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A posteriori error estimations for mixed finite element approximations to the Navier-Stokes equations based on Newton-type linearization

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Abstract

In this paper we derive a posteriori error estimates for inf-sup stable mixed finite element approximations to the evolutionary Navier-Stokes equations. We reduce the problem of getting a posteriori error estimations of a non-linear evolutionary problem to that of getting a posteriori estimations of a linear steady problem. The main idea is based on the fact that the solutions of some Newton-type linearized Navier-Stokes equations around the plain Galerkin approximations approach the solution of the original Navier-Stokes equations with a bigger rate of convergence than the plain Galerkin method. As a consequence, the difference between the Galerkin approximations and the solution of the linearized problem is an estimator of the Galerkin error. Moreover, since the Galerkin approximation of the evolutionary Navier-Stokes equations is also the Galerkin approximation of the linearized equations, any a posteriori estimator of the error in the Newton-type linearized Navier-Stokes equations is also an a posteriori error estimator of the full Navier-Stokes equations.

Keywords Incompressible Navier–Stokes equations; a posteriori error estimations; static two-grid methods

1 Introduction

We consider the incompressible Navier–Stokes equations

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \\ \operatorname{div}(u) &= 0, \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary, subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1), u is the velocity field, p the pressure, and f a given force field.

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Let u_h and p_h be the semidiscrete (in space) mixed finite element (MFE) approximations to the velocity u and pressure p , respectively, solution of (1) corresponding to a given initial condition

$$u(\cdot, 0) = u_0. \quad (2)$$

We study the a posteriori error estimation of these approximations in the L^2 and H^1 norm for the velocity and in the L^2/\mathbb{R} norm for the pressure. To do this, for a given time $t^* > 0$, we consider the solution (\tilde{u}, \tilde{p}) of the following Newton-type linearized Navier-Stokes equations

$$\begin{aligned} -\nu \Delta \tilde{u} + (u_h \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) u_h + \nabla \tilde{p} + \lambda \tilde{u} &= f - \frac{d}{dt} u_h + (u_h \cdot \nabla) u_h + \lambda u_h \\ \operatorname{div}(\tilde{u}) &= 0, \end{aligned} \quad (3)$$

where u_h is evaluated at the time t^* and λ is a positive parameter that is added to have coercivity in the bilinear form associated to problem (3) and whose value will be specified later.

We prove that the solution (\tilde{u}, \tilde{p}) of (3) has a rate of convergence one unit bigger (up to a logarithmic term) than the rate of convergence of the standard Galerkin approximation (u_h, p_h) . As a consequence, the quantities $\tilde{u} - u_h$ and $\tilde{p} - p_h$, are asymptotically exact indicators of the errors $u - u_h$ and $p - p_h$ in the Navier-Stokes problem (1)–(2). In this paper we will refer to (\tilde{u}, \tilde{p}) as infinite-dimensional Newton-type approximations (IDN-approximations). Of course, (\tilde{u}, \tilde{p}) are not computable in practice and they are only considered for the analysis of the a posteriori error estimators.

Now, the key point in the a posteriori error estimation is the observation that (u_h, p_h) is also the Galerkin approximation to problem (3). Then, instead of computing the quantities $\tilde{u} - u_h$ and $\tilde{p} - p_h$ (which cannot be done in practice) we can apply any a posteriori error estimator for the Newton-type linearized equations (3) to estimate the error in the non-linear evolutionary Navier-Stokes equations. Any of the existing methods in the literature to estimate the error in linearized Navier-Stokes equations or even in steady Navier-Stokes equations can be adapted to estimate the error in the non-linear evolutionary problem, see for example [17], [4], [2], [34], [33], [18], [3], [9].

The idea of the a posteriori error estimation we study in this paper is based on an analogous procedure developed in [21]. In [21] the a posteriori error estimation of the Navier-Stokes equations is reduced to the a posteriori error estimation of a Stokes problem. In both cases, [21] and the present paper, the idea is inherited from the so called postprocessed or two-grid mixed finite element method. The postprocessed or two-grid method follows two steps. In the first one, the standard mixed finite element method over a coarse mesh of size h is computed over a time interval $[0, T]$. In the second step, the Galerkin approximation at time T is postprocessed. A postprocessed or two-grid approximation is then computed at the fixed final time T over a fine mesh h' by solving a steady Stokes, Oseen or Newton-type problem with data based on the already computed Galerkin approximation, see [7], [5], [20], [22], [16]. The postprocessed or two-grid approximation has a rate of convergence in terms of h that is one unit bigger (up to a logarithmic term) than the rate of convergence of the plain Galerkin approximation based on the coarse mesh of size h and a rate of convergence in terms of h' that is the same as the Galerkin approximation

computed over $[0, T]$ based on the fine mesh h' . As a consequence, the postprocessed procedure allows to get an enhanced approximation at a computational cost that is almost negligible compared with the computational cost required to integrate in time on the fine mesh over the full interval $[0, T]$ to get the Galerkin approximation.

In [16] we show that the Newton-type two-grid algorithm while getting the same rate of convergence than the previous two-grid Stokes or Oseen based methods is able to produce better approximations specially for increasing values of the Reynolds numbers, see the numerical experiments of [16]. As a consequence, an a posteriori error estimation based on this procedure will be more robust and then more interesting in practice than the previous procedure that was developed in [21].

Although any a posteriori error estimator of the Newton-type linearized Navier-Stokes equations can be applied to estimate the error in the Navier-Stokes equations in this paper we consider the simplest a posteriori error estimator that is based on computing the difference between the Newton-type two grid approximation and the plain Galerkin approximation. We prove that computing this difference we can get a reliable, efficient an asymptotically exact a posteriori error estimator.

Finally, we want to mention that, as a future research, we could try to extend the techniques of the present paper to more complicated problems such as the Navier-Stokes equations with nonlinear viscosity, using the error estimators for the stationary case that are developed in [23], and the time-dependent Boussinesq equations using the estimators for the stationary Boussinesq model developed in [12].

The outline of the paper is as follows. In Section 2 we introduce some preliminaries and notations. Some auxiliary results are written in Section 3. In Section 4 we describe the a posteriori error estimator and prove it is robust and asymptotically exact. Finally, some numerical experiments are shown in the last section.

2 Preliminaries and notations

In the sequel we will assume that Ω is a bounded domain in \mathbb{R}^d , $d = 2, 3$, of class \mathcal{C}^m , for $m \geq 2$. When dealing with linear elements Ω may also be a convex polygonal or polyhedral domain. We consider the Hilbert spaces

$$\begin{aligned} H &= \{u \in L^2(\Omega)^d \mid \operatorname{div}(u) = 0, u \cdot n|_{\partial\Omega} = 0\}, \\ V &= \{u \in H_0^1(\Omega)^d \mid \operatorname{div}(u) = 0\}, \end{aligned}$$

endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively. For $l \geq 0$ integer and $1 \leq q \leq \infty$, we consider the standard spaces, $W^{l,q}(\Omega)^d$, of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d = W^{l,2}(\Omega)^d$. We will denote by $\|\cdot\|_l$ the norm in $H^l(\Omega)^d$, and $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}} = \inf\{\|p + c\|_l \mid c \in \mathbb{R}\}$.

We recall the following Sobolev's embedding [1]: For $q \in [1, \infty)$, there exists a constant $C = C(\Omega, q)$ such that

$$\|v\|_{L^{q'}} \leq C\|v\|_{W^{s,q}}, \quad \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad v \in W^{s,q}(\Omega)^d. \quad (4)$$

For $q' = \infty$, (4) holds with $\frac{1}{q} < \frac{s}{d}$. In particular, we will do extensive use of the following cases, which hold simultaneously for both two and three spatial dimensions.

$$\|v\|_{L^{2d}} \leq C\|v\|_s, \quad s \geq 1, \quad \|v\|_{L^{2d/(d-1)}} \leq C\|v\|_s, \quad s \geq 1/2. \quad (5)$$

The following inf-sup condition is satisfied (see [27, Theorem 5.1]): there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(\Omega)/\mathbb{R}} \sup_{v \in H_0^1(\Omega)^d} \frac{(q, \nabla \cdot v)}{\|v\|_1 \|q\|_{L^2/\mathbb{R}}} \geq \beta, \quad (6)$$

where, here and in the sequel, (\cdot, \cdot) denotes the standard inner product in $L^2(\Omega)$ or in $L^2(\Omega)^d$.

