UNIVERSIDAD AUTÓNOMA DE MADRID
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS

On o-minimal homotopy

Memoria presentada para optar al grado de
Doctor en Matemáticas

Elías Baro González

Dirigida por
Margarita Otero Domínguez

Madrid, Marzo 2009

Contents

Agradecimientos

Siempre que me acerco a mi padre para comentarle que creo haberme equivocado en algo, él me dice que incluso el más genio entre los genios acierta, como mucho, una de cada tres decisiones. Después de escuchar semejante sentencia, suelo marcharme cabizbajo y pensativo mientras cuento con los dedos de las manos cada cuánto acierta entonces una persona normal como yo. En cualquier caso, el día que escogí a Margarita como directora de tesis sabía que no iba a equivocarme. Y es que, siendo sincero, tengo que confesar que no escogí la Lógica Matemática para hacer el doctorado, sino que la elegí a ella. No nos engañemos, escogemos entre los distintos caminos en función del maestro encargado de mostrarnos su belleza. Si me gustó la asignatura de Lógica Matemática no fue sólo mérito de los teoremas de Gödel, sino por la forma en que los expuso Margarita. Son muchos los papeles que ha tenido que desempeñar Margarita en estos años. Ha sido mucho el tiempo y el esfuerzo que me ha dedicado. Ha sido mucho lo que he aprendido de ella, y no sólo de matemáticas. Me ha apoyado incondicionalmente. Se lo agradezco muchísimo. Creo ser el primer alumno de Margarita, lo cierto es que me siento orgulloso de ello.

Toda mi gratitud también para Jose. Tengo que reconocer que tanto apoyo ha llegado incluso a desconcertarme. Desde prácticamente el primer día que le conocí no ha parado de ayudarme: me dió mi primera oportunidad de dar una charla en un congreso, me ha escuchado siempre que lo he necesitado y ha aceptado ser el lector de esta tesis.

A Anand Pillay y Dugald Macpherson por el tiempo que pasé en Leeds. Aquella fue mi primera estancia en el extranjero, de la cual guardo un recuerdo fantástico gracias, en buena medida, a ellos. A mi compañero en Leeds, Ricardo, cuyo cocido italojapocastellano ha marcado un antes y un después en mi vida culinaria. También a Aniceto Murillo y Antonio Viruel (y sus estudiantes) por lo bien que me trataron en Málaga. De hecho, toda una sección de esta tesis surgió a raíz de lo que aprendí de ellos. A Kobi Peterzil, por el tiempo que dedicó a mis preguntas. A Mario Edmundo, por su paciencia en la elaboración del corrigendum. A María Ángeles Zurro por el tiempo que me dedicó con el trabajo del DEA. Gracias también a Jesús Ruíz y a Antonio Díaz-Cano.

Quiero dar las gracias también a mis compañeros del departamento, son muchos y me da miedo dejar de nombrar a alguno. Aún así, quiero agradecérselo especialmente a los que han estado más cerca de mí. A Nati, que es la auténtica voz de la razón y la eficiencia. Luego resulta que además es la persona más valiente que he conocido nunca, capaz de venirse con sus tres hijos de Luanda a Madrid para hacer el doctorado pero, como digo, eso viene después. A Charro, que me ha hecho reir mucho con sus mundos paralelos. A Adrián, que cuando no estaba Charro para molestarle, ya estaba yo. A Mariluz, que me ha ayudado con todos los papeleos habidos y por haber. A Angélica, por ser sincera, por ser siempre ella misma.

Normalmente se dice que cuando los niños pasan de la infancia a la pubertad dejan de pensar que sus padres son superhéroes. Yo pasé de creerlo a tener el total convencimiento. Mis padres son una parte importante de esta tesis, y no sólo por su atención y su cariño. En cierta manera, no me puede resultar raro haber tenido el impulso casi instintivo de llevar mis estudios lo más lejos posible. Y es que le tengo cariño a aquellos años en los que mi madre decidió estudiar (¡ya con tres hijos!) la carrera que no tuvo la oportunidad de hacer cuando era joven. Lo recuerdo, entre otras cosas, porque cuando ella se metía en la cocina con sus libros (ya se sabe que las cocinas son las mejores salas de estudios de una casa) yo no podía interrumpirla, aún cuando mis hermanos mayores estuviesen haciéndome de rabiar (injustamente, por supuesto). Las enseñanzas de mi padre me vienen a la cabeza continuamente, en cualquier situación, basta ver cómo he empezado estos agradecimientos. Tengo suerte de haber tenido tal ejemplo en mi propio hogar.

A mi hermano Pepe, que con el ejemplo me ha enseñado qué es un científico: una persona que, estando a solas en una habitación con un destornillador en la mano y enfrente de una máquina de incalculable valor de la que cuelga un cartel advirtiendo ”No tocar”, una extraña y grave patología le impide tardar más de un nanosegundo en destriparla para descubrir cómo funciona. Así pues, para mi hermano Pepe, la única persona en el mundo capaz de unir dos cables con una servilleta de papel.

A mi hermano Nacho, cuyo camino ha marcado siempre el mío. Supongo que esta tesis supone una divergencia en nuestros caminos. Tengo que reconocer que a veces me he llegado a plantear si esta tesis no era más que una forma de diferenciarme de él o incluso un intento (sano) de superarle, lo cual, por fin he comprendido, es imposible.

Para toda mi familia, que son muchos y no tengo espacio suficiente para todos. Yo creía que una familia tan unida como la mía era algo común, sin embargo con el tiempo me voy dando cuenta de que es un bien preciado que no todo el mundo posee. Sí que quiero mencionar especialmente a mis abuelos Paqui, Salvador, Pepa y Julio, porque si ya dije que mis padres me parecen superhéroes, ellos (y toda su generación) son para mi directamente inexplicables.

Para mis colegas del barrio Anacleta, Dani, Manolo y Nacho, por tantos buenos ratos.

Para la familia y la ”familia” de Ana, con la que he pasado casi tanto tiempo como con la mía.

Para esos enanos con los que tanto me divierto: para Marquitos, Luisa, Charlie y la Melocotona.

Mi número favorito es el 101. Lo cierto es que no sé qué curiosas propiedades matemáticas (más allá de las evidentes) tendrá este número. Lo que sí sé es que es un número que me da tranquilidad. Pase lo que pase, el 101 siempre está ahí para calmarme, para ayudarme, para apoyarme. El 101 siempre me ofrece las palabras que necesito oír. El 101 cree en mí; me hace pensar, como dice aquel, que un brillo mágico alumbra mi camino. Ha habido días en que ningún maldito símbolo ha tenido la más mínima piedad conmigo. Sin embargo, esos símbolos sabían que no podían ganar. Sabían que, al final del día, rabiosos, verían cómo inevitablemente yo cogería el teléfono, marcaría el 101, esperaría unos pocos tonos...¿Ana?.

A los que no están, estén donde estén.

Introducción

Presentación

A principios de los años ochenta, L. van den Dries, A. Pillay y C. Steinhorn, entre otros, observaron que, dada una expansión de un cuerpo realmente cerrado, podían ser deducidas propiedades similares a las de los conjuntos y aplicaciones semialgebraicas tan sólo exigiendo a dicha expansión que los conjuntos definibles en dimensión uno fuesen una unión finita de intervalos y puntos. A tales estructuras se las conocería más tarde como estructuras o-minimales, una apasionante área de investigación donde confluyen la teoría de modelos y la geometría algebraica real. Existen dos líneas fundamentales de investigación sobre estructuras o-minimales: una se centra en la búsqueda de nuevos ejemplos de estructuras o-minimales, la otra se ocupa del estudio de los conjuntos y estructuras definibles. El trabajo de A. Wilkie ha inspirado un gran número de resultados en la primera línea, siendo de especial importancia [37] donde el autor prueba que el cuerpo de los reales expandido con la función exponencial es o-minimal. También caben destacar en esta dirección los resultados de van den Dries, Macintyre y Marker en [16] y los de Rolin, Speissegger y Wilkie en [34].

Esta memoria está dedicada a la segunda línea de investigación. Resultados básicos en esta línea son el Teorema de descomposición celular y el Teorema de triangulación, probados por van den Dries, Pillay, Steinhorn y Knight. Más recientemente se ha llevado a cabo un profundo estudio de las propiedades topológicas de conjuntos definibles, como la existencia de teorías de homología y cohomología en el contexto o-minimal. El desarrollo de estas herramientas de topología algebraica en un cuerpo realmente cerrado no arquimediano no es inmediato: recuérdese que ya incluso para la definición clásica de aplicación inducida en homología por una función continua es necesario usar el lema de Lebesgue (véase la introducción de la Sección 1.6). El tema principal de esta tesis es la homotopía o-minimal. H. Delfs y M. Knebusch desarrollaron en [13] una teoría de homotopía semialgebraica. Recordemos brevemente los principales resultados de dicho trabajo.

Dado un conjunto semialgebraico punteado (X, x) sobre un cuerpo realmente cerrado R, definimos de forma natural el n-ésimo grupo de homotopía semialgebraica $\pi_n(X, x)^{\text{sa}}$ al igual que en el caso clásico pero usando tan sólo aplicaciones semialgebraicas y homotopías semialgebraicas. Al contrario de lo que ocurre en el desarrollo de la homología, con la definición anterior no surge ningún problema a la hora de definir la aplicación inducida en homotopía por una aplicación semialgebraica continua. Sin embargo, en esta ocasión aparecen otras cuestiones muy naturales: ¿tienen estos grupos de homotopía semialgebraica alguna relación con los clásicos?, ¿tienen un buen comportamiento bajo extensiones de cuerpos realmente cerrados? Los autores responden afirmativamente ambas preguntas con los siguientes dos interesantes resultados (véanse los enunciados generales en 2.1.1 y 2.1.2). En el primero de ellos, prueban que dado un cuerpo realmente cerrado S extendiendo a R, la aplicación $\pi_n(X, x)^{\text{sa}} \to \pi_n(X(S), x)^{\text{sa}} : [f] \mapsto [f(S)]$ es un isomorfismo, donde $X(S)$ y $f(S)$ denotan la realización en S de X y f respectivamente. En el segundo resultado, muestran que si $R = \mathbb{R}$ entonces la aplicación $\pi_n(X, x)^{\text{sa}} \to \pi_n(X, x) : [f] \mapsto [f]$ es un isomorfismo. Para probar estos teoremas Delfs y Knebusch hacen uso tanto de la modelo completitud de la teoría de cuerpos realmente cerrados (o principio de Tarski-Seidenberg) como del número de Lebesgue, los cuales no están disponibles en el contexto o-minimal. Por tanto, no podemos aplicar estos métodos para desarrollar una teoría de homotopía o-minimal. De hecho, tan sólo el grupo fundamental o-minimal pudo ser considerado en [20] y [7], eso sí, con fuertes consecuencias en el estudio de grupos definibles como veremos en el siguiente párrafo.

El estudio de grupos definibles se enmarca también en la segunda línea de investigación antes mencionada. En [32], A. Pillay prueba que todo grupo definible G en una estructura o-minimal está dotado de una estructura de variedad definible que convierte a G en un grupo topológico. Por tanto, si el cuerpo de nuestra estructura o-minimal es el cuerpo de los reales, entonces obtenemos un grupo de Lie. Este resultado supuso el punto de partida del estudio de los grupos definibles y, en particular, del estudio de sus analogías con los grupos de Lie. Para este propósito, las herramientas de topología algebraica han resultado ser muy útiles. Por ejemplo, el siguiente resultado de M. Otero y M. Edmundo en [20] es uno de los primeros mostrando la mencionada analogía. Sea G un grupo definible abeliano definiblemente compacto y definible conexo de dimensión n en una estructura o-minimal. Entonces tanto el álgebra de cohomología sobre $\mathbb{Q}$ y el grupo fundamental o-minimal de G son isomorfos respectivamente al álgebra de cohomología sobre $\mathbb{Q}$ y al grupo fundamental de un toro de dimensión n. A partir de estos isomorfismos deducen que la torsión de G es isomorfa a la torsión de un toro de dimensión n. Actualmente, las conjeturas de Pillay, formuladas en [33] y resueltas positivamente en [9] y [23] (basándose en el trabajo de diversos autores), han conseguido estrechar aún más la relación existente entre los grupos de Lie y los grupos definibles. En particular, se conoce que todo grupo definible definiblemente compacto es una extensión de un grupo de Lie compacto por un subgrupo normal divisible y sin torsión. Además la dimensión o-minimal del grupo definible coincide con la dimensión del grupo de Lie.

Por otro lado, en [13], H. Delfs y M. Knebusch introducen una nueva categoría que extiende a la semialgebraica y lo suficientemente general como para trabajar con objetos naturales como pueden ser los recubrimentos de "grado infinito". Intuitivamente, los autores definen los espacios locamente semialgebraicos como aquellos que se obtienen pegando adecuadamente infinitos conjuntos semialgebraicos. En el contexto o-minimal tenemos la situación correspondiente, es decir, la categoría definible no es lo suficientemente grande como para trabajar con objetos naturales como ocurre con los recubrimentos de grado infinito. Sin embargo, la teoría de espacios localmente semialgebraicos no ha sido extendida formalmente al ambiente o-minimal, aunque a lo largo del estudio de grupos definibles han surgido algunas nociones relacionadas. Este es el caso de los grupos $\bigvee$-definibles, usados por Y. Peterzil, A. Pillay y S. Starchenko en [31] y [30] como una herramienta para el estudio de problemas de interpretabilidad. Después, M. Edmundo introduce en [17] una noción restringida de grupo $\bigvee$-definible, los grupos "localmente definibles", y desarrolla toda una teoría sobre ellos. Sin embargo, para trabajar con conceptos topológicos, la noción de grupo $\bigvee$-definible es demasiado rígida, como prueba la existencia de tres definiciones no equivalentes de conexión (véase la Sección 3.6 más abajo) establecidas para estos grupos en la literatura (véanse [17],[27] y [31]).

Conclusiones

Los principales resultados de esta memoria se centran en el Teorema de las triangulaciones normales y sus aplicaciones, y la homotopía o-minimal de conjuntos definibles y su generalización a los espacios locamente definibles. Pasemos ahora a describirlos brevemente. Fijemos una expansión o-minimal $\mathcal{R}$ de un cuerpo realmente cerrado R y denotemos por $\mathcal{R}_0$ la estructura de cuerpo ordenado de R. Cuando escribimos *definible* queremos decir definible en $\mathcal{R}$ con parámetros. Todas las aplicaciones definibles se suponen continuas.

En el Capítulo 1 probaremos el siguiente refinamiento del Teorema de triangulación, el cual es nuevo también en el contexto semialgebraico (véase el Teorema 1.1.5).

Teorema (Teorema de las triangulaciones normales). *Sea K un com-*

plejo simplicial y sean $S_1, \ldots, S_l$ subconjuntos definibles de su realización $|K|$. Entonces, existen una subdivisión K' de K y un homeomorfismo definible $\phi' : |K'| \to |K|$ tales que
(i) (K', ϕ') *parte todos los subconjuntos $S_1, \ldots, S_l$ y cada $\sigma \in K$,*
(ii) *para todo $\tau \in K'$ y $\sigma \in K$, si $\tau \subset \sigma$ entonces $\phi'(\tau) \subset \sigma$.*

Independientemente del papel esencial que desempeña este resultado en el desarrollo de la homotopía o-minimal en el Capítulo 2, el Teorema de triangulaciones normales puede ser de interés por sí mismo como muestra la siguiente situación. Como ya sabemos, una herramienta básica para el estudio de conjuntos definibles es el Teorema de triangulación: dado un conjuntos definible S y dados algunos subconjuntos definibles $S_1, \ldots, S_l$ de S, existe un complejo simplical K y un homeomorfismo definible $\phi : |K| \to S$ que parte $S_1, \ldots, S_l$. Un análisis más profundo de S podría llevarnos a considerar nuevos subconjuntos definibles $S'_1, \ldots, S'_{l'}$ de S. En esta situación, nos gustaría preservar de alguna forma la triangulación ya obtenida y partir al mismo tiempo los nuevos subconjuntos. Sin embargo, un nuevo uso del Teorema de triangulación no puede ayudarnos en este caso. Es más, técnicas como la repetición de subdivisiones baricéntricas no funcionan en nuestro contexto ya que, ni disponemos de un número de Lebesgue, ni los subconjuntos S_i tienen por qué ser abiertos. Obviamente, el Teorema de la triangulaciones normales resuelve este problema. Observamos también que el Teorema de las triangulaciones normales puede darnos nueva información incluso en el caso de que el cuerpo base de la estructura o-minimal sea $\mathbb{R}$.

Aplicaremos primero el Teorema de las triangulaciones normales para probar el siguiente resultado (véase el Teorema 1.5.5).

Corolario (**Hauptvermutung semialgebraico**). *Sean K y L dos complejos simpliciales cerrados en R y sea $f : |K| \to |L|$ un homeomorfismo semialgebraico. Entonces f es semialgebraicamente homótopo a un isomorfismo simplicial $g : |K'| \to |L'|$ entre subdivisiones K' y L' de K y L, respectivamente.*

Como segunda aplicación del Teorema de las triangulaciones normales mostraremos una prueba alternativa a aquella de A. Woerheide de la existencia de un funtor de homología simplicial o-minimal (véase la Sección 1.6).

El resultado principal del Capítulo 2 establece una relación entre los conjuntos de homotopía o-minimal y los conjuntos de homotopía semialgebraica (véase el Teorema 2.3.4). Para evitar un exceso de notación, mostramos aquí tan sólo una versión débil que, esperamos, dé una idea clara del resultado.

Teorema. *Sean X e Y conjuntos semialgebraicos. Entonces,*
(i) *toda aplicación definible $f : X \to Y$ es definiblemente homotópica a una*

aplicación semialgebraica $g : X \to Y$, y
(ii) *cualesquiera aplicaciones semialgebraicas $g_1 : X \to Y$ y $g_2 : X \to Y$ que son definiblemente homotópicas son también semialgebraicamente homotópicas.*

Como primera aplicación de este teorema estudiaremos la categoría de Lusternik-Schnirelmann (MSC 2000: 55M30) de conjuntos definibles. Por lo que sabemos, esta es la primera vez que se estudia la LS-categoría en los contextos o-minimal y semialgebraico. La LS-categoría de un conjunto definible X, $\mathrm{cat}(X)^{\mathcal{R}}$, es el menor entero m tal que X tiene un recubrimiento de $m+1$ abiertos definibles, cada uno de ellos definiblemente contractible en X a un punto. Probaremos el siguiente resultado (véase el Corolario 2.6.9).

Corolario 1. *Sea X un conjunto semialgebraico definido sin parámetros. Entonces $cat(X)^{\mathcal{R}} = cat(X(\mathbb{R}))$, donde $cat(X(\mathbb{R}))$ denota la LS-categoría clásica.*

Como segunda aplicación del Teorema 2.3.4 (mencionado más arriba), probamos que los grupos de homotopía semialgebraica $\pi_n(X,x)^{\mathcal{R}_0}$ y los grupos de homotopía o-minimal $\pi_n(X,x)^{\mathcal{R}}$ de un conjunto semialgebraico punteado (X,x) son isomorfos. Es decir, la aplicación $\rho : \pi_n(X,x)^{\mathcal{R}_0} \to \pi_n(X,x)^{\mathcal{R}} : [f] \mapsto [f]$ es un isomorfismo natural (véase el Teorema 2.4.1). Usando este último resultado y aquellos de H. Delfs y M. Knebusch sobre homotopía semialgebraica obtendremos lo siguiente (véase el Corolario 2.4.4).

Corolario 2. *Sea (X,x) un conjunto punteado semialgebraico definido sin parámetros. Entonces existe un isomorfismo natural entre el grupo de homotopía clásico $\pi_n(X(\mathbb{R}),x)$ y el grupo de homotopía o-minimal $\pi_n(X(R),x)^{\mathcal{R}}$ para todo $n \geq 1$.*

Las hipótesis del corolario anterior no suponen ningún impedimento ya que, recuérdese, gracias al Teorema de triangulación cualquier conjunto definible punteado es definiblemente homeomorfo a un conjunto semialgebraico definido sin parámetros.

Otras aplicaciones del Teorema 2.3.4 son las siguientes (véanse los teoremas 2.5.3 y 2.5.7).

Corolario 3 (Los teoremas o-minimales de Hurewicz). *Sea (X,x) un conjunto definible punteado y sea $n \geq 2$. Supongamos que $\pi_r(X,x)^{\mathcal{R}} = 0$ para todo $0 \leq r \leq n-1$. Entonces el homomorfismo de Hurewicz o-minimal*

$$h_{n,\mathcal{R}} : \pi_n(X,x_0)^{\mathcal{R}} \to H_n(X)^{\mathcal{R}}$$

es un isomorfismo.

Corolario 4 (**El Teorema o-minimal de Whitehead**). *Sean X e Y dos conjuntos definiblemente conexos. Sea $\psi : X \to Y$ una aplicación definible tal que para algún $x_0 \in X$, $\psi_* : \pi_n(X, x_0)^{\mathcal{R}} \to \pi_n(Y, \psi(x_0))^{\mathcal{R}}$ es un isomorfismo para todo $n \geq 1$. Entonces ψ es una equivalencia de homotopía definible.*

Dentro del estudio de homotopía o-minimal, probaremos las siguientes versiones o-minimales de los correspondientes resultados clásicos (véanse la Proposición 2.4.10 y el Corolario 2.4.11), los cuales son nuevos también en el caso semialgebraico.

Teorema. *Sean E y B dos conjuntos definibles. Entonces todo recubrimento definible $p : E \to B$ es una fibración definible.*

Teorema. *Sea $p : E \to B$ un recubrimento definible y sea $p(e_0) = b_0$. Entonces $p_* : \pi_n(E, e_0)^{\mathcal{R}} \to \pi_n(B, b_0)^{\mathcal{R}}$ es inyectiva para $n = 1$ y un isomorfismo para todo $n > 1$.*

Las demostraciones de estos dos últimos resultados son independientes del Teorema 2.3.4. Efectivamente, nótese que el hecho de ser una fibración definible es una propiedad que no se preserva bajo homotopías definibles y por tanto no podemos utilizar el Teorema 2.3.4 en esta ocasión.

Finalmente, en el Capítulo 3 introduciremos los espacios localmente definibles (abreviado ld-espacios). Los espacios localmente definibles de especial interés son los regulares y paracompactos (abreviado LD-espacios). Recogeremos aquellos hechos relevantes de [13] necesarios para el desarrollo de la homotopía localmente definible. En particular, probaremos el Teorema de triangulación para LD-espacios. Con todas estas herramientas a mano, mostraremos las generalizaciones para los LD-espacios de los resultados sobre homotopía del Capítulo 2, en particular, los Teoremas de Hurewicz y el Teorema de Whitehead. Las demostraciones de estos resultados en [13] están basadas en propiedades de los conjuntos semialgebraicos que comparten los conjuntos definibles y por tanto pueden ser adaptadas directamente a nuestro contexto. Por lo tanto, hemos nombrado todos estos resultados con el apelativo *Hecho*. Sin embargo, hacemos notar que todos ellos son nuevos en el contexto o-minimal. También mostraremos la existencia de una teoría de homología para LD-espacios por medio de un acercamiento alternativo al de [13] para espacios localmente semialgebraicos (el cual se desarrolla, este último, a través de una cohomología de haces).

Como mostramos en el siguiente resultado, los grupos $\bigvee$-definibles como en [31, Def.2.1] y los grupos “localmente definibles“ como en [17, Def.2.1], son ejemplos de espacios localmente definibles (véanse los Teoremas 3.4.9 y 3.4.10).

Teorema. *Todo grupo $\bigvee$-definible es un ld-espacio. Es más, todo grupo "locamente definible" es un LD-espacio.*

Una vez que hayamos puesto los grupos $\bigvee$-definibles en su categoría correcta, tendremos una noción natural de conexión para ellos. Con respecto a las tres nociones de conexión para grupos $\bigvee$-definibles existentes en la literatura, E-conexión en [17], PS-conexión en [31] y OP-conexión en [27], probamos lo siguiente (véase la Sección 3.6).

Teorema. *Para todo grupo $\bigvee$-definible tenemos que*

$$\textit{Conexo} \Leftrightarrow \textit{OP-conexo} \Rightarrow \textit{PS-conexo} \Rightarrow \textit{E-conexo},$$

donde la segunda y la tercera implicaciones son estrictas.

Por otro lado, nos gustaría remarcar que los resultados de esta tesis ya han sido aplicados al estudio de grupos definibles. Efectivamente, usando distintos resultados de esta memoria, en [6] se prueba que todo grupo definible abeliano, definiblemente conexo y definiblemente compacto de dimensión d es definible homotópicamente equivalente a un toro de dimensión d. Este último resultado nos permite, a su vez, mostrar lo siguiente (véase el Corolario 2.6.11).

Teorema. *Sea G un grupo definible abeliano, definiblemente conexo, definiblemente compacto de dimensión d. Entonces $cat(G)^{\mathcal{R}} = d$.*

Esta tesis ha dado lugar a los siguientes artículos de investigación,

[1] E. Baro, Normal triangulations in o-minimal structures, aparecerá en J. Symb. Log.

[2] E. Baro and M.J. Edmundo, Corrigendum to Locally definable groups in o-minimal structures, J. Algebra 320 (7) (2008), 3079–3080.

[3] E. Baro and M. Otero, Locally definable homotopy, aparecerá en Ann. Pure Appl. Logic., 31pp.

[4] E. Baro and M. Otero, On o-minimal homotopy groups, aparecerá en Quart. J. Math., 18pp.

Terminamos con algunos comentarios referentes a un posible trabajo futuro derivado de esta memoria. Primero, señalamos una conexión entre el Teorema de las triangulaciones normales y el Hauptvermutung o-minimal. El Hauptvermutung o-minimal sobre el cuerpo de los reales fue probado por M. Shiota y M. Yokoi en [35], pero sobre cuerpos realmente cerrados en general sigue aún abierto. Recordemos brevemente una de las situaciones motivadoras del Teorema de las triangulaciones normales. Sea (K, ϕ) una triangulación de un conjunto definible S partiendo ciertos subconjuntos definibles $S_1, \ldots, S_l$ de S. Dados nuevos subconjuntos definibles $S'_1, \ldots, S'_{l'}$ de S, nos gustaría preservar la triangulación ya obtenida y partir al mismo tiempo estos nuevos subconjuntos. Como ya dijimos, el Teorema de las triangulaciones normales puede resolver este problema. Lo que observamos ahora es que si el Hauptvermung o-minimal fuese cierto entonces también podríamos resolver fácilmente la anterior situación. Efectivamente, por el Teorema de triangulación, existe una triangulación (L, ψ) de $|K|$ partiendo los subconjuntos definibles $\phi^{-1}(S'_1), \dots, \phi^{-1}(S'_{l'})$ y cada símplice $\sigma \in K$. Como $\psi : |L| \to |K|$ es un homeomorphismo definible, por el Hauptvermutung o-minimal existirían subdivisiones L' y K' de L y K respectivamente y un isomorfismo simplicial $g : |L'| \to |K'|$ tal que $\psi \sim g$. Por tanto, $(K', \phi \circ \psi \circ g^{-1})$ es una triangulación de S partiendo $S_1, \dots, S_l, S'_1, \dots, S'_{l'}$ y tal que $\phi \circ \psi \circ g^{-1} \sim \phi \circ g \circ g^{-1} \sim \phi$. Por otro lado, parece que el argumento anterior podría ser modificado para probar el Teorema de las triangulaciones normales a partir del Hauptvermutung o-minimal, no obstante obsérvese que difícilmente podríamos recuperar la propiedad (iii) de las triangulaciones normales (Definición 1.1.1). Por tanto, el Teorema de las triangulaciones normales no es, al menos de forma obvia, más débil que el Hauptvermutung o-minimal. Es más, en el contexto semialgebraico, nos ayudó a probar el Hauptvermutung semialgebraico. Así pués, *parece natural pensar que el Teorema de las triangulaciones normales es equivalente al Hauptvermutung o-minimal*, aunque, por supuesto, estamos interesados en la implicación de izquierda a derecha.

En segundo lugar, el siguiente paso natural es intentar aplicar los resultados de esta tesis al estudio de los grupos definibles. De hecho, como comentamos anteriormente, dicho estudio ya ha comenzado en [6], donde se prueba que

$$\pi_n(G)^{\mathcal{R}} \cong \pi_n(G/G^{00})$$

para todo $n \geq 1$ y para cualquier grupo definible definiblemente compacto G. Ahora bien, dado un grupo definible definiblemente compacto G, por el Teorema de triangulación, podemos suponer que $G = |K|$ para cierto complejo simplicial cerrado K cuyos vértices viven en $\overline{\mathbb{Q}}$. Por tanto, por el Corolario 2.4.4 y el resultado en [6] anterior, sabemos que $\pi_n(|K|(\mathbb{R})) \cong \pi_n(G)^{\mathcal{R}} \cong \pi_n(G/G^{00})$. Así pues, sería razonable esperar que $|K|(\mathbb{R})$ fuese homotópicamente equivalente a G/G^{00}; de hecho, por el Teorema de Whitehead, tan sólo necesitamos una aplicación adecuada entre $|K|(\mathbb{R})$ y G/G^{00}. En particular, nótese que esto implicaría que $\mathrm{cat}(G)^{\mathcal{R}} = \mathrm{cat}(G/G^{00})$ para todo grupo definible definiblemente compacto, lo cual extendería los resultados de la Sección 2.6.

Introduction

Background

In the 1980s, L. van den Dries, A. Pillay, C. Steinhorn and others showed that given an expansion of a real closed field, many properties of semialgebraic sets and maps could be derived from the fact that the one dimensional definable sets are a finite union of intervals and points. These structures would be known as o-minimal structures, a fascinating research area where model theory and real algebraic geometry come together. Roughly, there are two fundamental research lines in o-minimal structures: one is focused in the construction of new o-minimal structures, the other one on the study of definable sets and structures. The work of A. Wilkie has inspired a large body of results in the first line, most notably [37], where he proves that the real field expanded by the exponential function is o-minimal. Of particular note in this line are also the results of van den Dries, Macintyre and Marker in [16] and of Rolin, Speissegger and Wilkie in [34]. This dissertation is included in the second research area. The basic results in this line are the Cell decomposition theorem and the Triangulation theorem proved by van den Dries, Pillay, Steinhorn and Knight. Recently, there has been a profound study of topological properties of definable sets, including the existence of both homology and cohomology theories in the o-minimal setting. The development of these tools from algebraic topology in a non-archimedean real closed field is not obvious: recall that in the classical setting we have to use the Lebesgue lemma even in the definition of the induced map in homology by a continuous map (see also the introduction to Section 1.6). The main theme of this thesis is o-minimal homotopy. H. Delfs and M. Knebusch developed semialgebraic homotopy in [13]. We next mention some of their results. Given a semialgebraic pointed set (X, x) over a real closed field R, we define naturally the nth semialgebraic homotopy group $\pi_n(X, x)^{\mathrm{sa}}$ as in the classical case but using semialgebraic maps and semialgebraic homotopies. Contrary to the development of semialgebraic homology, with this definition there is no problem to define the induced map in homotopy by a continuous semialgebraic map. Now, natural questions arise: have these semialgebraic homotopy groups any relation with the classical ones in the relevant cases?, have they a good behaviour under real closed field extensions? The authors answer positively both questions with the following two interesting results (see Facts 2.1.1 and 2.1.2 below for the general statements). In the first one, they prove that given a real closed field extension S of R, the map $\pi_n(X,x)^{\text{sa}} \to \pi_n(X(S),x)^{\text{sa}} : [f] \mapsto [f(S)]$ is an isomorphism, where $X(S)$ and $f(S)$ denotes the realization in S of X and f respectively. In the second one, they show that if $R = \mathbb{R}$ then the map $\pi_n(X,x)^{\text{sa}} \to \pi_n(X,x) : [f] \mapsto [f]$ is an isomorphism. To prove these results Delfs and Knebusch make use of both the model completeness of the theory of real closed fields (or Tarski-Seidenberg principle) and the Lebesgue number, which are not available in the o-minimal setting. Hence, we cannot apply their methods to develop an o-minimal homotopy theory. Actually, only the o-minimal fundamental group was considered in [20] and [7] (with strong consequences in the study of definable groups as we will see next).

The study of definable groups lies within this second line of research we are talking about. In [32], A. Pillay proves that every definable group G in an o-minimal structure can be equipped with a definable manifold structure making G a topological group. Hence, if the underlying field of the o-minimal structure is the real field, we then obtain a Lie group. This result is the starting point of the study of definable groups and, in particular, the study of their analogies with Lie groups. In this sense, algebraic topology tools have been proved to be useful. For instance, the following result of M. Otero and M. Edmundo in [20] is one of the first in showing the mentioned analogy. Let G be a definably connected definably compact n-dimensional abelian group in an o-minimal structure. Then both the o-minimal cohomology algebra over $\mathbb{Q}$ and the o-minimal fundamental group of G are isomorphic to the cohomology algebra over $\mathbb{Q}$ and the fundamental group of an n-dimensional torus respectively. From these isomorphisms they deduce that the torsion of G is isomorphic to the torsion of an n-dimensional torus. Nowadays, Pillay's conjectures stated in [33] and positively solved in [9] and [23] based on the work of several authors, have joined even more the Lie groups and the definable ones. In particular, it is known that every definably compact definable group is an extension of a Lie group by a divisible torsion-free normal subgroup. Moreover, the o-minimal dimension of the definable group equals the one of the Lie group.

On the other hand, in [13], H. Delfs and M. Knebusch introduce a new category extending the semialgebraic one and large enough to be able to deal with objects such as covering maps of "infinite degree". The authors define locally semialgebraic spaces, roughly, as those obtained by glueing infinitely many affine semialgebraic sets. In the o-minimal setting we have the corresponding situation, the definable category is not large enough to deal with certain natural objects. Even though the theory of locally semialgebraic spaces had not been formally extended to the o-minimal framework, some related notions have already appeared (always carrying a group structure). This is the case of $\bigvee$-definable groups which were used by Y. Peterzil, A. Pillay and S. Starchenko in [31] and [30] as a tool for the study of interpretability problems. Later, M. Edmundo introduces in [17] a restricted notion of $\bigvee$-definable groups, the "locally definable" groups, and he develops a whole theory around them. However, to deal with topological concepts, the notion of $\bigvee$-definable group is too rigid as it shows the three non-equivalent definitions of connectedness (see Section 3.6 below) established for these groups which appear in the literature (see [17],[27] and [31]).

Main results

The main results of this thesis are focused on the Normal triangulation theorem and its applications, as well as the o-minimal homotopy of definable sets and its generalization to locally definable spaces. We now are going to briefly describe them. We fix an o-minimal expansion $\mathcal{R}$ of a real closed field R. We denote by $\mathcal{R}_0$ the field structure of R. By *definable* we mean definable in $\mathcal{R}$ with parameters. All definable maps are assumed to be continuous.

In Chapter 1 we will prove the following refinement of the triangulation theorem, which is also new in the semialgebraic context (see Theorem 1.1.5).

Theorem (Normal triangulation theorem). *Let K be a simplicial complex and let $S_1, \dots, S_l$ be definable subsets of its realization $|K|$. Then, there is a subdivision K' of K and a definable homeomorphism $\phi' : |K'| \to |K|$ such that*
(i) *(K', ϕ') partitions all $S_1, \dots, S_l$ and each $\sigma \in K$,*
(ii) *for every $\tau \in K'$ and $\sigma \in K$, if $\tau \subset \sigma$ then $\phi'(\tau) \subset \sigma$.*

Independently of the essential role of this result in the development of o-minimal homotopy in Chapter 2, the Normal triangulation theorem may be of interest by itself as it shows the following situation. As we know, a basic tool to study definable sets is the Triangulation Theorem: given a definable set S and some definable subsets $S_1, \dots, S_l$ of S, there exists a simplicial complex K and a definable homeomorphism $\phi : |K| \to S$ partitioning $S_1, \dots, S_l$. Further study of S may lead to consider new definable subsets $S'_1, \dots, S'_{l'}$ of S. In this situation, we would like to both preserve the already obtained triangulation and partition the new sets. However, a further application of the Triangulation Theorem cannot help us in this case. Moreover, techniques as repeated barycentric subdivisions are not available in this context because of the lack of Lebesgue number or just because the sets S_i's may not be open. Clearly, the Normal triangulation solves this problem. We also point out that the Normal triangulation theorem give us new information even if the base field is $\mathbb{R}$.

We will first apply the Normal triangulation theorem to prove the following (see Theorem 1.5.5).

Corollary (**Semialgebraic Hauptvermutung**). *Let K and L be two closed simplicial complexes in R and let $f : |K| \to |L|$ be a semialgebraic homeomorphism. Then f is semialgebraically homotopic to a simplicial isomorphism $g : |K'| \to |L'|$ between subdivisions K' and L' of K and L, respectively.*

As a second application of the Normal triangulation theorem, we will show an alternative proof to that of A. Woerheide of the existence of a functor for o-minimal simplicial homology (see Section 1.6).

The main result of Chapter 2 establishes a relation between the o-minimal and semialgebraic homotopy sets (see Theorem 2.3.4). In order to avoid the introduction of cumbersome notation, we mention here a weak version of this result which gives an idea of it.

Theorem. *Let X and Y be semialgebraic sets. Then,*
(i) *every definable map $f : X \to Y$ is definably homotopic to a semialgebraic map $g : X \to Y$, and*
(ii) *any two semialgebraic maps $g_1 : X \to Y$ and $g_2 : X \to Y$ which are definably homotopic are also semialgebraically homotopic.*

As a first application of this theorem, we will study the Lusternik-Schnirelmann category (MSC 2000: 55M30) of definable sets. As far as we know, this is the first time that LS-category is studied in both the o-minimal and semialgebraic setting. The LS-category of a definable set X, $\text{cat}(X)^{\mathcal{R}}$, is the least integer m such that X has a definable open cover of $m + 1$ elements with each of them definably contractible to a point in X. We prove the following (see Corollary 2.6.9).

Corollary 1. *Let X be a semialgebraic set defined without parameters. Then $cat(X)^{\mathcal{R}} = cat(X(\mathbb{R}))$, where $cat(X(\mathbb{R}))$ denotes the classical LS-category.*

As a second application of Theorem 2.3.4 (mentioned above) we prove that (the semialgebraic homotopy group) $\pi_n(X, x)^{\mathcal{R}_0}$ and (the o-minimal homotopy group) $\pi_n(X, x)^{\mathcal{R}}$ of a semialgebraic pointed set (X, x) are isomorphic. Namely, $\rho : \pi_n(X, x)^{\mathcal{R}_0} \to \pi_n(X, x)^{\mathcal{R}} : [f] \mapsto [f]$ is a natural isomorphism (see Theorem 2.4.1). Now using the latter result and those of H. Delfs and M. Knebusch we obtain (see Corollary 2.4.4).

Corollary 2. *Let (X,x) be a semialgebraic pointed set defined without parameters. Then there exists a natural isomorphism between the classical homotopy group $\pi_n(X(\mathbb{R}),x)$ and the o-minimal homotopy group $\pi_n(X(R),x)^{\mathcal{R}}$ for every $n \geq 1$.*

Note that thanks to the Triangulation theorem any definable pointed set is definably homeomorphic to a semialgebraic one defined without parameters.

Further applications of Theorem 2.3.4 are the following (see Theorem 2.5.3 and Theorem 2.5.7).

Corollary 3 (The o-minimal Hurewicz theorems). *Let (X,x) be a definable pointed set and $n \geq 2$. Suppose that $\pi_r(X,x)^{\mathcal{R}} = 0$ for every $0 \leq r \leq n-1$. Then the o-minimal Hurewicz homomorphism*

$$h_{n,\mathcal{R}} : \pi_n(X,x_0)^{\mathcal{R}} \to H_n(X)^{\mathcal{R}}$$

is an isomorphism for $n \geq 2$.

Corollary 4 (The o-minimal Whitehead theorem). *Let X and Y be two definably connected sets. Let $\psi : X \to Y$ be a definable map such that for some $x_0 \in X$, $\psi_* : \pi_n(X,x_0)^{\mathcal{R}} \to \pi_n(Y,\psi(x_0))^{\mathcal{R}}$ is an isomorphism for all $n \geq 1$. Then ψ is a definable homotopy equivalence.*

Within the study of o-minimal homotopy, we will prove the following o-minimal versions of the corresponding classical results (see Proposition 2.4.10 and Corollary 2.4.11) which are also new in the semialgebraic context.

Theorem. *Let E and B definable sets. Then every definable covering $p : E \to B$ is a definable fibration.*

Theorem. *Let $p : E \to B$ be a definable covering and let $p(e_0) = b_0$. Then $p_* : \pi_n(E,e_0)^{\mathcal{R}} \to \pi_n(B,b_0)^{\mathcal{R}}$ is an isomorphism for every $n > 1$ and injective for $n = 1$.*

The proofs of these latter results are independent of Theorem 2.3.4. Indeed, to be a definable fibration is a property that is not invariant under definably homotopies and hence Theorem 2.3.4 cannot be applied here.

Finally, in Chapter 3 we will introduce the locally definable spaces (in short ld-spaces). Locally definable spaces of special interest are the regular paracompact ones (in short LD-spaces). We collect the relevant facts from [13] needed for the development of locally definable homotopy. In particular, we prove the Triangulation theorem for LD-spaces. With all these tools at hand, we prove the generalizations to LD-spaces of the homotopy results in Chapter 2, in particular the Hurewicz theorems and the Whitehead theorem. The proofs of these results in [13] are based on properties of semialgebraic sets which are shared by definable sets and hence can be directly adapted to our context. Therefore, we have labelled all these results with *Fact*. However, we point out that all of them are new in the o-minimal setting. We also show the existence of a homology theory for LD-spaces via an alternative approach to that of [13] for locally semialgebraic spaces.

As we have already mentioned in the background section the $\bigvee$-definable groups in the sense of [31, Def.2.1] and the "locally definable" groups in the sense of [17, Def.2.1], are examples of locally definable spaces as we show in the following result (see Theorem 3.4.9 and Theorem 3.4.10).

Theorem. *Every $\bigvee$-definable group is an ld-space. Moreover, every "locally definable" group is an LD-space.*

Once we have put the $\bigvee$-definable groups in their natural category, we have a natural concept of connectedness for them. With respect to the three different notions of connected $\bigvee$-definable group which appear in the literature, E-connected in [17], PS-connected in [31] and OP-connected [27], we prove the following (see Section 3.6).

Theorem. *For every $\bigvee$-definable group we have that*

$$\textit{Connected} \Leftrightarrow \textit{OP-connected} \Rightarrow \textit{PS-connected} \Rightarrow \textit{E-connected},$$

where the second and third implications are strict.

We finish this outline of the main results by pointing out that the results in this thesis has already been applied to the study of definable groups. Indeed, using several results of this dissertation, it is proved in [6] that every definably connected definably compact d-dimensional abelian group is definably homotopy equivalent to the d-dimensional torus over R. The latter, in turn, will allow us to show the following (see Corollary 2.6.11).

Theorem. *Let G be a definably connected definably compact d-dimensional abelian group. Then $cat(G)^{\mathcal{R}} = d$.*

This dissertation has led to the following research papers,

[1] E. Baro, Normal triangulations in o-minimal structures, to appear in J. Symb. Log.

[2] E. Baro and M.J. Edmundo, Corrigendum to Locally definable groups in o-minimal structures, J. Algebra 320 (7) (2008), 3079–3080.

[3] E. Baro and M. Otero, Locally definable homotopy, to appear in Ann. Pure Appl. Logic., 31pp.

[4] E. Baro and M. Otero, On o-minimal homotopy groups, to appear in Quart. J. Math., 18pp.

Notation and Prerequisites

For the rest of the paper we fix an o-minimal expansion $\mathcal{R}$ of a real closed field R. For definitions and basic results on o-minimal structures, we refer to [15]. We always take 'definable' to mean 'definable in $\mathcal{R}$ with parameters'. We take the order topology on R and the product topology on R^n for $n > 1$. All definable maps are assumed to be continuous except otherwise stated.

In general, given a real closed field S we will denote by $\mathcal{S}_0$ its ordered field structure. The only exception will be $\mathbb{R}$, whose ordered field structure we denote by $\overline{\mathbb{R}}$. We denote by $\overline{\mathbb{Q}}$ the real algebraic numbers.

If a set X is definable with parameters in some structure $\mathcal{M}$ we denote by $X(M)$ the realization of X in $\mathcal{M}$.

Recall that given a definable set S the **frontier** of S is the set $\partial S = \overline{S} \setminus S$ and the **boundary** of S is the set $bd(S) = \overline{S} \setminus int(S)$. We use the standard notation $I := [0, 1] = \{x \in R : 0 \leq x \leq 1\}$. In Chapter 3, we will also denote by I a set of indexes, hopefully without confusion. We denote the **graph** of a definable map $f : X \to Y$ by $\Gamma(f) := \{(x, f(x)) : x \in X\}$.

If X is a definable set and $A_1, \ldots, A_k$ are definable subsets of X then $(X, A_1, \ldots, A_k)$ is called a **system** (or **pair**, if $k = 1$) of definable sets. A definable map $f : (X, A_1, \ldots, A_k) \to (Y, B_1, \ldots, B_k)$ between systems of definable sets is a definable map $f : X \to Y$ such that $f(A_i) \subset B_i$ for each $i = 1, \ldots, k$.

Since this dissertation uses (and develops) in an essential way simplicial complexes and triangulations, we have included here a brief summary of the basic notions:

The **k-simplex** spanned by the affine independent points $v_0, \ldots, v_k \in R^p$ is the set $(v_0, \ldots, v_k) := \{\sum t_i v_i : \text{ all } t_i > 0, \sum t_i = 1\}$, we call $v_0, \ldots, v_k$ the **vertices** of $(v_0, \ldots, v_k)$. A **face** of $(v_0, \ldots, v_k)$ is a simplex spanned by a nonempty subset of $\{v_0, \ldots, v_k\}$. A **simplicial complex** K is a finite collection of simplices in R^p such that for all $\sigma_1, \sigma_2 \in K$, either $cl(\sigma_1) \cap cl(\sigma_2) = \emptyset$ or $cl(\sigma_1) \cap cl(\sigma_2) = cl(\tau)$ for some common face τ of σ_1 and σ_2. We denote the **realization** of a simplicial complex K in R^p by $|K| :=$

$\bigcup_{\sigma\in K}\sigma \subset R^p$. We say that K is closed if it contains with each simplex all its faces. Note that K is closed if and only if $|K|$ is a closed set. We will denote by $\overline{K}$ the simplicial complex which is the set of all faces of simplices of K, so that $|\overline{K}| = \overline{|K|}$. We denote by $\mathrm{Vert}(K)$ the set of all **vertices** of simplices in K. Note that given $v \in \mathrm{Vert}(K)$, $\{v\}$ might not be a simplex of K. On the other hand, if $v \in \mathrm{Vert}(K) \cap |K|$ then $\{v\}$ is a simplex of K. Recall that given a subset A of $|K|$, the **star** of A in K, denoted by $\mathrm{St}_K(A)$, is the union of all the simplices $\sigma \in K$ such that $\overline{\sigma} \cap A \neq \emptyset$. We say that a simplicial complex K' is a **subdivision** of a simplicial complex K if each simplex of K' is contained in a simplex of K and each simplex of K equals the union of finitely many simplices of K'. We will use the standard notion of **barycentric subdivision** of a simplicial complex (see [15, Ch.8, §1.8]).

