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How to properly decompose economic efficiency using technical and allocative criteria with non-homothetic DEA technologies

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Abstract

We discuss how to properly decompose economic efficiency when the underlying technology is non-homothetic using alternative allocative and technical efficiency criteria. We first show that only under the production of one output and assuming the particular case of constant returns to scale homotheticity, we may claim that the standard radial models correctly measure pure technical efficiency. Otherwise, when non-homotheticity is assumed, we then show that these traditional estimations would measure an undetermined mix of technical and allocative efficiency. To restore a consistent measure of technical efficiency in the non-homothetic case we introduce a new methodology that takes as reference for the economic efficiency decomposition the preservation of the allocative efficiency of firms producing in the interior of the technology. This builds upon the so-called reversed approach recently introduced by Bogetoft et al. (2006) that allows estimating allocative efficiency without presuming that technical efficiency has been already accomplished. We illustrate our methodology within the Data Envelopment Analysis framework adopting the most simple non-homothetic BCC model and a numerical example. We show that there are significant differences in the allocative and technical efficiency scores depending on the approach.

Keywords: Data Envelopment Analysis, Overall efficiency, Technical efficiency, Allocative efficiency, Homotheticity.

JEL Classification: C61, D21, D24.

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1. Introduction

Economic (overall) efficiency measurement based on the approach initiated by Farrell (1957) has received great attention from academics and practitioners. Since Farrell, economic efficiency originates from two different sources, viz. technical efficiency and allocative efficiency. In the spirit of his renowned decomposition, technical efficiency is estimated in first place as some measure of the gains obtained from moving the evaluated firm to the frontier of the production possibility set. The main argument behind this approach is that the measurement of allocative efficiency presumes technical efficiency since only on the production isoquant the rate of substitution between production inputs is well-defined and comparable with the ratio of market prices. Therefore, under the Farrell's approach, the analysis focuses on the isoquant corresponding to the observed output before estimating allocative efficiency. Specifically, Farrell (1957) resorted to radial movements in order to measure technical efficiency, relating this particular component to both the coefficient of resource utilization of Debreu (1951) and the inverse of the Shephard's distance function (Shephard, 1953). Indeed, and thanks to duality results (Shephard, 1953), allocative efficiency can be derived as a residual between the overall economic efficiency and its technical efficiency component As a result of this residual nature of the allocative efficiency term, where its technical efficiency counterpart is the driving component, the former has received much less attention in the literature. While there are many ways to define and calculate technical efficiency (oriented and non-oriented models, radial, additive, directional-based measures, etc.), the allocative efficiency problem of the firm in relation to the overall economic efficiency has been neglected.

However, this is changing nowadays. In contrast to Farrell's approach, Bogetoft et al. (2006) introduced a new method for estimating the potential gains from improving allocative efficiency without presuming that technical efficiency has already been accomplished. In particular, they propose to use a 'reversed' Farrell approach, first correcting for allocative efficiency and next for technical efficiency and, consequently, changing the traditional order to decompose overall efficiency. The rationale is that when a firm is inefficient, both the input and output orientations are feasible choices to gain efficiency, and allocative efficiency can be evaluated in alternative input or output isoquants. Following this thread, we show that approaching the problem of decomposing overall economic efficiency dealing with allocative efficiency in the first place sheds new light on the analysis of the economic behavior of inefficient firms, which must be taken into account by researchers.

Particularly, the fact that firms may solve technical inefficiencies by either reducing inputs or expanding outputs has relevant implications in empirical research, as these two alternative dimensions normally used by researchers when measuring technical efficiency—output or input—pass on to the concept of allocative efficiency. This is indeed central in our analysis since an inefficient firm situating inside the technology may have used inputs in excess for the observed level of output (input perspective), or may have fallen short of potential output given its observed level of inputs (output perspective). This theoretical or conceptual ambivalence that the applied researcher faces when choosing a particular orientation has an immediate implication in a cost minimizing analytical framework because a firm, when demanding its optimal input quantities, may take as reference its actually observed output level that the firm has not been capable of producing efficiently by incurring in input excesses (an input perspective), or the intended —and unknown—potential output level (an output perspective).

In this respect, the analytical implications of the choice of the output benchmark are clear. Given the observed market prices for inputs, the first order conditions for cost minimization subject to a given output level determine whether the firm is allocative efficient or not; particularly, if the marginal rates of technical substitution are equal to the price ratios. As a result, a firm will demand different input mixes depending on its *ex-ante* planned output level, which may not be realized latter on resulting in technical inefficiency. Assuming perfect competition in the input markets results in price taking firms, and therefore alternative input mixes imply different allocative efficiency levels. As a result both technical and allocative efficiency will differ depending on the chosen orientation when assessing overall cost efficiency.

Relevant for this discussion, Bogetoft et al. (2006) prove that if the technology is homothetic then both decompositions based on the standard and reversed Farrell approaches are equivalent. Therefore, researchers do not have to worry about whether the subjective analytical choice of orientation yields alternative decompositions of overall economic efficiency, as they are the same. This is because from an economic theory perspective, one remarkable result of homotheticity is that least cost expansion paths are vectors passing through the origin and, therefore, this property preserves marginal rates of substitution or transformation as one moves along rays from the origin, and, consequently, as it is well-known in the standard Farrell approach, radial measures preserve the value of allocative efficiency along the contracting paths given by the input mix. Since market prices are exogenous, allocative efficiency remains constant along radial projections of technically inefficient firms. As marginal rates of substitution do not change, whatever the difference between the ratios of market prices and marginal rates might be (when they are equal the firm is allocative efficient), it does not change regardless of the input or output isoquant that is considered to evaluate allocative efficiency—formally the marginal rates of substitution are independent of the output levels. Moreover, it is this normally overlooked property of homothetic technologies what guarantees that the radial movements associated to the traditional input and output measures can be rightly interpreted as pure technical efficiency gains, since allocative efficiency remains unchanged, resulting in a consistent decomposition of overall economic efficiency. In this framework, and not surprisingly, Chamber and Mitchell (2001) established the advantages of assuming homotheticity as the most common functional restriction used in economics. Specifically, the level sets for a homothetic function are radial expansions ("blow ups") of a reference level set.

One interesting byproduct of the reversed Farrell decomposition proposed by Bogetoft et al. (2006) is that it opens the way to determine allocative efficiency without first projecting the evaluated firm on the isoquant corresponding to the observed level of output. In this respect, a point that has received little attention in the production economics literature and that stems from the above discussion is that if one is interested in measuring the technical efficiency corresponding to a firm producing in the interior of the production possibility set through movements to the frontier, then it is necessary to assure that the allocative efficiency does not change along this process—as in the standard Farrell approach for homothetic technologies. In other words, if we determine the 'starting' allocative efficiency of the assessed firm before projecting it on the frontier of the technology, applying the reversed approach, this value should coincide with the estimation of the allocative efficiency at the projected point after moving the original production plan of the firm to the corresponding isoquant. Only in this way we could be sure that the gains in moving from the original to the projected plan are waste due to exclusively technical reasons. In a homothetic setting researches do not have to worry about how to measure the residual allocative efficiency, either by the standard or reversed approaches since both methodologies coincide, but this would not be the case for non-homothetic technologies. Keeping in mind that true technologies will not generally follow the stylized assumptions underlying theoretical analyses, we believe that to define, interpret and correctly measure technical efficiency, it is necessary to keep constant the allocative efficiency so as to rightly and unambiguously decompose overall efficiency.

