

## **CURVES WITH CONSTANT WIDTH: A PROPOSAL FOR A GEOMETRY UNIT**

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### **Resumen:**

En este trabajo se propone una actividad de matemática para la educación primaria y una posible extensión a la escuela secundaria, basada en resolución de problemas y debates, empezando con monedas de todo el mundo de formas diferentes de las comunes.

**Palabras clave:** Geometría, curvas con altura constante.

### **Abstract:**

An activity for primary school and its extension to high school in geometry is proposed, based on guided classroom discussion and problem solving, starting from unusual shapes of coins worldwide.

**Keywords:** Geometry, curves with constant width.

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We all know that education is valuable not only because it conveys various kinds of skills and knowledge that may be applied in practical life, but also because it encourages the development of reasoning skills, as the work of Vygotsky has strongly pointed out (Vygotsky, 1986). Mathematics and geometry in particular seem to be notably useful to this aim. Within geometry the teacher can provide attractive activities which lead to abstract mathematics, along a path that encourages students towards discussions, sharing ideas, experimenting by hands, and finding solutions. In this short article, we propose a classroom activity which does not require technicalities and starts from a practical and curious observation, in order to catch the students' interests.

The argument is proposed for primary school, but hints are given to adapt it to high school, where of course more technical results can be stimulated.

We propose to start a classroom discussion by simply inviting the students to the description of the properties of the circle: a circle can rotate around its center, can also move rolling inside a strip, that is, between parallel lines, the borders touching them while moving. This property is extremely useful, for instance, for the coffee vending machines in order that they may recognize the inserted coins and release the desired beverage. By the way, it was a mathematician and scientist, Hero of Alexandria, who invented the first vending machine, as one can find in his book *The Pneumatics*, in the remote first century A.D., much before coffee was known in Greece- and in fact it was a holy water vending machine (cf. Hero of Alexandria, & Woodcroft, B. 1851).

But not all countries have round coins! What about the twenty or fifty pence English coins, or coins in use on attractions at amusement parks? Neither square nor round: it is clear that something needs an exploration.

At this stage a little research on coins with unusual shapes can be a strategy to stimulate interest and observational skills in the students, and give room to interdisciplinary classes.

Indeed, the vending machine is not the first use to think at when designing a coin: the shape is usually functional to the massive production and the recognizability by touch, thus round coin, sometimes with a little hole in the center, is the most common choice worldwide. Nevertheless it is not rare to find odd shaped coins. For instance, several countries that have historical links to the United Kingdom have seven-sided coins, as it is the case for some English coins. Other countries have minted

polygonal shaped coins with 6, 8, 10, sides. Flowered shaped coins also exists as regular coins.

Some examples of unusual shaped regular coins are shown in the picture.

					
Bermuda triangular 3 dollar	Australia 12-sided fifty cents	Cook Island 2 dollars	Belize flowered shape 1 cent	Indian diamond shaped 5 paise (till 1994)	Forthcoming English 12-sided 1 pound (from 2017)

				
English 50 pence	Chile 5 pesos octagonal	Canadian 11-sided 1 dollar	Czech 13-sided Koruna	Yemen 21-sided 1 riyal

Now, discussion in the classroom is open: every student is invited to observe the coins, to describe them, to measure, to draw. This activity implies a certain consideration of the appropriate mathematical properties: when can we use terms such as edges, angles and diameters?

In the following session, experiments are welcome: students can be invited to construct a physical model of track in which a round coin can be inserted and roll. Two wooden sticks fixed on a support would be a good choice, but also cardboard can be used.

The problem is now to find new shapes that can move rolling inside the track and always touching it, as the round coins do: how many of them can we find? How to construct them? If some odd shaped coin is available, students can test them, besides, they are invited to construct their own odd shaped tokens to make experiments on the track.

The teacher can help the students to formulate the mathematical questions by themselves, stimulating the discussion and suggesting the right terminology: they are now in fact in the process of finding what are called *curvilinear polygons with constant width*. The curves are supposed to be *closed* and *convex*, the *width* is the gauge of any

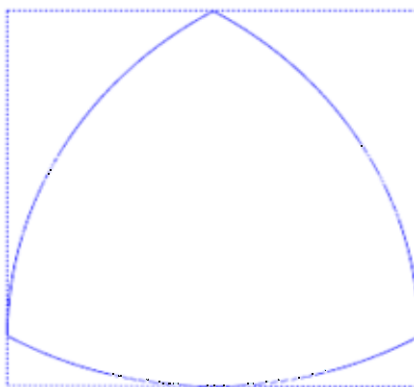
track in which a curve can move, not necessarily rolling. Mathematically, it is defined as the distance between two parallel lines each having at least one point in common with the border of the polygon and none with its interior. We recall that the distance of a pair of parallel lines is defined as the measure of any segment which meets the lines orthogonally in its extremal points.

The students are here invited to check (and in fact it can be proved) that for any given closed and convex curve, and any given direction there is always a pair of parallel lines with the given direction each of which having at least a point in common with the curve, and no point in common with its interior. In general, the distance between these two lines, that is the width of the curve, varies with the chosen direction, as one can test on some polygon. For instance, two lines containing the opposite sides of a rectangle have distance the length of the two other sides, but there are many different pairs of parallel lines intersecting the rectangle only in the two opposite vertices. It could be proposed here to try to find out which of these pairs of parallel lines has the shortest distance.