Given two simplicial complexes K and L we say that $g : |K| \to |L|$ is a **simplicial map** if it is the restriction to $|K|$ of a map $\widetilde{g} : |\overline{K}| \to |\overline{L}|$ which sends each simplex of $\overline{K}$ to a simplex of $\overline{L}$ by a linear map taking vertices to vertices and each simplex in K to a simplex in L. Alternatively, we can say that $g : |K| \to |L|$ is a simplicial map if it is piecewise-linearly induced by a map $g_{\mathrm{v}} : \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ such that if $(v_0, \ldots, v_k) \in K$ then $(g_{\mathrm{v}}(v_0), \ldots, g_{\mathrm{v}}(v_k)) \in L$.

Given a definable set S and some definable subsets $S_1, \ldots, S_l$ of S we say that (K, ϕ) is a **triangulation of S partitioning** $S_1, \ldots, S_l$, and denoted by $(K, \phi) \in \Delta(S; S_1, \ldots, S_l)$, if K is a simplicial complex and $\phi : |K| \to S$ is a definable homeomorphism such that each S_i is the union of the images by ϕ of some simplices of K. Note that for a triangulation (K, ϕ) of S we will use the classical notation $\phi : |K| \to S$ instead of the notation $\phi : S \to |K|$ used in [15].

Without mention we will use the following fact: given a definable set S and some definable subsets of S, a triangulation of S which partitions this subsets also partitions their closures and their frontiers both in S.

Chapter 1

Normal Triangulations

1.1 Introduction

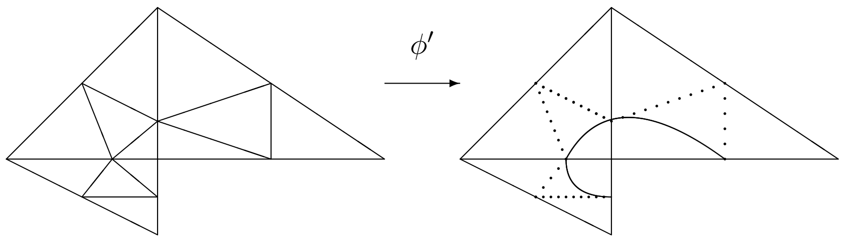
In this chapter we introduce and study the following notion.

Definition 1.1.1. *Let K be a simplicial complex in R^m and $S_1, \dots, S_l$ definable subsets of $|K|$. A triangulation $(K', \phi') \in \Delta(|K|; S_1, \dots, S_l)$ is a* ***normal triangulation of the complex*** K ***partitioning*** $S_1, \dots, S_l$*, denoted by $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$, if it satisfies the following conditions:*
(i) $(K', \phi') \in \Delta(|K|; S_1, \dots, S_l, \sigma)_{\sigma \in K}$
(ii) *K' is a subdivision of K, and*
(iii) *for every $\tau \in K'$ and $\sigma \in K$, if $\tau \subset \sigma$ then $\phi'(\tau) \subset \sigma$.*

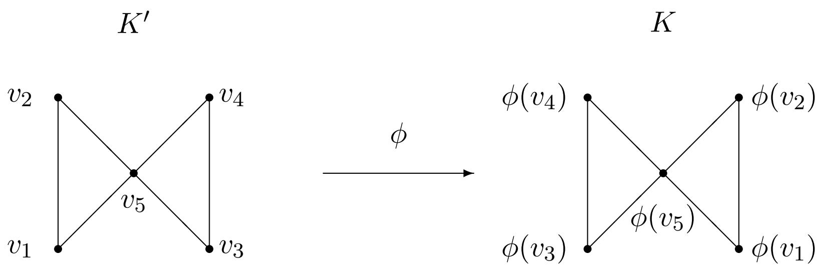
Note that the definition depends not only on $|K|$ but also on K and that the interesting case is when the subsets of $|K|$ are nonempty (otherwise $(K, \mathrm{id}_{|K|})$ is a normal triangulation).

Remark 1.1.2. (i) It is easy to prove that given a normal triangulation (K', ϕ') as in Definition 1.1.1, $\phi'(|L|) = |L|$ for any subcomplex L of K. Indeed, we fix $\sigma \in K$ and we show that $\phi'(\sigma) = \sigma$. Since K' is a subdivision of K, there are $\tau_1, \dots, \tau_k \in K'$ such that $\sigma = \tau_1 \cup \cdots \cup \tau_k$. By normality of (K', ϕ') we have that $\phi'(\sigma) = \phi'(\tau_1) \cup \cdots \cup \phi'(\tau_k) \subset \sigma$. Suppose there is $x \in \sigma$ such that $x \notin \phi'(\sigma)$. Take $\tilde{\tau} \in K'$ and $\tilde{\sigma} \in K$ such that $x \in \phi'(\tilde{\tau})$ and $\tilde{\tau} \subset \tilde{\sigma}$. Since $x \in \phi'(\tilde{\tau}) \cap \sigma$ and (K', ϕ') partitions the simplices of K, we deduce that $\phi'(\tilde{\tau}) \subset \sigma$. On the other hand, by normality of (K', ϕ'), we have that $\phi'(\tilde{\tau}) \subset \tilde{\sigma}$. Therefore $\phi'(\tilde{\tau}) \subset \sigma \cap \tilde{\sigma}$, that is, $\sigma = \tilde{\sigma}$. Hence $x \in \phi'(\tilde{\tau}) \subset \tilde{\sigma} = \sigma$, which is a contradiction.
(ii) Let $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$. Then ϕ' is definably homotopic to $\mathrm{id}_{|K|}$ via the canonical definable homotopy $H : |K'| \times I :\to |K| : (x, t) \to (1-t)\phi'(x) + tx$. The map H is well-defined because, by normality of (K', ϕ'), we have that $\phi'(x) \in \sigma$ for each $x \in \sigma \in K$. Note also that H is indeed continuous. Therefore H is a definable homotopy from ϕ' to $\mathrm{id}_{|K|}$.

Example 1.1.3. We show a normal triangulation in dimension 2. Let K be the closed simplicial complex on the right hand of the figure (with continuous lines) and let S_1 be the curve drawn inside. Then (K', ϕ') is a normal triangulation of K partitioning S_1, where K' is the simplicial complex on the left hand of the figure.



Example 1.1.4. In Definition 1.1.1, condition (iii) cannot be deduced from (i) and (ii), as the following example shows;



where $K' = K$ and ϕ is a symmetry.

Our aim is to prove the existence of normal triangulations.

Theorem 1.1.5 (Normal Triangulation Theorem). *Let K be a simplicial complex and let $S_1, \dots, S_l$ be definable subsets of $|K|$. Then there exists a triangulation $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$.*

The chapter is organized as follows. Section 1.2 contains some definitions and results from [15] that we will need in the following section. Section 1.3 is devoted to prove the existence of independent triangulations (see Definition 1.3.1 and Theorem 1.3.2). The existence of independent triangulations allows us to prove the Normal Triangulation Theorem in Section 1.4, where we also prove an extension lemma for this kind of triangulations (see Theorem 1.4.3). The reader may skip Sections 1.2 and 1.3 at a first reading, only the statement of Theorem 1.3.2 is used later. In Section 1.5 and 1.6 we give the applications described in the Main results section of the Introduction.

1.2 Preliminaries and Notation

We will make extensively use of the following notions and results from Chapter 8 of [15]. We include them here to both make this chapter readable and be able to change slightly the notation.

We recall that a *triangulated set* is a pair $(S, \phi(K))$, where S is a definable set, $(K, \phi) \in \Delta(S)$ and $\phi(K) = \{\phi(\sigma) : \sigma \in K\}$. Given a triangulated set $(S, \mathcal{P})$ and $C, D \in \mathcal{P}$ we call D a *face* of C if $D \subset cl(C)$, a *proper face* of C if $D \subset cl(C) \setminus C$ and a *vertex* of C if it has dimension 0.

A multivalued function F on a triangulated set $(S, \mathcal{P})$ is a finite collection of functions, $F = \{f_{C,i} : C \in \mathcal{P}, 1 \le i \le k(C)\}$, $k(C) \ge 0$, each function $f_{C,i} : C \to R$ definable and $f_{C,1} < \ldots < f_{C,k(C)}$. We set

$$F|_C = \{f_{C,i} : 1 \le i \le k(C)\}, \text{ for } C \in \mathcal{P},$$
$$\mathcal{P}^F = \{\Gamma(f) : f \in F\} \cup \{(f_{C,i}, f_{C,i+1}) : C \in \mathcal{P}, 1 \le i < k(C)\},$$
$$S^F = \text{the union of the sets in } \mathcal{P}^F.$$

Given $f_{C,i} \in F|_C$ we shall omit the subscript C of $f_{C,i}$ and we will denote it just by f_i if there is no ambiguity. Such a multivalued function F is called *closed* if for each pair $C, D \in \mathcal{P}$ with D a proper face of C and each $f \in F|_C$ there is $g \in F|_D$ such that $g(y) = \lim_{x \to y} f(x)$ for all $y \in D$. Note that then each $f \in F$, say $f \in F|_C$, extends continuously to a definable function $cl(f) : cl(C) \cap S \to R$ such that the restrictions of $cl(f)$ to the faces of C in $\mathcal{P}$ belong to F.

Fact 1.2.1. [15, Ch.8,Lem.2.6] *Let F be a multivalued function on the triangulated set $(S, \mathcal{P})$ such that $\Gamma(F)$ is closed in $S \times R$, and there is $M > 0$ such that $\Gamma(F) \subset S \times [-M, M]$. Then F is closed.*

Definition 1.2.2. *Let $(S, \mathcal{P})$ be a triangulated set such that each $C \in \mathcal{P}$ has a vertex. We call a multivalued function F on $(S, \mathcal{P})$* ***superfull*** *if it is closed, $k(C) \ge 1$ for all $C \in \mathcal{P}$, and it satisfies the following two conditions:*
(A) *for each pair $C, D \in \mathcal{P}$ with D a proper face of C and each $g \in F|_D$ we have $g = cl(f)|_D$ for some $f \in F|_C$, where $cl(f)$ is the continuous extension of f to $cl(C) \cap S$, and*
(B) *if $f_1, f_2 \in F|_C$, $f_1 \ne f_2$, then there exists at least one vertex of C where $cl(f_1)$ and $cl(f_2)$ take different values.*

Note that our definition of superfull is stronger of that of *full* in [15] (where only condition (A) is required –see Definition VIII.2.5–). In general, we can convert a full multivalued function in a superfull one by taking the first barycentric subdivision. However, the properties of triangulations we want to consider are not preserved by taking barycentric subdivisions (see Example 1.3.4).

Recall that given a triangulation (K, ϕ) in R^n of a definable set $S \subset R^m$ and given a definable set $S' \subset S \times R$, a triangulation $(L, \psi) \in \Delta(S')$ in R^{n+1}

is said to be a *lifting* of (K, ϕ) if $K = \{\pi_n(\sigma) : \sigma \in L\}$ and the diagram

$$\begin{array}{ccc} |L| & \xrightarrow{\psi} & S' \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_m} \\ |K| & \xrightarrow[\phi]{} & S \end{array}$$

commutes where π_m and π_n are the projections maps on the first m and n coordinates, respectively.

For the proof of the following technical fact see [15, Ch.8,Lem.1.10]. We will show in Lemma 1.3.7 how to lift a triangulation via a superfull multivalued function using this fact.

Fact 1.2.3. *Let $(a_0, \ldots, a_n)$ be an n-simplex in R^p and let $r_j, s_j \in R$, $r_j \leqslant s_j$, for $j = 0, \ldots, n$. Write $b_j = (a_j, r_j)$, $c_j = (a_j, s_j) \in R^{p+1}$. Then $(b_0, \ldots, b_j, c_j, \ldots, c_n)$ is an $(n+1)$-simplex in R^{p+1} for any $0 \leqslant j \leqslant n$ such that $b_j \neq c_j$ (so $r_j < s_j$). Moreover, the collection of all $(n+1)$-simplices $(b_0, \ldots, b_j, c_j, \ldots, c_n)$ with $b_j \neq c_j$ and all their faces is a closed simplicial complex.*

1.3 Independent Triangulations

In order to prove the existence of normal triangulations, we now introduce triangulations satisfying an independence property which may be of interest by itself.

Definition 1.3.1. *Let $(K, \phi) \in \Delta(S)$, where S is a closed and bounded definable set in R^m. We say that (K, ϕ) is an* ***independent triangulation*** *if*
(i) for every n-simplex $\tau = (v_0, \ldots, v_n) \in K$ we have that $\phi(v_0), \ldots, \phi(v_n) \in R^m$ are affinely independent, that is, they span an n-simplex $\tau^\phi := (\phi(v_0), \ldots, \phi(v_n))$ in R^m, and
(ii) if τ_1 and τ_2 are different simplices of K then τ_1^ϕ and τ_2^ϕ are disjoint.

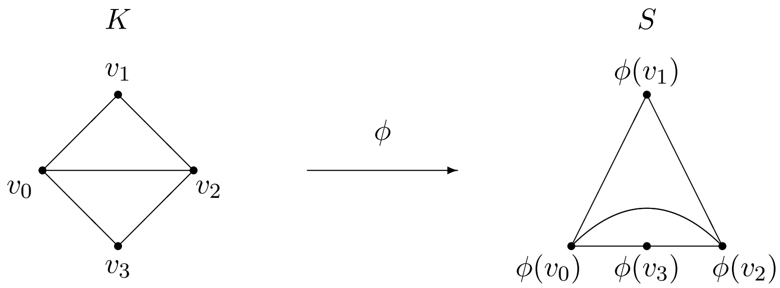
Note that an independent triangulation induces in R^m –via the images by ϕ of the vertices of K– a copy of K. The aim of this section is to prove the following.

Theorem 1.3.2 (Independent Triangulation Theorem). *Let $S \subset R^m$ be a closed and bounded definable set and let $S_1, \ldots, S_l$ be definable subsets of S. Then there exists an independent triangulation $(K, \phi) \in \Delta(S; S_1, \ldots, S_l)$.*

We will show that any closed and bounded definable set has an independent triangulation by an induction argument and following closely the scheme of the proof of the Triangulation Theorem in [15]. In the induction step we will need the existence of triangulations with the following technical property: a triangulation $(K,\phi) \in \Delta(S; S_1, \dots, S_l)$ of a closed and bounded definable set S and some definable subsets $S_1, \dots, S_l$ of S is said to be **small with respect to** $S_1, \dots, S_l$ if for every $\tau = (v_0, \dots, v_n) \in K$ with $\phi(v_0), \dots, \phi(v_n) \in \overline{S_j}$ we have that $\phi(\tau) \subset \overline{S_j}$.

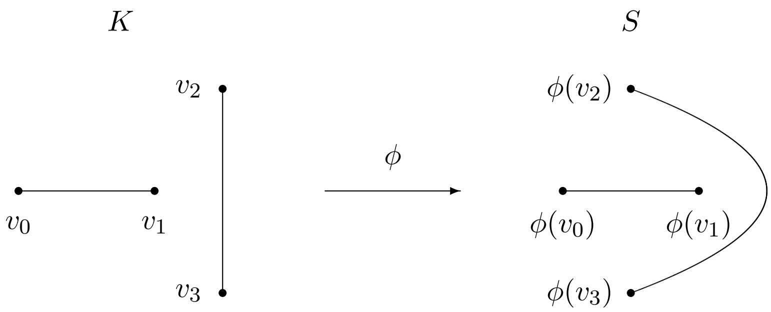
Example 1.3.3. All the examples are in dimension 2.

(1) Example of a triangulation (K,ϕ) of a closed and bounded definable set S which is not independent because (i) fails.

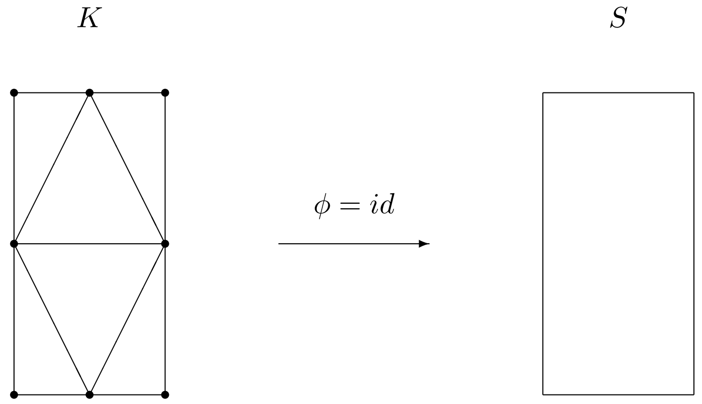


Moreover, if we denote by $S_1 = \phi((v_0, v_2))$ then (K,ϕ) is small w.r.t. S_1. For, the only simplex which has all its vertices in $\overline{S_1}$ is (v_0, v_2). On the other hand, $\phi((v_0, v_2)) = S_1 \subset \overline{S_1}$, as required. This shows that there is no relation between the independence and smallness properties.

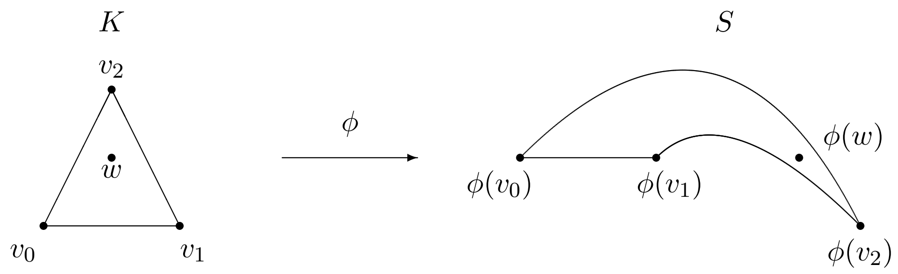
(2) Example of a triangulation (K,ϕ) of a closed and bounded definable set S which is not independent because (ii) fails. However, note that it satisfies (i).



(3) Our last example shows an independent triangulation (K,ϕ) of a closed and bounded definable set S which is not small w.r.t. $bd(S)$.



Example 1.3.4. This example shows that the independence property of triangulations is not preserved by taking barycentric subdivisions.



Remark 1.3.5. Let S be a closed and bounded definable set. Let $S_1, \dots, S_l$ be definable subsets of S. Let $(K, \phi) \in \Delta(S; S_1, \dots, S_l)$ and let $(\widetilde{K}, \widetilde{\phi}) \in \Delta(S; \phi(\sigma))_{\sigma \in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$. Then,
(a) if $\tau \in \widetilde{K}$ and $\sigma \in K$ are such that $\widetilde{\phi}(\tau) \subset \phi(\sigma)$ then there exists a vertex $v \in \mathrm{Vert}(\tau)$ with $\widetilde{\phi}(v) \in \phi(\sigma)$ (in particular if σ is not a vertex then $\widetilde{\phi}(\tau) \neq \phi(\sigma)$), and
(b) $(\widetilde{K}, \widetilde{\phi}) \in \Delta(S; S_1, \dots, S_l)$ and is small w.r.t. $S_1, \dots, S_l$.

Proof. (a) Let $\tau \in \widetilde{K}$ and $\sigma \in K$. Suppose that $\widetilde{\phi}(\tau) \subset \phi(\sigma)$. Exclude the trivial case of τ being a vertex and assume for all $v \in \mathrm{Vert}(\tau)$ we have that $\widetilde{\phi}(v) \notin \phi(\sigma)$, i.e., $\widetilde{\phi}(v) \in \partial\phi(\sigma)$. By smallness $\widetilde{\phi}(\tau) \subset \overline{\partial\phi(\sigma)} = \partial\phi(\sigma)$, a contradiction.
(b) It is clear that $(\widetilde{K}, \widetilde{\phi}) \in \Delta(S; S_1, \dots, S_l)$. To show smallness w.r.t. $S_1, \dots, S_l$, let $\tau = (v_0, \dots, v_n) \in \widetilde{K}$ and $j \in \{1, \dots, l\}$ be such that $\widetilde{\phi}(v_i) \in \overline{S_j}$, for all $i = 0, \dots, n$. Let $\sigma \in K$ be such that $\widetilde{\phi}(\tau) \subset \phi(\sigma)$. By (a) there

exists a vertex $v_{i_0} \in \mathrm{Vert}(\tau)$ with $\widetilde{\phi}(v_{i_0}) \in \phi(\sigma)$. Hence, since (K, ϕ) also partitions $\overline{S_j}$ and $\widetilde{\phi}(v_{i_0}) \in \phi(\sigma) \cap \overline{S_j}$, we deduce that $\phi(\sigma) \subset \overline{S_j}$. Therefore $\widetilde{\phi}(\tau) \subset \overline{S_j}$. □

Lemma 1.3.6. *Let $(K, \phi) \in \Delta(S; S_1, \dots, S_l)$ be a triangulation of a closed and bounded definable set $S \subset R^n$ and some definable subsets $S_1, \dots, S_l \subset S$.*
(i) *If (K, ϕ) is small w.r.t. $S_1, \dots, S_l$ and $f : R^n \to R^n$ is a definable homeomorphism then $(K, f \circ \phi) \in \Delta(f(S); f(S_1), \dots, f(S_l))$ is small w.r.t. $f(S_1), \dots, f(S_l)$.*
(ii) *If (K, ϕ) is independent and $f : R^n \to R^n$ is a linear automorphism, then $(K, f \circ \phi) \in \Delta(f(S); f(S_1), \dots, f(S_l))$ is independent.*

Proof. (i) Let $\sigma = (v_0, \dots, v_m) \in K$ such that $f \circ \phi(v_0), \dots, f \circ \phi(v_m) \in \overline{f(S_j)}$ for some fixed $1 \leq j \leq l$. Since f is a definable homeomorphism, $\overline{f(S_j)} = f(\overline{S_j})$ and therefore $\phi(v_0), \dots, \phi(v_m) \in \overline{S_j}$. Hence $\phi(\sigma) \subset \overline{S_j}$ because (K, ϕ) is small w.r.t. $S_1, \dots, S_l$. Finally, $f(\phi(\sigma)) \subset f(\overline{S}_j) = \overline{f(S_j)}$, as required.
(ii) Consider a simplex $\sigma = (v_0, \dots, v_m) \in K$. Since (K, ϕ) is independent, $\phi(v_0), \dots, \phi(v_m)$ are affinely independent. Hence, since f is linear, $f \circ \phi(v_0), \dots, f \circ \phi(v_m)$ are also affinely independent. Now, take $\sigma_1, \sigma_2 \in K$ such that $\sigma_1^{f \circ \phi} \cap \sigma_2^{f \circ \phi} \neq \emptyset$. Again, since f is linear, $\sigma_1^{\phi} \cap \sigma_2^{\phi} \neq \emptyset$ and hence, by the independence property of (K, ϕ), $\sigma_1 = \sigma_2$. □

The following lemma will be useful in the induction step. Its proof is an adaptation –taking care of independence– of that of [15, Ch.8,Lem.2.8].

Lemma 1.3.7. *Let $A \subset R^m$ be a closed and bounded definable set and let $(K, \phi) \in \Delta(A)$ be an independent triangulation in R^p. Let F be a superfull multivalued function on $(A, \phi(K))$. Then (K, ϕ) can be lifted to an independent triangulation $(L, \psi) \in \Delta(A^F; D)_{D \in \phi(K)^F}$ in R^{p+1}.*

Proof. Firstly, note that since A is closed and bounded and F is superfull, A^F is also closed and bounded. The construction of (L, ψ) is that of [15, Ch.8,Lem.2.8]. Unfortunately we will need to introduce the notation to check that (L, ψ) is independent. We construct L and ψ above each $C \in \phi(K)$. Let $C \in \phi(K)$ and let $a_0, \dots, a_n$ be the vertices of $\phi^{-1}(C)$ listed in some prefixed order on $\mathrm{Vert}(K)$. Let $f < g$ be two successive members of $F|_C$. Put $r_j = cl(f)(\phi(a_j))$, $s_j = cl(g)(\phi(a_j))$, $b_j = (a_j, r_j)$, $c_j = (a_j, s_j)$. Let $L(f, g)$ be the complex in R^{p+1} constructed in Fact 1.2.3 (since F satisfies condition (B) of superfullness we do not need to take a barycentric subdivision as in [15]). Define the homeomorphism $\psi_{f,g}^{-1} : [cl(f), cl(g)] \to |L(f, g)|$ by $\psi_{f,g}^{-1}(x, t \cdot cl(f)(x) + (1-t)cl(g)(x)) = t\Phi_b(x) + (1-t)\Phi_c(x), 0 \leq t \leq 1$, where $\Phi_b(x)$ and $\Phi_c(x)$ are the points of $(b_0, \dots, b_n)$ and $(c_0, \dots, c_n)$ with the same affine coordinates with respect to $b_0, \dots, b_n$ and $c_0, \dots, c_n$ as $\phi^{-1}(x)$ has with respect to $a_0, \dots, a_n$. Note that it follows from the superfullness that $\psi_{f,g}^{-1}$ is injective: if $cl(f)(x)$ and $cl(g)(x)$ are distinct, then so are $\Phi_b(x)$ and $\Phi_c(x)$. We also define for each $f \in F|_C$ the simplicial complex $L(f)$ in R^{p+1} as the n-simplex $(b_0, \ldots, b_n)$ with $b_j = (a_j, cl(f)(\phi(a_j)))$, and all its faces. Then $\psi_f^{-1} : \Gamma(cl(f)) \to |L(f)|$ is by definition the homeomorphism given by $\psi_f^{-1}(x, cl(f)(x)) = \Phi_b(x)$, where $\Phi_b(x)$ is defined as before. Finally we consider the simplicial complex L which is the union of all simplicial complexes $L(f,g)$ and $L(f)$ and the definable homeomorphism $\psi : |L| \to A^F$ such that $\psi|_{L(f,g)} = \psi_{f,g}$ and $\psi|_{L(f)} = \psi_f$.

Let us show that the triangulation (L, ψ) is independent. Let $C \in \phi(K)$ and two successive functions $f, g \in F|_C$. Consider the triangulation $(L(f,g), \psi_{f,g})$ constructed at the beginning of the proof (and with the same notation). Given $v \in \mathrm{Vert}(L(f,g))$ then either $v = b_j$ or $v = c_j$ for some j. Hence either $\psi_{f,g}(v) = \psi_{f,g}(b_j) = (\phi(a_j), r_j)$ or $\psi_{f,g}(v) = \psi_{f,g}(c_j) = (\phi(a_j), s_j)$. Denote $(\phi(a_j), r_j)$ by $\widetilde{b}_j$ and $(\phi(a_j), s_j)$ by $\widetilde{c}_j$. Observe that by independence of (K, ϕ) we have that $\phi(a_0), \ldots, \phi(a_n)$ are affinely independent and therefore, by Fact 1.2.3, $(\widetilde{b}_0, \ldots, \widetilde{b}_j, \widetilde{c}_j, \ldots, \widetilde{c}_n)$ are also affinely independent for any j such that $c_j \neq b_j$. Moreover, also by Fact 1.2.3, the $(n+1)$-simplices $(\widetilde{b}_0, \ldots, \widetilde{b}_j, \widetilde{c}_j, \ldots, \widetilde{c}_n)$ with $\widetilde{b}_j \neq \widetilde{c}_j$ and all their faces is a closed simplicial complex which we will denote by $L(f,g)^{\psi_{f,g}}$. In a similar way, we construct a closed simplicial complex $L(f)^{\psi_f}$ for each $f \in F|_C$, $C \in \phi(K)$. Since (K, ϕ) is independent, a routine argument shows that the collection L^ψ of all simplices of all closed simplicial complexes $L(f,g)^{\psi_{f,g}}$ and $L(f)^{\psi_f}$ is a closed simplicial complex. To finish the proof it is enough to observe, using the notation of Definition 1.3.1, that σ^ψ is a simplex of L^ψ for every $\sigma \in L$. □

From the proof of Lemma 1.3.7 we get the following.

Corollary 1.3.8. *There is a lifting (L, ψ) as in the conclusion of Lemma 1.3.7 satisfying the following: For any $\tau \in L$ with $\psi(\tau) \subset D$ for some $D \in \phi(K)^F$, $C = \pi(D) \in \phi(K)$, and $\phi^{-1}(C) = (a_0, \ldots, a_n) \in K$, we have*
(i) $\pi(\psi(\tau)) = C$,
(ii) if $D = (f,g)_C$ for some successive $f, g \in F|_C$, $f < g$, then τ is either an $(n+1)$-simplex $(b_0, \ldots, b_j, c_j, \ldots, c_n)$, $b_j \neq c_j$, or is an n-simplex $(b_0, \ldots, b_{j-1}, c_j, \ldots, c_n)$, $b_j \neq c_j$, $j \neq 0$, where $b_i = (a_i, cl(f)(\phi(a_i)))$ and $c_i = (a_j, cl(g)(\phi(a_i)))$, $i = 0, \ldots, n$, and
(iii) if $D = \Gamma(f)$ for some $f \in F|_C$, then τ is an n-simplex $(b_0, \ldots, b_n)$, where $b_i = (a_i, cl(f)(\phi(a_i)))$, $i = 0, \ldots, n$. Moreover, $\psi(\tau) = \Gamma(f) = D$.

Proof. Let (L, ψ) be the lifting constructed in the proof of Lemma 1.3.7. Then (i) and (iii) are clear by construction. Let us prove (ii). Firstly, note that the dimension of τ is either $n+1$ or n (otherwise we get a contradiction to (i)). If τ is an $(n+1)$-simplex then the statement is clear by construction. So suppose that τ is an n-simplex, say, $\tau = (v_0, \ldots, v_n)$. Since $\pi(\psi(\tau)) = C$ we have that $\{\pi(v_0), \ldots, \pi(v_n)\} = \{a_0, \ldots, a_n\}$. Without loss of generality, $\pi(v_i) = a_i$ for all $i = 0, \ldots, n$. Since $\psi(\tau) \subset D$, τ is a face of some $(n+1)$-simplex of the form $(b_0, \ldots, b_j, c_j, \ldots, c_n)$ with $b_j \neq c_j$. Hence, either $\tau = (b_0, \ldots, b_{j-1}, c_j, \ldots, c_n)$ for some $j \neq 0$ with $b_j \neq c_j$, or $\tau = (b_0, \ldots, b_j, c_{j+1}, \ldots, c_n)$ for some $j \neq n$ with $b_j \neq c_j$. In the first case, the statement (ii) clearly holds. So assume that $\tau = (b_0, \ldots, b_j, c_{j+1}, \ldots, c_n)$ for some $j \neq n$ with $b_j \neq c_j$. If $b_{j+1} \neq c_{j+1}$ then $\tau = (b_0, \ldots, b_j, c_{j+1}, \ldots, c_n)$ with $b_{j+1} \neq c_{j+1}$, as required. Otherwise, let $i_0 > j+1$ be the maximum such that $b_i = c_i$ for all $j+1 \leq i \leq i_0$. If $i_0 < n$ then we are done, since $\tau = (b_0, \ldots, b_{i_0}, c_{i_0+1}, \ldots, c_n)$ with $b_{i_0+1} \neq c_{i_0+1}$. If $i_0 = n$, then $\tau = (b_0, \ldots, b_j, c_{j+1}, \ldots, c_n) = (b_0, \ldots, b_n)$, which is a contradiction to $\psi(\tau) \subset D$. □

In the next lemma we apply the smallness condition to get a superfull multivalued function. We will apply this lemma in the induction step.

Lemma 1.3.9. *Let A be a closed and bounded definable set, let $(K, \phi) \in \Delta(A)$ and F be a closed multivalued function on the triangulated set $(A, \phi(K))$ satisfying condition (A) of superfull. Let $(K_0, \phi_0) \in \Delta(A; \phi(\sigma))_{\sigma \in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$. Then, the multivalued function F_0 on $(A, \phi_0(K_0))$, obtained by the restrictions of the functions in F to the sets of $\phi_0(K_0)$, is superfull.*

Proof. By Fact 1.2.1, the multivalued function F_0 is closed. Let us show that F_0 satisfies (A) of superfullness. Consider a pair $D_0, C_0 \in \phi_0(K_0)$ with D_0 a proper face of C_0 and consider also $g_0 \in F_0|_{D_0}$. Since $(K_0, \phi_0) \in \Delta(A; \phi(\sigma))_{\sigma \in K}$, there are $C, D \in \phi(K)$, D a face of C, such that $C_0 \subset C$ and $D_0 \subset D$. On the other hand, F_0 is by definition the restriction of F to $\phi_0(K_0)$ and therefore there exists $g \in F|_D$ such that $g|_{D_0} = g_0$. If $C = D$ then obviously $g|_{C_0} \in F_0$ and $g_0 = cl(g|_{C_0})|_{D_0}$. If D is a proper face of C then, since F satisfies (A) of superfullness, there exists $f \in F|_C$ such that $g = cl(f)|_D$. Hence $g_0 = cl(f_0)|_{D_0}$, where $f_0 := f|_{C_0} \in F_0$.

Now, let us check that F_0 satisfies (B) of superfullness. Let $C = \phi_0(\tau)$, where $\tau \in K_0$, and let $f_1, f_2 \in F_0|_C$ be two different functions. By construction, there exist $\sigma \in K$ and two different $\widetilde{f}_1, \widetilde{f}_2 \in F|_{\phi(\sigma)}$ such that $C \subset \phi(\sigma)$ and $\widetilde{f}_i|_C = f_i$, $i = 1, 2$. By Remark 1.3.5(a), there exists a vertex $\phi_0(v)$ of C such that $\phi_0(v) \in \phi(\sigma)$. Then $cl(f_i)(\phi_0(v)) = \widetilde{f}_i(\phi_0(v))$, $i = 1, 2$, so $cl(f_1)$ and $cl(f_2)$ take different values on $\phi_0(v)$. □

We are now ready to prove the main result of this section.

Proof of Theorem 1.3.2. In fact, we will prove by induction on m a stronger result: Given a closed and bounded definable set $S \subset R^m$ and some definable subsets $S_1, \ldots, S_l$ of S, there exists a triangulation $(K, \phi) \in \Delta(S; S_1, \ldots, S_l)$ which is both independent and small w.r.t. $S_1, \ldots, S_l$.

The case $m = 0$ is trivial. Suppose that the theorem holds for a certain m and let us prove it for $m+1$. Consider the closed and bounded definable sets $T = bd(S) \cup bd(S_1) \cup \cdots \cup bd(S_l)$ and $A = \pi(T)$, where π denotes the projection on the first m coordinates. Note that by [15, Ch.4, Cor.1.10], T has dimension less than $m+1$.

Claim 1. *We can assume that:*
(a) *for every $a \in A$ the fiber $T_a = \{x \in R : (a,x) \in T\}$ is finite, and*
(b) *there exist an independent triangulation $(K,\phi) \in \Delta(A; \pi(S_1), \ldots, \pi(S_l))$ and a closed multivalued function F on the triangulated set $(A, \phi(K))$ satisfying condition (A) of superfullness and such that S, S_i, $\overline{S_i}$ and $\overline{S}$ are finite unions of sets in $\phi(K)^F$.*

The proof of this claim, which has been included below for completeness, is just an adaptation of the proof of the Triangulation Theorem in [15]. In our case –aiming for independence– we now first modify the multivalued function (see M1 below) to obtain a superfull one and hence being able to apply our lifting Lemma 1.3.7. Unfortunately, to be able to execute M1 we make essential use of the smallness property of the induction hypothesis. A trivial second modification (see M2 below) will allows us to prove smallness (see Example 1.3.11 for a justification of this modification).
M1. By induction hypothesis there exists an independent triangulation $(K_1, \phi_1) \in \Delta(A; \phi(\sigma))_{\sigma \in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$. Let F_1 be the multivalued function on $(A, \phi_1(K_1))$ obtained by the restrictions of the functions in F to the sets of $\phi_1(K_1)$. By Lemma 1.3.9 the multivalued function F_1 is superfull.
M2. Given $C \in \phi_1(K_1)$ and two successive functions $f, g \in F_1|_C$ with $f < g$, we consider the function $\frac{f+g}{2}$ on C. Let F_2 be the multivalued function obtained by adding to F_1 the new functions $\frac{f+g}{2}$ for each pair of successive functions $f, g \in F_1|_C$, $C \in \phi_1(K_1)$, $f < g$. The new multivalued function F_2 on $\phi_1(K_1)$ is also superfull.

By Lemma 1.3.7, we can lift (K_1, ϕ_1) to an independent triangulation $(L_0, \psi_0) \in \Delta(A^{F_2}; D)_{D \in \phi_1(K_1)^{F_2}}$. Finally, let $L = \{\sigma \in L_0 : \psi_0(\sigma) \subset S\}$. Clearly $(L, \psi) \in \Delta(S; S_1, \ldots, S_l)$ is independent, where $\psi = \psi_0|_{|L|}$.

It remains to prove that (L, ψ) is small w.r.t. $S_1, \ldots, S_l$. Let $\tau = (v_0, \ldots, v_n) \in L$ and $i \in \{1, \ldots, l\}$ be such that $\psi(v_r) \in \overline{S_i}$ for all $r = 0, \ldots, n$. We show that $\psi(\tau) \subset \overline{S_i}$. Let $D \in \phi_1(K_1)^{F_2}$ be such that $\psi(\tau) \subset D$ and denote by $C = \pi(D) \in \phi_1(K_1)$. Note that by Corollary 1.3.8(i), $\pi(\psi(\tau)) = \pi(D) = C \in \phi_1(K_1)$. We first consider the case that D is the graph of a function of F_2. By Corollary 1.3.8(iii), we have that $D = \psi(\tau)$.

Claim 2. *For any $C \in \phi_1(K_1)$, $f \in F_2|_C$ and $i \in \{1, \ldots, l\}$ if $(w, cl(f)(w)) \in \overline{S}_i$ for every vertex w of C then $\Gamma(f) \subset \overline{S}_i$.*

Then $D \subset \overline{S_i}$ modulo Claim 1. Now consider the case that $D = (f,g)_C$ for two successive functions $f, g \in F_2|_D$, $f < g$, and suppose that $D \not\subseteq \overline{S_i}$. Then $D \subset \overline{S_i}^c$. By definition of F_2 we can assume that there exist two successive functions $f_1, g_1 \in F_1|_C$, $f_1 < g_1$, such that $f = f_1$ and $g = \frac{f_1|_C + g_1|_C}{2}$. Therefore $D \subset \widetilde{D} := (f_1, g_1)_C$. Since $D \subset \overline{S_i}^c$ then $\widetilde{D} \subset \overline{S_i}^c$. At this point we claim that

Claim 3. *The set*
$\widetilde{D}_{cil} = \{(x, y) : x \in \overline{C} \setminus C, cl(f_1)(x) \neq cl(g_1)(x), cl(f_1)(x) < y < cl(g_1)(x)\}$
is contained in $\overline{S_i}^c$.

Assume also that we have proved Claim 2. Then by Corollary 1.3.8(ii), and following its notation, we have two cases: either τ is an n-simplex $(b_0, \ldots, b_{j-1}, c_j, \ldots, c_n)$ with $b_j \neq c_j$ and $\phi_1^{-1}(C) = (a_0, \ldots, a_n)$, or τ is an n-simplex $(b_0, \ldots, b_j, c_j, \ldots, c_{n-1})$ with $b_j \neq c_j$ and $\phi_1^{-1}(C) = (a_0, \ldots, a_{n-1})$. In both cases, since $b_j \neq c_j$, we have that $cl(f)(\phi_1(a_j)) \neq cl(g)(\phi_1(a_j))$ and therefore $cl(f_1)(\phi_1(a_j)) \neq cl(g_1)(\phi_1(a_j))$. Since

$$cl(g)(\phi_1(a_j)) = \frac{cl(f_1)|_C + cl(g_1)|_C}{2}(\phi_1(a_j)),$$

we deduce that $cl(f_1)(\phi_1(a_j)) < cl(g)(\phi_1(a_j)) < cl(g_1)(\phi_1(a_j))$ and hence $\psi(c_j) = (\phi_1(a_j), cl(g)(\phi_1(a_j))) \in \widetilde{D}_{cil}$. By Claim 2, $\psi(c_j) \notin \overline{S_i}$, a contradiction. We conclude that $\psi(\tau) \subset D \subset \overline{S_i}$ as required.

It remains to prove the three claims.

Proof of Claim 1. By the good directions lemma (see [15, Ch.7,Thm.4.2]) there is a linear automorphism of R^{m+1} such that the image of T by this automorphism satisfies (a). Hence, since independence and smallness properties are preserved by linear automorphisms (see Lemma 1.3.6), we can assume that condition (a) holds. By the Cell decomposition theorem (see [15, Ch.3,Thm.2.11]), T is the disjoint union of $\Gamma(f)$'s for finitely many definable functions f on cells A_h that form a finite partition of A. By inductive hypothesis there exists an independent triangulation (K_1, ϕ_1) partitioning the subsets A_h. The restrictions of the functions f to the sets of $\phi(K_1)$ form a multivalued function F_1 on $(A, \phi(K_1))$ such that $\Gamma(F_1) = T$. Since T is closed and bounded, by Fact 1.2.1 the multivalued function F_1 is closed. However, F_1 may not satisfy condition (A) of superfullness. We achieve it with a modification of the multivalued function F_1. Since F_1 is closed, each function $f \in F_1$ extends definably and continuously to the closure of its domain and then, by [15, Ch.8,Lem.2.2], it extends definably and continuously to a function $\tilde{f} : A \to R$. Let $\widetilde{T}$ be the union of the graphs $\Gamma(\tilde{f})$ for $f \in F_1$. By inductive hypothesis there exists an independent triangulation

$$(K, \phi) \in \Delta(A; \phi(\sigma), \pi(S \cap \Gamma(\tilde{f})), \pi(S_i \cap \Gamma(\tilde{f})))_{\sigma \in K_1, f \in F_1, i=1,\ldots,k}$$

such that the restrictions of the $\tilde{f}$'s to the sets of $\phi(K)$ form a multivalued function F on $(A, \phi(K))$. Since $\Gamma(F) = \widetilde{T}$ and $\widetilde{T}$ is closed and bounded then, by Lemma 1.2.1, F is closed. Moreover, since every function $f \in F_1$

has been extended, F clearly satisfies (A) of superfullness. Finally, (K,ϕ) partitions $\pi(S\cap\Gamma(\tilde{f})),\pi(S_i\cap\Gamma(\tilde{f}))$, for all $f\in F$ and $i=1,\ldots,k$, and therefore $\phi(K)^F$ partitions the sets S, $\overline{S}$, S_i and $\overline{S_i}$, for all $i=1,\ldots,k$. That is, S, $\overline{S}$, S_i and $\overline{S_i}$ are finite disjoint unions of sets of $\phi(K)^F$.

Proof of Claim 2. Let $f\in F_2|_C$, $C\in\phi_1(K_1)$ and $i\in\{1,\ldots,l\}$ be such that $(w,cl(f)(w))\in\overline{S}_i$ for every vertex w of C. Suppose first that $f\in F_1|_C$ and $\Gamma(f)\not\subseteq\overline{S_i}$. Since F_1 is the restrictions of the functions of F to the sets of $\phi_1(K_1)$, there exists $\widetilde{f}\in F|_{\widetilde{C}}$, $\widetilde{C}\in\phi(K)$, $C\subset\widetilde{C}$, such that $\widetilde{f}|_C=f$. Since $\Gamma(f)\not\subseteq\overline{S_i}$, we have that $\Gamma(\widetilde{f})\not\subseteq\overline{S_i}$. Therefore $\Gamma(\widetilde{f})\subset\overline{S_i}^c$. Since (K_1,ϕ_1) is small w.r.t. $\partial\phi(\sigma)$, $\sigma\in K$, by Remark 1.3.5(a) there exists one vertex w_0 of C such that $w_0\in\widetilde{C}$. Therefore $(w_0,cl(f)(w_0))=(w_0,\widetilde{f}(w_0))\in\Gamma(\widetilde{f})$ does not lie in $\overline{S_i}$, a contradiction. Suppose now that $f=\frac{f_0+g_0}{2}$, for two successive functions $f_0,g_0\in F_1|_C$, $f_0<g_0$. Then $\Gamma(f)\subset(f_0,g_0)_C$. Since F_1 is the restrictions of the functions of F to the sets of $\phi_1(K_1)$, there exist $\widetilde{f}_0,\widetilde{g}_0\in F|_{\widetilde{C}}$, $\widetilde{C}\in\phi(K)$, $C\subset\widetilde{C}$, such that $\widetilde{f}_0|_C=f_0$ and $\widetilde{g}_0|_C=g_0$. Suppose $\Gamma(f)\not\subseteq\overline{S_i}$. Then $(f_0,g_0)_C\not\subseteq\overline{S_i}$ and therefore $(\widetilde{f}_0,\widetilde{g}_0)_{\widetilde{C}}\not\subseteq\overline{S_i}$. Hence $(\widetilde{f}_0,\widetilde{g}_0)_{\widetilde{C}}\subset\overline{S_i}^c$. By Remark 1.3.5(a), there exists one vertex w_0 of C such that $w_0\in\widetilde{C}$. Therefore $(w_0,cl(f)(w_0))=(w_0,\frac{cl(f_0)+cl(g_0)}{2}(w_0))=(w_0,\frac{\widetilde{f}_0+\widetilde{g}_0}{2}(w_0))\in(\widetilde{f}_0,\widetilde{g}_0)_{\widetilde{C}}$ does not lie in $\overline{S_i}$, a contradiction.

Proof of Claim 3. Suppose there is $(x_0,y_0)\in\widetilde{D}_{cil}$ such that $(x_0,y_0)\in\overline{S_i}$. Since $\phi_1(K_1)^{F_2}$ partitions $\overline{S_i}$, for all $y\in(cl(f_1)(x_0),cl(g_1)(x_0))$ we have that $(x_0,y)\in\overline{S_i}$. Moreover, we will show that $(x_0,y)\in bd(S_i)$ for every $y\in(cl(f_1)(x_0),cl(g_1)(x_0))$. Fix $y\in(cl(f_1)(x_0),cl(g_1)(x_0))$. Since $x_0\in\overline{C}\setminus C$, by the Curve Selection Lemma (see [15, Ch.6,Lem.1.5]), there exists a definable curve $\gamma(t)$, $t\in(0,1)$, such that $\lim_{t\to1}\gamma(t)=x_0$ and $\gamma(t)\in C$, for all $t\in(0,1)$. Consider the curve

$$\gamma_y(t)=(\gamma(t),f_1(\gamma(t))+(g_1(\gamma(t))-f_1(\gamma(t)))\left(\frac{y-cl(f_1)(x_0)}{cl(g_1)(x_0)-cl(f_1)(x_0)}\right)),t\in(0,1).$$

Note that $\gamma_y(t)\in\widetilde{D}$ for all $t\in(0,1)$ and $\lim_{t\to1}\gamma_y(t)=(x_0,y)$. Therefore $(x_0,y)\in\overline{\widetilde{D}}\subset\overline{\overline{S_i}^c}\subset\overline{int(S_i)^c}=int(S_i)^c$. We conclude that $(x_0,y)\in bd(S_i)$. We have shown that $(x_0,y)\in bd(S_i)$ for all $y\in(cl(f_1)(x_0),cl(g_1)(x_0))$, which is a contradiction because the fiber of T in x_0 is finite (see (a) at the beginning of the proof). □

We have the following corollary to the proof of Theorem 1.3.2.

