

Research Article

Umberto Biccari, Mahamadi Warma and Enrique Zuazua*

Local Elliptic Regularity for the Dirichlet Fractional Laplacian

DOI: 10.1515/ans-2017-0014

Received February 6, 2017; revised April 13, 2017; accepted April 19, 2017

Abstract: We prove the $W_{loc}^{2s,p}$ local elliptic regularity of weak solutions to the Dirichlet problem associated with the fractional Laplacian on an arbitrary bounded open set of \mathbb{R}^N . The key tool consists in analyzing carefully the elliptic equation satisfied by the solution locally, after cut-off, to later employ sharp regularity results in the whole space. We do it by two different methods. First working directly in the variational formulation of the elliptic problem and then employing the heat kernel representation of solutions.

Keywords: Fractional Laplacian, Dirichlet Boundary Condition, Weak Solutions, Local Regularity

MSC 2010: 35B65, 35R11, 35S05

Communicated by: Antonio Ambrosetti and David Arcoya

Dedicated to Ireneo Peral on the occasion of his 70th birthday: Gracias Ireneo por tantos años de amistad y ejemplo

1 Introduction

The aim of the present paper is to study the local elliptic regularity of weak solutions to the following Dirichlet problem:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded open set and $s \in (0, 1)$.

Here, f is a given distribution and $(-\Delta)^s$ denotes the fractional Laplace operator, which is defined as the following singular integral:

$$(-\Delta)^s u(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \quad (1.2)$$

In (1.2), $C_{N,s}$ is a normalization constant, given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma(\frac{2s+N}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

Γ being the usual Gamma function. Moreover, we have to mention that, for having a completely rigorous definition of the fractional Laplace operator, it is necessary to introduce also the class of functions u for which computing $(-\Delta)^s u$ makes sense. We postpone this discussion to the next section.

Umberto Biccari: DeustoTech, University of Deusto, 48007 Bilbao, Basque Country; and Facultad de Ingeniería, Universidad de Deusto, Avda Universidades 24, 48007 Bilbao, Basque Country, Spain, e-mail: umberto.biccari@deusto.es

Mahamadi Warma: Department of Mathematics, College of Natural Sciences, University of Puerto Rico (Rio Piedras Campus), PO Box 70377, San Juan, PR 00936-8377, USA, e-mail: mahamadi.warma1@upr.edu

***Corresponding author: Enrique Zuazua:** DeustoTech, University of Deusto, 48007 Bilbao, Basque Country; and Facultad de Ingeniería, Universidad de Deusto, Avda Universidades 24, 48007 Bilbao, Basque Country; and Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain, e-mail: enrique.zuazua@deusto.es

Models involving the fractional Laplacian or other types of non-local operators have been recently used in the description of several complex phenomena for which the classical local approach turns up to be inappropriate or limited. Among others, we mention applications in elasticity [8], turbulence [2], anomalous transport and diffusion [5, 21], porous media flow [32], image processing [14], wave propagation in heterogeneous high contrast media [33]. Also, it is well known that the fractional Laplacian is the generator of s -stable processes, and it is often used in stochastic models with applications, for instance, in mathematical finance [20, 23].

One of the main differences between these non-local models and classical partial differential equations is that the fulfilment of a non-local equation at a point involves the values of the function far away from that point.

Our concern in this article is the study of the local elliptic regularity for weak solutions of the Dirichlet problem (1.1). For this purpose, we firstly remind that, according to [19], we have the following definition of weak solutions.

Definition 1.1. Let $f \in W^{-s,2}(\bar{\Omega})$. A function $u \in W_0^{s,2}(\bar{\Omega})$ is said to be a finite energy solution of the Dirichlet problem (1.1) if for every $v \in W_0^{s,2}(\bar{\Omega})$, the equality

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy = \langle f, v \rangle_{W^{-s,2}(\bar{\Omega}), W_0^{s,2}(\bar{\Omega})}$$

holds.

We notice that, when $1 < p < 2$, it is not natural to consider finite energy solutions for (1.1), and we shall rather introduce an alternative notion of solution. This will be given by duality with respect to the following class of test functions:

$$\mathcal{T}(\Omega) = \{\phi : (-\Delta)^s \phi = \psi \text{ in } \Omega, \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \psi \in C_0^\infty(\Omega)\}.$$

Definition 1.2. Let $1 < p < 2$. We say that $u \in L^1(\Omega)$ is a weak solution to (1.1) if, for $f \in L^1(\Omega)$ we have that

$$\int_{\Omega} u \psi dx = \int_{\Omega} f \phi dx$$

for any $\phi \in \mathcal{T}(\Omega)$ with $\psi \in C_0^\infty(\Omega)$.

The following $W_{\text{loc}}^{2s,2}(\Omega)$ -regularity property is our first main result.

Theorem 1.3 (L^2 -Local Regularity). *Let $f \in W^{-s,2}(\bar{\Omega})$ and let $u \in W_0^{s,2}(\bar{\Omega})$ be the unique weak solution to the Dirichlet problem (1.1). If $f \in L^2(\Omega)$, then $u \in W_{\text{loc}}^{2s,2}(\Omega)$.*

This result can be extended to the L^p setting as follows.

Theorem 1.4 (L^p -Local Regularity). *Let $f \in W^{-s,2}(\bar{\Omega})$ and let $u \in W_0^{s,2}(\bar{\Omega})$ be the unique weak solution to the Dirichlet problem (1.1). If $f \in L^p(\Omega)$ with $1 < p < \infty$, then $u \in W_{\text{loc}}^{2s,p}(\Omega)$.*

We have to notice that these two theorems are already known when Ω is the whole space \mathbb{R}^N . In fact, they follow by combining several results on Fourier transform and singular integrals contained in [29, Chapter V]. This combination has been done in the reference [3]. On the other hand, when Ω is a bounded open set, to the best of our knowledge, such results are not yet available in the literature.

In Theorems 1.3 and 1.4, $W_0^{s,2}(\bar{\Omega})$ denotes the fractional order Sobolev space which consists of all functions $u \in W^{s,2}(\mathbb{R}^N)$ which are zero on $\mathbb{R}^N \setminus \Omega$, while $W^{-s,2}(\bar{\Omega})$ is its dual. We will give a more exhaustive description of these spaces in Appendix A at the end of this paper. Moreover, we comment that our results tell that, as it is for the classical Laplace operator (which corresponds to the case $s = 1$), when the right-hand side is in $L^p(\Omega)$ the corresponding solution of (1.1) gains locally the maximum possible regularity, that is, it gains locally up to $2s$ derivatives in $L^p(\Omega)$.

Our results complement some previous ones on local and global Sobolev regularity.

- In [16, Section 7], Grubb proves that, under the restriction $s > \frac{N}{p}$, the assumption $f \in W^{t,p}(\Omega)$ for some $t \geq 0$ implies that the corresponding solution u of (1.1) belongs to $W_{\text{loc}}^{t+2s,p}(\Omega)$.
- In [19, Theorem 17], Leonori, Peral, Primo and Soria show that, if $f \in L^m(\Omega)$ for some $m \geq \frac{2N}{N+2s}$, then the weak solution u of (1.1) belongs to $W_0^{s\theta,p}(\Omega)$ for some $0 < \theta < 1$ and p such that

$$\frac{1}{p} = \frac{1}{m} + \theta \left(\frac{1}{2} - \frac{1}{m} \right) - \frac{2s(1-\theta)}{N}.$$

This is proved by means of an interpolation argument between $W^{s,2}(\Omega)$ and $L^{mN/(N-2ms)}(\Omega)$. Note however that this global regularity result does not achieve the maximal gain of regularity since $0 < s\theta < s$. On the other hand, a well-known example shows that the optimal global regularity fails (for more details, see [25, Remark 7.2]).

- In [6], it is proved that if we take $f \in L^2(\Omega)$, then the corresponding weak solution of (1.1) satisfies $u \in W_{\text{loc}}^{2s-\varepsilon,2}(\Omega)$ for all $\varepsilon > 0$.

We conclude that our results complement those mentioned above, showing that optimal regularity holds locally away from the boundary, for all $s \in (0, 1)$.

Finally, we remind that, for the classical Laplace operator, maximum regularity holds globally provided that the open set is smooth enough. This result fails for the fractional Laplace operator. We refer to Section 5 for a full discussion on this topic and the possible remedies that should involve weighted estimates to take into account the boundary singularities.

The strategy that we will employ to prove our local regularity theorems does not involve interpolation techniques, as in the proof of [19, Theorem 17]. Instead, it will be based on a cut-off argument that will allow us to reduce the problem to the whole space case, for which, as we have mentioned above, the result is already known (see for example Theorem 2.7 below).

In order to develop this technique, the following proposition, which provides a formula for the fractional Laplacian of the product of two functions, will be fundamental (see, e.g., [24] and the references therein).

Proposition 1.5. *Let u and v be such that $(-\Delta)^s u$ and $(-\Delta)^s v$ exist and*

$$\int_{\mathbb{R}^N} \frac{|(u(x) - u(y))(v(x) - v(y))|}{|x - y|^{N+2s}} dy < \infty. \quad (1.3)$$

Then $(-\Delta)^s(uv)$ exists and is given by

$$(-\Delta)^s(uv) = u(-\Delta)^s v + v(-\Delta)^s u - I_s(u, v), \quad (1.4)$$

where

$$I_s(u, v)(x) := C_{N,s} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \quad (1.5)$$

Remark 1.6. *We mention that for example if $u, v \in W^{s,2}(\mathbb{R}^N)$ with $(-\Delta)^s u, (-\Delta)^s v \in L^2(\mathbb{R}^N)$, then one has (1.3) and thus formula (1.4) holds for such functions.*

Formula (1.4), applied to the product of u with a cut-off function η , will be the principal tool for transforming our original problem (1.1) to one in the whole \mathbb{R}^N . Then, for the proof of our main results, we will need to carefully analyze the regularity of the remainder term I_s . This analysis will be developed following two different approaches. In the first one, we will consider the fractional Laplacian $(-\Delta)^s$ as defined in (1.2). In the second one, we will instead use the equivalent characterization of the fractional Laplace operator through the heat semigroup $(e^{t\Delta})_{t \geq 0}$, given by

$$(-\Delta)^s u := \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta} u - u) \frac{dt}{t^{1+s}} \quad (1.6)$$

(see for instance [30, Section 2.1] and the references therein). We recall that here, $\Gamma(1-s) := -s\Gamma(-s)$.

Finally, we mention that this careful analysis of the regularity of the remainder term had been partially developed already in [4, Lemma B1], as a technical tool for obtaining the results therein presented. This has been one of the main motivations that led to the development of the present work.

The paper is organized as follows. In Section 2, we present some preliminary tools that we shall use in the proof of our main results. In Section 3, we give the proof of Theorems 1.3 and 1.4 using the integral representation of the fractional Laplacian. In Section 4, we use the second approach which is based on the representation (1.6). Finally, in Section 5, we present some open problems and perspectives that are closely related to our work.

2 Preliminaries

In this section, we introduce some preliminary result that will be useful for the proof of our main Theorems 1.3 and 1.4.