As a result of these reflections, in this paper we maintain that the interpretation of the scores in the well-known radial Data Envelopment Analysis models (the CCR by

Charnes et al., 1978 and the BBC by Banker et al., 1984) as technical efficiency is unclear unless we can assume that the underlying technology is homothetic, a scenario that is verified only for the production of one output when the technology exhibits constant returns to scale (CRS). This implies that unless researchers are certain of the mistakes made by the managers of the firm resulting in input excesses or output deficits (and note that individual firms in the evaluated sample could differ in their production errors), the decomposition of overall economic efficiency may be erroneous. Additionally, we propose a simple solution for properly measuring technical efficiency and decomposing overall efficiency when the technology is non-homothetic, which should result in an improvement of the strategies prescribed to managers when adopting both technical and economic decisions aimed at improving their efficiency.

The paper is organized as follows. In Section 2, we briefly recall the standard and reversed Farrell approaches and, in addition, we study under which technological assumptions the radial models actually measure technical efficiency in Data Envelopments Analysis, DEA. Section 3 is devoted to introduce the correct decomposition of economic efficiency into its technical and allocative components when the technology is not homothetic. In particular, we show that any DEA measure of economic performance would not only convey technical shortcomings, but also would be related to allocative criteria and, consequently, we suggest a method to overcome this problem. In Section 4, we illustrate the new methodology using a numerical example. Section 5 concludes.

2. The standard and reversed cost efficiency decompositions for homothetic and non-homothetic technologies

In this section, we first formalize some key notions about the technology and recall how overall economic efficiency has been traditionally decomposed. In a second stage we show the main characteristics of the approach introduced by Bogetoft et al. (2006) and, finally, we prove that the Shephard's distance function, related to the inverse of the radial models in DEA, properly measure technical efficiency if the technology satisfies the particular homotheticity property.

Let us consider *n* firms (or decision making units, DMUs) to be evaluated, which consume *m* inputs to produce *s* outputs. Firm *j* consumes $X_j = \left(x_{1j},...,x_{mj}\right) \in R_+^m$ amounts of inputs to produce the following amounts of outputs: $Y_j = \left(y_{1j},...,y_{sj}\right) \in R_+^s$. As usual, the relative efficiency of each firm in the sample is assessed with reference to the so-called production possibility set, which can be empirically constructed using

DEA from the observations by assuming several postulates. In particular, if we impose constant returns to scale on the technology including the postulate of 'ray-unboundedness', Banker et al. (1984) proved that the production possibility set can be characterized as follows:

$$T_{CRS} = \left\{ \begin{array}{l} (X,Y) \in R_+^{m+s} : \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, i = 1,...,m, \\ \sum_{j=1}^{s} \lambda_j y_{rj} \ge y_r, r = 1,...,s, \lambda_j \ge 0, j = 1,...,n \end{array} \right\}. \tag{1}$$

On the other hand, if we assume variable returns to scale, then the corresponding characterization of the production possibility set, denoted by $T_{\text{VRS},}$ is the same as (1) but adding the additional constraint $\sum_{j=1}^{n} \lambda_j = 1$. Hereafter, we will use the corresponding subscripts when needed.

As in the introduction, and seeking simplicity, we state most of our discussions in the input space. To do so, we need to introduce the input requirement set $L(Y)$ defined as the set of inputs that can produce output Y. Formally, $L(Y) = \{X \in R_+^m : (X,Y) \in T\}$. On the other hand, in order to measure technical efficiency it is necessary to isolate certain subset of $L(Y)$ that serves as benchmark for the evaluation of efficiency. We are referring to the isoquant of $L(Y)$: $IsoqL(Y) = \{X \in L(Y) : \varepsilon < 1 \Rightarrow \varepsilon X \notin L(Y)\}$.

Since we are concerned with overall efficiency in the input space, and following standard economic theory, we assume that firms minimize production costs while facing exogenously determined input prices. This implies that if firms succeed in choosing the inputs combination (bundle) resulting in the minimum cost of producing a given output level at the existing market prices, they are allocative efficient. Let us denote by $C(Y,W)$ the minimum cost of producing the output level Y given the input price vector $W = (w_1,...,w_m) \in R_{++}^m$: $C(Y,W) = \min\left\{ \sum_{i=1}^{m} w_i x_i : X \in L(Y) \right\}$.

2.1. The standard and reversed approaches based on the radial input distance function

The standard Farrell approach (Farrell, 1957) views the overall (cost) efficiency as originating from technical efficiency and allocative efficiency. Specifically, Farrell quantified, and therefore defined each of these terms as follows. Technical efficiency corresponds to the largest feasible equiproportional contraction of the observed input vector X_0, capable of producing the observed output vector Y_0, by moving from X_0 to $\theta_0^* X_0 \in IsoqL(Y_0)$, where θ_0^* is determined as the optimal value of the following linear program:

$$\begin{aligned} \min \quad & \theta_0 \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_0 x_{i0}, \quad i = 1,\dots,m, \\ & \sum_{j=1}^{s} \lambda_j y_{rj} \ge y_{r0}, \qquad r = 1,\dots,s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \ge 0, \qquad\qquad j = 1,\dots,n. \end{aligned} \tag{2}$$

This program is known in the DEA literature as the BCC model (Banker et al., 1984) and it is closely related to the Shephard's input distance function (Shephard, 1953), which defines as $D_i(Y,X) = \sup\{\delta > 0 : X/\delta \in L(Y)\}$, thereby coinciding with the inverse of θ_0^*.

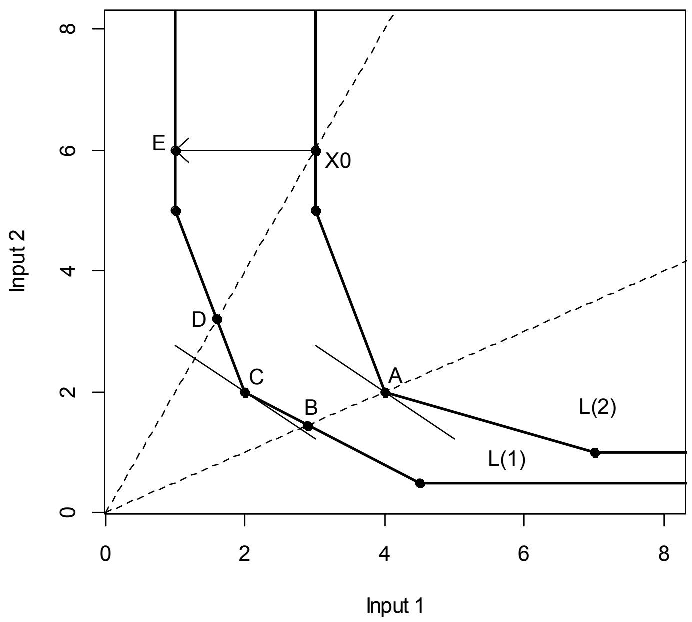
Regarding the allocative efficiency component, following Farrell's tradition, it corresponds to the adjustment of the projected input vector to the minimum cost input combination; i.e., from $\theta_0^* X_0$ to $X^*(Y_0,W)$, where $X^*(Y_0,W)$ = $\arg\min\left\{\sum_{i=1}^{m} w_i x_i : X \in L(Y_0)\right\}$. As for the decomposition of cost efficiency, the following well-known inequality holds:

$$\underbrace{\frac{C(Y_0,W)}{\sum_{i=1}^{m} w_i x_{i0}}}_{\substack{\text{Cost} \\ \text{Efficiency } (CE)}} \le \underbrace{\theta_0^*}_{\substack{\text{Technical} \\ \text{Efficiency } (TE)}} \Leftrightarrow \underbrace{\frac{C(Y_0,W)}{\sum_{i=1}^{m} w_i x_{i0}}}_{CE} \le \underbrace{\frac{1}{D_i(Y_0,X_0)}}_{TE}. \tag{3}$$

Finally, the residual allocative efficiency (AE) component is derived from (3) rendering it an equality, i.e., *AE=CE/TE*.