Corollary 1.3.10. *Let $S\subset R^m$ be a closed and bounded definable set and let $S_1,\ldots,S_l$ be definable subsets of S. Then there exists a triangulation $(K,\phi)\in\Delta(S;S_1,\ldots,S_l)$ which is both independent and small with respect to $S_1,\ldots,S_l$.*

Example 1.3.11. Let S be the closed and bounded 2-dimensional definable set of Figure 1, where the union of the curves in its interior is the subset S_1. If we follow the proof of the Independent Triangulation Theorem and we make the modification M1 but we do not make the modification M2 (see Figure 2) then we obtain the following triangulation.

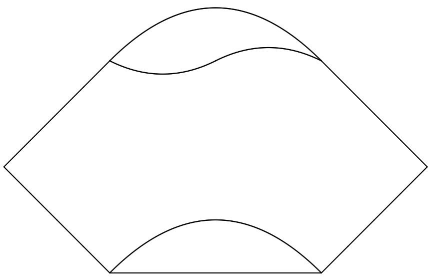


Figure 1

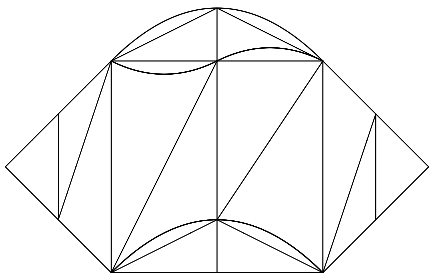


Figure 2

Figure 2 represents a triangulation in which there is a vertical line with both extremes in S_1 but not included in the closure of S_1, witnessing the non-smallness.

1.4 Proof of the Normal Triangulation Theorem

We begin this section with a key lemma for the proof of the Normal Triangulation Theorem. It can be easily proved making use of o-minimal homology. However, we only use the Open Mapping Theorem which has been proved (independently of o-minimal homology) by J. Johns in [24].

Lemma 1.4.1. *Let $\sigma \subset R^n$ be an n-simplex and let $f : \overline{\sigma} \to \overline{\sigma}$ be an injective definable map. If $f(\partial\sigma) \subset \partial\sigma$ then $f(\sigma) = \sigma$ (and hence $f(\partial\sigma) = \partial\sigma$).*

Proof. Since $\overline{\sigma}$ is closed and bounded, $f(\overline{\sigma})$ is closed. By the Open Mapping Theorem, $f(\sigma)$ is open in R^n. This implies that $\sigma \cap f(\sigma)$, which is closed in σ since $\sigma \cap f(\sigma) = \sigma \cap f(\overline{\sigma})$, is also open in σ. On the other hand σ is definably connected and, since f is injective, $dim(f(\sigma)) = n$, so $f(\sigma)$ cannot be included in $\partial\sigma$ which has smaller dimension. Hence $\sigma \cap f(\sigma) = \sigma$. Since $f(\sigma) \subset int(f(\overline{\sigma})) \subset \sigma$, we conclude that $f(\sigma) = \sigma$. Finally, let us show that $f(\partial\sigma) = \partial\sigma$. Given $x \in \partial\sigma$, since f is injective, if $f(x) \in f(\sigma)$ then $x \in \sigma$, which is a contradiction. Hence $f(x) \notin f(\sigma) = \sigma$, that is, $f(x) \in \partial\sigma$. On the other hand, $\overline{\sigma} = \overline{f(\sigma)} \subset f(\overline{\sigma})$ and therefore $\partial\sigma \subset f(\partial\sigma)$. □

Remark 1.4.2. It is enough to prove the Normal Triangulation Theorem for closed simplicial complexes. For, let K be a simplicial complex and $S_1, \dots, S_l$ some definable subsets of $|K|$. Let $(K_0, \phi_0) \in \Delta^{NT}(|\overline{K}|; S_1, \dots, S_l)$, where $\overline{K}$ denotes the closed simplicial complex which is the collection of all faces of simplices of K. Then $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$, where $K' = \{\sigma \in K_0 : \sigma \subset |K|\}$ and $\phi' = \phi_0|_{|K'|}$.

Proof of Theorem 1.1.5. By Remark 1.4.2 we can assume that K is closed. We first apply Theorem 1.3.2 to get an independent triangulation $(K_0, \phi_0) \in \Delta(|K|; S_1, \dots, S_l, \sigma)_{\sigma \in K}$. Now consider the collection of simplices $K' = \{\tau^{\phi_0} : \tau \in K_0\}$ (with the notation of Definition 1.3.1). By definition of independent triangulation, K' is a closed simplicial complex. Moreover, the map between the set of vertices $g_{vert} : \mathrm{Vert}(K_0) \to \mathrm{Vert}(K') : v \mapsto \phi_0(v)$ induce a simplicial isomorphism $g : |K_0| \to |K'|$. Note that $g(\tau) = \tau^{\phi_0}$ for $\tau \in K_0$. We also observe that given $\tau \in K_0$ if $\phi_0(\tau) \subset \sigma \in K$ then the images by ϕ_0 of the vertices of τ lie in $\overline{\sigma}$ and therefore $g(\tau) = \tau^{\phi_0} \subset \overline{\sigma}$. Hence, given $\sigma \in K$ take $\tau_1, \dots, \tau_m \in K_0$ be such that $\sigma = \phi_0(\tau_1)\dot\cup \cdots \dot\cup \phi_0(\tau_m)$ and then we get $g(\tau_1)\dot\cup \cdots \dot\cup g(\tau_m) \subset \overline{\sigma}$.

Claim. $g(\tau_1)\dot\cup \cdots \dot\cup g(\tau_m) = \sigma$.

Once we have proved the Claim, we can assure that (a) K' is clearly a subdivision of K, and (b) for every $\tau \in K_0$ and every $\sigma \in K$ we have that $\phi_0(\tau) \subset \sigma$ if and only if $g(\tau) \subset \sigma$. Indeed, to prove (b), let $\tau \in K_0$ and $\sigma \in K$. By the Claim, if $\phi_0(\tau) \subset \sigma$ then clearly $g(\tau) \subset \sigma$. On the other hand, assume that $g(\tau) \subset \sigma$ and $\phi_0(\tau) \nsubseteq \sigma$. Then there is $\widetilde{\sigma} \in K$, $\sigma \neq \widetilde{\sigma}$, such that $\phi_0(\tau) \subset \tilde{\sigma}$. By the Claim, $g(\tau) \subset \widetilde{\sigma}$ and therefore $g(\tau) \subset \widetilde{\sigma} \cap \sigma = \emptyset$, which is a contradiction.

Now, we consider the following map

$$\phi' := \phi_0 \circ g^{-1} : |K'| \to |K|.$$

Finally we are ready to show that $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$ as required. We have to check the three conditions of Definition 1.1.1. The fact that g is a simplicial isomorphism and that the triangulation $(K_0, \phi_0) \in \Delta(|K|, S_1, \dots, S_l, \sigma)_{\sigma \in K}$ give us (i); by (a) above we get (ii), and to check (iii), given a simplex $\tau^{\phi_0} \in K'$ such that $\tau^{\phi_0} \subset \sigma \in K$ we have that $\phi'(\tau^{\phi_0}) = \phi_0 \circ g^{-1}(\tau^{\phi_0}) = \phi_0(\tau) \subset \sigma$ by (b). It remains to prove the Claim.
Proof of the Claim. By induction on the dimension n of the simplex $\sigma \in K$, the case $n = 0$ being trivial. Let $\sigma \in K$ be an $(n+1)$-simplex. Let $\tau_1, \dots, \tau_m \in K_0$ be such that $\sigma = \phi_0(\tau_1)\dot\cup \cdots \dot\cup \phi_0(\tau_m)$. We may assume $\sigma \subset R^{n+1}$. Now consider the injective definable map $(g \circ \phi_0^{-1})|_{\overline{\sigma}} : \overline{\sigma} \to \overline{\sigma}$. Applying the induction hypothesis to each simplex in $\partial\sigma$ we get $(g \circ \phi_0^{-1})(\partial\sigma) = \partial\sigma$. Therefore by Lemma 1.4.1 we have that $(g \circ \phi_0^{-1})(\sigma) = \sigma$, i.e., $\sigma = g(\tau_1)\dot\cup \cdots \dot\cup g(\tau_m)$. □

Extending a given triangulation is a technical tool used in the construction of triangulations (see Lemma II.4.3 in [13]). We next prove that the extension process can be done preserving normality. We will make use of this tool in the proof of Theorem 2.3.1.

Lemma 1.4.3. *Let K be a closed simplicial complex and K_Z a closed simplicial subcomplex of K. Let (K_0, ϕ_0) be a normal triangulation of K_Z. Then there exists a normal triangulation (K', ϕ') of K such that $K_0 \subset K'$ and $\phi'|_{|K_0|} = \phi_0$.*

Proof. Note that $|K_0| = |K_Z|$, since K_0 is a subdivision of K_Z. For every $m \geq 0$ we denote by SK^m the closed complex which is the union of K_Z and all the simplices of K of dimension $\leq m$. We will show that there exists a normal triangulation (K^m, ϕ^m) of SK^m such that $K_0 \subset K^m$ and $\phi^m|_{|K_0|} = \phi_0$. Hence for $m = dim(K)$ we will obtain the required normal triangulation.

For $m = 0$ let K^0 be the union of K_0 and all vertices of K. Let ϕ^0 be equal to ϕ_0 on $|K_0|$ and the identity on the vertices of K that does not lie in $|K_0|$. Clearly (K^0, ϕ^0) is a normal triangulation of SK^0, $K_0 \subset K^0$ and $\phi^0|_{|K_0|} = \phi_0$.

Suppose we have constructed (K^m, ϕ^m). Let Σ_{m+1} be the collection of simplices in $K \setminus K_0$ of dimension $m+1$. Hence, for every $\sigma \in \Sigma_{m+1}$, $\partial\sigma$ is contained in SK^m. On the other hand, K^m is a subdivision of SK^m and so, for each $\sigma \in \Sigma_{m+1}$, there exists a finite collection of indices J_σ and simplices τ_j^σ of K^m, $j \in J_\sigma$, such that $\partial\sigma = \dot{\bigcup}_{j \in J_\sigma} \tau_j^\sigma$. For each $j \in J_\sigma$ denote by $[\tau_j^\sigma, \hat{\sigma}]$ the cone over τ_j^σ with vertex the barycentre $\hat{\sigma}$ of σ, that is, $[\tau_j^\sigma, \hat{\sigma}] = \{(1-t)u + t\hat{\sigma} : u \in \tau_j^\sigma, t \in [0,1]\}$. For each $\sigma \in \Sigma_{m+1}$ and $j \in J_\sigma$ we define

$$\begin{array}{rrcl} h_j^\sigma : & [\tau_j^\sigma, \hat{\sigma}] & \to & \overline{\sigma} \\ & (1-t)u + t\hat{\sigma} & \to & (1-t)\phi^m(u) + t\hat{\sigma}. \end{array}$$

Note that h_j^σ is well-defined because given $u \in \tau_j^\sigma$ there exists a proper face $\sigma_0 \in K$ of σ such that $\tau_j^\sigma \subset \sigma_0$ and therefore, since $\sigma_0 \in SK^m$ and (K^m, ϕ^m) is a normal triangulation, we have that $\phi^m(u) \in \phi^m(\tau_j^\sigma) \subset \sigma_0 \subset \partial\sigma$. Hence $h_j^\sigma((1-t)u + t\hat{\sigma}) \in \overline{\sigma}$ for all $t \in [0,1]$ and $u \in \tau_j^\sigma$. Note that the map h_j^σ is injective and it is indeed continuous. Let K^{m+1} be the collection of simplices in K^m together with the collection of simplices $(\tau_j^\sigma, \hat{\sigma}) = \{(1-t)u + t\hat{\sigma} : u \in \tau_j^\sigma, t \in (0,1)\}$ and all their faces for $\sigma \in \Sigma_{m+1}$ and τ_j^σ as described above. Finally, let ϕ^{m+1} be the extension of ϕ^m to K^{m+1} such that $\phi^{m+1}|_{[\tau_j^\sigma, \hat{\sigma}]} = h_j^\sigma$. We show that ϕ^{m+1} is well-defined. It is enough to prove that for a fixed $\sigma \in \Sigma_{m+1}$, the sets $h_j^\sigma((\tau_j^\sigma, \hat{\sigma}))$, $j \in J_\sigma$, are pairwise disjoint. Indeed, $h_j^\sigma((\tau_j^\sigma, \hat{\sigma})) = (\phi^m(\tau_j^\sigma), \hat{\sigma})$, where $(\phi^m(\tau_j^\sigma), \hat{\sigma}) = \{(1-t)x + t\hat{\sigma} : x \in \phi^m(\tau_j^\sigma), t \in (0,1)\}$ and since the sets $\phi^m(\tau_j^\sigma)$ are pairwise disjoint, the sets $(\phi^m(\tau_j^\sigma), \hat{\sigma})$ are also pairwise disjoint. Note that ϕ^{m+1} is continuous.

We now show that (K^{m+1}, ϕ^{m+1}) is a normal triangulation of SK^{m+1}. To prove that (K^{m+1}, ϕ^{m+1}) partitions the simplices of SK^{m+1} it is enough to consider each $\sigma \in \Sigma_{m+1}$ (since $K^m \subset K^{m+1}$, $\phi^{m+1}|_{|K^m|} = \phi^m$, and (K^m, ϕ^m) is normal). Now, for each of these $\sigma \in \Sigma_{m+1}$, the image of h_j^σ is contained in $\overline{\sigma}$ and, since $\partial\sigma = \bigcup_{j \in J_\sigma} \phi^m(\tau_j^\sigma)$, we have that $\sigma =$

$\bigcup_{j\in J_\sigma}(\phi^m(\tau_j^\sigma),\hat{\sigma})\cup\{\hat{\sigma}\}$. Clearly K^{m+1} is a subdivision of SK^{m+1} because for the relevant simplices of SK^{m+1}, i.e, those $\sigma\in\Sigma_{m+1}$, the cones $(\tau_j^\sigma,\hat{\sigma})$ and their faces form a triangulation of $\overline{\sigma}$. Since we have always worked inside each simplex $\sigma\in\Sigma_{m+1}$, property *(iii)* of normality also holds. □

1.5 Applications I: the semialgebraic Hauptvermutung and other topics

In this section we will give applications of the Normal Triangulation Theorem.

Corollary 1.5.1. *Let S be a definable set and let $S_1,\dots,S_l$ be some definable subsets of S. Let (K,ϕ) be a definable triangulation of S partitioning $S_1,\dots,S_l$. Then for any $S'_1,\dots,S'_{l'}$ definable subsets of S there exist a subdivision K' of K and a definable triangulation $\phi':|K'|\to S$ partitioning $S_1,\dots,S_l,S'_1,\dots,S'_{l'}$ such that ϕ' is definably homotopic to ϕ.*

Proof. By the Normal Triangulation Theorem there exists a triangulation $(K',\psi)\in\Delta^{NT}(|K|;\phi^{-1}(S'_1),\dots,\phi^{-1}(S'_{l'}))$. By Remark 1.1.2.(ii), ψ is definably homotopic to $id_{|K|}$. We define $\phi'=\phi\circ\psi$. Clearly $(K',\phi')\in\Delta(S;S_1,\dots,S_l,S'_1,\dots,S'_{l'})$ and ϕ' is definably homotopic to ϕ. □

Corollary 1.5.2. *Let K be a simplicial complex and $S_1,\dots,S_l$ definable subsets of $|K|$. Then there exists $(K',\phi')\in\Delta(|K|;S_1,\dots,S_l,\sigma)_{\sigma\in K}$ such that K' is a subdivision of K and ϕ' is definably homotopic to $id_{|K|}$.*

Proof. By the Normal Triangulation Theorem there exists a triangulation $(K',\phi')\in\Delta^{NT}(|K|;S_1,\dots,S_l)$. By Remark 1.1.2.(ii), ϕ' and $id_{|K|}$ are definably homotopic. □

Recall that given a definable map $f:|K|\to|L|$ between the realizations of two simplicial complexes K and L we say that f is **compatible** if for every $\sigma\in K$ there is $\tau\in L$ such that $f(\sigma)\subset\tau$.

Corollary 1.5.3. *Let K and L be two simplicial complexes and let $f:|K|\to|L|$ be a definable map. Then there exist a subdivision K' of K and a definable homeomorphism $\phi:|K'|\to|K|$ definably homotopic to $id_{|K|}$ such that both ϕ and $f\circ\phi$ are compatible.*

Proof. By the Normal Triangulation Theorem there exists a triangulation $(K',\phi)\in\Delta^{NT}(|K|,f^{-1}(\tau))_{\tau\in L}$. The definable map ϕ is clearly compatible since (K',ϕ) partitions the simplices of K. Let us show that $f\circ\phi$ is also compatible. Consider a simplex $\sigma'\in K'$. Since (K',ϕ) partitions $f^{-1}(\tau)$ for each $\tau\in L$ and $|K|=\bigcup_{\tau\in L}f^{-1}(\tau)$, there exists $\tau_0\in L$ such that $\phi(\sigma')\subset f^{-1}(\tau_0)$ and therefore $f\circ\phi(\sigma')\subset\tau_0$. □

Theorem 1.5.4. *Let R be a real closed field. Let $X \subset R^n$ and $Y \subset R^m$ be two semialgebraic sets defined without parameters. Then any semialgebraic map (homeomorphism) $f : X \to Y$ is semialgebraically homotopic to a semialgebraic map (resp. homeomorphism) $g : X \to Y$ defined without parameters.*

Proof. We denote by $\overline{\mathbb{Q}}$ the real algebraic numbers. We can assume that X and Y are the realization of two simplicial complexes K and L respectively, whose vertices lie in $\overline{\mathbb{Q}}$.

Claim. *We can also assume that for each $\sigma \in K$ there exists $\tau_\sigma \in L$ such that $f(\sigma) \subset \tau_\sigma$.*

Now, we denote f by f_c to stress the fact that c is a tuple of parameters in R such that f is defined over. Consider the first order formula $\psi(y)$ defined without parameters which says that f_y is a map (resp. homeomorphism) between $|K|$ and $|L|$ and for each $\sigma \in K$, $f_y(\sigma) \subset \tau_\sigma$. By completeness of the theory of real closed fields, since R satisfies $\exists y \psi(y)$, then $\overline{\mathbb{Q}}$ satisfies $\exists y \psi(y)$. Therefore there exists a tuple of parameters a in $\overline{\mathbb{Q}}$ such that $f_a : |K| \to |L|$ is a semialgebraic map (resp. homeomorphism) and for each $\sigma \in K$, $f_a(\sigma) \subset \tau_\sigma$. Denote f_a by g. Finally the map $H : |K| \times I \to |L| : (x,t) \mapsto (1-t)f_c(x) + tg(x)$ is well-defined because for each $\sigma \in K$ we have that both $g(\sigma)$ and $f_c(\sigma)$ are contained in τ_σ. Hence H is a semialgebraic homotopy between f and g.
Proof of the Claim. We first apply Corollary 1.5.2 to get a semialgebraic triangulation $(K', \phi') \in \Delta(|K|; f^{-1}(\tau), \sigma)_{\sigma \in K, \tau \in L}$ such that K' is a subdivision of K and ϕ' is semialgebraically homotopic to $id_{|K|}$. Note that the map $f' = f \circ \phi'$ is semialgebraically homotopic to f and that for each $\sigma' \in K'$ there exists $\tau_{\sigma'} \in L$ such that $f'(\sigma') \subset \tau_{\sigma'}$. Unfortunately, the vertices of K' may not lie in $\overline{\mathbb{Q}}$. However, by completeness of the theory of real closed fields, there exists a subdivision K'' of K whose vertices lie in $\overline{\mathbb{Q}}$ and such that there is a simplicial isomorphism $\phi : |K''| \to |K'|$ with $\phi(\sigma) = \sigma$ for all $\sigma \in K$. The semialgebraic map $F : |K| \times I \to |K| : (x,t) \mapsto (1-t)x + t\phi(x)$ is well-defined because $\phi(\sigma) = \sigma$ for all $\sigma \in K$. Hence, since F is clearly continuous, ϕ is semialgebraically homotopic to $id_{|K|}$. Finally, it suffices to consider the map $f'' = f' \circ \phi : |K''| \to |L|$. Clearly f'' is semialgebraically homotopic to f' (and therefore to f) and for each $\sigma'' \in K''$ there exists $\tau_{\sigma''} \in L$ with $f''(\sigma'') \subset \tau_{\sigma''}$, as required. □

Theorem 1.5.5 (**Semialgebraic Hauptvermutung**)**.** *Let R be a real closed field. Let K and L be two closed simplicial complexes in R and let $f : |K| \to |L|$ be a semialgebraic homeomorphism. Then f is semialgebraically homotopic to a simplicial isomorphism $g : |K'| \to |L'|$ between subdivisions K' and L' of K and L, respectively.*

This last result is proved for the real field by M.Shiota and M.Yokoi in [36].

In [11] M. Coste proves a weaker version of the semialgebraic Hauptvermutung, but strong enough to prove the unicity and strong effectiveness of semialgebraic triangulations. Namely, he proves that under the hypotheses of Theorem 1.5.5 there exists a simplicial isomorphism g between two subdivisions of K and L. However no relation between f and g is established.

Proof of Theorem 1.5.5. We can assume that the vertices of K and L lie in $\overline{\mathbb{Q}}$ (the real algebraic numbers). By Theorem 1.5.4 we can also assume that f is defined without parameters. We now show that, since f is defined without parameters, the theorem follows from the real Semialgebraic Hauptvermutung. Indeed, by Theorem 4.1 in [36] there exists a simplicial isomorphism $\widetilde{g} : |K'|(\mathbb{R}) \to |L'|(\mathbb{R})$ between the realizations in $\mathbb{R}$ of two subdivision K' and L' of K and L respectively, such that $\widetilde{g}$ is semialgebraically homotopic to the map $f^{\mathbb{R}} : |K|(\mathbb{R}) \to |L|(\mathbb{R})$ induced by f. Note that we can express with a first order sentence ψ defined without parameters the existence of some parameters and some points in K and L which are the vertices of two simplicially isomorphic subdivisions of K and L and that the simplicial isomorphism is semialgebraically homotopic to f. By completeness of the theory of real closed fields, since $\mathbb{R}$ satisfies ψ, then $\overline{\mathbb{Q}}$ also satisfies ψ. □

1.6 Applications II: a new approach to o-minimal simplicial homology

We finish this chapter with another application of the Normal Triangulation Theorem. In both the semialgebraic and o-minimal setting it is possible to develop a homology theory over real closed fields as it was proved by M. Knebusch–H. Delfs and A. Woerheide respectively (see e.g. [13] and [38]). When adapting the classical development to the o-minimal setting, the lack of the Simplicial Approximation Theorem lead us to the problem of verifying the excision axiom in the singular case and to construct a well-defined functor in the simplicial one. M. Knebusch and H. Delfs avoid this problem developing their semialgebraic homology theory via cohomology of sheaves. In his PhD dissertation A. Woerheide returns to the classical line solving the problem of constructing a well-defined o-minimal simplicial homology functor applying the Triangulation Theorem and the Acyclic Models Theorem. Then he uses this o-minimal simplicial homology theory to verify the excision axiom of the o-minimal singular homology. On the other hand, our Normal Triangulation Theorem fills the gap left by the lack of the Simplicial Approximation Theorem, which allows us to follow the classical proof. *Therefore it gives an alternative proof of the existence of a functor for o-minimal simplicial homology.*

Given a closed simplicial complex K in $\mathcal{R}$, we define the nth o-minimal (simplicial) homology group $H_n(K)$ as the nth (simplicial) homology group of K as an abstract simplicial complex (see [25, Ch.1,§5]). *We fix two closed simplicial complexes K and L. We also fix a definable map $f : |K| \to |L|$.* Our purpose is to define (naturally) an induced homomorphism $f_* : H_*(K) \to H_*(L)$. We will use the machinery of simplicial approximations developed in Section 2.2 below. We first suppose that f is *compatible* (recall the definition of a compatible definable map given before Corollary 1.5.3 and the star condition in Definition 2.2.5). We have labelled some of the following results with *Fact* because they are adaptations of classical results.

Fact 1.6.1. *Every compatible definable map $h : |K| \to |L|$ satisfies the star condition.*

Proof. Fix a vertex $v \in K$. We have to show that there is a vertex $w \in L$ such that $h(\mathrm{St}_K(v)) \subset \mathrm{St}_L(w)$. Since h is compatible, there is a simplex $\widetilde{\tau} \in L$ such that $h(v) \in \widetilde{\tau}$. We check that for any vertex w of $\widetilde{\tau}$, $h(\mathrm{St}_K(v)) \subset \mathrm{St}_L(w)$. Indeed, let $\sigma \in \mathrm{St}_K(v)$, i.e., $v \in \overline{\sigma}$. Since h is compatible, there is $\tau \in L$ such that $h(\sigma) \subset \tau$. Therefore, $h(v) \in h(\overline{\sigma}) \subset \overline{h(\sigma)} \subset \overline{\tau}$. Since $h(v) \in \widetilde{\tau}$, we have that $\widetilde{\tau} \cap \overline{\tau} \neq \emptyset$ and hence $w \in \widetilde{\tau} \subset \overline{\tau}$. That is, $h(\sigma) \subset \tau \subset \mathrm{St}_L(w)$, as required. □

Since f is compatible, by Fact 1.6.1 it satisfies the star condition and therefore f has a simplicial approximation (see Proposition 2.2.6).

Fact 1.6.2. *Let $h : |K| \to |L|$ be a definable map. If $F, F' : |K| \to |L|$ are simplicial approximations to h then F and F' are contiguous, i.e, for each simplex $\sigma = (v_0, \dots, v_n) \in K$ the points $F(v_0), \dots, F(v_n), F'(v_0), \dots, F'(v_n)$ span a simplex of L.*

Proof. We use the notation of the statement. Since F and F' are simplicial approximations to h, we have that $h(\sigma) \subset \bigcap_{i=0}^n \mathrm{St}_L F(v_i)$ and $h(\sigma) \subset \bigcap_{i=0}^n \mathrm{St}_L F'(v_i)$. Let $\tau \in L$ be such that $h(\sigma) \cap \tau \neq \emptyset$. Then $\tau \subset \bigcap_{i=0}^n \mathrm{St}_L F(v_i)$ and $\tau \subset \bigcap_{i=0}^n \mathrm{St}_L F'(v_i)$. Therefore $F(v_0), \dots, F(v_n), F'(v_0), \dots, F'(v_n) \in \overline{\tau}$, so that they span a face of $\overline{\tau}$. Since L is closed, this face lies in L. □

As in the classical setting, a simplicial map $F : |K| \to |L|$ induces naturally a homomorphism $F_* : H_*(K) \to H_*(L)$ (see [25, Ch.1,§12]). Hence we can and do define $f_* : H_*(K) \to H_*(L)$ as the homomorphism induced by a simplicial approximation to f. Moreover, we can adapt (or even transfer) easily the proof of the following fact: two contiguous simplicial maps induce the same homomorphism in homology (see [25, Thm.12.5]). Hence, by Fact 1.6.2 the definition of f_* does not depend on the choice of the simplicial approximation.

Remark 1.6.3. Given a closed simplicial complex M, if $g : |L| \to |M|$ is a compatible definable map then $g \circ f$ is clearly compatible. Moreover, $(g \circ f)_* = g_* \circ f_*$. Indeed, if F and G are simplicial approximations to

f and g respectively, then $G \circ F$ is a simplicial approximation to $g \circ f$. Indeed, for every $v \in K$, $f(St_K(v)) \subset St_L F(v)$, and hence $g(f(St_K(v))) \subset g(St_L F(v)) \subset St_M G(F(v))$.

Now, let us define f_* in the general case. By Corollary 1.5.3, there exist a subdivision K' of K and a definable homeomorphism $\phi : |K'| \to |K|$ definably homotopic to $\mathrm{id}_{|K|}$ such that both ϕ and $f \circ \phi$ are compatible. We will call such a triangulation (K', ϕ) a **compatible triangulation** of f. Since $f \circ \phi$ and ϕ are compatible, by the previous case there are homomorphisms $(f \circ \phi)_* : H_*(K') \to H_*(L)$ and $\phi_* : H_*(K') \to H_*(K)$. Moreover, let us prove that ϕ_* is an isomorphism. Firstly, we show that the compatible map $\mathrm{id}_{K'} : |K'| \to |K| : x \mapsto x$ induces an isomorphism in homology. This is natural since $|K'| = |K|$ and the map $\mathrm{id}_{K'}$ is just the identity. However, note that it is not obvious because $\mathrm{id}_{K'}$ is not a simplicial isomorphism (not even a simplicial map).

Fact 1.6.4 (**The algebraic subdivision theorem**). *The homomorphism $(id_{K'})_* : H_*(K') \to H_*(K)$ induced by the identity map $id_{K'} : |K'| \to |K|$ is an isomorphism.*

Proof. Let K_0 and K_0' be closed simplicial complexes, with K_0' a subdivision of K_0, whose vertices lie in the real algebraic numbers $\overline{\mathbb{Q}}$ and such that there exist simplicial isomorphisms $F_1 : |K| \to |K_0|$ and $F_2 : |K'| \to |K_0'|$. Both K_0 and K_0' exists because of the Tarski-Seidenberg principle. We show that $(\mathrm{id}_{K_0'})_* : H_*(K_0') \to H_*(K_0)$ induced by $\mathrm{id}_{K_0'} : |K_0'| \to |K_0|$ is an isomorphism. By the classical analogue of Fact 1.6.4 (see [25, Thm. 17.2]), $(\mathrm{id}_{K_0'}(\mathbb{R}))_* : H_*(K_0') \to H_*(K_0)$ induced by $\mathrm{id}_{K_0'}(\mathbb{R}) : |K_0'|(\mathbb{R}) \to |K_0|(\mathbb{R})$ is an isomorphism. On the other hand, $(\mathrm{id}_{K_0'}(\mathbb{R}))_*$ is exactly $(\mathrm{id}_{K_0'})_*$, so that $(\mathrm{id}_{K_0'})_*$ is an isomorphism. Finally, since both F_1 and F_2 are simplicial isomorphisms, $(F_1)_*$ and $(F_2)_*$ are isomorphisms and hence $(\mathrm{id}_{K'})_* = (F_1)_*^{-1} \circ (\mathrm{id}_{K_0'})_* \circ (F_2)_*$ is an isomorphism. □

Since (K', ϕ) is a compatible triangulation of f, we have that $\phi(\sigma') \subset \sigma$ for each pair of simplexes $\sigma' \in K'$ and $\sigma \in K$ with $\sigma' \subset \sigma$, and therefore any simplicial approximation to ϕ is a simplicial approximation to $\mathrm{id}_{K'} : |K'| \to |K|$, so that $\phi_* = (\mathrm{id}_{K'})_*$. Hence, by Fact 1.6.4, ϕ_* is an isomorphism, as required. Now, we can and do define the homomorphism

$$f_*^{\phi} : H_*(K) \to H_*(L)$$

by $f_*^{\phi} = (f \circ \phi)_* \circ \phi_*^{-1}$. A priori, and as it stress the notation, f_*^{ϕ} depends on the choice of (K', ϕ). The following lemma and proposition show that this is not the case.

Lemma 1.6.5. *Let K_1, K_2 and K_3 be closed simplicial complexes and let $h : |K_1| \to |K_2|$ and $g : |K_2| \to |K_3|$ be definable maps. Let (K_2', ϕ_2) be a* *compatible triangulation of g and let (K_1', ϕ_1) be a compatible triangulation of both h and $\phi_2^{-1} \circ h$. Then (K_1', ϕ_1) is a compatible triangulation of $g \circ h$ and moreover $(g \circ h)_*^{\phi_1} = g_*^{\phi_2} \circ h_*^{\phi_1}$.*

Proof. Firstly, let us check that (K_1', ϕ_1) is a compatible triangulation of $g \circ h$. Let $\sigma' \in K_1'$. Since (K_1', ϕ_1) is a compatible triangulation of $\phi_2^{-1} \circ h$, there is $\tau' \in K_2'$ such that $\phi_2^{-1}(h(\phi_1(\sigma'))) \subset \tau'$. Therefore $h(\phi_1(\sigma')) \subset \phi_2(\tau')$. Since (K_2, ϕ_2) is a compatible triangulation of g, there is $\gamma \in K_3$ such that $g(\phi_2(\tau')) \subset \gamma$ and hence $g(h(\phi_1(\sigma'))) \subset g(\phi_2(\tau')) \subset \gamma$, as required. Now, since (K_1', ϕ_1) is a compatible triangulation of both $\phi_2^{-1} \circ h$ and h, by Remark 1.6.3 we have that $\phi_{2*} \circ (\phi_2^{-1} \circ h \circ \phi_1)_* = (h \circ \phi_1)_*$ and therefore $(\phi_2^{-1} \circ h \circ \phi_1)_* = (\phi_2)_*^{-1} \circ (h \circ \phi_1)_*$. Hence, again using Remark 1.6.3, $g_*^{\phi_2} \circ h_*^{\phi_1} = (g \circ \phi_2)_* \circ (\phi_2)_*^{-1} \circ (h \circ \phi_1)_* \circ (\phi_1)_*^{-1} = (g \circ \phi_2)_* \circ (\phi_2^{-1} \circ h \circ \phi_1)_* \circ (\phi_1)_*^{-1} = (g \circ h \circ \phi_1)_* \circ (\phi_1)_*^{-1} = (g \circ h)_*^{\phi_1}$. □

Proposition 1.6.6. *Let K_1 and K_2 be closed simplicial complexes. Let $h, g : |K_1| \to |K_2|$ be definably homotopic definable maps and let (K_1', ϕ) and (K_1'', ψ) be compatible triangulations of h and g respectively. Then,*
(i) *if h is compatible, we have that $h_* = h_*^{\phi}$, and*
(ii) $h_*^{\phi} = g_*^{\psi}$.

Proof. (i) By Remark 1.6.3, $h_*^{\phi} = (h \circ \phi)_* \circ \phi_*^{-1} = h_* \circ \phi_* \circ \phi_*^{-1} = h_*$.
(ii) Without loss of generality, we can assume that $K_1' = K_1''$. Indeed, take a subdivision K_1''' of both K_1' and K_1''. We prove that we can replace (K_1', ϕ) by $(K_1''', \phi \circ i)$, where $i : |K_1'''| \to |K_1'|$ is the identity map. Clearly, $(K_1''', \phi \circ i)$ is also a compatible triangulation of h. By Remark 1.6.3 we have that $(\phi \circ i)_* = \phi_* \circ i_*$ and $(h \circ \phi \circ i)_* = (h \circ \phi)_* \circ i_*$. Hence, $h_*^{\phi \circ i} = (h \circ \phi \circ i)_* \circ (\phi \circ i)_*^{-1} = (h \circ \phi)_* \circ i_* \circ i_*^{-1} \circ \phi_*^{-1} = (h \circ \phi)_* \circ \phi_*^{-1} = h_*^{\phi}$. Similarly, we prove that we can replace (K_1'', ψ) by $(K_1''', \psi \circ \widetilde{i})$, where $\widetilde{i} : |K_1'''| \to |K_1''|$ is the identity map, as required.

Now, it follows from a straightforward adaptation of [25, Lem. 19.1] that there is a closed simplicial complex M such that $|M| = |K_1'| \times I$ and for each $\sigma \in K_1'$, both $\sigma \times \{0\}$ and $\sigma \times \{1\}$ are simplices of M, and $\sigma \times I$ is the realization of a subcomplex of M (we could also transfer [25, Lem. 19.1] via the Tarski-Seidenberg principle). Since both ϕ and ψ are definably homotopic to $\mathrm{id}_{|K_1|}$ and since h and g are definably homotopic, $h \circ \phi$ and $g \circ \psi$ are also definably homotopic. Let $F : |K_1'| \times I \to |K_2|$ be a homotopy from $h \circ \phi$ to $g \circ \psi$. Let $j_0, j_1 : |K_1'| \to |K_1'| \times I$ be the maps $j_0(x) = (x, 0)$ and $j_1(x) = (x, 1)$ for all $x \in |K_1'|$. Clearly, j_0 and j_1 are simplicial maps from K_1' into M. Furthermore, $F \circ j_0 = h \circ \phi$ and $F \circ j_1 = g \circ \psi$. It is easy to prove that $(j_0)_* = (j_1)_*$. Indeed, we can transfer the classical result in [25, Lem. 19.1] as we did in the proof of Fact 1.6.4. By Corollary 1.5.3 there exists a compatible triangulation (M', ρ) of F. Consider the simplicial complexes $K_{10} = \{\pi(\sigma' \times \{0\}) : \sigma' \times \{0\} \in M'\}$ and $K_{11} = \{\pi(\sigma' \times \{1\}) :$

$\sigma' \times \{1\} \in M'\}$, where $\pi : |K_1'| \times I \to |K_1'| : (x,t) \mapsto x$. Both K_{10} and K_{11} are subdivisions of K_1'. Consider also the definable homeomorphism $\rho_i : |K_{1i}| \to |K_1'| : x \mapsto \pi(\rho(x,i))$ for each $i = 0,1$. Note that (K_{1i}, ρ_i) is a compatible triangulation of both j_i and $\rho^{-1} \circ j_i$ for each $i = 0,1$. Hence, by (i) and Lemma 1.6.5,

$$\begin{aligned} (h \circ \phi)_* &= (F \circ j_0)_* = (F \circ j_0)_*^{\rho_0} = F_*^{\rho} \circ (j_0)_*^{\rho_0} = F_*^{\rho} \circ (j_0)_* = \\ &= F_*^{\rho} \circ (j_1)* = F_*^{\rho} \circ (j_1)_*^{\rho_1} = (F \circ j_1)_*^{\rho_1} = (F \circ j_1)_* = (g \circ \psi)_*. \end{aligned}$$

On the other hand, as we showed after Remark 1.6.4, $\phi_* = \psi_* = (\mathrm{id}_{|K_1|})_*$. Finally, $h_*^{\phi} = (h \circ \phi)_* \circ \phi_*^{-1} = (g \circ \psi)_* \circ \psi_*^{-1} = g_*^{\psi}$, as required. □

Hence, given two compatible triangulations (K', ϕ) and (K'', ψ) of f, by Proposition 1.6.6 we have that $f_*^{\phi} = f_*^{\psi}$. Finally, we define $f_* := f_*^{\phi}$ for some (any) compatible triangulation (K', ϕ) of f. Note that from Lemma 1.6.5 it is easy to deduce the functorial properties of these induced homomorphisms. Moreover, from Proposition 1.6.6 it is easy to deduce the homotopy axiom.

A similar approach allow us to define the o-minimal simplicial functor in the relative case and, adapting the classical techniques, it is easy to verify the o-minimal Eilenberg-Steenrod homology axioms. Since the subject of this PhD dissertation is not o-minimal homology but o-minimal homotopy, we have avoided the details. Our purpose in this section was just to propose an alternative, and in some sense, more natural definition of induced homomorphism in o-minimal simplicial homology. Furthermore, note that our definition of induced homomorphisms is, a posteriori, that of A. Woerheide and therefore we know that the Eilenberg-Steenrod axioms are fulfilled.

Chapter 2

o-minimal homotopy

2.1 Introduction

In [13], H. Delfs and M. Knebusch study semialgebraic homotopy. The following comparison theorems are the main results of their work (recall the definition of semialgebraic homotopy set in Section 2.3).

Fact 2.1.1. [13, Thm.III.4.1] *Let (X, A) and (Y, B) be two pairs of semialgebraic sets over a real closed field R. Let S be a real closed field extension of R. Let C be a closed semialgebraic subset of X and let $h : C \to Y$ be a semialgebraic map such that $h(A \cap C) \subset B$. Then, if A is closed in X, the map*

$$\begin{array}{rcl} \rho : [(X, A), (Y, B)]_h^{\mathcal{R}_0} & \to & [(X(S), A(S)), (Y(S), B(S))]_h^{\mathcal{S}_0} \\ [f] & \mapsto & [f(S)] \end{array}$$

is a bijection.

Fact 2.1.2. [13, Thm III.5.1] *Let (X, A) and (Y, B) be two pairs of semialgebraic sets of $\overline{\mathbb{R}}$. Let C be a closed semialgebraic subset of X and let $h : C \to Y$ be a semialgebraic map such that $h(A \cap C) \subset B$. Then, if A is closed in X, the map*

$$\begin{array}{rcl} \rho : [(X, A), (Y, B)]_h^{\overline{\mathbb{R}}} & \to & [(X, A), (Y, B)]_h \\ [f] & \mapsto & [f] \end{array}$$

is a bijection, where $[(X, A), (Y, B)]_h$ denotes the classical homotopy set.

Semialgebraic versions of Hurewicz and Whitehead theorems are also deduced from the classical ones via the above comparison results.

In Section 2.3 we will prove a comparison theorem concerning o-minimal and semialgebraic homotopy (see Theorem 2.3.1 and 2.3.4). To do this, we will follow the scheme of the proofs of the results in [13], however the core of their proofs cannot be adapted to our context since they make use of both the Lebesgue number and the Tarski-Seidenberg principle via the polynomial description of semialgebraic sets, which are not available in the o-minimal setting. Instead, we use the results on normal triangulations obtained in Chapter 1. In Section 2.4 we restrict our attention to the study of the o-minimal homotopy groups. In particular, we will show o-minimal versions of some classical results, such as the o-minimal fibration property or the fact that every definable covering is a definable fibration (see Theorem 2.4.9 and Proposition 2.4.10). In Section 2.5, we will deduce o-minimal versions of Hurewicz and Whitehead theorems from the semialgebraic ones (see Theorem 2.5.3 and Theorem 2.5.7). Finally, we will introduce the Lusternik-Schnirelmann category of a definable set in Section 2.6 and we will show its relation with the classical one and its invariance under elementary extensions and o-minimal expansions.

2.2 Preliminaries

We start this section with a basic result for the study of homotopy.

Lemma 2.2.1 (o-minimal homotopy extension lemma). *Let X, Z and A be definable sets with $A \subset X$ closed in X. Let $f : X \to Z$ be a definable map and $H : A \times I \to Z$ a definable homotopy such that $H(x, 0) = f(x)$, $x \in A$. Then there exists a definable homotopy $G : X \times I \to Z$ such that $G(x, 0) = f(x)$, $x \in X$, and $G|_{A\times I} = H$.*

Proof. Let (K, ϕ) be a triangulation of X partitioning A and let $K_A = \{\sigma \in K : \phi(\sigma) \subset A\}$. Note that $|K_A|$ is closed in $|K|$. By [14, Thm.5.1], there exists a semialgebraic retract $r : |K| \times I \to (|K_A| \times I) \cup (|K| \times \{0\})$. This retract naturally induces a definable retract $r' : X \times I \to (A \times I) \cup (X \times \{0\})$. Let $H' : (A \times I) \cup (X \times \{0\}) \to Z$ be the following definable map

$$H'(x,t) = \begin{cases} H(x,t) & \text{for all} \quad (x,t) \in A \times I, \\ f(x) & \text{for all} \quad (x,0) \in X \times \{0\}. \end{cases}$$

Then $G = H' \circ r'$ is the required definable homotopy. □

Now, we briefly develop the approximation simplicial machinery in the o-minimal context.

Definition 2.2.2. *Let K and L be simplicial complexes with K closed. Let $f : |K| \to |L|$ be a definable map. We say that a simplicial map $g : |K| \to |L|$ is a **simplicial approximation to** f if $f(\mathrm{St}_K(w)) \subset \mathrm{St}_L(g(w))$ for each $w \in \mathrm{Vert}(K)$.*

Fact 2.2.3. *Let K and L be simplicial complexes with K closed. Let $f : |K| \to |L|$ be a definable map and let $g : |K| \to |L|$ be a simplicial approximation to f. Then,*
(i) *for every $x \in |K|$ and $\rho \in L$ with $f(x) \in \rho$, we have that $g(x) \in \overline{\rho}$, and*
(ii) *f and g are canonically definably homotopic via the map $(x,s) \mapsto (1-s)f(x) + sg(x)$ for all $(x,s) \in |K| \times I$.*

Proof. (i) Let $x \in |K|$ and $\rho \in L$ with $f(x) \in \rho$. Let $\sigma = (v_0, \dots, v_n) \in K$ such that $x \in \sigma$. Since $\sigma \subset \bigcap_{i=0}^{n} \mathrm{St}_K(v_i)$ then $f(x) \in f(\bigcap_{i=0}^{n} \mathrm{St}_K(v_i)) \subset \bigcap_{i=0}^{n} \mathrm{St}_L(g(v_i)) = \mathrm{St}_L(\tau)$, where $\tau = (g(v_0), \dots, g(v_n))$. Therefore $f(x) \in \mathrm{St}_L(\tau)$. As the star of a set is the smallest open subcomplex containing it, we deduce that $\mathrm{St}_L(f(x)) \subset \mathrm{St}_L(\tau)$. Hence, since $f(x) \in \rho$, we have that $\rho \subset \mathrm{St}_L(f(x)) \subset \mathrm{St}_L(\tau)$. In particular, $\overline{\rho} \cap \tau \neq \emptyset$. As τ is an open simplex, $\tau \subset \overline{\rho}$. Therefore $g(x) \in \tau \subset \overline{\rho}$.
(ii) Clearly $H : |K| \times I \to |L| : (x,s) \mapsto f(x) + s(g(x) - f(x))$ is a definable map. Let us show that H is well-defined. By the previous lemma for every $x \in |K|$ and $\rho \in L$ such that $f(x) \in \rho \in L$, $g(x) \in \overline{\rho}$. Therefore $[f(x), g(x)) \subset \rho$, as required. Finally, note that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in |K|$. □

Fact 2.2.4. [13, Ch.III, Rmk.1.5] *Let L be a simplicial complex. Consider the first barycentric subdivision L' of L. Then,*
(i) *for every simplex σ of L' at least one vertex of σ lies in L',*
(ii) *if every vertex of $\sigma \in \overline{L'}$ is a vertex of L' then $\sigma \in L'$.*

Definition 2.2.5. *Let K and L be simplicial complexes with K closed. We say that a definable map $f : |K| \to |L|$ satisfies the **star condition** if there is a map $\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L) \cap |L|$ such that $f(\mathrm{St}_K(v)) \subset \mathrm{St}_L(\varphi(v))$ for every vertex $v \in \mathrm{Vert}(K)$.*

Fact 2.2.6. *Let K and L be simplicial complexes with K closed. Let $f : |K| \to |L|$ be a definable map and let $\varphi : \mathrm{Vert}(K) \to \mathrm{Vert}(L) \cap |L|$ be a map such that $f(\mathrm{St}_K(v)) \subset \mathrm{St}_L(\varphi(v))$ for every vertex $v \in \mathrm{Vert}(K)$ (so f satisfy the star condition). Then, if either L is the first barycentric subdivision of some simplicial complex or L is closed, the map φ induces a simplicial map which is a simplicial approximation to f.*

Proof. Let us show that φ induce a simplicial map. It is enough to show that $(\varphi(v_0), \dots, \varphi(v_r)) \in L$ for every $\sigma = (v_0, \dots, v_r) \in K$. It follows from $\sigma \subset \bigcap_{i=1}^{r} \mathrm{St}_K(v_i)$ that

$$\emptyset \neq f(\bigcap_{i=1}^{r} \mathrm{St}_K(v_i)) \subset \bigcap_{i=1}^{r} f(\mathrm{St}_K(v_i)) \subset \bigcap_{i=1}^{r} \mathrm{St}_L(\varphi(v_i)).$$

Hence there is $\tau \in L$ such that $\tau \subset \bigcap_{i=1}^{r} \mathrm{St}_L(\varphi(v_i))$. We deduce that $\varphi(v_0), \dots, \varphi(v_r)$ are vertices of τ and therefore $(\varphi(v_0), \dots, \varphi(v_r))$ is a simplex of $\overline{L}$. If L is the first barycentric subdivision of some simplicial complex then, by Lemma 2.2.4, $(\varphi(v_0), \dots, \varphi(v_r))$ is a simplex of L. If L is closed then obviously $(\varphi(v_0), \dots, \varphi(v_r)) \in \overline{L} = L$. Finally, it follows immediately from the hypotheses that the simplicial map induced by φ is a simplicial approximation. □

Definition 2.2.7. *The* ***core*** *of a simplicial complex K is the subcomplex $\mathrm{co}(K)$ of K consisting of all simplices $\sigma \in K$ such that $cl_{|\overline{K}|}(\sigma) \subset |K|$. The* ***core*** *of a system of complexes $(K, K_1, \dots, K_n)$ is the system of complexes $\mathrm{co}(K, K_1, \dots, K_n) = (\mathrm{co}(K), K_1 \cap \mathrm{co}(K), \dots, K_n \cap \mathrm{co}(K))$.*

Remark 2.2.8. (i) Given a simplicial complex K, $\mathrm{co}(K)$ is the maximal subcomplex of K which is a closed simplicial complex.
(ii) By Fact 2.2.4, if K is the first barycentric subdivision of some simplicial complex then $\mathrm{co}(K)$ is non-empty.

