

SPHERICAL TUPLES OF HILBERT SPACE OPERATORS

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ABSTRACT. We introduce and study a class of operator tuples in complex Hilbert spaces, which we call spherical tuples. In particular, we characterize spherical multi-shifts, and more generally, multiplication tuples on RKHS. We further use these characterizations to describe various spectral parts including the Taylor spectrum. We also find a criterion for the Schatten S_p -class membership of cross-commutators of spherical m -shifts. We show, in particular, that cross-commutators of non-compact spherical m -shifts cannot belong to S_p for $p \leq m$.

We specialize our results to some well-studied classes of multi-shifts. We prove that the cross-commutators of a spherical joint m -shift, which is a q -isometry or a 2-expansion, belongs to S_p if and only if $p > m$. We further give an example of a spherical jointly hyponormal 2-shift, for which the cross-commutators are compact but not in S_p for any $p < \infty$.

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1. INTRODUCTION

The motivation for the present paper comes from different directions. Firstly, as there is considerable literature on circular operators (refer to [44], [23], [4], [38], [40]), it is natural to look for the higher-dimensional analogs of circular operators. There are of course two possible analogs, namely, *poly-circular tuples* and *spherical tuples*. Multi-variable weighted shifts (for short, multi-shifts) form a subclass of the class of poly-circular tuples, and indeed, there are some important papers on this latter class (see, for instance, [32], [15], [27], [20]). There are also several important papers on multivariable weighted shifts that are spherical, see [8], [6], [2], [7], [1], [27], [25]. However, the higher-dimensional counter-parts of many important results in the masterful exposition [44] by A. Shields are either unknown or not formulated. The main objective of this paper is to introduce spherical operator tuples in an abstract way and to study some of their basic properties, as well as properties of spherical multi-variable weighted shifts, which form a subclass of this class.

One of our motivations is the following phenomenon concerning multi-dimensional cross-commutators and Hankel operators, which is often referred to as “cut-off”: in several particular situations, these operators cannot be too small unless they are zero, see [31], [48], [51], [19], [37], [50]. More recently, related questions have been studied in relation with the multi-variable Berger-Shaw theory and the so-called Arveson conjecture, see, for instance, [18], [3], [21], [22], [28], [34] and others. In our context, we prove that cross-commutators of non-compact spherical m -shifts do not belong to S_p for $p \leq m$.

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If \mathbb{N} stands for the set of non-negative integers, we denote by \mathbb{N}^m for the cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). Let $p = (p_1, \dots, p_m)$ and $n = (n_1, \dots, n_m)$ be in \mathbb{N}^m . We write $p \leq n$ if $p_j \leq n_j$ for $j = 1, \dots, m$. For $p \leq n$, $\binom{n}{p}$ is understood to be the product $\binom{n_1}{p_1} \cdots \binom{n_m}{p_m}$ and $|p|$ is understood to be $p_1 + \cdots + p_m$.

For a Hilbert space \mathcal{H} , let $\mathcal{H}^{(m)}$ denote the orthogonal direct sum of m copies of \mathcal{H} . Let $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators on \mathcal{H} . If the opposite is not specified, all the operators we consider will be assumed linear and bounded.

If $T = (T_1, \dots, T_m)$ is an m -tuple of commuting bounded linear operators T_j ($1 \leq j \leq m$) on \mathcal{H} then we set T^* to be (T_1^*, \dots, T_m^*) and T^p to be $T_1^{p_1} \cdots T_m^{p_m}$.

The main object of our interest in this paper is the class of so-called spherical tuples.

Definition 1.1 : Let T be an m -tuple of commuting bounded linear operators T_1, \dots, T_m on an infinite dimensional Hilbert space \mathcal{H} . Let $\mathcal{U}(m)$ denote the group of complex $m \times m$ unitary matrices. For $U = (u_{jk})_{1 \leq j, k \leq m} \in \mathcal{U}(m)$, the commuting operator m -tuple T_U is given by

$$(1.1) \quad (T_U)_j = \sum_{k=1}^m u_{jk} T_k \quad (1 \leq j \leq m).$$

We say that T is *spherical* if for every $U \in \mathcal{U}(m)$, there exists a unitary operator $\Gamma(U) \in B(\mathcal{H})$ such that $\Gamma(U)T_j = (T_U)_j\Gamma(U)$ for all $j = 1, \dots, m$. If, further, Γ can be chosen to be a strongly continuous unitary representation of $\mathcal{U}(m)$ on \mathcal{H} then we say that T is *strongly spherical*.

Remark 1.2 : Let $T = (T_1, \dots, T_m)$ be a spherical m -tuple.

- (1) Any permutation of T is unitarily equivalent to T . In particular, T_j is unitarily equivalent to T_k for any $1 \leq j, k \leq m$.
- (2) For any unital $*$ -representation $\pi : B(\mathcal{H}) \rightarrow B(\mathcal{H}_1)$, $\pi(T) := (\pi(T_1), \dots, \pi(T_m))$ is also a spherical m -tuple. Indeed, since π sends unitaries to unitaries, $\pi(\Gamma(U))\pi(T_j) = (\pi(T)_U)_j\pi(\Gamma(U))$ for all $j = 1, \dots, m$, and $\pi(\Gamma(U))$ is unitary. We also observe that $T^* = (T_1^*, \dots, T_m^*)$ is spherical.

Let $T = (T_1, \dots, T_m)$ be an m -tuple of commuting bounded linear operators on a Hilbert space \mathcal{H} . Let D_T denote the linear transformation from \mathcal{H} to $\mathcal{H}^{(m)}$, given by

$$D_T h := (T_1 h, \dots, T_m h) \quad (h \in \mathcal{H}).$$

Note that $\ker(D_T) = \bigcap_{i=1}^m \ker(T_i)$.

Next, we need to invoke the basics of the theory of multi-shifts [32]. First a definition.

Let T be an m -tuple of commuting operators T_1, \dots, T_m on a Hilbert space \mathcal{H} . A closed subspace \mathcal{M} of \mathcal{H} is said to be *cyclic* for T if

$$\mathcal{H} = \bigvee \{T^n x : x \in \mathcal{M}, n \in \mathbb{N}^m\}.$$

We say that T is *cyclic* with *cyclic vector* x if the subspace spanned by x is cyclic for T .

Let $\{w_n^{(j)} : 1 \leq j \leq m, n \in \mathbb{N}^m\}$ be a multi-sequence of complex numbers. An *m-variable weighted shift* $T = (T_1, \dots, T_m)$ with respect to an orthonormal basis $\{e_n\}_{n \in \mathbb{N}^m}$ of a Hilbert space \mathcal{H} is defined by

$$T_j e_n := w_n^{(j)} e_{n+\varepsilon_j} \quad (1 \leq j \leq m),$$

where ε_j is the m -tuple with 1 in the j th place and zeros elsewhere. The notation $T : \{w_n^{(j)}\}_{n \in \mathbb{N}^m}$ will mean that T is the m -variable weighted shift tuple with weight multi-sequence $\{w_n^{(j)} : 1 \leq j \leq m, n \in \mathbb{N}^m\}$. Notice that T_j commutes with T_k if and only if $w_n^{(j)} w_{n+\varepsilon_j}^{(k)} = w_n^{(k)} w_{n+\varepsilon_k}^{(j)}$ for all $n \in \mathbb{N}^m$. By [32, Corollary 9], T is bounded if and only if

$$\sup \{|w_n^{(j)}| : 1 \leq j \leq m, n \in \mathbb{N}^m\} < \infty.$$

We always assume that the weight multi-sequence of T consists of positive numbers and that T is commuting. Note that $T : \{w_n^{(j)}\}_{n \in \mathbb{N}^m}$ is cyclic with cyclic vector e_0 .

Let $T : \{w_n^{(j)}\}_{n \in \mathbb{N}^m}$ be an m -variable weighted shift. Define $\beta_n = \|T^n e_0\|$ ($n \in \mathbb{N}^m$) and consider the Hilbert space $H^2(\beta)$ of formal power series

$$f(z) = \sum_{n \in \mathbb{N}^m} a_n z^n$$

such that

$$\|f\|_\beta^2 = \sum_{n \in \mathbb{N}^m} |a_n|^2 \beta_n^2 < \infty.$$

It follows from [32, Proposition 8] that any m -variable weighted shift T is unitarily equivalent to the m -tuple $M_z = (M_{z_1}, \dots, M_{z_m})$ of multiplication by the co-ordinate functions z_1, \dots, z_m on the corresponding space $H^2(\beta)$. Notice that the linear set of polynomials in z_1, \dots, z_m (that is, formal power series with finitely many non-zero coefficients) is dense in $H^2(\beta)$. Equivalently, M_z is cyclic with cyclic vector the constant formal power series 1 (that is, the formal series $\sum_{n \in \mathbb{N}^m} a_n z^n$, for which $a_n = 0$ for all non-zero $n \in \mathbb{N}^m$ and $a_0 = 1$). The relation between weights $w_n^{(j)}$ and the sequence β_n is given by

$$(1.2) \quad w_n^{(j)} = \beta_{n+\varepsilon_j} / \beta_n, \quad 1 \leq j \leq m, \quad n \in \mathbb{N}^m.$$

Note further that $\ker(D_{M_z^*})$ is spanned by the constant formal power series 1.

Recall that all formal power series in $H^2(\beta)$ converge absolutely on the point-spectrum $\sigma_p(M_z^*)$ of the adjoint m -tuple M_z^* of M_z [32, Propositions 19 and 20]. In particular, $H^2(\beta)$ may be realized as a *reproducing kernel Hilbert space* (RKHS) with *reproducing kernel* $\varkappa : \sigma_p(T^*) \times \sigma_p(T^*) \rightarrow \mathbb{C}$ given by

$$(1.3) \quad \varkappa(z, w) = \sum_{n \in \mathbb{N}^m} \overline{w}^n z^n / \beta_n^2 \quad (z, w \in \sigma_p(T^*)),$$

assuming that $\sigma_p(M_z^*)$ has non-empty interior. Conversely, as it follows from Theorem 2.11 below, the multiplication m -tuple M_z acting in a RKHS \mathcal{H} with reproducing kernel \varkappa of the form (1.3) is unitarily equivalent to an m -variable weighted shift on $H^2(\beta)$ if all complex polynomials in z_1, \dots, z_m are contained in \mathcal{H} . Notice that the norm in $H^2(\beta)$ has polycircular symmetry: $\|f(\zeta \cdot z)\|_\beta = \|f(z)\|_\beta$ for any $f \in H^2(\beta)$ and any $\zeta \in \mathbb{T}^m$, where $\zeta \cdot z = (\zeta_1 z_1, \dots, \zeta_m z_m)$. So if the largest open set where all series in $H^2(\beta)$ converge is not empty, it is a Reinhardt domain.

We denote by \mathbb{B}_R the open ball centered at the origin and of radius $R > 0$:

$$\mathbb{B}_R := \{z = (z_1, \dots, z_m) \in \mathbb{C}^m : \|z\|_2^2 = |z_1|^2 + \dots + |z_m|^2 < R^2\}.$$

The sphere centered at the origin and of radius $R > 0$ is denoted by $\partial\mathbb{B}_R$. For simplicity, the unit ball \mathbb{B}_1 and the unit sphere $\partial\mathbb{B}_1$ are denoted respectively by \mathbb{B} and $\partial\mathbb{B}$.

Let us discuss three basic examples of (spherical) weighted m -variable shifts, with which we are primarily concerned.

Example 1.3. For any real number $p > 0$, let \mathcal{H}_p be the RKHS of holomorphic functions on the unit ball \mathbb{B} with reproducing kernel

$$\varkappa_p(z, w) = \frac{1}{(1 - \langle z, w \rangle)^p} \quad (z, w \in \mathbb{B}).$$

If $M_{z,p}$ denotes the multiplication tuple on \mathcal{H}_p then it is unitarily equivalent to the weighted shift m -tuple with weight sequence

$$(1.4) \quad w_{n,p}^{(i)} = \sqrt{\frac{n_i + 1}{|n| + p}} \quad (n \in \mathbb{N}^m, i = 1, \dots, m).$$

The RKHS's $\mathcal{H}_m, \mathcal{H}_{m+1}, \mathcal{H}_1$ are, respectively, the *Hardy space* $H^2(\partial\mathbb{B})$, the *Bergman space* $A^2(\mathbb{B})$, the *Drury-Arveson space* H_m^2 . The multiplication tuples $M_{z,m}, M_{z,m+1}, M_{z,1}$ are commonly known as the *Szegő m -shift*, the *Bergman m -shift*, the *Drury-Arveson m -shift* respectively. The spaces \mathcal{H}_p have been studied in many papers. In [47], a characterization of Carleson measures in these spaces has been given. In [33], the spaces $\mathcal{F}_q = \mathcal{H}_{1+m+q}$ have been studied; in particular, a kind of model theorem and von Neumann inequalities related to these spaces for row contractions is established there and some K-theory results are proved for the corresponding Toeplitz algebras. In this work, a scale of Dirichlet-type spaces corresponding to $q < 0$ is also considered, but their definition is different, and they do not belong to the collection of spaces \mathcal{H}_p .