We now illustrate the standard decomposition through Figure 1 and a set of seven firms that consume two inputs producing a single output (x_1,x_2,y): (3,6,1), (2,2,1), (1,5,1), (4.5,0.5,1), (3,5,2), (4,2,2) and (7,1,2). Additionally, we consider $w_1 = 1$ and $w_2 = 1.3$. In Figure 1, variable returns to scale have been assumed in order to estimate the piecewise linear input requirement sets and their corresponding isoquants. For the particular firm (X_0, Y_0) = (3,6,1), Figure 1 shows that it is producing in the interior of the technology represented by the input requirement set $L(1)$. Resorting to the standard equiproportional projection for solving technical efficiency, this firm should reduce input quantities matching those used by D on the $IsoqL(1)$. Afterwards, the firm should correct for allocative efficiency by changing its input bundle from D to C, the production plan where cost is minimized for $IsoqL(1)$.

Figure 1. The traditional and the reversed decompositions of cost efficiency with non-homothetic DEA technologies.



Complementing the classic approach, Bogetoft et al. (2006) recently introduced an alternative decomposition to *CE* in (3). In their method, allocative efficiency is corrected in first place and technical efficiency is calculated in a second stage. In order to undertake the new approach it is necessary to consider a reference output vector Y^* such that $X_0 \in IsoqL(Y^*)$. Following this approach, they define the 'reversed' allocative efficiency AE^R and the 'reversed' technical efficiency TE^R as:

$$AE^R = \frac{C\left(Y^*, W\right)}{\sum_{i=1}^{m} w_i x_{i0}}, \text{ and } TE^R = \frac{1}{D_i\left(Y_0, X^*\left(Y^*, W\right)\right)}. \tag{4}$$

In this way, they first correct for allocative efficiency by changing the input bundle form X_0 to $X^*\left(Y^*,W\right) = \arg\min\left\{\sum_{i=1}^{m} w_i x_i : X \in L\left(Y^*\right)\right\}$ on the $IsoqL\left(Y^*\right)$, and, later remove technical inefficiency by reducing input quantities from $X^*\left(Y^*,W\right)$ to $\gamma X^*\left(Y^*,W\right) \in IsoqL\left(Y_0\right)$, where $\gamma = \frac{1}{D_i\left(Y_0, X^*\left(Y^*,W\right)\right)}$.

The reversed decomposition is also shown in Figure 1. In the example, $Y^* = 2$ since X_0 = (3,6) belongs to $IsoqL\left(2\right)$. Therefore, X_0 reduces cost by adopting A's production plan, and mirroring the input mix that minimizes the cost of producing $Y^* = 2$. Afterwards, technical inefficiency should be corrected by moving from A to B, where B is the efficient projection of A on $IsoqL\left(1\right)$ obtained by way of an equiproportional—radial—contraction of inputs.

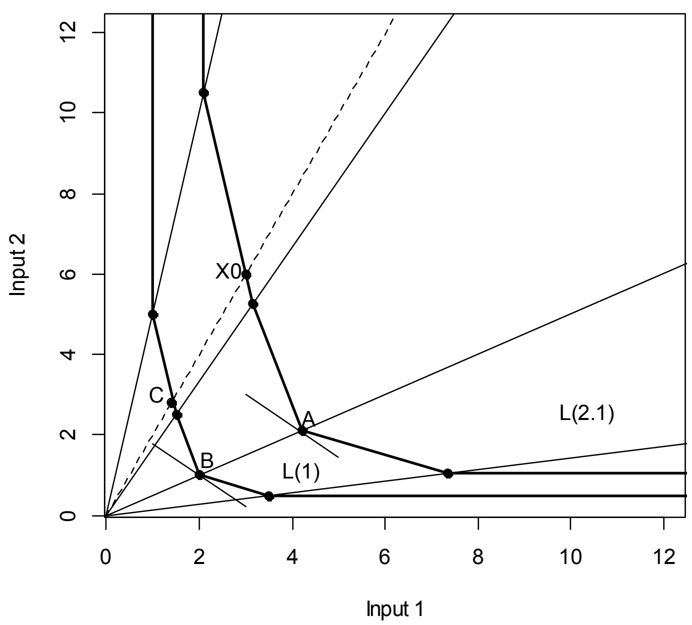
Bogetoft et al. (2006) proved that the reversed decomposition coincides with the standard Farrell approach if and only if the technology is input homothetic, a property that geometrically establishes that the input requirement sets for different output vectors are "parallel"; i.e. $L\left(Y\right) = H\left(Y\right) \cdot L\left(1_s\right)$, where $H\left(Y\right) : R^s \to R_+$ and $1_s = \left(1,...,1\right) \in R^s$ (Jacobsen, 1970).

Proposition 1 [Bogetoft et al., 2006]. $AE = AE^R$ and $TE = TE^R$ and, consequently, $CE = AE \cdot TE = AE^R \cdot TE^R$ if and only if the technology is input homothetic.

Note that in Figure 1, proposition 1 is not verified as the projection of firm A in to B does not correspond to firm C minimizing cost for $IsoqL\left(1\right)$. Nevertheless, as argued in the introduction, if firm X_0 = (3,6) had planned producing Y^* = 2, the right reference for allocative efficiency measurement is indeed firm A, and Bogetoft's approach yields the decomposition that is consistent with the production plan of firm (X_0, Y^*) = (3,6,2). While information on the firm's planned output level (either Y_0 or Y^*) is necessary to choose the correct decomposition (standard or reversed) for non-homothetic technologies (as exemplified above with the popular variable returns to scale BCC formulation), it is not a matter of concern for homothetic technologies since proposition 1 holds.

Indeed, the equivalence of the standard and reversed approaches under homotheticity is illustrated in Figure 2. In this example the same dataset used for Figure 1 is exploited for estimating two isoquants (those corresponding to Y_0 and Y^*) but assuming now the constant returns to scale case. Considering again for evaluation the firm X_0 = (3,6), its output benchmarks are either Y_0 = 1 or Y^* = 2.1, with the latter being a suitable transformation of $L(1_s)$ in accordance with the usual structure of input homothetic technologies and such that $X_0 \in IsoqL(2.1)$. Applying Farrell's approach technical efficiency is solved by projecting X_0 = (3,6) to C and, subsequently, allocative inefficiency is corrected by shifting the input mix from C to B. On the other hand, following Bogetoft et al.'s approach allocative efficiency is corrected by matching A's input bundle, while technical inefficiency is solved by projecting A onto B. By input homotheticity, it can be proved that $AE = AE^R$, $TE = TE^R$ and $CE = AE \cdot TE = AE^R \cdot TE^R$ as Proposition 1 states.

Figure 2. The reversed approach with homothetic DEA technologies. Distance function with G=X_0, and the new approach.



2.2. The standard and reversed approaches based on the directional input distance function

After Farrell's work, and particularly during the last two decades, part of the economic theory literature has focused on duality theory and distance functions, being Chambers et al. (1996), Briec and Lesourd (1999) and Briec and Garderes (2004) good examples. In particular, Chambers et al. (1996) introduced the notion of directional input distance functions as a way of generalizing the Shephard's input distance function and showed by duality how cost efficiency may be decomposed into the usual technical and allocative components.