Fact 2.2.9. [13, Ch.III, Prop.1.6, 1.8] *Let K and K_1 be simplicial complexes with K_1 a subcomplex of K. Suppose that K is the first barycentric subdivision of some simplicial complex. Then*
(a) *there exist a semialgebraic retraction $r_K : |K| \to |\mathrm{co}(K)|$ such that $(1-t)x + t \cdot r_K(x) \in |K|$ for all $(x,t) \in |K| \times I$ and hence $H_K : |K| \times I \to |K| : (x,t) \mapsto (1-t)x + t \cdot r_K(x)$ is a canonical semialgebraic strong deformation retraction, and*
(b) *if $|K_1|$ is closed in $|K|$, the retraction $r_{K_1} : |K_1| \to |\mathrm{co}(K_1)|$ is the restriction of r_K to $|K_1|$ and hence $H_K|_{|K_1| \times I} = H_{K_1}$.*

2.3 The o-minimal homotopy sets

Let (X, A) and (Y, B) be two pairs of definable sets. Let C be a relatively closed definable subset of X and let $h : C \to Y$ be a definable map such that $h(A \cap C) \subset B$. We say that two definable maps $f, g : (X, A) \to (Y, B)$ with $f|_C = g|_C = h$, are **definably homotopic relative to** h, denoted by $f \sim_h g$, if there exists a definable map $H : (X \times I, A \times I) \to (Y, B)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in X$ and $H(x,t) = h(x)$ for all $x \in C$ and $t \in I$. The **o-minimal homotopy set of** (X, A) **and** (Y, B) **relative to** h is the set

$$[(X, A), (Y, B)]_h^{\mathcal{R}} = \{f : f : (X, A) \to (Y, B) \text{ definable in } \mathcal{R}, f|_C = h\}/ \sim_h .$$

If $C = \emptyset$ we omit all references to h. Recall that we denote by $\mathcal{R}_0$ the field structure of the real closed field R of our o-minimal structure $\mathcal{R}$. Note that if we take $\mathcal{R}$ to be $\mathcal{R}_0$ above, then we obtain the definition of a semialgebraic homotopy set (see Section 2 of Chapter 3 in [13]).

Our main result is the following theorem (see also Theorem 2.3.4).

Theorem 2.3.1. *Let (X, A) and (Y, B) be two pairs of semialgebraic sets with X closed and bounded. Let C be a closed semialgebraic subset of X and $h : C \to Y$ a semialgebraic map such that $h(A \cap C) \subset B$. Then, if A is closed in X, the map*

$$\begin{array}{rcl} \rho : [(X,A),(Y,B)]_h^{\mathcal{R}_0} & \to & [(X,A),(Y,B)]_h^{\mathcal{R}} \\ {[f]} & \mapsto & [f] \end{array}$$

is a bijection.

We are specially interested in the case $C = \emptyset$. However, in order to reduce Theorem 2.3.1 to the following proposition, we will need to consider the general case.

Proposition 2.3.2. *Let K, K_C and L be simplicial complexes with K closed and K_C a closed subcomplex of K. Let $h : |K_C| \to |L|$ be a semialgebraic map. Then the map*

$$\rho : [|K|, |L|]_h^{\mathcal{R}_0} \to [|K|, |L|]_h^{\mathcal{R}}$$

is surjective.

Fact 2.3.3. [13, Thm.III.4.2] *Theorem 2.3.1 can be reduced to Proposition 2.3.2.*

Proof. We will denote ρ by $\rho_{\mathcal{S}}$, where $\mathcal{S} := (X, A, Y, B, h)$, to stress the fact that it depends on (X, A), (Y, B) and h. Granted the Proposition 2.3.2, Theorem 2.3.1 is deduced from the following three reductions.
Reduction 1. It is enough to prove that $\rho_{\mathcal{S}}$ is surjective for every tuple $\mathcal{S}$ satisfying the hypotheses of Theorem 2.3.1. Indeed, fix a tuple $\mathcal{S} := (X, A, Y, B, h)$ satisfying the hypotheses of Theorem 2.3.1. We have to show that $\rho_{\mathcal{S}}$ is injective. Let $f, g : (X, A) \to (Y, B)$, $f|_C, g|_C = h$, be semialgebraic maps such that $\rho_{\mathcal{S}}([f]) = \rho_{\mathcal{S}}([g])$. Then there exists a definable homotopy $F_1 : (X \times I, A \times I) \to (Y, B)$ such that $F_1(x, 0) = f$, $F_1(x, 1) = g$, for all $x \in X$, and $F_1(x, t) = h(x)$ for all $x \in C$ and all $t \in I$. Consider the closed semialgebraic set $\widetilde{C} = (C \times I) \cup (X \times \partial I)$ and the semialgebraic map $H_1 : \widetilde{C} \to Y$,

$$H_1(x,t) = \begin{cases} h(x) & \text{for all} \quad (x,t) \in C \times I, \\ f(x) & \text{for all} \quad x \in X, t = 0, \\ g(x) & \text{for all} \quad x \in X, t = 1. \end{cases}$$

By hypothesis, for $\mathcal{S}_1 := (X \times I, A \times I, Y, B, H_1)$ the map

$$\rho_{\mathcal{S}_1} : [(X \times I, A \times I), (Y, B)]_{H_1}^{\mathcal{R}_0} \to [(X \times I, A \times I), (Y, B)]_{H_1}^{\mathcal{R}}$$

is surjective. Hence, there exists a semialgebraic map $G_1 : (X \times I, A \times I) \to (Y, B)$ such that $G_1|_{\widetilde{C}} = H_1$ and $\rho_{\mathcal{S}_1}([G_1]) = [F_1]$. In particular, G_1 is a semialgebraic homotopy of f to g relative to h. Therefore $[f] = [g]$ in $[(X, A), (Y, B)]_h^{\mathcal{R}_0}$.

Reduction 2. It is enough to prove that $\rho_{\mathcal{S}}$ is surjective for every tuple $\mathcal{S} = (X, \emptyset, Y, \emptyset, h)$ satisfying the hypotheses of Theorem 2.3.1. Indeed, fix a tuple $\mathcal{S} = (X, A, Y, B, h)$ satisfying the hypotheses of Theorem 2.3.1. By Reduction 1, it suffices to show that $\rho_{\mathcal{S}}$ is surjective. Let $f : (X, A) \to (Y, B)$ be a definable map with $f|_C = h$. Consider the restriction $f|_A : A \to B$. By hypothesis, for $\mathcal{S}_2 = (A, \emptyset, B, \emptyset, h|_{A\cap C})$ the map

$$\rho_{\mathcal{S}_2} : [A, B]_{h|_{A\cap C}}^{\mathcal{R}_0} \to [A, B]_{h|_{A\cap C}}^{\mathcal{R}}$$

is surjective. Hence, there is a definable map $G_2 : A \times I \to B$ such that

$$G_2(x, t) = \begin{cases} h(x) & \text{for all} \quad (x, t) \in (A \cap C) \times I, \\ f(x) & \text{for all} \quad x \in A \text{ and } t = 0, \end{cases}$$

and such that the map $k = G_2(x, 1) : A \to B$ is semialgebraic. Consider the closed semialgebraic set $C_2 = C \cup A$ and the semialgebraic map $h_2 : C_2 \to Y$,

$$h_2(x) = \begin{cases} h(x) & \text{for all} \quad x \in C, \\ k(x) & \text{for all} \quad x \in A. \end{cases}$$

Note that indeed the map h_2 is continuous because h and k are continuous and are equal on $A \cap C$. Now, consider the definable map $H_2 : C_2 \times I \to Y$,

$$H_2(x, t) = \begin{cases} G_2(x, t) & \text{for all} \quad (x, t) \in A \times I, \\ h(x) & \text{for all} \quad (x, t) \in C \times I. \end{cases}$$

Note that indeed H is continuous because both G_2 and h are continuous and $G_2(x, t) = h(x)$ for all $(x, t) \in (A \cap C) \times I$. Furthermore, $H_2(x, 0) = f|_{C_2}(x)$ and $H_2(x, 1) = h_2(x)$ for all $x \in C_2$. Since C_2 is closed in X, by Lemma 2.2.1 there exists a definable extension $F_2 : X \times I \to Y$ of H_2 such that $F_2(x, 0) = f(x)$ for all $x \in X$. Since F_2 extends H_2,

$$\begin{cases} F_2(x, t) = H_2(x, t) = h(x) & \text{for all} \quad (x, t) \in C \times I, \\ F_2(a, t) = G_2(a, t) \in B & \text{for all} \quad (a, t) \in A \times I, \end{cases}$$

and the definable map $F_2(x, 1) : X \to Y$ is such that $F_2(x, 1) = h_2(x)$ for all $x \in C_2$. By hypothesis, for $\mathcal{S}_3 = (X, \emptyset, Y, \emptyset, h_2)$ the map

$$\rho_{\mathcal{S}_3} : [X, Y]_{h_2}^{\mathcal{R}_0} \to [X, Y]_{h_2}^{\mathcal{R}}$$

is surjective. Hence there is a definable map $F_3 : X \times I \to Y$ such that

$$F_3(x, t) = \begin{cases} F_3(x, 0) = F_2(x, 1) & \text{for all} \quad x \in X, \\ F_3(x, t) = h_2(x) & \text{for all} \quad (x, t) \in C_2 \times I. \end{cases}$$

and the map $F_3(x,1)$ is semialgebraic. Therefore, $[f] = [F_2(x,1)] = [F_3(x,1)]$ in $[(X,A),(Y,B)]_h^{\mathcal{R}}$, as required.

Reduction 3. It is enough to prove Proposition 2.3.2. By Reduction 1 and 2, it suffices to prove that $\rho_{\mathcal{S}}$ is surjective for a tuple $\mathcal{S} = (X, \emptyset, Y, \emptyset, h)$, X and Y semialgebraic sets with X closed and bounded, C a closed semialgebraic subset of X and $h : C \to Y$ a semialgebraic map. Since X, Y and C are semialgebraic, by the Triangulation theorem there exist triangulations (K, ϕ_1) and (L, ϕ_2) of X and Y respectively such that ϕ_1 partitions C and both ϕ_1 and ϕ_2 are semialgebraic. In particular, since X is closed and bounded, K is a closed simplicial complex. Denote by K_C the closed subcomplex of K such that $\phi^{-1}(C) = |K_C|$. Consider the semialgebraic map $h' = \phi_2^{-1} \circ h \circ \phi_1|_{|K_C|}$ and the natural bijections $\kappa_1 : [X,Y]_h^{\mathcal{R}_0} \to [|K|,|L|]_{h'}^{\mathcal{R}_0} : [f] \mapsto [\phi_2^{-1} \circ f \circ \phi_1]$ and $\kappa_2 : [X,Y]_h^{\mathcal{R}} \to [|K|,|L|]_{h'}^{\mathcal{R}} : [f] \mapsto [\phi_2^{-1} \circ f \circ \phi_1]$. Clearly, the following diagram

$$\begin{array}{ccc}
[X,Y]_h^{\mathcal{R}_0} & \xrightarrow{\rho_{\mathcal{S}}} & [X,Y]_h^{\mathcal{R}} \\
\kappa_1 \downarrow & & \downarrow \kappa_2 \\
[|K|,|L|]_{h'}^{\mathcal{R}_0} & \xrightarrow[\rho_{\mathcal{S}'}]{} & [|K|,|L|]_{h'}^{\mathcal{R}}
\end{array}$$

commutes, where $\mathcal{S}' := (|K|, \emptyset, |L|, \emptyset, h')$. Hence, since $\rho_{\mathcal{S}'}$ is surjective because of Proposition 2.3.2, $\rho_{\mathcal{S}}$ is also surjective. □

Proof of Proposition 2.3.2. Without loss of generality we can assume that L is the first barycentric subdivision of some simplicial complex. Let $[f] \in [|K|,|L|]_h^{\mathcal{R}}$. We will find a semialgebraic map definably homotopic to f relative to h.

Claim. *We can assume that*

(a) *there exist two closed subcomplexes K_D of K and K_E of K_D such that*

$$|K_C| \subset \text{int}_{|K|}(|K_E|) \subset |K_E| \subset \text{int}_{|K|}(|K_D|), \text{ and}$$

(b) *the map f satisfies $f|_{|K_D|} = \widetilde{h}$, where $\widetilde{h} : |K_D| \to |L|$ is a semialgebraic map such that $\widetilde{h}|_{|K_C|} = h$ and for each simplex $\sigma \in K_D$ there is a simplex of L containing $\widetilde{h}(\sigma)$.*

These assumptions allow us to protect $|K_C|$ with two successives "barriers", $|K_D|$ and $|K_E|$. We shall use these barriers in two different places in the following proof to transform the map f without modifying it on $|K_C|$.

We divide the proof in two steps. In Step 1 we will make use of the Normal triangulation theorem (see Theorem 1.1.5) to show that there exists a definable map g satisfying the star condition such that $f \sim_h g$. In Step 2 we will use a simplicial approximation to g (whose existence is ensured by the star condition) to find a semialgebraic map definably homotopic to f relative to h.

Step 1: Let K_Z be the closed subcomplex of K whose polyhedron is $|K_Z| = |K| \setminus \mathrm{int}_{|K|}(|K_D|)$. By the Normal Triangulation Theorem (see Theorem 1.1.5) there exists a normal triangulation (K_0, ϕ_0) of $K_Z \cup K_E$ partitioning $f^{-1}(\sigma) \cap |K_Z|$, $\sigma \in L$. Moreover, since $|K_E| \cap |K_Z| = \emptyset$ and (K_E, id) is a normal triangulation of K_E, we can assume that $\phi_0|_{|K_E|} = \mathrm{id}$. Next we extend (K_0, ϕ_0) to a triangulation of the whole of $|K|$. By Lemma 1.4.3 there exists a normal triangulation (K', ϕ') of K such that $K_0 \subset K'$ and $\phi'|_{|K_0|} = \phi_0$. In particular, $\phi'|_{|K_E|} = \mathrm{id}$. Note that (K', ϕ') partitions the sets $f^{-1}(\sigma)$, $\sigma \in L$. Indeed, it suffices to show that for each $\sigma' \in K'$, $\phi'(\sigma')$ is contained in the preimage by f of some simplex of L. If $\sigma' \subset |K_Z|$ this is clear since ϕ' extends ϕ_0, which in turn partitions the subsets $f^{-1}(\sigma) \cap |K_Z|$ for $\sigma \in L$. On the other hand, if $\sigma' \subset |K| \setminus |K_Z| \subset |K_D|$ then $\phi'(\sigma')$ is contained in some simplex of K_D because (K', ϕ') partitions the simplices of K and, by (b), each simplex of K_D is contained in the preimage by f of some simplex of L.

By Remark 1.1.2.(ii), ϕ' and $id_{|K'|}$ are definably homotopic via the canonical homotopy $H_1 : |K'| \times I \to |K| : (x, s) \mapsto (1-s)x + s\phi'(x)$. The map $H_2 := f \circ H_1$ is clearly a definable homotopy of f to $g := f \circ \phi'$ relative to $\widetilde{h}|_{|K_E|}$. Note also that since (K', ϕ') partitions $f^{-1}(\tau)$ for $\tau \in L$ we have that for every $\sigma \in K'$ there exists $\tau \in L$ such that $g(\sigma) \subset \tau$. Therefore, for every $v \in \mathrm{Vert}(K')$ there exist $w \in \mathrm{Vert}(L)$ with $w \in L$ such that $g(St_{K'}(v)) \subset St_L(w)$. Indeed, take $v \in \mathrm{Vert}(K')$ and $\tau \in L$ such that $g(v) \in \tau$. Since L is the first barycentric subdivision of some simplicial complex, there exists a vertex w of τ with $w \in L$. Since $g^{-1}(St_L(w))$ is the realization of a subcomplex of K', it is open in $|K'|$ and contains the vertex v, we deduce that $St_{K'}(v) \subset g^{-1}(St_L(w))$.

Step 2: Consider the map $\mu_{\mathrm{vert}} : \mathrm{Vert}(K') \to \mathrm{Vert}(L) : v \mapsto \mu_{\mathrm{vert}}(v)$, where (as in Step 1) $\mu_{\mathrm{vert}}(v)$ is such that $\mu_{\mathrm{vert}}(v) \in L$ and $g(\mathrm{St}_{K'}(v)) \subset \mathrm{St}_L(\mu_{\mathrm{vert}}(v))$. By Fact 2.2.6 the map μ_{vert} induces a simplicial approximation μ to g. However neither μ nor the canonical homotopy between μ and g (see Fact 2.2.3(ii)) are good enough for us since we need a map definably homotopic to f relative to h. We do as follows. Since $|K_C|$ and $|K| \setminus \mathrm{int}_{|K|}(|K_E|)$ are closed and disjoint, by Theorem 1.6 in [14], there exists a semialgebraic function $\lambda : |K| \to [0, 1]$ such that $\lambda^{-1}(0) = |K_C|$ and $\lambda^{-1}(1) = |K| \setminus \mathrm{int}_{|K|}(|K_E|)$. Consider the map $H : |K| \times I \to |L| : (x, s) \mapsto (1 - s\lambda(x))g(x) + s\lambda(x)\mu(x)$. The definable map H is indeed continuous and is well-defined. Note that

$$\begin{cases} H(x,0) = g(x) \quad \text{for all} \quad x \in |K|, \\ H(x,s) = g(x) = h(x) \quad \text{for all} \quad x \in |K_C| \text{ and } s \in I. \end{cases}$$

Furthermore, observe that

$$H(x,1) = \begin{cases} \mu(x) \quad \text{for all} \quad x \in |K| \setminus \mathrm{int}_{|K|}(|K_E|), \\ (1-\lambda(x))\widetilde{h}(x) + \lambda(x)\mu(x) \quad \text{for all} \quad x \in |K_E|, \end{cases}$$

is semialgebraic. Hence $f \sim_h g \sim_h H(x,1)$, as required.

Proof of the Claim. By [14, Thm 2.7], $|K_C|$ is a semialgebraic strong deformation retract of $\overline{\mathrm{St}_{K''}(|K_C|)}$, where K'' is the second barycentric subdivision. Without loss of generality, we replace K by K'' and hence we can assume that there is a closed subcomplex K_D of K such that $|K_C| \subset \mathrm{int}(|K_D|)$ and $|K_C|$ is a semialgebraic strong deformation retract of $|K_D|$, i.e., there is a semialgebraic map $H_2 : |K_D| \times I \to |K_D|$ such that

$$\begin{cases} H_2(x,0) = x \quad \text{for all} \quad x \in |K_D|, \\ H_2(x,1) \in |K_C| \quad \text{for all} \quad x \in |K_D|, \\ H_2(x,t) = x \quad \text{for all} \quad (x,t) \in |K_C| \times I. \end{cases}$$

In particular, $\widetilde{h} : |K_D| \to |L| : x \mapsto \widetilde{h}(x) = h(H_2(x,1))$ is a semialgebraic extension of h. Moreover, the map $G : |K_D| \times I \to |L| : (x,t) \mapsto f(H_2(x,t))$ is a definable homotopy such that

$$\begin{cases} G(x,0) = f(x) \quad \text{for all} \quad x \in |K_D|, \\ G(x,1) = f(H_2(x,1)) = h(H_2(x,1)) = \widetilde{h}(x) \quad \text{for all} \quad x \in |K_D|, \\ G(x,t) = h(x) \quad \text{for all} \quad x \in |K_C| \times I. \end{cases}$$

By Lemma 2.2.1, there is a definable homotopy $F : |K| \times I \to |L|$ such that $F(x,0) = f(x)$ for all $x \in |K|$ and $F|_{|K_D| \times I} = G$. Consider the definable map $f'(x) := F(x,1)$, which satisfies $f'(x) = \widetilde{h}(x)$ for all $x \in |K_D|$. Clearly, $[f] = [f']$ in $[|K|,|L|]^{\mathcal{R}}_h$. Without loss of generality, we replace f by f' and hence we can assume that the map f satisfies $f|_{|K_D|} = \widetilde{h}$, where $\widetilde{h} : |K_D| \to |L|$ is a semialgebraic map such that $\widetilde{h}|_{|K_C|} = h$. By using the first barycentric subdivision of K, which we may denote again by K, we can assume that the closed subcomplex K_E of K with $|K_E| = \overline{\mathrm{St}_K(K_C)}$ satisfies $|K_C| \subset \mathrm{int}(|K_E|)$ and $|K_E| \subset \mathrm{int}(|K_D|)$.

Now, we show that we can assume that for every $\sigma \in K_D$, $\widetilde{h}(\sigma)$ is contained in a simplex of L. Let $(\widehat{K}, \widehat{\phi})$ be a semialgebraic triangulation of $|K|$ partitioning the simplices of K and the semialgebraic subsets $\widetilde{h}^{-1}(\tau)$, for all $\tau \in L$. Consider the subcomplexes $\widehat{K}_D := \{\widehat{\phi}^{-1}(\sigma) : \sigma \in K_D\}$, $\widehat{K}_E := \{\widehat{\phi}^{-1}(\sigma) : \sigma \in K_E\}$ and $\widehat{K}_C = \{\widehat{\phi}^{-1}(\sigma) : \sigma \in K_C\}$. Consider also the semialgebraic map $\widehat{h} := h \circ \widehat{\phi}|_{|\widehat{K}_C|} : |\widehat{K}_C| \to |L|$. Note that $\widehat{K}_D$, $\widehat{K}_E$ and $\widehat{K}_C$ are closed subcomplexes of $\widehat{K}$ such that $|\widehat{K}_C| \subset \mathrm{int}(|\widehat{K}_E|)$ and $|\widehat{K}_E| \subset \mathrm{int}(|\widehat{K}_D|)$. Moreover, the map $\widehat{f} := f \circ \widehat{\phi}$ satisfies $\widehat{f}|_{|K_D|} = \widetilde{h} \circ \widehat{\phi}|_{|\widehat{K}_D|}$, where $\widetilde{h} \circ \widehat{\phi}|_{|\widehat{K}_D|} : |\widehat{K}_D| \to |L|$ is a semialgebraic map such that $\widetilde{h} \circ \widehat{\phi}|_{|\widehat{K}_C|} = \widehat{h}$. Finally, note that since $\widehat{\phi}$ is semialgebraic, to prove that f is definably homotopic to a semialgebraic map relative to h it is enough to prove that $\widehat{f}$ is definably homotopic to a semialgebraic map relative to $\widehat{h}$, as required. □

Proof of Theorem 2.3.1. By Fact 2.3.2 and Proposition 2.3.2. □

As an immediate consequence of Theorem 2.3.1 we prove a more general result.

Theorem 2.3.4. *Let* $(X, A_1, \dots, A_k)$ *and* $(Y, B_1, \dots, B_k)$ *be two systems of semialgebraic sets. Let* C *be a relatively closed semialgebraic subset of* X *and* $h : C \to Y$ *a semialgebraic map such that* $h(C \cap A_i) \subset B_i$, $i = 1, \dots, k$. *Then, if the subsets* $A_1, \dots, A_k$ *are relatively closed in* X, *the map*

$$\begin{array}{rcl} \rho : [(X, A_1, \dots, A_k), (Y, B_1, \dots, B_k)]_h^{\mathcal{R}_0} & \to & [(X, A_1, \dots, A_k), (Y, B_1, \dots, B_k)]_h^{\mathcal{R}} \\ [f] & \mapsto & [f] \end{array}$$

is a bijection.

Proof. Firstly, note that the hypotheses of Lemma 2.2.1 do not include X closed. Therefore, the reductions in Fact 2.3.3 apply to this setting. Hence, it suffices to show that $\rho : [|K|, Y]_h^{\mathcal{R}_0} \to [|K|, Y]_h^{\mathcal{R}} : [f] \mapsto [f]$ is surjective, where K is a simplicial complex, K_C is a relatively closed subcomplex of K and $h : |K_C| \to Y$ is a semialgebraic map. Without loss of generality we can assume that K is the first barycentric subdivision of a simplicial complex. Let $f : |K| \to Y$ be a definable map. Denote by $K^0 := \mathrm{co}(K)$ and $K_C^0 := \mathrm{co}(K_C)$ (see Definition 2.2.7 and Remark 2.2.8). Since K^0 is closed, by Theorem 2.3.1 there is a definable homotopy $F_1' : |K^0| \times I \to Y$ such that $F_1'(-, 0) = f|_{|K^0|}$, $g' := F_1'(-, 1)$ is semialgebraic and $F_1'(x, t) = h(x)$ for all $(x, t) \in |K_C^0| \times I$. Let $r : |K| \to |K^0|$ be the semialgebraic retract of Fact 2.2.9(a). We define the semialgebraic map $F_2' : |K_C| \times I \to Y : (x, t) \mapsto h((1-t)r(x) + tx)$ (see Fact 2.2.9(b)). By the homotopy extension lemma 2.2.1 there is a semialgebraic homotopy $F_2 : |K| \times I \to Y$ such that $F_2|_{|K_C| \times I} = F_2'$ and $F_2(x, 0) = g' \circ r(x)$ for all $x \in |K|$. Note that $g := F_2(-, 1)$ is a semialgebraic map with $g|_{|K_C|} = h$. We show that f is definably homotopic to g relative to h. The maps $f \circ r$ and $g' \circ r$ are definably homotopic via $F_1 : |K| \times I \to Y : (x, t) \mapsto F_1'(r(x), t)$. Note that by Fact 2.2.9(b), $F_1(x, t) = h \circ r(x)$ for all $(x, t) \in |K_C| \times I$. Consider the closed definable subset $A := |K| \times \{0\} \cup |K| \times \{1\} \cup |K_C| \times I$ of $|K| \times I$. We consider also the definable homotopy $H' : A \times I \to Y$ such that

$$H'(x, t, s) = \begin{cases} f((1-s)r(x) + sx) & \text{for all} \quad (x, t, s) \in |K| \times \{0\} \times I, \\ F_2(x, s) & \text{for all} \quad (x, t, s) \in |K| \times \{1\} \times I, \\ h((1-s)r(x) + sx) & \text{for all} \quad (x, t, s) \in |K_C| \times I \times I. \end{cases}$$

By the homotopy extension lemma 2.2.1 there is a definable homotopy $H : |K| \times I \times I \to Y$ such that $H|_{A \times I} = H'$ and $H(x, t, 0) = F_1(x, t)$ for all $(x, t) \in |K| \times I$. In particular, $F(x, t) := H(x, t, 1) : |K| \times I \to Y$ is a definable homotopy of f to g relative to h. □

Corollary 2.3.5. *Let* X *and* Y *be two pairs of semialgebraic sets defined without parameters. Then there exist a bijection*

$$\rho : [X(\mathbb{R}), Y(\mathbb{R})] \to [X, Y]^{\mathcal{R}},$$

where $[X(\mathbb{R}), Y(\mathbb{R})]$ *denotes the classical homotopy set. Moreover, if the real closed field* R *is a field extension of* $\mathbb{R}$*, then the result remains true allowing parameters from* $\mathbb{R}$*.*

Proof. By Facts 2.1.1 and 2.1.2, there exits a canonical bijection between $[X(\mathbb{R}), Y(\mathbb{R})]$ and the semialgebraic homotopy set over the real algebraic numbers $[X(\overline{\mathbb{Q}}), Y(\overline{\mathbb{Q}})]^{\overline{\mathbb{Q}}}$. By Fact 2.1.1, there exists a canonical bijection between $[X(\overline{\mathbb{Q}}), Y(\overline{\mathbb{Q}})]^{\overline{\mathbb{Q}}}$ and $[(X, A), (Y, B)]^{\mathcal{R}_0}$. The result then follows by Theorem 2.3.1. The proof of the second part is similar. □

Remark 2.3.6. This corollary remains true for systems of semialgebraic sets satisfying the hypotheses of Theorem 2.3.4.

Corollary 2.3.7. *Let* X *and* Y *be two definable sets defined without parameters. Then any definable map* $f : X \to Y$ *is definably homotopic to a definable map* $g : X \to Y$ *defined without parameters. If moreover* X *and* Y *are semialgebraic then* g *can also be taken semialgebraic.*

Proof. By the Triangulation Theorem there are triangulations of X and Y defined without parameters and therefore it suffices to prove the case in which both X and Y are semialgebraic. By Theorem 2.3.1, f is definably homotopic to a semialgebraic map g_1. Finally, it follows from Fact 2.1.1 applied to $\mathcal{R}_0$ and $\overline{\mathbb{Q}}$ that g_1 is semialgebraically homotopic to a semialgebraic map g defined without parameters. □

2.4 The o-minimal homotopy groups

We begin this section with a general discussion of homotopy groups in the o-minimal setting. Then we will relate the semialgebraic and the o-minimal homotopy groups via Theorem 2.3.1. Finally, we will prove the usual properties related to homotopy in the o-minimal framework.

We will work with the category whose objects are the **definable pointed sets**, i.e., (X, x_0), where X is a definable set with $x_0 \in X$, and whose morphisms are the definable continuous maps between definable pointed sets. In a similar way, we define the categories of **definable pointed pairs**, i.e., (X, A, x_0), where X is a definable set, A is a definable subset of X and $x_0 \in A$.

Let (X, x_0) be a definable pointed set. The **o-minimal homotopy group** of dimension n, $n \geq 1$, is the set $\pi_n(X, x_0)^{\mathcal{R}} = [(I^n, \partial I^n), (X, x_0)]^{\mathcal{R}}$. We define $\pi_0(X, x_0)$ as the set of definably connected components of X. The **o-minimal relative homotopy group** of dimension n, $n \geq 1$, of a definable pointed pair (X, A, x_0) is the homotopy set $\pi_n(X, A, x_0)^{\mathcal{R}} = [(I^n, I^{n-1}, J^{n-1}), (X, A, x_0)]^{\mathcal{R}}$, where $I^{n-1} = \{(t_1, \ldots, t_n) \in I^n : t_n = 0\}$ and $J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$.

As in the classical case, we can define a group operation in the o-minimal homotopy groups $\pi_n(X,x_0)^{\mathcal{R}}$ and $\pi_m(X,A,x_0)^{\mathcal{R}}$ via the usual concatenation of maps for $n \geq 1$ and $m \geq 2$. Specifically, given $[f],[g] \in \pi_n(X,x_0)^{\mathcal{R}}$, $n \geq 1$, we define the operation $[f][g] = [f \cdot g]$, where

$$f \cdot g = \begin{cases} f(2t_1, t_2, \ldots, t_n) & \text{for all} \quad t_1 \in [0, \frac{1}{2}], \\ g(2t_1 - 1, t_2, \ldots, t_n) & \text{for all} \quad t_1 \in [\frac{1}{2}, 1]. \end{cases}$$

Likewise, given $[f],[g] \in \pi_n(X,A,x_0)^{\mathcal{R}}$, $n \geq 2$, we define the operation $[f][g] = [f \cdot g]$. Moreover, these groups are abelian for $n \geq 2$ and $m \geq 3$ (see pp. 340 and pp. 343 in [21]). Also, given a definable map between definable pointed sets (or pairs), we define the induced map in homotopy by the usual composing, which will be a homomorphism in the case we have a group structure. It is easy to check that with these definitions of o-minimal homotopy group and induced map, both the absolute and relative o-minimal homotopy groups $\pi_n(-)$ are covariant functors (see pp. 342 in [21]).

As a consequence of Theorem 2.3.1, we deduce the following relation between the semialgebraic and the o-minimal homotopy groups.

Theorem 2.4.1. *Let (X,x_0) be a semialgebraic pointed set and let $n \geq 1$. Then the map $\rho : \pi_n(X,x_0)^{\mathcal{R}_0} \to \pi_n(X,x_0)^{\mathcal{R}} : [f] \mapsto [f]$, is a natural isomorphism.*

Proof. By Theorem 2.3.1 ρ is a bijection and its clearly a homomorphism. For the naturality condition, just observe that by definition the following diagram

$$\begin{array}{ccc} \pi_n(X,x_0)^{\mathcal{R}_0} & \xrightarrow{\psi_*} & \pi_n(Y,y_0)^{\mathcal{R}_0} \\ \downarrow \rho & & \downarrow \rho \\ \pi_n(X,x_0)^{\mathcal{R}} & \xrightarrow[\psi_*]{} & \pi_n(Y,y_0)^{\mathcal{R}} \end{array}$$

commutes, for every semialgebraic map $\psi : (X,x_0) \to (Y,y_0)$. □

Remark 2.4.2. This last result remains true in the relative case and its proof is similar.

Corollary 2.4.3. *The o-minimal homotopy groups are invariants under elementary extensions and o-minimal expansions.*

Proof. The invariance under o-minimal expansions follows from the Triangulation Theorem and Theorem 2.4.1. The invariance under elementary extensions follows from the Triangulation Theorem, Theorem 2.4.1 and the invariance of the semialgebraic homotopy sets under real closed field extensions (see Theorem III.6.3 in [13]). □

The following result gives us a relation between the classical and the o-minimal homotopy groups (the case $n = 1$ was already treated in [7]).

Corollary 2.4.4. *Let (X, x_0) be a semialgebraic pointed set defined without parameters. Then there exists a natural isomorphism between the classical homotopy group $\pi_n(X(\mathbb{R}), x_0)$ and the o-minimal homotopy group $\pi_n(X(R), x_0)^{\mathcal{R}}$ for every $n \geq 1$.*

Proof. Either by Corollary 2.4.3 and Theorem III.6.4 in [13] or by Theorem 2.3.4 noting that the bijections involved are isomorphisms. □

Remark 2.4.5. These last results remain true in the relative case and their proofs are similar. Moreover, the analogue of Corollary 2.4.3 is true for homotopy sets of definable systems satisfying the hypotheses of Theorem 2.3.4.

Properties 2.4.6. We now list some characteristic properties of the homotopy groups that remain true in the o-minimal setting.

(1) *The homotopy property:* It is immediate that given two definably homotopic maps $\psi, \phi : (X, A, x_0) \to (Y, B, y_0)$, their induced homomorphisms $\psi_*, \phi_* : \pi_n(X, A, x_0)^{\mathcal{R}} \to \pi_n(Y, B, y_0)^{\mathcal{R}}$ are equal for every $n \geq 1$. Note that for $A = \{x_0\}$ and $B = \{y_0\}$ we have the absolute case.

(2) *The exactness property:* Let (X, A, x_0) be a pointed pair. For every $n \geq 2$ we define the **boundary operator**

$$\begin{array}{rcl} \partial : \pi_n(X, A, x_0)^{\mathcal{R}} & \to & \pi_{n-1}(A, x_0)^{\mathcal{R}} \\ [f] & \mapsto & [f|_{I^{n-1}}]. \end{array}$$

For $n = 1$, we define $\partial([u])$, $[u] \in \pi_1(X, A, x_0)^{\mathcal{R}}$, as the definably connected component of A which contains $u(0)$. It is easy to prove that the boundary operator is a natural well-defined homomorphism for $n > 1$. Moreover, if we denote by $i : (A, x_0) \to (X, x_0)$ and $j : (X, x_0, x_0) \to (X, A, x_0)$ the inclusion maps, then the following sequence is exact

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(A, x_0),$$

where the superscript $\mathcal{R}$ has been omitted. Indeed, by the triangulation theorem we can assume that (X, A, x_0) is the realization of a simplicial complex with vertices in the real algebraic numbers. Then the exactness property follows from Corollary 2.4.4, the obvious fact that ∂ commutes with the isomorphism defined there and the classical exactness property.

(3) *The action of π_1 on π_n:* Given a definable pointed set (X, x_0) we can define the usual action $\beta : \pi_1(X, x_0)^{\mathcal{R}} \times \pi_n(X, x_0)^{\mathcal{R}} \to \pi_n(X, x_0)^{\mathcal{R}}$. Given $[u] \in \pi_1(X, x_0)^{\mathcal{R}}$, we denote by $\beta_{[u]}$ the isomorphism $\pi_n(X, x_0)^{\mathcal{R}} \to \pi_n(X, x_0)^{\mathcal{R}} : [f] \mapsto \beta([u], [f])$. In a similar way, given a definable pointed pair (X, A, x_0) there is an action $\beta : \pi_1(A, x_0)^{\mathcal{R}} \times \pi_n(X, A, x_0)^{\mathcal{R}} \to \pi_n(X, A, x_0)^{\mathcal{R}}$. The existence of both actions can be proved just adapting what is done in pp. 268 in [13] to the o-minimal setting. We briefly recall the construction of this action

in the absolute case. Given $[f] \in \pi_n(X,x_0)^{\mathcal{R}}$, by the o-minimal homotopy extension lemma 2.2.1 there is a definable homotopy $H : I^n \times I \to X$ such that $H(x,1) = f(x)$ for all $x \in I^n$ and $H(x,t) = u(t)$ for all $(x,t) \in \partial I^n \times I$. We define $\beta_{[u]}([f]) := [H(-,0)]$. Let us check that $\beta_{[u]}$ is well-defined. Firstly, given a definable curve $v : I \to X$ with $v(0) = v(1) = x_0$ and such that $[u] = [v]$, we show that $\beta_{[u]} = \beta_{[v]}$. Let $F : I \times I \to X$ be a definable homotopy from u to v with $F(0,s) = F(1,s) = x_0$ for all $s \in I$. Given $[f] \in \pi_n(X,x_0)^{\mathcal{R}}$, let H and H' be definable homotopies as before such that $\beta_{[u]}([f]) = [H(-,0)]$ and $\beta_{[v]}([f]) = [H'(-,0)]$. By the o-minimal homotopy extension lemma 2.2.1 there is a definable homotopy $G : I^n \times I \times I \to X$ such that

$$G(x,t,s) = \begin{cases} f(x) & \text{for all} \quad (x,t,s) \in I^n \times I \times \{1\}, \\ H(x,s) & \text{for all} \quad (x,t,s) \in I^n \times \{0\} \times I, \\ H'(x,s) & \text{for all} \quad (x,t,s) \in I^n \times \{1\} \times I, \\ F(s,t) & \text{for all} \quad (x,t,s) \in \partial I^n \times I \times I. \end{cases}$$

Hence, $G(x,t,0)$ is a definable homotopy from $H(-,0)$ to $H'(-,0)$ relative to ∂I^n, as required. In particular, the latter also proves that the definition of $\beta_{[u]}([f])$ above does not depend on the choice of H as well as that $\beta_{[u]}([f])$ depends only on the homotopy class of f. Let us check that $\beta_{[u]}$ is a homomorphism. Given $[f],[g] \in \pi_n(X,x_0)$, we consider the definable homotopy $H : I^n \times I \to X$ such that $H(x_1,\dots,x_n,t) = \beta_{[u]}([f][0])((2-t)x_1,\dots,x_n)$ for all $(x_1,\dots,x_n,t) \in [0,\frac{1}{2}] \times I^n$ and $H(x_1,\dots,x_n,t) = \beta_{[u]}([0][g])((2-t)x_1+t-1,\dots,x_n)$ for all $(x_1,\dots,x_n,t) \in [\frac{1}{2},1] \times I^n$. Using this homotopy we have $\beta_{[u]}([f][g]) = \beta_{[u]}([f][0])\beta_{[u]}([0][g]) = \beta_{[u]}([f])\beta_{[u]}([g])$. Finally, the remain properties
$\beta_{[c_{x_0}]} = \mathrm{id}$, for c_{x_0} the constant curve, and $\beta_{[u][v]} = \beta_{[u]} \circ \beta_{[v]}$ for all $[u],[v] \in \pi_1(X,x_0)^{\mathcal{R}}$, are obvious.

We will need the following technical lemma in the proof of the o-minimal Hurewicz theorem (see Section 2.5).

Lemma 2.4.7. *Let $\psi : (X,x_0) \to (Y,y_0)$ be a definable map between definable pointed sets and let $[u] \in \pi_1(X,x_0)^{\mathcal{R}}$. Then for all $[f] \in \pi_n(X,x_0)^{\mathcal{R}}$, $\psi_*(\beta_{[u]}([f])) = \beta_{\psi_*([u])}(\psi_*([f]))$.*

Proof. It is enough to observe that if $H : I^n \times I \to X$ is a definable homotopy such that $H(t,1) = f(t)$ for all $t \in I^n$ and $H(t,s) = u(s)$ for all $t \in \partial I^n$ and $s \in I$, then $\psi \circ H : I^n \times I \to Y$ is a definable homotopy such that $\psi \circ H(t,1) = \psi \circ f(t)$ for all $t \in I^n$ and $\psi \circ H(t,s) = \psi \circ u(s)$ for all $t \in \partial I^n$ and $s \in I$. □

(4) *The fibration property:* We say that a definable map $p : E \to B$ is a **definable (Serre) fibration** if it has the definable *homotopy lifting property* for every (resp. closed and bounded) definable set X, i.e., if for each

definable homotopy $H : X \times I \to B$ and each definable map $\widetilde{f} : X \to E$ with $p \circ \widetilde{f}(x) = H(x, 0)$ for all $x \in X$, there exists a definable homotopy $\widetilde{H} : X \times I \to E$ with $p \circ \widetilde{H} = H$ and $\widetilde{H}(x, 0) = \widetilde{f}(x)$ for all $x \in X$.

Remark 2.4.8. We say that a definable map $p : E \to B$ has the *definable homotopy lifting property for a definable set X relative to a definable subset A of X* if for each definable homotopy $H : X \times I \to B$, each definable map $\widetilde{f} : X \to E$ with $p \circ \widetilde{f}(x) = H(x, 0)$ for all $x \in X$, and each definable homotopy $F : A \times I \to E$ with $p \circ F = H|_{A\times I}$ there exists a definable homotopy $\widetilde{H} : X \times I \to E$ with $p \circ \widetilde{H} = H$, $\widetilde{H}|_{A\times I} = F$ and $\widetilde{H}(x, 0) = \widetilde{f}(x)$ for all $x \in X$. As in the classical setting, the definable homotopy lifting property for a *closed* simplex σ is equivalent to the definable homotopy lifting property for σ relative to $\partial\sigma$. Indeed, it suffices to show that there is a semialgebraic homeomorphism of $\sigma \times I$ onto itself which carries $\sigma \times \{0\}$ homeomorphically onto $(\sigma \times \{0\}) \cup (\partial\sigma \times I)$. We adapt [22, Ch.III,Thm.3.1]. Without loss of generality, we can assume that σ is the standard simplex $\Delta \subset R^m$ for some $m \in \mathbb{N}$. Firstly, consider the semialgebraic homeomorphism $h_0 : (\Delta \times \{0\}) \cup (\partial\Delta \times I) \to \Delta \times \{0\} : (x_0, \dots, x_m, t) \mapsto (y_0, \dots, y_m, 0)$, where

$$y_i = \begin{cases} \frac{1}{2}(x_i + \frac{1}{m+1}) & \text{for all} \quad (x_0, \dots, x_m, t) \in \Delta \times \{0\}, \\ \frac{1}{2}(x_i + \frac{1}{m+1}) + \frac{1}{2}(x_i - \frac{1}{m+1})t & \text{for all} \quad (x_0, \dots, x_m, t) \in \partial\Delta \times I. \end{cases}$$

Note that $h_0(x_0, \dots, x_m, 1) = (x_0, \dots, x_m, 0)$ for each $(x_0, \dots, x_m) \in \partial\Delta$. Similarly, we can define a semialgebraic homeomorphism $h_1 : (\Delta \times \{1\}) \cup (\partial\Delta \times I) \to \Delta \times \{1\}$ such that $h_1(x_0, \dots, x_m, 0) = (x_0, \dots, x_m, 1)$ for each $(x_0, \dots, x_m) \in \partial\Delta$. Next, consider the semialgebraic homeomorphism $h_2 : \partial(\Delta\times I) \to \partial(\Delta\times I)$ such that $h_2(w) = h_0(w)$ for all $w \in (\Delta\times\{0\})\cup(\partial\Delta\times I)$ and $h_2(w) = h_1^{-1}(w)$ for all $w \in \Delta \times \{1\}$. Finally, the semialgebraic homeomorphism h_2 of $\partial(\Delta\times I)$ can be extended to a semialgebraic homeomorphism h of $\Delta\times I$ by radial extension from the point $(c, \frac{1}{2})$ of $\Delta\times I$, where c denotes the barycentre of Δ.

Furthermore, the homotopy lifting property for closed simplices is equivalent to the homotopy lifting property for closed and bounded definable sets X relative to closed subsets A of X. For, by the triangulation theorem we can assume that X is the realization of a closed simplicial complex and A is the realization of a closed subcomplex of X. By induction over the skeleta of X it suffices to construct a lifting over the closure of each open simplex contained in $X \setminus A$ at a time (and relative to the lifting constructed previously over its frontier).

With the above remark it is easy to adapt to the o-minimal setting the corresponding classical proof of the following fact (see Theorem 4.41 in [21]).

Theorem 2.4.9 (The fibration property). *Let E and B definable sets and let $p : E \to B$ be a definable Serre fibration. Then the induced map*

$p_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ *is a bijection for* $n = 1$ *and an isomorphism for all* $n \geq 2$*, where* $e_0 \in F = p^{-1}(b_0)$.

