Weighted inequalities in Fluid Mechanics and General Relativity: Carleman estimates and cusped travelling waves

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Abstract

The results in this thesis can be divided into two blocks, respectively devoted to the study of certain aspects of geometric evolution partial differential equations (PDEs) and travelling waves with singularities. While the problems in the first block appear naturally in the context of General Relativity, in the second part we will deal with singular solutions to a model of shallow water waves and prove a recent conjecture in Fluid Mechanics. The underlying main topic is then the study of evolution PDEs with singular behavior, understood in a broad sense. In this fashion, the questions studied here can be related through some analytical techniques: the study of nonlocal operators, the use of weighted inequalities and the treatment of singularities of very different nature.

The first block revolves around geometric PDEs linked to General Relativity. First, we shall see how to employ Carleman estimates techniques to show some boundary observability properties for wave equations with strong geometric motivation. Roughly speaking, observability is tantamount to quantitative uniqueness; it defines a stronger notion of unique continuation in which the prescribed Cauchy data controls a meaningful energy norm of the solution of the equation, often an energy for which the PDE is well-posed. The key feature of the equations studied in the first part of the thesis is that they become very singular at the boundary of a set. Typically, they can be written as a regular part plus a strongly singular potential depending on the distance to the boundary. The most interesting case occurs when the potential scales exactly as it does the Laplace–Beltrami operator (in which case it is called critically singular) and when it blows up in a manifold of codimension one. The main objective in this part is the development of new Carleman-type inequalities adapted to the geometry of the singularity. These estimates thus provide a better understanding of the observability and uniqueness properties of evolution PDEs with critically singular potentials.

Somewhat related to the above, an interesting problem in conformal geometry is to show a Lorentzian analog of the relationship between fractional Laplacians and conformally covariant operators defined on the boundary of a Riemannian conformally compact Einstein manifold. Here we will see that the fractional powers of the wave operator (in the flat space) can be constructed as Dirichlet-to-Neumann maps associated with certain wave equations in anti-de Sitter backgrounds. This construction is then not only interesting from a pure mathematical point of view, but also because the deep connection with some questions in theoretical physics. In fact, as recalled later on, anti-de Sitter spaces play a major role in cosmology due to their connection with the celebrated AdS/CFT conjecture in string theory. On the other hand, the relationship with the above evolution PDEs is clear as the very construction of these operators boils down to the study of a

boundary value problem consisting in a massive wave equation with a critically singular potential and some boundary data at (conformal) infinity.

The other central part of the thesis is devoted to the study of fluid mechanics problems and, more specifically, the analysis of a nonlocal dispersive equation known as Whitham equation. This is a model of surface water waves in one dimension with rich mathematical properties, including the existence of travelling and solitary waves as well as wave breaking and singularities. In relation to the latter, our interest in Whitham's equation comes from a conjecture on the highest cusped travelling wave solution. Similarly to the famous statement for Stokes water waves, it was conjectured that Whitham waves of extreme form, namely the ones of greatest height, have a convex profile between consecutive stagnation points.

In the final part of the thesis we prove this conjecture by exploiting some unexplored structural properties of the equation and a very careful asymptotic analysis close to the singularity. In addition, the conjecture also suggested that near the crests the highest cusped waves satisfied a certain asymptotic expansion, which is showed by constructing a rather complicated approximate solution with the desired properties. Besides the singular behavior, the key feature of the Whitham equation is precisely the linear term that is governed by a smoothing, nonlocal, nonhomogeneous operator that makes the equation to be weakly dispersive.

Resumen y Conclusiones

Los resultados de esta tesis están divididos en dos bloques dedicados, respectivamente, al estudio de ciertos aspectos relativos a ecuaciones de evolución geométricas en derivadas parciales y de ondas viajeras con singularidades. Mientras que los problemas del primer bloque aparecen naturalmente en el contexto de la Relatividad General, en la segunda parte vamos a tratar con soluciones singulares a un modelo de ondas de agua superficiales y probaremos una conjetura reciente en Mecánica de Fluidos. Los dos bloques están entonces conectados a través de un tema central que se puede describir como ecuaciones de evolución con comportamiento singular, entendido de manera amplia. De este modo, las cuestiones estudiadas aquí se pueden relacionar mediante algunas técnicas analíticas: el estudio de operadores no locales, el uso de desigualdades con pesos y el tratamiento de singularidades de naturaleza muy diferente.

El primer bloque trata ecuaciones en derivadas parciales que están relacionadas con la Relatividad General. En primer lugar veremos como emplear técnicas de estimaciones de Carleman para demostrar propiedades de observabilidad de frontera para ecuaciones de onda con una fuerte motivación geométrica. En términos generales, observabilidad es sinónimo de unicidad cuantitativa; define una noción de continuación única en la que los datos de Cauchy prescritos controlan una energía significativa de la solución, a menudo una para la cual la ecuación define un problema bien planteado. La principal característica de las ecuaciones estudiadas en la primera parte de la tesis es que son muy singulares en el borde de un conjunto. Típicamente, estas se pueden escribir como una parte regular más un potencial fuertemente singular que depende de la distancia a la frontera. El caso más interesante ocurre cuando el potencial escala exactamente igual que el operador de Laplace–Beltrami y cuando diverge en una variedad de codimensión uno, en cuyo caso se denomina críticamente singular. El principal objetivo en esta parte es desarrollar nuevas estimaciones de tipo Carleman adaptadas a la geometría de la singularidad. Estas estimaciones nos proveen así de un mejor entendimiento de las propiedades de observabilidad de frontera de ecuaciones de evolución con potenciales críticamente singulares.

Relacionado con lo anterior, un problema interesante en geometría conforme es probar un análogo Lorentziano de la relación entre Laplacianos fraccionarios y operadores covariantes conformes definidos en el borde de una variedad Riemanniana conforme compacta de tipo Einstein. Aquí veremos que las potencias fraccionarias del operador de ondas estándar (en el espacio plano) se pueden construir como operadores de Dirichlet-Neumann asociados con ciertas ecuaciones de onda en geometrías de tipo anti-de Sitter. Esta construcción es interesante no solo desde un punto de vista puramente matemático, sino que también por la profunda conexión con algunas preguntas de física teórica. De hecho, como recordaremos mas tarde, los espacios de tipo anti-de Sitter juegan un papel fundamental en cosmología debido a su conexión con la celebrada conjetura AdS/CFT de teoría de cuerdas. Por otra parte, la conexión con las ecuaciones de antes es clara viendo que la construcción de estos operadores se reduce a manejar un problema de frontera que consiste en una ecuación de ondas con masa en el que hay un potencial con una singularidad crítica y datos de borde en el infinito (conforme).

La otra parte central de la tesis está dedicada al estudio de Mecánica de Fluidos y, más específicamente, al análisis de un modelo de ecuación dispersiva no local conocida como ecuación de Whitham. Este es un modelo de ondas superficiales en una dimensión con propiedades matemáticas ricas, incluyendo la existencia de ondas viajeras y solitarias, así como ruptura de ondas y singularidades. En relación con esto último, nuestro interés en la ecuación de Whitham proviene de una conjetura sobra la onda de altura máxima con cúspide. De manera similar al famoso caso de las ondas de agua de Stokes, se conjeturó que las ondas de Whitham de forma extrema, esto es la que alcanzan máxima altura, tienen un perfil convexo entre picos consecutivos.

En la parte final de la tesis probaremos que esta conjetura es cierta utilizando algunas propiedades estructurales de la ecuación y un análisis asintótico cerca de la singularidad muy cuidadoso. Por otra parte, la conjetura sugería que cerca de las crestas de las ondas de altura máxima se satisfacía una cierta expansión asintótica, la cual probamos construyendo una solución aproximada bastante complicada con las propiedades deseadas. Además del comportamiento singular, una característica fundamental de la ecuación de Whitham es precisamente el término lineal que viene dado por un operador que suaviza, es no local y además inhomogéneo, que hace que la ecuacón sea débilmente dispersiva.

Contents

Introduction

Here we give a brief summary of the results contained in this thesis and how they are structured in the three following chapters.

In the first block of results we have two chapters that treat different aspects of wave equations with coefficients that are singular at the boundary of a manifold. Carleman estimates and Boundary Observability, on the one hand, and Dirichlet–to–Neumann maps and fractional wave operators, on the other, are the central objects of study in this part of the thesis strongly motivated by the study of geometric wave equations in General Relativity.

The third chapter corresponds to the block devoted to Fluid Mechanics. In this part we will be focused on Whitham's model of shallow water waves and a recent conjecture on waves of greatest height. Here we will develop a new strategy to construct singular solutions to equations with very low regularity that, in particular, shows the existence of a highest, cusped, periodic travelling wave solution to the Whitham equation that is convex between consecutive crests.

Carleman estimates and Boundary Observability for waves with critically singular potentials

The breadth of applications of Carleman estimates to a wide range of PDEs [26, 67] is remarkable. Examples include unique continuation, control theory, inverse problems, as well as showing the absence of embedded eigenvalues in the continuous spectrum of Schrödinger operators. In Chapter 1 we derive a novel family of Carleman-type estimates for the wave operator associated to the following initial boundary value problem in a cylindrical spacetime domain:

$$\begin{aligned} \Box_\kappa u := \Box u + \frac{\kappa(1-\kappa)}{(1-|x|)^2} u = 0 \quad & \text{in } (-T,T)\times B_1\,, \\ u(0,x) = u_0(x), \qquad & \partial_t u(0,x) = u_1(x)\,. \end{aligned}$$

where $\Box := -\partial_{tt} + \Delta$ denotes the flat wave operator, the spatial domain is the unit ball B_1 of $\mathbb{R}^n$, and the constant parameter $\kappa \in \mathbb{R}$ measures the strength of the potential. In spherical coordinates, the equation simply reads as

$$-\partial_{tt} u + \partial_{rr} u + \frac{n-1}{r}\partial_r u + \frac{\kappa(1-\kappa)}{(1-r)^2} u + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} u = 0\,,$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on the unit sphere. The potential is then critically singular at the boundary of the ball $\partial B_1 := \{r = 1\}$, where, according to the classical theory of Frobenius for ODEs, the characteristic exponents of this equation are κ and $1-\kappa$. Therefore, if κ is not a half-integer (which ensures that logarithmic branches will not appear), solutions to the equation are expected to behave either like $(1-r)^\kappa$ or $(1-r)^{1-\kappa}$ as $r \nearrow 1$.

As one can easily infer by plugging these powers in the energy associated with this equation,

$$\int_{B_1} \Big[(\partial_t u)^2 + (1-r)^{2\kappa}\big|\nabla_x[(1-r)^{-\kappa}u]\big|^2\Big]$$

the equation admits exactly one finite-energy solution when $\kappa \leqslant -\frac{1}{2}$, no finite-energy solutions when $\kappa \geqslant \frac{1}{2}$, and infinitely many finite-energy solutions when $-\frac{1}{2} < \kappa < \frac{1}{2}$ (see [75] for the details on the well-posedness theory). In this range one must impose a (Dirichlet, Neumann or Robin) boundary condition on ∂B_1. This is constructed in terms of the natural Dirichlet and Neumann traces, which now include weights and are defined as the limits

$$\mathcal{D}_\kappa u := (1-r)^{-\kappa} u|_{r=1}\,, \qquad \mathcal{N}_\kappa u := (1-r)^{2\kappa}\partial_r\big[(1-r)^{-\kappa}u\big]|_{r=1}\,.$$

Notice that singular weights depending on κ appear everywhere in this problem, and that all the associated quantities reduce to the standard ones in the absence of the singular potential, i.e., when $\kappa = 0$.

The dispersive properties of wave equations with potentials that diverge as an inverse square at one point [6, 14] or an a (timelike) hypersurface [3] have been thoroughly studied, as critically singular potentials are notoriously difficult to analyze. In general, one would not expect Carleman estimates to behave well with singular potentials such as $\kappa(1-\kappa)(1-r)^{-2}$. Since the singularity in the potential scales just as $\Box$, there is no hope in absorbing it into the estimates by means of a perturbative argument. Indeed, Carleman estimates generally assume [68, 25] that the potential is at least in $L^{(n+1)/2}_{\text{loc}}$, but this condition is not satisfied here.

A setting which is closely related to the one described above is that of linear wave equations on asymptotically anti-de Sitter spacetimes (AdS). These equations are conformally equivalent to analogues of our model Cauchy problem on curved backgrounds. In the next section we will elaborate on linear waves on AdS backgrounds from a more geometric perspective. Here it is worth mentioning that they have attracted considerable attention in the recent years due to their connection to cosmology, see e.g. [3, 32, 33, 40, 75] and the references therein.

Carleman estimates for linear waves were established in this asymptotically AdS setting in [40, 41], for the purposes of studying their unique continuation properties from the conformal boundary. In particular, these estimates capture the natural Dirichlet and Neumann data (i.e., the analogues of the $\mathcal{D}_\kappa$ and $\mathcal{N}_\kappa$

defined before). On the other hand, the Carleman estimates in [40, 41] are local in nature and apply only to a neighborhood of the conformal boundary, and they do not capture the naturally associated H^1-energy. As a result, these estimates would not translate into corresponding observability results.

The main result of Chapter 1 is a novel family of Carleman inequalities for the wave operator $\Box_\kappa$ with our critically singular potential that capture both the natural boundary weights and the natural H^1-energy described above. To the best of our knowledge, these are the first available Carleman estimates for an operator with such a strongly singular potential that also captures the natural boundary data and energy. Moreover, our estimates hold in all spatial dimensions, except for $n = 2$. An informal statement of the theorem is the following:

Theorem 1. *Let B_1 denote the unit ball in $\mathbb{R}^n$, with $n \neq 2$, and fix $-\frac{1}{2} < \kappa < 0$. Moreover, let $u : (-T, T) \times B_1 \to \mathbb{R}$ be a smooth function, and assume that u "has the boundary asymptotics of a sufficiently regular, finite energy solution to $\Box_\kappa u = 0$". In particular, u has zero Dirichlet data, $\mathcal{D}_\kappa u = 0$, and the Neumann trace $\mathcal{N}_\kappa u$ of u exists and is finite. Moreover, suppose there exists $\delta > 0$ such that $u(t) = 0$ for all $T - \delta \leqslant |t| < T$. Then, for $\lambda \gg 1$ large enough, independently of u, the following inequality holds:*

$$\lambda \int_{(-T,T)\times\partial B_1} e^{2\lambda f} (\mathcal{N}_\kappa u)^2 + \int_{(-T,T)\times B_1} e^{2\lambda f} (\Box_\kappa u)^2 \gtrsim \lambda \int_{(-T,T)\times B_1} e^{2\lambda f} \left[(\partial_t u)^2 + y^{2\kappa} \left| \nabla_x (y^{-\kappa} u) \right|^2 + |\kappa| \lambda^2 y^{6\kappa - 1} u^2 \right],$$

where $y(r) := 1 - r$ denotes the distance to ∂B_1 and f is the weight

$$f(t, r) := -\frac{1}{1 + 2\kappa} y(r)^{1+2\kappa} - ct^2 ,$$

with a suitably chosen positive constant c.

The main ingredients of the proof of Theorem 1 are presented at the beginning of Chapter 1. Here we would like to point out that one of these ideas is a generalization of the classical Morawetz multiplier estimate for the standard wave equation. This estimate was originally developed in [58] in order to establish integral decay properties for waves in 3 spatial dimensions. Analogous estimates hold in higher dimensions as well; see [66], as well as [60] and references therein for more recent extensions of Morawetz estimates. In fact, at the heart of the proof of Theorem 1 lies a generalization of the classical Morawetz estimate from $\Box$ to $\Box_\kappa$ that builds upon the use of "twisted" derivatives (a particular form of weighted derivative), in the place of the usual derivatives. This produces a number of additional singular terms, which we must arrange so that they have the required positivity. Finally, our generalized Morawetz bound is encapsulated within a larger Carleman estimate, which is proved using geometric multiplier arguments (see, e.g., [1, 40, 41, 45, 52]).

One particular consequence of Theorem 1 is the boundary observability of linear waves involving a critically singular potential. Roughly speaking, a boundary observability estimate shows that the energy of a wave confined in a bounded region can be estimated quantitatively only by measuring its boundary data over a large enough time interval. In the following theorem we show that to control the natural H^1-energy associated to our Cauchy problem in a cylindrical spacetime domain, it is enough to "observe" the Neumann data $\mathcal{N}_\kappa u$ during a sufficiently large time:

Theorem 2. *Let $y(r)$, B_1, n, and κ be as in the previous theorem. Moreover, let u be a smooth and real-valued solution of the wave equation*

$$\Box_\kappa u = X \cdot \nabla u + V u$$

on the cylinder $(-T, T) \times B_1$, where X is a bounded (spacetime) vector field, and where V is a bounded scalar potential. Furthermore, suppose u has zero Dirichlet data, $\mathcal{D}_\kappa u = 0$, and that it "has the boundary asymptotics of a sufficiently regular, finite energy solution of the above equation", so that $\mathcal{N}_\kappa u$ exists and is finite. Then, for sufficiently large T, the following observability estimate holds for u:

$$\int_{(-T,T)\times\partial B_1} (\mathcal{N}_\kappa u)^2 \gtrsim \int_{\{0\}\times B_1} \left[(\partial_t u)^2 + |y^\kappa \nabla_x (y^{-\kappa} u)|^2 + u^2 \right] .$$

Let us conclude this introduction to the first chapter by mentioning that analogous results can be proved for parabolic equations featuring our inverse square potential $\kappa(1-\kappa)y^{-2}$. For the corresponding evolution operators one can prove similar Carleman-type inequalities with sharp weights that capture the natural boundary conditions and the natural L^2-energy. As in the case of waves, these estimates also translate into boundary observability results, but here the geometric motivation stems from the study of heat-like equations in asymptotically hyperbolic spaces. The latter are Riemannian analogues of the aforementioned asymptotically AdS backgrounds, that is compact manifolds solutions to the Einstein equations with negative sectional curvature.

In the above context, the interest in evolution equations with critically singular potentials has in creased over the last decades due to their connection with a wide range of phenomena including combustion theory [9] and quantum cosmology [10] In the mathematical side, they have also been widely studied in the context of PDE [13], specially in control theory, where one can highlight among other results the proof of new sharp Hardy-type inequalities that are central in the well-posedness theory for these equations [73].

Fractional wave operators

It is a classical result in potential theory that the Dirichlet-to-Neumann map of the harmonic extension problem in the upper half-space is given by the square root of the Laplacian. This relation can be generalized to encompass all fractional powers of the Laplacian, and has recently made a major impact in the theory of nonlocal elliptic equations [15].

Indeed, this relation connects the multiplier $\widehat{(-\Delta)^{1/2-\kappa} f}(\xi) := |\xi|^{1-2\kappa}\widehat{f}(\xi)$ on $\mathbb{R}^n$, which is a nonlocal operator of the form

$$(-\Delta)^{1/2-\kappa} f(x) = c_{n,\kappa} \int_{\mathbb{R}^n} \frac{f(x) - f(x')}{|x - x'|^{n+1-2\kappa}}\, dx' \,, \qquad \kappa \in (-\tfrac{1}{2}, \tfrac{1}{2})\,,\ c_{n,\kappa} \in \mathbb{R}\,,$$

with a local elliptic equation in $n + 1$ variables. Specifically, given a function f on $\mathbb{R}^n$, let us consider the function u on $\mathbb{R}^n \times \mathbb{R}_+$ that solves the boundary value problem

$$\begin{aligned} &\Delta_x u + \partial_{yy} u + \frac{\kappa(1-\kappa)}{y^2} u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+\,, \\ &\lim_{y \searrow 0} y^{-\kappa} u(x, 0) = f(x)\,, \qquad \lim_{y\to\infty} u(x, y) = 0\,; \end{aligned}$$

it was shown in [15] that

$$(-\Delta)^{1/2-\kappa} f(x) = c_\kappa \lim_{y\searrow 0} y^\kappa \partial_y (y^{-\kappa} u(x, y))\,.$$

This reduces to the ordinary derivative $\partial_y u(x, 0)$ in the case of the square root of the Laplacian ($\kappa = \frac{1}{2}$), and in fact for all values of α it provides the natural Neumann datum associated with the above elliptic operator $y^{-\kappa}\nabla(y^{2\kappa}\nabla(y^{-\kappa}\cdot))$. Note that a generalized formula for higher powers $\alpha \in (0, \frac{n}{2})$ has been established in [18, 19].

In Chapter 2 we show an analogous relationship for the fractional powers of the wave operator, by which we will always mean a fractional power of the usual wave operator $-\Box := \partial_{tt} - \Delta$ and not an evolution equation driven by a fractional power of the Laplacian or, more generally, a generator of a suitable semigroup (for fine information on the latter in various contexts, cf. [47, 64, 65]). Note that this cannot be seen as an analytic continuation of the elliptic case and that in fact several nontrivial choices need to be made, starting with the very definition of the fractional wave operator. This is due to the fact that the symbol of the wave operator, $|\xi|^2 - \tau^2$, is not positive definite, so one cannot immediately define $(-\Box)^\alpha$ through this quantity to the power of α, and can also be seen in the integral formula of the fractional Laplacian, since formally replacing the squared Euclidean distance $|x - x'|^2$ by its Minkowskian counterpart, $|x - x'|^2 - (t - t')^2$, in the denominator leads to an integral that is too singular to be well defined.

In particular, the fractional wave operator $(-\Box)^\alpha$ is defined for all noninteger α as the multiplier $\widehat{(-\Box)^\alpha f}(\tau, \xi) := \sigma_\alpha(\tau, \xi)\, \widehat{f}(\tau, \xi)$, where the symbol σ_α is defined as,

$$\sigma_\alpha(\tau, \xi) := \lim_{\epsilon \searrow 0} \left(|\xi|^2 - (\tau - i\epsilon)^2 \right)^\alpha ,$$

and where we have chosen the principal branch of the complex logarithm. It should be noted that this is in fact a natural definition of $(-\Box)^\alpha$ in the sense that $\sigma_\alpha(\tau, \xi)$ raised to the power of $1/\alpha$ gives $|\xi|^2 - \tau^2$, i.e., the symbol of $-\Box$.

A simplified version of the main result of Chapter 2 is presented below. As suggested by its elliptic analog, one can show that the fractional wave operator is given by the Dirichlet-to-Neumann map operator associated to a wave equation with an inverse square potential that blows up on all the boundary of the half-space:

Theorem 3. *Let $u(t, x, y)$ be the solution of the wave equation*

$$-\partial_{tt} u + \Delta_x u + \partial_{yy} u + \frac{\kappa(1 - \kappa)}{y^2} u = 0$$

in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+$ with Dirichlet boundary condition $\lim_{y \searrow 0} y^{-\kappa} u = f \in C_0^\infty$ and trivial initial data at $-\infty$: $u(-\infty, x, y) = u_t(-\infty, x, y) = 0$. Assume moreover that the parameter $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then, up to some numerical constant,

$$(-\Box)^{1/2 - \kappa} f(t, x) = \lim_{y \searrow 0} y^{2\kappa}\, \partial_y \big(y^{-\kappa} u(t, x, y) \big) .$$

Notice that the above extension problem on the half-space can be seen as a wave equation localized near the conformal boundary of an asymptotically AdS manifold. The latter, in turn, is a curved analog of our model problem on the cylinder as described in Chapter 1.

In this way, a second motivation in this part of the thesis is of a more geometric and physical nature. Specifically, in Chapter 2 we show that the fractional wave operator, as considered in the theory of hypersingular integrals, does arise in gravitational physics as the Dirichlet-to-Neumann map of a certain Klein–Gordon equation in AdS spacetimes. As earlier mentioned, hyperbolic equations for fields in AdS backgrounds with nontrivial data on their conformal boundaries have attracted much attention over the last two decades, especially in connection with the celebrated AdS/CFT correspondence in string theory [56, 79]. Indeed, this conjectural relation establishes a connection between conformal field theories in n dimensions and gravity fields on an $(n+1)$-dimensional spacetime of AdS type, to the effect that correlation functions in conformal field theory are given by the asymptotic behavior at infinity of the supergravity action. Mathematically, this involves describing the solution to the gravitational field equations in $(n+1)$ dimensions (which, in the simplest case of a scalar field reduces to the Klein–Gordon equation) in terms of a conformal field, which plays the role of the boundary data imposed on the (timelike) conformal "infinity".

Nonlocal dispersive equations: Whitham's model

Chapter 3 is devoted to the study of a model of water waves which features both dispersive and nonlinear effects, the so called Whitham equation [77]. This is a nonlocal shallow water wave model in one space dimension with a quadratic nonlinearity that reads as

$$\partial_t v + \partial_x (Lv + v^2) = 0\,,$$

where L is the Fourier multiplier defined in terms of the full dispersion relation for gravity water waves $m(\xi) := (\tanh \xi/\xi)^{1/2}$,

$$\widehat{Lf}(\xi) := m(\xi)\,\widehat{f}(\xi)\,.$$

Whitham proposed this equation in 1967 as an alternative to the well-known KdV equation, as the latter does not accurately describe the dynamics of short waves.

The key feature of Whitham's equation is its very weak dispersion, which is due to the fact that the symbol $m(\xi)$ has a completely different behavior for large frequencies than equations such as KdV, whose corresponding Fourier multiplier is precisely defined by the second-order Taylor series of $m(\xi)$. This very weak dispersion allows Whitham's equation to exhibit both smooth periodic and solitary solutions on the one hand [27, 29, 30], and singular solutions on the other.

For large frequencies the Whitham equation can be seen as a weakly dispersive perturbation of the Burgers equation in which the dispersive term acts like a smoothing fractional operator. In that regime the multiplier becomes homogeneous and essentially behaves as $|\xi|^{-1/2}$, so it is roughly like the inverse of the fractional Laplacian operator $(-\Delta)^{1/4}$ that we presented in connection to the results of Chapter 2.

Singular solutions for the Whitham equation appear as wave breaking [21, 43] (i.e., as bounded solutions whose derivative blows up in finite time) and as traveling waves with sharp crests, which are only of $C^{1/2}$ regularity. Here we will be concerned with the latter, whose existence was conjectured some forty years ago by Whitham [77] and established by Ehrnström and Wahlén [31] just recently. Interestingly, if one replaces the Whitham equation by a related fully dispersive model that contains both branches of the full Euler dispersion relation instead of just one, non-smooth traveling waves have been found too [28], but the solutions are in C^α for all $\alpha < 1$ (and not in C^1).

Let us elaborate on the existence of sharp crests. With the ansatz $v(x,t) := \varphi(x - \mu t)$, the study of traveling waves for the Whitham equation reduces to the analysis of the equation

$$L\varphi - \mu\varphi + \varphi^2 = 0\,,$$

where the positive constant μ represents the speed of the traveling wave. Whitham himself conjectured [78, p. 479] that the equation should admit traveling waves with a sharp crest, and provided a heuristic argument suggesting that the crest should be cusped with $\varphi(x) \sim \frac{\mu}{2} - c|x|^{1/2}$.

Ehrnström and Wahlén's proof of this conjecture [31] is based on a remarkable global bifurcation argument, where cusped solutions of any period were shown to exist by continuing off a local branch of small amplitude periodic traveling waves bifurcating from the zero state. These solutions were shown to be smooth away from their highest point (the crest) and behave like $|x|^{1/2}$ near the crest, so their sharp Hölder regularity is $C^{1/2}$. These authors also conjectured that, just as in the celebrated case of the highest traveling water waves (which present a corner of 120 degrees) [2, 61], Whitham's highest cusped waves must be convex between consecutive crests, where they have a very specific asymptotic behavior.

Our objective in Chapter 3 is to prove the following theorem which shows the previous conjecture to be true (see Figure 1 below):

Theorem 4. *The 2π-periodic highest cusped traveling wave $\varphi \in C^{1/2}(\mathbb{T})$ of the Whitham equation is a convex function and behaves asymptotically as*

$$\varphi(x) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x|^{1/2} + O(|x|^{1+\eta})$$

for some $\eta > 0$. Furthermore, φ is even and strictly decreasing on $[0, \pi]$.

At this stage it is worth discussing why the strategy of the proof of Theorem 4 is so different from the celebrated proof of the convexity of the highest Stokes waves. In short, the reason is that, although Nekrasov's equation for the interface reduces the problem to the analysis of a nonlinear, nonlocal equation similar to Whitham's, the actual proof of the conjecture due to Plotnikov and Toland [61]

$$\varphi(x) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}|x|} + o(|x|)$$

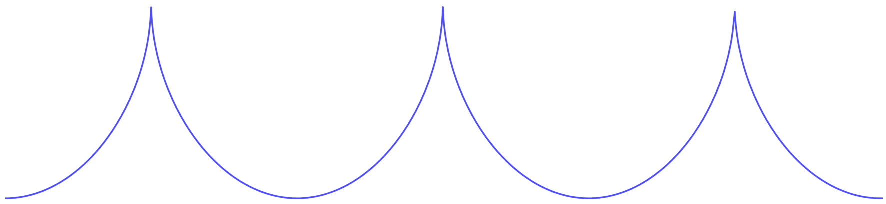

Figure 1: A representation of a travelling wave of greatest height solution to the Whitham equation. It corresponds to a 2π-periodic solution with speed $\mu \approx 0.77$, crested at points $x_n = 2\pi n$, $n \in \mathbb{Z}$, where $\varphi(x_n) = \mu/2$. Away from these singularities the solution has been shown to be smooth. In Chapter 3 we prove both the conjectured asymptotic behavior near the cusps as well as the convexity of the profile.

hinges on the equivalent formulation of the problem in terms of a harmonic function on the half-strip $(-\infty, 0) \times (-\pi, \pi)$ satisfying certain Dirichlet and Neumann boundary conditions. In a tour de force of complex analysis, this overdetermined boundary condition is shown to imply that the boundary conditions of the above harmonic function can be written in terms of a holomorphic function satisfying a certain ODE in the complex plane. Via the Poisson kernel, this leads to writing the derivative of the function $\theta(x)$ that describes the water interface as

$$\theta'(x) = \int_0^\infty \frac{\Phi'(y)\,\sinh y}{\cosh y - \cos x}\,dy\,,$$

with $\Phi'(y) > 0$. Hence $\theta'(x) > 0$, and this automatically implies that the interface is convex. In our case, the problem does not admit a local description and is not amenable to the use of complex-analytic methods, so one needs to work directly with the Whitham equation using real-variable techniques.

Let us emphasize the significant difference between showing the existence of these waves of extreme form and proving the convexity of the profiles, as the latter problem should not seen as a mere technical gap that one can easily close. In fact, while the existence result due to Amick, Fraenkel, Plotnikov and Toland was proved in the mid 80's, the conjectured convexity avoid a proof for almost 20 years and remained open until the last two authors combined several ideas coming from the analytic bifurcation theory and complex analysis.

The proof of Theorem 4 is the result of a concoction of a very careful asymptotic analysis near the singularities together with some computer assisted estimates. In Chapter 3 we detail the strategy to tackle down the problem. Undoubtedly, one crucial piece is the use of some special functions known as Clausen functions,

$$C_z(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^z}\,, \quad S_z(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^z}\,,$$

in terms of which we construct a rather complicated approximate solution to the equation. Moreover, it will be clear from the proof that these functions are well adapted to construct approximate solutions to PDEs in C^α-regularity for $0 < \alpha < 1$.

Hence our approach is expected to be useful in the construction of singular solutions in other low-regularity situations. To offer some perspective as to why the proof is so demanding without getting bogged down in technicalities, suffice it to say that this is the first computer-assisted proof of the existence of truly low-regularity (e.g., continuous but not C^1) solutions of any (ordinary or partial, even local) differential equation. See [16, 23] for computer-assisted proofs of periodic solutions or KAM tori of ill-posed PDE.

Final remarks and organization of the thesis

In this introduction we have presented the state of the art of the problems considered in the two central blocks of the thesis. We have focused mainly on some of the difficulties in the proofs of our results and discussed the interest, from both the mathematical and physical point of view, in these PDE problems which are motivated by the study of singular waves in General Relativity and Fluid Mechanics. Also, we have discussed the novelties in our work, and why the available results and the standard PDE methods do not provide with answers to the questions here considered.

For the sake of clarity, we will start each chapter with a brief presentation of the key ideas behind the proofs of the theorems that conform the central results of this thesis. As for the organization of the thesis, the three following chapters appear in the same order as presented in this introduction: in Chapter 1 we introduce the model wave equations with singular coefficients that blow up critically in all the boundary of a cylindrical spacetime domain and then derive Carleman inequalities for the associated wave operators. Using some standard arguments, boundary observability properties are showed as a consequence of the previous estimates. More geometrically, in Chapter 2 we work on closely related equations with the same kind of singularities. Introducing the notion of fractional wave operator, here we prove a Lorentzian analog of the relationship between conformally covariant operators and Dirichlet-to-Neumman maps that holds for conformally compact manifold. Finally, Chapter 3 settles on the question of whether periodic travelling waves of greatest height solutions to the Whitham equation are, as in the celebrated case of the Stokes water waves, convex between consecutive peaks of $C^{1/2}$-Hölder regularity. We will see that the method of the proof constitutes a new strategy to construct possibly singular solutions to other low-regularity PDE problems.

Each result contained in this thesis corresponds to a paper communicated in a scientific journals or to a preprint, as follows:

Chapter 1:

1. A. Enciso, A. Shao and B. Vergara, Carleman estimates with sharp weights and boundary observability for wave equations with critically singular potentials, *arXiv preprint*: `https://arxiv.org/abs/1902.00068`.

2. A. Enciso and B. Vergara, Carleman inequalities for parabolic equations with inverse square potentials and boundary observability, *In preparation.*

Chapter 2:

1. A. Enciso, M.d.M González and B. Vergara, Fractional powers of the wave operator via Dirichlet-to-Neumann maps in anti-de Sitter spaces, *Journal of Functional Analysis*, 273, no. 6, 2144–2166.

Chapter 3:

1. A. Enciso, J. Gómez-Serrano and B. Vergara, Convexity of Whithams highest cusped wave, *arXiv preprint*: `https://arxiv.org/abs/1810.10935`.

Chapter 1

Sharp Carleman estimates for waves with critically singular potentials and boundary observability

Preliminaries

Our objective in this chapter is to derive Carleman estimates for wave operators with critically singular potentials, that is, with potentials that scale like the principal part of the operator. More specifically, we are interested in the case of potentials that diverge as an inverse square on a convex hypersurface. For this purpose, here we consider the model operator

$$\Box_\kappa := \Box + \frac{\kappa(1-\kappa)}{(1-|x|)^2}\,, \tag{1.0.1}$$

where $\Box := -\partial_{tt} + \Delta$ denotes the standard wave operator on $\mathbb{R} \times B_1$, with B_1 the unit ball of $\mathbb{R}^n$, and $\kappa \in \mathbb{R}$ is the strength parameter henceforth assumed to be in the range

$$-\frac{1}{2} < \kappa < \frac{1}{2}\,. \tag{1.0.2}$$

As emphasized in the introduction, the Carleman estimates that we will derive next are sharp, in that the weights that appear capture both the optimal decay rate of the solutions near the boundary, as well as the natural energy that appears in the well-posedness theory for the equation

$$\Box_\kappa u = 0 \tag{1.0.3}$$

on the cylinder $(-T, T) \times B_1$ (or the equation (1.4.1), more generally). As we will see, this property is not only desirable but also essential for applications such as boundary observability.

In the case of one spatial dimension, the observability and controllability of wave equations with critically singular potentials has also received considerable attention in the guise of the degenerate wave equation

$$\partial_{tt} v - \partial_z (z^\alpha \partial_z v) = 0\,,$$

where the variable z takes values in the positive half-line and the parameter α ranges over the interval $(0, 1)$ (see [39] and the references therein). Indeed, it is not difficult to show that one can relate equations in this form with the operator $\Box_\kappa$ in one dimension through a suitable change of variables, with the parameter κ being now some function of the power α. The methods employed in those references, which rely on the spectral analysis of a one-dimensional Bessel-type operator, provide a very precise controllability result.

On the other hand, no related Carleman estimates that are applicable to observability results have been found. This manifests itself in two important limitations: firstly, the available inequalities are not robust under perturbations on the coefficients of the equation, and secondly, the method of proof cannot be extended to higher-dimensional situations. Recent results for different notions of observability for parabolic equations with inverse square potentials, which are based on Carleman and multiplier methods, can be found, e.g., in [12, 71]. Related questions for wave equations with singularities all over the boundary have been presented as very challenging in the open problems section of [12]. As stressed there, the boundary singularity makes the multiplier approach extremely tricky.

A few remarks about the informal Theorem 1, which are also applicable to the main Theorem 1.3.1, are given now in order:

Remark 1.0.1.

i) To begin with, notice that our results hold in the range $-\frac{1}{2} < \kappa < 0$ of the strength parameter. This is imposed for several reasons: first, a restriction to the values (1.0.2) is needed, as this is the range for which a robust well-posedness theory exists [75] for the equation (1.0.3). Moreover, the case $\kappa = 0$ is simply the standard free wave equation, for which the existence of Carleman and observability estimates is well-known. On the other hand, the aforementioned spectral results [39] in the $(1+1)$-dimensional setting suggest that the analogue of the estimate in Theorem 1 is false when $\kappa > 0$. See also the Appendix 1.A for more details on one-dimensional spectral results.

ii) About assuming the"expected boundary asymptotics" for solutions to (1.0.3) in the statement of Theorem 1, the precise formulation for u is given in Section 1.1 and is briefly justified in the discussion following Definition 1.1.2 on boundary admissible solutions.

iii) A final remark is that one can further strengthen the estimates in the informal theorem to include additional positive terms on the right-hand side that depend on n; see Theorem 1.3.1. It should be also observed that the constant c in the weight function is closely connected to the total timespan needed for an observability estimate to hold; in Theorem 1, this c depends on n, as well as on κ when $n = 3$.

We can now discuss the main ideas behind the proof of Theorem 1 (as well as the more precise Theorem 1.3.1). In particular, the proof is primarily based around three ingredients.

The first ingredient is to adopt derivative operations that are well-adapted to our operator $\Box_\kappa$. In particular, we make use of the "twisted" derivatives that were pioneered in [75]. The main observation here is that $\Box_\kappa$ can be written as

$$\Box_\kappa = -\overline{D} D + \text{l.o.t.} \,,$$

where D is the conjugated (spacetime) derivative operator,

$$D = D_{t,x} = (1 - |x|)^\kappa \nabla_{t,x} (1 - |x|)^{-\kappa} \,,$$

where $-\overline{D}$ is the (L^2-)adjoint of D, and where "l.o.t." represents lower-order terms that can be controlled by more standard means.

As a result, we can view D as the natural derivative operation for $\Box_\kappa$. For instance, the twisted H^1-energy associated with the Cauchy problem (1.0.3) is best expressed purely in terms of D (in fact, this energy is conserved for the equation $\overline{D} D u = 0$). Similarly, in our Carleman estimates of Theorem 1 and their proofs, we will always work with D-derivatives, rather than the usual derivatives, of u. This helps us to better exploit the structure of $\Box_\kappa$.

The second main ingredient in the proof of Theorem 1 is the classical Morawetz multiplier estimate for the wave equation discussed in the introduction. In this derivation we adopt the above twisted operators, so that an essential part of our work in the Section 1.3 is to obtain positivity for many additional singular terms that now appear.