Let $G = (g_1,...,g_m) \in R_+^m$ be a vector such that $G \neq 0_m$, then the directional input distance function defines as $\vec{D}_i(X,Y;G) = \sup\{\beta : X - \beta G \in L(Y)\}$ (see Chambers et al., 1996). It can be proved that if $G = X_0$ then $\vec{D}_i(X_0,Y_0;X_0) = 1 - 1/D_i(X_0,Y_0) = 1 - \theta_0^*$, and from this relationship and the flexibility of *G*, the directional input distance function encompasses the Shephard's input distance function. Moreover, Chambers et al. (1996) were able to establish a dual correspondence between the cost function and the directional input distance function, depending on the value of the reference vector G:

$$\underbrace{\frac{\sum_{i=1}^{m} w_i x_{i0} - C(Y_0,W)}{\sum_{i=1}^{m} w_i g_i}}_{\substack{\text{Cost} \\ \text{Inefficiency (CI)}}} \geq \underbrace{\vec{D}_i(X_0,Y_0;G)}_{\substack{\text{Technical} \\ \text{Inefficiency (TI)}}}. \tag{5}$$

From (5), allocative inefficiency may be computed as a residual: *AI* = *CI* – *TI*. Note that in this case the decomposition is additive instead of multiplicative due to the nature of each of the corresponding distance functions used in (3) and (5). Also, *AI* can be explicitly expressed as:

$$AI(X_0,Y_0,W,G) = \frac{\sum_{i=1}^{m} w_i \left(x_{i0} - \vec{D}_i(X_0,Y_0;G) g_i\right) - C(Y_0,W)}{\sum_{i=1}^{m} w_i g_i}, \tag{6}$$

since, from (5), *AI* = *CI* – *TI* and *TI* can be equivalently rewritten as

$$\vec{D}_i\left(X_0,Y_0;G\right)=\frac{\sum_{i=1}^{m} w_i \vec{D}_i\left(X_0,Y_0;G\right) g_i}{\sum_{i=1}^{m} w_i g_i}.$$

Next, we show that if $G = X_0$ the generalized decomposition (5) encompasses (3). In this case *CI* can be expressed as $CI=\frac{\sum_{i=1}^{m} w_i x_{i0}-C\left(Y_0,W\right)}{\sum_{i=1}^{m} w_i x_{i0}}=1-\frac{C\left(Y_0,W\right)}{\sum_{i=1}^{m} w_i x_{i0}}$ $=1-CE$, and regarding the technical component: $TI=\vec{D}_i\left(X_0,Y_0;X_0\right)=1-1/D_i\left(X_0,Y_0\right)=1-\theta_0^*=1-TE$.

It is worth noting that although (5) is highly flexible in measuring cost and technical inefficiency through a wide set of potential reference vectors *G*, in practice researchers usually resort to $G = X_0$ as their only choice. The reason is apparent; When $G = X_0$ the overall cost inefficiency decomposition (5) in the additive framework is completely equivalent to the well-known multiplicative setting of the Shephard's input distance function (3) and, therefore, the value of the technical measure has a clear interpretation in terms of equiproportional reduction in inputs. However, dismissing the flexibility of the directional distance function by adopting a decomposition dating back to the fifties is unjustified as it holds back theoretical breakthroughs. In fact, we believe that thanks to the flexibility of the directional distance function we here can consistently extend the notion of cost efficiency decomposition to non-homothetic technologies and, in doing so, show that the current practice, results, and interpretations of allocative and technical efficiency obtained in many empirical studies using common DEA models can be questioned. In this respect, until now, one of the most attractive features of the directional input distance function, its flexibility associated with the direction *G*, has been underutilized. Finally, besides its convenience when interpreting results, the only explanation for the systematic adoption of $G = X_0$ is the absence of a criterion to *a priori* select a vector different from that related to the standard scenario. Here we set the ground for a far reaching application of the directional distance function and the choice of a different vector *G* as a consistent and interpretable measure of technical inefficiency in non-homothetic technologies.

Once the definition of the standard overall economic efficiency decomposition in terms of the directional distance function has been presented, we can now extend Bogetoft's et al. (2006) reversed efficiency contribution to the case of the directional input distance function. As already presented, these authors introduced a way to estimate the "starting" allocative efficiency of X_0, which does not need the initial projection of this vector to the input requirement set defined by the observed output vector Y_0, $IsoqL\left(Y_0\right)$. As previously discussed in the motivation we believe that if one is interested in measuring the technical efficiency corresponding to X_0 in the input space by means of movements to the frontier associated with the production of Y_0, then it seems appropriate to assure that the allocative efficiency does not change along this process. Only in this way, one could be sure that the cost savings derived from these input adjustments are consequence of exclusively technical—engineering—issues not related to allocative efficiency; i.e., changes in the input mix.

To formalize these ideas let us define the reversed allocative inefficiency associated with an arbitrary output vector Y^* and an input vector $X_0 \in IsoqL\left(Y^*\right)$ as the normalized difference between the optimal cost given a set of market prices W and the observed cost at X_0:

$$AI^R\left(X_0,Y^*,W,G\right)=\frac{\sum_{i=1}^{m} w_i\left(x_{i0}-\vec{D}_i\left(X_0,Y^*;G\right)g_i\right)-C\left(Y^*,W\right)}{\sum_{i=1}^{m} w_i g_i}=\frac{\sum_{i=1}^{m} w_i x_{i0}-C\left(Y^*,W\right)}{\sum_{i=1}^{m} w_i g_i}. \tag{7}$$

The second equality in (7) is true thanks to $\vec{D}_i\left(X_0,Y^*;G\right)=0$ for all $X_0 \in IsoqL\left(Y^*\right)$ and $G=\left(g_1,\ldots,g_m\right)\in R_+^m$ such that $G \neq 0_m$.

Now, returning to the idea of properly interpreting and measuring technical efficiency, we contend that it is necessary to keep constant allocative efficiency along projections of the observed input vector X_0. In this case the following question arises naturally: Is there a reference vector G that actually measures technical inefficiency through the directional input distance function while leaving allocative efficiency unchanged? We now undertake this question. In particular, we show that under the assumption of input homotheticity the directional input distance function with reference vector $G = X_0$, always satisfies this desired property. In words, in the case of using this specific direction the "starting" allocative inefficiency, measured with respect to $IsoqL\left(Y^*\right)$, *i.e.,* before projecting the original input vector X_0 to the isoquant of Y_0, and the "final" allocative inefficiency, after projecting the original input vector, are the same.

Proposition 2. Let T be an input-homothetic technology and $Y_0, Y^* \in R_+^s$. Then, $AI^R\left(X_0, Y^*, W, X_0\right) = AI\left(X_0, Y_0, W, X_0\right)$, for all $X_0 \in L\left(Y_0\right) \cap IsoqL\left(Y^*\right)$.

Proof. If $G = X_0$ then $AI^R\left(X_0, Y^*, W, G\right) = 1 - \dfrac{C\left(Y^*, W\right)}{\sum_{i=1}^{m} w_i x_{i0}}$ and

$$AI\left(X_0, Y_0, W, G\right) = \frac{\sum_{i=1}^{m} w_i\left(x_{i0} - \vec{D}_i\left(X_0, Y_0; X_0\right) x_{i0}\right) - C\left(Y_0, W\right)}{\sum_{i=1}^{m} w_i x_{i0}} = 1 - \frac{C\left(Y_0, W\right)}{\sum_{i=1}^{m} w_i\left(\theta_0^* x_{i0}\right)}$$ since

$\vec{D}_i\left(X_0, Y_0; X_0\right) = 1 - 1/D_i\left(X_0, Y_0\right) = 1 - \theta_0^*$. Finally, based on the assumption of input-homotheticity, $\dfrac{C\left(Y^*, W\right)}{\sum_{i=1}^{m} w_i x_{i0}} = \dfrac{C\left(Y_0, W\right)}{\sum_{i=1}^{m} w_i\left(\theta_0^* x_{i0}\right)}$ (see Bogetoft et al., 2006, p. 456) and, therefore, $AI^R\left(X_0, Y^*, W, X_0\right) = AI\left(X_0, Y_0, W, X_0\right)$. ■