Proof. Firstly, we show that p_* is onto. Let $f : (I^n, \partial I^n) \to (B, b_0)$ be a definable map. By Remark 2.4.8, there is a definable homotopy $\widetilde{H} : I^{n-1} \times I \to E$ such that $\widetilde{H}(x,t) = e_0$ for all $(x,t) \in \partial I^{n-1} \times I$, $\widetilde{H}(x,0) = e_0$ for all $x \in I^{n-1}$ and $p \circ \widetilde{H}(x_1, \ldots, x_{n-1}, t) = f(x_1, \ldots, x_{n-1}, 1-t)$ for all $(x_1, \ldots, x_{n-1}, t) \in I^n$. Consider the definable map $\widetilde{f} : (I^n, I^{n-1}, J^{n-1}) \to (E, F, e_0) : (x_1, \ldots, x_{n-1}, t) \mapsto \widetilde{H}(x_1, \ldots, x_{n-1}, 1-t)$. Clearly, $p_*([\widetilde{f}]) = [p \circ \widetilde{f}] = [f]$. Now, we show that p_* is injective. Given definable maps $\widetilde{f}, \widetilde{g} : (I^n, I^{n-1}, J^{n-1}) \to (E, F, e_0)$ such that $p_*([\widetilde{f}]) = p_*([\widetilde{g}])$, let $H : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a definable homotopy from $p \circ \widetilde{f}$ to $p \circ \widetilde{g}$. By Remark 2.4.8, there is a definable homotopy $\widetilde{G} : I^n \times I \to E$ such that $\widetilde{G}(0, x_2, \ldots, x_n, t) = \widetilde{f}(x_2, \ldots, x_n, 1-t)$ for all $(x_2, \ldots, x_n, t) \in I^n$, $\widetilde{G}(1, x_2, \ldots, x_n, t) = \widetilde{g}(x_2, \ldots, x_n, 1-t)$ for all $(x_2, \ldots, x_n, t) \in I^n$, $\widetilde{G}(x_1, \ldots, x_n, t) = e_0$ for all $(x_1, x_2, \ldots, x_n, 1-t) \in I \times J^{n-1}$ and $p \circ \widetilde{G}(x_1, \ldots, x_n, t) = G(x_2, \ldots, x_n, 1-t, x_1)$ for all $(x_1, \ldots, x_n, t) \in I^n \times I$. In particular, the map $\widetilde{H} : (I^n, , I^{n-1}, J^{n-1}) \to (E, F, e_0) : (x_1, \ldots, x_n, t) \mapsto \widetilde{G}(t, x_1, \ldots, x_{n-1}, 1-x_n)$ is a definable homotopy from $\widetilde{f}$ to $\widetilde{g}$. □

As a consequence of the fibration property and the following proposition, we can extend Corollary 2.8 in [20], concerning coverings and the fundamental group, to all the homotopy groups (see Corollary 2.4.11 below). Recall that given two definable sets E and B, a definable map $p : E \to B$ is a **definable covering map** if p is onto and there is a finite family $\{U_l : l \in L\}$ of definably connected open definable subsets of B such that $B = \bigcup_{l \in L} U_l$, and for each $l \in L$ and for each definably connected component V of $p^{-1}(U_l)$, the map $p|_V : V \to U_l$ is a definable homeomorphism (see also Section 2 in [20].

Proposition 2.4.10. *Let* E *and* B *definable sets. Then every definable covering* $p : E \to B$ *is a definable fibration.*

Proof. Let X be a definable set. Let $H : X \times I \to B$ a definable homotopy and $\widetilde{f}$ a definable map $\widetilde{f} : X \to E$ with $p \circ \widetilde{f}(x) = H(x,0)$ for all $x \in X$. Consider the definable family of paths $\{H_x : x \in X\}$, where $H_x : I \to B : t \mapsto H(x,t)$. Since p has the path lifting property (see Proposition 2.6 in [20]), for each $x \in X$ there is a (unique) lifting $\widetilde{H}_x : I \to E$ of H_x such that $\widetilde{H}_x(0) = \widetilde{f}(x)$. Moreover, an easy modification of the proof of Proposition 2.6 in [20] shows that the family of paths $\{\widetilde{H}_x : x \in X\}$ is definable. Therefore, the map $\widetilde{H} : X \times I \to E : (x,t) \mapsto \widetilde{H}_x(t)$ is definable, $p \circ \widetilde{H} = H$ and $\widetilde{H}(x,0) = \widetilde{f}(x)$ for all $x \in X$. It remains to prove that $\widetilde{H}$ is indeed continuous. Fix $(x_0, s_0) \in X \times I$. It is enough to prove that for each definable path $u : I \to X \times I$ with $u(1) = (x_0, s_0)$ we have that $\widetilde{H}(u(t)) \to \widetilde{H}(x_0, s_0)$ when $t \to 1$. We will prove it for $s_0 = 1$, but the same proof works for every $s_0 \in I$.

Claim. *We can assume that* $u(0) = (x_0, 0)$, *that* u *is definably homotopic to the canonical path* $I \to X \times I : t \mapsto (x_0, t)$ *and that* $\widetilde{H} \circ u : [0, 1) \to E$ *is continuous.*

Granted the Claim, the path homotopy lifting property of p (see Proposition 2.7 in [20]) implies that the respective liftings $\widetilde{H \circ u}$ and $\widetilde{H}_{x_0}$ of $H \circ u$ and H_{x_0} starting at $\widetilde{f}(x_0)$, satisfy $\widetilde{H \circ u}(1) = \widetilde{H}_{x_0}(1)$. On the other hand, by the unicity of liftings of paths of p, we have that for every $\epsilon \in [0, 1)$, $\widetilde{H}(u(t)) = \widetilde{H \circ u}(t)$ for all $t \in [0, \epsilon]$. Therefore, $\widetilde{H}(u(t)) = \widetilde{H \circ u}(t)$ for all $t \in [0, 1)$. Hence, $\widetilde{H}(u(t)) \to \widetilde{H \circ u}(1) = \widetilde{H}_{x_0}(1) = \widetilde{H}(x_0, 1)$ when $t \to 1$, as required.

Proof of the Claim. Since X is definable, there exist a definably connected neighbourhood U of x_0 which is definably contractible. Since $\widetilde{H} \circ u$ is definable, without loss of generality, $\widetilde{H} \circ u$ is continuous in $[\frac{2}{3}, 1)$ and $u(t) \in U \times I$ for all $t \in [\frac{2}{3}, 1)$. Let $(x_1, s_1) = u(\frac{2}{3})$. Take a definable path $w : [0, \frac{1}{3}] \to U$ such that $w(0) = x_0$ and $w(\frac{1}{3}) = x_1$. We define the path $\hat{u}(t) : I \to X \times I$ such that $\hat{u}(t) := (w(t), 0)$ for all $t \in [0, \frac{1}{3}]$, $\hat{u}(t) := (x_1, 3s_1(t - \frac{1}{3}))$ for all $t \in [\frac{1}{3}, \frac{2}{3}]$ and $\hat{u}(t) := u(t)$ for all $t \in [\frac{2}{3}, 1]$. The definable path $\widetilde{H}(\hat{u}(t))$ is continuous for all $t \in [0, 1)$ because $\widetilde{f}$ is continuous and because of the construction of $\widetilde{H}$. Since U is definably contractible, $\{x_0\} \times I$ is a definable deformation retract of $U \times I$ and therefore $\hat{u}$ is definably homotopic to the canonical path $I \to X \times I : t \mapsto (x_0, t)$. Finally, since we are just interested in the behaviour of the definable path u when t is near 1, we can replace u by $\hat{u}$. □

Corollary 2.4.11. *Let* $p : E \to B$ *be a definable covering and let* $p(e_0) = b_0$. *Then* $p_* : \pi_n(E, e_0)^{\mathcal{R}} \to \pi_n(B, b_0)^{\mathcal{R}}$ *is an isomorphism for every* $n > 1$ *and injective for* $n = 1$.

Proof. Since $p^{-1}(b_0)$ is finite, we have that $\pi_n(p^{-1}(b_0), e_0) = 0$ for every $n \geq 1$. Then the result follows from Proposition 2.4.10 and both the exactness and the fibration properties. □

2.5 The o-minimal Hurewicz and Whitehead theorems

Next we will prove both the absolute and relative Hurewicz theorems in the o-minimal setting by transferring from the semialgrebraic setting via Theorem 2.3.1.

First let us define the o-minimal Hurewicz homomorphism. Recall that there exists an o-minimal singular homology theory $H_*(-)^{\mathcal{R}}$ on the category of definable sets (see [38]). Moreover, by Proposition 3.2 in [7] there exists a natural isomorphism θ between the functors $H_*(-)^{\mathcal{R}_0}$ and $H_*(-)^{\mathcal{R}}$ on the category of (pairs of) semialgebraic sets (note that the notation used in the above paper is different from ours, where $H_*(-)^{\mathcal{R}_0} = H_*^{sa}(-)$ and $H_*(-)^{\mathcal{R}} = H_*^{def}(-)$). Fix $n \geq 1$. By Proposition 3.2 in [7],

$$H_n(I^n, \partial I^n)^{\mathcal{R}_0} \cong H_n(I^n(\mathbb{R}), \partial I^n(\mathbb{R})) \cong \mathbb{Z}.$$

We fix a generator $z_n^{\mathcal{R}_0}$ of $H_n(I^n, \partial I^n)^{\mathcal{R}_0}$ and we define $z_n^{\mathcal{R}} := \theta(z_n^{\mathcal{R}_0})$. Now, given a definable pointed set (X, x_0), the **o-minimal Hurewicz homomorphism**, for $n \geq 1$, is the map $h_{n,\mathcal{R}} : \pi_n(X, x_0)^{\mathcal{R}} \to H_n(X)^{\mathcal{R}} : [f] \mapsto h_{n,\mathcal{R}}([f]) = f_*(z_n^{\mathcal{R}})$, where $f_* : H_n(I^n, \partial I)^{\mathcal{R}} \to H_n(X)^{\mathcal{R}}$ denotes the map in o-minimal singular homology induced by f. Note that by the homotopy axiom of o-minimal singular homology if $f \sim g$ then $f_* = g_*$, hence $h_{n,\mathcal{R}}$ is well-defined. We define the relative Hurewicz homomorphism adapting in the obvious way what was done in the absolute case. Now, following the classical proof, it is easy to check that $h_{n,\mathcal{R}}$ is a natural transformation between the functors $\pi_n(-)^{\mathcal{R}}$ and $H_n(-)^{\mathcal{R}}$ (see [22, Ch.V,Prop.4.1]). This fact can also be deduced from the semialgebraic setting (see Remark 2.5.2).

The following result give us a relation between the semialgebraic and the o-minimal Hurewicz homomorphisms.

Proposition 2.5.1. *Let (X, x_0) be a semialgebraic pointed set. Then the following diagram commutes*

$$\begin{array}{ccc} \pi_n(X, x_0)^{\mathcal{R}_0} & \xrightarrow{h_{n,\mathcal{R}_0}} & H_n(X)^{\mathcal{R}_0} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \theta} \\ \pi_n(X, x_0)^{\mathcal{R}} & \xrightarrow[h_{n,\mathcal{R}}]{} & H_n(X)^{\mathcal{R}} \end{array}$$

for all $n \geq 1$.

Proof. Let $[f] \in \pi_n(X, x_0)^{\mathcal{R}_0}$. By definition $z_n^{\mathcal{R}} = \theta(z_n^{\mathcal{R}_0})$ and by the naturality of θ we have that $\theta(f_*(z_n^{\mathcal{R}_0})) = f_*(\theta(z_n^{\mathcal{R}_0}))$. Therefore $\theta(h_{n,\mathcal{R}_0}([f])) = \theta(f_*(z_n^{\mathcal{R}_0})) = f_*(\theta(z_n^{\mathcal{R}_0})) = f_*(z_n^{\mathcal{R}}) = h_{n,\mathcal{R}}(\rho([f]))$. □

Remark 2.5.2. (1) This last result remains true in the relative case and its proof is similar.
(2) Since $h_{n,\mathcal{R}_0}$ is a homomorphism for $n \geq 1$ (see [13, Ch.III,Thm.7.3]), it follows from Proposition 2.5.1 and the Triangulation theorem that $h_{n,\mathcal{R}}$ is also a homomorphism for $n \geq 1$.

Recall the definition of the action of π_1 on π_n defined in Properties 2.4.6.

Theorem 2.5.3 (The o-minimal Hurewicz theorems). *Let (X, x_0) be a definable pointed set and $n \geq 1$. Suppose that $\pi_r(X, x_0)^{\mathcal{R}} = 0$ for every $0 \leq r \leq n-1$. Then the o-minimal Hurewicz homomorphism*

$$h_{n,\mathcal{R}} : \pi_n(X, x_0)^{\mathcal{R}} \to H_n(X)^{\mathcal{R}}$$

is surjective and its kernel is the subgroup generated by $\{\beta_{[u]}([f])[f]^{-1} : [u] \in \pi_1(X, x_0)^{\mathcal{R}}, [f] \in \pi_n(X, x_0)^{\mathcal{R}}\}$*. In particular,* $h_{n,\mathcal{R}}$ *is an isomorphism for* $n \geq 2$*.*

Proof. Note that X is definably connected since $\pi_0(X, x_0)^{\mathcal{R}} = 0$. Let (K, ϕ) be a definable triangulation of X and $y_0 = \phi^{-1}(x_0)$. Since $\pi_r(-)^{\mathcal{R}}$ is a covariant functor, $\pi_r(|K|, y_0)^{\mathcal{R}} = 0$ for $0 \leq r \leq n-1$. Moreover, as ρ is a natural isomorphism, $\pi_r(|K|, y_0)^{\mathcal{R}_0} \cong \pi_r(|K|, y_0)^{\mathcal{R}} = 0$ for $0 \leq r \leq n-1$. Since $h_{n,\mathcal{R}}$ is a natural transformation, the following diagram

$$\begin{array}{ccc} \pi_n(|K|, y_0)^{\mathcal{R}} & \xrightarrow{h_{n,\mathcal{R}}} & H_n(|K|)^{\mathcal{R}} \\ \phi_* \downarrow & & \downarrow \phi_* \\ \pi_n(X, x_0)^{\mathcal{R}} & \xrightarrow[h_{n,\mathcal{R}}]{} & H_n(X)^{\mathcal{R}} \end{array}$$

commutes. Furthermore, since ϕ is a homeomorphism, the induced map ϕ_* in both homology and homotopy are isomorphism. Hence, by Lemma 2.4.7, it is enough to prove that $h_{n,\mathcal{R}} : \pi_n(|K|, y_0)^{\mathcal{R}} \to H_n(|K|)^{\mathcal{R}}$ is surjective and that its kernel is the subgroup generated by $\{\beta_{[u]}([f])[f]^{-1} : [u] \in \pi_1(|K|, y_0)^{\mathcal{R}}, [f] \in \pi_n(|K|, y_0)^{\mathcal{R}}\}$. By Proposition 2.5.1, the following diagram

$$\begin{array}{ccc} \pi_n(|K|, y_0)^{\mathcal{R}_0} & \xrightarrow{h_{n,\mathcal{R}_0}} & H_n(|K|)^{\mathcal{R}_0} \\ \rho \downarrow & & \downarrow \theta \\ \pi_n(|K|, y_0)^{\mathcal{R}} & \xrightarrow[h_{n,\mathcal{R}}]{} & H_n(|K|)^{\mathcal{R}} \end{array}$$

commutes. Since ρ and θ are natural isomorphisms, it is enough to prove that $h_{n,\mathcal{R}_0} : \pi_n(|K|, y_0)^{\mathcal{R}_0} \to H_n(|K|)^{\mathcal{R}_0}$ is surjective and that its kernel is the subgroup generated by $\{\beta_{[u]}([f])[f]^{-1} : [u] \in \pi_1(|K|, y_0)^{\mathcal{R}_0}, [f] \in \pi_n(|K|, y_0)\}^{\mathcal{R}_0}$. But this fact follows from the semialgebraic Hurewicz theorems (see [13, Ch.III,Thm.7.4]). Finally, the second part of the theorem follows immediately from the first one since for $n \geq 2$, by hypothesis, $\pi_1(X, x_0)^{\mathcal{R}} = 0$. □

Theorem 2.5.4 (**The o-minimal relative Hurewicz theorems**). *Let* (X, A, x_0) *be a definable pointed pair and let* $n \geq 2$ *such that* $\pi_r(X, A, x_0)^{\mathcal{R}} = 0$ *for every* $1 \leq r \leq n-1$*. Then the o-minimal Hurewicz homomorphism* $h_{n,\mathcal{R}} : \pi_n(X, A, x_0)^{\mathcal{R}} \to H_n(X, A)^{\mathcal{R}}$ *is surjective and its kernel is the subgroup generated by* $\{\beta_{[u]}([f])[f]^{-1} : [u] \in \pi_1(A, x_0)^{\mathcal{R}}, [f] \in \pi_n(X, A, x_0)^{\mathcal{R}}\}$*. In particular,* $h_{n,\mathcal{R}}$ *is an isomorphism for* $n \geq 3$*.*

Proof. It is enough to adapt the proof of the o-minimal absolute Hurewicz theorems to the relative case. Note that at some point, we need the relative version of Lemma 2.4.7 (whose proof is similar), i.e., that given a definable map $\psi : (X, A, x_0) \to (Y, B, y_0)$ and $[u] \in \pi_1(A, x_0)^{\mathcal{R}}$, we have that $\psi_*(\beta_{[u]}([f])) = \beta_{\psi_*([u])}(\psi_*([f]))$ for all $[f] \in \pi_n(X, A, x_0)^{\mathcal{R}}$. □

Remark 2.5.5. (1) With the hypotheses of Theorem 2.5.3, for $n = 1$, we have that $\mathrm{Ker}(h_{1,\mathcal{R}})$ is the subgroup generated by $\{\beta_{[u]}([v])[v]^{-1} : [u] \in \pi_1(X, x_0)^{\mathcal{R}}, [v] \in \pi_1(X, x_0)^{\mathcal{R}}\}$. On the other hand, using the definable homotopy $H(t, s) = u(ts)v(t)u(s - ts)$, we can prove that $\beta_{[u]}([v])[v]^{-1} = [u][v][u]^{-1}[v]^{-1}$. Hence, $\mathrm{Ker}(h_{1,\mathcal{R}})$ is the commutator of $\pi_1(X, x_0)^{\mathcal{R}}$. In particular, if $\pi_1(X, x_0)^{\mathcal{R}}$ is abelian then $h_{1,\mathcal{R}}$ is an isomorphism. This result was already proved in [20, Thm.5.1].
(2) With the hypotheses of Theorem 2.5.4, $\pi_2(X, A, x_0)^{\mathcal{R}}/\mathrm{Ker}(h_{2,\mathcal{R}}) \cong H_2(X, A)^{\mathcal{R}}$ is abelian and therefore $\mathrm{Ker}(h_{2,\mathcal{R}})$ contains the commutator subgroup of $\pi_2(X, A, x_0)^{\mathcal{R}}$. This fact can also be shown directly by proving that for every $[f], [g] \in \pi_2(X, A, x_0)^{\mathcal{R}}$, $[g][f][g]^{-1} = \beta_{[u]}([f])$, where $u(t) = g(t, 0)$ for $t \in I$.

We finish this section with the proof of the o-minimal Whitehead theorem. We say that a definable map $\psi : X \to Y$ is a **definable homotopy equivalence** if there exists a definable map $\psi' : Y \to X$ such that $\psi \circ \psi' \sim \mathrm{id}_Y$ and $\psi' \circ \psi \sim \mathrm{id}_X$.

Remark 2.5.6. If a definable map ψ is a definable homotopy equivalence then it is a definable homotopy equivalence relative to a point. Indeed, by the Triangulation Theorem we can assume that both (X, x_0) and $(Y, \psi(x_0))$ are semialgebraic pairs defined without parameters. Then, by Theorem 2.3.1 and Fact 2.1.2, there exists a semialgebraic map ψ' defined without parameters such that $[\psi] = [\psi']$ in $[(X, x_0), (Y, \psi(x_0))]$. Finally, by the classical version of the present remark (see [21, Prop. 0.19]), Fact 2.1.1 and Fact 2.1.2, ψ' is a semialgebraic homotopy equivalence relative to a point and hence so is ψ.

Theorem 2.5.7 (The o-minimal Whitehead theorem). *Let X and Y be two definably connected sets. Let $\psi : X \to Y$ be a definable map such that for some $x_0 \in X$, $\psi_* : \pi_n(X, x_0)^{\mathcal{R}} \to \pi_n(Y, \psi(x_0))^{\mathcal{R}}$ is an isomorphism for all $n \geq 1$. Then ψ is a definable homotopy equivalence.*

Proof. Let (K, ϕ_1) and (L, ϕ_2) be definable triangulations of X and Y, respectively. Consider the points $x_1 = \phi_1^{-1}(x_0)$ and $y_1 = \phi_2^{-1}(\psi(x_0))$. It suffices to prove that the definable map $\widetilde{\psi} = \phi_2^{-1} \circ \psi \circ \phi_1 : |K| \to |L|$ is a definable homotopy equivalence provided $\widetilde{\psi}_* : \pi_n(|K|, x_1)^{\mathcal{R}} \to \pi_n(|L|, y_1)^{\mathcal{R}}$ is an isomorphism for all $n \geq 1$. By Theorem 2.3.1 there exists a semialgebraic map $\varphi : (|K|, x_1) \to (|L|, y_1)$ such that $\varphi \sim \widetilde{\psi}$. By the homotopy property it follows that $\varphi_* = \widetilde{\psi}_* : \pi_n(|K|, x_1)^{\mathcal{R}} \to \pi_n(|L|, y_1)^{\mathcal{R}}$ is an isomorphism for all $n \geq 1$. Therefore by Theorem 2.4.1, $\varphi_* : \pi_n(|K|, x_1)^{\mathcal{R}_0} \to \pi_n(|L|, y_1)^{\mathcal{R}_0}$ is an isomorphism for all $n \geq 1$. Hence, by the semialgebraic Whitehead theorem (see [13, Ch.III,Thm.6.6]), φ is a semialgebraic homotopy equivalence, that is, there exists a semialgebraic map $\varphi' : |L| \to |K|$ such that $\mathrm{id}_{|K|} \sim_0 \varphi' \circ \varphi$ and $\mathrm{id}_{|L|} \sim_0 \varphi \circ \varphi'$, where $\sim_0$ means "semialgebraically homotopic". Hence $\mathrm{id}_{|K|} \sim_0 \varphi' \circ \varphi \sim \varphi' \circ \widetilde{\psi}$ and so $\mathrm{id}_{|K|} \sim \varphi' \circ \widetilde{\psi}$. In a similar way we prove that $\mathrm{id}_{|L|} \sim \widetilde{\psi} \circ \varphi'$. Therefore $\widetilde{\psi}$ is a definable homotopy equivalence, as required. □

Corollary 2.5.8. *Let X be a definable set and let $x_0 \in X$. If $\pi_n(X, x_0)^{\mathcal{R}} = 0$ for all $n \geq 0$ then X is definably contractible.*

Proof. This follows from Theorem 2.5.7 applied to a constant map. □

Next result follows the transfer approach developed in [8].

Corollary 2.5.9. *Let X be a semialgebraic set defined without parameters. Then X is definably contractible if and only if $X(\mathbb{R})$ is contractible in the classical sense.*

Proof. This follows from Corollary 2.4.4 and Corollary 2.5.8. □

2.6 Lusternik-Schnirelmann category of definable sets

In this section we introduce the Lusternik-Schnirelmann category (in short LS-category) for definable sets. We apply the results of Section 2.3 and the Normal triangulation theorem to prove some comparison theorems concerning the LS-category. For a general reference on the classical Lusternik-Schnirelmann category see [10].

Definition 2.6.1. *Let X be a definable set. We say that a definable subset A of X is **definably categorical** in X if A is definably contractible in X. We say that a definable cover $\{V_i\}_{i=1}^m$ of X is a **definable categorical cover** of X if each V_i is definably categorical in X.*

Fact 2.6.2. [10, Lem.1.29] *Let X and Y be definable sets. Let $f : X \to Y$ and $g : Y \to X$ be definable maps such that $f \circ g \sim id_Y$. Then $g^{-1}(U)$ is a definable categorical subset of Y for each definable categorical subset U of X.*

Proof. Let $F : Y \times I \to Y$ be a definable homotopy from id_Y to $f \circ g$. Let $H : U \times I \to X$ be a definable homotopy and $x_0 \in X$ such that $H(x, 0) = x$ for all $x \in U$ and $H(x, 1) = x_0$ for all $x \in U$. Let $V = g^{-1}(U)$ and consider the definable map $G : V \times I \to Y$ defined by $G(y, t) = F(y, 2t)$ for all $(y, t) \in V \times [0, \frac{1}{2}]$ and $G(y, t) = f(H(g(y), 2t - 1))$ for all $(x, t) \in V \times [\frac{1}{2}, 1]$. Note that $G(y, 0) = y$ and $G(y, 1) = f(x_0)$ for all $y \in V$, i.e., V is a definable categorical subset of X. □

Note that every definable set X has a definable categorical open cover. Indeed, by the Triangulation theorem and Fact 2.6.2 we can assume that $X = |K|$ for a simplicial complex K. We denote by K'' the second barycentric subdivision of K. By [14, Prop 2.2], the open definable subset $\mathrm{St}_{K''}(v)$ of $|K|$ is definably categorical in $|K|$ for each $v \in \mathrm{Vert}(K)$. Therefore, $\{\mathrm{St}_{K''}(v) : v \in \mathrm{Vert}(K)\}$ is a definable categorical open cover of $|K|$.

Definition 2.6.3. *The **definable LS-category** of a definable set X, denoted by $cat(X)^{\mathcal{R}}$, is the least integer m such that X has a definable categorical open cover of $m+1$ elements.*

For example, by definition we have that a definable set X is definably contractible if and only if $\mathrm{cat}(X)^{\mathcal{R}} = 0$. Now, we prove that the definable LS-category is homotopy invariant.

Fact 2.6.4. [10, Lem.1.30] *Let X and Y be definably homotopy equivalent definable sets. Then $cat(X)^{\mathcal{R}} = cat(Y)^{\mathcal{R}}$.*

Proof. Let $f : X \to Y$ and $g : Y \to X$ be definable maps such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. We show that $\mathrm{cat}(Y)^{\mathcal{R}} \leq \mathrm{cat}(X)^{\mathcal{R}}$, the other inequality by symmetry. Let $\{U_i\}_{i=1}^{m+1}$ be a definable categorical open cover of X. By Fact 2.6.2, $g^{-1}(U_i)$ is definably categorical in Y for each $i = 1, \dots, m+1$. Hence $\{g^{-1}(U_i)\}_{i=1}^{m+1}$ is a categorical open cover of Y. □

Theorem 2.6.5. *Let X be a semialgebraic set. Then $cat(X)^{\mathcal{R}_0} = cat(X)^{\mathcal{R}}$.*

Proof. Clearly, $\mathrm{cat}(X)^{\mathcal{R}_0} \geq \mathrm{cat}(X)^{\mathcal{R}}$. We show that $\mathrm{cat}(X)^{\mathcal{R}_0} \leq \mathrm{cat}(X)^{\mathcal{R}}$. By the Triangulation theorem we can assume that $X = |K|$ for some simplicial complex K. Let $m = \mathrm{cat}(X)^{\mathcal{R}}$ and let $U_1, \dots, U_{m+1}$ be a definable categorical open cover of $|K|$. By the Normal triangulation theorem 1.1.5 there is a subdivision K' of K and a definable homeomorphism $\phi : |K'| \to |K|$ such that (K', ϕ) partitions $U_1, \dots, U_{m+1}$. Therefore the open subsets $V_i := \phi^{-1}(U_i)$ of X are semialgebraic for each $i = 1, \dots, m+1$. Since ϕ is a definable homeomorphism, V_i is a definably categorical subset of $|K|$ for all $i = 1, \dots, m+1$ (see Fact 2.6.2). Now, by Theorem 2.3.4 if a semialgebraic subset A of a semialgebraic set B is definably contractible in B, then it is semialgebraically contractible in B. Hence, V_i is a semialgebraic categorical subset of $|K|$ for all $i = 1, \dots, m+1$. Then $\{V_i\}_{i=1}^{m+1}$ is a semialgebraic categorical open cover of $|K|$ and hence $\mathrm{cat}(X)^{\mathcal{R}_0} \leq m$. □

Theorem 2.6.6. *Let $X \subset R^n$ be a semialgebraic set. Let S be a real closed field extension of R. Then $cat(X)^{\mathcal{R}_0} = cat(X(S))^{\mathcal{S}_0}$.*

Proof. It is immediate that $\mathrm{cat}(X)^{\mathcal{R}_0} \geq \mathrm{cat}(X(S))^{\mathcal{S}_0}$. For, given a semialgebraic categorical open cover $\{U_i\}_{i=1}^{m+1}$ of X, $\{U_i(S)\}_{i=1}^{m+1}$ is clearly a semialgebraic categorical open cover of $X(S)$. We show that $\mathrm{cat}(X)^{\mathcal{R}_0} \leq \mathrm{cat}(X(S))^{\mathcal{S}_0}$. Let $\{V_i\}_{i=1}^{m+1}$ be a semialgebraic categorical open cover of

$X(S)$. Let $H_i : V_i \times I \to X(S)$ be a semialgebraic homotopy such that $H_i(x,0) = x$ for all $x \in V_i$ and $H_i(x,1) = x_i \in X(S)$ for all $x \in V_i$ for each $i = 1,\dots,m+1$. Without loss of generality we can assume that $x_i \in R^n$. Indeed, let $x'_i \in X(S) \cap R^n$ be a point which lies in the same semialgebraically connected component of x_i. Consider a semialgebraic curve $\alpha_i : I \to X(S)$ such that $\alpha_i(0) = x_i$ and $\alpha_i(1) = x'_i$. Then, we can replace H_i by the semialgebraic homotopy $H'_i : V_i \times I \to X(S)$ with $H'_i(x,t) = H_i(x,2t)$ for all $(x,t) \in V_i \times [0,\frac{1}{2}]$ and $H'_i(x,t) = \alpha_i(2t-1)$ for all $(x,t) \in V_i \times [\frac{1}{2},1]$. We denote V_i and H_i by $V_{i,c}$ and $H_{i,c}$ respectively to stress the fact that c is a tuple of parameters of S such that V_i and H_i are defined over, for all $i = 1,\dots,m+1$. Consider the first order formula $\psi(y)$ with parameters over R which says that $\{V_{i,y}\}_{i=1}^{m+1}$ is a semialgebraic open cover of X and each $H_{i,y} : V_{i,y} \times I \to X$ is a semialgebraic map such that $H_{i,y}(x,0) = x$ and $H_{i,y}(x,1) = x_i$ for all $x \in V_{i,y}$. By completeness of the theory of real closed fields, since $\mathcal{S}_0$ satisfies $\exists y \psi(y)$, $\mathcal{R}_0$ satisfies $\exists y \psi(y)$. Hence, there is a tuple b in R such that $\{V_{i,b}(R)\}_{i=1}^{m+1}$ is a semialgebraic categorical open cover of X. Hence, $\mathrm{cat}(X)^{\mathcal{R}_0} \leq \mathrm{cat}(X(S))^{\mathcal{S}_0}$, as required. □

Theorem 2.6.7. *Let X be a semialgebraic set of $\overline{\mathbb{R}}$. Then $cat(X)^{\overline{\mathbb{R}}} = cat(X)$, where $cat(X)$ denotes the classical LS-category of X.*

Proof. Clearly, $\mathrm{cat}(X)^{\overline{\mathbb{R}}} \geq \mathrm{cat}(X)$. We show that $\mathrm{cat}(X)^{\overline{\mathbb{R}}} \leq \mathrm{cat}(X)$. By the Triangulation theorem we can assume that $X = |K|$ for some simplicial complex K. Moreover, by Fact 2.2.9 and Fact 2.6.4, we can assume that K is closed. Let $\{U_i\}_{i=1}^{m+1}$ be a categorical open cover of $|K|$. We will construct a semialgebraic categorical open cover $\{V_i\}_{i=1}^{m+1}$ of $|K|$. Firstly, by the Shrinking lemma (see [26, §36, Ex.4]) we can assume that each $\overline{U_i}$ is contractible in $|K|$. Furthermore, by the Lebesgue's number lemma (see the proof of [25, Ch.2, Thm.16.1]) we can also assume that for each $\sigma \in K$ there is $i \in \{1,\dots,m+1\}$ such that $\sigma \subset U_i$. We define $A_i := \bigcup_{\sigma \in \mathcal{F}_i} \sigma$ for each $i = 1,\dots,m+1$, where $\mathcal{F}_i = \{\sigma \in K : \sigma \subset U_i\}$. Note that (i) $|K| = A_1 \cup \dots \cup A_{m+1}$ and (ii) each $\overline{A_i}$ is contractible in $|K|$. On the other hand, each $\overline{A_i}$ is a semialgebraic strong deformation retract of the open semialgebraic set $V_i := \mathrm{St}_{K''}(\overline{A_i})$, where K'' is the second barycentric subdivision of K (see [14, Prop 2.2]). Therefore, by (ii), V_i is (not necessarily semialgebraically) contractible in $|K|$. Now, by Theorem 2.1.2 if a semialgebraic subset A of a semialgebraic set B is contractible in B, then it is semialgebraically contractible in B. Hence, each V_i is semialgebraically contractible in $|K|$ and hence, by (i), $\{V_i\}_{i=1}^{m+1}$ is a semialgebraic categorical open cover of $|K|$. We deduce that $\mathrm{cat}(X)^{\overline{\mathbb{R}}} \leq \mathrm{cat}(X)$, as required. □

Corollary 2.6.8. *The definable LS-category is invariant under elementary extensions and o-minimal expansions.*

Proof. This follows from Theorem 2.6.5 and 2.6.6. □

Corollary 2.6.9. *Let X be a semialgebraic set defined without parameters. Then $cat(X)^{\mathcal{R}} = cat(X(\mathbb{R}))$, where $cat(X(\mathbb{R}))$ denotes the classical LS-category.*

Proof. It follows from Corollary 2.6.8 and Theorem 2.6.7 that $\mathrm{cat}(X)^{\mathcal{R}} = \mathrm{cat}(X)^{\mathcal{R}_0} = \mathrm{cat}(X(\overline{\mathbb{Q}}))^{\overline{\mathbb{Q}}} = \mathrm{cat}(X(\mathbb{R})^{\overline{\mathbb{R}}} = \mathrm{cat}(X(\mathbb{R}))$. □

Using the above comparison theorems 2.6.5, 2.6.6 and 2.6.7 we can transfer some results concerning the classical LS-category to the o-minimal setting.

Corollary 2.6.10. *Let X be a definably connected definable set. Then*

$$cat(X)^{\mathcal{R}} \leq dim(X).$$

Proof. By the Triangulation theorem, Fact 2.2.9 and 2.6.4, we can assume that $X = |K|$ for a closed simplicial complex K whose vertices lie in $\overline{\mathbb{Q}}$. Now, it follows from Theorem 2.6.5, 2.6.6 and 2.6.7 that $\mathrm{cat}(X)^{\mathcal{R}} = \mathrm{cat}(|K|(\mathbb{R}))$. By the classical version of Corollary 2.6.10 (see [10, Thm. 1.7]),

$$\mathrm{cat}(|K|(\mathbb{R})) \leq \dim(|K|(\mathbb{R}))^{\mathrm{top}},$$

where $\dim(|K|(\mathbb{R}))^{\mathrm{top}}$ denotes the covering dimension of $|K|(\mathbb{R})$. On the other hand, since K is a simplicial complex, $\dim(|K|(\mathbb{R}))^{\mathrm{top}}$ is exactly the dimension of K as a simplicial complex, i.e., $\dim(|K|(\mathbb{R}))^{\mathrm{top}} = \dim(|K|)$, the latter being the o-minimal dimension, as required. □

We conclude this section by applying the above results to the study of definable groups. We assume that our o-minimal structure $\mathcal{R}$ is sufficiently saturated. A group G is definable if both the set and the graph of the group operation are definable sets. The following essential fact is well-known: every definable group G can be equipped with a definable manifold structure making G a topological group (see [32]). Since topological groups are regular spaces, we can assume that the manifold topology is induced by that of the ambient space (see the o-minimal version of Robson's embedding theorem [15, Ch.10,Thm.1.8]). By the work of several authors on the so called "Pillay's conjecture", every definably compact group G has a canonical type-definable divisible subgroup G^{00} such that G/G^{00} with the "logic topology" is a compact Lie group (see [28]). For instance, if G is a definably compact definable abelian group, then G/G^{00} is a torus of dimension $\dim(G)$. *Our purpose is to compare the definable LS-category of a definable group G and the classical LS-category of G/G^{00}.* We say that a definable group G has a **very good reduction** if and only if it is definably isomorphic in $\mathcal{R}$ to a group G_1 which can be defined over $\mathbb{R}$ in the following sense: there is a sublanguage $\mathcal{L}_0$ of the language $\mathcal{L}$ of $\mathcal{R}$ which contains $+, \cdot$ and there is an elementary substructure $\mathcal{R}_0$ of $\mathcal{R}|_{\mathcal{L}_0}$ of the form $\langle \mathbb{R}, +, \cdot, ... \rangle$ such that G is defined over $\mathcal{R}_0$. Every definably connected centreless semisimple group has a very good reduction by [29, Thm.2.37] (furthermore, in [18, Thm.3.1] the centreless condition is not needed).

Corollary 2.6.11. *Let G be a definably connected definably compact group. Then*
(i) *if G is abelian then $cat(G)^{\mathcal{R}} = cat(G/G^{00}) = dim(G)$, and*
(ii)*if G has a very good reduction then $cat(G)^{\mathcal{R}} = cat(G/G^{00})$.*

Proof. (i) We denote by d the dimension of G. Then G is a definably homotopy equivalent to the d-dimensional torus $\mathbb{T}^d(R)$. This last result is proved in [6, Thm.3.4] using some results of this dissertation. Hence, by Fact 2.6.4 and Corollary 2.6.9, $\operatorname{cat}(G)^{\mathcal{R}} = \operatorname{cat}(\mathbb{T}^d(R))^{\mathcal{R}} = \operatorname{cat}(\mathbb{T}^d)$. On the other hand, by [10, Ex.1.8] we have $\operatorname{cat}(\mathbb{T}^d) = d$, as required.
(ii) Let G_1 be a very good reduction of G. By Theorem 1.6 and Proposition 5.1 in [5], $G_1(\mathbb{R}) \cong G_1/G_1^{00}$ and $G/G^{00} \cong G_1/G_1^{00}$ as Lie groups. Therefore, by Theorem 2.6.5, 2.6.6 and 2.6.7,

$$\operatorname{cat}(G)^{\mathcal{R}} = \operatorname{cat}(G_1)^{\mathcal{R}} = \operatorname{cat}(G_1(\mathbb{R})) = \operatorname{cat}(G_1/G_1^{00}) = \operatorname{cat}(G/G^{00}).$$

□

Chapter 3

Locally definable homotopy

3.1 Introduction

Once the bases of o-minimal homotopy have been established in Chapter 2, we now extend them to the locally definable setting.

As already mentioned in the introduction of this memory, the locally definable category needs to be developed which is the aim of Section 3.2. We first introduce the category of locally definable spaces (in short ld-spaces). We have avoided the presentation style of Delfs and Knebusch in [13] with "sheaf" flavour, using instead the natural generalization of definable spaces of L. van den Dries in [15]. Locally definable spaces of special interest are the regular paracompact ones (in short LD-spaces). In Section 3.3 we prove the Triangulation theorem for LD-spaces. The proof uses certain glueing results as well as locally definable versions of well-known results such as partition of unity and shrinking of coverings. Since the latter may be skipped at a first reading, we have decided to include them in Appendix 3.8. The proofs of all these results in [13] are based on properties of semialgebraic sets which are shared by definable sets and hence can be directly adapted to our context. Therefore, we have labelled all these results with *Fact*, however we point out that all of them are new in the o-minimal setting. We also point out that new results concerning o-minimal expansions (which were obviously not treated in [13]) appear along these sections.

In Section 3.5 we develop a homology theory for LD-spaces via an alternative approach to that of [13] for locally semialgebraic spaces (the latter going through sheaf cohomology). With all these tools at hand, we prove in Section 3.7 the generalizations to LD-spaces of the homotopy results in Chapter 2, in particular the Hurewicz theorems and the Whitehead theorem, as well as the locally definable versions of the results concerning fibrations.

On the other hand, in Sections 3.4 and 3.6 we use the locally definable category to unify and clarify existing notions in the o-minimal literature. In particular, in Section 3.4 we have tried to unify the related notions of $\bigvee$-definable groups and "locally definable" groups via the theory of locally definable spaces. We show that $\bigvee$-definable groups are examples of ld-spaces, the "locally definable" groups are moreover LD-spaces. In Section 3.6 we clarify the relation among the different notions of connectedness used for $\bigvee$-definable groups which appear in the literature, pointing out the inadequacy of some of them.

3.2 Locally definable spaces

We shall briefly discuss the category of locally definable spaces.

Definition 3.2.1. *Let M be a set. An* ***atlas*** *on M is a family of* ***charts*** *$\{(M_i, \phi_i)\}_{i\in I}$, where M_i is a subset of M and $\phi_i : M_i \to Z_i$ is a bijection between M_i and a definable set Z_i of $R^{n(i)}$ for all $i \in I$, such that $M = \bigcup_{i\in I} M_i$ and for each pair $i, j \in I$ the set $\phi_i(M_i \cap M_j)$ is a relatively open definable subset of Z_i and the map*

$$\phi_{ij} := \phi_j \circ \phi_i^{-1} : \phi_i(M_i \cap M_j) \to M_i \cap M_j \to \phi_j(M_i \cap M_j)$$

is definable. We say that $(M, M_i, \phi_i)_{i\in I}$ is a ***locally definable space****. The* ***dimension*** *of M is $dim(M) := sup\{dim(Z_i) : i \in I\}$. If Z_i and ϕ_{ij} are defined over A for all $i, j \in I$, $A \subset R$, we say that M is a locally definable space over A.*

We say that two atlases $(M, M_i, \phi_i)_{i\in I}$ and $(M, M'_j, \psi_j)_{j\in J}$ on a set M are ***equivalent*** *if and only if for all $i \in I$ and $j \in J$ we have that* (i) *$\phi_i(M_i \cap M'_j)$ and $\psi_j(M_i \cap M'_j)$ are relatively open definable subsets of $\phi_i(M_i)$ and $\psi_j(M'_j)$ respectively,* (ii) *the map $\psi_j \circ \phi_i^{-1}|_{\phi_i(M_i\cap M'_j)} : \phi_i(M_i \cap M'_j) \to M_i \cap M'_j \to \psi_j(M_i \cap M'_j)$ and its inverse are definable and* (iii) *$M_i \subset \bigcup_{k\in J_0} M'_k$ and $M'_j \subset \bigcup_{s\in I_0} M_s$ for some finite subsets J_0 and I_0 of J and I respectively.*

Note that in the above definition if we take I to be finite then M is just a definable space in the sense of [15]. In fact, some of the notions that we are going to introduce in this section are generalizations of the corresponding ones in the category of definable spaces.

Even though the above definition seems different from its semialgebraic analogue (see [13, Def.I.3]), they are actually equivalent. In [13] it is (implicitly) proved that Definition I.3 is equivalent to the semialgebraic analogue of our definition here (see [13, Lem.I.2.2] and the remark after [13, Lem.I.2.1]). The same proofs can be adapted to the o-minimal setting.

Given a locally definable space (M, M_i, ϕ_i), there is a unique topology in M for which M_i is open and ϕ_i is a homeomorphism for all $i \in I$. For the rest of the paper any topological property of locally definable spaces refers to this topology. We are mainly interested in Hausdorff topologies. *Henceforth, an* ***ld-space*** *means a Hausdorff locally definable space.*

We now introduce the subsets of interest in the category of ld-spaces.

Definition 3.2.2. *Let $(M, M_i, \phi_i)_{i\in I}$ be an ld-space. We say that a subset X of M is a **definable subspace** of M (over A) if there is a finite $J \subset I$ such that $X \subset \bigcup_{j\in J} M_j$ and $\phi_j(M_j \cap X)$ is definable (resp. over A) for all $j \in J$. A subset $Y \subset M$ is an **admissible subspace** of M (over A) if $\phi_i(Y \cap M_i)$ is definable (resp. over A) for all $i \in I$, or equivalently, $Y \cap X$ is a definable subspace of M (resp. over A) for every definable subspace X of M (resp. over A).*

The admissible subspaces of an ld-space are closed under complements, finite unions and finite intersections. Moreover, the interior and the closure of an admissible subspace is an admissible subspace.

Every definable subspace of an ld-space is admissible. The definable subspaces of an ld-space are closed under finite unions and finite intersections, but not under complements. The interior of a definable subspace is a definable subspace. However, the closure of a definable subspace might not be a definable subspace (see Example 3.4.2).

Remark 3.2.3. Given an ld-space $(M, M_i, \phi_i)_{i\in I}$ we have that every admissible subspace Y of M inherits in a natural way a structure of an ld-space, whose atlas is $(Y, Y_i, \psi_i)_{i\in I}$, where $Y_i := M_i \cap Y$ and $\psi_i := \phi_i|_{Y_i}$. In particular, if Y is a definable subspace then it inherits the structure of a definable space.

Now, we introduce the maps that we will use in the locally definable category. First, note that given two ld-spaces M and N, with their atlas $(M_i, \phi_i)_{i\in I}$ and $(N_j, \psi_j)_{j\in J}$, respectively, the atlas $(M_i \times N_j, (\phi_i, \psi_j))_{i\in I, j\in J}$ makes $M \times N$ into an ld-space. In particular, if M and N are definable spaces, then $M \times N$ is a definable space. Recall that a map f from a definable space M into a definable space N is a *definable map* over A, $A \subset R$, if its graph is a definable subset of $M \times N$ over A.

Definition 3.2.4. *Let $(M, M_i, \phi_i)_{i\in I}$ and $(N, N_j, \phi_j)_{j\in J}$ be ld-spaces. A map $f : M \to N$ is an **ld-map** over A, $A \subset R$, if $f(M_i)$ is a definable subspace of N and the map $f|_{M_i} : M_i \to f(M_i)$ is definable over A for all $i \in I$.*

The behaviour of admissible subspaces and ld-maps in the locally definable category is different from that of definable subsets and definable maps in the definable category. For, even though the preimage of an admissible subspace by an ld-map is an admissible subspace, the image of an admissible subspace by an ld-map might not be an admissible subspace (see comments after Example 3.4.1). Nevertheless, the image of a definable subspace by an ld-map is a definable subspace. In particular, let us note that every ld-map between definable spaces is a definable map and therefore the category of definable spaces is a full subcategory of the category of ld-spaces. On the other hand, given two ld-spaces M and N, the graph of an ld-map $f : M \to N$ is an admissible subspace of $M \times N$. However, as the following example shows, not every continuous map $f : M \to N$ whose graph is admissible in $M \times N$ is an ld-map.

Example 3.2.5. Let $R = \mathbb{R}$. We denote by $\widetilde{\mathbb{R}}$ the ld-space whose underlying set is $\mathbb{R}$ and its atlas is $\{(-, n, n), \mathrm{id}|_{(-n,n)}\}_{n\geq 1}$ (see Example 3.4.1). If we consider $\mathbb{R}$ as definable space, then the identity map $\mathrm{id} : \mathbb{R} \to \widetilde{\mathbb{R}}$ is not an ld-map because $\mathrm{id}(\mathbb{R})$ is not a definable subspace of $\widetilde{\mathbb{R}}$. However, the graph of id is clearly an admissible subspace of $\mathbb{R} \times \widetilde{\mathbb{R}}$ since $\Gamma(\mathrm{id}) \cap (\mathbb{R} \times (-n, n)) = \{(x, x) \in \mathbb{R} \times \widetilde{\mathbb{R}} : -n < x < n\}$ is a definable subspace of $\mathbb{R} \times \widetilde{\mathbb{R}}$ for all $n > 0$.

The notion of connectedness in the locally definable category which we now introduce is a subtle issue. It extends the natural concept of "definably connected" for definable spaces. In Section 3.6 below we will analyze this concept and we will compare it with other definitions introduced by different authors in the study of $\bigvee$-definable groups.