As it is proved in [6], $M_{z,p}$ is subnormal for any $p \geq m$. In fact, $M_{z,p}$ is jointly subnormal if and only if $p \geq m$, see the discussion after Theorem 5.3.

The paper is organized as follows. In the second section, we present various characterizations of spherical tuples. The main results of this section are Theorem 2.1, where we characterize m -variable weighted shifts (equivalently, multiplication m -tuples), which are spherical, and Theorem 2.5, which gives abstract conditions, when an arbitrary spherical operator m -tuple is unitarily equivalent to a multiplication m -tuple. We also discuss some examples. In Section 3, we describe various spectral parts of spherical multi-shifts, including the Taylor spectrum. In particular, we obtain refinements of some results in [27]. In Section 4, we provide a sufficient and necessary condition for the Schatten p -class membership of cross-commutators of spherical m -shifts. We deduce that for a noncompact m -tuple M_z , if $[M_{z_j}^*, M_{z_k}] \in S_p$ for all j, k , then $p > m$ (which is a manifestation of the cut-off). Here $[A, B]$ stands for the commutator $AB - BA$ of operators A, B on a space \mathcal{H} . The results of Sections 3 and 4 rely heavily on the results of Section 2. In the last Section 5, we mainly discuss the cut-off phenomenon for some special classes of spherical multi-shifts, such as q -expansions, q -isometries and jointly hyponormal tuples.

2. SPHERICAL TUPLES

Let $\mathbb{C}[z]$ stand for the vector space of analytic polynomials in z_1, \dots, z_m . We define

$$\text{Hom}(k) = \left\{ p \in \mathbb{C}[z] : p(z) = \sum_{|n|=k} a_n z^n \right\}.$$

For a polynomial $p \in \mathbb{C}[z]$ and an integer $k \geq 0$, we denote by $p_{[k]} \in \text{Hom}(k)$ the homogeneous part of p of degree k . More generally, $f_{[k]}$ stands for the homogeneous part $\sum_{|n|=k} a_n z^n$ of a formal power series $f(z) = \sum_{n \in \mathbb{N}^m} a_n z^n$.

Let σ denote the normalized surface area measure on the unit sphere $\partial\mathbb{B}$. We often use the short notation $L^2(\partial\mathbb{B})$ for the Hilbert space $L^2(\partial\mathbb{B}, \sigma)$ of σ -square-integrable “functions” on $\partial\mathbb{B}$.

The first theorem of this section provides a handy characterization of spherical multi-shifts. The multi-shifts with weight multi-sequence given by (2.6) arise naturally in the study of reproducing $\mathbb{C}[z_1, \dots, z_m]$ -modules with $\mathcal{U}(m)$ -invariant kernels, refer to [27, Section 4].

Theorem 2.1. *Let M_z be a bounded multiplication m -tuple in $H^2(\beta)$. Then M_z is spherical if and only if the norm $\|\cdot\|_\beta$ on $H^2(\beta)$ can be expressed as*

$$(2.5) \quad \|f\|_\beta^2 = \sum_{k=0}^{\infty} \tilde{\beta}_k^2 \|f_{[k]}\|_{L^2(\partial\mathbb{B})}^2 \quad (f \in H^2(\beta))$$

for a sequence $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \dots$, of positive numbers. If this happens then M_z is unitarily equivalent to the m -variable weighted shift $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ with the weight sequence

$$(2.6) \quad w_n^{(i)} = \frac{\tilde{\beta}_{|n|+1}}{\tilde{\beta}_{|n|}} \sqrt{\frac{n_i + 1}{|n| + m}} \quad (n \in \mathbb{N}^m, 1 \leq i \leq m).$$

In this case, the sequence $\beta_n = \|z^n\|_\beta$ can be expressed as

$$(2.7) \quad \beta_n = \tilde{\beta}_{|n|} \sqrt{\frac{(m-1)! n!}{(m-1+|n|)!}} \quad (n \in \mathbb{N}^m).$$

Remark 2.2 : Whenever $\{\beta_n\}_{n \in \mathbb{N}^m}$ is a multi-sequence, which gives rise to a spherical tuple M_z , we will denote by $\{\tilde{\beta}_k\}_{k \in \mathbb{N}}$ the corresponding scalar weight sequence, related to β via formula (2.7).

Definition 2.3 : Let $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ be a spherical m -variable weighted shift and let $\{\tilde{\beta}_k\}_{k \in \mathbb{N}}$ be the corresponding scalar weight sequence. Then the *shift associated with T* is the one-variable weighted shift $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$, where

$$\delta_k := \frac{\tilde{\beta}_{k+1}}{\tilde{\beta}_k}, \quad k \in \mathbb{N}.$$

It is easy to see that the following statements are equivalent:

- (1) A scalar weight sequence $\{\tilde{\beta}_k\}$ gives rise to a *bounded* spherical m -tuple M_z on $H^2(\beta)$, where β is given by (2.7);
- (2) The spherical m -variable shift $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ is bounded;
- (3) $\sup_{k \geq 0} \delta_k < \infty$;
- (4) The one-variable shift T_δ , associated with T , is bounded.

When dealing with a spherical multiplication m -tuple M_z and with the corresponding m -variable weighted shift T , we will always assume that the condition (3) above holds.

For an m -tuple T of commuting bounded linear operators T_1, \dots, T_m on \mathcal{H} , let

$$Q_T(I) := \sum_{j=1}^m T_j^* T_j.$$

Remark 2.4 : Let M_z be a bounded multiplication m -tuple in $H^2(\beta)$. Then $Q_{M_z}(I) = I$ if and only if M_z is the Szegő m -shift. Further, the defect operator $I - Q_{M_z^*}(I)$ is an orthogonal projection if and only if M_z is the Drury-Arveson m -shift.

The next result characterizes all multi-shifts within the whole class of spherical tuples and should be combined with the above Theorem 2.1. Recall that for an m -tuple $S = (S_1, \dots, S_m)$, $\ker(D_{S^*}) = \bigcap_{i=1}^m \ker(S_i^*)$.

Theorem 2.5. *Let T be a commuting, bounded spherical operator m -tuple on a Hilbert space \mathcal{H} . Then the following assertions are equivalent.*

- (1) $\ker(D_{T^*})$ is a one-dimensional cyclic subspace for T ;
- (2) T is unitarily equivalent to an m -variable weighted shift;
- (3) T is unitarily equivalent to a multiplication m -tuple M_z on a space $H^2(\beta)$.

Before we turn to the proofs of Theorems 2.1 and 2.5, let us see a couple of instructive examples.

Example 2.6. Let M_z be a bounded spherical multiplication m -tuple on a space $H^2(\beta)$ and suppose that the ball B_R , where all power series in $H^2(\beta)$ converge has positive radius (see

Theorem 3.4(2) below for the description of $R = r(M_z)$ in terms of β_k 's). Fix an integer $s > 0$. Then the set

$$H^2(\beta)_s = \{f \in H^2(\beta) : D^\alpha f(0) = 0 \text{ for all } \alpha, |\alpha| < s\}$$

is a closed subspace of $H^2(\beta)$, which is invariant under M_z . Let T be the restriction of the m -tuple M_z to $H^2(\beta)_s$. Then T is a spherical m -tuple (see Theorem 2.12 below), but the dimension of $\ker(D_{T^*})$ is greater than one. This gives an example of an m -tuple of operators of multiplication by the co-ordinate functions z_1, \dots, z_m on a Hilbert space of *scalar* power series in z_1, \dots, z_m , which does not satisfy the equivalent conditions (1)-(3) of Theorem 2.5.

Example 2.7. Here we show that the existence of a cyclic vector for a commuting spherical operator m -tuple T also does not imply the above conditions (1)-(3). Namely, take any integer $\ell > m - \frac{1}{2}$. Consider the Sobolev space $\mathcal{H} = W^{\ell,2}(\partial\mathbb{B})$; we refer to [29] for a definition. We will need also the dual space $\mathcal{H}' = W^{-\ell,2}(\partial\mathbb{B})$; its elements are complex-valued distributions, defined on the unit sphere $\partial\mathbb{B}$. Both spaces are Hilbert, and infinitely differential functions are dense both in \mathcal{H}' and in \mathcal{H} . The pairing between \mathcal{H} and \mathcal{H}' is a continuation of the L^2 pairing $\langle f, g \rangle = \int_{\partial\mathbb{B}} f \bar{g}$, defined for C^∞ functions.

Let T be the multiplication tuple M_z on \mathcal{H}' . Since the spaces \mathcal{H} and \mathcal{H}' and their norms are invariant under unitary rotations in \mathbb{C}^n , T is a (strongly) spherical tuple.

It is easy to see that T is not unitarily equivalent to a spherical m -variable weighted shift. Indeed, if S denotes the m -tuple of multiplication by \bar{z} on \mathcal{H}' then $\sum_{j=1}^m T_j S_j = I$. It follows that $\ker(D_{T^*}) = \{0\}$, and hence T cannot be unitarily equivalent to a weighted shift.

Nevertheless, T has a cyclic vector. Indeed, choose any dense sequence $\{a_n\}$ of points on $\partial\mathbb{B}$ such that their first coordinates $z_1(a_n)$ are all distinct. The adjoint tuple to T coincides with the multiplication tuple M_z , acting on \mathcal{H} . By the Sobolev embedding theorem, \mathcal{H} is continuously embedded into $C(\partial\mathbb{B})$. Hence for any sequence $\{c_n\}$ in ℓ^1 , the linear functional

$$\psi(f) \stackrel{\text{def}}{=} \sum_n c_n f(a_n)$$

is bounded on \mathcal{H} and therefore is an element of \mathcal{H}' . We assert that if the sequence $\{c_n\}$ does not vanish and decays sufficiently fast, then the vector $\psi \in \mathcal{H}'$ is cyclic for M_{z_1} and therefore for the whole tuple T .

Indeed, suppose that some function $f \in \mathcal{H}$ satisfies

$$\psi((\lambda - M_{z_1})^{-1} f) = \sum_n \frac{c_n f(a_n)}{\lambda - z_1(a_n)} = 0$$

for any λ with $|\lambda| > 1$. Suppose that $c_n \neq 0$ for all n , and $\sum_n n^{-2} \log |c_n| = -\infty$. Since the points $z_1(a_n)$ are all distinct, it follows from a theorem by Sibilev [46] that $c_n f(a_n) = 0$ for all n , which implies that f is zero. Hence ψ is cyclic for the operator M_{z_1} on \mathcal{H}' .

We remark that a similar construction of a cyclic vector for a family of normal operators is given in [42].

Before proving Theorems 2.1 and 2.5, we need several lemmas.

Lemma 2.8. *Let L be a finite-dimensional Hilbert space and let $\pi : \mathcal{U}(m) \rightarrow B(L)$ be an irreducible unitary representation with respect to two unitary structures defined by scalar products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ on L . Then there is a constant $\gamma > 0$ such that*

$$\langle x, y \rangle_2 = \gamma \langle x, y \rangle_1 \quad (x, y \in L).$$

Proof. By the Riesz Representation Theorem, there exists a positive operator A on L such that $\langle x, y \rangle_2 = \langle Ax, y \rangle_1$ for every $x, y \in L$. Since L is finite-dimensional and A is positive, the

point-spectrum of A is a non-empty finite subset of $(0, +\infty)$. Let γ be the minimal eigenvalue of A . We claim that $\ker(A - \gamma I) = L$.

Since $A - \gamma I$ is a nonnegative operator, for $x \in L$, one has

$$(2.8) \quad \langle x, x \rangle_2 = \gamma \langle x, x \rangle_1 \text{ iff } \langle (A - \gamma I)x, x \rangle_1 = 0 \text{ iff } x \in \ker(A - \gamma I).$$

Let $U \in \mathcal{U}(m)$ and $x \in \ker(A - \gamma I)$. By assumption, $\pi(U)$ preserves both scalar products, and hence by (2.8),

$$\langle \pi(U)x, \pi(U)x \rangle_2 = \langle x, x \rangle_2 = \gamma \langle x, x \rangle_1 = \gamma \langle \pi(U)x, \pi(U)x \rangle_1.$$

It follows that $\ker(A - \gamma I)$ is invariant under $\pi(U)$. Since $\pi(U)^* = \pi(U^{-1})$, $\ker(A - \gamma I)$ is indeed a reducing subspace for $\pi(U)$. Since $\ker(A - \gamma I) \neq \{0\}$ and π is irreducible by assumption, we must have $\ker(A - \gamma I) = L$. Thus the claim stands verified. The desired conclusion now follows from (2.8) and the polarization identity. \square

We also need an analogue of this lemma for reducible representations.

