



Repositorio Institucional de la Universidad Autónoma de Madrid

<https://repositorio.uam.es>

Esta es la **versión de autor** del artículo publicado en:
This is an **author produced version** of a paper published in:

Linear Algebra and Its Applications 582 (2019): 103-113

DOI: <https://doi.org/10.1016/j.laa.2019.07.038>

Copyright: © 2019 Elsevier Inc. All rights reserved.

El acceso a la versión del editor puede requerir la suscripción del recurso

Access to the published version may require subscription

Morphisms and period matrices

Fernando Chamizo¹

*Departamento de Matemáticas and ICMAT. Facultad de Ciencias. Universidad
Autónoma de Madrid. 28049 Madrid. Spain.*

Abstract

Bounding the number of morphisms between compact Riemann surfaces is a long standing problem coming from complex and algebraic geometry. We show that linear algebra techniques allow to improve the known results when we assume a kind of condition number bound for the period matrix.

Keywords: Riemann surface, period matrix, morphism, matrix norm
2000 MSC: 30F30, 14H55, 15A45

1. Statement of the problem

Given two compact Riemann surfaces S and S' with genera $g > g' > 1$, consider the (holomorphic non-constant) morphisms $S \rightarrow S'$. If we equivalently think in plane algebraic curve models [9, §1.3], they are rational maps and in principle they are objects not related in any apparent way to linear algebra. For instance, $(X, Y) = (x/(1-x), y^2/(1-x)^2)$ defines a morphism between $S = \{x^6 + y^6 = 1\}$ and $S' = \{X^6 + Y^3 = (X+1)^6\}$.

We are interested in upper bounds for

$$\mathcal{N}(S, S') = \#\{f \text{ morphism } S \rightarrow S'\}.$$

A classical result due to Hurwitz [13] that can be proved in a more or less combinatorial way [9, Th.2.41] states that the number of automorphisms (bijective morphisms $S \rightarrow S$) is less or equal than $84(g-1)$. It is known to be sharp meaning that the equality is reached for infinitely many values of the

¹The author is partially supported by the MTM2017-83496-P project of the MCINN (Spain) and the “Severo Ochoa Programme for Centres of Excellence in R&D” (SEV-2015-0554).

genus [16]. Twenty years later, in 1913, de Franchis [3] proved that $\mathcal{N}(S, S')$ is finite, in fact $\sum_{S'} \mathcal{N}(S, S')$ is finite when S' runs over all possible target surfaces (see [8, Remark 3.4] for a short modern proof). Since then, several authors [17], [11], [14], [20] have given effective bounds, but in contrast with the behavior when $S = S'$, all of these bounds show a super-exponential growth in g . Without entering into details, the best known result [2] slightly improves a bound due to Tanabe [20] that implies

$$\log \mathcal{N}(S, S') < 2g \log(4g) + \log(6g). \quad (1)$$

There are not known examples of Riemann surfaces with a large number of morphisms (see [18] for very low genus) and a fascinating question arises unsettled after a century of research: Either we are missing very peculiar examples or bounds like (1) are very far from truth. To make the situation more perplexing, a simple construction included in [14] proves that there are not polynomial bounds in g for $\sum_{S'} \mathcal{N}(S, S')$.

This paper is organized as follows. Section 2 describes the link with linear algebra and states the main results. Section 3 includes some known auxiliary results. Section 4 is devoted to the proof of the main results. Finally, the last section illustrates the strength of the bound in a numerical example.