Definition 3.2.6. *Let M be an ld-space and X an admissible subspace of M. We say that X is **connected** if there is no admissible subspace U of M such that $X \cap U$ is both open and closed in X.*

We now introduce ld-spaces with some special properties. As we will see below, in the ld-spaces with these properties there is a good relation between the topological and the definable settings. Moreover, they form an adequate framework to develop a homotopy theory.

Definition 3.2.7. *We say that an ld-space (M, M_i, ϕ_i) is **regular** if every $x \in M$ has a fundamental system of closed (definable) neighbourhoods, i.e., for every open U of M with $x \in U$ there is a closed (definable) subspace C of M such that $C \subset U$ and $x \in int(C)$. Equivalently, an ld-space M is regular if for every closed subset C of M and every point $x \in M \setminus C$ there are open (admissible) disjoint subsets U_1 and U_2 with $C \subset U_1$ and $x \in U_2$.*

Remark 3.2.8. If M is a regular ld-space then every definable subspace of M can be considered as an affine set, i.e, as a definable set of R^n for some $n \in \mathbb{N}$. For, suppose that X is a definable subspace of M. Then, X inherits a structure of definable space from M (see Remark 3.2.3). Since M is regular then X is also regular. Finally, by the o-minimal version of Robson's embedding theorem, X is affine (see [15, Ch.10,Thm. 1.8]).

Let $(M, M_i, \phi_i)_{i\in I}$ be an ld-space. A family $\{X_j\}_{j\in J}$ of admissible subspaces of M is an ***admissible covering*** of $X := \bigcup_{j\in J} X_j$ if for all $i \in I$, $M_i \cap X = M_i \cap (X_{j_1} \cup \cdots \cup X_{j_l})$ for some $j_1, \ldots, j_l \in J$ (note that in particular X is an admissible subspace). A family $\{Y_j\}_{j\in J}$ of admissible subspaces of M is ***locally finite*** if for all $i \in I$ we have that $M_i \cap Y_j \neq \emptyset$ for only a finite number of $j \in J$ (note that in particular it is an admissible covering of their union). In general, not every admissible covering is locally finite (see Example 3.4.2).

Definition 3.2.9. *We say that an ld-space M is* ***paracompact*** *if there exists a locally finite covering of M by open definable subspaces. We say that an ld-space M is* ***Lindelöf*** *if there exist an admissible covering of M by countably many open definable subspaces.*

Remark 3.2.10. (1) Note that the above notion of paracompactness is "weaker" than the classical one.
(2)[13, Prop.I.4.5] If M is paracompact ld-space then every admissible covering of M has a locally finite refinement. Indeed, let $\{U_i\}_{i\in I}$ be a locally finite covering of M by open definable subspaces. We fix an admissible covering $\{Y_j\}_{j\in J}$ of M. Since U_i is a definable subspace of M, there is a finite subset of indexes $J(i) \subset J$ such that $U_i \subset \bigcup_{j\in J(i)} Y_j$, for each $i \in I$. Then $\{U_i \cap Y_j\}_{i\in I, j\in J(i)}$ is a locally finite covering of M by definable subspaces which refines $\{Y_j\}_{j\in J}$. *In particular, given a paracompact ld-space M, we can assume that its atlas is locally finite.*

Paracompactness provides us with a good relation between the topological and definable setting.

Fact 3.2.11. *Let M be an ld-space.*
(1) [13, Prop. I.4.6] *If M is paracompact then for every definable subspace X, the closure $\overline{X}$ is also a definable subspace of M.*
(2) [13, Thm. I.4.17] *If M is connected and paracompact then M is Lindelöf.*
(3) [13, Prop. I.4.18] *If M is Lindelöf and for every definable subspace X its closure $\overline{X}$ is also a definable subspace, then M is paracompact.*
(4) [13, Prop. I.4.7] *If M is paracompact and every open definable subspace of M is regular then M is regular.*

Proof. We denote by $\{(M_i, \phi_i)\}_{i\in I}$ an atlas of M.
(1) It is enough to proof that each $\overline{M_j}$ is contained in the union of a finite number of charts M_i. Since M is paracompact, we can assume that $\{M_i\}_{i\in I}$ is locally finite. We define the set of indexes $\Gamma(j) = \{i \in I : M_i \cap M_j \neq \emptyset\}$. Since the atlas is locally finite, $\Gamma(j)$ is finite. On the other hand, it is easy to check that $\overline{M_j} \subset \bigcup_{i\in\Gamma(j)} M_i$, as required.
(2) Since M is paracompact, we can assume that $\{(M_i, \phi_i)\}_{i\in I}$ is locally finite. Let us show that I must be countable. For each $j \in I$ we define by induction the following sets of indexes $\Gamma_0(i) = \{i\}$, $\Gamma_{n+1}(i) := \{j \in I : M_k \cap M_j \neq \emptyset$ for some $k \in \Gamma_n(i)\}$ for $n \in \mathbb{N}$. Since $\{M_i\}_{i\in I}$ is locally finite, $\Gamma_n(i)$ is finite for all $i \in I$ and $n \in \mathbb{N}$. Let $\Gamma(i) = \bigcup_{n=0}^{\infty} \Gamma_n(i)$ and $N_i = \bigcup_{j\in\Gamma(i)} M_j$. Note that (a) if $M_j \cap N_i \neq \emptyset$ then $M_j \subset N_i$ and $\Gamma(i) = \Gamma(j)$ (in particular $N_i = N_j$) and (b) if $M_j \cap N_i = \emptyset$ then $\Gamma(i) \cap \Gamma(j) = \emptyset$.

Moreover, from (a) and (b) we deduce that each N_i is an open and closed admissible subspace. Since M is connected, $N_i = M$ for all $i \in I$. Hence, $\Gamma(i) = I$ for all $i \in I$, i.e., I is countable.
(3) Since M is Lindelöf, we can assume that $I = \mathbb{N}$. Furthermore, we can assume that $M_n \subset M_{n+1}$ for all $n \in \mathbb{N}$. By (1) $\overline{M_n}$ is a definable subspace for each $n \in \mathbb{N}$. Therefore we can also assume that $M_n \subset \overline{M_n} \subset M_{n+1}$ for all $n \in \mathbb{N}$. Consider the family $U_0 = M_0$, $U_1 = M_1$, $U_n = M_n \setminus \overline{M_{n-2}}$ for each $n \geq 2$. Then, $\{U_n : n \in \mathbb{N}\}$ is a locally finite covering of M by open definable subspaces.
(4) Let $x \in M$ and let V be an open definable subspace of M with $x \in V$. Since M is paracompact, by (1) above, there are finite subsets J and K of I such that $V \subset \bigcup_{i\in J} M_i \subset \bigcup_{i\in J} \overline{M_i} \subset \bigcup_{i\in K} M_i$. By hypothesis, $N := \bigcup_{i\in K} M_i$ is regular and hence there exist an open definable subspace W of N such that $x \in W$ and $\overline{W} \cap N \subset V$. Then $W \subset \overline{W} \cap N \subset V \subset \bigcup_{i\in J} M_i$, so that $\overline{W} \subset \bigcup_{i\in J} \overline{M_i} \subset N$. Therefore, W is an open definable subspace of M such that $x \in W \subset \overline{W} \subset V$, as required. □

The fact that definable subspaces are affine together with paracompactness permits to establish a Triangulation Theorem for regular and paracompact ld-spaces (which will be essential for the proof of the Hurewicz and Whitehead theorems below). Fix a cardinal κ. We denote by R^κ the R-vector space generated by a fixed basis of cardinality κ. A ***generalized simplicial complex*** K in R^κ is a usual simplicial complex except that we may have infinitely many (open) simplices. The ***locally finite*** generalized simplicial complexes are those ones for which the star of each simplex is a finite subcomplex. On the latter we can define in an obvious way an ld-space structure. Indeed, given a locally finite generalized simplicial complex K, for each $\sigma \in K$ we have that $\mathrm{St}_K(\sigma)$ is a finite subcomplex and therefore $\mathrm{St}_K(\sigma) \subset R^{n_\sigma} \subset R^\kappa$ for some $n_\sigma \in \mathbb{N}$. Now, giving each $\mathrm{St}_K(\sigma)$ the topology it inherits from R^{n_σ}, it suffices to consider the atlas $\{(\mathrm{St}_K(\sigma), \mathrm{id}|_{\mathrm{St}_K(\sigma)}\}_{\sigma\in K}$. With this ld-space structure, a locally finite generalized simplicial complex is regular and paracompact (see Fact 3.2.11.(4)). *Henceforth, all the locally definable concepts about locally finite generalized simplicial complexes refer to the regular and paracompact ld-space structure we mention above.* As in the definable setting, it is easy to prove that a locally finite generalized simplicial complex K is connected if and only if there is no proper subcomplex L of K such that $|L|$ (which is clearly an admissible subspace) is both open and closed in $|K|$.

The next result is a sort of converse of the fact that locally finite generalized simplicial complexes are regular and paracompact ld-spaces.

Fact 3.2.12 (Triangulation Theorem). [13, Thm. II.4.4] *Let M be a regular and paracompact ld-space and let $\{A_j : j \in J\}$ be a locally finite family of admissible subspaces of M. Then, there exists an **ld-triangulation** of*

M partitioning $\{A_j : j \in J\}$, i.e., there is a locally finite generalized simplicial complex K and a ld-homeomorphism $\psi : |K| \to M$, where $|K|$ is the realization of K, such that $\psi^{-1}(A_j)$ is the realization of a subcomplex of K for every $j \in J$.

We will prove the Triangulation Theorem in Section 3.3.

Remark 3.2.13. In the Triangulation theorem above, and as in the definable case, we can find the generalized simplicial complex K with its vertices tuples of real algebraic numbers. For, as in the classical theory, if we consider K as an abstract complex and we denote by κ the cardinal of the set of vertices of K, then we obtain a "canonical realization" of K in R^{κ} whose vertices are the standard basis of R^{κ}. Moreover, if the ld-space M is defined over A, $A \subset R$, then we can find the locally definable homeomorphism ψ defined over A.

Henceforth, we denote a regular and paracompact ld-space by ***LD-space***. Note that by Fact 3.2.11.(2) a connected LD-space is Lindelöf.

We finish this section studying the behaviour of ld-spaces with respect to model theoretic operators. Firstly, let us show that given an elementary extension $\mathcal{R}_1$ of an o-minimal structure $\mathcal{R}$ and given an ld-space M in $\mathcal{R}$, there is a natural ***realization*** $M(\mathcal{R}_1)$ of M over $\mathcal{R}_1$ as an ld-space. For, denote by $\{\phi_i : M_i \to Z_i\}_{i\in I}$ the atlas of M and consider the set $Z = \bigcup_{i\in I} Z_i/\sim$, where $x \sim y$ for $x \in Z_i$ and $y \in Z_j$ if and only if $\phi_{ij}(x) = y$. Note that we can define an ld-space structure on Z in a natural way and that Z with this ld-space structure is isomorphic to M (see Definition 3.2.4). Now, we define the realization $Z(\mathcal{R}_1)$ as the ld-space whose underlying set is $\bigcup_{i\in I} Z_i(\mathcal{R}_1)$ modulo the relation $\sim_{\mathcal{R}_1}$, where $x \sim_{\mathcal{R}_1} y$ for $x \in Z_i(\mathcal{R}_1)$ and $y \in Z_j(\mathcal{R}_1)$ if and only if $\phi_{ij}(\mathcal{R}_1)(x) = y$, and with the obvious atlas. If X is a (admissible) definable subspace of M then $X(\mathcal{R}_1)$ is clearly a (resp. admissible) definable subspace of M in $\mathcal{R}_1$. The following result concerning the behaviour of several properties under elementary extensions is an adaptation of those from [13].

Fact 3.2.14. *Let $\mathcal{R}_1$ be an elementary extension of $\mathcal{R}$ and let M be an ld-space in $\mathcal{R}$. Then,*
(i) *M is connected if and only if $M(\mathcal{R}_1)$ is connected,*
(ii) *M is Lindelöf if and only if $M(\mathcal{R}_1)$ is Lindelöf,*
(iii) *M is paracompact if and only if $M(\mathcal{R}_1)$ is paracompact,*
(iv) *M is regular and paracompact if and only if $M(\mathcal{R}_1)$ is regular and paracompact.*

Proof. Let $(M_i, \phi_i)_{i\in I}$ be an atlas of M. *(i)* For the nontrivial part, note that the connected components (see section 3.6) of the ld-space $M(\mathcal{R}_1)$ are actually defined over R. *(ii)* It is enough to note that a Lindelöf ld-space is covered by a countable subcovering of its atlas. *(iii)* For the nontrivial part, by (i), (ii) and Fact 3.2.11, it suffices to prove that for all definable subspace X of M the closure $\overline{X}$ is also a definable subspace of M, i.e., $\overline{X}$ is contained in a finite union of M_i's. Indeed, given a definable subspace X of M, the latter follows from the fact that $\overline{X(\mathcal{R}_1)} = \overline{X}(\mathcal{R}_1)$ is contained in a finite union of $M_i(\mathcal{R}_1)$'s by Fact 3.2.11.(1). *(iv)* By (iii) and Fact 3.2.11.(4), this follows from the fact that every finite union of M_i's is regular if and only if every finite union of $M_i(\mathcal{R}_1)$'s is regular. □

On the other hand, note that given an o-minimal expansion $\mathcal{R}'$ of $\mathcal{R}$ and an ld-space M in $\mathcal{R}$, we can consider M as an ld-space in $\mathcal{R}'$. Clearly, if X is a (admissible) definable subspace of M in $\mathcal{R}$ then X is a (resp. admissible) definable subspace of M in $\mathcal{R}'$.

Proposition 3.2.15. *Let $\mathcal{R}'$ be an o-minimal expansion of $\mathcal{R}$ and let M be an ld-space in $\mathcal{R}$. Then,*
(i) *M is regular in $\mathcal{R}$ if and only if it is regular in $\mathcal{R}'$,*
(ii) *M is connected in $\mathcal{R}$ if and only if it is connected in $\mathcal{R}'$,*
(iii) *M is Lindelöf in $\mathcal{R}$ if and only if it is Lindelöf in $\mathcal{R}'$,*
(iv) *M is paracompact in $\mathcal{R}$ if and only if it is paracompact in $\mathcal{R}'$.*

Proof. *(i)* This follows from the fact that an ld-space is regular if and only if each point has a fundamental system of closed neighbourhoods.
(ii) If M is connected in $\mathcal{R}'$ then it is clearly connected in $\mathcal{R}$. On the other hand, if M is connected in $\mathcal{R}$ then by Fact 3.6.1 any two points are connected by an ld-path definable in $\mathcal{R}$. In particular, any two points are connected by an ld-path definable in $\mathcal{R}'$ and hence, again by Fact 3.6.1, M is connected in $\mathcal{R}'$.
(iii) Let us show that if M is Lindelöf in $\mathcal{R}'$ then M is Lindelöf in $\mathcal{R}$ (the converse is trivial). Indeed, let $(M_i, \phi_i)_{i \in I}$ be an atlas of M in $\mathcal{R}$ and let $\{U_n : n \in \mathbb{N}\}$ be a countable admissible covering of M by open definable subspaces in $\mathcal{R}'$ of M. Since each U_n is a definable subspace, it is contained in a finite union of charts M_i. Therefore, there exists a countable subcovering of $\{M_i : i \in I\}$ which already covers M and hence M is Lindelöf in $\mathcal{R}$.
(iv) Let us show that if M is paracompact in $\mathcal{R}'$ then M is paracompact in $\mathcal{R}$ (the converse is trivial). Without loss of generality we can assume that M is connected. Therefore, by the above equivalences and Fact 3.2.11.(2), M is Lindelöf in $\mathcal{R}$. Then, by Fact 3.2.11.(3), it suffices to prove that for every definable subspace X of M in $\mathcal{R}$, its closure $\overline{X}$ is also a definable subspace of M in $\mathcal{R}$. Since M is paracompact in $\mathcal{R}'$, the latter is clear by Fact 3.2.11.(1). □

3.3 Triangulation of LD-spaces

This section is devoted to the Triangulation theorem for LD-spaces (see Fact 3.2.12). The hardest part of this proof, which may be of interest by itself, is to show that we can embed an LD-space in another one with good properties.

Definition 3.3.1. *We say that an ld-space M is* ***partially complete*** *if every closed definable subspace X of M is definably compact, i.e., every definable curve in X is completable in X.*

Fact 3.3.2. [13, Thm. II.2.1] *Let M be an LD-space. Then, there exist an embedding of M into a partially complete LD-space, i.e, there is partially complete ld-space N and a ld-map $i : M \to N$ such that $i(M)$ is an admissible subspace of N and $i : M \to i(M)$ is an ld-homeomorphism (where $i(M)$ has the LD-space structure inherited from M).*

The proof of Fact 3.3.2 will be given at the end of the current section. The advantage of working with a partially complete LD-space M is that given a triangulation of a closed definable subspace X of M, we know that the corresponding simplicial complex must be closed. This allow us to prove the following "glueing" principle of triangulations for partially complete LD-spaces. Firstly, recall that given two ld-triangulations (K, ϕ) and (L, ψ) of an LD-space M, we say that (K, ϕ) ***refines*** (L, ψ) if for every $\tau \in L$ there is $\sigma \in K$ such that $\phi(\sigma) \subset \psi(\tau)$. We say that (K, ϕ) is a ***equivalent ld-triangulation*** to (L, ψ) if each ld-triangulation is a refinement of the other.

Fact 3.3.3. [13, Thm. II.4.1] *Let M be a partially complete LD-space and $\{C_j : j \in J\}$ a locally finite covering of M by closed definable subspaces. Let (K_j, ϕ_j) be a triangulation of C_j for each $j \in J$. Moreover, assume that ϕ_i and ϕ_j are equivalent ld-triangulations on $C_i \cap C_j$ for every $i, j \in J$ with $C_i \cap C_j \neq \emptyset$. Then, there is a ld-triangulation (K, ϕ) of M such that ϕ is equivalent to ϕ_j on C_j for each $j \in J$.*

Proof. Since M is partially complete and each C_j is closed, we have that K_j is a closed simplicial complex for each $j \in J$. Denote by E the quotient of the disjoint union of the sets $\mathrm{Vert}(K_j)$ of vertices of each K_j by the equivalence relation such that $v \in \mathrm{Vert}(K_i)$ and $w \in \mathrm{Vert}(K_j)$ are related if and only if $\phi_i(v) = \phi_j(w)$. Clearly, for each $j \in J$ we have an injective map $I_j : \mathrm{Vert}(K_j) \to E$. Let $S := \{\{I_j(v_0), \dots, I_j(v_n)\} : (v_0, \dots, v_n) \in K_j, j \in J\}$. It is easy to check that (E, S) is an abstract simplicial complex. In fact, since the covering $\{C_j : j \in J\}$ is locally finite, the complex (E, S) is locally finite. Consider a realization $|K|$ of (E, S) and for each $j \in J$ denote by $|I_j| : |K_j| \to |K|$ the simplicial map generated by I_j. Finally, consider the map $\phi : |K| \to M$ such that $\phi|_{Y_j} = \phi_j \circ |I_j|^{-1}$. Since the triangulations

(K_j, ϕ_j) are equivalent on the intersections, ϕ is a well-defined ld-map. It is not difficult to check that (K, ϕ) is the required ld-triangulation of M. □

Proof of the Triangulation theorem 3.2.12. By Fact 3.3.2 we can assume that M is partially complete. We can also assume that M is connected. Therefore, since M is paracompact, M is Lindelöf (see Fact 3.2.11). Hence, there is a covering $\{C_n : n \in \mathbb{N}\}$ of M by closed definable subspaces such that $C_n \cap C_m = \emptyset$ if $|n-m| > 1$. Indeed, it suffices to apply the shrinking of coverings (see Fact 3.8.6) to the locally finite covering constructed in the proof of Fact 3.2.11.(3). Note that for each $n \in \mathbb{N}$ there is only a finite number of $j \in J$ such that $A_j \cap C_n \neq \emptyset$. Therefore, by the (affine) Triangulation theorem and applying an iteration process, there are triangulations (K_n, ϕ_n) of C_n partitioning $C_n \cap M_{n+1}$, $\{C_n \cap A_j\}_{j \in J}$ and $\{\phi_{n-1}(\sigma) \cap C_n\}_{\sigma \in K_{n-1}}$. Note that (K_n, ϕ_n) refines (K_{n-1}, ϕ_{n-1}) on $C_n \cap C_{n-1}$. Now, since $C_n \cap C_{n-1}$ and $C_n \cap C_{n+1}$ are disjoint, it follows from the o-minimal version of [13, Lem.II.4.3] (see Remark below) that there is a triangulation (L_n, ψ_n) of C_n refining (K_n, ϕ_n) and equivalent to (K_n, ϕ_n) and (K_{n+1}, ϕ_{n+1}) on $C_n \cap C_{n-1}$ and $C_n \cap C_{n+1}$ respectively. Finally, by Fact 3.3.3, there is an ld-triangulation (L, ψ) of M partitioning $\{A_j : j \in J\}$. □

Remark. We do not include the proof of the o-minimal version of [13, Lem.II.4.3] because it is a straightforward adaptation of the semialgebraic one. Moreover, to prove Lemma 1.4.3 we used the ideas involved in the proof of [13, Lem.II.4.3] .

Now, we prove Fact 3.3.2. We will need some results concerning glueing of definable spaces with closed intersections whose proofs have been included in Appendix 3.8. We will also need the following easy embedding result.

Fact 3.3.4. *Let M be an LD-space and let U be an open definable subspace of M. Then there exist an ld-map $f : M \to S^n$ such that $f|_U$ is an embedding and $f^{-1}(p) = M \setminus U$, where $S^n = \{x \in R^{n+1} : |x| = 1\}$ and $p = (0, \dots, 0, 1)$.*

Proof. It is enough to prove the fact in the case that M is a definable space. Indeed, by Fact 3.2.11, $\overline{U}$ is a definable subspace of M and hence we can replace M by any open definable subspace V of M such that $\overline{U} \subset V$. Moreover, since M is regular, we can assume that M is a definable set of R^n for some $n \in \mathbb{N}$. Consider the definable function $d : M \to R : x \mapsto d(x) = \inf\{|x-y| : y \in M \setminus U\}$. Note that $d^{-1}(0) = M \setminus U$. Let $F : R^{n+1} \to S^{n+1} \setminus \{p\}$ be the inverse of the stereographic projection with center p. Finally, we take the definable map $f : M \to S^{n+1}$ such that $f(x) = F(x, \frac{1}{d(x)})$ for all $x \in U$ and $f(x) = p$ for all $x \in M \setminus U$. □

Proof of Fact 3.3.2. Let $\{M_i : i \in I\}$ be a locally finite covering of M by open definable subspaces. By Fact 3.3.4, there is an sphere $S_i := S^{n_i}$,

$n_i \in \mathbb{N}$, and an ld-map $g_i : M \to S_i$ for each $i \in I$ with $g_i^{-1}(p_i) = M \setminus M_i$, p_i the north pole of S^{n_i}, and such that $g_i|_{M_i}$ is an embedding. On the other hand, we define the finite subsets of indexes $\Gamma_1(i) := \{j \in I : M_i \cap M_j \neq \emptyset\}$, $\Gamma_2(i) := \bigcup_{j\in\Gamma_1(i)} \Gamma_1(j)$ and $\Gamma_2^*(i) = \Gamma_2(i) \setminus \{i\}$ for each $i \in I$. Consider the set $Z := \prod_{i\in I}(S_i \times [0,1])$ and the family of subsets $N_i := \prod_{j\in I} N_{i,j} \subset Z$, where $N_{i,i} := S_i \times \{1\}$, $N_{i,j} := (p_j, 0)$ if $j \in I \setminus \Gamma_2(i)$ and $N_{i,j} := S_j \times [0,1]$ if $j \in \Gamma_2^*(i)$. We regard each N_i in the obvious way as a definably compact definable space isomorphic to the product $(S_i \times \{1\}) \times \prod_{j\in\Gamma_2^*(i)}(S_j \times [0,1])$. Now, the family $\{N_i : i \in I\}$ satisfies the hypotheses of Fact 3.8.5. Indeed, it suffices to check that given $i \in I$, there are only a finite number of $j \in I$ with $N_i \cap N_j \neq \emptyset$ and, in this case, $N_i \cap N_j$ is closed in both N_i and N_j with the inherited definable space structures equivalent. Clearly, $N_i \cap N_j \neq \emptyset$ if and only if $i \in \Gamma_2(j)$ and in this case $N_i \cap N_j = \prod_{k\in I} N_{i,j,k}$, where $N_{i,j,k} := S_k \times \{1\}$ if $k = i$ or $k = j$, $N_{i,j,k} := S_k \times [0,1]$ if $k \in \Gamma_2^*(i) \cap \Gamma_2^*(j)$ and $N_{i,j,k} := (p_k, 0)$ in other case, as required. Hence, by Fact 3.8.5, we have an LD-space structure on

$$N := \bigcup_{i\in I} N_i$$

such that each N_i is closed in N (with the inherited structure of definable space from N equal to the original one) and such that $\{N_i : i \in I\}$ is a locally finite covering of N. Moreover, N is partially complete because given a closed definable subspace X of N we have that $X = (X \cap N_{i_1}) \cup \cdots \cup (X \cap N_{i_m})$ for some $i_1, \ldots, i_m \in I$ and each $X \cap N_{i_1}, \ldots, X \cap N_{i_m}$ is a definably compact definable space because it is a closed subspace of a definably compact one.

Next, we construct the embedding of M in N. Consider the open definable subspace $W_i := \bigcup_{j\in\Gamma_1(i)} M_j$ of M for each $i \in I$. Note that $\overline{M_i} \subset W_i$ for each $i \in I$. By Fact 3.8.8, there is an ld-function $f_i : M \to [0,1]$ for each $i \in I$ such that $f_i^{-1}(1) = \overline{M_i}$ and $f_i^{-1}(0) = M \setminus W_i$. We consider the map

$$\psi : M \to N : x \mapsto (g_i(x), f_i(x))_{i\in I}.$$

Now, we prove that ψ is a locally definable embedding.

(A) ψ is a well-defined: Indeed, given $x \in M$, take $i \in I$ such that $x \in M_i \subset \overline{M_i}$. Then $(g_i(x), f_i(x)) \in S_i \times \{1\}$. On the other hand, if $j \notin \Gamma_2(i)$ then $x \notin W_j$, so that $(g_j(x), f_j(x)) = (p_j, 0)$. Hence $\psi(x) \in N_i \subset N$, as required.
(B) ψ is injective: Let $x, y \in M$ be such that $\psi(x) = \psi(y)$. If $x \in M_i$ and $y \notin M_i$ then $(\psi(x))_i = (g_i(x), 1)$ with $g_i(x) \neq p_i$ and $(\psi(y))_i = (p_i, f_i(y))$, i.e., $\psi(x) \neq \psi(y)$, which is a contradiction. If $x, y \in M_i$ then $(\psi(x))_i = (\psi(y))_i$ and hence $g_i(x) = g_i(y)$, so that $x = y$.
(C) ψ is an ld-map: By Fact 3.8.4, it suffices to prove that $\psi|_{M_i}$ is definable for all $i \in I$. Firstly, note that it follows from (A) that $\psi(M_i) \subset N_i$ and hence $\psi(M_i)$ is definable. We regard N_i as the affine definable set $(S_i \times \{1\}) \times \prod_{j\in\Gamma_2^*(i)}(S_j \times [0,1])$. Now, the map $\psi|_{M_i} : M_i \to (S_i \times \{1\}) \times$

$\prod_{j\in\Gamma_2^*(i)}(S_j \times [0,1]) : x \mapsto (g_j|_{M_i}, f_j|_{M_i})_{j\in\Gamma_2(i)}$ is clearly definable.
(D) ψ is an sld-map (see Definition 3.8.9): Firstly, note that $\psi(M) \cap N_i = \psi(\overline{M_i})$ for all $i \in I$. Clearly, $\psi(\overline{M_i}) \subset \psi(M) \cap \overline{\psi(M_i)} \subset \psi(M) \cap \overline{N_i} = \psi(M) \cap N_i$. On the other hand, given $x \in M$ such that $\psi(x) \in N_i$, we have $f_i(x) = 1$, so that $x \in \overline{M_i}$, as required. Hence, by Fact 3.8.11 and (C), ψ is an sld-map. In particular, it follows from Fact 3.8.10 that $\psi(M)$ is an admissible subspace of N.
(E) The map $\psi_i := \psi|_{\overline{M_i}} : \overline{M_i} \to N_i$ is a homeomorphism from $\overline{M_i}$ to $\psi(M) \cap N_i$: It is enough to prove that for every closed definable subspace A of $\overline{M_i}$, $\psi_i(A)$ is a closed definable subspace of $\psi(M) \cap N_i$. We regard N_i as the affine definable set $(S_i \times \{1\}) \times \prod_{j\in\Gamma_2^*(i)}(S_j \times [0,1])$. Then $\psi_i = (g_j|_{M_i}, f_j|_{M_i})_{j\in\Gamma_2(i)}$. Let $x \in \overline{M_i}$ such that $\psi_i(x) \in \overline{\psi_i(A)}$. We have to prove that $x \in A$. Let $k \in I$ such that $x \in M_k$. Firstly, we show that $g_k(x) \in \mathrm{cl}_{g_k(M_k)}(g_k(A \cap M_k))$. We denote by $|\cdot|$ Euclidean distance. Since $x \in \overline{M_i}$, we have $k \in \Gamma_2(i)$. By definition, $g_k(x) \neq p_k$. Therefore, there exist $\tilde{\epsilon} > 0$ such that if $|g_k(x) - z| < \tilde{\epsilon}$ then $z \neq p_k$. On the other hand, since $\psi_i(x) \in \overline{\psi_i(A)}$, for all $\epsilon \in (0, \tilde{\epsilon})$ there is $y \in A$ such that $|g_j(x) - g_j(y)| < \epsilon$ for all $j \in \Gamma_2(i)$. In particular, since $|g_k(x) - g_k(y)| < \epsilon < \tilde{\epsilon}$ we have that $g_k(y) \neq p_k$, so that $y \in M_k$. Hence, for all $\epsilon \in (0, \tilde{\epsilon})$ there is $y \in A \cap M_k$ such that $|g_k(x) - g_k(y)| < \epsilon$, as required. Finally, by definition $g_k|_{M_k}$ is a homeomorphism and $A \cap M_k$ is a closed definable subspace of M_k and hence $g_k(x) \in g_k(A \cap M_k)$, i.e., $x \in A \cap M_k \subset A$.

It follows from the claims (A)-(E) that $\psi : M \to N$ is an ld-homeomorphism from M to $\psi(M)$ with the structure of LD-space inherited from N, as required. □

3.4 Examples of locally definable spaces

We begin this section discussing some natural examples of subsets of R^n carrying a special ld-space structure. In the second subsection we will consider $\bigvee$-definable groups as ld-spaces. Another important class of examples will be shown in Section 3.4.3, where we prove the existence of covering maps for LD-spaces.

3.4.1 Subsets of R^n as ld-spaces

Example 3.4.1. Fix an $n \in \mathbb{N}$ and a collection $\{M_i\}_{i\in I}$ of definable subsets of R^n such that $M_i \cap M_j$ is open in both M_i and M_j (with the topology they inherit from R^n) for all $i, j \in I$. Then, clearly $(M_i, id|_{M_i})_{i\in I}$ is an atlas for $M := \bigcup_{i\in I} M_i$ and hence M is an ld-space.

Let $M \subset R^n$ be an ld-space as in Example 3.4.1. Then it is easy to prove that a definable subspace of M is a definable subset of R^n. However, consider the particular example where $M_i := (-i, i) \subset R$ for $i \in \mathbb{N}$, so that $M = \bigcup_{i \in \mathbb{N}} M_i = Fin(R)$. Note that if $R = \mathbb{R}$ then $\mathbb{R}$ is not a definable subspace of $Fin(\mathbb{R})$ $(= \mathbb{R})$. This also shows that the structures of $\mathbb{R}$ as ld-space and definable set are different. The latter example can be used also to show that the image of an admissible subspace of an ld-space by an ld-map might not be admissible. For, take R a non-archimedean real closed field and the ld-map $id : Fin(R) \to R : x \mapsto x$. Clearly, $Fin(R)$ is not an admissible subspace of R since the admissible subspaces of R are exactly the definable ones.

Nevertheless, we point out that if $M \subset R^n$ is as in Example 3.4.1 with each M_i defined over A, $A \subset R$, $|A| < \kappa$, and $\mathcal{R}$ is κ-saturated, then a definable subset of R^n contained in M is a definable subspace of M. For, if $X \subset M$ is a definable subset, to prove that it is a definable subspace it suffices to show that it is contained in a finite union of charts M_i, which is clear by saturation.

In general, the topology of an ld-space $M \subset R^n$ as in Example 3.4.1 does not coincide with the topology it inherits from R^n. Consider the following example in $\mathbb{R}$. Take $M_0 := \{0\}$ and $M_i := \{\frac{1}{i}\}$ for $i \in \mathbb{N} \setminus \{0\}$. M_0 is open in the topology of M as ld-space but it is non-open with the topology that M inherits from $\mathbb{R}$. It is well known that this also happen at the definable space level (see Robson's example of a non-regular semialgebraic space –Chapter 10 in [15]–). Moreover, Robson's example shows that even in the presence of saturation the topologies might not coincide.

Finally, let $M \subset R^n$ is as in Example 3.4.1 with each M_i defined over A, $A \subset R$, $|A| < \kappa$. Furthermore, assume that $\mathcal{R}$ is κ-saturated and that the topology of M as ld-space coincides with the topology it inherits from R^n. Then let us note that in this case a definable subspace of M (which as we have seen is also a definable subset of R^n) is definably connected if and only if it is connected as an ld-space (see Definition 3.2.6).

Next, we show that an ld-space M as in Example 3.4.1 might not be paracompact.

Example 3.4.2. Let M be as in Example 3.4.1 with $M_i = \{(x, y) \in R^2 : y < 0\} \cup \{(x, y) \in R^2 : x = i\}$ for each $i \in \mathbb{N}$. The set $X = \{(x, y) \in R^2 : y < 0\}$ is a definable subspace of $M = \bigcup_{i \in \mathbb{N}} M_i \subset R^2$. However, $\overline{X} = X \cup \{(i, 0) \in R^2 : i \in \mathbb{N}\}$ is not a definable subspace of M. In particular, M is not paracompact (see Fact 3.2.11.(1)).

We finish by showing that another class of subsets that classically has been considered as "locally semialgebraic subsets" (for example, by S. Lojasiewicz) can be treated inside the theory of ld-spaces.

Example 3.4.3. Let M be a subset of R^n such that for every $x \in M$ there is an open definable neighbourhood U_x of x in R^n with $U_x \cap M$ definable subset. Let $M_x := U_x \cap M$ for each $x \in M$. Then M is an ld-space with the atlas $(M_x, id|_{M_x})_{x \in M}$.

Using the notation of Example 3.4.3, it is clear that $M_x \cap M_y$ is definable and open in both M_x and M_y for all $x, y \in M$ and therefore M is an ld-space as in Example 3.4.1. Moreover, the topology of M as ld-space equals the one it inherits from R^n.

3.4.2 $\bigvee$-definable groups

Throughout this subsection we will assume $\mathcal{R}$ is $\aleph_1$-saturated. The $\bigvee$-definable groups have been considered by several authors as a tool for the study of definable groups in o-minimal structures. Y. Peterzil and S. Starchenko give the following definition in [31]. A group $(G, \cdot)$ is a $\bigvee$-*definable group* over A, $A \subset R$, if $|A| < \aleph_1$ and there is a collection $\{X_i : i \in I\}$ of definable subsets of R^n over A such that $G = \bigcup_{i \in I} X_i$ and for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subset X_k$ and the restriction of the group multiplication to $X_i \times X_j$ is a (not necessarily continuous) definable map into R^n. M. Edmundo introduces in [17] a notion of restricted $\bigvee$-definable group which he calls "locally definable" group. Our purpose in this section is to include both notions within the theory of ld-spaces.

In [31], some (topological) topics of $\bigvee$-definable groups are discussed to study the definable homomorphisms of abelian groups in o-minimal structures and, in particular, they prove the following result.

Fact 3.4.4. [31, Prop. 2.2] *Let $G \subset R^n$ be a $\bigvee$-definable group. Then, there is a uniformly definable family $\{V_a : a \in S\}$ of subsets of G containing the identity element e and a topology τ on G such that $\{V_a : a \in S\}$ is a basis for the τ-open neighbourhoods of e and G is a topological group. Moreover, every generic $h \in G$ has an open neighbourhood $U \subset N^n$ such that $U \cap G$ is τ-open and the topology which $U \cap G$ inherits from τ agrees with the topology it inherits from R, and the topology τ is the unique one with the above properties.*

Because of the above fact is natural to introduce the following concept.

Definition 3.4.5. *We say that a group $(G, \cdot)$ is an* ***ld-group*** *if G is an ld-space and both $\cdot : G \times G \to G$ and ${}^{-1} : G \to G$ are ld-maps. If G is moreover paracompact as ld-space we say that G is an* ***LD-group****.*

Remark 3.4.6. (i) Every ld-group G is regular because it is a topological group. We recall the standard proof. Let $g \in G$ and let U be an open neighbourhood of g in G. We show that there is an open neighbourhood V of g such that $\overline{V} \subset U$. Firstly, since $G \to G : x \mapsto g^{-1}x$ is a homeomorphism, without loss of generality we can assume that $g = e$, where e is the identity element of G. Now, since $\cdot : G \times G \to G$ is continuous and $ee^{-1} = e$, there is an open neighbourhood V of e such that $VV^{-1} \subset U$. We prove that $\overline{V} \subset U$. Let $y \in \overline{Y}$. Since yV is an open neighbourhood of y in G, we have that $yV \cap V \neq \emptyset$. Therefore, $y \in VV^{-1} \subset U$, as required.

(ii) We show that the dimension of an ld-group G is finite. Given $g \in G$, we define $\dim_G(g)$ as the least integer n such that there is an open definable subspace U of G of dimension n with $g \in U$. Clearly, $\dim_G(g) \leq \dim(G)$ for every $g \in G$ and $\dim(G) = \sup\{\dim_G(g) : g \in G\}$. We show that $\dim_G(g) = \dim_G(h)$ for all $g, h \in G$. Fix $g, h \in G$. By symmetry, it suffices to show that $\dim_G(g) \leq \dim_G(h)$. Let U be an open definable subspace of G such that $h \in U$ and $\dim(U) = \dim_G(h)$. Since the map $G \to G : x \mapsto gh^{-1}x$ is an ld-isomorphism, we have that $gh^{-1}U$ is an open definable subspace of G with $g \in gh^{-1}U$ and $\dim(gh^{-1}U) = \dim(U)$. We deduce that $\dim_G(g) \leq \dim(gh^{-1}U) = \dim(U) = \dim_G(h)$, as required. Finally, we have that $\dim(G) = \sup\{\dim_G(g) : g \in G\} = \dim_G(h)$ for some (any) $h \in G$, so that $\dim(G)$ is finite.

We will see that every $\bigvee$-definable group (with its group topology) is an ld-group. We begin with the following result.

Lemma 3.4.7. *Let $G \subset R^n$ be a $\bigvee$-definable group over A and let τ be the topology of Fact 3.4.4. Then, for every generic $g \in G$ there is a definable* OVER *A subset $U_g \subset G$ which is τ-open and such that the topology which U_g inherits from τ agrees with the topology it inherits from R^n.*

Proof. By Fact 3.4.4 it suffices to prove that the parameter set A is preserved. Write $G = \bigcup_{i \in I} X_i$. The dimension of G is defined as $\max\{\dim(X_i) : i \in I\}$. Fix an X_i of maximal dimension and a generic $g \in X_i$. We can assume that $X_i^{-1} = X_i$. Let X_j be such that $X_i X_i X_i \subset X_j$. All the definable sets we shall consider in the proof are definable subsets of X_j. For each $a \in X_i$ we consider the definable set

$$W_a = \{x \in X_i : \forall \delta > 0 \exists \epsilon > 0 \; B(x, \epsilon) \subset xa^{-1}B(a, \delta) \wedge \\ \forall \epsilon > 0 \exists \delta > 0 \; xa^{-1}B(a, \delta) \subset B(x, \epsilon)\},$$

where $B(x, \epsilon) = \{y \in X_i : |y - x| < \epsilon\}$. We also consider the definable set

$$V = \{y \in X_i : W_y \text{ is large in } X_i\}.$$

By Claim 2.3 of [31, Prop. 2.2], for every $h \in X_i$ generic over A, g we have that $h \in W_g$ and therefore $g \in V$. Moreover, since g is generic, we have that $g \in U := int_{X_i}(V)$ (the interior with respect to the topology of the ambient space R^n), which is a definable over A subset of X_i. Fix $a \in U$. We shall prove that

(i) for every $\epsilon > 0$ there is $\delta > 0$ such that $ag^{-1}B(g, \delta) \subset B(a, \epsilon)$, and
(ii) for every $\epsilon > 0$ there is $\delta > 0$ such that $ga^{-1}B(a, \delta) \subset B(g, \epsilon)$.

Granted (i) and (ii), note that $U_g := U$ is the desired neighbourhood of g. Let us show (i). Consider a generic $h \in X_i$ over A, a. Since $h \in W_a$, there is $\widetilde{\delta} > 0$ such that $ah^{-1}B(h, \widetilde{\delta}) \subset B(a, \epsilon)$. By Claim 2.3 of [31, Prop. 2.2], there is $\delta > 0$ such that $g^{-1}B(g, \delta) \subset h^{-1}B(h, \widetilde{\delta})$. Hence $ag^{-1}B(g, \delta) \subset ah^{-1}B(h, \widetilde{\delta}) \subset B(a, \epsilon)$. The proof of (ii) is similar. □

The following technical fact can be easily deduced from the proof of [17, Prop 2.11]. We include its proof for completeness.

Fact 3.4.8. *Let $G = \bigcup_{i \in I} X_i$ be an $\bigvee$-definable group over A. Let $V = \bigcup_{k \in \Lambda} V_k$ (directed union) be a subset of G such that each V_k is definable over A and V is large in G over A, i.e, every generic point of G over A is contained in V. Then there is a collection of elements $\{b_j \in G : j \in J\}$ with each b_j definable over A, such that each X_i is contained in a finite union of subsets of the form $b_j V_k$. In particular, $G = \bigcup_{j \in J} b_j V$.*

Proof. We fix $i \in I$. Let $\mathcal{K}$ be the prime model of $\mathrm{Th}_A(\mathcal{R})$. It is enough to show that given $a \in X_i$ and a generic point $c \in X_i$ of G over K with $\mathrm{tp}(c|K, a)$ finitely satisfiable in K, there is $k \in \Lambda$ such that $a \in c^{-1}V_k$. Indeed, since $\mathrm{tp}(c|K, a)$ is finitely satisfiable over K, there is $b \in X_i(K)$ such that $a \in b^{-1}V_k$ for some $k \in \Lambda$. Therefore, by saturation, for each $i \in I$, there are $b_1, \ldots, b_{r_i} \in X_i(K)$ and $k_1, \ldots, k_{r_i} \in \Lambda$ such that $X_i \subset \bigcup_{j=1}^{r_i} b_j^{-1} V_{k_j}$, as required. Now, if $c \in X_i$ is a generic point of G over K with $\mathrm{tp}(c|K, a)$ finitely satisfiable in K then c is a generic point of G over K, a (see the proof of [32, Lem.2.4]). Since the directed union V is large in G over A, the directed union Va^{-1} is large in G over A, a. Then, by genericity, $c \in Va^{-1}$ and hence $a \in c^{-1}V_k$ for some $k \in \Lambda$. This finishes the proof. □

As it was pointed out by Y. Peterzil to us, a stronger version of the above fact can be proved. In particular, and using the notation of Fact 3.4.8, there exist $b_0, \ldots, b_n \in G$, $n = \dim(G)$, such that $G = \bigcup_{i=0}^{n} b_n V$ (it is enough to adapt the proof of [28, Fact. 4.2]). However, in this case we do not know if $b_0, \ldots, b_n$ are definable over A. Since we are interested in preserving the parameter set we will use the above Fact 3.4.8.

Theorem 3.4.9. *Let $G \subset R^n$ be a $\bigvee$-definable group over A. Let $A \subset C \subset R$. Then*
(i) *G with its group topology (from Fact 3.4.4) is an ld-group over A,*
(ii) *a subset X of G is a definable subset of R^n over C if and only if it is a definable subspace of G over C, and*
(iii) *given a definable subspace X of G over C, its closure $\overline{X}$ (with respect to the group topology) is a definable subspace of G over C.*

Proof. (i) Let $\mathcal{G}$ be the collection of all generics points of G. For each $g \in \mathcal{G}$, let U_g be the definable over A subset of G of Lemma 3.4.7. Consider the subset $V = \bigcup_{g \in \mathcal{G}} U_g$ of G, which is large in G. By Fact 3.4.8, there is a collection $\{b_j \in G : j \in J\}$, with each b_j definable over A, such that $G = \bigcup_{j \in J} b_j V$. For each $j \in J$ and $g \in \mathcal{G}$, consider the definable set $V_{j,g} := b_j U_g$ and the bijection $\psi_{j,g} : V_{j,g} \to U_g : y \mapsto b_j^{-1} y$. Finally, it is easy to check that $\{(V_{j,g}, \psi_{j,g})\}_{j \in J, g \in \mathcal{G}}$ is an atlas of G and therefore G is an ld-group over A.
(ii) It is clear that if $X \subset G$ is a definable subspace over C then it is a

definable subset of R^n over C. So, let X be a definable subset of R^n over C and consider the atlas $\{(V_{j,g},\psi_{j,g})\}_{j\in J,g\in\mathcal{G}}$ of G constructed in the proof of (i). Since X is definable over C we have that $\psi_{j,g}(X\cap V_{j,g})=b_j^{-1}X\cap U_g$ is also definable over C for every $j\in J$ and $g\in\mathcal{G}$. Hence, it is enough to show that X is contained in a finite union of the sets $V_{j,g}$ (which are defined over A) and, since they cover G, this is clear by saturation.
(iii) Let X be a definable subspace of G over C and write $G=\bigcup_{i\in I}X_i$. By (ii) X is a definable subset of R^n over C. We will show that $\overline{X}$ is a definable subset of R^n over C (this is enough also by (ii)). Fix a generic point g of G and let U_g as in Lemma 3.4.7. Firstly, let us show that $\overline{X}\subset X_j$ for some $j\in I$. Since $\{X_i\}_{i\in I}$ is a directed family and X and U_g are definable, there is $j\in I$ such that $XU_g^{-1}g\subset X_j$. Now, if $y\in\overline{X}$ then $yg^{-1}U_g\cap X\neq\emptyset$ and hence $y\in XU_g^{-1}g\subset X_j$. Finally, $\overline{X}=\{y\in X_j : g\in cl_{U_g}(gy^{-1}X\cap U_g)\}$ is clearly a definable subset of R^n over C, where $cl_{U_g}(-)$ denotes the closure in U_g with respect to the inherited topology from the ambient space R^n. □

Theorem 3.4.9.(iii) states that in a $\bigvee$-definable group we have a good relation between the topological and the definable setting as it happens with LD-spaces (see Fact 3.2.11(1)). However, as we will see not every $\bigvee$-definable group is paracompact or Lindelöf as ld-group. Firstly, let $\mathcal{R}$ be an $\aleph_1$-saturated elementary extension of the o-minimal structure $\langle\mathbb{R},<,+,-,\cdot,r\rangle_{r\in\mathbb{R}}$. Consider the collection $\mathcal{F}$ of finite subsets of $\mathbb{R}$. Then $(G,+)$, where $G=\bigcup_{F\in\mathcal{F}}F\subset R$ and $+$ is the usual addition, is a $\bigvee$-definable group over $\emptyset$ which is not Lindelöf as ld-group. However, G is paracompact (note that the group topology of G as $\bigvee$-definable group is the discrete one). Secondly, let S be a real closed field such that there is no countable subset $C\subset S_+:=\{s\in S : s>0\}$ with $S=\bigcup_{x\in C}(-x,x)$ (e.g. if S is $\aleph_1$-saturated). Let $\mathcal{R}$ be an $\aleph_1$–saturated elementary extension of the o-minimal structure $\langle S,<,+,-,\cdot,s\rangle_{s\in S}$. Consider $(G,+)$, where $G=\bigcup_{s\in S_+}(-s,s)\subset R$ and $+$ is the usual addition. The group $(G,+)$ is a $\bigvee$-definable group over $\emptyset$ which is not Lindelöf as ld-group. Since it is connected, $(G,+)$ is not paracompact (see Fact 3.2.11.(2)).