Lemma 2.9. *Let L be a finite-dimensional Hilbert space and let $\pi : \mathcal{U}(m) \rightarrow B(L)$ be an unitary representation with respect to a unitary structure defined by a scalar product $\langle \cdot, \cdot \rangle_1$. Let $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$ be the corresponding decomposition of L into irreducible subspaces L_j and suppose these subspaces are of distinct dimensions. Suppose that we are given another semidefinite sesquilinear product $\langle \cdot, \cdot \rangle_2$ on L , which is invariant with respect to π : $\langle \pi(U)x, \pi(U)y \rangle_2 = \langle x, y \rangle_2$ for all $x, y \in L$ and all $U \in \mathcal{U}(m)$. Then there are nonnegative constants $\tilde{\beta}_1, \dots, \tilde{\beta}_k$ such that the following statements hold:*

- (1) $\langle x, y \rangle_2 = \tilde{\beta}_j \langle x, y \rangle_1$ ($x, y \in L_j$);
- (2) $\langle x, y \rangle_2 = 0$ if $x \in L_p, y \in L_r, p \neq r$.

Proof. Similarly to the previous proof, there is a nonnegative operator A on L such that $\langle x, y \rangle_2 = \langle Ax, y \rangle_1$ for every $x, y \in L$. By the assumption, one has a decomposition $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$, where $\pi_j : \mathcal{U}(m) \rightarrow B(L_j)$ are irreducible representations. We obtain assertion (1) by applying Lemma 2.8 to representations π_j (if the product $\langle \cdot, \cdot \rangle_2$ is not definite, one can apply Lemma 2.8 to positive definite products $\langle \cdot, \cdot \rangle_1$ and $\langle x, y \rangle_3 = \langle x, y \rangle_1 + \langle x, y \rangle_2$). To see (2), note that π_j are all inequivalent representations and apply [43, Corollary 2.21]. \square

Next lemma will be crucial in the proof of Theorem 2.5.

Lemma 2.10. *Let T be a commuting, bounded spherical operator m -tuple on \mathcal{H} . Suppose that $\ker(D_{T^*})$ is one-dimensional and is spanned by a vector $e \in \mathcal{H}$. Suppose that e is cyclic for T . Then there is sequence of positive weights $\{\tilde{\beta}_k\}_{k \geq 0}$ such that for any polynomial $p \in \mathbb{C}[z_1, \dots, z_m]$,*

$$(2.9) \quad \|p(T)e\|^2 = \sum_{k=0}^{\deg p} \tilde{\beta}_k \|p_{[k]}\|_{L^2(\partial \mathbb{B})}^2$$

where $\|p\|_{L^2(\partial \mathbb{B})}^2 = \int_{\partial \mathbb{B}} |p(z)|^2 d\sigma(z)$ for the surface area measure σ on the unit sphere $\partial \mathbb{B}$. The sequence $\{\tilde{\beta}_k\}$ is defined uniquely.

Proof. Notice first that $\ker D_{T^*} = (T_1 \mathcal{H} + \cdots + T_m \mathcal{H})^\perp$ is invariant under the action of $\mathcal{U}(m)$. Hence for any U in $\mathcal{U}(m)$, there is a scalar constant $\zeta(U)$, $|\zeta(U)| = 1$, such that $\Gamma(U)e = \zeta(U)e$.

Fix a positive integer N , and denote by H_N the space of polynomials in $\mathbb{C}[z]$ of degree less or equal to N . Clearly, H_N is a closed subspace of $L^2(\partial \mathbb{B})$; the corresponding scalar product will be denoted as $\langle \cdot, \cdot \rangle_1$. Define a second semidefinite sesquilinear product on H_N by

$$\langle p, q \rangle_2 = \langle p(T)e, q(T)e \rangle_{\mathcal{H}}.$$

Both products are invariant under the action of $\mathcal{U}(m)$. Indeed, $p(T_U)e = \zeta(U)\Gamma(U)^{-1}p(T)e$ for all $p \in \mathbb{C}[z]$ and $U \in \mathcal{U}(m)$. Hence

$$\begin{aligned} \langle p(Uz), q(Uz) \rangle_2 &= \langle p(T_U)e, q(T_U)e \rangle_{\mathcal{H}} \\ &= \langle \zeta(U)\Gamma(U)^{-1}p(T)e, \zeta(U)\Gamma(U)^{-1}q(T)e \rangle_{\mathcal{H}} = \langle p, q \rangle_2 \end{aligned}$$

for all $p, q \in H_N$. It follows from [43, pg. 175] that the decomposition of $(H_N, \langle \cdot, \cdot \rangle_1)$ into irreducible subspaces with respect to the action of $\mathcal{U}(m)$ on H_N is given by $H_N = \text{Hom}(0) \oplus \text{Hom}(1) \oplus \cdots \oplus \text{Hom}(N)$. This fact and Lemma 2.9 imply formula (2.9) for some nonnegative constants $\tilde{\beta}_0, \dots, \tilde{\beta}_N$. If a constant $\tilde{\beta}_j$ were zero, it would follow that $p(T)e = 0$ for any homogeneous polynomial $p \in \text{Hom}(j)$, which would imply that $p(T)e = 0$ for all $p \in \text{Hom}(k, 0)$ with $k > j$. Since e is cyclic, this would imply that \mathcal{H} is finite dimensional, which gives a contradiction.

Since N is arbitrary, the statement of Lemma follows. \square

Proof of Theorem 2.1. First of all, we mention that $\langle z^n, z^k \rangle_{L^2(\partial\mathbb{B})} = 0$ for any distinct multi-indices $n, k \in \mathbb{N}^m$ see [52, formula (1.21), page 13]. So the functions z^n , $n \in \mathbb{N}^m$ form an orthogonal sequence in $L^2(\partial\mathbb{B})$. It follows that the norm, defined by (2.5), is an $H^2(\beta)$ norm for certain multi-sequence β_n . It is clear that the multiplication tuple M_z on the Hilbert space with the norm (2.5) is spherical. This gives the “if” part of the first statement.

Conversely, for each multiplication tuple M_z , the space $\ker(D_{M_z^*})$ is one-dimensional and is spanned by the formal power series 1. So we can apply Lemma 2.10 to get the “only if” part of the first statement.

Finally, one can make use of (2.5) and of the formula

$$(2.10) \quad \int_{\partial\mathbb{B}} |z^n|^2 d\sigma(z) = \frac{(m-1)! n!}{(m-1+|n|)!} \quad (n \in \mathbb{N}^m),$$

(see [52, Lemma 1.11]) to derive the expressions (2.6) and (2.7) for $w_n^{(i)}$ and $\tilde{\beta}_n$ respectively. \square

Proof of Theorem 2.5. The equivalence of (2) and (3) has been noted already. If (3) holds, then $\ker(D_{T^*})$ is one-dimensional and is spanned by the image in \mathcal{H} of the formal power series 1 under the unitary equivalence. This implies (1). Finally, suppose that (1) holds, and let e be a unit vector that spans $\ker(D_{T^*})$. Then it follows from Lemma 2.10 that there is a sequence $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \dots$ such the map $p \mapsto p(T)e$, $p \in \mathbb{C}[z]$ extends to a unitary map from $H^2(\beta)$ to \mathcal{H} , which intertwines T with M_z . \square

Let $\Lambda \subset \mathbb{Z}_+^m$ be a set of multi-indices. In what follows, we will say that Λ is *inductive* if for any $n \in \Lambda$, the multi-indices $n + \varepsilon_j$ are also in Λ for $j = 1, \dots, m$.

Theorem 2.11. *Let Ω be a Reinhardt domain in \mathbb{C}^m such that $0 \in \Omega$. Let \mathcal{H} be a M_z -invariant RKHS of functions on Ω such that $\mathcal{H} \subset \text{Hol}(\Omega)$, the inclusion being continuous. Let $\kappa(z, w)$ ($z, w \in \Omega$) denote the reproducing kernel of \mathcal{H} .*

Then the following statements are equivalent.

(1) *For every $\zeta \in \mathbb{T}^m$,*

$$(2.11) \quad \kappa(\zeta \cdot z, \zeta \cdot w) = \kappa(z, w) \quad (z, w \in \Omega),$$

where $\zeta \cdot z = (\zeta_1 z_1, \dots, \zeta_m z_m) \in \mathbb{C}^m$.

(2) *For every $\zeta \in \mathbb{T}^m$, $f(\zeta \cdot) \in \mathcal{H}$ whenever $f \in \mathcal{H}$, and*

$$\langle f(\zeta \cdot), g(\zeta \cdot) \rangle = \langle f, g \rangle \quad (f, g \in \mathcal{H}).$$

- (3) There exist a multi-sequence $\{\beta_n\}_{n \in \mathbb{Z}_+^m}$ and an inductive set $\Lambda \subset \mathbb{Z}_+^m$ such that $\mathcal{H} = H^2(\beta)_\Lambda$, where

$$H^2(\beta)_\Lambda = \{f \in H^2(\beta) : D^n f(0) = 0 \text{ for all } n \in \mathbb{Z}_+^m, n \notin \Lambda\}.$$

- (4) There exists an inductive set $\Lambda' \subset \mathbb{Z}_+^m$ such that the functions z^n , $n \in \Lambda'$, are contained in \mathcal{H} and form there an orthogonal basis.
- (5) There exist an inductive set $\Lambda'' \subset \mathbb{Z}_+^m$ and a family $\{\alpha_n\}_{n \in \Lambda''}$ of positive numbers such that

$$(2.12) \quad \kappa(z, w) = \sum_{n \in \Lambda''} \alpha_n z^n \overline{w}^n \quad (z, w \in \Omega).$$

Moreover, if (1)–(5) hold, then $\Lambda = \Lambda' = \Lambda''$.

In (3), in the equality $\mathcal{H} = H^2(\beta)_\Lambda$ we identify analytic functions in Ω with the corresponding formal power series centered at the origin. This equality means that these two Hilbert spaces consist of the same functions and the norms in these two spaces are identical.

Theorem 2.12. Let \mathcal{H} be a M_z -invariant RKHS of functions on \mathbb{B}_R in \mathbb{C}^m . Suppose $\mathcal{H} \subset \text{Hol}(\mathbb{B}_R)$, the inclusion being continuous. Let $\kappa(z, w)$ ($z, w \in \mathbb{B}_R$) denote the reproducing kernel of \mathcal{H} .

Then the following statements are equivalent.

- (1) For every $U \in \mathcal{U}(m)$,

$$(2.13) \quad \kappa(Uz, Uw) = \kappa(z, w) \quad (z, w \in \mathbb{B}_R).$$

- (2) For every $U \in \mathcal{U}(m)$, $f(U \cdot) \in \mathcal{H}$ whenever $f \in \mathcal{H}$, and

$$\langle f(U \cdot), g(U \cdot) \rangle = \langle f, g \rangle \quad (f, g \in \mathcal{H}).$$

- (3) There exist $s \in \mathbb{Z}_+$ and a scalar sequence $\{\tilde{\beta}_k\}_{k \in \mathbb{N}}$ such that $\mathcal{H} = H^2(\beta)_s$, where the multi-sequence β is given by (2.7) and

$$H^2(\beta)_s = \{f \in H^2(\beta) : D^n f(0) = 0 \text{ for all } n \in \mathbb{Z}_+^m, |n| < s\}.$$

If any of the conditions (1) – (3) holds, then M_z is a strongly spherical tuple.

Remark 2.13 : Some statements close to the above Theorem 2.12 are given in the beginning of Section 4 of the paper [27] by Guo, Hu and Xu, though they do not discuss the continuity of the representations Γ . As follows from their discussion, the spaces $H^2(\beta)_s$ are defined uniquely by their generating function $F(t)$, analytic in the disc $|t| < R^2$ in the complex plane, such that

$$\kappa(z, w) = F(\langle z, w \rangle) \quad (z, w \in \mathbb{B}_R).$$

Such representation always exists, all the coefficients a_n in the expansion $F(t) = \sum_{k=s}^{\infty} a_k t^k$ are positive and are given by

$$(2.14) \quad a_k = \frac{(m-1+k)!}{(m-1)! k!} \frac{1}{\tilde{\beta}_k^2}, \quad k \geq s$$

(it follows from (1.3) and (2.7)).

Lemma 2.14. Let G be a subgroup of the group $GL_m(\mathbb{C})$ of invertible, complex $m \times m$ matrices and let Ω be a G -invariant (that is, $\mathfrak{g}z \in \Omega$ whenever $\mathfrak{g} \in G$ and $z \in \Omega$) domain in \mathbb{C}^m such that $0 \in \Omega$. Let \mathcal{H} be a M_z -invariant RKHS of functions on Ω such that $\mathcal{H} \subset \text{Hol}(\Omega)$, the inclusion being continuous. Let $\kappa(z, w)$ ($z, w \in \Omega$) denote the reproducing kernel of \mathcal{H} . Let $M_z = (M_{z_1}, \dots, M_{z_m})$ be the bounded m -tuple of multiplication by the co-ordinate functions z_1, \dots, z_m . Then the following statements are equivalent:

(1) For every $\mathfrak{g} \in G$,

$$(2.15) \quad \kappa(\mathfrak{g}z, \mathfrak{g}w) = \kappa(z, w) \quad (z, w \in \Omega).$$

(2) For every $\mathfrak{g} \in G$, $f(\mathfrak{g}\cdot) \in \mathcal{H}$ whenever $f \in \mathcal{H}$, and

$$\langle f(\mathfrak{g}\cdot), h(\mathfrak{g}\cdot) \rangle = \langle f, h \rangle \quad (f, h \in \mathcal{H}).$$

If this happens then the representation $\Gamma : G \rightarrow B(\mathcal{H})$ of G on \mathcal{H} given by

$$(2.16) \quad \Gamma(\mathfrak{g})f(z) = f(\mathfrak{g}z) \quad (z \in \Omega, \mathfrak{g} \in G)$$

is strongly continuous, unitary and satisfies $\Gamma(\mathfrak{g})M_{z_j} = M_{(\mathfrak{g}z)_j}\Gamma(\mathfrak{g})$ ($j = 1, \dots, m$). In particular, if $G = \mathcal{U}(m)$, then M_z is strongly spherical.