Indeed, by Proposition 2, the directional input distance function with reference vector $G = X_0$, or equivalently the Shephard's input distance function, is the unique—exact—measure that yields allocative efficiency-preserving estimations of technical efficiency when input homotheticity is assumed. Some comments on this result are in order. First, we have established that for properly measuring technical efficiency we previously need to make sure that allocative efficiency does not change between the "starting" technical inefficient input vector and its "final" technical efficient projection. That is, allocative efficiency must be the same with respect to Y^* and Y_0 at the existing input market prices (regardless of whether we decide for the standard or the reversed approach). Therefore, the correct estimation of the technical efficiency component so as to ensure that allocative efficiency remains unchanged requires knowledge of the applicable input prices. However, the significance of Proposition 2 is that, *even without knowing actual* input prices, the *true* technical efficiency can be computed from the observed quantity data using the directional input distance function taking as reference vector $G = X_0$, irrespective of input prices and ignoring the allocative efficiency of the firm. Secondly, since allocative efficiency remains constant along the radial projections of the input vector, it turns out that the actually planned—but unrealized—output level Y^* that the inefficient firm might had in mind when setting its production schedule, does not need to be known by the researcher when assessing the relevant allocative efficiency, since it is the same regardless that output level. That is, input homotheticity guarantees that the cost efficiency decomposition into its technical and allocative terms is always correct, as the actually planned level of output that should be taken as benchmark to measure allocative efficiency is irrelevant, because allocative efficiency is the same across the whole set of possible reference output levels. Thereby, the radial input measure constitutes a precise measure of technical efficiency. From an empirical perspective, assuming (even if wrongly) input-homotheticity simplifies the whole evaluation process in terms of the information required to achieve a correct decomposition of cost inefficiency, as both knowledge of input market prices and the actually planned output level are unnecessary to properly estimate technical efficiency (but not, obviously, for calculating cost efficiency).

2.3. The most common DEA technologies are non-homothetic yielding an inconsistent decomposition of cost efficiency

We now turn to the analysis of the most usual DEA technologies in order to explore whether their technological characteristics ensure a correct decomposition of cost efficiency by satisfying the desirable allocative efficiency-preserving property. In this respect, the usual constant returns to scale (CRS) assumption is an example of an input homothetic technology, but only in the case of a single output.

Proposition 3. Let $s = 1$. Then, T_{CRS} in (1) is input homothetic.

Proof. $H(y)L(1) = \{H(y)X : X \in R_+^m, (X,1) \in T\} = \left\{Z : Z \in R_+^m, \left(H(y)^{-1}X, 1\right) \in T\right\}$,

where the last equality is true thanks to the following change of variables: $Z = H(y)X$. Finally, considering that $T_{CRS} = \delta T_{CRS}$, for all $\delta > 0$, we have that

$\left\{Z : Z \in R_+^m, \left(H(y)^{-1} Z, 1\right) \in T\right\}$ = $\left\{Z : Z \in R_+^m, H(y)\left(H(y)^{-1} Z, 1\right) \in T\right\}$ = $\left\{Z : Z \in R_+^m, (Z, H(y)) \in T\right\}$. Defining $H(y) = y$, then $\left\{Z : Z \in R_+^m, (Z, H(y)) \in T\right\}$ = $\left\{Z : Z \in R_+^m, (Z, y) \in T\right\} = L(y)$. ■

Figure 2 illustrates Proposition 3, where a CRS DEA technology is generated using the dataset with two inputs and one output.

By Propositions 1 and 2, the above result ensures that the only measure that properly estimates technical inefficiency under the restricted scenario of a DEA technology exhibiting CRS and one single output is the well-known CCR measure (Charnes et al., 1978), which yields the same projections that the Shephard's input distance function. Additionally, we conclude that even under this so simple scenario, alternative measures as, for example, the input-oriented Russell measure (Färe and Lovell, 1978), the input-oriented additive models (Lovell and Pastor, 1995), or even the directional input distance function itself with a reference vector different from X_0 (Chambers et al., 1996), would not correctly measure technical efficiency. This is because these measures cannot assure that the starting allocative efficiency, assessed at X_0 with $IsoqL\left(Y^*\right)$ as reference, and the final allocative efficiency, evaluated in its corresponding projection on $IsoqL\left(Y_0\right)$, coincide.

Let us now to prove that in general the DEA technologies under CRS are not input homothetic. To do that, we next show a numerical counterexample. Let us assume that we have observed two firms using two inputs to produce two outputs $\left(x_1, x_2, y_1, y_2\right)$: A = (2,1,2,1) and B = (1,1,1,1). Resorting to expression (1), the DEA technology under CRS is estimated for this example as follows:

$$T_{CRS} = \left\{ \begin{matrix} \left(x_1, x_2, y_1, y_2\right) \in R_+^4 : 2\lambda_A + \lambda_B \le x_1, \lambda_A + \lambda_B \le x_2, \\ 2\lambda_A + \lambda_B \ge y_1, \lambda_A + \lambda_B \ge y_2, \ \lambda_A \ge 0, \ \lambda_B \ge 0 \end{matrix} \right\}. \quad (8)$$

The input set for the output vector $\left(y_1, y_2\right) = (1,1)$ corresponds to $L(1,1) = \left\{\left(x_1, x_2\right) \in R_+^2 : x_1 \ge 1, x_2 \ge 1\right\}$. However, for the output vector $\left(y_1, y_2\right) = (1,2)$, $L(1,2) = \left\{\left(x_1, x_2\right) \in R_+^2 : x_1 \ge 1, x_2 \ge 2\right\} \ne k\, L(1,1)$, for any $k > 0$. As a consequence, the technology is not input homothetic since $L(Y)$ cannot be written as $L(Y)=H(Y)\cdot L(1_s)$, with $H(Y):R^s \rightarrow R_+$ for all $Y=(y_1,y_2)$.

Likewise, it can be shown that the variable returns to scale DEA production possibility set characterizes technologies that are not in general input homothetic. In fact, Figure 1, which corresponds to this kind of technology, depicts two isoquants that do not follow 'parallel' expansions. Consequently, we conclude that radial projections linked to the CCR-CRS and BCC-VRS models do not always keep allocative efficiency constant in moving from X_0 to its projected benchmark on the isoquant, except in the case of CRS DEA technologies restricted to exclusively one output. Therefore, it cannot be guaranteed that the values provided by the CCR and BCC models could be unambiguously interpreted as true technical efficiency scores, since they depend on the relevant output level targeted by the firms.

Finally, as a historical note, we point out that the methodology introduced by the seminal paper of Farrell (1957) to decompose cost efficiency was well defined in the sense of this paper. It is due to the input homotheticity setting that was implicitly assumed by the author, who restricted his analysis to CRS technologies with one single output.

3. Decomposing cost efficiency with non-homothetic DEA technologies

These findings call for the introduction of new DEA models that estimate technical efficiency in the general case of non-homothetic technologies. These models would keep constant allocative efficiency as in the classic approach, thereby allowing for a correct interpretation of the distance function as a measure of technical efficiency. Having shown that the classic radial measures in DEA are not a suitable tool for estimating the true technical efficiency under non-homothetic technologies, it is necessary to overcome their inadequacy by adopting the flexibility offered by the directional distance function. To do that, we resort to Bogetoft et al.'s approach for estimating the starting allocative efficiency of the firm, and from there we determine its technical efficiency subject to the condition that when projected to its benchmark on the isoquant, its final allocative efficiency must be the same as the starting one.