2. The linear algebra approach and main results

A compact Riemann surface of genus g can be considered as a subvariety of its Jacobian which is a complex torus \mathbb{C}^g/Λ with Λ a lattice. A morphism $f : S \rightarrow S'$ lifts to a linear map $F : \mathbb{C}^g/\Lambda \rightarrow \mathbb{C}^{g'}/\Lambda'$ preserving the lattice quotients.

$$\begin{array}{ccc} \mathbb{C}^g/\Lambda & \xrightarrow{F} & \mathbb{C}^{g'}/\Lambda' \\ \uparrow a & & \uparrow a' \\ S & \xrightarrow{f} & S' \end{array} \quad a, a' = \text{Abel-Jacobi maps.} \quad (2)$$

After a change of basis, Λ and Λ' can be moved to the standard lattice and the matrices of these linear maps have integral entries. Bounding $\mathcal{N}(S, S')$ counting linear maps with integral matrices has been present in all the previous approaches to the problem in a somehow hidden way. In the usual setting

these integral matrices are interpreted as homomorphisms on the homology or cohomology groups.

The hardness of the problem stems from the fact that it is very difficult to impose conditions assuring that these linear maps preserve the conformal structure and can be restricted to the Riemann surfaces reversing the Abel-Jacobi maps in the commutative diagram (2). Conditions of this kind allow to pass from the infinitely many linear maps to a finite bound for $\mathcal{N}(S, S')$. As an aside, the case of genus one (elliptic curves) is special because the Riemann surface coincides with the complex torus and then the linear maps may give infinitely many morphisms.

Although lattice preserving linear maps commonly do not correspond to a morphisms, the conformal structure of a Riemann surface is determined by the lattice (Torelli theorem). With this idea in mind, in this paper we give bounds for $\mathcal{N}(S, S')$ involving the “size” of the period matrix that represents this lattice. When this size is fixed, the logarithm of our bound is essentially linear in g .

The canonical period matrix of a Riemann surface (see [5] for the actual definition) is usually written in its standard imaginary form, say Ω_i , but we find convenient to use the equivalent real form, Ω_r , obtained through the natural isomorphism $\mathbb{C}^g \rightarrow \mathbb{R}^{2g}$, $\vec{x} \mapsto (\Re \vec{x}, \Im \vec{x})$. In this way

$$\Omega_i = (I, A + \imath B) \quad \longleftrightarrow \quad \Omega_r = \begin{pmatrix} I & A \\ O & B \end{pmatrix}$$

where A and B are real symmetric $g \times g$ matrices and B is positive definite. We write $\imath = \sqrt{-1}$ to avoid confusion with the use of i as a subscript for entries of vectors and matrices.

Let $\|\cdot\|$ be the usual matrix norm induced by the Euclidean norm in \mathbb{R}^n that we denote with the same symbol,

$$\|A\| = \sup_{\|\vec{x}\|=1} \|A\vec{x}\|$$

Then our main result in terms of Ω_r reads

Theorem 2.1. *Let $M = 1/3 + \|\Omega_r^{-1} + \Omega_r^t\| + \|\Omega_r^{-1}\Omega_r^t\|$. Then*

$$\mathcal{N}(S, S') < (12e\sqrt{2}M)^g \sqrt{\frac{32g}{\pi}}.$$

In fact, this result is a consequence of a less symmetric, but slightly stronger, bound.

Theorem 2.2. *Let $M = \max(\|B\| + \|B^{-1}\| + \|AB^{-1}A\|, 8)$. Then*

$$\mathcal{N}(S, S') < (4e\sqrt{2}M)^g \sqrt{\frac{32g}{\pi}}.$$

Using non-uniqueness of the period matrix it is possible to give some variants of this result, for instance the action of $\begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$ on Ω_i implies that the previous theorem still holds if A is replaced by $A - K$ where K is an integral symmetric matrix.