Proof. (1) implies (2): Suppose that (1) holds. Set

$$\Gamma(\mathfrak{g})\kappa(\cdot, w) = \kappa(\cdot, \mathfrak{g}^{-1}(w)) \quad (w \in \Omega, \mathfrak{g} \in G).$$

We check that Γ extends to a unitary representation of G on \mathcal{H} . By the reproducing property of κ and (2.15),

$$\langle \Gamma(\mathfrak{g})\kappa(\cdot, z), \Gamma(\mathfrak{g})\kappa(\cdot, w) \rangle = \langle \kappa(\cdot, z), \kappa(\cdot, w) \rangle.$$

Since $\bigvee \{\kappa(\cdot, w) : w \in \Omega\} = \mathcal{H}$, $\Gamma(\mathfrak{g})$ extends isometrically to the entire \mathcal{H} . Since $\mathfrak{g}(\Omega) = \Omega$, $\Gamma(\mathfrak{g})$ is surjective, and hence unitary. Finally, since $\Gamma(\mathfrak{g})^* = \Gamma(\mathfrak{g}^{-1})$, it follows that

$$\Gamma(\mathfrak{g})f(z) = \langle \Gamma(\mathfrak{g})f, \kappa(\cdot, z) \rangle = \langle f, \Gamma(\mathfrak{g}^{-1})\kappa(\cdot, z) \rangle = f(\mathfrak{g}z)$$

for any $z \in \Omega$ and any $f \in \mathcal{H}$.

(2) implies (1): Assume that (2) is true. By the uniqueness of the reproducing kernel, it suffices to check that $\kappa(\mathfrak{g}z, \mathfrak{g}w)$ is a reproducing kernel for \mathcal{H} for every $\mathfrak{g} \in G$. However,

$$\langle f, \kappa(\mathfrak{g}\cdot, \mathfrak{g}w) \rangle = \langle f(\mathfrak{g}^{-1}\cdot), \kappa(\cdot, \mathfrak{g}w) \rangle = f(w) \quad (w \in \Omega),$$

which gives (1).

The fact that Γ is a unitary representation of G on \mathcal{H} follows from (2). It follows from the closed graph theorem that the operators M_{z_j} are bounded. Notice that by Hartogs' separate analyticity theorem [35], $\kappa(z, \bar{w})$ is holomorphic in z, w , and it follows that the map $w \rightarrow \kappa(w, w)$ is continuous. Since $\|\kappa(\cdot, w) - \kappa(\cdot, w_0)\|^2 = \kappa(w, w) + \kappa(w_0, w_0) - 2\operatorname{Re} \kappa(w, w_0)$, the function $w \mapsto \kappa(\cdot, w) \in \mathcal{H}$ is norm continuous. Therefore $\Gamma(\mathfrak{g})\kappa(\cdot, w)$ depends continuously on \mathfrak{g} for any w . Since the reproducing kernels are complete, Γ is strongly continuous. The remaining part is a routine verification. \square

Remark 2.15 : We are particularly interested in the subgroups $\mathcal{UD}(m)$ and $\mathcal{U}(m)$ of $GL_m(\mathbb{C})$, where $\mathcal{UD}(m)$ denotes the subgroup of unitary diagonal $m \times m$ matrices.

Proof of Theorem 2.11. By Lemma 2.14, (1) and (2) are equivalent. It is clear that (3) and (4) are equivalent, and the corresponding sets Λ and Λ' coincide whenever (3) and (4) hold. It is also clear that (3) implies (2).

(2) implies (4). Assume that (2) holds. Define the set $\Lambda' \subset \mathbb{N}^m$ by

$$\Lambda' = \{n_0 \in \mathbb{N}^m : \exists f = \sum a_n z^n \in \mathcal{H} : a_{n_0} \neq 0\}.$$

Since \mathcal{H} is M_z -invariant, Λ' is inductive.

Put $S(t)f(z) = f(e^{it}z)$, $t \in \mathbb{R}^m$, where $e^{it}z = (e^{it_1}z_1, \dots, e^{it_m}z_m)$. By applying Lemma 2.14, we get that S is a unitary strongly continuous m -parameter group. Given any function $f(z) = \sum a_n z^n \in \mathcal{H}$ and any $n_0 \in \mathbb{N}^m$ such that $a_{n_0} \neq 0$, we notice that

$$a_{n_0} z^{n_0} = \frac{1}{(2\pi)^m} \int_{[0, 2\pi]^m} e^{-in_0 t} S(t) f \, dt.$$

(The integral is understood in the Bochner sense. The equality is true because it holds pointwise for any $z \in \Omega$.) It follows that for any $n_0 \in \Lambda'$, $z^{n_0} \in \mathcal{H}$.

Now take any $p, q \in \mathbb{N}^m$ such that $p \neq q$. Then for some $1 \leq j \leq m$, $p_j \neq q_j$. Let $\zeta = w\varepsilon_j + \sum_{i \neq j} \varepsilon_i$, where $w \in \mathbb{T}$. Then $\langle z^p, z^q \rangle = \langle \zeta z^p, \zeta z^q \rangle = w^{p_j - q_j} \langle z^p, z^q \rangle$, which is possible for all $w \in \mathbb{T}$ only if $\langle z^p, z^q \rangle = 0$. We have checked that the functions z^n , $n \in \Lambda'$ form an orthogonal sequence in \mathcal{H} . Any $f \in \mathcal{H}$ has a Taylor series representation $f(z) = \sum_{n \in \Lambda'} a_n z^n$, which converges weakly in \mathcal{H} . Therefore the sequence $\{z^n\}_{n \in \Lambda'}$ is in fact an orthogonal basis in \mathcal{H} .

Given any orthonormal basis $\{\phi_k\}_{k \in \mathcal{K}}$ in \mathcal{H} , the reproducing kernel of \mathcal{H} can be expressed by the well-known formula $\kappa(z, w) = \sum_{k \in \mathcal{K}} \overline{\phi_k(w)} \phi_k(z)$. It follows that (3) implies (5) (with $\Lambda'' = \Lambda$).

It is immediate that (5) implies (1), which concludes the proof of the fact that conditions (1)–(5) are all equivalent. It also has been shown already that if (1)–(5) are fulfilled, then $\Lambda = \Lambda' = \Lambda''$. \square

Proof of Theorem 2.12. By Lemma 2.14, (1) is equivalent to (2). It is clear that (3) implies (2). It remains to prove that (2) implies (3). Suppose that (2) holds. Then we can apply Theorem 2.11 and deduce that $\mathcal{H} = H^2(\beta)_\Lambda$ for an inductive set Λ . Let $s = \min\{|n| : n \in \Lambda\}$, then the intersection R of \mathcal{H} with the space $\text{Hom}(s)$ of analytic homogeneous polynomials of order s is non-zero, and the group $\mathcal{U}(m)$ acts on R . Since the action of $\mathcal{U}(m)$ on $\text{Hom}(s)$ is irreducible (we already have used it in Lemma 2.10), it follows that $R = \text{Hom}(s)$. Since Λ is inductive, $\Lambda = \{n \in \mathbb{N}^m : |n| \geq s\}$, which gives (3).

By Lemma 2.14, if any of the equivalent conditions (1)–(3) holds, then the tuple M_z consists of bounded operators. Now (3) implies that M_z is strongly spherical. \square

3. SPECTRAL THEORY FOR MULTI-SHIFTS

For a masterful exposition of various notions of invertibility, Fredholmness and multi-parameter spectral theory, the reader is referred to [13]. For $T \in B(\mathcal{H})$, we reserve the symbols $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_e(T)$ for the Taylor spectrum, point-spectrum, approximate-point spectrum, essential spectrum of T respectively. It is well known that the spectral mapping theorem for polynomial mappings holds for both the Taylor and the approximate-point spectra. Except the point-spectrum, all spectra mentioned above are always non-empty.

Given a commuting m -tuple $T = (T_1, \dots, T_m)$ of operators on \mathcal{H} , set

$$(3.17) \quad Q_T(X) := \sum_{i=1}^m T_i^* X T_i \quad (X \in B(\mathcal{H})).$$

We define inductively $Q_T^0(I) = I$ and $Q_T^k(I) = Q_T(Q_T^{k-1}(I))$ for $k \geq 1$. Then we have

$$(3.18) \quad Q_T^k(I) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{*\alpha} T^\alpha.$$

Lemma 3.1. *Let T be a spherical commuting, bounded m -variable weighted shift with respect to an orthonormal basis $\{e_n\}_{n \in \mathbb{N}^m}$. Let $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$ be the (one-variable) shift associated with T with respect to an orthonormal basis $\{f_k\}_{k \in \mathbb{N}}$. Then*

$$(3.19) \quad \langle Q_T^k(I) e_n, e_n \rangle = \|T_\delta^k f_n\|^2 \quad (k \in \mathbb{N}, n \in \mathbb{N}^m).$$

Proof. It is easy to see that the operator $Q_T^k(I)$ is diagonal with respect to the basis $\{e_n\}$, and

$$(3.20) \quad Q_T^k(I) e_n = \delta_{|n|}^2 \delta_{|n|+1}^2 \cdots \delta_{|n|+k-1}^2 e_n \quad (k \in \mathbb{N}, n \in \mathbb{N}^m).$$

The desired conclusion is now immediate. \square

Proposition 3.2. *Let T be a spherical commuting, bounded m -variable weighted shift with respect to the orthonormal basis $\{e_n\}_{n \in \mathbb{N}^m}$. Let $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$ be the shift associated with T with respect to the orthonormal basis $\{f_k\}_{k \in \mathbb{N}}$. Then the geometric spectral radius $r(T) := \sup\{\|z\|_2 : z \in \sigma(T)\}$ of T is equal to the spectral radius of T_δ .*

Proof. By [39, Theorem 1] and [10, Theorem 1], the geometric spectral radius R of T is given by

$$R = \lim_{k \rightarrow \infty} \|Q_T^k(I)\|^{\frac{1}{2k}}.$$

It is easy to see that the orthogonal basis $\{e_n\}_{n \in \mathbb{N}^m}$ diagonalizes the positive operator $Q_T^k(I)$. Also, by (3.19), $\langle Q_T^k(I)e_n, e_n \rangle = \|T_\delta^k f_n\|^2$ for every $k \in \mathbb{N}$ and $n \in \mathbb{N}^m$. It follows that

$$\lim_{k \rightarrow \infty} \|Q_T^k(I)\|^{1/2k} = \sup_{k \geq 0} \|T_\delta^k\|^{1/k} = r(T_\delta),$$

by the well-known general formula for the spectral radius of a linear operator. \square

Let $\mathcal{C}(\mathcal{H})$ denote the norm-closed ideal of compact operators on \mathcal{H} . Since $B(\mathcal{H})/\mathcal{C}(\mathcal{H})$ is a unital C^* -algebra, the *Calkin algebra*, there exist a Hilbert space \mathcal{K} and an injective unital $*$ -representation $\pi : B(\mathcal{H})/\mathcal{C}(\mathcal{H}) \rightarrow B(\mathcal{K})$ [11, Chapter VIII]. In particular, $\pi \circ q : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ is a unital $*$ -representation, where $q : B(\mathcal{H}) \rightarrow B(\mathcal{H})/\mathcal{C}(\mathcal{H})$ stands for the *quotient (Calkin) map*. Set $\pi \circ q(T) := (\pi \circ q(T_1), \dots, \pi \circ q(T_m))$.

We recall that a tuple $T = (T_1, \dots, T_m)$ is called *essentially normal* if all commutators $[T_j, T_k]$ and $[T_j^*, T_k]$, $j, k = 1, \dots, m$ are compact. The following (known) version of the Fuglede–Putman commutativity theorem holds: given operators A and N on a Hilbert space H , if N is essentially normal and the commutator $[A, N]$ is compact, then the commutator $[A, N^*]$ also is compact. This follows by applying the classical Fuglede–Putman theorem to operators $\pi \circ q(N)$ and $\pi \circ q(A)$. We refer to [49] for an additional information.

It follows that a commutative tuple T is essentially normal whenever $[T_j^*, T_j]$ are compact for $j = 1, \dots, m$.

Remark 3.3 : Let $T : \{w_n^{(i)}\}$ be a bounded spherical m -variable weighted shift and let $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$ be the one-variable shift associated with T . As it follows from [27, Corollary 4.4], T is essentially normal if and only if T_δ is essentially normal if and only if $\lim_{k \rightarrow \infty} (\delta_k^2 - \delta_{k-1}^2) = 0$ (observe that by (2.14), $\frac{a_k}{a_{k+1}} = \frac{k+1}{k+m} \frac{\tilde{\beta}_{k+1}^2}{\tilde{\beta}_k^2}$).