For $u \in V$, $v, w \in H_0^1(\Omega)^d$, it also holds

$$((u \cdot \nabla)v, w) = -((u \cdot \nabla)w, v) = -(\nabla \bar{w} \cdot v, u) \quad (7)$$

where $(\nabla \bar{w})_{ij} = \partial_i w_j$.

Let $\Pi : L^2(\Omega)^d \rightarrow H$ be the $L^2(\Omega)^d$ projector onto H . We denote by A the Stokes operator on Ω :

$$A : \mathcal{D}(A) \subset H \rightarrow H, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap V.$$

Applying Leray's projector Π to (1), the equations can be written in the form

$$u_t + \nu Au + B(u, u) = \Pi f \quad \text{in } \Omega,$$

where $B(u, v) = \Pi(u \cdot \nabla)v$ for u, v in $H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot, \cdot, \cdot)$ defined by

$$b(u, v, w) = (F(u, v), w) \quad \forall u, v, w \in H_0^1(\Omega)^d,$$

where

$$F(u, v) = (u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v \quad \forall u, v \in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys skew-symmetry:

$$b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in H_0^1(\Omega)^d. \quad (8)$$

Let us observe that $B(u, v) = \Pi F(u, v)$ for $u \in V$, $v \in H_0^1(\Omega)^d$.

We shall assume, as in [28], that $u_0 \in V \cap H^2(\Omega)^d$, that there exists a constant \tilde{M}_1 such that $\|f\|_0 + \|f_t\|_0 \leq \tilde{M}_1$, for $t \in [0, T]$, and that the solution u of (1)-(2) exists on an interval $[0, T]$ and satisfies

$$\|u(t)\|_1 \leq M_1, \quad 0 \leq t \leq T, \quad (9)$$

for some constant M_1 . Then, following Theorem 2.3 in [28] we get

$$\|u(t)\|_2 + \|u_t(t)\|_0 + \|p(t)\|_{H^1/\mathbb{R}} \leq M_2, \quad 0 \leq t \leq T. \quad (10)$$

Moreover, assuming that there exists a constant \tilde{M}_2 such that

$$\|f\|_1 + \|f_t\|_1 + \|f_{tt}\|_1 \leq \tilde{M}_2, \quad 0 \leq t \leq T, \quad (11)$$

and that for some $k \geq 2$

$$\sup_{0 \leq t \leq T} \|\partial_t^{[k/2]} f\|_{k-1-2[k/2]} + \sum_{j=0}^{\lfloor (k-2)/2 \rfloor} \sup_{0 \leq t \leq T} \|\partial_t^j f\|_{k-2j-2} < +\infty,$$

according to Theorems 2.4 and 2.5 in [28], there exist positive constants M_k and K_k such that the following bounds hold:

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \leq M_k \tau(t)^{1-k/2}, \quad (12)$$

$$\int_0^t \sigma_{k-3}(s) (\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2) ds \leq K_k^2, \quad (13)$$

where $\tau(t) = \min(t, 1)$ and $\sigma_n(s) = e^{-\alpha(t-s)} \tau^n(s)$ for some $\alpha > 0$. Observe that for $t \leq T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n .

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i \in I_h}$, $h > 0$, be a family of partitions of suitable domains Ω_h , where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$, and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h .

For $r \geq 2$, we consider the finite-element spaces

$$\begin{aligned} S_{h,r} &= \left\{ \chi_h \in \mathcal{C}(\overline{\Omega}_h) \mid \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0) \right\} \subset H^1(\Omega_h), \\ S_{h,r}^0 &= S_{h,r} \cap H_0^1(\Omega_h), \end{aligned}$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0 .

It will be assumed that the family of meshes is quasi-uniform and that the following inverse inequality holds, see [14, Theorem 3.2.6], for each $v_h \in (S_{h,r}^0)^d$:

$$\|v_h\|_{W^{l,q}(\Omega_h)^d} \leq Ch^{l-m-d(\frac{1}{q'} - \frac{1}{q})} \|v_h\|_{W^{l,q'}(\Omega_h)^d}, \quad (14)$$

where $0 \leq l \leq m \leq 1$, and $1 \leq q' \leq q \leq \infty$.

We shall denote by $(X_{h,r}, Q_{h,r-1})$ the mixed finite-elements spaces that we consider, which are, when $r \geq 3$, the so-called Hood–Taylor element [10], [32], given by

$$X_{h,r} = (S_{h,r}^0)^d, \quad Q_{h,r-1} = S_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \quad r \geq 3,$$

and, when $r = 2$, the so-called mini-element [11], for which $Q_{h,1} = S_{h,2} \cap L^2(\Omega_h)/\mathbb{R}$, and $X_{h,2} = (S_{h,2}^0)^d \oplus \mathbb{B}_h$. Here, \mathbb{B}_h is spanned by the bubble functions b_τ , $\tau \in \mathcal{T}_h$, defined by $b_\tau(x) = (d+1)^{d+1} \lambda_1(x) \cdots \lambda_{d+1}(x)$, if $x \in \tau$ and 0 elsewhere, where $\lambda_1(x), \dots, \lambda_{d+1}(x)$ denote the barycentric coordinates of x . For these elements a uniform inf-sup condition is satisfied, that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \geq \beta, \quad (15)$$

see [10], [11]. We remark that our analysis can also be applied to other pairs of LBB-stable mixed finite elements (see [20, Remark 2.1]).

The approximate velocity belongs to the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \{\chi_h \in H_0^1(\Omega_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1}\},$$

which is not a subspace of V . We shall frequently write V_h instead of $V_{h,r}$ whenever the value of r plays no particular role.

We will denote by $\Pi_h : L^2(\Omega)^d \rightarrow V_{h,r}$ the discrete Leray's projection defined by

$$(\Pi_h u, \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will denote by $A_h : V_h \rightarrow V_h$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla \phi_h) = (A_h v_h, \phi_h) = (A_h^{1/2} v_h, A_h^{1/2} \phi_h) \quad \forall v_h, \phi_h \in V_h.$$

Let $(u, p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$ be the solution of a Stokes problem with right-hand side $g \in L^2$, we will denote by $s_h = S_h(u) \in V_h$ the so-called Stokes projection (see [29]) defined as the velocity component of the solution of the following Stokes problem: find $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ such that

$$(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \quad \forall \phi_h \in X_{h,r}, \quad (16)$$

$$(\nabla \cdot s_h, \psi_h) = 0 \quad \forall \psi_h \in Q_{h,r-1}. \quad (17)$$

The following bound holds for $2 \leq l \leq r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \leq Ch^l (\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}). \quad (18)$$

The proof of (18) for $\Omega = \Omega_h$ can be found in [29, Lemma 5.3]. For the general case, Ω_h must be such that the value of $\delta(h) = \max_{x \in \partial\Omega_h} \text{dist}(x, \partial\Omega)$ satisfies $\delta(h) = O(h^{2(r-1)})$. This can be achieved if, for example, $\partial\Omega$ is piecewise of class $\mathcal{C}^{2(r-1)}$, and superparametric approximation at the boundary is used [6]. Under the same conditions, the bound for the pressure is [27, Theorem 1.1]

$$\|p - q_h\|_{L^2/\mathbb{R}} \leq C_\beta h^{l-1} (\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}), \quad (19)$$

where the constant C_β depends on the constant β in the inf-sup condition (15). Following [13], one can also obtain the following bound for s_h

$$\|\nabla(u - s_h)\|_\infty \leq C\|\nabla u\|_\infty. \quad (20)$$

We consider the semidiscrete finite-element approximation (u_h, p_h) to (u, p) , solution of (1)–(2). That is, given $u_h(0) = \Pi_h u_0$, we compute $u_h(t) \in X_{h,r}$ and $p_h(t) \in Q_{h,r-1}$, $t \in (0, T]$, satisfying

$$(\dot{u}_h, \phi_h) + \nu(\nabla u_h, \nabla \phi_h) + b(u_h, u_h, \phi_h) - (p_h, \nabla \cdot \phi_h) = (f, \phi_h) \quad \forall \phi_h \in X_{h,r}, \quad (21)$$

$$(\nabla \cdot u_h, \psi_h) = 0 \quad \forall \psi_h \in Q_{h,r-1}. \quad (22)$$

For $2 \leq r \leq 5$, provided that (18)–(19) hold for $l \leq r$, and (12)–(13) hold for $k = r$, we have

$$\|u(t) - u_h(t)\|_0 + h\|u(t) - u_h(t)\|_1 \leq C \frac{h^r}{t^{(r-2)/2}}, \quad 0 \leq t \leq T, \quad (23)$$

(see, e.g., [20, 28, 29]), and also,

$$\|p(t) - p_h(t)\|_{L^2/\mathbb{R}} \leq C \frac{h^{r-1}}{t^{(r'-2)/2}}, \quad 0 \leq t \leq T, \quad (24)$$

where $r' = r$ if $r \leq 4$ and $r' = r + 1$ if $r = 5$.

