{H}(w) = 0\}$ the constrained velocity-momentum space $W_0^{\mathcal{C}} = pr_1^{-1}(\mathcal{C})$, which is a submanifold of W_0, whose induced embedding and whose constraint functions will still be denoted $i : W_0^{\mathcal{C}} \hookrightarrow W$ and Ψ^μ, where $1 \leq \mu \leq l$. The first order case $k = 1$ is treated in [33].

The following proposition allows us to work in local coordinates on the unconstrained velocity-momentum space W, as it is done in [11].

Proposition 4.56. *Given a point $w \in W_0^{\mathcal{C}}$, let $X \in \Lambda_d^m(T_wW_0^{\mathcal{C}})$ be a decomposable multivector and denote its image, $i_*(X) \in \Lambda^m(T_wW)$, by $\bar{X}$. The following statements are equivalent:*

1. *$i_X\Omega_0^{\mathcal{C}}(Y) = 0$ for every $Y \in T_wW_0^{\mathcal{C}}$;*
2. *$i_{\bar{X}}\Omega \in T_w^0W_0^{\mathcal{C}}$;*

where $T_w^0W_0^{\mathcal{C}}$ is the annihilator of $i_*(T_wW_0^{\mathcal{C}})$ in T_wW.

We therefore look for solutions of the *constrained dynamical equation*

$$(-1)^m i_{\bar{X}}\Omega = -\lambda_\mu\,\mathrm{d}\Psi^\mu - \lambda\,\mathrm{d}H, \tag{4.124}$$

where $\bar{X}$ is a tangent multivector field along $W_0^{\mathcal{C}}$, the λ^μ's and λ are Lagrange multipliers to be determined. Here, the coefficient $(-1)^m$ is used for technical purposes.

Remark 4.57. It should be said that the Lagrange multipliers that appear in the dynamical equation (4.124) have a different nature that the ones that appear in Proposition 4.43. The former are locally defined on W, while the latter are locally defined on M. Although they coincide on the integral sections $\sigma \in \Gamma\pi_{W,M}$ of a solution X of the dynamical equation (4.124), since its "Lagrangian part" $\sigma_1 = pr_1 \circ \sigma$ satisfies the constrained Euler-Lagrange equation (4.46) with $\tilde{\lambda}_\mu = \lambda_\mu \circ \sigma$ (*cf.* Proposition 4.59).

Let $\bar{X} \in \Lambda^m(T_wW)$ be a decomposable m-vector at a given point $w \in W$, that is, $\bar{X} = \bar{X}_1 \wedge \cdots \wedge \bar{X}_m$ for m tangent vectors $\bar{X}_i \in T_wW$, which have the form

$$\bar{X}_j = \frac{\partial}{\partial x^j} + A^\alpha_{Jj}\frac{\partial}{\partial u^\alpha_J} + B^{Ii}_{\alpha j}\frac{\partial}{\partial p^{Ii}_\alpha} + C_j\frac{\partial}{\partial p} \tag{4.125}$$

in a given adapted chart $(x^i, u^\alpha_J, p^{Ii}_\alpha, p)$. A straightforward computation gives us

$$(-1)^m i_{\bar{X}}(\,\mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i) = (A^\alpha_{Ii}B^{Ij}_{\alpha j} - A^\alpha_{Ij}B^{Ij}_{\alpha i})\,\mathrm{d}x^i + A^\alpha_{Ii}\,\mathrm{d}p^{Ii}_\alpha - B^{Ii}_{\alpha i}\,\mathrm{d}u^\alpha_I \tag{4.126}$$

and

$$(-1)^m i_{\bar{X}}(\,\mathrm{d}p \wedge \mathrm{d}^m x) = \mathrm{d}p - C_i\,\mathrm{d}x^i. \tag{4.127}$$

Applying the above equations to the dynamical one (4.124), we obtain the relations

$$\begin{array}{lrcl}
\text{coefficients in } \mathrm{d}p: & 1 & = & \lambda; \\
\text{coefficients in } \mathrm{d}p^{Ii}_\alpha: & A^\alpha_{Ii} & = & \lambda u^\alpha_{I+1_i}; \\
\text{coefficients in } \mathrm{d}u^\alpha_J & B_{\alpha i}^{\ i} & = & \lambda\frac{\partial L}{\partial u^\alpha} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha}; \\
 & B^{Ii}_{\alpha i} & = & \lambda\left(\frac{\partial L}{\partial u^\alpha_I} - \sum_{J+1_j=I} p^{Jj}_\alpha\right) - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_I}; \\
 & 0 & = & \lambda\left(\frac{\partial L}{\partial u^\alpha_K} - \sum_{J+1_j=K} p^{Jj}_\alpha\right) - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_K}; \\
\text{coefficients in } \mathrm{d}x^j: & A^\alpha_{Ii}B^{Ii}_{\alpha j} - A^\alpha_{Ij}B^{Ii}_{\alpha i} + C_j & = & \lambda\frac{\partial L}{\partial x^j} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial x^j}.
\end{array}$$

Thus, a decomposable m-vector $\bar{X} \in \Lambda^m(T_wW)$ at a point $w \in W$ is a solution of the dynamical equation

$$(-1)^m i_{\bar{X}}\Omega = -\lambda_\mu\,\mathrm{d}\Psi^\mu - \mathrm{d}H, \tag{4.128}$$

if for any adapted chart $(x^i, u^\alpha_J, p^{Ii}_\alpha, p)$, the coefficients of $\bar{X}$ and the point w satisfy the equations

$$A^\alpha_{Ii} = u^\alpha_{I+1_i},\ \text{with } |I| = 0,\ldots,k-1,\ i = 1,\ldots; \tag{4.129}$$

$$0 = \frac{\partial L}{\partial u^\alpha} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha} - B_{\alpha j}^{\ j}; \tag{4.130}$$

$$\sum_{I+1_i=J} p^{Ii}_\alpha = \frac{\partial L}{\partial u^\alpha_J} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_J} - B^{Jj}_{\alpha j},\ \text{with } |J| = 1,\ldots,k-1; \tag{4.131}$$

$$\sum_{I+1_i=K} p^{Ii}_\alpha = \frac{\partial L}{\partial u^\alpha_K} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_K},\ \text{with } |K| = k; \tag{4.132}$$

$$C_j = \frac{\partial L}{\partial x^j} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial x^j} + \sum_{|J|=0}^{k-1} u^\alpha_{J+1_j}\left(\frac{\partial L}{\partial u^\alpha_I} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_J} - \sum_{I+1_i=J} p^{Ii}_\alpha\right) - u^\alpha_{I+1_i}B^{Ii}_{\alpha j}. \tag{4.133}$$

Because of the Lagrange multipliers λ_μ, we cannot describe the submanifold of $W_0^{\mathcal{C}}$ where solutions X of the constrained dynamical equation (4.128) exist, like it has been done in (4.67) for the unconstrained dynamical equation (4.52) . Therefore, we need to get rid off of them. Consider the more concise expression for the equations of dynamics (4.130), (4.131) and (4.132)

$$\sum_{I+1_i=J} p^{Ii}_\alpha = \frac{\partial L}{\partial u^\alpha_J} - \lambda_\mu\frac{\partial \Psi^\mu}{\partial u^\alpha_J} - B^{Jj}_{\alpha j},\ \text{with } |J| = 0,\ldots,k, \tag{4.134}$$

where the first summation term is understood to be void when $|J| = 0$, as well as it is the last one when $|J| = k$. We now suppose that the constraints Ψ^μ are of the type $u^{\hat\alpha}_{\hat J} = \Phi^{\hat\alpha}_{\hat J}(x^i, u^{\check\alpha}_{\check J})$, where $u^{\hat\alpha}_{\hat J}$ are some constrained coordinates which depend on the free coordinates $(x^i, u^{\check\alpha}_{\check J})$ through the functions $\Phi^{\hat\alpha}_{\hat J}$. Thus, the constraint have the form $\Psi^{\hat\alpha}_{\hat J}(x^i, u^\alpha_J) = u^{\hat\alpha}_{\hat J} - \Phi^{\hat\alpha}_{\hat J}(x^i, u^{\check\alpha}_{\check J}) = 0$. So, writing again the previous equation (4.134) for the different sets of coordinates, the ones that are free and the ones that are not, we obtain

$$\sum_{I+1_i=\hat J} p^{Ii}_{\hat\alpha} = \frac{\partial L}{\partial u^{\hat\alpha}_{\hat J}} - \lambda^{\hat J}_{\hat\alpha} \qquad - B^{\hat J j}_{\hat\alpha j}, \text{ with } |\hat J| = 0, \ldots, k; \tag{4.135}$$

$$\sum_{I+1_i=\check J} p^{Ii}_{\check\alpha} = \frac{\partial L}{\partial u^{\check\alpha}_{\check J}} + \lambda^{\hat J}_{\hat\alpha} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}} - B^{\check J j}_{\check\alpha j}, \text{ with } |\check J| = 0, \ldots, k. \tag{4.136}$$

Substituting $-\lambda^{\hat J}_{\hat\alpha}$ from (4.135) into (4.136), we have that

$$\sum_{I+1_i=\check J} p^{Ii}_{\check\alpha} + \left(\sum_{I+1_i=\hat J} p^{Ii}_{\hat\alpha} \right) \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}} = \frac{\partial L}{\partial u^{\check\alpha}_{\check J}} + \frac{\partial L}{\partial u^{\hat\alpha}_{\hat J}} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}} - B^{\check J j}_{\check\alpha j} - B^{\hat J j}_{\hat\alpha j} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}}, \text{ with } |\check J| = 0, \ldots, k. \tag{4.137}$$

Note that, when $|\check J| = k$, the term $B^{\check J j}_{\check\alpha j}$ disappears, but $B^{\hat J j}_{\hat\alpha j} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}}$ do not necessarily. This is circumvent by supposing that, if $|\hat J| < k$, then $\frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{K}} = 0$ for any $|K| = k$. That is the case when the constraint submanifold $\mathcal{C}$ has no constraint of higher order, *i.e.* $\mathcal{C} = \pi^{-1}_{k,k-1}(\pi_{k,k-1}(\mathcal{C}))$, or, more generally, when $\mathcal{C}$ fibers by $\pi_{k,k-1}$ over its image.

Taking this into account, we expand the previous equation (4.137), obtaining then constrained equations of dynamics freed of the Lagrange multipliers

$$\sum_{I+1_i=\hat J} p^{Ii}_{\hat\alpha} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}} = \frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}} - B^{j}_{\check\alpha j} - B^{\hat J j}_{\hat\alpha j} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}}; \tag{4.138}$$

$$\sum_{I+1_i=\check J} p^{Ii}_{\check\alpha} + \sum_{I+1_i=\hat J} p^{Ii}_{\hat\alpha} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}} = \frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}_{\check J}} - B^{\check J j}_{\check\alpha j} - B^{\hat J j}_{\hat\alpha j} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}}, \text{ with } |\check J| = 1, \ldots, k-1 \tag{4.139}$$

$$\sum_{I+1_i=\check K} p^{Ii}_{\check\alpha} + \sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha} \frac{\partial \Phi^{\hat\alpha}_{\hat K}}{\partial u^{\check\alpha}_{\check K}} = \frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}_{\check K}}, \text{ with } |\check K| = k; \tag{4.140}$$

where $\frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}_{\check J}} = \frac{\partial L}{\partial u^{\check\alpha}_{\check J}} + \frac{\partial L}{\partial u^{\hat\alpha}_{\hat J}} \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}}$, being $\mathcal{L}^{\mathcal{C}} = \mathcal{L} \circ i : \mathcal{C} \longrightarrow \Lambda^m M$ the restricted Lagrangian.

We are now in disposition to define the submanifold $W_2^{\mathcal{C}}$ along to which solutions of the constrained dynamical equation (4.128) exist,

$$W_2^{\mathcal{C}} = \left\{ w \in W_0^{\mathcal{C}} \ : \ (4.140) \right\} = \left\{ w \in W \ : \ \begin{array}{c} u^{\hat\alpha}_{\hat J} = \Phi^{\hat\alpha}_{\hat J}(x^i, u^{\check\alpha}_{\check J}) \\ p = L(x^i, u^\alpha_J) - p^{Ii}_\alpha u^\alpha_{I+1_i} \\ \displaystyle\sum_{I+1_i=\check K} p^{Ii}_{\check\alpha} + \sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha} \frac{\partial \Phi^{\hat\alpha}_{\hat K}}{\partial u^{\check\alpha}_{\check K}} = \frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}_{\check K}} \end{array} \right\} \tag{4.141}$$

Tangency conditions on X with respect to $W_2^{\mathcal{C}}$ will give us the constrained equations of tangency

$$A^{\hat\alpha}_{\hat J j} = \frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial x^j} + A^{\check\alpha}_{\check J j}\frac{\partial \Phi^{\hat\alpha}_{\hat J}}{\partial u^{\check\alpha}_{\check J}}, \tag{4.142}$$

$$\begin{aligned} C_j = {} & \frac{\partial L^{\mathcal{C}}}{\partial x^j} + \sum_{|\check J|=0}^{k-1} u^{\check\alpha}_{\check J+1_j}\left(\frac{\partial L^{\mathcal{C}}}{\partial u^{\check\alpha}_{\check J}} - \sum_{I+1_i=\check J} p^{Ii}_{\check\alpha}\right) \\ & - \sum_{|\hat J|=0}^{k-1} u^{\hat\alpha}_{\hat J+1_j}\sum_{I+1_i=\hat J} p^{Ii}_{\hat\alpha} - \sum_{|\hat K|=k}\frac{\partial \Phi^{\hat\alpha}_{\hat K}}{\partial x^j}\sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha} - \sum_{|I|=0}^{k-1} u^{\alpha}_{I+1_i}B^{Ii}_{\alpha j}, \end{aligned} \tag{4.143}$$

$$\begin{aligned} \sum_{I+1_i=\check K} B^{Ii}_{\check\alpha j} = {} & \frac{\partial^2 L^{\mathcal{C}}}{\partial x^j \partial u^{\check\alpha}_{\check K}} - \sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha}\frac{\partial^2 \Phi^{\hat\alpha}_{\hat K}}{\partial x^j \partial u^{\check\alpha}_{\check K}} \\ & + A^{\check\beta}_{\check J j}\left(\frac{\partial^2 L^{\mathcal{C}}}{\partial u^{\check\beta}_{\check J}\partial u^{\check\alpha}_{\check K}} - \sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha}\frac{\partial^2 \Phi^{\hat\alpha}_{\hat K}}{\partial u^{\check\beta}_{\check J}\partial u^{\check\alpha}_{\check K}}\right) - \sum_{I+1_i=\hat K} B^{Ii}_{\hat\alpha j}\frac{\partial \Phi^{\hat\alpha}_{\hat K}}{\partial u^{\check\alpha}_{\check K}},\ |\check K| = k. \end{aligned} \tag{4.144}$$

Proposition 4.58. *Let $\Omega_2^{\mathcal{C}}$ be the pullback of the premultisymplectic form $\Omega_{\mathcal{H}}$ to $W_2^{\mathcal{C}}$ by the natural inclusion $i : W_2^{\mathcal{C}} \hookrightarrow W$, that is $\Omega_2^{\mathcal{C}} = i^*(\Omega_{\mathcal{H}})$. Suppose that $m = \dim M > 1$, then the $(m+1)$-form $\Omega_2^{\mathcal{C}}$ is multisymplectic if and only if $\mathcal{L}$ is* regular along $W_2^{\mathcal{C}}$, i.e. *if and only if the matrix*

$$\left(\frac{\partial^2 L^{\mathcal{C}}}{\partial u^{\check\beta}_{\check R}\partial u^{\check\alpha}_{\check K}} - \sum_{I+1_i=\hat K} p^{Ii}_{\hat\alpha}\frac{\partial^2 \Phi^{\hat\alpha}_{\hat K}}{\partial u^{\check\beta}_{\check R}\partial u^{\check\alpha}_{\check K}}\right)_{|\check R|=|\check K|=k} \tag{4.145}$$

is non-degenerate along $W_2^{\mathcal{C}}$.

Proof. First of all, let us make some considerations. By definition, $\Omega_2^{\mathcal{C}}$ is multisymplectic whenever $\Omega_2^{\mathcal{C}}$ has trivial kernel, that is,

$$\text{if } v \in TW_2,\ i_v\Omega_2^{\mathcal{C}} = 0 \iff v = 0\ .$$

This is equivalent to say that

$$\text{if } v \in i_*(TW_2),\ i_v\Omega_{\mathcal{H}}|_{i_*(TW_2)} = 0 \iff v = 0\ .$$

Let $v \in TW$ be a tangent vector whose coefficients in an adapted basis are given by

$$v = \gamma^i\frac{\partial}{\partial x^i} + A^\alpha_J\frac{\partial}{\partial u^\alpha_J} + B^{Ii}_\alpha\frac{\partial}{\partial p^{Ii}_\alpha} + C\frac{\partial}{\partial p}.$$

Using the expression (4.51), we may compute the contraction of $\Omega_{\mathcal{H}}$ by v,

$$\begin{aligned} i_v\Omega_{\mathcal{H}} = {} & -B^{Ii}_\alpha\, \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + A^\alpha_I\, \mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}^{m-1}x_i - \gamma^j\, \mathrm{d}p^{Ii}_\alpha \wedge \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-2}x_{ij} \\ & + \left(A^\alpha_{I+1_i}p^{Ii}_\alpha + B^{Ii}_\alpha u^\alpha_{I+1_i} - A^\alpha_J\frac{\partial L}{\partial u^\alpha_J}\right)\mathrm{d}^m x \\ & - \gamma^j\left(p^{Ii}_\alpha\, \mathrm{d}u^\alpha_{I+1_i} + u^\alpha_{I+1_i}\, \mathrm{d}p^{Ii}_\alpha - \frac{\partial L}{\partial u^\alpha_J}\, \mathrm{d}u^\alpha_J\right) \wedge \mathrm{d}^{m-1}x_j. \end{aligned} \tag{4.146}$$

In addition to this, let us consider a vector $v \in TW$ tangent to W_2, that is $v \in i_*(TW_2)$, we then have that

$$\mathrm{d}(u^{\hat{\alpha}}_{\hat{J}} - \Phi^{\hat{\alpha}}_{\hat{J}})(v) = 0, \quad \mathrm{d}\left(\sum_{I+1_i = \check{K}} p^{Ii}_{\check{\alpha}} + \sum_{I+1_i=\hat{K}} p^{Ii}_{\hat{\alpha}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial u^{\check{\alpha}}_{\check{K}}} - \frac{\partial L^{\mathcal{C}}}{\partial u^{\check{\alpha}}_{\check{K}}} \right)(v) = 0 \quad \text{and} \quad \mathrm{d}H(v) = 0,$$

which leads us to the following relations for the coefficients of v:

$$\begin{aligned}
A^{\hat{\alpha}}_{\hat{J}} &= \gamma^j \frac{\partial \Phi^{\hat{\alpha}}_{\hat{J}}}{\partial x^j} + A^{\check{\alpha}}_{\check{J}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{J}}}{\partial u^{\check{\alpha}}_{\check{J}}}, &(4.147)\\
\sum_{I+1_i=\check{K}} B^{Ii}_{\check{\alpha}} &= \gamma^j \left(\frac{\partial^2 L^{\mathcal{C}}}{\partial x^j \partial u^{\check{\alpha}}_{\check{K}}} - \sum_{I+1_i=\hat{K}} p^{Ii}_{\hat{\alpha}} \frac{\partial^2 \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial x^j \partial u^{\check{\alpha}}_{\check{K}}} \right) &(4.148)\\
&\quad + A^{\check{\beta}}_{\check{J}} \left(\frac{\partial^2 L^{\mathcal{C}}}{\partial u^{\check{\beta}}_{\check{J}} \partial u^{\check{\alpha}}_{\check{K}}} - \sum_{I+1_i=\hat{K}} p^{Ii}_{\hat{\alpha}} \frac{\partial^2 \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial u^{\check{\beta}}_{\check{J}} \partial u^{\check{\alpha}}_{\check{K}}} \right) - \sum_{I+1_i=\hat{K}} B^{Ii}_{\hat{\alpha}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial u^{\check{\alpha}}_{\check{K}}}\\
C &= \gamma^j \left(\frac{\partial L^{\mathcal{C}}}{\partial x^j} - \sum_{I+1_i=\hat{K}} p^{Ii}_{\hat{\alpha}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial x^j} \right) &(4.149)\\
&\quad + A^{\check{\alpha}}_{\check{J}} \left(\frac{\partial L^{\mathcal{C}}}{\partial u^{\check{\alpha}}_{\check{J}}} - \sum_{I+1_i=\check{J}} p^{Ii}_{\check{\alpha}} - \sum_{I+1_i=\hat{J}} p^{Ii}_{\hat{\alpha}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{J}}}{\partial u^{\check{\alpha}}_{\check{J}}} \right) - B^{Ii}_{\alpha} u^{\alpha}_{I+1_i}.
\end{aligned}$$

It is important to note that, even though in all the previous equations (4.146), (4.147), (4.148) and (4.149) explicitly appear A's with multi-index of length k, for such a vector $v \in i_*(TW_2)$, the terms associated to these A's cancel out in the development of $i_v \Omega_{\mathcal{H}}$, Equation (4.146), and the third tangency relation (4.149). Thus, a tangent vector $v \in i_*(TW_2)$ would kill $\Omega_{\mathcal{H}}$ if and only if its coefficients satisfy the following relations

$$\gamma^j = 0, \; A^{\alpha}_I = 0, \; B^{Ii}_{\alpha} = 0, \; C = 0,$$
$$A^{\hat{\alpha}}_{\hat{K}} = A^{\check{\alpha}}_{\check{K}} \frac{\partial \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial u^{\check{\alpha}}_{\check{K}}} \text{ and } A^{\check{\beta}}_{\check{R}} \left(\frac{\partial^2 L^{\mathcal{C}}}{\partial u^{\check{\beta}}_{\check{R}} \partial u^{\check{\alpha}}_{\check{K}}} - \sum_{I+1_i=\hat{K}} p^{Ii}_{\hat{\alpha}} \frac{\partial^2 \Phi^{\hat{\alpha}}_{\hat{K}}}{\partial u^{\check{\beta}}_{\check{R}} \partial u^{\check{\alpha}}_{\check{K}}} \right) = 0.$$

These considerations being made, the assertion is now clear. □

Proposition 4.59. . *Let $\sigma \in \Gamma \pi_{W,M}$ be an integral section of a solution X of the constrained dynamical equation (4.128). Then, its "Lagrangian part" $\sigma_1 = pr_1 \circ \sigma$ is holonomic, $\sigma_1 = j^k \phi$ for some section $\phi \in \Gamma\pi$, which furthermore satisfies the constrained higher-order Euler-Lagrange equations (4.46).*

Proof. If X is locally expressed as in (4.125), we know that it must satisfy the equations of dynamics (4.130), (4.132) and (4.132), for unknown Lagrange multipliers λ_μ. If we note $\lambda'_\mu = \lambda_\mu \circ \sigma$ and $L' = L - \lambda'_\mu \Psi^\mu$, it suffices to follow the proof for L' of Theorem 4.50. □

Example: Controlled Fluid Mechanics

Here, we study an incompressible fluid under control as in [3]. The corresponding equations are the Navier-Stokes one plus the divergence-free condition:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla_{\mathbf{v}}\mathbf{v} + \nabla\Pi = \nu\Delta\mathbf{v} + \mathbf{f} \tag{4.150}$$
$$\nabla\cdot\mathbf{v} = 0 \tag{4.151}$$

where the vector field $\mathbf{v}$ is the velocity of the fluid, $\mathbf{f}$ is the field of exterior forces acting on the fluid, which will be our controls, and the scalar functions Π and ν are the pressure and the viscosity, respectively. In particular, our case of interest is the two dimensional case on $\mathbb{R}^2$ endowed with the standard metric. If we fix global Cartesian coordinates (x, y) on $\mathbb{R}^2$ and adapted coordinates (x, y, u, v) on its tangent $T\mathbb{R}^2 = \mathbb{R}^4$, the previous equations become

$$u_t + u\cdot u_x + v\cdot u_y + \partial_x\Pi = \nu\cdot(u_{xx} + u_{yy}) + F \tag{4.152}$$
$$v_t + u\cdot v_x + v\cdot v_y + \partial_y\Pi = \nu\cdot(v_{xx} + v_{yy}) + G \tag{4.153}$$
$$u_x + v_y = 0 \tag{4.154}$$

where, with some abuse of notation, $\mathbf{v}(t, x, y) = (u, v)$ and $\mathbf{f} = (F, G)$.