Recall that in the standard Carleman-based proofs of observability for wave equations, one employs Carleman weights of the form

$$f_*(t, x) = |x|^2 - ct^2 \,, \qquad 0 < c < 1 \,.$$

For our present estimates, we make use of a novel Carleman weight

$$f(t, x) := -\frac{1}{1 + 2\kappa} (1 - |x|)^{1+2\kappa} - ct^2 \,, \tag{1.0.4}$$

that is especially adapted to the operator $\Box_\kappa$. In particular, the $(1-|x|)^{1+2\kappa}$-term in (1.0.4), which has rather singular derivatives at $r = 1$, is needed precisely in order to capture the Neumann data in the boundary data of Theorem 1.

Both the Carleman estimates in Theorem 1 and the underlying Morawetz estimates can be viewed as “centered about the origin”, and both estimates crucially depend on the domain being spherically symmetric. As a result, Theorem 1 only holds when the spatial domain is an open ball. We defer questions of whether Theorem 1 is extendible to more general spatial domains to future works. Moreover, that Theorem 1 fails to hold for $n = 2$ can be traced to the fact that the classical Morawetz breaks down for $n = 2$. In this case, the usual multiplier computations yield a boundary term at $r = 0$ that is divergent.

To conclude this preliminaries section, let us give a few remarks about the boundary observability Theorem 2:

Remark 1.0.2.

i) First of all, the required timespan $2T$ in the statment of the theorem can be shown to depend on n, as well as on κ when $n = 3$. This is in direct parallel to the dependence of c in Theorem 1. See Theorem 1.4.1 for more precise statements. Once again, a precise statement of the expected boundary asymptotics for u in Theorem 2 is given in Definition 1.1.2.

ii) If $\Box_\kappa$ in Theorem 2 is replaced by $\Box$ (that is, we consider non-singular wave equations), then observability holds for any $T > 1$. This can be deduced from either the geometric control condition of [7] (see also [13, 55]) or from standard Carleman estimates [8, 52, 80]. To our knowledge, the optimal timespan for the observability result in Theorem 2 is not known.

iii) For non-singular wave equations, standard observability results also involve observation regions that contain only part of the boundary [7, 13, 52, 54]. On the other hand, as our Carleman estimates are centered about the origin, they only yield observability results from the entire boundary. Whether partial boundary observability results also hold for the singular wave equation in Theorem 2 is a topic of further investigation.

The chapter is organized as follows. In Section 1.1, we list some definitions that will be pertinent to our setting, and we establish some general properties that will be useful later on. Section 1.2 is devoted to the multiplier inequalities that are fundamental to our main theorems. In particular, these generalize the classical Morawetz estimates to wave equations with critically singular potentials. In Section 1.3, we give a precise statement and a proof of our main Carleman estimates (see Theorem 1.3.1), while our main boundary observability result (see Theorem 1.4.1) is stated and proved in Section 1.4. At the end of the chapter in Appendix 1.A, we provide an overview of some well-known results for waves in $(1 + 1)$ dimensions.

1.1 Geometric Background and Asymptotics

In this section, we record some basic definitions, and we establish the notations that we will use in the rest of the chapter. In particular, we define weights that capture the boundary behavior of solutions to wave equations rendered by $\Box_\kappa$. We also define twisted derivatives constructed using the above weights, and we recall their basic properties. Furthermore, we prove pointwise inequalities in terms of these twisted derivatives that will later lead to Hardy-type estimates.

Our background setting is the spacetime $\mathbb{R}^{1+n}$. As usual, we let t and x denote the projections to the first and the last n components of $\mathbb{R}^{1+n}$, respectively, and we let $r := |x|$ denote the radial coordinate.

In addition, we let g denote the Minkowski metric on $\mathbb{R}^{1+n}$. Recall that with respect to polar coordinates, we have that

$$g = -dt^2 + dr^2 + r^2 g_{\mathbb{S}^{n-1}} \,,$$

where $g_{\mathbb{S}^{n-1}}$ denotes the metric of the $(n-1)$-dimensional unit sphere. Henceforth, we use the symbol ∇ to denote the g-covariant derivative, while we use $\nabla\!\!\!\!/$ to represent the induced angular covariant derivative on level spheres of (t, r). As before, the wave operator (with respect to g) is defined as

$$\Box = g^{\alpha\beta} \nabla_{\alpha\beta} \,.$$

As it is customary, we use lowercase Greek letters for spacetime indices over $\mathbb{R}^{n+1}$ (ranging from 0 to n), lowercase Latin letters for spatial indices over $\mathbb{R}^n$ (ranging from 1 to n), and uppercase Latin letters for angular indices over $\mathbb{S}^{n-1}$ (ranging from 1 to $n-1$). We always raise and lower indices using g, and we use the Einstein summation convention for repeated indices.

As in the previous section, we use B_1 to denote the open unit ball in $\mathbb{R}^n$, representing the spatial domain for our wave equations. We also set

$$\mathcal{C} := (-T, T) \times B_1 \,, \qquad T > 0 \,, \tag{1.1.1}$$

corresponding to the cylindrical spacetime domain. In addition, we let

$$\Gamma := (-T, T) \times \partial B_1 \tag{1.1.2}$$

denote the timelike boundary of $\mathcal{C}$.

To capture singular boundary behavior, we will make use of weights depending on the radial distance from ∂B_1. Toward this end, we define the function

$$y : \mathbb{R}^{1+n} \to \mathbb{R} \,, \qquad y := 1 - r \,. \tag{1.1.3}$$

From direct computations, we obtain the following identities for y:

$$\begin{aligned} \nabla^\alpha y \nabla_\alpha y &= 1\,, & \nabla^{\alpha\beta} y \nabla_\alpha y \nabla_\beta y &= 0\,, \\ \Box y &= -(n-1) r^{-1}\,, & \nabla^\alpha y \nabla_\alpha (\Box y) &= -(n-1) r^{-2}\,, \\ \Box^2 y &= (n-1)(n-3) r^{-3}\,, & \nabla^{\alpha\beta} y \nabla_{\alpha\beta} y &= (n-1) r^{-2}\,. \end{aligned} \tag{1.1.4}$$

From here on, let us fix a constant

$$-\frac{1}{2} < \kappa < 0\,, \tag{1.1.5}$$

and let us define the twisted derivative operators

$$\begin{aligned} D\Phi &:= y^\kappa \nabla (y^{-\kappa} \Phi) = \nabla \Phi - \frac{\kappa}{y} \nabla y \cdot \Phi\,, \\ \overline{D}\Phi &:= y^{-\kappa} \nabla (y^\kappa \Phi) = \nabla \Phi + \frac{\kappa}{y} \nabla y \cdot \Phi\,, \end{aligned} \tag{1.1.6}$$

where Φ is any spacetime tensor field. Observe that $-\overline{D}$ is the formal (L^2-)adjoint of D. Moreover, the following (tensorial) product rules hold for D and $\overline{D}$:

$$D(\Phi \otimes \Psi) = \nabla \Phi \otimes \Psi + \Phi \otimes D\Psi\,, \qquad \overline{D}(\Phi \otimes \Psi) = \nabla \Phi \otimes \Psi + \Phi \otimes \overline{D}\Psi\,. \tag{1.1.7}$$

In addition, let $\Box_y$ denote the y-twisted wave operator:

$$\Box_y := g^{\alpha\beta} \overline{D}_\alpha D_\beta\,. \tag{1.1.8}$$

A direct computation shows that $\Box_y$ differs from the singular wave operator $\Box_\kappa$ from (1.0.1) by only a lower-order term. More specifically, by (1.1.4) and (1.1.6),

$$\begin{aligned} \Box_y &= \Box + \frac{\kappa(1-\kappa) \cdot \nabla^\alpha y \nabla_\alpha y}{y^2} - \frac{\kappa \cdot \Box y}{y} \\ &= \Box_\kappa + \frac{(n-1)\kappa}{ry}\,. \end{aligned} \tag{1.1.9}$$

In particular, (1.1.9) shows that, up to a lower-order correction term, $\Box_y$ and $\Box_\kappa$ can be used interchangeably. In practice, the derivation of our estimates will be carried out in terms of $\Box_y$, as it is better adapted to the twisted operators.

Finally, we remark that since y is purely radial,

$$D_t \phi = \nabla_t \phi = \partial_t \phi\,, \qquad D_A \phi = \nabla\!\!\!/_A \phi = \partial_A \phi$$

for scalar functions ϕ. Thus, we will use the above notations interchangeably whenever convenient and whenever there is no risk of confusion. Moreover, we will write

$$D_X \phi = X^\alpha D_\alpha \phi$$

to denote derivatives along a vector field X.

Next, we establish a family of pointwise Hardy-type inequalities in terms of the twisted derivative operator D:

Proposition 1.1.1. *For any $q \in \mathbb{R}$ and any $u \in C^1(\mathcal{C})$, the following holds:*

$$\begin{aligned} y^{q-1}(D_r u)^2 \geqslant \frac{1}{4}(2\kappa + q - 2)^2 y^{q-3} \cdot u^2 - (n-1)\left(\kappa + \frac{q-2}{2}\right) y^{q-2} r^{-1} \cdot u^2 \\ - \nabla^\beta \left[\left(\kappa + \frac{q-2}{2}\right) y^{q-2} \nabla_\beta y \cdot u^2\right] . \end{aligned} \tag{1.1.10}$$

Proof. First, for any $p, b \in \mathbb{R}$, we have the inequality

$$\begin{aligned} 0 &\leqslant (y^p \cdot \nabla^\alpha y D_\alpha u + b y^{p-1} \cdot u)^2 \\ &= y^{2p} \cdot (\nabla^\alpha y D_\alpha u)^2 + b^2 y^{2p-2} \cdot u^2 + 2b y^{2p-1} \cdot u \nabla^\alpha y D_\alpha u \\ &= y^{2p} \cdot (D_r u)^2 + b(b - 2\kappa - 2p + 1) y^{2p-2} \cdot u^2 \\ &\qquad - b y^{2p-1} \Box y \cdot u^2 + \nabla^\beta (b y^{2p-1} \nabla_\beta y \cdot u^2) , \end{aligned}$$

where we used (1.1.6) in the last step. Setting $2p = q - 1$, the above becomes

$$\begin{aligned} y^{q-1}(D_r u)^2 \geqslant -b(b - 2\kappa - q + 2) y^{q-3} \cdot u^2 + b y^{q-2} \Box y \cdot u^2 \\ - \nabla^\beta (b y^{q-2} \nabla_\beta y \cdot u^2) . \end{aligned}$$

Taking $b = \kappa + \frac{q-2}{2}$ (which extremizes the above) yields (1.1.10). □

We conclude this section by discussing the precise boundary limits for our main results. First, given $u \in C^1(\mathcal{C})$, we define its Dirichlet and Neumann traces on Γ with respect to $\Box_y$ (or equivalently, $\Box_\kappa$) by

$$\begin{aligned} \mathcal{D}_\kappa u : \Gamma \to \mathbb{R} , \qquad \mathcal{D}_\kappa u &:= \lim_{r \nearrow 1} (y^{-\kappa} u) , \\ \mathcal{N}_\kappa u : \Gamma \to \mathbb{R} , \qquad \mathcal{N}_\kappa u &:= \lim_{r \nearrow 1} y^{2\kappa} \partial_r (y^{-\kappa} u) . \end{aligned} \tag{1.1.11}$$

Note in particular that the formulas (1.1.11) are directly inspired from the boundary traces of the introduction.

Now, the subsequent definition lists the main assumptions we will impose on boundary limits in our Carleman estimates and observability results:

Definition 1.1.2. *A function $u \in C^1(\mathcal{C})$ is called* boundary admissible *with respect to $\Box_y$ (or $\Box_\kappa$) when the following conditions hold:*

i) $\mathcal{N}_\kappa u$ *exists and is finite.*

ii) The following Dirichlet limits hold for u*:*

$$(1-2\kappa)\mathcal{D}_\kappa(y^{-1+2\kappa}u) = -\mathcal{N}_\kappa u\,, \qquad \mathcal{D}_\kappa(y^{2\kappa}\partial_t u) = 0\,. \tag{1.1.12}$$

Here, the Dirichlet and Neumann limits are in an L^2*-sense on* $(-T,T)\times\mathbb{S}^{n-1}$*.*

The main motivation for Definition 1.1.2 is that *it captures the expected boundary asymptotics for solutions of the equation* $\Box_y u = 0$ *that have vanishing Dirichlet data.* (In particular, note that u being boundary admissible implies $\mathcal{D}_\kappa u = 0$.) To justify this statement, we must first recall some results from [75].

For $u \in C^1(\mathcal{C})$ and $\tau \in (-T,T)$, we define the following twisted H^1-norms:

$$E_1[u](\tau) := \int_{\mathcal{C}\cap\{t=\tau\}} (|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2 + u^2)\,, \tag{1.1.13}$$

$$\overline{E}_1[u](\tau) := \int_{\mathcal{C}\cap\{t=\tau\}} (|\partial_t u|^2 + |\overline{D}_r u|^2 + |\nabla\!\!\!/ u|^2 + u^2)\,. \tag{1.1.14}$$

Moreover, if $u \in C^2(\mathcal{C})$ as well, then we define the twisted H^2-norm,

$$E_2[u](\tau) := \overline{E}_1[D_r u](\tau) + E_1[\partial_t u](\tau) + E_1[\nabla\!\!\!/_t u](\tau) + E_1[u](\tau)\,. \tag{1.1.15}$$

The results of [75] show that both $E_1[u]$ and $E_2[u]$ are natural energies associated with the operator $\Box_y$, in that their boundedness is propagated in time for solutions of $\Box_y u = 0$ with Dirichlet boundary conditions.

The following proposition shows that functions with uniformly bounded E_2-energy are boundary admissible, in the sense of Definition 1.1.2. In particular, the preceding discussion then implies that boundary admissibility is achieved by sufficiently regular (in a twisted H^2-sense) solutions of the singular wave equation $\Box_y u = 0$, with Dirichlet boundary conditions.

Proposition 1.1.3. *Let* $u \in C^2(\mathcal{C})$*, and assume that:*

i) $\mathcal{D}_\kappa u = 0$.

ii) $E_2[u](\tau)$ *is uniformly bounded for all* $\tau \in (-T,T)$*.*

Then, u *is boundary admissible with respect to* $\Box_y$*, in the sense of Definition 1.1.2.*

Proof. Fix $\tau \in (-T,T)$ and $\omega \in \mathbb{S}^{n-1}$, and let $0 < y_1 < y_0 \ll 1$. Applying the fundamental theorem of calculus and integrating in y yields

$$y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y_1,\omega)} - y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y_0,\omega)} = \int_{y_1}^{y_0} y^{\kappa}\overline{D}_r(D_r u)|_{(\tau,1-y,\omega)}dy\,,$$

where we have described points in $\overline{\mathcal{C}}$ using polar (t,r,ω)-coordinates.

We now integrate the above over $\Gamma = (-T,T)\times\mathbb{S}^{n-1}$, and we let $y_1 \searrow 0$. In particular, observe that for $\mathcal{N}_\kappa u$ to be finite, it suffices to show that

$$I := \int_\Gamma \left[\int_0^{y_0} y^{\kappa}\overline{D}_r(D_r u)|_{(\tau,1-y,\omega)}dy\right]^2 d\tau d\omega < \infty\,.$$

However, by Hölder's inequality and (1.1.5), we have

$$I \leqslant \int_\Gamma \left[\int_0^{y_0} y^{2\kappa}dy \int_0^{y_0} |\overline{D}_r(D_r u)|^2|_{(\tau,1-y,\omega)}dy\right] d\tau d\omega \lesssim \int_{-T}^{T} E_2[u](\tau)\, d\tau\,.$$

Thus, the assumptions of the proposition imply that I, and hence $\mathcal{N}_\kappa u$, is finite.

Next, to prove the first limit in (1.1.12), it suffices to show that

$$J_{y_0} := \int_\Gamma \left(y^{-1+\kappa}u|_{(\tau,1-y_0,\omega)} + \frac{1}{1+2\kappa}\mathcal{N}_\kappa u|_{(\tau,\omega)}\right)^2 d\tau d\omega \to 0\,, \tag{1.1.16}$$

as $y_0 \searrow 0$. Since $\mathcal{D}_k u = 0$, then fundamental theorem of calculus implies

$$\begin{aligned}
J_{y_0} &= \int_\Gamma \left[-y_0^{-1+2\kappa}\int_0^{y_0} y^{-2\kappa}y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y,\omega)}dy + \frac{1}{1+2\kappa}\mathcal{N}_\kappa u|_{(\tau,\omega)}\right]^2 d\tau d\omega \\
&= \int_\Gamma \left\{y_0^{-1+2\kappa}\int_0^{y_0} y^{-2\kappa}[y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y,\omega)} - \mathcal{N}_\kappa u|_{(\tau,\omega)}]dy\right\}^2 d\tau d\omega\,.
\end{aligned}$$

Moreover, the Minkowski integral inequality yields

$$\begin{aligned}
\sqrt{J_{y_0}} &\leqslant y_0^{-1+2\kappa}\int_0^{y_0} y^{-2\kappa}\left\{\int_\Gamma [y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y,\omega)} - \mathcal{N}_\kappa u|_{(\tau,\omega)}]^2 d\tau d\omega\right\}^{\frac{1}{2}} dy \\
&\lesssim \sup_{0<y<y_0}\left\{\int_\Gamma [y^{2\kappa}\partial_r(y^{-\kappa}u)|_{(\tau,1-y,\omega)} - \mathcal{N}_\kappa u|_{(\tau,\omega)}]^2 d\tau d\omega\right\}^{\frac{1}{2}}\,.
\end{aligned}$$

By the definition of $\mathcal{N}_\kappa u$, the right-hand side of the above converges to 0 when $y_0 \searrow 0$. This implies (1.1.16), and hence the first part of (1.1.12).

For the remaining limit in (1.1.12), we first claim that $\mathcal{D}_\kappa(\partial_t u)$ exists and is finite. This argument is analogous to the first part of the proof. Note that since

$$y^{-\kappa}\partial_t u|_{(\tau,1-y_1,\omega)} - y^{-\kappa}\partial_t u|_{(\tau,1-y_0,\omega)} = \int_{y_1}^{y_0} y^{-\kappa}D_r\partial_t u|_{(\tau,1-y,\omega)}dy\,,$$

then the claim immediately follows from the fact that

$$\int_\Gamma \left[\int_0^{y_0} y^{-\kappa} D_r \partial_t u|_{(\tau,1-y,\omega)} dy\right]^2 d\tau d\omega \lesssim \int_{-T}^{T} E_2[u](\tau)\, d\tau < \infty\,.$$

Moreover, to determine $\mathcal{D}_\kappa(\partial_t u)$, we see that for any test function $\varphi \in C_0^\infty(\Gamma)$,

$$\int_\Gamma \mathcal{D}_\kappa(\partial_t u)\cdot\varphi = -\lim_{y\searrow 0}\int_\Gamma y^{-\kappa} u|_{r=1-y}\cdot \partial_t\varphi = -\int_\Gamma \mathcal{D}_\kappa u\cdot\partial_t\varphi = 0\,.$$

It then follows that $\mathcal{D}_\kappa(\partial_t u) = 0$.

Finally, to prove the second limit of (1.1.12), it suffices to show

$$K_{y_0} := \int_\Gamma (y^{-\frac{1}{2}}\partial_t u)^2|_{(\tau,1-y_0,\omega)} d\tau d\omega \to 0\,, \qquad y_0 \searrow 0\,. \tag{1.1.17}$$

Using that $\mathcal{D}_\kappa(\partial_t u) = 0$ along with the fundamental theorem of calculus yields

$$\begin{aligned} K_{y_0} &= \int_\Gamma \left[y_0^{-\frac{1}{2}+\kappa}\int_0^{y_0} y^{-\kappa} D_r\partial_t u|_{(\tau,1-y,\omega)} dy\right]^2 d\tau d\omega \\ &\leqslant y_0^{-1+2\kappa}\int_\Gamma\left[\int_0^{y_0} y^{-2\kappa} dy \int_0^{y_0} (D_r\partial_t u)^2|_{(\tau,1-y,\omega)} dy\right] d\tau d\omega \\ &\lesssim \int_0^{y_0}\int_\Gamma (D_r\partial_t u)^2|_{(\tau,1-y,\omega)} d\tau d\omega dy\,. \end{aligned}$$

The integral on the right-hand side is (the time integral of) $E_2[u](\tau)$, restricted to the region $1-y_0 < r < 1$. Since $E_2[u](\tau)$ is uniformly bounded, it follows that K_{y_0} indeed converges to zero as $y_0 \searrow 0$, completing the proof. □

Remark 1.1.4. From the intuitions of [39], one may conjecture that Proposition 1.1.3 could be further strengthened, with the boundedness assumption on $E_2[u]$ replaced by a sharp boundedness condition on an appropriate fractional $H^{1+\kappa}$-norm. However, we will not pursue this question here.

1.2 Multiplier Inequalities

In this section, we derive some multiplier identities and inequalities, which form the foundations of the proof of the main Carleman estimates, Theorem 1.3.1. As mentioned before, these can be viewed as extensions to singular wave operators of the classical Morawetz inequality for wave equations.

In what follows, we fix $0 < \varepsilon \ll 1$, and we define the cylindrical region

$$\mathcal{C}_\varepsilon := (-T, T) \times \{\varepsilon < r < 1 - \varepsilon\} \,. \tag{1.2.1}$$

Moreover, let Γ_ε denote the timelike boundary of $\mathcal{C}_\varepsilon$:

$$\Gamma_\varepsilon := \Gamma_\epsilon^- \cup \Gamma_\epsilon^+ := [(-T, T) \times \{r = \varepsilon\}] \cup [(-T, T) \times \{r = 1 - \varepsilon\}] \,. \tag{1.2.2}$$

We also let ν denote the unit outward-pointing (g-)normal vector field on Γ_ε.

Finally, we fix a constant $c > 0$, and we define the functions

$$f := -\frac{1}{1 + 2\kappa} \cdot y^{1+2\kappa} - ct^2 \,, \qquad z := -4c \,, \tag{1.2.3}$$

which will be used to construct the multiplier for our upcoming inequalities.

We begin by deriving a preliminary form of our multiplier identity, for which the multiplier is defined using f and z:

Proposition 1.2.1. *Let $u \in C^\infty(\mathcal{C})$, and assume u is supported on $\mathcal{C} \cap \{|t| < T - \delta\}$ for some $0 < \delta \ll 1$. Then, we have the identity,*

$$\begin{aligned} -\int_{\mathcal{C}_\varepsilon} \Box_y u \cdot S_{f,z} u = \int_{\mathcal{C}_\varepsilon} (\nabla^{\alpha\beta} f + z \cdot g^{\alpha\beta}) D_\alpha u D_\beta u + \int_{\mathcal{C}_\varepsilon} \mathcal{A}_{f,z} \cdot u^2 \\ - \int_{\Gamma_\varepsilon} S_{f,z} u \cdot D_\nu u + \frac{1}{2} \int_{\Gamma_\varepsilon} \nabla_\nu f \cdot D_\beta u D^\beta u \\ + \frac{1}{2} \int_{\Gamma_\varepsilon} \nabla_\nu w_{f,z} \cdot u^2 \,, \end{aligned} \tag{1.2.4}$$

for any $0 < \varepsilon \ll 1$, where

$$\begin{aligned} w_{f,z} &:= \frac{1}{2} \left(\Box f + \frac{2\kappa}{y} \nabla_\alpha y \nabla^\alpha f \right) + z \,, \\ \mathcal{A}_{f,z} &:= -\frac{1}{2} \left(\Box w_{f,z} + \frac{2\kappa}{y} \nabla_\alpha y \nabla^\alpha w_{f,z} \right) \,, \\ S_{f,z} &:= \nabla^\alpha f \cdot D_\alpha + w_{f,z} \,. \end{aligned} \tag{1.2.5}$$

Proof. Integrating the left-hand side of (1.2.4) by parts twice reveals that

$$\begin{aligned}
-\int_{\mathcal{C}_\varepsilon} \Box_y u \cdot \nabla^\alpha f D_\alpha u &= \int_{\mathcal{C}_\varepsilon} D_\beta u \cdot \overline{D}^\beta (\nabla^\alpha f D_\alpha u) - \int_{\Gamma_\varepsilon} \nabla^\alpha f D_\alpha u \cdot D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} \nabla^{\alpha\beta} f \cdot D_\alpha u D_\beta u + \int_{\mathcal{C}_\varepsilon} \nabla^\alpha f \cdot D_\beta u D_\alpha{}^\beta u \\
&\quad - \int_{\Gamma_\varepsilon} \nabla^\alpha f D_\alpha u \cdot D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} \nabla^{\alpha\beta} f \cdot D_\alpha u D_\beta u + \frac{1}{2} \int_{\mathcal{C}_\varepsilon} \nabla^\alpha f \cdot \nabla_\alpha (D_\beta u D^\beta u) \\
&\quad - \int_{\mathcal{C}_\varepsilon} \frac{\kappa}{y} \nabla_\alpha y \nabla^\alpha f \cdot D_\beta u D^\beta u - \int_{\Gamma_\varepsilon} \nabla^\alpha f D_\alpha u \cdot D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} \left[\nabla^{\alpha\beta} f - \frac{1}{2} \left(\Box f + \frac{2\kappa}{y} \nabla_\alpha y \nabla^\alpha f \right) g^{\alpha\beta} \right] \cdot D_\alpha u D_\beta u \\
&\quad - \int_{\Gamma_\varepsilon} \nabla^\alpha f D_\alpha u \cdot D_\nu u + \frac{1}{2} \int_{\Gamma_\varepsilon} \nabla_\nu f \cdot D_\beta u D^\beta u \,,
\end{aligned}$$

where in the above steps, we also applied the identities (1.1.6), (1.1.7), (1.1.8), as well as the observation that $\overline{D}$ is the adjoint of D.

A similar set of computations also yields

$$\begin{aligned}
-\int_{\mathcal{C}_\varepsilon} \Box_y u \cdot w_{f,z} u &= \int_{\mathcal{C}_\varepsilon} D^\alpha u \overline{D}_\alpha (w_{f,z} u) - \int_{\Gamma_\varepsilon} w_{f,z} \cdot u D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} \nabla_\alpha w_{f,z} \cdot u D^\alpha u + \int_{\mathcal{C}_\varepsilon} w_{f,z} \cdot D^\alpha u D_\alpha u - \int_{\Gamma_\varepsilon} w_{f,z} \cdot u D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} w_{f,z} \cdot D^\alpha u D_\alpha u + \frac{1}{2} \int_{\mathcal{C}_\varepsilon} \nabla_\alpha w_{f,z} \cdot \nabla^\alpha (u^2) \\
&\quad - \int_{\mathcal{C}_\varepsilon} \frac{\kappa}{y} \nabla^\alpha y \nabla_\alpha w_{f,z} \cdot u^2 - \int_{\Gamma_\varepsilon} w_{f,z} \cdot u D_\nu u \\
&= \int_{\mathcal{C}_\varepsilon} w_{f,z} \cdot D^\alpha u D_\alpha u - \frac{1}{2} \int_{\mathcal{C}_\varepsilon} \left(\Box w_{f,z} + \frac{2\kappa}{y} \nabla^\alpha y \nabla_\alpha w_{f,z} \right) \cdot u^2 \\
&\quad - \int_{\Gamma_\varepsilon} w_{f,z} \cdot u D_\nu u + \frac{1}{2} \int_{\Gamma_\varepsilon} \nabla_\nu w_{f,z} \cdot u^2 \,.
\end{aligned}$$

Adding the above two identities results in (1.2.4). □

In the following proposition, we collect some computations involving the functions f and z that will be useful later on.

Proposition 1.2.2. *f, $w_{f,z}$, and $\mathcal{A}_{f,z}$ (defined as in* (1.2.3) *and* (1.2.5)*) satisfy*

$$\begin{aligned} \nabla_{\alpha\beta} f &= y^{2\kappa} \cdot \nabla_{\alpha\beta} r - 2\kappa y^{2\kappa-1} \cdot \nabla_\alpha r \nabla_\beta r - 2c \cdot \nabla_\alpha t \nabla_\beta t \,, \\ w_{f,z} &= -2\kappa \cdot y^{2\kappa-1} + \frac{1}{2}(n-1) \cdot y^{2\kappa} r^{-1} - 3c \,, \\ \mathcal{A}_{f,z} &= 2\kappa(2\kappa-1)^2 \cdot y^{2\kappa-3} - \frac{1}{2}(n-1)\kappa(8\kappa-3) \cdot y^{2\kappa-2} r^{-1} \\ &\qquad + \frac{1}{2}(n-1)(n-4)\kappa \cdot y^{2\kappa-1} r^{-2} + \frac{1}{4}(n-1)(n-3) \cdot y^{2\kappa} r^{-3} \,. \end{aligned} \tag{1.2.6}$$

Proof. First, we fix $q \in \mathbb{R} \setminus \{-1\}$, and we let

$$f_q := -\frac{y^{1+q}}{1+q} \,. \tag{1.2.7}$$

Note that f_q satisfies

$$\begin{aligned} \nabla_\alpha f_q &= -y^q \cdot \nabla_\alpha y \,, \\ \nabla_{\alpha\beta} f_q &= -y^q \cdot \nabla_{\alpha\beta} y - q y^{q-1} \cdot \nabla_\alpha y \nabla_\beta y \,, \\ \Box f_q &= -y^q \cdot \Box y - q y^{q-1} \cdot \nabla^\alpha y \nabla_\alpha y \,, \\ \frac{2\kappa}{y} \cdot \nabla^\alpha y \nabla_\alpha f_q &= -2\kappa y^{q-1} \cdot \nabla^\alpha y \nabla_\alpha y \,. \end{aligned} \tag{1.2.8}$$

Next, using the notations from (1.2.5), along with (1.1.4) and (1.2.8), we have

$$\begin{aligned} w_{f_q,0} &= -\frac{1}{2} y^q \cdot \Box y - \left(\kappa + \frac{q}{2}\right) y^{q-1} \cdot \nabla^\alpha y \nabla_\alpha y \\ &= -\left(\kappa + \frac{q}{2}\right) \cdot y^{q-1} + \frac{n-1}{2} \cdot y^q r^{-1} \,. \end{aligned} \tag{1.2.9}$$

Moreover, further differentiating (1.2.9) and again using (1.1.4), we see that

$$\begin{aligned} \Box w_{f_q,0} &= -\frac{1}{2}(q+2\kappa)(q-1)(q-2) y^{q-3} \cdot (\nabla^\alpha y \nabla_\alpha y)^2 \\ &\quad - (q-1)[(q+\kappa)\Box y \nabla^\alpha y \nabla_\alpha y + 2(q+2\kappa)\nabla^{\alpha\beta} y \nabla_\alpha y \nabla_\beta y] \cdot y^{q-2} \\ &\quad - 2(q+\kappa) y^{q-1} \cdot \nabla^\alpha y \nabla_\alpha (\Box y) - (q+2\kappa) y^{q-1} \cdot \nabla^{\alpha\beta} y \nabla_{\alpha\beta} y \\ &\quad - \frac{1}{2} q y^{q-1} \cdot (\Box y)^2 - \frac{1}{2} y^q \cdot \Box^2 y \,, \\ \frac{2\kappa}{y} \nabla^\alpha y \nabla_\alpha w_{f_q,0} &= -\kappa(q+2\kappa)(q-1) y^{q-3} \cdot (\nabla^\alpha y \nabla_\alpha y)^2 - \kappa q y^{q-2} \cdot \Box y \nabla^\alpha y \nabla_\alpha y \\ &\quad - 2\kappa(q+2\kappa) y^{q-2} \cdot \nabla^{\alpha\beta} y \nabla_\alpha y \nabla_\beta y - \kappa y^{q-1} \cdot \nabla^\alpha y \nabla_\alpha (\Box y) \,. \end{aligned}$$

We can then use the above to compute the coefficient $\mathcal{A}_{f_q,0}$:

$$\begin{aligned}
\mathcal{A}_{f_q,0} &= \frac{1}{4}(q+2\kappa)(q+2\kappa-2)(q-1)y^{q-3}\cdot(\nabla^\alpha y\nabla_\alpha y)^2 \\
&\quad + \frac{1}{2}(q^2-q+2\kappa q-\kappa)y^{q-2}\cdot\Box y\nabla^\alpha y\nabla_\alpha y \\
&\quad + (q+2\kappa)(q+\kappa-1)y^{q-2}\cdot\nabla^{\alpha\beta}y\nabla_\alpha y\nabla_\beta y \\
&\quad + \frac{1}{2}(2q+3\kappa)y^{q-1}\cdot\nabla^\alpha y\nabla_\alpha(\Box y) + \frac{1}{2}(q+2\kappa)y^{q-1}\cdot\nabla^{\alpha\beta}y\nabla_{\alpha\beta}y \\
&\quad + \frac{1}{4}qy^{q-1}\cdot(\Box y)^2 + \frac{1}{4}y^q\cdot\Box^2 y \\
&= \frac{1}{4}(q+2\kappa)(q+2\kappa-2)(q-1)\cdot y^{q-3} \\
&\quad - \frac{1}{2}(n-1)(q^2-q+2\kappa q-\kappa)\cdot y^{q-2}r^{-1} \\
&\quad + \frac{1}{4}(n-1)[q(n-3)-2\kappa]\cdot y^{q-1}r^{-2} + \frac{1}{4}(n-1)(n-3)\cdot y^q r^{-3}\,.
\end{aligned} \tag{1.2.10}$$

Notice from (1.2.3) and (1.2.7) that we can write

$$f = f_{2\kappa} - ct^2\,,$$

Thus, substituting $q = 2\kappa$ in (1.2.7), we see that the Hessian of f satisfies

$$\begin{aligned}
\nabla_{\alpha\beta}f &= \nabla_{\alpha\beta}f_{2\kappa} - c\nabla_{\alpha\beta}t^2 \\
&= y^{2\kappa}\cdot\nabla_{\alpha\beta}r - 2\kappa y^{2\kappa-1}\cdot\nabla_\alpha r\nabla_\beta r - 2c\nabla_\alpha t\nabla_\beta t\,,
\end{aligned}$$

which is precisely the first part of (1.2.6).

Moreover, noting that

$$w_{-ct^2,0} = c\,,$$

then we also have

$$\begin{aligned}
w_{f,z} &= w_{f_{2\kappa},0} + w_{-ct^2,0} + z \\
&= -2\kappa\cdot y^{2\kappa-1} + \frac{1}{2}(n-1)\cdot y^{2\kappa}r^{-1} - 3c\,,
\end{aligned}$$

which gives the second equation in (1.2.6). Finally, noting that

$$\mathcal{A}_{-ct^2,0} = 0\,, \qquad -\frac{1}{2}\left(\Box z + \frac{2\kappa}{y}\cdot\nabla^\alpha y\nabla_\alpha z\right) = 0\,,$$

we obtain, with the help of (1.1.4), the last equation of (1.2.6):

$$\begin{aligned}
\mathcal{A}_{f,z} &= \mathcal{A}_{f_{2\kappa},0} + \mathcal{A}_{-ct^2,0} - \frac{1}{2}\left(\Box z + \frac{2\kappa}{y}\cdot\nabla^\alpha y\nabla_\alpha z\right)\\
&= 2\kappa(2\kappa-1)^2 y^{2\kappa-3}\cdot(\nabla^\alpha y\nabla_\alpha y)^2 + \frac{1}{2}\kappa(8\kappa-3)y^{2\kappa-2}\cdot\Box y\nabla^\alpha y\nabla_\alpha y\\
&\qquad + 4\kappa(3\kappa-1)y^{2\kappa-2}\cdot\nabla^{\alpha\beta}y\nabla_\alpha y\nabla_\beta y + \frac{7}{2}\kappa y^{2\kappa-1}\cdot\nabla^\alpha y\nabla_\alpha(\Box y)\\
&\qquad + 2\kappa y^{2\kappa-1}\cdot\nabla^{\alpha\beta}y\nabla_{\alpha\beta}y + \frac{1}{2}\kappa y^{2\kappa-1}\cdot(\Box y)^2 + \frac{1}{4}y^{2\kappa}\cdot\Box^2 y\\
&= 2\kappa(2\kappa-1)^2\cdot y^{2\kappa-3} - \frac{1}{2}(n-1)\kappa(8\kappa-3)\cdot y^{2\kappa-2}r^{-1}\\
&\qquad + \frac{1}{2}(n-1)(n-4)\kappa\cdot y^{2\kappa-1}r^{-2} + \frac{1}{4}(n-1)(n-3)\cdot y^{2\kappa}r^{-3}\,. \qquad \Box
\end{aligned}$$

We conclude this section with the multiplier inequality that will be used to prove our main Carleman estimate:

Proposition 1.2.3. *Let f and z be as in* (1.2.3)*, and let $u \in C^\infty(\mathcal{C})$ be supported on $\mathcal{C}\cap\{|t| < T-\delta\}$ for some $0 < \delta \ll 1$. Then, we have the inequality*

$$\begin{aligned}
-\int_{\mathcal{C}_\varepsilon}\Box_y u\cdot S_{f,z}u \geqslant &\int_{\mathcal{C}_\varepsilon}[(1-4c)\cdot|\nabla\!\!\!/ u|^2 + 2c\cdot(\partial_t u)^2 - 4c\cdot(D_r u)^2] \qquad (1.2.11)\\
&- \frac{1}{2}(n-1)\kappa\int_{\mathcal{C}_\varepsilon} y^{2\kappa-2}r^{-2}[r-(n-4)y]\cdot u^2\\
&+ \frac{1}{4}(n-1)(n-3)\int_{\mathcal{C}_\varepsilon} y^{2\kappa}r^{-3}\cdot u^2 - \int_{\Gamma_\varepsilon} S_{f,z}u\cdot D_\nu u\\
&+ \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu f\cdot D_\beta u D^\beta u + \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu w_{f,z}\cdot u^2\\
&+ 2\kappa(2\kappa-1)\int_{\Gamma_\varepsilon} y^{2\kappa-2}\nabla_\nu y\cdot u^2\,,
\end{aligned}$$

for any $0 < \varepsilon \ll 1$, where $w_{f,z}$ and $S_{f,z}$ are defined as in (1.2.5).