3 Some auxiliary results

In [16], it has been proved that the coercivity of the Newton postprocess can be ensured taking a spatial resolution h small enough. In this section, we develop an alternative analysis of the coercivity based on the parameter λ . It is proved that there exists a certain $\lambda_0 > 0$ such that, for all $\lambda \geq \lambda_0$, the Newton postprocess is coercive for any value of h .

Let us denote by $B(u; v, w)$ the Newton-type bilinear form:

$$B(u; v, w) = \nu(\nabla v, \nabla w) + ((u \cdot \nabla)v, w) + ((v \cdot \nabla)u, w) + \lambda(v, w), \quad (25)$$

where $u, v, w \in H_0^1$ and λ is a positive parameter. The bilinear form $B(u; v, w)$ is continuous in H_0^1 . Using (5) we get

$$\begin{aligned} |B(u; v, w)| &\leq \nu \|v\|_1 \|w\|_1 + \|u\|_{L^{2d/(d-1)}} \|\nabla v\|_0 \|w\|_{L^{2d}} \\ &\quad + \|v\|_{L^{2d/(d-1)}} \|\nabla u\|_0 \|w\|_{L^{2d}} + \lambda \|v\|_1 \|w\|_1 \\ &\leq (\nu + 2C\|u\|_1 + \lambda) \|v\|_1 \|w\|_1. \end{aligned} \quad (26)$$

Coercivity of (25) can be derived from

$$|((v \cdot \nabla)u, v)| \leq \|\nabla u\|_\infty \|v\|_0^2,$$

and

$$|((u \cdot \nabla)v, v)| = \left| -\frac{1}{2}((\nabla \cdot u)v, v) \right| \leq \frac{1}{2} \|\nabla \cdot u\|_\infty \|v\|_0^2 \leq \frac{d}{2} \|\nabla u\|_\infty \|v\|_0^2.$$

Hence, if we assume not only that $u \in H_0^1$ but also that $\|\nabla u\|_\infty$ is finite and that

$$\lambda \geq \lambda_0 = \left(1 + \frac{d}{2}\right) \|\nabla u\|_\infty, \quad (27)$$

we get that $B(u; v, w)$ is coercive:

$$B(u; v, v) \geq \nu \|\nabla v\|_0^2 + \left(\lambda - \left(1 + \frac{d}{2}\right) \|\nabla u\|_\infty\right) \|v\|_0^2 \geq \nu \|\nabla v\|_0^2. \quad (28)$$

Let (u, p) be the solution of the Navier-Stokes equations (1). We consider the following Newton-type linearized equations with $\varphi \in L^2(\Omega)^d$:

$$\left. \begin{aligned} -\nu \Delta v + (u \cdot \nabla)v + (v \cdot \nabla)u + \lambda v + \nabla j &= \varphi \\ \operatorname{div}(v) &= 0 \end{aligned} \right\} \quad \begin{aligned} &\text{in } \Omega \\ &\text{on } \partial\Omega, \end{aligned} \quad (29)$$

Let us observe that the norm $\|\nabla u\|_\infty$ is bounded. By (4) and (12) we get

$$\|\nabla u(t)\|_\infty \leq C \|\nabla u(t)\|_{3/2+\alpha} \leq C \|u(t)\|_{5/2+\alpha} \leq M_3 t^{-1/2} < \infty, \quad t > 0. \quad (30)$$

Since $\nabla u \in L^\infty$, the bilinear form $B(u; v, w)$ is continuous and coercive in H_0^1 for all λ satisfying (27). This fact and the inf-sup condition (6) allow us to apply the Babuška-Brezzi theory from which we deduce that there exists a unique solution (v, j) of the problem (29). A regularity condition can be obtained by standard compactness arguments (see e.g. [24, Lemma 9.17]:

$$\|v\|_2 + \|j\|_{H^1(\Omega)/\mathbb{R}} \leq C \|\varphi\|_0. \quad (31)$$

Taking into account (7), the dual problem of (29) is

$$\begin{aligned} -\nu\Delta v - (u \cdot \nabla)v - \nabla \bar{v} \cdot u + \lambda v + \nabla j &= \varphi, \\ \operatorname{div}(v) &= 0, \\ v &= 0, \end{aligned} \quad \left. \vphantom{\begin{aligned} -\nu\Delta v - (u \cdot \nabla)v - \nabla \bar{v} \cdot u + \lambda v + \nabla j \\ \operatorname{div}(v) \\ v \end{aligned}} \right\} \begin{array}{l} \text{in } \Omega \\ \\ \text{on } \partial\Omega \end{array} \quad (32)$$

Let $D(u; v, w)$ be the bilinear form associated to the problem (32), with u the velocity in the solution of the Navier-Stokes equations (1), that in view of (30) satisfies $\nabla u \in L^\infty$,

$$D(u; v, w) = \nu(\nabla v, \nabla w) - ((u \cdot \nabla)v, w) - (\nabla \bar{v} \cdot u, w) + \lambda(v, w). \quad (33)$$

It is satisfied that $D(u; v, w) = B(u; w, v)$. Hence, $D(u; v, w)$ is also continuous and coercive so existence and uniqueness can be obtained arguing exactly as above. The regularity condition (31) also holds for the dual problem.

Let (u_h, p_h) be the MFE approximations to the solution (u, p) of the Navier-Stokes equations (1). The bilinear form $B(u_h; v, w)$ associated to the problem (3) is

$$B(u_h; v, w) = \nu(\nabla v, \nabla w) + ((u_h \cdot \nabla)v, w) + ((v \cdot \nabla)u_h, w) + \lambda(v, w), \quad (34)$$

with $v, w \in H_0^1$.

We will show that $\|\nabla u_h\|_\infty$ is bounded. Let s_h the Stokes projection (16). Applying (14), (20) and (30), we get

$$\begin{aligned} \|\nabla u_h\|_\infty &\leq \|\nabla(u_h - s_h)\|_\infty + \|\nabla s_h\|_\infty \\ &\leq Ch^{-d/2}\|\nabla(u_h - s_h)\|_0 + C\|\nabla u\|_\infty \\ &\leq Ch^{-d/2}\|\nabla(u_h - s_h)\|_0 + CM_3 t^{-1/2}. \end{aligned}$$

Taking into account the super-convergence between the Stokes projection and the Galerkin approximation to the velocity, proved in [20, Remark 4.2]:

$$\|\nabla(u_h - s_h)\|_0 \leq Ch^2 |\log(h)|^2$$

we get

$$Ch^{-d/2}\|\nabla(u_h - s_h)\|_0 \leq Ch^{-d/2}h^2 |\log(h)|^2 \leq C,$$

with C depending on M_2 in (10). Then, for all λ satisfying

$$\lambda \geq \lambda_0 = \left(1 + \frac{d}{2}\right) \|\nabla u_h\|_\infty, \quad (35)$$

the bilinear form (34) is continuous and coercive. Due to the discrete inf-sup condition (15), problem (3) has a unique solution.

The following two lemmas establish some bounds for the temporal derivative of the Galerkin error and for the dual norm or the Galerkin error. Their proofs can be found in [21, Lemma 4] for the case $r = 2$, [20, Lemma 5.1] for $r = 3, 4$ and in [20, p. 226], respectively.

Lemma 1 *Let (u, p) be the solution of (1) and let u_h be the mixed finite-element approximation to u defined in (21)-(22). Let A be the Stokes operator. Then, there exists a positive constant C such that*

$$\|u_t(t) - \dot{u}_h(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} h^r |\log(h)|^{r'}, \quad t \in (0, T], \quad r = 2, 3, 4, \quad (36)$$

$$\|A^{-1}\Pi(u_t(t) - \dot{u}_h(t))\|_0 \leq \frac{C}{t^{(r-1)/2}} h^{r+1} |\log(h)|, \quad t \in (0, T], \quad r = 3, 4, \quad (37)$$

where $r' = 2$ when $r = 2$ and $r' = 1$ otherwise.