The main theorem of this section describes several spectral parts of spherical m -shifts.

Theorem 3.4. *Let M_z be a bounded spherical multiplication m -tuple in $H^2(\beta)$, so that the norm in $H^2(\beta)$ is given by (2.5) for a certain sequence $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \dots$, of positive numbers. Let $R(M_z), r(M_z), i(M_z)$ be given by*

$$(3.21) \quad R(M_z) = \lim_{j \rightarrow \infty} \sup_{k \geq 0} \sqrt[j]{\frac{\tilde{\beta}_{k+j}}{\tilde{\beta}_k}},$$

$$(3.22) \quad r(M_z) = \liminf_{j \rightarrow \infty} \sqrt[j]{\tilde{\beta}_j},$$

$$(3.23) \quad i(M_z) = \lim_{j \rightarrow \infty} \inf_{k \geq 0} \sqrt[j]{\frac{\tilde{\beta}_{k+j}}{\tilde{\beta}_k}}.$$

Then $i(M_z) \leq r(M_z) \leq R(M_z)$, and the following statements are true:

- (1) *The Taylor spectrum of T is the closed ball $\overline{\mathbb{B}}_{R(M_z)}$ in \mathbb{C}^m .*
- (2) *The ball $\mathbb{B}_{r(M_z)}$ in \mathbb{C}^m is the largest open ball in which all the power series in $H^2(\beta)$ converge.*

- (3) Either $\sigma_p(T^*) = \mathbb{B}_{r(M_z)}$ or $\sigma_p(T^*) = \overline{\mathbb{B}}_{r(M_z)}$.
- (4) $\sigma_{ap}(M_z) = \overline{A}_{i(M_z), R(M_z)}$, where $\overline{A}_{i(M_z), R(M_z)}$ stands for the closed ball shell in \mathbb{C}^m of inner-radius $i(M_z)$ and outer-radius $R(M_z)$.
- (5) If in addition, $\lim_{k \rightarrow \infty} \frac{\tilde{\beta}_{k+1}}{\tilde{\beta}_k} - \frac{\tilde{\beta}_k}{\tilde{\beta}_{k-1}} = 0$ then the essential spectrum of M_z is the closed ball shell of inner-radius $\liminf_{k \rightarrow \infty} \frac{\tilde{\beta}_{k+1}}{\tilde{\beta}_k}$ and outer-radius $\limsup_{k \rightarrow \infty} \frac{\tilde{\beta}_{k+1}}{\tilde{\beta}_k}$.

The first part of the Theorem 3.4 is obtained in [27, Theorem 4.5(1)], under the additional assumption of essential normality, by entirely different methods. An upper estimate of the geometric joint spectral radius of M_z is given in [33, Theorem 9.6].

Note further that statement (5) of the theorem is precisely [27, Theorem 4.5(2)]. We can give a more general version of this statement.

Lemma 3.5. *Let T be an essentially normal spherical m -tuple. Then the essential spectrum of T is given by*

$$\sigma_e(T) = \{z \in \mathbb{C}^m : \|z\|_2^2 \in \sigma_e(Q_T(I))\}.$$

Proof. We adapt the proof of [27, Theorem 4.5(2)] to the present situation. Suppose T is essentially normal. Equivalently, $(q(T_1), \dots, q(T_m))$ is a commuting normal m -tuple in the Calkin Algebra. Let \mathcal{M} be the maximal ideal space of the commutative C^* -algebra $C^*(q(T))$ generated by $q(T_1), \dots, q(T_m)$. By [12, Corollary 3.10], the essential spectrum of T is given by

$$\sigma_e(T) = \{(\phi(q(T_1)), \dots, \phi(q(T_m))) : \phi \in \mathcal{M}\}.$$

If $\lambda \in \sigma_e(T)$ then for some $\phi \in \mathcal{M}$,

$$q(Q_T(I) - \|\lambda\|^2 I) = \sum_{j=1}^m q(T_j^*)q(T_j) - |\phi(q(T_j))|^2.$$

Clearly, ϕ annihilates $q(Q_T(I) - \|\lambda\|^2 I) \in C^*(q(T))$. Thus $q(Q_T(I) - \|\lambda\|^2 I)$ is not invertible, and hence $\|\lambda\|^2 \in \sigma_e(Q_T(I))$. Conversely, suppose $\|\lambda\|^2 \in \sigma_e(Q_T(I))$ for some $\lambda \in \mathbb{C}^m$. Thus $q(Q_T(I) - \|\lambda\|^2 I)$ is not invertible in the Calkin algebra, and hence in $C^*(q(T))$. Thus there exists some $\phi_\lambda \in \mathcal{M}$ annihilating $q(Q_T(I) - \|\lambda\|^2 I)$. This gives $\|\lambda\|^2 = \sum_{j=1}^m |\phi_\lambda(q(T_j))|^2$. On the other hand, $(\phi_\lambda(q(T_1)), \dots, \phi_\lambda(q(T_m))) \in \sigma_e(T)$. By the spherical symmetry of the essential spectrum, we must have $\lambda \in \sigma_e(T)$. \square

Let us pass to the proof of Theorem 3.4. It involves several lemmas and propositions. The first lemma is a multi-variable analog of a well-known fact about the approximate point spectrum [44, Proposition 13].

Lemma 3.6. *Let T be a commuting m -tuple of operators on a Hilbert space. Then the approximate point-spectrum of T is disjoint from the open ball $\mathbb{B}_{m_\infty(T)}$, where*

$$(3.24) \quad m_\infty(T) = \sup_{k \geq 1} \inf_{h \in \mathcal{H}, \|h\|=1} \|Q_T^k(I)h\|^{\frac{1}{2k}}.$$

Proof. Take any $\lambda \in \mathbb{C}^m$ such that $\|\lambda\|_2 < m_\infty(T)$. Then there exist some $\mu > 0$ and integer $k > 0$ such that $\|\lambda\|_2 < \mu$ and $\|Q_T^k(I)h\| \geq \mu^{2k}\|h\|$ for any $h \in \mathcal{H}$.

Put $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_m^{\alpha_m}$ for $\alpha \in \mathbb{N}^m$. Notice that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} |\lambda^\alpha|^2 = (|\lambda_1|^2 + \dots + |\lambda_m|^2)^k = \|\lambda\|_2^{2k}.$$

Hence, by the Cauchy-Schwarz inequality, for any unit vector $h \in \mathcal{H}$,

$$\begin{aligned} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha h - \lambda^\alpha h\|^2 &\geq \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\|T^\alpha h\| - |\lambda^\alpha|)^2 \\ &\geq \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha h\|^2 - 2 \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha h\|^2 \right)^{1/2} \|\lambda\|_2^k + \|\lambda\|_2^{2k} \\ &= \|Q_T^k(I)h\|^2 - 2\|Q_T^k(I)h\| \|\lambda\|_2^k + \|\lambda\|_2^{2k} \geq (\mu^k - \|\lambda\|_2^k)^2 > 0. \end{aligned}$$

Therefore $\lambda \notin \sigma_{ap}(T)$. \square

Proposition 3.7. *The Taylor spectrum, approximate point-spectrum, point-spectrum, essential spectrum of a spherical m -tuple are spherically symmetric.*

Proof. The spherical symmetry of Taylor spectrum and approximate point-spectrum follows immediately from the spectral mapping property for polynomials. On the other hand, the spherical symmetry of the point spectrum follows from the definition. We now check the assertion for the essential spectrum. Let T be our spherical m -tuple and let π, q, \mathcal{K} be as in the discussion following Proposition 3.2. One may deduce from [13, Theorem 6.2] and spectral permanence for the Taylor spectrum that $\sigma_e(T) = \sigma(\pi \circ q(T))$. Since the Taylor spectrum of a spherical tuple has spherical symmetry, it now suffices to show that $\pi \circ q(T)$ is spherical. This follows from Remark 1.2(2). \square

In the single variable case, the following result was obtained by R. L. Kelley (refer to [41]).

Lemma 3.8. *Let M_z be a bounded multiplication m -tuple in $H^2(\beta)$. Then the Taylor spectrum of M_z is connected.*

Proof. By [32, Corollary 3], $\sigma(M_z)$ has a poly-circular symmetry (that is, $\zeta \cdot w \in \sigma(M_z)$ for any $w \in \sigma(M_z)$ and any $\zeta \in \mathbb{T}^m$). Note that 0 belongs to the point spectrum $\sigma_p(M_z^*)$ of M_z^* , and hence to the Taylor spectrum of M_z in view of $\sigma_p(M_z^*) \subseteq \sigma(M_z^*) = \{\bar{z} : z \in \sigma(M_z)\}$ (for $z = (z_1, \dots, z_m)$, we put $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m)$). It suffices to check that $\sigma(M_z^*)$ is connected. Let K_1 be the connected component of $\sigma(M_z^*)$ containing 0 and let $K_2 = \sigma(M_z^*) \setminus K_1$. By the Shilov Idempotent Theorem [13, Application 5.24], there exist invariant subspaces $\mathcal{M}_1, \mathcal{M}_2$ of M_z^* such that $\mathcal{H} = \mathcal{M}_1 \dot{+} \mathcal{M}_2$ and $\sigma(M_z^*|_{\mathcal{M}_i}) = K_i$ for $i = 1, 2$.

Let $h \in \ker(D_{S_k^*})$, where $S_k := (M_{z_1}^k, \dots, M_{z_m}^k)$ for a positive integer k . Then $h = x + y$ for $x \in \mathcal{M}_1$ and $y \in \mathcal{M}_2$. It follows that $(M_{z_j}^*)^k x = 0$ and $(M_{z_j}^*)^k y = 0$ for all $j = 1, \dots, m$. If y is non-zero then $0 \in \sigma_p(S_k^*) \subseteq \sigma(S_k^*)$, and hence by the spectral mapping property, $0 \in \sigma(M_z^*|_{\mathcal{M}_2})$. Since $0 \notin K_2$, we must have $y = 0$. It follows that \mathcal{M}_1 contains the dense linear manifold $\bigcup_{k \geq 1} \ker(D_{S_k^*})$, and hence $\mathcal{M}_1 = \mathcal{H}$. Thus the Taylor spectrum of M_z^* is equal to K_1 . In particular, the Taylor spectrum of M_z is connected. \square

Lemma 3.9.

$$\begin{aligned} \lim_{j \rightarrow \infty} \sup_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}} \left(\frac{(k+2) \cdots (k+j+1)}{(k+m+1) \cdots (k+j+m)} \right)^{\frac{1}{2j}} &= \lim_{j \rightarrow \infty} \sup_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}}, \\ \lim_{j \rightarrow \infty} \inf_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}} \left(\frac{(k+2) \cdots (k+j+1)}{(k+m+1) \cdots (k+j+m)} \right)^{\frac{1}{2j}} &= \lim_{j \rightarrow \infty} \inf_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}}. \end{aligned}$$

Proof. For $k \geq 0$ and $j \geq 1$, put

$$\rho_{kj} := \frac{(k+2) \cdots (k+j+1)}{(k+m+1) \cdots (k+j+m)} = \frac{(k+2) \cdots (k+m)}{(k+j+2) \cdots (k+j+m)}.$$

It is easy to see that ρ_{kj} is an increasing function of k (for a fixed j). Hence $\rho_{0j}^{\frac{1}{2j}} \leq \rho_{kj}^{\frac{1}{2j}} \leq 1$ for all $k \geq 0$. Since $\rho_{0j}^{\frac{1}{2j}} \rightarrow 1$ as $j \rightarrow \infty$, both statements of the Lemma follow. \square

Proof of Theorem 3.4. (1): Suppose M_z is spherical. We already recorded that the Taylor spectrum of a spherical tuple has spherical symmetry. By Lemma 3.8, the Taylor spectrum of M_z is connected. It is easy to see that the only bounded closed connected subsets of \mathbb{C}^m with spherical symmetry are balls and ball shells. This follows from the fact that the unitary group $\mathcal{U}(m)$ acts transitively on any sphere. Since 0 belongs to the spectrum $\sigma(M_z)$, it must be a ball centered at the origin. The formula for the spectral radius of M_z now follows from Proposition 3.2 and the known formula for the spectral radius of T_δ [44].

(2): Let $w \in \mathbb{B}_{r(M_z)}$. We claim that any power series in $H^2(\beta)$ converges absolutely at w . It suffices to check that w belongs to the point spectrum $\sigma_p(M_z^*)$ of T^* , or, equivalently, $\sum_{n \geq 0} |w^n|^2 / \|z^n\|_\beta^2 < \infty$ (see [32, Propositions 18–20]). Since $\sigma_p(M_z^*)$ has spherical symmetry, it suffices to check that $\tilde{w} = (|w|, 0, \dots, 0) \in \mathbb{B}_{r(M_z)}$ belongs to $\sigma_p(M_z^*)$. But $\sum_{n \geq 0} |\tilde{w}^n|^2 / \|z^n\|_\beta^2 = \sum_{n_1 \geq 0} |w|^{2n_1} / \|z^{n_1}\|_\beta^2 = \sum_{n_1=0}^\infty \binom{m-1+n_1}{n_1} \tilde{\beta}_{n_1}^{-2} |w|^{2n_1} < \infty$, and the claim follows. This also shows that $\mathbb{B}_{r(M_z)} \subseteq \sigma_p(M_z^*)$. Finally, note that the maximal ball contained in the domain of convergence of the above series is precisely $\mathbb{B}_{r(M_z)}$.