We start by giving a more rigorous definition of the fractional Laplace operator, as we have anticipated in Section 1. Let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable, } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}} dx < \infty \right\}.$$

For $u \in \mathcal{L}_s^1(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)_\varepsilon^s u(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N.$$

The *fractional Laplace operator* $(-\Delta)^s$ is then defined by the following singular integral:

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s u(x), \quad x \in \mathbb{R}^N, \quad (2.1)$$

provided that the limit exists.

We notice that if $0 < s < \frac{1}{2}$ and u is smooth, for example bounded and Lipschitz continuous on \mathbb{R}^N , then the integral in (2.1) is in fact not really singular near x (see, e.g., [7, Remark 3.1]). Moreover, $\mathcal{L}_s^1(\mathbb{R}^N)$ is the right space for which $v := (-\Delta)_\varepsilon^s u$ exists for every $\varepsilon > 0$, v being also continuous at the continuity points of u .

The following result of existence and uniqueness of weak solutions to the Dirichlet problem (1.1) is by now well known (see, e.g., [19, Theorem 12]).

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $0 < s < 1$. Then for every $f \in W^{-s,2}(\overline{\Omega})$, the Dirichlet problem (1.1) has a unique finite energy solution $u \in W_0^{s,2}(\overline{\Omega})$. In addition, there exists a constant $C > 0$ such that*

$$\|u\|_{W_0^{s,2}(\overline{\Omega})} \leq C \|f\|_{W^{-s,2}(\overline{\Omega})}. \quad (2.2)$$

Proof. For the sake of completeness we include the proof. We recall that a complete description of the functional setting in which we are working is presented in Appendix A. Moreover, we recall that, according to Definition 1.1, a function $u \in W_0^{s,2}(\overline{\Omega})$ is said to be a weak solution of the Dirichlet problem (1.1) if for every $v \in W_0^{s,2}(\overline{\Omega})$, the equality

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dx dy = \langle f, v \rangle_{W^{-s,2}(\overline{\Omega}), W_0^{s,2}(\overline{\Omega})} \quad (2.3)$$

holds. Hence, given $u, v \in W_0^{s,2}(\overline{\Omega})$ let us consider the bilinear form

$$\mathcal{E}(u, v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} dx dy, \quad (2.4)$$

which is symmetric, continuous and coercive.

Thus by the classical Lax–Milgram theorem, for every

$$f \in (W_0^{s,2}(\bar{\Omega}))^* =: W^{-s,2}(\bar{\Omega}),$$

there exists a unique $u \in W_0^{s,2}(\bar{\Omega})$ such that the equality (2.3) holds for every $v \in W_0^{s,2}(\bar{\Omega})$. We have shown that (1.1) has a unique weak solution $u \in W_0^{s,2}(\bar{\Omega})$. Taking $v = u$ as a test function in (2.3) and using (A.5), we get that

$$C\|u\|_{W_0^{s,2}(\bar{\Omega})}^2 = \langle f, u \rangle_{W^{-s,2}(\bar{\Omega}), W_0^{s,2}(\bar{\Omega})} \leq \|f\|_{W^{-s,2}(\bar{\Omega})} \|u\|_{W_0^{s,2}(\bar{\Omega})}.$$

We have shown (2.2) and the proof is finished. \square

Remark 2.2. Notice that also for $1 < p < 2$ existence and uniqueness of a weak solution to problem (1.1) are guaranteed by [19, Theorem 28].

Remark 2.3. Notice that [19, Theorem 12] holds for a more general non-local operator where the kernel $|x - y|^{-N-2s}$ is replaced by a general symmetric kernel $K(x, y)$ satisfying $\lambda \leq K(x, y)|x - y|^{N+2s} \leq \lambda^{-1}$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $x \neq y$, and for some constant $0 < \lambda \leq 1$.

Remark 2.4. Let $f \in W^{-s,2}(\bar{\Omega})$ and let $u \in W_0^{s,2}(\bar{\Omega})$ be the weak solution of the Dirichlet problem (1.1). We notice that it follows from the Sobolev embeddings (A.2) and (A.1) that if $N < 2s$, then $u \in C^{0,s-N/2}(\bar{\Omega})$ and if $N = 2s$, then $u \in L^q(\Omega)$ for every $1 \leq q < \infty$.

The following lemma, giving a precise L^q -regularity of weak solutions and complementing the results in [19, Theorem 16] will be useful in the sequel.

Lemma 2.5. Assume that $N > 2s$ and let $f \in L^p(\Omega)$ for some $p \geq \frac{2N}{N+2s}$. Then (1.1) has a unique weak solution u . In addition the following assertions hold.

(a) If $p > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

(b) If $\frac{2N}{N+2s} \leq p \leq \frac{N}{2s}$, then $u \in L^q(\Omega)$ for every q satisfying $p \leq q < \frac{Np}{N-2sp}$ and there exists a constant $C > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Proof. Let $f \in L^p(\Omega)$ for some $p \geq \frac{2N}{N+2s}$. Since $W_0^{s,2}(\bar{\Omega}) \hookrightarrow L^{2N/(N-2s)}(\Omega)$ (see (A.1)), we have that

$$L^p(\Omega) \hookrightarrow L^{\frac{2N}{N+2s}}(\Omega) \hookrightarrow W^{-s,2}(\bar{\Omega}).$$

Thus (1.1) has a unique weak solution u .

Let \mathcal{E} with domain $D(\mathcal{E}) = W_0^{s,2}(\bar{\Omega})$ be the bilinear, symmetric, continuous and coercive form defined in (2.4). As we have shown in the proof of Proposition 2.1, for every $f \in W^{-s,2}(\bar{\Omega})$ there exists a unique $u \in W_0^{s,2}(\bar{\Omega})$ such that

$$\mathcal{E}(u, v) = \langle f, v \rangle_{W^{-s,2}(\bar{\Omega}), W_0^{s,2}(\bar{\Omega})} \quad \text{for all } v \in W_0^{s,2}(\bar{\Omega}).$$

This defines an operator $\mathcal{A} : W_0^{s,2}(\bar{\Omega}) \rightarrow W^{-s,2}(\bar{\Omega})$ which is continuous and coercive. Let A_D be the part of \mathcal{A} in $L^2(\Omega)$, in the sense that

$$D(A_D) := \{u \in W_0^{s,2}(\bar{\Omega}) : \mathcal{A}u \in L^2(\Omega)\}, \quad A_D u = \mathcal{A}u.$$

Using an integration by parts argument, one can show that A_D is given precisely by

$$D(A_D) = \{u \in W_0^{s,2}(\bar{\Omega}) : (-\Delta)^s u \in L^2(\Omega)\}, \quad A_D = (-\Delta)^s u.$$

Then A_D is the realization in $L^2(\Omega)$ of the operator $(-\Delta)^s$ with the Dirichlet boundary condition $u = 0$ on $\mathbb{R}^N \setminus \Omega$. The operator A_D has a compact resolvent (this follows from the compactness of the embedding from $W_0^{s,2}(\bar{\Omega})$ into $L^2(\Omega)$, see (A.1)) and its first eigenvalue $\lambda_1 > 0$.

In addition $-A_D$ generates a submarkovian strongly continuous semigroup $(e^{-tA_D})_{t \geq 0}$ which is also ultracontractive in the sense that the semigroup maps $L^r(\Omega)$ into $L^m(\Omega)$ for every $t > 0$ and $1 \leq r \leq m \leq \infty$. More precisely, following line by line the proof of [13, Theorem 2.16] by using the appropriate estimates, we get that, for every $1 \leq r \leq m \leq \infty$ there exists a constant $C > 0$ such that for every $f \in L^r(\Omega)$ and $t > 0$,

$$\|e^{-tA_D}f\|_{L^m(\Omega)} \leq C e^{-\lambda_1(\frac{1}{r}-\frac{1}{m})t} t^{-\frac{N}{2s}(\frac{1}{r}-\frac{1}{m})} \|f\|_{L^r(\Omega)}. \quad (2.5)$$

Since the operator A_D is invertible, it follows from the abstract result in [10, p. 55, Theorem 1.10] that for every $f \in L^1(\Omega) \cap W^{-s,2}(\bar{\Omega})$, the unique solution u of the Dirichlet problem (1.1) is given by

$$u = A_D^{-1}f = \int_0^\infty e^{-tA_D}f \, dt.$$

(a) Assume that $p > \frac{N}{2s}$. Then applying (2.5) with $r = p$ and $m = \infty$, we get that

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq \int_0^\infty \|e^{-tA_D}f\|_{L^\infty(\Omega)} \, dt \leq C \int_0^\infty e^{-\frac{\lambda_1}{p}t} t^{-\frac{N}{2sp}} \|f\|_{L^p(\Omega)} \, dt \\ &= C \left(\int_1^\infty e^{-\frac{\lambda_1}{p}t} t^{-\frac{N}{2sp}} \, dt + \int_0^1 e^{-\frac{\lambda_1}{p}t} t^{-\frac{N}{2sp}} \, dt \right) \|f\|_{L^p(\Omega)}. \end{aligned}$$

The first integral in the right-hand side of the previous estimate is always finite. The second integral will be finite if $1 - \frac{N}{2sp} > 0$. This is equivalent to $p > \frac{N}{2s}$ and we have shown part (a).

(b) Assume that $\frac{2N}{N+2s} \leq p \leq \frac{N}{2s}$ and let $p \leq q < \frac{Np}{N-2sp}$. Then applying (2.5) with $r = p$ and $m = q$, we get that

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\leq \int_0^\infty \|e^{-tA_D}f\|_{L^q(\Omega)} \, dt \leq C \int_0^\infty e^{-\lambda_1(\frac{1}{p}-\frac{1}{q})t} t^{-\frac{N}{2s}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)} \, dt \\ &= C \left(\int_1^\infty e^{-\lambda_1(\frac{1}{p}-\frac{1}{q})t} t^{-\frac{N}{2s}(\frac{1}{p}-\frac{1}{q})} \, dt + \int_0^1 e^{-\lambda_1(\frac{1}{p}-\frac{1}{q})t} t^{-\frac{N}{2s}(\frac{1}{p}-\frac{1}{q})} \, dt \right) \|f\|_{L^p(\Omega)}. \end{aligned}$$

As above, the first integral is always finite and the second integral will be finite if $1 - \frac{N}{2s}(\frac{1}{p} - \frac{1}{q}) > 0$. This is equivalent to $q < \frac{Np}{N-2sp}$. We have shown part (b) and the proof is finished. \square

Remark 2.6. Assertion (b) has been also proved in [19, Theorem 16]. There, Leonori, Peral, Primo and Soria obtained the result adapting Moser's method in [22], which allows to obtain the $L^q(\Omega)$ -regularity of the solution u to (1.1) employing functions depending nonlinearly on the solution. To the best of our knowledge, our approach to the proof of Lemma 2.5 (b) is new.

In our discussion, the following result of regularity on the whole space \mathbb{R}^N will play an important role.

Theorem 2.7. Let $F \in W^{-s,2}(\mathbb{R}^N) := (W^{s,2}(\mathbb{R}^N))^*$ and let $u \in W^{s,2}(\mathbb{R}^N)$ be the weak solution to the fractional Poisson type equation

$$(-\Delta)^s u = F \quad \text{in } \mathbb{R}^N. \quad (2.6)$$

If $F \in L^p(\mathbb{R}^N)$ with $1 < p < \infty$, then $u \in W^{2s,p}(\mathbb{R}^N)$.