Accordingly, in a first step we need to determine a level of output Y^*, at which to evaluate allocative efficiency, such that $X_0 \in IsoqL\left(Y^*\right)$ (e.g., $Y^* = 2$ in Figure 1).[1] To identify a valid level of output Y^* for X_0, it is necessary to solve the following linear programming model:

$$
\begin{aligned}
&\underset{\lambda,\beta}{Max} \quad \beta \\
&s.t. \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0}, && i = 1,\dots,m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0} + \beta y_{r0}, && r = 1,\dots,s, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0, && j = 1,\dots,n.
\end{aligned}
\tag{9}
$$

Model (9) coincides with the directional output distance function with the reference direction $G_y = Y_0$. It is well-known that if $\left(\lambda^*, \beta^*\right)$ is an optimal solution of (9), then $Y_0 + \beta^* Y_0 \in IsoqP\left(X_0\right)$, where $P\left(X_0\right)$ denotes the set of outputs producible from X_0. Less known is the fact that X_0 belongs to the isoquant of the optimal level of output $Y^* := Y_0 + \beta^* Y_0$.

Proposition 4. Let $\left(\lambda^*, \beta^*\right)$ be an optimal solution of (9). Then, $X_0 \in IsoqL\left(Y^*\right)$, where $Y^* = Y_0 + \beta^* Y_0$.

Proof. This results is derived from Lemma 2.2.10 in Färe et al. (1985). ■

Given input market prices W and Y^* from (9), it is trivial to determine $C\left(Y_0, W\right)$ and $C\left(Y^*, W\right)$, the minimum cost at output levels Y_0 and Y^*, respectively. However, in order to calculate the starting allocative inefficiency at $IsoqL\left(Y^*\right)$ using expression (7) we need to choose a value for the endogenous reference vector G. However, prefixing its value has both implications on the calculation of technical inefficiency, as

[1] Hereafter and seeking simplicity we assume a variable returns to scale technology—the case of constant returns to scale with s>1 can be implemented analogously.

well as the compliance of the desired property: allocative inefficiency must remain constant along the projection of X_0 on $IsoqL\left(Y_0\right)$. For example, if we determine Y^* by means of (9) for $X_0 = \left(3,6\right)$ using the dataset in Figure 1, we obtain a value of 2 for the optimal output, with $\left(3,6\right) \in IsoqL\left(2\right)$ by proposition 4. Given input market prices $w_1 = 1$ and w_2 =1.3, we first calculate $C\left(Y^*,W\right)$ and, subsequently, $AI^R\left(X_0,Y^*,W,G\right)$. If we define $G = X_0$, then we obtain $AI^R\left(X_0,Y^*,W,X_0\right) = 0.389$ using (7) but, additionally, we are measuring technical inefficiency along that specific input vector, projecting X_0 on point D=(1.6,3.2) by contracting inputs equiproportionally (see Figure 1). In this respect, as argued above, radial projections may not keep allocative efficiency constant in the case of non-homothetic technologies. Indeed, using (6) $AI\left(X_0,Y_0,W,X_0\right) = 0.107 \neq 0.389$. Therefore, the question is, what reference peer in the input space is generated by removing technical inefficiency through reductions in X_0 to its corresponding projection on the frontier of $L\left(Y_0\right)$ while ensuring that allocative inefficiency remains constant?

Next, we introduce the reversed directional input distance function that simultaneously yields the reference vector that preserves allocative inefficiency, and provides a consistent measure of technical inefficiency. In particular, we take advantage of the flexibility of the directional distance function allowing the reference vector G be a decision variable of the following optimization problem:[2]

$$
\begin{aligned}
\vec{D}_i^R\left(X_0,Y_0,Y^*,W\right) = \quad & \underset{\lambda,\beta,G}{Max} \quad \beta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{i0} - \beta g_i, && i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}, && r = 1,\ldots,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& AI\left(X_0,Y_0,W,G\right) = AI^R\left(X_0,Y^*,W,G\right), \\
& \lambda_j \geq 0, && j = 1,\ldots,n, \\
& \beta \geq 0, \\
& g_i \geq 0, && i = 1,\ldots,m.
\end{aligned}
\tag{10}
$$

[2] See Zofio et al. (2013) and Färe et al. (2013) as examples of directional distance function approaches where the reference vector is a decision variable of the corresponding model.

Model (10) is similar to the directional input distance function but in this case g_i, $i=1,...,m$, is a set of decision variables and, additionally, we force that the final allocative inefficiency at $L(Y_0)$, $AI(X_0,Y_0,W,G)$, matches the starting allocative inefficiency as calculated by $AI^R(X_0,Y^*,W,G)$ at $L(Y^*)$.

From (6) and (7), we have that the constraint $AI(X_0,Y_0,W,G)=AI^R(X_0,Y^*,W,G)$ is equivalent to $\frac{\sum_{i=1}^{m} w_i(x_{i0}-\beta g_i)-C(Y_0,W)}{\sum_{i=1}^{m} w_i g_i}$ = $\frac{\sum_{i=1}^{m} w_i x_{i0}-C(Y^*,W)}{\sum_{i=1}^{m} w_i g_i}$, which can be written as $\sum_{i=1}^{m} w_i \beta g_i = C(Y^*,W)-C(Y_0,W)$.

Before we engage in the corresponding maximization process, we remark that (10) yields an infinite number of optimal solutions; i.e., it is affected by an arbitrary positive multiplicative scalar since given a feasible solution $(\bar{\lambda},\bar{\beta},\bar{G})$, the transformation $(\bar{\lambda},k\bar{\beta},\bar{G}/k)$ is also a feasible solution for any $k>0$. So, if a feasible solution exists, then (10) is unbounded as we may consider $k=2,3,4,...$ To control for this possibility and settle for a single solution we propose the following restriction to be incorporated as an additional constraint in (10):

$$\sum_{i=1}^{m} g_i = \sum_{i=1}^{m} x_{i0}. \tag{11}$$

This formulation presents several advantages. First it is linear and simple. Second, it does not eliminate any *a priori* feasible direction for model (10) since the constraint is only a normalization of the reference vector. And third, it makes comparable the solution of model (10), the optimal value of β, with the value of the most usual directional input distance function with $G=X_0 \Leftrightarrow g_i = x_{i0}$, $\forall i=1,...,m$, as in this case the sum of the components of the reference vector coincides with $\sum_{i=1}^{m} x_{i0}$. In other words, both reference vectors have the same size. Note, however, that model (10) enhanced with (11) is not linear. Fortunately, it may be transformed into a linear model by adopting the following change of variables: $\psi_i = \beta g_i$, $i = 1,...,m$.

$$\begin{aligned}
&\underset{\lambda,\psi}{Max} \quad \frac{\sum_{i=1}^{m} \psi_i}{\sum_{i=1}^{m} x_{i0}} \\
&s.t. \\
&\qquad \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{i0} - \psi_i, && i = 1,...,m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{r0}, && r = 1,...,s, \qquad (12)\\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \\
&\qquad \sum_{i=1}^{m} w_i \psi_i = C\left(Y^*, W\right) - C\left(Y_0, W\right), \\
&\qquad \lambda_j \ge 0, && j = 1,...,n, \\
&\qquad \psi_i \ge 0, && i = 1,...,m.
\end{aligned}$$

Proposition 5. Let $\left(\lambda^*, \psi^*\right)$ be an optimal solution for (12). If $C\left(Y^*, W\right) \ne C\left(Y_0, W\right)$, then $\left(\lambda^*, \beta^*, G^*\right)$ with $\beta^* = \frac{\sum_{i=1}^{m} \psi_i^*}{\sum_{i=1}^{m} x_{i0}}$ and $G^* = \psi^* / \beta^*$ is an optimal solution of (10) plus (11).

