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FOCAL RADIUS, RIGIDITY, AND LOWER CURVATURE BOUNDS

LUIS GUIJARRO AND FREDERICK WILHELM

ABSTRACT. We prove a new comparison lemma for Jacobi fields that exploits Wilking's transverse Jacobi equation. In contrast to standard Riccati and Jacobi comparison theorems, there are situations when our technique can be applied after the first conjugate point.

Using it we show that the focal radius of any submanifold N of positive dimension in a manifold M with sectional curvature greater than or equal to 1 does not exceed $\frac{\pi}{2}$. In the case of equality, we show that N is totally geodesic in M and the universal cover of M is isometric to a sphere or a projective space with their standard metrics, provided N is closed.

Our results also hold for k^{th} -intermediate Ricci curvature, provided the submanifold has dimension $\geq k$. Thus in a manifold with Ricci curvature $\geq n - 1$, all hypersurfaces have focal radius $\leq \frac{\pi}{2}$, and space forms are the only such manifolds where equality can occur, if the submanifold is closed.

Example 2.38 and Remark 3.4 show that our results cannot be proven using standard Riccati or Jacobi comparison techniques.

A Riemannian manifold M has k^{th} -intermediate Ricci curvature $\geq \ell$ if for any orthonormal $(k + 1)$ -frame $\{v, w_1, w_2, \dots, w_k\}$, the sectional curvature sum, $\sum_{i=1}^k \sec(v, w_i)$, is $\geq \ell$ ([33], [27]). For brevity we write $\text{Ric}_k M \geq \ell$. Motivated by Myers theorem we show that if $\text{Ric}_k M \geq k$, then all submanifolds with dimension $\geq k$ have focal radius $\leq \frac{\pi}{2}$.

Theorem A. *Let M be a complete Riemannian n -manifold with $\text{Ric}_k \geq k$ and N be any submanifold of M with $\dim(N) \geq k$.*

1. *Every unit speed geodesic γ that leaves N orthogonally at time 0 has at least $\dim(N) - k + 1$ focal points for N in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, counting multiplicities. In particular, the focal radius of N is $\leq \frac{\pi}{2}$.*
2. *If the focal radius of N is $\frac{\pi}{2}$, then N is totally geodesic.*

Since $\text{Ric}_1 M \geq \ell$ means that all sectional curvatures of M are $\geq \ell$ and $\text{Ric}_{n-1} M \geq \ell$ means that M has Ricci curvature $\geq \ell$, the theorem applies to $N \subset M$ if either the Ricci curvature of M is $\geq n - 1$ and N is a hypersurface, or the sectional curvature of M is ≥ 1 and $\dim(N) \geq 1$.

We emphasize that N need not be closed or even complete, and there is no hypothesis about its second fundamental form. On the other hand, if N happens to be closed and have focal radius $\frac{\pi}{2}$, then we determine M up to isometry.

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Theorem B. *Let M be a complete Riemannian n -manifold with $\text{Ric}_k \geq k$. If M contains a closed, embedded, submanifold N with focal radius $\frac{\pi}{2}$ and $\dim(N) \geq k$, then N is totally geodesic in M , and the universal cover of M is isometric to the sphere or a projective space with the standard metrics.*

In Section 3.1, we provide examples showing that the hypothesis on the dimension of N can not be dropped from either Theorem A or B.

In the course of proving Theorem B we will also establish the following corollary (see Theorem 5.17, below.)

Corollary C. *If the submanifold N of Theorem B is a hypersurface, then the universal cover of M is isometric to the unit sphere.*

It is reasonable to compare the Ricci curvature versions of Theorems A and B with the Bonnet-Myers Theorem and Cheng's Maximal Diameter Theorem (cf also Theorem 3 in [6] and Theorem 1 in [10]). While an analogy can be made between the sectional curvature version of Theorem B and the Diameter Rigidity Theorem ([14],[32]), the following example shows that Theorem B applies to more nonsimply connected manifolds.

Example D. *Let \mathbb{S}^3 be the unit sphere in $\mathbb{C} \oplus \mathbb{C}$, and embed \mathbb{S}^1 as the unit circle in the first copy of \mathbb{C} . Let Q be the quaternion group of order 8 in $SO(4)$. Then the focal radius of $N = Q(\mathbb{S}^1)/Q$ in $M = \mathbb{S}^3/Q$ is $\frac{\pi}{2}$, and N is its own focal set. On the other hand, M has diameter strictly smaller than $\frac{\pi}{2}$.*

More generally, let $\pi : \mathbb{S}^n \rightarrow \mathbb{S}^n/G$ be the quotient map of a properly discontinuous action by G on \mathbb{S}^n , and let N be any closed geodesic in \mathbb{S}^n/G . Then $\pi^{-1}(N)$ is the disjoint union of closed geodesics in \mathbb{S}^n , and hence both $\pi^{-1}(N)$ and N have focal radius $\frac{\pi}{2}$.

Theorem B implies that the standard unit metric is the only one on any topological sphere with sectional curvature ≥ 1 that has a closed submanifold with focal radius $\frac{\pi}{2}$. In contrast, the conclusion of the Diameter Rigidity Theorem is softer, since there are many metrics on \mathbb{S}^n with curvature ≥ 1 and diameter $\geq \frac{\pi}{2}$, and there is even the possibility of such a metric on an exotic sphere.

It is also reasonable to compare the sectional curvature version of Theorem B to the ‘‘rank rigidity’’ results of Schmidt and Shankar–Spatzier–Willing in [24] and [26]. Shankar, Spatzier, and Wilking obtained the conclusion of Theorem B for manifolds with curvature *less* than or equal to 1 and minimal conjugate radius π . Schmidt proves that if M has sectional curvature ≥ 1 and conjugate radius $\geq \frac{\pi}{2}$, then its universal cover is homeomorphic to S^n or isometric to a projective space. The conjugate radius hypotheses of these theorems apply to every geodesic in M . In contrast, the focal radius hypothesis of Theorem B only concerns the geodesics that meet a single submanifold orthogonally.

To prove Theorems A and B, we exploit Wilking's transverse Jacobi equation ([31]) to get a new comparison lemma for Jacobi fields. To state it, we let $\gamma : (-\infty, \infty) \rightarrow M$ be a unit speed geodesic in a complete Riemannian n -manifold M . We call an $(n-1)$ -dimensional subspace Λ of normal Jacobi fields along γ , *Lagrangian*, if the restriction of the Riccati operator to Λ is self adjoint, that is, if

$$\langle J_1(t), J_2'(t) \rangle = \langle J_1'(t), J_2(t) \rangle$$

for all t and for all $J_1, J_2 \in \Lambda$ (see (1.2) below for the formal definition of the Riccati operator on Λ).

In Sections 1 and 2, we review Wilking's transverse Jacobi equation, justify the name Lagrangian, and prove a comparison lemma for intermediate Ricci curvature. In the special case when the sectional curvature is bounded from below our comparison result becomes the following.

Lemma E. (*Sectional Curvature Comparison*) *For $\kappa = -1, 0$, or 1 , let $\gamma : (-\infty, \infty) \rightarrow M$ be a unit speed geodesic in a complete Riemannian n -manifold M with $\sec(\dot{\gamma}, \cdot) \geq \kappa$. Let J_0 be a nonzero, normal Jacobi field along γ , and let Λ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S such that $J_0 \in \Lambda$.*

For $t_0 < t_{\max}$, suppose that Λ has no singularities on (t_0, t_{\max}) , and that $\tilde{\lambda}_\kappa : [t_0, t_{\max}) \rightarrow \mathbb{R}$ is a solution of

$$\tilde{\lambda}'_\kappa + \tilde{\lambda}_\kappa^2 + \kappa = 0 \quad (1)$$

with

$$\langle S(J_0), J_0 \rangle|_{t_0} \leq \tilde{\lambda}_\kappa(t_0) |J_0(t_0)|^2. \quad (2)$$

Then for each $t_1 \in [t_0, t_{\max})$ there is a $J_1 \in \Lambda \setminus \{0\}$ so that

$$\langle S(J_1), J_1 \rangle|_{t_1} \leq \tilde{\lambda}_\kappa(t_1) |J_1(t_1)|^2. \quad (3)$$

In particular, if $\kappa = 1$, $\alpha \in [0, \pi)$, $\tilde{\lambda}_1(t) = \cot(t + \alpha)$, and $t_0 \in [0, \pi - \alpha)$, then Λ has a singularity by time $\pi - \alpha$, that is, there is a $J \in \Lambda \setminus \{0\}$ with $J(t_2) = 0$ for some $t_2 \in (t_0, \pi - \alpha]$.

Lemma E holds in certain situations where Λ has singularities on $[t_0, t_{\max})$, for example when $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$. We describe another such situation in Lemma 2.23, where the reader will also find a discussion of the equality case.

The reader is probably familiar with the Riccati comparison theorem of Eschenburg-Heintze in [9]. It requires the initial condition (2) to hold for *all* $J_0 \in \Lambda$, while Lemma E only demands that the initial condition holds for a *single* Jacobi field. This comes at the expense that the derived future inequality (3) is only guaranteed to hold for a single Jacobi field, which moreover, is not likely to be the original field. In Examples 2.37 and 2.38 (below), we show that J_1 can in fact be different from J_0 . A similar example can be found on page 463 of [18]. This phenomenon is tied to the nonvanishing of Wilking's generalized A -tensor (see (1.8)).

The difference between Lemma E and the theorem of [9] is starker if one considers the contrapositives: Lemma E implies that if Inequality (3) fails for all $J_1 \in \Lambda$, then Inequality (2) fails for *all* $J \in \Lambda$. In contrast, the theorem of [9] only gives that Inequality 2 fails for *some* $J \in \Lambda$.

The main tool to prove Theorem A is Lemma 2.23, which is a generalization of Lemma E to intermediate Ricci curvature. So that we can prove Theorem B, Lemma 2.23 also includes an analysis of the rigid situation. Other cases when rigidity occurs are given in Lemmas 2.26 and 2.27.

The proof of Theorem B begins by establishing Proposition 4.4, which draws a strong analogy between N and one of the dual sets in the proof of the Diameter Rigidity Theorem. Example D shows that we can only push this analogy so far. The dual sets of [14] are disjoint while

Example D shows that N can be its own focal set. In fact, one of the challenges of the proof of Theorem B is showing that phenomena like Example D do not occur in the simply connected case. In spite of the differences, our overall strategy is similar to that of [14], and our proof employs ideas from there. To keep the exposition tight, we will often refrain from giving further specific references to [14] and have made our exposition reasonably self-contained.

After the introduction, we establish notations and conventions. The remainder of the paper is divided into two parts and eight sections. The sections are subordinate to the parts. Each part and many of the sections begin with a detailed summary of the contents, so the outline immediately below is only meant to indicate where each result is proven.

Part 1 contains Sections 1 to 3. In Section 1, we review Wilking's transverse Jacobi equation; in Section 2 we state and prove Lemma 2.23, which is the main tool of the paper. Subsection 2.4 provides examples showing that J_0 and J_1 can indeed be different in Lemma E. In Section 3, we prove Theorem A and give examples showing its optimality.

In Part 2, we prove Theorem B in Sections 4—8. In the special case of Theorem B, when the sectional curvature is ≥ 1 , the argument can be completed a little faster by an appeal to the Diameter Rigidity Theorem. We do this in Section 7, and we complete the proof of the general case of Theorem B in Section 8.

Remark F. The reader may have noticed that the hypotheses $\text{Ric}_k \geq k \cdot \kappa$ of Theorems A and B are global, whereas in Lemma E, we only assumed that $\sec(\dot{\gamma}, \cdot) \geq \kappa$.

For the conclusion of Theorems A to hold, we in fact, only need $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$ for all unit speed geodesics γ that leave N orthogonally at time zero. That is,

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa$$

for any orthonormal set $\{\dot{\gamma}, E_1, \dots, E_k\}$.

On the other hand, our proof of Theorem B uses the global hypothesis $\text{Ric}_k \geq k \cdot \kappa$ and also the fact that Lemma 2.23 and its rigidity case are valid with only the radial curvature lower bound.

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NOTATIONS AND CONVENTIONS

Unless otherwise specified, all curves are parameterized at unit speed. Given $v \in TM$, we denote the unique geodesic with $\gamma'_v(0) = v$ by γ_v .

Let N be a submanifold of the Riemannian manifold M . Let $\nu(N)$ be the normal bundle of $N \subset M$. For every unit $v \in \nu(N)$, there is a first time $t_1 \in (0, \infty]$ at which $\gamma_v(t_1)$ is focal for N along γ_v . We set

$$\text{reg}_N \equiv \{tv \in \nu(N) \mid |v| = 1 \text{ and } t \in [0, t_1)\}.$$

We let g^* be the metric on the domain reg_N obtained from pulling back (M, g) via the normal exponential map. We use the term *tangent focal point* for a critical point of $\exp_N^\perp : \nu(N) \rightarrow M$ and the term *focal point* for a critical value of \exp_N^\perp .

$\pi : \nu(N) \longrightarrow N$ will denote the projection of the normal bundle; N_0 will be the 0-section of $\nu(N)$, and $\nu^1(N)$ will be the unit normal bundle of N . The fibers of $\nu(N)$ and $\nu^1(N)$ over $x \in N$ will be called $\nu_x(N)$ and $\nu_x^1(N)$.

We let Λ be any Lagrangian family of normal Jacobi fields along a geodesic γ , and for any subspace $\mathcal{W} \subset \Lambda$ we write

$$\mathcal{W}(t) \equiv \{J(t) \mid J \in \mathcal{W}\} \oplus \{J'(t) \mid J \in \mathcal{W} \text{ and } J(t) = 0\}. \quad (4)$$

When γ is a geodesic that leaves N orthogonally at time 0, we will write Λ_N for the Lagrangian family of normal Jacobi fields along γ corresponding to variations by geodesics that leave N orthogonally at time 0. We call the elements of Λ_N , N -Jacobi fields. According to Lemma 4.1 on page 227 of [7], Λ_N consists of the following *normal* Jacobi fields J along γ :

$$\Lambda_N \equiv \{J \mid J(0) = 0, J'(0) \in \nu_{\gamma(0)}(N)\} \oplus \{J \mid J(0) \in T_{\gamma(0)}N \text{ and } J'(0) = S_{\gamma'(0)}J(0)\}, \quad (5)$$

where $S_{\gamma'(0)}$ is the shape operator of N determined by $\gamma'(0)$, that is,

$$\begin{aligned} S_{\gamma'(0)} &: T_{\gamma(0)}N \longrightarrow T_{\gamma(0)}N \text{ is} \\ S_{\gamma'(0)} &: w \longmapsto (\nabla_w \gamma'(0))^{TN}. \end{aligned}$$

We write \mathbb{S}^n for the unit sphere in \mathbb{R}^{n+1} , and for $\kappa = -1, 0$, or 1 , we let \mathcal{S}_κ^2 be the simply connected 2-dimensional space form of constant curvature κ .