We therefore look for time-dependent vector fields $\mathbf{v} = (u, v)$ on $\mathbb{R}^2$ that satisfy the Navier-Stokes equations (4.152) and (4.153) for a prescribed control $\mathbf{f} = (F, G)$ and submitted to the free divergence condition (4.154). Moreover, we look for such vector fields $\mathbf{v} = (u, v)$ that are in addition optimal in the controls for the integral action

$$\mathcal{A}_{\mathcal{L}}(\mathbf{v}, R) = \frac{1}{2}\int_R \|\mathbf{f}\|^2\, \mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y. \tag{4.155}$$

In order to apply the development of the present jet bundle framework, all of this is restated in the following way: We set a fiber bundle $\pi : E \longrightarrow M$ by putting $M = \mathbb{R}\times\mathbb{R}^2$, $E = \mathbb{R}\times T\mathbb{R}^2$ and $\pi = (pr_1, pr_{\mathbb{R}^2})$. We fix global adapted coordinates (t, x, y, u, v) on E, which induce the corresponding global adapted coordinates on $J^k\pi$ and $J^k\pi^\dagger$. Besides, we choose the volume form η on M to be $\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y$. Thus, the Lagrangian function $L : J^2\pi \to \mathbb{R}$ is nothing else but

$$L = \frac{1}{2}(F^2 + G^2),$$

where we obtain F and G as functions on $J^2\pi$ using the equations (4.152) and (4.153).

To make the reading easier, we change slightly the coordinate notation of jet bundles to fit in this example: The coordinate "velocities" associated to u and v will still be labeled u and v, respectively, with symmetric subindexes (as in the original equations); the coordinate "momenta" associated to u and v will now be labeled p and q, respectively, with non-symmetric subindexes. Finally and as we will focus on the equations of dynamics (4.138), (4.139) and (4.140), the coefficients in the local expression (4.125) of a multivector X associated to the coordinate momenta p and q will be labeled B and D, respectively.

Example 4.60 (The Euler equation). We will first suppose that the fluid is Eulerian, that is, it has null viscosity. In this case, the Lagrangian function $L = (F^2 + G^2)/2$ associated to the integral action (4.155) is of first order when the "Euler equations", (4.152) and (4.153) with $\nu = 0$, are taken into account. In $J^1\pi$, we consider the divergence-free constraint submanifold $\mathcal{C} = \{z \in J^1\pi \,:\, u_x + u_y = 0\}$, which introduces a single Lagrange multiplier λ.

Proceeding with the theoretical machinery, we compute the bottom level equations of dynamics corresponding to those of (4.130)

$$\begin{aligned} 0 &= u_x \cdot F + v_x \cdot G - (B_t^t + B_x^x + B_y^y) \\ 0 &= u_y \cdot F + v_y \cdot G - (D_t^t + D_x^x + D_y^y) \end{aligned}$$

and the top level equations of dynamics (there are no middle ones) corresponding to those of (4.132)

$$\begin{aligned} p^t &= F & \qquad q^t &= G \\ p^x &= u \cdot F - \lambda & \qquad q^x &= u \cdot G \\ p^y &= v \cdot F & \qquad q^y &= v \cdot G - \lambda \end{aligned}$$

We can dispose of the only Lagrange multiplier λ by putting

$$p^x - q^y = u \cdot F - v \cdot G,$$

what defines $W_1^{\mathcal{C}}$ together with the top level equations of dynamics with no Lagrange multiplier.

From here, we may compute also the constrained Euler-Lagrange equations (4.46) for this problem, which are

$$\begin{aligned} \mathrm{d}_t F + u \cdot \mathrm{d}_x F + v \cdot \mathrm{d}_y F + v_y \cdot F - v_x \cdot G = \partial_x \lambda \\ \mathrm{d}_t G + u \cdot \mathrm{d}_x G + v \cdot \mathrm{d}_y G + u_x \cdot G - u_y \cdot F = \partial_y \lambda \end{aligned}$$

where $\mathrm{d}_* = \frac{\mathrm{d}}{\mathrm{d}*}$.

Finally, we note that L is not regular along $W_2^{\mathcal{C}}$ since the square matrix, that correspond to (4.145),

$$\begin{pmatrix} 1 & u & v & 0 & 0 \\ u & u^2 + v^2 & u \cdot v & -v & -u \cdot v \\ v & u \cdot v & v^2 & 0 & 0 \\ 0 & -v & 0 & 1 & u \\ 0 & -u \cdot v & 0 & u & u^2 \end{pmatrix}$$

has obviously rank 2. Here we have used as u_x as independent ("check") coordinate and v_y as dependent ("hat") coordinate.

Example 4.61 (The Navier-Stokes equation). Now, we tackle the full problem of the Navier-Stokes equations. In this case, the Lagrangian function $L = (F^2 + G^2)/2$ is of second order. In $J^2\pi$, we consider the constraint submanifold

$$\mathcal{C} = \left\{ z \in J^2\pi \,:\, u_x + u_y = 0,\; u_{tx} + v_{ty} = 0,\; u_{xx} + v_{xy} = 0,\; u_{xy} + v_{yy} = 0 \right\}$$

which comes from the first order constraint (4.151), free divergence, and its consequences to second order (see Remark 4.22). These constraints introduce for Lagrange multiplier λ, λ_t, λ_x and λ_y that are associated to them respectively.

We now proceed like in the previous example by computing the equations of dynamics. In first place, we have the bottom level ones corresponding to those of (4.130)

$$\begin{aligned} 0 &= u_x \cdot F + v_x \cdot G - (B_t^t + B_x^x + B_y^y) \\ 0 &= u_y \cdot F + v_y \cdot G - (D_t^t + D_x^x + D_y^y) \end{aligned}$$

Note that they are formally the same as before. In second place, the mid level equations corresponding to those of (4.131)

$$\begin{aligned} p^t &= F - (B_t^{tt} + B_x^{tx} + B_y^{ty}) & q^t &= G - (D_t^{tt} + D_x^{tx} + D_y^{ty}) \\ p^x &= u \cdot F - (B_t^{xt} + B_x^{xx} + B_y^{xy}) + \lambda & q^x &= u \cdot G - (D_t^{xt} + D_x^{xx} + D_y^{xy}) \\ p^y &= v \cdot F - (B_t^{yt} + B_x^{yx} + B_y^{yy}) & q^y &= v \cdot G - (D_t^{yt} + D_x^{yx} + D_y^{yy}) + \lambda \end{aligned}$$

Note that formally they also coincide with the top level ones of the previous example but for the coefficients that now appear in them. And in third place, the top level equations corresponding to those of (4.132)

$$\begin{aligned} p^{tt} &= 0 & q^{tt} &= 0 \\ p^{xx} &= -\nu \cdot F - \lambda_x & q^{xx} &= -\nu \cdot G \\ p^{yy} &= -\nu \cdot F & q^{yy} &= -\nu \cdot G - \lambda_y \\ p^{tx} + p^{xt} &= -\lambda_t & q^{tx} + q^{xt} &= 0 \\ p^{ty} + p^{yt} &= 0 & q^{ty} + q^{yt} &= -\lambda_t \\ p^{xy} + p^{yx} &= -\lambda_y & q^{xy} + q^{yx} &= -\lambda_x \end{aligned}$$

We can again get rid easily of the Lagrange multipliers by putting

$$p^{tx} + p^{xt} = q^{ty} + q^{yt} \qquad p^{xx} + \nu \cdot F = q^{xy} + q^{yx} \qquad p^{xy} + p^{yx} = q^{yy} + \nu \cdot G$$

what defines $W_1^{\mathcal{C}}$ together with the top level equations of dynamics with no Lagrange multiplier.

From here, we may compute also the constrained Euler-Lagrange equations (4.46) for this problem, which are

$$\begin{aligned} 2\partial_{tx}^2\lambda_t + \partial_{xx}^2\lambda_x + 2\partial_{xy}^2\lambda_y - \partial_x\lambda &= \partial_{xx}^2\nu \cdot F + 2\partial_x\nu \cdot \mathrm{d}_x F + \nu \cdot \mathrm{d}_{xx}^2 F + \\ &\quad + \partial_{yy}^2\nu \cdot F + 2\partial_y\nu \cdot \mathrm{d}_y F + \nu \cdot \mathrm{d}_{yy}^2 F - \\ &\quad - \mathrm{d}_t F - u \cdot \mathrm{d}_x F - v \cdot \mathrm{d}_y F - v_y \cdot F + v_x \cdot G \\ 2\partial_{ty}^2\lambda_t + 2\partial_{xy}^2\lambda_x + \partial_{yy}^2\lambda_y - \partial_y\lambda &= \partial_{xx}^2\nu \cdot G + 2\partial_x\nu \cdot \mathrm{d}_x G + \nu \cdot \mathrm{d}_{xx}^2 G + \\ &\quad + \partial_{yy}^2\nu \cdot G + 2\partial_y\nu \cdot \mathrm{d}_y G + \nu \cdot \mathrm{d}_{yy}^2 G - \\ &\quad - \mathrm{d}_t G - u \cdot \mathrm{d}_x G - v \cdot \mathrm{d}_y G - u_x \cdot G + u_y \cdot F \end{aligned}$$

As before, the Lagrangian is not regular along $W_2^{\mathcal{C}}$, what seems to be clear if we observe that L is highly non-degenerate: It depends only on 4 of the 12 coordinates of second order. It is worthless to show its "Hessian", even though it is interesting to say that it is null only when ν is.

4.2.5 Hamilton-Pontryagin Principle

We next show how the higher-order Euler-Lagrange equations for unconstrained systems can be derived from a Hamilton-Pontryagin principle (see [156]).

Definition 4.62. Let $\mathcal{L} : J^k\pi \longrightarrow \Lambda^m M$ be a Lagrangian density. The associated *(extended) Hamiltonian-Pontryagin action* is the map $\mathcal{A}_{\mathcal{L}} : \Gamma\pi_{W,M} \times \mathcal{K} \to \mathbb{R}$ given by

$$\mathcal{A}_{\mathcal{L}}(\sigma, R) := \int_R \mathcal{L} \circ \sigma_k + \left\langle \sigma_k^\dagger, j^1\sigma_{k-1} \right\rangle - \left\langle \sigma_k^\dagger, \sigma_k \right\rangle \tag{4.156}$$

where $\mathcal{K}$ is the collection of smooth compact regions of M.

Theorem 4.63. *A section $\sigma : M \to W$ of $\pi_{W,M} : W \to M$ is a critical point of the Hamiltonian-Pontryagin action $\mathcal{A}_{\mathcal{L}}$ if and only if σ_k is holonomic, being $\sigma_k = j^k\sigma_0$, and σ satisfies the local equations*

$$\begin{aligned} 0 &= \frac{\partial L}{\partial u^\alpha} - \frac{\partial \sigma_\alpha^{\ j}}{\partial x^j}; && (4.157)\\ \sum_{I+1_i=J} \sigma_\alpha^{Ii} &= \frac{\partial L}{\partial u_J^\alpha} - \frac{\partial \sigma_\alpha^{Jj}}{\partial x^j}, \ \textit{with } |J| = 1, \dots, k-1; && (4.158)\\ \sum_{I+1_i=K} \sigma_\alpha^{Ii} &= \frac{\partial L}{\partial u_K^\alpha}, \ \textit{with } |K| = k. && (4.159) \end{aligned}$$

on M, and

$$\sigma_\alpha^{Ii} = 0, \ \textit{with } |I| = 0, \dots, k-1. \tag{4.160}$$

on the boundary ∂M of M, where $(x^i, u_J^\alpha, p, p_\alpha^{Ii})$ denotes adapted coordinates on W and $\sigma = (x^i, \sigma_J^\alpha, \tilde{\sigma}, \sigma_\alpha^{Ii})$.

Proof. Given a section $\sigma \in \Gamma\pi_{W,M}$ and a compact region $R \subseteq M$, we have that the variation of the Hamiltonian-Pontryagin action $\mathcal{A}_{\mathcal{L}}$ with respect to a vertical variation $\delta\sigma$ of σ is given by

$$\begin{aligned} \left.\frac{\delta \mathcal{A}_{\mathcal{L}}}{\delta\sigma}\right|_{(\sigma,R)} \cdot \delta\sigma &= \int_R \frac{\delta}{\delta\sigma}\left[L(x^i, \sigma_J^\alpha) + \sigma_\alpha^{Ii}\left(\frac{\partial \sigma_I^\alpha}{\partial x^i} - \sigma_{I+1_i}^\alpha\right)\right]\Bigg|_\sigma \cdot \delta\sigma \, \mathrm{d}^m x \\ &= \int_R \left[\frac{\partial L}{\partial u_J^\alpha}\delta\sigma_J^\alpha + \delta\sigma_\alpha^{Ii}\left(\frac{\partial \sigma_I^\alpha}{\partial x^i} - \sigma_{I+1_i}^\alpha\right) + \sigma_\alpha^{Ii}\left(\frac{\partial}{\partial x^i}\delta\sigma_I^\alpha - \delta\sigma_{I+1_i}^\alpha\right)\right] \mathrm{d}^m x \\ &= \int_R \left[\frac{\partial L}{\partial u_J^\alpha}\delta\sigma_J^\alpha + \delta\sigma_\alpha^{Ii}\left(\frac{\partial \sigma_I^\alpha}{\partial x^i} - \sigma_{I+1_i}^\alpha\right) - \frac{\partial \sigma_\alpha^{Ii}}{\partial x^i}\delta\sigma_I^\alpha - \sigma_\alpha^{Ii}\delta\sigma_{I+1_i}^\alpha\right] \mathrm{d}^m x \\ &\quad + \int_{\partial R} \sigma_\alpha^{Ii}\delta\sigma_I^\alpha \, \mathrm{d}^{m-1}x_i \\ &= \int_R \Bigg[\left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial \sigma_\alpha^{\ j}}{\partial x^j}\right)\delta\sigma^\alpha + \sum_{|J|=1}^{k-1}\left(\frac{\partial L}{\partial u_J^\alpha} - \frac{\partial \sigma_\alpha^{Jj}}{\partial x^j} - \sum_{I+1_i=J}\sigma_\alpha^{Ii}\right)\delta\sigma_J^\alpha \\ &\qquad + \sum_{|K|=k}\left(\frac{\partial L}{\partial u_K^\alpha} - \sum_{I+1_i=K}\sigma_\alpha^{Ii}\right)\delta\sigma_K^\alpha + \left(\frac{\partial \sigma_I^\alpha}{\partial x^i} - \sigma_{I+1_i}^\alpha\right)\delta\sigma_\alpha^{Ii}\Bigg] \mathrm{d}^m x \\ &\quad + \int_{\partial R} \sigma_\alpha^{Ii}\delta\sigma_I^\alpha \, \mathrm{d}^{m-1}x_i \end{aligned}$$

where $(x^i, u^\alpha_J, p, p^{Ii}_\alpha)$ denotes adapted coordinates on W and $\sigma = (x^i, \sigma^\alpha_J, \tilde{\sigma}, \sigma^{Ii}_\alpha)$. We thus infer that σ is a critical point of $\mathcal{A}_\mathcal{L}$, *i.e.* $\delta\mathcal{A}_\mathcal{L}/\delta\sigma = 0$, if and only if the relations (4.157–4.160) are satisfied and $\sigma_k = j^k\sigma_0$, what is derived from the last term of the first integrand. □

4.2.6 The space of symmetric multimomenta

Lemma 4.64. *Let (x^i, u^α) and (y^j, v^β) be adapted coordinates on E, whose domains have a non-empty intersection, and let $(x^i, u^\alpha_I, p, p^{Ii}_\alpha)$ and $(y^j, v^\beta_J, q, q^{Jj}_\beta)$ be the corresponding induced coordinates on the space of forms $\Lambda^m_2 J^k\pi$, where $0 \leq |I|, |J| \leq k$.*

1. *For any pair of multi-indexes $I, J \in \mathbb{N}^m$ of length k and any pair of indexes $1 \leq \alpha, \beta \leq n$, the following holds:*

$$\frac{\partial v^\beta_J}{\partial u^\alpha_I} = \sum_{\pi\in\Sigma_k} \frac{1}{I!}\frac{\partial v^\beta}{\partial u^\alpha}\frac{\partial x^{i_{\pi(1)}}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_{\pi(k)}}}{\partial y^{j_k}}, \tag{4.161}$$

where Σ_k denotes the collection of permutations π of k elements and the indexes $1 \leq i_1, \ldots, i_k \leq m$ and $1 \leq j_1, \ldots, j_k \leq m$ are such that $I = 1_{i_1} + \cdots + 1_{i_k}$ and $J = 1_{j_1} + \cdots + 1_{j_k}$.

2. *For any multi-index $I \in \mathbb{N}^m$ of length k and any indexes $1 \leq \alpha \leq n$ (and $1 \leq \beta \leq n$), the following holds:*

$$\sum_{|J|=k} \frac{\partial v^\beta_J}{\partial u^\alpha_I} q^{Jj}_\beta = \sum_{j_1,\ldots,j_k} \frac{J_k!}{I!}\frac{\partial v^\beta}{\partial u^\alpha}\frac{\partial x^{i_1}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_k}}{\partial y^{j_k}} q^{J_k j}_\beta, \tag{4.162}$$

where $J_k := 1_{j_1} + \cdots + 1_{j_k}$ and the indexes $1 \leq i_1, \ldots, i_k \leq m$ are such that $I = 1_{i_1} + \cdots + 1_{i_k}$.

Proof. The first equation is proven by induction on k. The case $k = 0$ is trivial thus, let us suppose that the result is true for $k-1 \geq 0$ and show that it is also true for k. Thanks to Equation (4.4) and the identity (A.6), we may write

$$\begin{aligned}
\frac{\partial v^\beta_J}{\partial u^\alpha_I} &= \sum_{S+1_s=I} \frac{\partial v^\beta_{J_{k-1}}}{\partial u^\alpha_S}\frac{\partial x^s}{\partial y^{j_k}} \\
&= \sum_{s=1}^k \frac{1}{I(i_s)}\frac{\partial v^\beta_{J_{k-1}}}{\partial u^\alpha_{I_{\hat{s}}}}\frac{\partial x^{i_s}}{\partial y^{j_k}},
\end{aligned}$$

where we have used the fact that v^β_J only depends on u^α_I's of order $|I| \geq |J|$, which is in this case on u^α_I's of order k. We therefore have by the hypothesis of induction

$$\begin{aligned}
\frac{\partial v^\beta_J}{\partial u^\alpha_I} &= \sum_{s=1}^k \sum_{\pi\in\Sigma_{k-1}} \frac{1}{I(i_s)}\frac{1}{I_{\hat{s}}!}\frac{\partial v^\beta}{\partial u^\alpha}\frac{\partial x^{i_{s_{\pi(1)}}}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_{s_{\pi(k-1)}}}}{\partial y^{j_{k-1}}}\frac{\partial x^{i_s}}{\partial y^{j_k}} \\
&= \sum_{\pi\in\Sigma_k} \frac{1}{I!}\frac{\partial v^\beta}{\partial u^\alpha}\frac{\partial x^{i_{\pi(1)}}}{\partial y^{j_1}}\cdots\frac{\partial x^{i_{\pi(k)}}}{\partial y^{j_k}}.
\end{aligned}$$

The second statement is easily proved using the first one.

$$\begin{aligned}\sum_{|J|=k} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} &= \sum_{j_1,\dots,j_k} \frac{J_k!}{k!} \frac{\partial v_{J_k}^\beta}{\partial u_I^\alpha} q_\beta^{J_k j} \\ &= \sum_{\pi\in\Sigma_k} \sum_{j_1,\dots,j_k} \frac{J_k!}{k!} \frac{1}{I!} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_{\pi(1)}}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_{\pi(k)}}}{\partial y^{j_k}} q_\beta^{J_k j}.\end{aligned}$$

Now, for each permutation $\pi \in \Sigma_k$, we relabel the indexes j_s in such a way its subindexes s coincide with those of i_s, *i.e.*

$$\begin{aligned}\sum_{|J|=k} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} &= \sum_{\pi\in\Sigma_k} \sum_{j_1,\dots,j_k} \frac{J_k!}{k!} \frac{1}{I!} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_{\pi(1)}}}{\partial y^{j_{\pi(1)}}} \cdots \frac{\partial x^{i_{\pi(k)}}}{\partial y^{j_{\pi(k)}}} q_\beta^{J_k j} \\ &= \sum_{\pi\in\Sigma_k} \sum_{j_1,\dots,j_k} \frac{J_k!}{k!} \frac{1}{I!} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} q_\beta^{J_k j} \\ &= \sum_{j_1,\dots,j_k} \frac{J_k!}{k!} \frac{k!}{I!} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} q_\beta^{J_k j}.\end{aligned}$$

Note that in any moment J_k is affected by the relabelling. □

Theorem 4.65. *Let $(x^i, u_I^\alpha, p, p_\alpha^{Ii})$ be an adapted system of coordinates on $\Lambda_2^m J^k\pi$. The relation*

$$I! \cdot p_\alpha^{Ii} = I'! \cdot p_\alpha^{I'i'}, \text{ whenever } I + 1_i = I' + 1_{i'} \text{ and } |I| = |I'| = k, \tag{4.163}$$

is invariant under change of coordinates.