(3): This is clear from the proof of (2) and the spherical symmetry of $\sigma_p(M_z^*)$.

(4): First of all, it follows from (3.19) that

$$m_\infty(T) = \lim_{k \rightarrow \infty} \inf_{\|h\|=1} \|T_\delta^k h\| = i(M_z)$$

(see (3.24)). By Lemma 3.6, the open ball $\mathbb{B}_{i(M_z)}$ is disjoint from $\sigma_{ap}(T)$. Since the approximate point-spectrum of M_z is contained in the Taylor spectrum, it follows that $\sigma_{ap}(T) \subset \overline{A}_{i(M_z), R(M_z)}$.

To prove the converse inclusion, consider the bounded linear operator $S_1 := M_{z_1}|_{\mathcal{M}}$, where \mathcal{M} is the closure of $\mathbb{C}[z_1]$ in $H^2(\beta)$. It is a one-variable weighted shift in the basis $\{z_1^j / \|z_1^j\|\}$, and we can apply to it [41, Theorem 1]. We get $\sigma_{ap}(S_1) = \{\alpha \in \mathbb{C} : i(S_1) \leq |\alpha| \leq R(S_1)\}$, where $R(S_1), i(S_1)$ are given by

$$R(S_1) = \lim_{j \rightarrow \infty} \sup_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}} \left(\frac{(k+2) \cdots (k+j+1)}{(k+m+1) \cdots (k+j+m)} \right)^{\frac{1}{2j}}$$

$$i(S_1) = \lim_{j \rightarrow \infty} \inf_{k \geq 0} \left(\frac{\tilde{\beta}_{k+j+1}}{\tilde{\beta}_{k+1}} \right)^{\frac{1}{j}} \left(\frac{(k+2) \cdots (k+j+1)}{(k+m+1) \cdots (k+j+m)} \right)^{\frac{1}{2j}}.$$

By Lemma 3.9, $i(S_1) = i(M_z)$ and $R(S_1) = R(M_z)$.

The proof is divided into two cases:

$i(M_z) = R(M_z)$: If this happens then by the preceding discussion, $i(S_1) = i(M_z) = R(M_z) = R(S_1)$. In particular, $\{w \in \mathbb{C} : |w| = R(M_z)\} = \sigma_{ap}(S_1) \subseteq \sigma_{ap}(M_{z_1})$. Now by the projection property for the approximate point-spectrum [13, Pg. 18], the projection of $\sigma_{ap}(M_z)$ onto the z_1 -axis is precisely $\sigma_{ap}(M_{z_1})$. Since $R(M_z) \in \sigma_{ap}(M_{z_1})$, there exists $w_2, \dots, w_m \in \mathbb{C}$ such that $(R(M_z), w_2, \dots, w_m) \in \sigma_{ap}(M_z)$. By (1) above, $\sigma_{ap}(M_z) \subseteq \sigma(M_z) = \overline{B}_{R(M_z)}$. It follows that $w_2 = \dots = w_m = 0$, and hence $(R(M_z), 0, \dots, 0) \in \sigma_{ap}(M_z)$. Since $\sigma_{ap}(M_z)$ has spherical symmetry, it contains the degenerate ball-shell $\overline{A}_{i(M_z), R(M_z)}$.

$i(M_z) < R(M_z)$: Since the approximate point-spectrum is always closed, it suffices to check that $A_{i(M_z), R(M_z)} \subseteq \sigma_{ap}(M_z)$. Let $w \in A_{i(M_z), R(M_z)}$. By the spherical symmetry of $\sigma_{ap}(M_z)$, we may take w of the form $(\|w\|, 0, \dots, 0)$. We adapt the argument of [41, Theorem 1] to the present situation. Choose numbers a, b such that $i(M_z) < a < \|w\| < b < R(M_z)$. Let $\varepsilon > 0$ be given. Choose positive integers n, k such that $(\|w\|/b)^n < \varepsilon$, $1/k < \varepsilon$ and $\|z_1^{n+k+1}\| / \|z_1^{k+1}\| > b^n$. We further choose positive integers p and q such that $(a/\|w\|)^p < \varepsilon$, $q > n+k$ and $\|z_1^{p+q+1}\| / \|z_1^{q+1}\| < a^p$. Now the argument in the ([41], Proof of Theorem 1)

actually yields $\|(S_1 - \|w\|I)f\| < \varepsilon\|S_1\|\|f\|$, where $f \in \mathcal{M}$ is given by

$$f(z_1) = \sum_{l=k+1}^{p+q+1} \|w\|^{k+1-l} z_1^l.$$

Set $g(z) = f(z_1)$, and note that $g \in H^2(\beta)$ and $\|(M_{z_1} - \|w\|I)g\| < \varepsilon\|M_{z_1}\|\|g\|$. Further, since $1/k < \varepsilon$, we have $\|M_{z_j}g\| < \varepsilon\|M_{z_1}\|\|g\|$ for $j = 2, \dots, m$. Since $\varepsilon > 0$ is arbitrary, $w \in \sigma_{ap}(T)$.

(5): The assertion about the essential spectrum is already obtained in [27, Theorem 4.5(2)]. Alternatively, it may be deduced from [27, Lemma 4.7], Remark 3.3 and Lemma 3.5. \square

We remark that for an essentially commuting tuple T satisfying $\sigma_e(T) = \partial\mathbb{B}$, the C^* algebra it generates can be described using the results of [26].

Let T be an essentially normal, spherical m -tuple. It follows from Lemma 3.5 that the essential spectrum of T is connected if and only if the essential spectrum of $\sum_{j=1}^m T_j^* T_j$ is connected. If in addition, T is a multi-shift, then this always happens as seen above. In view of this, it is interesting to note that there exists a jointly hyponormal 2-shift with disconnected essential spectrum [17, Theorem 2.5].

We close the section with the following question.

Question 3.10. *Calculate the essential spectrum of any spherical multiplication m -tuple M_z . Is it always connected?*

As it is shown in [36, Example 3.7.7], one always has $\sigma_e(M_z) = \sigma_{ap}(M_z) = \bar{A}_{i(M_z), R(M_z)}$ if $M_z = M_{z_1}$ is a 1-tuple.

4. THE MEMBERSHIP OF CROSS-COMMUTATORS IN THE SCHATTEN CLASSES

In this section, we discuss the so-called p -essential normality of spherical tuples. Recall that an m -tuple T of commuting bounded linear operators T_1, \dots, T_m is p -essentially normal if the cross-commutators $[T_i^*, T_l]$ belong to the Schatten p -class for all $j, l = 1, \dots, m$. As before, we put $\delta_k = \tilde{\beta}_{k+1}/\tilde{\beta}_k$, $k \in \mathbb{N}$. Throughout this section, we assume that $m \geq 2$.

Remark 4.1 : An m -variable weighted shift $T : \{w_n^{(i)}\}$ is compact if and only if $\lim_{|n| \rightarrow \infty} w_n^{(i)} = 0$ for all indices $i = 1, \dots, m$ [32, Proposition 6]. It follows that a spherical m -variable weighted shift is compact if and only if $\lim_{k \rightarrow \infty} \delta_k = 0$.

The main part of this Section is devoted to the proof of the following criterion of when the cross-commutators of a spherical weighted shift belong to the Schatten class \mathcal{S}^p .

Theorem 4.2. *Let M_z be a bounded spherical multiplication m -tuple in $H^2(\beta)$, so that the norm in $H^2(\beta)$ is given by (2.5) for a certain sequence $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \dots$, of positive numbers. Let $1 \leq p \leq \infty$. Then the following statements are equivalent:*

- (1) *The self-commutators $[M_{z_j}^*, M_{z_j}]$ belong to the Schatten class \mathcal{S}^p for all j , $1 \leq j \leq m$;*
- (2) *The cross-commutators $[M_{z_j}^*, M_{z_l}]$ belong to the Schatten class \mathcal{S}^p for all indices j, l ;*
- (3)

$$(4.25) \quad \sum_{k=1}^{\infty} \delta_k^{2p} k^{m-p-1} + \sum_{k=1}^{\infty} |\delta_k^2 - \delta_{k-1}^2|^p k^{m-1} < \infty.$$

We refer to [33] for some related results, such as the membership of $I - \sum M_{z_j}^* M_{z_j}$ in classes \mathcal{S}_p , see Proposition 6.9 and other results in Section 6 of the cited work.

The following notation will be used. We will say that two quantities F_k, G_k , depending on $k \in \mathbb{N}$, are comparable, and write $F_k \approx G_k$ ($k \rightarrow \infty$) if there exist positive constants A, B, k_0 (that may depend on m and p) such that $AG_k \leq F_k \leq BG_k$ for all $k \geq k_0$.

Lemma 4.3. *Let $1 \leq p < \infty$. Then for $j \neq l$,*

$$\sum_{\substack{n \in \mathbb{N}^m, \\ |n| = k, n_j > 0}} n_j^{p/2} n_l^{p/2} \approx k^{p+m-1} \text{ as } k \rightarrow \infty.$$

Proof. By symmetry, it suffices to consider the case $j = 1, l = 2$. One has

$$\sum_{\substack{n \in \mathbb{N}^m, \\ |n| = k, n_1 > 0}} n_1^{p/2} n_2^{p/2} = \frac{1}{m-1} \sum_{r=2}^m \sum_{\substack{n \in \mathbb{N}^m, \\ |n| = k, n_1 > 0}} n_1^{p/2} n_r^{p/2} \approx \sum_{\substack{n \in \mathbb{N}^m, \\ |n| = k, n_1 > 0}} n_1^{p/2} \left(\sum_{r=2}^m n_r \right)^{p/2}.$$

Hence

$$\begin{aligned} \sum_{\substack{n \in \mathbb{N}^m, \\ |n| = k, n_1 > 0}} n_1^{p/2} n_2^{p/2} &\approx \sum_{|n|=k, n_1 > 0} n_1^{p/2} (k - n_1)^{p/2} = \sum_{j=1}^k \binom{k-j+m-2}{k-j} j^{p/2} (k-j)^{p/2} \\ &\approx \sum_{j=1}^k j^{p/2} (k-j)^{m-2+(p/2)} \approx k^{p+m-1} \int_0^1 x^{p/2} (1-x)^{m-2+(p/2)} dx, \end{aligned}$$

which gives the assertion of the Lemma. \square

Lemma 4.4. *Let $1 \leq p < +\infty$. For any $j, 1 \leq j \leq m$, and any $s \in \mathbb{R}$, one has*

$$(4.26) \quad \sum_{n \in \mathbb{N}^m, |n|=k} |sn_j - 1|^p \approx k^{p+m-1} |s|^p + k^{m-1},$$

where the constants involved in the relation \approx can depend on p, m but not on k and s .

Proof. Denoting $l = n_j$, we get

$$\sum_{n \in \mathbb{N}^m, |n|=k} |sn_j - 1|^p = \sum_{l=0}^k \binom{k-l+m-2}{m-2} |sl - 1|^p \approx \sum_{l=0}^k (k-l+1)^{m-2} |sl - 1|^p.$$

So the estimate in one direction is trivial:

$$\sum_{n \in \mathbb{N}^m, |n|=k} |sn_j - 1|^p \leq \sum_{l=0}^k (k+1)^{m-2} |sl - 1|^p \leq C \sum_{l=0}^k (k+1)^{m-2} (|sl|^p + 1) \leq C_1 (k^{p+m-1} |s|^p + k^{m-1}).$$

To prove the reverse estimate, define the interval $I(s)$ as follows: $I(s) = [k/2, k]$ if $|s|k \geq 6$ and $I(s) = [0, k/18]$ if $|s|k < 6$. One has

$$|sl - 1| \geq \frac{1 + |s|l}{2} \text{ for } l \in I(s).$$

Indeed, in the first case $|sl - 1| \geq |sl| - 1 \geq (1 + |s|l)/2$ and in the second case, $|sl - 1| \geq 1 - |sl| \geq (1 + |s|l)/2$. So for some positive constants C_2, C_3 one gets

$$\begin{aligned} \sum_{n \in \mathbb{N}^m, |n|=k} |sn_j - 1|^p &\geq \sum_{l \in \mathbb{Z} \cap I(s)} (k-l+1)^{m-2} |sl - 1|^p \\ &\geq C_2 \sum_{l \in \mathbb{Z} \cap I(s)} (k-l+1)^{m-2} (1 + |s|^p l^p) \geq C_3 (k^{p+m-1} |s|^p + k^{m-1}). \end{aligned} \quad \square$$

Proof of Theorem 4.2. We will use the following formulas, which are easy to deduce from (2.6).