3. Notation and auxiliary results

Consider the $2g$ -dimensional real vector space $\mathcal{H}(S)$ generated by the real harmonic forms on S , and choose a basis $\{\omega_1, \omega_2, \dots, \omega_{2g}\}$ dual to a canonical basis for $H_1(S, \mathbb{Z})$. Let $[\cdot, \cdot]$ be the inner product in \mathbb{R}^{2g} induced by $\langle \omega, \eta \rangle = \int \eta \wedge * \omega$ through the linear isomorphism $j_S : \mathcal{H}(S) \rightarrow \mathbb{R}^{2g}$ with $j_S(\omega_j) = \vec{e}_j$,

$$[\vec{x}, \vec{y}] = \int j_S^{-1}(\vec{y}) \wedge * j_S^{-1}(\vec{x}) = \sum x_i y_j \int \omega_j \wedge * \omega_i.$$

With a similar construction, replacing S by S' we define $[\cdot, \cdot]'$ and $j_{S'}$. We write $[[\cdot]]$ and $[[\cdot]]'$ for the corresponding norms.

The following result embodies the aforementioned relation of the problem to linear maps and it is the basis of the linear algebra approach.

Lemma 3.1. *If $f : S \rightarrow S'$ is a morphism of degree d and $F : \mathbb{R}^{2g'} \rightarrow \mathbb{R}^{2g}$ is given by $F = j_S \circ f^* \circ j_{S'}^{-1}$ (where f^* denotes the pullback of f), then F has an integral matrix with respect to the canonical basis and satisfies $[[F\vec{x}]] = \sqrt{d} [[\vec{x}]]'$. Furthermore distinct morphisms cannot give the same F .*

Proof. By the choice of the basis of $\mathcal{H}(S')$ and $\mathcal{H}(S)$ as the dual of a canonical homology basis, f^* induces a linear map between the lattices generated by them, hence F has an integral matrix. Since $\int f^* \eta \wedge f^* \omega = d \int \eta \wedge \omega$, we get $[F\vec{x}, F\vec{y}] = d[\vec{x}, \vec{y}]'$. Finally, a morphism is completely determined by its action on the Jacobian varieties, by its pullback, and $j_S, j_{S'}$ are fixed linear isomorphisms. \square

Now we state two results on geometry of numbers. The first one is a trivial remark in lattice point theory (cf. §1 [7]) and the second is a form of Minkowski's theorem.

Lemma 3.2. *Let $D_0 \subset \mathbb{R}^n$ be a bounded convex domain containing $\vec{0}$. If $D_1 \supset D_0$ is another bounded convex domain with $d(D_0, \mathbb{R}^n - D_1) \geq \sqrt{n}/2$ where d is the Euclidean distance, then*

$$\#\{\vec{x} \in D_0 \cap \mathbb{Z}^n\} < \text{Vol}(D_1).$$

Proof. For each $\vec{x} \in D_0 \cap \mathbb{Z}^n$, the unit cube $[x_1 - 1/2, x_1 + 1/2] \times \cdots \times [x_n - 1/2, x_n + 1/2]$ is contained in the closure of D_1 and all of these cubes have disjoint interiors. \square

Lemma 3.3. *Let Q be a quadratic form in \mathbb{R}^{2n} with unit determinant. Then*

$$\min_{\vec{x} \in \mathbb{Z}^{2n} - \{\vec{0}\}} Q(\vec{x}) \leq \frac{4}{\pi} (n!)^{1/n}.$$

Proof. This is a direct consequence of Theorem 4.1 and Theorem 4.2 of [12, §20]. \square

Finally, we state a complex analysis result.

Lemma 3.4. *Let $\eta \in \mathcal{H}(S) \setminus \{0\}$, $\omega \in \mathcal{H}(S')$; then there are at most $8(g-1)$ morphisms $f : S \rightarrow S'$ such that $f^*\omega = \eta$.*

Proof. This is Corollary 3.2 of [19]. \square

4. Proof of the main results

Let us compare firstly $[[\cdot]]$ and the standard Euclidean norm $\|\cdot\|$.