Proof. Consider adapted coordinates $(x^i, u_I^\alpha, p, p_\alpha^{Ii})$ and $(y^j, v_J^\beta, q, q_\beta^{Jj})$ on $\Lambda_2^m J^k\pi$, where $0 \leq |I|, |J| \leq k$, whose domains have a non-empty intersection. Let p_α^{Ii} a fixed coordinate where $I \in \mathbb{N}^m$ is a multi-index of length k, there must be k integers $1 \leq i_1, \dots, i_k \leq m$ such that we have the decomposition $I = 1_{i_1} + \cdots + 1_{i_k}$. Using the dual coordinate change formula (4.29) and the previous Lemma 4.64, we obtain

$$\begin{aligned}\mathrm{Jac}(x(y)) p_\alpha^{Ii} &= \sum_j \sum_{|J|=k} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} \\ &= \sum_j \sum_{j_1,\dots,j_k} \frac{J_k!}{I!} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{j_k}} q_\beta^{J_k j}.\end{aligned}$$

Let (I, i), (I', i') such that $I + 1_i = I' + 1_{i'}$ and $|I| = |I'| = k$. Then $I = \tilde{I} + 1_{i'}$ and $I' = \tilde{I} + 1_i$, for some multi-index $\tilde{I}$ of length $|\tilde{I}| = k - 1$. If $I! \cdot p_\alpha^{Ii} = I! \cdot p_\alpha^{I'i'}$, by the previous reasoning, we do have

$$\begin{aligned}&\sum_{j_1,\dots,j_{k+1}} J_k! \cdot \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_{k-1}}}{\partial y^{j_{k-1}}} \frac{\partial x^{i'}}{\partial y^{j_k}} \frac{\partial x^{i}}{\partial y^{j_{k+1}}} q_\beta^{J_k j_{k+1}} = \\ &\qquad = \sum_{j'_1,\dots,j'_{k+1}} J'_k! \cdot \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i_1}}{\partial y^{j'_1}} \cdots \frac{\partial x^{i_{k-1}}}{\partial y^{j'_{k-1}}} \frac{\partial x^{i}}{\partial y^{j'_k}} \frac{\partial x^{i'}}{\partial y^{j'_{k+1}}} q_\beta^{J'_k j_{k+1}}\end{aligned}$$

Note that $(\partial v^\beta/\partial u^\alpha)$ is regular and $\partial v^\beta/\partial u^\alpha \cdot \partial u^\alpha/\partial v^{\beta'} = \delta^\beta_{\beta'}$, thus

$$\sum_{j_1,\dots,j_{k+1}} J_k(j_k)J_{k-1}! \cdot \frac{\partial x^{i_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{i_{k-1}}}{\partial y^{j_{k-1}}} \frac{\partial x^{i'}}{\partial y^{j_k}} \frac{\partial x^{i}}{\partial y^{j_{k+1}}} q_\beta^{J_k j_{k+1}} =$$
$$= \sum_{j'_1,\dots,j'_{k+1}} J'_k(j'_k)J'_{k-1}! \cdot \frac{\partial x^{i_1}}{\partial y^{j'_1}} \cdots \frac{\partial x^{i_{k-1}}}{\partial y^{j'_{k-1}}} \frac{\partial x^{i}}{\partial y^{j'_k}} \frac{\partial x^{i'}}{\partial y^{j'_{k+1}}} q_\beta^{J'_k j_{k+1}}$$

As $(\partial x^i/\partial y^j)$ is also regular, we have

$$\sum_{j_k,j_{k+1}} (J_{k-1}(j_k)+1)J_{k-1}! \cdot \frac{\partial x^{i'}}{\partial y^{j_k}} q_\beta^{J_k j_{k+1}} \frac{\partial x^{i}}{\partial y^{j_{k+1}}} =$$
$$= \sum_{j'_k,j'_{k+1}} (J_{k-1}(j'_k)+1)J_{k-1}! \cdot \frac{\partial x^{i}}{\partial y^{j'_k}} q_\beta^{J'_k j_{k+1}} \frac{\partial x^{i'}}{\partial y^{j'_{k+1}}},$$

thus

$$(J_{k-1}(j')+1)J_{k-1}! q_\beta^{J_{k-1}+1_{j'}j} = (J_{k-1}(j)+1)J_{k-1}! q_\beta^{J_{k-1}+1_j j'}.$$

Which is equivalent to

$$J! q_\beta^{Jj} = J'! q_\beta^{J'j'},$$

whenever $J+1_j = J'+1_j$ and $|J| = |J'| = k$. □

Corollary 4.66. *The* space of $(k+1)$-symmetric multimomenta

$$J^{k+1}\pi^\ddagger := \{\omega \in \Lambda^m_2 J^k\pi \ : \ I! \cdot p_\alpha^{Ii} = I'! \cdot p_\alpha^{I'i'},\ I+1_i = I'+1_{i'},\ |I| = |I'| = k\} \quad (4.164)$$

is an embedded submanifold of $J^{k+1}\pi^\dagger$*. A system of adapted coordinates* (x^i, u^α) *on* E *induces coordinates* $(x^i, u^\alpha_I, p, p_\alpha^{I'i}, p_\alpha^K)$ *on* $J^{k+1}\pi^\ddagger$*, where* $0 \le |I'| < |I| \le |k|$ *and* $|K| = k+1$*. The natural embedding* $J^{k+1}\pi^\ddagger \hookrightarrow J^{k+1}\pi^\dagger$ *is then given in coordinates by* $p_\alpha^{Ii} = p_\alpha^{I+1_i}/(I(i)+1)$*, for* $|I| = k$*. This manifold is transverse to* $\pi^\dagger_{k+1}$ *and therefore fibers over* $J^k\pi$*.*

Remark 4.67. For the second order case, there is an intrinsic definition of this space that involves the use of the semi-holonomic jets (see Definition 4.26) and which was presented by Saunders and Crampin in [140]. Moreover, if one considers the equivalence class of affine maps in $J^1\pi_k \to J^k\pi$ that are equal along the fibers of $J^{k+1}\pi \to J^k\pi$, then $J^{k+1}\pi^\ddagger$ is a distinguished section of the resultant fibration $J^{k+1}\pi^\dagger \to J^{k+1}\pi^\dagger/\sim$.

Note that the k-symmetric multimomenta space $J^k\pi^\ddagger$ coincides with the whole dual $J^k\pi^\dagger$ whenever we are considering a first order theory ($k=1$) or a unidimensional one ($m=1$). Thus, in the forthcoming discussion we may assume that we are not in any of these cases ($k, m \ge 2$).

Remark 4.68. Unfortunately, the restriction $I! \cdot p_\alpha^{Ii} = I'! \cdot p_\alpha^{I'i'}$, $I+1_i = I'+1_{i'}$, is no longer invariant under a change of coordinates when $|I| = |I'| < k$. For instance, if $|I| = k-1$, we have that

$$\begin{aligned}
\mathrm{Jac}(x(y)) p_\alpha^{Ii} &= \sum_{|J|=k-1}^{k} \frac{\partial v^\beta_J}{\partial u^\alpha_I} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} \\
&= \sum_{|J|=k-1} \frac{\partial v^\beta_J}{\partial u^\alpha_I} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} + \sum_{|J|=k} \frac{\partial v^\beta_J}{\partial u^\alpha_I} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j}.
\end{aligned}$$

The first term will be easily expandable to the form (4.162) and, following the proof of Theorem 4.65, it is invariant. However, this is not true for the second term, which depends on the chosen coordinates (see Example 4.70 below).

Example 4.69 (Second order case). Consider the dual space $\Lambda_2^m J^1\pi$ of $J^2\pi$ and let (x^i, u^α, u_i^α, p, p_α^i, p^{Ii}) and $(y^j, v^\beta, v_j^\beta, q, q_\beta^j, q_\beta^{Jj})$ denote adapted coordinates on it. As the multi-indexes I and J have unitary length, we may view them as a regular indexes. In this case, the higher momenta transform accordingly to

$$\begin{aligned}
\mathrm{Jac}(x(y))p_\alpha^{Ii} &= \sum_{|J|=1} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} \\
&= \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^I}{\partial y^J} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j}.
\end{aligned}$$

Indeed, the relation (4.163) that defines the space of 2-symmetric multimomenta is invariant,

$$p_\alpha^{Ii} = p_\alpha^{iI} \implies q_\beta^{Jj} = q_\beta^{jJ},$$

as stated by Theorem 4.65.

Example 4.70 (Third order case). Consider the dual space $\Lambda_2^m J^2\pi$ of $J^3\pi$ and consider the induced coordinates from adapted ones (x^i, u^α) and (y^j, v^β) on E. Consider a fixed multimomentum coordinate p_α^{Ii} where I has length $|I| = 2$. If we assume that $I = 1_{i''} + 1_{i'}$, then the change of coordinates (4.30) reads

$$\begin{aligned}
\mathrm{Jac}(x(y))p_\alpha^{1_{i''}+1_{i'},i} &= \sum_j \sum_{|J|=2} \frac{\partial v_J^\beta}{\partial u_{1_{i''}+1_{i'}}^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} \\
&= \sum_{j,j',j''} \frac{\delta_{j''}^{j'}+1}{\delta_{i''}^{i'}+1} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i''}}{\partial y^{j''}} \frac{\partial x^{i'}}{\partial y^{j'}} q_\beta^{1_{j''}+1_{j'},j} \frac{\partial x^i}{\partial y^j}.
\end{aligned}$$

Which proof that the relation $I! \cdot p_\alpha^{Ii} = I'! \cdot p_\alpha^{I'i'}$, for $I + 1_i = I' + 1_{i'}$, is invariant whenever $|I| = |I'| = 2$.

Let I now denote a multi-index of length $|I| = 1$. In this case, the rule (4.30) is written

$$\mathrm{Jac}(x(y))p_\alpha^{Ii} = \sum_{|J|=1} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} + \sum_{|J|=2} \frac{\partial v_J^\beta}{\partial u_I^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j}.$$

The first term, may be treated as in the previous example 4.69. The second term is

$$\sum_{|J|=2} \frac{\partial v_J^\beta}{\partial u_{i'}^\alpha} q_\beta^{Jj} \frac{\partial x^i}{\partial y^j} = \sum_{j,j',j''} \left[\frac{\mathrm{d}}{\mathrm{d}x^{i''}} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial x^{i''}}{\partial y^{j''}} \frac{\partial x^{i'}}{\partial y^{j'}} \frac{\partial x^i}{\partial y^j} + \frac{1}{2} \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial^2 x^{i''}}{\partial y^{j''} \partial y^{j'}} \frac{\partial x^i}{\partial y^j} \right] q_\beta^{J_3}.$$

From here, we see that the relation $p_\alpha^{i'i} = p_\alpha^{ii'}$ fails to be coordinate independent in $\Lambda_2^m J^2\pi$, while it is in $\Lambda_2^m J^1\pi$ (see Example 4.69 and Remark 4.68 above).

Proposition 4.71. *Assume that $k, m \geq 2$ and consider the pullback Ω^s of the canonical multisymplectic form Ω of $\Lambda_2^m J^k\pi$ to the space of $(k+1)$-symmetric multimomenta. We have that Ω^s is still multisymplectic.*

Proof. From the local description (4.34) of Ω, we have that

$$\begin{aligned}\Omega^s &= -\mathrm{d}p\wedge \mathrm{d}^m x - \sum_{|I|=0}^{k-1} \mathrm{d}p_\alpha^{Ii}\wedge \mathrm{d}u_I^\alpha \wedge \mathrm{d}^{m-1}x_i - \sum_{|K|=k} \mathrm{d}p_\alpha^{K+1_i}\wedge \mathrm{d}u_K^\alpha\wedge \mathrm{d}^{m-1}x_i \qquad (4.165)\\ &= -\mathrm{d}p\wedge \mathrm{d}^m x - \sum_{|I|=0}^{k-1} \mathrm{d}p_\alpha^{Ii}\wedge \mathrm{d}u_I^\alpha \wedge \mathrm{d}^{m-1}x_i - \sum_{|K_+|=k+1} \mathrm{d}p_\alpha^{K_+}\wedge \sum_{K+1_i=K_+} \mathrm{d}u_K^\alpha\wedge \mathrm{d}^{m-1}x_i.\end{aligned}$$

Let $V\in TJ^{k+1}\pi^\ddagger$ be of the form

$$V=\gamma^i\frac{\partial}{\partial x^i}+A_J^\alpha\frac{\partial}{\partial u_J^\alpha}+B_\alpha^{Ii}\frac{\partial}{\partial p_\alpha^{Ii}}+B_\alpha^{K_+}\frac{\partial}{\partial p_\alpha^{K_+}}+C\frac{\partial}{\partial p},$$

then

$$\begin{aligned}i_V\Omega^s = -C\,\mathrm{d}^m x - \sum_{|I|=0}^{k-1} B_\alpha^{Ii}\,\mathrm{d}u_I^\alpha\wedge \mathrm{d}^{m-1}x_i + \sum_{|I|=0}^{k-1} A_I^\alpha\,\mathrm{d}p_\alpha^{Ii}\wedge \mathrm{d}^{m-1}x_i\\ - \sum_{|K|=k} B_\alpha^{K+1_i}\,\mathrm{d}u_K^\alpha\wedge \mathrm{d}^{m-1}x_i + \sum_{|K_+|=k+1}\mathrm{d}p_\alpha^{K_+}\wedge\sum_{K+1_i=K_+} A_K^\alpha\,\mathrm{d}^{m-1}x_i.\end{aligned}$$

We deduce from this expression that Ω^s has a trivial kernel ($i_V\Omega^s=0$ *iff* $V=0$), thus Ω^s is multisymplectic. $\square$

This result turns to be trivial for a first order theory or a unidimensional one since, as stated earlier, in either cases the space of symmetric multimomenta coincides with the whole dual space.

Symmetric multimomenta constraints within the Skinner-Rusk formalism

We are now in disposition to introduce the k-symmetric multimomenta within the Skinner-Rusk formalism. Two options are available here: First, we could consider the fibered product $J^k\pi\times_{J^{k-1}\pi}J^k\pi^\ddagger$ and work directly there following the schema of the Skinner-Rusk formalism presented in Section §4.2.3; The second option is to mimic the constrained version of it, presented in Section §4.2.4, but considering the k-symmetric multimomenta constraints in $J^k\pi^\dagger$ instead of an arbitrary constraint submanifold of $J^k\pi$. We will stick to the latter.

Let $W=J^k\pi\times_{J^{k-1}\pi}J^k\pi^\dagger$ be the mixed space of velocities and momenta and $W^s=J^k\pi\times_{J^{k-1}\pi}J^k\pi^\ddagger=\pi_2^{-1}(J^k\pi^\ddagger)$ be the mixed space of velocities and k-symmetric multimomenta. There is a natural embedding $W^s\hookrightarrow W$ which is described in coordinates by $p_\alpha^{Ii}=p_\alpha^{I+1i}/(I(i)+1)$, where $|I|=k-1$ (see Corollary 4.66). Therefore W^s is defined by the constraints $I!p_\alpha^{Ii}=I'p_\alpha^{I'i'}$, where $I+1_i=I'+1_{i'}$ and $|I|=|I'|=k-1$. As usual, we consider in addition the constraint $\mathcal{H}=0$ that defines the Hamiltonian submanifold W_0 of W. Thus, we will work on $W_0^s=W^s\cap W_0$ rather than on W^s. If $\Omega_{\mathcal{H}}=\Omega+\mathrm{d}\mathcal{H}$ denotes the Cartan $(m+1)$-form of W associated to a Lagrangian density $\mathcal{L}:J^k\pi\to\Lambda^m M$, we write $\Omega_{\mathcal{H}}^s$ and Ω_0^s for their pullbacks to W^s and W_0^s, respectively.

In order to be able to use the free coordinates of W, we use again the Proposition 4.56 (more precisely, an adaptation of it) that establishes that the dynamical equation in terms of multivectors

$$i_X\Omega_0^s=0,\quad X\in\mathfrak{X}_d^m(TW_0^s),$$

is equivalent to

$$i_{i_* X}\Omega_{\mathcal{H}} \in T^0 W_0^s, \quad X \in \mathfrak{X}^m_d(TW_0^s),$$

where $T^0 W_0^s$ is the annihilator of $i_*(TW_0^s)$ in TW. To write down in coordinates the last equation, we first have to describe properly de set of constraints. For each multi-index K of length k, we fix a pair (I_K, i_K) where I_K is a multi-index of length $k-1$ and $1 \leq i_k \leq m$ such that $I_K + 1_{i_K} = K$. The set of constraints is $I! \cdot p^{Ii}_\alpha = (I_{I+1_i})! \cdot p_\alpha^{I_{I+1_i} i_{I+1_i}}$, for arbitrary pairs (I, i) where I is a multi-index of length $k-1$ and $1 \leq i \leq m$. Note that in this set, for each multi-index K of length k, there is a trivial identity for the fixed pair (I_K, i_k). We therefore look for solutions of the *dynamical equation*

$$(-1)^m i_X \Omega_{\mathcal{H}} = \sum_{\substack{(I,i)\neq(I_{I+1_i}, i_{I+1_i})\\ |I|=k-1}} \lambda^\alpha_{Ii}\left(I! \cdot \mathrm{d}p^{Ii}_\alpha - (I_{I+1_i})! \cdot \mathrm{d}p_\alpha^{I_{I+1_i} i_{I+1_i}}\right) + \lambda\, \mathrm{d}H, \tag{4.166}$$

where X is a multivector field tangent along W_0^s and the λ's are Lagrange multipliers to be determined.

If we assume that the locally decomposable m-multivector fields $X \in \mathfrak{X}(W)$ have the form

$$X = X_1 \wedge \cdots \wedge X_m = \frac{\partial}{\partial x^j} + A^\alpha_{Jj}\frac{\partial}{\partial u^\alpha_J} + B^{Ii}_{\alpha j}\frac{\partial}{\partial p^{Ii}_\alpha} + C_j \frac{\partial}{\partial p}, \tag{4.167}$$

expanding the first member in local coordinates and equating coefficients, we obtain:

$$\begin{aligned}
&\text{coeffs. in } \mathrm{d}p: & 0 =& \lambda;\\
&\text{coeffs. in } \mathrm{d}p^{Ii}_\alpha: & A^\alpha_{Ii} =& u^\alpha_{I+1_i}, \quad |I| = 0, \ldots, k-2;\\
&& A^\alpha_{Ii} + I! \cdot \lambda^\alpha_{Ii} =& u^\alpha_{I+1_i}, \quad |I| = k-1, \ (I,i) \neq (I_{I+1_i}, i_{I+1_i});\\
&& A^\alpha_{I_K i_K} - \sum_{\substack{(I,i)\neq(I_K,i_K)\\ I+1_i=K}} I_K! \cdot \lambda^\alpha_{Ii} =& u^\alpha_K, \quad |K| = k;\\
&\text{coeffs. in } \mathrm{d}u^\alpha_J: & B_{\alpha i}^{\ i} =& \frac{\partial L}{\partial u^\alpha};\\
&& B^{Ii}_{\alpha i} =& \frac{\partial L}{\partial u^\alpha_I} - \sum_{J+1_j=I} p^{Jj}_\alpha, \quad |I| = 1, \ldots, k-1;\\
&& 0 =& \frac{\partial L}{\partial u^\alpha_K} - \sum_{J+1_j=K} p^{Jj}_\alpha, \quad |K| = k;\\
&\text{coeffs. in } \mathrm{d}x^j: & A^\alpha_{Ii}B^{Ii}_{\alpha j} - A^\alpha_{Ij}B^{Ii}_{\alpha i} =& \left(\frac{\partial L}{\partial u^\alpha_J} - \sum_{I+1_i=J} p^{Ii}_\alpha\right) A^\alpha_{Jj} + u^\alpha_{I+1_i} B^{Ii}_{\alpha j}.
\end{aligned}$$

To get rid off the Lagrange multipliers that appear in the equations coming from the coefficients of $\mathrm{d}p^{Ii}_\alpha$ with $|I| = k-1$, we multiply the corresponding equations by $I(i)+1/|I|+1$ and sum over $I + 1_i = K$. Besides, the last equation turns to be null thanks to the other ones and the tangency equations (see below) which come from the k-symmetric restriction of the multimomenta. So, we do have

$$A^\alpha_{Ii} = u^\alpha_{I+1_i}, \text{ with } |I| = 0, \dots, k-2,\ i = 1, \dots, m; \tag{4.168}$$

$$\sum_{I+1_i=K} \frac{I(i)+1}{|I|+1} A^\alpha_{Ii} = u^\alpha_K, \text{ with } |K| = k; \tag{4.169}$$

$$0 = \frac{\partial L}{\partial u^\alpha} - B^{\ j}_{\alpha j}; \tag{4.170}$$

$$\sum_{I+1_i=J} p^{I,i}_\alpha = \frac{\partial L}{\partial u^\alpha_J} - B^{Jj}_{\alpha j}, \text{ with } |J| = 1, \dots, k-1; \tag{4.171}$$

$$\sum_{I+1_i=K} p^{I,i}_\alpha = \frac{\partial L}{\partial u^\alpha_K}, \text{ with } |K| = k. \tag{4.172}$$

Furthermore, we have the tangency conditions

$$I! \cdot B^{Ii}_{\alpha j} = I'! \cdot B^{I'i'}_{\alpha j},\ I+1_i = I'+1_{i'},\ |I| = |I'| = k-1; \tag{4.173}$$

$$\sum_{I+1_i=K} B^{Ii}_{\alpha j} = \frac{\partial^2 L}{\partial x^j \partial u^\alpha_K} + \sum_{|I|=0}^{k-1} A^\beta_{Ij} \frac{\partial^2 L}{\partial u^\beta_I \partial u^\alpha_K} + \sum_{|J|=k} A^\beta_{Jj} \frac{\partial^2 L}{\partial u^\beta_J \partial u^\alpha_K}; \tag{4.174}$$

$$C_j = \frac{\partial L}{\partial x^j} + A^\alpha_{Jj} \frac{\partial L}{\partial u^\alpha_J} - A^\alpha_{I+1_i j} p^{I,i}_\alpha - B^{Ii}_{\alpha j} u^\alpha_{I+1_i}. \tag{4.175}$$

with respect to the k-symmetry restriction, the equation (4.172) and the zero-level set of $\mathcal{H}$, respectively. Note that, the Lagrange multipliers are hidden in Equation (4.174) and (4.175) through the coefficients A^α_{Jj} of X of degree $k-1$.