We use the acronym CROSS for Compact Rank One Symmetric Space. For convenience, we normalize the nonspherical CROSSes so that their curvatures are in $[1, 4]$, and we normalize the spherical CROSSes to have constant curvature 4.

We write \sec for sectional curvature and κ for our lower curvature bound. After rescaling, we may always assume that κ is either $-1, 0$, or 1 .

Given $r > 0$ and $A \subset M$ we set

$$\begin{aligned} B(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) < r\}, \\ D(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) \leq r\}, \text{ and} \\ S(A, r) &\equiv \{x \in M \mid \text{dist}(x, A) = r\}. \end{aligned}$$

Finally, we write $D_v(f)$ the derivative of f in the direction v .

Part 1: Bounding the Focal Radius

Part 1 is divided in three sections. Section 1 reviews Wilking's transverse equation. In Section 2, we state and prove Lemma 2.23, which is a generalization of Lemma E and is the main tool of the paper; in subsection 2.4 we give an example that shows that J_1 need not equal J_0 in Lemma E. Finally, in Section 3, we prove Theorem A, and give some examples showing its optimality.

1. WILKING'S TRANSVERSE JACOBI EQUATION

In this section, we review Lagrangian families and Wilking's transverse Jacobi equation.

1.1. Lagrangian Families. Let γ be a unit speed geodesic in a complete Riemannian n -manifold M , and let \mathcal{J} be the vector space of normal Jacobi fields along γ . Using symmetries of the curvature tensor, we see that for $J_1, J_2 \in \mathcal{J}$,

$$\omega(J_1, J_2) = \langle J'_1, J_2 \rangle - \langle J_1, J'_2 \rangle,$$

is constant along γ and hence defines a symplectic form on \mathcal{J} .

Thus an $(n - 1)$ -dimensional subspace Λ of \mathcal{J} on which ω vanishes is called Lagrangian. Of course this is equivalent to saying that the restriction of the Riccati operator to Λ is self-adjoint. Examples of Lagrangian families include the Jacobi fields that are 0 at time 0 and those that correspond to variations by geodesics that leave a submanifold orthogonally at time 0.

The set of times t so that

$$\{J(t) \mid J \in \Lambda\} = \dot{\gamma}(t)^\perp \quad (1.1)$$

is open and dense (cf Lemma 1.7 of [15]). For these t we get a well-defined Riccati operator

$$\begin{aligned} S_t &: \dot{\gamma}(t)^\perp \longrightarrow \dot{\gamma}(t)^\perp \\ S_t &: v \longmapsto J'_v(t), \end{aligned} \quad (1.2)$$

where J_v is the unique $J_v \in \Lambda$ so that $J_v(t) = v$. The Jacobi equation then decomposes into the two first order equations

$$S_t(J) = J', \quad S'_t + S_t^2 + R = 0,$$

where S'_t is the covariant derivative of S_t along γ and R is the curvature along γ , that is $R(\cdot) = R(\cdot, \dot{\gamma})\dot{\gamma}$ (see Equation 1.7.1 in [15]). We will omit the dependence on t if it is clear from the context.

Remark 1.3. Given any $\mathcal{W} \subset \Lambda$, and some t such that no Jacobi field in $\mathcal{W} \setminus \{0\}$ vanishes at t , Equation (1.2) gives a well defined Riccati operator

$$S_t : \mathcal{W}(t) \longrightarrow \gamma'(t)^\perp.$$

This S_t agrees with the restriction of S_t defined in (1.2) when Λ has no zeros.

1.2. Singularities in the Lagrangian and the Riccati operator. The set of times t when

$$\dim \{J(t) \mid J \in \Lambda\} < n - 1 \quad (1.4)$$

corresponds to the moments where some of the Jacobi fields in Λ vanish. They are important since, in general, they correspond to moments when the Riccati operator S_t is not defined.

Definition. Let \mathcal{V} be a subspace of Λ . We will say that \mathcal{V} has full index at \bar{t} if any $J \in \Lambda$ with $J(\bar{t}) = 0$ belongs to \mathcal{V} ; we will also say that \mathcal{V} has full index on an interval I if it has full index at each point of I .

There is a different way of stating the above condition: for fixed $t \in \mathbb{R}$, define the *evaluation map* as

$$\begin{aligned}\mathcal{E}_t &: \Lambda \longrightarrow T_{\gamma(t)}M \\ \mathcal{E}_t &: J \longmapsto J(t).\end{aligned}$$

Observe that, for given $t \in I$, the kernel of \mathcal{E}_t is the set of those $J \in \Lambda$ vanishing at t . Thus a subspace $\mathcal{V} \subset \Lambda$ has full index in an interval if and only if \mathcal{V} contains the kernel of the evaluation map \mathcal{E}_t for every t in the interval.

1.3. Wilking's Transverse Jacobi Equation. Let \mathcal{V} be any subspace of Λ . Set

$$\mathcal{V}(t) \equiv \{J(t) \mid J \in \mathcal{V}\} \oplus \{J'(t) \mid J \in \mathcal{V}, J(t) = 0\}. \quad (1.5)$$

Then $\mathcal{V}(t)$ is a smooth vector bundle along γ (Lemma 1.7.1 in [15], or [31]). Set

$$H(t) \equiv \mathcal{V}(t)^\perp \cap \dot{\gamma}(t)^\perp.$$

Proposition 1.6. *Fix $t \in I$ and suppose that \mathcal{V} has full index at t .*

1. *For $x \in H(t)$, there is a $J \in \Lambda$ so that $J(t) = x$.*

2. *We have a well-defined Riccati operator*

$$\hat{S}_t : H(t) \longrightarrow H(t)$$

given by

$$\hat{S}_t(x) = J'^H(t), \quad (1.7)$$

where J is an element of Λ so that $J(t) = x$, and $J'^H(t)$ is the $H(t)$ -component of $J'(t)$; in other words, \hat{S}_t is the $H(t)$ -projection of $S_t|_{H(t)}$.

Proof. Since Λ is Lagrangian, the splitting

$$\Lambda(t) = \{J(t) \mid J \in \Lambda\} \oplus \{J'(t) \mid J \in \Lambda, J(t) = 0\}$$

is orthogonal. Since the kernel of \mathcal{E}_t lies in \mathcal{V} , $H(t)$ is contained in the first summand, and Part 1 follows.

For the second part, suppose $x = J_1(t) = J_2(t) \in H(t)$ and $J_1, J_2 \in \Lambda$. Since $J_1 - J_2$ vanishes at t and $\text{Kernel}(\mathcal{E}_t) \subset \mathcal{V}$, we have $J_1 - J_2 \in \mathcal{V}$. Together with $(J_1 - J_2)(t) = 0$, this implies that $(J_1 - J_2)'(t) \in \mathcal{V}(t)$. Thus $((J_1 - J_2)'(t))^H = 0$, and $\hat{S}_t(x)$ is independent of the choice of $J \in \Lambda$ so that $J(t) = x$. \square

We will call \hat{S} the *Riccati operator associated to \mathcal{V}* , if it is clear which Lagrangian Λ is being used.

Wilking also defined maps

$$\begin{aligned}A_t &: \mathcal{V}(t) \longrightarrow H(t) \text{ given by,} \\ A_t(v) &= (J')^h(t), \text{ where } J \in \mathcal{V}, J(t) = v.\end{aligned} \quad (1.8)$$

A priori, A_t is only defined at points where Λ has no zeros; however, A extends smoothly to \mathbb{R} (cf. [31]). Indeed, let $A_t^* : H(t) \rightarrow \mathcal{V}(t)$ be the adjoint of A_t , and let X be a field in H so that $(X')^H \equiv 0$. According to Equation 1.7.6 on page 38 of [15],

$$X' = -A^*X.$$

Since the left-hand side is smooth, A^* is smooth, and it follows that A is smooth.

Theorem 1.9 (Wilking [31]). *\hat{S} is self-adjoint, and*

$$\hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma})\dot{\gamma}\}^h + 3AA^* = 0. \quad (1.10)$$

Equation (1.10) is known as the Transverse Jacobi Equation. It is a vast generalization of the Horizontal Curvature Equation of [12] and [21]. For details see [16] or [19].

Proposition 1.6 only gives us that \hat{S} is defined almost everywhere. However, $\hat{S}' + \hat{S}^2$ has a smooth extension to all of \mathbb{R} , because $\{R(\cdot, \dot{\gamma})\dot{\gamma}\}^h + 3AA^*$ is smooth everywhere (see [31] for an interpretation of $\hat{S}' + \hat{S}^2$ as a second order differential operator $H(t) \rightarrow H(t)$).

1.4. Splitting of Lagrangians. Like the Gray-O'Neill A -tensor, the Wilking A -tensor vanishes identically along a geodesic γ if and only if the distributions $\mathcal{V}(t)$ and $H(t)$ are parallel along γ . In this case, it follows that the subspaces of Λ ,

$$\{J \in \Lambda \mid J(t) \in H(t)\},$$

are independent of t , and the parallel, orthogonal splitting $\mathcal{V}(t) \oplus H(t)$ is given by Jacobi fields. We make this more rigorous in what follows.

Lemma 1.11. *With the above notation, assume that $A_t = 0$ for every $t \in I$. Then*

1. $\mathcal{V}(t)$ and $H(t)$ are parallel distributions along γ .
2. If for some $\bar{t} \in I$, a Jacobi field $J \in \Lambda$ has $J(\bar{t}) \in H(\bar{t})$, then $J(t) \in H(t)$ for every t .
3. There is a subspace $\mathcal{H} \subset \Lambda$ such that $\mathcal{H}(t) = H(t)$ for every t .

Proof. By continuity, it is enough to check the first part at times $t \in I$ where Λ has no zeros. Since any section of the bundle $\mathcal{V}(t)$ can be written as

$$Y = \sum_i f_i \cdot J_i,$$

where J_i are a basis of \mathcal{V} , we have that

$$Y'^H = \sum_i f_i \cdot J_i'^H = 0$$

since $A_t \equiv 0$. Therefore $\mathcal{V}(t)$, and consequently $\mathcal{V}^\perp = \mathcal{H}$, are both parallel, proving the first part of the Lemma.

Since $\mathcal{V}(t)$ is parallel and spanned by Jacobi fields, it follows that $R(\cdot, \gamma')\gamma'$ leaves $\mathcal{V}(t)$ invariant. From this it follows that $R(\cdot, \gamma')\gamma'$ leaves $H(t)$ invariant. Combining this with the fact that $H(t)$ is parallel, we get Part 2.

For the last part, choose a set $\{J_1, \dots, J_\ell\}$ in Λ such that for some $\bar{t} \in I$, $\{J_1(\bar{t}), \dots, J_\ell(\bar{t})\}$ is a basis of $H(\bar{t})$. As previously shown, $\{J_1(t), \dots, J_\ell(t)\}$ are in $H(t)$ for any $t \in I$, and it is a basis of $H(t)$ whenever Λ has no zeros at that t . By continuity, the subspace \mathcal{H} spanned by $\{J_1, \dots, J_\ell\}$ satisfies the third part of the Lemma. \square

2. COMPARISON THEORY FOR THE TRANSVERSE JACOBI EQUATION

2.1. Riccati Comparison. In this subsection, we review the Riccati comparison results of Eschenburg ([8]) and Eschenburg–Heintze ([9]). For $\kappa = -1, 0$, or 1 , let $\tilde{\lambda}_\kappa$ be a solution of the ODE

$$\tilde{\lambda}'_\kappa + \tilde{\lambda}_\kappa^2 + \kappa = 0. \quad (2.1)$$

The possible $\tilde{\lambda}_\kappa$ are the logarithmic derivatives of the functions

$$\tilde{f}(t) = \begin{cases} (c_1 \sin t + c_2 \cos t) & \text{if } \kappa = 1, \\ (c_1 t + c_2) & \text{if } \kappa = 0, \\ (c_1 \sinh t + c_2 \cosh t) & \text{if } \kappa = -1, \end{cases} \quad (2.2)$$

where $c_1, c_2 \in \mathbb{R}$. There are explicit formulas for $\tilde{\lambda}_\kappa$ in page 302 of [8].

Theorem 2.3. (*Eschenburg–Heintze*, [9], cf Proposition 2.3 in [8]) *Let $r : \mathbb{R} \rightarrow \mathbb{R}$ be a C^∞ -function with $r \geq \kappa$. Let s be a smooth solution of the initial value problem*

$$s' + s^2 + r = 0, \quad s(t_0) \leq \tilde{\lambda}_\kappa(t_0)$$

on the interval $[t_0, t_{\max})$, where $\tilde{\lambda}_\kappa$ is as in (2.1). Then

1.

$$s(t) \leq \tilde{\lambda}_\kappa(t) \quad (2.4)$$

on $[t_0, t_{\max})$.

2. *If $s(t_1) = \tilde{\lambda}_\kappa(t_1)$ for some $t_1 \in (t_0, t_{\max})$, then for all $t \in [t_0, t_1]$*

$$s(t) = \tilde{\lambda}_\kappa(t) \text{ and } r|_{[t_0, t_1]} \equiv \kappa. \quad (2.5)$$

When s is the trace of the Riccati operator of a Lagrangian family in $\text{Ric} \geq \kappa(n-1)$, the rigidity of Part 2 in Theorem 2.3 also yields rigidity of S and $R(\cdot, \dot{\gamma})\dot{\gamma}$. This idea goes back at least as far as the Splitting Theorem ([4]) and Cheng’s Maximal Diameter Theorem, ([5]). It also appears in Croke and Kleiner’s paper on rigidity of warped products ([6]), in Theorem 1.7.1 of [15], and in Theorem H of [16]. Since our applications will be to the transverse Jacobi equation, we formulate them in terms of abstract Riccati equations.

Lemma 2.6. *Let $\hat{S}(t), \hat{R}(t) : V \rightarrow V$ be symmetric endomorphisms of a k -dimensional vector space V so that on $[t_0, t_{\max})$*

$$\hat{S}' + \hat{S}^2 + \hat{R} = 0.$$

Choose $\tilde{\lambda}_\kappa$ a solution of $\tilde{\lambda}'_\kappa + \tilde{\lambda}_\kappa^2 + \kappa = 0$ defined on $[t_0, t_{\max})$. In addition, assume that

$$\text{Trace } \hat{S}(t_0) \leq k \cdot \tilde{\lambda}_\kappa(t_0), \text{ and} \quad (2.7)$$

$$\text{Trace } \hat{R}(t) \geq k \cdot \kappa$$

for all $t \in [t_0, t_{\max})$. Then

1. *For all $t \in [t_0, t_{\max})$*

$$\text{Trace } \hat{S}(t) \leq k \cdot \tilde{\lambda}_\kappa(t).$$

2. If $\text{Trace } \hat{S}(t_1) = k\tilde{\lambda}_\kappa(t_1)$ for some $t_1 \in (t_0, t_{\max}]$, then

$$\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id} \text{ and } \hat{R} = \kappa \cdot \text{id} \quad (2.8)$$

on $[t_0, t_1]$, and the solutions of the Jacobi equation $J'' + \hat{R}J = 0$ on $[t_0, t_1]$, have the form

$$J(t) = \tilde{f}(t) \cdot E, \quad (2.9)$$

where E is a constant vector in V and \tilde{f} is the function from (2.2) that satisfies $\tilde{f}(t_0) = |J(t_0)|$.