Lemma 2 *Let (u, p) be the solution of (1) and let u_h be the mixed finite-element approximation to u defined in (21)–(22). Then, there exists a positive constant C such that*

$$\|u(t) - u_h(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} h^{r+1} |\log(h)|, \quad t \in (0, T], \quad r = 3, 4. \quad (38)$$

4 A posteriori error estimator

Let us consider the MFE approximation (u_h, p_h) to $(u(t^*), p(t^*))$ at any time $t^* \in (0, T]$, obtained by solving (21)–(22). We consider the IDN-approximation $(\tilde{u}(t^*), \tilde{p}(t^*))$ in $(V, L^2(\Omega)/\mathbb{R})$ which is the solution of the following Newton-type linearized problem written in weak form

$$\begin{aligned} \nu(\nabla \tilde{u}(t^*), \nabla \phi) + ((u_h(t^*) \cdot \nabla) \tilde{u}(t^*), \phi) + ((\tilde{u}(t^*) \cdot \nabla) u_h(t^*), \phi) + \lambda(\tilde{u}(t^*), \phi) \\ - (\tilde{p}(t^*), \nabla \cdot \phi) = (f(t^*) - \dot{u}_h(t^*), \phi) + ((u_h(t^*) \cdot \nabla) u_h(t^*), \phi) + \lambda(u_h(t^*), \phi) \quad (39) \\ (\nabla \cdot \tilde{u}(t^*), \psi) = 0. \end{aligned}$$

for all $\phi \in H_0^1(\Omega)^d$ and $\psi \in L^2(\Omega)/\mathbb{R}$. Let us observe that in view of (21)–(22) taking in (39) $\phi = \phi_h \in X_{h,r}$ and $\psi = \psi_h \in Q_{h,r-1}$ (39) holds changing $(\tilde{u}(t^*), \tilde{p}(t^*))$ by $(u_h(t^*), p_h(t^*))$. Then, the MFE approximation $(u_h(t^*), p_h(t^*))$ to $(u(t^*), p(t^*))$ is also the MFE approximation to the solution $(\tilde{u}(t^*), \tilde{p}(t^*))$ of the Newton-type linearized problem (39). In Theorems 1 and 2 below we prove that the IDN-approximation $(\tilde{u}(t^*), \tilde{p}(t^*))$ is an improved approximation to the solution (u, p) of the evolutionary Navier-Stokes equations (1)–(2) at time t^* . Although, as it is obvious, $(\tilde{u}(t^*), \tilde{p}(t^*))$ is not computable in practice, it is however a useful tool to provide a posteriori error estimates for the MFE approximation (u_h, p_h) at any desired time $t^* > 0$. In Theorem 1 we obtain the error bounds for the velocity and, in Theorem 2, the bounds for the pressure. The improvement is achieved in both the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms when $r = 3, 4$, and only in the $H^1(\Omega)^d$ norm when using the mini-element ($r = 2$).

Theorem 1 *Let (u, p) be the solution of (1)–(2). Then, there exist positive constants C, λ_0 such that, for all $\lambda \geq \lambda_0$, the IDN-approximate velocity \tilde{u} , defined in (39), satisfies the following bounds:*

(i) *If $r = 2$ then*

$$\|u(t^*) - \tilde{u}(t^*)\|_1 \leq \frac{C}{t^{*(1/2)}} h^2 |\log(h)|^2. \quad (40)$$

(ii) *If $r = 3, 4$ then*

$$\|u(t^*) - \tilde{u}(t^*)\|_m \leq \frac{C}{t^{*(r-1)/2}} h^{r+1-m} |\log(h)|, \quad m = 0, 1. \quad (41)$$

Proof The proof can be reached arguing similarly as in [16, Theorem 1] with the main difference that now we can avoid the requirement of having h small enough.

Subtracting (39) from (1) and denoting by $e_h = u - \tilde{u}$ and $r_h = p - \tilde{p}$ we get the error equation

$$\begin{aligned} & \nu(\nabla e_h, \nabla \phi) + ((u_h \cdot \nabla) e_h, \phi) + ((e_h \cdot \nabla) u_h, \phi) + \lambda(e_h, \phi) + (\nabla r_h, \phi) = \\ & (\dot{u}_h - u_t, \phi) + (((u_h - u) \cdot \nabla) u, \phi) + ((u \cdot \nabla)(u_h - u), \phi) + ((u \cdot \nabla) u, \phi) \\ & - ((u_h \cdot \nabla) u_h, \phi) + \lambda(u - u_h, \phi), \quad \forall \phi \in H_0^1(\Omega)^d, \end{aligned}$$

that can be written, projecting on the divergence-free space, as

$$\begin{aligned} & \nu(\nabla e_h, \nabla \phi) + ((u_h \cdot \nabla) e_h, \phi) + ((e_h \cdot \nabla) u_h, \phi) + \lambda(e_h, \phi) = \\ & (\dot{u}_h - u_t, \phi) + (((u_h - u) \cdot \nabla)(u - u_h), \phi) + \lambda(u - u_h, \phi), \quad \forall \phi \in V. \end{aligned} \quad (42)$$

The following bounds are valid for $r = 2, 3, 4$.

$$\begin{aligned} |(\dot{u}_h - u_t, \phi)| & \leq \|\dot{u}_h - u_t\|_{-1} \|\phi\|_1, \\ |(((u_h - u) \cdot \nabla)(u - u_h), \phi)| & \leq C \|u - u_h\|_1^2 \|\phi\|_1, \\ |\lambda(u - u_h, \phi)| & \leq \lambda \|u - u_h\|_0 \|\phi\|_0 \leq \lambda \|u - u_h\|_0 \|\phi\|_1. \end{aligned}$$

Taking $\phi = e_h$ and using (28) we get

$$\nu \|e_h\|_1 \leq \|u_t - \dot{u}_h\|_{-1} + C \|u - u_h\|_1^2 + \lambda \|u - u_h\|_0. \quad (43)$$

From (36) and (23) it can be seen that the biggest term is the temporal derivative and then

$$\|e_h\|_1 \leq \frac{C}{t^{(r-1)/2}} h^r |\log(h)|^{r'}, \quad r = 2, 3, 4, \quad (44)$$

where $r' = 2$ for $r = 2$ and $r' = 1$ for $r = 3, 4$.

To obtain a bound for the L^2 norm, we argue by duality. Let us consider the dual problem:

$$\begin{aligned} -\nu \Delta v - (u_h \cdot \nabla) v - \nabla \bar{v} \cdot u_h + \lambda v + \nabla j &= \varphi, \\ \operatorname{div}(v) &= 0, \\ v &= 0. \end{aligned} \quad \left. \begin{array}{l} \text{in } \Omega \\ \text{on } \partial\Omega \end{array} \right\} \quad (45)$$

Using (42) and taking into account that $e_h = u - \tilde{u}$ belongs to V , it can be checked that

$$\begin{aligned} (\varphi, e_h) &= \nu(\nabla v, \nabla e_h) - ((u_h \cdot \nabla) v, e_h) - (\nabla \bar{v} \cdot u_h, e_h) + \lambda(v, e_h) \\ &= \nu(\nabla e_h, \nabla v) + ((u_h \cdot \nabla) e_h, v) + ((e_h \cdot \nabla) u_h, v) + \lambda(e_h, v) \\ &= (\dot{u}_h - u_t, v) + (((u_h - u) \cdot \nabla)(u - u_h), v) + \lambda(u - u_h, v). \end{aligned}$$

We will apply the following bounds:

$$\begin{aligned} |(\dot{u}_h - u_t, v)| & \leq \|A^{-1} \Pi(u_t(t) - \dot{u}_h(t))\|_0 \|Av\|_0 \\ & \leq \|A^{-1} \Pi(u_t(t) - \dot{u}_h(t))\|_0 \|v\|_2, \\ |(((u_h - u) \cdot \nabla)(u - u_h), v)| & \leq C \|u - u_h\|_1^2 \|v\|_1 \leq C \|u - u_h\|_1^2 \|v\|_2, \\ |\lambda(u - u_h, v)| & \leq \lambda \|u - u_h\|_{-1} \|v\|_1 \leq \lambda \|u - u_h\|_{-1} \|v\|_2. \end{aligned}$$

Taking into account the regularity condition (31) we get

$$\|e_h\|_0 \leq \|A^{-1} \Pi(u_t(t) - \dot{u}_h(t))\|_0 + C \|u - u_h\|_1^2 + \lambda \|u - u_h\|_{-1}.$$

From (37), (38) and (23) it can be seen that the biggest term is again the temporal derivative so that we conclude

$$\|e_h\|_0 \leq \frac{C}{t^{(r-1)/2}} h^{r+1} |\log(h)|, \quad r = 3, 4.$$

□

In the following theorem we obtain the error bounds for the pressure \tilde{p} .

