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This is an **author produced version** of a paper published in:

Journal of Computational and Applied Mathematics 368 (2020): 112516

DOI: <https://doi.org/10.1016/j.cam.2019.112516>

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Generalized postprocessed approximations to the Navier-Stokes equations based on two grids

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September 23, 2019

Abstract

In this paper we generalize previous postprocessed approximations to the Navier-Stokes equations. The idea of postprocessing in the context of mixed finite element approximations is the following. Once a standard Galerkin mixed finite element approximation is computed using a mesh (we will call coarse) of size H at a fixed time T , we postprocess the approximation using a finer mesh of size $h < H$, by solving a steady problem with data based on the Galerkin approximation. In the literature, one can find different ways to postprocess the Galerkin approximation depending on the problem that is solved over the fine mesh at the final time T . Basically, one can solve a Stokes problem, an Oseen type problem or a Newton type problem. In the present paper, we present a method based on several parameters that generalize the above postprocessing techniques. Depending on the values chosen for the different parameters one can recover one of the old (known) postprocessing procedures but also produce a new (different) method. We get error bounds for the generalized method valid for any of the values of the different parameters. In all the cases, the postprocessed method has a rate of convergence one unit bigger than the rate of convergence of the plain Galerkin method in terms of the coarse mesh H and optimal in terms of the fine mesh h . The computational added cost of postprocessing is however negligible. For the error analysis we do not assume nonlocal compatibility conditions for the true solutions of the Navier-Stokes equations. Some numerical experiments show the performance of the method for different values of the parameters.

Keywords. Evolutionary Navier–Stokes equations; mixed finite elements; two-grids methods, postprocessing

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with polyhedral and Lipschitz boundary $\partial\Omega$. The incompressible Navier–Stokes equations model the conser-

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vation of linear momentum and the conservation of mass (continuity equation) by

$$\begin{aligned} u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, & \text{in } (0, T] \times \Omega \\ \operatorname{div}(u) &= 0, & \text{in } (0, T] \times \Omega \\ u(\cdot, 0) &= u_0, & \text{in } \Omega. \end{aligned} \tag{1}$$

We assume for simplicity homogeneous Dirichlet boundary conditions.

In this paper we propose and analyze a generalization of some previous two-grid postprocessing techniques. In these techniques, a mixed finite-element approximation (u_H, p_H) is computed over a coarse mesh of size H at a fixed final time $T > 0$. Then, the post-processed approximation (\tilde{u}, \tilde{p}) at time $T > 0$ is the mixed finite element approximation of a concrete steady problem over a finer mesh of size $h < H$.

Several procedures have been proposed in the literature. In [3], [2] (see also [10], [11]) the problem solved at the final time is a Stokes-type problem:

$$\begin{aligned} -\nu \Delta \tilde{u} + \nabla \tilde{p} &= f - \frac{d}{dt} u_H - (u_H \cdot \nabla) u_H, \\ \operatorname{div}(\tilde{u}) &= 0, \end{aligned}$$

where the right-hand side is evaluated at time $T > 0$. In [3] the error analysis for mixed finite element approximations of higher order (from quadratics in the approximation to the velocity) and assuming nonlocal compatibility conditions for the true solution is considered. The rate of convergence of the posprocessed approximation is proved to be one order higher than the rate of convergence of the Galerkin approximation in terms of the coarse mesh size H while being of optimal order respect to the fine mesh h . The results of [3] are valid for the L^2 and H^1 norms of the errors in the velocity and L^2 norm of the errors of the pressure. In [2] the linear finite element case is considered. For this case the improvement in the rate of convergence can only be observed in the H^1 norm of the error of the velocity and L^2 norm of the error of the pressure. In [10] the assumption on the satisfaction of nonlocal compatibility conditions for the solution is removed and in [11] the results are extended to fully discrete methods.

In [13] the problem solved at the final time is an Oseen-type problem

$$\begin{aligned} -\nu \Delta \tilde{u} + (u_H \cdot \nabla) \tilde{u} + \nabla \tilde{p} &= f - \frac{d}{dt} u_H, \\ \operatorname{div}(\tilde{u}) &= 0, \end{aligned} \tag{2}$$

where the right-hand side is also evaluated at the final time. Analogous results to those in [10] are proved in [13].