Proof. Set

$$\begin{aligned} s &\equiv \frac{1}{k} \text{Trace } \hat{S}, \\ \hat{S}_0 &\equiv \hat{S} - \frac{\text{Trace } \hat{S}}{k} \cdot \text{id}, \text{ and} \\ r &\equiv \frac{1}{k} \left(\text{Trace } \hat{R} + |\hat{S}_0|^2 \right). \end{aligned} \quad (2.10)$$

Taking the trace of

$$\hat{S}' + \hat{S}^2 + \hat{R} = 0$$

yields

$$s' + s^2 + r = 0.$$

From inequalities (2.4) and (2.7), we get that

$$s(t) \leq \tilde{\lambda}_\kappa(t) \quad (2.11)$$

for all $t \in (t_0, t_{\max})$, and the first part follows.

For the second part, if $\text{Trace } \hat{S}(t_1) = k\tilde{\lambda}_\kappa(t_1)$ for some $t_1 \in (t_0, t_{\max}]$, then Equation (2.5) gives us $s(t) \equiv \tilde{\lambda}_\kappa(t)$ and $r \equiv \kappa$ in the subinterval $[t_0, t_1]$.

Consequently,

$$\kappa = r = \frac{\text{Trace } \hat{R} + |\hat{S}_0|^2}{k} \geq \frac{\kappa k + |\hat{S}_0|^2}{k} = \kappa + \frac{|\hat{S}_0|^2}{k}.$$

Thus $|\hat{S}_0| \equiv 0$ and

$$\hat{S} = \frac{\text{Trace } \hat{S}}{k} \cdot \text{id} = s \cdot \text{id} = \tilde{\lambda}_\kappa(t) \cdot \text{id}.$$

Substituting $\hat{S} = \tilde{\lambda}_\kappa(t) \cdot \text{id}$ into the Riccati equation, $\hat{S}^2 + \hat{S}' + \hat{R} = 0$, gives

$$\begin{aligned} (\tilde{\lambda}_\kappa^2 + \tilde{\lambda}_\kappa') \cdot \text{id} + \hat{R} &= 0, \\ -\kappa \cdot \text{id} + \hat{R} &= 0, \text{ and} \\ \hat{R} &= \kappa \cdot \text{id}. \end{aligned}$$

So the Jacobi fields have the form in equation (2.9). \square

Remark 2.12. When $\kappa = 0$ and $t_{\max} = \infty$, the above Lemma states that if $\text{Trace } \hat{S}(t_0) \leq 0$, then $\text{Trace } \hat{S}(t) \leq 0$ for any $t \geq t_0$, since in this case, $\tilde{\lambda}_0 \equiv 0$ satisfies condition (2.7). The following result improves this observation.

Lemma 2.13 (Long geodesics in nonnegative curvature). *For \hat{S} and \hat{R} as in Lemma 2.6, suppose that*

$$\text{Trace } \hat{S}(t_0) \leq 0, \text{ and } \text{Trace } \hat{R}(t) \geq 0 \quad (2.14)$$

for all $t \in [t_0, \infty)$. If \hat{S} is defined on $[t_0, \infty)$, then

$$\hat{S} \equiv 0 \text{ and } \hat{R} = 0, \quad (2.15)$$

on $[t_0, \infty)$.

Proof. As in the previous proof, (2.4) gives

$$s(t) \leq 0 \quad (2.16)$$

for all $t \in [t_0, \infty)$. If for some $t_1 > t_0$, $s(t_1) < 0$, then there is some $c > t_1$ such that

$$s(t_1) = \frac{1}{t_1 - c}.$$

Thus, for

$$\tilde{\lambda}_0(t) = \frac{1}{t - c},$$

we get from (2.4) that

$$s(t) \leq \tilde{\lambda}_0(t)$$

for all $t \in [t_1, c)$, and in particular, $s(t)$ could not be defined after c . Since this contradicts our hypothesis on \hat{S} being defined on $[t_0, \infty)$, we obtain that $s \equiv 0$ and $r \equiv 0$ for all $t \in (t_0, \infty)$. The rest of the proof follows as in Lemma 2.6. \square

For arbitrary curvature, there is also a rigidity statement:

Lemma 2.17. *Let $\tilde{\lambda}_\kappa$ be as in (2.1), and have no singularities on (t_0, t_{\max}) . Suppose that*

$$\text{Trace } \hat{S}(t_0) \leq k \cdot \tilde{\lambda}_\kappa(t_0), \text{ and } \text{Trace } \hat{R}(t) \geq k \cdot \kappa \quad (2.18)$$

for all $t \in [t_0, t_{\max})$. If \hat{S} is defined on $[t_0, \infty)$ and

$$\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty,$$

then

$$\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id} \text{ and } \hat{R} = \kappa \cdot \text{id} \quad (2.19)$$

holds on $[t_0, t_{\max})$.

Proof. The hypothesis $\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty$ implies that

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot(\pi + t - t_{\max}) & \text{if } \kappa = 1, \\ \frac{1}{t - t_{\max}} & \text{if } \kappa = 0, \\ \coth(t - t_{\max}) & \text{if } \kappa = -1 \end{cases}$$

(see, e.g., page 302 of [8]). Since $\tilde{\lambda}_\kappa$ has no singularities on (t_0, t_{\max}) , it follows that $\tilde{\lambda}_\kappa(t)$ is strictly decreasing on (t_0, t_{\max}) . So if $s(t_1) < \tilde{\lambda}_\kappa(t_1)$ for some $t_1 \in (t_0, t_{\max})$, then there is an $\alpha \in (0, t_{\max} - t_1)$ so that

$$s(t_1) \leq \tilde{\lambda}_\kappa(t_1 + \alpha).$$

Thus by (2.4),

$$s(t) \leq \tilde{\lambda}_\kappa(t + \alpha)$$

for all $t \in (t_1, t_{\max})$. In particular, for some $\tilde{t}_{\max} \in (t_0, t_{\max} - \alpha]$, $\lim_{t \rightarrow \tilde{t}_{\max}^-} s(t) = -\infty$. Since this contradicts our hypothesis that \hat{S} is defined on (t_0, t_{\max}) , Inequality (2.11) must be an equality for all $t \in (t_0, t_{\max})$ and $r \equiv \kappa$. \square

Remark 2.20. In the event that $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$, Theorem 2.3 and Lemmas 2.6, 2.13, and 2.17, hold with the hypothesis $s(t_0) = \frac{1}{k} \text{Trace } \hat{S} \leq \tilde{\lambda}_\kappa(t_0)$ replaced by

$$\liminf_{t \rightarrow t_0^+} \left(\tilde{\lambda}_\kappa(t) - s(t) \right) \geq 0. \quad (2.21)$$

If s is the trace of the Riccati operator of the Lagrangian family $\{J \mid J(t_0) = 0\}$ along a geodesic in a Riemannian manifold, then Inequality (2.21) is satisfied with

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot(t - t_0) & \text{if } \kappa = 1 \\ \frac{1}{t - t_0} & \text{if } \kappa = 0 \\ \coth(t - t_0) & \text{if } \kappa = -1 \end{cases}$$

(see Theorem 27 on page 175 of [23]). So, for example, in this case, Theorem 2.3 implies the classical Rauch Comparison Theorem for 2-manifolds.

2.2. Statements of Comparison Lemmas. For a subspace $\mathcal{W} \subset \Lambda$, write

$$\mathcal{W}(t) = \{J(t) \mid J \in \mathcal{W}\} \oplus \{J'(t) \mid J \in \mathcal{W} \text{ and } J(t) = 0\},$$

and

$$P_{\mathcal{W},t} : \Lambda(t) \longrightarrow \mathcal{W}(t)$$

for orthogonal projection. For simplicity of notation we will write

$$\text{Trace } S_t|_{\mathcal{W}} \text{ for } \text{Trace } (P_{\mathcal{W},t} \circ S_t|_{\mathcal{W}(t)}).$$

Remark 2.22. Choose a fixed $t_0 \in \mathbb{R}$; given any subspace $W_{t_0} \perp \gamma'(t_0)$, W_{t_0} becomes the horizontal subspace $H(t_0)$ for Wilking's equation when we choose \mathcal{V} as the subset of Λ formed by Jacobi fields J with $J(t_0) \perp W_{t_0}$.

By considering 1-dimensional subspaces, we see that Lemma E is a special case of the following result. In its statement we write $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$ to mean that the radial intermediate Ricci curvatures along γ are bounded from below by $k \cdot \kappa$, that is,

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa$$

for any orthonormal set $\{\dot{\gamma}, E_1, \dots, E_k\}$.

Lemma 2.23 (Intermediate Ricci Comparison). *For $\kappa = -1, 0$, or 1 , let $\gamma : (-\infty, \infty) \longrightarrow M$ be a unit speed geodesic in a complete Riemannian n -manifold M with $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$. Let Λ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S , and let $W_{t_0} \perp \gamma'(t_0)$ be some k -dimensional subspace such that*

$$\text{Trace}(S_{t_0})|_{W_{t_0}} \leq k \cdot \tilde{\lambda}_\kappa(t_0), \quad (2.24)$$

where $\tilde{\lambda}_\kappa$ is as in (2.1). Denote by \mathcal{V} the subspace of Λ formed by those Jacobi fields that are orthogonal to W_{t_0} at t_0 and by $H(t)$ the subspace of γ'^\perp that is orthogonal to $\mathcal{V}(t)$ at each $t \in (t_0, t_{\max})$. Assume that \mathcal{V} is of full index in the interval $[t_0, t_{\max})$. Then

1. For all $t \in [t_0, t_{\max})$,

$$\text{Trace } S_t|_{H(t)} \leq k \cdot \tilde{\lambda}_\kappa(t). \quad (2.25)$$

2. If for some $t_1 \in [t_0, t_{\max})$,

$$\text{Trace } S_{t_1}|_{H(t_1)} = k \cdot \tilde{\lambda}_\kappa(t_1),$$

then the Jacobi equation splits orthogonally along γ in the interval $[t_0, t_1]$ as

$$\Lambda = \mathcal{V} \oplus \mathcal{H}$$

where every nonzero Jacobi field in \mathcal{H} is equal to $J = \tilde{f} \cdot E$, where E is a unit parallel field with $E(t_0) \in H(t_0)$, and \tilde{f} is the function from (2.2) that satisfies $\tilde{f}(t_0) = |J(t_0)|$.

Lemma 2.26. Under the hypothesis of the first part of Lemma 2.23, if $\lim_{t \rightarrow t_{\max}^-} \tilde{\lambda}_\kappa(t) = -\infty$ then the Jacobi equation splits orthogonally along γ in the interval $[t_0, t_{\max})$ as

$$\Lambda = \mathcal{V} \oplus \mathcal{H}.$$

Moreover, every nonzero Jacobi field $J \in \mathcal{H}$ is equal to $J = \tilde{f} \cdot E$, where E is a unit parallel field with $E(t_0) \in W_{t_0}$, and \tilde{f} is the function from (2.2) that satisfies $\tilde{f}(t_0) = |J(t_0)|$.

Lemma 2.27. Let $\gamma : [t_0, \infty) \rightarrow M$ be a unit speed geodesic in a complete Riemannian n -manifold M with $\text{Ric}_k(\dot{\gamma}, \cdot) \geq 0$. Let Λ be a Lagrangian subspace of normal Jacobi fields along γ with Riccati operator S . Suppose that for some k -dimensional subspace $W_{t_0} \perp \gamma'(t_0)$,

$$\text{Trace } S_{t_0}|_{W_{t_0}} \leq 0. \quad (2.28)$$

With \mathcal{V} and $H(t)$ as in Lemma 2.23, the Jacobi equation splits orthogonally along γ in the interval $[t_0, \infty)$ as

$$\Lambda = \mathcal{V} \oplus \mathcal{H}.$$

Moreover, every nonzero Jacobi field $J \in \mathcal{H}$ is equal to $J = \tilde{f} \cdot E$, where E is a unit parallel field with $E(t_0) \in W_{t_0}$, and \tilde{f} is the function from (2.2) that satisfies $\tilde{f}(t_0) = |J(t_0)|$.

2.3. Proof of the comparison Lemmas. In this subsection, we combine Riccati comparison with the Transverse Jacobi Equation to prove Lemmas 2.23, 2.26, and 2.27.

Recall that, for a Lagrangian Λ and for fixed $t \in \mathbb{R}$, we defined the *evaluation map* as

$$\begin{aligned} \mathcal{E}_t &: \Lambda \longrightarrow T_{\gamma(t)}M \\ \mathcal{E}_t &: J \longmapsto J(t). \end{aligned}$$

Lemma 2.29. The image of \mathcal{E}_t is the orthogonal complement of the subspace

$$\{ K'(t) : K \in \ker \mathcal{E}_t \}.$$

Proof. Since both subspaces have the same dimension, it suffices to check that for any $J \in \Lambda$ and any $K \in \ker \mathcal{E}_t$, $\langle J(t), K'(t) \rangle = 0$; but $\langle J(t), K'(t) \rangle = \langle J'(t), K(t) \rangle = 0$ since $K(t) = 0$. \square

Lemma 2.30. (*Eigenvalue Transfer Lemma*) Let $\gamma : [0, l] \rightarrow M$ and Λ be as in Lemma 2.23. Let \mathcal{V} be an $(n - 1 - k)$ -dimensional subspace of Λ with full index in $[0, l]$. For any subspace \mathcal{W} of Λ , define $\mathcal{W}(t)$ as in (4).

1. For each fixed $\bar{t} \in [0, l]$, there is a k -dimensional subspace \mathcal{W} of Λ so that $\mathcal{W}(\bar{t})$ is the orthogonal complement of $\mathcal{V}(\bar{t})$. If $\mathcal{E}_{\bar{t}}$ is one-to-one, then \mathcal{W} is unique.
2. Let $\hat{S}_t : H(t) \rightarrow H(t)$ be the Riccati operator defined in (1.7). Then for any \mathcal{W} as in Part 1,

$$\text{Trace } \hat{S}_{\bar{t}} = \text{Trace } S_{\bar{t}}|_{\mathcal{W}}.$$

where $\text{Trace } S_{\bar{t}}|_{\mathcal{W}} = \text{Trace } (P_{\mathcal{W}, \bar{t}} \circ S_{\bar{t}}|_{\mathcal{W}(\bar{t})})$, and $P_{\mathcal{W}, \bar{t}} : \Lambda(\bar{t}) \rightarrow \mathcal{W}(\bar{t})$ is orthogonal projection.

