

Research Article

Anisotropic Weak Hardy Spaces and Wavelets

B. Barrios¹ and J. J. Betancor²

¹ Departamento de Matemáticas, Universidad Autónoma de Madrid and Instituto de Matemáticas (ICMAT, CSIC-UAM, UC3M, UCM), C/Nicolás Cabrera 15, 28049 Madrid, Spain

² Departamento de Análisis Matemático, Universidad de la Laguna, Campus de Anchieta, Avenida Astrofísico Francisco Sánchez s/n, Santa Cruz de Tenerife, 38271 La Laguna, Spain

Correspondence should be addressed to J. J. Betancor, jbetanco@ull.es

Received 17 January 2012; Accepted 5 April 2012

Academic Editor: Quanhua Xu

Copyright © 2012 B. Barrios and J. J. Betancor. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We characterize the anisotropic weak Hardy spaces $H_A^{p,\infty}(\mathbb{R}^n)$ associated with an expansive matrix A by using square functions involving wavelets coefficients.

1. Introduction

Bownik, in a series of papers [1–5], studied anisotropic function spaces associated with dilations. In the monography [1] he investigated anisotropic Hardy spaces. Suppose that A is an expansive matrix (also called dilation) in \mathbb{R}^n , that is, A is an $n \times n$ -matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. If ϕ is a function in the Schwartz class $S(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$ and f is a distribution in $S(\mathbb{R}^n)'$, the dual space of $S(\mathbb{R}^n)$, the radial maximal function $M_\phi(f)$ is defined by

$$M_\phi(f) = \sup_{k \in \mathbb{Z}} |f * \phi_k|, \quad (1.1)$$

where $\phi_k(x) = |\det A|^{-k} \phi(A^{-k}x)$, $x \in \mathbb{R}^n$, and $k \in \mathbb{Z}$. For every $0 < p < \infty$, the Hardy space $H_A^p(\mathbb{R}^n)$ associated with A consists of all those $f \in S(\mathbb{R}^n)'$ such that $M_\phi(f) \in L^p(\mathbb{R}^n)$. The space $H_A^p(\mathbb{R}^n)$ does not depend on the function ϕ and the $\|\cdot\|_{H_A^p(\mathbb{R}^n)}$ -norm is defined by

$$\|f\|_{H_A^p(\mathbb{R}^n)} = \|M_\phi(f)\|_p, \quad f \in H_A^p(\mathbb{R}^n). \quad (1.2)$$

Like in the classical case [6], the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ can be characterized by nontangential or grand maximal functions [1, Theorem 7.1, page 42]. Also, atomic representations of the distributions in $H_A^p(\mathbb{R}^n)$ are obtained in [1, Theorem 6.5, page 39]. Wavelets for a dilation A are studied in [1, Chapter 2]. The author proved that r -wavelets associated with the expansive matrix A form an unconditional basis for the anisotropic Hardy spaces defined by A . Recently, Bownik et al. [7, 8] have investigated weighted anisotropic Hardy spaces.

The weak anisotropic Hardy space $H_A^{p,\infty}(\mathbb{R}^n)$, $0 < p \leq 1$, was introduced by Ding and Lan [9]. A distribution $f \in S(\mathbb{R}^n)'$ is in $H_A^{p,\infty}(\mathbb{R}^n)$ if and only if $M_\phi(f) \in L^{p,\infty}(\mathbb{R}^n)$, where $L^{p,\infty}(\mathbb{R}^n)$ denotes the weak L^p -space. We define

$$\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)} = \|M_\phi f\|_{L^{p,\infty}(\mathbb{R}^n)}, \quad f \in H_A^{p,\infty}(\mathbb{R}^n). \quad (1.3)$$

As the case $H_A^p(\mathbb{R}^n)$, the space $H_A^{p,\infty}(\mathbb{R}^n)$ does not depend on the election of ϕ and it can be described by nontangential and grand maximal functions. Atomic representations of distributions in $H_A^{p,\infty}(\mathbb{R}^n)$ were established in [9, Theorem 1.1]. In this paper we study wavelets for $H_A^{p,\infty}(\mathbb{R}^n)$. We characterize in Theorem 2.2 below the distributions in $H_A^{p,\infty}(\mathbb{R}^n)$ by square functions involving wavelets coefficients. Our result can be seen as an anisotropic version of the one showed in [10] (see also [11]).

This paper is organized as follows. In Section 2 we recall the main definitions and properties about the anisotropic setting that we need throughout the paper. We also state our result (Theorem 2.2). The proof of Theorem 2.2 is presented in Section 3.

2. Preliminaries and Results

We now recall the main definitions and properties concerning the analysis in the anisotropic setting. We refer the reader to [1] where the anisotropic theory associated with expansive matrixes was developed. Suppose that A is an expansive matrix in \mathbb{R}^n . We denote by Δ an ellipsoid for A such that the Lebesgue measure $|\Delta|$ of Δ is equal to 1. For every $k \in \mathbb{Z}$, we define $B_k = A^k \Delta$. We consider the mapping $\rho : \mathbb{R}^n \rightarrow [0, \infty)$ given by

$$\rho(x) = \begin{cases} |\det A|^j, & x \in B_{j+1} \setminus B_j, \quad j \in \mathbb{Z}, \\ 0, & x = 0. \end{cases} \quad (2.1)$$

Thus, ρ is a homogeneous quasinorm associated with A in the sense of [1, Definition 2.3, page 6]. This means that ρ satisfies the following properties:

- (a) $\rho(x) > 0$, $x \in \mathbb{R}^n \setminus \{0\}$,
- (b) $\rho(Ax) = |\det A| \rho(x)$, $x \in \mathbb{R}^n$,
- (c) $\rho(x + y) \leq H_\rho(\rho(x) + \rho(y))$, $x, y \in \mathbb{R}^n$,

where $H_\rho \geq 1$. In [1, Lemma 2.4, page 6] it was proved that if ρ_1 and ρ_2 are two homogeneous quasinorms associated with A then,

$$\frac{1}{C} \rho_1(x) \leq \rho_2(x) \leq C \rho_1(x), \quad x \in \mathbb{R}^n, \quad (2.2)$$

for a certain $C > 0$. As it was mentioned above a tempered distribution $f \in H_A^{p,\infty}(\mathbb{R}^n)$, $0 < p \leq 1$, when $M_\phi(f) \in L^{p,\infty}(\mathbb{R}^n)$, where $\phi \in S(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi(x) dx \neq 0$. In [9, Theorem 1.1] Ding and Lan proved the following atomic representation for the distributions in $H_A^{p,\infty}(\mathbb{R}^n)$ that will be useful in the sequel.

