

Adiabatic renormalization of the stress-energy tensor for spin-1/2 fields.

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Abstract. We review the adiabatic renormalization method for spin-1/2 fields in FLRW spacetimes, emphasizing its similarities and differences with the scalar case. With it, we obtain a generic expression for the renormalized expectation value of the stress-energy tensor and analyze its properties. We particularize this result to de Sitter and radiation-dominated universes, recovering in the second case the equations of cold matter for late times.

1. Introduction

The renormalization of quantum fields in curved spacetime is an extensively studied issue [1, 2]. When one tries to compute the vacuum expectation value of quantities such as the stress-energy tensor, new additional ultraviolet divergences appear which are otherwise absent in Minkowski spacetime. Therefore, one needs to apply a well-motivated renormalization procedure specifically designed to work in curved spacetime.

One of the most useful for FLRW metrics is adiabatic regularization. In this method, the divergent terms are identified and subtracted through an expansion in momenta of the field modes defining the quantum state. By dimensionality, this is equivalent to an expansion in derivatives of the scale factor, hence the *adiabatic* denomination. This method was originally developed for scalar fields [3], and we recently generalized it to deal with spin-1/2 fields [4]. The key difference between both cases is that in the first one, the adiabatic expansion that identifies the divergent terms is of WKB type, while in the second one it is not. In order to check the validity of our approach, we renormalized adiabatically the trace and chiral anomalies and checked that our results were coincident with those obtained with other renormalization procedures.

In [5], we used this construction to obtain a full renormalized expression for the stress-energy tensor of a massive spin-1/2 field. We review here this work. After writing the equations of motion for spin-1/2 and spin-0 fields in a FLRW metric, we show how to obtain the adiabatic expansion of the fermion field modes and the similarities and differences with the well-known scalar case. We then renormalize the stress-energy tensor and analyze some of its properties, including the potential ambiguities associated to the renormalization program. We finally apply this result to two particular examples: de Sitter and radiation-dominated universes. In the second case, we obtain that the created particles obey at late-times the state equation of cold matter.



2. Basic equations

We first review the basic equations for spin-1/2 and spin-0 fields in the FLRW metric $ds^2 = dt^2 - a^2(t)d\vec{x}^2$. The notation used here is explained in more detail in [5].

2.1. Spin-1/2 fields

A spin-1/2 field ψ of mass m propagating in this metric can be written as

$$\psi(\vec{x}, t) = \int d^3\vec{k} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{(2\pi)^3 a^3}} \sum_{\lambda} \left[B_{\vec{k}\lambda} u_{\vec{k}\lambda}(t) + D_{-\vec{k}\lambda}^\dagger v_{-\vec{k}\lambda}(t) \right], \quad (1)$$

where $B_{\vec{k}\lambda}$ and $D_{\vec{k}\lambda}$ are destruction operators, and $u_{\vec{k}\lambda}(t)$ and $v_{\vec{k}\lambda}(t)$ are two time-dependent spinors defined as

$$u_{\vec{k}\lambda}(t) = \begin{pmatrix} h_k^I(t) \xi_{\lambda}(\vec{k}) \\ h_k^{II}(t) \frac{\vec{\sigma} \cdot \vec{k}}{k} \xi_{\lambda}(\vec{k}) \end{pmatrix}, \quad v_{\vec{k}\lambda}(t) = \begin{pmatrix} -h_k^{II*}(t) \xi_{-\lambda}(\vec{k}) \\ -h_k^{I*}(t) \frac{\vec{\sigma} \cdot \vec{k}}{k} \xi_{-\lambda}(\vec{k}) \end{pmatrix}. \quad (2)$$

Here, $h_k^I(t)$ and $h_k^{II}(t)$ are the field modes that define the quantum state, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual Pauli matrices, $k \equiv |\vec{k}|$, and ξ_{λ} are two-component helicity eigenstates with $\lambda = \pm 1$. In this decomposition, the field modes satisfy

$$h_k^{II} = \frac{ia}{k} (\partial_t + im) h_k^I, \quad h_k^I = \frac{ia}{k} (\partial_t - im) h_k^{II}, \quad |h_k^I|^2 + |h_k^{II}|^2 = 1 \quad (3)$$

where the first two are the equations of motion and the third one is the normalization condition.