Theorem 2 *Let (u, p) be the solution of (1)-(2). Then, there exist positive constants C, λ_0 such that, for all $\lambda \geq \lambda_0$, the IDN-approximate pressure \tilde{p} , satisfies the following bounds:*

$$\|p(t^*) - \tilde{p}(t^*)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{*(r-1)/2}} h^r |\log(h)|^{r'}, \quad (46)$$

where $r' = 2$ if $r = 2$ and $r' = 1$ if $r = 3, 4$.

Proof The proof follows the same steps as [16, Theorem 1]. Let us denote by $r_h = p - \tilde{p}$ and, as before, by $e_h = u - \tilde{u}$. We subtract again (39) from (1) to get

$$\begin{aligned} \nu(\nabla e_h, \nabla \phi) + ((u_h \cdot \nabla) e_h, \phi) + ((e_h \cdot \nabla) u_h, \phi) + \lambda(e_h, \phi) - (r_h, \nabla \cdot \phi) = \\ (\dot{u}_h - u_t, \phi) + (((u_h - u) \cdot \nabla)(u - u_h), \phi) + \lambda(u - u_h, \phi), \end{aligned}$$

for all $\phi \in H_0^1(\Omega)^d$. Using the continuity condition (26) and applying the inf-sup condition (15) we get

$$\beta \|r_h\|_{L^2/(R)} \leq C \|e_h\|_1 + \|u_t - \dot{u}_h\|_{-1} + C \|u - u_h\|_1^2 + \lambda \|u - u_h\|_0.$$

Applying (44) to bound $\|e_h\|_1$ together with (36) and (23) we conclude

$$\beta \|r_h\|_{L^2/(R)} \leq \frac{C}{t^{(r-1)/2}} h^r |\log(h)|^{r'}, \quad r = 2, 3, 4,$$

and $r' = 2$ if $r = 2$ and $r' = 1$ if $r = 3, 4$.

□

Remark 1 As a consequence of Theorems 1 and 2, in the proof of Theorem 3 we obtain that $(\tilde{u} - u_h)$ is an asymptotically exact estimator of the error $(u - u_h)$, while $(\tilde{p} - p_h)$ is an asymptotically exact estimator of the error $(p - p_h)$. However, as we have already observed \tilde{u} and \tilde{p} are not computable in practice. In Theorems 3, 4 and 5 we present different procedures to get computable error estimators.

As we pointed out before, the MFE approximations (u_h, p_h) to the velocity and the pressure of the solution (u, p) of the evolutionary Navier-Stokes equations (1)-(2) at any fixed time t^* are also the approximations to the velocity and pressure of the Newton-type linearized equations (3). In Theorem 3 below, we show that any a posteriori error estimator of the error in the Newton-type linearized equations (3) is also an a posteriori indicator of the error in the approximations to the evolutionary Navier-Stokes equations.

Theorem 3 Let (u, p) be the solution of (1)-(2) and fix any positive time $t^* > 0$. Assume that the Galerkin approximation (u_h, p_h) satisfies

$$\|u(t^*) - u_h(t^*)\|_j \geq C_r h^{r-j}, \quad j = 0, 1. \quad (47)$$

for some positive constant $C_r = C_r(t^*)$, $r = 2, 3, 4$.

(i) Let us denote by $\xi(t^*)$ any reliable and efficient a posteriori error estimator of the error in the Newton-type linearized equations (3). That is, we assume that there exist positive constants C_1 and C_2 , that are independent of the mesh size h , such that the following bound holds

$$C_2 \xi(t^*) \leq \|\tilde{u}(t^*) - u_h(t^*)\|_1 + \|\tilde{p}(t^*) - p_h(t^*)\|_0 \leq C_1 \xi(t^*). \quad (48)$$

Let us assume that h is small enough so that the following condition holds

$$\frac{C t^{*-((r-1)/2)}}{C_r} h |\log(h)|^{r'} \leq 1/2. \quad (49)$$

Then, $\xi(t^*)$ is also a reliable and efficient estimator of the error in the evolutionary Navier-Stokes equations, i.e., the following bound holds for h small enough

$$\frac{2}{3} C_2 \xi(t^*) \leq \|u(t^*) - u_h(t^*)\|_1 + \|p(t^*) - p_h(t^*)\|_0 \leq 2C_1 \xi(t^*). \quad (50)$$

(ii) If $\xi_{\text{vel}}^j(t^*)$, $j = 0, 1$ is an asymptotically exact estimator of the norm $\|\tilde{u}(t^*) - u_h(t^*)\|_j$ of the error in the velocity Newton-type linearized equations (3), then, it is also an asymptotically exact estimator of the norm of the error $\|u(t^*) - u_h(t^*)\|_j$ in the velocity in the evolutionary Navier-Stokes equations. The same result holds for the pressure in the L^2 norm.

Proof The proof is the same as [21, Theorem 3] but we include it here for convenience of the reader. Let us first observe that

$$\|u_h(t^*) - u(t^*)\|_1 \leq \|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|\tilde{u}(t^*) - u(t^*)\|_1$$

and

$$\|p_h(t^*) - p(t^*)\|_0 \leq \|p_h(t^*) - \tilde{p}(t^*)\|_0 + \|\tilde{p}(t^*) - p(t^*)\|_0,$$

so that adding the two above inequalities we get

$$\begin{aligned} \|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 &\leq \|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0 \\ &\quad + \|\tilde{u}(t^*) - u(t^*)\|_1 + \|\tilde{p}(t^*) - p(t^*)\|_0. \end{aligned}$$

Dividing by $\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0$, using (47) and applying Theorems 1 and 2 we obtain

$$1 \leq \frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} + \frac{C t^{*-((r-1)/2)}}{C_r} h |\log(h)|^{r'},$$

where $r' = 2$ for $r = 2$ and $r' = 1$ for $r = 3, 4$. Now, using (48) we get

$$\frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \leq \frac{C_1 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0}.$$

Applying condition (49) we get

$$\frac{1}{2} \leq \frac{C_1 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0},$$

and then

$$\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 \leq 2C_1 \xi(t^*). \quad (51)$$

Now, we use the decompositions

$$\|u_h(t^*) - \tilde{u}(t^*)\|_1 \leq \|u_h(t^*) - u(t^*)\|_1 + \|u(t^*) - \tilde{u}(t^*)\|_1,$$

and

$$\|p_h(t^*) - \tilde{p}(t^*)\|_0 \leq \|p_h(t^*) - p(t^*)\|_0 + \|p(t^*) - \tilde{p}(t^*)\|_0,$$

and add both inequalities as before to get

$$\begin{aligned} \|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0 &\leq \|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 \\ &\quad + \|u(t^*) - \tilde{u}(t^*)\|_1 + \|p(t^*) - \tilde{p}(t^*)\|_0. \end{aligned}$$

Reasoning as before and applying condition (49) again we get

$$\frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \leq 1 + \frac{C t^{*-((r-1)/2)}}{C_r} h |\log(h)|^{r'} \leq \frac{3}{2},$$

for h small enough. Using again (48) we obtain

$$\frac{C_2 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \leq \frac{3}{2},$$

so that

$$\frac{2}{3} C_2 \xi(t^*) \leq \|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0. \quad (52)$$

From (51) and (52) we conclude (50).