Theorem 2.1 (see [9, Theorem 1.1]). *Suppose that $f \in H_A^{p,\infty}(\mathbb{R}^n)$, $0 < p \leq 1$, and $s \geq [(1/p - 1) \log |\det A| / \log m_1]$, where $m_1 = \min\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. Then, there exists a sequence of bounded functions $\{f_k\}_{k \in \mathbb{Z}}$ satisfying, for a certain $C > 0$,*

- (a) $f = \sum_{k=-\infty}^{\infty} f_k$ in $S(\mathbb{R}^n)'$, and $\|f_k\|_{\infty} \leq C2^k$, for every $k \in \mathbb{Z}$,
- (b) for every $k \in \mathbb{Z}$, $f_k = \sum_{i=0}^{\infty} \beta_i^k$, in $S(\mathbb{R}^n)'$, where, for each $i \in \mathbb{N}$,

- (i) there exist $x_{i,k} \in \mathbb{R}^n$ and $j_{i,k} \in \mathbb{Z}$ such that $\text{supp } \beta_i^k \subset \tilde{B}_i^k = x_{i,k} + A^{j_{i,k}} \Delta$, and

$$\sum_{i=0}^{\infty} |\tilde{B}_i^k| \leq C2^{-kp} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p, \quad (2.3)$$

- (ii) $\|\beta_i^k\|_{\infty} \leq C2^k$,
- (iii) $\int_{\mathbb{R}^n} \beta_i^k(x) P(x) dx = 0$, for every polynomial P whose degree is less or equal to s .

Conversely, if $f \in S(\mathbb{R}^n)'$ satisfying (a) and (b) where in (i) $C\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p$ is replacing by $M > 0$, then $f \in H_A^{p,\infty}(\mathbb{R}^n)$ and $M : \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p$.

Theorem 2.1 is an anisotropic version of the isotropic result established by Fefferman and Soria [12, Proposition, page 8].

Let $\varphi \in L^2(\mathbb{R}^n)$. For every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we define

$$\varphi_{j,k}(x) = |\det A|^{j/2} \varphi(A^j x - k), \quad x \in \mathbb{R}^n. \quad (2.4)$$

We say that φ is a Bessel A -wavelet if there exists $C > 0$ such that, for every $f \in L^2(\mathbb{R}^n)$,

$$\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \varphi_{j,k} \rangle|^2 \leq C \|f\|_2^2. \quad (2.5)$$

φ is a frame A -wavelet when, for a certain $C > 0$,

$$\frac{1}{C} \|f\|_2^2 \leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \varphi_{j,k} \rangle|^2 \leq C \|f\|_2^2, \quad f \in L^2(\mathbb{R}^n). \quad (2.6)$$

We say that φ is a tight frame A -wavelet when (2.6) holds with $C = 1$.

As usual if $g \in L^2(\mathbb{R}^n)$ we denote by \hat{g} the Fourier transform of g .

We now establish our result where the distributions in $H_A^p(\mathbb{R}^n)$, $0 < p \leq 1$, are characterized by using square functions involving A -wavelets.

Theorem 2.2. Let $f \in S(\mathbb{R}^n)'$ and $0 < p \leq 1$. Assume that $\psi \in S(\mathbb{R}^n)$ is a tight frame A -wavelet where A is an expansive matrix in \mathbb{R}^n , such that $\psi(0) \neq 0$, $\text{supp } \hat{\psi}$ is compact, and $0 \notin \text{supp } \hat{\psi}$. Then, the following properties are equivalent:

- (a) $W(f) = (\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}|^2)^{1/2} \in L^{p,\infty}(\mathbb{R}^n)$,
- (b) $G(f) = (\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 \chi_{Q_{j,k}} |Q_{j,k}|^{-1})^{1/2} \in L^{p,\infty}(\mathbb{R}^n)$,
where $Q_{j,k} = A^{-j}([0, 1]^n + k)$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}^n$,
- (c) $S(f) = (\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 \chi_{R_{j,k}} |Q_{j,k}|^{-1})^{1/2} \in L^{p,\infty}(\mathbb{R}^n)$,
where for every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, $R_{j,k}$ is a measurable set such that $R_{j,k} \subset Q_{j,k}$ and $|R_{j,k}| \geq \gamma |Q_{j,k}|$, for a certain $\gamma \in (0, 1)$,
- (d) the distribution $f = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} \langle f, \psi_{j,k} \rangle \psi_{j,k}$ is in $H_A^{p,\infty}(\mathbb{R}^n)$.

Moreover, if one of (and then all) the above conditions is satisfied, then

$$\|f\|_{H_A^{p,\infty}(\mathbb{R}^n)} \sim \|W(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \sim \|G(f)\|_{L^{p,\infty}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p,\infty}(\mathbb{R}^n)}. \quad (2.7)$$

Bownik proved in [1, Theorem 4.2, page 94] that there exists a tight frame A -wavelet $\psi \in S(\mathbb{R}^n)$ satisfying the conditions in Theorem 2.2 such that $\int_{\mathbb{R}^n} f(x)P(x)dx = 0$, for every polynomial P in \mathbb{R}^n .

3. Proof of Theorem 2.2

In this section we present a proof of Theorem 2.2. Throughout this section with C we denote a positive constant that can change in each occurrence.