Theorem 2.7 is a classical result whose proof can be done by combining several results on singular integrals and Fourier transform contained in [29, Chapter V, Section 3.3]; see, in particular, formulas (38), (40) and Theorem 3 therein. These mentioned results have been put together in the reference [3]. In particular, our Theorem 2.7 is obtained as a consequence of [3, Corollary 6.7 and Lemma 6.9].

3 Proof of Theorems 1.3 and 1.4: First Approach

3.1 Proof of the L^2 -Local Regularity Theorem

Proof of Theorem 1.3. As we have mentioned above, our strategy is based on a cut-off argument that will allow us to show that the solutions of the fractional Dirichlet problem in Ω , after cut-off, are solutions of the elliptic problem on the whole space \mathbb{R}^N , for which Theorem 2.7 holds. For this purpose, given two open subsets ω and $\bar{\omega}$ of the domain Ω such that $\bar{\omega} \Subset \omega \Subset \Omega$, we introduce a cut-off function $\eta \in \mathcal{D}(\omega)$ such that

$$\begin{cases} \eta(x) \equiv 1 & \text{if } x \in \bar{\omega}, \\ 0 \leq \eta(x) \leq 1 & \text{if } x \in \omega \setminus \bar{\omega}, \\ \eta(x) = 0 & \text{if } x \in \mathbb{R}^N \setminus \omega. \end{cases} \quad (3.1)$$

Let $f \in W^{-s,2}(\bar{\Omega})$ and let $u \in W_0^{s,2}(\bar{\Omega})$ be the unique weak solution to the Dirichlet problem (1.1).

Let ω and $\eta \in \mathcal{D}(\omega)$ be respectively the set and the cut-off function constructed in (3.1). We consider the function $u\eta$. It is clear that $u\eta \in W^{s,2}(\mathbb{R}^N)$. It follows from Proposition 1.5 and Remark 1.6 that

$$(-\Delta)^s(u\eta) - \eta(-\Delta)^s u = u(-\Delta)^s \eta - I_s(u, \eta). \quad (3.2)$$

Let

$$g := u(-\Delta)^s \eta - I_s(u, \eta).$$

We claim that $g \in L^2(\mathbb{R}^N)$. In fact, there exists a constant $C > 0$, independent of u , such that

$$\|g\|_{L^2(\mathbb{R}^N)} \leq C\|u\|_{W_0^{s,2}(\bar{\Omega})}. \quad (3.3)$$

Since $u = 0$ on $\mathbb{R}^N \setminus \Omega$ and $(-\Delta)^s \eta \in L^\infty(\mathbb{R}^N)$, we have that

$$\|u(-\Delta)^s \eta\|_{L^2(\mathbb{R}^N)}^2 = \int_{\Omega} |u(-\Delta)^s \eta|^2 dx \leq \|(-\Delta)^s \eta\|_{L^\infty(\Omega)}^2 \|u\|_{L^2(\Omega)}^2. \quad (3.4)$$

Now, recall from (1.5) that for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned} I_s(u, \eta)(x) &:= C_{N,s} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\ &= \underbrace{C_{N,s} \int_{\Omega} \frac{(u(x) - u(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy}_{=: \mathbb{I}_1(x)} + \underbrace{C_{N,s} \eta(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy}_{=: \mathbb{I}_2(x)}, \quad x \in \mathbb{R}^N. \end{aligned}$$

Let us start to estimate the term $\mathbb{I}_1(x)$. Using the Cauchy–Schwarz inequality, we get that

$$|\mathbb{I}_1(x)| \leq C_{N,s} \left(\int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{(|\eta(x) - \eta(y)|^2)}{|x - y|^{N+2s}} dy \right)^{\frac{1}{2}}. \quad (3.5)$$

Let $x \in \Omega$ be fixed and $R > 0$ such that $\Omega \subset B(x, R)$. Since η is a smooth function (in particular Lipschitz continuous on \mathbb{R}^N), we have that there exists a constant $C > 0$ (depending on η) such that

$$\int_{\Omega} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dy \leq C \int_{\Omega} \frac{dy}{|x - y|^{N+2s-2}} \leq C \int_{B(x,R)} \frac{dy}{|x - y|^{N+2s-2}} \leq C.$$

Using the preceding estimate and (3.5), we get that

$$\int_{\mathbb{R}^N} |\mathbb{I}_1(x)|^2 dx \leq C \int_{\mathbb{R}^N} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dy dx \leq C\|u\|_{W_0^{s,2}(\bar{\Omega})}^2. \quad (3.6)$$

Concerning the term \mathbb{I}_2 , we notice that $\mathbb{I}_2 = 0$ on $\mathbb{R}^N \setminus \omega$. In addition, using the Cauchy–Schwarz inequality, we get that

$$|\mathbb{I}_2(x)|^2 \leq C_{N,s}^2 \int_{\mathbb{R}^N \setminus \Omega} \frac{\eta^2(x) dy}{|x-y|^{N+2s}} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dy. \quad (3.7)$$

For any $y \in \mathbb{R}^N \setminus \Omega$, we have that

$$\frac{\eta^2(x)}{|x-y|^{N+2s}} = \frac{\chi_{\bar{\omega}}(x)\eta^2(x)}{|x-y|^{N+2s}} \leq \chi_{\bar{\omega}}(x)\eta^2(x) \sup_{x \in \bar{\omega}} \frac{1}{|x-y|^{N+2s}}.$$

Thus there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{\eta^2(x) dy}{|x-y|^{N+2s}} \leq \chi_{\bar{\omega}}(x)\eta^2(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{\text{dist}(y, \partial\bar{\omega})^{N+2s}} \leq C\chi_{\bar{\omega}}(x)\eta^2(x), \quad (3.8)$$

where we have used that the integral is finite which follows from the facts that $\text{dist}(\partial\Omega, \partial\bar{\omega}) \geq \delta > 0$, that the distance function $\text{dist}(y, \partial\bar{\omega})$ grows linearly as y tends to infinity and that $N + 2s > N$.

Since $\chi_{\bar{\omega}}\eta^2 \in L^\infty(\omega)$ and using (3.7) and (3.8), we get that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} |\mathbb{I}_2(x)|^2 dx = \int_{\omega} |\mathbb{I}_2(x)|^2 dx \leq C \int_{\omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dy dx \leq C\|u\|_{W_0^{s,2}(\bar{\Omega})}^2. \quad (3.9)$$

Now estimate (3.3) follows from (3.4), (3.6), (3.9), and the claim is proved. We have shown that ηu is a weak solution to the Poisson equation (2.6) with F given by $F = \eta(-\Delta)^s u + g \in L^2(\mathbb{R}^N)$. It follows from Theorem 2.7 that $(\eta u) \in W^{2s,2}(\mathbb{R}^N)$. Thus $u \in W_{\text{loc}}^{2s,2}(\Omega)$ and the proof is complete. \square

3.2 Proof of the L^p -Local Regularity Theorem

We will now use Theorem 1.3 to prove our local regularity result in the general L^p setting.

Proof of Theorem 1.4. We start by noticing that, assuming $f \in L^p(\Omega) \cap W^{-s,2}(\bar{\Omega})$, we have that (1.1) has a unique weak solution $u \in W_0^{s,2}(\bar{\Omega})$. We divide the proof into two steps.

Step 1: $1 < p < 2$. If $1 < p < 2$, then, according to Theorem A.2, $u \in W^{s,p}(\Omega)$. In particular, $u \in L^p(\Omega)$.

Let ω and $\eta \in \mathcal{D}(\omega)$ be respectively the set and the cut-off function constructed in (3.1). We consider the function $u\eta \in W^{s,p}(\mathbb{R}^N)$. As in the proof of Theorem 1.3, we have that $(-\Delta)^s(u\eta)$ is given by

$$(-\Delta)^s u\eta = \eta f + u(-\Delta)^s \eta - I_s(u, \eta),$$

where the term $I_s(u, \eta)$ has been introduced in (1.5). Let ω_1, ω_2 be open sets such that

$$\bar{\omega} \subset \omega_1 \subset \bar{\omega}_1 \subset \omega_2 \subset \bar{\omega}_2 \subset \Omega.$$

Since the function η and the set ω in (3.1) are arbitrary, it follows that $u \in W^{s,p}(\omega_2)$. Thus we have $u \in W^{s,p}(\omega_2) \cap L^p(\Omega)$. Let

$$g := u(-\Delta)^s \eta - I_s(u, \eta).$$

We now claim that $g \in L^p(\mathbb{R}^N)$ and there exists a constant $C > 0$ such that

$$\|g\|_{L^p(\mathbb{R}^N)} \leq C(\|u\|_{W^{s,p}(\omega_2)} + \|u\|_{L^p(\Omega)}). \quad (3.10)$$

Indeed, it is clear that g is defined on all \mathbb{R}^N . Moreover,

$$\|u(-\Delta)^s \eta\|_{L^p(\mathbb{R}^N)}^p = \int_{\Omega} |u(-\Delta)^s \eta|^p dx \leq \|(-\Delta)^s \eta\|_{L^\infty(\Omega)}^p \|u\|_{L^p(\Omega)}^p. \quad (3.11)$$

For estimating the term I_s , we use the decomposition

$$\begin{aligned} I_s(u, \eta)(x) &:= C_{N,s} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy \\ &= C_{N,s} \underbrace{\int_{\omega_1} \frac{(u(x) - u(y))(\eta(x) - \eta(y))}{|x - y|^{N+2s}} dy}_{=: \mathbb{I}_1(x)} + C_{N,s} \eta(x) \underbrace{\int_{\mathbb{R}^N \setminus \omega_1} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy}_{=: \mathbb{I}_2(x)}, \quad x \in \mathbb{R}^N. \end{aligned}$$

Let $p' := \frac{p}{p-1}$. Using the Hölder inequality, we get that for a.e. $x \in \mathbb{R}^N$,

$$|\mathbb{I}_1(x)| \leq C_{N,s} \left(\int_{\omega_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy \right)^{\frac{1}{p}} \left(\int_{\omega_1} \frac{|\eta(x) - \eta(y)|^{p'}}{|x - y|^{N+sp'}} dy \right)^{\frac{1}{p'}}. \quad (3.12)$$

Let $x \in \omega_1$ be fixed and $R > 0$ such that $\omega_1 \subset B(x, R)$. Using the Lipschitz continuity of the function η , we obtain that there exists a constant $C > 0$ such that

$$\int_{\omega_1} \frac{|\eta(x) - \eta(y)|^{p'}}{|x - y|^{N+sp'}} dy \leq C \int_{\omega_1} \frac{dy}{|x - y|^{N+sp'-p'}} \leq C \int_{B(x,R)} \frac{dy}{|x - y|^{N+sp'-p'}} \leq C. \quad (3.13)$$