Proof. Applying the multiplier identity (1.2.4), with f and z from (1.2.3), and recalling the formulas (1.2.6) for $\nabla^2 f$, $w_{f,z}$, and $\mathcal{A}_{f,z}$, we obtain that

$$I := -\int_{\mathcal{C}_\varepsilon}\Box_y u\cdot S_{f,z}u$$

satisfies the identity

$$I = \int_{\mathcal{C}_\varepsilon} (y^{2\kappa}\nabla^{\alpha\beta}r - 2\kappa y^{-1+2\kappa}\nabla^\alpha r\nabla^\beta r - 2c\nabla^\alpha t\nabla^\beta t - 4cg^{\alpha\beta})D_\alpha u D_\beta u \tag{1.2.12}$$
$$+ 2\kappa(2\kappa-1)^2\int_{\mathcal{C}_\varepsilon} y^{2\kappa-3}u^2 - \frac{1}{2}(n-1)\kappa(8\kappa-3)\int_{\mathcal{C}_\varepsilon} y^{2\kappa-2}r^{-1}u^2$$
$$+ \frac{1}{2}(n-1)(n-4)\kappa\int_{\mathcal{C}_\varepsilon} y^{2\kappa-1}r^{-2}u^2 + \frac{1}{4}(n-1)(n-3)\int_{\mathcal{C}_\varepsilon} y^{2\kappa}r^{-3}u^2$$
$$- \int_{\Gamma_\varepsilon} S_{f,z}u\cdot D_\nu u + \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu f\cdot D_\beta u D^\beta u + \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu w_{f,z}\cdot u^2\,.$$

For the first-order terms in the multiplier identity, we notice that

$$\nabla^{\alpha\beta}r\cdot D_\alpha u D_\beta u = r^{-1}|\nabla\!\!\!/ u|^2\,, \qquad |\nabla\!\!\!/ u|^2 = g^{AB}\nabla\!\!\!/_A u\nabla\!\!\!/_B u\,,$$

and we hence expand

$$(y^{2\kappa}\cdot\nabla^{\alpha\beta}r - 2\kappa y^{-1+2\kappa}\nabla^\alpha r\nabla^\beta r - 2c\cdot\nabla^\alpha t\nabla^\beta t - 4c\cdot g^{\alpha\beta})D_\alpha u D_\beta u \tag{1.2.13}$$
$$\geqslant -2\kappa y^{-1+2\kappa}(D_r u)^2 + (y^{2\kappa}r^{-1} - 4c)|\nabla\!\!\!/ u|^2 + 2c(\partial_t u)^2 - 4c(D_r u)^2$$
$$\geqslant -2\kappa y^{-1+2\kappa}(D_r u)^2 + (1-4c)|\nabla\!\!\!/ u|^2 + 2c(\partial_t u)^2 - 4c(D_r u)^2\,.$$

Moreover, applying the Hardy inequality (1.1.10), with $q = 2\kappa$, yields

$$-2\kappa y^{2\kappa-1}(D_r u)^2 \geqslant -2\kappa(2\kappa-1)^2 y^{2\kappa-3}u^2 + (n-1)2\kappa(2\kappa-1)y^{2\kappa-2}r^{-1}u^2 \tag{1.2.14}$$
$$+ 2\kappa(2\kappa-1)\nabla^\beta(y^{2\kappa-2}\nabla_\beta y\cdot u^2)\,.$$

The desired inequality (1.2.11) now follows by combining (1.2.12)–(1.2.14) and applying the divergence theorem to the last term in (1.2.14). □

1.3 The Carleman Estimates

In this section, we apply the preceding multiplier inequality to obtain our main Carleman estimates. The precise statement of our estimates is the following:

Theorem 1.3.1. *Assume $n \neq 2$, and fix $-\frac{1}{2} < \kappa < 0$. Also, let $u \in C^\infty(\mathcal{C})$ satisfy:*

i) u is boundary admissible (see Definition 1.1.2).

ii) u is supported on $\mathcal{C}\cap\{|t| < T-\delta\}$ for some $\delta > 0$.

Then, there exists some sufficiently large $\lambda_0 > 0$, depending only on n and κ, such that the following Carleman inequality holds for all $\lambda \geqslant \lambda_0$:

$$\lambda \int_\Gamma e^{2\lambda f} (\mathcal{N}_\kappa u)^2 + \int_\mathcal{C} e^{2\lambda f} (\Box_\kappa u)^2 \tag{1.3.1}$$
$$\geqslant C_0 \lambda \int_\mathcal{C} e^{2\lambda f} [(\partial_t u)^2 + |\nabla\!\!\!/ u|^2 + (D_r u)^2] + C_0 \lambda^3 \int_\mathcal{C} e^{2\lambda f} y^{6\kappa - 1} u^2$$
$$+ C_0 \lambda \cdot \begin{cases} \int_\mathcal{C} e^{2\lambda f} y^{2\kappa-2} r^{-3} & n \geqslant 4 \\ \int_\mathcal{C} e^{2\lambda f} y^{2\kappa-2} r^{-2} & n = 3 \\ 0 & n = 1 \end{cases} .$$

where the constant $C_0 > 0$ depends on n and κ, where

$$f = -\frac{1}{1 + 2\kappa} \cdot y^{1+2\kappa} - ct^2 ,$$

as in (1.2.3)*, and where the constant c satisfies*

$$0 < c < \frac{1}{5} , \qquad \begin{cases} c \leqslant \frac{1}{4\sqrt{3} \cdot T} & n \geqslant 4 \\ c \leqslant \min \left\{ \frac{1}{4\sqrt{15} \cdot T}, \frac{|\kappa|}{120} \right\} & n = 3 \\ c \leqslant \frac{1}{4\sqrt{15} \cdot T} & n = 1 \end{cases} . \tag{1.3.2}$$

The proof of Theorem 1.3.1 is carried out in remainder of this section.

Remark 1.3.2. We note that parts of this proof will treat the cases $n = 1$, $n = 3$, and $n \geqslant 4$ separately. This accounts for the difference in the assumptions for c in (1.3.2), which will affect the required timespan in our upcoming observability inequalities.

From here on, let us assume the hypotheses of Theorem 1.3.1. Let us also suppose that λ_0 is sufficiently large, with its precise value depending only on n and κ. In addition, we define the following:

$$v := e^{\lambda f} u , \qquad \mathcal{L} v := e^{\lambda f} \Box_y (e^{-\lambda f} v) . \tag{1.3.3}$$

The objective of this subsection is to establish the following inequality for v:

Lemma 1.3.3. *For any $\lambda \geqslant \lambda_0$, we have the inequality*

$$\begin{aligned}\frac{1}{4\lambda}\int_{\mathcal{C}_\varepsilon}(\mathcal{L}v)^2 \geqslant{}& \frac{c}{2}\int_{\mathcal{C}_\varepsilon}\left[(\partial_t v)^2 + |\nabla\!\!\!/ v|^2 + (D_r v)^2\right] - \frac{1}{2}\kappa\lambda^2\int_{\mathcal{C}_\varepsilon} y^{6\kappa-1}v^2 \\ &+ \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu f\cdot D_\beta v D^\beta v - \int_{\Gamma_\varepsilon} S_{f,z}v\cdot D_\nu v \\ &- \frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2 \\ &+ \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu w_{f,z}\cdot v^2 + 2\kappa(2\kappa-1)\int_{\Gamma_\varepsilon} y^{2\kappa-2}\nabla_\nu y\cdot v^2 \\ &+ \begin{cases} c_1\int_{\mathcal{C}_\varepsilon} y^{2\kappa-2}r^{-3}\cdot v^2 & n\geqslant 4 \\ c_1\int_{\mathcal{C}_\varepsilon} y^{2\kappa-2}r^{-2}\cdot v^2 + c_2\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2 & n=3 \\ c_2\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2 & n=1 \end{cases},\end{aligned} \tag{1.3.4}$$

where $S_{f,z}$ and $w_{f,z}$ are defined as in (1.2.5) *and* (1.2.6)*, where the constant $c_1 > 0$ depends on n and κ, and where the constant $c_2 > 0$ depends on n.*

Proof. First, observe that by (1.1.6)–(1.1.8), we can expand $\mathcal{L}v$ as follows:

$$\begin{aligned}\mathcal{L}v &= e^{\lambda f}\overline{D}^\alpha D_\alpha(e^{-\lambda f}v) \\ &= e^{\lambda f}\overline{D}^\alpha(e^{-\lambda f}D_\alpha v) - \lambda e^{\lambda f}\overline{D}^\alpha(e^{-\lambda f}\nabla_\alpha f\cdot v) \\ &= \Box_y v - \lambda\nabla^\alpha f(D_\alpha\psi + \overline{D}_\alpha v) - \lambda\Box f\cdot v + \lambda^2\nabla^\alpha f\nabla_\alpha f\cdot v \\ &= \Box_y v - 2\lambda S_{f,z}v + \mathcal{A}_0 v\,,\end{aligned} \tag{1.3.5}$$

where $\mathcal{A}_0$ is given by

$$\mathcal{A}_0 := \lambda^2\nabla^\alpha f\nabla_\alpha f + 2\lambda z = \lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda\,. \tag{1.3.6}$$

Multiplying (1.3.5) by $S_{f,z}v$ yields

$$-\mathcal{L}vS_{f,z}v = -\Box_y vS_{f,z}v + 2\lambda(S_{f,z}v)^2 - \mathcal{A}_0\cdot vS_{f,z}v\,. \tag{1.3.7}$$

For the last term, we apply (1.1.6) and the product rule:

$$\begin{aligned}-\mathcal{A}_0\cdot vS_{f,z}v &= -\mathcal{A}_0\cdot v(\nabla^\alpha fD_\alpha v + w_{f,z}v) \\ &= -\mathcal{A}_0\cdot\left[\frac{1}{2}\nabla^\alpha f\nabla_\alpha(v^2) - \frac{\kappa}{y}\nabla^\alpha f\nabla_\alpha y\cdot v^2 + w_{f,z}v^2\right] \\ &= -\nabla^\alpha\left(\frac{1}{2}\mathcal{A}_0\nabla_\alpha f\cdot v^2\right) + \frac{1}{2}\nabla^\alpha f\nabla_\alpha\mathcal{A}_0\cdot v^2 - z\mathcal{A}_0\cdot v^2\,.\end{aligned} \tag{1.3.8}$$

Moreover, recalling (1.2.3) and (1.3.6) yields

$$\begin{aligned} -z\mathcal{A}_0 &= 4c\lambda^2(y^{4\kappa} - 4c^2t^2) - 32\lambda c^2\,, \\ \frac{1}{2}\nabla^\alpha f \nabla_\alpha \mathcal{A}_0 &= \lambda^2(-2\kappa y^{6\kappa-1} - 8c^3t^2)\,. \end{aligned} \tag{1.3.9}$$

Combining (1.3.7)–(1.3.9) results in the identity

$$-\mathcal{L}vS_{f,z}v = -\Box_y v S_{f,z}v + 2\lambda(S_{f,z}v)^2 + \mathcal{A}_{f,z}\cdot v^2 - \nabla^\alpha\left(\frac{1}{2}\mathcal{A}_0\nabla_\alpha f\cdot v^2\right)\,, \tag{1.3.10}$$

where the coefficient $\mathcal{A}_{f,z}$ is given by

$$\begin{aligned} \mathcal{A}_{f,z} &:= \frac{1}{2}\nabla^\alpha f \nabla_\alpha \mathcal{A}_0 - z\mathcal{A}_0 \\ &= \lambda^2(-2\kappa y^{6\kappa-1} + 4cy^{4\kappa} - 24c^3t^2) - 32\lambda c^2\,. \end{aligned} \tag{1.3.11}$$

Integrating (1.3.10) over $\mathcal{C}_\varepsilon$ and recalling (1.3.11) then yields

$$\begin{aligned} -\int_{\mathcal{C}_\varepsilon}\mathcal{L}vS_{f,z}v = &-\int_{\mathcal{C}_\varepsilon}\Box_y v S_{f,z}v + 2\lambda\int_{\mathcal{C}_\varepsilon}(S_{f,z}v)^2 \\ &+\int_{\mathcal{C}_\varepsilon}[\lambda^2(-2\kappa y^{6\kappa-1} + 4cy^{4\kappa} - 24c^3t^2) - 32\lambda c^2]\cdot v^2 \\ &-\frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2\,. \end{aligned} \tag{1.3.12}$$

Notice that the bound (1.3.2) for c implies (for all values of n)

$$48c^2t^2 \leqslant 48c^2T^2 \leqslant 1 \leqslant y^{4\kappa}\,. \tag{1.3.13}$$

Then, with large enough λ_0 (depending on n and κ), we obtain

$$\begin{aligned} \lambda^2(-2\kappa y^{6\kappa-1} + 4cy^{4\kappa} - 24c^3t^2) - 32\lambda c^2 &\geqslant -2\kappa\lambda^2\cdot y^{6\kappa-1} - 32\lambda c^2 \\ &\geqslant -\kappa\lambda^2\cdot y^{6\kappa-1}\,. \end{aligned} \tag{1.3.14}$$

Noting in addition that

$$|\mathcal{L}vS_{f,z}v| \leqslant \frac{1}{4\lambda}(\mathcal{L}v)^2 + \lambda(S_{f,z}v)^2\,,$$

then (1.3.12) and (1.3.14) together imply

$$\begin{aligned} \frac{1}{4\lambda}\int_{\mathcal{C}_\varepsilon}(\mathcal{L}v)^2 \geqslant &-\int_{\mathcal{C}_\varepsilon}\Box_y v S_{f,z}v + \lambda\int_{\mathcal{C}_\varepsilon}(S_{f,z}v)^2 - \kappa\lambda^2\int_{\mathcal{C}_\varepsilon}y^{6\kappa-1}\cdot v^2 \\ &-\frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2\,. \end{aligned} \tag{1.3.15}$$

At this point, the proof splits into different cases, depending on n.

Case 1: $n \geqslant 4$. First, note that for large λ_0, we have

$$\begin{aligned}\frac{1}{9}\lambda(S_{f,z}v)^2 &\geqslant cy^{-4\kappa}(S_{f,z}v)^2 \\ &\geqslant c(D_r v)^2 + c(2cty^{-2\kappa}\cdot\partial_t v + y^{-2\kappa}w_{f,z}\cdot v)^2 \\ &\quad + 2c(D_r v)(2cty^{-2\kappa}\cdot\partial_t v + y^{-2\kappa}w_{f,z}\cdot v) \\ &\geqslant \frac{1}{2}c(D_r v)^2 - c(2cty^{-2\kappa}\cdot\partial_t v + y^{-2\kappa}w_{f,z}\cdot v)^2 \\ &\geqslant \frac{1}{2}c(D_r v)^2 - 8c^3t^2y^{-4\kappa}\cdot(\partial_t v)^2 - 2cy^{-4\kappa}w_{f,z}^2\cdot v^2 \\ &\geqslant \frac{1}{2}c(D_r v)^2 - \frac{1}{6}c\cdot(\partial_t v)^2 - 2cy^{-4\kappa}w_{f,z}^2\cdot v^2\,,\end{aligned} \tag{1.3.16}$$

where we also recalled (1.3.13) and the definitions (1.2.3) and (1.2.5) of f, z, and $S_{f,z}$. Moreover, recalling the formula (1.2.6) for $w_{f,z}$, we obtain that

$$-18cy^{-4\kappa}w_{f,z}^2\cdot v^2 \geqslant -C(y^{-2}+r^{-2})\cdot v^2\,, \tag{1.3.17}$$

for some constant $C > 0$, depending on n and κ. Thus, for sufficiently large λ_0, it follows from (1.3.16) and (1.3.17) that

$$\lambda(S_{f,z}v)^2 \geqslant \frac{9}{2}c(D_r v)^2 - \frac{3}{2}c\cdot(\partial_t v)^2 - C(y^{-2}+r^{-2})\cdot v^2\,. \tag{1.3.18}$$

Combining (1.3.15) with (1.3.18), we obtain

$$\begin{aligned}\frac{1}{4\lambda}\int_{\mathcal{C}_\varepsilon}(\mathcal{L}v)^2 \geqslant &-\int_{\mathcal{C}_\varepsilon}\Box_y v S_{f,z}v + \frac{9}{2}c\int_{\mathcal{C}_\varepsilon}(D_r v)^2 - \frac{3}{2}c\int_{\mathcal{C}_\varepsilon}(\partial_t v)^2 \\ &- \kappa\lambda^2\int_{\mathcal{C}_\varepsilon}y^{6\kappa-1}\cdot v^2 - C\int_{\mathcal{C}_\varepsilon}(y^{-2}+r^{-2})\cdot v^2 \\ &- \frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa}-4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2\,.\end{aligned} \tag{1.3.19}$$

Applying the multiplier inequality (1.2.11) to (1.3.19) then results in the bound

$$\begin{aligned}\frac{1}{4\lambda}\int_{\mathcal{C}_\varepsilon}(\mathcal{L}v)^2 \geqslant \int_{\mathcal{C}_\varepsilon}&\left[(1-4c)\cdot|\nabla\!\!\!/ v|^2+\frac{1}{2}c\cdot(\partial_t v)^2+\frac{1}{2}c\cdot(D_r v)^2\right] \\ &-\kappa\lambda^2\int_{\mathcal{C}_\varepsilon}y^{6\kappa-1}\cdot v^2-\frac{1}{2}(n-1)\kappa\int_{\mathcal{C}_\varepsilon}y^{2\kappa-2}r^{-1}\cdot v^2 \\ &+\frac{1}{4}(n-1)(n-3)\int_{\mathcal{C}_\varepsilon}y^{2\kappa}r^{-3}\cdot v^2 \\ &-C\int_{\mathcal{C}_\varepsilon}(y^{-2}+y^{2\kappa-1}r^{-2})\cdot v^2-\int_{\Gamma_\varepsilon}S_{f,z}v\cdot D_\nu v \\ &+\frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu f\cdot D_\beta v D^\beta v+\frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu w_{f,z}\cdot v^2 \\ &-\frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa}-4c^2t^2)-8c\lambda]\nabla_\nu f\cdot v^2 \\ &+2\kappa(2\kappa-1)\int_{\Gamma_\varepsilon}y^{2\kappa-2}\nabla_\nu y\cdot v^2\,.\end{aligned} \tag{1.3.20}$$

(Here, C may differ from previous lines, but still depends only on n and κ.)

Let $d>0$, and define now the (positive) quantities

$$\begin{aligned} J &:= dy^{2\kappa-2}r^{-3}+C(y^{-2}+y^{2\kappa-1}r^{-2})\,, \qquad J_0 := -\kappa\lambda^2 y^{6\kappa-1}\,, \\ J_1 &:= -\frac{1}{2}(n-1)\kappa y^{2\kappa-2}r^{-1}\,, \qquad J_2 := \frac{1}{4}(n-1)(n-3)y^{2\kappa}r^{-3}\,. \end{aligned} \tag{1.3.21}$$

Observe that for sufficiently small d (depending on n and κ), there is some $0<\delta\ll 1$ (also depending on n and κ) such that:

i) $J\leqslant J_2$ whenever $0<r<\delta$.

ii) $J\leqslant J_1$ whenever $1-\delta<r<1$.

iii) For sufficiently large λ_0, we have that $J\leqslant J_0$ whenever $\delta\leqslant r\leqslant 1-\delta$.

Combining the above with (1.3.20) yields the desired bound (1.3.4), in the case $n\geqslant 4$.

Case 2: $n\leqslant 3$. For the cases $n=1$ and $n=3$, we first note that (1.3.2) implies

$$240c^2t^2\leqslant 240c^2T^2\leqslant 1\leqslant y^{4\kappa}\,. \tag{1.3.22}$$

In this setting, we must deal with $(S_{f,z}v)^2$ a bit differently. To this end, we use (1.2.5), the fact that λ_0 is sufficiently large, and the inequality

$$(A+B)^2\geqslant(1-2\epsilon)A^2-\frac{1}{2\epsilon}(1-2\epsilon)B^2$$

(with the values $\epsilon := \frac{1}{3}$, $A := y^{2\kappa} D_r v$, and $B := 2ct(\partial_t v) + w_{f,z} v$) in order to obtain

$$\lambda (S_{f,z} v)^2 \geqslant 60c \left[\frac{1}{3} y^{4\kappa} (D_r v)^2 - 4c^2 t^2 (\partial_t v)^2 - w_{f,z}^2 v^2 \right] . \tag{1.3.23}$$

Moreover, expanding $w_{f,z}^2$ using (1.2.6) and excluding terms with favorable sign yields

$$\begin{aligned} \lambda (S_{f,z} v)^2 \geqslant{}& 20c y^{4\kappa} (D_r v)^2 - 240 c^3 t^2 (\partial_t v)^2 - 540 c^3 v^2 \\ &- 60c \left[4\kappa^2 y^{4\kappa-2} + \frac{(n-1)^2}{4r^2} y^{4\kappa} - \frac{2\kappa(n-1)}{r} y^{4\kappa-1} \right] v^2 . \end{aligned} \tag{1.3.24}$$

The pointwise Hardy inequality (1.1.10), with $q := 4\kappa + 1$, yields

$$\begin{aligned} y^{4\kappa} (D_r v)^2 \geqslant{}& \frac{1}{4} (1 - 6\kappa)^2 y^{4\kappa-2} \cdot v^2 + \frac{(1-6\kappa)(n-1)}{2r} y^{4\kappa-1} \cdot v^2 \\ &+ \nabla^\beta \left[\frac{(1-6\kappa)}{2} y^{4\kappa-1} \nabla_\beta y \cdot v^2 \right] . \end{aligned}$$

Combining the above with (1.3.22) and (1.3.24), and noting that

$$\frac{15}{4} (1 - 6\kappa)^2 > 240 \kappa^2 ,$$

we then obtain the bound

$$\begin{aligned} \lambda (S_{f,z} v)^2 \geqslant{}& 5c (D_r v)^2 - c (\partial_t v)^2 - 15 c (n-1)^2 y^{4\kappa} r^{-2} v^2 \\ &- C (n-1) y^{4\kappa-1} r^{-1} v^2 + \nabla^\beta \left[\frac{15c(1-6\kappa)}{2} y^{4\kappa-1} \nabla_\beta y \cdot v^2 \right] , \end{aligned} \tag{1.3.25}$$

where $C > 0$ depends on n and κ.

Now, applying the multiplier inequality (1.2.11) and (1.3.25) to (1.3.15), we see that

$$
\begin{aligned}
\frac{1}{4\lambda}\int_{\mathcal{C}_\varepsilon}(\mathcal{L}v)^2 \geqslant & \int_{\mathcal{C}_\varepsilon}\left[(1-4c)|\nabla\!\!\!/ v|^2 + c(\partial_t v)^2 + c(D_r v)^2\right] \\
& - \kappa\lambda^2\int_{\mathcal{C}_\varepsilon} y^{6\kappa-1}\cdot v^2 - \frac{1}{2}(n-1)\kappa\int_{\mathcal{C}_\varepsilon} y^{2\kappa-2}r^{-1}\cdot v^2 \\
& + \frac{1}{2}(n-1)(n-4)\kappa\int_{\mathcal{C}_\varepsilon} y^{2\kappa-1}r^{-2}\cdot v^2 \\
& - 15c(n-1)^2\int_{\mathcal{C}_\varepsilon} y^{4\kappa}r^{-2}\cdot v^2 \\
& - C(n-1)\int_{\mathcal{C}_\varepsilon} y^{4\kappa-1}r^{-1}\cdot v^2 \\
& - \int_{\Gamma_\varepsilon} S_{f,z}v\cdot D_\nu v + \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu f\cdot D_\beta v D^\beta v \\
& + \frac{1}{2}\int_{\Gamma_\varepsilon}\nabla_\nu w_{f,z}\cdot v^2 + 2\kappa(2\kappa-1)\int_{\Gamma_\varepsilon} y^{2\kappa-2}\nabla_\nu y\cdot v^2 \\
& - \frac{1}{2}\int_{\Gamma_\varepsilon}[\lambda^2(y^{4\kappa}-4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2 \\
& + c_2\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2\,.
\end{aligned}
\tag{1.3.26}
$$

For $n = 1$, the bound (1.3.26) immediately implies (1.3.4).

For the remaining case $n = 3$, we also note from (1.3.2) that

$$
\frac{1}{2}(n-1)(n-4)\kappa y^{2\kappa-1}r^{-2} - 15c(n-1)^2 y^{4\kappa}r^{-2} \geqslant -\frac{1}{2}\kappa y^{2\kappa-1}r^{-2}\,. \tag{1.3.27}
$$

To control the remaining bulk integrand $-C(n-1)y^{4\kappa-1}r^{-1}\cdot v^{-2}$, we define

$$
\begin{aligned}
K &:= dy^{2\kappa-2}r^{-2} + C(n-1)y^{4\kappa-1}r^{-1}\,, \qquad K_0 := -\kappa\lambda^2 y^{6\kappa-1}\,, \\
K_1 &:= -\frac{1}{2}(n-1)\kappa y^{2\kappa-2}r^{-1}\,, \qquad K_2 := -\frac{1}{2}\kappa y^{2\kappa-1}r^{-2}\,.
\end{aligned}
\tag{1.3.28}
$$

Like for the $n \geqslant 4$ case, as long as d is sufficiently small (depending on n and κ), then there exists $0 < \delta \ll 1$ (depending on n and κ) such that:

i) $K \leqslant K_2$ whenever $0 < r < \delta$.

ii) $K \leqslant K_1$ whenever $1 - \delta < r < 1$.

iii) For large enough λ_0, we have that $K \leqslant K_0$ whenever $\delta \leqslant r \leqslant 1 - \delta$.

Combining the above with (1.3.26) and (1.3.27) yields (1.3.4) for $n = 3$. □

In this subsection, we derive and control the limits of the boundary terms in (1.3.4) when $\varepsilon \searrow 0$. More specifically, we show the following:

Lemma 1.3.4. *Let $\Gamma_\varepsilon^\pm$ be as in* (1.A). *Then, for $\lambda \geqslant \lambda_0$,*

$$\begin{aligned} -c_3 \int_\Gamma e^{2\lambda f} (\mathcal{N}_\kappa u)^2 \leqslant \liminf_{\varepsilon \searrow 0} \left[\int_{\Gamma_\varepsilon^+} \nabla_\nu f \cdot D_\beta v D^\beta v - 2 \int_{\Gamma_\varepsilon^+} S_{f,z} v D_\nu v \right] \\ - \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^+} [\lambda^2 (y^{4\kappa} - 4c^2 t^2) - 8c\lambda] \nabla_\nu f \cdot v^2 \\ + \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^+} \nabla_\nu w_{f,z} \cdot v^2 \\ + 4\kappa(2\kappa - 1) \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^+} y^{2\kappa-2} \nabla_\nu y \cdot v^2 \,, \\ 0 = \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^+} y^{4\kappa-1} \nabla_\nu y \cdot v^2 \,, \end{aligned} \tag{1.3.29}$$

where the constant $c_3 > 0$ depends on κ. In addition, for $\lambda \geqslant \lambda_0$,

$$\begin{aligned} 0 \leqslant \lim_{\varepsilon \searrow 0} \left[\int_{\Gamma_\varepsilon^-} \nabla_\nu f \cdot D_\beta v D^\beta v - 2 \int_{\Gamma_\varepsilon^-} S_{f,z} v D_\nu v \right] \\ - \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} [\lambda^2 (y^{4\kappa} - 4c^2 t^2) - 8c\lambda] \nabla_\nu f \cdot v^2 \\ + \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} \nabla_\nu w_{f,z} \cdot v^2 + 4\kappa(2\kappa - 1) \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} y^{2\kappa-2} \nabla_\nu y \cdot v^2 \,, \\ 0 \leqslant \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} y^{4\kappa-1} \nabla_\nu y \cdot v^2 \,. \end{aligned} \tag{1.3.30}$$

Proof. First, note that on $\Gamma_\varepsilon^\pm$, we have

$$\nu|_{\Gamma_\varepsilon^\pm} = \pm \partial_r \,, \qquad \nabla_\nu y|_{\Gamma_\varepsilon^\pm} = \mp 1 \,, \qquad \nabla_\nu f|_{\Gamma_\varepsilon^\pm} = \pm y^{2\kappa}|_{\Gamma_\varepsilon^\pm} \,. \tag{1.3.31}$$

Moreover, note that (1.2.3) and (1.2.5) imply

$$S_{f,z} v = y^{2\kappa} D_r v + 2ct \cdot \partial_t v + w_{f,z} \cdot v \,. \tag{1.3.32}$$

We begin with the outer limits (1.3.29). The main observation is that by (1.2.3) and by the assumption that u is boundary admissible (see Definition 1.1.2), we have

$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{2\kappa}(\partial_t v)^2 = 0\,, \tag{1.3.33}$$
$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{2\kappa}(D_r v)^2 = \int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,,$$
$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{-2+2\kappa} v^2 = (1-2\kappa)^{-2}\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,.$$

We also recall that we have assumed $-\frac{1}{2}<\kappa<0$.

For the first boundary term, we apply (1.3.31) and (1.3.33) to obtain

$$\liminf_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+}\nabla_\nu f\cdot D_\beta v D^\beta v \geqslant \lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{2\kappa}[-(\partial_t v)^2+(D_r v)^2] \tag{1.3.34}$$
$$= \int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,.$$

Next, expanding $S_{f,z}v$ using (1.3.32), noting from (1.2.6) that the leading-order behavior of $w_{f,z}$ near Γ is $-2\kappa\cdot y^{2\kappa-1}$, and applying (1.3.33), we obtain that

$$-2\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} S_{f,z}vD_\nu v = -2\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+}[y^{2\kappa}(D_r v)^2+2ct\partial_t vD_r v+w_{f,z}vD_r v] \tag{1.3.35}$$
$$= -2\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2+4\kappa\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{2\kappa-1}vD_r v$$
$$= \left(-2+\frac{4\kappa}{1-2\kappa}\right)\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,.$$

The remaining outer boundary terms are treated similarly. By (1.3.31) and (1.3.33),

$$-\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+}[\lambda^2(y^{4\kappa}-4c^2t^2)-8c\lambda]\nabla_\nu f\cdot v^2 = -\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{6\kappa}v^2 = 0\,, \tag{1.3.36}$$
$$4\kappa(2\kappa-1)\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+} y^{2\kappa-2}\nabla_\nu y\cdot v^2 = \frac{4\kappa}{1-2\kappa}\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,.$$

Moreover, by (1.2.6) and (1.3.31), we see that the leading-order behavior of $\partial_r w_{f,z}$ is given by $-2\kappa(1-2\kappa)y^{2\kappa-2}$. Combining this with (1.3.31) and (1.3.33) yields

$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+}\nabla_\nu w_{f,z}\cdot v^2 = -2\kappa(1-2\kappa)\lim_{\varepsilon\searrow 0}\int_\Gamma y^{2\kappa-2}v^2 \tag{1.3.37}$$
$$= -\frac{2\kappa}{1-2\kappa}\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2\,.$$

Summing (1.3.34)–(1.3.37) yields the first part of (1.3.29). The second part of (1.3.29) similarly follows by applying (1.3.31) and (1.3.33).

Next, for the interior limits (1.3.30), we split into two cases:

Case 1: $n \geqslant 3$. In this case, we begin by noting that the volume of Γ_ε^- satisfies

$$|\Gamma_\varepsilon^-| \lesssim_{T,n} \varepsilon^{n-1} . \tag{1.3.38}$$

Furthermore, since u is smooth on $\mathcal{C}$, then (1.2.3) and (2.2.7) imply that $\partial_t v$, $\nabla\!\!\!/ v$, $D_r v$, and v are all uniformly bounded whenever r is sufficiently small. Combining the above with (1.2.6), (1.3.31), (1.3.32), we obtain that the following limits vanish:

$$\begin{aligned} 0 &= \lim_{\varepsilon \searrow 0} \left[\int_{\Gamma_\varepsilon^-} \nabla_\nu f \cdot D_\beta v D^\beta v - 2 \int_{\Gamma_\varepsilon^-} S_{f,z} v D_\nu v \right] \\ &\quad - \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} [\lambda^2 (y^{4\kappa} - 4c^2 t^2) - 8c\lambda] \nabla_\nu f \cdot v^2 \\ &\quad + 4\kappa(2\kappa - 1) \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} y^{2\kappa-2} \nabla_\nu y \cdot v^2 , \\ 0 &= \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} y^{4\kappa-1} \nabla_\nu y \cdot v^2 . \end{aligned} \tag{1.3.39}$$

This leaves only one remaining limit in (1.3.30); for this, we note, from (1.2.6), that the leading-order behavior of $-\partial_r w_{f,z}$ near $r = 0$ is $\frac{1}{2}(n-1) r^{-2} y^{2\kappa}$. As a result,

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} \nabla_\nu w_{f,z} \cdot v^2 &= \frac{n-1}{2} \lim_{\varepsilon \searrow 0} \int_{\Gamma_\varepsilon^-} r^{-2} y^{2\kappa} v^2 \\ &= \begin{cases} 0 & n > 3 \\ C \int_{-T}^{T} |v(t,0)|^2 dt & n = 3 \end{cases} , \end{aligned} \tag{1.3.40}$$

where the last integral is over the line $r = 0$, and where the constant C depends only on n. Combining (1.3.39) and (1.3.40) yields (1.3.30) in this case.

Case 2: $n = 1$. Here, we can no longer rely on (1.3.38) to force most limits to vanish, so we must examine all the terms more carefully.

First, from (1.2.6), (1.3.31), (1.3.32), we have that

$$\begin{aligned} &\int_{\Gamma_\varepsilon^-} \nabla_\nu f \cdot D_\beta v D^\beta v - 2 \int_{\Gamma_\varepsilon^-} S_{f,z} v D_\nu v \\ &\quad = \int_{\Gamma_\varepsilon^-} y^{2\kappa} [(\partial_t v)^2 + (D_r v)^2] + \int_{\Gamma_\varepsilon^-} [4ct \cdot \partial_t v D_r v - 4\kappa y^{2\kappa-1} v D_r v] . \end{aligned}$$

Recalling also our assumption (1.3.2) for c, we conclude from the above that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\left[\int_{\Gamma_\varepsilon^-}\nabla_\nu f\cdot D_\beta v D^\beta v-2\int_{\Gamma_\varepsilon^-}S_{f,z}vD_\nu v\right]&\geqslant -C\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-}y^{2\kappa-2}v^2 \qquad (1.3.41)\\ &=-C\int_{-T}^{T}|v(t,0)|^2dt\,,\end{aligned}$$

where the last integral is over the line $r=0$, and where C depends only on κ. Moreover, letting λ_0 be sufficiently large and recalling (1.3.2) and (1.3.31), we obtain

$$-\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-}[\lambda^2(y^{4\kappa}-4c^2t^2)-8c\lambda]\nabla_\nu f\cdot v^2\geqslant\tilde{C}\lambda^2\int_{-T}^{T}|v(t,0)|^2dt\,, \qquad (1.3.42)$$

for some constant $\tilde{C}>0$.

Next, applying (1.2.6) and (1.3.31) in a similar manner as before, we obtain inequalities for the remaining limits in the right-hand side of (1.3.30):

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-}\nabla_\nu w_{f,z}\cdot v^2&\geqslant -C\int_{-T}^{T}|v(t,0)|^2dt\,, \qquad (1.3.43)\\ 4\kappa(2\kappa-1)\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-}y^{2\kappa-2}\nabla_\nu y\cdot v^2&\geqslant -C\int_{-T}^{T}|v(t,0)|^2dt\,,\\ \lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-}y^{4\kappa-1}\nabla_\nu y\cdot v^2&=2\int_{-T}^{T}|v(t,0)|^2dt\,.\end{aligned}$$

Here, C denotes various positive constants that depend on κ. Finally, combining (1.3.41)–(1.3.43) and taking λ_0 to be sufficiently large results in (1.3.30). □

We are now in position to complete the proof of Theorem 1.3.1. First, recalling the definitions (1.2.3) and (2.2.7) of f and v and the fact that $c^2t^2\lesssim 1$ by our assumption (1.3.2), we have that

$$\begin{aligned}e^{2\lambda f}(\partial_t u)^2&\lesssim(\partial_t v)^2+\lambda^2c^2t^2v^2\lesssim(\partial_t v)^2+\lambda^2y^{6\kappa-1}v^2\,, \qquad (1.3.44)\\ e^{2\lambda f}(D_r u)^2&\lesssim(D_r v)^2+\lambda^2y^{4\kappa}v^2\lesssim(D_r v)^2+\lambda^2y^{6\kappa-1}v^2\,,\\ e^{2\lambda f}|\nabla\!\!\!/ u|^2&=|\nabla\!\!\!/ v|^2\,.\end{aligned}$$

Furthermore, by (1.1.9) and (2.2.7), we observe that

$$(\mathcal{L}v)^2\leqslant 2e^{2\lambda f}[(\Box_\kappa u)^2+\kappa(n-1)y^{-2}r^{-2}\cdot u^2]\,. \qquad (1.3.45)$$

Therefore, using these bounds in Lemma 1.3.3, it follows that

$$\begin{aligned}
&2\int_{\mathcal{C}_\varepsilon} e^{2\lambda f}(\Box_\kappa u)^2 + 2\kappa(n-1)\int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{-1} r^{-1}\cdot u^2 \\
&\quad\geqslant C\lambda \int_{\mathcal{C}_\varepsilon} e^{2\lambda f}[(\partial_t u)^2 + |\nabla\!\!\!/ u|^2 + (D_r u)^2] + C\lambda^3 \int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{6\kappa-1} u^2 \\
&\qquad + 2\lambda\int_{\Gamma_\varepsilon} \nabla_\nu f\cdot D_\beta v D^\beta v - 4\lambda\int_{\Gamma_\varepsilon} S_{f,z} v\cdot D_\nu v \\
&\qquad - 2\lambda\int_{\Gamma_\varepsilon} [\lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2 \\
&\qquad + 2\lambda\int_{\Gamma_\varepsilon} \nabla_\nu w_{f,z}\cdot v^2 + 8\lambda\kappa(2\kappa-1)\int_{\Gamma_\varepsilon} y^{2\kappa-2}\nabla_\nu y\cdot v^2 \\
&\qquad + \begin{cases} C\lambda\int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{2\kappa-2} r^{-3}\cdot u^2 & n\geqslant 4 \\ C\lambda\int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{2\kappa-2} r^{-2}\cdot u^2 + 4c_2\lambda\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2 & n=3 \\ 4c_2\lambda\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2 & n=1 \end{cases},
\end{aligned} \tag{1.3.46}$$

for some constant $C > 0$ depending on n and κ. Note that if λ_0 is sufficiently large, then the last term on the left-hand side of (1.3.46) can be absorbed into the last term on the right-hand side of (1.3.46) (for all values of n). From this, we obtain

$$\begin{aligned}
\int_{\mathcal{C}_\varepsilon} e^{2\lambda f}(\Box_\kappa u)^2 &\geqslant C\lambda\int_{\mathcal{C}_\varepsilon} e^{2\lambda f}[(\partial_t u)^2 + |\nabla\!\!\!/ u|^2 + (D_r u)^2 + \lambda^2 y^{6\kappa-1}u^2] \\
&\quad + \begin{cases} C\lambda\int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{2\kappa-2} r^{-3}\cdot u^2 & n\geqslant 4 \\ C\lambda\int_{\mathcal{C}_\varepsilon} e^{2\lambda f} y^{2\kappa-2} r^{-2}\cdot u^2 & n=3 \\ 0 & n=1 \end{cases} \\
&\quad + \lambda\int_{\Gamma_\varepsilon} \nabla_\nu f\cdot D_\beta v D^\beta v - 2\lambda\int_{\Gamma_\varepsilon} S_{f,z} v\cdot D_\nu v \\
&\quad - \lambda\int_{\Gamma_\varepsilon} [\lambda^2(y^{4\kappa} - 4c^2t^2) - 8c\lambda]\nabla_\nu f\cdot v^2 \\
&\quad + \lambda\int_{\Gamma_\varepsilon} \nabla_\nu w_{f,z}\cdot v^2 + 4\lambda\kappa(2\kappa-1)\int_{\Gamma_\varepsilon} y^{2\kappa-2}\nabla_\nu y\cdot v^2 \\
&\quad + \begin{cases} 0 & n\geqslant 4 \\ 2c_2\lambda\int_{\Gamma_\varepsilon} y^{4\kappa-1}\nabla_\nu y\cdot v^2 & n\leqslant 3 \end{cases}.
\end{aligned} \tag{1.3.47}$$

Finally, the desired inequality (1.3.1) follows by taking the limit $\varepsilon \searrow 0$ in (1.3.47) and applying all the inequalities from Lemma 1.3.4.