(a) \Rightarrow (c) According to [1, equation (2.2), page 5], $A^j x \rightarrow 0$, as $j \rightarrow -\infty$, uniformly in $x \in [0, 1]^n$. Moreover, there exist $0 < \eta < 1$ and $\delta > 0$ for which

$$|\psi(x)| \geq \delta, \quad x \in [0, \eta]^n. \quad (3.1)$$

Then, for every $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we have

$$|\psi_{j,k}(x)| \geq \frac{\delta}{\sqrt{|Q_{j,k}|}}, \quad x \in R_{j,k}, \quad (3.2)$$

where $R_{j,k} = A^{-j}([0, \eta]^n + k) \subseteq Q_{j,k}$. Hence, we can write

$$\left(\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}(x)|^2 \right)^{1/2} \geq \delta \left(\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 \frac{1}{|Q_{j,k}|} \chi_{R_{j,k}}(x) \right)^{1/2}. \quad (3.3)$$

Note that

$$|R_{j,k}| = \left| A^{-j} [0, \eta]^n \right| = |\det A|^{-j} \eta^n = |Q_{j,k}| \eta^n. \quad (3.4)$$

Thus we show that (a) implies (b).

(b) \Rightarrow (c) It is clear.

(c) \Rightarrow (b) Assume that (c) holds. For every $k \in \mathbb{Z}$, we define

$$E_k = \left\{ x \in \mathbb{R}^n : Sf(x) > 2^k \right\}. \quad (3.5)$$

According to our assumption we have that $\sup_{k \in \mathbb{Z}} 2^{kp} |E_k| < \infty$. Fix $\beta \in (0, \gamma)$ and $k \in \mathbb{N}$. We denote by D_k the set defined by

$$D_k = \left\{ (j, l) \in \mathbb{Z} \times \mathbb{Z}^n : |Q_{-j,l} \cap E_k| \geq \beta |Q_{-j,l}| \right\}. \quad (3.6)$$

By $D_{k,\max}$ we represent the set that consists of all $(j, l) \in D_k$ such that $Q_{-j,l}$ is maximal with respect to the order in \leq_{D_k} introduced in [1, Definition 6.4, page 105].

It is clear that

$$|Q_{-j,l}| \leq \frac{1}{\beta} |Q_{-j,l} \cap E_k| \leq C 2^{-kp}, \quad (j, l) \in D_k. \quad (3.7)$$

Moreover, $|Q_{-j,l}| = |\det A|^j$, $j \in \mathbb{Z}$, and $l \in \mathbb{Z}^n$. Hence, there exists $j_0 \in \mathbb{Z}$ such that $j \leq j_0$ provided that $(j, l) \in D_k$. Then, for every $(j, l) \in D_k$, there exists $(j_1, l_1) \in D_{k,\max}$ for which $Q_{-j,l} \leq_{D_k} Q_{-j_1,l_1}$ (see [1, page 105]).

For every $(j, l) \in D_{k,\max}$ we define

$$D_k(j, l) = \left\{ (m, s) \in D_k : Q_{-m,s} \leq_{D_k} Q_{-j,l} \right\}. \quad (3.8)$$

By [1, Lemma 6.5, page 105], we can find $\eta \in \mathbb{N}$ such that for every $(j, l) \in D_{k,\max}$ we have

$$\bigcup_{(m,s) \in D_k(j,l)} Q_{-m,s} \subseteq \bigcup_{|l-l_1| < \eta} Q_{-j,l_1}. \quad (3.9)$$

If $E_k^* = \bigcup_{(j,l) \in D_k} Q_{-j,l}$, it follows that

$$\begin{aligned}
 |E_k^*| &= \left| \bigcup_{(j,l) \in D_{k,\max}} \bigcup_{(m,s) \in D_k(j,l)} Q_{-m,s} \right| \\
 &\leq \sum_{(j,l) \in D_{k,\max}} \left| \bigcup_{(m,s) \in D_k(j,l)} Q_{-m,s} \right| \\
 &\leq \sum_{(j,l) \in D_{k,\max}} \left| \bigcup_{|l-l_1| < \eta} Q_{-j,l_1} \right| \\
 &\leq (2\eta + 1)^n \sum_{(j,l) \in D_{k,\max}} |\det A|^j \\
 &= (2\eta + 1)^n \sum_{(j,l) \in D_{k,\max}} |Q_{-j,l}| \\
 &\leq \frac{(2\eta + 1)^n}{\beta} \sum_{(j,l) \in D_{k,\max}} |Q_{-j,l} \cap E_k| \\
 &\leq \frac{(2\eta + 1)^n}{\beta} |E_k|.
 \end{aligned} \tag{3.10}$$

In the last inequality we have used $|Q_{-j,l} \cap Q_{-j_1,l_1}| = 0$ provided that $(j,l), (j_1,l_1) \in D_{k,\max}$, $(j,l) \neq (j_1,l_1)$.

By proceeding as in [1, page 107] by induction on $(j,l) \in D_{k,\max}$ we find, for every $(j,l) \in D_{k,\max}$, a set

$$I(j,l) \subseteq D_k(j,l), \tag{3.11}$$

satisfying the following properties:

$$D_k = \bigcup_{(j,l) \in D_{k,\max}} I(j,l), \tag{3.12}$$

$$I(j,l) \cap I(j_1,l_1) = \emptyset, \quad (j,l), (j_1,l_1) \in D_{k,\max}, \quad (j,l) \neq (j_1,l_1).$$

Note that $D_{k+1} \subseteq D_k$. We can consider the sets $Z_k = D_k \setminus D_{k+1}$, $Z_k(j,l) = Z_k \cap I(j,l)$ and $M_k(j,l) = \bigcup_{(m,s) \in Z_k(j,l)} Q_{-m,s}$, for every $(j,l) \in D_{k,\max}$.