Let us now assume that $\xi_{\text{vel}}^j(t^*)$ is an asymptotically exact error estimator for the velocity. Using

$$\|u_h(t^*) - \tilde{u}(t^*)\|_j \leq \|u_h(t^*) - u(t^*)\|_j + \|u(t^*) - \tilde{u}(t^*)\|_j, \quad j = 0, 1,$$

we have

$$\lim_{h \rightarrow 0} \frac{\|u_h(t^*) - \tilde{u}(t^*)\|_j}{\|u_h(t^*) - u(t^*)\|_j} = 1 + \lim_{h \rightarrow 0} \frac{\|u(t^*) - \tilde{u}(t^*)\|_j}{\|u_h(t^*) - u(t^*)\|_j} = 1,$$

the last equality being a consequence of Theorem 1 and the saturation hypothesis (47). As we pointed out before, this limit implies that $(\tilde{u} - u_h)$ is an asymptotically exact estimator of the error $(u - u_h)$. Then

$$\lim_{h \rightarrow 0} \frac{\xi_{\text{vel}}^j(t^*)}{\|u_h(t^*) - u(t^*)\|_j} = \lim_{h \rightarrow 0} \frac{\xi_{\text{vel}}^j(t^*)}{\|u_h(t^*) - \tilde{u}(t^*)\|_j} \frac{\|u_h(t^*) - \tilde{u}(t^*)\|_j}{\|u_h(t^*) - u(t^*)\|_j} = 1,$$

and $\xi_{\text{vel}}^j(t^*)$ is also an asymptotically exact estimator of the error in the approximation to the velocity of the evolutionary Navier-Stokes equations. The proof for the pressure can be obtained arguing exactly in the same way. \square

Remark 2 We remark that with hypothesis (47) we are merely assuming that the term of order h^{r-j} is really present in the asymptotic expansion of the Galerkin error. The same assumption is also assumed in [25], [26], [19], [36]. As argued in [15], this is not a very restrictive condition in practice. Respect to condition (49), in the numerical experiments of Section 5, we observed a good behaviour of the estimator for all the values of the mesh size h , so that, at least in our experience, no special effort is required to make condition (49) be satisfied.

We now propose a simple procedure to estimate the error in the Galerkin finite element approximation to the Navier-Stokes equations (1). The idea is to compare the two-grid Newton-type approximation analyzed in [16] with the Galerkin approximation. More precisely, the two-grid or postprocessed approximation is defined in the following way. In the first level, we choose a mesh of size h and compute the mixed finite-element approximation (u_h, p_h) to (u, p) defined by (21)-(22). In the second level, the discrete velocity and pressure $(u_h(t), p_h(t))$ are postprocessed by solving the following linear Newton-type problem: find $(\tilde{u}_{h'}(t), \tilde{p}_{h'}(t)) \in (X_{h',r}, Q_{h',r-1})$, $h' < h$, satisfying for all $\phi_{h'} \in X_{h',r}$ and $\psi_{h'} \in Q_{h',r-1}$

$$\begin{aligned} & \nu(\nabla \tilde{u}_{h'}(t), \nabla \phi_{h'}) + ((u_h(t) \cdot \nabla) \tilde{u}_{h'}(t), \phi_{h'}) + ((\tilde{u}_{h'}(t) \cdot \nabla) u_h(t), \phi_{h'}) \\ & + \lambda(\tilde{u}_{h'}(t), \phi_{h'}) - (\tilde{p}_{h'}(t), \nabla \cdot \phi_{h'}) \\ & = (f(t) - \dot{u}_h(t), \phi_{h'}) + ((u_h(t) \cdot \nabla) u_h(t), \phi_{h'}) + \lambda(u_h, \phi_{h'}) \\ & (\nabla \cdot \tilde{u}_h(t), \psi_{h'}) = 0. \end{aligned} \quad (53)$$

Equations (53) can also be solved over a higher order mixed finite-element space over the same grid. For simplicity in the exposition we will only consider the case in which we refine the mesh at the postprocessing step.

The following theorem has been proved in [16].

Theorem 4 *Let (u, p) be the solution of (1) and $(\tilde{u}_{h'}, \tilde{p}_{h'})$ be the solution of (53). Then, for h and h' small enough the following bounds hold for $t \in (0, T]$, $j = 0, 1$:*

$$\|u(t) - \tilde{u}_{h'}(t)\|_1 \leq Ch' + \frac{C}{t^{1/2}} h^2 |\log(h)|^2, \quad r = 2, \quad (54)$$

$$\|u(t) - \tilde{u}_{h'}(t)\|_j \leq \frac{C}{t^{(r-2)/2}} (h')^{r-j} + \frac{C}{t^{(r-1)/2}} h^{r+1-j} |\log(h)|, \quad r = 3, 4. \quad (55)$$

$$\|p(t) - \tilde{p}_{h'}(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-2)/2}} (h')^{r-1} + \frac{C}{t^{(r-1)/2}} h^r |\log(h)|^{r'}, \quad r = 2, 3, 4, \quad (56)$$

where $r' = 2$ for $r = 2$ and $r' = 1$ otherwise.

To estimate the error in $(u_h(t^*), p_h(t^*))$ we propose to take the difference between the postprocessed and the Galerkin approximations:

$$\tilde{\eta}_{h,\text{vel}}(t^*) = \tilde{u}_{h'}(t^*) - u_h(t^*), \quad \tilde{\eta}_{h,\text{pres}}(t^*) = \tilde{p}_{h'}(t^*) - p_h(t^*).$$

In the following theorem we prove that this error estimator is efficient and asymptotically exact both in the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms, and it has the advantage of providing an improved approximation when added to the Galerkin MFE approximation.

Theorem 5 Let (u, p) be the solution of (1)-(2) and fix any positive time $t^* > 0$. Assume that condition (47) is satisfied. Then for h small enough such that condition (49) holds and

$$h' < \gamma h, \quad 0 < \gamma \leq \gamma_0, \quad \frac{C}{C_r}(t^*)^{-(r-2)/2}(\gamma_0)^{r-j} < 1/4, \quad (57)$$

the error estimators $\tilde{\eta}_{h,\text{vel}}(t^*)$, $\tilde{\eta}_{h,\text{pres}}(t^*)$ satisfy the following bounds:

$$\frac{1}{4} \leq \frac{\|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j}{\|(u - u_h)(t^*)\|_j} \leq \frac{3}{4}, \quad j = 0, 1, \quad \frac{1}{4} \leq \frac{\|\tilde{\eta}_{h,\text{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p - p_h)(t^*)\|_{L^2/\mathbb{R}}} \leq \frac{3}{4}. \quad (58)$$

Furthermore, in case we take $h' = h^{1+\epsilon}$, $\epsilon > 0$ then

$$\lim_{h \rightarrow 0} \frac{\|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j}{\|(u - u_h)(t^*)\|_j} = 1, \quad j = 0, 1, \quad \lim_{h \rightarrow 0} \frac{\|\tilde{\eta}_{h,\text{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p - p_h)(t^*)\|_{L^2/\mathbb{R}}} = 1. \quad (59)$$

For the mini element, the case $j = 0$ in (58) and (59) must be excluded.

Proof The proof is the same as [21, Theorem 6] but we include it for convenience of the reader. We will prove the estimates for the velocity in the case $r = 3, 4$, since the estimates for the pressure and the case $r = 2$ are obtained by similar arguments but with obvious changes. Let us observe that for $j = 0, 1$,

$$\begin{aligned} \|u(t^*) - u_h(t^*)\|_j &\leq \|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j + \|\tilde{u}_{h'}(t^*) - u(t^*)\|_j \\ &\leq \|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} \\ &\quad + \frac{C}{(t^*)^{(r-1)/2}}h^{r+1-j}|\log(h)|. \end{aligned}$$

On the other hand

$$\begin{aligned} \|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j &\leq \|u(t^*) - u_h(t^*)\|_j + \|\tilde{u}_{h'}(t^*) - u(t^*)\|_j \\ &\leq \|u(t^*) - u_h(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} \\ &\quad + \frac{C}{(t^*)^{(r-1)/2}}h^{r+1-j}|\log(h)|. \end{aligned} \quad (60)$$

Using (47) we get

$$\left| \frac{\|\tilde{\eta}_{h,\text{vel}}(t^*)\|_j}{\|(u - u_h)(t^*)\|_j} - 1 \right| \leq \frac{C}{C_r} \left((t^*)^{-(r-2)/2} \left(\frac{h'}{h} \right)^{r-j} + (t^*)^{-(r-1)/2} |\log(h)|h \right). \quad (61)$$

Then for $h' \leq \gamma h$, $\gamma \leq \gamma_0$ satisfying (57) and applying condition (49) the bound (58) is readily obtained. The proof of (59) follows straightforwardly from (61), since for $h' = h^{1+\epsilon}$, $\epsilon > 0$, the term $(h'/h)^{r-j} \rightarrow 0$ when h tends to zero. \square

Remark 3 We want to remark that in practice we can apply the estimators for values of h and h' different from those satisfying (49) and (57) (and even also for λ not satisfying condition (35)). As a consequence, we will have an indicator that will probably satisfy (58) but with unknown constants instead of the values $1/4$ and $3/4$.