In [17], M. Edmundo considers $\bigvee$-definable groups $G=\bigcup_{i\in I}X_i$ over A with the restriction $|I|<\aleph_1$ (which already implies the restriction $|A|<\aleph_1$), he calls them *"locally definable" groups*. This restriction on the cardinality of I allows Edmundo to prove results using techniques which are not available in the general setting of $\bigvee$-definable groups. As he notes the main examples of $\bigvee$-definable groups are of this form: the subgroup of a definable group generated by a definable subset and the coverings of definable groups. The restriction on the cardinality of $|I|$ of the "locally definable" groups has also the following consequences on them as ld-spaces.

Theorem 3.4.10. (i) *Every "locally definable" group over A with its group topology is a Lindelöf LD-group over A.*

(ii) *Moreover, every Lindelöf LD-group over A is ld-isomorphic to a "locally definable" group over A (considered as an LD-group by (i)).*

Proof. (i) Let G be a "locally definable" group over A. By Theorem 3.4.9.(i), G is an ld-group over A. We first show that G is Lindelöf. Recall the notation of Theorem 3.4.9.(i). Write $G = \bigcup_{i \in I} X_i$, with $|I| < \aleph_1$. Since I is countable, to prove that G is Lindelöf we can assume that the language is countable (recall that Lindelöf property is invariant under o-minimal expansions by Proposition 3.2.15). Now, since for each generic $g \in G$ the definable subset U_g of Lemma 3.4.7 is definable over A, the collection $\{U_g : g \in G \text{ generic}\}$ is countable. Hence, the atlas $\{(V_{j,g}, \psi_{j,g})\}_{j \in J, g \in \mathcal{G}}$ of the proof of Theorem 3.4.9.(i) is also countable and so G is Lindelöf. Having proved the latter, the paracompacity follows from Theorem 3.4.9.(iii) and Fact 3.2.11.
(ii) Let G be a Lindelöf LD-group over A. Since G is regular and paracompact, by Fact 3.2.12 and Remark 3.2.13 there is an ld-triangulation $f : |K| \to G$ over A. Moreover, we can assume that $\overline{K}$ is also a locally finite generalized simplicial complex (in this case we say that K is a strictly locally finite generalized simplicial complex). Indeed, the semialgebraic Triangulation theorem [13, Thm. II.4.4] is stronger than the locally definable version we have proved here: it states that given a regular and paracompact locally semialgebraic space M there is a locally semialgebraic triangulation $f : |K| \to M$ with K a strictly locally finite generalized simplicial complex. However, note that we can deduce this stronger version in the locally definable setting from the semialgebraic one. For, given an LD-space M, by (the weaker) locally definable version of the Triangulation theorem 3.2.12, there is an ld-triangulation $f : |K| \to M$ with K a locally finite generalized simplicial complex. Now, since $|K|$ is a regular and paracompact locally semialgebraic space, by [13, Thm. II.4.4] there is a locally semialgebraic triangulation $g : |L| \to |K|$ partitioning all the simplices of K and with L a strictly locally finite generalized simplicial complex L. Therefore, it suffices to take the ld-triangulation $f \circ g : |L| \to M$.

Now, since G is an LD-group, the dimension of K is finite (see Remark 3.4.6). Furthermore, since G is Lindelöf, the admissible covering $\{St_{|K|}(\sigma) : \sigma \in K\}$ of $|K|$ has a countable subcovering of $|K|$. From this fact we deduce that K is countable. Then, since K is countable, has finite dimension and is strictly locally finite, by [13, Prop.II.3.3] we can assume that the realization $|K|$ lie in R^{2n+1}, $n = \dim(K)$, and that the topology it inherits from R^{2n+1} coincides with its topology as LD-space. Now, define in $|K|$ a group operation via the ld-isomorphism ψ and the group operation of G. With this group operation, $|K|$ is an LD-group which we will denote by H. Of course, G is ld-isomorphic to H via ψ. On the other hand, we can consider $|K|$ as a "locally definable" group. For, let $\mathcal{F}$ the collection of all finite simplicial subcomplexes of K. Clearly, $|K| = \bigcup_{L \in \mathcal{F}} |L|$ with the group operation obtained via ψ is a "locally definable" group over A.

Indeed, since the group operation is an ld-map, its restriction to $|L_1| \times |L_2|$ is a definable map into R^{2n+1} for all $L_1, L_2 \in \mathcal{F}$. Finally, since the group operation is already continuous and the topology of $|K|$ as ld-space coincides with the one inherited form R^{2n+1}, the "locally definable" group $|K|$ with the ld-group structure obtained in part (i) is exactly H. □

Corollary 3.4.11. *Let G be a "locally definable" group over A. Then, there is an ld-triangulation $\psi : |K| \to G$ of G over A with $|K| \subset R^{2n+1}$, $n = dim(G)$, and such that the topology of $|K|$ as LD-space coincides with the one inherited from R^{2n+1}. Moreover, $|K|$ with the group operation inherited from G via ψ is also a "locally definable" group over A whose group topology equals the one inherited from R^{2n+1}.*

Let us point out that there are important examples of $\bigvee$-definable groups which are not Lindelöf LD-spaces (and hence not "locally definable" groups). The group of definable homomorphisms between abelian groups were used in [31] as a tool to study interpretability problems. In particular, given to abelian definable groups A and B over C, $C \subset R$, it is proved there that the group of definable homomorphisms $\mathcal{H}(A, B)$ from A to B is a $\bigvee$-definable group over C (see [31, Prop. 2.20]). Note that $\mathcal{H}(A, B)$ might not be a "locally definable" group (see the Examples at the end of Section 3 in [31]). Nevertheless, we have the following corollary to Theorem 3.4.9.

Corollary 3.4.12. *$\mathcal{H}(A, B)$ is an LD-group.*

Proof. We have already seen in Theorem 3.4.9.(i) that $\mathcal{H}(A, B)$ is an ld-group (and hence regular). To prove paracompactness, consider its connected component $\mathcal{H}(A, B)^0$, which is a definable group by [31, Thm. 3.6]. Then, by Theorem 3.4.9.(ii), $\mathcal{H}(A, B)^0$ is a definable subspace of $\mathcal{H}(A, B)$. Hence, $\{g\mathcal{H}(A, B)^0 : g \in \mathcal{H}(A, B)\}$ is a locally finite covering of $\mathcal{H}(A, B)$ by open definable subspaces and therefore $\mathcal{H}(A, B)$ is paracompact. As we will see in the next section, the notion of connectedness used in [31] for $\bigvee$-definable groups differs from the one used here. However, in this particular case, since $\mathcal{H}(A, B)^0$ is definable, both notions coincide. □

3.4.3 Covering maps for LD-spaces

In this section we deal with one of the motivations for considering the locally definable category.

Fact 3.4.13. [14, Thm.5.11] *Let B be a connected ld-space, $b_0 \in B$ and let L be a subgroup of $\pi_1(B, b_0)^{\mathcal{R}}$. Then, there exists connected ld-space E and a covering $p : E \to B$ with $p_*(\pi_1(E, e_0)^{\mathcal{R}}) = L$ for some $e_0 \in p^{-1}(b_0)$. Moreover, if B is an LD-space then E is also an LD-space.*

We give a proof of the above result just for the case of LD-spaces because this is enough for our purposes (see Subsection 3.7.2). However, the general case can be proved with a straightforward adaptation of the semialgebraic proof.

Proof of Fact 3.4.13. Consider the collection $\mathcal{P}$ of all ld-curves $\alpha : I \to B$ such that $\alpha(0) = b_0$. Let $\sim$ be the equivalence relation on $\mathcal{P}$ such that $\alpha \sim \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha * \beta^{-1}] \in L$, where $*$ denotes the usual concatenation of curves. We will denote by $\alpha^{\#}$ the class of $\alpha \in \mathcal{P}$. Let $E = \mathcal{P}/\sim$ and $p : E \to B : \alpha^{\#} \mapsto \alpha(1)$. Now, we divide the proof in several steps.
(1) E is an ld-space: Firstly, note that every definable subspace of B has a finite covering by open connected definable subspaces which are simply connected (because of Remark 3.2.8, the Triangulation theorem and the fact that the star of a vertex is definably simply connected). Therefore, since B is an LD-space, there exist a locally finite covering $\{U_j : j \in J\}$ of B such that each U_j is a connected and simply connected (i.e, $\pi_1(U_j)^{\mathcal{R}} = 0$) definable open subspace of B. Now, for each $j \in J$ and $\alpha \in \mathcal{P}$ with $\alpha(1) \in U_j$, we define $W_{j,\alpha} := \{(\alpha * \delta)^{\#} : \delta : I \to U_j \text{ ld-map}, \delta(0) = \alpha(1)\}$. Henceforth, when we write $W_{j,\alpha}$, we assume that $\alpha(1) \in U_j$. Consider the map $\phi_{j,\alpha} : W_{j,\alpha} \to U_j : (\alpha * \delta)^{\#} \mapsto \delta(1)$ for each $j \in J$ and $\alpha \in \mathcal{P}$. Since U_j is connected and simply connected, $\phi_{j,\alpha}$ is a well-defined bijection for every $j \in J$ and $\alpha \in \mathcal{P}$. The family $(W_{j,\alpha}, \phi_{j,\alpha})_{j\in J,\alpha\in\mathcal{P}}$ is an atlas of E. Indeed, fix $i, j \in J$ and $\alpha, \beta \in \mathcal{P}$ with $W_{i,\alpha} \cap W_{j,\beta} \neq \emptyset$. Then, $\phi_{i,\alpha}(W_{i,\alpha} \cap W_{j,\beta})$ is the union of some connected components of $U_i \cap U_j$. Moreover, $\phi_{j,\beta}(W_{i,\alpha} \cap W_{j,\beta})$ is the union of exactly the same connected components of $U_i \cap U_j$, i.e., $\phi_{j,\beta}(W_{i,\alpha} \cap W_{j,\beta}) = \phi_{i,\alpha}(W_{i,\alpha} \cap W_{j,\beta})$. This shows that both $\phi_{i,\alpha}(W_{i,\alpha} \cap W_{j,\beta})$ and $\phi_{j,\beta}(W_{i,\alpha} \cap W_{j,\beta})$ are open in U_i and U_j respectively and that each change of charts is the identity, hence definable.
(2) The map p is an ld-map: since $p|_{W_{j,\alpha}} : W_{j,\alpha} \to U_j \subset B$ is a definable map of definable spaces, for all $W_{j,\alpha}$.
(3) E is paracompact: Fix $i \in J$ and $\alpha \in \mathcal{P}$. We prove that $\#\{W_{j,\beta} : W_{i,\alpha} \cap W_{j,\beta} \neq \emptyset, j \in J, \beta \in \mathcal{P}\}$ is finite. Firstly, note that if $W_{i,\alpha} \cap W_{j,\beta} \neq \emptyset$ then $U_i \cap U_j \neq \emptyset$. Therefore, since the covering $\{U_j : j \in J\}$ is locally finite, it suffices to prove that the family $\{W_{j,\beta} : W_{i,\alpha} \cap W_{j,\beta} \neq \emptyset, \beta \in \mathcal{P}\}$ is finite for a fixed $j \in J$. Indeed, we will show that given W_{j,β_1} and W_{j,β_2} with $W_{i,\alpha} \cap W_{j,\beta_1} \neq \emptyset$ and $W_{i,\alpha} \cap W_{j,\beta_2} \neq \emptyset$, if $p(W_{i,\alpha} \cap W_{j,\beta_1}) \cap p(W_{i,\alpha} \cap W_{j,\beta_2}) \neq \emptyset$ then $W_{j,\beta_1} = W_{j,\beta_2}$. The latter is enough because for each $\beta \in \mathcal{P}$, $p(W_{i,\alpha} \cap W_{j,\beta})$ $(= \phi_{i,\alpha}(W_{i,\alpha} \cap W_{j,\beta}))$ is the union of some connected components of $U_i \cap U_j$, which has only a finite number of them. Firstly, since U_j is connected, it is easy to prove that if $\gamma^{\#} \in W_{j,\beta_1}$ then $W_{j,\gamma} = W_{j,\beta_1}$. The same holds for W_{j,β_2}. So, if $W_{j,\beta_1} \cap W_{j,\beta_2} \neq \emptyset$ then $W_{j,\beta_1} = W_{j,\beta_2}$. On the other hand, since $p|_{W_{i,\alpha}} = \phi_{i,\alpha}$ and $\phi_{i,\alpha}$ is a bijection, from $p(W_{i,\alpha} \cap W_{j,\beta_1}) \cap p(W_{i,\alpha} \cap W_{j,\beta_2}) \neq \emptyset$ we deduce that $\emptyset \neq W_{i,\alpha} \cap W_{j,\beta_1} \cap W_{j,\beta_2} \subset W_{j,\beta_1} \cap W_{j,\beta_2}$ and hence $W_{j,\beta_1} = W_{j,\beta_2}$.

(4) The ld-map $p : E \to B$ is a covering map: By to proof of (3), we have that $p^{-1}(U_j) = \dot\bigcup_{\alpha \in \mathcal{P}} W_{j,\alpha}$ for every $j \in J$. On the other hand, $p|_{W_{j,\alpha}} : W_{j,\alpha} \to U_j$ is an ld-homeomorphism for every $j \in J$ and $\alpha \in \mathcal{P}$.
(5) E is an LD-space: Indeed, the regularity of E can be deduced from the regularity of B and (4).
(6) E is path-connected, hence connected: Let $e_0 := c_{b_0}^{\#} \in E$ for the ld-curve $c_{b_0} : I \to B : t \mapsto b_0$ (recall $b_0 \in B$ is a fixed point). Given $\alpha \in \mathcal{P}$, we will show that there is and ld-map from e_0 to $\alpha^{\#}$. Consider the map $\widetilde{\alpha} : I \to E : s \mapsto \alpha_s^{\#}$, where $\alpha_s : I \to B : t \mapsto \alpha_s(t) = \alpha(ts)$ is clearly an ld-curve. Note that $p \circ \widetilde{\alpha}(s) = \alpha(s)$, $\widetilde{\alpha}(0) = e_0$ and $\widetilde{\alpha}(1) = \alpha^{\#}$. Let us check that $\widetilde{\alpha}$ is an ld-curve. Since α is an ld-curve, there are $s_0 = 0 < s_1 < \cdots < s_m = 1$ such that $\alpha([s_k, s_{k+1}]) \subset U_{i_k}$ for every $k = 0, \ldots, m-1$. Hence $\widetilde{\alpha}(I) \subset \bigcup_{k=0}^{m-1} W_{i_k,\alpha_{s_k}}$. On the other hand, $\phi_{i,\alpha_{s_k}} \circ \widetilde{\alpha}|_{[s_k,s_{k+1}]} = \alpha|_{[s_k,s_{k+1}]}$ for every $k = 0, \ldots, m-1$ and therefore $\widetilde{\alpha}$ is an ld-curve as required.
(7) Finally, let us show that $p_*(\pi_1(E, e_0)^{\mathcal{R}}) = L$. Let α be an ld-loop of B at b_0. By the proof of (6), $\widetilde{\alpha} : I \to E : s \mapsto \alpha_s^{\#}$, where $\alpha_s : I \to B : t \mapsto \alpha_s(t) = \alpha(ts)$, is an ld-curve. Now, as in the classical case, we have that $[\alpha] \in p_*(\pi_1(E, e_0)^{\mathcal{R}})$ if and only if $\widetilde{\alpha}(1) = \alpha^{\#} = e_0$. Indeed, the latter can be proved using both the path and homotopy lifting properties of covering maps (see the proof of Proposition 3.7.10). Hence $[\alpha] \in p_*(\pi_1(E, e_0)^{\mathcal{R}})$ if and only if $[\alpha] \in L$. □

Note that if B is an LD-group (see Definition 3.4.5), then it is possible to define a group operation in the covering space E. Using the notation of the proof of Fact 3.4.13, given $\alpha, \beta \in \mathcal{P}$, we define $\alpha^{\#}\beta^{\#} := (\alpha\beta)^{\#}$. Note that with this group operation E becomes an LD-group. This was also proved in [19] for the particular case of the universal covering map of a definable group for o-minimal expansions of ordered groups.

3.5 Homology of locally definable spaces

We fix for the rest of this section an LD-space M. We consider the abelian group $S_k(M)^{\mathcal{R}}$ freely generated by the *singular locally definable simplices* $\sigma : \Delta_k \to M$, where Δ_k is the standard k-dimensional simplex in R. Note that since σ is locally definable and Δ_k is definable, the image $\sigma(\Delta_k)$ is a definable subspace of M. As we will see, this fact allows us to use the o-minimal homology developed by A. Woerheide in [38] (see also Section 1.6 for an alternative development of simplicial o-minimal homology). The boundary operator $\delta : S_{k+1}(M)^{\mathcal{R}} \to S_k(M)^{\mathcal{R}}$ is defined as in the classical case, making $S_*(M)^{\mathcal{R}} = \bigoplus_k S_k(M)^{\mathcal{R}}$ into a chain complex. We similarly define the chain complex of a pair of locally definable spaces. The graded group $H_*(M)^{\mathcal{R}} = \bigoplus_k H_k(M)^{\mathcal{R}}$ is defined as the homology of the complex $S_*(M)^{\mathcal{R}}$. Locally definable maps induce in a natural way homomorphisms in homology. Similarly for relative homology. Note that if M is just a definable set then we obtain the usual o-minimal homology groups (see e.g. [20]).

It remains to check that the functor we have just defined satisfies the locally definable version of the Eilenberg-Steenrod axioms. We shall check them making use of the corresponding axioms for definable sets through an adaptation of a classical result in homology that (roughly) states that the homology commutes with direct limits. Note that each definable subspace $Y \subset M$ is a definable regular space and hence affine (see Remark 3.2.8). Therefore, the o-minimal homology groups of Y as definable set are the ones we have just defined as (locally) definable space. Denote by $\mathcal{D}_M$ the set

$$\{Y \subset M : Y \text{ definable subspace}\}.$$

Note that M can be written as the directed union $M = \bigcup_{Y \in \mathcal{D}_M} Y$. Now, consider the direct limit

$$\underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} = \dot{\bigcup}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} / \sim,$$

where $c_1 \sim c_2$ for $c_1 \in H_n(Y_1)^{\mathcal{R}}$ and $c_2 \in H_n(Y_2)^{\mathcal{R}}$, $Y_1, Y_2 \in \mathcal{D}_M$, if and only if there is $Y_3 \in \mathcal{D}_M$ with $Y_1, Y_2 \subset Y_3$ such that $(i_1)_*(c_1) = (i_2)_*(c_2)$ for $(i_1)_* : H_n(Y_1)^{\mathcal{R}} \to H_n(Y_3)^{\mathcal{R}}$ and $(i_2)_* : H_n(Y_2)^{\mathcal{R}} \to H_n(Y_3)^{\mathcal{R}}$ the homomorphisms in homology induced by the inclusions. On the other hand, we have a well-defined homomorphism $(i_Y)_* : H_n(Y)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}}$ for each $Y \in \mathcal{D}_M$, where $i_Y : Y \to M$ is the inclusion. Hence, there exists a well-defined homomorphism

$$\psi : \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}},$$

where $\psi(\overline{c}) = (i_Y)_*(c)$ for $c \in H_n(Y)^{\mathcal{R}}$. In a similar way, given an admissible subspace A of M, we have a well-defined homomorphism

$$\widetilde{\psi} : \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y, A \cap Y)^{\mathcal{R}} \to H_n(M, A)^{\mathcal{R}},$$

where $\widetilde{\psi}(\overline{c}) = i_*(c)$ for $c \in H_n(Y, A \cap Y)^{\mathcal{R}}$ and $i : (Y, Y \cap A) \to (M, A)$ the inclusion map.

Theorem 3.5.1. (i) $\psi : \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}}$ *is an isomorphism.*
(ii) *Let A be an admissible subspace of M. Then* $\widetilde{\psi} : \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y, A \cap Y)^{\mathcal{R}} \to H_n(M, A)^{\mathcal{R}}$ *is an isomorphism.*

Proof. (i) Firstly, we show that ψ is surjective. Let $c \in H_n(M)^{\mathcal{R}}$ and α be a finite sum of singular ld-simplices of M which represents c. Consider the definable subspace X of M which is the union of the images of the singular ld-simplices in α. Then $[\alpha] \in H_n(X)^{\mathcal{R}}$ and therefore it suffices to consider $\overline{[\alpha]} \in \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}}$ because $\psi(\overline{[\alpha]}) = c$. Now, let us show that ψ is injective. Let $\overline{c} \in \underrightarrow{lim}_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}}$, $c \in H_n(X)^{\mathcal{R}}$, $X \in \mathcal{D}_M$, such that

$\psi(\overline{c}) = 0$. Since $\psi(\overline{c}) = 0$, there is a finite sum β of singular ld-simplices of M such that $\delta\beta = \alpha$. Consider the definable subspace Z of M which is the union of X and the images of the singular ld-simplices in β. Then we have that $[\alpha] = 0$ in $H_n(Z)^{\mathcal{R}}$ and therefore $\overline{c} = 0$ in $\varinjlim_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}}$. The proof of (ii) is similar. □

Remark 3.5.2. Let M be an LD-space and $\mathcal{D}$ a collection of definable subspaces of M such that for every $Y \in \mathcal{D}_M$ there is $X \in \mathcal{D}$ with $Y \subset X$. Then Theorem 3.5.1 remains true if we replace $\mathcal{D}_M$ by $\mathcal{D}$.

Now, with the above result, we verify the Eilenberg-Steenrod axioms.

Proposition 3.5.3 (Homotopy axiom)**.** *Let M and N be LD-spaces and let A and B be admissible subspaces of M and N respectively. If $f : (M, A) \to (N, B)$ and $g : (M, A) \to (N, B)$ are ld-homotopic ld-maps then $f_* = g_*$.*

Proof. Let $[\alpha] \in H_n(M, A)^{\mathcal{R}}$. Consider the definable subspace X of M which is the union of the images of the singular ld-simplices in α. By Theorem 3.5.1 and the homotopy axiom for definable sets, it is enough to prove that there is a definable subspace Z of N such that $f(X), g(X) \subset Z$ and that the definable maps $f|_X : (X, A \cap X) \to (Z, B \cap Z)$ and $g|_X : (X, A \cap X) \to (Z, B \cap Z)$ are definably homotopic. Let $F : (M \times I, A \times I) \to (N, B)$ be a ld-homotopy from f to g. Then, it suffices to take Z as the definable subspace $F(X \times I)$ of N and the definable homotopy $F|_{X \times I} : (X \times I, A \cap X \times I) \to (Z, B \cap Z)$ from $f|_X$ to $g|_X$. □

Proposition 3.5.4 (Exactness axiom)**.** *Let A be an admissible subspace of M and let $i : (A, \emptyset) \to (M, \emptyset)$ and $j : (M, \emptyset) \to (M, A)$ be the inclusions. Then the following sequence is exact*

$$\cdots \to H_n(A)^{\mathcal{R}} \xrightarrow{i_*} H_n(M)^{\mathcal{R}} \xrightarrow{j_*} H_n(M, A)^{\mathcal{R}} \xrightarrow{\partial} H_{n-1}(A)^{\mathcal{R}} \to \cdots,$$

where $\partial : H_n(M, A)^{\mathcal{R}} \to H_{n-1}(A)^{\mathcal{R}}$ is the natural boundary map, i.e, $\partial[\alpha]$ is the class of the cycle $\partial\alpha$ in $H_{n-1}(A)^{\mathcal{R}}$.

Proof. It is easy to check that for every $Y \in \mathcal{D}_M$ the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots H_n(A \cap Y) & \xrightarrow{(i_Y)_*} & H_n(Y) & \xrightarrow{(j_Y)_*} & H_n(Y, A \cap Y) & \xrightarrow{\partial} & H_{n-1}(A \cap Y) & \xrightarrow{(i_Y)_*} & H_{n-1}(Y) \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots H_n(A) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(M, A) & \xrightarrow{\partial} & H_{n-1}(A) & \xrightarrow{i_*} & H_{n-1}(M) \cdots
\end{array}$$

where $i_Y : (A \cap Y, \emptyset) \to (Y, \emptyset)$ and $j_Y : (Y, \emptyset) \to (Y, A \cap Y)$ are the inclusions (and the superscript $\mathcal{R}$ has been omitted). By the o-minimal exactness axiom the first sequence is exact for every $Y \in \mathcal{D}_M$. Hence, if we take the direct limit, the sequence remains exact. The result then follows from Theorem 3.5.1. □

Proposition 3.5.5 (Excision axiom)**.** *Let M be an LD-space and let A be an admissible subspace of X. Let U be an admissible open subspace of M such that $\overline{U} \subset int(A)$. Then the inclusion $j : (M - U, A - U) \to (M, A)$ induces an isomorphism $j_* : H_n(M - U, A - U)^{\mathcal{R}} \to H_n(M, A)^{\mathcal{R}}$.*

Proof. By Theorem 3.5.1.(ii), it is enough to prove that for each definable subspace Y of M the inclusion $j_Y : (Y - U_Y, A_Y - U_Y) \to (Y, A_Y)$ induces an isomorphism in homology, where $U_Y = U \cap Y$ and $A_Y = A \cap Y$. So let Y be a definable subspace of M. Since M is regular then we can regard Y as a definable set. Now, $cl_Y(U_Y) \subset \overline{U \cap Y} \cap Y \subset \overline{U} \cap Y \subset int(A) \cap Y \subset int_Y(A_Y)$. Finally, by the o-minimal excision axiom, j_Y induces an isomorphism in homology. □

The proof of the dimension axiom is trivial.

Proposition 3.5.6 (Dimension axiom)**.** *If M is a one point set, then $H_n(M)^{\mathcal{R}} = 0$ for all $n > 0$.*

Once we have a well-defined homology functor in the locally definable category, we now see that this functor has a good behavior with respect to model theoretic operators. The following result will be used in Section 3.7 in the proof of the Hurewicz theorems for LD-spaces.

Theorem 3.5.7. *The homology groups of LD-spaces are invariant under elementary extension and o-minimal expansions.*

Proof. We prove the invariance by o-minimal expansions. So let $\mathcal{R}'$ be an o-minimal expansion of $\mathcal{R}$ and let M be an LD-space in $\mathcal{R}$. Denote by $\mathcal{D}_M$ the collection of all definable subspaces of M. Recall that since M is regular each $Y \in \mathcal{D}_M$ can be regarded as an affine definable space (see Remark 3.2.8). Now, since the o-minimal homology groups are invariant under o-minimal expansions (see [7, Prop.3.2]), for each $Y \in \mathcal{D}_M$ there is a natural isomorphism $F_Y : H_n(Y)^{\mathcal{R}} \to H_n(Y)^{\mathcal{R}'}$. Hence, there exist a natural isomorphism $F : \varinjlim_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} \to \varinjlim_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}'}$. On the other hand, by Theorem 3.5.1 and Remark 3.5.2 we have natural isomorphisms $\psi_1 : \varinjlim_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}}$ and $\psi_2 : \varinjlim_{Y \in \mathcal{D}_M} H_n(Y)^{\mathcal{R}'} \to H_n(M)^{\mathcal{R}'}$. Finally, we consider the natural isomorphism $\psi_2 \circ F \circ \psi_1^{-1} : H_n(M)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}'}$. The proof of the invariance by elementary extensions is similar. □

Notation 3.5.8. We will denote by θ the natural isomorphism given by Theorem 3.5.7 between the semialgebraic and the o-minimal homology groups of a regular and paracompact locally semialgebraic space. Note that if we restrict θ above to the definable category then we obtain the natural isomorphism of [7, Prop.3.2].

3.6 Connectedness

Recall that an ld-space M is connected if there is no admissible nonempty proper clopen subspace U of M. We can also extend the natural concept of "path connected" for definable spaces to the locally definable ones. Specifically, we say that an admissible subspace X of an ld-space M is ***path connected*** if for every $x_0, x_1 \in X$ there is an ld-path $\alpha : [0,1] \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Naturally, the (path) ***connected components*** of an ld-space are the maximal (path) connected subsets.

Fact 3.6.1. [13, Prop. I.3.18] *Every path connected component of an ld-space is a clopen admissible subspace.*

Proof. Let M be an ld-space and let C be a path connected component of M. Let $(M_i, \phi_i)_{i \in I}$ be an atlas of M. It suffices to show that $\phi_i(C \cap M_i)$ is a clopen definable subset of $\phi_i(M_i)$. The latter is obvious because $\phi_i(C \cap M_i)$ is the (finite) union of some definably path-connected components of $\phi_i(M_i)$, which are clopen definable subsets of $\phi_i(M_i)$ (see [15, Ch.3,Prop.2.18] and [15, Ch.6,Prop.3.2]). □

From the above fact we deduce that the connected and path-connected components of an ld-space are admissible subspaces and coincide. In particular, every connected ld-space is path connected (the converse is trivial).

Remark 3.6.2. Let M be an ld-space. Then M is connected if and only if every ld-map from M to a discrete ld-space is constant. Indeed, suppose M is connected and let $f : M \to N$ be an ld-map, where N is a discrete ld-space. Since N is discrete, $\{y\}$ is a clopen definable subspace of N for all $y \in N$. Therefore the admissible subspace $f^{-1}(y)$ of M is clopen for all $y \in N$. Hence, since M is connected, $M = f^{-1}(y_0)$ for some $y_0 \in N$. To prove the right-to-left implication, suppose that M is not connected. Then there are two proper clopen admissible subspaces U_0 and U_1 of M such that $U_0 \cap U_1 = \emptyset$ and $U_0 \cup U_1 = M$. Finally, it suffices to consider the ld-map $f : M \to \{0,1\}$ such that $f(x) = i$ for all $x \in U_i$.

Since $\bigvee$-definable groups were first considered several non equivalent notions of connectedness have been used. As we will see here some of them are not really adequate and lead to pathological examples. Fix a $\bigvee$-definable group $G = \bigcup_{i \in I} X_i \subset R^n$ over A, $A \subset R$, $|A| < \aleph_1$, in an $\aleph_1$-saturated o-minimal expansion $\mathcal{R}$ of a real closed field R. Here, we say that G is connected if it is so as ld-group (see Theorem 3.4.9). In [31], G is said to be $\mathcal{M}$-connected (*PS-connected*, for us) if there is no definable set U in R^n such that $U \cap G$ is a nonempty proper clopen subset with the group topology of G. In [17], G is said to be connected (*E-connected*, for us) if there is no definable set $U \subset G$ such that U is a nonempty proper clopen subset with the group topology of G. Finally, in [27], G is said to be connected (*OP-connected*, for us) if all the X_i can be chosen to be definably connected with respect to the definable subspace structure it inherits from G as ld-group. Notice that in [27] the situation is simpler because G is a subgroup of a definable group and hence embedded in some R^n, so each X_i is connected with respect to the ambient R^n (see Section 3.4.1).

For $\bigvee$-definable groups the relation of the above notions is as follows:

$$\text{OP-connected} \Leftrightarrow \text{Connected} \Rightarrow \text{PS-connected} \Rightarrow \text{E-connected}.$$

The second and third implications are clear by definition. Furthermore, the following examples show that these implications are strict.

Example 3.6.3. Let R be a non archimedean real closed field. Consider the definable set $B = \{(t, -t) \in R^2 : t \in [0, 1]\} \cup \{(t, t-2) \in R^2 : t \in [1, 2]\}$. For each $n \in \mathbb{N}$, consider the definable set $X_n = (\bigcup_{i=-n}^{n}(2i, 0) + B) \cup (\bigcup_{i=-n}^{n}(2i, -\frac{1}{2}) + B) \subset R^2$. Define a group operation on $G = \bigcup_{n\in\mathbb{N}} X_n$ via the natural bijection of G with $\mathrm{Fin}(R)\times\mathbb{Z}/2\mathbb{Z}$, where $\mathrm{Fin}(R)= \{x \in R : |x| < n \text{ for some } n \in \mathbb{N}\}$. Then, G with this group operation is a $\bigvee$-definable group.

Note that the topology of G inherited from R^2 coincides with its group topology. G is not connected as an ld-space because it has two connected components. However, G is PS-connected because any definable subset of R^2 which contains one of these connected components must have a nonempty intersection with the other component.

Example 3.6.4. [2] Let R be a non archimedean real closed field and consider the definable sets $X_n = (-n, -\frac{1}{n}) \cup (\frac{1}{n}, n)$ for $n \in \mathbb{N}$, $n > 1$. Then, $G = \bigcup_{n>1} X_n$ is a $\bigvee$-definable group with the multiplicative operation of R.

Here, again, the topology G inherits from R^2 coincides with its group topology. The $\bigvee$-definable group G is not PS-connected since it is the disjoint union of the clopen subsets $\{x \in R : x > 0\}\cap G$ and $\{x \in R : x < 0\}\cap G$. But neither of these subsets is definable and therefore G is E-connected.

Note that in both examples we can define in an obvious way an ld-map $f : G \to \{0, 1\}$ which is not constant and therefore Remark 3.6.2 is not true if we replace connectedness by PS-connectedness or E-connectedness.

Even though there are pathological examples, the results in [31] are correct for PS-connectedness. For the results in [17], one should substitute E-connectedness by connectedness (see [2]).

We now prove the equivalence between OP-connectedness and connectedness.

Proposition 3.6.5. *Let G be a $\bigvee$-definable group over A. Then, G is OP-connected if and only if G is connected.*

Proof. Firstly, recall that by Theorem 3.4.9 a subset of G is a definable subspace if and only if it is a definable subset of R^n. Let G be an OP-connected $\bigvee$-definable group, i.e, such that $G = \bigcup_{i \in I} X_i$ with X_i definably connected for all $i \in I$. Consider a nonempty admissible clopen subspace U of G. Since U is not empty and each X_i is definably connected, there is $i_0 \in I$ such that $X_{i_0} \subset U$. Now, for every $i \in I$ there is $j \in I$ with $X_{i_0} \cup X_i \subset X_j$. Since X_j is definably connected and $\emptyset \neq X_{i_0} \subset X_j \cap U$ we have that $X_j \subset U$ and, in particular, $X_i \subset U$. So we have proved that for every $i \in I$, $X_i \subset U$. Hence $U = G$, as required.

Now, let G be a connected $\bigvee$-definable group over A. Let $\mathcal{C}$ be the collection of all connected definable subspaces over A of G which are connected and contain the unit element of G. It is enough to show that $G = \bigcup_{X \in \mathcal{C}} X$. Note that we just consider the connected definable subspaces of G which are definable over A because we need to preserve the parameter set. So let $x \in G$. By Fact 3.6.1, G is also path connected and hence there is an ld-curve $\alpha : I \to G$ such that $\alpha(0) = x$ and $\alpha(1) = e$. Since $\alpha(I)$ is definable and G is an ld-group over A, a finite union of charts (which are definable over A) contains $\alpha(I)$. Hence $\alpha(I)$ is contained in a definable over A subset X of G. Taking the adequate connected component, we can assume that X is connected. Hence $x \in X \in \mathcal{C}$. □

Corollary 3.6.6. *A $\bigvee$-definable group is OP-connected if and only if is path-connected.*

Proof. By Fact 3.6.1 and Proposition 3.6.5. □

3.7 Homotopy theory in LD-spaces

Once we have defined the category of locally definable spaces, in the following section we will develop a homotopy theory for LD-spaces, that is, regular and paracompact locally definable spaces. This section is divided in Subsections 3.7.1, 3.7.2 and 3.7.3, which are the locally definable analogues of Sections 2.3, 2.4 and 2.5 respectively.

3.7.1 Homotopy sets of locally definable spaces

The homotopy sets in the locally definable category are defined as in the definable one just substituting the definable maps by the locally definable ones (see Section 2.3). Specifically, let (M, A) and (N, B) be two pairs of LD-spaces, i.e., M and N are LD-spaces and A and B are admissible subspaces of M and N respectively. Let C be a closed admissible subspace of M and let $h : C \to N$ be an ld-map such that $h(A \cap C) \subset B$. We say that two ld-maps $f, g : (M, A) \to (N, B)$ with $f|_C = g|_C = h$, are ***ld-homotopic relative to*** h, denoted by $f \sim_h g$, if there exists an ld-homotopy H :

$(M \times I, A \times I) \to (N, B)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ for all $x \in M$ and $H(x, t) = h(x)$ for all $x \in C$ and $t \in I$. The ***homotopy set of*** (M, A) ***and*** (N, B) ***relative to*** h is the set

$$[(M, A), (N, B)]_h^{\mathcal{R}} = \{f : f : (M, A) \to (N, B) \text{ ld-map in } \mathcal{R}, f|_C = h\}/\sim_h .$$

If $C = \emptyset$ we omit all references to h.

The next theorem is the main result of this section and it establishes a strong relation between the locally definable and the locally semialgebraic homotopy. It is the locally definable analogue of Corollary 2.3.4 for pairs of LD-spaces. Recall the behavior of the ld-spaces under o-minimal expansions in Proposition 3.2.15.

Theorem 3.7.1. *Let (M, A) and (N, B) be two pairs of regular paracompact locally semialgebraic spaces. Let C be a closed admissible semialgebraic subspace of M and $h : C \to N$ a locally semialgebraic map such that $h(A \cap C) \subset B$. Suppose A is closed in M. Then, the map*

$$\begin{array}{rcl} \rho : [(M, A), (N, B)]_h^{\mathcal{R}_0} & \to & [(M, A), (N, B)]_h^{\mathcal{R}} \\ {[f]} & \mapsto & [f], \end{array}$$

which sends the locally semialgebraic homotopic class of a locally semialgebraic map to its locally definable homotopic class, is a bijection.

An important tool for the proof of the above theorem (and in general, for the study of homotopy properties of LD-spaces) is the following homotopy extension lemma. Even though the proof for locally semialgebraic spaces (see [13, Cor.III.1.4]) can be adapted to the locally definable setting, we have included here an alternative proof which, in particular, does not make use of the Triangulation Theorem of LD-spaces (see Fact 3.2.12).

Lemma 3.7.2 (Homotopy extension lemma). *Let M, N be two LD-spaces and let A be a closed admissible subspace of M. Let $f : M \to N$ be an ld-map and $H : A \times I \to N$ a ld-homotopy such that $H(x, 0) = f(x)$ for all $x \in A$. Then, there exists a ld-homotopy $G : M \times I \to N$ such that $G(x, 0) = f(x)$ for all $x \in M$ and $G|_{A \times I} = H$.*

Proof. Without loss of generality, we can assume that M is connected and hence, by Fact 3.2.11.(2), that M is Lindelöf. Let $(M_k, \phi_k)_{k \in \mathbb{N}}$ be an atlas of M. Consider $X_n := \bigcup_{k=0}^{n} \overline{M_k}$ for each $n \in \mathbb{N}$. By Fact 3.2.11.(1) each X_n is a closed definable subspace of M and hence $\{X_n : n \in \mathbb{N}\}$ is an admissible covering by closed definable subspaces such that $X_n \subset X_{n+1}$ for all $n \in \mathbb{N}$. Take the restrictions $f_n := f|_{X_n}$ and $H_n := H|_{A_n \times I}$, where A_n is the closed definable subspace $A \cap X_n$. Moreover, since M is regular, we can regard each X_n as an affine definable space (see Remark 3.2.8). Now, by the o-minimal homotopy extension lemma 2.2.1 and applying an induction process, we can find a collection of definable homotopies $G_n : X_n \times I \to N$ such that $G_n(x,0) = f_n(x)$ for all $x \in X_n$, $G_n|_{X_{n-1}\times I} = G_{n-1}$ and $G_n|_{A_n\times I} = H_n$. Finally, we define the map $G : M \times I \to N$ such that $G|_{X_n\times I} = G_n$ for every $n \in \mathbb{N}$. By Fact 3.8.4, the map G is locally definable and, by definition, $G|_{A\times I} = H$ and $G(x,0) = f(x)$ for all $x \in M$. □

Proof of Theorem 3.7.1. With the above tools at hand we can follow the lines of the proof of [13, Thm. III.4.2]. Here are the details. As in the definable case, it suffices to prove that ρ is surjective when $A = B = \emptyset$. Indeed, we can do here similar reductions than the ones we followed in Fact 2.3.3 just applying the homotopy extension lemma 3.7.2 for LD-spaces instead of its definable version. Now, we divide the proof in two cases.
Case M is a semialgebraic space: Since M is regular, we can assume that it is affine (see Remark 3.2.8). Let $f : M \to N$ be an ld-map such that $f|_C = h$. Since M is semialgebraic, $f(M)$ is a definable subspace of the locally semialgebraic space N and therefore it is contained in the union of a finite number of semialgebraic charts. Hence, there is a semialgebraic subspace N' of N such that $f(M) \subset N'$. Now, since N is regular, we can regard N' also as an affine definable space and therefore we can see the map $f : M \to N'$ as a definable map between semialgebraic sets (see comments after Definition 3.2.4). By Corollary 2.3.4, there exist a definable homotopy $H' : M \times I \to N'$ such that $H'(x,0) = f(x)$ for all $x \in M$, $H'(x,t) = h(x)$ for all $x \in C$ and $t \in I$ and $H'(-,1) : M \to N'$ is semialgebraic. Hence, it suffices to consider the definable homotopy $H = i \circ H'$ where $i : N' \to N$ is the inclusion, to get $\rho([H(-,1)]) = [f]$.
General Case: Let $f : M \to N$ be an ld-map such that $f|_C = h$. We have to show that f is ld-homotopic relative to h to a locally semialgebraic map. Without loss of generality, we can assume that M is connected and hence, by Fact 3.2.11.(2), that M is Lindelöf. Furthermore, by the shrinking of covering property (see Fact 3.8.6) there is a locally finite covering $\{X_n : n \in \mathbb{N}\}$ of M by closed semialgebraic subspaces. Consider the closed semialgebraic subspace $Y_n := X_0 \cup \cdots \cup X_n$ and the closed admissible subspace $C_n := Y_n \cup C$ for each $n \in \mathbb{N}$. By the previous case, there exist a definable homotopy $\widetilde{H}_0 : Y_0 \times I \to N$ such that $\widetilde{H}_0(x,0) = f(x)$ for all $x \in Y_0$, $\widetilde{H}_0(-,1) : Y_0 \to N$ is a locally semialgebraic map and $\widetilde{H}_0(x,t) = h(x)$ for all $x \in C \cap Y_0$ and $t \in I$. Moreover, by Lemma 3.7.2, there exist an ld-homotopy $H_0 : M \times I \to N$ with $H_0(x,0) = f(x)$ for all $x \in M$, $H_0(x,t) = h(x)$ for all $x \in C$ and $t \in I$ and such that $H_0|_{Y_0\times I} = \widetilde{H}_0$. In particular, $g_0 := H_0|_{C_0\times\{1\}}$ is a locally semialgebraic map with $g_0|_C = h$. Now, by iteration we obtain a sequence of ld-homotopies $\{H_n : M \times I \to N : n \in \mathbb{N}\}$ such that

(i) $g_n := H_n|_{C_n\times\{1\}}$ is a locally semialgebraic map,
(ii) $H_{n+1}(x,t) = g_n(x)$ for all $(x,t) \in C_n \times I$ (so $g_{n+1}|_{C_n} = g_n$), and
(iii) $H_{n+1}|_{M\times\{0\}} = H_n|_{M\times\{1\}}$.

Note that in particular $H_n(x,t) = g_0(x) = h(x)$ for all $(x,t) \in C \times I$ and $n \in \mathbb{N}$. By Fact 3.8.4, the map $g : M \to N$ such that $g|_{C_n} = g_n$ for $n \in \mathbb{N}$, is a locally semialgebraic map. Let us show that f is ld-homotopic to g relative to h. The idea is to glue all the homotopies H_n in a correct way. Let $t_n := 1 - 2^{-n}$ for each $n \in \mathbb{N}$. Consider the map $G : M \times I \to N$ such that (a) $G(x,t) = H_n(x, \frac{t-t_n}{t_{n+1}-t_n})$ for all $x \in M$ and $t \in [t_n, t_{n+1}]$ and (b) $G(x,t) = g(x)$ otherwise. By construction it is clear that $G(x,t) = h(x)$ for all $(x,t) \in C \times I$. It remains to check that G is indeed an ld-map. By Fact 3.8.4, it suffices to show that the restriction $G|_{Y_n \times I}$ is definable for each $n \in \mathbb{N}$. So fix $n \in \mathbb{N}$. By definition, $G|_{Y_n \times [0,t_n]}$ is clearly definable. On the other hand, take $(x,t) \in Y_n \times [t_n, 1]$. If $t > t_m$ for every $m \in \mathbb{N}$, then $G(x,t) = g(x)$ by definition. If $t \in [t_m, t_{m+1}]$ for some $m \geq n$, then $G(x,t) = H_m(x,t) = g_n(x) = g(x)$. Therefore $G|_{Y_n \times [t_n,1]} = g|_{Y_n}$, which is also a definable map. Hence $G|_{Y_n \times I}$ is definable, as required. □

The following corollary is the analogue (and it can be proved adapting its proof) of Corollary 2.3.5 for LD-spaces. Recall the definition of the realization of an LD-space in an elementary extension given before Fact 3.2.14.

Corollary 3.7.3. *Let M and N be two pairs of regular paracompact locally semialgebraic spaces defined without parameters. Then, there exist a bijection*

$$\rho : [M(\mathbb{R}), N(\mathbb{R})] \to [M, N]^{\mathcal{R}},$$

where $[M(\mathbb{R}), N(\mathbb{R})]$ denotes the classical homotopy set. Moreover, if the real closed field R is a field extension of $\mathbb{R}$, then the result remains true allowing parameters from $\mathbb{R}$.

Note that both Theorem 3.7.1 and Corollary 3.7.3 remain true for systems of LD-spaces (see Corollary 2.3.4). Thanks to the Triangulation Theorem for LD-spaces (see Fact 3.2.12), we have also the following corollary (see the proof of Corollary 2.3.7, noting that the finiteness of the simplicial complexes plays an irrelevant role).