Let $(j, l) \in D_{k, \max}$. We can write

$$\begin{aligned}
 \int_{M_k(j, l) \setminus E_{k+1}} |S(f)(x)|^2 dx &= \int_{M_k(j, l) \setminus E_{k+1}} \sum_{m \in \mathbb{Z}, s \in \mathbb{Z}^n} |\langle f, \psi_{-m, s} \rangle|^2 \frac{1}{|Q_{-m, s}|} \chi_{R_{-m, s}}(x) dx \\
 &\geq \int_{M_k(j, l) \setminus E_{k+1}} \sum_{(m, s) \in Z_k(j, l)} |\langle f, \psi_{-m, s} \rangle|^2 \frac{1}{|Q_{-m, s}|} \chi_{R_{-m, s}}(x) dx \\
 &\geq \sum_{(m, s) \in Z_k(j, l)} |\langle f, \psi_{-m, s} \rangle|^2 \frac{1}{|Q_{-m, s}|} |R_{-m, s} \cap E_{k+1}^c|.
 \end{aligned} \tag{3.13}$$

Also, for every $(m, s) \in Z_k(j, l)$, we get

$$|R_{-m, s} \cap E_{k+1}^c| = |R_{-m, s}| - |R_{-m, s} \cap E_{k+1}| \geq (\gamma - \beta) |Q_{-m, s}|, \tag{3.14}$$

because $|R_{-m, s}| \geq \gamma |Q_{-m, s}|$, and, since $(m, s) \notin D_{k+1}$,

$$|R_{-m, s} \cap E_{k+1}| \leq |Q_{-m, s} \cap E_{k+1}| < \beta |Q_{-m, s}|. \tag{3.15}$$

Hence

$$\int_{M_k(j, l) \setminus E_{k+1}} |S(f)(x)|^2 dx \geq \sum_{(m, s) \in Z_k(j, l)} |\langle f, \psi_{-m, s} \rangle|^2 (\gamma - \beta). \tag{3.16}$$

On the other hand, we have that

$$\int_{M_k(j, l) \setminus E_{k+1}} |S(f)(x)|^2 dx \leq 4^{k+1} |M_k(j, l) \setminus E_{k+1}|. \tag{3.17}$$

We conclude that

$$\sum_{(m, s) \in Z_k(j, l)} |\langle f, \psi_{-m, s} \rangle|^2 \leq \frac{4^{k+1}}{\gamma - \beta} |M_k(j, l)|. \tag{3.18}$$

Again according to [1, Lemma 6.5] (see (3.9)) it follows that

$$M_k(j, l) \subseteq \bigcup_{|s-l| < \eta} Q_{-j, s}. \tag{3.19}$$

Then

$$|M_k(j, l)| \leq \sum_{|s-l| < \eta} |Q_{-j, s}| \leq (2\eta + 1)^n |\det A|^j. \tag{3.20}$$

Therefore

$$\sum_{(m,s) \in Z_k(j,l)} |\langle f, \varphi_{-m,s} \rangle|^2 \leq \frac{(2\eta+1)^n}{\gamma-\beta} 4^{k+1} |Q_{-j,l}|. \quad (3.21)$$

Let $\nu > 0$. We choose $k_0 \in \mathbb{N}$ such that $2^{k_0} \leq \nu < 2^{k_0+1}$ and we define the functions G_1 and G_2 by

$$\begin{aligned} G_1(f) &= \left(\sum_{k=-\infty}^{k_0} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} |\langle f, \varphi_{-m,s} \rangle|^2 \frac{1}{|Q_{-m,s}|} \chi_{Q_{-m,s}} \right)^{1/2}, \\ G_2(f) &= \left(\sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} |\langle f, \varphi_{-m,s} \rangle|^2 \frac{1}{|Q_{-m,s}|} \chi_{Q_{-m,s}} \right)^{1/2}. \end{aligned} \quad (3.22)$$

Note that if $(m,s) \in \mathbb{Z} \times \mathbb{Z}^n$ and $\langle f, \varphi_{-m,s} \rangle \neq 0$ then there exists $k_1 \in \mathbb{Z}$ such that $R_{-m,s} \subseteq E_{k_1}$. Hence $(m,s) \in D_k$, for every $k \in \mathbb{Z}$, $k \leq k_1$. Moreover, there exists $k_2 \in \mathbb{Z}$ such that $(m,s) \notin D_k$, when $k \geq k_2$. We deduce that $G^2 = G_1^2 + G_2^2$. By (3.10) and (3.21), since $\sum_{(j,l) \in D_{k,\max}} |Q_{-j,l}| \leq |E_k^*|$, we infer

$$\begin{aligned} \|G_1 f\|_2^2 &\leq \frac{(2\eta+1)^n}{\gamma-\beta} \sum_{k=-\infty}^{k_0} \sum_{(j,l) \in D_{k,\max}} 4^{k+1} |Q_{-j,l}| \\ &\leq \frac{(2\eta+1)^n}{\gamma-\beta} \sum_{k=-\infty}^{k_0} 4^{k+1} |E_k^*| \\ &\leq \frac{(2\eta+1)^{2n}}{\beta(\gamma-\beta)} \sum_{k=-\infty}^{k_0} 4^{k+1} |E_k| \\ &\leq \frac{4(2\eta+1)^{2n}}{\beta(\gamma-\beta)} \sum_{k=-\infty}^{k_0} 2^{(2-p)k} \\ &\leq \frac{4(2\eta+1)^{2n}}{\beta(\gamma-\beta)} \frac{2^{(2-p)k_0}}{1-2^{-(2-p)}} \\ &\leq C\nu^{2-p}. \end{aligned} \quad (3.23)$$

Moreover, since $G_2(f)(x) = 0$ when $x \notin \bigcup_{k \geq k_0+1} E_k^*$, by (3.10), it follows that

$$\begin{aligned} |\{x \in \mathbb{R}^n : G_2(f)(x) \neq 0\}| &\leq \sum_{k=k_0+1}^{\infty} |E_k^*| \leq \frac{(2\eta+1)^n}{\beta} \sum_{k=k_0+1}^{\infty} |E_k| \\ &\leq \frac{C(2\eta+1)^n}{\beta} \sum_{k=k_0+1}^{\infty} 2^{-kp} \leq C\nu^{-p}. \end{aligned} \quad (3.24)$$

Then, we conclude that

$$\begin{aligned}
& |\{x \in \mathbb{R}^n : G(f)(x) > \nu\}| \\
& \leq \left| \left\{ x \in \mathbb{R}^n : G_1(f)(x) > \frac{\nu}{2} \right\} \right| + |\{x \in \mathbb{R}^n : G_2(f)(x) \neq 0\}| \\
& \leq \int_{\{x \in \mathbb{R}^n : G_1(f)(x) > \nu/2\}} \left(\frac{2G_1(f)(x)}{\nu} \right)^2 dx + C\nu^{-p} \\
& \leq C\nu^{-p}.
\end{aligned} \tag{3.25}$$

Thus (b) is proved.