2.2. Spin-0 field

A scalar field of mass m and coupling ξ to the curvature can be written in this metric as

$$\phi(\vec{x}, t) = \int d^3\vec{k} \frac{e^{i\vec{k}\vec{x}}}{\sqrt{(2\pi)^3 a^3}} \left[A_{\vec{k}} f_k(t) + A_{-\vec{k}}^\dagger f_k^*(t) \right], \quad (4)$$

where $A_{\vec{k}}$ are destruction operators and $f_k(t)$ are the field modes. They satisfy

$$\frac{d^2 f_k}{dt^2} + \left(\frac{k^2}{a^2} + m^2 + \sigma \right) f_k = 0, \quad f_k \dot{f}_k^* - f_k^* \dot{f}_k = i \quad (5)$$

with $\sigma \equiv (6\xi - 3/4)(\dot{a}/a)^2 + (6\xi - 3/2)\ddot{a}/a$. The first one is the equation of motion and the second one is the corresponding normalization condition.

3. Adiabatic expansion

The adiabatic regularization method is based on an expansion in momenta of the field modes (h_k^I and h_k^{II} for spin-1/2 fields, and f_k for scalars). This construction allows afterwards to identify and subtract the divergent terms from the original quantity. By dimensionality, this is equivalent to an expansion in derivatives of the scale factor, i.e. an adiabatic expansion. In any case, the expansion must recover in the adiabatic limit the solutions

$$f_k(t) \sim \frac{1}{\sqrt{2\omega}} e^{-i \int^t \omega(t') dt'}, \quad h_k^I(t) \sim \sqrt{\frac{\omega + m}{2\omega}} e^{-i \int^t \omega(t') dt'}, \quad h_k^{II}(t) \sim \sqrt{\frac{\omega - m}{2\omega}} e^{-i \int^t \omega(t') dt'} \quad (6)$$

with $\omega \equiv \sqrt{(k/a(t))^2 + m^2}$. In [4], we found that for spin-1/2 fields, there exists the following self-consistent expansion for the field modes

$$h_k^I(t) \sim \sqrt{\frac{\omega + m}{2\omega}} F(t) e^{-i \int^{t'} W(t') dt'} , \quad h_k^{II}(t) \sim \sqrt{\frac{\omega - m}{2\omega}} G(t) e^{-i \int^{t'} W(t') dt'} \quad (7)$$

where $\Omega(t)$, $F(t)$ and $G(t)$ are time-dependent functions that are expanded adiabatically as

$$W(t) = \sum_{n=0}^{\infty} \omega^{(n)}(t) , \quad F(t) = \sum_{n=0}^{\infty} F^{(n)}(t) , \quad G(t) = \sum_{n=0}^{\infty} G^{(n)}(t) . \quad (8)$$

Here, the superindex (n) in a function indicates that it is of adiabatic order n , which means that it contain n derivatives of the scale factor¹. Condition (6) imposes that the zeroth order terms are $\omega^{(0)}(t) = \omega$ and $F^{(0)} = G^{(0)} = 1$. The following terms are found by substituting (7) into (3) and solving iteratively the system of equations order by order. The terms obtained in this way contain ambiguities, which in any case do not appear in the final expressions of the renormalized quantities $\langle \bar{\psi}\psi \rangle_{ren}$ and $\langle T_{\mu\nu} \rangle_{ren}$. By simplicity, we can impose at all orders the additional condition $F^{(n)}(m) = G^{(n)}(-m)$ under the change of mass sign, which eliminates all the ambiguities. We obtain and display explicit expressions for $\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ up to fourth order in [4].

A natural question to ask is what is the relation of this approach to the one used for scalar fields. For these, the adiabatic expansion of the field modes is of the WKB form [3]

$$f_k(t) = \frac{1}{\sqrt{2\chi(t)}} e^{-i \int^t \chi(t') dt'} , \quad \chi(t) = \sum_{n=0}^{\infty} \chi^{(n)} . \quad (9)$$

We have $\chi^{(0)} = \omega$ from (6), and the other terms of the expansion are found by solving iteratively order by order the equation of motion (5) (note that the normalization condition is automatically satisfied). The key difference between (7) and (9) is that, in the first case, the adiabatic expansions of the multiplicative and exponent terms are independent, while in the second case they are not. In fact, one can try to obtain the adiabatic expansion for scalar fields with the more generic ansatz

$$f_k(t) = \frac{1}{\sqrt{2\omega}} H(t) e^{-i \int^t \Omega(t') dt'} , \quad \Omega(t) = \sum_{i=0}^{\infty} \Omega^{(i)}(t) , \quad H(t) = \sum_{i=0}^{\infty} H^{(i)}(t) \quad (10)$$

which mimics the one used for fermions in (7). Here, $H^{(0)} = 1$ and $\Omega^{(0)} = \omega$. If we substitute (10) into (5) and solve iteratively order by order, we find $H^{(n)} = \sqrt{\omega}(1/\sqrt{\chi})^{(n)}$ and $\Omega^{(n)} = \chi^{(n)}$, rediscovering this way the scalar WKB-type expansion.