1.4 Observability

Our aim in this section is to show that the Carleman estimates of Theorem 1.3.1 imply a boundary observability property for solutions to wave equations on the cylindrical spacetime $\mathcal{C}$ containing potentials that are critically singular at the boundary Γ. More specifically, we establish the following result, which is a precise and a slightly stronger version of the result stated in Theorem 2.

Theorem 1.4.1. *Assume $n \neq 2$, and fix $-\frac{1}{2} < \kappa < 0$. Let u be a solution to*

$$\Box_\kappa u = D_X u + V u \,, \tag{1.4.1}$$

on $\overline{\mathcal{C}}$, where the vector field $X : \mathcal{C} \to \mathbb{R}^{1+n}$ and the potential $V : \mathcal{C} \to \mathbb{R}$ satisfy

$$|X| \lesssim 1 \,, \qquad |V| \lesssim \frac{1}{y} + \frac{n-1}{r} \,, \tag{1.4.2}$$

In addition, assume that:

i) u is boundary admissible (in the sense of Definition 1.1.2).

ii) u has finite twisted H^1-energy for any $\tau \in (-T, T)$:

$$E_1[u](\tau) = \int_{\mathcal{C}\cap\{t=\tau\}} ((\partial_t u)^2 + (D_r u)^2 + |\nabla\!\!\!/ u|^2 + u^2) < \infty \,. \tag{1.4.3}$$

Then, for sufficiently large observation time T satisfying

$$\begin{cases} T > \frac{4\sqrt{3}}{1+2\kappa} & n \geqslant 4 \\ T > \max\left\{ \frac{4\sqrt{15}}{1+2\kappa}, \frac{2\sqrt{30}}{\sqrt{|\kappa|(1+2\kappa)}} \right\} & n = 3 \\ T > \frac{4\sqrt{15}}{1+2\kappa} & n = 1 \end{cases} , \tag{1.4.4}$$

we have the boundary observability inequality

$$\int_\Gamma (\mathcal{N}_\kappa u)^2 \gtrsim E_1[u](0) \,, \tag{1.4.5}$$

where the constant of the inequality depends on n, κ, T, X, and V.

In order to prove Theorem 1.4.1, we require preliminary estimates. The first is a Hardy estimate to control singular integrands:

Lemma 1.4.2. *Assume the hypotheses of Theorem 1.4.1. Then,*

$$\int_{\mathcal{C}\cap\{t_0<t<t_1\}} \left(\frac{1}{y^2} + \frac{n-1}{r^2} \right) u^2 \lesssim \int_{\mathcal{C}\cap\{t_0<t<t_1\}} (D_r u)^2 \,, \tag{1.4.6}$$

for any $-T \leqslant t_0 < t_1 \leqslant T$, where the constant depends only on n and κ.

Proof. The inequality (1.1.10), with $q = 1$, yields

$$(D_r u)^2 \geqslant \frac{1}{8}(1-2\kappa)^2 \frac{u^2}{y^2} + \frac{(n-1)}{9} \frac{u^2}{r^2} + \frac{(1-2\kappa)}{2} \nabla^\beta (\nabla_\beta y \cdot y^{-1} u^2) \,.$$

Letting $0 < \varepsilon \ll 1$ and integrating the above over $\mathcal{C} \cap \{t_0 < t < t_1\}$ yields

$$\begin{aligned} \int_{\mathcal{C}_\varepsilon\cap\{t_0<t<t_1\}} (D_r u)^2 \geqslant C & \int_{\mathcal{C}_\varepsilon\cap\{t_0<t<t_1\}} \left(\frac{1}{y^2} + \frac{n-1}{r^2} \right) u^2 \\ & - \frac{(1-2\kappa)}{2} \int_{\Gamma_\varepsilon^+\cap\{t_0<t<t_1\}} y^{-1} u^2 \\ & + \frac{(1-2\kappa)}{2} \int_{\Gamma_\varepsilon^-\cap\{t_0<t<t_1\}} y^{-1} u^2 \\ \geqslant C & \int_{\mathcal{C}_\varepsilon\cap\{t_0<t<t_1\}} \left(\frac{1}{y^2} + \frac{n-1}{r^2} \right) u^2 \\ & - \frac{(1-2\kappa)}{2} \int_{\Gamma_\varepsilon^+\cap\{t_0<t<t_1\}} y^{-1} u^2 \,. \end{aligned}$$

(Here, we have also made use of the identities (1.3.31).) Letting $\varepsilon \searrow 0$ and recalling that u is boundary admissible results in the estimate (1.4.6). □

We will also need the following energy estimate for solutions to (1.4.1):

Lemma 1.4.3. *Assume the hypotheses of Theorem 1.4.1. Then,*

$$E_1[u](t_1) \leqslant e^{M|t_1-t_0|} E_1[u](t_0) \,, \qquad t_0, t_1 \in (-T, T) \,, \tag{1.4.7}$$

where the constant M depends on n, κ, X, and V.

Proof. We assume for convenience that $t_0 < t_1$; the opposite case can be proved analogously. By a standard density argument, we can assume u is smooth within $\mathcal{C}$. Fix now a sufficiently small $0 < \varepsilon \ll 1$, and define

$$E_{1,\varepsilon}[u](\tau) = \int_{\mathcal{C}_\varepsilon\cap\{t=\tau\}} ((\partial_t u)^2 + (D_r u)^2 + |\nabla\!\!\!/ u|^2 + u^2) \,. \tag{1.4.8}$$

Differentiating $E_{1,\varepsilon}[u]$ and integrating by parts, we obtain, for any $\tau \in (-T, T)$,

$$\begin{aligned}\frac{d}{d\tau}E_{1,\varepsilon}[u](\tau) &= 2\int_{\mathcal{C}_\varepsilon\cap\{t=\tau\}}(\partial_{tt}u\partial_t u + D^j u D_j \partial_t u + u\partial_t u) \\ &= -2\int_{\mathcal{C}_\varepsilon\cap\{t=\tau\}}\partial_t u(\Box_y u - u) + 2\int_{\Gamma_\varepsilon\cap\{t=\tau\}}\partial_t u D_\nu u\,.\end{aligned} \tag{1.4.9}$$

Note that (1.1.9), (1.4.1), and (1.4.2) imply

$$\begin{aligned}|\Box_y u| &\lesssim \left|D_X u + Vu + \frac{(n-1)\kappa}{ry}u\right| \\ &\lesssim |\partial_t u| + |\nabla\!\!\!/ u| + |D_r u| + \left(\frac{1}{y} + \frac{n-1}{r}\right)|u|\,.\end{aligned}$$

Combining the above with (1.4.9) yields

$$\begin{aligned}\frac{d}{d\tau}E_{1,\varepsilon}[u](\tau) \leqslant C\cdot E_1[u](\tau) + C\cdot E_1^{\frac{1}{2}}[u](\tau)\left[\int_{\mathcal{C}\cap\{t=\tau\}}\left(\frac{1}{y^2}+\frac{n-1}{r^2}\right)u^2\right]^{\frac{1}{2}} \\ + 2\int_{\Gamma_\varepsilon\cap\{t=\tau\}}\partial_t u D_\nu u\,.\end{aligned}$$

Next, integrating the above in τ and applying Lemma 1.4.2, we obtain

$$E_{1,\varepsilon}[u](t_1) \leqslant E_1[u](t_0) + C\int_{t_0}^{t_1}E_1[u](\tau)\,d\tau + 2\int_{\Gamma_\varepsilon\cap\{t_0<t<t_1\}}\partial_t u D_\nu u\,. \tag{1.4.10}$$

Since u is boundary admissible, it follows that

$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^+\cap\{t_0<t<t_1\}}\partial_t u D_\nu u = 0\,. \tag{1.4.11}$$

Moreover, since ν points radially along Γ_ε^-, then by symmetry,

$$\lim_{\varepsilon\searrow 0}\int_{\Gamma_\varepsilon^-\cap\{t_0<t<t_1\}}\partial_t u D_\nu u = 0\,. \tag{1.4.12}$$

(Alternatively, when $n > 1$, we can also use (1.3.38).)

Letting $\varepsilon \searrow 0$ in (1.4.10) and applying (1.4.11)–(1.4.12), we conclude that

$$E_1[u](t_1) \leqslant E_1[u](t_0) + C\int_{t_0}^{t_1}E_1[u](\tau)\,d\tau\,.$$

The estimate (1.4.7) now follows from the Grönwall inequality. □

Assume the hypotheses of Theorem 1.4.1, and set

$$c = \begin{cases} \frac{1}{4\sqrt{3}\cdot T} & n \geqslant 4 \\ \min\left\{\frac{1}{4\sqrt{15}\cdot T}, \frac{|\kappa|}{120}\right\} & n = 3 \\ \frac{1}{4\sqrt{15}\cdot T} & n = 1 \end{cases} . \tag{1.4.13}$$

Note, in particular, that (1.4.13) and (1.4.4) imply that the conditions (1.3.2) hold.

Moreover, we define the function f as in the statement of Theorem 1.3.1, with c as in (1.4.13). Then, direct computations, along with (1.4.4), imply that

$$\inf_{\mathcal{C}\cap\{t=0\}} f \geqslant -(1+2\kappa)^{-1}, \qquad \sup_{\mathcal{C}\cap\{t=\pm T\}} f < -(1+2\kappa)^{-1}.$$

Hence, one can find constants $0 < \delta \ll T$ and $\mu_\kappa > (1+2\kappa)^{-1}$ such that

$$\begin{cases} f \leqslant -\mu_\kappa & \text{when } t \in (-T, -T+\delta) \cup (T-\delta, T) \\ f \geqslant -\mu_\kappa & \text{when } t \in (-\delta, \delta) \end{cases} . \tag{1.4.14}$$

In addition, we define the shorthands

$$I_\delta = [-T+\delta, T-\delta], \qquad J_\delta = (-T, -T+\delta) \cup (T-\delta, T). \tag{1.4.15}$$

We also let $\xi \in C^\infty(\overline{\mathcal{C}})$ be a cutoff function satisfying:

i) ξ depends only on t.

ii) $\xi = 1$ when $t \in I_\delta$.

iii) $\xi = 0$ near $t = \pm T$.

We can then apply the Carleman inequality in Theorem 1.3.1, with our above choice (1.4.13) of c and to the function ξu, in order to obtain

$$\begin{aligned} \lambda \int_\Gamma e^{2\lambda f} \xi^2 (\mathcal{N}_\kappa u)^2 + \int_{\mathcal{C}} e^{2\lambda f} |\Box_\kappa (\xi u)|^2 & \\ \gtrsim \lambda \int_{\mathcal{C}} e^{2\lambda f} [|\partial_t(\xi u)|^2 + \xi^2 |\nabla\!\!\!/ u|^2 + \xi^2 (D_r u)^2 + \lambda^2 y^{-1+6\kappa} \xi^2 u^2] & \\ \gtrsim \lambda \int_{I_\delta \times B_1} e^{2\lambda f} [(\partial_t u)^2 + |\nabla\!\!\!/ u|^2 + (D_r u)^2 + \lambda^2 y^{-1+6\kappa} u^2]. & \end{aligned} \tag{1.4.16}$$

Moreover, noting that

$$\begin{aligned}|\Box_\kappa(\xi u)| &\lesssim |\xi\Box_\kappa u| + |\partial_t\xi||\partial_t u| + |\partial_t^2\xi||u|\\ &\lesssim |\Box_\kappa u| + |\partial_t u| + |u|\,,\end{aligned}$$

and recalling (1.4.2) and (1.4.14), we derive that

$$\begin{aligned}\int_{\mathcal{C}} e^{2\lambda f}|\Box_\kappa(\xi u)|^2 &\lesssim \int_{I_\delta\times B_1} e^{2\lambda f}|\Box_\kappa u|^2 + \int_{J_\delta\times B_1} e^{2\lambda f}(|\Box_\kappa u| + |\partial_t u| + |u|)\\ &\lesssim \int_{I_\delta\times B_1} e^{2\lambda f}(|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2)\\ &\quad + \int_{I_\delta\times B_1}\left(\frac{1}{y^2} + \frac{n-1}{r^2}\right)(e^{\lambda f}u)^2\\ &\quad + e^{-2\lambda\mu_\kappa}\int_{J_\delta\times B_1}(|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2)\\ &\quad + e^{-2\lambda\mu_\kappa}\int_{J_\delta\times B_1}\left(\frac{1}{y^2} + \frac{n-1}{r^2}\right)u^2\,,\end{aligned}$$

where the implicit constants of the inequalities depend also on X and V. Applying Lemma 1.4.2 and recalling the definition of f, the above becomes

$$\begin{aligned}\int_{\mathcal{C}} e^{2\lambda f}|\Box_\kappa(\xi u)|^2 &\lesssim \int_{I_\delta\times B_1}[e^{2\lambda f}(|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2) + |D_r(e^{\lambda f}u)|^2] \qquad (1.4.17)\\ &\quad + e^{-2\lambda\mu_\kappa}\int_{J_\delta\times B_1}(|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2)\\ &\lesssim \int_{I_\delta\times B_1} e^{2\lambda f}(|\partial_t u|^2 + |D_r u|^2 + |\nabla\!\!\!/ u|^2 + \lambda^2 y^{4\kappa}u^2)\\ &\quad + e^{-2\lambda\mu_\kappa}\int_{J_\delta} E_1[u](\tau)\,d\tau\,,\end{aligned}$$

Combining the inequalities (1.4.16) and (1.4.17) and letting λ be sufficiently large (depending also on X and V), we then arrive at the bound

$$\begin{aligned}&\lambda\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2 + e^{-2\lambda\mu_\kappa}\int_{J_\delta} E_1[u](\tau)\,d\tau\\ &\quad\gtrsim \lambda\int_{I_\delta\times B_1} e^{2\lambda f}(|\partial_t u|^2 + |\nabla\!\!\!/ u|^2 + |D_r u|^2 + \lambda^2 y^{6\kappa-1}u^2)\end{aligned}$$

Further restricting the domain of the integral in the right-hand side to $(-\delta,\delta)\times B_1$ and recalling the lower bound in (1.4.14), the above becomes

$$\lambda\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2 + e^{-2\lambda\mu_\kappa}\int_{J_\delta} E_1[u](\tau)\,d\tau \gtrsim \lambda e^{-2\lambda\mu_\kappa}\int_{-\delta}^{\delta} E_1[u](\tau)\,d\tau\,. \qquad (1.4.18)$$

Finally, the energy estimate (1.4.7) implies

$$e^{-MT}E_1[u](0) \leqslant E_1[u](t) \leqslant e^{MT}E_1[u](0)\,,$$

which, when combined with (1.4.18), yields

$$\lambda\int_\Gamma e^{2\lambda f}(\mathcal{N}_\kappa u)^2 + \delta e^{-2\lambda\mu_\kappa}e^{MT}\cdot E_1[u](0) \gtrsim \lambda\delta e^{-2\lambda\mu_\kappa}e^{-MT}\cdot E_1[u](0)\,. \quad (1.4.19)$$

Taking λ in (1.4.19) large enough such that $e^{2MT} \ll \lambda$ results in (1.4.5).

1.A One-dimensional Results

To highlight even more the differences with the classical case, in this appendix we would like to illustrate the multiplier method in the simplest model of the free wave equation in one spatial dimension. We also recall the concept of *pseudoconvexity* and show that an observability estimate is precisely a necessary condition for *controllability*. At the end of the appendix, we describe the spectral approach of [39], which suggests that a one-dimensional analog of our Theorem 1.3.1 is false for positive values of the strength parameter κ.

Free Wave Equation

Let us begin by considering a regular enough solution to the $(1+1)$-dimensional wave equation

$$\Box u = -\partial_{tt}u + \partial_{yy}u = 0 \quad (1.A.1)$$

in $(-T,T)\times[0,1]$, together with trivial boundary conditions $u|_{\widehat{\Gamma}} = 0$ on

$$\widehat{\Gamma} := [(-T,T)\times\{y=0\}]\cup[(-T,T)\times\{y=1\}]\,,$$

and initial data $u(0,\cdot) = u_0 \in H^1([0,1])$ and $\partial_t u(0,\cdot) = u_1 \in L^2([0,1])$.

As in the rest of the chapter, here we adopt the summation convention for repeated indices, contracted always with the Minkowski metric $\eta = -dt^2 + dy^2$. Of course, in this simple case we employ the usual partial derivatives instead of covariant or twisted derivatives. Taking the classical weight

$$f_0(t,y) = (y-1)^2 - ct^2\,,$$

we perform the following integration by parts:

$$\begin{aligned}0 &= \int_{(-T,T)\times[0,1]} \Box u \cdot \partial^\alpha f_0 \partial_\alpha u \\ &= \int_{\widehat{\Gamma}} \partial^\alpha f_0 \partial_\alpha u \partial_\beta u \cdot \nu^\beta - \frac{1}{2}\int_{(-T,T)\times[0,1]} \partial_\alpha(\partial^\beta u \partial_\beta u)\cdot \partial^\alpha f_0 \\ &\quad - \int_{(-T,T)\times[0,1]} \partial^{\alpha\beta} f_0 \cdot \partial_\alpha u \partial_\beta u \\ &= \int_{\widehat{\Gamma}} \Big(\partial^\alpha f_0 \partial_\beta u - \frac{1}{2}\partial_\beta f_0 \partial^\alpha u\Big)\nu^\beta \partial_\alpha u - \int_{(-T,T)\times[0,1]} \Big(\partial^{\alpha\beta} f_0 - \frac{1}{2}\Box f_0 \eta^{\alpha\beta}\Big)\partial_\alpha u \partial_\beta u \,.\end{aligned}$$

This computation immediately implies that, when $0 < c < 1$,

$$\int_{\{y=0\}} (\partial_y u)^2 \gtrsim \|u_0\|^2_{H^1([0,1])} + \|u_1\|^2_{L^2([0,1])} \tag{1.A.2}$$

by the prescribed boundary conditions and energy estimates analogues to the ones in Lemma 1.4.3. Here one also needs to assume that $T > 1$ so that there is time for the wave to travel from $y = 1$ to $y = 0$ with velocity 1.

The main point here is that one can exploit the pseudoconvexity of f_0 with respect to $\Box$ to obtain positivity for the bulk term, which here simply means that

$$\Big(\partial^{\alpha\beta} f_0 - \frac{1}{2}\Box f_0 \eta^{\alpha\beta}\Big) X_\alpha X_\beta \gtrsim |X|^2$$

for all $X \in \mathbb{R}^{n+1}$. This property has the geometric interpretation of null geodesics remaining in the region $f_0 < 0$ after hitting tangentially the surface $f_0 = 0$ (see [42, Ch. 28.4] for more details on the role of pseudoconvexity in the uniqueness of linear PDEs). Notice that more robust Carleman inequalities for $\Box$ can be easily obtained from estimates for the commutator $e^{\lambda f_0}\Box e^{-\lambda f_0}$, by taking the parameter λ to be large enough. Again, these estimates rely ultimately on the positivity that stems from the pseudoconvexity of the classical weight f_0.

Boundary controllability

With the notation of the previous subsection, consider now the following initial boundary value problem:

$$\begin{cases} -w_{tt} + w_{yy} = 0 & \text{in} \quad (-T,T)\times[0,1] \\ w = F & \text{on} \quad \widehat{\Gamma} \\ w(0,\cdot) = w_0\,,\ w_t(0,\cdot) = w_1 & \text{in} \quad [0,1]\,, \end{cases} \tag{1.A.3}$$

where $(w_0, w_1) \in L^2([0,1]) \times H^1([0,1])$, and F is a "control function" supported in $(-T,T)\times\{y=0\} \subset \widehat{\Gamma}$. Let us also recall the definition of controllability:

Definition 1.A.1. *The system* (1.A.3) *is said to be (null) boundary controllable in time T if for every initial data $(w_0, w_1) \in L^2([0,1]) \times H^{-1}([0,1])$ there exists $F \in L^2(\widehat{\Gamma})$ such that the corresponding solution w satisfies that $w(T,\cdot) = w_t(T,\cdot) = 0$.*

It is well-known that the Hilbert uniqueness method (HUM) [54] allows us to derive a necessary and sufficient condition for controllability. In this duality argument one defines a mapping

$$\theta : H^1([0,1]) \times L^2([0,1]) \to L^2([0,1]) \times H^{-1}([0,1])$$

through $\theta(u_0, u_1) = (w_0, -w_1)$, where u_0, u_1 denote initial data of (1.A.1). In particular, it is clear that

$$\|\theta(u_0, u_1)\|_{L^2([0,1]) \times H^{-1}([0,1])} \lesssim \|F\|_{L^2([0,1])}$$

holds by the well-posedness of (1.A.3).

Moreover, it is not difficult to show the equivalence between the L^2-norm of the normal derivative and the H^1-energy, so that

$$\|\theta(u_0, u_1)\|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \lesssim \|u_0\|^2_{H^1([0,1])} + \|u_1\|^2_{L^2([0,1])} \,. \tag{1.A.4}$$

Hence the linear map θ is bounded.

On the other hand, denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1([0,1]) \times L^2([0,1])$ and $L^2([0,1]) \times H^{-1}([0,1])$, the observability inequality (1.A.2) implies that θ is also injective,

$$\langle \theta(u_0, u_1), (u_0, u_1) \rangle = \int_{\{y=0\}} (\partial_y u)^2 \gtrsim \|u_0\|^2_{H^1} + \|u_1\|^2_{L^2} \,, \tag{1.A.5}$$

so there exists a unique (u_0^*, u_1^*) such that $\theta(u_0^*, u_1^*) = (w_0, -w_1)$. The geometric control is given then by $F = \lim_{y \searrow 0} \partial_y u^*$, where u^* is solution to (1.A.1) with initial data (u_0^*, u_1^*).

Spectral Approach

Let us finally discuss the boundary observability properties of the one-dimensional equation

$$\begin{cases} -\partial_{tt} u + \overline{D}_y D_y u = 0 \,, & \text{in } [-T, T] \times [0,1] \\ \mathcal{D}_\kappa u = 0 \,, & \text{on } \{y = 0\} \\ u(t,1) = 0 \,, & \text{on } \{y = 1\} \,, \end{cases}$$

where $\overline{D}_y, D_y$ are twisted derivatives (1.1.6) and where $\mathcal{D}_\kappa u = \lim_{y\searrow 0} y^{-\kappa}u$ denotes the Dirichlet trace.

Note that according to Sturm-Liouville theory, the Bessel-Laplacian $-\overline{D}_y D_y$ is a self-adjoint operator in $L^2([0,1])$ when $\kappa < 1/2$, and also positive by the Hardy inequality (1.1.10). Consequently, the eigenfunctions of $-\overline{D}_y D_y$ satisfy the equation

$$-\overline{D}_y D_y \psi_l = \rho_l^2 \psi_l \,,$$

where $\{\rho_l\}_{l\in\mathbb{N}}$ denote the corresponding L^2-eigenvalues. A straightforward computation using the homogeneous boundary conditions shows that the number ρ_l agrees with the l-th zero of the first kind Bessel function of order $\frac{1}{2} - \kappa$, i.e. $J_{\frac{1}{2}-\kappa}(\rho_l) = 0$ with $\rho_0 = 0$ and $l \geqslant 0$, while the eigenfunctions are given by

$$\psi_l(y) = \frac{C\sqrt{y}}{J'_{\frac{1}{2}-\kappa}(\rho_l)} J_{\frac{1}{2}-\kappa}(\rho_l y) \,,$$

where we have fixed the normalization factor using the orthogonality relation

$$\int_0^1 y J_{\frac{1}{2}-\kappa}(y\rho_l) J_{\frac{1}{2}-\kappa}(y\rho_m)\, dy = \frac{\delta_{l,m}}{2} \left(J'_{\frac{1}{2}-\kappa}(\rho_l)\right)^2 .$$

We can then separate variables and expand u in Fourier-Bessel series,

$$u(t,y) = \sum_{l\geqslant 0} u_l(t)\psi_l(y) \,, \qquad u_l^0 := u_l(0) \,,\; u_l^1 := \partial_t u_l(0) \,,$$

to find that the Neumann data can be written as

$$\mathcal{N}_\kappa u(t) = \sum_{l\geqslant 0} \Big(\cos(\rho_l t) u_l^0 + \frac{\sin(\rho_l t)}{\rho_l} u_l^1 \Big) \frac{\rho_l^{\frac{1}{2}-\kappa}}{J'_{\frac{1}{2}-\kappa}(\rho_l)} \,,$$

where we have used the asymptotic behavior of the Bessel functions [76]

$$J_\alpha(z) = \frac{1}{\Gamma(1+\alpha)} \left(\frac{z}{2}\right)^\alpha , \qquad \alpha \notin \mathbb{Z}_- \,,\; 0 < z \ll \sqrt{1+\alpha} \,.$$

Finally, since the smallest eigengap is at least some positive constant γ when the observation time T is large, we can make use of the following Ingham inequality [44]:

Lemma 1.A.2. *Let $\{\rho_l\}_{l\in\mathbb{Z}}$ be an increasing sequence of positive real numbers such that $\rho_{l+1} - \rho_l \geqslant \gamma > 0$, $\forall l \geqslant 0$, and let $\{c_l\}_{l\in\mathbb{Z}}$ be any sequence of complex numbers. Then, for $T > 0$ such that $\gamma T > 2\pi$, it follows that*

$$\int_{-T}^{T} \big| \sum_l c_l e^{i\rho_l t} \big|^2 \, dt \gtrsim \sum_l |c_l|^2 \,.$$

By using the above estimate together with the asymptotic formula

$$|J'_\alpha(z)| \leqslant \frac{C}{\sqrt{z}}\,, \quad z \gg 1\,,$$

and the expansion of large eigenvalues $\rho_l = \pi l + O(\frac{1}{l})$ (see [76]), we then arrive at the boundary observability inequality

$$\|\mathcal{N}_\kappa u\|^2_{L^2([-T,T])} \geqslant C \sum_{l \geqslant 0} \rho_l^{2-2\kappa} (u_l^0)^2 + \rho_l^{-2\kappa} (u_l^1)^2\,.$$

As a final remark, it should be noticed that this spectral approach is in general unsuitable for higher-dimensional problems. However, the regularity in the above observability estimate strongly suggests that analogues of the Carleman Theorem 1.3.1 for $\kappa > 0$ are false.

Chapter 2

Fractional wave operators

Preliminaries

There is a considerable body of classical work on fractional wave operators, which have never appeared naturally in a physical problem but which are studied in detail in the theory of hypersingular integrals (see e.g. [63] and references therein). In particular, the fractional wave operator $(-\Box)^\alpha$ is defined for all noninteger α as the multiplier

$$\widehat{(-\Box)^\alpha f}(\tau,\xi) := \sigma_\alpha(\tau,\xi)\,\widehat{f}(\tau,\xi)\,, \tag{2.0.1}$$

where the symbol σ_α is defined as

$$\sigma_\alpha(\tau,\xi) := (|\xi|^2-\tau^2)^\alpha\chi_+(\tau,\xi) + e^{i\pi\alpha\,\mathrm{sgn}(\tau)}(\tau^2-|\xi|^2)^\alpha\chi_-(\tau,\xi), \tag{2.0.2}$$

with $\chi_\pm(\tau,\xi)$ denotes the indicator function of the set $\pm(|\xi|^2-\tau^2)>0$. Equivalently, choosing the principal branch of the complex logarithm one can write

$$\sigma_\alpha(\tau,\xi) := \lim_{\epsilon\searrow 0}\left(|\xi|^2-(\tau-i\epsilon)^2\right)^\alpha. \tag{2.0.3}$$

Our aim in this chapter is to show the connection between fractional wave operators and scattering operators in asymptotically AdS backgrounds. To this end, let us start by recalling the basic structure of the AdS spacetime of dimension $n+1$, which is a Lorentzian manifold of negative constant sectional curvature (which we set here to -1) and, as such, satisfies the Einstein equations with a negative cosmological constant. For concreteness, we will consider the AdS half-space (see e.g. [3] for a mathematical analysis of the problem and [4] for the physics of this space). The metric can be written using Poincaré coordinates $(t,x,y)\in\mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ as

$$g^+ := \frac{dt^2-dy^2-|dx|^2}{y^2}, \tag{2.0.4}$$

where $|dx|^2$ denotes the standard flat metric on $\mathbb{R}^{n-1}$.

Analytically, here one can take boundary conditions on the set $y=0$ (which is conformal to the n-dimensional Minkowski space) and decay conditions at $y=\infty$. This makes it the obvious generalization of the half-space model of the

$(n+1)$-dimensional hyperbolic space, which one describes in terms of the Poincaré coordinates $(y,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ through the metric

$$g_{\mathbb{H}}^+ := \frac{dy^2 + |dx|^2}{y^2},$$

which is the natural setting for the elliptic equation considered in the introduction after a conformal change.

While here we prefer to stick to the AdS half-space, let us mention that in Section 2.3 we will also consider in detail the problem for the usual AdS space, whose conformal timelike infinity is the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. This is sometimes referred to as the global AdS space in the context of the AdS/CFT conjecture and can be covered by two half-space AdS charts. The results are qualitatively the same but the algebra is less transparent. We will also encounter the same behavior when we analyze more general stationary asymptotically anti-de Sitter metrics, again in Section 2.3.

Let us then consider the Klein–Gordon equation with parameter μ in the AdS space (2.0.4),

$$\Box_{g^+}\phi + \mu\phi = 0\,, \tag{2.0.5}$$

where $\Box_{g^+}$ is the wave operator associated with the AdS metric:

$$\Box_{g^+}\phi := y^2(\partial_{tt}\phi - \Delta_x\phi - \partial_{yy}\phi) - (1-n)y\partial_y\phi.$$

Physically, μ is the mass of the particle modeled by the Klein–Gordon equation plus a negative contribution from the scalar curvature of the underlying space [74, Section 4.3]. For our purposes, it is convenient to assume that the parameter

$$\alpha := \left(\frac{n^2}{4} + \mu\right)^{1/2}$$

takes values in the interval $(0, \frac{n}{2})$.

We then have that Equation (2.0.5) can be rewritten as a wave equation with coefficients critically singular at $y = 0$,

$$\partial_{tt}\phi - \Delta_x\phi - \partial_{yy}\phi - \frac{1-n}{y}\partial_y\phi + \frac{4\alpha^2 - n^2}{4y^2}\phi = 0\,. \tag{2.0.6}$$

A simple look at the singularities of the equation reveals that the solutions are expected to scale at conformal infinity as $y^{\frac{n}{2}\pm\alpha}$. Then, since we are assuming $\alpha > 0$, the natural Dirichlet condition for this problem is

$$\lim_{y\searrow 0} y^{\alpha - \frac{n}{2}}\phi(t,x,y) = f(t,x)\,. \tag{2.0.7}$$

If one prefers to prescribe a Neumann condition [75], one must instead impose

$$\lim_{y\searrow 0} y^{1-2\alpha}\,\partial_y\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big) = h(t,x)\,,$$

for $\alpha \in (0,1)$, and a generalized Neumann condition $\alpha \in (1,\frac{n}{2})$, as we will see below. With this notation in place, the main result of this chapter is that the fractional wave operator $(-\Box)^\alpha$ (in flat space) is the Dirichlet-to-Neumann map associated to switching on a nontrivial boundary datum in an anti-de Sitter space:

Theorem 2.0.1. *For any function $f \in C_0^\infty(\mathbb{R}^n)$, let ϕ be the solution of the Klein–Gordon equation* (2.0.5) *with this Dirichlet boundary condition (Eq.* (2.0.7)*) and trivial initial data at $-\infty$: $\phi(-\infty,x,y)=\phi_t(-\infty,x,y)=0$. Assume moreover that the mass parameter α takes values in* $(0,1)$*. Then, one has the identity*

$$(-\Box)^\alpha f(t,x) = c_\alpha \lim_{y\searrow 0} y^{1-2\alpha}\,\partial_y\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big)$$

for an explicit constant $c_\alpha = -2^{2\alpha-1}\Gamma(\alpha)/\Gamma(1-\alpha)$. More generally, if $\alpha \in (0,\frac{n}{2})$ is not an integer and we write $\alpha = m+\alpha_0$, with $m := \lfloor \alpha \rfloor$ the integer part, then

$$(-\Box)^\alpha f(t,x) = c_\alpha \lim_{y\searrow 0} y^{2(1-\alpha_0)}\Big(\frac{1}{y}\,\partial_y\Big)^{m+1}\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big),$$

with $c_\alpha := (-1)^{m+1}2^{\alpha+\alpha_0-1}\dfrac{\Gamma(\alpha)}{\Gamma(1-\alpha_0)}$.

Several remarks are in order. First, an elementary observation is that this identity, which obviously applies to much for general data by the bounds that we establish here, implies that the well-known estimates for the fractional wave operator immediately translate into assertions about the Dirichlet datum of the solution and its associated Neumann condition, and viceversa. Second, it is worth emphasizing that this relation can be generalized to more general asymptotically AdS metrics, as we will do it Section 2.3. A third comment is that Theorem 2.0.1 implies for $\alpha \in (1,\frac{n}{2})$ the operator $(-\Box)^\alpha$ can be naturally interpreted as the scattering operator of the manifold. In Riemannian signature, this connection is discussed in detail in [18], and quite remarkably the interest in the the conformally covariant operators on the boundary that it defines (see e.g. [38, 57, 35]) was originally fueled by work of Newman, Penrose and LeBrun [53] on gravitational physics quite in the spirit of Maldacena's AdS/CFT correspondence.

Recall that, in the Lorentzian case, the construction of the scattering operator for a general asymptotically AdS metric was carried out in [72], but the resulting operator was not characterized.

The chapter is organized as follows. In Section 2.1 we will recall some basic facts and definitions about fractional wave operators in a form that is particularly convenient for our purposes, and in particular, a convolution formula with a singular kernel. In Section 2.2 we will analyze the Klein–Gordon equation and establish the main result in the half-space region of anti-de Sitter space. This identity will be extended to more general asymptotically AdS spaces, and to the global AdS space, in Section 2.3.

2.1 Fractional Powers of the Wave Operator

In this section we will recall some basic facts about fractional powers of the wave operator, as defined in (2.0.1). In the following we will see that, just as in the case of the fractional Laplacian [50], one can represent $(-\Box)^\alpha f$ as a regularized integral depending analytically on the power α when f is a sufficiently smooth function. To derive this result in full generality, we will need the analytic continuation of the classical hyperbolic Riesz potential [62], which will be explicitly constructed when the power is not a positive half-integer.

A careful analysis of the poles of the multiplier (2.0.2) shows that one can regard σ_α as a tempered distribution on $\mathbb{R}^n$, analytic in the parameter α for $\alpha \neq -\frac{n}{2} - k$ with k a non-negative integer (cf. e.g. [36, Chapter III]). This property is of crucial importance to recover the fractional wave operator as a convolution

$$(-\Box)^\alpha f = k_\alpha * f,$$

where the kernel k_α coincides with the inverse Fourier transform of σ_α in a sense that will be specified later on. Notice that, while this relation holds true distributionally, it is not easy to transform it into a pointwise converging formula due to the singularities that the multiplier presents on the light cone.

To regularize the integrals that appear, we will use Riesz distributions. Let us recall that, for any complex parameter α with $\operatorname{Re}\alpha > \frac{n}{2} - 1$, R_α is the distribution whose action on a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ is given by the absolutely convergent integral

$$\langle R_\alpha, \varphi \rangle = C_{n,\alpha} \int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} \varphi(t,x)\, dt\, dx, \quad \varphi \in C_0^\infty(\mathcal{K}_+^+),$$

where we write the points in $\mathbb{R}^n$ as $(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the set

$$\mathcal{K}_+^+ := \{(t,x) \in \mathbb{R}^n : t^2 \geqslant |x|^2,\ t \geqslant 0\}$$

is the forward causal cone and the constant $C_{n,\alpha}$ takes the value

$$C_{n,\alpha} := \frac{2^{1-2\alpha}\pi^{1-\frac{n}{2}}}{\Gamma(\alpha)\,\Gamma\big(\alpha+1-\frac{n}{2}\big)}. \tag{2.1.1}$$

A straightforward computation shows that the map $\alpha \mapsto \langle R_\alpha, \varphi\rangle$ is analytic in the half-plane $\operatorname{Re}\alpha > \frac{n}{2}-1$. It is well-known (see e.g. [48]) that this mapping can be analytically continued to the whole complex plane by means of the identity $(-\Box)R_{\alpha+1} = R_\alpha$. Hence for any complex number α it makes sense to consider the convolution of the distribution R_α a smooth compactly supported function $f \in C_0^\infty(\mathbb{R}^n)$, which we will denote by $I_\alpha f$ and call the *hyperbolic Riesz potential* of f. Notice that for $\operatorname{Re}\alpha > \frac{n}{2}-1$ the Riesz potential simply reads as

$$I_\alpha f(t,x) = \int_{\mathcal{K}_+^+} (s^2-|y|^2)^{\alpha-\frac{n}{2}} f(t-s,x-y)\,ds\,dy. \tag{2.1.2}$$

In the following proposition we establish the relationship among the powers of the wave operator, the Riesz distribution and its associated potential, showing that, as in the Euclidean case, it is possible to understand $(-\Box)^\alpha f$ as a regularized integral represented by the analytic extension of the Riesz potential. This relation was essentially stated in [63] using the results of [36] on the Fourier transform of analytically continued quadratic forms, but we prefer to sketch the proof here rather than to refer to vague variations on results from the above references.

Proposition 2.1.1. *Let α be a complex number such that $\alpha+\frac{n}{2}$ is not a non-positive integer. Then the Fourier transform of R_α is the function*

$$\widehat{R}_\alpha(\tau,\xi) = \sigma_{-\alpha}(\tau,\xi), \tag{2.1.3}$$

so, in particular, for any function $f \in C_0^\infty(\mathbb{R}^n)$ the α^{th} power of the wave operator is given by

$$(-\Box)^\alpha f = I_{-\alpha} f. \tag{2.1.4}$$

Proof. Let us first assume that $\operatorname{Re}\alpha > \frac{n}{2}-1$. The Fourier transform $\widehat{R}_\alpha$ is then the distribution given by

$$\begin{aligned}
\langle \widehat{R}_\alpha, \varphi\rangle = \langle R_\alpha, \widehat{\varphi}\rangle &= C_{n,\alpha}\int_{\mathcal{K}_+^+} (t^2-|x|^2)^{\alpha-\frac{n}{2}}\widehat{\varphi}(t,x)\,dt\,dx \\
&= C_{n,\alpha}\lim_{\epsilon\to 0^+}\int_{\mathcal{K}_+^+} (t^2-|x|^2)^{\alpha-\frac{n}{2}} e^{-\epsilon t}\widehat{\varphi}(t,x)\,dt\,dx \\
&= C_{n,\alpha}\int_{\mathbb{R}^n}\lim_{\epsilon\to 0^+}\left(\int_{\mathcal{K}_+^+} (t^2-|x|^2)^{\alpha-\frac{n}{2}} e^{-i\xi x} e^{-t(\epsilon+i\tau)}\,dt\,dx\right)\varphi(\tau,\xi)\,d\tau\,d\xi,
\end{aligned}$$

where we have used dominated convergence and Fubini's theorem.