Let $1 \leq j \leq m$. Then

$$(4.27) \quad [T_j^*, T_j] e_n = \begin{cases} \left[\frac{(n_j+1)\delta_{|n|}^2}{|n|+m} - \frac{n_j\delta_{|n|-1}^2}{|n|+m-1} \right] e_n, & n_j > 0, \\ \frac{\delta_{|n|}^2}{|n|+m} e_n, & n_j = 0. \end{cases}$$

If $1 \leq j, l \leq m$ and $j \neq l$, then

$$(4.28) \quad [T_j^*, T_l] e_n = \begin{cases} \sqrt{n_j(n_l+1)} \left[\frac{\delta_{|n|}^2}{|n|+m} - \frac{\delta_{|n|-1}^2}{|n|+m-1} \right] e_{n+\varepsilon_l-\varepsilon_j}, & n_j > 0, \\ 0, & n_j = 0. \end{cases}$$

By symmetry, to prove that **(1)** is equivalent to **(3)**, it suffices to give two-sided estimates of the self-commutator $[T_m^*, T_m]$. Equation (4.27) gives

$$\begin{aligned} \|[T_m^*, T_m]\|_{S^p}^p &= \sum_{k=1}^{\infty} \sum_{\substack{n \in \mathbb{N}^m, \\ |n|=k, n_m > 0}} \left| \frac{(n_m+1)\delta_k^2}{k+m} - \frac{n_m\delta_{k-1}^2}{k+m-1} \right|^p + \sum_{n \in \mathbb{N}^m, n_m=0} \frac{\delta_{|n|}^{2p}}{(|n|+m)^p} \\ &\approx \sum_{k=1}^{\infty} \sum_{\substack{n \in \mathbb{N}^m, \\ |n|=k, n_m > 0}} \left| n_m \left(\frac{\delta_k^2}{k+m} - \frac{\delta_{k-1}^2}{k+m-1} \right) + \frac{\delta_k^2}{k+m} \right|^p + \sum_{k=0}^{\infty} (k+1)^{m-2-p} \delta_k^{2p}. \end{aligned}$$

By substituting s/t for s in (4.26), one gets

$$\sum_{n \in \mathbb{N}^m, |n|=k} |n_m s - t|^p \approx k^{p+m-1} |s|^p + k^{m-1} |t|^p$$

(where the constants involved in the relation \approx do not depend on $s, t \in \mathbb{R}$ and k). It follows that

$$\|[T_m^*, T_m]\|_{S^p}^p \approx \sum_{k=1}^{\infty} \left| \frac{\delta_k^2}{k+m} - \frac{\delta_{k-1}^2}{k+m-1} \right|^p k^{p+m-1} + \sum_{k=0}^{\infty} \frac{\delta_k^{2p}}{(k+m)^p} k^{m-1}.$$

Now, by applying the two-sided estimate

$$\begin{aligned} (4.29) \quad C_1 \left(\frac{|\delta_k^2 - \delta_{k-1}^2|^p}{(k+m)^p} - \frac{\delta_{k-1}^{2p}}{(k+m)^p(k+m-1)^p} \right) &\leq \left| \frac{\delta_k^2}{k+m} - \frac{\delta_{k-1}^2}{k+m-1} \right|^p \\ &\leq C_2 \left(\frac{|\delta_k^2 - \delta_{k-1}^2|^p}{(k+m)^p} + \frac{\delta_{k-1}^{2p}}{(k+m)^p(k+m-1)^p} \right), \end{aligned}$$

where C_1 and C_2 are positive constants, it is easy to see that the relation $\|[T_m^*, T_m]\|_{S^p} < \infty$ is equivalent to **(3)**.

It is obvious that **(2)** implies **(1)**, so it only remains to prove that **(3)** implies that the commutators $[T_j^*, T_l]$ belong to \mathcal{S}_p whenever the indices j and l are distinct. This follows easily from (4.28), Lemma 4.3 and (4.29). \square

Next we derive several consequences of Theorem 4.2, which are motivated by the so-called cut-off phenomenon in the Berger-Shaw theory [2, Proposition 5.3], [19, Proposition 3], [22, Theorem 1.1] (refer to [50] for a detailed account of this phenomenon).

Corollary 4.5. *Let M_z be a bounded spherical multiplication m -tuple on $H^2(\beta)$. If all commutators $[M_{z_j}^*, M_{z_k}]$ belong to \mathcal{S}_p , where $1 \leq p < \infty$, then either the operators M_{z_j} are compact or $p > m$.*

Proof. To simplify notation, we put $\tau_k = \delta_k^2$. It suffices to show that if all commutators $[M_{z_j}^*, M_{z_k}]$ are in \mathcal{S}_m , then $\tau_k \rightarrow 0$ as $k \rightarrow \infty$ (see Remark 4.1). Suppose, to the contrary, that $[M_{z_j}^*, M_{z_k}]$ are in \mathcal{S}_m , but τ_k do not tend to zero. Then there exist an $\varepsilon > 0$ and a sequence k_1, k_2, \dots such that $\tau_{k_j} > \varepsilon$ for all $j \in \mathbb{N}$. By Theorem 4.2,

$$(4.30) \quad \sum_k \tau_k^m k^{-1} + \sum_k |\tau_{k+1} - \tau_k|^m k^{m-1} < \infty,$$

and, as we will now show, it leads to a contradiction. Choose N so large that

$$\sum_{l=k_N}^{\infty} |\tau_{l+1} - \tau_l|^m l^{m-1} < \left(\frac{\varepsilon}{2}\right)^m.$$

Take any $j \geq N$ and any k such that $k_j \leq k \leq 2k_j$. Since $\sum_{l=k_j}^{2k_j-1} \frac{1}{l} \leq 1$, we get

$$\begin{aligned} |\tau_k - \tau_{k_j}| &\leq \sum_{l=k_j}^{k-1} |\tau_{l+1} - \tau_l| \\ &\leq \left(\sum_{l=k_j}^{k-1} |\tau_{l+1} - \tau_l|^{m-1} \right)^{\frac{1}{m}} \cdot \left(\sum_{l=k_j}^{k-1} 1 \right)^{\frac{m-1}{m}} < \frac{\varepsilon}{2} \cdot 1 = \frac{\varepsilon}{2}. \end{aligned}$$

Hence $\tau_k > \varepsilon/2$ for all k in the range $k_j \leq k \leq 2k_j$ for all $j \geq N$. This implies that

$$\sum_{k=k_j}^{\infty} \tau_k^m k^{-1} \geq \sum_{k=k_j}^{2k_j} \tau_k^m k^{-1} \geq \left(\frac{\varepsilon}{2}\right)^m \sum_{k=k_j}^{2k_j} k^{-1} \geq \frac{1}{2} \left(\frac{\varepsilon}{2}\right)^m$$

for all $j \geq N$. Therefore the first sum in (4.30) diverges.

This contradiction implies that, in fact, τ_k should tend to 0. \square

Corollary 4.6. *Suppose that the sequence $\{\delta_k\}$ does not tend to zero and $|\delta_{k+1} - \delta_k| \leq C/k$ for some constant C . Then the commutators $[M_{z_j}^*, M_{z_k}]$ belong to S_p if and only if $p > m$.*

Proof. Since $\{\delta_k\}$ does not tend to zero, by Remark 4.1, none of M_{z_1}, \dots, M_{z_m} is compact. If $[M_{z_j}^*, M_{z_l}] \in S_p$ for all j, l , then by Corollary 4.5, $p > m$. The converse statement follows immediately from Theorem 4.2. \square

In particular, the statement of this Corollary holds if $|\delta_k - 1| \leq C/k$ for some constant C . It can also be applied to sequences $\tilde{\beta}_k$ like $\tilde{\beta}_k = C_1 \exp(C_2 k^\alpha)$, where C_1, C_2 and α are constants.

We end this section with the following question.

Question 4.7. *Give a characterization of all (strongly) spherical m -tuples T such that $\ker(D_{T^*})$ is finite-dimensional and is cyclic for T , in terms of some free parameters (similarly to Theorems 2.1 and 2.5). Can our results on the calculation of parts of the spectrum and on membership of cross-commutators $[T_j, T_j^*]$ in S_p be generalized to this subclass of spherical m -tuples?*

5. SPECIAL CLASSES OF SPHERICAL TUPLES

Recall that an m -tuple $S = (S_1, \dots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *jointly subnormal* if there exist a Hilbert space \mathcal{K} containing \mathcal{H} and an m -tuple $N = (N_1, \dots, N_m)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i h = S_i h$ for every $h \in \mathcal{H}$ and $1 \leq i \leq m$.

An m -tuple $S = (S_1, \dots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *jointly hyponormal* if the $m \times m$ matrix $([T_j^*, T_i])_{1 \leq i, j \leq m}$ is positive definite, where $[A, B]$ stands for the commutator $AB - BA$ of A and B . It is not difficult to see that a jointly subnormal tuple is always jointly hyponormal [5], [14].

Definition 5.1 : Fix an integer $q \geq 1$ and put

$$(5.31) \quad B_q(Q_T) := \sum_{s=0}^q (-1)^s \binom{q}{s} Q_T^s(I)$$

(see (3.17), (3.18)). We say that T is a *joint q -contraction* (respectively, *joint q -expansion*) if $B_q(Q_T) \geq 0$ (respectively, $B_q(Q_T) \leq 0$).

We say that T is a *joint q -hyperexpansion* if T is a joint k -expansion for all $k = 1, \dots, q$. Also, T is said to be a *joint complete hyperexpansion* if T is a joint q -hyperexpansion for all $q \geq 1$. If $B_q(Q_T) = 0$, then T is a *joint q -isometry*. If $m = 1$ then we drop the prefix 1- and term joint in all the above definitions.

The Bergman m -shift is jointly subnormal while the Drury-Arveson m -shift is a joint m -isometry [24]. The Szegő m -shift being a joint isometry is jointly subnormal. It is also a joint q -isometry for any $q \geq 1$.

Remark 5.2 : Let T be a spherical m -tuple. Assume further that T is a joint p -isometry or a joint 2-hyperexpansion. Then the approximate point-spectrum $\sigma_{ap}(T)$ of T is a subset of the unit sphere [9, Proposition 3.4]. Since $\sigma_{ap}(T)$ is always non-empty, by its spherical symmetry, it must be the entire unit sphere.

Theorem 5.3. *Let $T : \{w_n^{(i)}\}$ be a spherical m -variable weighted shift. Let $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$ be the one-variable weighted shift associated with T (see the Definition 2.3). Then we have the following statements:*

- (1) T is jointly subnormal if and only if T_δ is subnormal.
- (2) T is a joint q -isometry if and only if T_δ is a q -isometry.
- (3) T is a joint q -expansion if and only if T_δ is a q -expansion.
- (4) T is a joint complete hyperexpansion if and only if T_δ is a complete hyperexpansion.
- (5) T is jointly hyponormal if and only if T_δ is hyponormal.

Proof. The desired conclusions in (2)-(4) follow immediately from (3.19).

To see (1), without loss of generality, we may assume that $Q_T(I) = T_1^*T_1 + \cdots + T_m^*T_m \leq I$. By [6, Theorem 5.2], an operator m -tuple T such that $Q_T(I) \leq I$ is jointly subnormal if and only if

$$\sum_{j=0}^p (-1)^j \binom{p}{j} \sum_{|\alpha|=j} \frac{j!}{\alpha!} \|T^\alpha f\|^2 \geq 0$$

for every $f \in \mathcal{H}$ and every $p, k \in \mathbb{N}$. Now (1) may be derived from (3.19).

To see (5), let us recall first that T_δ is hyponormal if and only if and only if $\{\delta_k\}_{k \in \mathbb{N}}$ is an increasing sequence. Suppose first that T_δ is hyponormal. By [14, Theorem 6.1], T is jointly hyponormal if and only if the matrix

$$P = \left(w_{n+\varepsilon_j}^{(i)} w_{n+\varepsilon_i}^{(j)} - w_n^{(i)} w_n^{(j)} \right)_{1 \leq i, j \leq m}$$

is positive definite for every $n \in \mathbb{N}^m$. By (2.6),

$$w_n^{(i)} = \delta_{|n|} \alpha_n^{(i)} \quad (n \in \mathbb{N}^m, i = 1, \dots, m),$$

where $\alpha_n^{(i)} := \sqrt{\frac{n_i+1}{|n|+m}}$ is the weight multi-sequence of the Szegő m -shift. It is easy to see that the matrix

$$Q = \left(\alpha_{n+\varepsilon_j}^{(i)} \alpha_{n+\varepsilon_i}^{(j)} - \alpha_n^{(i)} \alpha_n^{(j)} \right)_{1 \leq i, j \leq m}$$

is positive definite for every $n \in \mathbb{N}^m$. Let P_{ij}, Q_{ij} denote the (i, j) th entry of $m \times m$ matrices P, Q respectively. It follows that

$$P_{ij} = \delta_{|n|+1}^2 Q_{ij} + (\delta_{|n|+1}^2 - \delta_{|n|}^2) \alpha_n^{(i)} \alpha_n^{(j)}$$

Since $\{\delta_k\}_{k \in \mathbb{N}}$ is an increasing sequence, P is positive definite.