Proposition 4.1. *For M as in Theorem 2.2, we have*

$$\frac{2}{3M} \|\vec{x}\|^2 \leq [[\vec{x}]]^2 \leq \frac{3M}{2} \|\vec{x}\|^2.$$

Proof. Write $[[\vec{x}]]^2 = \vec{x}^t \Gamma \vec{x}$ with Γ positive definite. Then III.2.4 and III.2.8 in [5] imply

$$\Gamma = \begin{pmatrix} \Lambda_2 & -\Lambda_1 \\ -\Lambda_1^t & -\Lambda_3 \end{pmatrix} \quad \text{and} \quad \Gamma^{-1} = \begin{pmatrix} -\Lambda_3 & \Lambda_1^t \\ \Lambda_1 & \Lambda_2 \end{pmatrix} \quad (3)$$

with $A = -\Lambda_1\Lambda_3^{-1}$ and $B = -\Lambda_3^{-1}$.

Of course Λ_2 and $-\Lambda_3$ are positive definite. The symmetry of Γ and the identity $\Gamma\Gamma^{-1} = I$ also imply

$$\Lambda_1^2 + \Lambda_2\Lambda_3 + I = 0, \quad \Lambda_1\Lambda_2 = \Lambda_2\Lambda_1^t \quad \text{and} \quad \Lambda_3\Lambda_1 = \Lambda_1^t\Lambda_3.$$

Using these relations, we have

$$\Lambda_2 = -\Lambda_3^{-1} - \Lambda_1\Lambda_1\Lambda_3^{-1} = -\Lambda_3^{-1} - (\Lambda_1\Lambda_3^{-1})\Lambda_3(\Lambda_1\Lambda_3^{-1}).$$

Hence

$$\|\Lambda_2\| \leq \|B\| + \|AB^{-1}A\|.$$

Given $\vec{v}, \vec{w} \in \mathbb{R}^g - \{\vec{0}\}$ take $\vec{x} \in \mathbb{R}^{2g}$, $\vec{x}^t = (\vec{v}^t/\|\vec{v}\|, \pm\vec{w}^t/\|\vec{w}\|)$ where the sign \pm is chosen in such a way that $\vec{v}^t\Lambda_1\vec{w} \geq 0$. Then

$$0 < \vec{x}^t\Gamma\vec{x} = \|\vec{v}\|^{-2}\vec{v}^t\Lambda_2\vec{v} - \|\vec{w}\|^{-2}\vec{w}^t\Lambda_3\vec{w} - 2\|\vec{v}\|^{-1}\|\vec{w}\|^{-1}|\vec{v}^t\Lambda_1\vec{w}|,$$

and hence

$$2|\vec{v}^t\Lambda_1\vec{w}| \leq (\|\Lambda_2\| + \|\Lambda_3\|)\|\vec{v}\|\|\vec{w}\|. \quad (4)$$

For any $\vec{x} \in \mathbb{R}^{2g} - \{\vec{0}\}$ write $|\cos t|$ and $|\sin t|$ for the norms of the vectors consisting of the first and the last g coordinates of $\vec{x}/\|\vec{x}\|$. Then according to (3) and (4)

$$\begin{aligned} \frac{[[\vec{x}]]^2}{\|\vec{x}\|^2} &\leq \|\Lambda_2\| \cos^2 t + \|\Lambda_3\| \sin^2 t + (\|\Lambda_2\| + \|\Lambda_3\|) |\sin t \cos t| \\ &\leq \frac{3}{2}(\|\Lambda_2\| + \|\Lambda_3\|) \leq \frac{3M}{2}. \end{aligned}$$

By a similar argument with Γ replaced with Γ^{-1} , after changing Λ_2 by Λ_3 , and Λ_1 by $-\Lambda_4$, we also have

$$\frac{\vec{y}^t\Gamma^{-1}\vec{y}}{\|\vec{y}\|^2} \leq \frac{3M}{2} \quad \forall \vec{y} \in \mathbb{R}^{2g} - \{\vec{0}\}.$$

Writing $\vec{y} = R\vec{x}$ with R symmetric such that $R^2 = \Gamma$, we obtain

$$\|\vec{x}\|^2 \leq \frac{3M}{2} [[\vec{x}]]^2 \quad \forall \vec{x} \in \mathbb{R}^{2g} - \{\vec{0}\}$$

and the proof is complete. \square

Now we relate the bounds for $\mathcal{N}(S, S')$ with a lattice point problem.