The inner integral is computed (for $n > 3$; the case $n = 3$ is much simpler) by first passing to polar coordinates $(r, \phi_1, \phi_2, \ldots, \phi_{n-2})$ with $r > 0$, $\phi_j \in [0, \pi]$ for $j = 1, \ldots, n-3$ and $\phi_{n-2} \in [0, 2\pi]$:

$$x_1 = r\cos(\phi_1), \qquad x_j = r\cos(\phi_j)\prod_{i=1}^{j-1}\sin(\phi_i), \qquad x_{n-2} = r\prod_{i=1}^{n-2}\sin(\phi_i),$$

with $dx = r^{n-2}\sin^{n-3}(\phi_1)\sin^{n-4}(\phi_2)\cdots\sin(\phi_{n-3})\,dr\,d\phi_1\,d\phi_2\cdots d\phi_{n-2}$ the volume element. On the other hand, we also need the well-known representation formulas [76] for the Bessel functions

$$\begin{aligned} J_\nu(z) &= \frac{z^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\int_{-1}^{1} e^{izs}(1-s^2)^{\nu-\frac{1}{2}}\,ds, && \mathrm{Re}(\nu) > -\tfrac{1}{2},\ z \in \mathbb{C} \\ K_\nu(z) &= \frac{\sqrt{\pi}z^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_{1}^{\infty} e^{-zw}(w^2-1)^{\nu-\frac{1}{2}}\,dw, && \mathrm{Re}(\nu) > -\tfrac{1}{2},\ |\operatorname{Arg}(z)| < \tfrac{\pi}{2}, \end{aligned} \tag{2.1.5}$$

and the Bessel integral

$$\int_0^\infty r^{\mu+\nu+1}K_\mu(ar)J_\nu(br)dr = \frac{(2a)^\mu(2b)^\nu\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}},$$

for $\mathrm{Re}\,\nu + 1 > |\,\mathrm{Re}\,\mu|$ and $\mathrm{Re}\,a > \mathrm{Im}\,b$. Making then the change $s = \cos(\phi_1)$ and using the above integral formulas, it is not difficult to check that in the half-plane $\mathrm{Re}\,\alpha > \frac{n}{2} - 1$,

$$\begin{aligned} &\int_{\mathcal{K}_+^+}(t^2-|x|^2)^{\alpha-\frac{n}{2}}e^{-i\xi x}e^{-t(\epsilon+i\tau)}\,dt\,dx \\ &= \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})}\cdot\int_0^\infty dr\ r^{n-2}\int_{\{t^2\geqslant r^2\}}(t^2-r^2)^{\alpha-\frac{n}{2}}e^{-t(\epsilon+i\tau)}dt\int_{-1}^{1}(1-s^2)^{\frac{n}{2}-2}e^{-i|\xi|rs}\,ds \\ &= 2^\alpha\pi^{\frac{n}{2}-1}\Gamma(\alpha+1-\tfrac{n}{2})\cdot|\xi|^{\frac{3-n}{2}}(\epsilon+i\tau)^{\frac{n-1}{2}-\alpha} \\ &\qquad\qquad\cdot\int_0^\infty dr\ r^\alpha K_{\alpha+\frac{1-n}{2}}(r(\epsilon+i\tau))J_{\frac{n-3}{2}}(r|\xi|) \\ &= \frac{1}{C_{n,\alpha}}(|\xi|^2+(\epsilon+i\tau)^2)^{-\alpha}, \end{aligned}$$

where we have used in the second line that the surface of the $(n-3)$-sphere is given by $\frac{1}{\Gamma(n/2-1)}\cdot 2\pi^{\frac{n}{2}-1}$. Taking now the limit as $\epsilon \searrow 0$ we get (2.1.3).

To complete the proof we recall that, as pointed out right before introducing the concept of Riesz distribution, σ_α defines a tempered distribution that is analytic for any complex α except for $\alpha = -\frac{n}{2} - k$ with $k \in \mathbb{N}$. Therefore, although we have proved (2.1.3) when the parameter takes values in a certain open set of the complex plane, this relation must hold in the whole domain where R_α can be analytically continued. Moreover, since by definition $\widehat{(-\Box)^\alpha f} = \sigma_\alpha \widehat{f}$ for all α, it is also true that $(-\Box)^\alpha f = R_{-\alpha} * f = I_{-\alpha} f$. The proposition then follows. $\Box$

The above proposition is roughly the analog of the formula for the fractional Laplacian discussed in the introduction [50], which for $0 < \alpha < n$ allows us to write the fractional Laplacian $(-\Delta)^{-\alpha} f$, up to a multiplicative constant, as the convolution of f with the locally integrable function $|x|^{2\alpha - n}$. Notice, however, that the singularities in the integral kernel are much stronger here.

For the benefit of the reader we shall next record an explicit formula for $(-\Box)^\alpha f$ in terms of integrals regularized via suitable finite difference operators, which we borrow from [63, Eq. (9.93)], and connect it with the previous proposition. Before we can state the result, let us first introduce some further notation. For q a real parameter and $k, l \in \mathbb{N}$, let us denote the q-number of k, its q-factorial and the q-binomial coefficient as

$$[k]_q = \frac{1 - q^k}{1 - q}, \qquad [k]_q! = [1]_q [2]_q \cdots [k]_q, \qquad \binom{l}{k}_q = \frac{[l]_q!}{[k]_q! \, [l-k]_q!} \quad (k \leqslant l),$$

respectively. In addition, let us define the q-functions

$$C^l_k = q^{k(\frac{k+1}{2} - l)} \binom{l}{k}_q, \qquad A^l_\mu = \sum_{k=0}^{l} (-1)^k q^{k\mu} C^l_k, \tag{2.1.6}$$

with $\mu \in \mathbb{C}$ and where we omit the dependence on q for notational simplicity.

We are now ready to write down the integral formula for $(-\Box)^\alpha f$. For simplicity we will assume that $f \in C^\infty_0(\mathbb{R}^n)$, but the result it is still true e.g. for $C^l(\mathbb{R}^n)$ functions that decay fast enough at infinity, with $l > 2\alpha$.

Proposition 2.1.2. *Let us take a real $\alpha \in (0, \frac{l}{2})$, where $l \in \mathbb{N}$ and we assume that α is not a half-integer. Then for any $f \in C^\infty_0(\mathbb{R}^n)$ the fractional wave operator $(-\Box)^\alpha$ is given by the absolutely convergent integral*

$$(-\Box)^\alpha f(t, x) = C_{n,-\alpha} \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} \frac{\Delta^{l,\alpha}_{s,y} f(t,x)}{s^{\frac{n}{2}+\alpha} |y|^{n+2\alpha-1}} ds \; dy, \tag{2.1.7}$$

where $\Delta^{n,\alpha}_{s,y}$ stands for the difference operator

$$\Delta^{l,\alpha}_{s,y} f(t,x) = \frac{1}{A^{l^*}_{\frac{n}{2}-1+\alpha} A^l_{2\alpha}} \sum_{j=0}^{l^*} \sum_{k=0}^{l} (-1)^{j+k} C^{l^*}_j C^l_k \frac{(1+q^j s)^{2\alpha}}{(2+q^j s)^{\frac{n}{2}+\alpha}} f\Big(t - q^k |y|, x - \frac{q^k y}{1+q^j s}\Big),$$

l^* is the integer part of $\frac{n+l-1}{2}$, $q \neq 1$ is a positive constant, C^l_k and A^l_μ are the q-functions defined in (2.1.6)*, and the constant $C_{n,\alpha}$ is given by* (2.1.1)*.*

Proof. By Proposition 2.1.1 the result is equivalent to showing that the integral (2.1.7) represents the extension of $I_{-\alpha}f$ to the strip $0 < \alpha < l/2$. Firstly, observe that this integral converges absolutely on this interval except for integers and semi-integers values of α. Indeed, a tedious but straightforward computation of the Taylor series of $\Delta^{l,\alpha}_{s,y}f$ about zero combined with the fact that $A^l_m = 0$ for $0 \leqslant m \leqslant l-1$ as a consequence of the well-known identity [20]

$$\sum_{k=0}^{l} q^{k\frac{(k-1)}{2}} z^k \binom{l}{k}_q = \prod_{k=0}^{l-1}(1+zq^k)$$

applied to $z = -q^{1+m-l}$, reveal that

$$|\Delta^{l,\alpha}_{s,y}f| \leqslant C_q s^{l^*}|y|^l \|D^l f\|_\infty + \mathcal{O}\big(s^{l^*}|y|^{l+1}\|D^{(l+1)}f\|_\infty\big)$$

for (s, y) close to zero and where C_q is a real constant that depends only on q. From this bound and since f vanishes at infinity, it immediately follows that the integral is finite when $\alpha \in (0, \frac{l}{2})$ except for the zeros of $A^{l^*}_{\frac{n}{2}-1+\alpha}$ and $A^l_{2\alpha}$, which correspond to the points of the form $\alpha = \frac{k}{2},\ k \in \mathbb{N}$ when $q \neq 1$ is a positive real number.

Now we need to prove that the formula above actually represents the analytic continuation of the Riesz potential from the half-plane $\operatorname{Re}\alpha > \frac{n}{2} - 1$ to the range $\alpha \in (0, \frac{l}{2})$. Note that by (2.1.4) and the form of the differences operator, it suffices to show that (2.1.7) coincides with the expression of the potential given in (2.1.2) replacing α by $-\alpha$. This assertion can be readily checked by introducing new variables

$$u := q^k|y|, \qquad v := q^k\frac{|y|}{1+q^js}$$

and changing to polar coordinates in y. Incidentally, since the above substitutions also remove the dependence of the operator $(-\Box)^\alpha f$ on the parameter q, then one can safely choose any positive q other than the limit value $q = 1$. □

Remark 2.1.3. It is worth noticing that even in the simplest case, $0 < \alpha < 1$ and $n = 2$, the formula for the fractional wave operator is much more involved than its Euclidean counterpart and cannot be deduced from it. This reflects the different nature of the singularities of the corresponding kernels: the entire light cone in the hyperbolic case and a single point in the Euclidean setting. Remarkably, in the simple case that we are now discussing ($n = 2$, $0 < \alpha < 1$) there is another realization of $(-\Box)^\alpha$ (cf. [63, Theorem 9.30]) that is easier to compare with its fractional Laplacian analog:

$$(-\Box)^\alpha f(t,x) = \frac{C_{2,-\alpha}}{2^{1+2\alpha}} \int_{\mathcal{K}_+^+} \frac{T_{s,y} f(t,x)}{(s^2-y^2)^{1+\alpha}} ds\, dy, \tag{2.1.8}$$

where we are using the finite difference operator

$$T_{s,y} f(t,x) := f(t,x) - f\big(t - \tfrac{s+y}{2}, x - \tfrac{s+y}{2}\big) - f\big(t - \tfrac{s-y}{2}, x + \tfrac{s-y}{2}\big) + f(t-s, x-y).$$

Note that one can readily use this formula and the crude approximations

$$T_{s,y} f(t,x) = sy\, \Box f(t,x) + O(s^2+y^2)\,, \qquad \Gamma(-1+\epsilon)^{-1} = -\epsilon + O(\epsilon^2)$$

to prove the pointwise convergence of $(-\Box)^\alpha f$ to $(-\Box) f$ as $\alpha \to 1$.

2.2 The Klein–Gordon Equation in AdS Spaces

In this section we will connect the fractional powers of the wave operator with the solutions to the mixed initial-boundary problem corresponding to the Klein–Gordon equation in an anti-de Sitter space. Our specific goal here is to give a local realization of the fractional wave operator as a Dirichlet-to-Neumann map that is the Lorentzian counterpart of the relation for the fractional Laplacian derived in [15] and extended in [18]. For this purpose, we will use a Laplace–Fourier transform in order to transform our wave equation into an ODE that contains the relevant information about the solution at infinity, which will enable us to derive the Dirichlet-to-Neumann map in Fourier space.

Recall that our starting point was the Klein–Gordon equation

$$\Box_{g^+}\phi + \Big(\alpha^2 - \frac{n^2}{4}\Big)\phi = 0,$$

where $\Box_{g^+}$ denotes the wave operator associated to the AdS metric (2.0.4). As discussed in the Introduction, this equation can be thought as a wave equation with coefficients that are singular at conformal infinity and whose indicial roots are $n/2 \pm \alpha$. For simplicity, we will henceforth assume that α is not a half-integer (i.e., $2\alpha \notin \mathbb{N}$) in order to ensure that the solution does not have a logaritheoremic branch cut. The argument carries over to the case that $\alpha \notin \mathbb{N}$ with minor modifications.

Let us begin with the analysis of the wave equation (2.0.6). For this, is convenient to define the function

$$u(t,x,y) := y^{\alpha - \frac{n}{2}} \phi(t,x,y)\,,$$

which satisfies the equation

$$\partial_{tt}u - \Delta_x u - \partial_{yy}u - \frac{1-2\alpha}{y}\partial_y u = 0 \tag{2.2.1}$$

with Dirichlet datum

$$u(t,x,0) = f(t,x)\,. \tag{2.2.2}$$

Notice that the above equation agrees with the leading part of (1.0.3) upon making the substitution

$$\alpha = \frac{1}{2} - \kappa\,,$$

with κ the strength parameter on which the estimates of Chapter 1 depend on. Nonetheless, in this chapter it is slightly more convenient to use α as the comparison with the analogous elliptic result is clearer in terms of this parameter.

We shall next prove a bound for u that will be needed later in this section. To state it, let us consider the weighted Lebesgue space

$$L^2_\alpha := L^2(\mathbb{R}^n_+, y^{1-2\alpha} dx\, dy)\,,$$

endowed with the norm

$$\|v\|^2_{L^2_\alpha} := \int_{\mathbb{R}^n_+} v^2\, y^{1-2\alpha} dx\, dy\,,$$

and denote by $\dot{H}^1_\alpha$ (respectively, $\dot{H}^1_{\alpha,0}$) the closure of $C^\infty_0(\overline{\mathbb{R}^n_+})$ (respectively, $C^\infty_0(\mathbb{R}^n_+)$) with respect to the norm

$$\|v\|^2_{\dot{H}^1_\alpha} := \int_{\mathbb{R}^n_+} \left(|\nabla_x v|^2 + (\partial_y v)^2\right) y^{1-2\alpha} dx\, dy\,.$$

In the following theorem we only state a qualitative result for boundary data $f \in C^\infty_0(\mathbb{R}^n)$, which is what we need here, but in the proof we provide quantitative estimates that obviously extend the result to data in more general function spaces.

Lemma 2.2.1. *Let k be the lowest integer such that $k > \frac{1+\alpha}{2}$. Given a boundary datum $f \in C^\infty_0(\mathbb{R}^n)$ and an integer $j < k$, define*

$$u_j(t,x,y) := y^{2j}\, \chi(y)\, (-\Box)^j f(t,x)\,,$$

where $\chi(y)$ is a fixed smooth cutoff function identically 1 in $y < 1$ and 0 in $y > 2$, and $(-\Box)^j$ denotes the j-th power of the wave operator $-\Box = \partial_{tt} - \Delta_x$. Then there are real numbers c_j and a function $v \in L^\infty(\mathbb{R}, \dot{H}^1_\alpha)$ such that

$$u(t,x,y) := \sum_{j=0}^{k-1} c_j u_j(t,x,y) + v(t,x,y)\,, \tag{2.2.3}$$

is the unique solution of Equation (2.2.1) *with trivial initial data* $u(-\infty, x, y) = u_t(-\infty, x, y) = 0$ *and boundary condition* $u(t, x, 0) = f(t, x)$.

Proof. Let us consider the initial condition

$$u(t_0, \cdot) = u_t(t_0, \cdot) = 0\,, \tag{2.2.4}$$

where t_0 is any number such that $f(t, x) = 0$ for all $t < t_0$. We shall see in the proof that the solution is independent of the choice of t_0, so it is equivalent to imposing $u(-\infty, \cdot) = u_t(-\infty, \cdot) = 0$.

We shall start by considering an auxiliary non-homogeneous Cauchy problem of the form

$$\partial_{tt} v - \Delta_x v - \partial_{yy} v - \frac{1-2\alpha}{y}\partial_y v = F(t, x, y), \tag{2.2.5a}$$

$$v(t_0, \cdot) = v_t(t_0, \cdot) = 0. \tag{2.2.5b}$$

We shall next show that if $F \in L^1(\mathbb{R}, L^2_\alpha)$, there is a unique solution

$$v \in L^2_{\mathrm{loc}}(\mathbb{R}, \dot{H}^1_{\alpha,0}) \cap H^1_{\mathrm{loc}}(\mathbb{R}, L^2_\alpha)$$

to this equation. Notice that the fact that $v(t, \cdot)$ takes values in $\dot{H}^1_{\alpha,0}$ means that we are imposing the boundary condition $v|_{y=0} = 0$.

To prove this, we will use an a priori estimate for the energy associated to the solution v, which we define as

$$E_v(t) := \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} (v_t^2 + |\nabla_x v|^2 + v_y^2)\, y^{1-2\alpha}\, dx\, dy.$$

To prove this estimate, it is standard that by a density argument one can assume that v is in $C^\infty_0(\mathbb{R}^{n+1}_+)$, differentiate under the integral sign and integrate by parts to find that

$$\begin{aligned}
\frac{d}{dt}E_v(t) &= \int_{\mathbb{R}^n_+} (v_t v_{tt} + \nabla_x v_t \cdot \nabla_x v + v_y v_{yt})\, y^{1-2\alpha} dx\, dy \\
&= \int_{\mathbb{R}^n_+} v_t \left(v_{tt} - \Delta_x v - v_{yy} - \frac{1-2\alpha}{y} v_y \right) y^{1-2\alpha} dx\, dy \\
&= \int_{\mathbb{R}^n_+} F\, v_t\, y^{1-2\alpha} dx\, dy \\
&\leqslant C\|F(t, \cdot)\|_{L^2_\alpha}\, E_v(t)^{1/2}\,.
\end{aligned}$$

Using now Grönwall's inequality, we arrive at

$$E_v(t)^{1/2} \leqslant E_v(t_0)^{1/2} + C\left|\int_{t_0}^{t} \|F(t',\cdot)\|_{L^2_\alpha} dt'\right|,$$

which, by the trivial initial conditions, readily implies the estimate

$$\sup_{t\in\mathbb{R}} (\|v(t,\cdot)\|_{\dot{H}^1_\alpha} + \|v_t(t,\cdot)\|_{L^2_\alpha}) \leqslant C \int_{-\infty}^{\infty} \|F(t',\cdot)\|_{L^2_\alpha}\, dt', \tag{2.2.6}$$

thereby ensuring that $v \in L^\infty(\mathbb{R}, \dot{H}^1_{\alpha,0})$. It is standard that this estimate leads to the existence of a unique solution $v \in L^2_{\mathrm{loc}}(\mathbb{R}, \dot{H}^1_{\alpha,0}) \cap H^1_{\mathrm{loc}}(\mathbb{R}, L^2_\alpha)$ to the problem (2.2.5).

To apply the estimate (2.2.6) in our problem, let us set

$$v(t,x,y) := u(t,x,y) - \sum_{j=0}^{k-1} \frac{(-1)^j y^{2j}\chi(y)}{\prod_{l=1}^{j} 4l(l-\alpha)} (-\Box)^j f(t,x)\,,$$

where the product is to be taken as 1 when $l = 0$. Using now that

$$\Big(\partial_{yy} + \frac{1-2\alpha}{y}\partial_y\Big) y^j = j(j-2\alpha)\, y^{j-2}\,,$$

a direct calculation shows that

$$\begin{aligned} &\partial_{tt} v - \Delta_x v - \partial_{yy} v - \frac{1-2\alpha}{y}\partial_y v \\ &\qquad = \frac{(-1)^k y^{2k-2}\chi(y)}{\prod_{l=1}^{k-1} 4l(l-\alpha)} (-\Box)^k f + \sum_{j=0}^{k-1} \chi_j(y)\,(-\Box)^j f\,, \end{aligned} \tag{2.2.7}$$

where $\chi_j(y)$ is a smooth function whose support is contained in the interval $[1,2]$. Moreover, by construction v satisfies the initial and boundary conditions

$$v(t_0,x,y) = v_t(t_0,x,y) = v(t,x,0) = 0.$$

The point now is that, as the right hand side of (2.2.7) behaves as y^{2k-2} and vanishes for $y > 2$, it is easy to see that it is in $L^1(\mathbb{R}, L^2_\alpha)$, so the estimate (2.2.6) ensures that u written as in (2.2.3) is the unique solution of (2.2.1) satisfying the boundary condition $u|_{y=0} = f$ and vanishing initial data. □

Remark 2.2.2. Arguing as in [33, Proposition 5.2], we could have proved higher regularity estimates for the solution in suitable weighted spaces, but we will not need that result. The global L^∞ bound in time, on the contrary, will be essential and is not proved in the aforementioned paper, as it does not hold for the more general equations there considered.

Given a positive real parameter α that is not a half-integer, let us write it as $\alpha = \alpha_0 + m$, where m denotes its integer part and $\alpha_0 \in (0,1)$. The generalized Dirichlet-to-Neumann map Λ_α is then defined as in [18, Theorem 3.3]

$$\Lambda_\alpha f(t,x) = c_\alpha \lim_{y\searrow 0} y^{2(1-\alpha_0)} \Big(\frac{1}{y}\,\partial_y\Big)^{m+1} u(t,x,y), \tag{2.2.8}$$

with

$$c_\alpha := (-1)^{m+1} 2^{\alpha+\alpha_0-1} \frac{\Gamma(\alpha)}{\Gamma(1-\alpha_0)}$$

an inessential normalizing factor.

The following theorem, which is the central result of this chapter, is essentially a rewording of Theorem 2.0.1:

Theorem 2.2.3. *Given a positive real number α which is not an integer and a function $f \in C_0^\infty(\mathbb{R}^n)$, let u be the solution of the initial-boundary problem in the Poincaré-AdS space* (2.2.1) *with Dirichlet datum* (2.2.2) *and initial data $u(-\infty,\cdot) = u_t(-\infty,\cdot) = 0$. Then the map Λ_α defined in* (2.2.8) *is given by the α^{th} power of the wave operator:*

$$\Lambda_\alpha f = (-\Box)^\alpha f. \tag{2.2.9}$$

Proof. To begin with, let us start by noticing that the solution with the initial condition $u(-\infty,\cdot) = u_t(-\infty,\cdot) = 0$ is well defined, and it can be equivalently defined by the condition $u(t_0,\cdot) = u_t(t_0,\cdot) = 0$ where t_0 is any number such that $f(t,x) = 0$ for all $t < t_0$. In what follows, t_0 will denote a number with this property.

Consider the Laplace transform of $u(t,\cdot)$,

$$U(s,\cdot) = \int_{t_0}^\infty e^{-s(t-t_0)}\, u(t,\cdot)\, dt,$$

where $s = \epsilon + i\tau$ with $\epsilon > 0$ and $\tau \in \mathbb{R}$. Notice this expression converges as a vector-valued function because u is a Banach-space valued L^∞ function of time by Lemma 2.2.1. (When $\alpha < 1$, the norm is simply that of $\dot H^1_\alpha$, while for $\alpha \geqslant 1$ the norm must also control the terms u_j appearing in (2.2.3), for example by decomposing

$$u(t,x,y) =: \chi(y) \sum_{j=0}^{k-1} y^{2j}\, \overline{u}_j(t,x) + v(t,x,y)$$

and adding the L^∞ norm of the functions $\overline{u}_j$ and the $L^\infty(\mathbb{R}, \dot H^1_\alpha)$ norm of v).

Furthermore, observe that one can then recover u through the inverse Laplace transform formula

$$u(t,\cdot) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{s(t-t_0)}\, U(s,\cdot)\, ds,$$

where the integration contour is the vertical line of numbers whose real part is $\epsilon > 0$.

We tackle the problem of finding explicit solutions to problem (2.2.1)-(2.2.2) as follows. First, we apply the Laplace transform on the equation for u to remove the time derivatives by integration by parts and then use the trivial initial conditions in the LHS above,

$$\int_{t_0}^{\infty} e^{-s(t-t_0)}\, u_{tt}(t,\cdot)\, dt = s^2 U(s,\cdot) - u_t(t_0,\cdot) - s u(t_0,\cdot) = s^2 U(s,\cdot)$$

to find that $U(s,x,y)$ satisfies the equation

$$\partial_{yy} U(s,x,y) + \frac{1-2\alpha}{y}\partial_y U(s,\xi,y) + (\Delta_x - s^2)U(s,x,y) = 0.$$

Next we take the Fourier transform in space with respect to the variable x, which here is denoted with a tilde to avoid confusions with the space-time Fourier transform. This yields the ODE

$$\partial_{yy}\widetilde{U}(s,\xi,y) + \frac{1-2\alpha}{y}\partial_y \widetilde{U}(s,\xi,y) - (|\xi|^2 + s^2)\widetilde{U}(s,\xi,y) = 0.$$

The general solution of this equation can be written as a linear combination of Bessel functions multiplied by a certain power of y, and spans a two-dimensional vector space. However, by Lemma 2.2.1 the solution $u(t,\cdot)$ is given by the sum of a function bounded in $\dot{H}^1_\alpha$ and other terms that are uniformly bounded in y, so we may discard the independent solution that grows exponentially in y as $y \to \infty$ to arrive at the formula

$$\widetilde{U}(s,\xi,y) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} y^\alpha (|\xi|^2 + s^2)^{\frac{\alpha}{2}} K_\alpha\big(y\sqrt{|\xi|^2 + s^2}\big)\widetilde{F}(s,\xi), \tag{2.2.10}$$

where K_α denotes the modified Bessel function of the second kind defined in (2.1.5). The constant (which depends on s and ξ) has been chosen to ensure that the Dirichlet condition is satisfied:

$$\widetilde{U}(s,\xi,0) = \widetilde{F}(s,\xi).$$

Here and in what follows, $F(s,x)$ denotes the Laplace transform of $f(t,x)$, computed as above, and the tilde denotes the Fourier transform in space.

Therefore, using the identities

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k (z^\nu K_\nu(z)) = (-1)^k z^{\nu-k} K_{\nu-k}(z), \qquad k = 0,1,2,\dots$$
$$K_\nu(z) \simeq \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}, \qquad 0 < |z| \ll \sqrt{\nu+1},\ \nu > 0$$

for modified Bessel functions (cf. [76]), we readily infer that the Dirichlet-to-Neumann map (2.2.8) reads in the Laplace-Fourier space as

$$\widetilde{\Lambda_\alpha F}(s,\xi) := c_\alpha \lim_{y\searrow 0} y^{2(1-\alpha_0)}\left(\frac{1}{y}\partial_y\right)^{m+1}\widetilde{U}(s,\xi,y) = (|\xi|^2+s^2)^\alpha \widetilde{F}(s,\xi).$$

The key point now is that one can relate the inverse Laplace transform with the inverse Fourier transform in time using the freedom in the choice of the parameter ϵ. To show this, notice that by the Laplace inversion formula, we have that the Dirichlet-to-Neumann map in the variables (t,ξ) can be written as

$$\begin{aligned}\widetilde{\Lambda_\alpha f}(t,\xi) &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{s(t-t_0)}(|\xi|^2+s^2)^\alpha \widetilde{F}(s,\xi)\, ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{t_0}^{\infty} e^{(\epsilon+i\tau)(t-t')}(|\xi|^2+(\epsilon+i\tau)^2)^\alpha \widetilde{f}(t',\xi)\, d\tau\, dt'.\end{aligned}$$

Since the latter integral does not depend on the value of ϵ and f is smooth with compact support in $\{t \geqslant t_0\}$, by the dominated convergence theorem one can take the limit as $\epsilon \searrow 0$ inside the integral to find that

$$\widetilde{\Lambda_\alpha f}(t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\sigma_\alpha(\tau,\xi)\widehat{f}(\tau,\xi)\, d\tau,$$

with σ_α as in (2.0.3) and $\widehat{f}(\tau,\xi)$ denoting the Fourier transform of f with respect to both time and space variables as in Section 2. Taking now the Fourier transform with respect to the time, we obtain

$$\widehat{\Lambda_\alpha f}(\tau,\xi) = \sigma_\alpha(\tau,\xi)\widehat{f}(\tau,\xi),$$

proving our claim. □

A consequence of the proof is an explicit formula for the spacetime energy of the solution in terms of its boundary datum that is analogous to the result in Euclidean signature from [15]:

Corollary 2.2.4. *With u and f be as in Theorem 2.2.3, the total energy of u is*

$$\int_{\mathbb{R}^{n+1}_+} \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2\right] y^{1-2\alpha}\, dt\, dx\, dy = C_\alpha \int_{\mathbb{R}^n} \sigma_\alpha(\tau,\xi)\, |\widehat{f}(\tau,\xi)|^2\, d\tau\, d\xi\,,$$

with C_α a nonzero constant.

Proof. We first note that the Laplace transform of the function $u(s,\cdot)$ at $s=\epsilon+i\tau$,

$$U(\epsilon+i\tau) = \int_0^\infty e^{-i\tau t} e^{-\epsilon t} u(t,\cdot)\, dt,$$

is the Fourier transform of the function $e^{-\epsilon t}u(t,\cdot)$. Thus by Plancherel theorem we can write

$$\int_{\epsilon-i\infty}^{\epsilon+i\infty} |U(s,\cdot)|^2\, ds = \int_{-\infty}^{+\infty} |U(\epsilon+i\tau,\cdot)|^2\, d\tau = \int_0^\infty e^{-2\epsilon t}|u(t,\cdot)|^2\, dt. \qquad (2.2.11)$$

Now we consider the energy for equation (2.2.1), given by

$$\begin{aligned} E &= \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2\right] dt\, dx\, dy \\ &= \lim_{\epsilon\searrow 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2\right] e^{-2\epsilon t}\, dt\, dx\, dy, \end{aligned}$$

which, using Plancherel identity (2.2.11) for the Laplace transform, becomes

$$\begin{aligned} E = \lim_{\epsilon\searrow 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} &\big[|\partial_y U(s,x,y)|^2 + |\nabla_x U(s,x,y)|^2 \\ &+ s^2 |U(s,x,y)|^2\big]\, ds\, dx\, dy. \end{aligned}$$

Now we take Fourier transform in the variable x, yielding

$$E = \lim_{\epsilon\searrow 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \left[|\partial_y \widetilde{U}(s,\xi,y)|^2 + (|\xi|^2+s^2)|\widetilde{U}(s,\xi,y)|^2\right] ds\, d\xi\, dy.$$

Substituting the explicit expression (2.2.10) we arrive at

$$\begin{aligned} E = \lim_{\epsilon\searrow 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} &(|\xi|^2+s^2)\big[|K_1'(y\sqrt{|\xi|^2+s^2})|^2 \\ &+ |K_1(y\sqrt{|\xi|^2+s^2})|^2\big]\, |\widetilde{F}(s,\xi)|^2\, ds\, d\xi\, dy, \end{aligned}$$

where, for simplicity, we have set $K_1(y) := \dfrac{2^{1-\alpha}}{\Gamma(\alpha)} y^\alpha K_\alpha(y)$. A change of variable allows us to integrate in y, obtaining

$$E = \lim_{\epsilon\searrow 0} C_\alpha \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} (|\xi|^2+s^2)^\alpha |\widetilde{F}(s,\xi)|^2\, ds\, d\xi$$

with C_α an explicit constant, as claimed. □

2.3 Applications

Our point in this section is to show that the identities established in Theorem 2.0.1 remain valid in considerably more general situations. We will illustrate this fact by connecting other fractional wave operators with the Dirichlet-to-Neumann map (or, more generally, the scattering operator) of two simple classes of static asymptotically AdS manifolds:

Fractional waves in product spaces

Consider a compact Riemannian manifold $\mathcal{M}$ of dimension $n-1$ endowed with a Riemannian metric g_0 and take the natural wave operator on $\mathbb{R}\times\mathcal{M}$, which is

$$\Box_0 := \partial_{tt} - \Delta_{g_0},$$

where Δ_{g_0} stands for the Laplace-Beltrami operator on $\mathcal{M}$.

Since $\mathcal{M}$ is compact, we can take an orthonormal basis $\{Y_j\}_{j\in\mathbb{N}}$ of eigenfunctions of the Laplacian Δ_{g_0}, which satisfy

$$-\Delta_{g_0} Y_j = \lambda_j^2 Y_j\,,$$

and write any L^2 function f on M (depending on t as a parameter) as the L^2-convergent series $f(t,\cdot) = \sum_j f_j(t) Y_j(\cdot)$. Let us denote by $(\Box_0 f)_j(t)$ the j^{th} component of the function $\Box_0 f$ in this basis. Taking the Fourier transform with respect to t we obtain that

$$\widehat{\Box_0 f_j}(\tau) = (\lambda_j^2 - \tau^2)\widehat{f_j}(\tau)$$

and, given a real parameter α, we can define here the α^{th} power of the wave operator as the pseudo-differential operator that in the Fourier space reads as

$$\widehat{\Box_0^\alpha f_j}(\tau) := \sigma_\alpha(\tau,\lambda_j)\widehat{f_j}(\tau), \tag{2.3.1}$$

with σ_α the function defined in (2.0.3).

Consider now an $(n+1)$-dimensional Lorentzian spacetime with the metric

$$g^+ := \frac{dt^2 - dy^2 - g_0}{y^2}, \tag{2.3.2}$$

where $t\in\mathbb{R}$ is the time coordinate and $y\in\mathbb{R}_+$ is a spatial coordinate. Comparing with the metric defined in (2.0.4), it is clear that the Klein-Gordon equation with parameter $\mu := (\alpha^2 - n^2/4)$ associated to the metric (2.3.2) then takes the form

$$\partial_{tt}\phi - \Delta_{g_0}\phi - \partial_{yy}\phi - \frac{1-n}{y}\partial_y\phi + \frac{4\alpha^2 - n^2}{4y^2}\phi = 0\,,$$

where one must prescribe some suitable initial-boundary conditions for the scalar field ϕ. As in the last section, the above equation can be rewritten in terms of the rescaled function $u := y^{\alpha-\frac{n}{2}}\phi$ as

$$\partial_{tt}u = \Delta_{g_0}u + \frac{1-2\alpha}{y}\partial_y u + \partial_{yy}u, \tag{2.3.3}$$

where we take trivial initial data at time $-\infty$ and prescribe the boundary condition at timelike conformal infinity: $u|_{y=0} = f$.

We can next define the (generalized) Dirichlet-to-Neumann map through its coefficients

$$(\Lambda_\alpha f)_j = c_\alpha \lim_{y\searrow 0} y^{2(1-\alpha_0)}\Big(\frac{1}{y}\partial_y\Big)^{m+1} u_j, \tag{2.3.4}$$

with c_α as before, $\alpha = \alpha_0 + m$, $m = \lfloor\alpha\rfloor$ the integer part of α and $\alpha_0 \in (0,1)$.

By means of expansion in eigenfunctions of the Laplacian Δ_{g_0} and the Laplace transform in time (together with the vanishing initial conditions), the equation (2.3.3) can be transformed into our well-known ordinary equation

$$\partial_{yy}U_j(s,y) + \frac{1-2\alpha}{y}\partial_y U_j(s,y) - (\lambda_j^2 + s^2)U_j(s,y) = 0,$$

with boundary condition $F_j(s) = U_j(s,0)$, where

$$U_j(s,y) := \int_{t_0}^{\infty} e^{-s(t-t_0)}\, u_j(s,y)\, dt$$

is the Laplace transform of the coefficient u_j with $s = \epsilon + i\tau$ and ϵ a fixed positive constant. Notice that U_j can be shown to be well defined for $\epsilon > 0$ by an L^∞ bound in time that goes exactly as in Lemma 2.2.1.

Arguing just as in the previous section, we find that the Dirichlet-to-Neumann map in the transformed space reads as

$$\Lambda_\alpha F_j := c_\alpha \lim_{y\searrow 0} y^{2(1-\alpha_0)}\Big(\frac{1}{y}\partial_y\Big)^{m+1} U_j(s,y) = (\lambda_j^2 + s^2)^\alpha F_j(s),$$

and therefore, by the inverse Laplace transform formula and the Fourier transform in time,

$$\widehat{\Lambda f}_j(\tau) = \sigma_\alpha(\tau,\lambda_j)\widehat{f}_j(\tau).$$

Thereby, we can identify the Dirichlet-to-Neumann map in the Lorenztian space with metric (2.3.2) with the powers of the wave operator $\Box_0$.

The global anti-de Sitter space

Consider now the global anti-de Sitter mentioned in the introduction, which is diffeomorphic to $\mathbb{R}^{n+1}$ and one can describe through spherical coordinates

$$(t, r, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^{n-2}\,,$$

which cover the whole manifold modulo the usual abuse of notation at the origin. In these coordinates the metric of AdS_{n+1} reads as

$$g^+ := (1+r^2)dt^2 - \frac{1}{1+r^2}dr^2 - r^2 g_{\mathbb{S}^{n-2}}\,, \tag{2.3.5}$$

where $g_{\mathbb{S}^{n-2}}$ is the canonical metric on the unit $(n-2)$-dimensional sphere, associated with the coordinate θ.

In this space we can picture the spatial limit $r \to +\infty$ as the cylinder $\mathbb{R}_t \times \mathbb{S}^{n-2}$ with the standard metric

$$g_0 := dt^2 - g_{\mathbb{S}^{n-2}}\,,$$

which defines the timelike conformal infinity of the spacetime.

Let us now focus on the Klein–Gordon equation on this anti-de Sitter space with the natural Dirichlet datum

$$\lim_{r\to\infty} r^{\frac{n}{2}-\alpha}\phi(t, r, \theta) = f(t, \theta),$$

$f \in C_0^\infty(\mathbb{R}^{n+1})$ and trivial initial conditions at time $-\infty$. Upon expanding the operator $\Box_{g^+}$ associated to the metric (2.3.5), we obtain the Klein–Gordon equation

$$\partial_{tt}\phi = \frac{1+r^2}{r^2}\Delta_\theta\phi + (1+r^2)^2\partial_{rr}\phi + (1+r^2)\Big(\frac{n-1}{r} + (n+1)r\Big)\partial_r\phi - (1+r^2)\Big(\alpha^2 - \frac{n^2}{4}\Big)\phi\,. \tag{2.3.6}$$

As before, in order obtain the scattering operator we take a basis $\{\mathcal{Y}_j\}_{j\in\mathbb{N}}$ of spherical harmonics of energy $\lambda_j^2 := j(j+n-3)$. They satisfy the equation

$$-\Delta_\theta \mathcal{Y}_j = \lambda_j^2 \mathcal{Y}_j,$$

with Δ_θ the Laplace-Beltrami operator on $\mathbb{S}^{n-2}$. Introducing the coefficients

$$\phi_j(t, r) := \int_{\mathbb{S}^{n-2}} \phi(t, r, \theta)\,\mathcal{Y}_j(\theta)\,d\theta\,,$$

one can apply the Laplace transform and expand the Klein–Gordon equation in the basis of spherical harmonics to obtain an ODE in the variable r,

$$(1+r^2)\partial_{rr}\Phi_j+\Big(\frac{n-1}{r}+(n+1)r\Big)\partial_r\Phi_j+\Big(\frac{s^2}{1+r^2}+\frac{\lambda_j^2}{r^2}+\alpha^2-\frac{n^2}{4}\Big)\Phi_j=0\,, \quad (2.3.7)$$

where

$$\Phi_j(s,r):=\int_{t_0}^{\infty} e^{-s(t-t_0)}\;\phi_j(t,r)\;dt$$

is the Laplace transform of $\phi_j(r,\theta)$.