(c) \Rightarrow (d) Suppose that (c) holds. We keep the notation that it was used in the proof of (c) \Rightarrow (b).

Let $\nu > 0$. We choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \nu < 2^{k_0+1}$ and we define

$$f_1 = \sum_{k=-\infty}^{k_0} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} \langle f, \psi_{-m,s} \rangle \psi_{-m,s}. \tag{3.26}$$

By proceeding as in the proof of (c) \Rightarrow (b) we get

$$\sum_{k=-\infty}^{k_0} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} |\langle f, \psi_{-m,s} \rangle|^2 \leq C\nu^{2-p}. \tag{3.27}$$

Then, since ψ is a tight frame A -wavelet, by using duality, we obtain that

$$\|f_1\|_2^2 \leq C \sum_{k=-\infty}^{k_0} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} |\langle f, \psi_{-m,s} \rangle|^2 \leq C\nu^{2-p}. \tag{3.28}$$

We now consider

$$f_2 = \sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} \sum_{(m,s) \in Z_k(j,l)} \langle f, \psi_{-m,s} \rangle \psi_{-m,s}. \tag{3.29}$$

We choose $\varphi \in S(\mathbb{R}^n)$ such that $\text{supp } \hat{\varphi}$ is compact and bounded away from the origin and that $\sum_{l \in \mathbb{Z}} \hat{\varphi}((A^*)^l y) = 1$, $y \in \mathbb{R}^n \setminus \{0\}$. We are going to prove that

$$\left| \left\{ x \notin \mathbb{R}^2 : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_2(x)| > \nu \right\} \right| \leq \frac{C}{\nu^p}. \tag{3.30}$$

For every $(j, l) \in D_{k, \max}$, we define

$$C(j, l) = \bigcup_{|l-l_1| \leq b\eta} Q_{-j, l_1}, \quad (3.31)$$

where $\eta > 0$ is the one appeared in (3.9) and $b > 2$ will be chosen later. We also consider

$$\Omega = \bigcup_{k=k_0+1}^{\infty} \bigcup_{(j, l) \in D_{k, \max}} C(j, l). \quad (3.32)$$

If $(j, l) \in D_{k, \max}$, then

$$|C(j, l)| \leq \sum_{|l-l_1| \leq b\eta} |Q_{-j, l_1}| \leq (2b\eta + 1)^n |Q_{-j, l}|. \quad (3.33)$$

Hence, since $\sum_{(j, l) \in D_{k, \max}} |Q_{-j, l}| \leq |E_k^*|$, by (3.10) it follows that

$$\begin{aligned} |\Omega| &= \left| \bigcup_{k=k_0+1}^{\infty} \bigcup_{(j, l) \in D_{k, \max}} C(j, l) \right| \leq (2b\eta + 1)^n \sum_{k=k_0+1}^{\infty} |E_k^*| \leq C \sum_{k=k_0+1}^{\infty} |E_k| \\ &\leq C \sum_{k=k_0+1}^{\infty} 2^{-kp} \leq C 2^{-k_0 p} \leq \frac{C}{\nu^p}. \end{aligned} \quad (3.34)$$

By (3.34), (3.30) will be proved when we obtain that

$$\left| \left\{ x \notin \Omega : \sup_{\alpha \in \mathbb{Z}} |\varphi_{\alpha} * f_2(x)| > \nu \right\} \right| \leq \frac{C}{\nu^p}. \quad (3.35)$$

Since $\text{supp } \hat{\varphi}$ and $\text{supp } \hat{\varphi}$ are compact and $0 \notin \text{supp } \hat{\varphi} \cup \text{supp } \hat{\varphi}$, there exists $M > 0$ such that $\text{supp } \hat{\varphi}_{-m, s} \cap \text{supp } \hat{\varphi}_{\alpha} = \emptyset$, provided that $|m - \alpha| > M$, $m, \alpha \in \mathbb{Z}$, and $s \in \mathbb{Z}^n$.

Let $\alpha \in \mathbb{Z}$. We can write

$$\varphi_{\alpha} * f_2 = \sum_{k=k_0+1}^{\infty} \sum_{(j, l) \in D_{k, \max}} \sum_{m=\alpha-M}^{\alpha+M} \sum_{s: (m, s) \in Z_k(j, l)} \langle f, \varphi_{-m, s} \rangle \varphi_{\alpha} * \varphi_{-m, s}. \quad (3.36)$$

Using Hölder's inequality we get

$$\begin{aligned} &\left| \sum_{m=\alpha-M}^{\alpha+M} \sum_{s: (m, s) \in Z_k(j, l)} \langle f, \varphi_{-m, s} \rangle \varphi_{\alpha} * \varphi_{-m, s}(x) \right| \\ &\leq \left(\sum_{m=\alpha-M}^{\alpha+M} \sum_{s: (m, s) \in Z_k(j, l)} |\langle f, \varphi_{-m, s} \rangle|^2 \right)^{1/2} \left(\sum_{m=\alpha-M}^{\alpha+M} \sum_{s: (m, s) \in Z_k(j, l)} |(\varphi_{\alpha} * \varphi_{-m, s})(x)|^2 \right)^{1/2}. \end{aligned} \quad (3.37)$$