Finally, in [8], a Newton problem is solved at time T , assuming as before

that the right-hand side is evaluated at the final time:

$$\begin{aligned} -\nu\Delta\tilde{u} + \lambda\tilde{u} + (u_H \cdot \nabla)\tilde{u} + (\tilde{u} \cdot \nabla)u_H + \nabla\tilde{p} &= f - \frac{d}{dt}u_H + \lambda u_H \\ &+ (u_H \cdot \nabla)u_H, \\ \operatorname{div}(\tilde{u}) &= 0. \end{aligned} \quad (3)$$

Let us observe that in (3) a positive parameter $\lambda \geq 0$ is added to the bilinear form that defines the postprocessed method to achieve coercivity. Then, we have the extra term $\lambda\tilde{u}$ on the left-hand side of the equation and the extra term λu_H on the right-hand side. Related references are [20] and [21] where fully discrete schemes are considered to approximate the Navier-Stokes equations with a Fourier method in the first reference and a mixed finite element method in the second one. In both references the coarse and the fine approximations are computed together along the full time integration and for the approximation of the fine level the nonlinear term is linearized as in the Newton postprocess (3).

In the present paper, we generalize the previous procedures defining the following postprocessed method in which the right-hand side is evaluated at time $T > 0$:

$$\begin{aligned} -\nu\Delta\tilde{u} + \lambda\tilde{u} + \mu(u_H \cdot \nabla)\tilde{u} + \tau(\tilde{u} \cdot \nabla)u_H + \nabla\tilde{p} &= f - \frac{d}{dt}u_H + \lambda u_H \\ &+ \psi(u_H \cdot \nabla)u_H, \\ \operatorname{div}(\tilde{u}) &= 0, \end{aligned} \quad (4)$$

and μ , τ and ψ are (not necessary positive) parameters satisfying $\mu + \tau - \psi = 1$ and $\lambda \geq 0$ is a positive parameter required for coercivity, as in (3). This postprocess generalizes the Stokes, Oseen and Newton post-processes, with the appropriate choices of the parameters μ , τ and ψ . We can observe that the Stokes postprocess is recovered for $\mu = 0$, $\tau = 0$ and $\psi = -1$; the Oseen postprocess for $\mu = 1$, $\tau = 0$ and $\psi = 0$ (both with $\lambda = 0$) and the Newton postprocess for $\mu = 1$, $\tau = 1$ and $\psi = 1$. With the error analysis of the present paper we open the possibility to consider problem (4) with values of the parameters other than those above that could produce better approximations depending on the concrete problem being approached. Also, following [12] (see also [9]) the postprocessed approximation based on (4) could be used to a posteriori estimate the error in the plain Galerkin approximation by taking the difference between the postprocessed approximation and the standard Galerkin approximation. However, we do not explore the a posteriori capabilities of the postprocessing technique in the present paper.

To finish this section we mention two related references for two-grid methods for the evolutionary Navier-Stokes equations. In [16] the authors consider the case of non-smooth initial data for a two level method that is based on a linearized time dependent Stokes problem. In [22] three kinds of two-level consistent splitting algorithms based on a linearized time dependent Stokes, Oseen and Newton corrections for the time-dependent Navier-Stokes equations are discussed.

The outline of the paper is as follows. In Section 2 we state some preliminaries and notations and prove some auxiliary results. In Section 3 we consider the steady linear problem that defines the postprocessed method and get a bound for the error of its mixed finite element approximation. In Section 4 we bound the error of the new method. Some numerical experiments are shown in Section 5.

2 Preliminaries and notations

Using the standard notation $W^{m,q}(\Omega)^d$ the following Sobolev inequality holds for $q \in [1, \infty)$, [1]

$$\|u\|_{L^{q'}} \leq c_s \|u\|_{W^{s,q}}, \quad \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty, \quad u \in W^{s,q}(\Omega)^d, \quad (5)$$

where the constant $c_s = c_s(\Omega, q)$. In the case $q' = \infty$, (5) can be applied with $\frac{1}{q} < \frac{s}{d}$.

Along this paper we will apply the following particular inequalities that are valid both in 2 and 3 dimensions.

$$\|u\|_{L^{2d}} \leq c_s \|u\|_s, \quad s \geq 1, \quad \|u\|_{L^{2d/(d-1)}} \leq c_s \|u\|_s, \quad s \geq 1/2. \quad (6)$$

Then for $d = 2$ the $\|\cdot\|_{L^4}$ norm is bounded by the $\|\cdot\|_{1/2}$ norm and for $d = 3$ the $\|\cdot\|_{L^3}$ norm is bounded by the $\|\cdot\|_{1/2}$ norm and the $\|\cdot\|_{L^6}$ norm is bounded by the $\|\cdot\|_1$ norm.