The explicit solution of this equation is a combination of certain powers of r multiplied by ordinary hypergeometric functions. Discarding the solution that is not locally in H^1 at the origin, $r=0$, we then obtain

$$\Phi_j(s,r)=c_j(s)r^{\beta}e^{-\frac{i}{2}\log(1+r^2)^s}\,{}_2F_1\Big(\tfrac{1}{2}(\beta-is+\tfrac{n}{2}-\alpha),\tfrac{1}{2}(\beta-is+\tfrac{n}{2}+\alpha),\beta+\tfrac{n}{2},-r^2\Big),$$

where

$$\beta:=\frac{1}{2}\left(2-n+\sqrt{4\lambda^2+(n-2)^2}\right).$$

The coefficient $c_j(s)$ is readily computed using that $F_j(s):=\lim_{r\to\infty}r^{\frac{n}{2}-\alpha}\Phi_j(s,r)$ must be the Laplace transform of the j^{th} component of the boundary datum.

The transformed Dirichlet-to-Neumann operator is then readily shown to be

$$\Lambda_\alpha F_j(s):=\lim_{r\to\infty}r^{1+2\alpha}\partial_r(r^{\frac{n}{2}-\alpha}\Phi_j(s,r)).$$

Using now the explicit solution, we obtain that

$$\Lambda_\alpha F_j(s)=\frac{\Gamma(-\alpha)\Gamma\big(\frac{1}{2}(\beta-is+\frac{n}{2}+\alpha)\big)\Gamma\big(\frac{1}{2}(\beta+is+\frac{n}{2}+\alpha)\big)}{\Gamma(\alpha)\Gamma\big(\frac{1}{2}(\beta-is+\frac{n}{2}-\alpha)\big)\Gamma\big(\frac{1}{2}(\beta+is+\frac{n}{2}-\alpha)\big)}\cdot(\beta-is+\tfrac{n}{2}-\alpha)F_j(s),$$

which can be written using the Fourier transform in time as

$$\widehat{\Lambda f}_j(\tau):=\lim_{\epsilon\searrow 0}\Lambda_\alpha F_j(\epsilon+i\tau).$$

It should be noticed that in the limit of large frequencies of the multiplier $\sigma_\alpha(\tau,\lambda_j)$ is, up to some numerical factor, the principal symbol of the scattering operator in this globally defined AdS space, as one would have expected.

Chapter 3

Whitham's highest waves

Preliminaries

In this chapter we prove the existence of a limiting periodic travelling wave solution to the Whitham equation

$$L\varphi - \mu\varphi + \varphi^2 = 0\,, \tag{3.0.1}$$

that features cusps of $C^{1/2}$-regularity and is convex between consecutive crests as in Figure 1. Our work, which is analogous to the Stokes waves result for the Euler equation [61], answers to the following conjecture due to Ehrnström and Wahlén:

Conjecture 3.0.1. *(Convexity of Whitham's highest cusped wave [31, p. 4]) Whitham's highest wave φ is everywhere convex and its asymptotic behavior at $x = 0$ is*

$$\varphi(x) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x|^{1/2} + o(|x|)\,.$$

For the benefit of the reader, here we record again the statement of Theorem 4, which addresses the above conjecture and is the main result of this chapter. Taking for concreteness φ as a function on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, we have:

Theorem 3.0.2. *The 2π-periodic highest cusped traveling wave $\varphi \in C^{1/2}(\mathbb{T})$ of the Whitham equation is a convex function and behaves asymptotically as*

$$\varphi(x) = \frac{\mu}{2} - \sqrt{\frac{\pi}{8}}|x|^{1/2} + O(|x|^{1+\eta}) \tag{3.0.2}$$

for some $\eta > 0$. Furthermore, φ is even and strictly decreasing on the interval $[0, \pi]$.

Before presenting the main ingredients behind the proof of this theorem, let us recall that solutions to the Whitham equation are smooth away from the cusps. In fact, one should note that if φ solves (3.0.1), then for all $x_1, x_2 \in \mathbb{T}$,

$$(\mu - \varphi(x_1) - \varphi(x_2))(\varphi(x_1) + \varphi(x_2)) = L(\varphi(x_1) - \varphi(x_2))\,.$$

A bootstrapping argument then guarantees the smoothness of solutions when $\varphi < \mu/2$, while the sharp $C^{1/2}$-regularity can be only attained if $\varphi(0) = \mu/2$. Of course, this Hölder regularity is expected in view of the balance between the nonlinearity and the order of the operator (quadratic and $-1/2$, respectively).

The proof of Theorem 3.0.2 is rather involved and relies in part on computer-assisted estimates. We start off by noticing that the function

$$u(x) = \frac{\mu}{2} - \varphi(x) \tag{3.0.3}$$

satisfies an equation that does not explicitly depend on the parameter μ, which can nonetheless be recovered from u. Making a guess of what u should look like, we then write

$$u(x) = u_0(x) + |x|v_0(x)\,,$$

where $u_0(x) \sim \sqrt{\pi/8}|x|^{1/2}$ is an explicit, carefully chosen approximate solution of the equation and the correction term $v_0(x)$ should then be obtained via an inverse function theorem on $L^\infty(\mathbb{T})$. Up to a technicality (namely, that $v_0(x)$ appears in this formula with a factor of $|x|$ instead of $|x|^{1+\eta}$), this proves the easier part of Theorem 3.0.2, namely, the asymptotic formula (3.0.2).

We should emphasize, however, that this description hides three key difficulties that make the proof much harder than it looks. A first, fairly obvious one is that the argument boils down to estimates on $L^\infty([-\pi,\pi])$ for the linear operator

$$T_0 f(x) := \frac{1}{2|x|u_0(x)}\big[L(|\cdot|f)(x) + L(|\cdot|f(-\cdot))(x) - 2L(|\cdot|f)(0)\big]\,,$$

whose kernel is rather difficult to control. Indeed, the convolution kernel of the operator L acting on $L^\infty(\mathbb{T})$ that appears in the definition of T_0 has the rather awkward expression [31]

$$Lf(x) = \int_{-\pi}^{\pi} K(y)\, f(x-y)\, dy\,, \tag{3.0.4}$$

$$K(x) = \sum_{n=1}^{\infty} \int_{(n-\frac{1}{2})\pi}^{n\pi} \frac{\cosh[s(\pi - |x|)]}{\pi \sinh(s\pi)} \left(\frac{|\tan s|}{s}\right)^{1/2} ds \tag{3.0.5}$$

for $x \in (-\pi,\pi)$.

A second, less obvious difficulty is that the operator norm of T_0 turns out to be very slightly smaller than 1. Therefore, the bound for the norm of $(I - T_0)^{-1}$ that we need in the argument is large, and this has the crucial consequence that it becomes very hard to construct an approximate solution u_0 such that the associated error $T_0 u_0 - \frac{u_0}{2|x|}$ is small enough in $L^\infty(\mathbb{T})$.

Finally, the third difficulty is that, as the solution φ is not smooth at the origin, one cannot effectively use ordinary or trigonometric polynomials to construct the approximate solution u_0 (which would interact well with the operator L), as is customary in computer-assisted proofs, and plain powers $|x|^s$ cannot be used to approximate the 2π-periodic function u properly either as they do not glue well at $x = \pm\pi$ and do not have simple representations whenever L acts on them. Instead, to construct u_0 we utilize information about the asymptotic behavior of the solutions at 0 and carefully concoct a linear combination of trigonometric polynomials and Clausen functions of different orders, defined as

$$C_z(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^z}\,, \quad S_z(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^z}\,. \tag{3.0.6}$$

Suitable estimates for Clausen functions are derived in order to obtain the required uniform bounds for the approximate solution. Two relevant additional remarks are that choosing u_0 just from asymptotic information at 0 is not possible, as the approximation that one obtains away from zero is poor, and that u_0 is a combination of 20 different terms, so carrying out the estimates without a computer seems unwise. The need of so many terms is due to the almost non-invertibility of $(I - T_0)$, which results in the need for a very accurate approximate solution that cannot be constructed using just a few explicit terms.

It should be stressed that the hardest part of Theorem 3.0.2, that is the proof of the convexity of the solution, is considerably more technical but is based on the same principles, suitably strengthened to control two derivatives of the function u. This is ultimately accomplished by solving an extended system of equations that is controlled by three linear operators: the aforementioned T_0 and two new, more complicated operators T_1 and T_2 that involve up to two derivatives of the (extremely messy) approximate solution u_0. Just as before, one needs to invert $I - T_i$ for $0 \leqslant i \leqslant 2$ and the norms of the three operators T_i are very close but strictly less than 1.

A major theme of our work is the interplay between rigorous computer calculations and traditional mathematics; in this work we use interval arithmetics as part of a proof whenever they are needed. Lately, computer-assisted proofs have been made possible due to the increment of computational resources. Naturally, floating-point operations can result in numerical errors. In order to overcome these, we will employ interval arithmetics to deal with this issue. The main paradigm is the following: instead of working with arbitrary real numbers, we perform computations over intervals which have representable numbers by the computer as endpoints in order to guarantee that the true result at any point belongs to the interval by which is represented. On these objects, an arithmetic is defined in such a way that we are guaranteed that for every $x \in X, y \in Y$

$$x \star y \in X \star Y,$$

for any operation $\star$. For example,

$$\begin{aligned}
[\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \\
[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}].
\end{aligned}$$

We can also define the interval version of a function $f(X)$ as an interval I that satisfies that for every $x \in X$, $f(x) \in I$. Rigorous computation of integrals has been theoretically developed since the seminal works of Moore and many others [11, 17, 22, 49, 51]. We also refer the reader to the books [59, 70] and to the survey [37] for a more specific treatment of computer-assisted proofs in PDE.

Of course, the proof of Theorem 3.0.2 would be much easier if one could come up with a simpler strategy where soft analysis could be used to bypass the need for hard estimates, but it is difficult to imagine what such a strategy could be based on. Let us briefly comment on this important point. For instance, a first idea would be to try to adapt the proof of Ehrnström and Wahlén to include as part of the functional space the additional fact that the functions are convex. However, some work shows that this philosophy cannot be easily implemented since it is by no means clear how to carry out the local or global bifurcation argument within this framework. Another obvious idea is to carry out the global bifurcation argument directly in a $C^{1/2}$ Hölder space. Alas, showing that $C^{1/2}$ is indeed the sharp Hölder regularity of the resulting solution, meaning that it does not belong to a higher space in the Hölder scale, turns out to be highly nontrivial in this approach.

The chapter is organized as follows. In Section 3.1 we give some technical results concerning generalized Clausen functions and their asymptotic behavior at $x = 0$. Equipped with these formulas, in Section 3.2 we construct an approximate solution (3.2.7) to the equation verified by (3.0.3). A linearized version of the Whitham equation is studied in Section 3.3. Here we use a fixed point argument together with the invertibility of $1 - T_0$ to show the existence of a solution which is an L^∞ small perturbation of our approximate solution that displays (almost) the right asymptotic behavior claimed in Conjecture 3.0.1. Section 3.4 is devoted to the proof of the main Theorem 3.0.2. We exploit the bounds for the norms of linear operators T_0, T_1 and T_2 to obtain a priori estimates that help us to conclude the convexity of a highest cusped Whitham wave. Two appendices are given at the end of the chapter. For convenience, we leave the study of the norm of T_2 for Appendix 3.A meanwhile in Appendix 3.B, we give details on the computer assisted proofs.

3.1 Technical Lemmas About Clausen Functions

In this section we provide all the estimates for the generalized Clausen functions introduced in (3.0.6) that we use in the rest of the chapter. In particular, as mentioned in the introduction, these functions play a fundamental role in the proof of Theorem 3.0.2 as they are the building blocks of the approximate solution that we shall present in the next section.

Let us begin with the relationship between Clausen functions and the polylogarithm $\mathrm{Li}_z(s)$ [24, Eq. 25.12.10]:

$$\mathrm{Li}_z(s) := \sum_{n=1}^{\infty} \frac{s^n}{n^z}. \tag{3.1.1}$$

This series defines an analytic function for all complex z whenever $|s| < 1$ and it can be analytically continued for other values. Further, recalling the definition of Clausen functions (3.0.6), it is clear now that

$$C_z(x) := \frac{1}{2}\big(\mathrm{Li}_z(e^{ix}) + \mathrm{Li}_z(e^{-ix})\big) = \mathrm{Re}\,\big(\mathrm{Li}_z(e^{ix})\big)\,,$$
$$S_z(x) := \frac{1}{2i}\big(\mathrm{Li}_z(e^{ix}) - \mathrm{Li}_z(e^{-ix})\big) = \mathrm{Im}\,\big(\mathrm{Li}_z(e^{ix})\big)\,.$$

By the well-known identity [24, Eq. 25.12.12],

$$\mathrm{Li}_z(s) = \Gamma(1-z)\big(\log(s^{-1})\big)^{z-1} + \sum_{n=0}^{\infty} \zeta(z-n)\frac{(\log s)^n}{n!}\,, \quad z \notin \mathbb{Z}_+\,, |\log(s)| < 2\pi\,.$$

one has the following series representations for C_z and S_z:

$$C_z(x) = \Gamma(1-z)\sin(\tfrac{\pi}{2}z)|x|^{z-1} + \sum_{m=0}^{\infty}(-1)^m\zeta(z-2m)\frac{x^{2m}}{(2m)!} \tag{3.1.2}$$

$$S_z(x) = \Gamma(1-z)\cos(\tfrac{\pi}{2}z)\mathrm{sgn}\,(x)|x|^{z-1} + \sum_{m=0}^{\infty}(-1)^m\zeta(z-2m-1)\frac{x^{2m+1}}{(2m+1)!}\,, \tag{3.1.3}$$

where $\zeta(z)$ is the Riemann zeta function. Observe that these formulas (analytically) extend the definition (3.0.6) when $\mathrm{Re}(z) < 1$ for all x real.

As it will be useful later on, in the following lemma we give uniform bounds for the lower order terms in the above series:

Lemma 3.1.1. *Let z be a positive real number and let $M := \left\lceil \frac{z+1}{2} \right\rceil$. Then the Clausen functions can be expressed as*

$$\begin{aligned} C_z(x) &= \Gamma(1-z)\sin(\tfrac{\pi}{2}z)|x|^{z-1} + \zeta(z) \\ &\qquad + \sum_{m=1}^{M-1} (-1)^m \zeta(z-2m) \frac{x^{2m}}{(2m)!} + E_{C_z}(x) \\ S_z(x) &= \Gamma(1-z)\cos(\tfrac{\pi}{2}z)\mathrm{sgn}\,(x)|x|^{z-1} \\ &\qquad + \sum_{m=0}^{M-1} (-1)^m \zeta(z-2m-1) \frac{x^{2m+1}}{(2m+1)!} + E_{S_z}(x)\,, \end{aligned}$$

where the error terms satisfy

$$|E_{C_z}(x)| \leqslant 2(2\pi)^{1+z-2M} \zeta(2M+1-z) \frac{x^{2M}}{4\pi^2 - x^2}\,, \tag{3.1.4}$$

$$|E_{S_z}(x)| \leqslant 2(2\pi)^{z-2M} \zeta(2M+2-z) \frac{|x|^{2M+1}}{4\pi^2 - x^2}\,. \tag{3.1.5}$$

Proof. As they are similar, for simplicity we only prove the estimate for $C_z(x)$. From (3.1.2) we have that for any positive integer M,

$$\begin{aligned} C_z(x) &= \Gamma(1-z)\sin(\tfrac{\pi}{2}z)|x|^{z-1} + \zeta(z) + \sum_{m=1}^{M-1} (-1)^m \zeta(z-2m) \frac{x^{2m}}{(2m)!} \\ &\qquad + \sin(\tfrac{\pi}{2}z) \sum_{m=M}^{\infty} \frac{\Gamma(2m+1-z)}{\Gamma(2m+1)} 2^{z-2m} \pi^{z-2m-1} \zeta(2m+1-z) x^{2m}\,. \end{aligned}$$

Here we have used that $\Gamma(2m+1) = (2m)!$ and the functional identity

$$\zeta(s) = 2^s \pi^{s-1} \sin(\tfrac{\pi}{2}s) \Gamma(1-s) \zeta(1-s)\,, \tag{3.1.6}$$

which is valid for all $s \in \mathbb{C}$.

Since $\zeta(s)$ and $\Gamma(s)$ are, respectively, monotonically decreasing and increasing functions on $s > 2$, by taking $M = \left\lceil \frac{z+1}{2} \right\rceil$, we arrive at

$$|E_{C_z}(x)| \leqslant \zeta(2M+1-z) |\sin(\tfrac{\pi}{2}z)| \sum_{m=M}^{\infty} 2^{z-2m} \pi^{z-2m-1} x^{2m}\,.$$

By computing the above infinite sum in closed form,

$$\sum_{m=M}^{\infty} 2^{z-2m} \pi^{z-2m-1} x^{2m} = 2(2\pi)^{1+z-2M} \frac{x^{2M}}{4\pi^2 - x^2}\,,$$

the estimate for C_z follows. □

3.2 Approximate Solution

Our objective here is to introduce an approximate solution u_0 to the Whitham equation and study its asymptotic behavior at $x = 0$. Making use of the estimates derived in the previous section, we will be able to control the L^∞-norm of the error and prove that it is sufficiently small for the purposes of the fixed point iteration scheme that we set up later in the chapter.

Let us begin by introducing the linear operator

$$\mathcal{L}u(x) := \frac{1}{2}\int_{-\pi}^{\pi} \big(K(x-y) + K(x+y) - 2K(y)\big)u(y)\,dy\,, \tag{3.2.1}$$

with $K(x)$ as in (3.0.4). Furthermore, it will be useful to express the kernel K alternatively as the Fourier series [31]

$$K(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} m(n)e^{inx} = \frac{1}{\pi}\sum_{n=0}^{\infty} m(n)\cos(nx)\,, \tag{3.2.2}$$

where in the second equality we have used the parity of $m(n) := \sqrt{\dfrac{\tanh(n)}{n}}$.

Remark 3.2.1. Notice that by the representation of the Clausen function $C_{1/2}$ given in Lemma 3.1.1, the above kernel satisfies

$$K(x) = \frac{1}{\sqrt{2\pi|x|}} + E_{\text{reg}}(x)\,, \qquad E_{\text{reg}}(x) = E_{C_{\frac{1}{2}}}(x) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\sqrt{\tanh(n)}}{\sqrt{n}}\cos(nx)\,,$$

which agrees with the description given in [31, Prop. 3.1].

Through this chapter we will take advantage of the fact that u defined as in (3.0.3) satisfies a quadratic equation that does not depend explicitly on the parameter μ:

Proposition 3.2.2. *Let $\varphi(x)$ be a solution of* (3.0.1)*. Then, the function $u(x) := \frac{\mu}{2} - \varphi(x)$ satisfies the reduced Whitham equation*

$$(u(x))^2 = \mathcal{L}u(x)\,, \tag{3.2.3}$$

where the wavespeed μ is recovered through

$$\mu\Big(1 - \frac{\mu}{2}\Big) = 4\int_0^{\pi} K(y)u(y)\,dy\,. \tag{3.2.4}$$

Remark 3.2.3. Notice that in view of the Galilean transformation

$$\mu \mapsto 2-\mu\,, \quad \varphi \mapsto \varphi + 1 - \mu\,,$$

solutions φ to (3.0.1) with wavespeed $\mu \in [1,2]$ are mapped bijectively to solutions for $\mu \in [0,1]$ with maxima in $[0,\mu/2]$. Since K and u are positive by (3.0.4) and $\varphi < \frac{\mu}{2}$, the quadratic equation (3.2.4) for μ has only one root in $[0,1]$, which is precisely the value of μ associated to the highest wave φ.

As we will show in the next section, (3.2.3) imposes strong restrictions on the asymptotic behavior of the solution u. In particular, since Remark 3.2.1 implies that

$$\int_0^\pi \big(C_{\frac{1}{2}}(x-y) + C_{\frac{1}{2}}(x+y) - 2C_{\frac{1}{2}}(y)\big)\sqrt{y}\,dy = \frac{\pi}{2}|x| + O(x^2)\,,$$

by using the formula (3.1.2) one can easily show the following asymptotic formula:

Proposition 3.2.4. *Let $\lambda > 0$ and assume that u is a solution of* (3.2.3) *with the asymptotic behavior*

$$u(x) = \lambda\sqrt{|x|} + O(|x|^{\frac{1}{2}+p})\,, \quad p > 0\,,$$

close to $|x| = 0$. Then the constant λ must take the value

$$\lambda := \sqrt{\frac{\pi}{8}}\,. \tag{3.2.5}$$

An approximate solution to the reduced Whitham equation is given now in terms of Clausen functions and trigonometric polynomials. We postpone to the next section the construction of an actual solution of (3.2.3) with the desired behavior at $x = 0$.

Definition 3.2.5. *Let $p_0 \in (0,1)$ and $p_1 \in (2,3)$ be numbers such that*

$$\frac{\Gamma(-1/2-p_j)}{\Gamma(-1-p_j)}\big(1-\cot(\tfrac{\pi}{2}p_j)\big) = \frac{2}{\sqrt{\pi}}\,. \tag{3.2.6}$$

Then, we define

$$u_0(x) = \sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\big(\zeta\big(3/2 + kp_0 + jp_1\big) - C_{\frac{3}{2}+kp_0+jp_1}(x)\big) + \sum_{n=1}^{N_2} b_n\big(\cos(nx) - 1\big)\,, \tag{3.2.7}$$

where the coefficients a_{jk} and b_n are real and N_0, N_1 and N_2 are fixed non-negative integers.

In view of this definition and the formulas (3.1.2) and (3.1.3), we have in addition that the derivatives of u_0 can be written as

$$u_0'(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} S_{\frac{1}{2}+kp_0+jp_1}(x) - \sum_{n=1}^{N_2} n b_n \sin(nx)\,, \tag{3.2.8}$$

$$u_0''(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} C_{-\frac{1}{2}+kp_0+jp_1}(x) - \sum_{n=1}^{N_2} n^2 b_n \cos(nx)\,. \tag{3.2.9}$$

Moreover, the explicit values of the numbers p_0, p_1 can be enclosed with high precision, as shown in the following Lemma. Its proof will be given in Appendix 3.B.

Lemma 3.2.6. *The only solutions p_0, p_1 of the equation (3.2.6) in the above intervals are*

$$p_0 = 0.611\ldots\,, \quad p_1 = 2.762\ldots,.$$

A key feature of the approximate solution u_0 is precisely its asymptotic behavior near $x = 0$. In fact, the bounds shown in Lemma 3.1.1 imply the following asymptotic expansions that we give without proof as they involve tedious but largely standard computations:

Lemma 3.2.7. *Let u_0 be a function of the form (3.2.7) and let M be the smallest integer such that $M \geqslant 3/2 + \max\{N_0 p_0, N_1 p_0 + p_1\}$. Then, the following asymptotic expansions hold near $x = 0$:*

$$u_0(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk}^0 |x|^{\frac{1}{2}+kp_0+jp_1} + \Big(a_1^0 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big) x^2 + \Big(a_2^0 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big) x^4 + E_{u_0}(x)\,, \tag{3.2.10}$$

$$u_0'(x) = \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk}^1 |x|^{-\frac{1}{2}+kp_0+jp_1} + \Big(a_0^1 - \sum_{n=1}^{N_2} n^2 b_n\Big) |x| + \Big(a_1^1 + \frac{1}{6}\sum_{n=1}^{N_2} n^4 b_n\Big) |x|^3 + E_{u_0'}(x)\,, \tag{3.2.11}$$

and

$$u_0''(x) = \sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}^2 |x|^{-\frac{3}{2}+kp_0+jp_1} + \Big(a_1^2 - \sum_{n=1}^{N_2} n^2 b_n\Big) + \Big(a_2^2 + \frac{1}{2}\sum_{n=1}^{N_2} n^4 b_n\Big)x^2 + E_{u_0''}(x)\,, \quad (3.2.12)$$

where

$$\begin{aligned}
a_{jk}^0 &:= -\Gamma(-1/2 - kp_0 - jp_1)\sin\big(\tfrac{\pi}{2}(\tfrac{3}{2} + kp_0 + jp_1)\big)a_{jk}\,,\\
a_m^0 &:= \frac{(-1)^{m+1}}{(2m)!}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\zeta(3/2 + kp_0 + jp_1 - 2m)\,, \quad m = 1, 2\,,\\
a_{jk}^1 &:= \Gamma(1/2 - kp_0 - jp_1)\cos\big(\tfrac{\pi}{2}(\tfrac{1}{2} + kp_0 + jp_1)\big)a_{jk}\,,\\
a_m^1 &:= \frac{(-1)^{m}}{(2m+1)!}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\zeta(-1/2 + kp_0 + jp_1 - 2m)\,, \quad m = 0, 1\,,\\
a_{jk}^2 &:= -\Gamma(3/2 - kp_0 - jp_1)\sin\big(\tfrac{\pi}{2}(-\tfrac{1}{2} + kp_0 + jp_1)\big)a_{jk}\\
a_m^2 &:= \frac{(-1)^{m}}{(2m)!}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\zeta(-1/2 + kp_0 + jp_1 - 2m)\,, \quad m = 1, 2\,,
\end{aligned}$$

and

$$|E_{u_0}(x)| \leqslant 2(2\pi)^{5/2-2M}\sum_{k=0}^{N_j}\sum_{j=0}^{1}(2\pi)^{kp_0+jp_1}|\zeta(2M - 1/2 - kp_0 - jp_1)a_{jk}|\frac{x^{2M}}{4\pi^2 - x^2} + \frac{x^6}{6!}\sum_{n=1}^{N_2} n^6|b_n| + \sum_{m=3}^{M-1}|a_m^0|x^{2m}\,, \quad (3.2.13)$$

$$|E_{u_0'}(x)| \leqslant 2(2\pi)^{3/2-2M}\sum_{k=0}^{N_j}\sum_{j=0}^{1}(2\pi)^{kp_0+jp_1}|\zeta(2M + 3/2 - kp_0 - jp_1)a_{jk}|\frac{|x|^{2M+1}}{4\pi^2 - x^2} + \frac{|x|^5}{5!}\sum_{n=1}^{N_2} n^6|b_n| + \sum_{m=2}^{M-1}|a_m^1||x|^{2m+1}\,. \quad (3.2.14)$$

$$|E_{u_0''}(x)| \leqslant 2(2\pi)^{1/2-2M} \sum_{k=0}^{N_j} \sum_{j=0}^{1} (2\pi)^{kp_0+jp_1} |\zeta(2M+5/2-kp_0-jp_1)a_{jk}| \frac{x^{2M}}{4\pi^2-x^2} + \frac{x^4}{4!} \sum_{n=1}^{N_2} n^6 |b_n| + \sum_{m=2}^{M-1} |a_m^2| x^{2m} . \quad (3.2.15)$$

Moreover, the derivatives of the error term $E_{u_0}(x)$ are trivially bounded as

$$|E'_{u_0}(x)| \leqslant |E_{u_0'}(x)| , \quad |E''_{u_0}(x)| \leqslant |E_{u_0''}(x)| .$$

Analogously, we will need asymptotics for $\mathcal{L}u_0$ with $\mathcal{L}$ the linear operator introduced in (3.2.1). We also omit the proof, which is tedious but elementary.

Lemma 3.2.8. *The asymptotic expansion of $\mathcal{L}u_0$ close $x=0$ is*

$$\begin{aligned}\mathcal{L}u_0 &= \sum_{k=0}^{N_j} \sum_{j=0}^{1} A_{jk}^0 |x|^{1+kp_0+jp_1} \\ &+ \Big(A_1^0 - \frac{1}{2} \sum_{n=1}^{N_2} b_n n^{3/2} \sqrt{\tanh(n)} + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} \frac{\sqrt{\tanh(n)}}{n^{kp_0+jp_1}} \Big) x^2 \\ &+ \Big(A_2^0 + \frac{1}{24} \sum_{n=1}^{N_2} b_n n^{7/2} \sqrt{\tanh(n)} - \frac{1}{24} \sum_{n=1}^{\infty} \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} \frac{\sqrt{\tanh(n)}}{n^{kp_0+jp_1-2}} \Big) x^4 + E_{\mathcal{L}u_0}(x) ,\end{aligned} \quad (3.2.16)$$

where

$$A_{jk}^0 := \Gamma(-1-kp_0-jp_1) \sin\big(\tfrac{\pi}{2}(kp_0+jp_1)\big) a_{jk} .$$

$$A_m^0 := \frac{(-1)^{m+1}}{(2m)!} \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk} \zeta(2+kp_0+jp_1-2m) ,$$

and

$$|E_{\mathcal{L}u_0}(x)| \leqslant 2(2\pi)^{3-2M} \sum_{k=0}^{N_j} \sum_{j=0}^{1} (2\pi)^{kp_0+jp_1} |\zeta(2M-1-kp_0-jp_1)a_{jk}| \frac{x^{2M}}{4\pi^2-x^2} + \frac{x^6}{6!} \sum_{n=1}^{N_2} n^{3/2} \sqrt{\tanh(n)} |b_n| + \sum_{m=3}^{M-1} |A_m^0| x^{2m} .$$

Moreover, the derivatives of the error term have the following bounds:

$$|E'_{\mathcal{L}u_0}(x)| \leqslant 2(2\pi)^{2-2M} \sum_{k=0}^{N_j} \sum_{j=0}^{1} (2\pi)^{kp_0+jp_1} |\zeta(2M+1+kp_0-jp_1)a_{jk}| \frac{|x|^{2M+1}}{4\pi^2-x^2} + \frac{|x|^5}{5!} \sum_{n=1}^{N_2} n^{3/2}\sqrt{\tanh(n)}|b_n| + \sum_{m=2}^{M-1} |A^1_m||x|^{2m+1}\,,$$

$$|E''_{\mathcal{L}u_0}(x)| \leqslant 2(2\pi)^{1-2M} \sum_{k=0}^{N_j} \sum_{j=0}^{1} (2\pi)^{kp_0+jp_1} |\zeta(2M+3+kp_0-jp_1)a_{jk}| \frac{x^{2M}}{4\pi^2-x^2} + \frac{x^4}{4!} \sum_{n=1}^{N_2} n^{3/2}\sqrt{\tanh(n)}|b_n| + \sum_{m=3}^{M-1} |A^2_m|x^{2m}\,,$$

with

$$A^1_m = \frac{(-1)^m}{(2m+1)!} \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk}\zeta(kp_0+jp_1-2m)\,,$$
$$A^2_m = \frac{(-1)^m}{(2m)!} \sum_{k=0}^{N_j} \sum_{j=0}^{1} a_{jk}\zeta(kp_0+jp_1-2m)\,.$$

At this point we can now understand the construction of the approximate solution u_0 and how it helps us to address Conjecture 3.0.1. Indeed, let $u(x) = u_0(x)+|x|v_0(x)$ be a solution of the reduced Whitham equation (3.2.3) with $u_0(x)$ as before and $v_0(x) \in L^\infty(\mathbb{T})$. In terms of the perturbation v_0 the equation can be recast as

$$(I-T_0)v_0 = F_0 - \frac{|x|}{2u_0}v_0^2\,, \tag{3.2.17}$$

with $T_0 : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ the operator

$$T_0v_0(x) := \frac{1}{2|x|u_0} \int_0^\pi \big(K(x-y)+K(x+y)-2K(y)\big)y\, v_0(y)\, dy\,, \tag{3.2.18}$$

and where we have defined

$$F_0 := \frac{1}{2|x|u_0}(\mathcal{L}u_0 - u_0^2)\,. \tag{3.2.19}$$

Since we aim to show the existence of a small v_0 in $L^\infty(\mathbb{T})$, the idea we uso to pick u_0 in (3.2.7) becomes apparent: we choose the coefficients a_{jk} so that the defect term F_0 is bounded in $L^\infty(\mathbb{T})$ and arbitrarily small close to $x = 0$, while the constants b_n are chosen to control the norm globally.

For notational convenience and before we give a uniform bound for F_0, let us introduce an auxiliary function

$$\widehat{u}_0(x) := \frac{\lambda\sqrt{x} - u_0(x)}{u_0(x)}, \quad x \in [0,\pi], \tag{3.2.20}$$

which is small close to $x = 0$ as the following lemma shows:

Lemma 3.2.9. *Let $\widehat{u}_0$ be as before and take $\epsilon > 0$ a small fixed number. Then, for $0 \leqslant x \leqslant \epsilon$,*

$$\widehat{u}_0(x) \leqslant c_{\epsilon,\widehat{u}_0} x^{p_0}.$$

Proof. By the definition of $\widehat{u}_0(x)$,

$$\begin{aligned}\widehat{u}_0(x) \leqslant \Big(&\sum_{j+k>0} |a^0_{jk}||x|^{(k-1)p_0+jp_1} + \Big|a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big||x|^{\frac{3}{2}-p_0} \\ &+ \Big|a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big||x|^{\frac{7}{2}-p_0} + |x|^{-1/2}|E_{u_0}(x)|\Big) \cdot \Big(\lambda - \sum_{j+k>0} |a^0_{jk}||x|^{kp_0+jp_1} \\ &- \Big|a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big||x|^{3/2} - \Big|a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big||x|^{7/2} - |x|^{-1/2}|E_{u_0}(x)|\Big)^{-1} |x|^{p_0}.\end{aligned}$$

Using the monotonicity of all the terms in the above expression, by evaluating the fraction at $|x| = \epsilon$ we obtain the constant $c_{\epsilon,\widehat{u}_0}$:

$$\begin{aligned}c_{\epsilon,\widehat{u}_0} := \Big(&\epsilon^{-1/2}|E_{\epsilon,u_0}| + \sum_{j+k>0} |a^0_{jk}|\epsilon^{(k-1)p_0+jp_1} + \Big|a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big|\epsilon^{\frac{3}{2}-p_0} \\ &+ \Big|a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big|\epsilon^{\frac{7}{2}-p_0}\Big) \cdot \Big(\lambda - \sum_{j+k>0} |a^0_{jk}|\epsilon^{kp_0+jp_1} - \Big|a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big|\epsilon^{3/2} \\ &- \Big|a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big|\epsilon^{7/2} - |x|^{-1/2}E_{\epsilon,u_0}|\Big)^{-1},\end{aligned}$$

where

$$\begin{aligned}E_{\epsilon,u_0} := 2(2\pi)^{5/2-2M} &\sum_{k=0}^{N_j}\sum_{j=0}^{1} (2\pi)^{kp_0+jp_1} |\zeta(2M - 1/2 - kp_0 - jp_1)a_{jk}| \frac{\epsilon^{2M}}{4\pi^2 - \epsilon^2} \\ &+ \frac{\epsilon^6}{6!}\sum_{n=1}^{N_2} n^6|b_n| + \sum_{m=3}^{M-1} |a^0_m|\epsilon^{2m}\end{aligned}$$

denotes the RHS of (3.2.13) at $x = \epsilon$. □

Lemma 3.2.10. *Let* u_0 *be as in* (3.2.7) *with* $a_{00} = \frac{1}{4}$. *Then* $F_0 \in L^\infty(\mathbb{T})$ *and*

$$\delta_0 := \|F_0\|_{L^\infty(\mathbb{T})} \leqslant 9.1 \cdot 10^{-8} . \tag{3.2.21}$$

Proof. A long but straightforward computation shows that

$$\begin{aligned}
\mathcal{L}u_0(x) - u_0^2(x) = & \big(A^0_{00} - (a^0_{00})^2\big)|x| + (A^0_{01} - 2a^0_{00}a^0_{01})|x|^{1+p_0} + (A^0_{10} - 2a^0_{00}a^0_{10})|x|^{1+p_1} \\
& + (A^0_{11} - a^0_{00}a^0_{11} - a^0_{01}a^0_{10})|x|^{1+p_0+p_1} \\
& + \sum_{k=2}^{N_0}\Big(A^0_{0k} - \frac{1}{2}((-1)^k + 1)(a^0_{0\lfloor\frac{k}{2}\rfloor})^2 - 2\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} a^0_{0j}a^0_{0(k-j)}\Big)|x|^{1+kp_0} \\
& + \Big[A^0_1 - \frac{1}{2}\Big(\sum_{n=1}^{N_2} b_n n^{3/2}\sqrt{\tanh(n)} - \sum_{n=1}^{\infty}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\frac{\sqrt{\tanh(n)}}{n^{kp_0+jp_1}}\Big)\Big]x^2 \\
& + \Big[A^0_2 + \frac{1}{24}\Big(\sum_{n=1}^{N_2} b_n n^{7/2}\sqrt{\tanh(n)} - \sum_{n=1}^{\infty}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\frac{\sqrt{\tanh(n)}}{n^{kp_0+jp_1-2}}\Big) \\
& - \Big(a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big)^2\Big]x^4 - \Big(a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big)^2 x^8 \\
& - 2\Big(a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Big)\sum_{k=0}^{N_j}\sum_{j=0}^{1} a^0_{jk}|x|^{\frac{5}{2}+kp_0+jp_1} \\
& - 2\Big(a^0_1 - \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Big)\sum_{k=0}^{N_j}\sum_{j=0}^{1} a^0_{jk}|x|^{\frac{9}{2}+kp_0+jp_1} \\
& - 2E_{u_0}(x)\sum_{k=0}^{N_j}\sum_{j=0}^{1} a^0_{jk}|x|^{\frac{1}{2}+kp_0+jp_1} + E_{\mathcal{L}u_0}(x) - (E_{u_0}(x))^2 .
\end{aligned} \tag{3.2.22}$$

Using now Lemma 3.2.9 to write

$$\frac{1}{u_0(x)} = \frac{1 + \widehat{u}_0(x)}{\lambda\sqrt{x}} ,$$

by (3.2.19) the coefficient $A^0_{00} - (a^0_{00})^2$ must then vanish identically to ensure that $F_0 \in L^\infty(\mathbb{T})$. This is the case when a_{00} takes the value $\frac{1}{4}$ by Lemma 3.2.7. The rest of the proof is computer assisted. See Appendix 3.B. □

Remark 3.2.11. Notice that $a_{00} = \frac{1}{4}$ is equivalent to fixing $a^0_{00} = \lambda$ in (3.2.10), with λ the constant of (3.2.5). Since we write a solution of (3.2.3) as $u(x) =$

$u_0(x) + |x|v_0(x)$ for some $v_0 \in L^\infty(\mathbb{T})$, this condition is naturally expected by Proposition 3.2.4

Lemma 3.2.12. *Let u_0 be the approximate solution* (3.2.7) *and take $\epsilon = 0.1$. Then, the following inequalities hold for $0 \leqslant x \leqslant \epsilon$:*

$$\frac{1}{\lambda\sqrt{x}}(\lambda\sqrt{x} - u_0(x)) \leqslant \frac{1}{\lambda} c_{\epsilon,p_0} x^{p_0}\,, \tag{3.2.23}$$

$$\frac{1}{2x} - \frac{u_0'(x)}{u_0(x)} \leqslant \frac{1}{\lambda} c'_{\epsilon,p_0} x^{p_0-1}\,, \tag{3.2.24}$$

$$\frac{3}{4x^2} - \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big) \leqslant \frac{c''_{\epsilon,p_0}}{\lambda} x^{p_0-2}\,, \tag{3.2.25}$$

where the constants $c_{\epsilon,p_0}, c'_{\epsilon,p_0}, c''_{\epsilon,p_0}$ verify

$$c_{\epsilon,p_0} < 0.142\,, \quad c'_{\epsilon,p_0} < 0.16\,, \quad c''_{\epsilon,p_0} < 0.178\,. \tag{3.2.26}$$

Proof. We only show the first two bounds as the third is obtained in the same way. To start, observe that by (3.2.20),

$$\frac{1}{2x} - \frac{u_0'(x)}{u_0(x)} = \frac{(\lambda\sqrt{x} - u_0(x))' - \widehat{u}_0(x)u_0'(x)}{\lambda\sqrt{x}}\,.$$

By the monotonicity of all the quantities involved, we also have that

$$\lambda\sqrt{x} - u_0(x) \leqslant \Bigg(\sum_{j+k>0} |a^0_{jk}|\epsilon^{(k-1)p_0 + jp_1} + \Bigg|a^0_1 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Bigg|\epsilon^{\frac{3}{2}-p_0} + \Bigg|a^0_2 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Bigg|\epsilon^{\frac{7}{2}-p_0} + \epsilon^{-\frac{1}{2}-p_0}E_{\epsilon,u_0}\Bigg) x^{\frac{1}{2}+p_0} \leqslant c_{\epsilon,p_0} x^{\frac{1}{2}+p_0}\,, \tag{3.2.27}$$

$$(\lambda\sqrt{x} - u_0(x))' - \widehat{u}_0(x)u_0'(x) \leqslant \Bigg(\sum_{j+k>0} |a^1_{jk}|\epsilon^{(k-1)p_0 + jp_1} + \Bigg|a^1_0 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n\Bigg|\epsilon^{\frac{3}{2}-p_0} + \Bigg|a^1_1 + \frac{1}{24}\sum_{n=1}^{N_2} n^4 b_n\Bigg|\epsilon^{\frac{7}{2}-p_0} + \epsilon^{\frac{1}{2}-p_0}E_{\epsilon,u_0'}\Bigg) x^{p_0-\frac{1}{2}} \leqslant c'_{\epsilon,p_0} x^{p_0-\frac{1}{2}}\,, \tag{3.2.28}$$

where E_{ϵ,u_0} (resp. $E_{\epsilon,u_0'}$) denotes the RHS of (3.2.13) (resp. (3.2.14)) evaluated at $x = \epsilon$ and where the numbers $c_{\epsilon,p_0}, c'_{\epsilon,p_0}$ are obtained letting $\epsilon = 0.1$ in the above bounds. □

For the arguments of the subsequent sections, we need to show that not only F_0, but also its first and second order (weighted) derivatives

$$F_1(x) = F_0'(x)\,, \quad F_2(x) = |x|F_0''(x)\,, \tag{3.2.29}$$

are bounded and small near $x = 0$. This is the content of the following lemma, whose proof is omitted as it follows the same scheme of Lemma 3.2.10, that is, it relies on the asymptotic analysis of $\mathcal{L}u_0 - u_0^2$ given in (3.2.22), and the estimates of the Lemmas 3.2.7, 3.2.8 and 3.2.12. See Appendix 3.B for more details.