From (3.21) we deduce that

$$\left(\sum_{m=\alpha-M}^{\alpha+m} \sum_{s:(m,s) \in Z_k(j,l)} |\langle f, \varphi_{-m,s} \rangle|^2 \right)^{1/2} \leq \left(\sum_{(m,s) \in Z_k(j,l)} |\langle f, \varphi_{-m,s} \rangle|^2 \right)^{1/2} \leq C 2^k |Q_{-j,l}|^{1/2}. \quad (3.38)$$

According to [13, page 1482] for any $\lambda > 1$ there exists $C = C(\lambda) > 0$ for which

$$|(\varphi_\beta * \varphi_{-m,s})(x)| \leq C |\det A|^{-m/2} (1 + \rho_A(A^{-m}x - s))^{-\lambda}, \quad x \in \mathbb{R}^n, \quad (3.39)$$

when $|\beta - m| \leq M$.

Let $\lambda > 0$ that will be fixed later. We can write

$$\begin{aligned} & \left(\sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} |(\varphi_\alpha * \varphi_{-m,s})(x)|^2 \right)^{1/2} \\ & \leq C \left(\sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} \frac{|\det A|^{-m}}{(1 + \rho_A(A^{-m}x - s))^{2\lambda}} \right)^{1/2} \\ & \leq C \sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} \frac{|\det A|^{-m/2}}{(1 + \rho_A(A^{-m}x - s))^\lambda}, \quad x \in \mathbb{R}^n. \end{aligned} \quad (3.40)$$

Let $k \in \mathbb{Z}$, $k > k_0$, and $(j, l) \in D_{k, \max}$. Assume that $x \notin \Omega$. Then $x \notin C(j, l)$. By (3.9) we know that

$$\bigcup_{(m,s) \in Z_k(j,l)} Q_{-m,s} \subseteq \bigcup_{|l-l_1| < \eta} Q_{-j,l_1}. \quad (3.41)$$

Let $(m, s) \in Z_k(j, l)$. We define

$$x' = A^j x, \quad x_{m,s} = A^m s, \quad x'_{m,s} = A^j x_{m,s}, \quad x_{j,l} = A^{-j} l, \quad x'_{j,l} = l. \quad (3.42)$$

Note that

$$\begin{aligned} x' & \notin \bigcup_{|l-l_1| < b\eta} ([0, 1]^n + l_1), \\ x'_{m,s} & \in \bigcup_{|l-l_1| < \eta} ([0, 1]^n + l_1), \\ x'_{j,l} & \in \bigcup_{|l-l_1| < \eta} ([0, 1]^n + l_1). \end{aligned} \quad (3.43)$$

We have that

$$\rho_A(x' - x'_{m,s}) \geq \frac{1}{H} \rho_A(x' - x'_{j,l}) - \rho_A(x'_{m,s} - x'_{j,l}). \quad (3.44)$$

By [1, Lemma 3.2] it follows that, for certain $\gamma > 0$,

$$\rho_A(x'_{m,s} - x'_{j,l}) \leq C\eta^\gamma, \quad (3.45)$$

and, for some $\gamma' > 0$,

$$\rho_A(x' - x'_{j,l}) \geq C(b\eta)^{\gamma'}. \quad (3.46)$$

Choosing b large enough we get that

$$\rho_A(x' - x'_{m,s}) \geq \frac{1}{2H} \rho_A(x' - x'_{j,l}). \quad (3.47)$$

Then,

$$\begin{aligned} \rho_A(x - x_{m,s}) &= \rho_A(A^{-j}(x' - x'_{m,s})) = |\det A|^{-j} \rho_A(x' - x'_{m,s}) \\ &\geq \frac{|\det A|^{-j}}{2H} \rho_A(x' - x'_{j,l}) = \frac{1}{2H} \rho_A(x' - x'_{j,l}). \end{aligned} \quad (3.48)$$

By (3.40) and (3.48) with $\lambda > 3/2$ we obtain that

$$\begin{aligned} &\left(\sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} |(\varphi_\alpha * \varphi_{-m,s})(x)|^2 \right)^{1/2} \\ &\leq \frac{C}{\rho_A(x - x_{j,l})^\lambda} \sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} |\det A|^{-m(1/2-\lambda)} \\ &\leq \frac{C}{\rho_A(x - x_{j,l})^\lambda} \sum_{m=\alpha-M}^{\alpha+M} \left(\sum_{s:(m,s) \in Z_k(j,l)} |\det A|^m \right)^{-1/2+\lambda} \\ &\leq \frac{C}{\rho_A(x - x_{j,l})^\lambda} \sum_{m=\alpha-M}^{\alpha+M} \left| \bigcup_{s:(m,s) \in Z_k(j,l)} Q_{-m,s} \right|^{-1/2+\lambda} \\ &\leq C \frac{|Q_{-j,l}|^{-1/2+\lambda}}{\rho_A(x - x_{j,l})^\lambda}. \end{aligned} \quad (3.49)$$

In the last inequality we have used (3.20). From (3.37), (3.38), and (3.49) it follows that

$$\left| \sum_{m=\alpha-M}^{\alpha+M} \sum_{s:(m,s) \in Z_k(j,l)} \langle f, \psi_{-m,s} \rangle \varphi_\alpha * \psi_{-m,s}(x) \right| \leq C \frac{2^k |Q_{-j,l}|^\lambda}{\rho_A(x - x_{j,l})^\lambda}, \quad \lambda \geq \frac{3}{2}. \quad (3.50)$$

Whence

$$\begin{aligned} \left| \left\{ x \notin \Omega : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_2(x)| > \nu \right\} \right| &\leq \left| \left\{ x \notin \Omega : \sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} \frac{C2^k |Q_{-j,l}|^\lambda}{\rho_A(x - x_{j,l})^\lambda} > \nu \right\} \right| \\ &\leq \left| \left\{ x \notin \Omega : \sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} c_{j,l}^k g_{j,l}^k > \nu \right\} \right|, \end{aligned} \quad (3.51)$$

where $\lambda > 3/2$, $c_{j,l}^k = C2^k |Q_{-j,l}|^\lambda$ and

$$g_{j,l}^k(x) = \frac{\chi_{\Omega^c}(x)}{\rho_A(x - x_{j,l})^\lambda}, \quad x \in \mathbb{R}^n. \quad (3.52)$$