Remark 4.72. We have obtained the same equations than in the free case, *cf.* equations (4.64–4.67), but with a slight difference in the highest order equations of holonomy (4.169). What does that imply? An integral section $\sigma \in \Gamma\pi_{W,M}$ of a solution X of the dynamical equation (4.166) will no longer be holonomic (at order k) as happens in the free case, *cf.* Proposition 4.50 and we will have to require it.

Proposition 4.73. *Given a solution $X \in \mathfrak{X}^m_d(i_*(TW^s_0))$ of the dynamical equation*

$$i_X \Omega_{\mathcal{H}} \in T^0 W^s_0,$$

let $\sigma \in \Gamma\pi_{W,M}$ be an integral section of X and denote its Lagrangian part $\sigma_k = pr_1 \circ \sigma$. If $j^1(\pi_{k,k-1} \circ \sigma_k) = \sigma_k$, then σ_k is holonomic, i.e. *$\sigma_k = j^k\phi$, and $\sigma_0 = \pi_{k,0} \circ \sigma_k$ satisfies the higher order Euler-Lagrange equations.*

Proof. The hypotesis $\sigma_k = j^1(\pi_{k,k-1} \circ \sigma_k)$ directly implies that σ is holonomic, *i.e.* $\sigma_k = j^k(\sigma_0)$ (and that the Lagrange multipliers are null along the image of σ). The rest of the proof is the same than the one of Proposition 4.50 (note that equations (4.170), (4.171) and (4.172) coincide with (4.65), (4.66) and (4.67)). □

This result ensures that, even with the addition of the k-symmetric multimomenta constraints, the holonomic integral sections of a solution of the dynamical equation are still solutions of the Euler-Lagrange equations. Furthermore, there is an improvement with respect to the free case, Section 4.2.3. If we consider the system of linear equations in terms of coefficients B's with multi-indexes of length $k-1$, the highest one, given by

Equation (4.171), (4.173) and (4.174), then the system is overdetermined in oposition to the free case (see Proposition 4.46). This is because now we have added the tangency condition with respect to the k-symmetry, Equation (4.173). If we put $B^{I+1_i}_{\alpha j} = B^{Ii}_{\alpha j}/(I(i)+1)$, for $|I| = k-1$, which is well defined thanks to (4.173), then Equation (4.171) and (4.174) is rewritten to

$$\sum_{I+1_i=J} p^{I,i}_\alpha = \frac{\partial L}{\partial u^\alpha_J} - B^{J+1_j}_{\alpha j}, \text{ with } |J| = k-1; \tag{4.176}$$

$$B^K_{\alpha j} = \frac{\partial^2 L}{\partial x^j \partial u^\alpha_K} + \sum_{|I|=0}^{k-1} A^\beta_{Ij} \frac{\partial^2 L}{\partial u^\beta_I \partial u^\alpha_K} + \sum_{|J|=k} A^\beta_{Jj} \frac{\partial^2 L}{\partial u^\beta_J \partial u^\alpha_K}, \text{ with } |K| = k \tag{4.177}$$

Now, the new unknowns $B^K_{\alpha j}$ are explecitely given in Equation (4.177). Thus, for fixed coefficients A^α_{Kj} with $|K| = k$, we may consider Equation (4.176) as an extra constraint on W. Tangency conditions on it will then give conditions on the B's of order $k-2$ but, since there are no $(k-1)$-symmetric constraints on the multimomenta (see Remark 4.68), we have to deal again with an undeterminacy on the coefficients of a solution X of the dynamical equation.

Let us recover some examples to clarify this.

Example 4.74 (First order Lagrangian as second order). In Example 4.74, we set up a first order Lagrangian $\mathcal{L} : J^1\pi \to \Lambda^m M$ as a second order one $\bar{\mathcal{L}} : J^2\pi \to \Lambda^m M$ by putting $\bar{\mathcal{L}} = \mathcal{L} \circ \pi_{2,1}$. We saw that, we cannot go pass the first constraint manifold even though $\bar{\mathcal{L}}$ is completely degenerate (in the second order sense). The space of solutions $\bar{X}$ of the second order dynamical equation is too big and the natural first order solutions cannot been determined from it since the system of linear equations of the coefficients $\bar{B}$ of $\bar{X}$ is underdetermined.

We considered the first and second order velocity-momenta mixed spaces $W^1 = J^1\pi \times_E J^1\pi^\dagger$ and $W^2 = J^2\pi \times_{J^1\pi} J^2\pi^\dagger$, together with the premultisymplectic forms $\Omega_{\mathcal{H}}$ and $\Omega_{\bar{\mathcal{H}}}$, where $\mathcal{H}$ and $\bar{\mathcal{H}}$ are the corresponding Hamiltonian functions associated to the Lagrangians $\mathcal{L}$ and $\bar{\mathcal{L}}$. Adapted coordinates are denoted (x^i, u, u_i, p, p^i) and $(x^i, u, u_i, u_K, p, p^i, p^{ij})$ (with $|K| = 2$) on W^1 and W^2, respectively. For the sake of simplicity, we assume that the fibers of $\pi : E \to M$ have dimension $n = 1$.

$$\begin{tikzcd}
W^2 \arrow[r] \arrow[d] \arrow[dr] & W^1 \arrow[d] \arrow[dr] & \\
J^2\pi \arrow[r, "\pi_{2,1}"] \arrow[dr, "\bar{\mathcal{L}}"] & J^1\pi \arrow[r] \arrow[d, "\mathcal{L}"] \arrow[dr, "\pi_1"] & E \arrow[d, "\pi"] \\
& \Lambda^m M & M
\end{tikzcd}$$

Figure 4.7: The 1st and 2nd order Lagrangian settings

If the multivector field $X \in \mathfrak{X}^m_d(W^1)$ and $\bar{X} \in \mathfrak{X}^m_d(W^2)$, solutions of the respective (free) dynamical equations $i_X\Omega_{\mathcal{H}} = 0$ and $i_{\bar{X}}\Omega_{\bar{\mathcal{H}}} = 0$, have the form

$$\begin{aligned} X &= \bigwedge_{j=1}^{m}\left(\frac{\partial}{\partial x^j} + A_j\frac{\partial}{\partial u} + A_{ij}\frac{\partial}{\partial u_i} + B^i_j\frac{\partial}{\partial p^i} + C_j\frac{\partial}{\partial p}\right), \\ \bar{X} &= \bigwedge_{j=1}^{m}\left(\frac{\partial}{\partial x^j} + \bar{A}_j\frac{\partial}{\partial u} + \bar{A}_{ij}\frac{\partial}{\partial u_i} + \bar{A}_{Kj}\frac{\partial}{\partial u_K} + \bar{B}^i_j\frac{\partial}{\partial p^i} + \bar{B}^{ki}_j\frac{\partial}{\partial p^{ki}} + \bar{C}_j\frac{\partial}{\partial p}\right), \end{aligned}$$

where $|K| = 2$. We then obtain the relations

$$\begin{aligned} A_i &= u_i, \\ 0 &= \frac{\partial L}{\partial u} - B^j_j, \\ p^i &= \frac{\partial L}{\partial u_i}, \end{aligned} \tag{4.178}$$

for $(W^1, \Omega_{\mathcal{H}}, X)$; and

$$\begin{aligned} \bar{A}_i &= u_i, \\ \bar{A}_{ij} &= u_{1_i+1_j}, \\ 0 &= \frac{\partial L}{\partial u} - \bar{B}^j_j, \\ p^i &= \frac{\partial L}{\partial u_i} - \bar{B}^{ij}_j, \\ p^{ij} + p^{ji} &= (1_i + 1_j)! \cdot \frac{\partial \bar{L}}{\partial u_{1_i+1_j}} = 0, \end{aligned} \tag{4.179}$$

for $(W^2, \Omega_{\bar{\mathcal{H}}}, \bar{X})$. Equations (4.178) and (4.179), together with $\mathcal{H} = 0$ and $\bar{\mathcal{H}} = 0$, define the corresponding submanifolds W^1_1 and W^2_1 of W^1 and W^2. The tangency condition to (4.179) is

$$\bar{B}^{ij}_k + \bar{B}^{ji}_k = 0,$$

which is not enough to overdetermine the $\bar{B}$'s of highest order.

We therefore introduce the 2-symmetric multimomentum constraint $p^{ij} = p^{ji}$ in W^2 and denote the resulting submanifold $W^{2,s}$. If we use adapted coordinates $(x^i, u, u_i, u_K, p, p^i, p^K)$ (with $|K| = 2$) on $W^{2,s}$, then the embedding is given by $p^{ij} = p^{1_i+1_j}$. Now, a solution $\bar{X} \in \mathfrak{X}^m_d(W^2)$ of $i_{\bar{X}}\Omega_{\bar{\mathcal{H}}} = 0$ along $W^{2,s}$ is governed by

$$\begin{aligned} \bar{A}_i &= u_i, \\ \bar{A}_{ij} + \bar{A}_{ji} &= u_{1_i+1_j}, \\ 0 &= \frac{\partial L}{\partial u} - \bar{B}^j_j, \\ p^i &= \frac{\partial L}{\partial u_i} - \bar{B}^{ij}_j, \\ p^{ij} + p^{ji} &= (1_i + 1_j)! \cdot \frac{\partial \bar{L}}{\partial u_{1_i+1_j}} = 0. \end{aligned}$$

This, together with the 2-symmetric multimomentum constraint $p^{ij} = p^{ji}$ and the tangency contitions

$$\bar{B}^{ij}_k + \bar{B}^{ji}_k = 0 \quad \text{and} \quad \bar{B}^{ij}_k = \bar{B}^{ji}_k,$$

reduces the previous system to

$$\begin{array}{rcl}
\bar{A}_i &=& u_i,\\
\bar{A}_{ij}+\bar{A}_{ji} &=& u_{1_i+1_j},\\
0 &=& \dfrac{\partial L}{\partial u}-\bar{B}_j^j,\\
p^i &=& \dfrac{\partial L}{\partial u_i},\\
p^{ij} &=& 0,\\
\bar{B}_k^{ij} &=& 0,
\end{array}$$

which is precisely the first order one (if we ignore the second order terms).

If we compare this example with Example 4.55), we have again that the Euler-Lagrange equations appear as a constraint at the second step of the reduction algorithm (combine the tangency condition to $p^i = \partial L/\partial u_i$ with $0 = \partial L/\partial u - \bar{B}_j^j$). But, in this case, and in contrast to the free setting, now it manages to detect if a second order Lagrangian is actually a first order one.

Example 4.75 (The second order case). Given a second order Lagrangian $\mathcal{L} : J^2\pi \to \Lambda^m M$, let $\mathcal{H} : W = J^2\pi \times_{J^1\pi} J^2\pi^\dagger \to \Lambda^m M$ be the associated Hamiltonian and let $\Omega_{\mathcal{H}} = \Omega - \mathrm{d}\mathcal{H}$ denote the Cartan $(m+1)$-form. If we consider the dynamical equation $i_X\Omega_{\mathcal{H}} = 0$ "in" the space of mixed velocities and 2-symmetric momenta W^s (defined by $p_\alpha^{ij} = p_\alpha^{ji}$) instead of "along" W^s, then a solution $X \in \mathfrak{X}_d^m(W^s)$ will be governed by the equations

$$\begin{array}{rcl}
A_i^\alpha &=& u_i,\\
A_{ij}^\alpha+A_{ji}^\alpha &=& u_{1_i+1_j}^\alpha,\\
0 &=& \dfrac{\partial L}{\partial u^\alpha}-B_{\alpha j}^{\ j},\\
p_\alpha^i &=& \dfrac{\partial L}{\partial u_i^\alpha}-B_{\alpha j}^{1_i+1_j},
\end{array} \tag{4.180}$$

$$p_\alpha^K = \frac{\partial L}{\partial u_K^\alpha},\ |K|=2, \tag{4.181}$$

where $(x^i, u, u_i, u_K, p, p^i, p^K)$, $|K| = 2$, denote adapted coordinates on W^s and X has the form

$$X = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + A_j^\alpha \frac{\partial}{\partial u^\alpha} + A_{ij}^\alpha \frac{\partial}{\partial u_i^\alpha} + A_{Kj}^\alpha \frac{\partial}{\partial u_K^\alpha} + B_{\alpha j}^{\ i} \frac{\partial}{\partial p_\alpha^i} + B_{\alpha j}^K \frac{\partial}{\partial p_\alpha^K} + C_j \frac{\partial}{\partial p} \right).$$

The tangency condition to Equation (4.181) explicitly gives the coefficients $B_{\alpha j}^K$, $|K| = 2$, of X. Thus, Equation (4.180) is a space constraint from which we may determine the coefficients $B_{\alpha j}^{\ i}$ of X. Moreover, if $\mathcal{L}$ was degenerate, then further constraints would be determined, so reducing the space of possible solutions.

So far, we have seen that the introduction of the k-symmetric momentum constraints not only removes the ambiguity in the simple case of a 1st order Lagrangian viewed from a 2nd order setting, but also the full general problem within the 2nd order setting. All this ambiguity was one of the reasons why it was not possible to define a Legendre transform nor a Poincaré-Cartan form in higher-order field theories, problem of furthermost importance. Having removed this ambiguity, is it possible now to define such objects?

Given a solution $X \in \mathfrak{X}^m_d(W^s)$ of the dynamical equation $i_X\Omega_{\mathcal{H}} = 0$, let $\sigma \in \Gamma\pi_{W^s,M}$ be a holonomic integral section of X, meaing that its Lagrangian part $\sigma_k = pr_1 \circ \sigma$ is holonomic. Then,

$$B^K_{\alpha j} \circ \sigma = \frac{\partial \sigma^K_\alpha}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial u^\alpha_K} \circ \sigma_k\right) = \left(\frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_K}\right) \circ j^1\sigma_k.$$

We therefore define the 2nd order (extended) Legendre transform as the fibered map $\mathrm{Leg}_{\mathcal{L}} : J^3\pi \to J^2\pi^\ddagger$ locally given by

$$p^K_\alpha = \frac{\partial L}{\partial u^\alpha_K}, \ |K| = 2, \tag{4.182}$$

$$p^i_\alpha = \frac{\partial L}{\partial u^\alpha_i} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}, \tag{4.183}$$

$$p = L - u^\alpha_i \cdot \left(\frac{\partial L}{\partial u^\alpha_i} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}\right) - u^\alpha_K \cdot \frac{\partial L}{\partial u^\alpha_K}. \tag{4.184}$$

The Poincaré-Cartan form is then the $(m+1)$-form $\Omega_{\mathcal{L}}$ along $\pi_{3,2}$ locally given by

$$\begin{aligned} \Omega_{\mathcal{L}} = &- \mathrm{d}\left(L - u^\alpha_i \cdot \left(\frac{\partial L}{\partial u^\alpha_i} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}\right) - u^\alpha_K \cdot \frac{\partial L}{\partial u^\alpha_K}\right) \wedge \mathrm{d}^m x \\ &- \mathrm{d}\left(\frac{\partial L}{\partial u^\alpha_i} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}\right) \wedge \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i \\ &- \mathrm{d}\left(\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}\right) \wedge \mathrm{d}u^\alpha_j \wedge \mathrm{d}^{m-1}x_i. \end{aligned} \tag{4.185}$$

For a similar approach, the paper [140] by Saunders and Crampin is strongly recommended.

Example 4.76 (The third order case). In this example, we are going to see that the improvements we got in the second order case by introducing the 2-symmetric momentum constraints are only partial for the third order case. We fix a third order Lagrangian $\mathcal{L} : J^3\pi \to \Lambda^m M$ and look for solutions $X \in \mathfrak{X}^m_d(W)$, where $W = J^3\pi \times_{J^2\pi} J^3\pi^\dagger$, of the dynamical equation $i_X\Omega_{\mathcal{H}} \in T^0W^s_0$, where W^s is the Hamiltonian mixed space of velocities and 3-symmetric momenta given by $\mathcal{H} = 0$ and $I! \cdot p^{Ii}_\alpha = J! \cdot p^{Jj}_\alpha$, for $I + 1_i = J + 1_j$ and $|I| = |J| = 2$. Recall that, as usual, we denote adapted coordinates on W by $(x^i, u^\alpha, u^\alpha_i, u^\alpha_I, u^\alpha_K, p, p^i_\alpha, p^{i'i}_\alpha, p^{Ii}_\alpha)$, with $|I| = 2$ and $|K| = 3$. Thus, we take coordinates $(x^i, u^\alpha, u^\alpha_i, u^\alpha_I, u^\alpha_K, p, p^i_\alpha, p^{i'i}_\alpha, p^K_\alpha)$, with $|I| = 2$ and $|K| = 3$, on W^s such that the embedding $W^s \hookrightarrow W$ is given by $p^{Ii}_\alpha = p^{I+1_i}_\alpha/(I(i)+1)$.

If $X \in \mathfrak{X}^m_d(W^s)$ has the form

$$\begin{aligned} X = \bigwedge_{j=1}^m \Bigg(&\frac{\partial}{\partial x^j} + A^\alpha_j \frac{\partial}{\partial u^\alpha} + A^\alpha_{ij}\frac{\partial}{\partial u^\alpha_i} + A^\alpha_{Ij}\frac{\partial}{\partial u^\alpha_K} + A^\alpha_{Kj}\frac{\partial}{\partial u^\alpha_K} \\ &+ B_{\alpha j}{}^{i}\frac{\partial}{\partial p^i_\alpha} + B^{i'i}_{\alpha j}\frac{\partial}{\partial p^{i'i}_\alpha} + B^{Ii}_{\alpha j}\frac{\partial}{\partial p^{Ii}_\alpha} + C_j\frac{\partial}{\partial p}\Bigg), \end{aligned}$$

in order to be a solution of the dynamical equations in W^s, its coefficients must satisfy the following relations

$$
\begin{aligned}
A_i^\alpha &= u_i, \\
A_{ij}^\alpha &= u^\alpha_{1_i+1_j}, \\
\sum_{J+1_j=I} A^\alpha_{Jj} &= u^\alpha_I, \ |I|=2, \\
0 &= \frac{\partial L}{\partial u^\alpha} - B^{\,j}_{\alpha j}, \\
p^i_\alpha &= \frac{\partial L}{\partial u^\alpha_i} - B^{ij}_{\alpha j}, \\
\sum_{1_{i'}+1_i=I} p^{i'i}_\alpha &= \frac{\partial L}{\partial u^\alpha_I} - B^{I+1_j}_{\alpha j}, \ |I|=2, \qquad (4.186) \\
p^K_\alpha &= \frac{\partial L}{\partial u^\alpha_K}, \ |K|=3. \qquad (4.187)
\end{aligned}
$$

Tangency conditions on Equation (4.187) gives explicitly all the coefficients $B^K_{\alpha j}$, with $|K|=3$. This turns Equation (4.186) into a space constraint in W^s; however, tangency conditions on it do not give enough conditions on the coefficients $B^{i'i}_{\alpha j}$ to determined them, like in the free second order case. In general, the top level coefficients $B^K_{\alpha j}$ are overdetermined inducing a new space constraint on W^s, but the subsequent coefficients $B^{Ii}_{\alpha j}$ (of order $k-1$) with $|I|=k-2$ are always undetermined, unless $k=2$.

This example is of furthermost importance since it is the key step to solve the ambiguity that exists in the solutions of the dynamical equation for higher order field theories. Moreover, to solve or describe this ambiguity will also do it for the definition of the higher-order Legendre transform and, consequently, higher-order Poincaré-Cartan form.

Chapter 5

Conclusions and future work

As for conclusion, I summarize the main results obtained in this memory.

- First, we have given a description without ambiguity of the higher-order classical field theory within a formulation of Skinner and Rusk type, which has permitted to define a premultisymplectic form and a unique Hamiltonian function; and in consequence a global and unique formulation of the dynamics. This part of the treatise has been published in Journal of Physics A: Mathematical and Theoretical Vol. 42 (2009).