Lemma 3.2.13. *Let u_0 be as in* (3.2.7)*, in which the coefficients a_{jk} and b_n satisfy the relations*

$$\begin{aligned}
&a_{00} - \frac{1}{4} = 0\,,\\
&A_{01}^0 - 2a_{00}^0 a_{01}^0 = 0\,,\\
&A_1^0 - \frac{1}{2}\Big(\sum_{n=1}^{N_2} b_n n^{3/2}\sqrt{\tanh(n)} - \sum_{n=1}^{\infty}\sum_{k=0}^{N_j}\sum_{j=0}^{1} a_{jk}\frac{\sqrt{\tanh(n)}}{n^{kp_0+jp_1}}\Big) = 0\,,\\
&A_{02}^0 - (a_{02}^0)^2 - 2a_{00}^0 a_{02}^0 = 0\,,\\
&a_1^0 - \frac{1}{2}\sum_{n=1}^{N_2} n^2 b_n = 0\,.
\end{aligned}$$

Then $F_1, F_2 \in L^\infty(\mathbb{T})$ and

$$\delta_1 := \|F_1\|_{L^\infty(\mathbb{T})} \leqslant 9.2\cdot 10^{-7}\,, \quad \delta_2 := \|F_2\|_{L^\infty(\mathbb{T})} \leqslant 1.2\cdot 10^{-5}. \tag{3.2.30}$$

3.3 Analysis of the Linearized Equation

As we discussed in the introduction, one of the key elements in this work is that we are able to exploit the (rather nontrivial) invertibility of the linear operator that renders the reduced Whitham equation. The linearized equations for the derivatives of the solution are also given by operators that one can invert and that we will study in the following section.

Indeed, it is clear that Equation (3.2.17) suggests to invert the linear operator $I - T_0$ to show the existence of a function $v_0 \in L^\infty(\mathbb{T})$ that allows us to express a solution of the reduced Whitham equation (3.2.3) as

$$u(x) = u_0(x) + |x|v_0(x)\,, \tag{3.3.1}$$

with $u_0(x)$ our approximate solution (3.2.7). Although this ansatz by itself is not sufficient to prove the first part of the Conjecture 3.0.1 on the asymptotic behavior of Whitham waves, as we shall see in the next section one can argue that the continuity of all the estimates with respect to a small parameter $\eta > 0$ associated to the weight $|x|^{1+\eta}$ is sufficient to obtain the conclusion.

To begin with this analysis, in the following lemma we show that the norm of the operator T_0 is smaller than 1. For notational simplicity, here and in what follows we will denote by $\|T\|$ the $L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ norm of a linear operator T.

Lemma 3.3.1. *Let C_B be the constant given by*

$$C_B := \frac{1}{\pi} \int_0^\infty \Big| \frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{1+t}} - 2 \Big| t^{-5/2} \, dt = 0.997362\dots . \tag{3.3.2}$$

The number C_B, which can be computed explicitly as the root of a quartic polynomial, coincides with the norm of the operator T_0:

$$\|T_0\| = C_B . \tag{3.3.3}$$

Proof. Let us start with the computation of C_B. For convenience we split $C_B = c_B^1 + c_B^2$ as

$$c_B^1 := \frac{1}{\pi} \int_0^1 \Big(\frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{1+t}} - 2 \Big) t^{-5/2} \, dt ,$$
$$c_B^2 := \frac{1}{\pi} \int_1^\infty \Big| \frac{1}{\sqrt{t-1}} + \frac{1}{\sqrt{1+t}} - 2 \Big| t^{-5/2} \, dt .$$

Notice now that the first integral is immediate,

$$c_B^1 = \frac{1}{\pi} \int_0^1 \Big(\frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{1+t}} - 2 \Big) t^{-5/2} \, dt = \frac{2}{3\pi}(\sqrt{2} + 2) = 0.724519\dots .$$

Furthermore, a simple analysis of the sign of the integrand

$$I(t) := \frac{1}{\pi} \Big(\frac{1}{\sqrt{t-1}} + \frac{1}{\sqrt{1+t}} - 2 \Big) t^{-5/2}$$

reveals that $I(t)$ is positive when $1 < t < t^*$, where $t^* := 1.531623\dots$ denotes the largest (real) root of the quartic polynomial $4t^4 - 4t^3 - 8t^2 + 4t + 5$. Then,

$$c_B^2 = \int_1^{t^*} I(t) \, dt - \int_{t^*}^\infty I(t) \, dt = 0.272842\dots ,$$

where this number can be obtained with high precision using that a primitive of $I(t)$ is

$$\frac{2}{3\pi t^{3/2}} \left(\sqrt{t-1} - \sqrt{t+1} + 2t(\sqrt{t-1} + \sqrt{t+1}) + 2 \right) .$$

Summing both contributions we see that C_B takes the value of (3.3.2).

From the expression of the kernel of T_0 (which is even by equation (3.2.18)), it is standard that the norm of T_0 is

$$\|T_0\| := \sup_{0<x<\pi} \frac{1}{2|x|u_0(x)} \int_0^\pi \big|K(x-y)+K(x+y)-2K(y)\big| y\, dy \, .$$

Here we have used a simple parity argument to ensure we can take $x, y > 0$ and analyze separately the integral in (3.3.3) over the regions $x < y < \pi$ and $0 < y < x$. We will show next that the supremum of the above expression in the interval $0 < x \leqslant \epsilon$, where $\epsilon \in (0,1)$ is a certain number, is attained at $x = 0$, where the above integral takes the value C_B. To compute the supremum in the interval $\epsilon < x < \pi$ we then proceed as explained in Appendix 3.B.

By the formula (3.2.2) of the Whitham kernel K, we notice that

$$K(x-y)+K(x+y)-2K(y) = \frac{2}{\pi}\sum_{n=1}^{\infty} m(n)(\cos(nx)-1)\cos(ny) \, .$$

Moreover, this expression is positive when $y > x$ by Lemma 3.4.1. Therefore, by the definition (3.0.6) of Clausen functions,

$$\begin{aligned}
&\frac{1}{2xu_0}\int_x^\pi \big|K(x-y)+K(x+y)-2K(y)\big| y\, dy \\
&\qquad = \frac{1}{\pi x u_0}\sum_{n=1}^{\infty}\frac{m(n)}{n^2}(\cos(nx)-1)\big((-1)^n - nx\sin(nx)-\cos(nx)\big) \\
&\qquad = \frac{1}{\pi x u_0}\Big(x S_{\frac{3}{2}}(x) - \frac{x}{2} S_{\frac{3}{2}}(2x) + \frac{\sqrt{2}-2}{4}(C_{\frac{5}{2}}(2x)-\zeta(5/2)) \\
&\quad + \sum_{n=1}^{\infty}\frac{1-\sqrt{\tanh(n)}}{n^{5/2}}(1-\cos(nx))\big((-1)^n - nx\sin(nx)-\cos(nx)\big)\Big)
\end{aligned} \tag{3.3.4}$$

Using now the estimates proved in (3.1.1), we readily find that

$$\begin{aligned}
&\frac{1}{2xu_0}\int_x^\pi \big|K(x-y)+K(x+y)-2K(y)\big| y\, dy \\
&\qquad =: c_B^1 - \frac{2(1-\sqrt{2})\zeta(1/2)}{\pi^{3/2}}(1+\widehat{u}_0(x))\sqrt{x} + E^1_{T_0}(x) \, ,
\end{aligned} \tag{3.3.5}$$

with $\widehat{u}_0$ the auxiliary function (3.2.20) and where the error term $E^1_{T_0}(x)$ can be estimated as

$$
\begin{aligned}
|E^1_{T_0}(x)| \leqslant c^1_B\widehat{u}_0(x) + \frac{1}{4\lambda}\Big(\frac{10\sqrt{2}\zeta(5/2)}{4\pi^2 - x^2} + \frac{3}{\pi}\sum_{n=1}^{\infty} n^{3/2}(1 - \sqrt{\tanh(n)})\Big) \\
\cdot (1 + \widehat{u}_0(x))x^{5/2} = c^1_B\widehat{u}_0(x) + \frac{1}{\lambda}c^1_{T_0}(1 + \widehat{u}_0(x))x^{5/2} \,.
\end{aligned}
$$

For the region $0 < y < x$, we rewrite the integrand in terms of Clausen functions and then make use of the asymptotic formulas in order to obtain an explicit error term that is small when $x < \epsilon$. In fact, observe first that

$$
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}(\cos(nx) - 1)\cos(ny) = \frac{1}{2}\big(C_{\frac{1}{2}}(x - y) + C_{\frac{1}{2}}(x + y) - 2C_{\frac{1}{2}}(y)\big) \,.
$$

Hence, by the series representation (3.1.2),

$$
\begin{aligned}
&\sqrt{\frac{2}{\pi}}\frac{1}{x^{3/2}}\int_0^x \big|K(x - y) + K(x + y) - 2K(y)\big|y\,dy \\
&= \frac{1}{\pi x^{3/2}}\int_0^x \Big|\frac{1}{\sqrt{x - y}} + \frac{1}{\sqrt{x + y}} - \frac{2}{\sqrt{y}} + \sqrt{\frac{2}{\pi}}\big(E_{C_{\frac{1}{2}}}(x - y) + E_{C_{\frac{1}{2}}}(x + y) - 2E_{C_{\frac{1}{2}}}(y)\big) \\
&\qquad + 2\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{1 - \sqrt{\tanh(n)}}{\sqrt{n}}(1 - \cos(nx))\cos(ny)\Big|y\,dy
\end{aligned}
$$

Using now the fact that $E_{C_{\frac{1}{2}}}(x - y) + E_{C_{\frac{1}{2}}}(x + y) - 2E_{C_{\frac{1}{2}}}(y) > 0$ and the formula of $E_{C_{\frac{1}{2}}}(x)$, we obtain that

$$
\begin{aligned}
&\int_0^x \big(E_{C_{\frac{1}{2}}}(x - y) + E_{C_{\frac{1}{2}}}(x + y) - 2E_{C_{\frac{1}{2}}}(y)\big)y\,dy \\
&\qquad = \sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)!}\zeta(1/2 - 2m)\int_0^x \big(|x - y|^{2m} + |x + y|^{2m} - 2y^{2m}\big)y\,dy \\
&= 2\sqrt{2}\sum_{m=1}^{\infty}\zeta(1/2 + 2m)\frac{\Gamma(1/2 + 2m)}{\Gamma(1 + 2m)}\frac{m(4^m - 1)}{(m + 1)(2m + 1)}(2\pi)^{-1/2-2m}x^{2m+2} \leqslant c_\epsilon x^4
\end{aligned}
$$

where the constant c_ϵ is given by

$$
\begin{aligned}
c_\epsilon &:= \frac{1}{4!}f''''(\epsilon)\,, \\
f(x) &= \frac{2}{3}\sqrt{\pi}\Big(\sqrt{2}\sqrt{\pi^2 - x^2}\sqrt{\sqrt{\pi^2 - x^2} + \pi} - 5\sqrt{2}\pi\sqrt{\sqrt{\pi^2 - x^2} + \pi} \\
&\quad - 2\sqrt{4\pi^2 - x^2}\sqrt{\sqrt{4\pi^2 - x^2} + 2\pi} + 20\pi\sqrt{\sqrt{4\pi^2 - x^2} + 2\pi} - 24\pi^{3/2}\Big)\zeta(5/2)\,.
\end{aligned}
\tag{3.3.6}
$$

Furthermore, since

$$\Big|\sum_{n=1}^{\infty}\frac{1-\sqrt{\tanh(n)}}{\sqrt{n}}(1-\cos(nx))\cos(ny)\Big| \leqslant \frac{x^2}{2}\sum_{n=1}^{\infty} n^{3/2}\big(1-\sqrt{\tanh(n)}\big)\,,$$

we arrive at the estimate

$$\frac{1}{2|x|u_0}\int_0^x \big|K(x-y)+K(x+y)-2K(y)\big|y\,dy \leqslant c_B^2 + E_{T_0}^2(x), \tag{3.3.7}$$

where

$$|E_{T_0}^2(x)| \leqslant c_B^2\widehat{u}_0(x) + \frac{1}{4\pi\lambda}\Big(2c_\epsilon + \sum_{n=1}^{\infty} n^{3/2}\big(1-\sqrt{\tanh(n)}\big)\Big)(1+\widehat{u}_0(x))|x|^{5/2}$$
$$=: c_B^2\widehat{u}_0(x) + \frac{1}{\lambda}c_{T_0}^2(1+\widehat{u}_0(x))|x|^{5/2}$$

In this way, to obtain that $\|T_0\| = C_B$, we need first to verify that

$$E_{T_0}^1(x) + E_{T_0}^2 - \frac{2(1-\sqrt{2})\zeta(1/2)}{\pi^{3/2}}(1+\widehat{u}_0(x))\sqrt{x} \leqslant 0$$

in the range $0 < x < \epsilon$ for sufficiently small ϵ. This in turn follows from the bounds that we have derived here together with Lemma 3.2.12 and the numerical inequality

$$C_B c_{\epsilon,p_0}\epsilon^{p_0-1/2} + (c_{T_0}^1 + c_{T_0}^2)\epsilon^2 < \frac{\sqrt{2}-2}{2\pi}\zeta(1/2)\,. \tag{3.3.8}$$

See Appendix 3.B to see how to deal with the case $x \geqslant \epsilon$. □

Using this lemma, the inverse on $L^\infty(\mathbb{T})$ of the operator $I - T_0$ can be written as a Neumann series with norm bounded as $\|(I-T_0)^{-1}\|_{L^\infty} \leqslant \beta_0$, where

$$\beta_0 := \frac{1}{1-C_B} = 379.017\ldots \tag{3.3.9}$$

is a parameter that we will use hereafter. It is well-known that if we show that the mapping $G_0 : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$,

$$v_0 \mapsto G_0(v_0) := (I-T_0)^{-1}\Big(F_0 - \frac{|x|}{2u_0}v_0^2\Big)\,, \tag{3.3.10}$$

is contractive and takes the ball of a certain radius ϵ_0 in $L^\infty(\mathbb{T})$ into itself, then the existence of a solution v_0 of (3.2.17) is guaranteed by the Banach fixed point theorem. More precisely, letting

$$X_{\epsilon_0} := \{v_0 \in L^\infty(\mathbb{T}) :\ v_0(x) = v_0(-x)\,,\ \|v_0\|_{L^\infty(\mathbb{T})} \leqslant \epsilon_0\}$$

be the functional space on which we consider (3.2.17), the next result holds for the constants β_0 and δ_0 of before:

Proposition 3.3.2. *Let u_0 be the approximate solution* (3.2.7) *of the reduced Whitham equation* (3.2.3) *for which its associated defect $\delta_0 := \|F_0\|_{L^\infty(\mathbb{T})}$ is bounded as*

$$\delta_0 \leqslant \frac{1}{4\alpha_0\beta_0^2}\,, \qquad \alpha_0 := \sup_{x\in\mathbb{T}}\left|\frac{x}{2u_0(x)}\right|.$$

Then, for a radius ϵ_0 such that

$$\frac{1-\sqrt{1-4\alpha_0\beta_0^2\delta_0}}{2\alpha_0\beta_0} \leqslant \epsilon_0 \leqslant \frac{1}{2\alpha_0\beta_0}\,, \tag{3.3.11}$$

the following statements are true:

(i) $G_0(X_{\epsilon_0}) \subseteq X_{\epsilon_0}$.

(ii) $\|G_0(v_0) - G_0(w_0)\|_{L^\infty(\mathbb{T})} \leqslant k_0\|v_0 - w_0\|_{L^\infty(\mathbb{T})}$ *with $k_0 < 1$ for all $v_0\,, w_0$ in X_{ϵ_0}.*

Proof. As shown in Lemma 3.4.4, the estimate for $\widehat{u}_0$ given in Lemma 3.2.9 yields that $\alpha_0 \leqslant 2.696$. Moreover, by Lemma 3.2.10,

$$\delta_0 \leqslant 9.1\cdot 10^{-8} < \frac{1}{4\alpha_0\beta_0^2} = 5.2\cdot 10^{-7}\,.$$

Using now (3.3.10), it is not difficult to show that the first condition above is equivalent to the inequality $\beta_0\big(\delta_0 + \alpha_0\epsilon_0^2\big) \leqslant \epsilon_0$, which holds in view of the bound from below for ϵ_0, and the fact that the operator T_0 takes even functions into even functions, with u_0 and F_0 also even functions by construction.

Moreover,

$$\|G_0(v_0) - G_0(w_0)\|_{L^\infty(\mathbb{T})} \leqslant \beta_0 \sup_{x\in\mathbb{T}}\left|\frac{x}{2u_0(x)}\big(v_0^2 - w_0^2\big)\right| \leqslant 2\alpha_0\beta_0\epsilon_0\|v_0 - w_0\|_{L^\infty(\mathbb{T})}\,,$$

which by the bound from above for ϵ_0 makes $k_0 < 1$ and completes the proof. □

3.4 Convexity

In this section we prove the convexity part of the conjecture on Whitham waves, namely the existence of a highest cusped traveling wave solution to (3.0.1) with convex profile. To this end, we prove a priori estimates for the first and second order derivatives of the solution of the reduced Whitham equation (3.2.3). The conclusion of the main Theorem 3.0.2 will then follow from the smallness of the perturbation and the convexity of our approximate solution u_0.

Let us begin by considering operators $T_i : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ with $i = 1, 2$ that will play an analogous role as T_0 in (3.2.17):

$$T_1 v_1(x) := \frac{1}{2u_0(x)} \int_0^\pi \Big(K(x-y) - K(x+y) + \frac{u_0'(x)}{u_0(x)} K_1(x,y) \Big) v_1(y)\, dy\,, \tag{3.4.1}$$

$$\begin{aligned} T_2 v_2(x) := \frac{|x|}{2u_0(x)} \int_0^\pi \Big(& K(x-y) + K(x+y) + \frac{2u_0'(x)}{u_0(x)} K_2(x,y) \\ & + \frac{1}{(u_0(x))^2} \big(2(u_0'(x))^2 - u_0(x) u_0''(x) \big) \overline{K}_2(x,y) - \chi(x,y) f(x) \Big) \frac{v_2(y)}{y}\, dy\,. \end{aligned} \tag{3.4.2}$$

Here $\chi(x,y)$ denotes a step function that is 1 when $y < x$ and zero otherwise,

$$f(x) := 2K(x) + \frac{2u_0'(x)}{u_0(x)} K_2(x,0) + \frac{1}{(u_0(x))^2} \big(2(u_0'(x))^2 - u_0(x) u_0''(x) \big) \overline{K}_2(x,0)\,, \tag{3.4.3}$$

and we have introduced the following functions of the Whitham kernel K:

$$K_1(x,y) := \int_0^{x+y} K(t)\, dt - \int_0^{x-y} K(t)\, dt - 2 \int_0^y K(t)\, dt\,, \tag{3.4.4}$$

$$K_2(x,y) := - \int_0^{x+y} K(t)\, dt - \int_0^{x-y} K(t)\, dt\,, \tag{3.4.5}$$

$$\overline{K}_2(x,y) = \int_0^{x-y} \int_0^s K(t)\, dt\, ds + \int_0^{x+y} \int_0^s K(t)\, dt\, ds - 2 \int_0^y \int_0^s K(t)\, dt\, ds\,, \tag{3.4.6}$$

In the next lemma we show that the kernels of the three operators T_i have a definite sign when $y > x$, in the above notation. This feature will be remarkably useful in the computation of the norms $\|T_i\|$.

Lemma 3.4.1. *Let u_0 be our approximate solution* (3.2.7). *Then,*

$$K(x-y) + K(x+y) - 2K(y)\,, \tag{3.4.7}$$

$$K(x-y) - K(x+y) + \frac{u_0'(x)}{u_0(x)} K_1(x,y)\,, \tag{3.4.8}$$

$$\begin{aligned} & K(x-y) + K(x+y) + \frac{2u_0'(x)}{u_0(x)} K_2(x,y) \\ & \qquad + \frac{1}{(u_0(x))^2} \big(2(u_0'(x))^2 - u_0(x) u_0''(x) \big) \overline{K}_2(x,y) \end{aligned} \tag{3.4.9}$$

are positive functions for $y > x$.

Proof. By parity considerations and the representation formula of the Whitham Kernel (3.0.4) that stems from [31, Eq. 2.18], it is enough to check that

$$\sinh\big(s(\pi-y)\big)\Big(\sinh(sx)+\frac{(1-\cosh(sx))}{s}\frac{u_0'(x)}{u_0(x)}\Big)\geqslant 0\,,$$
$$\cosh\big(s(\pi-y)\big)\Big(\cosh(sx)-2\frac{u_0'(x)}{u_0(x)}\frac{\sinh(sx)}{s} + \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\frac{(\cosh(sx)-1)}{s^2}\Big)\geqslant 0$$

for $y>x>0$ and $s>0$. In fact, the proof relies on the fact that the sign in the three expressions (3.4.7), (3.4.8) and (3.4.9) depends on the sign of some combinations of the function $\cosh[s(\pi-|x|)]$ (and its derivatives) that appears in the integrand of (3.0.4).

Notice first that (3.4.7) is positive as

$$\cosh\big(s(\pi-x-y)\big)+\cosh\big(s(\pi+x-y)\big)-2\cosh\big(s(\pi-y)\big) = 4\cosh\big(s(\pi-y)\big)\sinh(\tfrac{1}{2}sx)^2>0$$

for all $s>0$. Furthermore, since the functional inequality

$$\alpha\sinh(z)+\frac{1-\cosh(z)}{2z}\geqslant 0$$

holds for all $z>0$ and $\alpha>1/4$, the positivity of (3.4.8) follows immediately by the bound (3.2.24) stated in Lemma 3.2.12. Analogously, for the last expression we use (3.2.25), the above inequality and the fact that

$$\cosh(z)-\frac{1}{z}\sinh(z)+\frac{3}{4z^2}\big(\cosh(z)-1\big)\geqslant 0\,.$$

□

Lemma 3.4.2. *The norm of the operator T_1 is*

$$\|T_1\|=C_B\,, \tag{3.4.10}$$

where C_B is the constant defined in (3.3.2).

Proof. As in the proof of the norm of T_0, we divide the integral (3.4.10) into two pieces and make use of the bounds for the Clausen functions to show that the integral is bounded by C_B for $x\leqslant\epsilon$. For $x>\epsilon$ the proof is detailed in Appendix 3.B.

Let us first analyze the integral when $x < y < \pi$ (as before, we can assume that x and y are positive by parity):

$$\begin{aligned}
&\frac{1}{2u_0(x)}\int_x^{\pi}\Big(K(x-y)-K(x+y)+\frac{u_0'(x)}{u_0(x)}K_1(x,y)\Big)\,dy\\
&\quad=\frac{1}{\pi u_0(x)}\Big[S_{\frac{3}{2}}(x)+\frac{1-\sqrt{2}}{2}S_{\frac{3}{2}}(2x)+\frac{2-\sqrt{2}}{4}\big(C_{\frac{5}{2}}(2x)-\zeta(5/2)\big)\frac{u_0'(x)}{u_0(x)}\\
&\quad+\sum_{n=1}^{\infty}\frac{1-\sqrt{\tanh(n)}}{n^{3/2}}\Big(\sin(nx)+\frac{u_0'(x)}{nu_0(x)}\big(\cos(nx)-1\big)\Big)\big((-1)^n-\cos(nx)\big)\Big]\,.
\end{aligned}\tag{3.4.11}$$

Noticing that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1-\sqrt{\tanh(n)}}{n^{3/2}}\Big(\sin(nx)+\frac{u_0'(x)}{nu_0(x)}\big(\cos(nx)-1\big)\Big)\big((-1)^n-\cos(nx)\big)\\
&\qquad\leqslant\frac{5}{4}x\sum_{n=1}^{\infty}\frac{|1-(-1)^n|}{\sqrt{n}}\big(1-\sqrt{\tanh(n)}\big)+\frac{5}{8}x^3\sum_{n=1}^{\infty}n^{3/2}\big(1-\sqrt{\tanh(n)}\big)\,,
\end{aligned}$$

and using Lemma 3.1.1 combined with Lemma 3.2.12, we then have that

$$\begin{aligned}
&\frac{1}{2u_0(x)}\int_x^{\pi}\Big(K(x-y)-K(x+y)+\frac{u_0'(x)}{u_0(x)}K_1(x,y)\Big)\,dy\\
&\qquad\leqslant c_B^1-\frac{1}{4\pi\lambda}\Big(3(\sqrt{2}-2)\zeta(1/2)+5\sum_{n=1}^{\infty}\frac{|1-(-1)^n|}{\sqrt{n}}\big(1-\sqrt{\tanh(n)}\big)\Big)\\
&\qquad\cdot(1+\widehat{u}_0(x))\sqrt{x}+E_{T_1}^1(x)=:c_B^1-\frac{1}{\lambda}c_{\frac{1}{2}}(1+\widehat{u}_0(x))\sqrt{x}+E_{T_1}^1(x)\,,
\end{aligned}\tag{3.4.12}$$

with

$$\begin{aligned}
|E_{T_1}^1(x)|&\leqslant c_B^1\widehat{u}_0(x)+\frac{c'_{\epsilon,p_0}(\sqrt{2}-1)}{\lambda\pi\sqrt{\pi}}\Big(\frac{2\sqrt{2\pi}}{3}+\frac{|\zeta(1/2)|}{2}\sqrt{x}+\frac{|\zeta(5/2)|}{\sqrt{\pi}}\frac{x^{5/2}}{4\pi^2-x^2}\\
&\quad+\frac{5\sqrt{\pi}}{8c'_{\epsilon,p_0}(\sqrt{2}-1)}\sum_{n=1}^{\infty}n^{3/2}\big(1-\sqrt{\tanh(n)}\big)x^{5/2-p_0}\Big)(1+\widehat{u}_0(x))x^{p_0}\\
&\leqslant c_B^1\widehat{u}_0(x)+\frac{1}{\lambda}c'_{\epsilon,p_0}c_{T_1}^1(1+\widehat{u}_0(x))x^{p_0}\,.
\end{aligned}$$

Here we have used that

$$\frac{1}{\pi\sqrt{x}}\int_x^{\pi}\Big(\frac{1}{\sqrt{y-x}}-\frac{1}{\sqrt{x+y}}+\frac{1}{x}\big(\sqrt{x+y}+\sqrt{y-x}-2\sqrt{y}\big)\Big)\,dy=:c_B^1\,.$$

As for the integral in the region $0 < y < x$, the proof relies on the formula

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin(ny)\Big(\sin(nx) + \frac{1}{2nx}\big(\cos(nx) - 1\big)\Big) \\ = \frac{1}{2}\Big(C_{\frac{1}{2}}(x-y) - C_{\frac{1}{2}}(x+y) + \frac{1}{2x}\big(S_{3/2}(x+y) - S_{3/2}(x-y) - 2S_{3/2}(y)\big)\Big) ,$$

the estimates for the Clausen functions stemming from Lemma 3.1.1 and the value of the integral

$$\frac{1}{\pi\sqrt{x}} \int_0^x \Big| \frac{1}{\sqrt{x-y}} - \frac{1}{\sqrt{x+y}} + \frac{1}{x}\big(\sqrt{x+y} - \sqrt{x-y} - 2\sqrt{y}\big) \Big| \, dy = c_B^2 .$$

In fact, we have

$$\frac{1}{2u_0(x)} \int_0^x \Big| K(x-y) - K(x+y) + \frac{u_0'(x)}{u_0(x)} K_1(x,y) \Big| \, dy \\ = \frac{1 + \widehat{u}_0(x)}{\pi\sqrt{x}} \int_0^x \Big| \frac{1}{\sqrt{x-y}} - \frac{1}{\sqrt{x+y}} + \frac{1}{x}\big(\sqrt{x+y} - \sqrt{x-y} - 2\sqrt{y}\big) \\ + 2\Big(\frac{u_0'(x)}{u_0(x)} - \frac{1}{2x}\Big)\big(\sqrt{x+y} - \sqrt{x-y} - 2\sqrt{y}\big) \\ + \sqrt{\frac{2}{\pi}}\Big(E_{C_{1/2}}(x-y) - E_{C_{1/2}}(x+y) \\ + \frac{u_0'(x)}{u_0(x)}\big(E_{S_{3/2}}(x+y) - E_{S_{3/2}}(x-y) - 2E_{S_{3/2}}(x+y)\big)\Big) \\ + 2\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{1 - \sqrt{\tanh(n)}}{\sqrt{n}} \sin(ny)\Big(\sin(nx) + \frac{u_0'(x)}{nu_0(x)}\big(\cos(nx) - 1\big)\Big) \Big| \, dy \\ \leqslant c_B^2 + E_{T_1}^2(x) . \quad (3.4.13)$$

In the above bound,

$$|E_{T_1}^2(x)| \leqslant c_B^2 \widehat{u}_0 + \frac{8c_{\epsilon,p_0}'}{15\pi\lambda}(5 - 5\sqrt{2} + 2\sqrt{5})(1 + \widehat{u}_0(x))x^{p_0} \\ + \frac{1}{\lambda}\Big(\frac{5}{4\pi} \sum_{n=1}^{\infty} n^{1/2}\big(1 - \sqrt{\tanh(n)}\big) + c_\epsilon' x\Big)(1 + \widehat{u}_0(x))x^{3/2} \\ \leqslant c_B^2 \widehat{u}_0 + \frac{1}{\lambda} c_{\epsilon,p_0}' c_{T_1}^2 (1 + \widehat{u}_0(x))x^{p_0} + \frac{1}{\lambda} c_{T_1}^3 (1 + \widehat{u}_0(x))x^{3/2} , \quad (3.4.14)$$

where the number c_ϵ' is a bound for the integrals coming from the Clausen error terms:

$$
\begin{aligned}
c'_\epsilon &:= \frac{g'''(\epsilon)}{3!}\,, \\
g(x) &:= \frac{x}{\sqrt{\pi}}\Big(\frac{\sqrt{2}}{\sqrt{\sqrt{\pi^2-x^2}+\pi}} - \frac{2}{\sqrt{\sqrt{4\pi^2-x^2}+2\pi}}\Big)\zeta(5/2) \\
&+ \frac{2}{3\sqrt{\pi}x}\Big(x\big(-\sqrt{\pi-x}+\sqrt{x+\pi}-\sqrt{2}\sqrt{x+2\pi}+\sqrt{4\pi-2x}\big) \\
&+ \pi\big(\sqrt{\pi-x}+\sqrt{x+\pi}-2\sqrt{2}\sqrt{x+2\pi}-2\sqrt{4\pi-2x}\big)+6\pi^{3/2}\Big)\zeta(5/2)\,.
\end{aligned}
$$

Then, since the numerical inequality

$$
(C_B c_{\epsilon,p_0} + c'_{\epsilon,p_0}(c^1_{T_1} + c^2_{T_1}))\epsilon^{p_0-1/2} + c^3_{T_1}\epsilon < c_{\frac{1}{2}} \tag{3.4.15}
$$

holds for small enough ϵ, it follows that $\|T_1\| = C_B$. □

Likewise, using Lemma 3.4.1 we can prove that the norm of the operator T_2 on $L^\infty(\mathbb{T})$ is also less than 1. As the proof involves some careful computations related to the singularity y^{-1} in the integrand of (3.4.3), we leave it for Appendix 3.A:

Lemma 3.4.3. *The norm of T_2 is also given by the constant C_B, i.e.*

$$
\|T_2\| = C_B\,. \tag{3.4.16}
$$

For the following, it will be also useful to have explicit control of the following numerical constants obtained from the approximate solution u_0:

Lemma 3.4.4. *Let u_0 be the approximation given by* (3.2.7) *and let*

$$
\alpha_0 := \sup_{x\in\mathbb{T}}\Big|\frac{x}{2u_0(x)}\Big|, \quad \alpha_1 := \sup_{x\in\mathbb{T}}\Big|\frac{x^2u_0'(x)}{2u_0^2(x)}\Big|, \quad \alpha_f := \sup_{x\in\mathbb{T}}\Big|\frac{x}{2u_0(x)}f(x)\Big|,
$$
$$
\alpha_2 := \sup_{x\in\mathbb{T}}\Big|\frac{x^3}{2(u_0(x))^2}\Big(u_0''(x) - \frac{2(u_0'(x))^2}{u_0(x)}\Big)\Big|, \quad \overline{\alpha}_2 := \sup_{x\in\mathbb{T}}\Big|\frac{2x^2u_0'(x)}{(u_0(x))^2}\Big|.
$$

Then, the values of these constants are

$$
\alpha_0 \leqslant 2.696\,, \alpha_1 \leqslant 0.32\,, \alpha_2 \leqslant 1.382\,, \overline{\alpha}_2 \leqslant 1.280\,, \alpha_f \leqslant 0.448\,.
$$

Proof. As for the rest of quantities that we estimate through the chapter, the bounds near $x = 0$ follow from the asymptotic analysis carried out in Section 3.2 and particularly from Lemma 3.2.12. The proof of the estimates away from 0 is presented in Appendix 3.B.