To end with, we note that adiabatic regularization can be extended to spin-1 fields [6]. In this case, the adiabatic expansion of the field modes turn out to be WKB again.

4. Renormalization of the stress-energy tensor

We now show how to renormalize the vacuum expectation value of the stress-energy tensor of a spin-1/2 field using the formalism we have explained. As the FLRW universe is homogeneous and spatially isotropic, we only have two independent components: the 00-component (energy density) and the ii-component (pressure)

$$\langle T_{00} \rangle = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 \rho_k(t) , \quad \langle T_{ii} \rangle = \frac{1}{2\pi^2 a} \int_0^\infty dk k^2 p_k(t) \quad (11)$$

¹ For example, \dot{a} is of adiabatic order 1 and $\ddot{a}\dot{a}^2$ is of adiabatic order 4.

where

$$\rho_k = i \left(h_k^I \frac{\partial h_k^{I*}}{\partial t} + h_k^{II} \frac{\partial h_k^{II*}}{\partial t} - h_k^{I*} \frac{\partial h_k^I}{\partial t} - h_k^{II*} \frac{\partial h_k^{II}}{\partial t} \right), \quad p_k = -\frac{2k}{3a} (h_k^I h_k^{II*} + h_k^{I*} h_k^{II}). \quad (12)$$

The vacuum state is defined as $B_{\vec{k},\lambda}|0\rangle = D_{\vec{k},\lambda}|0\rangle = 0$. Both quantities in (11) contain quartic, quadratic and logarithmic ultraviolet divergences. To renormalize them, we must first expand their integrands adiabatically

$$\rho_k = \rho_k^{(0)} + \rho_k^{(1)} + \rho_k^{(2)} + \rho_k^{(3)} + \rho_k^{(4)} + \dots, \quad p_k = p_k^{(0)} + p_k^{(1)} + p_k^{(2)} + p_k^{(3)} + p_k^{(4)} + \dots \quad (13)$$

where here $\rho_k^{(n)}$ and $p_k^{(n)}$ are of n th adiabatic order (note that $\rho_k^{(odd)} = 0 = p_k^{(odd)}$). The different terms of the expansion are found by substituting (7) into (12), and are written explicitly in [5]. After this, we must subtract the terms of the expansion that cause the divergences. As for scalar fields [3, 1, 2], we must subtract up to fourth adiabatic order. Therefore, if we define the adiabatic subtraction terms as

$$\langle T_{00} \rangle_{Ad}^{(n)} \equiv \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 \rho_k^{(n)}, \quad \langle T_{ii} \rangle_{Ad}^{(n)} \equiv \frac{1}{2\pi^2 a} \int_0^\infty dk k^2 p_k^{(n)}, \quad (14)$$

the renormalized stress-energy tensor is

$$\langle T_{\mu\nu} \rangle_{ren} \equiv \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle_{Ad}^{(0)} - \langle T_{\mu\nu} \rangle_{Ad}^{(2)} - \langle T_{\mu\nu} \rangle_{Ad}^{(4)}. \quad (15)$$

We now analyze some of its properties.

4.1. Finite fourth-order subtraction terms

One finds that the fourth-order subtraction term can be written as

$$\langle T_{\mu\nu} \rangle_{Ad}^{(4)} = \frac{2}{2880\pi^2} \left[-\frac{1}{2} {}^{(1)}H_{\mu\nu} + \frac{11}{2} {}^{(3)}H_{\mu\nu} \right], \quad (16)$$

where ${}^{(1)}H_{\mu\nu}$ and ${}^{(3)}H_{\mu\nu}$ are covariant geometric tensors of fourth adiabatic order [5]. This tensor is mass-independent and finite. Therefore, one could naturally ask if it is really necessary to subtract this term in (15). However, for an arbitrary spacetime, the fourth adiabatic order do contain ultraviolet divergences, they just disappear accidentally for spatially flat FLRW metrics [7]. Therefore, by consistency, we must also subtract in this case the fourth adiabatic order. Note that this is similar to the renormalization of a scalar field with conformal coupling $\xi = 1/6$: in this case the fourth order subtraction term is also finite, but we must subtract it anyway [1].

4.2. Conservation

The renormalized tensor is conserved, $\nabla^\mu \langle T_{\mu\nu} \rangle_{ren} = 0$ as a consequence of the independent conservation laws $\nabla^\mu \langle T_{\mu\nu} \rangle_{Ad}^{(n)} = 0$, for $n = 0, 2, 4$.