Remark 2.31. For any \mathcal{W} as in Part 1, $S_{\bar{t}}|_{\mathcal{W}}$ is well defined via Remark 1.3.

Proof. Since $\ker \mathcal{E}_{\bar{t}} \subset \mathcal{V}$, we have that

$$\{J'(\bar{t}) : J \in \ker \mathcal{E}_{\bar{t}}\} \subset \mathcal{V}(\bar{t}),$$

and by Lemma 2.29,

$$\mathcal{V}(\bar{t})^\perp \subset \text{image } \mathcal{E}_{\bar{t}}.$$

Thus there exist some k -dimensional subspace $\mathcal{W} \subset \Lambda$ with $\mathcal{W}(t) = \mathcal{V}(t)^\perp$, and if $\mathcal{E}_{\bar{t}}$ is one-to-one, then it is an isomorphism onto $\mathcal{V}(\bar{t})^\perp$, so \mathcal{W} is unique.

To prove Part 2, for $J \in \mathcal{W}$, we write

$$J^\perp = J - J^\mathcal{V},$$

where $J^\mathcal{V}$ is the component of J that lies in $\mathcal{V}(t)$. Then for all t ,

$$0 = \frac{d}{dt} \langle J^\mathcal{V}, J^\perp \rangle = \langle (J^\mathcal{V})', J^\perp \rangle + \langle J^\mathcal{V}, J^{\perp'} \rangle.$$

Since $J \in \mathcal{W}$, $J^\mathcal{V}(\bar{t}) = 0$, and $\langle J^\mathcal{V}, J^{\perp'} \rangle|_{\bar{t}} = 0$. So the previous display evaluated at \bar{t} becomes

$$\langle (J^\mathcal{V})', J^\perp \rangle|_{\bar{t}} = 0.$$

For $J \in \mathcal{W}$, it follows that

$$\begin{aligned} \langle \hat{S}(J^\perp), J^\perp \rangle|_{\bar{t}} &= \langle (J' - (J^\mathcal{V})'), J^\perp \rangle|_{\bar{t}} \\ &= \langle J', J^\perp \rangle|_{\bar{t}} \\ &= \langle S(J), J^\perp \rangle|_{\bar{t}} \\ &= \langle S(J), J \rangle|_{\bar{t}}. \end{aligned} \tag{2.32}$$

So

$$\text{Trace } \hat{S}_{\bar{t}} = \text{Trace } S_{\bar{t}}|_{\mathcal{W}(\bar{t})}.$$

□

Proof of Lemma 2.23. We combine Theorem 2.3 and Lemma 2.6 with the Transverse Jacobi Equation, and the Eigenvalue Transfer Lemma 2.30.

Observe first that Theorem 2.3 holds on intervals where the function s is smooth. In our context, this happens as long as \hat{S} is well-defined. According to Proposition 1.6, \hat{S} is well-defined at all times t where \mathcal{V} has full index at t , and therefore we can apply it in the situation of Lemma 2.23.

Recall that

$$\mathcal{V} \equiv \{X \in \Lambda \mid X(t_0) \perp J(t_0) \text{ for all } J \in W_{t_0}\}.$$

Let $\hat{S} : H(t) \rightarrow H(t)$ be as in Equation (1.7). It follows from the Eigenvalue Transfer Lemma 2.30 that

$$\text{Trace } \hat{S}_{t_0} \leq k \cdot \tilde{\lambda}_\kappa(t_0).$$

The Transverse Jacobi Equation says,

$$\hat{S}' + \hat{S}^2 + \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^* = 0. \quad (2.33)$$

Since $\text{Ric}_k \geq k$, AA^* is nonnegative, and W_{t_0} is k -dimensional, when we take the trace of Equation (2.33), divide by k , and make the substitutions of (2.10), we get an equation that satisfies the hypotheses of Theorem 2.3. Thus for all $t \in [t_0, t_{\max})$,

$$\frac{1}{k} \text{Trace } \hat{S}_t \leq \tilde{\lambda}_\kappa(t). \quad (2.34)$$

By combining this with the Eigenvalue Transfer Lemma 2.30 and the fact that \mathcal{V} has full index on (t_0, t_{\max}) , we have

$$\text{Trace } S|_{H(t)} \leq k \cdot \lambda_\kappa,$$

as claimed.

To prove the rigidity statement, suppose that

$$\text{Trace } S|_{H(t_1)} = k \cdot \tilde{\lambda}_\kappa(t_1)$$

for some $t_1 \in (t_0, t_{\max})$.

It follows from Lemma 2.30 that

$$\text{Trace } \hat{S}_{t_1} = k \cdot \tilde{\lambda}_\kappa(t_1).$$

Writing \hat{R} for $\{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h + 3AA^*$, we see from Theorem 2.3 that

$$\text{Trace } \hat{S}_t \equiv k \cdot \tilde{\lambda}_\kappa(t) \text{ and } \text{Trace } \hat{R} \equiv k \cdot \kappa$$

for all $t \in [t_0, t_1]$.

Our hypothesis that $\text{Ric}_k \geq k \cdot \kappa$ implies that $\text{Trace } \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq k \cdot \kappa$. Combining this with $\text{Trace } \hat{R} \equiv k \cdot \kappa$ and the fact that AA^* is nonnegative, we see that $A \equiv 0$. So Lemma 1.11 guarantees the existence of a subspace \mathcal{H} in Λ such that $\mathcal{H}(t) = H(t)$ at every $t \in [t_0, t_1]$, and Λ splits orthogonally as

$$\Lambda = \mathcal{V} \oplus \mathcal{H}.$$

By Part 2 of Lemma 2.6, $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id}$ and $\hat{R} = \kappa \cdot \text{id}$. So it follows that \mathcal{H} consists of Jacobi fields whose restrictions to $[t_0, t_1]$ have the form

$$J = \tilde{f}E,$$

where E is a parallel field and \tilde{f} is the function from (2.2) that satisfies $\tilde{f}(t_0) = |J(t_0)|$. \square

Proof of Lemma 2.26. Since \mathcal{V} has full index, Proposition 1.6 implies that \hat{S} is defined on $[t_0, t_{\max})$. As above, the Eigenvalue Transfer Lemma 2.30 gives us that

$$\text{Trace } \hat{S}_{t_0} \leq k \cdot \tilde{\lambda}_\kappa(t_0).$$

Once again, $\text{Ric}_k \geq k \cdot \kappa$ implies that $\text{Trace } \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq k \cdot \kappa$ and $\hat{R} \geq k \cdot \kappa$. So by Lemma 2.17, $\hat{S} \equiv \tilde{\lambda}_\kappa \cdot \text{id}$ and $\hat{R} = \kappa \cdot \text{id}$ on $[t_0, t_{\max})$. This implies, as in the proof of Part 2 of Lemma 2.23, that $A = 0$ in $[t_0, t_{\max})$. The remainder of the argument is exactly the same as the proof of Part 2 of Lemma 2.23. \square

Proof of Lemma 2.27. Since \mathcal{V} has full index, Proposition 1.6 gives that \hat{S} is defined on $[t_0, \infty)$. As above, the Eigenvalue Transfer Lemma 2.30 gives us that

$$\text{Trace } \hat{S}_{t_0} \leq 0.$$

So by Lemma 2.13, $\hat{S} \equiv 0$ and $\hat{R} \equiv 0$ on $[t_0, \infty)$. As before, our hypothesis that $\text{Ric}_k \geq 0$ implies that $\text{Trace } \{R(\cdot, \dot{\gamma}(t)) \dot{\gamma}(t)\}^h \geq 0$. Combining this with $\hat{R} \equiv 0$ and the fact that AA^* is nonnegative, we see that $A \equiv 0$. The remainder of the argument is exactly the same as the proof of Part 2 of Lemma 2.23. \square

Remark 2.35. If $\lim_{t \rightarrow t_0^+} \tilde{\lambda}_\kappa(t) = \infty$, then, using Remark 2.20, Lemmas 2.23 to 2.27 hold with the hypothesis $\text{Trace}(S_{t_0})|_{W_{t_0}} \leq k \tilde{\lambda}_\kappa(t_0)$ replaced with

$$\liminf_{t \rightarrow t_0^+} \left(\text{Trace } S_t|_{H(t)} - k \tilde{\lambda}_\kappa(t) \right) \geq 0, \quad (2.36)$$

and

$$\tilde{\lambda}_\kappa(t) = \begin{cases} \cot(t - t_0) & \text{if } \kappa = 1 \\ \frac{1}{t - t_0} & \text{if } \kappa = 0 \\ \coth(t - t_0) & \text{if } \kappa = -1. \end{cases}$$

If N is a smooth submanifold of M , then Inequality (2.36) holds for

$$W_0 = \{J | J(0) = 0, J'(0) \in \nu_{\gamma(0)}(N)\} \subset \Lambda_N$$

(see Part 3 of Lemma 2.7 in [25] and also Remark 3 in [9]).

2.4. Why J_1 need not be J_0 . This subsection neither depends on nor is used in the rest of the paper. In it we give examples showing that the field J_1 in Lemma E can indeed be different from the field J_0 . A similar example can be found on page 463 of [18].

Example 2.37. Let E_1 and E_2 be parallel orthonormal fields along a geodesic γ in \mathbb{R}^3 with $E_1, E_2 \perp \gamma$. Let Λ be the Lagrangian family

$$\Lambda = \text{span} \{tE_1, (t+1)E_2\}.$$

Let

$$J_0 = tE_1 + (t+1)E_2.$$

Then

$$\langle J'_0(0), J_0(0) \rangle = \tilde{\lambda}_0(0) \langle J_0(0), J_0(0) \rangle = 1,$$

where $\tilde{\lambda}_0 = \frac{1}{t+1}$ comes from the model Jacobi field on \mathbb{R}^2 given by $\tilde{J} = (t+1) \tilde{E}$ with \tilde{E} a parallel field. In particular, J_0 satisfies Inequality (2) with $t_0 = 0$.

On the other hand,

$$\langle J'_0(t), J_0(t) \rangle = \langle E_1 + E_2, tE_1 + (t+1)E_2 \rangle = 2t+1,$$

and for $t > 0$,

$$\tilde{\lambda}_0(t) \langle J_0(t), J_0(t) \rangle = \frac{1}{t+1} (t^2 + (t+1)^2) = \frac{t^2}{t+1} + t+1 < 2t+1 = \langle J'_0(t), J_0(t) \rangle.$$

To verify the validity of Lemma E for this example, take $J_1(t) = (t+1)E_2$ and note that Inequality 3 is an equality for all $t > 0$.

Example 2.38. Let E_1 and E_2 be parallel orthonormal fields along a geodesic γ in \mathbb{S}^3 with $E_1, E_2 \perp \gamma$. Let Λ be the Lagrangian family

$$\Lambda = \text{span} \{ \sin t E_1, \cos t E_2 \}.$$

Let

$$J = \sin t E_1 + \cos t E_2.$$

Then

$$\langle J'(0), J(0) \rangle = 0.$$

So J satisfies Inequality 2 where $\tilde{\lambda} = \cot(t + \frac{\pi}{2})$ comes from the model Jacobi field on \mathbb{S}^2 given by $\tilde{J} = \cos(t) \tilde{E}$ with \tilde{E} a parallel field. On the other hand, for $t \in (0, \frac{\pi}{2})$,

$$\begin{aligned} \langle J'(t), J(t) \rangle &\equiv 0 \\ &> \cot\left(t + \frac{\pi}{2}\right) \langle J(t), J(t) \rangle, \end{aligned}$$

and Inequality 3 does not hold with $J_0 = J_1 = J$, $\tilde{\lambda} = \cot(t + \frac{\pi}{2})$, and $t \in (0, \frac{\pi}{2})$.

In contrast, the field $\cos(t + \frac{\pi}{2}) E_2$ satisfies Inequality 3 for all $t \in (0, \frac{\pi}{2})$.

3. FOCAL RADIUS AND POSITIVE CURVATURE

In this section, we prove Theorem A, and give examples showing the hypotheses on the dimension of N can not be removed.

Proof of Theorem A (cf Theorem 3.5 in [11]). Let $v \in \nu(N)$ be any unit vector. Recall that we denoted by Λ_N the Lagrangian of normal Jacobi fields along γ_v given by

$$\Lambda_N = \{ J \mid J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0) \}.$$

It suffices to show that for the subspace

$$\begin{aligned} \mathcal{K} &\equiv \text{span} \left\{ J \in \Lambda_N \mid J(t_i) = 0 \text{ for some nonzero } t_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}, \\ \dim \mathcal{K} &\geq \dim(N) - k + 1. \end{aligned}$$

The definition of \mathcal{K} implies that \mathcal{K} has full index for all $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Suppose, by way of contradiction, that $\dim \mathcal{K} \leq \dim(N) - k$, and set

$$\mathcal{K}(t) \equiv \{ J(t) \mid J \in \mathcal{K} \} \oplus \{ J'(t) \mid J \in \mathcal{K} \text{ and } J(t) = 0 \}.$$

Since $\dim \mathcal{K} \leq \dim(N) - k$, there is a k -dimensional subspace $W_0 \subset T_{\gamma_v(0)}N$ orthogonal to $\mathcal{K}(0)$. Replacing γ_v with γ_{-v} if necessary we may assume that

$$\text{Trace}(S_0|_{W_0}) \leq 0. \quad (3.1)$$

Let $\mathcal{V} \subset \Lambda_N$ be the subspace so that $\mathcal{V}(0) \perp W(0)$, and notice that $\mathcal{K} \subset \mathcal{V}$. From (3.1), we see that Lemma 2.23 applies to Λ_N and W_0 on $[0, \frac{\pi}{2}]$. So for all $t \in [0, \frac{\pi}{2}]$, there is a k -dimensional space $H(t) \subset \gamma'_v(t)^\perp$ so that

$$\text{Trace } S_t|_{H(t)} \leq k \cot\left(t + \frac{\pi}{2}\right) \text{ and } H(t) \perp \mathcal{V}(t). \quad (3.2)$$

It follows from Inequality (3.2) that there is a $Z \in \Lambda \setminus \mathcal{V}$ with

$$Z(t) = 0 \text{ for some } t \in \left(0, \frac{\pi}{2}\right]. \quad (3.3)$$

Since $Z \notin \mathcal{V}$, it follows that $Z \notin \mathcal{K}$, and (3.3) contradicts the definition of \mathcal{K} .