□

Our objective now is to control the derivatives of the solution u to the reduced Whitham equation (3.2.3) in terms of our knowledge about the approximate solution u_0. To this end we will derive a priori estimates to bound in L^∞ the functions v_1, v_2 defined as

$$v_1(x) := u'(x) - u_0'(x) \tag{3.4.17}$$
$$v_2(x) := |x|\big(u''(x) - u_0''(x)\big)\,. \tag{3.4.18}$$

For convenience, we also recall the definition of the function

$$v_0(x) := \frac{1}{|x|}\big(u(x) - u_0(x)\big)\,, \tag{3.4.19}$$

which are introduced in (3.3.1) in order to control the difference between the exact and the approximate solutions. Furthermore, in terms of $\beta_0 := (1 - C_B)^{-1}$ and $\delta_0 := \|F_0\|_{L^\infty(\mathbb{T})}$, we have

$$\|v_0\|_{L^\infty(\mathbb{T})} \leqslant \epsilon_0 := \frac{1 - \sqrt{1 - 4\alpha_0\beta_0^2\delta_0}}{2\alpha_0\beta_0} \leqslant 3.59 \cdot 10^{-5}\,, \tag{3.4.20}$$

by Proposition 3.3.2. It is not difficult to show that the errors v_1 and v_2 are small in L^∞, with bounds depending on the radius ϵ_0:

Lemma 3.4.5. *Let $v_1\,, v_2$ be defined as above and let $F_0(x), F_1(x)$ and $F_2(x)$ be the error terms given by Lemmas 3.2.10 and 3.2.13:*

$$F_0(x) := \frac{1}{2|x|u_0}(\mathcal{L}u_0 - u_0^2)\,, \quad F_1(x) := F_0'(x)\,, \quad F_2(x) := |x|F_0''(x)\,,$$

with $\delta_j := \|F_j\|_{L^\infty(\mathbb{T})}$ for $j = 0, 1, 2$. Then, for $\alpha_0, \alpha_1, \alpha_2, \overline{\alpha}_2, \alpha_f$ as in Lemma 3.4.4, the functions v_1 and v_2 satisfy the estimates

$$\|v_1\|_{L^\infty(\mathbb{T})} \leqslant \epsilon_1\,, \quad \|v_2\|_{L^\infty(\mathbb{T})} \leqslant \epsilon_2\,, \tag{3.4.21}$$

where the constants ϵ_j are

$$\epsilon_1 := \beta\big(\delta_1 + \alpha_1\epsilon_0^2\big) \leqslant 3.77 \cdot 10^{-4}\,, \tag{3.4.22}$$
$$\epsilon_2 := \beta\big(\delta_2 + \alpha_f\epsilon_1 + \alpha_2\epsilon_0^2 + 2\alpha_0\epsilon_1^2 + \overline{\alpha}_2\epsilon_0\epsilon_1\big) \leqslant 7.46 \cdot 10^{-2}\,, \tag{3.4.23}$$

and where

$$\beta := \frac{1}{1 - C_B - 2\alpha_0\epsilon_0}\,. \tag{3.4.24}$$

Proof. Let us write $u(x) := u_0(x) + \overline{u}(x)$. Since

$$\overline{u}(x) - \frac{1}{2u_0(x)}\mathcal{L}\overline{u}(x) = F_0(x) - \frac{1}{2u_0(x)}\overline{u}^2(x)\,,$$

it is clear that by taking $\overline{u}(x) := |x|v_0(x)$ we obtain (3.2.17). On the other hand, differentiating in the above equation we have that

$$\begin{aligned}
F_1(x)+&\frac{u_0'(x)}{2u_0^2(x)}\overline{u}^2(x) - \frac{1}{u_0(x)}\overline{u}'(x)\overline{u}(x)\\
&= \overline{u}'(x) + \frac{u_0'(x)}{2u_0^2(x)}\int_0^\pi \big(K(x+y)+K(x-y)-2K(y)\big)\overline{u}(y)\,dy\\
&\quad - \frac{1}{2u_0(x)}\int_0^\pi \big(K'(x+y)+K'(x-y)\big)\overline{u}(y)\,dy\\
&= \overline{u}'(x) + \frac{u_0'(x)}{2u_0^2(x)}\int_0^\pi K_1(x,y)\overline{u}(y)\,dy\\
&\quad + \frac{1}{2u_0(x)}\int_0^\pi \big(K(x-y)-K(x+y)\big)\overline{u}(y)\,dy\,.
\end{aligned}$$

Integrating by parts again (where the boundary terms are zero by parity), by the definition of the operator T_1 and letting $v_1(x) := \overline{u}'(x)$, one has that

$$\Big(I + \frac{|x|}{u_0(x)}v_0(x) - T_1\Big)v_1(x) = F_1(x) + \frac{x^2u_0'(x)}{2u_0^2(x)}v_0^2(x)\,, \qquad (3.4.25)$$

Furthermore, defining $v_2(x) := \overline{u}''(x)/|x|$, differentiating twice and integrating by parts, we obtain the equation

$$\begin{aligned}
\Big(I + \frac{|x|}{u_0(x)}v_0(x) - T_2\Big)v_2(x) = F_2(x) &+ \frac{|x|^3}{2u_0^3(x)}\big(u_0(x)u_0''(x) - 2(u_0'(x))^2\big)v_0^2(x)\\
&+ \frac{|x|}{2u_0(x)}f(x)v_1(x) - \frac{|x|}{u_0(x)}v_1(x)^2 + \frac{2x^2u_0'(x)}{u_0^2(x)}v_0(x)v_1(x)\,.
\end{aligned} \qquad (3.4.26)$$

Inverting the linear operators on the LHS of (3.4.25) and (3.4.26), by using the fact that $\|T_i\| = C_B$ for $i = 0, 1, 2$ and the above definitions one readily obtains L^∞-bounds for v_1 and v_2 which immediately yield the estimates (3.4.21). □

Now we are ready to prove the main result of this chapter. Firstly, we use the fact that there exists a negative constant c such that $u_0''(x) < c/|x| < 0$ everywhere. Since ϵ_2 is sufficiently small, then the second derivative of the solution u of (3.2.3) has a sign too. More precisely:

Lemma 3.4.6. *Let* ϵ_2 *be the bound* (3.4.23) *for the perturbation* v_2 *solution to* (3.4.26)*. Then,*

$$u''(x) \leqslant u_0''(x) + \frac{\epsilon_2}{|x|} < 0 \tag{3.4.27}$$

for $x \in (-\pi, \pi)$.

Proof. Sufficiently close to $x = 0$ the proof follows by the asymptotic formula (3.2.9) and the bound ϵ_2 obtained in Lemma 3.4.5. For x bounded away from zero the proof is done as explained in Appendix 3.B. □

Finally, we prove the convexity of the highest cusped Whitham wave:

Proof of Theorem 3.0.2: From Proposition 3.3.2 and Lemma 3.4.5, it follows that there exists a solution u such that the associated errors v_0, v_1 and v_2 defined in Equations (3.4.17) and (3.4.19) are bounded as

$$\|v_j\| \leqslant \epsilon_j\,, \qquad j = 0, 1, 2\,,$$

with ϵ_j as in (3.4.21) and (3.4.20). In particular, Lemma 3.4.6 ensures that for this solution we have that $u''(x) < 0$ for all $x \in (-\pi, \pi)$.

To complete the proof of the theorem, it is enough to show that $|u(x) - u_0(x)| \leqslant C|x|^{1+\eta}$ for some $\eta > 0$, so that $u(x) - u_0(x) = o(|x|)$. For this purpose, let us write the error $v_0(x) =: |x|^\eta v_\eta(x)$. Thus, it is clear that the function

$$v_\eta(x) := \frac{1}{|x|^{1+\eta}}\big(u(x) - u_0(x)\big)$$

satisfies the equation

$$(I - T_\eta)v_\eta = F_\eta - \frac{|x|^{1+\eta}}{2u_0}v_\eta^2\,.$$

Here the defect is $F_\eta(x) = |x|^{-\eta}F_0(x)$ and the linear operator that controls the equation is $T_\eta v_\eta(x) := |x|^{-\eta}T_0(|x|^\eta v_\eta(x))$, whose $L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ norm is

$$\|T_\eta\| := \sup_{0<x<\pi} \frac{1}{2|x|^{1+\eta}u_0(x)} \int_0^\pi \big|K(x-y) + K(x+y) - 2K(y)\big| y^{1+\eta}\, dy\,.$$

Observe now that the dominated convergence theorem easily implies that $\|T_\eta\|$ is continuous in η at $\eta = 0$. Therefore, as $\|T_0\| = \|T_\eta\||_{\eta=0} = C_B < 1$, there exists a sufficiently small $\eta > 0$ such that $\|T_\eta\| < 1$. Hence, by a completely analogous fixed point argument (based again on the smallness of the defect F_η), we find the estimate $\|v_\eta\|_{L^\infty(\mathbb{T})} \leqslant C$, which readily implies that

$$|u(x) - u_0(x)| = ||x|^{1+\eta}v_\eta(x)| = o(|x|)\,.$$

The theorem then follows.

□

3.A The Norm of T_2

This Appendix is devoted to the computation of the norm of the operator T_2 given in (3.4.2). As in the cases of T_0 and T_1, using the asymptotic analysis carried out in Section 3.2 we show that $\|T_2\|$ is precisely the constant C_B.

Proof of Lemma 3.4.3. Arguing as in Lemma 3.4.2, let us take $x, y > 0$ and let ϵ be a small positive number. In this way, notice that the integrand of (3.4.2) can be expressed as

$$
\begin{aligned}
&\frac{2}{\pi}\sum_{n=1}^{\infty} m(n)\Big[\cos(nx) - 2\frac{u_0'(x)}{nu_0(x)}\sin(nx) \\
&\qquad + \frac{1}{n^2(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big)\big(1-\cos(nx)\big)\Big] \\
&\qquad\quad \cdot \frac{\big(\cos(ny) - \chi(x,y)\big)}{y} \\
&\qquad = \frac{1}{\pi y}\Big[C_{\frac{1}{2}}(x-y) + C_{\frac{1}{2}}(x+y) - 2\chi(x,y)C_{\frac{1}{2}}(x) \\
&\qquad - 2\frac{u_0'(x)}{u_0(x)}\cdot\big(S_{\frac{3}{2}}(x-y) + S_{\frac{3}{2}}(x+y) - 2\chi(x,y)S_{\frac{3}{2}}(x)\big) \\
&\qquad - \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big) \\
&\qquad \cdot\big(C_{\frac{5}{2}}(x-y) + C_{\frac{5}{2}}(x+y) - 2C_{\frac{5}{2}}(y) - 2\chi(x,y)(C_{\frac{5}{2}}(x) - \zeta(5/2))\big)\Big] \\
&\qquad + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\big(\sqrt{\tanh(n)}-1\big)}{\sqrt{n}}\Big[\cos(nx) - 2\frac{u_0'(x)}{nu_0(x)}\sin(nx) \\
&\qquad + \frac{1}{n^2(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big)\big(1-\cos(nx)\big)\Big] \\
&\qquad\quad \cdot \frac{\big(\cos(ny) - \chi(x,y)\big)}{y}.
\end{aligned}
$$

Then, the norm of T_2 is obtained by taking the supremum in $x \in [0, \pi]$ of the integral

$$
\begin{aligned}
&\frac{\sqrt{x}}{\pi}(1+\widehat{u}_0(x)) \int_0^\pi \Big| \frac{1}{\sqrt{|x-y|}} + \frac{1}{\sqrt{x+y}} - \frac{2}{x}\big(\sqrt{x+y} + \operatorname{sgn}(x-y)\sqrt{|x-y|}\big) \\
&\qquad + \frac{1}{x^2}\big((x+y)^{3/2} + |x-y|^{3/2} - 2y^{3/2}\big) \\
&\quad + 4\Big(\frac{1}{2x} - \frac{u_0'(x)}{u_0(x)}\Big)\big(\sqrt{x+y} + \operatorname{sgn}(x-y)\sqrt{|x-y|} - 2\chi(x,y)\sqrt{x}\big) \\
&\qquad - \frac{4}{3}\Big(\frac{3}{4x^2} - \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big)\Big) \\
&\qquad \big((x+y)^{3/2} + |x-y|^{3/2} - 2y^{3/2} - 2\chi(x,y)x^{3/2}\big) \\
&\quad + \sqrt{\frac{2}{\pi}}\Big[E_{C_{\frac12}}(x-y) + E_{C_{\frac12}}(x+y) - 2\chi(x,y)E_{C_{\frac12}}(x) \\
&\quad - \frac{2u_0'(x)}{u_0(x)}\big(E_{S_{\frac32}}(x-y) + E_{S_{\frac32}}(x+y) - 2\chi(x,y)E_{S_{\frac32}}(x)\big) \\
&\qquad - \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big) \\
&\quad \big(E_{C_{\frac52}}(x-y) + E_{C_{\frac52}}(x+y) - 2E_{C_{\frac52}}(y) - 2\chi(x,y)E_{C_{\frac52}}(x)\big) \\
&\quad + 2\sum_{m=1}^{\infty} \frac{1}{\sqrt{n}}\big(1-\sqrt{\tanh(n)}\big)\Big(\cos(nx) - 2\frac{u_0'(x)}{nu_0(x)}\sin(nx) \\
&\quad + \frac{1}{n^2(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big)\big(1-\cos(nx)\big)\Big) \\
&\qquad\qquad \big(\chi(x,y) - \cos(ny)\big)\Big]\Big| \frac{dy}{y}\,.
\end{aligned}
$$

As before, we carry out the analysis of the norm by dividing the above integral into two pieces: the regions $x < y < \pi$, in which $\chi(x,y) = 0$ so the integrand is positive by Lemma 3.4.1, and $0 < y < x$.

Notice that the integral on $x < y < \pi$ yields

$$
\begin{aligned}
&\frac{x}{\pi u_0(x)} \sum_{n=1}^{\infty} m(n)\Big(\cos(nx) - 2\frac{u_0'(x)}{nu_0(x)}\sin(nx) \\
&\quad + \frac{1}{n^2(u_0(x))^2}\big(2(u_0'(x))^2 - u_0(x)u_0''(x)\big)\big(1-\cos(nx)\big)\Big)\big(\mathrm{Ci}(n\pi) - \mathrm{Ci}(nx)\big)\,,
\end{aligned}
$$

where $\mathrm{Ci}(n\pi)$ denotes the cosine integral function (see e.g. [24, Chapter 6] for the definition of $\mathrm{Ci}(x)$ and its properties). In this region, we will use the integral estimates

$$4\Big(\frac{1}{2x}-\frac{u_0'(x)}{u_0(x)}\Big)\int_x^\pi\big(\sqrt{x+y}-\sqrt{y-x}\big)\,\frac{dy}{y}$$
$$-\frac{4}{3}\Big(\frac{3}{4x^2}-\frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\Big)\int_x^\pi\big((x+y)^{3/2}+(y-x)^{3/2}\big)\,\frac{dy}{y}$$
$$\leqslant\frac{\pi\widehat{c}_{p_0}}{\lambda}x^{p_0-1/2}\,,$$

and

$$-\frac{2}{\pi}\sum_{n=1}^\infty\frac{1}{\sqrt{n}}\big(1-\sqrt{\tanh(n)}\big)\Big(\cos(nx)-2\frac{u_0'(x)}{nu_0(x)}\sin(nx)$$
$$+\frac{1}{n^2(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\big(1-\cos(nx)\big)\Big)\int_x^\pi\frac{\cos(ny)}{y}\,dy$$
$$\leqslant\frac{3}{4\pi}\sum_{n=1}^\infty\frac{1-\sqrt{\tanh(n)}}{\sqrt{n}}\big(\log(x)+\log(n)+\gamma-\mathrm{Ci}(n\pi)\big)+2\pi(c_{p_0}^1-\log(x)c_{p_0}^2)x^{p_0}$$
$$\leqslant-\frac{1}{12}\log(\pi)\zeta(1/2)+\frac{3}{4\pi}\sum_{n=1}^\infty\frac{1-\sqrt{\tanh(n)}}{\sqrt{n}}\log(x)+2\pi(c_{p_0}^1-\log(x)c_{p_0}^2)x^{p_0}\,,$$

which holds for small constants $\widehat{c}_{p_0}, c_{p_0}^1, c_{p_0}^2$ that only depend on the bounds obtained in Lemma 3.2.12. In the last estimate we have used that

$$\int_x^\pi\frac{\cos(ny)}{y}\,dy=\mathrm{Ci}(n\pi)-\mathrm{Ci}(nx)\geqslant\mathrm{Ci}(n\pi)-\gamma-\log(n)-\log(x)\,,$$

and the numerical inequality

$$\sum_{n=1}^\infty\frac{\sqrt{\tanh(n)}-1}{\sqrt{n}}(\mathrm{Ci}(n\pi)-\log(n)-\gamma)+\frac{\log(\pi)}{9\pi}\zeta(1/2)<0\,,\tag{3.A.1}$$

with γ the Euler constant.

Since in addition

$$\int_x^\pi\Big(E_{C_{\frac12}}(x-y)+E_{C_{\frac12}}(x+y)-\frac{2u_0'(x)}{u_0(x)}\big(E_{S_{\frac32}}(x-y)+E_{S_{\frac32}}(x+y)\big)$$
$$-\frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\big(E_{C_{\frac52}}(x-y)+E_{C_{\frac52}}(x+y)-2E_{C_{\frac52}}(y)\big)\,\frac{dy}{y}$$
$$\leqslant\frac{3}{4}\zeta(1/2)\log\Big(\frac{\pi}{x}\Big)+2\pi c_{T_2}^1x^2\,,$$

putting together the above estimates we arrive at

$$\frac{x}{2u_0(x)}\int_x^\pi \Big(K(x-y)+K(x+y)+\frac{2u_0'(x)}{u_0(x)}K_2(x,y) + \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\overline{K}_2(x,y)\Big)\,\frac{dy}{y} \leqslant c_B^1 - \frac{1}{\lambda}c_{\frac12}''\sqrt{x}(1+\widehat{u}_0(x)) + E_{T_2}^1(x)\,, \quad (3.A.2)$$

where $c_{\frac12}'' := -\frac{1}{3\pi}\log(\pi)\zeta(1/2)$,

$$\begin{aligned}|E_{T_2}^1(x)| &\leqslant c_B^1\widehat{u}_0(x) + \frac{3}{8\pi^2\lambda}\log(x)\Big(\sum_{n=1}^\infty \frac{1-\sqrt{\tanh(n)}}{\sqrt{n}} - \pi\zeta(1/2)\Big)\sqrt{x}(1+\widehat{u}_0(x)) \\ &\quad + \frac{1}{\lambda}\widehat{c}_{p_0}'x^{p_0}(1+\widehat{u}_0(x)) + \frac{1}{\lambda}(c_{p_0}^1 - \log(x)c_{p_0}^2)x^{p_0+1/2}(1+\widehat{u}_0(x)) \\ &\quad + \frac{c_{T_2}^1}{\lambda}x^{5/2}(1+\widehat{u}_0(x)) \\ &=: c_B^1\widehat{u}_0(x) + \frac{1}{\lambda}\tilde{c}_{1/2}\log(x)\sqrt{x}(1+\widehat{u}_0(x)) \\ &\quad + \frac{1}{\lambda}\Big(\widehat{c}_{p_0}' + (c_{p_0}^1 - \log(x)c_{p_0}^2)\sqrt{x}\Big)x^{p_0}(1+\widehat{u}_0(x)) + \frac{c_{T_2}^1}{\lambda}x^{5/2}(1+\widehat{u}_0(x)).\end{aligned} \quad (3.A.3)$$

On the other hand, analogous estimates yield

$$\frac{x}{2u_0(x)}\int_0^x \Big|K(x-y)+K(x+y)+\frac{2u_0'(x)}{u_0(x)}K_2(x,y) + \frac{1}{(u_0(x))^2}\big(2(u_0'(x))^2-u_0(x)u_0''(x)\big)\overline{K}_2(x,y)\Big|\,\frac{dy}{y} \leqslant c_B^2 + E_{T_2}^2(x)\,, \quad (3.A.4)$$

with

$$|E_{T_2}^2(x)| \leqslant c_B^2\widehat{u}_0(x) + \frac{1}{\lambda}\widehat{c}_{p_0}''x^{p_0}(1+\widehat{u}_0(x)) + \frac{c_{T_2}^2}{\lambda}x^{5/2}(1+\widehat{u}_0(x))\,. \quad (3.A.5)$$

Then, since for sufficiently small ϵ

$$\Big(C_B c_{p_0} + \widehat{c}_{p_0}' + \widehat{c}_{p_0}'' + (c_{p_0}^1 - \log(\epsilon)c_{p_0}^2)\sqrt{\epsilon}\Big)\epsilon^{p_0-1/2} + (c_{T_2}^1 + c_{T_2}^2)\epsilon^{2-p_0} < c_{\frac12}'' - \tilde{c}_{\frac12}\log(\epsilon)\,, \quad (3.A.6)$$

the proof follows in the same manner as in Lemmas 3.3.1 and 3.4.2. □

3.B Technical Details Concerning Computer Assisted Estimates

In this section we discuss the technical details about the implementation of the different rigorous numerical computations such as the integrals that appear in the proofs along the chapter. We remark that we are computing explicit (but complicated) functions on a one dimensional domain. In order to perform the rigorous computations we used the Arb library [46]; the code can be found in the supplementary material.

The implementation is split into several files, and many of the headers of the functions (such as the integration methods) contain pointers to functions (the integrands) so that they can be reused for an arbitrary number of integrals with minimal changes and easy and safe debugging. We first describe the data structures that will appear in the different parts of the code and later get to the specific algorithms of each lemma.

There is a basic class that encloses all the necessary information used throughout the computations in Lemmas 3.2.10, 3.2.13, 3.3.1, 3.4.2 and 3.4.3. It is called `Integration_params_struct` and has the following members: three integers, `N_0`, `N_1` and `N_2`; three vectors of intervals `a0k`, `a1k` and `bi` of sizes N_0, N_1 and N_2 respectively, containing the coefficients that describe the approximate solution u_0 of (3.2.7). There is also an interval called `x`, which is used only in Lemmas 3.3.1, 3.4.2 and 3.4.3, indicating the value of x used for the integration.

We had to implement the Clausen functions, since they are not part of the Arb library. A naive implementation of $C_z(x)$ (resp. $S_z(x)$) would be to evaluate the real (resp. imaginary) parts of $\mathrm{Li}_z(e^{ix})$. When x is an interval, this gives a disastrous error. Instead, we will make use of the following lemma:

Lemma 3.B.1. *Let z be a fixed non-integer real number. Then, the Clausen function $C_z(x)$ is strictly monotonic for $x \in (0, \pi]$.*

Proof. Notice that $C_z'(x) = -S_{z-1}(x)$ for all x and assume first that $z > 1$. By [24, Eq. 25.12.11], we have that

$$S_{z-1}(x) = \frac{\sin(x)}{\Gamma(z-1)} \int_0^\infty t^{z-1} \frac{e^t}{\left(e^t - \cos(x)\right)^2 + \sin(x)^2} \, dt \, .$$

Since $\Gamma(z-1) > 0$ for $z > 1$ and $\sin(x) > 0$ in $[0, \pi]$, $C_z'(x) < 0$ in that range.

Likewise, when $z < 1$ we can use the representation formula

$$S_{z-1}(x) = \sin(\tfrac{\pi}{2} z) \int_0^\infty t^{1-z} \frac{\sinh\left(t(\pi - x)\right)}{\sinh(\pi t)} \, dt$$

that follows from the well-known relationship between zeta functions and polylogarithms, cf. [24, Eq. 25.11.25]. □

This shows that if $X = [\underline{x}, \overline{x}] \subset (0, \pi]$, then $C_z(X)$ is contained in the convex hull of $C_z(\underline{x})$ and $C_z(\overline{x})$. That is exactly how we implement it. We compute C_z at the endpoints using the polylogarithm function and we take their convex hull. In order to implement S_z (which is not monotonic in $(0, \pi]$) we use that $S_z'(x) = C_{z-1}(x)$ and $S_z(X) = S_z(x_0) + (X - x_0)C_{z-1}(X)$ by virtue of the mean value theorem, choosing x_0 as the midpoint of X.

It is also important to remark that given the delicate set of calculations that need to be performed, working with double precision is not enough and multi-precision is needed. In all our calculations we worked with 100 bits (as opposed to the usual 53).

Proof of Lemma 3.2.6. We will enclose a solution to (3.2.6) by applying a Newton method to the difference of the LHS and the RHS of the equation. We discuss the details of the algorithm below.

The first step of the algorithm is to isolate the roots. This is done by checking the signs of the endpoints and ensuring that the derivative of the function has a definite sign between the endpoints. On the contrary, if the signs of the function at the endpoints are the same and the function is monotone, there is no root in that interval and it is discarded. Finally, if none of these two conditions are met, the interval is split by the midpoint in two and the isolating function is called recursively with the two resulting subintervals. The second step is to refine the interval even more using a bisection method. Finally, a Newton zero-finding method is applied. The code can be found in the file `Lemma_p0_p1.c`. The total execution time was a few seconds. The initial intervals for p_0 and p_1 were $[0.5125, 0.75]$ and $[2.625, 2.875]$, and the final enclosures were $0.61120158988884395 \pm 7.01 \cdot 10^{-19}$ and $2.7624011603378232 \pm 2.00 \cdot 10^{-17}$, respectively.

□

Proof of Lemmas 3.2.9, 3.2.12. This concerns the proof of the inequalities 3.3.8 and 3.4.15, and all the Lemmas such as 3.2.9 which involve evaluations at a single point. We refer the reader to the file `Constant_checking.c`. □

Proof of Lemmas 3.2.10, 3.2.13 3.4.4 and 3.4.6. This describes the bounding of the quantities α_0, α_1, α_2, $\overline{\alpha}_2$, α_f and δ_0, δ_1, δ_2, which are all done the same way.

We start by splitting $I = [0, \pi]$ into two pieces, $I_1 = [0, \varepsilon]$ and $I_2 = [\varepsilon, \pi]$, with $\varepsilon = 10^{-2}$. The bounds of the different quantities over $x \in I_1$ were obtained using asymptotics for small x (see e.g. Lemmas 3.2.7 and 3.2.8). In order to deal with the case $x \in I_2$, we constructed a function called `compute_bound_Linfty_norm_C1` that takes as arguments a function `func`, its derivative `deriv`, a bound `bound`, an interval `min_width` and an interval `inp` and performs recursively the following branch and bound algorithm: we first compute an enclosure of `func` (which we call `F`). The enclosure is a C^1 one, given by

$$F(X) = F(x_0) + (X - x_0)F'(X),$$

taking x_0 as the midpoint of X. Given `F`, the function performs the following algorithm:

- If `F` $>$ `bound` it returns false
- If `F` $<$ `bound` it returns true
- If none of the two conditions are met:
  - If $width$(`inp`) $<$ `min_width`, split into two pieces and return true if both true, otherwise false
  - Else return false

It is clear that if the algorithm returns true, then `bound` is a guaranteed upper bound of $f(x), x \in I_2$. For all the above quantities, the total time of computation was a few minutes. The code can be found in the file `Lemma_bound_functions.c`.
□

Proof of Lemmas 3.3.1, 3.4.2 and 3.4.3. We now explain how the integrals are calculated. For simplicity, we will explain how to calculate T_0 but the same method applies to T_1 and T_2. First, we split the interval $I = [0, \pi]$ into $I_1 = [0, \varepsilon]$ and $I_2 = [\varepsilon, \pi]$ (we take $\varepsilon = 0.1$ in all three cases T_0, T_1, T_2). Calling $T_0(x)$ the function in (3.3.3) whose supremum on I gives $\|T_0\|$, it is clear that when $x \in I_1$ then $T_0(x)$ is bounded using the asymptotic expansion described in Lemma 3.3.3.

We here explain the calculation when $x \in I_2$. The first step is to split the integral

$$\begin{aligned}T_0(x) &= \frac{1}{2xu_0}\int_0^\pi \big|K(x-y)+K(x+y)-2K(y)\big|y\,dy\\ &= \frac{1}{2xu_0}\int_0^x \big|K(x-y)+K(x+y)-2K(y)\big|y\,dy\\ &+\frac{1}{2xu_0}\int_x^\pi \big|K(x-y)+K(x+y)-2K(y)\big|y\,dy\\ &\qquad\qquad\qquad\qquad =: T_0^1(x)+T_0^2(x).\end{aligned}$$

The expression of $T_0^2(x)$ can be calculated explicitly (see Equation (3.3.4)) so we will focus on the calculation of $T_0^1(x)$. Changing variables, we write

$$T_0^1(x) = \frac{x}{2u_0}\int_0^1 \big|K(x(1-w))+K(x(1+w))-2K(xw)\big|w\,dw$$

We should note, however, that the integrand is singular (although integrable) at $w=0$ and $w=1$. The next step is to remove those singularities and treat them separately. We thus split $T_0^1(x)$ as

$$\begin{aligned}T_0^1(x) &= \frac{x}{2u_0}\int_0^{\delta_0} \big|K(x(1-w))+K(x(1+w))-2K(xw)\big|w\,dw\\ &+\frac{x}{2u_0}\int_{\delta_0}^{1-\delta_1} \big|K(x(1-w))+K(x(1+w))-2K(xw)\big|w\,dw\\ &+\frac{x}{2u_0}\int_{1-\delta_1}^{1} \big|K(x(1-w))+K(x(1+w))-2K(xw)\big|w\,dw\\ &=: T_0^{1,1}(x)+T_0^{1,2}(x)+T_0^{1,3}(x)\end{aligned}$$

with $\delta_0=\delta_1=10^{-6}$ for T_0, T_1, and $\delta_0 = 0.0625, \delta_1 = 10^{-4}$ for T_2. The values of $T_0^{1,1}(x)$ and $T_0^{1,3}(x)$ are calculated using asymptotic expansions at $w=0$ and $w=1$ respectively. We remark that the integrand of $T_1^1(x)$ (the analog of $T_0^1(x)$ for the operator T_1), is not singular at $w=0$ so we do not have to consider another splitting of the singularity. We are left with the calculation of $T_0^{1,2}(x)$, which we pass to explain now for a fixed interval x.

In this case, the integration is done recursively. For each subdomain, we compute an enclosure of the integral. Since the integrand is not smooth because of the absolute value, we first compute a C^0 enclosure (i.e. evaluating the integrand at the full integration region). If the enclosure is sign-definite, the integrand is

C^2 inside it, so we can improve on the width of the enclosure by performing a midpoint quadrature, given by:

$$\int_a^b f(y)dy \in (b-a)f\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)^3 f''([a,b])$$

We now decide to accept or reject the result, based on its width in an absolute and a relative (to the length of the integration region) way. Specifically, it has to be smaller than `abs_tol` and `rel_tol` respectively. In the latter case, we split the region and recompute the integral on both subregions. The splitting is done as the midpoint. We keep track of the regions over which we need to integrate in a queue, implemented using a circular array. In order to avoid infinite loops — which could potentially happen since there is an uncertainty in the value of x —, the size of the queue is limited at all times to `QSIZE` elements. In our code, `QSIZE` $= 1024$. If the program is not able to calculate an enclosure of the integral with the desired tolerances, we split in x (by the midpoint) and recalculate each part until the tolerances are met.

The integration region I_2 is further split into three regions. This is because the source of the error comes from different places: for x small, most of the error will come from the evaluation of u and its derivatives. For x large, it will come from the integral. If x is close to π, we do not decide based on relative tolerances since the result is very small (even 0). The different subregions were $I_{2,1} = [0.1, 1]$, $I_{2,2} = [1, 3]$ and $I_{2,3} = [3, \pi]$. The total runtime (for the three regions combined) was about 2 hours for T_0, about 8 hours for T_1 and about 50 hours for T_2.

□

References

[1] S. Alexakis, A. Shao, Global uniqueness theorems for linear and nonlinear waves, J. Func. Anal. 269 (2015) 3458–3499.

[2] C.J. Amick, L.E. Fraenkel, J.F. Toland, On the Stokes conjecture for the wave of extreme form, Acta Math. 148 (1982) 193–214.

[3] A. Bachelot, The Klein–Gordon equation in the anti-de Sitter cosmology, J. Math. Pures Appl. 96 (2011) 527-554.

[4] C.A. Ballón, N. Braga, Anti-de Sitter boundary in Poincaré coordinates, Gen. Rel. Grav. 39 (2007) 1367-1379.

[5] P. Baras, J. Goldstein, The heat equation with a singular potential, Trans. Amer. Math. Soc. 284 (1984) 121–139.

[6] J. Barceló, A. Ruiz, L. Vega, Some dispersive estimates for Schrödinger equations with repulsive potentials, J. Funct. Anal. 236 (2006) 1–24.

[7] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.

[8] L. Baudouin, M. de Buhan, S. Ervedoza, Global Carleman estimates for waves and applications, Comm. Partial Differential Equations 38 (2013) 823–859.

[9] J. Bebernes, D. Eberly, Mathematical Problems from Combustion Theory, Math. Sci. Vol. 83 Springer-Verlag (1989).

[10] H. Berestyckia, M. J. Esteban, Existence and bifurcation of solutions for an elliptic degenerate problem, J. Differential Equations 134 (1997) 1–25.

[11] M. Berz, K. Makino, New methods for high-dimensional verified quadrature, Reliable Computing, 5 (1999) 13–22.

[12] U. Biccari, E. Zuazua, Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function, J. Differential Equations 261 (2016) 2809–2853.

[13] N. Burq, P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, C. R. Acad. Sci. Paris Sér. I Math. 325 (1997) 749–752.

[14] N. Burq, F. Planchon, J. Stalker, S. Tahvildar-Zadeh, Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, Indiana Univ. Math. J. 3 (2004) 1665–1680.

[15] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.

[16] R. Castelli, M. Gameiro, J.-P. Lessard, Rigorous numerics for ill-posed PDEs: periodic orbits in the Boussinesq equation, Arch. Ration. Mech. Anal. 228 (2018) 129–157.

[17] A. Castro, D. Córdoba, J. Gómez-Serrano, Global smooth solutions for the inviscid SQG equation, Mem. Am Math. Soc. (2017), To appear.

[18] S.-Y. A. Chang, M.d.M. González, Fractional Laplacian in conformal geometry, Adv. Math. 226 (2011) 1410–1432.

[19] S.-Y. A. Chang, M.d.M. González, On a class of non-local operators in conformal geometry, preprint, 2016.

[20] J. Cigler, Operatormethoden für q-Identitäten, Monatsh. Math. 88 (1979) 87–105.

[21] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998) 229–243.

[22] D. Córdoba, J. Gómez-Serrano, A. Zlatoš, A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable, Anal. PDE 10 (2017) 367–378.

[23] R. de la Llave, Y. Sire, An a posteriori KAM theorem for whiskered tori in hamiltonian partial differential equations with applications to some ill-posed equations, Arch. Rat. Mech. Anal. (2018), To appear

[24] NIST *Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.20 of 2018-09-15. F.W. J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, and B.V. Saunders, eds.

[25] D. Dos Santos Ferreira, Sharp L^p Carleman estimates and unique continuation, Duke Math. J. 129 (2005) 503–550.

[26] T. Duyckaerts, X. Zhang, E. Zuazua, On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008) 1–41.

[27] M. Ehrnström, M.D. Groves, E. Wahlén, On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type, Nonlinearity 25 (2012) 1–34.

[28] M. Ehrnström, M.A. Johnson, K.M. Claassen, Existence of a highest wave in a fully dispersive two-way shallow water model, Arch. Rat. Mech. Anal., in press (arXiv:1610.02603).

[29] M. Ehrnström, H. Kalisch, Traveling waves for the Whitham equation, Differ. Integral Equ. 22 (2009) 1193–1210.

[30] M. Ehrnström, H. Kalisch, Global bifurcation for the Whitham equation, Math. Model. Nat. Phenom. 7 (2013).

[31] M. Ehrnström, E. Wahlén, On Whitham's conjecture of a highest cusped wave for a nonlocal dispersive equation, Ann. I.H. Poincaré–A.N., in press (arXiv:1602.05384).

[32] A. Enciso, M.d.M. González, B. Vergara, Fractional powers of the wave operator via Dirichlet-to-Neumann maps in anti-de Sitter spaces, J. Funct. Anal. 273 (2017) 2144–2166.

[33] A. Enciso, N. Kamran, A singular initial-boundary value problem for nonlinear wave equations and holography in asymptotically anti-de Sitter spaces, J. Math. Pures Appl. 103 (2015) 1053–1091.

[34] A. Enciso, N. Kamran, Lorentzian Einstein metrics with prescribed conformal infinity, J. Differential Geom., in press (arXiv:1412.4376).

[35] C. Fefferman and C. R. Graham, *The ambient metric*, Princeton University Press, Princeton, 2012.

[36] I.M. Gelfand, G.E. Shilov, *Generalized Functions I*, Academic Press, New York, 1964.

[37] J. Gómez-Serrano, Computer-assisted proofs in PDE: a survey, (arXiv:1810.00745).

[38] C. R. Graham, M. Zworski, Scattering matrix in conformal geometry, Invent. Math., 152 (2003) 89–118.

[39] M. Gueye, Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, SIAM J. Control Optim. 52 (2014) 2037–2054.

[40] G. Holzegel, A. Shao, Unique Continuation from Infinity in Asymptotically Anti-de Sitter Spacetimes, Comm. Math. Phys. 347 (2016) 723–775.

[41] G. Holzegel, A. Shao, Unique continuation from infinity in asymptotically Anti-de Sitter spacetimes II: Non-Static Boundaries, Comm. PDE 42 (2017) 1871–1922.

[42] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV*, Springer-Verlag, Berlin, 1985.

[43] V.M. Hur, Wave breaking in the Whitham equation, Adv. Math. 317 (2017) 410–437.

[44] A.E. Ingham, Some trigonometric inequalities with applications to the theory of series, Math. Z. 41 (1936), 367–379.

[45] A.D. Ionescu, S. Klainerman, On the uniqueness of smooth, stationary black holes in vacuum, Invent. Math. 175 (2009) 35–102.

[46] F. Johansson, Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, IEEE Trans. Comput. 66 (2017) 1281–1292.

[47] M. Kemppainen, P. Sjögren, J.L. Torrea, Wave extension problem for the fractional Laplacian. Discrete Contin. Dyn. Syst. 35 (2015) 4905–4929.

[48] J.A.C. Kolk, V.S. Varadarajan, Riesz distributions, Math. Scand. 68 (1991) 273–291.

[49] W. Krämer, S. Wedner, Two adaptive Gauss-Legendre type algorithms for the verified computation of definite integrals, Reliable Computing 2 (1996) 241–253.

[50] N.S. Landkoff, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin, 1972.

[51] B. Lang, Derivative-based subdivision in multi-dimensional verified gaussian quadrature. In G. Alefeld, J. Rohn, S. Rump, T. Yamamoto, editors, Symbolic Algebraic Methods and Verification Methods, (2001) 145–152, Springer Vienna.

[52] I. Lasiecka, R. Triggiani, X. Zhang, Nonconservative wave equations with unobserved Neumann BC: Global uniqueness and observability in one shot, Contemp. Math. 268 (2000) 227–326.

[53] C. LeBrun, H-space with a cosmological constant, Proc. Roy. Soc. London Ser. A 380 (1982) 171–185.

[54] J.L. Lions, *Controlabilité exacte perturbations et stabilisation de systèmes distribués*, Masson, Paris, 1988.

[55] F. Macia, E. Zuazua, On the lack of observability for wave equations: a Gaussian beam approach, Asympt. Anal. 32 (2002) 1–26.

[56] J. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231–252.

[57] R.R. Mazzeo, R.B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (1987) 260–310.

[58] C. S. Morawetz, Time decay for the nonlinear Klein-Gordon equations, Proc. Roy. Soc. Ser. A 306 (1968) 291–296.

[59] R. Moore, F. Bierbaum, Methods and applications of interval analysis, SIAM J. Appl. Math. 2 (1979).

[60] T. Ozawa, K. M. Rogers, Sharp Morawetz estimates, J. Anal. Math. 121 (2013) 163–175.

[61] P.I. Plotnikov, J.F. Toland, Convexity of Stokes waves of extreme form, Arch. Ration. Mech. Anal. 171 (2004) 349–416.

[62] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math. 81 (1951) 1–223.

[63] S.G. Samko, *Hypersingular integrals and their applications*, CRC Press, Boca Ratón, 2001.

[64] P.R. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators. Comm. Partial Differential Equations 35 (2010) 2092–2122.

[65] P. R. Stinga, J. L. Torrea. Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, (arXiv:1511.01945).

[66] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, American Mathematical Society, 2006.

[67] D. Tataru, Unique continuation for solutions to PDEs: between Hörmander's theorem and Holmgren's theorem, Comm. Partial Differential Equations. 20 (1995) 855–884.

[68] D. Tataru, The X^s spaces and unique continuation for solutions to the semi-linear wave equation, Comm. Partial Differential Equations. 21 (1996) 841–887.

[69] J.F. Toland, On the existence of a wave of greatest height and Stokes's conjecture, Proc. Roy. Soc. London Ser. A 363 (1978) 469–485.

[70] W. Tucker, Validated numerics, A short introduction to rigorous computations, Princeton University Press, Princeton (2011).

[71] J. Vancostenoble, E. Zuazua, Hardy Inequalities, Observability, and Control for the Wave and Schrdinger Equations with Singular Potentials, SIAM J. Math. Anal. 41 (2009) 1508–1532.

[72] A. Vasy, The wave equation on asymptotically anti de Sitter spaces. Anal. PDE 5 (2012) 81–144.

[73] J.L. Vázquez, E. Zuazua, The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (2000) 103–153.

[74] R.M. Wald, *General relativity*, University of Chicago Press, Chicago, 1984.

[75] C. Warnick, The massive wave equation in asymptotically AdS spacetimes, Comm. Math. Phys. 321 (2013) 85–111.

[76] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1966.

[77] G.B. Whitham, Variational methods and applications to water waves, Proc. R. Soc. Lond. Ser. A 299 (1967) 6–25.

[78] G.B. Whitham, Linear and nonlinear waves, Wiley, New York (1974).

[79] E. Witten, Anti de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291.

[80] X. Zhang, Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities, SIAM J. Control Optim. 39 (2001), 812–834.