

EULER CHARACTERISTIC FOR TEACHERS

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RESUMEN:

En este trabajo se propone la famosa fórmula de Euler para las superficies en el espacio, que consideramos ser un tema interesante tratar en las clase, para destacar cómo las matemáticas, especialmente en la geometría, hay muchas maneras diferentes de investigación de un mismo objeto. A través de algunos ejemplos, en la definición de la característica de Euler, se destaca la necesidad de introducir no una tessalación de superficies, sino una triangulación. También sugerimos que los profesores utilizan la característica de Euler para poner de relieve las diferencias entre sólidos topológicos con y sin agujeros.

Palabras clave: Poliedros, característica de Euler, triangulación, invariantes.

ABSTRACT:

In the present paper we explain the well-known Euler formula for solids, as an interesting argument to be dealt with students, in order to point out that in mathematics, and particularly in geometry, there are many different ways to investigate objects. It is also explained with examples the topological reason for which, in defining the Euler Characteristic of a polyhedral surface in the space, it is necessary to use a triangulation, instead of a general tessellation. Moreover, it is suggested to teacher to use the Euler Characteristic to enlight the topological difference between solids with or without holes.

Keywords: Polyedra, Euler characteristic, invariants, triangulation.

Among mathematic teachers, is quite famous the formula “vertices minus edges plus faces equals two”, valid for all the platonic solids, the so-called Euler Characteristic Formula for solids. It can be a nice activity to propose a fifth grade classroom (age 10-11) to discover it, eventually through a by-hand experiment with sticks and play-dough models, some of these activities can be found in Kennedy and Tipps.

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The famous formula has also inspired a successful novel (see Neuschwander), fashioned as a Tale of King Arthur style, among a series devoted to illustrate to younger students some mathematical results. The novel is addressed to age 6-9, while a deeper lecture devoted to check math materials inside can be performed also at age 9-12. Working on a mathematic activity not exclusively finalized to arithmetic computation is a valuable action, in order to catch a new interest from the students towards the subject, other than to give a more complete presentation of it. We all know in fact that mathematics, and geometry in particular, is not only the art of measuring and computing, but also is the art of comparing and finding similarities as well as differences among objects.

It is beneficial, nevertheless, to hit the core of the mathematical meaning of what may appear simply a fancy of nature. What does really mean the Euler formula for solids?

In what follows we try to clarify the formula starting from the definitions and give the taste of its geometric meaning.

In Euclidean geometry, a plane polygonal figure is signed with the number of its sides, the number of equal internal vertices, the length of the sides, the number of couple of opposite sides... The comparison of two plane polygonal figures at a glance goes firstly through counting its sides: if they are different in number you can say that the two pictures are not “the same” picture, if they are equal, you go on looking further for other data until every data is tested to be equal, or you find something different. In this way a child can say that a square and a rectangular are not the same geometric picture, or that two squares are different because they have different side length. Theorems come in support to shorten the check list of data to be compared, as it happens with equality criteria for triangles.

The Euclidean geometry studies geometric figures up to isometries, that is to say, two figures are “equal”, meaning they are “the same” figure, if there exists a rigid movement that brings one onto another: two squares are the same square if and only if they have the same side, no matter which is the position in the space.

All the data which do not change under isometries are taken in account, number of edges, vertices, measuring lengths, angles, areas... These are called “Euclidean invariants”, and are the descriptors of the objects under observations. It is evident that the more the object is complex, the more the number of data to describe it increases: just think of the number of data related to an irregular prism compared to the data

related to rectangle.

More generally, rightly the frame to keep in mind is the notion of invariants: this elementary principle is the skeleton of every classification method, in geometry as well as in biology, or every science: to put labels on objects to enlighten differences and similarities. In the study of shapes, if we want to focus other properties than the metric properties, we are using a geometric theory, other than the Euclidean geometry, with a definition of the notion of “equality” (technically, of “isomorphism”) different from the isometry, and are checking other list of data. These data, which are the “invariants” under the chosen isomorphisms, may be either numbers, either may be more complex algebraic structures. Likely, one can expect that the amount of invariants increases with the complexity of the structures under examination.