Remark 4 Let us observe that one could apply inequality (60) taking norms at every element $\tau_i^h \in \mathcal{T}_h$ instead of in the full domain Ω trying to prove a local lower bound. However, while global assumption (47) is realistic, as explained in Remark 2, the equivalent local assumption would not be realistic since it would essentially imply that the error of the Galerkin approximation is uniformly distributed (almost of the same size) at every element. Since our error estimator is based on the comparison between two approximations of different order the classical technique of employing bubble functions in the derivation of local lower a posteriori error estimates (see e.g. [35]) does not work. As a future research, it could be studied if the procedure of [25] to obtain local error estimates for the p version of the finite element method for non linear parabolic problems could also be applied in the context of the present paper.

5 Numerical experiments

In this section, we carry out some numerical experiments using MATLAB for studying the a posteriori error estimation of the Galerkin approximation, taking as estimator the difference between the post-processed and the Galerkin approximations. The experiments are computed in the unit square $\Omega = [0, 1] \times [0, 1]$, using the mini-element over a regular triangulation based on the set of nodes $(i/N, j/N)$ $i, j = 1, 2, \dots, N$, where $N = 1/H$ is the Galerkin spatial resolution. In the time integration we use a semi-implicit trapezoidal rule, where spatial derivatives are treated implicitly. The size of the time step is chosen so that temporal errors are negligible compared to spatial errors. Once the mini-element is obtained, the bubble part is removed and the errors of the linear part are computed, because it has been reported in [22] that the linear part is a better approximation to the solution than the complete linear-bubble velocity. The bubble part is considered only for stability reasons. In these experiments, we take the following functions

$$\begin{aligned} u^1(x, y, t) &= \pi t \sin^2(\pi x) \sin(2\pi y), \\ u^2(x, y, t) &= -\pi t \sin^2(\pi y) \sin(2\pi x), \\ p(x, y, t) &= 5tx^2y. \end{aligned} \tag{62}$$

and calculate the forcing term $f(x, t)$ so that (62) is the solution of the Navier Stokes equations for different values of the diffusion parameter ranging from $\nu = 1$ to $\nu = 0.001$. The Galerkin approximations are obtained integrating up to time $T = 1$ in the coarse mesh of size $H = 1/N$. Then, there are post-processed at the fixed final time over a finer mesh of size $h < H$ small enough to retain the asymptotic behaviour of the rate of convergence. Computational cost of the post-processed step is typically of the order of a single time step over the fine mesh h . In this experiment, the sizes of the coarse mesh are given by $N = 40, N = 50, N = 60, N = 70$ and the fine mesh by $n = 130, n = 175, n = 223$ and $n = 273$. Although for the mini-element, optimal values for the fine mesh are those obtained taking $h = H^2$, in practice the rate of convergence of the methods can be reached taking exponents less than 2.

We consider the error estimator

$$\tilde{\eta}_{h,\text{vel}} = \tilde{u}_{h'} - u_h, \quad \tilde{\eta}_{h,\text{pres}} = \tilde{p}_{h'} - p_h.$$

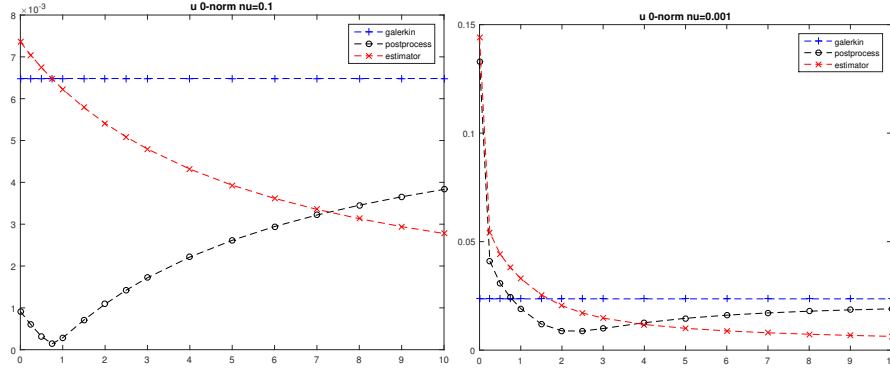


Figure 1: Galerkin, postprocessed and estimator L^2 errors for the first velocity component for $\nu = 0.1$ and $\nu = 0.001$.

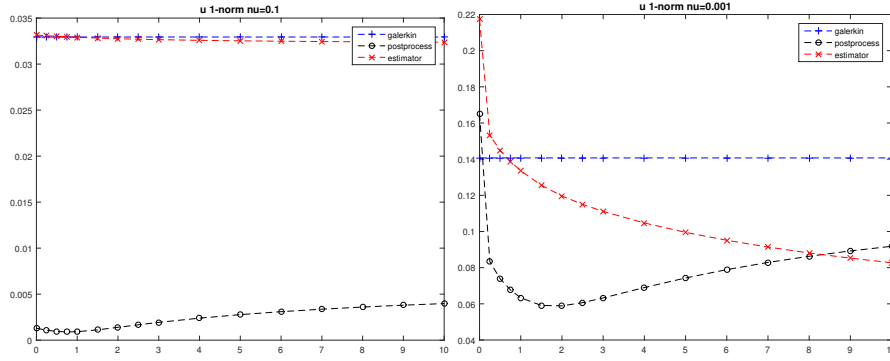


Figure 2: Galerkin, postprocessed and estimator H^1 errors for the first velocity component for $\nu = 0.1$ and $\nu = 0.001$.

The Newton postprocess includes an extra parameter λ to ensure coercivity. The same Galerkin approximation can be post-processed taking different values of λ . Since the choice of λ could in principle impacts the quality of the estimator we make a study of the results obtained for different values of λ . In the first experiment, the Galerkin approximation is post-processed for values of λ ranging from 0 to 10. We have represented this experiment in the Figures 1,2 and 3. Although for the coercivity we need a value of λ strictly positive we include in the studies the case $\lambda = 0$ since in practice we can also compute the approximations for this value of λ .

Figure 1 shows the L^2 errors for the Galerkin, postprocess and error estimations of the first component of the velocity, taking $N = 40$ fixed. The Galerkin approximation does not depend on λ , so it is constant. The post-processed solution seems to reach a minimal error value near to $\lambda = 0.75$ for $\nu = 0.1$ and near to $\lambda = 2$ for $\nu = 0.001$. For $\nu = 0.1$, the error of the post-processed solution is always better than the Galerkin approximation. For $\nu = 0.001$, the post-processed solution is worse than the Galerkin approximation in the neighborhood of zero. On the other hand, if λ increases, the λ -term becomes dominant in the postprocess equations and the post-processed velocity approximates

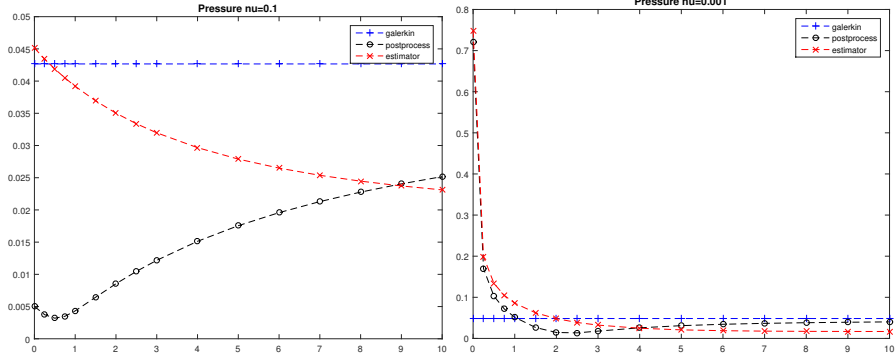


Figure 3: Galerkin, postprocessed and estimator L^2 errors for the pressure for $\nu = 0.1$ and $\nu = 0.001$.