Proposition 4.2. *We have*

$$\mathcal{N}(S, S') \leq 8g \# \{ \vec{n} \in \mathbb{Z}^{2g} : [[\vec{n}]]^2 < 4g\sqrt{2}/\pi \}.$$

Proof. As before, we denote with Γ the matrix of the scalar product $[\cdot, \cdot]$. Then (3) gives the relation $J^{-1}\Gamma J = \Gamma^{-1}$ with J the standard symplectic matrix and it implies $\det(\Gamma) = 1$. Then Lemma 3.3 for $Q(\vec{x}) = ([[\vec{x}]]')^2$ gives (note that $a_n = (n!)^{1/n}/n$ is a decreasing sequence)

$$([[\vec{x}_0]]')^2 \leq 2g'\sqrt{2}/\pi \quad \text{for some } \vec{x}_0 \in \mathbb{Z}^{2g'} - \{\vec{0}\}.$$

Consider the equivalence relation in the set of morphisms $f \sim g$ if $f^*\omega = g^*\omega$ where $\omega = j_{S'}^{-1}(\vec{x}_0)$ is the harmonic form corresponding to \vec{x}_0 . By Lemma 3.4 the cardinality of each equivalence class is bounded by $8g$ and the number of equivalence classes coincides with the number of possible values of $F\vec{x}_0$ (with F as in Lemma 3.1). On the other hand, by Lemma 3.1 and the Riemann-Hurwitz relation [9, Th.1.76]

$$[[F\vec{x}_0]] = \sqrt{d} [[\vec{x}_0]]' \leq \sqrt{\frac{g-1}{g'-1}} [[\vec{x}_0]]' < \sqrt{4g\sqrt{2}/\pi}.$$

As $F\vec{x}_0 \in \mathbb{Z}^{2g}$, the result follows. \square

The previous result suggests using lattice point theory to count integral points in the ellipsoid $\{[[\vec{x}]] \leq R\}$. There are several results of this kind [7] but the difficulty in our case is that the ellipsoid remains somehow unspecified and the usual volume approximation could fail if the eccentricity is large. We take control of the situation via Proposition 4.1.

Proof of Theorem 2.2. Let

$$\begin{aligned} D_0 &= \{ \vec{x} \in \mathbb{R}^{2g} : [[\vec{x}]]^2 < 4g\sqrt{2}/\pi \}, \\ D_1 &= \{ \vec{x} \in \mathbb{R}^{2g} : [[\vec{x}]]^2 < 4Mg\sqrt{2}/\pi \} \end{aligned}$$

and \vec{y}_0 and \vec{y}_1 points on the boundaries of D_0 and D_1 , respectively.

By Proposition 4.1 and the triangle inequality,

$$\begin{aligned}\|\vec{y}_1 - \vec{y}_0\| &\geq \lceil \|\vec{y}_1 - \vec{y}_0\| \rceil \sqrt{\frac{2}{3M}} \geq (\lceil \|\vec{y}_1\| \rceil - \lceil \|\vec{y}_0\| \rceil) \sqrt{\frac{2}{3M}} \\ &= \sqrt{\frac{8g\sqrt{2}}{3\pi} \frac{\sqrt{M} - 1}{\sqrt{M}}} > \frac{\sqrt{2g}}{2}.\end{aligned}$$

In the last inequality we have employed $M \geq 8$.