At this stage students have been induced to discover that, other than the circle, the curves with constant width, that is the curves that can move rolling inside a track, if any, are very special among the closed convex curves, and the question is to find some or all of them.

Students proposals are welcome: one could try to start with a regular polygon, but no one would like a square wheel on his bicycle, so a natural idea would be to start deforming the edges of a polygon to get the thing work.

Instead of a square, it is better to start with a smaller number of edges: an equilateral triangle can be sufficient. In fact, the answer is in the hands of a guitar player: some plectra have the right shape: its name is Reuleaux triangle.

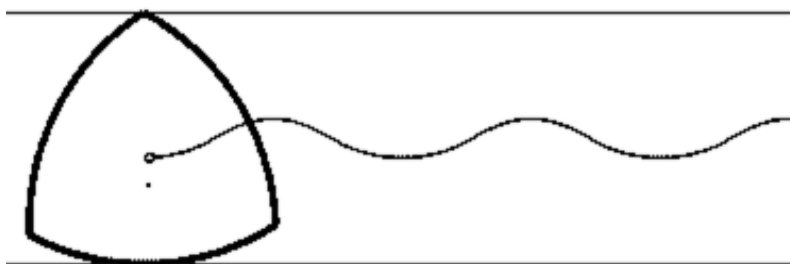


Finally, some student may come up with the right construction: take an equilateral triangle, call the vertices A, B and C, then center the compass in A and draw the arc BC, then center the compass in B and draw the arc CA, finally draw the arc AB on the circle centered at C. This figure has the required rolling property. In general, it is easy to extend this procedure to any regular polygon, of any odd number of sides. Some consideration is in order for the word odd: students are invited to play the game with a square and then an hexagon and check what happens, again having a look at the coins.

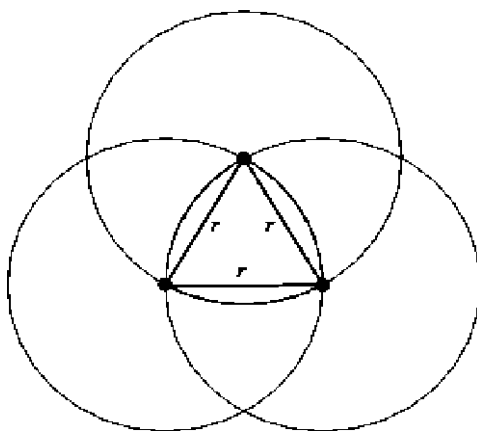
The following step is now to eliminate the request of being regular for the polygon we started the game with: an equilateral polygon, not necessarily equiangular is sufficient. Again, first experiment and observe, then conjecture, then finally try to prove!

For a high school class, more work can be proposed to the students. For instance, one can guess where the geometric centroid of the Reuleaux triangle is: while the center of a circle is equidistant from any of its points, this is not the case for the Reuleaux triangle. As in the case of its strict relative, the triangle, the geometric centroid C of the Reuleaux triangle has a varying distance with a point P which moves on the border. Thus, the centroid C draws a sort of wave while the Reuleaux triangle is rolling, as one can check experimentally:

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One can also propose to evaluate the perimeter and the area of a Reuleaux triangle, which is the union of an equilateral triangle and of three identical circular segments with opening angle  $\alpha = \pi/3$  inside circles with radius of length  $r$



One finds:

$$P = 3 \alpha r = \pi r$$

$$A = \sqrt{3} r^2/4 + 3 r^2 (\alpha - \sin \alpha)/2 = r^2 (\sqrt{3}/2 + 3 (\pi/3 - \sqrt{3}/2))/2 = (\pi - \sqrt{3})r^2/2.$$

Game: try to compute with a calculator  $A$  for  $r=1$ , and compare with the area of a square with side of length 1.

One can compare the above formulas with the ones of the circle of the same width, that means, a circle of diameter  $d=r$ , and observe that the perimeter is the same, while the area is smaller than the one of the circle with the same width, that is  $\pi r^2/4$ . Therefore, minting Reuleaux triangular coins would save metal.

Which is the radius of a circle with the same area? Considering curves with constant width drawn from above mentioned regular polygons, one can test that the perimeter depends only on the width, being always equal to  $\pi r$ . This is always the case, as it is proved by Barbier's theorem. The discussion in the class could now lead to another challenging question as the problem of maximizing the area for a fixed perimeter value, that is the isoperimetric problem, and other geometric problem of maximum and minimum. Further problems and theorems can be found in (Gardner, 1991), eventually without forgetting some aesthetic consideration, for instance referring to the shape of some gorgeous Gothic windows. Finally, it could be challenging to propose some observation for analogue shapes in higher dimension: the computation of

the volume and of the surface area of a Reuleaux tetrahedron (defined as the intersection of four spheres of equal radius) is not completely trivial, as it has been pointed out by B. Harbourne (Harbourne, 2010).

## CONCLUSION

To stimulate and keep alive the interest of our students in a math class is important to let them also experiment some practical aspects of the subject, that usually become more and more hidden, as the students advance in the school years, when often what Federigo Enriques called a *false concept of rigour* prevails and pervades the educational practice (Enriques, 1907). In the proposed activity, direct experiments and practical motivation are central, and the transition from the starting example towards following careful generalization, which is one of the typical mathematical line of reasoning, is encouraged.

## HISTORICAL NOTE

Franz Reuleaux was a German mechanical engineer of the Nineteenth Century, University Professor, President of the Berlin Royal Technical Academy, he is known as the father of kinematics.

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