Now, using (3.12), (3.13) and (A.7), we get that

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbb{I}_1(x)|^p dx &\leq C \left(\int_{\omega_2} \int_{\omega_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy dx + \int_{\mathbb{R}^N \setminus \omega_2} \int_{\omega_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy dx \right) \\ &\leq C \left(\|u\|_{W^{s,p}(\omega_2)}^p + \int_{\mathbb{R}^N \setminus \omega_2} \int_{\omega_1} \frac{|u(x)|^p + |u(y)|^p}{(1 + |x|)^{N+sp}} dy dx \right) \\ &\leq C (\|u\|_{W^{s,p}(\omega_2)}^p + \|u\|_{L^p(\Omega)}^p), \end{aligned} \quad (3.14)$$

where we have also used that $u = 0$ on $\mathbb{R}^N \setminus \Omega$. Recall that $\mathbb{I}_2 = 0$ on $\mathbb{R}^N \setminus \omega$. Then using the Hölder inequality, we get that

$$|\mathbb{I}_2(x)|^p \leq C \left(\int_{\mathbb{R}^N \setminus \omega_1} \frac{\eta^{p'}(x) dy}{|x - y|^{N+sp'}} \right)^{p-1} \int_{\mathbb{R}^N \setminus \omega_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy. \quad (3.15)$$

For any $y \in \mathbb{R}^N \setminus \omega_1$, we have that

$$\frac{\eta^{p'}(x)}{|x - y|^{N+sp'}} = \frac{\chi_{\bar{\omega}}(x) \eta^{p'}(x)}{|x - y|^{N+sp'}} \leq \chi_{\bar{\omega}}(x) \eta^{p'}(x) \sup_{x \in \bar{\omega}} \frac{1}{|x - y|^{N+sp'}}.$$

So there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N \setminus \omega_1} \frac{\eta^{p'}(x) dy}{|x - y|^{N+sp'}} \leq \chi_{\bar{\omega}}(x) \eta^{p'}(x) \int_{\mathbb{R}^N \setminus \omega_1} \frac{dy}{\text{dist}(y, \partial \bar{\omega})^{N+sp'}} \leq C \chi_{\bar{\omega}}(x) \eta^{p'}(x). \quad (3.16)$$

In (3.16) we have also used that the integral is finite which follows from the fact that $\text{dist}(\partial \omega_1, \partial \bar{\omega}) \geq \delta > 0$ together with the fact that $\text{dist}(y, \partial \bar{\omega})$ grows linearly as y tends to infinity and $N + sp' > N$.

Since $\chi_{\bar{\omega}} \eta^{p'} \in L^\infty(\omega)$ and using (3.15), (3.16) and (A.7), we also get that there exists a constant $C > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbb{I}_2(x)|^p dx &= \int_{\omega} |\mathbb{I}_2(x)|^p dx \leq C \int_{\omega} \int_{\mathbb{R}^N \setminus \omega_1} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy dx \\ &\leq C \int_{\omega} \int_{\mathbb{R}^N \setminus \omega_1} \frac{|u(x)|^p + |u(y)|^p}{(1 + |y|)^{N+sp}} dy dx \leq C \|u\|_{L^p(\Omega)}^p, \end{aligned} \quad (3.17)$$

where we have used again that $u = 0$ on $\mathbb{R}^N \setminus \Omega$. Estimate (3.10) follows from (3.11), (3.14), (3.17) and we have shown the claim. We therefore proved that ηu is a weak solution to the Poisson equation (2.6) with F given by $F = \eta f + g$. Since $F \in L^p(\mathbb{R}^N)$, it follows from Theorem 2.7 that $\eta u \in W^{2s,p}(\mathbb{R}^N)$. Thus $u \in W_{\text{loc}}^{2s,p}(\Omega)$ and the proof for $1 < p < 2$ is concluded.

Step 2: $p \geq 2$. Let $f \in W^{-s,2}(\overline{\Omega})$ and let $u \in W_0^{s,2}(\overline{\Omega})$ be the weak solution to the Dirichlet problem (1.1). Let ω and $\eta \in \mathcal{D}(\omega)$ be respectively the set and the cut-off function constructed in (3.1). We consider the function $u\eta \in W^{s,2}(\mathbb{R}^N)$. Assume that $f \in L^p(\Omega)$ with $p \geq 2$. As in the proof of Theorem 1.3, we have that $(-\Delta)^s(u\eta)$ is given by (3.2). Since by assumption $f \in L^p(\Omega) \hookrightarrow L^2(\Omega)$, it follows from Theorem 1.3 that $u\eta \in W^{2s,2}(\mathbb{R}^N)$.

(a) Applying Theorem A.2 (a) with $r = 2s$ and $p = 2$, we get that $W^{2s,2}(\mathbb{R}^N) \hookrightarrow W^{s,2N/(N-2s)}(\mathbb{R}^N)$. We have shown that $u\eta \in W^{s,2N/(N-2s)}(\mathbb{R}^N)$. Let ω_1, ω_2 be open sets such that

$$\overline{\omega} \subset \omega_1 \subset \overline{\omega}_1 \subset \omega_2 \subset \overline{\omega}_2 \subset \Omega.$$

Since the function η and the set ω in (3.1) are arbitrary, it follows from the observation $u\eta \in W^{s,2N/(N-2s)}(\mathbb{R}^N)$ that $u \in W^{s,2N/(N-2s)}(\omega_2)$. Let $q := \min\{p, \frac{2N}{N-2s}\}$. We notice that $q \geq 2$. Applying again Theorem A.2 (a) with $r = 2s$ and $p = 2$ and the above q , we also get that $W^{2s,2}(\omega_2) \hookrightarrow W^{s,q}(\omega_2)$. We have shown that $u \in W^{s,q}(\omega_2)$. Since by hypothesis, $u \in W_0^{s,2}(\overline{\Omega})$, it follows from the Sobolev embedding (A.1) that $u \in L^{2N/(N-2s)}(\Omega) \hookrightarrow L^q(\Omega)$. Thus $u \in W^{s,q}(\omega_2) \cap L^q(\Omega)$. Let

$$g := u(-\Delta)^s \eta - I_s(u, \eta).$$

Also in this case, it is possible to prove that $g \in L^q(\mathbb{R}^N)$ and there exists a constant $C > 0$ such that

$$\|g\|_{L^q(\mathbb{R}^N)} \leq C(\|u\|_{W^{s,q}(\omega_2)} + \|u\|_{L^q(\Omega)}). \quad (3.18)$$

We omit the proof of (3.18); it is totally analogous to the one we made in Step 1. As before, we have proved that ηu is a weak solution to the Poisson equation (2.6) with F given by $F = \eta(-\Delta)^s u + g$. Since $F \in L^q(\mathbb{R}^N)$, it follows from Theorem 2.7 that $\eta u \in W^{2s,q}(\mathbb{R}^N)$. Thus $u \in W_{\text{loc}}^{2s,q}(\Omega)$. If $2 \leq p \leq \frac{2N}{N-2s}$, then the proof is finished.

(b) Assume that $p > \frac{2N}{N-2s}$. Since $u \in W_{\text{loc}}^{2s,q}(\Omega)$, we have that $u \in W^{2s,q}(\omega_2)$. This implies that $u \in W^{s,q_1}(\omega_2)$ with $q_1 := \min\{p, \frac{Nq}{N-sq}\} = \min\{p, \frac{2N}{N-4s}\}$. It also follows from Lemma 2.5 that $u \in L^{q_1}(\Omega)$. We have shown that $u \in W^{s,q_1}(\omega_2) \cap L^{q_1}(\Omega)$. Now proceeding as in part (a), we get that $u \in W_{\text{loc}}^{2s,q_1}(\Omega)$. Here also if $2 \leq p \leq \frac{2N}{N-4s}$, then the proof is finished. Otherwise, iterating we will get that $u \in W_{\text{loc}}^{2s,q_j}(\Omega)$ with $q_j = \min\{p, \frac{2N}{N-sj}\}$ for all $j \geq 2$. Hence, we can find $j \in \mathbb{N}$, $j \geq 2$, such that $2 \leq p \leq \frac{2N}{N-sj}$. The proof of Theorem 1.4 is finished. \square

4 The Approach Using the Heat Semigroup Representation

One of the main passages in the proof of Theorems 1.3 and 1.4 has been to show that, after having applied the cut-off function η , the remainder g , that we obtain applying (1.4) to the product ηu , belongs to $L^p(\mathbb{R}^N)$ if f belongs to $L^p(\Omega)$. In this section, we present an alternative proof of this fact, using the characterization of the fractional Laplacian through the heat semigroup introduced in (1.6).

The heat equation representation of the operator looks a priori local and this will allow us to give a very precise information on the commutator, in particular in terms of the order of regularity and the localization properties.

Before going further into our discussion, we first need to describe how the operator introduced in (1.6) behaves when it is applied to the function ηu . For simplicity of notation, let us define

$$\varrho(t) := e^{t\Delta}(\eta u), \quad t \geq 0.$$

Then, by definition, we have that ϱ satisfies the following heat equation on \mathbb{R}^N :

$$\varrho_t - \Delta \varrho = 0, \quad t > 0, \quad \varrho(0) = \eta u. \quad (4.1)$$

Furthermore, the solution of (4.1) can be written in the form $\varrho = \phi \eta + z$ with

$$\phi_t - \Delta \phi = 0, \quad t > 0, \quad \phi(0) = u \quad (4.2)$$

and

$$z_t - \Delta z = 2 \operatorname{div}(\phi \nabla \eta) - \phi \Delta \eta, \quad t > 0, \quad z(0) = 0. \quad (4.3)$$

Finally, we can trivially compute

$$\begin{aligned} (-\Delta)^s(\eta u) &= \frac{1}{\Gamma(-s)} \int_0^{+\infty} (\varrho(t) - \varrho(0)) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^{+\infty} (\eta \phi(t) + z(t) - \eta u(t)) \frac{dt}{t^{1+s}} \\ &= \frac{\eta}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta} u - u) \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_0^{+\infty} \frac{z(t)}{t^{1+s}} dt. \end{aligned}$$

Therefore we find an expression of the type

$$(-\Delta)^s(\eta u) = \eta(-\Delta)^s u + g,$$

where the remainder term g is given by

$$g(x) := \frac{1}{\Gamma(-s)} \int_0^{+\infty} \frac{z(x, t)}{t^{1+s}} dt, \quad x \in \mathbb{R}^N. \quad (4.4)$$

4.1 Proof of the L^2 -Regularity of g

Keeping in mind the notations that we have just introduced, we can now prove the following result.