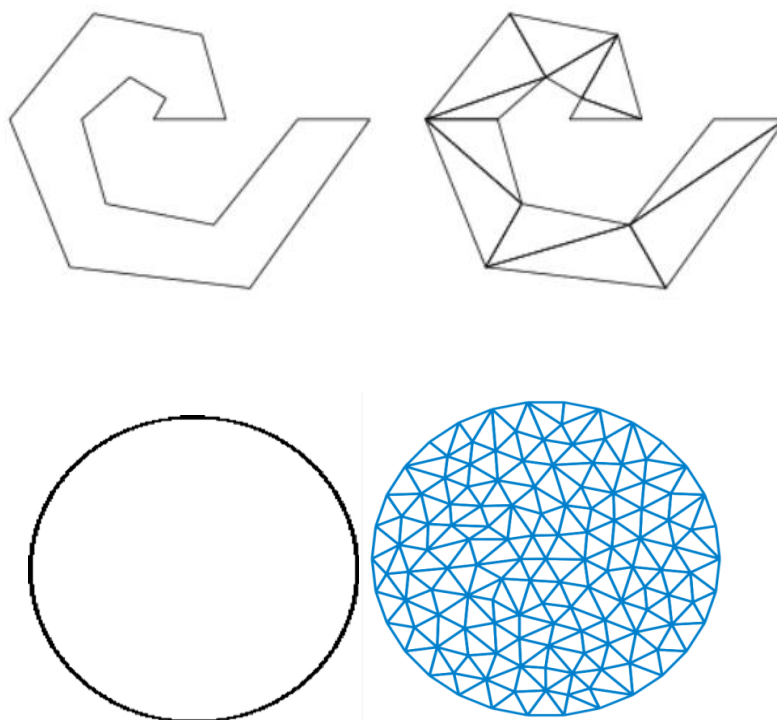
Thus, we have different geometry branches, each of which focuses on a specific kind of properties of the objects that are studied.

For example, the branch of geometry called topology is known as the geometry of the play-dough, intending that it studies properties that are preserved under continuous deformations, including stretching and bending, but not tearing or gluing. The movements allowed in topology are called “bicontinuous transformations”: this means continuous transformations which have a continuous inverse transformation. Measures such as lengths, areas, volumes, distances, are not relevant from a topological viewpoint, since those are not invariant under the action of a bicontinuous transformation. Convexity neither is a topological property, since, with a bicontinuous motion, starting from a circle, one can move a nylon loop on a plane to draw a U shaped region, thus losing the convexity property. Topology does care of connectivity properties, such as number of pieces of a figure, and of continuity, so that a mug is equal to a doughnut (“torus”), as well as a cube is equal to a ball or a rugby ball. Of course, topology has its basic definitions under which the above intuitive examples are rigorously expressed, but they are too technical to be given to elementary level.

The Euler characteristic of a geometric figure is nothing else than a basic topological invariant, that is, a label which remains unaltered under bicontinuous transformations, as well as the number of sides of a polygonal figure remains unaltered under isometries, in Euclidean geometry.

A general overview historical as well as mathematical can be found in the classical book of Courant and Robbins, and in Hartshorne the polyedra and their properties are treated starting from a Euclidean point of view, unlighted by modern geometry. A wide topological viewpoint, with a historical part dedicated specifically to Euler formula, and many illustrations and examples can be found in Richenson.

From a didactic viewpoint, it is possible to attack some elementary topological study using only intuition and without entering in technical definitions, nevertheless tasting the flavor of the power of the discipline. But some care is needed. To define the Euler characteristic we need to know what a triangulation is. For sake of simplicity, one can concentrate only on surfaces, but the notion can be generalized in any dimension. A triangulation is the division of a surface region into a set of triangles, such that each triangle side is entirely shared by two adjacent triangles, which in particular have two vertices in common. It is a theorem that every surface has a triangulation, but it might require an infinite number of triangles.



A surface with a finite number of triangles in its triangulation is called compact.

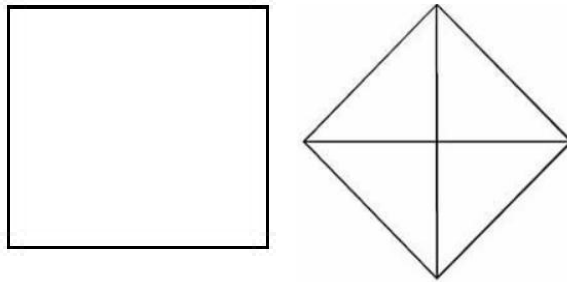
In higher dimension the triangles are substituted by generalized triangles, called simplexes.

Of course, for a given surface, there are many ways in drawing a triangulation, but it has proved (originally Euler did, but only for simple polyhedral surfaces) the following interesting result.

Theorem. The alternate sum

$$\chi = v - e + f$$

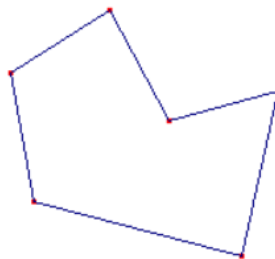
where v , e , and f are respectively the numbers of vertices, edges and faces, do not depend on the triangulation, moreover, it is a topological invariant of the surface.