Proof. If $\left(\lambda^*, \psi^*\right)$ is an optimal solution of (12), then $\left(\lambda^*, \beta^*, G^*\right)$ with $\beta^* = \frac{\sum_{i=1}^{m} \psi_i^*}{\sum_{i=1}^{m} x_{i0}}$ and $G^* = \psi^* / \beta^*$ is a feasible solution of (10) satisfying (11). Note, in fact, that G^* is well-defined because $\beta^* > 0$ since, by hypothesis, $\sum_{i=1}^{m} w_i \psi_i = C\left(Y^*, W\right) - C\left(Y_0, W\right) \ne 0$ and $w_i > 0$, $\forall i = 1,...,m$. Let us now assume that $\left(\lambda^*, \beta^*, G^*\right)$ is not an optimal solution of (10) plus (11). Then there exists a feasible solution $\left(\bar{\lambda}, \bar{\beta}, \bar{G}\right)$ such that $\bar{\beta} > \beta^*$. Now, it is not hard to prove that $\left(\bar{\lambda}, \bar{\psi}\right)$ with $\bar{\psi}_i = \bar{\beta}_i \bar{g}_i$, $\forall i = 1,...,m$, is a feasible solution of (12). But then, regarding the objective function value of model (12), we have that $\frac{\sum_{i=1}^{m}\bar{\psi}_i}{\sum_{i=1}^{m}x_{i0}} = \frac{\sum_{i=1}^{m}\bar{\beta}\bar{g}_i}{\sum_{i=1}^{m}x_{i0}} = \bar{\beta}\frac{\sum_{i=1}^{m}\bar{g}_i}{\sum_{i=1}^{m}x_{i0}} = \bar{\beta}\frac{\sum_{i=1}^{m}x_{i0}}{\sum_{i=1}^{m}x_{i0}} = \bar{\beta} > \beta^* = \frac{\sum_{i=1}^{m}\psi_i^*}{\sum_{i=1}^{m}x_{i0}}$, which is the contradiction we were seeking. ■

On the other hand, the standard directional input distance function with $G = X_0$ has the property of yielding technical inefficiency measures between zero and one. This desirable property is retained by the new approach as the following result establishes.

Proposition 6. Let $\left(\lambda^*,\beta^*,G^*\right)$ be an optimal solution of (10) plus (11), then $0 \le \beta^* \le 1$.

Proof. (i) $0 \le \beta^*$ is trivial by the constraints of (10). (ii) Let us assume that $\beta^* > 1$. Then if we sum the first m constraints of model (10) we have that $\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_j^* x_{ij} \le \sum_{i=1}^{m}x_{i0} - \beta^*\sum_{i=1}^{m}g_i^* = \left(1-\beta^*\right)\sum_{i=1}^{m}x_{i0}$ using (11). But then $\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_j^* x_{ij} < 0$ because $\left(1-\beta^*\right) < 0$. However, we get a contradiction with the fact that $\lambda_j^* \ge 0$, $j = 1,...,n$, and $x_{ij} \ge 0$, $j = 1,...,n$, $i = 1,...,m$. Consequently, $\beta^* \le 1$. ■

We now apply model (12) to the dataset portrayed in Figure 1. Resorting to the standard directional input distance function with $G = X_0$ for evaluating $X_0 = (3,6)$, the projection on $IsoqL\left(Y_0\right)$ coincides with point D. In this situation, as previously shown, $AI^R\left(X_0,Y^*,W,X_0\right) \ne AI\left(X_0,Y_0,W,X_0\right)$. However, solving (12) we obtain that $\psi_1^* = 2$ and $\psi_2^* = 0$ and, therefore, by Proposition 5 the reversed directional input distance function—technical inefficiency—equals $\vec{D}_i^R\left(X_0,Y_0,Y^*,W\right) = 0.222$ with $g_1^* = 9$ and $g_2^* = 0$, being in this particular case E the final projected benchmark. The main difference between both approaches is that the reversed directional input distance function keeps allocative inefficiency constant: $AI^R\left(X_0,Y^*,W,G^*\right) = 0.467 = AI\left(X_0,Y_0,W,G^*\right)$.

4. A numerical example

This section includes a numerical illustration of the use of the methodology proposed in this paper. We are particularly interested in showing that we may find substantial differences between the estimation of the technical inefficiency obtained when considering the usual directional input distance function with $G = X_0$ —or, equivalently, the traditional BBC model, and that obtained when implementing the new approach. This issue is quite relevant when decomposing cost inefficiency for non-homothetic technologies as researchers ignore the output level targeted by the firms. To illustrate the discrepancy between the standard and reversed approaches we use the data already explored in Ray (2004, pp. 222-223). A set of 51 production observations are assessed: 50 US states plus Washington D.C.. The production is characterized by one output, Gross value of production, and six inputs, Production workers (L), nonproduction workers or employees (EM), buildings and structures (BS), machinery and equipment (ME), materials consumed (MC) and energy (ENER).

The efficiency analysis performed by applying the standard input-oriented BCC model yields 17 technically efficient observations with a score of 1. Consequently, 34 states are found inefficient and our analysis focuses on the estimation of the technical inefficiency of these observations (see Table 1). Regarding the application of the reversed directional input distance function, we perform the steps described in Section 3. First, we solve (9) for each unit. In this way, we determine an output level Y^* such that the evaluated X_0 lays on the frontier of its corresponding input requirement set $L(Y^*)$. In this respect, for some observations we find significant differences between both possible target output levels Y^* and Y_0. Afterwards, we calculate the minimum cost of producing both Y_0 and Y^*, denoted as $C(Y_0,W)$ and $C(Y^*,W)$, respectively. Finally, we determine the allocative inefficiency-preserving estimation of technical inefficiency, which is solved through (12). Table 1 reports, for all technically inefficient units, all these values and, additionally, the optimal reference vector G obtained from (10) plus (11). Finally, in the last set of columns we show the optimal value of the reference vectors that yield the projections on each corresponding isoquant of $L(Y_0)$ associated with the reversed directional input distance function.