Corollary 3.7.4. *Let M and N be LD-spaces defined without parameters. Then, any ld-map $f : M \to N$ is ld-homotopic to an ld-map $g : M \to N$ defined without parameters. If moreover M and N are locally semialgebraic spaces then g can also be taken locally semialgebraic.*

3.7.2 Homotopy groups of locally definable spaces

The homotopy groups in the locally definable category are defined as in the definable setting using ld-maps instead of the definable ones (see Section 2.4). Specifically, given a **pointed LD-space** (M, x_0), i.e., M is an

LD-space and $x_0 \in M$, we define the n-**homotopy group** as the homotopy set $\pi_n(M,x_0)^{\mathcal{R}} := [(I^n,\partial I^n),(M,x_0)]^{\mathcal{R}}$. We define $\pi_0(M,x_0)$ as the collection of all connected components of M (which coincide with the collection of the path connected ones by Fact 3.6.1). We say that (M,A,x_0) is a **pointed pair of LD-spaces** if M is an LD-space, A is an admissible subspace of M and $x_0 \in A$. The **relative** n-**homotopy group**, $n \geq 1$, of a pointed pair (M,A,x_0) of LD-spaces is the homotopy set $\pi_n(X,A,x_0)^{\mathcal{R}} = [(I^n,I^{n-1},J^{n-1}),(X,A,x_0)]^{\mathcal{R}}$, where $I^{n-1} = \{(t_1,\dots,t_n) \in I^n : t_n = 0\}$ and $J^{n-1} = \overline{\partial I^n \setminus I^{n-1}}$.

As in the definable case, we can see that the homotopy groups $\pi_n(M,x_0)^{\mathcal{R}}$ and $\pi_m(M,A,x_0)^{\mathcal{R}}$ are indeed groups for $n \geq 1$ and $m \geq 2$, the group operation is defined via the usual concatenation of maps. Moreover, these groups are abelian for $n \geq 2$ and $m \geq 3$. Also, given an ld-map between pointed LD-spaces (or pointed pairs of LD-spaces), we define the induced map in homotopy, as usual, by composing. The latter will be a group homomorphism in the case we have a group structure. It is easy to check that with these definitions of homotopy group and induced map, both the absolute and relative homotopy groups $\pi_n(-)$ are covariant functors.

The following three results (and their relative versions) can be deduced from Theorem 3.7.1 (see the proofs of Theorem 2.4.1, Corollary 2.4.3 and Corollary 2.4.4 respectively).

Corollary 3.7.5. *Let (M,x_0) be a regular and paracompact locally semialgebraic pointed space. Then, the map $\rho : \pi_n(M,x_0)^{\mathcal{R}_0} \to \pi_n(M,x_0)^{\mathcal{R}} : [f] \mapsto [f]$, is a natural isomorphism for every $n \geq 1$.*

Corollary 3.7.6. *Let (M,x_0) be a regular paracompact locally semialgebraic pointed space defined without parameters. Then, there exists a natural isomorphism between the classical homotopy group $\pi_n(M(\mathbb{R}),x_0)$ and the homotopy group $\pi_n(M(R),x_0)^{\mathcal{R}}$ for every $n \geq 1$.*

Corollary 3.7.7. *The homotopy groups are invariants under elementary extensions and o-minimal expansions.*

All the results showed in Properties 2.4.6 remain true in the locally definable setting. We recall here briefly these results.

(1) *The homotopy property*: If two ld-maps are ld-homotopic then they induce the same homomorphism between the homotopy groups.

(2) *The exactness property*: Given a pointed pair (M,A,x_0) of LD-spaces, the following sequence is exact,

$$\cdots \to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(M,x_0) \xrightarrow{j_*} \pi_n(M,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to \cdots \to \pi_0(A,x_0),$$

where ∂ is the usual boundary map $\partial : \pi_n(M,A,x_0)^{\mathcal{R}} \to \pi_{n-1}(A,x_0)^{\mathcal{R}} : [f] \mapsto [f|_{I^{n-1}}]$ and $i : (A,x_0) \to (M,x_0)$ and $j : (M,x_0,x_0) \to (M,A,x_0)$ are the inclusions (and the superscript $\mathcal{R}$ has been omitted).

(3) *The action of π_1 on π_n:* Given a pointed LD-space (M, x_0), there is an action $\beta : \pi_1(M, x_0)^{\mathcal{R}} \times \pi_n(M, x_0)^{\mathcal{R}} \to \pi_n(M, x_0)^{\mathcal{R}}$. In a similar way, given a pointed pair (M, A, x_0) of LD-spaces, there is an action $\beta : \pi_1(A, x_0)^{\mathcal{R}} \times \pi_n(M, A, x_0)^{\mathcal{R}} \to \pi_n(M, A, x_0)^{\mathcal{R}}$. In the absolute (relative) case, we will denote by $\beta_{[u]}$ the isomorphism $\beta([u], -) : \pi_n(M, x_0)^{\mathcal{R}} \to \pi_n(M, x_0)^{\mathcal{R}}$ (resp. $\beta([u], -) : \pi_n(M, A, x_0)^{\mathcal{R}} \to \pi_n(M, A, x_0)^{\mathcal{R}}$) for each $[u] \in \pi_1(X, x_0)^{\mathcal{R}}$ (resp. $[u] \in \pi_1(A, x_0)^{\mathcal{R}}$).

The homotopy property is clear by definition. The exactness property can be proved with a straightforward adaptation of the proof of the classical one. Alternatively, we can also transfer the classical exactness property using the Triangulation Theorem 3.2.12 and Corollary 3.7.6. Finally, the existence of the action of π_1 on π_n is just an application of the homotopy extension lemma (see Lemma 3.7.2 and Properties 2.4.6.(3)). Furthermore, the following technical lemma is easy to prove (see the proof of Lemma 2.4.7).

Lemma 3.7.8. *Let (M, x_0) and (N, y_0) two pointed LD-spaces. Let $\psi : (M, x_0) \to (N, y_0)$ be an ld-map and let $[u] \in \pi_1(M, x_0)^{\mathcal{R}}$. Then, for all $[f] \in \pi_n(M, x_0)^{\mathcal{R}}$, $\psi_*(\beta_{[u]}([f])) = \beta_{\psi_*([u])}(\psi_*([f]))$.*

(4) *The fibration property:* The only part of Properties 2.4.6 which has not an obvious extension to LD-spaces is the one concerning fibrations. Naturally, we say that an ld-map $p : E \to B$ between LD-spaces is a **(Serre) fibration** if it has the homotopy lifting property for each (resp. closed and bounded) definable sets. As in Remark 2.4.8, the homotopy lifting property for closed simplices implies the homotopy lifting property for pairs of closed and bounded definable sets. Note that the restriction of a (Serre) fibration to the preimage of a definable subspace is not necessarily a definable (resp. Serre) fibration. Hence, we cannot deduce directly the fibration property for LD-spaces from the definable one (see Theorem 2.4.9). However, the fibration property for LD-spaces can be proved just adapting the definable proof.

Theorem 3.7.9 (The fibration property). *Let B and E be LD-spaces and let $p : E \to B$ be a Serre fibration. Then the induced map $p_* : \pi_n(E, F, e_0)^{\mathcal{R}} \to \pi_n(B, b_0)^{\mathcal{R}}$ is a bijection for $n = 1$ and an isomorphism for all $n \geq 2$, where $e_0 \in F = p^{-1}(b_0)$.*

As in the definable setting, the main examples of fibrations are the covering maps (see Proposition 2.4.10). Given two ld-spaces E and B, a **covering map** $p : E \to B$ is a surjective ld-map p such that there is an admissible covering $\{U_i : i \in I\}$ of B by open definable subspaces and for each $i \in I$ and each connected component V of $p^{-1}(U_i)$, the restriction $p|_V : V \to U_i$

is a locally definable homeomorphism (so in particular both V and $p|_V$ are definable).

Proposition 3.7.10. *Let B and E be LD-spaces. Then, every covering map $p : E \to B$ is a fibration.*

Proof. Firstly, note that coverings satisfy the unicity of liftings as in the definable case (see [20, Lem.2.5]). Indeed, given a connected LD-space Z and two ld-maps $\widetilde{f}_1, \widetilde{f}_2 : Z \to E$ with $p \circ \widetilde{f}_1 = p \circ \widetilde{f}_2$ and $\widetilde{f}_1(z) = \widetilde{f}_2(z)$ for some $z \in Z$, we have that $\widetilde{f}_1 = \widetilde{f}_2$. This is so because both $\{z \in Z : \widetilde{f}_1(z) = \widetilde{f}_2(z)\}$ and $\{z \in Z : \widetilde{f}_1(z) \neq \widetilde{f}_2(z)\}$ are clopen admissible subspaces of Z. The path lifting and the homotopy lifting properties also remain true for p (see the definable case in [20, Prop.2.6] and [20, Prop.2.7]). To see this for the path lifting property take an admissible covering $\{U_j : j \in J\}$ of B as in the definition of covering map. Let $\gamma : I \to B$ be an ld-curve. Since $\gamma(I)$ is a definable subspace of B, we have that $\gamma(I) \subset \bigcup_{j \in J_0} U_j$ for some finite subset J_0 of J. Now, by the shrinking covering property of definable sets, there are $0 = s_0 < s_1 < \cdots < s_r = 1$ such that for each $i = 0, \ldots, r-1$ we have $\gamma([s_i, s_{i+1}]) \subset U_{j(i)}$ and $\gamma(s_{i+1}) \in U_{j(i)} \cap U_{j(i+1)}$. Hence, by the unicity of liftings, it suffices to lift each $\gamma|_{[s_i, s_{i+1}]}$ step by step using the definable homeomorphism $p|_{V_{j(i)}} : V_{j(i)} \to U_{j(i)}$ for the suitable connected component $V_{j(i)}$ of $p^{-1}(U_{j(i)})$. The proof of the homotopy lifting property is similar.

Finally, the above properties and the fact that the images of definable sets by ld-maps are definable subspaces, give us the homotopy lifting property for definable sets as in Proposition 2.4.10. □

Corollary 3.7.11. *Let B and E be LD-spaces. Let $p : E \to B$ be a covering and let $p(e_0) = b_0$. Then, $p_* : \pi_n(E, e_0)^{\mathcal{R}} \to \pi_n(B, b_0)^{\mathcal{R}}$ is an isomorphism for every $n > 1$ and injective for $n = 1$.*

Proof. Since p is a covering, $p^{-1}(b_0)$ is discrete. Hence $\pi_n(p^{-1}(b_0), e_0) = 0$ for every $n \geq 1$. Then, the result follows from Proposition 3.7.10 and both the exactness and the fibration properties. □

3.7.3 The Hurewicz and Whitehead theorems for locally definable spaces

We define the Hurewicz homomorphism in a similar same way as in the definable case but using the homology groups developed in Section 3.5. We fix a generator $z_n^{\mathcal{R}_0}$ of $H_n(I^n, \partial I^n)^{\mathcal{R}_0}$ (recall that $H_n(I^n, \partial I^n)^{\mathcal{R}_0} \cong \mathbb{Z}$). Let $z_n^{\mathcal{R}} := \theta(z_n^{\mathcal{R}_0})$, where θ is the natural transformation of Notation 3.5.8 between the (locally) semialgebraic and the (locally) definable homology groups. Given a pointed LD-space (M, x_0), the ***Hurewicz homomorphism***, for $n \geq 1$, is the map $h_{n,\mathcal{R}} : \pi_n(M, x_0)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}} : [f] \mapsto h_{n,\mathcal{R}}([f]) = f_*(z_n^{\mathcal{R}})$, where $f_* : H_n(I^n, \partial I)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}}$ denotes the map in singular homology induced by f. Note that it follows from the homotopy axiom of singular homology that $h_{n,\mathcal{R}}$ is well-defined (see Proposition 3.5.3). We define the relative Hurewicz homomorphism adapting in the obvious way what was done in the absolute case. It is easy to check that $h_{n,\mathcal{R}}$ is a natural transformation between the functors $\pi_n(-)^{\mathcal{R}}$ and $H_n(-)^{\mathcal{R}}$. The following result can be easily deduced from the naturality of the isomorphisms ρ and θ introduced in Corollary 3.7.5 and Notation 3.5.8 respectively (see the proof of Proposition 2.5.1).

Proposition 3.7.12. *Let (M, x_0) be a pointed regular paracompact locally semialgebraic space. Then, the following diagram commutes*

$$\begin{array}{ccc} \pi_n(M,x_0)^{\mathcal{R}_0} & \xrightarrow{h_{n,\mathcal{R}_0}} & H_n(M)^{\mathcal{R}_0} \\ \rho \downarrow & & \downarrow \theta \\ \pi_n(M,x_0)^{\mathcal{R}} & \xrightarrow[h_{n,\mathcal{R}}]{} & H_n(M)^{\mathcal{R}} \end{array}$$

for all $n \geq 1$.

Now, the proofs in the definable setting of the Hurewicz and the Whitehead theorems (see Theorems 2.5.3 and 2.5.7) apply for LD-spaces just using (i) the locally definable category instead of the definable one, (ii) the respective isomorphisms ρ and θ of Theorem 3.7.1 and Notation 3.5.8 instead of the definable ones and (iii) the Triangulation Theorem for LD-spaces (see Fact 3.2.12). Note that in the proofs of the definable versions of the Hurewicz and Whitehead theorems, the finiteness of the simplicial complexes plays an irrelevant role. Specifically, we have the following results (recall the action β of π_1 on π_n defined after Corollary 3.7.7).

Theorem 3.7.13 (Hurewicz theorems). *Let (M, x_0) be a pointed LD-space and $n \geq 1$. Suppose that $\pi_r(M, x_0)^{\mathcal{R}} = 0$ for every $0 \leq r \leq n-1$. Then, the Hurewicz homomorphism*

$$h_{n,\mathcal{R}} : \pi_n(M, x_0)^{\mathcal{R}} \to H_n(M)^{\mathcal{R}}$$

is surjective and its kernel is the subgroup generated by $\{\beta_{[u]}([f])[f]^{-1} : [u] \in \pi_1(M, x_0)^{\mathcal{R}}, [f] \in \pi_n(M, x_0)^{\mathcal{R}}\}$. In particular, $h_{n,\mathcal{R}}$ is an isomorphism for $n \geq 2$.

Theorem 3.7.14 (Whitehead theorem). *Let M and N be two LD-spaces. Let $\psi : M \to N$ be an ld-map such that for some $x_0 \in M$, $\psi_* : \pi_n(M, x_0)^{\mathcal{R}} \to \pi_n(N, \psi(x_0))^{\mathcal{R}}$ is an isomorphism for all $n \geq 1$. Then, ψ is an ld-homotopy equivalence.*

Corollary 3.7.15. *Let M be an LD-space and let $x_0 \in M$. If $\pi_n(M, x_0)^{\mathcal{R}} = 0$ for all $n \geq 0$ then M is ld-contractible.*

3.8 Appendix

We show now those results needed in the proof of Fact 3.3.2.

Fact 3.8.1. *Let M be an ld-space and let $\{C_i : i \in I\}$ be a locally finite family of closed definable subspaces of M. Then the admissible subspace $C = \bigcup_{i\in I} C_i$ is closed in M.*

Proof. It suffices to prove that for every open definable subspace U of M, $U \cap C = U \cap \overline{C}$. Let U be an open definable subspace of M. Since $\{C_i\}_{i\in I}$ is locally finite, $U \cap C = U \cap (C_{i_1} \cup \cdots \cup C_{i_k})$ for some $i_1, \ldots, i_k \in I$. Hence $U \cap \overline{C} = U \cap \overline{U \cap C} = U \cap (\overline{C_{i_1}} \cup \cdots \cup \overline{C_{i_k}}) = U \cap (C_{i_1} \cup \cdots \cup C_{i_k}) = U \cap C$, as required. □

Fact 3.8.2. [13, Lem.II.1.1,II.1.2] *Let M be an ld-space and let $\{C_i : i \in I\}$ be a locally finite family of closed definable subspaces of M with $M = \bigcup_{i\in I} C_i$. Then,*
(i) *a subset U of M is an open admissible subspace of M if and only if $U \cap C_i$ is an open definable subspace of C_i for all $i \in I$,*
(ii) *M is paracompact, and*
(iii) *if C_i is regular for all $i \in I$, M is also regular.*

Proof. (i) Let $U \subset M$ such that $U \cap C_i$ is an open definable subspace of C_i for all $i \in I$. Then $C_i \setminus U$ is a closed definable subspace of C_i for all $i \in I$. Since the family $\{C_i\}_{i\in I}$ is locally finite, $\{C_i \setminus U\}_{i\in I}$ is a locally finite family of closed definable subspaces of M. It follows from Fact 3.8.1 that $M \setminus U = \bigcup_{i\in I}(C_i \setminus U)$ is a closed admissible subspace of M and hence U is an open admissible subspace of M.
(ii) We consider the finite subset of indexes $\Gamma(i) = \{j \in I : C_i \cap C_j \neq \emptyset\}$ and the admissible subspace $U_i := M \setminus \bigcup_{j\notin\Gamma(i)} C_j$ for each $i \in I$. By Fact 3.8.1, U_i is open. Moreover, since $U_i \subset \bigcup_{j\in\Gamma(i)} C_j$, U_i is actually a definable subspace of M. If $j \notin \Gamma(i)$ then $U_i \cap C_j = \emptyset$. Hence, since $\{C_i\}_{i\in I}$ is locally finite and $C_i \subset U_i$ for all $i \in I$, we have that $\{U_i\}_{i\in I}$ is a locally finite cover of M by open definable subspaces.
(iii) Firstly, note that since C_i is closed, the closure respect to C_i equals the one respect to M. Let $x \in M$ and let B be a closed admissible subspace of M with $x \notin B$. Since $\{C_i : i \in I\}$ is locally finite, the subset of indexes $I(x) := \{i \in I : x \in C_i\}$ is finite. Moreover, by Lemma 3.8.1, the set $A_x := \bigcup_{i\in I\setminus I(x)} C_i$ is a closed admissible subset of M. Replacing B by $B \cup A_x$ if necessary, we can assume that $A_x \subset B$. For each $i \in I(x)$, since C_i is regular, there is an open definable subspace U_i of C_i such that $x \in U_i$ and $\overline{U_i} \cap B = \emptyset$. Since U_i is an open definable subspace of C_i, there is an open definable subspace V_i of M such that $C_i \cap V_i = U_i$ for each $i \in I(x)$. Furthermore, we can assume that $V_i \subset M \setminus B$, otherwise it is enough to consider the intersection of V_i and the open admissible subspace

$M \setminus B$. Now, consider the open definable subspace $V := \bigcap_{i\in I(x)} V_i$ of M and the closed admissible subspace $Z := \bigcup_{i\in I(x)} \overline{U_i}$ of M. Clearly, $x \in V$ and $B \subset M \setminus Z$. On the other hand, since $A_x \subset B$ and $V \subset M \setminus B$, we have $V \cap A_x = \emptyset$ and hence

$$V = V \cap M = \bigcup_{i\in I} (V \cap C_i) = \bigcup_{i\in I(x)} (V \cap C_i) \subset \bigcup_{i\in I(x)} (V_i \cap C_i) = \bigcup_{i\in I(x)} U_i \subset Z.$$

Therefore, x and B are separated by V and the open admissible subspace $M \setminus Z$. □

Fact 3.8.3. [13, Lem.I.2.2] *Let M be a set and let $\{M_i : i \in I\}$ be a directed system of subsets of M such that $M = \bigcup_{i\in I} M_i$. Suppose that each M_i is a regular definable space and that for all $i, j \in I$, $i < j$, M_i is an open definable subspace of M_j whose definable space structure is equivalent to the one inherited from M_j. Then there is an ld-space structure in M such that M_i is an open definable subspace of M for all $i \in I$.*

Proof. Let $\psi_i : M_i \to E_i$ be a chart of M_i for each $i \in I$. We show that $\{(\psi_i, E_i)\}_{i\in I}$ is an atlas of M. It suffices to check that both $\psi_i(M_i \cap M_j)$ and $\psi_j(M_i \cap M_j)$ are open definable subsets of E_i and E_j respectively and the map $\psi_i \circ \psi_j^{-1}|_{\psi_j(M_i\cap M_j)}$ is definable for all $i, j \in I$. Let $k \in I$ such that $i, j < k$. By hypothesis, $\psi_k(M_i)$ and $\psi_k(M_j)$ are open definable subsets of E_k and $\psi_k \circ \psi_i^{-1}$ and $\psi_k \circ \psi_j^{-1}$ as well as their inverses are definable maps. Hence, $\psi_i(M_i \cap M_j) = (\psi_i \circ \psi_k^{-1})(\psi_k(M_i) \cap \psi_k(M_j))$ and $\psi_j(M_i \cap M_j) = (\psi_j \circ \psi_k^{-1})(\psi_k(M_i) \cap \psi_k(M_j))$ are open definable subsets of E_i and E_j respectively. Finally, $\psi_j \circ \psi_i^{-1}|_{\psi_i(M_i\cap M_j)} = (\psi_k \circ \psi_j^{-1})^{-1} \circ (\psi_k \circ \psi_i^{-1})|_{\psi_i(M_i\cap M_j)}$ and $\psi_i \circ \psi_j^{-1}|_{\psi_j(M_i\cap M_j)} = (\psi_k \circ \psi_i^{-1})^{-1} \circ (\psi_k \circ \psi_j^{-1})|_{\psi_j(M_i\cap M_j)}$ are definable maps. □

Fact 3.8.4. [13, Prop. I.3.16] *Let M be an ld-space and $\{C_j : j \in J\}$ be an admissible covering of M by closed definable subspaces. Let N be an ld-space and $f : M \to N$ be a map (not necessarily continuous) such that $f|_{C_j}$ is an ld-map for each $j \in J$. Then, f is an ld-map.*

Proof. Let $\{(M_i, \phi_i)\}_{i\in I}$ be an atlas of M. We have to prove that the conditions of Definition 3.2.4 are satisfied. Firstly, note that since the covering $\{C_j\}_{j\in J}$ is admissible, for each $i \in I$ there is a finite subset $J_i \subset J$ such that $M_i \subset \bigcup_{j\in J_i} C_j$. Therefore, since $f|_{M_i\cap C_j}$ is continuous and $M_i \cap C_j$ is a closed subset of M_i for all $j \in J_i$, $f|_{M_i}$ is also continuous for every $i \in I$. Now, to prove that $f(M_i)$ is a definable subspace of N for each $i \in I$, note that, since each $f|_{C_j}$ is an ld-map and C_j is a definable subspace of M, $f(M_i \cap C_j)$ is a definable subspace of N for all $i \in I$ and $j \in J$. Hence, $N_i := f(M_i) = \bigcup_{j\in J_i} f(M_i \cap C_j)$ is a definable subspace of N for each $i \in I$. Finally, the map $f|_{M_i} : M_i \to N_i$ is definable since $f|_{M_i\cap C_j} : M_i \cap C_j \to N_i$ is definable for all $j \in J_i$. □

Fact 3.8.5. [13, Thm.II.1.3] *Let M be a set and $\{M_i\}_{i\in I}$ a family of subsets of M. Assume that for each $i \in I$, M_i has an affine definable space structure satisfying that*
(i) *$M_i \cap M_j$ is a closed definable subspace of both M_i and M_j for every $i, j \in I$ and the structure that $M_i \cap M_j$ inherits from both M_i and M_j are equivalent and,*
(ii) *the family $\{M_i \cap M_j\}_{j\in I}$ is finite for every $i \in I$.*
Then, there is a (unique) LD-space structure in M such that
(a) *M_i is a closed definable subspace of M for every $i \in I$;*
(b) *the structure that M_i inherits from M is equivalent to its affine structure and,*
(c) *the family $\{M_i : i \in I\}$ is locally finite.*

Proof. Uniqueness, paracompactness and regularity follow clearly from Fact 3.8.2. We divide the proof in two cases.
(1) I is finite: It suffices to prove the case $I = \{1, 2\}$. Denote by $\psi_i : M_i \to E_i \subset R^n$ the chart of M_i for $i = 1, 2$. Let $A = M_1 \cap M_2$. By Tietze extension theorem (see [15, Ch.8,Cor.3.10]) there is a definable map $\chi_1 : M_1 \to R^n$ such that $\chi_1|_A = \psi_2$. Similarly, there is a definable map $\chi_2 : M_2 \to R^n$ such that $\chi_2|_A = \psi_1$. Consider the map $\phi_1 : M \to R^n$ such that $\phi_1|_{M_1} = \psi_1$ and $\phi_1|_{M_2} = \chi_2$. Consider also the map $\phi_2 : M \to R^n$ such that $\phi_2|_{M_2} = \psi_2$ and $\phi_2|_{M_1} = \chi_1$. On the other hand, by [15, Ch.6, Lemma 3.8], there are definable functions $h_1 : M_1 \to [-1, 0]$ and $h_2 : M_2 \to [0, 1]$ such that $h_1^{-1}(0) = h_2^{-1}(0) = A$. Consider the function $h : M \to [-1, 1]$ such that $h|_{M_1} = h_1$ and $h|_{M_2} = h_2$. Note that $h^{-1}(0) = A$, $h^{-1}([-1, 0]) = M_1$ and $h^{-1}([0, 1]) = M_2$. Finally, consider the map $f : M \to R^n \times R^n \times R : x \mapsto (\phi_1(x), \phi_2(x), h(x))$. Note that the function f is injective and the map $f \circ \psi_i^{-1} : E_i \to R^{2n+1}$ is definable for $i = 1, 2$. Hence $N_1 := f(M_1)$ and $N_2 := f(M_2)$ are definable. In particular, $E := f(M) = N_1 \cup N_2$ is definable. Using the bijection $f : M \to E$, we define a structure of affine definable space in M. Next, we check that properties (a) and (b) hold. Firstly, note that $N_1 = \{(x_1, \dots, x_{2n+1}) \in E : x_{2n+1} \leq 0\}$ and $N_2 = \{(x_1, \dots, x_{2n+1}) \in E : x_{2n+1} \geq 0\}$ and therefore N_1 and N_2 are closed subsets of E. Finally, $f|_{M_i} : M_i \to N_i$ is an embedding for $i = 1, 2$. Indeed, it suffices to observe that $f^{-1}|_{N_1} = \psi_1^{-1} \circ pr$ is definable, where pr denotes the projection over the first n coordinates. Similarly, we prove that $f^{-1}|_{N_2}$ is also definable.
(2) General case: We fix $i \in I$. By (ii), the set of indexes $\Gamma(i) := \{j \in I : M_i \cap M_j \neq \emptyset\}$ is finite. Consider the subset $U_i := M \setminus \bigcup_{j\notin\Gamma(i)} M_j$ of M. Note that U_i is the union of the finite family $\{U_i \cap M_k : k \in \Gamma(i)\}$. It follows from (i) that $U_i \cap M_k = \bigcup_{j\in\Gamma(k)\setminus\Gamma(i)} M_k \setminus M_j$ is an open definable subset of M_k for all $k \in \Gamma(i)$. We equip $U_i \cap M_k$ with the definable space structure inherited from M_k, for each $k \in \Gamma(i)$. Then by (1), U_i has a definable space structure such that $U_i \cap M_k$ is a closed definable subspace of U_i and the definable

space structure that $U_i \cap M_k$ inherits from both U_i and M_k are equivalent, for all $k \in \Gamma(i)$. Now, let L be a finite subset of indexes of I. We define $U_L := \bigcup_{i\in L} U_i$. Note that $U_L \cap M_k = \bigcup_{i\in\Gamma(k)\cap L}(U_i \cap M_k)$ is an open definable subspace of M_k for all $k \in \bigcup_{i\in L}\Gamma(i)$. We equip $U_L \cap M_k$ with the definable space structure inherited from M_k for all $k \in \bigcup_{i\in L}\Gamma(i)$. Then, by (1), U_L has a definable space structure such that $U_L \cap M_k$ is a closed definable subspace of U_L and the definable space structure that $U_L \cap M_k$ inherits from both U_L and M_k are equivalent for all $k \in \bigcup_{i\in L}\Gamma(i)$. Given two finite subsets of indexes L_1 and L_2 of I with $L_1 \subset L_2$, we have that U_{L_1} is an open definable subspace of U_{L_2}. For, by Lemma 3.8.2, it is enough to prove that $U_{L_1} \cap U_{L_2} \cap M_k$ is an open definable subspace of $U_{L_2} \cap M_k$ for every $k \in \bigcup_{i\in L_2}\Gamma(i)$. Moreover, since $U_{L_1} \subset U_{L_2}$ and $U_{L_1} \cap M_k = \emptyset$ for all $k \notin \bigcup_{i\in L_1}\Gamma(i)$, it suffices to check that $U_{L_1} \cap M_k$ is an open definable subspace of $U_{L_2} \cap M_k$ for all $k \in \bigcup_{i\in L_1}\Gamma(i)$. Indeed, for all $k \in \bigcup_{i\in L_1}\Gamma(i)$, we have showed above that $U_{L_1} \cap M_k$ and $U_{L_2} \cap M_k$ are open definable subspaces of M_k and hence $U_{L_1} \cap M_k$ is an open definable subspace of $U_{L_2} \cap M_k$, as required. It follows from this that the directed system $\{U_L : L \text{ finite subset of } I\}$ satisfies the hypotheses of Fact 3.8.3 and hence there is an ld-space structure in M such that each U_L is an open definable subspace of M and the definable space structure of U_L and the one it inherits from M are equivalent. Furthermore, for all $i, j \in I$ if $U_j \cap U_i \neq \emptyset$ then $j \in \bigcup_{k\in\Gamma(i)}\Gamma(k)$, and hence $\{U_i\}_{i\in I}$ is a locally finite cover of M by open definable subspaces. To finish the proof we have to check properties (a),(b) and (c). Note that $\{j \in I : U_j \cap M_i \neq \emptyset\}$ is a subset of $\Gamma(i)$ and therefore is finite for all $i \in I$. Hence, since $M_i \cap U_j$ is a closed definable subspace of U_j for every $i, j \in I$, (a) and (b) hold. To prove (c) is enough to note that the set of indexes $\{i \in I : M_i \cap U_j \neq \emptyset\}$ is also contained in the finite set $\Gamma(j)$ for all $j \in I$ and hence $\{M_i\}_{i\in I}$ is locally finite. □

Now, we prove some well-known results on o-minimal geometry in the locally definable setting also needed in the proof of Fact 3.3.2.

Fact 3.8.6 (**Shrinking of coverings**). [13, Thm.I.4.11] *Let M be a regular ld-space and let $\{U_i\}_{i\in I}$ be a locally finite cover of M by open definable subspaces (so in particular M is paracompact). Then there is a covering $\{V_i\}_{i\in I}$ of M by open definable subspaces such that $\overline{V_i} \subset U_i$ for all $i \in I$.*

Proof. Consider the finite set of indexes $\Gamma_1(i) = \{j \in I : U_j \cap U_i \neq \emptyset\}$ and the open definable subspace $W_i = \bigcup_{j\in\Gamma_1(i)} U_j$ of M for each $i \in I$. Note that $\overline{U_i} \subset W_i$. Consider also the finite set of indexes $\Gamma_2(i) = \{j \in I : U_j \cap W_i \neq \emptyset\}$ and the open definable subspace $\widetilde{W}_i = \bigcup_{k\in\Gamma_2(i)} U_k$ of M. Note that $\overline{W_i} \subset \widetilde{W}_i$ and that $\{\widetilde{W}_i\}_{i\in I}$ is also a locally finite cover of M by open definable subspaces. We denote by $W_{ik} := U_k$ for each $i \in I$ and each $k \in \Gamma_2(i)$. Then $\{W_{ik} : k \in \Gamma_2(i)\}$ is a finite cover of $\widetilde{W}_i$ for all $i \in I$. Since M is regular, each $\widetilde{W}_i$ is affine (see Remark 3.2.8) and hence, by the definable shrinking of coverings (see [15, Ch.6,Lem.3.6]), there is a cover $\{V_{ik} : k \in \Gamma_2(i)\}$ of $\widetilde{W}_i$ by open definable subspaces of $\widetilde{W}_i$ with $V_{ik} \subset \overline{V_{ik}} \cap \widetilde{W}_i \subset W_{ik}$ for all $k \in \Gamma_2(i)$. Now, consider the open definable subspace $V_k := \bigcup_{i\in\Gamma_1(k)} V_{ik} \subset U_k$ for each $k \in I$ (note that if $i \in \Gamma_1(k)$ then $k \in \Gamma_1(i) \subset \Gamma_2(i)$). We check that $M = \bigcup_{k\in I} V_k$. Let $x \in U_i$ for some $i \in I$. Then $x \in \widetilde{W}_i$ and hence $x \in V_{ik}$ for some $k \in \Gamma_2(i)$. Since $x \in U_i \cap V_{ik} \subset U_i \cap U_k$, we have that $i \in \Gamma_1(k)$, so that $x \in V_k$. Finally, we have to show that $\overline{V_k} \subset U_k$ for all $k \in I$. We fix $k \in I$. Clearly, $\overline{U_k} \subset \overline{W_i} \subset \widetilde{W}_i$ and therefore $\overline{V_{ik}} \subset \overline{W_{ik}} = \overline{U_k} \subset \widetilde{W}_i$ for all $i \in \Gamma_1(k)$. Hence,

$$\overline{V_k} = \bigcup_{i\in\Gamma_1(k)} \overline{V_{ik}} = \bigcup_{i\in\Gamma_1(k)} \overline{V_{ik}} \cap \widetilde{W}_i \subset \bigcup_{i\in\Gamma_1(k)} W_{ik} = U_k,$$

as required. □

Fact 3.8.7 (Partition of unity). [13, Thm.I.4.12] *Let M be an LD-space and let $\{U_i\}_{i\in I}$ be a locally finite cover of M by open definable subspaces. Then there is a family $\{g_i : M \to [0,1]\}_{i\in I}$ of ld-maps such that sup(g_i) := $\overline{\{x \in M : g_i(x) \neq 0\}} \subset U_i$ for all $i \in I$ and $\sum_{i\in I} g_i(x) = 1$ for all $x \in M$.*

Proof. By Fact 3.8.6, it suffices to find a family $\{g_i : M \to [0,1]\}$ of ld-maps such that $g_i^{-1}((0,1]) = U_i$ for all $i \in I$ and $\sum_{i\in I} g_i(x) = 1$ for all $x \in M$. Firstly, we construct a definable map $h_i : \overline{U_i} \to [0,1]$ such that $h_i^{-1}(0) = \overline{U_i} \setminus U_i$ for each $i \in I$. By Fact 3.2.11, $\overline{U_i}$ is a definable subspace of M for all $i \in I$. Furthermore, since M is regular, by the o-minimal version of Robson's theorem there is a definable embedding $\phi_i : \overline{U_i} \to R^n$ with $\phi_i(\overline{U_i})$ bounded for each $i \in I$. If $\overline{U_i} \neq U_i$, take the definable map

$$h_i : \overline{U_i} \to [0,1] : x \mapsto \frac{1}{C} \inf\{|\phi_i(x) - \phi_i(y)| : y \in \overline{U_i} \setminus U_i\},$$

where $C = \sup_{x\in\overline{U_i}} \inf\{|\phi_i(x) - \phi_i(y)| : y \in \overline{U_i} \setminus U_i\}$. If $\overline{U_i} = U_i$, take the definable map $h_i : \overline{U_i} \to [0,1] : x \mapsto 1$. By Fact 3.8.4, the map $H_i : M \to [0,1]$ such that $H_i(x) = h_i(x)$ for all $x \in \overline{U_i}$ and $H_i(x) = 0$ for all $x \in M \setminus \overline{U_i}$ is an ld-map. Since $\{U_i\}_{i\in I}$ is locally finite, the map $H(x) = \sum_{i\in I} H_i(x)$ is a well-defined ld-map. Moreover, note that $H(x) \neq 0$ for all $x \in M$. Finally, the ld-maps $g_i : M \to [0,1] : x \mapsto \frac{H_i(x)}{H(x)}$ fulfill the requirements of the fact. □

Fact 3.8.8. *Let M be an LD-space and let A and B be two closed disjoint admissible subsets of M. Then there is an ld-map $g : M \to [0,1]$ such that $g^{-1}(0) = A$ and $g^{-1}(1) = B$.*

Proof. Let $\{M_i\}_{i\in I}$ be a locally finite cover of M by open definable subsets. By 3.8.7, there is an ld-partition of unity $\{h_i : M \to [0,1]\}_{i\in I}$ with respect to $\{M_i\}_{i\in I}$. On the other hand, it follows from the definable version of Fact 3.8.8 that there is a definable map $g_i : M_i \to [0,1]$ for each $i \in I$ such that $g_i^{-1}(0) = M_i \cap A$ and $g_i^{-1}(1) = M_i \cap B$ (see [15, Ch.6,Lem.3.8]). Finally, it is enough to take the ld-map $g := \sum_{i\in I} h_i g_i$. □

We finish this appendix with a variation of locally definable maps (see [13, Ch.I,§5]).

Definition 3.8.9. *Let M and N be ld-spaces. We say that a map $f : M \to N$ is* ***strongly locally definable****, denoted by sld-map, if f is an ld-map and $f^{-1}(X)$ is a definable subspace of M for every definable subspace X of N.*

Fact 3.8.10. [13, Prop.I.5.3] *Let M and N be ld-spaces and let $f : M \to N$ be an sld-map. Then $f(X)$ is an admissible subspace of N for every admissible subspace X of M.*

Proof. It is enough to prove that for every open definable subspace U of N, $f(X) \cap U$ is also a definable subspace of N. Let U be an open definable subspace of N and let $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ be the restriction of f to $f^{-1}(U)$. Since f is an sld-map, $f^{-1}(U)$ is a definable subspace of M and hence $f|_{f^{-1}(U)}$ is a definable map. Moreover, since X is an admissible subspace and $f^{-1}(U)$ is a definable subspace, we have that $X \cap f^{-1}(U)$ is a definable subspace of M. Then $f|_{f^{-1}(U)}(X \cap f^{-1}(U)) = f(X) \cap U$ is a definable subspace of U. In particular, $f(X) \cap U$ is a definable subspace of N, as required. □

Fact 3.8.11. [13, Rmk.I.5.2] *Let M and N be ld-spaces and let $\{N_i\}_{i\in I}$ be a locally finite cover of N by closed definable subspaces. Then a map $f : M \to N$ is an sld-map if and only if* (i) *the family $\{f^{-1}(N_i) : i \in I\}$ is an admissible cover of M by closed definable subspaces and* (ii) *the map $f|_{f^{-1}(N_i)} : f^{-1}(N_i) \to N_i$ is definable for all $i \in I$.*

Proof. The left-to-right implication is clear. We show the other implication. It follows from (i), (ii) and Fact 3.8.4 that f is an ld-map. Let X be a definable subspace of N. Then $X = (X \cap N_{i_1}) \cup \cdots \cup (X \cap N_{i_m})$ for some $i_1, \ldots, i_m \in I$. By (ii), $f^{-1}(X \cap N_{i_l})$ is a definable subspace of $f^{-1}(N_{i_l})$ for each $l = 1, \ldots, m$. Hence $f^{-1}(X \cap N_{i_l})$ is a definable subspace of M for each $l = 1, \ldots, m$, so that $f^{-1}(X) = \bigcup_{l=1}^{m} f^{-1}(X \cap N_{i_l})$ is a definable subspace of M, as required. □

Conclusions

We now make some comments on possible further developments of the work in this thesis.

Firstly, we point out a possible connection between the Normal triangulation theorem and the o-minimal Hauptvermutung. The o-minimal Hauptvermutung over the real field was solved by M. Shiota in [35], but the general one is still open. We recall one of the motivations of the Normal triangulation theorem. Let (K, ϕ) be a triangulation of a definable set S and some definable subsets $S_1, \dots, S_l$ of S. Now, consider new definable subsets $S'_1, \dots, S'_{l'}$ of S. We would like to both preserve the already obtained triangulation and partition the new sets. Thanks to the Normal triangulation theorem we can solve this problem. Furthermore, we note that if the o-minimal Hauptvermung is true then we can also solve it positively in a rather simple way. For, by the Triangulation theorem, there is a triangulation $(L, \psi) \in \Delta(|K|; \phi^{-1}(S'_1), \dots, \phi^{-1}(S'_{l'}), \sigma)_{\sigma \in K}$. Now, since $\psi : |L| \to |K|$ is a definable homeomorphism, by the o-minimal Hauptvermutung there are subdivisions L' and K' of L and K respectively and a simplicial isomorphism $g : |L'| \to |K'|$ such that $\psi \sim g$. Therefore, $(K', \phi \circ \psi \circ g^{-1}) \in \Delta(S; S_1, \dots, S_l, S'_1, \dots, S'_{l'})$ and $\phi \circ \psi \circ g^{-1} \sim \phi \circ g \circ g^{-1} \sim \phi$, as required. On the other hand, using the argument above, we may be able to deduce the Normal triangulation theorem from the o-minimal Hauptvermutung. However, it does not seem possible to obtain property (iii) of normal triangulations (see Definition 1.1.1). Then, the Normal triangulation theorem is not weaker than the o-minimal Hauptvermutung (at least, in an obvious way). Moreover, in the semialgebraic context, it helped us to prove the semialgebraic Hauptvermutung. Hence, *it is natural to ask if the Normal triangulation theorem is equivalent to the o-minimal Hauptvermutung*. Of course, we are interested in the left-to-right implication.

Secondly, one obvious direction for further work is applying the results of this thesis to the study of definable groups. In fact, this study has been already started in [6], where it is proved that

$$\pi_n(G)^{\mathcal{R}} \cong \pi_n(G/G^{00})$$

for all $n \geq 1$ and all definably compact group G. Now, let G be a definably compact group. By the Triangulation theorem and the o-minimal version of Robson's embedding theorem, we can assume that $G = |K|$ for a closed simplicial complex K whose vertices lie in $\overline{\mathbb{Q}}$. Hence, by Corollary 2.4.4 and the result [6], we know that $\pi_n(|K|(\mathbb{R})) \cong \pi_n(G)^{\mathcal{R}} \cong \pi_n(G/G^{00})$. Hence, we might expect $|K|(\mathbb{R})$ being homotopic equivalent to G/G^{00}. Actually, by Whitehead theorem, we only need an adequate map from $|K|(\mathbb{R})$ to G/G^{00}. Note that this would imply that $\text{cat}(G)^{\mathcal{R}} = \text{cat}(G/G^{00})$ for every definably compact group G, extending the results in Section 2.6.

Bibliography

[1] E. Baro, Normal triangulations in o-minimal structures, to appear in J. Symb. Log., 17pp.

[2] E. Baro and M.J. Edmundo, Corrigendum to Locally definable groups in o-minimal structures, J. Algebra 320 (7) (2008), 3079–3080.

[3] E. Baro and M. Otero, Locally definable homotopy, to appear in Ann. Pure Appl. Logic., 31pp.

[4] E. Baro and M. Otero, On o-minimal homotopy groups, to appear in Quart. J. Math., 15pp.

[5] A. Berarducci, O-minimal spectra, infinitesimal subgroups and cohomology , J. Symb. Log. 72 (2007), no. 4, 1177-1193.

[6] A. Berarducci, M. Mamino and M. Otero, Higher homotopy of groups definable in o-minimal structures, to appear in Israel J. Math., 2008.

[7] A. Berarducci and M. Otero, o-minimal fundamental group, homology and manifolds, J. London Math. Soc. (2) 65 (2002), no. 2, 257–270.

[8] A. Berarducci and M. Otero, Transfer methods for o-minimal topology, J. Symb. Log. 68 (2003), 785–794.

[9] A. Berarducci, M. Otero, Y. Peterzil, A. Pillay, A descending chain condition for groups denable in o-minimal structures, Annals of Pure and Applied Logic 134 (2005) 303313.

[10] O. Cornea, G. Lupton, J. Oprea, D. Tanré, *Lusternik-Schnirelmann Category*, American Mathematical Society, Providence, 2003.

[11] M. Coste, Unicité des triangulations semi-algébriques: validité sur un corps réel clos quelconque, et effectivité forte, C. R. Acad. Sci. Paris Sr. I Math., 312 (1991), no. 5, 395–398.

[12] H. Delfs and M. Knebusch, An introduction to locally semialgebraic spaces, Rocky Mountain J. Math. (14) (1984), no. 4, 945–963.

[13] H. Delfs and M. Knebusch, *Locally semialgebraic spaces*, Lecture Notes in Mathematics, 1173, Springer-Verlag, Berlin, 1985.

[14] H. Delfs and M. Knebusch, Separation, retractions and homotopy extension in semialgebraic spaces, Pacific J. Math. 114 (1984), no. 1, 47–71.

[15] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, 248, Cambridge University Press, 1998.

[16] L. van den Dries, A. Macintyre and D. Marker, The elementary theory of restricted analytic fields with exponentation, Ann. Math. 140 (1994),183-205.

[17] M. Edmundo, Locally definable groups in o-minimal structures, J. Algebra 301 (1) (2006), 194-223.

[18] M. Edmundo, G.O. Jones and N.J. Peatfield, Hurewicz theorems for definable groups, Lie groups, and their cohomologies, preprint, 2007.

[19] M. Edmundo and P. Eleftheriou, The universal covering homomorphism in o-minimal expansions of groups, Math. Logic Quart. 53 (6)(2007) 571–582.

[20] M. Edmundo and M. Otero, Definably compact abelian groups, J. Math. Log. 4 (2004), no. 2, 163–180.

[21] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[22] S. Hu, *Homotopy theory*, Pure and Applied Mathematics, Vol. VIII Academic Press, New York-London 1959.

[23] E. Hrushovski, Y. Peterzil and A. Pillay, Groups, measures, and the NIP, J.Amer. Math. Soc., 21 (2008), no.2, 563-596.

[24] J. Johns, An open mapping theorem for o-minimal structures, J. Symbolic Logic, 66 (2001), 1817-1820.

[25] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[26] J.R. Munkres, *Topology: a first course*, second edition, Prentice-Hall, Inc., Englewood Cliffs, NJ, 2000.

[27] M. Otero and Y. Peterzil, G-linear sets and torsion points in definably compact groups, to appear in Arch. Math. Logic.

[28] Y. Peterzil, Pillay's conjecture and its solution-a survey, for the proceedings of the Logic Colloquium, Wroclaw, 2007.

[29] Y. Peterzil, A. Pillay and S. Starchenko, Definably simple groups in o-minimal structures, Trans.Am.Math.Soc. 352 (2000), 4397–4419.

[30] Y. Peterzil, A. Pillay and S. Starchenko, Simple algebraic groups and semialgebraic groups over real closed field, Trans.Am.Math.Soc. 352 (2000), 4421–4450.

[31] Y. Peterzil and S. Starchenko, Definable homomorphisms of abelian groups in o-minimal structures, Ann. Pure Appl. Logic (101) (2000), no. 1, 1–27.

[32] A. Pillay, On groups and fields definable in o-minimal structures, J. Pure Appl. Algebra 53 (1988), 239-255.

[33] A. Pillay, Type-definability, compact Lie groups, and o-minimality, J. Math. Logic 4 (2004), 147-162.

[34] J.P. Rolin, P. Speissegger and A. Wilkie, Quasianalytic Denjoy-Carleman classes and o-minimality, J. Amer. Math. Soc. 16 (2003), no. 4, 751-777.

[35] M. Shiota, *Geometry of Subanalytic and Semialgebraic Sets*, Birkhauser, 1997.

[36] M. Shiota and M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc., 286(1984), no.2, 727-750.

[37] A. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, J. Amer. Math. Soc. 9 (1996), no. 4, 1051-1094.

[38] A. Woerheide, O-minimal homology, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.