By Lemma 3.2,

$$\#\{\vec{n} \in \mathbb{Z}^{2g} : \lceil \|\vec{n}\| \rceil^2 < 4g\sqrt{2}/\pi\} < \text{Vol}(D_1) = \omega_{2g}(4Mg\sqrt{2}/\pi)^g$$

where $\omega_{2g} = \pi^g/g!$ is the volume of the $2g$ -dimensional unit ball (it equals the volume of $\{\vec{x} : \lceil \|\vec{x}\| \rceil < 1\}$ because $\det(\Gamma) = 1$).

Finally, using $g! > g^g e^{-g} \sqrt{2\pi g}$ and Proposition 4.2 we finish the proof. \square

Proof of Theorem 2.1. We obtain after some calculations

$$(\Omega_r^{-1})^t + \Omega_r + \Omega_r^{-1} \Omega_r^t = \begin{pmatrix} 3I - AB^{-1}A & O \\ O & I + B^{-1} + B \end{pmatrix}.$$

By the triangle inequality, we have

$$M \geq 1/3 + \max(\|3I - AB^{-1}A\|, \|I + B^{-1} + B\|).$$

Recall that B is positive definite. Then $\|I + B + B^{-1}\| \geq 3$ and in fact

$$\begin{aligned}\|I + B + B^{-1}\| &= 1 + \|B + B^{-1}\| \\ &= 1 + \max(\|B\| + \|B\|^{-1}, \|B^{-1}\| + \|B^{-1}\|^{-1}) \\ &\geq 1 + \frac{1}{2}(\|B\| + \|B^{-1}\|).\end{aligned}$$

Hence, if $\|AB^{-1}A\| \leq 5$,

$$3M \geq 1 + 3 + \frac{3}{2}(\|B\| + \|B^{-1}\|) \geq \|AB^{-1}A\| + \|B\| + \|B^{-1}\|.$$

(note that $\|B\| \cdot \|B^{-1}\| \geq 1 \Rightarrow \|B\| + \|B^{-1}\| \geq 2$). And if $\|AB^{-1}A\| \geq 5$, by the convexity bound $\max(|a|, |b|) \geq |a|/3 + 2|b|/3$,

$$\begin{aligned}3M &\geq 1 + (\|AB^{-1}A\| - 3) + 2\|I + B + B^{-1}\| \\ &\geq \|AB^{-1}A\| + \|B\| + \|B^{-1}\|.\end{aligned}$$

Then we can apply Theorem 2.2 renaming M as $3M$. Note that $M \geq 1/3 + \|I + B + B^{-1}\|$ assures $3M > 8$. \square

5. Some numerical calculations

Several authors have succeeded in finding explicit period matrices of some remarkable families of Riemann surfaces. For instance, Weil [21] worked out the case of Lefschetz surfaces $y^l = x^a(1 - x)$ and Rohrllich [10] the case of Fermat's curves $x^n + y^n = 1$ (see [15] for a detailed explanation and generalizations). On the other hand, if we are only interested in the numerical aspects of the problem, nowadays it is possible to calculate effortlessly and with great accuracy the period matrix of a compact Riemann surface from its equation as a plane algebraic curve using a suitable computer mathematical package [4], [6].

The action of the group $\mathrm{Sp}(2g, \mathbb{Z})$ on the Siegel upper half space \mathbb{H}_g does not leave $\|\Omega_r^{-1} + \Omega_r^t\| + \|\Omega_r^{-1}\Omega_r^t\|$ invariant. This means that our bound depends on the choice of the period matrix. Taking this into account, instead of picking randomly Riemann surfaces and choosing a period matrix from a black-box computer package, we are going to consider a simple family with closed and simple expressions for the period matrices. It is given by

$$C_n : y^2 = x^{2n} - 1 \quad \text{with } n \in \mathbb{Z}, \quad n > 1.$$

Each C_n is a two-sheeted covering of Riemann sphere ramified at $x = \zeta^k$, $0 \leq k < 2n$, with $\zeta = e^{i\pi/n}$. By the Riemann-Hurwitz relation its genus is $g = n - 1$. It is not difficult to describe a canonical homology basis [5] III.7 and relate the periods of the holomorphic differential forms $x^k dx/y$ thanks to the automorphism $(x, y) \mapsto (\zeta x, y)$. In [1] it is outlined this argument getting