4.3. Ambiguities and compatibility with other renormalization procedures

In the curved space renormalization program for the stress-energy tensor, there are in general four types of divergent subtraction terms that generate intrinsic ambiguities: $m^4 g_{\mu\nu}$, $m^2 G_{\mu\nu}$ (where $G_{\mu\nu}$ is the Einstein tensor), ${}^{(1)}H_{\mu\nu}$, and ${}^{(2)}H_{\mu\nu}$ (where ${}^{(2)}H_{\mu\nu}$ is another fourth-order curvature tensor) [2, 8]. We have checked in [5] that our renormalized tensor obeys this property.

In particular, in our FLRW spacetime, ${}^{(2)}H_{\mu\nu}$ is proportional to ${}^{(1)}H_{\mu\nu}$. Therefore, the most generic fourth-order subtraction term is, from (16),

$$\langle T_{\mu\nu} \rangle_{Ad}^{(4)} = \frac{2}{2880\pi^2} \left[c_1 {}^{(1)}H_{\mu\nu} + \frac{11}{2} {}^{(3)}H_{\mu\nu} \right] \quad (17)$$

where c_1 is an arbitrary coefficient. The different renormalization procedures can only potentially differ in the value of this coefficient. As we have seen, adiabatic regularization in particular leads to $c_1 = -1/2$. This ambiguity disappears for spacetime backgrounds for which the tensor ${}^{(1)}H_{\mu\nu}$ vanishes, such as de Sitter and radiation-dominated universes analyzed below. For more generic scale factors, this term rapidly vanishes at late-times.

From (16), the trace anomaly obtained by adiabatic regularization is

$$\langle T^\mu_\mu \rangle_{ren} = \langle T^\mu_\mu \rangle_{Ad}^{(4)} = \frac{2}{2880\pi^2} \left[-\frac{11}{2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + 3\Box R \right]. \quad (18)$$

This is in exact agreement with the conformal anomaly for spin-1/2 fields computed by point-splitting, zeta function, and dimensional regularization [2]. However, note that if one considers the general expression (17) instead of (16), the numerical coefficient of $\Box R$ is actually proportional to c_1 . Therefore, although we have not obtained the full renormalized stress-energy tensor with all these procedures, we know that they will yield $c_1 = -1/2$.

Finally, we would like to comment some recent results that give insight into the equivalence between adiabatic regularization and Schwinger-DeWitt approaches. In [9], the sixth-order contribution of the renormalized stress-energy tensor for scalar fields $\langle T_{\mu\nu} \rangle_{Ad}^{(6)}$ is computed analytically. This is a good approximation to the full tensor in the limit of high masses. In [10], this term is computed for spin-0, spin-1/2 and spin-1 fields using the Schwinger-DeWitt approach, obtaining for spin-0 fields the same result as in [9]. We have computed this term for spin-1/2 fields with the adiabatic regularization method described here, obtaining also the same result as in [10].

5. Examples

We now apply the formalism developed in the last sections to two particular scale factors: de Sitter and radiation-dominated universes. One must first solve the equations of motion (3). In practice, one can choose a set of two particular solutions $(h_{k,p}^I, h_{k,p}^{II})$ correctly normalized ($|h_{k,p}^I|^2 + |h_{k,p}^{II}|^2 = 1$) and then construct the full general solution with the Bogolubov-type rotation

$$h_k^I = E_k h_{k,p}^I + F_k h_{k,p}^{II*}, \quad h_k^{II} = E_k h_{k,p}^{II} - F_k h_{k,p}^{I*}. \quad (19)$$

Here, E_k and F_k are constants that must obey the following three conditions:

- (i) We require $|E_k|^2 + |F_k|^2 = 1$ so that the generic solution is also normalized.
- (ii) In the limit $k \rightarrow \infty$ the physical solutions must recover (6).
- (iii) The constants must not add extra ultraviolet divergences to $\langle T_{\mu\nu} \rangle$, so that the subtraction terms cancel all the divergences.

We now see an example of how this works below.