By [2, equation (2.4)] we know that $|B_{\rho_A}(a, r)| \approx r$, $a \in \mathbb{R}^n$, and $r > 0$. Therefore,

$$\left| \left\{ x \in \mathbb{R}^n : |g_{j,l}^k(x)| > \omega \right\} \right| = \left| \left\{ x \in \Omega^c : \rho_A(x - x_{j,l}) < \omega^{-1/\lambda} \right\} \right| \leq C\omega^{-1/\lambda}, \quad \omega > 0. \quad (3.53)$$

Hence, according to [9, Lemma 2.1], by taking $\lambda > 1/p$, we obtain

$$\begin{aligned} &\left| \left\{ x \notin \Omega : \sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} \frac{C2^k |Q_{-j,l}|^\lambda}{\rho_A(x - x_{j,l})^\lambda} > \nu \right\} \right| \\ &\leq C \frac{1}{\nu^{1/\lambda}} \sum_{k=k_0+1}^{\infty} \sum_{(j,l) \in D_{k,\max}} 2^{k/\lambda} |Q_{-j,l}| \\ &\leq C \frac{1}{\nu^{1/\lambda}} \sum_{k=k_0+1}^{\infty} 2^{k/\lambda} |E_k^*| \leq C \frac{1}{\nu^{1/\lambda}} \sum_{k=k_0+1}^{\infty} 2^{k/\lambda} |E_k| \\ &\leq C \frac{1}{\nu^{1/\lambda}} \sum_{k=k_0+1}^{\infty} 2^{k(1/\lambda-p)} \leq C \frac{1}{\nu^{1/\lambda}} \frac{2^{k_0(1/\lambda-p)}}{1 - 2^{1/\lambda-p}} \leq \frac{C}{\nu^p}. \end{aligned} \quad (3.54)$$

Hence,

$$\left| \left\{ x \notin \Omega : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_2(x)| > \nu \right\} \right| \leq \frac{C}{\nu^p}. \quad (3.55)$$

Combining (3.28), (3.34), and (3.55) we get

$$\begin{aligned}
& \left| \left\{ x \in \mathbb{R}^n : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f(x)| > \nu \right\} \right| \\
& \leq \left| \left\{ x \in \mathbb{R}^n : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_1(x)| > \frac{\nu}{2} \right\} \right| + |\Omega| + \left| \left\{ x \notin \Omega : \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_2(x)| > \frac{\nu}{2} \right\} \right| \\
& \leq \frac{4}{\nu^2} \left\| \sup_{\alpha \in \mathbb{Z}} |\varphi_\alpha * f_1(x)| \right\|_2^2 + \frac{C}{\nu^p} \\
& \leq C \left(\frac{1}{\nu^2} \|f_1\|_2^2 + \frac{1}{\nu^p} \right) \\
& \leq \frac{C}{\nu^p}.
\end{aligned} \tag{3.56}$$

Thus we conclude that $f \in H_A^{p,\infty}(\mathbb{R}^n)$ and, therefore, (d) is established.

(d) \Rightarrow (a) We define the sublinear operator W as follows:

$$Wf = \left(\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}(x)|^2 \right)^{1/2}, \quad f \in L^2(\mathbb{R}^n). \tag{3.57}$$

Since ψ is a tight frame A -wavelet, we have that

$$\begin{aligned}
\|Wf\|_2^2 &= \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}(x)|^2 dx \\
&= \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\det A|^j \int_{\mathbb{R}^n} |\psi(A^j x - k)|^2 dx \\
&= \int_{\mathbb{R}^n} |\psi(x)|^2 dx \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 \\
&= \|f\|_2^2 \int_{\mathbb{R}^n} |\psi(x)|^2 dx, \quad f \in L^2(\mathbb{R}^n).
\end{aligned} \tag{3.58}$$

Hence, W is a bounded operator from $L^2(\mathbb{R}^n)$ into itself.

According to [1, Theorem 6.7, page 109], for every $0 < q \leq 1$ and $f \in H_A^q(\mathbb{R}^n)$,

$$\|f\|_{H_A^q(\mathbb{R}^n)} \sim \left\| \left(\sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^n} |\langle f, \psi_{j,k} \rangle|^2 |\psi_{j,k}(x)|^2 \right)^{1/2} \right\|_q. \tag{3.59}$$

Then, the operator W defined as above is also bounded from the anisotropic Hardy space $H_A^q(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, for every $0 < q \leq 1$. Our proof will be finished when we see that

the operator T is bounded from $H_A^{p,\infty}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$. In order to do this we proceed interpolating by using the ideas developed in the proof of [9, Theorem 3.4].

Let $f \in H_A^{p,\infty}(\mathbb{R}^n)$ and $\nu > 0$. We choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \nu < 2^{k_0+1}$. We take the atomic decomposition of f given by $f = \sum_{k=-\infty}^{\infty} f_k$. Here $f_k = \sum_{i=0}^{\infty} \beta_i^k$, $k \in \mathbb{Z}$, as in Theorem 2.1 with $s = [(2/p - 1)((\log |\det A|) / \log m)]$, where $m = \min\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$. We decompose f as follows:

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k = F_1 + F_2. \quad (3.60)$$

We have that

$$\begin{aligned} \|F_1\|_2 &\leq \sum_{k=-\infty}^{k_0} \|f_k\|_2 \leq C \sum_{k=-\infty}^{k_0} 2^k \left(\sum_{i=0}^{\infty} |\tilde{B}_i^k| \right)^{1/2} \\ &\leq C \sum_{k=-\infty}^{k_0} 2^{k(1-p/2)} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^{p/2} \\ &\leq C 2^{k_0(1-p/2)} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^{p/2}. \end{aligned} \quad (3.61)$$