- Secondly, we have developed the previous work and exposed an intrinsic formulation of the variational problem equations subjected to constraints dependent on higher order partial derivatives of the fields with respect to the base coordinates. As a study case, we have apply this theory to optimal control systems of partial differential equations. This results are gathered in the proceedings of different congresses: "18th International Fall Workshop on Geometry and Physics" and "Variational Integratos in Nonholonomic and Vakonomic Mechanics"; and in a paper that has to appear in the Journal of Physics A: Mathematical and Theoretical.

- Finally, we have given an important step in order to answer the inherent ambiguity of the Hamiltonian formulation. This work has proven to define univocally the Hamiltonian formulation of classical field theories of second order; specifically, we have successfully established a space of momenta in which the reduction algorithm does not stop and continues giving the subsequent steps.

Besides, also some results have been obtained in continuum mechanics with of applying the developed work in classical field theory to it (see Section §5.4 below).

- Within the theory of constitutive equations of material, a new definition has been given for materials know as functionally grade media thanks to their inherent properties. This definition has been proven to generalize the classical one, which has been published in the proceedings of the "XVI International Fall Workshop on Geometry and Physics" and in the International Journal of Geometric Methods in Modern Physics.

In particular, this results are given by Equation (4.52), Proposition 4.46, Proposition 4.48, Theorem 4.50, Example 4.55, Equation (4.124), Proposition 4.58, Theorem 4.59, Theorem 4.63, Theorem 4.66, Theorem 4.73, Examples 4.74 and 4.75, Definition 5.15 and Theorems 5.20 and 5.22.

These results haven been published in

- C. M. Campos, M. Epstein y M. de León, *Functionally graded media.* Int. J. Geom. Methods Mod. Phys. 5 (2008), no. 3, 431-455.
- C. M. Campos y M. de León, *Functionally graded media.* Proceedings of the "XVI International Fall Workshop on Geometry and Physics" (2007)
- C. M. Campos, M. de León, D. Martín de Diego and J. Vankerschaver, *Unambiguous formalism for higher order Lagrangian field theories.* J. Phys. A: Math. Theor. 42 (2009) 475207 (24pp)
- C. M. Campos, *Vakonomic Constraints in Higher-Order Classical Field Theory.* Proceedings of the "XVIII International Fall Workshop on Geometry and Physics" (2010)
- C. M. Campos, *Higher-Order Field Theory with Constraints.* To appear in Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM
- C. M. Campos, *M. de León and D. Martín de Diego, Constrained Variational Calculus for Higher Order Classical Field Theories.* To appear in J. Phys. A: Math. Theor.

The geometrical framework of the developed field theory is already prepared for its application to different lines of research, for instance: continuum mechanics, media with microstructure, multisymplectic integrators in higher-order field theory with or without constraints, etc. I debrief some of them in the following sections.

5.1 Geometric integrators for higher-order field theories

During the last years, there was a great interest in developing of geometric integrators for mechanical systems using a discrete variational principle (see [123] and references therein). In particular, this effort has been concentrated for the case of discrete Lagrangian functions L_d on the cartesian product $Q \times Q$ of a differentiable manifold. This cartesian product plays the role of a "discretized version" of the standard velocity phase space TQ. Applying a natural discrete variational principle and assuming a regularity condition, one obtains a second order recursion operator $\Upsilon : Q \times Q \longrightarrow Q \times Q$ assigning to each input pair (q_0, q_1) the output pair (q_1, q_2). When the discrete Lagrangian is an approximation of a continuous Lagrangian function (more appropriately, when the discrete Lagrangian approximates the integral action for L_d) we obtain a numerical integrator which inherits some of the geometric properties of the continuous Lagrangian (symplecticity, momentum preservation). Although this type of geometric integrators have been mainly considered for conservative systems, the extension to geometric integrators for more involved situations is relatively easy, since, in some sense, many of the constructions mimic the corresponding ones for the continuous counterpart. In this sense, it has been recently shown how discrete variational mechanics can include forced or dissipative systems, holonomic constraints, explicitely time-dependent systems, frictional contact, nonholonomic constraints... All these geometric integrators have demonstrated, in worked examples, an exceptionally good longtime behavior and obviously this research is of great interest for numerical and geometric considerations (see [104, 135]).

These methods have also extended for lagrangian field theories (see [120] and references therein) of order 1. These methods start by discretizing the spacetime M and in many cases it is assumed for simplicity that $M = \mathbb{R}^2$, and $Y = \mathbb{R}^2 \times Q$, where Q is a vector space. Typically, it is considered a mesh as a discretized version of M. Remember that a mesh $\mathcal{X}$ is a discrete subset of $\mathbb{R}^2$. For instance, the quadrangular mesh $\mathcal{X} = h\mathbb{Z} \times k\mathbb{Z} = \{x_{i,j} = (hi, kj) \mid (i,j) \in \mathbb{Z} \times \mathbb{Z}\}$. In this sense a *discrete field* is a map $\phi_d : \mathcal{X} \longrightarrow Q$. In the following we will restrict ourselves to quadrangular mesh although it is easily generalizable to other types of meshes. Define the set of squares $\mathcal{X}^4$ whose elements are the ordered quadruples of the form

$$\square_{i,j} = (x_{i,j}, x_{i+1,j}, x_{i+1,j+1}, x_{i,j+1})$$

The idea behind these discretizations is that the values of the discrete field at the vertices of the squares can be used to define the concept of *discrete jet* as an approximation of the continuous jet. In the case of a first order field theory the discrete jet bundles is defined as

$$J^1_d\pi = \mathcal{X}^4 \times Q^4$$

and a discrete jet is a pair $(\square_{i,j}, [q_{i,j}, q_{i+1,j}, q_{i+1,j+1}, q_{i,j+1}])$.

For discretizing the theory it can be useful to define appropriate discretization maps $\Phi_d = J^1_d\pi \to J^1\pi$ as for instance:

$$\begin{aligned}&\phi_d((\square_{i,j}, [q_{i,j}, q_{i+1,j}, q_{i+1,j+1}, q_{i,j+1}])\\&= (\frac{x_{i,j} + x_{i+1,j} + x_{i+1,j+1} + x_{i,j+1}}{4}, \frac{q_{i,j} + q_{i+1,j} + q_{i+1,j+1} + q_{i,j+1}}{4}, V_1, V_2)\end{aligned}$$

where

$$\begin{aligned} V_1 &= \frac{1}{2}\left(\frac{q_{i+1,j} - q_{i,j}}{k} + \frac{q_{i+1,j+1} - q_{i,j+1}}{h}\right), \\ V_2 &= \frac{1}{2}\left(\frac{q_{i,j+1} - q_{i,j}}{h} + \frac{q_{i+1,j+1} - q_{i+1,j}}{k}\right), \end{aligned}$$

which are considered as an approximation of the partial derivatives of the field.

Then, given a lagrangian $L : J^1\pi \to \mathbb{R}$, we define the discrete lagrangian $L_d : J^1_d\pi \to \mathbb{R}$ by $L_d = hk\Phi^*_d$.

The discrete field equations are deduced extremizing an appropriate discrete sum. In this particular case, the discrete field equations are (see [120]:

$$\begin{aligned} 0 &= D_1L_d((\square_{i,j}, [q_{i,j}, q_{i+1,j}, q_{i+1,j+1}, q_{i,j+1}]) \\ &\quad +D_2L_d((\square_{i,j-1}, [q_{i,j-1}, q_{i+1,j-1}, q_{i+1,j}, q_{i,j}]) \\ &\quad +D_3L_d((\square_{i-1,j}, [q_{i-1,j}, q_{i,j}, q_{i,j+1}, q_{i-1,j+1}]) \\ &\quad +D_4L_d((\square_{i-1,j-1}, [q_{i-1,j-1}, q_{i,j-1}, q_{i,j}, q_{i-1,j}]). \end{aligned}$$

Of course we can generalize these methods for higher order field theories adding discretizations of the higher-order derivatives (the second order case is already studied in [112]). For instance, for a second-order lagrangian we can consider a discrete Lagrangian defined by

$$L_d^2 : (\mathcal{X}^4 \times \mathcal{X}^4)_2 \times Q^7 \to \mathbb{R}$$

where $(\mathcal{X}^4 \times \mathcal{X}^4)_2$ are rectangles such that the third right-upper vertex of the first rectangle is also, the left-bottom vertex of the second one. Now, a discretization of the second-order derivatives is given, for instance, by:

$$\begin{aligned} V_{11} &= \frac{q_{i+1,j} - 2q_{i.j} + q_{i-1,j}}{2h} \\ V_{22} &= \frac{q_{i,j+1} - 2q_{i.j} + q_{i,j-1}}{2k} \\ V_{12} &= V_{21} = \frac{q_{i+1,j+1} - q_{i,j+1} - q_{i,j-1} + q_{i-1,j-1}}{2hk} \end{aligned}$$

In future research we will study these methods for higher order lagrangian systems including their geometric preservation properties (multisymplecticity, etc.). Moreover, it is possible to extend these techniques for the case of Lagrangian systems with constraints (see [15]).

5.2 Space+Time Decomposition

As for the Skinner-Rusk formalism, another framework of interest is the so called "space+ time decomposition" originally developed by Gotay in [96] (see also [21]). This formalism is strongly based on the theory of Cauchy surfaces, in which ones assumes that there exists a space-like surface in the ambient space that evolves along the time line such that it covers the whole ambient space. This description allow us to consider any field theory in "frozen time" and then watch it evolve.

To be more precise, let as usual $\pi : E \to M$ be a fiber bundle whose fibers have dimension n but whose base manifold, which is assumed to be orientable and oriented with a provided volume form η, has now dimension $m+1$. We assume that there exists an m-dimensional manifold X that can be embedded into M. Let $\varepsilon \in \mathrm{Emb}(X, M)$ be one of such embeddings, we view $M_\varepsilon := \varepsilon(X)$ as a Cauchy surface. We now consider the field theory restricted to M_ε, that is me shall consider the fiber bundle $\pi_\varepsilon : E_\varepsilon \to M_\varepsilon$, where $E_\varepsilon = E_{M_\varepsilon} = \pi^{-1}(M_\varepsilon)$ and $\pi_\varepsilon = \pi|_{E_\varepsilon}$. The space of sections $\tilde{E}_\varepsilon = \Gamma\pi_\varepsilon$ is called the *instantaneous configuration space* at "time" ε.

In this setting, given a section $\sigma \in \tilde{E}_\varepsilon$ we have that the tangent space to $\tilde{E}_\varepsilon$ at σ is

$$T_\sigma \tilde{E}_\varepsilon = \{v : M_\varepsilon \to \mathcal{V}\pi_\varepsilon \mid v \text{ covers } \sigma\},$$

and the cotangent space to $\tilde{E}_\varepsilon$ at σ is

$$T_\sigma^* \tilde{E}_\varepsilon = \{\alpha : M_\varepsilon \to L(\mathcal{V}\pi_\varepsilon, \Lambda^m M_\varepsilon) \mid \alpha \text{ covers } \sigma\},$$

where $L(\mathcal{V}\pi_\varepsilon, \Lambda^m M_\varepsilon)$ is the vector bundle over E_ε whose fiber at $u \in (E_\varepsilon)_x$ is the set of linear maps form $\mathcal{V}_u E_\varepsilon$ to $\Lambda_x^m M_\varepsilon$. Thus the pairing between the elements of $T_\sigma^* \tilde{E}_\varepsilon$ and those of $T_\sigma \tilde{E}_\varepsilon$ is given by the integral expression:

$$\langle \alpha, v\rangle = \int_{M_\varepsilon} \alpha(v).$$

Furthermore, we can define a Liouville form on the ε-phase space $T^*\tilde{E}_\varepsilon$ in the usual manner:

$$\theta_\varepsilon(\alpha)(V) = \left\langle \alpha, T_\alpha \pi_{\tilde{E}_\varepsilon}(V) \right\rangle .$$

And, of course, the canonical symplectic form $\omega_\varepsilon = -\,\mathrm{d}\theta_\varepsilon$.

Before we give a Lagrange description of the field dynamics within this setting, we must introduce two concepts. First, we consider a slicing of M with section X, that is a time-dependent family of embeddings $\chi : I \times X \to M$, where $I \subset \mathbb{R}$, such that χ is in fact a diffeomorphism. We define the generator of χ as the push forward of $\partial/\partial t$, that is

$$\xi_\chi \circ \chi := T\chi \left(\frac{\partial}{\partial t} \right) .$$

Secondly, we assert that $T\tilde{E}_\varepsilon$ is isomorphic to the collection of restrictions of holonomic sections of $\pi_1 : J^1\pi \to M$ (see [96]).

Now, given a Lagrangian density $\mathcal{L} : J^1\pi \to \Lambda^m M$, we define a Lagrangian function $L : T\tilde{E}_\varepsilon \to \mathbb{R}$ in the following way:

$$L_{\varepsilon,\chi}(\sigma) = \int_{M_\varepsilon} i_{\xi_\chi} \mathcal{L}(j^1\phi),$$

where $j^1\phi$ is the holonomic section that corresponds to σ.

From here we could proceed in the standard ways but, we remark that we finally have the three basic elements to follow the Skinner-Rusk formalism: the Lagrangian, the pairing and the canonical form. The goals of this work is to study the space+time decomposition within the Skinner-Rusk formalism and extend it to higher-order theories. Since the base manifold adds a new data in the picture, the slicing, it could possibly reduce the ambiguity in the space of solutions.

5.3 Reduction

Among different extra structures that the fiber bundle $\pi : E \to M$ may carry, of particular interest is the case when π is a principal fiber bundle. In this context and under extra assumptions on the Lagrangian, one may seek for symmetries of the problem or use reduction techniques to eliminate variables and simplify the problem. This is a natural step when a dynamic formalism is well established, which is the case of first order classical field theories, and which has already started (see for instance [37, 38, 39]).

The aim of a future work is to study and develop a theory of multisymplectic reduction in higher-order field theories and, in view of example 4.75, particularly for the second order case.

5.4 Continuous Media

The study of the mechanics of continuous media constitutes a non-trivial example of theory of classical fields, whose structure and dynamics may be characterized geometrically. Nonetheless, there still is a long way in process to geometrize this study. From the use of Lie algebra to describe the movement of a rigid body, to the modeling of Cosserat media and liquid crystals by means of principal fiber bundles. Before focusing on the dynamical aspects of a continuum, one should start studying the behavior of a body under infinitesimal deformations in order to understand its internal structure, which is the basis of constitutive theory of materials. In this sense, what follows is the work developed in [28], which is a study of materials that gradually change its behavior from point to point, that is, functionally graded media.

The mechanical response at a point X of a simple (first-grade) local elastic body B depends on the first derivative F at $X \in B$ of the deformation. In other words, B obeys a constitutive law of the form:

$$W = W(F(X); X) \tag{5.1}$$

where W measures the strain energy per unit volume. The linear map $F(X)$ is called the deformation gradient at X. Of course, there are materials for which the constitutive equation implies higher order derivatives or even internal variables as it happens with the so-called Cosserat media or, more generally, media with microstructure, but such materials will not be considered here.

An important problem in Continuum Mechanics is to decide if the body is made of the same material at all its points. To handle this question in a proper mathematical way, one introduces the concept of material isomorphism, that is, a linear isomorphism $P_{XY} : T_X B \longrightarrow T_Y B$ such that

$$W(FP_{XY}; X) = W(F; Y)$$

for all deformation gradients F at Y. Intuitively, this means that we can extract a small piece of material around X and implant it into Y without any change in the mechanical response at Y. If such is the case for all pairs of body points, we say that the body B is uniform. This has been the starting point of the work by Noll and Wang [130, 146, 155, 154] in their approach to uniformity and homogeneity.

In this context, a material symmetry at X is nothing but a material automorphism of the tangent space $T_X B$. The collection of all the material symmetries at X forms a group, the material symmetry group $\mathcal{G}(X)$ at X. An important consequence of the uniformity property is that the material symmetry groups at two different points X and Y are conjugate.

A natural question arises: Is there a more general notion that permits to compare the material responses at two arbitrary points even if the body does not enjoy uniformity? An answer to this question is based on the comparison of the symmetry groups at different points. Indeed, we say that the body B is unisymmetric if the material symmetry groups at two different points are conjugate, whether or not the points are materially isomorphic. From the point of view of applications, this kind of body corresponds to certain types of the so-called functionally graded materials (FGM for short). The unisymmetry property was introduced in [81] with the objective to extend the notion of homogeneity to non-uniform material bodies. Let us recall that the homogeneity of a uniform body is equivalent to the integrability of the associated material G-structure [22, 80]. Roughly speaking, this material G-structure is obtained by attaching to each point of B the corresponding material symmetry group via the choice of a given linear reference at a fixed point; a change of the linear reference gives a conjugate G-structure. In a more sophisticated framework, the set of all material isomorphisms defines a Lie groupoid, which in some sense is a way to deal with all these conjugate G-structures at the same time.

In the case of unisymmetric materials the attached group is not the material symmetry group, but its normalizer within the whole general linear group. This implies a more difficult understanding of the generalized concept of homogeneity associated with unisymmetric materials. The main aim of the present paper is to provide a convenient characterization of this homogeneity property. In this sense, this work may be regarded as a continuation and improvement of the results obtained in [81].

The paper is organized as follows. Section §B.1 is devoted to a brief introduction to groupoids and Lie groupoids; in particular, we define the normalizoid of a subgroupoid within a groupoid, which is just the generalization of the notion of normalizer in the context of groups. An important family of examples is provided by the frame-groupoid, consisting of all the linear isomorphisms between the tangent spaces at all the points of a manifold M; if M is equipped with a Riemannian metric g, one can introduce the notion of orthonormal groupoid (taking the orthogonal part of the linear isomorphisms given by the polar decomposition). If, without necessarily possessing a distinguished Riemannian metric, M is endowed with a volume form, one obtains the Lie subgroupoid of unimodular isomorphisms. In Section §B.2 we analyze the relations between Lie groupoids and principal bundles; in particular, we examine the relation between the frame groupoid and G-structures on a manifold M. In Section §5.4.1 we study the concepts of material symmetry and material symmetry groups, and in Section §5.4.2 we discuss uniformity and homogeneity. Finally, Section §5.4.3 is devoted to study the case of FGM materials, and the geometric characterization of homogeneity in this case is obtained for both solid and fluids.

5.4.1 The Constitutive Equation

In the most general sense (see [119], for instance), a *body* is a manifold B that can be embedded in a Riemannian manifold (S, g) with the same dimension, the *ambient space*. Usually, the body B is a simply connected open set of $\mathbb{R}^3$ and the ambient space is $\mathbb{R}^3$ itself with the standard metric. Each embedding $K : B \to S$ is called a *configuration* and its tangent map $TK : TB \to TS$ is called an *infinitesimal configuration*. If we fix a configuration K (the *reference configuration*) and we pick an arbitrary configuration $\tilde{K}$, then the embedding compositon $\phi = \tilde{K} \circ K^{-1} : K(B) \subset S \to S$ is considered as a body *deformation* and we call its tangent map $T_X\phi$ at a point X in B an *infinitesimal deformation* or the *deformation gradient*, usually denoted by F. Since (S, g) is a Riemannian manifold, we can induce a Riemannian metric on B by the pull-back of g by a reference configuration K. Since the metric on B depends from a chosen reference configuration, it is not canonical. However, for solid materials, we are able to define an "almost" unique metric compatible with the material structure, as we will show in section §5.4.2.

Usually, points in the body or in the reference configuration (when they are identified) are denoted by capital letters X, Y, Z, etc., and by small letters x, y, z, etc., in the deformed configuration. At the moment we have the picture shown at Figure 5.1.

As stated by *the principle of determinism*, the mechanical and thermal behaviors of a material or substance are determined by a relation called *the constitutive equation*. It does not follow directly from physical laws but it is combined with other equations that do represent physical laws (the conservation of mass for instance) to solve some physical problems, like the flow of a fluid in a pipe, or the response of a crystal to an electric field. In our case of interest, *elastic materials*, the constitutive equation establishes that,

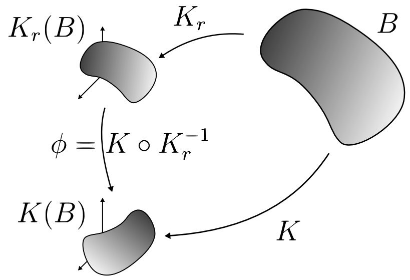

Figure 5.1: Deformation in a reference configuration.

in a given reference configuration, *the Cauchy stress tensor* depends only on the material points and on the infinitesimal deformations applied on them, that is

$$\sigma = \sigma(F_{K_r}, K_r(X)). \tag{5.2}$$

This relation is simplified in the particular case of *hyperelastic materials*, for which equation (5.2) becomes

$$W = W(F_{K_r}, K_r(X)). \tag{5.3}$$

where W is a scalar valued function which measures the stored energy per unit volume.

Among other postulates (*principle of determinism*, *principle of local action*, *principle of frame-indifference*, etc.), it is claimed that a constitutive equation must not depend on the reference configuration. It turns out that equation (5.2) (and (5.3)) now can be written in the form

$$\sigma = \sigma(F, X) \quad (W = W(F, X), \text{ respectively}), \tag{5.4}$$

where F stands for the tangent map at X of a local configuration (deformation).

Definition 5.1. A *material symmetry* at a given point $X \in B$ is a linear isomorphism $P : T_X B \to T_X B$ such that

$$\sigma(F \cdot P, X) = \sigma(F, X), \tag{5.5}$$

for any deformation F at X. The set of material symmetries at $X \in B$ is denoted by $\mathcal{G}(X)$ and it is called the *symmetry group* of B at X. Given a configuration K, we will denote by $\mathcal{G}_K(X)$ the symmetry group $\mathcal{G}(X)$ in the configuration K, that is

$$\mathcal{G}_K(X) = T_X K \cdot \mathcal{G}(X) \cdot (T_X K)^{-1}. \tag{5.6}$$

Different types of elastic materials are given in terms of their symmetry groups. For instance, a point is solid whenever its symmetry group in some reference configuration is a subgroup of the orthogonal group $O(3)$ and, fluid whenever the orthogonal group is a proper subgroup of the symmetry group. In [118, 154] it is possible to find a classification, due to Lie, of the connected Lie subgroups of Sl(3) and their corresponding Lie algebras.