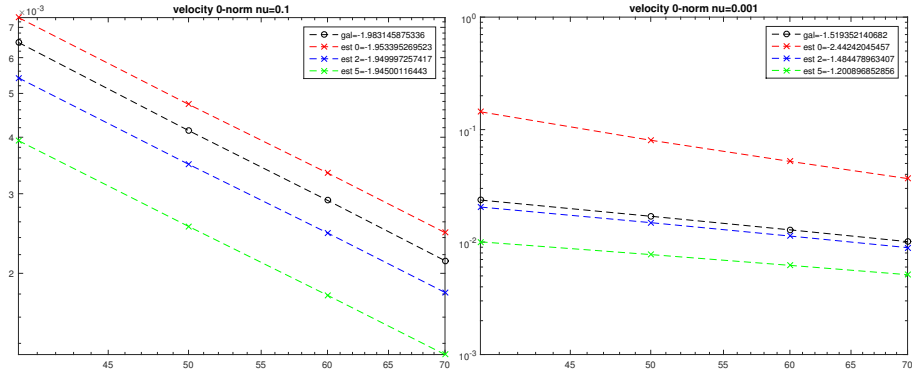


Figure 4: Galerkin and estimator L^2 errors for the first velocity component for $\lambda = 0, 2, 5$.

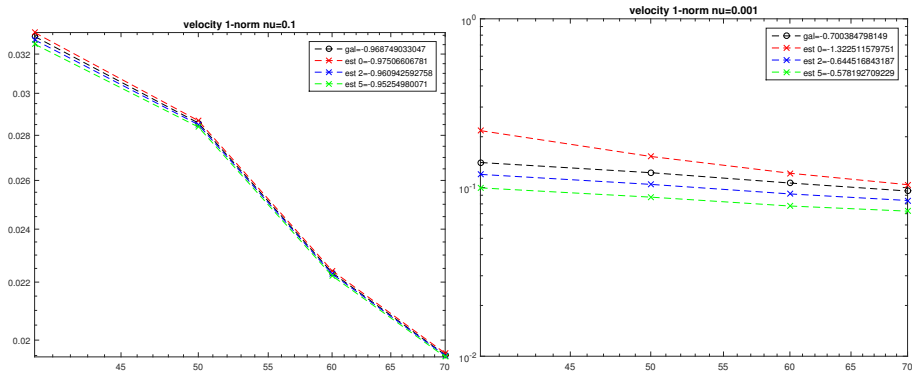


Figure 5: Galerkin and estimator H^1 errors for the first velocity component for $\lambda = 0, 2, 5$.

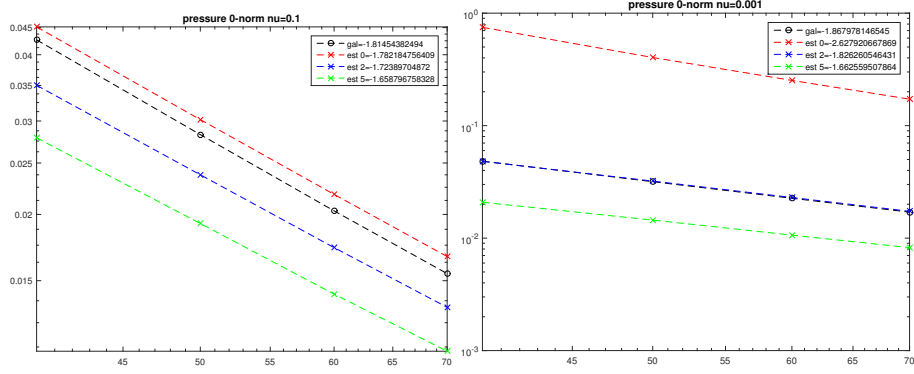


Figure 6: Galerkin and estimator L^2 errors for the pressure for $\lambda = 0, 2, 5$.

N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$	N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$
40	1.1350	1.0064	1.0579	40	6.0918	1.5454	15.4632
50	1.1429	1.0038	1.0673	50	4.7397	1.2479	12.7141
60	1.1496	1.0034	1.0742	60	4.0592	1.1431	11.1689
70	1.1538	1.0026	1.0768	70	3.6357	1.0902	10.1039

Table 1: Efficiency indexes for $\lambda = 0$. On the left, $\nu = 0.1$. On the right $\nu = 0.001$.

the Galerkin velocity. Figure 2 and 3 show the results for the H^1 norm of the velocity and the L^2 error of the pressure. Minimal values of the post-processed error are reached at values slightly different than those of the L^2 norm. In view of condition (35) and the above figures we could conclude that smaller values for λ than those predicted by the theory seem to be advisable for the numerical implementation of the method.

In the next experiment, we study the efficiency of the error estimators $\eta_{h,vel}$ and $\eta_{h,pres}$ for three values of λ . These values are $\lambda = 0, 2, 5$. Figures 4, 5 and 6 represent the Galerkin errors and the error estimations, calculated for the three selected values in the respective norms, and taking $N = 40, 50, 60$ and 70. Figure 4 represents the L^2 velocity errors. It can be seen that for $\nu = 0.1$ and $\nu = 0.001$, the accuracy of the estimator depends of λ . Although the optimal choice of λ is not known a priori we can observe that for all values of λ the estimations are efficient as Theorem 5 predicts. Figure 5 represents the H^1 velocity errors. For this experiment, the a posteriori estimator predicts essentially the same errors for the 3 values of lambda, matching also almost

N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$	N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$
40	0.8349	0.9944	0.8209	40	0.8668	0.8505	0.9934
50	0.8422	0.9972	0.8400	50	0.8773	0.8526	1.0126
60	0.8470	0.9981	0.8526	60	0.8808	0.8618	1.0162
70	0.8506	0.9988	0.8638	70	0.8843	0.8784	1.0172

Table 2: Efficiency indexes for $\lambda = 2$. On the left, $\nu = 0.1$. On the right $\nu = 0.001$.

N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$	N	$\ \theta_{vel}\ _0$	$\ \theta_{vel}\ _1$	$\ \theta_{pre}\ _{L^2/\mathbb{R}}$
40	0.6066	0.9874	0.6537	40	0.4251	0.7074	0.4304
50	0.6137	0.9934	0.6799	50	0.4563	0.7163	0.4556
60	0.6167	0.9950	0.6968	60	0.4833	0.7308	0.4703
70	0.6199	0.9966	0.7140	70	0.5082	0.7596	0.4833

Table 3: Efficiency indexes for $\lambda = 5$. On the left, $\nu = 0.1$. On the right $\nu = 0.001$.

exactly the true Galerkin errors in the case $\nu = 0.1$ and very precisely in the case $\nu = 0.001$. Figure 6 represents the L^2 pressure errors. On the right, we can see that for $\nu = 0.001$ and $\lambda = 2$ the estimated and the true errors are almost indistinguishable.

Let us now denote by

$$\theta_{vel} = \frac{\tilde{u}_{h'}^1 - u_h^1}{u^1 - u_h^1}, \quad \theta_{pre} = \frac{\tilde{p}_{h'} - p_h}{p - p_h}$$

the efficiency indexes for the pressure and the first component of the velocity for the estimator. Tables 1, 2 and 3 represent the index values for the different norms for $\lambda = 0, 2, 5$, respectively. As it could be observed in the figures, we can see in Table 1, that for increasing Reynolds numbers, the value $\lambda = 0$ becomes inadequate for estimating the true errors. For this reason, it is recommended to choose an strictly positive value of λ . On the other hand, if $\lambda = 2$ the efficiency indexes are pretty near to 1 and the proposed estimator based on the Newton postprocess is then a very efficient indicator of the true error. Finally, for $\lambda = 5$ the efficiency indexes are worse than those of the previous value of λ but also in this case the behaviour of the estimator is efficient and the values are not far away from the optimal value 1.

As mentioned in Remark 4, the question of getting local lower bounds is still not solved. However, in practice, one could compute the norm of the estimator at every element instead of in the full domain trying to get some insight into the size of the Galerkin error in that element. Using this procedure one could try to adapt the meshes. As a future research, the a posteriori error estimators proposed in the present paper could be applied to generate adapted meshes for more complicated examples.

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