To prove Part 2, assume that the focal radius of N is $\frac{\pi}{2}$. If necessary we replace γ_v with γ_{-v} to arrange that

$$\text{Trace}(S_0|_{W_0}) \leq 0.$$

This allows us to apply Lemma 2.26 with $W_0 = T_{\gamma_v(0)}N$, $\kappa = 1$, $t_0 = 0$, $t_{\max} = \frac{\pi}{2}$, and $\tilde{\lambda}_1 = \cot(t + \frac{\pi}{2})$, to conclude that

$$\{J \mid J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0)\}$$

is spanned by Jacobi fields of the form $\sin(t + \frac{\pi}{2})E$ where E is a parallel field. In particular, $S_v \equiv 0$, and since this holds for all unit vectors v orthogonal to N , N is totally geodesic. \square

Remark 3.4. Although the Ricci curvature version of Theorem A can be proven via standard Riccati comparison (see e.g. [9]), its statement does not seem to be in the literature. In contrast, it does not seem possible to prove the sectional curvature version of Theorem A with existing Jacobi or Riccati comparison results. In the special case when N is known to be totally geodesic, there are J in Λ_N with $J'(0) = 0$. Berger's version of the Rauch Comparison Theorem then gives $\langle J'(t), J(t) \rangle \leq \cot(\frac{\pi}{2} + t)$ for all $t \in (0, \frac{\pi}{2})$. In particular, γ would have a focal point in $[0, \frac{\pi}{2}]$ (see Theorem 1.29 in [3] and Theorem 4.9 on page 234 of [7]).

For a general submanifold, we can always flip the parameterization of a geodesic as in the proof of Theorem A, to obtain $\langle J'(0), J(0) \rangle \leq 0$ for some $J \in \Lambda_N$. However, Example 2.38 shows that $\langle J'(t), J(t) \rangle$ can exceed $\cot(\frac{\pi}{2} + t)$ if $J'(0) \neq 0$. In fact, the J of Example 2.38 never vanishes! Thus it does not seem possible to prove Theorem A using only Berger's Theorem in place of Lemma 2.23.

3.1. Examples. Next we give examples showing that the hypotheses about the dimension of the submanifolds in Theorems A and B cannot be removed. For the sectional curvature versions of the theorems, a point in small perturbation of \mathbb{S}^n shows that the conclusions can be false if N does not have positive dimension. For the Ricci curvature versions of the theorems, we have the following examples.

Example 3.5. Let S_k^n be the n -sphere with constant curvature k . The product metric on $S_{\frac{n+1}{n-1}}^n \times S_{n+1}^2$ satisfies

$$\begin{aligned} \text{Ric} \left(S_{\frac{n+1}{n-1}}^n \times S_{n+1}^2 \right) &= n + 1 = \text{Ric} \left(\mathbb{S}^{n+2} \right), \text{ and} \\ \text{FocalRadius} \left(\{pt\} \times S_{n+1}^2 \right) &= \pi \sqrt{\frac{n-1}{n+1}} \longrightarrow \pi \text{ as } n \rightarrow \infty. \end{aligned} \quad (3.6)$$

Thus the focal radius of N in the Ricci curvature version of Theorem A can converge to π if the hypothesis that N is a hypersurface is removed and the dimension of M is allowed to go to ∞ , while the dimension of N is fixed.

On the other hand, if we take $n = 2$ or 3 , then (3.6) becomes

$$\begin{aligned} \text{Ric} \left(S_3^2 \times S_3^2 \right) &\equiv 3 \equiv \text{Ric} \left(\mathbb{S}^4 \right), \\ \text{FocalRadius} \left(\{pt\} \times S_3^2 \right) &= \pi \sqrt{\frac{1}{3}} > \frac{\pi}{2}, \text{ and} \\ \text{Ric} \left(S_2^3 \times S_4^2 \right) &= 4 = \text{Ric} \left(\mathbb{S}^5 \right), \\ \text{FocalRadius} \left(S_2^3 \times \{pt\} \right) &= \frac{\pi}{2}. \end{aligned}$$

So the hypothesis that N is a hypersurface in Ricci curvature versions of Theorem A cannot be replaced with the hypothesis that N is a codimension 2 submanifold. Similarly, in the Ricci curvature version of Theorem B, the hypersurface can not be replaced with a codimension 2 submanifold.

For our intermediate Ricci curvature results we have

Example 3.7. For $k > \frac{4}{3}p$ and $p \geq 2$, $M = S_{\frac{k}{k-p}}^{k-1} \times S_k^p$ satisfies

$$\begin{aligned} \text{Ric}_k(M) &\geq k \text{ and} \\ \text{FocalRadius}(\{pt\} \times S_k^p) &= \pi \sqrt{\frac{k-p}{k}} > \frac{\pi}{2}, \text{ if } k > \frac{4}{3}p. \end{aligned}$$

Thus $\{pt\} \times S_k^p \subset S_{\frac{k}{k-p}}^{k-1} \times S_k^p$ is a closed submanifold of a $(k+p-1)$ -manifold with $\text{Ric}_k(M) \geq k$ and focal radius $> \frac{\pi}{2}$, and the focal radius of N in Theorem A can exceed $\frac{\pi}{2}$ if the hypothesis that $\dim(N) \geq k$ is replaced with $\dim(N) \geq p$ where $\frac{3}{4}k > p$.

By sending $k \rightarrow \infty$ while keeping p fixed, we see that

$$\text{FocalRadius}(\{pt\} \times S_k^p) = \pi \sqrt{\frac{k-p}{k}} \longrightarrow \pi.$$

So in Theorem A, the focal radius of N can converge to π , if there is no hypothesis about the dimension of N , and the dimension of M is allowed to go to ∞ .

Part 2: Focal Rigidity

Let M be a complete Riemannian manifold M with $\text{Ric}_k \geq k$, and let N be a closed submanifold of M of dimension at least k and focal radius $\frac{\pi}{2}$. Since each connected component of N has focal radius $\frac{\pi}{2}$, we may assume that N is connected.

In the second part of the paper, we prove Theorem B by showing that the universal cover of M is isometric to the unit sphere or to a projective space with the standard metric, with N totally geodesic in M .

In Section 4, we exploit Lemma 2.26 to prove a rigidity result for the Jacobi fields of Λ_N (see Proposition 4.4). This allows us to prove, in Section 5, that every first focal point of N is regular in the sense of [17]. With this it follows rather easily that F , the focal set of N , is a totally geodesic closed submanifold with focal radius $\frac{\pi}{2}$. We thus further the analogy between the pair (N, F) and the dual sets in the proof of the Diameter Rigidity Theorem. In particular, we establish, as in [14], that F (resp. N) is the base of a Riemannian submersion from the unit normal sphere to any point of N (resp. F). In Section 5, we also show that if $\dim(F) + \dim(N) = \dim(M) - 1$, then M has constant curvature 1, which in particular yields Corollary C.

To show that phenomena like Example D do not occur in the simply connected case, we prove, in Section 6, that our focal set F is *very regular* in the sense of Hebda ([17]). This allows us to appeal to Theorem 3.1 in [17] and conclude, in Theorem 6.2, that M is the union of two disk bundles. Using this we prove that if the codimension of F (resp. N) is ≥ 3 , then N (resp. F) is simply connected; hence the fibers of the Riemannian submersion to N (resp. F) are connected.

All of the above allows us to complete the proof of Theorem B along the lines of the proof of the Diameter Rigidity Theorem. In the sectional curvature case, the argument can be concluded more rapidly. We prove that the diameter of the universal cover of M is $\geq \frac{\pi}{2}$, and appeal to the Diameter Rigidity Theorem, after making a further topological argument that rules out exotic spheres and nonunit metrics on \mathbb{S}^n . We give the details of this in Section 7. In Section 8, we complete the proof of Theorem B for intermediate Ricci curvature.

4. THE DISTANCE FROM N

With the exception of Proposition 4.4, we assume throughout Sections 4–8 that M is a complete Riemannian manifold with $\text{Ric}_k \geq k$, and that N is a connected, closed submanifold of M of dimension at least k and focal radius $\frac{\pi}{2}$.

In this section, we apply Lemma 2.26 to prove Proposition 4.4, which says, among other things, that the radial sectional curvatures from N are all ≥ 1 .

We start by reviewing the notion of horizontally homothetic submersions.

Definition 4.1. ([1], [2]) *A submersion $\pi : M \rightarrow B$ of Riemannian manifolds is called horizontally homothetic if and only if there is a smooth function $\lambda : M \rightarrow (0, \infty)$ with vertical gradient so that for all horizontal vectors x and y*

$$\lambda^2 \langle x, y \rangle_M = \langle D\pi(x), D\pi(y) \rangle_B.$$

We also use the following result from [22].

Proposition 4.2. *Let $\pi : M \rightarrow B$ be a horizontally homothetic submersion with dilation λ and let r be a regular value of λ so that $\lambda^{-1}(r)$ is nonempty. Then*

$$\pi|_{\lambda^{-1}(r)} : (\lambda^{-1}(r), \langle \cdot, \cdot \rangle_M) \rightarrow \left(B, \frac{1}{\lambda(r)^2} \langle \cdot, \cdot \rangle_B \right)$$

is a Riemannian submersion.

For a unit speed geodesic γ_v that leaves N orthogonally at time 0, we set

$$\begin{aligned} \mathcal{Z}_N &\equiv \{J | J(0) = 0, J'(0) \perp \text{span} \{T_{\gamma_v(0)}N, \gamma'_v(0)\}\} \\ \mathcal{T}_N &\equiv \{J | J(0) \in T_{\gamma_v(0)}N \text{ and } J'(0) = S_v J(0)\}, \text{ and} \\ \Lambda_N &\equiv \mathcal{Z}_N \oplus \mathcal{T}_N, \end{aligned} \tag{4.3}$$

where S_v is the shape operator of N determined by v , that is, $S_v : T_{\gamma_v(0)}N \rightarrow T_{\gamma_v(0)}N$, is $(\nabla \cdot v)^{TN}$.

Our first consequence of $\text{FocalRadius}(N) = \frac{\pi}{2}$ holds even if N is not closed, and it only requires that the radial intermediate Ricci curvatures from N are $\geq k \cdot \kappa$. Theorem A and Remark F imply that such an N is totally geodesic.

Proposition 4.4. *Let M be a complete Riemannian n -manifold, and let N be a submanifold of M with focal radius $\frac{\pi}{2}$ and $\dim(N) \geq k$. Suppose that along each unit speed geodesic $\gamma : [0, \frac{\pi}{2}] \rightarrow M$ that leaves N orthogonally at time 0 we have $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k \cdot \kappa$, that is*

$$\sum_{i=1}^k \sec(\dot{\gamma}, E_i) \geq k \cdot \kappa,$$

for any orthonormal set $\{\dot{\gamma}, E_1, \dots, E_k\}$.

1. *All $J \in \mathcal{T}_N$ have the form $J(t) = \cos t E$ where E is a parallel field along γ .*
2. *$\mathcal{Z}_N \oplus \mathcal{T}_N$ is a parallel, orthogonal splitting along $[0, \frac{\pi}{2}]$.*
3. *Let g^* be the metric on $\text{reg}_N \subset \nu(N)$ obtained from pulling back (M, g) via the normal exponential map, and let $\pi : \text{reg}_N \rightarrow N$ be the projection of the normal bundle. Then with respect to g^* , π is a horizontally homothetic submersion with scaling function $\cos(\text{dist}(N_0, \cdot))$, where N_0 is the 0-section of the normal bundle, $\nu(N)$.*
4. *If $c : I \rightarrow N$ is a unit speed geodesic in N , and V is a parallel normal unit field along c , then*

$$\Phi : I \times \left(0, \frac{\pi}{2}\right) \rightarrow M, \quad \Phi(s, t) = \exp_{c(s)}^\perp(tV(s))$$

is a totally geodesic immersion whose image has constant curvature 1.

5. *With respect to g^* , every plane tangent to $\text{reg}_N \setminus N_0$ that contains $X \equiv \text{grad}\{\text{dist}(N_0, \cdot)\}$ has sectional curvature ≥ 1 .*

Proof. Parts 1 and 2 follow from the special case of Lemma 2.26 when $\kappa = 1$, $t_0 = 0$, $t_{\max} = \frac{\pi}{2}$, and $\tilde{\lambda}_1 = \cot(t + \frac{\pi}{2})$ (cf. also Theorem B in [16]).

For Part 3, we let \mathcal{Z}_N^* and \mathcal{T}_N^* be the pullbacks of \mathcal{Z}_N and \mathcal{T}_N to reg_N via \exp_N^\perp . Observe that by Part 2,

$$\text{grad}\{\text{dist}(N_0, \cdot)\} \oplus \mathcal{Z}_N^* \oplus \mathcal{T}_N^*$$

is an orthogonal splitting of $T\text{reg}_N$ with respect to g^* . Since the vertical space of π is spanned by $\text{grad}\{\text{dist}(N_0, \cdot)\} \oplus \mathcal{Z}_N^*$, the horizontal space of π with respect to g^* is spanned by \mathcal{T}_N^* . Since the fields of \mathcal{T}_N^* come from variations of geodesics that leave N_0 orthogonally, they are π -basic. Part 3 follows by combining this with Part 1.

For Part 4, observe that Part 1 gives us that Φ is an immersion. Let II be the second fundamental form of $\text{image}(\Phi)$. By construction, $\frac{\partial\Phi}{\partial t}$ is a geodesic field so $\text{II}\left(\frac{\partial\Phi}{\partial t}, \frac{\partial\Phi}{\partial t}\right) = 0$. It follows from Part 1 that $\text{II}\left(\frac{\partial\Phi}{\partial t}, \frac{\partial\Phi}{\partial s}\right) = 0$.

To see that $\text{II}\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right) = 0$, we note that, since $\frac{\partial\Phi}{\partial t}$ is normal to $\exp_N^\perp(S(N_0, r))$, it suffices to verify that

$$g\left\langle \text{II}\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right), Z \right\rangle = 0 \quad (4.5)$$

for all $Z \in T\exp_N^\perp(S(N_0, r))$ that are normal to the image of Φ . Since $\exp_N^\perp : (\text{reg}_N, g^*) \rightarrow (M, g)$ is a local isometry, to prove (4.5), it suffices to do the corresponding calculation in (reg_N, g^*) . From Part 3 we have that the restriction of $\pi : (\text{reg}_N, g^*) \rightarrow \left(N, \frac{1}{\cos(t)^2}g\right)$ to the t -level set of $\text{dist}(N_0, \cdot)$ is a Riemannian submersion. Let $\widetilde{\frac{\partial\Phi}{\partial s}}$ be a lift of $\frac{\partial\Phi}{\partial s}$ to reg_N via \exp_N^\perp . Then $\widetilde{\frac{\partial\Phi}{\partial s}}$ is a π -basic horizontal, geodesic field, so $\text{II}\left(\frac{\partial\Phi}{\partial s}, \frac{\partial\Phi}{\partial s}\right) = (D\exp_N^\perp)\left(\text{II}\left(\widetilde{\frac{\partial\Phi}{\partial s}}, \widetilde{\frac{\partial\Phi}{\partial s}}\right)\right) \equiv 0$. Hence the image (Φ) is totally geodesic. It follows from Part 1 that $\text{image}(\Phi)$ has constant curvature 1.