Definition 5.2. Given an elastic material B, let $X \in B$ and consider its symmetry group $\mathcal{G}(X)$. If there exists a configuration K such that:

1. $\mathcal{G}_K(X)$ is a subgroup of the orthogonal group of transformations $O(3)$, then X is said to be an *elastic solid point*. If furthermore

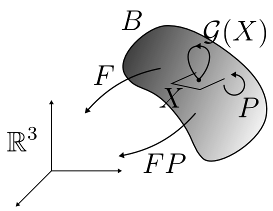

Figure 5.2: Material symmetry.

(a) $\mathcal{G}_K(X) = O(3)$, then we call X a *fully isotropic elastic solid point*;

(b) $\mathcal{G}_K(X)$ is a transverse orthogonal group (a group of rotations which fix an axis), then X is said to be a *transversely isotropic elastic solid point*;

(c) $\mathcal{G}_K(X)$ consists only of the identity element, then X will be a *triclinic elastic solid point*;

2. $\mathcal{G}_K(X)$ is a subgroup of the unimodular group of transformations $U(3)$ and has the orthogonal group $O(3)$ as a proper subgroup, then X is said to be an *elastic fluid point.* If furthermore

   (a) $\mathcal{G}_K(X) = \mathrm{Sl}(3)$ then we still call X an *elastic fluid*; and

   (b) $\mathcal{G}_K(X)$ is a transverse unimodular group (a group of unimodular transformations which fix an axis or a group of unimodular transformations which fix a plane) then we call X an *elastic fluid crystal.*

The infinitesimal configuration $T_X K$ or the induced frame $z = (T_X K)^{-1}$ is called an *undistorted state* of X.

This material classification is pointwise. A body is solid if every point is solid.

5.4.2 Uniformity and Homogeneity

To define the uniformity of a material, we first have to give a criterion that establishes when two points are made of the same material. To compare their symmetry groups is not sufficient since this is only a qualitative aspect. Indeed, consider two points in a rubber band, one point may be relaxed while another point may be under stress. But we are still able to release the stress on the second point and bring it to the same state as the first one, and then compare their responses.

Definition 5.3. We say that two points $X, Y \in B$ are *materially isomorphic*, if there exists a linear isomorphism $P_{XY} : T_X B \to T_Y B$ such that

$$\sigma(F \cdot P_{XY}, X) = \sigma(F, Y), \tag{5.7}$$

for any deformation F at Y. The linear map P_{XY} is called a *material isomorphism.*

Even if the definition of material isomorphism and material symmetries are mathematically similar, there is an important conceptual difference. While the symmetry group of a point characterizes the material behavior of that point, a material isomorphism establishes a relation between two different points. In fact, as already pointed out, a material

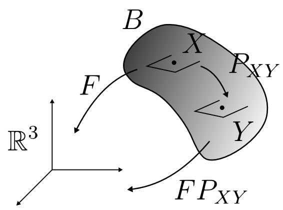

Figure 5.3: Material isomorphism.

symmetry can be viewed as a material automorphism by identifying X with Y in the above definition.

Definition 5.4. Given a material body B, the *material groupoid* is the set of all the material isomorphisms and symmetries, that is the set

$$\mathcal{G}(B) = \{P \in \Pi(B) \text{ satisfying Definition 5.3}\}. \tag{5.8}$$

It is easy to check that the material groupoid $\mathcal{G}(B)$ is actually a groupoid. Furthermore, it is a subgroupoid of the frame groupoid $\Pi(B)$, but note that it is not necessarily a Lie groupoid or even transitive as the frame groupoid. In fact, when all the points of a body are pairwise related by a material isomorphism, it means that the body consists only of one type of material. In this case, it is materially uniform.

Definition 5.5. Given a material body B, we say that it is *uniform* if the material groupoid $\mathcal{G}(B)$ is transitive, and *smoothly uniform* when the material groupoid is a transitive differential groupoid (and hence a Lie subgroupoid of $\Pi(B)$).

A simple but important property of uniform materials is that the groups of material symmetries are mutually conjugate by any material isomorphism between the respective base points. To be more precise, equation (B.2) reads in terms of elastic bodies:

$$\mathcal{G}(Y) = P \cdot \mathcal{G}(X) \cdot P^{-1}, \quad \forall P \in \mathcal{G}(B)_{X,Y}, \tag{5.9}$$

for any pair of materially isomorphic points $X, Y \in B$.

When we look a material through different configurations, there are prefered states of the material we want to distinguish: *e.g.* transversely isotropic solids have a fixed axis "invariant" under material isomorphisms that we prefer to align with the vertical axis. Such a state may be modelized in an infinitesimal configuration by a linear frame z. As we have just said, in the material paradigm, this frame of reference z has some behaviors that will be mainted by material isomorphisms. If we consider the set of all these distinguished references that arise from material transformations of the 'reference crystal' (see Figure 5.4), then we obtain the so called material G-structure of B. As far as we know, Wang was the first to realize that the uniformity of a material can be modeled by a G-structure [154], although this fact was emphasized by Bloom [22]. For definiteness,

Definition 5.6. A *material G-structure* of a smoothly uniform body B is any of the G_z-structures induced by the material groupoid $\mathcal{G}(B)$ as shown in Theorem B.17. The chosen frame of reference $z \in \mathcal{F}B$ is called the *reference crystal*.

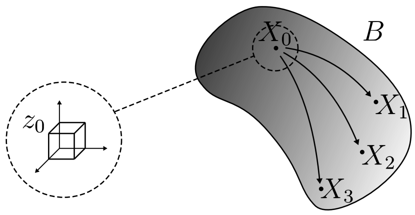

Figure 5.4: The reference crystal.

Definition 5.7. Given a smoothly uniform body B, a configuration K that induces a cross-section of a material G-structure will be called *uniform*. If there exists an atlas $\{(U_\alpha, K_\alpha)\}_{\alpha \in A}$ of B of local uniform configurations for a fixed material G-structure, the body B will be said *locally homogeneous*, and *(globally) homogeneous* if the body B may be covered by just one uniform configuration.

The material concept of homogeneity corresponds to the mathematical concept of integrability. By Theorem B.22, a smoothly uniform body B will be locally homogenous if and only if one (and therefore any) of the associated material G-structures is integrable. Let K a uniform configuration for a particular integrable G-structure $G(B)$ of a homogeneous elastic material B. If (X, v_1, v_2, v_3) denotes the cross section induced by K, thus the constitutive equation (5.2) may be written in the form

$$\sigma = \sigma(F_K, K(X)) = \sigma(F^i_j, x^i), \tag{5.10}$$

with obvious notation. Now note that, since through K any material isomorphism P may be considered as an element of the structure group G, which is clear for material symmetries, and since the body B is uniform, we have that

$$\sigma(F^i_j, y^i) = \sigma(F_K, K(Y)) = \sigma(F_K \cdot P_K, K(X)) = \sigma(F^i_k \cdot P^k_j, x^i) = \sigma(F^i_j, x^i). \tag{5.11}$$

Thus, we have just proved the following result:

Theorem 5.8. *If K is a uniform configuration of a homogeneous elastic body B, the constitutive equation* (5.2) *is independent of the material point and invariant under the right action of the structure group G of the G-structure $G(B)$ related to K. Thus,*

$$\sigma = \sigma(F^i_j) \quad \textit{and} \quad \sigma(F^i_k \cdot P^k_j) = \sigma(F^i_j) \textit{ for any } P \in G. \tag{5.12}$$

The physical interpretation of this theorem is that points of a homogenous elastic body B can be put by means of a configuration K in such a manner they are all at the same state, at least locally. This configuration K is uniform. Even if the material G-structures of a smoothly uniform body B are different (but equal via conjugation), there must be at least one of them in which the structure group G satisfies a condition of the material classification 5.2.

Definition 5.9. Accordingly to Definition 5.2, a smoothly uniform elastic body B is *solid* or *fluid*, if all the points are *solid* or *fluid*, respectively. Any of the material G-structures for which the structure group fulfills the classification is called *undistorted*.

Uniform Elastic Solids

The following result is due to Wang (*cf.* [154]). In his paper, Wang defines the material G-structures from the point of view of atlases, families of cross-sections of the frame bundle, instead of our approach through groupoids. These families are the cross-sections of the resulting G-structures. When a material is solid, it is possible to endow the body with a metric wich is compatible with the material structure. Wang calls such a metric an intrinsic metric.

Theorem 5.10. *Let B be a uniform elastic solid material; each undistorted material G-structure $G(M)$ defines a Riemannian metric g, invariant under material symmetries and isomorphisms.*

Proof. Given a cross-section (U,σ) of a fixed undistorted material G-structure $G(B)$, let $X \in U$ and define

$$g^\sigma_X(v,w) := \left\langle \sigma(X)^{-1}\cdot v, \sigma(X)^{-1}\cdot w \right\rangle, \quad \forall X \in U, \forall v,w \in T_XB, \tag{5.13}$$

where $\langle\,,\rangle$ is the Euclidean scalar product. Thus, g^σ is clearly a smoooth positive definite symmetric bilinear tensor field on U, since it is nothing more than the pullback of the Euclidean metric. Let us check that, in this manner, the metric g^σ does not depend on the chosen cross-section (U,σ). Given any other cross-section (V,τ), let $X \in B$ be in the intersection of their domains (if not empty, of course), then

$$\begin{array}{rcl} g^\sigma_X(v,w) & = & \langle \sigma(X)^{-1}\cdot v, \sigma(X)^{-1}\cdot w\rangle \\ & = & \langle Q\cdot\tau(X)^{-1}\cdot v, Q\cdot\tau(X)^{-1}\cdot w\rangle \\ & = & \langle \tau(X)^{-1}\cdot v, \tau(X)^{-1}\cdot w\rangle \\ & = & g^\tau_X(v,w), \end{array} \tag{5.14}$$

where we used the fact that, by hypothesis, $Q = \sigma(X)^{-1}\cdot\tau(X) \in G$ is orthogonal.

Now, let $P \in \mathcal{G}_{X,Y}(B)$ be a material isomorphism; there will exist cross-sections $(U,\sigma),(V,\tau)$ such that $P = \tau(Y)\cdot\sigma(X)^{-1}$. Then, we have

$$\begin{array}{rcl} g_Y(P\cdot v, P\cdot w) & = & \langle \tau(Y)^{-1}\cdot P\cdot v, \tau(Y)^{-1}\cdot P\cdot w\rangle \\ & = & \langle \sigma(X)^{-1}\cdot v, \sigma(X)^{-1}\cdot w\rangle \\ & = & g_Y(v,w). \end{array} \tag{5.15}$$

The metric we where looking for is just the metric g defined in (5.13). □

If we consider the orthogonal groupoid $\mathcal{O}(B)$ related to this metric, we have that the material groupoid is included in it, $\mathcal{G}(B) \subset \mathcal{O}(B)$. Reciprocally, if B is a smoothly uniform material such that it can be endowed with a Riemannian metric for which the material symmetries and isomorphisms are orthogonal transformations, $\mathcal{G}(B) \subset \mathcal{O}(B)$, then B must be an elastic solid. Thus, elastic solids are completely characterized by Riemannian metrics with the property of being invariant under material symmetries and isomorphisms.

Remark 5.11. Given two material G-structures, $G_1(B)$ and $G_2(B)$, of a uniform elastic solid B, we know that they must be related by the right action of a linear isomorphism $F \in \mathrm{Gl}(3)$, that is $G_2(B) = G_1(B)\cdot F$. Thus, if $G_1(B)$ is undistorted, the G-structure $G_2(B)$ will be undistorted if and only if the symmetric part V of the left polar decomposition of F, $F = V\cdot R$, lies in the centralizer of G_1, that is $V \in C(G_1)$ (*cf.* [154], proposition 11.3). But this does not imply that $G_1(B)$ and $G_2(B)$ define the same metric, which is true only if $V = I$.

Uniform Elastic Fluids

There are similar results for fluids as for solids. In this case, the fluid structure induces volume forms.

Proposition 5.12. *Let B be a uniform fluid material, then each undistorted material G-structure $G(B)$ defines a volume form ρ invariant under material symmetries and isomorphisms.*

Proof. Given a cross-section (U, σ) of a fixed undistorted material G-structure $G(B)$, let us define on U the volume form

$$\rho_\sigma = \sigma^{*1} \wedge \sigma^{*2} \wedge \sigma^{*3}, \tag{5.16}$$

where σ^* denotes the co-frame cross-section of σ, that is $\sigma^* : U \longrightarrow \mathcal{F}^*B$ such that $\sigma^{*i}(\sigma_j) \equiv \delta^i_j$ on U. Let us show that the volume form ρ_σ does not depend on the chosen cross-section (U, σ). In fact, let $(U, \sigma), (V, \tau)$ be two cross-sections with non-empty domain intersection, then for any n vectors $v_1, \ldots, v_n \in T_X B$, with $X \in U \cap V$, we have

$$\begin{aligned}
\rho_\sigma(v_1, \ldots, v_n) &= \det(v^j_i) \\
&= \det((\sigma^{-1}\tau)^k_i) \cdot \det(\tilde{v}^j_k) \\
&= \rho_\tau(v_1, \ldots, v_n),
\end{aligned}$$

where we have used $v_i = v^j_i \sigma_i = \tilde{v}^j_i \tau_i$, $v^j_i = (\sigma^{-1}\tau)^k_i \cdot \tilde{v}^j_k$ and $\sigma^{-1}\tau \in U(n)$. Since the tangent vectors $v_1, \ldots, v_n$ are arbitrary, ρ_σ and ρ_τ coincide on the intersection of their domains, $U \cap V$. Thus, the volume form given in (5.16) defines locally a volume form ρ on the whole material body B.

Let us see how ρ is invariant under material symmetries and isomorphisms. Given $P \in \mathcal{G}_{X,Y}(B)$, there must exist cross-sections $(U, \sigma), (V, \tau)$ such that $P = \tau(Y) \cdot \sigma(X)^{-1}$. Then, we have

$$\rho \circ P = (P^{-1}\tau)^{*1} \wedge (P^{-1}\tau)^{*2} \wedge (P^{-1}\tau)^{*3} = \sigma^{*1} \wedge \sigma^{*2} \wedge \sigma^{*3} = \rho, \tag{5.17}$$

which finishes the proof. □

Considering now the induced unimodular groupoid $\mathcal{U}(B)$, by the invariance we have the inclusion $\mathcal{G}(B) \subset \mathcal{U}(B)$ which also characterizes elastic fluids.

5.4.3 Unisymmetry and Homosymmetry

As we have seen, the concept of homogeneity must be understood within the framework of uniformity. But, there are materials that are not uniform by their very definition, the so called *functionally graded materials*, or FGM for short. This type of material can be made by techniques that accomplish a gradual variation of material properties from point to point: for instance, ceramic-metal composites, used in aeronautics, consist of a plate made of ceramic on one side that continuously change to some metal at the opposite face. The material properties are also given through a constitutive equation like (5.4). Therefore, we will have a notion of material symmetry and the symmetry groups will be non-empty as in the case of uniform materials. For a FGM material, the symmetry groups at two different points are still conjugate, accordingly to the following definition.

Definition 5.13. Given a functionally graded material B, let be $X, Y \in B$; we say that a linear map $A : T_X B \longrightarrow T_Y B$ is a *unisymmetric (material) isomorphism* if it conjugates the symmetry groups of X and Y, namely,

$$\mathcal{G}(Y) = A \cdot \mathcal{G}(X) \cdot A^{-1}. \tag{5.18}$$

As for uniform bodies, the material properties of a FGM are now characterized by the collection of all the possible unisymmetric isomorphisms.

Definition 5.14. Given a functionally graded material B, the set of unisymmetric isomorphisms, that is the set

$$\mathcal{N}(B) = \left\{ A \in \Pi(B) \ : \ \mathcal{G}(Y) = A \cdot \mathcal{G}(X) \cdot A^{-1} \right\}, \tag{5.19}$$

will be called the *FGM material groupoid* of B.

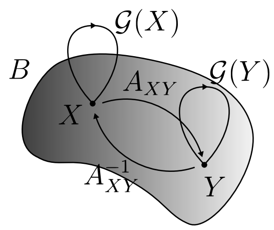

Figure 5.5: The FGM material groupoid.

We may now extend the ideas of section §5.4.2 using this new object. Then we obtain:

Definition 5.15. A functionally graded material B will be said *unisymmetric* if the FGM material groupoid $\mathcal{N}(B)$ is transitive and, *smoothly unisymmetric* if it is a Lie groupoid.

Note that the notion of unisymmetry covers a qualitative aspect in the sense that a unisymmetric FGM is made of only one "type" of material. For instance, it will be a fully isotropic solid everywhere or a fluid everywhere, but it cannot be a fully iscotropic solid at some point and a fluid at another point.

For this groupoid, we also have the associated G-structures.

Definition 5.16. Let B be a smoothly unisymmetric body. Any of the asociated G-strutures $\mathcal{N}_z(B)$, with $z \in \mathcal{F}B$, will be called a *material N-structure*. A cross-section of a material N-structure will be a *unisymmetric cross-section* and a configuration inducing such a cross-section will be a *unisymmetric configuration*. If for any of the material N-structures there exists a covering by unisymmetric configurations, the body B will be said *locally homosymmetric*, and *(globally) homosymmetric* if the covering consists of only one unisymmetric configuration.

As we may see, the homosymmetry property is equivalent to the integrability of any of the material N-structures. However, there is not an analogue result to Theorem 5.8 for homosymmetric bodies. Since, even if we have an N-structure and the group structure is the same for any point through any unisymmetric configuration, the symmetry groups may be represented by different subgroups of N at each point.

Functionally Graded Elastic Solids

Definition 5.17. We will say that a functionally graded elastic material B is a *functionally graded solid* if there is a Riemmanian metric on B invariant under material symmetries, that is every point is solid. Furthermore, B will be said

1. *fully isotropic* if every point is fully isotropic;

2. *transversely isotropic* if every point is transversely isotropic; and

3. *triclinic* if every point is triclinic.

The compatible metric is called a *material metric*.

We have not used the term "intrinsic" for the material metric, since it does not arise from the material structure as for uniform elastic solids (*cf.* Theorem 5.10). The material metric is an extra structures that ensures that the solid points are glued in a solid way.

If B is a FGM solid and we consider the orthonormal cross-sections (U, σ) of the $O(3)$-structure given by a solid metric, then they must verify:

$$\sigma(X)^{-1} \cdot \mathcal{G}(X) \cdot \sigma(X) \subseteq O(3) \quad \forall X \in U \quad \forall (U, \sigma), \tag{5.20}$$

$$\sigma(X)^{-1} \cdot \tau(X) \in O(3) \quad \forall X \in U \cap V \quad \forall (U, \sigma), (V, \tau); \tag{5.21}$$

where $\mathcal{G}(X)$ is the material symmetry group of B at X. In fact, these two conditions are necessary and sufficient to define a solid metric compatible with the material structure by means of a family of cross-sections of $\mathcal{F}B$.

On the other hand, if we consider another $O(3)$-structure, giving a second solid metric, the two structures are not a priori related by the right action of a linear isomorphism $F \in \mathrm{Gl}(3)$. But if they are, then the symmetric part of the polar decomposition of F must be spherical, a homothety. This can be interpreted as the material being in both cases in the same state but the measures of stress, or strain, are performed with different scales.

Definition 5.18. A solid FGM B will be said to be *relaxable* if the $O(3)$-structure given by some solid metric is integrable or, equivalently, if the Riemannian curvature (with respect to this metric) vanishes identically. We then say that the $O(3)$-structure is *relaxed*.

Definition 5.19. We say that a body B is *homosymmetrically relaxable* if B is an unisymmetric solid material for which there exists a covering Σ of local configuration that are both, unisymmetric and relaxed configurations.

Let B be a homosymmetrically relaxable elastic solid, then we have these two structures, the unisymmetric and the orthogonal, which are in certain manner interconnected. As B is a solid, intuitively we may perceive that only the orthogonal part of a unisymmetric isomorphism must be important. In what follows, we will explain this fact in more detail.

A direct consequence of the previous Lemma B.11 and Proposition B.23 is the following theorem, which implies a result proved by Epstein and de León [81].

Theorem 5.20. *If B is relaxable elastic solid that is also homosymmetric, we have*

$$\bar{\mathcal{N}}(B) = \mathcal{N}(B) \cap \mathcal{O}(B), \tag{5.22}$$

where $\bar{\mathcal{N}}(B)$ consits in the orthogonal part of the isomorphisms of $\mathcal{N}(B)$. Therefore, if $\bar{\mathcal{N}}_z(B)$ is a smooth $\bar{N}_z$-structure, B will be homosymmetrically relaxable if and only if the reduced material groupoid $\bar{\mathcal{N}}_z(B)$ *is integrable (where $z \in \mathcal{F}B$ is fixed).*

Let B a relaxable and homosymmetric elastic solid and let g denote the compatible material metric

- If B is *fully isotropic*, which means the symmetry group $\mathcal{G}(X)$ of each point $X \in B$ is equal to the orthogonal group $O(T_X B, g)$ itself, then the reduced FGM material groupoid $\bar{\mathcal{N}}(B)$ coincides with the orthogonal groupoid $\mathcal{O}(B)$.

- If B is *triclinic* (the only element of the symmetry group is the identity map), the FGM material groupoid $\mathcal{N}(B)$ is the full frame groupoid $\Pi(B)$, and thus $\bar{\mathcal{N}}(B) = \mathcal{O}(B)$ as before.