Lemma 4.1. *Let $u \in W_0^{s,2}(\bar{\Omega})$ and let η be the cut-off function introduced in (3.1). Moreover, let g be the remainder term in the expression*

$$(-\Delta)^s(\eta u) = \eta(-\Delta)^s u + g.$$

Then, there exists a constant $C > 0$ (independent of u) such that

$$\|g\|_{L^2(\mathbb{R}^N)} \leq C \|u\|_{W^{s,2}(\Omega)}. \quad (4.5)$$

Proof. According to the expression (4.4), to estimate the L^2 -norm of g , it will be enough to obtain suitable bounds of the L^2 -norm of z . For this purpose, we notice that the solution of (4.3) can be computed explicitly as

$$z(x, t) = \int_0^t \int_{\mathbb{R}^N} G(x - y, t - \tau) h(y, \tau) dy d\tau = \int_0^t [G(\cdot, t - \tau) * h(\cdot, \tau)](x) d\tau, \quad x \in \mathbb{R}^N, \quad (4.6)$$

where G is the Gaussian kernel

$$G(x, t) := (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^N, \quad t > 0,$$

and h is given by $h := 2 \operatorname{div}(\phi \nabla \eta) - \phi \Delta \eta$. Hence, in particular, we have

$$z(t) = 2 \int_0^t G(t - \tau) * \operatorname{div}(\phi(\tau) \nabla \eta) d\tau - \int_0^t G(t - \tau) * (\phi(\tau) \Delta \eta) d\tau := z_1(t) - z_2(t). \quad (4.7)$$

In (4.7), since we are only interested in the behavior of z with respect to the variable t , and for keeping the notations lighter, we have omitted the dependence of z on the variable x . We will maintain this convention

until the end of the proof. Finally, we have (recall that $\Gamma(1-s) = -s\Gamma(-s)$)

$$\begin{aligned}
 \|g\|_{L^2(\mathbb{R}^N)} &\leq \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \frac{\|z(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt \\
 &= \frac{s}{\Gamma(1-s)} \int_0^1 \frac{\|z(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt + \frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt \\
 &\leq \underbrace{\frac{s}{\Gamma(1-s)} \int_0^1 \frac{\|z_1(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt}_{=: A_1^1} + \underbrace{\frac{s}{\Gamma(1-s)} \int_0^1 \frac{\|z_2(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt}_{=: A_1^2} \\
 &\quad + \underbrace{\frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z_1(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt}_{=: A_2^1} + \underbrace{\frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z_2(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt}_{=: A_2^2}. \tag{4.8}
 \end{aligned}$$

We proceed now estimating the terms A_1^1 , A_1^2 , A_2^1 and A_2^2 separately.

Step 1: Preliminary Estimates. First of all, throughout the remainder of the proof, C will denote a generic positive constant depending only on Ω , η , s and N . This constant may change even from line to line.

Now, we observe that by using some classical energy estimates for solutions to the heat equation, we obtain that

$$\|\phi(t)\|_{L^2(\mathbb{R}^N)} \leq \|u\|_{L^2(\Omega)}, \tag{4.9}$$

$$\|\phi(t)\|_{W^{s,2}(\mathbb{R}^N)} \leq C\|u\|_{W^{s,2}(\Omega)} \quad \text{for all } s \in (0, 1). \tag{4.10}$$

These inequalities can be easily proved by multiplying (4.2) by ϕ and $(-\Delta)^s \phi$, respectively, and integrating by parts. Moreover, to obtain (4.10) we also took into account that, according to [31, Lemma 16.3], we have

$$\|(-\Delta)^{\frac{s}{2}} \phi(t)\|_{L^2(\mathbb{R}^N)} = C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(x, t) - \phi(y, t)|^2}{|x - y|^{N+2s}} dx dy.$$

In our proof, we will also need the following classical property of convolution (see, e.g., [12, Proposition 8.9]). For all $w_1 \in L^{q_1}(\mathbb{R}^N)$, $w_2 \in L^{q_2}(\mathbb{R}^N)$ and for all q_1, q_2 and q_3 satisfying

$$1 \leq q_1, q_2, q_3 < +\infty, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + 1, \tag{4.11}$$

we have that

$$\|w_1 * w_2\|_{L^{q_3}(\mathbb{R}^N)} \leq \|w_1\|_{L^{q_1}(\mathbb{R}^N)} \|w_2\|_{L^{q_2}(\mathbb{R}^N)}. \tag{4.12}$$

This is a straightforward consequence of the Young inequality. Finally, we recall that for all $1 \leq p < \infty$ and $k \geq 0$, the function G satisfies the following decay properties (see, e.g., [18]): there exists a constant $C > 0$ such that

$$\|D^k G(t)\|_{L^p(\mathbb{R}^N)} \leq C t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{k}{2}}.$$

Here, $k = (k_1, k_2, \dots, k_N)$ is a multi-index with modulus $|k| = k_1 + k_2 + \dots + k_N$ and we used the classical Schwartz notation

$$D^k \phi(x) = \frac{\partial^{|k|} \phi(x)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_N^{k_N}}.$$

In particular, we have that

$$\begin{aligned}
 \|G(t)\|_{L^2(\mathbb{R}^N)} &\leq C t^{-\frac{N}{4}}, & \|\nabla_x G(t)\|_{L^2(\mathbb{R}^N)} &\leq C t^{-\frac{N}{4}-\frac{1}{2}}, \\
 \|(G * h)(t)\|_{L^2(\mathbb{R}^N)} &\leq C \|h\|_{L^2(\Omega)}, & \|(\nabla_x G * h)(t)\|_{L^2(\mathbb{R}^N)} &\leq C t^{-\frac{1}{2}} \|h\|_{L^2(\Omega)}.
 \end{aligned}$$

Step 2: Upper Bound of $A_1 := A_1^1 + A_1^2$. We start by estimating the contribution of z_1 . Using (4.12) with $q_1 = 1$, $q_2 = q_3 = 2$, and (4.10), we get that

$$\begin{aligned} \|z_1(t)\|_{L^2(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * \operatorname{div}(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t \|D^{1-s}G(t-\tau) * D^s(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{1-s}{2}} \|\phi(\tau)\nabla\eta\|_{W^{s,2}(\mathbb{R}^N)} d\tau \\ &\leq C\|u\|_{W^{s,2}(\Omega)} \int_0^t (t-\tau)^{-\frac{1-s}{2}} d\tau = Ct^{\frac{1+s}{2}} \|u\|_{W^{s,2}(\Omega)}. \end{aligned}$$

In the previous computations, D^s denotes the differential operator with Fourier symbol $|\cdot|^s$, that is, $D^s\zeta(\cdot) = \mathcal{F}^{-1}\{|\cdot|^s\mathcal{F}\zeta(\cdot)\}$ for all functions ζ sufficiently smooth. Concerning the contribution of z_2 , instead, we have

$$\|z_2(t)\|_{L^2(\mathbb{R}^N)} \leq \int_0^t \|G(t-\tau) * (\phi(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)} d\tau \leq C \int_0^t \|\phi(\tau)\Delta\eta\|_{L^2(\mathbb{R}^N)} d\tau \leq Ct\|u\|_{L^2(\Omega)}.$$

Since $0 < s < 1$, we have that

$$A_1 \leq C\|u\|_{W^{s,2}(\Omega)} \int_0^1 \frac{dt}{t^{\frac{1+s}{2}}} + C\|u\|_{L^2(\Omega)} \int_0^1 \frac{dt}{t^s} \leq C\|u\|_{W^{s,2}(\Omega)} + C\|u\|_{L^2(\Omega)} \leq C\|u\|_{W^{s,2}(\Omega)}.$$

Step 3: Upper Bound of A_2^1 . We have to distinguish three cases: $N = 1$, $N = 2$ and $N \geq 3$.

Case 1: $N = 1$. Since $u \in L^2(\Omega)$ and Ω is bounded, we also have $u \in L^1(\Omega)$. Hence, the quantity

$$m := \int_{\mathbb{R}} u dx = \int_{\Omega} u dx$$

is well defined.

Let us now rewrite $u = (u - m\delta_0) + m\delta_0$, where δ_0 is the Dirac delta at $x = 0$. With this splitting in mind, we have that ϕ can be seen as the sum $\phi = \psi + mG$, with ψ solving

$$\psi_t - \psi_{xx} = 0, \quad t > 0, \quad \psi(0) = u - m\delta_0.$$

Therefore, we obtain

$$z_1(t) = \int_0^t G(t-\tau) * (\psi(\tau)\eta_x)_x d\tau + \int_0^t G(t-\tau) * (mG(\tau)\eta_x)_x d\tau := z_{1,\psi}(t) + z_{1,G}(t).$$

Let us consider firstly the term $z_{1,\psi}$. First of all, we notice that $\psi = \theta_x$ with θ solving

$$\theta_t - \theta_{xx} = 0, \quad t > 0, \quad \theta(0) = \int_{-\infty}^x (u - m\delta_0) d\xi,$$

and therefore,

$$z_{1,\psi}(t) = \int_0^t G(t-\tau) * (\theta_x(\tau)\eta_x)_x d\tau.$$

Now

$$\begin{aligned}\|z_{1,\psi}(t)\|_{L^2(\mathbb{R})} &\leq \int_0^t \|G(t-\tau) * (\theta_x(\tau)\eta_x)_x\|_{L^2(\mathbb{R})} d\tau \\ &= \int_0^t \|G_x(t-\tau) * (\theta_x(\tau)\eta_x)\|_{L^2(\mathbb{R})} d\tau \\ &\leq \int_0^t (t-\tau)^{-\frac{3}{4}} \|\theta_x(\tau)\eta_x\|_{L^1(\mathbb{R})} d\tau.\end{aligned}$$

Moreover, we have

$$\|\theta_x(\tau)\eta_x\|_{L^1(\mathbb{R})} \leq C\|\theta_x(\tau)\|_{L^1(\Omega)} \leq C\tau^{-\frac{1}{2}}\|\theta(0)\|_{L^1(\Omega)} \leq C\tau^{-\frac{1}{2}}\|u\|_{L^2(\Omega)},$$

where the last inequality is justified by the fact that the initial datum $\theta(0)$ is well defined as an L^1 -function compactly supported in Ω , and there exists a constant $C > 0$ such that

$$\|\theta(0)\|_{L^1(\Omega)} \leq C\|u\|_{L^2(\Omega)}.$$

See [9, Theorem 1] for more details. Hence,

$$\|z_{1,\psi}(t)\|_{L^2(\mathbb{R})} \leq C\|u\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau = Ct^{-\frac{1}{4}}\|u\|_{L^2(\Omega)}.$$

Let us now analyze the term $z_{1,G}$ which, we remind, is defined as

$$z_{1,G}(t) = m \int_0^t G(t-\tau) * (G(\tau)\eta_x)_x d\tau.$$

We have

$$\|z_{1,G}(t)\|_{L^2(\mathbb{R})} \leq m \int_0^t \|G(t-\tau) * (G(\tau)\eta_x)_x\|_{L^2(\mathbb{R})} d\tau = m \int_0^t \|G_x(t-\tau) * (G(\tau)\eta_x)\|_{L^2(\mathbb{R})} d\tau.$$

Now, since u is compactly supported in Ω , the Cauchy–Schwarz inequality yields

$$m \leq \|u\|_{L^1(\Omega)} \leq \sqrt{|\Omega|}\|u\|_{L^2(\Omega)},$$

where $|\Omega|$ is the Lebesgue measure of Ω ; hence

$$\|z_{1,G}(t)\|_{L^2(\mathbb{R})} \leq C\|u\|_{L^2(\Omega)} \int_0^t \|G_x(t-\tau) * (G(\tau)\eta_x)\|_{L^2(\mathbb{R})} d\tau.$$