To prove Part 5, we let $\{J_1^*, \dots, J_{k-1}^*\}$ be any $k-1$ linearly independent Jacobi fields in \mathcal{T}_N^* . It follows from Part 1 that for all i ,

$$\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, J_i^*) \equiv 1.$$

Together with Part 2 and our hypothesis that $\text{Ric}_k(\dot{\gamma}, \cdot) \geq k$, we conclude that for all $J^* \in \mathcal{Z}_N^*$, $\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, J^*) \geq 1$.

It follows from Part 2 that $\mathcal{Z}_N^* \oplus \mathcal{T}_N^*$ is a splitting of Λ_{N_0} into orthogonal, invariant subspaces for $R(\cdot, \text{grad}\{\text{dist}(N_0, \cdot)\})\text{grad}\{\text{dist}(N_0, \cdot)\}$. So

$$\sec(\text{grad}\{\text{dist}(N_0, \cdot)\}, Y) \geq 1$$

for all vectors Y orthogonal to $\text{grad}\{\text{dist}(N_0, \cdot)\}$. \square

5. THE STRUCTURE OF THE FOCAL SET

This section begins with a review of Hebda's notion of regular tangent focal points that generalizes a notion of Warner for conjugate points ([17], [30]). We next exploit the rigidity in Proposition 4.4 to show that every tangent focal point at time $\frac{\pi}{2}$ is regular. This allows us to apply a result of Hebda and conclude that our focal set $F \equiv \exp_N^\perp\left(S\left(N_0, \frac{\pi}{2}\right)\right)$ is a smooth submanifold of M . The rigid structure also yields that F has focal radius $\frac{\pi}{2}$. We then further the analogy between the pair (N, F) and the dual sets in the proof of the Diameter Rigidity Theorem by showing that F (resp. N) is the base of a Riemannian submersion from the unit normal sphere to any point of N (resp. F).

Definition 5.1. ([17], cf [30]) *A tangent focal point $v \in \nu(N)$ is called regular if and only if there is a neighborhood U of v so that every ray in $\nu(N)$ that intersects U has at most one tangent focal point in U , not counting multiplicities. Otherwise v is called singular.*

Continuity of the curvature tensor implies that every $v \in \nu(N)$ has a neighborhood U so that every ray meeting U has the same number of tangent focal points, counting multiplicities. So if v is a regular tangent focal point, then every ray tu in $\nu(N)$ that intersects U has exactly one focal point t_0u , and the multiplicities of t_0u and v coincide. Thus regular tangent focal points have locally maximal order. Using this and ideas of [30], Hebda showed the following.

Theorem 5.2. ([17], cf [30]) *The set of regular tangent focal points is a smooth codimension 1 submanifold of $\nu(N)$ that is an open, dense subset of the set of all tangent focal points.*

On $\text{reg}_N \subset \nu(N)$, we set

$$X \equiv \text{grad}(\text{dist}(N_0, \cdot)).$$

Along a fixed geodesic, focal points are isolated, so it follows that the set of regular, first-tangent focal points is an open, dense subset of the set of first-tangent focal points. It follows from the Gauss Lemma that $\ker(D \exp_N^\perp)_X \perp X$. Since the first-tangent focal set of our $N \subset M$ is $S(N_0, \frac{\pi}{2})$, it follows that $\ker(D \exp_N^\perp) \subset TS(N_0, \frac{\pi}{2})$. Combining this with the Rank Theorem we get

Corollary 5.3. *Let \tilde{F}_{reg} be the set of regular first-tangent focal points, and let*

$$F_{\text{reg}} \equiv \exp_N^\perp(\tilde{F}_{\text{reg}}).$$

Then F_{reg} is a smooth submanifold of M that is open and dense inside of F .

Lemma 5.4. *Let $v \in \nu(N)$ be a singular tangent focal point. For every neighborhood U of v , there is a regular tangent focal point $w \in U$ so that*

$$\dim(\ker(D \exp_N^\perp)_w) < \dim(\ker(D \exp_N^\perp)_v).$$

Proof. Let U be any neighborhood of v . Replacing U with a possibly smaller neighborhood, we may assume that the total multiplicity of the focal points on each ray that intersects U is constant, and that the ray tv contains only one focal point in U . Since v is singular, U contains a ray with more than one focal point $w_1 \neq w_2$, which by hypothesis is not the ray through v . Since the multiplicity of the focal points in $tw_1 \cap U$ and $tv \cap U$ is the same, it follows that

$$\dim(\ker(D \exp_N^\perp)_{w_1}) < \dim(\ker(D \exp_N^\perp)_v). \quad (5.5)$$

It might be that w_1 is not regular; however, since (5.5) holds for some w_1 in any neighborhood of v , by repeating this argument a finite number of times, we get the desired conclusion. \square

Let γ be a unit speed geodesic that leaves N orthogonally at time 0 with $\gamma(\frac{\pi}{2}) \in F_{\text{reg}}$. Recall that the elements of Λ_N are called N -Jacobi fields. We set

$$\begin{aligned} \mathcal{Z} &\equiv \left\{ J \in \Lambda_N \mid J(0) = J\left(\frac{\pi}{2}\right) = 0 \right\}, \\ \mathcal{T}_N &\equiv \left\{ J \in \Lambda_N \mid J(0) \in T_{\gamma(0)}N \text{ and } J'(0) = S_{\gamma'(0)}(J(0)) \right\}, \text{ and} \\ \mathcal{T}_{F_{\text{reg}}} &\equiv \left\{ J \mid J\left(\frac{\pi}{2}\right) \in T_{\gamma(\frac{\pi}{2})}F_{\text{reg}} \text{ and } J'\left(\frac{\pi}{2}\right) = -S\left(J\left(\frac{\pi}{2}\right)\right) \right\}, \end{aligned}$$

where $S_{\gamma'(0)}$ in the definition of \mathcal{T}_N is the shape operator of N and S in the definition of $\mathcal{T}_{F_{\text{reg}}}$ is the Riccati operator of Λ_N . The next lemma shows that the S in the definition of $\mathcal{T}_{F_{\text{reg}}}$ is also the shape operator of F_{reg} with respect to $\gamma'(\frac{\pi}{2})$.

Lemma 5.6. *For γ as above:*

1. $\gamma' \left(\frac{\pi}{2} \right) \in \nu_{\gamma \left(\frac{\pi}{2} \right)} F_{\text{reg}}$.
2. The N -Jacobi fields along γ are the F_{reg} -Jacobi fields along

$$\gamma^{-1} : t \mapsto \gamma \left(\frac{\pi}{2} - t \right).$$

3. The subspaces \mathcal{T}_N and $\mathcal{T}_{F_{\text{reg}}}$ are rigid, that is,

$$\begin{aligned} \mathcal{T}_N &= \{ \cos t E \mid E \text{ is parallel and tangent to } N \text{ at time } 0 \}, \text{ and} \\ \mathcal{T}_{F_{\text{reg}}} &= \left\{ \sin t E \mid E \text{ is parallel and tangent to } F_{\text{reg}} \text{ at time } \frac{\pi}{2} \right\}. \end{aligned}$$

4. Writing Λ_N for the N -Jacobi fields along γ , we have orthogonal splittings

$$\begin{aligned} \Lambda_N &= \mathcal{T}_N \oplus \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z} \text{ and} \\ \mathcal{Z}_N &= \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z}, \end{aligned}$$

where \mathcal{Z}_N is as in Equation (4.3).

Proof. Part 1 is a consequence of the Gauss Lemma and the fact that $F \equiv \exp_N^\perp \left(S \left(N_0, \frac{\pi}{2} \right) \right)$. The space Λ_N of N -Jacobi fields along γ are precisely the variation fields of variations by geodesics that leave N orthogonally at time 0. Similarly, the space $\Lambda_{F_{\text{reg}}}$ of F_{reg} -Jacobi fields along γ are precisely the variation fields of variations by geodesics that arrive at F_{reg} orthogonally at time $\frac{\pi}{2}$. It follows from Part 1 that $\Lambda_N \subset \Lambda_{F_{\text{reg}}}$. Since $\dim(\Lambda_N) = n - 1 = \dim(\Lambda_{F_{\text{reg}}})$, $\Lambda_N = \Lambda_{F_{\text{reg}}}$. This proves Part 2.

Since γ has no focal points for N on $(0, \frac{\pi}{2})$, it follows from Part 2 that $\gamma^{-1}(t) = \gamma(\frac{\pi}{2} - t)$ has no focal points for F_{reg} on $(0, \frac{\pi}{2})$. By Part 5 of Proposition 4.4, all the radial sectional curvatures along γ are ≥ 1 . Thus Parts 3 and 4 follow from Parts 1 and 2 of Proposition 4.4 and the fact that $\gamma^{-1}(t) = \gamma(\frac{\pi}{2} - t)$ has no focal points for F_{reg} on $(0, \frac{\pi}{2})$. \square

Lemma 5.7. $F_{\text{reg}} = F$.

Proof. We set

$$F_{\text{sng}} \equiv F \setminus F_{\text{reg}},$$

and suppose, by way of contradiction, that $F_{\text{sng}} \neq \emptyset$.

Let γ_{reg} and γ_{sng} be geodesics that leave N orthogonally at time 0 with

$$\gamma_{\text{reg}} \left(\frac{\pi}{2} \right) \in F_{\text{reg}} \text{ and } \gamma_{\text{sng}} \left(\frac{\pi}{2} \right) \in F_{\text{sng}}.$$

The idea of the proof is to examine how the splitting $\mathcal{Z}_N = \mathcal{T}_{F_{\text{reg}}} \oplus \mathcal{Z}$ behaves as a sequence of γ_{reg} 's approaches γ_{sng} . In particular, by Lemma 5.6, $\mathcal{T}_{F_{\text{reg}}}$ is spanned by constant curvature 1 Jacobi fields. By continuity, γ_{sng} inherits such a family, and this forces $\frac{\pi}{2} \gamma'_{\text{sng}}(0)$ to actually be regular. The details follow.

By appealing to Lemma 5.4, we can assume that

$$\dim \left(\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'_{\text{reg}}(0)} \right) < \dim \left(\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'_{\text{sng}}(0)} \right). \quad (5.8)$$

For either γ_{reg} or γ_{sng} we have the four spaces of Jacobi fields, Λ_N , \mathcal{T}_N , \mathcal{Z}_N , and \mathcal{Z} . We will distinguish the versions of the spaces along γ_{reg} from those along γ_{sng} with the superscripts $^{\text{reg}}$ and $^{\text{sng}}$. When no superscript is present, the statement applies to either case.

For either γ_{reg} or γ_{sng} ,

$$\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'(0)} = \{ J(0) \mid J \in \mathcal{T}_N \} \oplus \{ J'(0) \mid J \in \mathcal{Z} \}. \quad (5.9)$$

Thus

$$\begin{aligned} \dim \left(\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'(0)} \right) &= \dim(\mathcal{T}_N) + \dim(\mathcal{Z}) \\ &= \dim(N) + \dim(\mathcal{Z}). \end{aligned}$$

Since $\dim \left(\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'_{\text{sng}}(0)} \right) > \dim \left(\ker \left(D \exp_N^\perp \right)_{\frac{\pi}{2} \gamma'_{\text{reg}}(0)} \right)$, the dimensions of \mathcal{Z}^{reg} and \mathcal{Z}^{sng} satisfy

$$\dim(\mathcal{Z}^{\text{sng}}) > \dim(\mathcal{Z}^{\text{reg}}). \quad (5.10)$$

Along γ_{reg} , Lemma 5.6 gives us an orthogonal splitting

$$\mathcal{Z}_N^{\text{reg}} = \mathcal{T}_{F_{\text{reg}}}^{\text{reg}} \oplus \mathcal{Z}^{\text{reg}}, \quad (5.11)$$

where

$$\mathcal{T}_{F_{\text{reg}}}^{\text{reg}} = \left\{ \sin t E \mid E \text{ is parallel and tangent to } F_{\text{reg}} \text{ at time } \frac{\pi}{2} \right\}. \quad (5.12)$$

Combined with Inequality (5.10), this gives

$$\begin{aligned} \dim(\mathcal{Z}^{\text{sng}}) &> \dim(\mathcal{Z}^{\text{reg}}) \\ &= \dim(\mathcal{Z}_N^{\text{reg}}) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}), \text{ by 5.11} \\ &= (n-1) - \dim(N) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}). \end{aligned} \quad (5.13)$$

Note that γ_{sng} is a limit of γ_{reg} 's that satisfy (5.8). Further note that a $J \in \mathcal{T}_{F_{\text{reg}}}^{\text{reg}}$ together with γ'_{reg} spans a plane of constant curvature 1. Thus by continuity, $\mathcal{Z}_N^{\text{sng}}$ contains a subspace \mathcal{T}^{sng} of the form

$$\mathcal{T}^{\text{sng}} = \{ \sin t E \mid E \text{ is parallel} \} \subset \mathcal{Z}_N^{\text{sng}} \setminus \mathcal{Z}^{\text{sng}}$$

with

$$\dim(\mathcal{T}^{\text{sng}}) = \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}). \quad (5.14)$$

Moreover, by Remark 2.35, Part 2 of Lemma 2.23, and Part 5 of Proposition 4.4, Λ_N^{sng} splits orthogonally with one factor being \mathcal{T}^{sng} . Since \mathcal{T}^{sng} is a subspace of $\mathcal{Z}_N^{\text{sng}}$, we get

$$\mathcal{Z}_N^{\text{sng}} = \mathcal{T}^{\text{sng}} \oplus \mathcal{U}^{\text{sng}}, \quad (5.15)$$

where \mathcal{U}^{sng} is a space of Jacobi fields in $\mathcal{Z}_N^{\text{sng}}$ that is orthogonal to \mathcal{T}^{sng} throughout $(0, \frac{\pi}{2})$.

The splitting (5.15) combined with $\mathcal{T}^{\text{sng}} = \{ \sin t E \mid E \text{ is parallel} \}$ gives that \mathcal{Z}^{sng} is a subspace of \mathcal{U}^{sng} , so

$$\begin{aligned} \dim(\mathcal{Z}^{\text{sng}}) &\leq \dim(\mathcal{U}^{\text{sng}}) \\ &= \dim(\mathcal{Z}_N^{\text{sng}}) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}), \text{ by (5.14) and (5.15)} \\ &= (n-1) - \dim(N) - \dim(\mathcal{T}_{F_{\text{reg}}}^{\text{reg}}). \end{aligned}$$

Since this contradicts Inequality (5.13), the result is proven. \square

Lemma 5.16. *F is a totally geodesic closed submanifold of M with focal radius $\frac{\pi}{2}$.*

Proof. Since $F_{\text{reg}} = F$, it is a submanifold. Since $F = \exp_N^\perp(S(N_0, \frac{\pi}{2}))$, it is closed. It follows from Part 2 of Lemma 5.6 that $\Lambda_N = \Lambda_F$. Therefore the focal radius of F is $\frac{\pi}{2}$.