- If B is *transversally isotropic*, at each point $X \in B$ there exists a basis of $T_X B$ in which the material symmetries $g \in \mathcal{G}(X)$ may be represented by matrices of the form:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$
Thus, for this basis, the normalizer of $\mathcal{G}(X)$ is
$$\mathcal{N}(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix} \right\rangle$$
where the brackets denote the group generated by the elements enclosed, and where θ, α, β are real numbers, α, β being in addition positive. Therefore, the group at any base point of the reduced FGM material groupoid coincides with the respective symmetry group, that is
$$\bar{\mathcal{N}}(X) = \mathcal{G}(X) \quad \forall x \in B.$$
This means that, even if the material groupoid $\mathcal{G}(B)$ (the set consisting of material isomorphisms and symmetries) is not transitive (*i.e.* B is not uniform), the reduced FGM material groupoid $\bar{\mathcal{N}}(B)$ is, and it coincides with $\mathcal{G}(B)$ on the symmetry groups. Thus, there is some kind of uniformity that generalizes the classical one. Finally, note that any G-structure related to $\bar{\mathcal{N}}(B)$ will have a transversely isotropic structural group as mentioned before.

  Finally, note that we recover an analogue result to Theorem 5.8, which is also true for fully isotropic FGM solids. If B is homosymmetrically relaxable, then for a unisymmetric and relaxable configuration K, the constitutive equation will be invariant under the action of the structure group of the reduced N-stucture, related to the configuration K. In this case, the structure group will coincide through K with the symmetry group $\mathcal{G}_K(X)$ at any point X in the domain of K. However, the constitutive equation will not be independent of the point.

Functionally Graded Elastic Fluids

In the same way we have generalized the definition of elastic solids in section §5.4.3, we are going to give a new definition of elastic fluids. Classically, an elastic fluid is a uniform elastic material which posses a unimodular material structure, that is a $U(3)$-structure (see [146] for instance), even though there are smaller fluid structures as the ones of fluid crystals (*cf.* [118]).

Definition 5.21. We will say that a functionally graded elastic material B is a *functionally graded fluid* (or a *functionally graded fluid crystal*) if there is a volume form ρ on B invariant under material symmetries such that every point is fluid (or, respectively, if every point is a fluid crystal). The volume form is called a *material form*.

As in the case of functionally graded elastic solids, the following two conditions on cross-sections (U, σ) of the frame bundle $\mathcal{F}B$,

$$\sigma(X)^{-1} \cdot \mathcal{G}_x \cdot \sigma(X) \subseteq U(3) \quad \forall X \in U \quad \forall (U, \sigma) \tag{5.23}$$

$$\sigma(X)^{-1} \cdot \tau(X) \in U(3) \quad \forall X \in U \cap V \quad \forall (U, \sigma), (V, \tau) \tag{5.24}$$

characterize the fluid material structure.

Given a functionally graded elastic fluid B, consider the unimodular groupoid $\mathcal{U}(B)$ related to the volume form ρ (Example B.7). When two fluid points have conjugate symmetry groups, only the unimodular part of the conjugate transformation plays a role in the conjugation. That is, if P is the transformation that conjugates these two groups, then the unimodular transformation $P/\det_\rho(P)$ still realizes the conjugation.

Proposition 5.22. *If B is a unisymmetric elastic fluid, then*

$$\mathcal{N}^1(B) = \mathcal{N}(B) \cap \mathcal{U}(B), \tag{5.25}$$

where $\mathcal{N}^1(B)$ is the unimodular reduction of the FGM material groupoid.

Let B a fluid crystal of first kind (see [118, 154]), that is, an elastic fluid as in 5.21 such that, for each material point $X \in B$, the symmetry group $\mathcal{G}(X)$ may be represented for some reference z at X by matrices of the form

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & g \end{pmatrix}$$

with $\det(A) = \pm 1$. The normalizer in $\mathrm{Gl}(3)$ of this group of matrices is the set of matrices of the same form but with the restriction $\det(A) \neq 0$. Therefore, when we intersect the normalizer with $U(3)$ we obtain the original group of matrices. This means that $\mathcal{N}^1(X) = \mathcal{G}(X)$ for every material point $x \in B$.

The latter example shows us how a fluid material, which is not necessarily uniform, preserves uniformly the symmetry group structure across the body.

Appendix A

Multi-index properties

Given a function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$, its partial derivatives are classically denoted

$$f_{i_1 i_2 \cdots i_k} = \frac{\partial^k f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}.$$

When smooth functions are considered, their cross derivatives coincide. Thus, the order in which the derivatives are taken is no longer relevant, but the number of times with respect to each variable.

Another notation to denote partial derivatives is defined through "symmetric" multi-indexes (see [139]). A multi-index I will be an m-tuple of non-negative integers. The i-th component of I is denoted $I(i)$. Addition and subtraction of multi-indexes are defined component-wise (whenever the result is still a multi-index), $(I \pm J)(i) = I(i) \pm J(i)$. The length of I is the sum $|I| = \sum_i I(i)$, and its factorial $I! = \Pi_i I(i)!$. In particular, 1_i will be the multi-index that is zero everywhere except at the i-th component which is equal to 1.

Keeping in mind the above definition, we shall denote the partial derivatives of a function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ by:

$$f_I = \frac{\partial^{|I|} f}{\partial x^I} = \frac{\partial^{I(1)+I(2)+\cdots+I(m)} f}{\partial x_1^{I(1)} \partial x_2^{I(2)} \cdots \partial x_m^{I(m)}}.$$

Thus, given a multi-index I, $I(i)$ denotes the number of times the function is differentiated with respect to the i-th component. The former notation should not be confused with the latter one. For instance, the third order partial derivative $\frac{\partial^3 f}{\partial x_2 \partial x_3 \partial x_2}$ (with $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$) is denoted f_{232} and $f_{(0,2,1,0)}$, respectively.

Here we present some simple, but useful, results on multi-indexes.

Lemma A.1. *Given k integers $1 \leq i_1, \ldots, i_k \leq m$, with $k \geq 0$ and $m \geq 1$, define the function*

$$n(i_1, \ldots, i_k) := \Pi_{l=1}^{k} I_l(i_l) \quad (n(\emptyset) := 1), \tag{A.1}$$

where $I_l := 1_{i_1} + \cdots + 1_{i_l} \in \mathcal{N}^m$, for $l = 1, \ldots, k$. We have that n is invariant under permutations, that is, if $\pi \in \Sigma_k$ is a permutation of k elements, then

$$n(i_1, \ldots, i_k) = n(i_{\pi(1)}, \ldots, i_{\pi(k)}). \tag{A.2}$$

Moreover, $n(i_1, \ldots, i_k) = I_k!$.

Proof. We proceed by induction on k. The cases $k = 0$ and $k = 1$ are trivial thus, let us suppose that the result is true for some integer $k \geq 1$ to show that it is also true for $k+1$. Since $n(i_1, \ldots, i_k, i_{k+1}) = I_{k+1}(i_{k+1}) \cdot n(i_1, \ldots, i_k)$, by the hipotesys of induction, it suffies to show that $n(i_1, \ldots, i_{k-1}, i_k, i_{k+1}) = n(i_1, \ldots, i_{k-1}, i_{k+1}, i_k)$, which is equivalent to $I_{k+1}(i_{k+1}) \cdot I_k(i_k) = I'_{k+1}(i_k) \cdot I'_k(i_{k+1})$, where $I'_k = 1_{i_1} + \cdots + 1_{i_{k-1}} + 1_{i_{k+1}}$ and $I'_{k+1} = 1_{i_1} + \cdots + 1_{i_{k-1}} + 1_{i_{k+1}} + 1_{i_k}$.

$$\begin{aligned}
I_{k+1}(i_{k+1}) \cdot I_k(i_k) &= (\sum_{l=1}^{k-1} \delta_{i_l}^{i_k} + \delta_{i_k}^{i_k} + \delta_{i_{k+1}}^{i_k})(\sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} + \delta_{i_{k+1}}^{i_{k+1}}) \\
&= \sum_{l=1}^{k-1} \delta_{i_l}^{i_k} \cdot \sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} + \sum_{l=1}^{k-1} \delta_{i_l}^{i_k} + \sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} + \delta_{i_{k+1}}^{i_k} \cdot \sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} \\
&= \sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} \cdot \sum_{l=1}^{k-1} \delta_{i_l}^{i_k} + \sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} + \sum_{l=1}^{k-1} \delta_{i_l}^{i_k} + \delta_{i_k}^{i_{k+1}} \cdot \sum_{l=1}^{k-1} \delta_{i_l}^{i_k} \\
&= (\sum_{l=1}^{k-1} \delta_{i_l}^{i_{k+1}} + \delta_{i_{k+1}}^{i_{k+1}} + \delta_{i_k}^{i_{k+1}})(\sum_{l=1}^{k-1} \delta_{i_l}^{i_k} + \delta_{i_{k+1}}^{i_k}) \\
&= I'_{k+1}(i_k) \cdot I'_k(i_{k+1})
\end{aligned}$$

□

Note that $J! = J(i) \cdot I!$ for any pair (I, i) such that $I + 1_i = J$.

Lemma A.2. *1. Let $\{a_{I,i}\}_{I,i}$ be a family of real numbers indexed by a multi-index $I \in \mathcal{N}^m$ and by an integer i such that $1 \leq i \leq m$. Given an integer $k \geq 1$, we have that*

$$\sum_{|I|=k-1} \sum_{i=1}^{m} a_{I,i} = \sum_{|J|=k} \sum_{I+1_i=J} a_{I,i}. \tag{A.3}$$

2. More generally, let $\{a_{I_1,I_2}\}_{I_1,I_2}$ be a family of real numbers indexed by two multi-indexes $I_1, I_2 \in \mathcal{N}^m$. Given two integers $k \geq l \geq 0$, we have that

$$\sum_{|I_1|=l} \sum_{|I_2|=k-l} a_{I_1,I_2} = \sum_{|J|=k} \sum_{\substack{I_1+I_2=J \\ |I_1|=l}} a_{I_1,I_2}. \tag{A.4}$$

Proof. The proof is trivial when we realize that the sets $\{(I, i) : |I| = k-1, 1 \leq i \leq m\}$ and $\{(I, i) : I + 1_i = J, |J| = k\}$ are in bijective correspondence. For the general case, we shall consider the sets $\{(I_1, I_2) : |I_1| = l, |I_2| = k-l\}$ and $\{(I_1, I_2) : I_1 + I_2 = J, |I_1| = l, |J| = k\}$. □

$$\sum_{|I_1|+|I_2|=k} a_{I_1,I_2} = \sum_{l=0}^{k} \sum_{|I_1|=l} \sum_{|I_2|=k-l} a_{I_1,I_2} = \sum_{|J|=k} \sum_{I_1+I_2=J} a_{I_1,I_2}. \tag{A.5}$$

Lemma A.3. *Let $\{a_{I,i}\}_{I,i}$ be a family of real numbers indexed by a multi-index $I \in \mathcal{N}^m$ and by an integer i such that $1 \leq i \leq m$. If $J = 1_{i_1} + \cdots + 1_{i_k} \in \mathcal{N}^m$ is a multi-index of length $k \geq 0$, we then have that*

$$\sum_{I+1_i=J} a_{I,i} = \sum_{l=1}^{k} \frac{1}{J(i_l)} \cdot a_{J_{\hat{l}},i_l}, \tag{A.6}$$

where $J_{\hat{l}} := 1_{i_1} + \cdots + 1_{i_{l-1}} + 1_{i_{l+1}} + \cdots + 1_{i_k}$.

Proof. We proceed by induction on the dimension of the multi-indexes, m. The case $m = 1$ is clear thus, let us suppose that the result is true for $m - 1 \geq 1$ to show that it is also true for m. We first note without lose of generality that we may suppose that $i_1 \leq i_2 \leq \cdots \leq i_k$ and that $J(m) \neq$. Otherwise, we could easily reorder the indexes and the coordinates correspondingly before the computations and undo the changes at the end.

$$\begin{aligned}
\sum_{I+1_i=J} a_{I,i} &= \sum_{\substack{I+1_i=J \\ i\neq m}} a_{I,i} + a_{J-1_m,m} \\
&= \sum_{I+1_i=\tilde{J}} a_{I+J(m)1_m,i} + a_{J-1_m,m} \\
&\quad \text{where } \tilde{J} = J - J(m)1_m \\
&= \sum_{l=1}^{k-J(m)} \frac{1}{\tilde{J}(i_l)} \cdot a_{\tilde{J}_{\hat{l}}+J(m)1_m,i_l} + a_{J-1_m,m} \\
&\quad \text{where we have applied the hypothesis of induction} \\
&= \sum_{l=1}^{k-J(m)} \frac{1}{J(i_l)} \cdot a_{J_{\hat{l}},i_l} + \sum_{l=k-J(m)+1}^{k} \frac{1}{J(i_l)} \cdot a_{J_{\hat{l}},i_l}
\end{aligned}$$

□

Lemma A.4. *Let $J \in \mathcal{N}^m$ be a non-zero multi-index. We have that*

$$\sum_{I+1_i=J} \frac{I(i)+1}{|I|+1} = 1. \tag{A.7}$$

Proof.

$$1 = \sum_{i=1}^{m} \frac{J(i)}{|J|} = \sum_{I+1_i=J} \frac{J(i)}{|J|} = \sum_{I+1_i=J} \frac{I(i)+1}{|I|+1}$$

□

Lemma A.5. *Let $\{a_J\}_J$ be a family of real numbers indexed by a multi-index $J \in \mathcal{N}^m$. Given a positive integer $l \geq 1$, we have that*

$$\sum_{|J|=l} a_J = \sum_{|I|=l-1} \sum_{i=1}^{m} \frac{I(i)+1}{|I|+1} a_{I+1_i}, \tag{A.8}$$

Proof.

$$\begin{aligned}\sum_{|J|=l} a_J &= \sum_{|J|=l}\left(\sum_{I+1_i=J}\frac{I(i)+1}{|I|+1}\right)a_J\\ &= \sum_{|J|=l}\sum_{I+1_i=J}\frac{I(i)+1}{|I|+1}a_{I+1_i}\\ &= \sum_{|I|=l-1}\sum_{i=1}^{m}\frac{I(i)+1}{|I|+1}a_{I+1_i}.\end{aligned}$$

□

Lemma A.6. *Let $\{a_J\}_J$ be a family of real numbers indexed by a multi-index $J \in \mathcal{N}^m$. Given a positive integer $k \geq 1$, we have that*

$$\sum_{|J|=k} a_J = \sum_{1\leq j_1,\dots,j_k\leq m}\frac{J_k!}{k!}a_{J_k}, \tag{A.9}$$

where $J_k := 1_{j_1} + \cdots + 1_{j_k}$.

Proof. We proceed by induction on k. The case $k = 1$ is trivial thus, let us suppose that the result is true for some integer $k \geq 1$ to show that it is also true for $k + 1$.

$$\begin{aligned}\sum_{|J|=k+1} a_J &= \sum_{j_{k+1}=1}^{m}\sum_{|J|=k}\frac{J(j_{k+1})+1}{k+1}a_{J+1_{j_{k+1}}}\\ &= \sum_{j_{k+1}=1}^{m}\sum_{1\leq j_1,\dots,j_k\leq m}\frac{J_k!}{k!}\frac{J_k(j_{k+1})+1}{k+1}a_{J_k+1_{j_{k+1}}}\\ &= \sum_{1\leq j_1,\dots,j_{k+1}\leq m}\frac{J_{k+1}(j_{k+1})\cdot J_k!}{(k+1)!}a_{J_{k+1}}\end{aligned}$$

□

Lemma A.7. *Let $\{a_J, b^J\}_J$ be a family of real numbers indexed by a multi-index $J \in \mathcal{N}^m$. Given an integer $l \geq 1$, we have that*

$$\sum_{|J|=l} b^J a_J = \sum_{|I|=l-1}\sum_{i=1}^{m}\frac{I(i)+1}{|I|+1}(b^{I+1_i} + Q^{I,i})a_{I+1_i}, \tag{A.10}$$

where $\{Q^{I,i}\}_{I,i}$ is a family of real numbers such that for any multi-index $J \in \mathcal{N}^m$ (with $|J| \geq 1$) we have that

$$\sum_{I+1_i=J}\frac{I(i)+1}{|I|+1}Q^{I,i} = 0. \tag{A.11}$$

Proof.

$$\begin{aligned}\sum_{|J|=l} b^J a_J &= \sum_{|J|=l}\left(\sum_{I+1_i=J}\frac{I(i)+1}{|I|+1}\right)b^J a_J\\ &= \sum_{|J|=l}\sum_{I+1_i=J}\frac{I(i)+1}{|I|+1}(b^{I+1_i} + Q^{I,i})a_{I+1_i}.\end{aligned}$$

□

Appendix B

Groupoids and G-structures

B.1 Groupoids

Groupoids are a generalization of groups; indeed, they have a composition law with respect to which there are some identity elements and every element has an inverse. For a good reference on groupoids, the reader is refered to Mackenzie [117].

Definition B.1. Given two sets Ω and M, a *groupoid* Ω *over* M, the *base*, consists of these two sets together with two mappings $\alpha, \beta : \Omega \to M$, called the *source* and the *target projections*, and a composition law satisfying the following conditions:

1. The composition law is defined only for those $\eta, \xi \in \Omega$ such that $\alpha(\eta) = \beta(\xi)$ and, in this case, $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$. We will denote $\Omega_\Delta \subset \Omega \times \Omega$ the set of such pairs of elements.

2. The composition law is associative, that is $\zeta(\eta\xi) = (\zeta\eta)\xi$ for those $\zeta, \eta, \xi \in \Omega$ such that each member of the previous equality is well defined.

3. For each $x \in M$ there exists an element $1_x \in \Omega$, called the *unity over* x, such that

   (a) $\alpha(1_x) = \beta(1_x) = x$;

   (b) $\eta \cdot 1_x = \eta$, whenever $\alpha(\eta) = x$;

   (c) $1_x \cdot \xi = \xi$, whenever $\beta(\xi) = x$.

4. For each $\xi \in \Omega$ there exists an element $\xi^{-1} \in \Omega$, called the *inverse of* ξ, such that

   (a) $\alpha(\xi^{-1}) = \beta(\xi)$ and $\beta(\xi^{-1}) = \alpha(\xi)$;

   (b) $\xi^{-1}\xi = 1_{\alpha(\xi)}$ and $\xi\xi^{-1} = 1_{\beta(\xi)}$.

The groupoid Ω will be said *transitive* if, for every pair $x, y \in M$, the set of elements that have x as source and y as target, *i.e.* $\Omega_{x,y} = \alpha^{-1}(x) \cap \beta^{-1}(y)$, is not empty.

A subset $\Omega' \subset \Omega$ is said to be a *subgroupoid of* Ω *over* M if itself is a groupoid over M with the composition law of Ω.

The elements of M are often called *objects* and those of Ω *arrows* due to their graphical interpretation as we may see in the Figure B.1 or in the example B.2. By the very definition of groupoids, the unity over an object and the inverse of an arrow are unique. Note also that $\Omega_{x,x}$ is a group and the unity 1_x is the group identity.

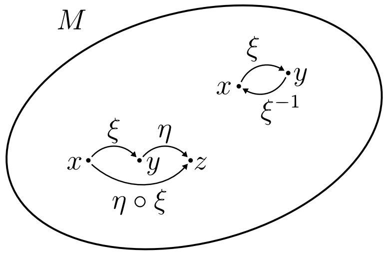

Figure B.1: The arrow picture.

Example B.2 (The trivial groupoid). Let M denote any non-empty set. The Cartesian product $M \times M$ is trivially a groupoid over M. The source of an arrow (x, y) is x and the target y, and the composition $(y', z) \cdot (x, y)$ is (x, z) if and only if $y' = y$.

Example B.3 (The action groupoid). Now, let G be a group acting on the left on M. Then the product $G \times M$ is a groupoid over M with the following structural maps:

- the source, $\alpha(g, x) = x$;
- the target, $\beta(g, x) = g \cdot x$;
- and the composition law, $(h, y) \cdot (g, x) = (h \cdot g, x)$ if and only if $y = g \cdot x$.

With these considerations, the unity over an element $x \in M$ and the inverse of an arrow $(g, x) \in G \times M$ are respectively given by $1_x = (e, x)$ and $(g^{-1}, g \cdot x)$, where $e \in G$ denotes the identity and g^{-1} the inverse of g.

Proposition B.4. *Let Ω be a groupoid over a set M. Then, given three points $x, y, z \in M$ such that they can be connected by arrows, we have the relation*

$$\Omega_{x,z} = g \cdot \Omega_{x,y} = \Omega_{y,z} \cdot f, \quad \forall g \in \Omega_{y,z}, \ \forall f \in \Omega_{x,y}; \tag{B.1}$$

in particular,

$$\Omega_{y,y} = g \cdot \Omega_{x,x} \cdot g^{-1}, \quad \forall g \in \Omega_{x,y}. \tag{B.2}$$

For the moment, we have only algebraic structures on groupoids. Let us endow them with differential structures.

Definition B.5. We say that a groupoid Ω over M is a *differential groupoid* if the groupoid Ω and the base M are equipped with respective differential structures such that:

1. the source and the target projections $\alpha, \beta : \Omega \to M$ are smooth surjective submersions;
2. the *unity* or *inclusion map* $i : x \in M \mapsto 1_x \in \Omega$ is smooth;
3. and the composition law, defined on Ω_Δ, is smooth.

Additionally if Ω is transitive, then we call it a *Lie groupoid.*

A subgroupoid Ω' of a differential (or Lie) groupoid Ω which is in turn a differential groupoid with the restricted differential structure is called a *differential subgroupoid* (resp. Lie subgroupoid).

Note that the condition (1) in Definition B.5 implies that the $\alpha\beta$-diagonal Ω_Δ is an embeded submanifold of $\Omega \times \Omega$, and then (3) makes sense. Ver Eecke showed (*cf.* [117]) that, even with more relaxed conditions, the inverse map $\xi \in \Omega \mapsto \xi^{-1} \in \Omega$ is smooth, and therefore a diffeomorphism. In fact, there is a more general way to define groupoids and subgroupoids (differentiable or not) as the reader may find in [117], but for our purposes these definitions will be sufficient.