Then,

$$\begin{aligned} |\{x \in \mathbb{R}^n : |WF_1(x)| > \nu\}| &\leq \frac{C}{\nu^2} \|F_1\|_2^2 \\ &\leq C \frac{2^{k_0(2-p)}}{\nu^2} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p \\ &\leq \frac{C}{\nu^p} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p. \end{aligned} \quad (3.62)$$

According to Theorem 2.1, there exists $C > 0$ such that, for every $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, $C2^{-k}|\tilde{B}_i^k|^{-2/p}\beta_i^k$ is a $(p/2, \infty, s)H_A^{p/2}(\mathbb{R}^n)$ -atom. Hence, for every $k \in \mathbb{Z}$, $f_k \in H_A^{p/2}(\mathbb{R}^n)$ and

$$\|f_k\|_{H_A^{p/2}(\mathbb{R}^n)}^{p/2} \leq C \sum_{i=0}^{\infty} 2^{kp/2} |\tilde{B}_i^k| \leq C 2^{-kp/2} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p. \quad (3.63)$$

Since W is bounded from $H_A^{p/2}(\mathbb{R}^n)$ into $L^{p/2}(\mathbb{R}^n)$, it follows that

$$\begin{aligned} |\{x \in \mathbb{R}^n : |W(f_k)(x)| > \nu\}| &\leq \nu^{-p/2} \|W(f_k)\|_{p/2}^{p/2} \\ &\leq C \nu^{-p/2} \|f_k\|_{H_A^{p/2}(\mathbb{R}^n)}^{p/2}, \quad k \in \mathbb{Z}. \end{aligned} \quad (3.64)$$

Therefore, using [9, Lemma 2.1] we get

$$\begin{aligned}
|\{x \in \mathbb{R}^n : |W(F_2)(x)| > \nu\}| &\leq \left| \left\{ x \in \mathbb{R}^n : \sum_{k=k_0+1}^{\infty} |W(f_k)(x)| > \nu \right\} \right| \\
&= \left| \left\{ x \in \mathbb{R}^n : \sum_{k=k_0+1}^{\infty} \|f_k\|_{H_A^{p/2}(\mathbb{R}^n)} \left| W\left(\frac{f_k}{\|f_k\|_{H_A^{p/2}(\mathbb{R}^n)}} \right)(x) \right| > \nu \right\} \right| \\
&\leq C\nu^{-p/2} \sum_{k=k_0+1}^{\infty} \|f_k\|_{H_A^{p/2}(\mathbb{R}^n)}^{p/2} \\
&\leq C\nu^{-p/2} \sum_{k=k_0+1}^{\infty} 2^{-kp/2} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p \\
&\leq C\nu^{-p/2} 2^{-k_0p/2} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p \\
&\leq \frac{C}{\nu^p} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p.
\end{aligned} \tag{3.65}$$

Putting together the above estimates we obtain

$$\begin{aligned}
|\{x \in \mathbb{R}^n : |W(f)(x)| > \nu\}| &\leq \left| \left\{ x \in \mathbb{R}^n : |W(F_1)(x)| > \frac{\nu}{2} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |W(F_2)(x)| > \frac{\nu}{2} \right\} \right| \\
&\leq \frac{C}{\nu^p} \|f\|_{H_A^{p,\infty}(\mathbb{R}^n)}^p.
\end{aligned} \tag{3.66}$$

Acknowledgment

B. Barrios is partially supported by MTM2010-16518 and J. J. Betancor is partially supported by MTM2010-17974.

References

- [1] M. Bownik, "Anisotropic Hardy spaces and wavelets," *Memoirs of the American Mathematical Society*, vol. 164, no. 781, 2003.
- [2] M. Bownik, "Atomic and molecular decompositions of anisotropic Besov spaces," *Mathematische Zeitschrift*, vol. 250, no. 3, pp. 539–571, 2005.
- [3] M. Bownik and K.-P. Ho, "Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces," *Transactions of the American Mathematical Society*, vol. 358, no. 4, pp. 1469–1510, 2006.
- [4] M. Bownik, "Anisotropic Triebel-Lizorkin spaces with doubling measures," *The Journal of Geometric Analysis*, vol. 17, no. 3, pp. 387–424, 2007.
- [5] M. Bownik, "Duality and interpolation of anisotropic Triebel-Lizorkin spaces," *Mathematische Zeitschrift*, vol. 259, no. 1, pp. 131–169, 2008.
- [6] C. Fefferman and E. M. Stein, " H^p spaces of several variables," *Acta Mathematica*, vol. 129, no. 3-4, pp. 137–193, 1972.

- [7] M. Bownik, B. Li, D. Yang, and Y. Zhou, "Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators," *Indiana University Mathematics Journal*, vol. 57, no. 7, pp. 3065–3100, 2008.
- [8] M. Bownik, B. Li, D. Yang, and Y. Zhou, "Weighted anisotropic product Hardy spaces and boundedness of sublinear operators," *Mathematische Nachrichten*, vol. 283, no. 3, pp. 392–442, 2010.
- [9] Y. Ding and S. Lan, "Anisotropic weak Hardy spaces and interpolation theorems," *Science in China. Series A. Mathematics*, vol. 51, no. 9, pp. 1690–1704, 2008.
- [10] H. P. Liu, "The wavelet characterization of the space weak H^1 ," *Studia Mathematica*, vol. 103, no. 1, pp. 109–117, 1992.
- [11] Y. Meyer, *Wavelets and Operators*, vol. 37 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1992.
- [12] R. Fefferman and F. Soria, "The space Weak H^1 ," *Studia Mathematica*, vol. 85, no. 1, pp. 1–16, 1986.
- [13] M. Bownik and K.-P. Ho, "Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces," *Transactions of the American Mathematical Society*, vol. 358, no. 4, pp. 1469–1510, 2006.