We have $F = F_{\text{reg}}$, so from Part 3 of Lemma 5.6,

$$\mathcal{T}_F = \left\{ \sin t E \mid E \text{ is parallel and tangent to } F \text{ at time } \frac{\pi}{2} \right\}.$$

In particular, for $J \in \mathcal{T}_F$, $J'(\frac{\pi}{2}) = 0$. So F is totally geodesic. \square

The next result will lead to the spherical rigidity portion of the conclusion of Theorem B, and also gives us Corollary C.

Theorem 5.17. *M has constant curvature 1 if either of the following holds:*

1. *F is not connected.*
2. *$\dim(F) + \dim(N) = \dim(M) - 1$.*

Proof. We have that $\exp^\perp : S(N_0, \frac{\pi}{2}) \rightarrow F$ is onto. So if F is not connected, then $S(N_0, \frac{\pi}{2})$ is not connected, and it follows that N is codimension 1 and has a trivial normal bundle. Since

$$\dim(F) + \dim(N) \leq \dim(\Lambda_N) = \dim(M) - 1,$$

to prove Part 1, it is enough to prove Part 2.

In general, we have an orthogonal splitting of $\Lambda_N = \mathcal{T}_N \oplus \mathcal{T}_F \oplus \mathcal{Z}$ along any one of our normal geodesics. Since $\dim(\mathcal{T}_F) = \dim(F)$, $\dim(\mathcal{T}_N) = \dim(N)$, and $\dim(F) + \dim(N) = \dim(M) - 1$, $\mathcal{Z} = 0$, and our geodesic is spanned by constant curvature 1 Jacobi fields. Moreover,

$$\mathcal{T}_N = \mathcal{Z}_F \text{ and } \mathcal{T}_F = \mathcal{Z}_N.$$

Combining this with Part 1 of Proposition 4.4, it follows that along a geodesic leaving F orthogonally at time 0,

$$\mathcal{Z}_F = \{ \sin t E \mid E \text{ is parallel} \}, \quad (5.18)$$

and along a geodesic leaving N orthogonally at time 0,

$$\mathcal{Z}_N = \{ \sin t E \mid E \text{ is parallel} \}. \quad (5.19)$$

It follows from Equation (5.18) that for all $x \in F$ and all $r \in (0, \frac{\pi}{2})$, the intrinsic metrics on

$$S_F(x, r) \equiv \exp_x^\perp \{ v \in \nu_x(F) \mid |v| = r \} \quad (5.20)$$

are locally isometric to $S^{\dim N}(\sin r)$, that is, to the sphere of radius $\sin r$ in $\mathbb{R}^{\dim N + 1}$. Similarly, it follows that for all $x \in N$ and all $r \in (0, \frac{\pi}{2})$, the intrinsic metrics on

$$S_N(x, r) \equiv \exp_x^\perp \{ v \in \nu_x(N) \mid |v| = r \} \quad (5.21)$$

are locally isometric to $S^{\dim F}(\sin r)$. Since $\mathcal{T}_N \oplus \mathcal{Z}_N$ is an orthogonal splitting, if γ leaves N orthogonally at time 0, then

$$S_N(\gamma(0), r) \text{ and } S_F\left(\gamma\left(\frac{\pi}{2}\right), \frac{\pi}{2} - r\right) \text{ intersect orthogonally at } \gamma(r). \quad (5.22)$$

Let

$$S_N(r) \equiv \exp_N^\perp \{ v \in \nu(N) \mid |v| = r \},$$

and let II_r be the second fundamental form of $S_N(r)$, that is

$$\text{II}_r(U, V) \equiv g(\nabla_U V, \gamma'(r)),$$

where γ leaves N orthogonally at time 0.

Combining (5.18), (5.19) and (5.22), we have for $Y \in \mathcal{T}_N$, and $W \in \mathcal{Z}_N$,

$$\begin{aligned} \Pi_r(Y, Y) &= |Y|^2 \tan(r) \\ \Pi_r(W, W) &= -|W|^2 \cot(r), \text{ and} \\ \Pi_r(Y, W) &= 0. \end{aligned} \tag{5.23}$$

Now view \mathbb{S}^n as a join, $\mathbb{S}^n = \mathbb{S}^{\dim N} * \mathbb{S}^{\dim F}$, and let $\tilde{\gamma}$ be a geodesic that leaves $\mathbb{S}^{\dim N}$ orthogonally at time 0. Setting $\tilde{M} \equiv \mathbb{S}^n$, $\tilde{N} \equiv \mathbb{S}^{\dim N}$, and $\tilde{F} \equiv \mathbb{S}^{\dim F}$, observe that (5.20), (5.21), (5.22), and (5.23) hold with M , N , and F replaced by \tilde{M} , \tilde{N} , and \tilde{F} . Observe further that Equations (5.20), (5.21), (5.22), and (5.23) together with the Gauss, Radial, and Codazzi-Mainardi Equations ([23]) determine the curvature tensor of $\tilde{M} \equiv \mathbb{S}^n$. Similarly, they determine the curvature tensor of M . Thus M has constant curvature 1. \square

Throughout the remainder of Part 2, we assume that F is connected and

$$\dim(F) + \dim(N) \leq \dim(M) - 2. \tag{5.24}$$

Lemma 5.25. *1. Let $x \in N$. With respect to the constant curvature 1 metric on the unit normal sphere, $\nu_x^1(N)$, the map*

$$\begin{aligned} \pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion onto F .

2. Let $x \in F$. With respect to the constant curvature 1 metric on the unit normal sphere, $\nu_x^1(F)$, the map

$$\begin{aligned} \pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_F\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion onto N .

Proof. The proofs are identical, except for notation. We give the details for Part 1.

Let γ be a geodesic that leaves N orthogonally at time 0. Then

$$\begin{aligned} T_{\gamma'(0)}(\nu_{\gamma(0)}^1(N)) &= \{J'(0) \mid J \in \mathcal{Z}_N\}, \text{ and} \\ D\pi_{\gamma(0)}(J'(0)) &= J\left(\frac{\pi}{2}\right) \end{aligned} \tag{5.26}$$

for all $J \in \mathcal{Z}_N$.

Since $\mathcal{T}_F \subset \mathcal{Z}_N$, the splittings $\Lambda_N = \mathcal{T}_N \oplus \mathcal{Z}_N = \mathcal{T}_N \oplus \mathcal{T}_F \oplus \mathcal{Z}$ give us an orthogonal splitting

$$\mathcal{Z}_N = \mathcal{Z} \oplus \mathcal{T}_F.$$

Combined with Equation (5.26) this gives an orthogonal splitting

$$T_{\gamma'(0)}(\nu_{\gamma(0)}^1(N)) = \{J'(0) \mid J \in \mathcal{Z}\} \oplus \{J'(0) \mid J \in \mathcal{T}_F\} \tag{5.27}$$

into the vertical and horizontal spaces, respectively, for $\pi_{\gamma(0)}$. Since $\dim(\mathcal{T}_F) = \dim(F)$, it follows from Equation (5.26) that $\pi_{\gamma(0)}$ is a submersion. By Part 1 of Proposition 4.4, with F

playing the role of N , the restriction of $D\pi_{\gamma(0)}$ to the second summand in Equation (5.27) is an isometry. Thus $\pi_{\gamma(0)}$ is a Riemannian submersion. \square

6. THE SIMPLY CONNECTED CASE

Let $\pi : \tilde{M} \longrightarrow M$ be the universal cover of M . Then each component $\pi^{-1}(N)$ is a submanifold with focal radius $\frac{\pi}{2}$ and dimension at least k . In particular, \tilde{M} contains a closed, connected, embedded submanifold with focal radius $\frac{\pi}{2}$ and dimension at least k . So to prove Theorem B, it suffices to consider the case when M is simply connected and N is connected.

In this section, we will combine our simply connected hypothesis with Hebda's theorem on "very regular" focal loci. This will allow us to assert that, topologically, M is the union of two disk bundles, and the fibers of our Riemannian submersions,

$$\begin{aligned} \pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

and

$$\begin{aligned} \pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_F\left(\frac{\pi}{2}v\right) \end{aligned}$$

are connected. We start with a review of Hebda's result.

Definition 6.1. (Hebda, [17]) *Consider geodesics γ that leave N orthogonally at time 0. N has a very regular first focal locus if the multiplicity of the first focal point is independent of γ and, in case the multiplicity is one, $\ker(D \exp_N^1)$ is contained in the tangent space to the tangent focal locus at every first-tangent focal point.*

Along a geodesic that leaves our N orthogonally at time 0, the multiplicity of the focal point at time $\frac{\pi}{2}$ is

$$\begin{aligned} \dim \mathcal{Z}_F &= \dim(\Lambda_N) - \dim \mathcal{T}_F \\ &= \dim(M) - 1 - \dim(F), \end{aligned}$$

and hence is constant. Since the focal radius of N along every geodesic is $\frac{\pi}{2}$, it follows from the Gauss Lemma that our N has a very regular first focal locus. Therefore, since M is simply connected, we can apply the following result of Hebda. (See Theorem 3.1 in [17] and the first line of its proof.)

Theorem 6.2. (Hebda, [17]) *Suppose M is a connected, compact Riemannian manifold, and N is a connected, compact submanifold having a very regular first focal locus such that the inclusion $\iota : N \hookrightarrow M$ induces a surjection of fundamental groups.*

If the multiplicity of the first focal points of N is $s - 1$, then the first focal locus F of N in M is a submanifold of codimension s that coincides with the cut locus of N in M . Moreover, the tangent cut locus of N coincides with the first-tangent focal locus of N , and M is the union of two disk bundles

$$M = D_N \cup_{\varphi} D_F$$

over N and F respectively, where

$$\varphi : \partial D_N \longrightarrow \partial D_F$$

is a diffeomorphism.

By combining transversality and Theorem 6.2 we get following.

Corollary 6.3. *Suppose that M is simply connected.*

1. *If $\text{codim}(F) \geq 3$, then N is simply connected.*
2. *If $\text{codim}(N) \geq 3$, then F is simply connected.*

Proof. The two statements have dual proofs. We give the details for Part 1.

Transversality gives us

$$\pi_1(M) \cong \pi_1(M \setminus F),$$

and by Theorem 6.2, $M \setminus F$ deformation retracts to N . □

Similarly, the cut locus statements in Theorem 6.2 gives us

Corollary 6.4. *If M is simply connected, then \exp_N^\perp is injective on $B(N_0, \frac{\pi}{2})$.*

Lemma 6.5. *Let M be simply connected.*

1. *If $\text{codim}(N) \geq 3$, then the Riemannian submersions*

$$\nu_x^1(N) \longrightarrow F, \quad x \in N$$

have connected fibers with positive dimension.

2. *If $\text{codim}(F) \geq 3$, then the Riemannian submersions*

$$\nu_x^1(F) \longrightarrow N, \quad x \in F$$

have connected fibers with positive dimension.

Proof. Since we have assumed that $\dim(F) + \dim(N) \leq \dim(M) - 2$,

$$\begin{aligned} \dim(\nu_x^1(N)) &= \dim(M) - \dim(N) - 1 \\ &> \dim(F). \end{aligned}$$

Thus the fibers of $\nu_x^1(N) \longrightarrow F$ have positive dimension.

By Corollary 6.3, F is simply connected if $\text{codim}(N) \geq 3$. In this case, the long exact homotopy sequence for $\nu_x^1(N) \longrightarrow F, x \in N$ gives

$$\pi_1(F) \longrightarrow \pi_0(\text{fiber}) \longrightarrow 0,$$

since $\pi_0(\nu_x^1(N))$ is trivial. Thus the first conclusion holds. A similar argument gives us the second conclusion, if $\text{codim}(F) \geq 3$. □

Since we have assumed that

$$\dim(F) + \dim(N) \leq \dim(M) - 2, \tag{6.6}$$

$\text{codim}(N) \geq 2$. Since $\dim(N) \geq 1$, we have $\text{codim}(F) \geq 3$. Combining this with Lemma 5.16, Corollary 6.4, and Lemma 6.5, we have

Theorem 6.7. *Let M be a complete Riemannian n -manifold with $\text{Ric}_k \geq k$ and N any closed, connected, submanifold of M with $\dim(N) \geq k$ and focal radius $\frac{\pi}{2}$. If M is simply connected and not isometric to the unit sphere, then:*

1.

$$\dim(N) + \dim(F) \leq n - 2.$$

2. N is totally geodesic and isometric to an even dimensional CROSS.

3. The focal set F of N is totally geodesic and is either a point or is isometric to an even dimensional CROSS.

4. The normal exponential maps of N and F are injective on the $\frac{\pi}{2}$ -balls around the zero sections of the normal bundles of N and F .

5. The conclusions of Proposition 4.4 hold with N replaced by F .

6. For every $x \in F$ the map

$$\begin{aligned} \pi_x &: \nu_x^1(F) \longrightarrow N \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion whose fibers are connected and have positive dimension.

7. For every $x \in N$ the map

$$\begin{aligned} \pi_x &: \nu_x^1(N) \longrightarrow F \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right) \end{aligned}$$

is a Riemannian submersion whose fibers are connected and have positive dimension.

7. RIGIDITY IN THE SECTIONAL CURVATURE CASE

In this section, we complete the proof of Theorem B in the case when the sectional curvature of M is ≥ 1 . Since M is simply connected, by Part 4 of Theorem 6.7, the diameter of M is $\geq \frac{\pi}{2}$. So combining the Diameter Rigidity Theorem with our dimension hypothesis (6.6) and Theorem 6.7 gives us the following.

Proposition 7.1. *If M does not have constant curvature 1, then the following hold.*

1. M is isometric to a compact, rank one, symmetric space or is homeomorphic to S^n .

2. N is even dimensional and isometric to the base of a Hopf fibration.

3. F is either a point or is isometric to the base of a Hopf fibration and is even dimensional

To conclude the proof of Theorem B, we show that conclusions 2 and 3 are not compatible with M being a topological sphere.

If M is a sphere, the long exact homology sequence of the pair (M, F) gives

$$H_q(M, F) \cong H_{q-1}^\#(F)$$

for $q \leq n - 1$.