Example B.6 (The frame groupoid). Let M be a smooth manifold with dimension n and consider the space of linear isomorphisms between tangent spaces to M at any pair of points, namely

$$\Pi(M) = \bigcup_{x,y \in M} \mathrm{Iso}(T_x M, T_y M). \tag{B.3}$$

This set is called the *frame groupoid* of M and, in fact, it is a Lie groupoid over M, as we are going to show.

First of all, we must give a manifold structure to $\Pi(M)$. Let (U, ϕ) and (V, ψ) be two charts of M and consider the map given by

$$\begin{array}{rcl} \chi : W & \longrightarrow & \phi(U) \times \mathrm{Gl}(n) \times \psi(V) \\ A & \longmapsto & (x^i, A_i^j, y^j) \end{array} \tag{B.4}$$

where $\mathrm{Gl}(n)$ denotes the general linear group on $\mathbb{R}^n$,

$$W = \bigcup_{x \in U, y \in V} \mathrm{Iso}(T_x M, T_y M) \quad \text{and} \quad A\left(\frac{\partial}{\partial x^i}\right) = A_i^j \frac{\partial}{\partial y^j}. \tag{B.5}$$

By means of the induced chart (W, χ) we endow $\Pi(M)$ with a differential structure of dimension $2n + n^2$.

The structural maps are given in the following way:

- the source and the target projections: if $A \in \mathrm{Iso}(T_x M, T_y M)$, then $\alpha(A) = x$ and $\beta(A) = y$;
- the composition law is the natural composition between isomorphisms when it is defined;
- and the inclusion: if $x \in M$, then the unity 1_x over x is the identity map of $\mathrm{Gl}(T_x M) = \mathrm{Iso}(T_x M, T_x M)$.

These maps define clearly a groupoid over M and, through (B.4) and (B.5), they are smooth for the differential structure naturally induced from the one of M.

Example B.7 (The unimodular groupoid). Let M be an orientable smooth manifold of dimension n and let ρ be a volume form on it (in a more general case, without the assumption of orientation, we can consider a volume density). We can use ρ to define a determinant function over the frame groupoid $\Pi(M)$ by the formula:

$$\rho(A \cdot v_1, \ldots, A \cdot v_n) = \det\nolimits_\rho(A) \cdot \rho(v_1, \ldots, v_n) \quad \forall A \in \Pi(M), \tag{B.6}$$

where $v_1, \ldots, v_n \in T_{\alpha(A)}M$. Now, it is easy to check that the set of unimodular transformations

$$\mathcal{U}(M) = \det_\rho^{-1}(\{-1, +1\}), \tag{B.7}$$

which is called the *unimodular groupoid*, is a transitive subgroupoid of $\Pi(M)$. In fact, it is a Lie subgroupoid of $\Pi(M)$, since $\det_\rho$ is a smooth submersion and thus $\mathcal{U}(M)$ is a closed submanifold.

Example B.8 (The orthogonal groupoid). Let (M, g) be a Riemannian manifold of dimension n and consider the space of orthogonal linear isomorphisms between tangent spaces to M at any pair of points, namely

$$\mathcal{O}(M) = \bigcup_{x,y \in M} O(T_xM, T_yM). \tag{B.8}$$

This set is called the *orthogonal groupoid* of M and, with the restriction to it of the structure maps of the frame groupoid $\Pi(M)$, $\mathcal{O}(M)$ is a subgroupoid of $\Pi(M)$. Since $\mathcal{O}(M)$ is defined by closed and smooth conditions, namely

$$\mathcal{O}(M) = \left\{ A \in \Pi(M) \ : \ A^{-1} = A^T \right\},$$

this set is a closed submanifold of $\Pi(M)$, and thus a Lie subgroupoid.

Furthermore, the orthogonal groupoid $\mathcal{O}(M)$ is also a Lie subgroupoid of the unimodular groupoid $\mathcal{U}(M)$ related to the Riemannian density induced by the metric.

Definition B.9. Let Ω be a groupoid over M; then the *normalizoid* of a subgroupoid $\tilde{\Omega}$ of Ω is the set defined by

$$N(\tilde{\Omega}) = \left\{ g \in \Omega_{x,y} \ : \ \tilde{\Omega}_{y,y} = g \cdot \tilde{\Omega}_{x,x} \cdot g^{-1}, \ x, y \in B \right\}. \tag{B.9}$$

From the definition, it is obvious that a subgroupoid $\tilde{\Omega}$ of a groupoid Ω is also a subgroupoid of its normalizoid $N(\tilde{\Omega})$ which is, in turn, a subgroupoid of the ambient groupoid Ω.

Note that the group over a base point in the normalizoid is the normalizer of the group over this point in the subgroupoid, that is

$$(N(\tilde{\Omega}))_{x,x} = N(\tilde{\Omega}_{x,x}), \tag{B.10}$$

which explains the used terminology. The difference between a subgroupoid and its normalizoid can be huge. For instance, given a transitive groupoid Ω over a set M, consider its base groupoid, that is the subgroupoid consisting of the groupoid unities:

$$1(\Omega) = \{1_x : x \in M\}. \tag{B.11}$$

Then, the normalizoid of $1(\Omega)$ in Ω is the whole groupoid Ω. From now on, we will focus on subgroupoids of the frame groupoid over a manifold and we will see how to reduce the normalizoid of a subgroupoid whenever an extra structure is avaible on the base manifold.

First of all, recall that there exists a unique decomposition of a linear isomorphism into an orthogonal part and a symmetric one. More precisely, let $F : E \longrightarrow E'$ be a linear isomorphism between two inner product vector spaces E and E'. There exist an orthogonal map $R : E \longrightarrow E'$ and positive definite symmetric maps $U : E \longrightarrow E$, $V : E' \longrightarrow E'$ such that:

$$F = R \cdot U \quad \text{and} \quad F = V \cdot R. \tag{B.12}$$

As we have mentioned, each of these decompositions is unique and they are called the left and right polar decompositions of F, respectively; the orthogonal part R will be denoted by $F^{\perp}$.

Proposition B.10. *Let Ω be a (transitive) subgroupoid of the frame groupoid $\Pi(M)$ of a Riemannian manifold (M, g). Denote by $\bar{\Omega}$ the set of the orthogonal part of elements of Ω, that is*

$$\bar{\Omega} = \left\{F^{\perp} \ : \ F \in \Omega\right\}. \tag{B.13}$$

Then $\bar{\Omega}$ is a (transitive) subgroupoid of the orthogonal groupoid $\mathcal{O}(M)$. We call $\bar{\Omega}$ the orthogonal reduction of Ω *(or the* reduced groupoid, *for the sake of simplicity).*

Proof. In order to show that $\bar{\Omega}$ is a subgroupoid of $\mathcal{O}(M)$, we only have to check that it is a groupoid over M with the restriction of the structure maps of $\Pi(M)$, which is clear once we note that for any three linear isomorphisms F_1, F_2, F_3, such that $F_3 = F_2 \cdot F_1$, we have by the uniqueness of the polar decomposition that $F_3^{\perp} = F_2^{\perp} \cdot F_1^{\perp}$. □

Note that the orthogonal reduction of a normalizoid is not necessarily a subgroupoid of the original one.

Proposition B.11. *In the hypotesis of Proposition B.10, if Ω is such that, for every base point $x \in M$, $\Omega_{x,x}$ is a subgroup of $\mathcal{O}_{x,x}(M)$ (the orthogonal group at x), then the orthogonal reduction of the normalizoid of Ω coincides with the intersection of the orthogonal groupoid and the normalizoid itself,* i.e.

$$\bar{\mathcal{N}}(\Omega) = \mathcal{N}(\Omega) \cap \mathcal{O}(M). \tag{B.14}$$

Proof. The inclusion $\bar{\mathcal{N}}(\Omega) \supset \mathcal{N}(\Omega) \cap \mathcal{O}(M)$ is clear and, from the above Proposition B.10, we have $\bar{\mathcal{N}}(\Omega) \subset \mathcal{O}(M)$, thus we only need to show that $\bar{\mathcal{N}}(\Omega) \subset \mathcal{N}(\Omega)$. Let $R \in \bar{\mathcal{N}}_{x,y}(\Omega)$, then there exist a linear isomorphism $F \in \mathcal{N}_{x,y}(\Omega)$ such that $F^{\perp} = R$. Since F conjugates the orthogonal subgroups $\Omega_{x,x}$ and $\Omega_{y,y}$, so does its orthogonal part (*cf.* [81], Lemma A.2). Hence, $R \in \mathcal{N}_{x,y}(\Omega)$ and $\bar{\mathcal{N}}(\Omega) \subset \mathcal{N}(\Omega) \cap \mathcal{O}(M)$. □

Similar results can be given whenever M is equipped with a volume form.

Proposition B.12. *Given a smooth manifold M, suppose it is endowed with a volume form (or density) ρ. If Ω denotes a (transitive) subgroupoid of the frame groupoid $\Pi(M)$, then the set*

$$\Omega^1 = \Omega / \det_{\rho}, \tag{B.15}$$

is a (transitive) subgroupoid of the unimodular groupoid $\mathcal{U}(M)$ associated with ρ and it will be called the unimodular reduction *of Ω.*

Even more, if Ω is such that, for every base point $x \in M$, $\Omega_{x,x}$ is a subgroup of $\mathcal{U}_{x,x}(M)$ (the unimodular group at x), then the unimodular reduction of the normalizoid of Ω coincides with the intersection of the unimodular groupoid and the normalizoid itself, i.e.

$$\mathcal{N}^1(\Omega) = \mathcal{N}(\Omega) \cap \mathcal{U}(M). \tag{B.16}$$

B.2 G-structures

Lie subgroupoids of the frame groupoid of a manifold are closely related to another geometric object: G-structures, which are a particular case of fiber bundles. For a comprehensive reference related to principal fiber bundles and G-structures see [84, 110, 111]. We give here their definition and some results about the interconnection with groupoids.

Definition B.13. Given two manifolds P, M and a Lie group G, we say that P is a *principal bundle* over M with *structure group* G if G acts on the right on P and the following conditions are satisfied:

1. the action of G is free, *i.e.* the fact that $ua = u$ for some $u \in P$ implies $a = e$, the identity element of G;
2. $M = P/G$, which implies that the canonical projection $\pi : P \longrightarrow M$ is differentiable;
3. P is locally trivial, *i.e.* P is locally isomorphic to the product $M \times G$, which means that for each point $x \in M$ there exists an open neighborhood U and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times G$ such that $\Phi = \pi \times \phi$, where the map $\phi : \pi^{-1}(U) \longrightarrow G$ has the property $\phi(ua) = \phi(u)a$ for all $u \in \pi^{-1}(U)$, $a \in G$.

A principal bundle is commonly denoted by $P(M, G)$, $\pi : P \longrightarrow M$ or simply by P, when there is no ambiguity. The manifold P is called the *total space*, M the *base space*, G the *structure group* and π the *projection*. The closed submanifold $\pi^{-1}(x)$, with $x \in M$, is called the *fiber over* x and is denoted P_x; if $u \in P$, $P_{\pi(u)}$ is called the *fiber through* u and is denoted P_u. The maps given in (3) are called *(local) trivializations*.

It should be remarked that a similar definition can be given for left principal bundles using left actions.

Notice that any fiber P_x is diffeomorphic to the structure group G, but not canonically so. On the other hand, if we fix $u \in P_x$, then $P_u = uG$. We may visualize a principal fiber bundle $P(M, G)$ as a copy of the structure Lie group G at each point of the base manifold M in a diffentiable way as it is stated by the trivialization property (3).

An elementary example of principal bundle is the *frame bundle* $\mathcal{F}M$ of a manifold M. This manifold consists of all the reference frames at all the point of M. The frame bundle $\mathcal{F}M$ is a principal bundle over M with structure group $\mathrm{Gl}(n)$, where n is the dimension of M. As it is obvious, the canonical projection π sends any frame $x \in \mathcal{F}M$ to the base point $x \in M$ where it lies. The right action of $\mathrm{Gl}(n)$ over M is defined in the following way:

$$\begin{array}{rcl} R : \mathcal{F}M \times \mathrm{Gl}(n) & \longrightarrow & \mathcal{F}M \\ (z, a) & \longmapsto & R_a z = z \cdot a = (a_i^j v_j), \end{array} \tag{B.17}$$

where (a_i^j) is the matrix representation of $a \in \mathrm{Gl}(n)$ in the canonical basis of $\mathbb{R}^n$ and (v_i) is the ordered basis given by $z \in \mathcal{F}M$.

Definition B.14. Let $P(M, G)$ and $Q(M, H)$ be two principal bundles such that Q is an embedded submanifold of P and H is a Lie subgroup of G. We say that $Q(M, H)$ is a *reduction* of the structure group G of P if the principal bundle structure of $Q(M, H)$ comes from the restriction of the action of G on P to H and Q. In this case, we call Q the *reduced bundle*.

Consider the following (non rigorous) construction: take a principal bundle $P(M, G)$, shrink its structure group to a Lie subgroup H of G, fix an element $u \in P$ in each fibre of the bundle and apply the action of H to each of these chosen elements; this gives us a subset $Q \subset P$. The obtained set Q is a reduced bundle when the selection of the u's is made smoothly and with certain compatibility.

Definition B.15. Let M be an n-dimensional smooth manifold and G a Lie subgroup of $\mathrm{Gl}(n)$; then a *G-structure* $G(M)$ is a G-reduction of the frame bundle $\mathcal{F}M$.

Note that there may exist different G-structures with the same structure group. As an example of G-structure, consider a Riemannian manifold (M, g). The set of orthonormal references of $\mathcal{F}M$ gives us an $O(n)$-structure. In fact, any $O(n)$-structure on M is equivalent to a Riemannian structure (see [84]).

Now let us introduce two results from [118] that show how a G-structure may arise from a Lie groupoid.

Proposition B.16. *Let Ω be a Lie groupoid over a smooth manifold M with source and target projections α and β, respectively. Given any point $x \in M$, we have that:*

1. *$\Omega_{x,x} = \alpha^{-1}(x) \cap \beta^{-1}(x)$ is a Lie group and*
2. *$\Omega_x = \alpha^{-1}(x)$ is a principal $\Omega_{x,x}$-bundle over M whose canonical projection is the restriction of β.*

Given a smooth manifold M of dimension n, any reference $z \in \mathcal{F}M$ (at a point $x \in M$) may be seen as the linear mapping $e_i \in \mathbb{R}^n \mapsto v_i \in T_xM$, where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{R}^n$ and $(v_1, \dots, v_n)$ the basis of T_xM defined by z.

Theorem B.17. *Suppose that M is a smooth n-dimensional manifold and Ω is a Lie subgroupoid of the frame groupoid $\Pi(M)$. If α and β denote the respective source and target projections of Ω, then we have that for any point $x \in M$ and any frame reference $z \in \mathcal{F}M$ at x:*

1. *$G_z = z^{-1} \cdot \Omega_{x,x} \cdot z$ is a Lie subgroup of $\mathrm{Gl}(n)$ and*
2. *the set Ω_z of all the linear frames obtained by translating z by Ω_x, that is*
$$\Omega_z = \{g_{x,y} \cdot z \ : \ g_{x,y} \in \Omega_x\}, \tag{B.18}$$
is a G_z-structure on M.

Once the reference z is fixed, the linear frames that lie in the G_z-structure are called adapted *or* distinguished *references.*

Even though the frame groupoid (and hence each of its subgroupoids) acts on the left on the frame bundle of the base manifold, the structural group that arises from a frame subgroupoid acts naturally on the right on any of the induced G-structures:

$$z_y \cdot g_{z_x} = (g_{x,y} \cdot z_x) \cdot (z_x^{-1} \cdot g_{x,x} \cdot z_x) = g_{x,y} \cdot g_{x,x} \cdot z_x = g'_{x,y} \cdot z_x = z'_y, \tag{B.19}$$

where $z_x \in \mathcal{F}_xM$, $z_y \in (\Omega_{z_x})_y$, $g_{z_x} \in G_{z_x}$, $g_{x,y} \in \Omega_{x,y}$ and so on.

Remark B.18. It is readily seen from equation (B.2) that two G-structures that come from the same Lie groupoid are equal if and only if they have a reference in common,

$$\Omega_{z_1} = \Omega_{z_2} \Leftrightarrow \Omega_{z_1} \cap \Omega_{z_2} \neq \emptyset. \tag{B.20}$$

Here "equal" means that the two G-structures are the same as sets and they have the same structure groups. By the above statement, given two G-structures Ω_{z_1} and Ω_{z_2} induced by a Lie groupoid Ω, we can suppose without loss of generality that z_1 and z_2 are linear frames at the same base point. Thus, it is easy to see that their respective structure groups G_{z_1} and G_{z_2} are conjugate; more precisely:

$$G_{z_2} = z_2^{-1} z_1 \cdot G_{z_1} \cdot z_1^{-1} z_2. \tag{B.21}$$

In short, given a Lie subgroupoid Ω of $\Pi(M)$, the frame bundle $\mathcal{F}M$ is the disjoint union of G-structures related to Ω by Theorem B.17. Moreover, they have conjugate group structures and one of these G-structures may be transformed to another by means of any element $g \in \mathrm{Gl}(n)$ that conjugates their structural groups. Hence, modulo these transformations, a G-structure related to a Lie subgroupoid Ω of $\Pi(M)$ is unique, which is clear since Ω is fixed.

A natural question is whether Theorem B.17 has a converse. Given a G-structure, it seems reasonable to be able to choose differentially isomorphisms that transform adapted references to their counterparts.

Theorem B.19. *Let ω be a G-structure over an n-dimensional smooth manifold M. Then the set of linear isomorphism that transforms distinguished frames into distinguished frames, that is the set*

$$\Omega = \left\{ A \in \Pi(M) \ : \ Az \in \omega, \ z \in \omega_{\alpha(A)} \right\}, \tag{B.22}$$

where $\Pi(M)$ is the frame groupoid of M and α the source projection, is a Lie soubgroupoid of $\Pi(M)$. Furthermore, for any reference frame $z \in \omega$, the G-structure associated to Ω and given by Theorem B.17 coincides with ω, i.e.

$$\Omega_z = \omega \quad \textit{and} \quad G_z = G. \tag{B.23}$$

Proof. The set defined by equation (B.22) is obviously a transitive subgroupoid of $\Pi(M)$. It remains only to show that it is a differential groupoid with the restriction of the structural maps. Given two local cross-sections (U, σ) and (V, τ) of ω, consider the set of isomorphisms in Ω with source in U and target in V, namely

$$\Omega_{U,V} = \alpha^{-1}(U) \cap \beta^{-1}(V), \tag{B.24}$$

where α and β are the restrictions to Ω of the source and the target projections of $\Pi(M)$. Given an isomorphism $A \in \Omega_{U,V}$, let $x = \alpha(A) \in U$ and $y = \beta(A) \in V$. If we denote the components of the ordered bases $\sigma(x)$ and $\tau(y)$ by $(\sigma_i(x))$ and $(\tau_j(y))$ respectively, we have that there exist coefficients A_i^j such that

$$A\sigma_i(x) = A_i^j \tau_j(y). \tag{B.25}$$

Since $\sigma(x) = (\sigma_i(x))$ is a linear frame at x in ω, $A\sigma(x) = (A_i^j \tau_j(y))$ is a linear frame at y in ω too. But $\tau(y) = (\tau_j(y))$ is also a linear frame at y in ω, thus $a = (A_i^j)$ must necessarily be an element of the structure group G. This consideration being made, we define the coordinate chart $\Phi_{\sigma,\tau}$ by

$$\begin{array}{rcl}\Phi_{\sigma,\tau} : \Omega_{U,V} & \longrightarrow & U \times G \times V \\ A & \longmapsto & (x, a, y)\end{array} . \tag{B.26}$$

Given a covering of M by local sections of ω, say Σ, the atlas

$$\{(\Omega_{U,V}, \Phi_{\sigma,\tau}) \ : \ (U,\sigma),(V,\tau) \in \Sigma\} \tag{B.27}$$

defines a smooth structure on Ω, from which it is a straightforward computation to show that the projections α and β and the composition law are smooth. □

Remark B.20. The result we have just proved, toghether with Theorem B.17, shows the equivalence between Lie subgroupoids of $\Pi(M)$ and reductions of the frame bundle $\mathcal{F}M$. In fact it is still true for principal bundles in general: by Proposition B.16 we are able to associate some principal bundles to a groupoid and, given a principal bundle $P(M,G)$, the set of maps $\phi_{x,y} : P_x \longrightarrow P_y$ such that $\phi_{x,y}(u \cdot g) = \phi_{x,y}(u) \cdot \phi(g)$, for a suitable group isomorphism $\phi : G \longrightarrow G$, is a Lie groupoid related to P by Proposition B.16.

Definition B.21. A G-structure $G(M)$ over a manifold M is said to be *integrable* if there exists an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ of the base manifold, such that the induced cross-sections $\sigma_\alpha(x) = (T_x\phi_\alpha)^{-1}$ take values in $G(M)$.

By the very definition, if a G-structure is integrable, the same happens to all its conjugate G-structures.

Theorem B.22. *A G-structure over a manifold M with dimension n is integrable if and only if it is locally isomorphic to the standard G-structure of $\mathbb{R}^n$, that is, to $\mathbb{R}^n \times G$.*

The following result will be useful in the next section.

Lemma B.23. *Let M be a manifold. If Ω and $\tilde{\Omega}$ are two subgroupoids of the frame groupoid $\Pi(M)$, then their intersection $\hat{\Omega} := \Omega \cap \tilde{\Omega}$ is again a subgroupoid of $\Pi(M)$ (and of Ω and $\tilde{\Omega}$). Furthermore, if they are Lie groupoids, then we have the following relations:*

$$\hat{\Omega}_z = \Omega_z \cap \tilde{\Omega}_z \quad \textit{and} \quad \hat{G}_z = G_z \cap \tilde{G}_z, \tag{B.28}$$

where $z \in \mathcal{F}M$ is a fixed frame and Ω_z, $\tilde{\Omega}_z$, $\hat{\Omega}_z$, G_z, $\tilde{G}_z$ and $\hat{G}_z$ are the respective G-structures and structural groups.

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