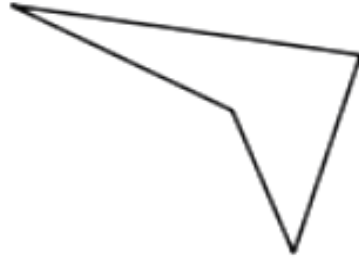


Note that drawing a segment joining two not consecutive vertices add a face to the picture, and an edge, but do not add any vertex, so the sum

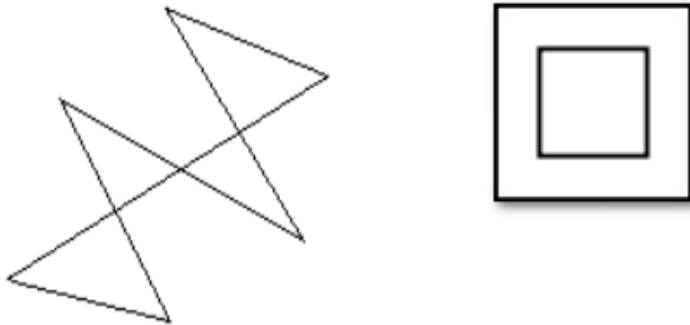
$$\chi = v - e + f$$

remains unchanged. Thus, for a plane polygon the sum χ coincide with the number of vertices of the polygon itself minus the number of its edges plus 1. The computation does not change changing the shape of the polygon:



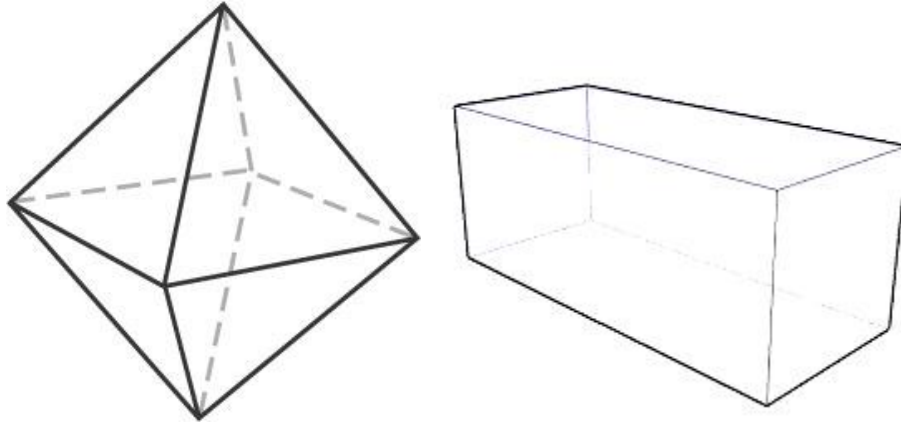


Here we point out that a polygon is simple by definition, that is, its boundary does not intersect itself. In this case the result for χ always turns out to be equal to 1. On the other hand, in the case of a complex polygon the number χ is not constantly equal to one, just check some examples as in the following pictures:

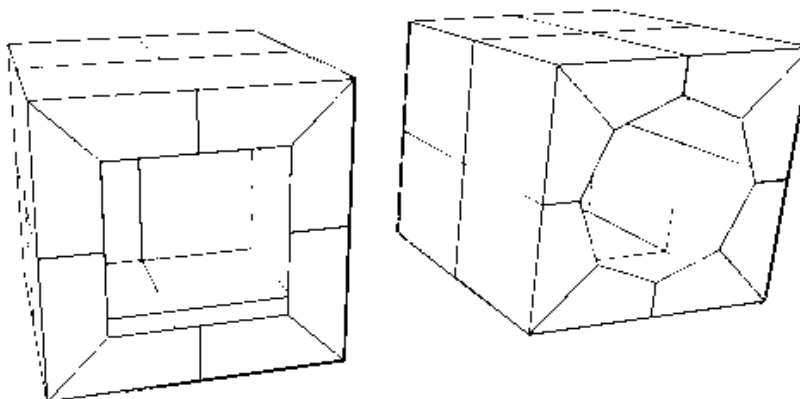


Note that only taking in account similar pictures one can experiment the necessity of introducing a triangulation and the value of the Euler characteristic as a topological invariant. In fact, in the picture of two concentric squares, considering the region between the two, one can test for instance that tessellations with quadrangular may give ambiguous values for χ . Moreover, χ in such pictures is different from 1, which is the value for polygons, and this is an evidence that the above pictures are not topologically equivalent to any polygon.

The same experience can be performed if one considers polyhedral surfaces: polyhedral figures have the didactic advantage of being more handy with respect to plane figures.



School experiments are proposed by teachers in order to discover that the alternate sum $\chi = v - e + f$ of vertices, edges and faces of a platonic solid surface always gives the same magic number 2, whatever of the five solids is chosen. As in the plane case, tessellation by polygons is sufficient to compute the number χ , but this works just because only simple figures are considered. A further step is needed for a complete comprehension, which could be performed proposing to the classroom to evaluate the number χ for not simple solids, as a cube with a polyhedral hole in the middle, using different triangular tessellation, and comparing the result with the computation of vertices, edges and faces of the figure with no tessellation.



Students will thus discover that these polyhedral surfaces, which are topologically different from a platonic polyhedral surface, needing to be glued to fit the hole, are labeled with a different Euler characteristic. The work can be further deepened in studying polyhedral surfaces obtained from prisms with more polyhedral holes. The last experiments lead the students to a deeper knowledge of topological properties of geometric figures versus metric properties, and of a correct way to classify them.

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