Use Theorem 6.2 to write

$$M = D_N \cup_\varphi D_F.$$

By excision,

$$H_q(M, F) \cong H_q(D_N, \partial D_N).$$

Thus for $q \leq n - 1$,

$$H_{q-1}^\#(F) \cong H_q(D_N, \partial D_N). \quad (7.2)$$

By Proposition 7.1, F is either an even dimensional CROSS or a point, and N is an even dimensional CROSS. It follows from Equation (7.2) that $H_q(D_N, \partial D_N) \cong 0$ if q is even and $\leq n - 1$. If q is odd and $q + 1 \leq n - 1$, then the sequence of the pair $(D_N, \partial D_N)$ gives

$$0 = H_{q+1}(D_N, \partial D_N) \longrightarrow H_q(\partial D_N) \longrightarrow H_q(N) = 0,$$

since N is an even dimensional CROSS. Thus

$$H_q(\partial D_N) \cong 0 \text{ if } q \text{ is odd and } \leq n - 2. \quad (7.3)$$

Since N is isometric to the base of a Hopf fibration with connected fibers, $\dim(N) \geq 2$. Since $\dim(N) + \dim(F) \leq n - 2$, we get $n \geq 4$. So $\dim(\partial D_N) \geq 3$. Since ∂D_N is a connected, compact, odd dimensional manifold, (7.3) implies, via Poincaré duality, that ∂D_N is a \mathbb{Z}_2 -homology sphere of dimension ≥ 3 . Thus the Mayer-Vietoris sequence with $q \in \{2, \dots, n - 2\}$ yields

$$0 = H_q(\partial D_N) \longrightarrow H_q(D_N) \oplus H_q(D_F) \longrightarrow H_q(M) \longrightarrow H_q(\partial D_N) = 0.$$

Since D_N has the homotopy type of the CROSS N , and $\dim(M) \geq \dim(N) + 2 \geq 4$, M cannot be homeomorphic to a sphere.

8. RIGIDITY AND INTERMEDIATE RICCI

In this section, we complete the proof of Theorem B. This is achieved by analyzing the radial geometry from N and F . Proposition 4.4, Lemma 5.6, and Lemma 5.16 give us rigid radial geometry along the distribution spanned by the Jacobi fields in \mathcal{T}_N and \mathcal{T}_F . To prove rigidity for the \mathcal{Z} -Jacobi fields, we show, in Proposition 8.8, that as in the proof of the Diameter Rigidity Theorem, there are enough other dual pairs in M to force the \mathcal{Z} -Jacobi fields to span projective lines. This is achieved via the next three results, wherein the hypotheses that N is connected and M is simply connected are still in force.

Proposition 8.1. *1. For any $p \in N$ the cut point along any geodesic emanating from p is at distance $\frac{\pi}{2}$ from p .*

2. For any $p, q \in N$, any minimal geodesic of M between p and q lies entirely in N .

Proof. Let $v \in T_p M \setminus \{T_p N, \nu_p(N)\}$ be any unit vector. Let v^T and v^\perp be the unit vectors that point in the same directions as the projections of v onto $T_p N$ and $\nu_p(N)$, respectively. By Part 4 of Proposition 4.4, $\text{span}\{v^T, v^\perp\}$ exponentiates to a totally geodesic immersed surface Σ of constant curvature 1 that contains γ_v . By Corollary 6.4, the restriction of \exp_p to the interior of the circular sector

$$\text{Sect}\left(v^T, v^\perp, \frac{\pi}{2}\right) \equiv \left\{ \exp_p\left(t(v^T \cos s + v^\perp \sin s)\right) \mid t, s \in \left[0, \frac{\pi}{2}\right] \right\}$$

of radius and angle $\frac{\pi}{2}$ spanned by v^T and v^\perp is an embedding. If $w \in T_p M$ is not in $T_p N, \nu_p(N)$, or $\text{span}\{v^T, v^\perp\}$, then Corollary 6.4 gives that interiors of $\text{Sect}(v^T, v^\perp, \frac{\pi}{2})$ and $\text{Sect}(w^T, w^\perp, \frac{\pi}{2})$ are disjoint. It follows that the cut-time of any unit $v \in T_p M \setminus \{T_p N, \nu_p(N)\}$ is $\geq \frac{\pi}{2}$, and by continuity, the cut-time of any unit $v \in T_p M$ is $\geq \frac{\pi}{2}$.

Since the focal radius of N is $\frac{\pi}{2}$, to complete the proof of Part 1, it suffices to check that any unit $v \in T_p M \setminus \nu_p(N)$ has a cut point at time $\frac{\pi}{2}$. Since N is a CROSS with curvature in $[1, 4]$, there is a unit vector $w^T \in T_p(N)$ with

$$\gamma_{w^T}\left(\frac{\pi}{2}\right) = \gamma_{v^T}\left(\frac{\pi}{2}\right) \text{ and } w^T \neq v^T.$$

Let $u \in \nu_{\gamma_{v^T}(\frac{\pi}{2})}(N)$ be a unit vector. Let U_v and U_w be the backwards parallel transports of u along γ_{v^T} and γ_{w^T} . Let Σ_{v^T} and Σ_{w^T} be the spherical sectors of radius and angle $\frac{\pi}{2}$ obtained via Part 4 of Proposition 4.4 by exponentiating U_v and U_w . That is

$$\Sigma_{v^T} \equiv \left\{ \exp_N(tU_v(s)) \mid s, t \in \left[0, \frac{\pi}{2}\right] \right\}$$

and

$$\Sigma_{w^T} \equiv \left\{ \exp_N(tU_w(s)) \mid s, t \in \left[0, \frac{\pi}{2}\right] \right\}.$$

Then Σ_{w^T} and Σ_{v^T} are different surfaces, but since

$$U_v\left(\frac{\pi}{2}\right) = U_w\left(\frac{\pi}{2}\right) = u \in \nu_{\gamma_{v^T}(\frac{\pi}{2})}(N),$$

Σ_{w^T} and Σ_{v^T} intersect along $\exp_N(tu)$. So every cut point from p occurs at distance $\frac{\pi}{2}$ from p .

By Corollary 6.4, the intersection of N with any of the sectors $\text{Sect}(v^T, v^\perp, \frac{\pi}{2})$ is precisely $\gamma_{v^T}[0, \frac{\pi}{2}] \subset N$. Part 2 follows. \square

Proposition 8.2. *For any $p \in N$, the set of points in M at distance $\frac{\pi}{2}$ from p ,*

$$A(p) \equiv S\left(p, \frac{\pi}{2}\right),$$

is a closed submanifold of dimension

$$\dim(N) + \dim F$$

and focal radius $\frac{\pi}{2}$.

Proof. Let

$$A_N(p) \equiv \left\{ x \in N \mid \text{dist}(p, x) = \frac{\pi}{2} \right\}.$$

Using the rigid hinges of Part 4 of Proposition 4.4 and the fact that $A(p) \equiv S(p, \frac{\pi}{2})$ is the cut locus of p we see that we can describe $A(p)$ in two ways:

$$A(p) = \left\{ \exp_N^\perp(tv) \mid v \in \nu^1(N) \mid_{A_N(p)} \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\},$$

and

$$A(p) = \left\{ \exp_F^\perp(tv) \mid v \in \nu^1(F), \exp_F\left(\frac{\pi}{2}v\right) \in A_N(p), \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\}. \quad (8.3)$$

Either description shows $A(p)$ is compact. Since both N and F have focal radius $\frac{\pi}{2}$, both descriptions show, via Corollary 6.4, that $A(p) \setminus \{F \cup N\}$ is a manifold. The first description shows that $A(p)$ is smooth near N .

Recall that for every $x \in F$, the map

$$\begin{aligned}\pi_x &: \nu_x^1(F) \longrightarrow N, \\ \pi_x &: v \mapsto \exp_N\left(\frac{\pi}{2}v\right)\end{aligned}$$

is a Riemannian submersion with connected fibers. Combining this with Part 4 of Proposition 4.4 and Proposition 8.1, we rewrite (8.3) as

$$A(p) = \cup_{x \in F} \left\{ \exp_F^\perp(tv) \mid v \in \nu_x^1(F), v \perp \pi_x^{-1}(p), \text{ and } t \in \left[0, \frac{\pi}{2}\right] \right\}, \quad (8.4)$$

where $\{v\}$ and $\pi_x^{-1}(p)$ are subsets of $\nu_x(F)$, and the notion of perpendicular comes from the inner product that g induces on $\nu_x(F)$. This shows $A(p)$ is smooth near F .

Next, we decompose $\nu_x^1(F)$ as a join

$$\begin{aligned}\nu_x^1(F) &= \pi_x^{-1}(p) * (\pi_x^{-1}(p))^\perp, \text{ where} \\ (\pi_x^{-1}(p))^\perp &\equiv \{v \in \nu_x^1(F) \mid v \perp \pi_x^{-1}(p)\}.\end{aligned}$$

Note that (8.4) gives that for any $x \in F$,

$$\dim(A(p)) = \dim F + \dim(\pi_x^{-1}(p))^\perp + 1. \quad (8.5)$$

Since $\pi_x : \nu_x^1(F) \longrightarrow N$ is a Riemannian submersion,

$$\dim(N) = \dim(H_v(\pi_x^{-1}(p))), \quad (8.6)$$

where $H_v(\pi_x^{-1}(p))$ is the horizontal space for π_x at any $v \in \pi_x^{-1}(p) \subset \nu_x^1(F)$.

The join decomposition $\nu_x^1(F) = \pi_x^{-1}(p) * (\pi_x^{-1}(p))^\perp$ identifies $(\pi_x^{-1}(p))^\perp$ with the unit vectors in $H_v(\pi_x^{-1}(p))$. So

$$\dim(\pi_x^{-1}(p))^\perp = \dim(H_v(\pi_x^{-1}(p))) - 1.$$

Combining with Equations (8.5) and (8.6), we get

$$\dim(A(p)) = \dim F + \dim(N) - 1 + 1.$$

Since $A(p) = S(p, \frac{\pi}{2})$ is a smooth submanifold, and every geodesic that leaves p has cut point at distance $\frac{\pi}{2}$ from p , it follows from 1st-variation that every geodesic leaving p arrives orthogonally at $A(p)$ at time $\frac{\pi}{2}$. This identifies the unit tangent sphere at p , S_p , with the unit normal bundle of $A(p)$, $\nu^1(A(p))$. Combined with Proposition 8.1, it follows that the focal radius of $A(p)$ along any normal geodesic is greater than or equal to $\frac{\pi}{2}$. So it follows from Theorem A that the focal radius of $A(p)$ is exactly $\frac{\pi}{2}$. \square

It follows that Theorem 6.7 applies with $N^p = A(p)$ and $F^p = p$.

Next, we apply Propositions 8.1 and 8.2 to $q \in A(p)$ and, with a further application of Theorem 6.7, get the following result.

Proposition 8.7. 1. Every cut point from q occurs at distance $\frac{\pi}{2}$ from q .
2. The set of points in M at distance $\frac{\pi}{2}$ from q ,

$$A(q) \equiv S\left(q, \frac{\pi}{2}\right),$$

is a closed submanifold with focal radius $\frac{\pi}{2}$ and dimension $\dim N + \dim F$.

3. $A(q)$ is totally geodesic and isometric to a CROSS.
4. For any $a, b \in A(q)$, any minimal geodesic of M between a and b lies entirely in $A(q)$.

Returning $N^p = A(p)$ and $F^p = p$, we now have the following refinement of Proposition 4.4.

Proposition 8.8. *Let γ be a unit speed geodesic that leaves N^p orthogonally at time 0 and let Λ_{N^p} , \mathcal{T}_{N^p} and \mathcal{Z}_{N^p} be as in (4.3).*

1. $\mathcal{Z}_{N^p} \oplus \mathcal{T}_{N^p}$ is a parallel, orthogonal splitting of Λ_{N^p} along $(0, \frac{\pi}{2})$.
2. \mathcal{T}_{N^p} and \mathcal{Z}_{N^p} have the forms

$$\begin{aligned} \mathcal{T}_{N^p} &\equiv \{ \cos t E \mid E \text{ is parallel and tangent to } N^p \text{ at time } 0 \}, \\ \mathcal{Z}_{N^p} &\equiv \left\{ \frac{1}{2} \sin 2t E \mid E \text{ is parallel and orthogonal to } N^p \text{ at time } 0 \right\}. \end{aligned}$$

Proof. Apart from the second equation in Part 2, this is a repeat of Parts 1 and 2 of Proposition 4.4. The second equation in Part 2 follows from Part 2 of Proposition 8.7. Indeed, let γ be a unit speed geodesic from $x \in N^p$ to p . Choose $q \in N^p$ at distance $\frac{\pi}{2}$ from x , and apply Proposition 8.7 to q . It follows that $\gamma \subset A(q)$, and $A(q)$ is a totally geodesic CROSS. In particular, the Jacobi fields along γ have the indicated form if they are tangent to $A(q)$. Since the normal space to $A(q)$ along γ is spanned by a subspace of \mathcal{T}_{N^p} , the result follows. \square

We finish the proof of Theorem B along the lines of the proof of Theorem 4.3 in [14] by using Cartan's Theorem (Theorem 2.1, page 157 of [7]).

Since Theorem 6.7 applies to $N^p = A(p)$, there is a CROSS P with

$$\dim(P) = \dim(M)$$

and

$$\dim(\mathcal{Z}_{N^p}) = \dim(\mathbb{F}) - 1,$$

where \mathbb{F} is the division algebra that defines P .

Choose a point $\tilde{p} \in P$. Since P is a CROSS, we have a Riemannian submersion

$$\tilde{\pi}_{\tilde{p}} : S_{\tilde{p}} \longrightarrow A(\tilde{p}) \equiv \left\{ x \in P \mid \text{dist}(x, P) = \frac{\pi}{2} \right\}$$

that is isometrically equivalent to a Hopf Fibration. Since $\dim(\mathcal{Z}_{N^p}) = \dim(\mathbb{F}) - 1$, we have, using [13] and [32], that $\tilde{\pi}_{\tilde{p}}$ is isometrically equivalent to

$$\pi_p : S_p \longrightarrow N^p \equiv A(p).$$

Let

$$I : S_{\tilde{p}} \longrightarrow S_p$$

be a linear isometric equivalence between $\tilde{\pi}_{\tilde{p}}$ and π_p . Then we have a commutative diagram

$$\begin{array}{ccc} S_{\tilde{p}} & \xrightarrow{I} & S_p \\ \downarrow \tilde{\pi}_{\tilde{p}} & & \downarrow \pi_p \\ A(\tilde{p}) & \xrightarrow{\hat{I}} & A(p) \end{array}$$

Since $\tilde{\pi}_{\tilde{p}}$ and π_p are Riemannian submersions and I is an isometry, \hat{I} is an isometry.

Via the Cartan Theorem and Proposition 8.8, we see that $\iota \equiv \exp_p \circ I \circ \exp_p^{-1}$ defines an isometry between $P \setminus A(\tilde{p})$ and $M \setminus A(p)$ that induces the isometry $\hat{I} : A(\tilde{p}) \rightarrow A(p)$, and thus ι extends to an isometry $P \rightarrow M$. This completes the proof of Theorem B.

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DEPARTMENT OF MATHEMATICS, UNIVERSIDAD AUTÓNOMA DE MADRID, AND ICMAT CSIC-UAM-UCM-UC3M

E-mail address: luis.guijarro@uam.es

URL: <http://www.uam.es/luis.guijarro>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE, CA 92521

E-mail address: fred@math.ucr.edu

URL: <https://sites.google.com/site/frederickhwilhelmjr/home>