Rewrite $G(\tau)\eta_x = (G(\tau)\eta)_x - G_x(\tau)\eta$. Then

$$\|z_{1,G}(t)\|_{L^2(\mathbb{R})} \leq \underbrace{C\|u\|_{L^2(\Omega)} \int_0^t \|G_x(t-\tau) * (G(\tau)\eta)_x\|_{L^2(\mathbb{R})} d\tau}_{=: J_1} + \underbrace{C\|u\|_{L^2(\Omega)} \int_0^t \|G_x(t-\tau) * (G_x(\tau)\eta)\|_{L^2(\mathbb{R})} d\tau}_{=: J_2}.$$

Concerning J_1 , we have

$$\begin{aligned}J_1 &\leq C\|u\|_{L^2(\Omega)} \int_0^t \|D^{1-s}G_x(t-\tau) * D^s(G(\tau)\eta)\|_{L^2(\mathbb{R})} d\tau \\ &\leq C\|u\|_{L^2(\Omega)} \int_0^t \|D^{1-s}G_x(t-\tau)\|_{L^1(\mathbb{R})} \|D^s(G(\tau)\eta)\|_{L^2(\mathbb{R})} d\tau \\ &\leq C\|u\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{2-s}{2}} \tau^{-\frac{1}{4}-\frac{s}{2}} d\tau = Ct^{-\frac{1}{4}}\|u\|_{L^2(\Omega)}.\end{aligned}$$

Finally, for J_2 we have

$$\begin{aligned} J_2 &\leq C \|u\|_{L^2(\Omega)} \int_0^t \|G_X(t-\tau)\|_{L^2(\mathbb{R})} \|G_X(\tau)\eta_X\|_{L^1(\mathbb{R})} d\tau \\ &\leq C \|u\|_{L^2(\Omega)} \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1}{2}} d\tau = Ct^{-\frac{1}{4}} \|u\|_{L^2(\Omega)}. \end{aligned}$$

Summarizing, we get that

$$\|z_{1,G}(t)\|_{L^2(\mathbb{R})} \leq Ct^{-\frac{1}{4}} \|u\|_{L^2(\Omega)}$$

which, combined with the estimate that we have obtained before for $z_{1,\psi}$, gives

$$\|z_1(t)\|_{L^2(\mathbb{R})} \leq Ct^{-\frac{1}{4}} \|u\|_{L^2(\Omega)}.$$

Therefore, since $s > 0$, we finally get that

$$A_2^1 = \frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z_1(t)\|_{L^2(\mathbb{R})}}{t^{1+s}} dt \leq C \|u\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{5}{4}}} = C \|u\|_{L^2(\Omega)}.$$

Case 2: $N = 2$. Using again (4.10), (4.12) with $q_1 = q_3 = 2$ and $q_2 = 1$ and the fact that η has compact support, we get that

$$\begin{aligned} \|z_1(t)\|_{L^2(\mathbb{R}^2)} &\leq 2 \int_0^t \|G(t-\tau) * \operatorname{div}(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^2)} d\tau \\ &\leq 2 \int_0^t \|D^{1-s}G(t-\tau) * D^s(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^2)} d\tau \\ &\leq C \|u\|_{W^{s,2}(\Omega)} \int_0^t (t-\tau)^{-1+\frac{s}{2}} d\tau \leq Ct^{\frac{s}{2}} \|u\|_{W^{s,2}(\Omega)}. \end{aligned}$$

Since $s > 0$, it follows that

$$A_2^1 = \frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z_1(t)\|_{L^2(\mathbb{R}^2)}}{t^{1+s}} dt \leq C \|u\|_{W^{s,2}(\Omega)} \int_1^{+\infty} \frac{dt}{t^{1+\frac{s}{2}}} = C \|u\|_{W^{s,2}(\Omega)}.$$

Case 3: $N \geq 3$. This case is more delicate and we need to proceed in a slightly different way. For a given $\varepsilon \in [0, 1]$, we will apply again (4.12) but this time by choosing

$$q_1 = \frac{4-2\varepsilon}{4-3\varepsilon}, \quad q_2 = 2-\varepsilon, \quad q_3 = 2. \quad (4.13)$$

It is straightforward to check that q_1, q_2 and q_3 given in (4.13) satisfy condition (4.11). In particular, we notice that $q_2 \in [1, 2]$. With this particular choice of the parameters we have

$$\begin{aligned} \|z_1(t)\|_{L^2(\mathbb{R}^2)} &\leq 2 \int_0^t \|G(t-\tau) * \operatorname{div}(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &= 2 \int_0^t \|D^{1-s}G(t-\tau) * D^s(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{N}{2}-\frac{\varepsilon}{4-2\varepsilon}-\frac{1-s}{2}} \|\phi(\tau)\nabla\eta\|_{L^{2-\varepsilon}(\mathbb{R}^N)} d\tau \leq Ct^{\frac{1+s}{2}-\frac{N}{2}-\frac{\varepsilon}{4-2\varepsilon}} \|u\|_{W^{s,2}(\Omega)}, \end{aligned}$$

provided that

$$\frac{1+s}{2} - \frac{N}{2} \frac{\varepsilon}{4-2\varepsilon} > 0 \implies \varepsilon < \frac{4+4s}{N+2+2s}.$$

Therefore,

$$A_2^1 = \frac{s}{\Gamma(1-s)} \int_1^{+\infty} \frac{\|z_1(t)\|_{L^2(\mathbb{R}^N)}}{t^{1+s}} dt \leq C\|u\|_{W^{s,2}(\Omega)} \int_1^{+\infty} \frac{dt}{t^{\frac{1+s}{2} + \frac{N}{2} \frac{\varepsilon}{4-2\varepsilon}}} = C\|u\|_{W^{s,2}(\Omega)},$$

if we impose that

$$\frac{1+s}{2} + \frac{N}{2} \frac{\varepsilon}{4-2\varepsilon} > 1 \implies \varepsilon > \frac{4-4s}{N+2-2s}.$$

Thus, we obtain a further condition on ε , namely

$$\varepsilon \in \left(\frac{4-4s}{N+2-2s}, \frac{4+4s}{N+2+2s} \right).$$

Furthermore, we can easily check that, for all $s \in (0, 1)$ and $N \geq 3$ the set

$$[0, 1] \cap \left(\frac{4-4s}{N+2-2s}, \frac{4+4s}{N+2+2s} \right) \neq \emptyset.$$

Therefore, for any given $s \in (0, 1)$ and $N \geq 3$, we can always choose q_1, q_2 and q_3 as in (4.13) such that

$$A_2^1 \leq C\|u\|_{W^{s,2}(\Omega)}.$$

Step 4: Upper Bound of A_2^2 . Using again (4.12), this time with $q_1 = 1, q_2 = q_3 = 2$ and the fact that η has compact support, for a given $\alpha \in (2-2s, 2)$ we can estimate

$$\begin{aligned} \|z_2(t)\|_{L^2(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * (\phi(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq \int_0^t \|D^\alpha G(t-\tau) * D^{-\alpha}(\phi(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq \int_0^t \|D^\alpha G(t-\tau)\|_{L^1(\mathbb{R}^N)} \|D^{-\alpha}(\phi(\tau)\Delta\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{\alpha}{2}} \|\phi(\tau)\Delta\eta\|_{W^{-\alpha,2}(\mathbb{R}^N)} d\tau \leq Ct^{1-\frac{\alpha}{2}} \|u\|_{L^2(\Omega)}. \end{aligned}$$

Hence

$$A_2^2 \leq C\|u\|_{L^2(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{\alpha}{2}}} = C\|u\|_{L^2(\Omega)}.$$

Step 5: Conclusion. Collecting all the above estimates, we can finally conclude that there exists a constant $C > 0$ such that (4.5) holds, and the proof of Lemma 4.1 is finished. \square

4.2 Proof of the L^p -Regularity of g

Lemma 4.1 provides an alternative proof of the $L^2(\mathbb{R}^N)$ -regularity of the remainder term g which appears in the formula for the fractional Laplacian of the product ηu . Moreover, as we did before in Section 3, also this result can be generalized to the L^p setting. In particular, we can prove the following.

Lemma 4.2. Let $u \in W_0^{s,2}(\bar{\Omega})$, $p \geq 2$, $N \geq 2$ and let η be the cut-off function introduced in (3.1). Moreover, let g be the remainder term in the expression

$$(-\Delta)^s(\eta u) = \eta(-\Delta)^s u + g.$$

Then, there exists a constant $C > 0$ (independent of u) such that

$$\|g\|_{L^p(\mathbb{R}^N)} \leq C(\|u\|_{L^p(\Omega)} + \|u\|_{W^{s,2}(\Omega)}). \quad (4.14)$$

Proof. We recall that, according to (4.4), to estimate the L^p -norm of g we only need an appropriate bound for the L^p -norm of the function z introduced in (4.6). Moreover, also in this case we have

$$\|g\|_{L^p(\mathbb{R}^N)} \leq A_1^1 + A_1^2 + A_2^1 + A_2^2,$$

where, with some abuse of notations, the terms A_1^1, A_1^2, A_2^1 and A_2^2 are the same ones as in (4.8), after having replaced $\|z(t)\|_{L^2(\mathbb{R}^N)}$ with $\|z(t)\|_{L^p(\mathbb{R}^N)}$.

Step 1: Preliminary Estimates. We recall that $u \in W_0^{s,2}(\bar{\Omega})$ follows from Proposition 2.1, and $u \in L^p(\Omega)$ follows from Lemma 2.5. Moreover, we observe that the classical energy decay estimates presented in (4.9) can be generalized to the L^p setting. In particular, we have

$$\|\phi(t)\|_{L^p(\mathbb{R}^N)} \leq \|u\|_{L^p(\Omega)}. \quad (4.15)$$

The proof of (4.15) is a straightforward application of (4.12), taking into account the fact that the solution of the heat equation (4.2) is given by the convolution $\phi(t) = G(t) * u$.

Step 2: Upper Bound of A_1^1 . First of all, throughout the remainder of the proof, C will denote a generic positive constant depending only on Ω, η, s, p and N . This constant may change even from line to line.