Conversely, suppose T is jointly hyponormal. Then it follows from [9, Lemma 4.10] that $Q_T^2(I) \geq Q_T(I)^2$, where $Q_T(X) = T_1^*XT_1 + \cdots + T_m^*XT_m$ ($X \in B(\mathcal{H})$). It is immediate from (3.20) that $\{\delta_k\}_{k \in \mathbb{N}}$ is an increasing sequence. \square

Let $p > 0$, and let $M_{z,p}$ be as introduced in Example 1.3. Note that the sequence δ_k there is given by $\delta_k^2 = (k+m)/(k+p)$. It is now easy to deduce from Theorem 5.3(5) that the tuple $M_{z,p}$ is jointly hyponormal if and only if $p \geq m$. As we already mentioned there, for $p \geq m$, $M_{z,p}$ is actually jointly subnormal (see [33, Theorem 9.8] for a closely related fact). Here are a few more consequences of Theorem 5.3:

- (1) $M_{z,p}$ is a spherical joint 2-expansion if and only if $m - 1 \leq p \leq m$.
- (2) $M_{z,p}$ is a spherical joint q -isometry if and only if p is a positive integer, $p \leq m$ and $q \geq m - p + 1$ (see Proposition 5.5 below).

It is a trivial consequence of Corollary 4.5 and Theorem 5.3(5) that for a spherical hyponormal m -variable weighted shift T , the cross-commutators can belong to S_p only if $p > m$. We give an example of a spherical, jointly hyponormal 2-variable weighted shift for which none of the self-commutators belongs to the Schatten class \mathcal{S}^p for any $p < \infty$.

Example 5.4. Define inductively the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ as follows:

$$\rho_0 = 1, \rho_{k+1} = \rho_k + \eta_k \quad (k \in \mathbb{N}),$$

where $\eta_k = 1/2^l$ if k is of the form 2^{2^l} ; and 0 otherwise. Obviously, $\{\rho_k\}_{k \in \mathbb{N}}$ is increasing. We next define $\tilde{\beta}_k$ inductively by setting $\tilde{\beta}_0 = 1$ and $\tilde{\beta}_{k+1} = \tilde{\beta}_k \sqrt{\rho_k}$ ($k \in \mathbb{N}$). Then the 2-variable weighted shift $T = (T_1, T_2)$ with weight multi-sequence

$$w_n^{(i)} = \frac{\tilde{\beta}_{|n|+1}}{\tilde{\beta}_{|n|}} \sqrt{\frac{n_i + 1}{|n| + 2}} \quad (n \in \mathbb{N}^2, i = 1, 2)$$

is bounded, spherical and jointly hyponormal. Note that

$$\sum_{k=1}^{\infty} k \left| \frac{\tilde{\beta}_{k+1}^2}{\tilde{\beta}_k^2} - \frac{\tilde{\beta}_k^2}{\tilde{\beta}_{k-1}^2} \right|^p = \sum_{k=1}^{\infty} k \eta_k^p \geq 2^{2^l} \frac{1}{2^{lp}} \rightarrow \infty$$

as $l \rightarrow \infty$. By Theorem 4.2, $[T_j^*, T_j]$ does not belong to the Schatten class \mathcal{S}^p for any $p < \infty$ and any $j = 1, 2$.

In a similar way, one give an example of a non-compact spherical 2-variable weighted shift T such that the corresponding one-variable shift T_δ is a contraction (or, equivalently, the sequence $\{\tilde{\beta}_k\}$ decays), but the cross-commutators do not belong to S_p for any p .

For a sequence $\{f_k\}_{k=0}^{\infty}$, we put $\nabla f_k = f_{k+1} - f_k$. In what follows, we denote

$$\gamma_k = \tilde{\beta}_k^2.$$

The following is certainly known (see, for instance, [45, pg 50]). We include a short proof for reader's convenience.

Proposition 5.5. Let $\{\tilde{\beta}_k\}_{k=0}^{\infty}$ be a 1-variable sequence and M_z the multiplication operator by z acting on the (1-variable) space $H^2(\tilde{\beta})$.

- (1) M_z is a q -isometry if and only if there is a polynomial S of degree $q - 1$ or less such that $\gamma_k = S(k)$ for all $k \in \mathbb{N}$.
- (2) M_z is a q -expansion if and only if $(-1)^q \nabla^q \gamma_k \leq 0$ for any $k \in \mathbb{N}$.

Proof. (1): By the definition, M_z is a q -isometry if and only if $\sum_{s=0}^q (-1)^s \binom{q}{s} M_z^{*s} M_z^s = 0$. The left hand part is a diagonal operator in the basis $\{z^k\}$ (for any weights $\{\tilde{\beta}_k\}$). Hence M_z is a q -isometry iff $\sum_{s=0}^q (-1)^s \binom{q}{s} \langle M_z^s z^k, M_z^s z^k \rangle = 0$ for any $k \in \mathbb{N}$, which happens iff $\nabla^q \gamma_k \equiv 0$. This gives (1).

(2): This can be proved along the lines of the verification of (1), and hence we skip it. \square

Now it follows from Theorem 5.3(2) that a spherical m -tuple M_z is a q -isometry if and only if the corresponding scalar sequence $\{\tilde{\beta}_k\}$ satisfies $\tilde{\beta}_k^2 = S(k)$, $k \in \mathbb{N}$, for some polynomial S , whose degree is less or equal than $q - 1$. In particular, we get the following fact.

Corollary 5.6. Let M_z be a spherical q -isometry. Then $[M_{z_j}^*, M_{z_l}]$ is in S_p for all j, l if and only if $p > m$.

Proof. Assuming that the m -tuple M_z is a q -isometry, we get a polynomial S of degree $d \leq q-1$ such that $\beta_k^2 = S(k)$, $k \in \mathbb{N}$. Then $\delta_k^2 - 1 \sim d/k$, and we can apply Corollary 4.6 to get our statement. \square

Proposition 5.7. *Let T be a m -variable spherical weighted shift and $\{\tilde{\beta}_k\}$, $\{\delta_k\}$ be the corresponding 1-variable sequences. Suppose that $\delta_k \leq C$ and that the sequence $\{\delta_k\}$ does not tend to zero. Put $\gamma_k = \tilde{\beta}_k^2$ (as before).*

- (1) *If $\nabla^2 \gamma_k \leq 0$, then all commutators $[T_j^*, T_k]$ are in S_p iff $p > m$.*
- (2) *If $\nabla^3 \gamma_k \leq 0$ and $\tilde{\beta}_k > C > 0$, then it is also true that all commutators $[T_j^*, T_k]$ belong to S_p iff $p > m$.*

It will follow from the proof that it suffices to assume that the inequality $\nabla^2 \gamma_k \leq 0$ or $\nabla^3 \gamma_k \leq 0$ holds except for a finite number of indices.

Proof of Proposition 5.7. First observe that

$$(5.32) \quad \delta_{k+1}^2 - \delta_k^2 = \frac{\gamma_{k+2}}{\gamma_{k+1}} - \frac{\gamma_{k+1}}{\gamma_k} = \frac{\nabla^2 \gamma_k}{\gamma_k} - \frac{\nabla \gamma_{k+1} \nabla \gamma_k}{\gamma_{k+1} \gamma_k}.$$

Proof of (1): Suppose that $\nabla^2 \gamma_k \leq 0$. It follows from Richter's Lemma [30, Lemma 6.9] that $\nabla \gamma_k \geq 0$ for all k . Since $\{\gamma_k\}$ is a concave sequence,

$$\nabla \gamma_k = \gamma_{k+1} - \gamma_k \leq \frac{\gamma_k - \gamma_0}{k} \leq \frac{\gamma_k}{k}$$

for all $k \geq 1$. Hence

$$|\nabla^2 \gamma_k| = -\nabla^2 \gamma_k = \nabla \gamma_k - \nabla \gamma_{k+1} \leq \nabla \gamma_k \leq \frac{\gamma_k}{k}.$$

Therefore, by (5.32),

$$|\delta_{k+1}^2 - \delta_k^2| \leq \frac{|\nabla^2 \gamma_k|}{\gamma_k} + \frac{|\nabla \gamma_{k+1}| |\nabla \gamma_k|}{\gamma_{k+1} \gamma_k} \leq \frac{1}{k} + \frac{1}{k} \cdot \frac{1}{k+1} \leq \frac{2}{k}, \quad k \geq 2.$$

So the assertion (1) follows from Corollary 4.6.

Proof of (2): We are assuming that $\nabla^3 \gamma_k \leq 0$ and that $\gamma_k \geq C > 0$. Then there is an index k_0 such that the sign of $\nabla^2 \gamma_k$ is constant for $k \geq k_0$. If $\nabla^2 \gamma_k \leq 0$ for all $k \geq k_0$, the proof is as above. So we can suppose that there is $k_0 \in \mathbb{N}$ such that $0 < \nabla^2 \gamma_k \leq \nabla^2 \gamma_{k_0}$ for $k \geq k_0$. Hence $\{\nabla \gamma_k\}_{k \geq k_0}$ is a growing sequence. Let us distinguish two opposite cases.

Case A: $\nabla \gamma_k > 0$ for large indices $k > k_0$. Then $\nabla \gamma_k > C_1 > 0$, and therefore $\gamma_k \geq C_2 k$ for large indices k , where $C_2 > 0$. Next let $k > 2k_0$. If k is even, then

$$\gamma_k > \gamma_k - \gamma_{k/2} = \sum_{\ell=k/2}^{k-1} \nabla \gamma_\ell \geq \frac{k}{2} \nabla \gamma_{k/2}$$

Since $\{\nabla \gamma_\ell\}$ is concave,

$$\frac{\nabla \gamma_k + \nabla \gamma_0}{2} \leq \nabla \gamma_{k/2} < \frac{2\gamma_k}{k}.$$

Therefore

$$\frac{\nabla \gamma_k}{\gamma_k} \leq \frac{4}{k} + \frac{|\nabla \gamma_0|}{\gamma_k} \leq \frac{C_3}{k}.$$

In the same way, one gets that $\frac{\nabla \gamma_k}{\gamma_k} \leq \frac{C_3}{k}$ for odd k , $k > 2k_0$ (just replace indices $0, k/2$ by $1, (k+1)/2$ in the above estimates). Since $\{\nabla^2 \gamma_k\}$ is bounded, (5.32) implies the estimate $|\delta_{k+1} - \delta_k| \leq C/k$, and we are done.

Case B: $\nabla \gamma_k \leq 0$ for all $k \geq k_0$. Hence $\{|\nabla \gamma_k|\}$ decays for $k \geq k_0$, and

$$(k - k_0) |\nabla \gamma_k| \leq \sum_{\ell=k_0}^{k-1} |\nabla \gamma_\ell| = \gamma_{k_0} - \gamma_k \leq \gamma_{k_0}$$

for $k > k_0$, so that $|\nabla\gamma_k| \leq C/k$. Next, for $k > k_0$, $\nabla^2\gamma_k = |\nabla\gamma_k| - |\nabla\gamma_{k+1}| \leq |\nabla\gamma_k| \leq C/k$, and once again, we obtain that $|\delta_{k+1} - \delta_k| \leq C/k$ by using (5.32) and the assumption $\tilde{\beta}_k \geq \text{const} > 0$.

Therefore in both cases A and B, Corollary 4.6 implies assertion (2). \square

Example 5.8. In the part 2) of the last Proposition, one cannot drop the assumption $\tilde{\beta}_k \geq C > 0$. Indeed, define $\{\tilde{\beta}_k\}$ by $\tilde{\beta}_{2k}^2 = 12^{-k}$ and $\tilde{\beta}_{2k+1}^2 = 12^{-k}/3$, $k \geq 0$. Then $(\nabla^3\gamma)_{2k} = -(2/9) \cdot 12^{-k}$ and $(\nabla^3\gamma)_{2k+1} = -(23/144) \cdot 12^{-k}$, $k \geq 0$, so that $\nabla^3\gamma_k < 0$ for all k . On the other side, $\delta_{k+1}^2 - \delta_k^2 = (-1)^{k+1}/12$ for any k , so that in this case, the tuple M_z is bounded, but is not essentially normal (and therefore the self-commutators do not belong to any S_p).

Remark 5.9 : Let $T : \{w_n^{(i)}\}_{n \in \mathbb{N}^m}$ be a spherical m -variable weighted shift and let $T_\delta : \{\delta_k\}_{k \in \mathbb{N}}$ be the shift associated with T . Suppose that $\{\delta_k\}_{k \in \mathbb{N}}$ converges to a non-negative number λ . Then by Remark 3.3, T is essentially normal. Moreover, by Theorem 3.4(5), the essential spectrum of T is $\partial\mathbb{B}_\lambda$. This happens whenever T is jointly hyponormal, a joint q -isometry or a joint 2-hyperexpansion. In case T is jointly hyponormal, $\lambda = \|T_\delta\|$ while in the remaining two cases λ is equal to 1.

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