Table 1. Results for the numerical example

Firm	θ_0^*	DDF G=X_0	Y_0	Y^*	$C(Y_0,W)$	$C(Y^*,W)$	New DDF	g_L	g_{EM}	g_{BS}	g_{ME}	g_{MC}	g_{ENER}
1	0.907	0.093	8.257	9.141	5.285	5.944	0.142	0.000	0.000	0.000	0.261	4.240	0.017
3	0.919	0.081	5.384	5.897	3.176	3.574	0.150	0.025	0.000	0.000	0.000	4.487	0.006
4	0.969	0.031	8.771	9.053	5.614	5.822	0.041	0.000	0.000	0.000	1.133	3.385	0.000
5	0.973	0.027	5.933	6.120	3.802	3.950	0.053	0.000	0.000	0.000	1.689	2.829	0.000
6	0.924	0.076	5.513	5.986	3.298	3.665	0.138	0.000	0.000	0.000	0.094	4.424	0.000
10	0.978	0.022	3.926	4.060	2.269	2.294	0.013	0.000	0.000	0.000	3.431	1.087	0.000
11	0.926	0.074	9.288	10.078	6.076	6.670	0.117	0.000	0.000	0.000	0.198	4.320	0.000
13	0.873	0.127	5.875	6.794	3.480	4.114	0.173	0.000	0.000	0.000	0.000	4.449	0.069
14	0.910	0.090	8.406	9.050	5.664	6.167	0.117	0.000	0.000	0.000	0.343	4.175	0.000
15	0.939	0.061	11.353	11.822	8.117	8.497	0.064	0.000	0.000	0.000	0.977	3.541	0.000
17	0.952	0.048	10.408	10.838	7.095	7.445	0.059	0.000	0.000	0.000	0.354	4.164	0.000
20	0.966	0.034	5.312	5.525	3.260	3.427	0.049	0.000	0.000	0.000	0.000	4.411	0.107
21	0.916	0.084	7.150	7.842	4.708	5.241	0.155	0.000	0.000	0.000	0.287	4.231	0.000
22	0.978	0.022	6.403	6.486	4.190	4.257	0.024	0.000	0.000	0.000	4.081	0.437	0.000
23	0.967	0.033	9.628	9.861	7.109	7.301	0.036	0.000	0.000	0.000	1.764	2.754	0.000
24	0.911	0.089	7.221	8.007	4.618	5.224	0.162	0.000	0.000	0.000	0.307	4.210	0.000
25	0.936	0.064	8.736	9.345	5.507	5.951	0.090	0.000	0.000	0.000	0.468	4.050	0.000
26	0.991	0.009	9.273	9.335	6.146	6.193	0.009	0.000	0.000	0.000	0.000	4.518	0.000
30	0.967	0.033	4.836	4.993	2.877	3.003	0.062	0.000	0.000	0.000	0.718	3.800	0.000
31	0.998	0.002	6.541	6.558	4.240	4.254	0.005	0.000	0.000	0.000	0.000	4.518	0.000
33	0.933	0.067	5.712	6.147	3.552	3.893	0.130	0.015	0.000	0.000	0.000	4.502	0.001
35	0.902	0.098	5.286	5.932	2.890	3.366	0.142	0.000	0.000	0.000	0.285	4.233	0.000
36	0.926	0.074	10.023	10.648	7.122	7.630	0.098	0.000	0.000	0.000	0.075	4.441	0.002
37	0.978	0.022	7.410	7.594	4.745	4.885	0.035	0.000	0.000	0.000	0.000	4.518	0.000
38	0.932	0.068	4.714	5.125	2.698	3.010	0.102	0.017	0.000	0.000	0.000	4.402	0.099
39	0.905	0.095	7.693	8.518	5.079	5.722	0.162	0.007	0.000	0.000	0.003	4.494	0.014
40	0.977	0.024	3.557	3.672	2.282	2.306	0.015	0.000	0.000	0.000	0.000	4.518	0.000
41	0.942	0.058	10.818	11.267	7.327	7.693	0.066	0.000	0.000	0.000	1.161	3.357	0.000
42	0.991	0.009	6.780	6.827	3.910	3.944	0.008	0.000	0.000	0.000	0.306	4.212	0.000
43	0.963	0.037	10.077	10.473	6.751	7.071	0.062	0.000	0.000	0.000	0.696	3.822	0.000
44	0.948	0.052	9.822	10.385	6.599	7.036	0.076	0.000	0.000	0.000	0.341	4.177	0.000
45	0.879	0.121	6.167	7.130	3.643	4.362	0.179	0.037	0.000	0.001	0.036	4.391	0.052
48	0.988	0.012	8.536	8.658	5.844	5.939	0.018	0.000	0.000	0.000	0.000	4.518	0.000
50	0.929	0.071	8.785	9.444	5.860	6.367	0.112	0.000	0.000	0.000	0.091	4.427	0.000

The first result to be highlighted is the large differences between the standard directional input distance function and the technical inefficiency obtained from the new reversed approach. Indeed, we find statistically significant differences between the technical inefficiencies estimated by both methods (p-value = $4.319 \cdot 10^{-8}$ running a Wilcoxon signed rank test on two paired samples). The second result is that in contrast to the standard directional input distance function, where the reference direction is *a priori* fixed, the new approach allows to know the main inputs responsible for technical inefficiency, as the individualized directions show what inputs must be reduced and the magnitude of such reduction. In this example, we obtain that the direction is mainly driven by ME (machinery and equipment) and MC (materials consumed), resulting in the most important sources of pure technical inefficiency.

Finally, it is clear that in the case of non-homothetic technologies, the overall cost efficiency decomposition is quite different depending on whether one is willing to assume that inefficient firms incur in input excesses when producing Y_0 from X_0 (the standard approach), or, alternatively, as this is unknown to the researcher, fall short of producing Y^* (the reverse approach). Our findings show that the difference between the results obtained by both approaches increases as the gap between the two possible target outputs gets larger. This is quite relevant when prescribing strategies aimed at improving both technical and allocative efficiency, since both the reductions in the observed input levels might not be equiproportional, but rather restricted to some particular inputs as in the example, and also the cost minimizing input mix may not be that corresponding to Y_0 but Y^*. It is apparent that in the most realistic case of non-homothetic technologies, researchers should make an effort to determine the intended output level of every individual firm lying inside the production possibility set since this information is critical when choosing the proper cost efficiency decomposition, and advice managers on what are the relevant sources of inefficiency and how to solve them.

5. Conclusions

Standard economic efficiency analysis assumes homothetic technologies when decomposing overall efficiency accounting for technical and allocative criteria. This assumption is rather convenient because it allows researcher to bypass the lack of information on whether inefficient firms use inputs in excess or fall short from a target output, which is critical to determine the firms' allocative efficiency with respect to the cost minimizing input bundle. When the technology is homothetic allocative efficiency is the same despite the output level that is considered for its measurement, and we show that the directional input distance function with $G = X_0$ —and equivalently the traditional radial distance functions—can be correctly interpreted as measures of technical efficiency, thereby yielding a consistent decomposition of overall efficiency into allocative and technical components. In particular, we show that only under constant returns to scale technologies producing a single output one can claim that these models measure true technical efficiency. Unfortunately the homotheticity assumption is quite restrictive from a theoretical perspective and generally untenable in empirical works.

The directional distance function constitutes the analytical tool that finally allows breaking the straight jacket represented by the classical and restrictive framework of the radially based decomposition of cost efficiency, and extend it to the case of non-homothetic technologies. The cornerstone for a correct decomposition of economic efficiency is the need for a flexible measure of technical efficiency that preserves allocative efficiency unchanged when prescribing reductions in the observed input vector. Resorting to Bogetoft et al. (2006) we are able to measure the starting allocative efficiency of interior points of the technology and, following the rationale behind a correct technical efficiency interpretation, the same allocative estimation must be observed at the final projected benchmark.

Based on these findings we conclude that all empirical work adopting the popular Data Envelopment Analysis models of constant (in the multiple output case) and variable returns to scale is prone to important errors of interpretation with regard to the sources of cost inefficiency, as these models characterize non-homothetic technologies where allocative efficiency depends on the chosen output level. In this case the value of the radial distance functions could be interpreted as an undetermined mix of sources of inefficiency instead of an estimate of true technical efficiency as traditionally assumed. Our new methodology overcomes these limitations by allowing the definition of consistent standard and reversed decompositions of cost efficiency that can be applied for every individual firm if information on its targeted level of output were available.

We illustrate our concerns regarding the limitations of the traditional decomposition based on radial distance functions and the new methodology using a numerical example. We show that for technically inefficient observations, substantial differences arise when decomposing cost efficiency by applying the standard non-homothetic BBC model, and that obtained when the new reversed approach is implemented. As a result, the alternative strategies that firms may undertake so as to improve their economic performance based on both allocative and technical criteria would be substantially different, and the decision making process informed by researchers may be misleading if information of the intended output levels is not at hand; particularly with respect to the reduction of the observed vector of inputs. While the standard approach prescribes equiproportional reductions that do not alter the input mix, the new approach may call for individual adjustments.

In summary, it can be concluded from our analysis that researchers must keep in mind that in the—usual—case of non-homothetic technologies, both the standard and reverse approaches yield different results, and that further information on the target output levels need to be brought into the analysis so as to correctly choose the appropriate method, and attain a consistent decomposition of overall cost efficiency.

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