5.1. de Sitter spacetime

For de Sitter spacetime $a(t) = e^{Ht}$ with H a constant, the general solution to the field equations can be conveniently expressed, using the technique of equation (19), in terms of Hankel functions

$$h_{k,p}^I = i\sqrt{\frac{\pi k}{4H}} e^{\frac{\pi m}{2H} - \frac{Ht}{2}} H_{\frac{1}{2} - i\frac{m}{H}}^{(1)} \left(\frac{k}{H} e^{-Ht} \right), \quad h_{k,p}^{II} = \sqrt{\frac{\pi k}{4H}} e^{\frac{\pi m}{2H} - \frac{Ht}{2}} H_{-\frac{1}{2} - i\frac{m}{H}}^{(1)} \left(\frac{k}{H} e^{-Ht} \right). \quad (20)$$

Condition (ii) leads to $E_k \sim 1$ and $F_k \sim 0$ in the limit $k \rightarrow \infty$, and condition (iii) leads to $|E_k|^2 - |F_k|^2 = 1 + O(k^{-8})$. In this case, due to the symmetries of de Sitter spacetime, there exists the preferred quantum state $E_k = 1$, $F_k = 0$. This determines a vacuum for spin one-half fields analogous to the Bunch-Davies vacuum [11] for scalar fields. With this choice, the stress-energy tensor can be computed analytically [4, 5].

5.2. Radiation-dominated Universe

For the scale factor $a(t) = a_0 t^{1/2}$, the spin-1/2 equations of motion (3) admit as a solution

$$h_{k,p}^I = \frac{N}{\sqrt{a(t)}} W_{\kappa, \frac{1}{4}}(i2mt) , \quad h_{k,p}^{II} = \frac{Nk}{2ma(t)^{3/2}} \left[W_{\kappa, \frac{1}{4}}(i2mt) + \left(\kappa - \frac{3}{4} \right) W_{\kappa-1, \frac{1}{4}}(i2mt) \right] \quad (21)$$

Here, $W_{\kappa, 1/4}$ are Whittaker functions, $\kappa = \frac{1}{4} - ik^2/(a_0^2 2m)$, and $N = (a_0^{1/2}/(2m)^{1/4})e^{-(\pi k^2/4a_0^2 m)}$.

Condition (ii) gives $E_k \sim 1$ and $F_k \sim 0$ for $k \rightarrow \infty$, while condition (iii) gives $|E_k|^2 - |F_k|^2 = 1 + O(k^{-5})$ [5]. However, in contrast with Sitter spacetime, the absence of extra symmetries in the radiation-dominated universe does not help us to fix a natural quantum state. Nevertheless, we can make generic predictions of the behavior of the renormalized stress-energy tensor at early and late times even without specifying the choice of constants. We have [5]:

- For late times ($t \gg m^{-1}$): $\langle T^{00} \rangle_{ren} \sim \frac{\rho_{0m}}{a^3}$ and $\langle T^{ii} \rangle_{ren} \sim 0$.
- For early times ($t \ll m^{-1}$): $\langle T^{00} \rangle_{ren} \sim \frac{\rho_{0r}}{a^4}$ and $\langle T^{ii} \rangle_{ren} \sim \frac{1}{3} \langle T^{00} \rangle_{ren}$.

Here, ρ_{0m} and ρ_{0r} are finite and positive constants. Therefore, at late and early times we get the classical equations of radiation and cold matter respectively.

6. Conclusions

In this work, we have obtained the renormalized expectation value of the stress-energy tensor for spin-1/2 fields within the framework of adiabatic regularization and studied its properties.

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References

- [1] L. Parker and D. J. Toms, *Quantum field theory in curved spacetime: quantized fields and gravity*, Cambridge University Press, Cambridge, England (2009).
- [2] N. D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, England (1982).
- [3] L. Parker and S. A. Fulling, *Phys. Rev. D* **9**, 341 (1974); S. A. Fulling and L. Parker, *Ann. Phys. (N.Y.)* **87**, 176 (1974); S. A. Fulling, L. Parker, and B. L. Hu, *Phys. Rev. D* **10**, 3905 (1974).
- [4] A. Landete, J. Navarro-Salas and F. Torrenti, *Phys. Rev. D* **88**, 061501(R) (2013); *Phys. Rev. D* **89**, 044030 (2014).
- [5] A. del Rio, J. Navarro-Salas and F. Torrenti, *Phys. Rev. D* **90**, 084017 (2014).
- [6] L.P. Chimento and A. E. Cossarini, *Phys. Rev. D* **41**, 3101 (1990); I. Agullo, A. Landete and J. Navarro-Salas, arXiv:gr-qc/1409.6406
- [7] S. M. Christensen, *Phys. Rev. D* **17**, 946 (1978).
- [8] R. M. Wald, *Quantum field theory in curved space-time and black hole thermodynamics*, University of Chicago Press, Chicago, (1994).
- [9] A. Kaya and M. Tarman, *J. Cosmol. Astropart. Phys.* **04** (2011) 040.
- [10] J. Matyjasek and P. Sadurski, *Phys. Rev. D* **88**, 104015 (2013).
- [11] T.S. Bunch and P.C.W. Davies, *Proc. P. Soc. London A* **360**, 117 (1978).