Now, using (4.12) with $q_1 = \frac{2p}{2+p}$, $q_2 = 2$ and $q_3 = p$, we get that

$$\begin{aligned} \|z_1(t)\|_{L^p(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * \operatorname{div}(\phi(\tau)\nabla\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t \|D^{1-s}G(t-\tau) * D^s(\phi(\tau)\nabla\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t \|D^{1-s}G(t-\tau)\|_{L^{q_1}(\mathbb{R}^N)} \|D^s(\phi(\tau)\nabla\eta)\|_{L^2(\mathbb{R}^N)} d\tau \\ &\leq C \|u\|_{W^{s,2}(\Omega)} \int_0^t (t-\tau)^{-\frac{N}{2}(1-\frac{1}{q_1})-\frac{1-s}{2}} d\tau = Ct^{\frac{1+s}{2}-\frac{N}{2}(1-\frac{1}{q_1})} \|u\|_{W^{s,2}(\Omega)}, \end{aligned}$$

provided that

$$\frac{1+s}{2} - \frac{N}{2} \left(1 - \frac{1}{q_1}\right) > 0 \quad \Rightarrow \quad q_1 < \frac{N}{N-1-s}.$$

In view of the previous estimate, we have

$$A_1^1 \leq C \|u\|_{W^{s,2}(\Omega)} \int_0^1 \frac{dt}{t^{\frac{1+s}{2} + \frac{N}{2}(1-\frac{1}{q_1})}} = C \|u\|_{W^{s,2}(\Omega)},$$

provided that

$$\frac{1+s}{2} + \frac{N}{2} \left(1 - \frac{1}{q_1}\right) < 1 \quad \Rightarrow \quad q_1 < \frac{N}{N-1+s}.$$

Finally, we notice that by hypothesis we have $p \geq 2$; this, according to the definition of q_1 that we are considering, corresponds to the further condition $1 \leq q_1 < 2$. Hence, recollecting the conditions on q_1 that we have encountered, we conclude that we have to impose

$$1 \leq q_1 < \min \left\{ 2, \frac{N}{N-1+s}, \frac{N}{N-1-s} \right\} = \frac{N}{N-1+s} = 1 + \frac{1-s}{N-1+s}.$$

Summarizing, we have

$$A_1^1 \leq C \|u\|_{W^{s,2}(\Omega)},$$

if in our computations we assume

$$1 \leq q_1 < 1 + \frac{1-s}{N-1+s}.$$

Step 3: Upper Bound of A_1^2 . We have

$$\begin{aligned}\|z_2(t)\|_{L^p(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * (\phi(\tau)\Delta\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t \|\phi(\tau)\Delta\eta\|_{L^p(\mathbb{R}^N)} d\tau \leq Ct\|u\|_{L^p(\Omega)}.\end{aligned}$$

Since $0 < s < 1$, we have that

$$A_1^2 \leq C\|u\|_{L^p(\Omega)} \int_0^1 \frac{dt}{t^s} \leq C\|u\|_{L^p(\Omega)}.$$

Step 4: Upper Bound of A_2^1 . Repeating the same computations that we did in Step 2, we get that

$$\|z_1(t)\|_{L^p(\mathbb{R}^N)} Ct^{\frac{1+s}{2} - \frac{N}{2}(1-\frac{1}{q_1})} \|u\|_{W^{s,2}(\Omega)},$$

provided that

$$\frac{1+s}{2} - \frac{N}{2}\left(1 - \frac{1}{q_1}\right) > 0 \implies q_1 < \frac{N}{N-1-s}.$$

Therefore

$$A_2^1 \leq C\|u\|_{W^{s,2}(\Omega)} \int_1^{+\infty} \frac{dt}{t^{\frac{1+s}{2} + \frac{N}{2}(1-\frac{1}{q_1})}} = C\|u\|_{W^{s,2}(\Omega)},$$

provided that

$$\frac{1+s}{2} + \frac{N}{2}\left(1 - \frac{1}{q_1}\right) > 1 \implies q_1 > \frac{N}{N-1+s}.$$

Finally, we notice that by hypothesis we have $p \geq 2$; this, according to the definition of q_1 that we are considering, corresponds to the further condition $1 \leq q_1 < 2$. Hence, recollecting the conditions on q_1 that we encountered, we conclude that we have to impose

$$q_1 \in \left[1, \min\left\{2, \frac{N}{N-1-s}\right\}\right) \cap \left(\frac{N}{N-1+s}, +\infty\right) = \left(\frac{N}{N-1+s}, \min\left\{2, \frac{N}{N-1-s}\right\}\right).$$

Summarizing, we have

$$A_1^2 \leq C\|u\|_{W^{s,2}(\Omega)},$$

if in our computations we assume

$$q_1 \in \left(\frac{N}{N-1+s}, \min\left\{2, \frac{N}{N-1-s}\right\}\right).$$

Step 5: Upper Bound of A_2^2 . Using again (4.12), this time with $q_1 = 1$, $q_2 = q_3 = p$ and the fact that η has compact support, for a given $\alpha \in (2-2s, 2)$ we can estimate

$$\begin{aligned}\|z_2(t)\|_{L^p(\mathbb{R}^N)} &\leq \int_0^t \|G(t-\tau) * (\phi(\tau)\Delta\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq \int_0^t \|D^\alpha G(t-\tau) * D^{-\alpha}(\phi(\tau)\Delta\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq \int_0^t \|D^\alpha G(t-\tau)\|_{L^1(\mathbb{R}^N)} \|D^{-\alpha}(\phi(\tau)\Delta\eta)\|_{L^p(\mathbb{R}^N)} d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{\alpha}{2}} \|\phi(\tau)\Delta\eta\|_{W^{-\alpha,p}(\mathbb{R}^N)} d\tau \\ &\leq Ct^{1-\frac{\alpha}{2}} \|u\|_{L^p(\Omega)}.\end{aligned}$$

Hence

$$A_2^2 \leq C \|u\|_{L^p(\Omega)} \int_1^{+\infty} \frac{dt}{t^{s+\frac{q}{2}}} = C \|u\|_{L^p(\Omega)}.$$

Step 6: Conclusion. Recollecting all the above estimates, we can finally conclude that there exists a constant $C > 0$ such that (4.14) holds. The proof of Lemma 4.2 is finished. \square

Remark 4.3. Lemma 4.2 provides an alternative proof of the $L^p(\mathbb{R}^N)$ -regularity of the remainder term g which appears in the formula for the fractional Laplacian of the product ηu . However, in its proof, we are able to deal only with the case $N \geq 2$. When $N = 1$, instead, we encounter some difficulties that, at the present stage, we are not able to overcome. We will present these difficulties with more details in Section 5, dedicated to open problems and perspectives. Nevertheless, we do not exclude that this regularity lemma could be extended also to the case of one-space dimension.

5 Open Problems and Perspectives

In the present paper we proved that weak solutions to the Dirichlet problem for the fractional Laplacian with a non-homogeneous right-hand side $f \in L^p(\Omega)$ ($1 < p < \infty$) belong to $W_{\text{loc}}^{2s,p}(\Omega)$.

The following comments are worth considering.

(a) In the proof of Lemma 4.2, which provides the $L^p(\mathbb{R}^N)$ -regularity of the remainder term g following the approach that employs the heat kernel characterization of the fractional Laplacian, we were not able to treat the cases $1 < p < 2$ and $N = 1$. In more detail, we cannot encounter appropriate bounds for the terms A_1^1 and A_1^2 (see (4.7) for more details on the notation). These difficulties are most likely related to the fact that, in this lower dimension case or for lower values of p , there is less diffusion and the decay rates that we shall employ are slower. On the other hand, we believe that there has to be a way to solve this problem.

(b) A natural interesting extension would be the analysis of the global elliptic regularity for weak solutions to (1.1). The problem is delicate however.

For the classical Dirichlet problem associated with the Laplace operator (the case $s = 1$), it is well known that if Ω is smooth, say of class C^2 , then weak solutions to the associated problem belong to $W^{2,p}(\Omega)$.

But, unfortunately, at least for large p , this maximal global elliptic regularity is not true for the fractional Laplacian. That is, for problem (1.1), weak solutions do not necessarily belong to $W^{2s,p}(\Omega)$. In fact, if this were the case, then for large p and $\frac{1}{2} < s < 1$, weak solutions would be at least β -Hölder continuous up to the boundary of Ω of order $\beta > s$. One can see that the latter property is not true by applying the Pohozaev identity obtained in [26] to the eigenfunctions of the Dirichlet fractional Laplacian. Indeed, let $\lambda_k > 0$ be an eigenvalue of A_D and u_k the associated eigenfunction. Then, rewriting the identity in [26, Proposition 1.6] with u_k by using the fact that $A_D u_k = \lambda_k u_k$, we get

$$\lambda_k \int_{\Omega} u_k (x \cdot \nabla u_k) dx = \frac{2s-N}{2} \lambda_k \int_{\Omega} u_k^2 dx - \frac{\Gamma(s+1)^2}{2} \int_{\partial\Omega} \left(\frac{u_k}{\rho^s} \right)^2 (x \cdot \nu) d\sigma.$$

Integrating the term on the left-hand side by parts and using that $u_k = 0$ on $\partial\Omega$, we get that

$$s\lambda_k \int_{\Omega} u_k^2 dx = \frac{\Gamma(s+1)^2}{2} \int_{\partial\Omega} \left(\frac{u_k}{\rho^s} \right)^2 (x \cdot \nu) d\sigma. \quad (5.1)$$

Now if u_k were β -Hölder continuous up to the boundary $\partial\Omega$ of order $\beta > s$, then since $0 < s < 1$ and $s\lambda_k > 0$, it would follow from (5.1) that $\int_{\Omega} u_k^2 dx = 0$. Thus $u_k = 0$ on Ω , which contradicts the fact that u_k is an eigenfunction. We have shown that u_k cannot be β -Hölder continuous up to the boundary $\partial\Omega$ of order $\beta > s$. A direct proof that u_k cannot be Lipschitz continuous up to the boundary is also contained in [28] and the references therein, where it has been shown that the eigenfunctions are s -Hölder continuous up to the boundary and this regularity is optimal. Finally, a concrete example has been given in [25, Section 7].

(c) It has been shown in [24] that if $f \in L^\infty(\Omega)$ with Ω of class C^2 and u is a weak solution of (1.1), then $u \in C^{0,s}(\mathbb{R}^N)$ and the function $\rho^{-s}u$, where $\rho = \text{dist}(x, \partial\Omega)$ is the distance of a point x to the boundary of the domain Ω , belongs to $C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < \min\{s, 1-s\}$. In addition one has the following precise regularity.

- If Ω is of class C^∞ and $f \in C^\infty(\overline{\Omega})$, then $\rho^{-s}u \in C^\infty(\overline{\Omega})$ (see, e.g., [24]).
- If Ω is of class $C^{2,\beta}$ and $f \in C^\beta(\overline{\Omega})$, then $\rho^{-s}u \in C^{s+\beta}(\overline{\Omega})$ (see, e.g., [27]).

Roughly speaking, these results just mentioned tell us that, if the domain Ω is regular enough, the solution u to (1.1) can be seen as $u = \rho^s v$, where v is a function regular up to the boundary. By part (b), weak solutions are in general not in $W^{2s,p}(\Omega)$. Nevertheless, compared with the above mentioned results, one could expect both $\rho^{-s}u$ and $\rho^{1-s}u$ to be smooth in the $L^p(\Omega)$ context, i.e. to belong to $W^{2s,p}(\Omega)$. In view of this, it would be natural to analyze whether this regularity property, which is not available in the literature, is actually true. Finally, more generally, it is also interesting to investigate for which $\beta > 0$ we have $\rho^\beta u \in W^{2s,p}(\Omega)$. Following our approach, we think that it is possible to show that $\rho^\beta u \in W^{2s,p}(\Omega)$ for every $\beta > s$. However, the most interesting case is $0 < \beta \leq s$. We mention that in this situation we have that $\rho^\beta u$ is also a solution of a certain Dirichlet problem.

A Appendix

For the sake of completeness, we introduce some well-known facts about the fractional order Sobolev spaces, which are not so familiar as the classical integral order Sobolev spaces.

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. For $p \in [1, \infty)$ and $s \in (0, 1)$, we denote by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy < \infty \right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_{\Omega} |u|^p dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.$$

We set

$$W_0^{s,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)},$$

where $\mathcal{D}(\Omega)$ is the space of all continuously infinitely differentiable functions with compact support in Ω .