$$A + \imath B = C^{-1}D \quad \text{with} \quad c_{ij} = \zeta^{2(j-1)i} \quad \text{and} \quad d_{ij} = \sum_{k=j}^g \zeta^{(2k-1)i},$$

where $1 \leq i, j \leq g$. Let $(x_{ij})_{i,j=1}^g = A + \imath B$, then the relation $\sum c_{ij} x_{jk} = d_{ik}$, after computing the finite sum d_{ik} , leads to

$$\sum_{j=1}^g \zeta^{2ij} x_{jk} = -\zeta^{ik} \sin\left(\frac{\pi ik}{n}\right) \csc\left(\frac{\pi i}{n}\right).$$

If we replace i by $n - i$ in this formula and conjugate, we get the same equation replacing x_{jk} by the opposite of its complex conjugate, $-\bar{x}_{jk}$. Subtracting both equations one obtains $\sum_j \zeta^{2ij} (x_{jk} + \bar{x}_{jk}) = 0$, hence x_{jk} is purely

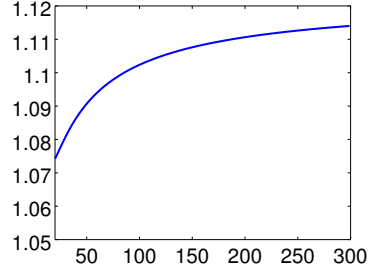
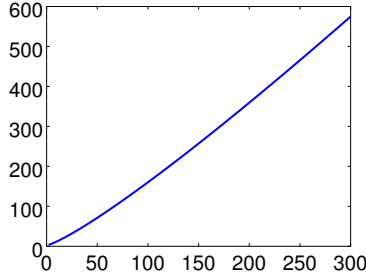
imaginary and we conclude

$$A = O \quad \text{and} \quad B = \tilde{C}^{-1} \tilde{D} \quad \text{with} \quad \tilde{c}_{ij} = \zeta^{2ij}, \quad \tilde{d}_{ij} = \imath \zeta^{ij} \sin\left(\frac{\pi ij}{n}\right) \csc\left(\frac{\pi i}{n}\right).$$

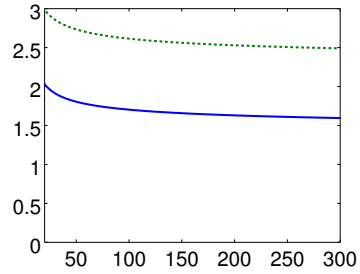
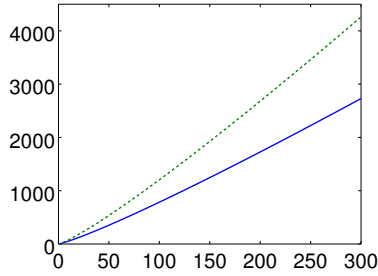
As B and B^{-1} are positive definite, their norms are their largest eigenvalues. With these ideas in mind and standard mathematical packages one can get easily M in Theorem 2.2 for large genera in this example. We have employed the following simple Octave code

```
U = pi*(1:g)'*(1:g)/(g+1);
C = exp(2*I*U);
D = -exp(I*U).*sin(U).*csc(pi*(1:g)'*ones(1,g)/(g+1));
M = eigs(imag(C\D),1) + eigs(-imag(D\C),1);
```

The following figures show the graphics of $M(g) = \|B\| + \|B^{-1}\|$ for $2 \leq g < 300$ and of $\log M(g)/\log g$ in the range $20 \leq g < 300$. Note that the latter is almost constant, it increases less than 0.04.