The following result is taken from [15, p. 25, Theorem 1.4.2.4].

Theorem A.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary and $1 < p < \infty$. Then for every $0 < s \leq \frac{1}{p}$, we have that $W^{s,p}(\Omega) = W_0^{s,p}(\Omega)$ with equivalent norm.*

It is well known (see, e.g., [7, 15]) that if $\Omega \subset \mathbb{R}^N$ is a bounded open set with a Lipschitz continuous boundary then

$$W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{with} \quad \begin{cases} 1 \leq q \leq \frac{Np}{N-sp} & \text{if } N > sp, \\ 1 \leq q < \infty & \text{if } N = sp. \end{cases} \quad (\text{A.1})$$

If $N < sp$, then

$$W^{s,p}(\Omega) \hookrightarrow C^{0,s-\frac{N}{p}}(\overline{\Omega}). \quad (\text{A.2})$$

Next, for $1 < p < \infty$ and $0 < s < 1$ we define

$$W_0^{s,p}(\overline{\Omega}) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\}.$$

It has been shown in [7, Lemma 6.1] that for an arbitrary bounded open set $\Omega \subset \mathbb{R}^N$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x - y|^{N+sp}} \geq C |\Omega|^{-\frac{sp}{N}}. \quad (\text{A.3})$$

Using (A.3), we get that there exists a constant $C > 0$ such that for every $u \in W_0^{s,p}(\bar{\Omega})$,

$$\int_{\mathbb{R}^N} |u|^p dx = \int_{\Omega} |u|^p dx \leq C \int_{\mathbb{R}^N} |u(x)|^p \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x-y|^{N+sp}} \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy. \quad (\text{A.4})$$

It follows from (A.4) that for every $1 < p < \infty$ and $0 < s < 1$,

$$\|u\|_{W_0^{s,p}(\bar{\Omega})} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dx dy \right)^{\frac{1}{p}} \quad (\text{A.5})$$

defines an equivalent norm on $W_0^{s,p}(\bar{\Omega})$. We shall denote by $W^{-s,p'}(\bar{\Omega})$ the dual of the reflexive Banach space $W_0^{s,p}(\bar{\Omega})$, that is,

$$W^{-s,p'}(\bar{\Omega}) := (W_0^{s,p}(\bar{\Omega}))^* \quad \text{where } p' := \frac{p}{p-1}.$$

We remark that there is no obvious inclusion between $W_0^{s,p}(\Omega)$ and $W_0^{s,p}(\bar{\Omega})$. In fact, for an arbitrary bounded open set $\Omega \subset \mathbb{R}^N$, the two spaces are different, since $\mathcal{D}(\Omega)$ is not always dense in $W_0^{s,p}(\bar{\Omega})$ (see, e.g., [11]). But if Ω has a continuous boundary, then by [11, Theorem 6], $\mathcal{D}(\Omega)$ is dense in $W_0^{s,p}(\bar{\Omega})$ and in addition we have that

$$W_0^{s,p}(\bar{\Omega}) = W_0^{s,p}(\Omega) \quad \text{for every } \frac{1}{p} < s < 1. \quad (\text{A.6})$$

In fact, (A.6) follows by using the Hardy inequality for fractional order Sobolev spaces and the following estimate (see, e.g., [15, formula (1.3.2.12)]): there exist two constants $0 < C_1 \leq C_2$ such that

$$\frac{C_1}{(\rho(x))^{ps}} \leq \int_{\mathbb{R}^N \setminus \Omega} \frac{dy}{|x-y|^{N+sp}} \leq \frac{C_2}{(\rho(x))^{ps}}, \quad x \in \Omega,$$

where $\rho(x) := \text{dist}(x, \partial\Omega)$, $x \in \Omega$.

We also notice that the continuous embeddings (A.1) and (A.2) hold with $W^{s,p}(\Omega)$ replaced with $W_0^{s,p}(\Omega)$ or $W_0^{s,p}(\bar{\Omega})$ and this case without any regularity assumption on the open set Ω .

Next, if $s > 1$ and is not an integer, then we write $s = m + \sigma$ where m is an integer and $0 < \sigma < 1$. In this case

$$W^{s,p}(\Omega) := \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha \text{ such that } |\alpha| = m\}.$$

Then $W^{s,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{W^{m,p}(\Omega)}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{W^{\sigma,p}(\Omega)}^p \right)^{\frac{1}{p}}.$$

If $s = m$ is an integer, then $W^{s,p}(\Omega)$ coincides with the Sobolev space $W^{m,p}(\Omega)$. Comparing with (A.1), we have the following general embedding.

Theorem A.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary. Then the following assertions hold:*

- (a) *If $0 < s \leq r$ and $1 < p \leq q < \infty$ are real numbers such that $r - \frac{N}{p} = s - \frac{N}{q}$, then $W^{r,p}(\mathbb{R}^N) \hookrightarrow W^{s,q}(\mathbb{R}^N)$.*
- (b) *If $0 < s \leq r$ and $1 < p \leq q < \infty$ are real numbers such that $r - \frac{N}{p} \geq s - \frac{N}{q}$, then $W^{r,p}(\Omega) \hookrightarrow W^{s,q}(\Omega)$.*

For more information on fractional order Sobolev spaces, we refer to [1, 7, 15, 17] and references therein.

Finally, we mention the following estimate that has been used previously. Let $A \subset \mathbb{R}^N$ be a bounded set and $B \subset \mathbb{R}^N$ an arbitrary set. Then there exists a constant $C > 0$ (depending on A and B) such that

$$|x - y| \geq C(1 + |y|) \quad \text{for all } x \in A, y \in \mathbb{R}^N \setminus B, \text{ dist}(A, \mathbb{R}^N \setminus B) = \delta > 0. \quad (\text{A.7})$$

Funding: All authors were supported by the Air Force Office of Scientific Research through award no. FA9550-15-1-0027. Umberto Biccari and Enrique Zuazua were supported by Ministerio de Economía y Competitividad (Spain) through grant MTM2014-52347 and by the European Research Council Executive Agency through Advanced Grant DYCON (Dynamic Control). Enrique Zuazua was supported by EOARD-AFOSR through award no. FA9550-14-1-0214 and by Agence Nationale de la Recherche (France) through ICON.

References

- [1] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren Math. Wiss. 314, Springer, Berlin, 1996.
- [2] O. G. Bakunin, *Turbulence and Diffusion: Scaling Versus Equations*, Springer, Berlin, 2008.
- [3] C. Bernard, Regularity of solutions to the fractional Laplace equation, preprint (2014), <http://math.uchicago.edu/~may/REU2014/REUPapers/Bernard.pdf>.
- [4] U. Biccari, Internal control for non-local Schrödinger and wave equations involving the fractional Laplace operator, preprint (2017), <https://arxiv.org/abs/1411.7800v2>.
- [5] M. Bologna, C. Tsallis and P. Grigolini, Anomalous diffusion associated with non-linear fractional derivative Fokker–Planck-like equation: Exact time-dependent solutions, *Phys. Rev. E* **62** (2000), 2213–2218.
- [6] M. Cozzi, Interior regularity of solutions of non-local equations in Sobolev and Nikol’skii spaces, *Ann. Mat. Pura Appl. (2)* **196** (2017), 555–578.
- [7] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521–573.
- [8] S. Dipierro, G. Palatucci and E. Valdinoci, Dislocation dynamics in crystals: A macroscopic theory in a fractional Laplace setting, *Comm. Math. Phys.* **333** (2015), no. 2, 1061–1105.
- [9] J. Duoandikoetxea and E. Zuazua, Moments, masses de Dirac et décomposition de fonctions, *C. R. Acad. Sci. Paris Sér. 1* **315** (1992), no. 6, 693–698.
- [10] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer, New York, 2000.
- [11] A. Fiscella, R. Servadei and E. Valdinoci, Density properties for fractional Sobolev spaces, *Ann. Acad. Sci. Fenn. Math.* **40** (2015), 235–253.
- [12] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, New York, 2013.
- [13] C. G. Gal and M. Warma, Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces, *Comm. Partial Differential Equations* (2017), DOI 10.1080/03605302.2017.1295060.
- [14] G. Gilboa and S. Osher, Nonlocal operators with applications to image processing, *Multiscale Model. Simul.* **7** (2008), no. 3, 1005–1028.
- [15] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monogr. Stud. Math. 24, Pitman, Boston, 1985.
- [16] G. Grubb, Fractional Laplacians on domains, a development of Hörmander’s theory of μ -transmission pseudodifferential operators, *Adv. Math.* **268** (2015), 478–528.
- [17] A. Jonsson and H. Wallin, *Function Spaces on Subsets of \mathbb{R}^N* , Math. Rep. 2, Harwood Academic Publishers, Reading, 1984.
- [18] T. Kato, Strong L^p solutions of the Navier–Stokes equation in \mathbb{R}^m , with applications to weak solutions, *Math. Z.* **187** (1984), no. 4, 471–480.
- [19] T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, *Discrete Contin. Dyn. Syst.* **35** (2015), 6031–6068.
- [20] S. Levendorski, Pricing of the American put under Lévy processes, *Int. J. Theor. Appl. Finance* **7** (2004), no. 3, 303–335.
- [21] M. M. Meerschaert, Fractional calculus, anomalous diffusion, and probability, in: *Fractional Dynamics*, World Scientific, Hackensack (2012), 265–284.
- [22] J. Moser, A new proof of de Giorgi’s theorem concerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.* **13** (1960), 457–468.
- [23] H. Pham, Optimal stopping, free boundary, and American option in a jump-diffusion model, *Appl. Math. Optim.* **35** (1997), no. 2, 145–164.
- [24] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary, *J. Math. Pures Appl.* **101** (2014), no. 9, 275–302.
- [25] X. Ros-Oton and J. Serra, The extremal solution for the fractional Laplacian, *Calc. Var. Partial Differential Equations* **50** (2014), 723–750.
- [26] X. Ros-Oton and J. Serra, [The Pohozaev identity for the fractional Laplacian](#), *Arch. Ration. Mech. Anal.* **213** (2014), 587–628.
- [27] X. Ros-Oton and J. Serra, Boundary regularity for fully nonlinear integro-differential equations, *Duke Math. J.* **165** (2016), 2079–2154.

- [28] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, *Proc. Roy. Soc. Edinburgh Sect. A* **144** (2014), 831–855.
- [29] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.
- [30] P. R. Stinga, *Fractional powers of second order partial differential operators: Extension problem and regularity theory*, Ph.D. Dissertation, Universidad Autónoma de Madrid, Spain, 2010.
- [31] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, Berlin, 2007.
- [32] J. L. Vázquez, Nonlinear diffusion with fractional Laplacian operators, in: *Nonlinear Partial Differential Equations* (Oslo 2010), Abel Symp. 7, Springer, Heidelberg (2012), 271–298.
- [33] T. Zhu and J. M. Harris, Modeling acoustic wave propagation in heterogeneous attenuating media using decoupled fractional Laplacians, *Geophysics* **79** (2014), no. 3, T105–T116.