The comparison with the best known general bound is shown in the next figure (to the left) where they are plotted the right hand side of (1) (dashed line) and the logarithm of the bound from Theorem 2.2 (solid line) in the range $2 \leq g < 300$.



The last figure shows both bounds applied to $\log \mathcal{N}(S, S')/(g \log g)$ for $20 \leq g < 300$.

The outcome of this analysis is that for this family and these ranges our upper bound for $\mathcal{N}(S, S')$ is less than the best known bound to the power α with $\alpha \approx 2/3$. We have checked it also for $300 \leq g < 400$ and probably it also applies for any g .

Acknowledgements: I am deeply indebted to E. Valenti for the tireless patience and encouraging help.

References

- [1] M. Bernstein and N. J. A. Sloane. Some lattices obtained from Riemann surfaces. In *Extremal Riemann surfaces (San Francisco, CA, 1995)*, volume 201 of *Contemp. Math.*, pages 29–32. Amer. Math. Soc., Providence, RI, 1997.
- [2] F. Chamizo and Y. Fuertes. The number of mappings between compact Riemann surfaces. *Osaka J. Math.*, 48(3):743–748, 2011.
- [3] M. de Franchis. Un teorema sulle involuzioni irrazionali. *Rend. Circ. Mat. Palermo*, 36:368, 1913.
- [4] B. Deconinck and M. van Hoeij. Computing Riemann matrices of algebraic curves. *Phys. D*, 152/153:28–46, 2001. Advances in nonlinear mathematics and science.
- [5] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.
- [6] J. Frauendiener and C. Klein. Computational approach to compact Riemann surfaces. *Nonlinearity*, 30(1):138–172, 2017.
- [7] F. Fricker. *Einführung in die Gitterpunktlehre*, volume 73 of *Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften. Mathematische Reihe*. Birkhäuser Verlag, Basel-Boston, Mass., 1982.
- [8] Y. Fuertes and G. González Díez. On the number of coincidences of morphisms between closed Riemann surfaces. *Publ. Mat.*, 37(2):339–353, 1993.

- [9] E. Gironde and G. González-Diez. *Introduction to compact Riemann surfaces and dessins d'enfants*, volume 79 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2012.
- [10] B. H. Gross. On the periods of abelian integrals and a formula of Chowla and Selberg. *Invent. Math.*, 45(2):193–211, 1978. With an appendix by D. E. Rohrlich.
- [11] A. Howard and A. J. Sommese. On the theorem of de Franchis. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 10(3):429–436, 1983.
- [12] L. K. Hua. *Introduction to number theory*. Springer-Verlag, Berlin, 1982. Translated from the Chinese by P. Shiu.
- [13] A. Hurwitz. Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.
- [14] E. Kani. Bounds on the number of nonrational subfields of a function field. *Invent. Math.*, 85(1):185–198, 1986.
- [15] S. Lang. *Introduction to algebraic and abelian functions*, volume 89 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1982.
- [16] A. M. Macbeath. On a theorem of Hurwitz. *Proc. Glasgow Math. Assoc.*, 5:90–96 (1961), 1961.
- [17] H. H. Martens. Observations on morphisms of closed Riemann surfaces. II. *Bull. London Math. Soc.*, 20(3):253–254, 1988.
- [18] A. D. Mednykh and I. A. Mednykh. On the equivalence classes of holomorphic mappings of a Riemann surface of genus three onto a Riemann surface of genus two. *Sibirsk. Mat. Zh.*, 57(6):1346–1360, 2016.
- [19] J. C. Naranjo and G. P. Pirola. Bounds of the number of rational maps between varieties of general type. *Amer. J. Math.*, 129(6):1689–1709, 2007.
- [20] M. Tanabe. A bound for the theorem of de Franchis. *Proc. Amer. Math. Soc.*, 127(8):2289–2295, 1999.
- [21] A. Weil. Sur les périodes des intégrales abéliennes. *Comm. Pure Appl. Math.*, 29(6):813–819, 1976.