Let $Q = L_0^2(\Omega)$. We will use the following notation for the divergence free Hilbert spaces

$$\begin{aligned} \mathcal{H} &= \{u \in L^2(\Omega)^d \mid \operatorname{div}(u) = 0, u \cdot n|_{\partial\Omega} = 0\}, \\ V &= \{u \in H_0^1(\Omega)^d \mid \operatorname{div}(u) = 0\}. \end{aligned}$$

It is known that the standard inf-sup condition

$$\inf_{p \in Q} \sup_{u \in H_0^1(\Omega)^d} \frac{(p, \nabla \cdot u)}{\|p\|_0 \|u\|_1} \geq \beta_c > 0 \quad (7)$$

is satisfied, [15].

We denote by $\Pi : L^2(\Omega)^d \rightarrow \mathcal{H}$ the L^2 projection onto \mathcal{H} and by A the so-called Stokes operator, $A = -\Pi\Delta : \mathcal{D}(A) = H^2(\Omega)^d \cap V \subset \mathcal{H} \rightarrow \mathcal{H}$.

We will assume that u is a strong solution up to time $t = T$ so that the following inequality holds:

$$\|u(t)\|_i \leq M_i, \quad i = 1, 2, \quad 0 \leq t \leq T. \quad (8)$$

Assuming enough regularity for the forcing term f and for some of its time derivatives we can apply Theorems 2.4 and 2.5 in [17]. Then, apart from (8),

the following bounds also hold for the solution of the Navier-Stokes equations (1)

$$\begin{aligned} \|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{k-1} &\leq M_k t^{1-k/2}, \quad (9) \\ \int_0^t s^{k-3} (\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{k-1}^2 \|p_s(s)\|_{k-3}^2) ds &\leq K_k^2. \end{aligned}$$

Let $X_{h,r} \subset V$ and $Q_{h,r-1} \subset Q$ be two families of finite element spaces (corresponding to a family of partitions \mathcal{T}_h of Ω into mesh cells with maximal diameter h) composed of continuous piecewise polynomials of degrees $r-1$ and $r-2$ respectively. More precisely, we consider the so-called Hood-Taylor element [4, 19], when $r \geq 3$ and the so-called mini-element [5] when $r = 2$. In this case, linear polynomials plus bubble functions are used to approximate the velocity and linear polynomials for the pressure although we keep the notation $(X_{h,2}, Q_{h,1})$ also for this pair for simplicity. These pairs of finite element spaces satisfy the discrete inf-sup condition

$$\inf_{q_h \in Q_{h,r-1}} \sup_{v_h \in X_{h,r}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_1 \|q_h\|_0} \geq \beta, \quad (10)$$

where β is a positive constant that does not depend on h .

It will be assumed that the family of meshes is quasi-uniform and that the following inverse inequality holds for each $v_h \in X_{h,r}$, see e.g., [6, Theorem 3.2.6],

$$\|v_h\|_{W^{m,p}(K)} \leq Ch^{n-m-d(\frac{1}{q}-\frac{1}{p})} \|v_h\|_{W^{n,q}(K)}, \quad (11)$$

where $0 \leq n \leq m \leq 1$, $1 \leq q \leq p \leq \infty$.

We will denote by $V_{h,r}$ the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \{ \chi_h \in H_0^1(\Omega_h)^d \mid (q_h, \nabla \cdot \chi_h) = 0 \quad \forall q_h \in Q_{h,r-1} \},$$

to which the approximate velocity belongs.

In the sequel we consider the linear mini-element case for $r = 2$ and Hood-Taylor elements for $r = 3$ (quadratics) and $r = 4$ (cubics) since it is known that the loss of regularity at $t = 0$ makes that no better than fifth order bounds can be proved. This limitation in the rate of convergence was previously found in [18], [10], [13] and [8].

Let $(u, p) \in V \times Q$ we will denote by $s_h \in V_{h,r}$ the Stokes projection [18], i.e., the velocity approximation in the Stokes problem: find $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ such that

$$\begin{aligned} \nu(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) &= \nu(\nabla u, \nabla \phi_h) + (\nabla p, \phi_h), \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot s_h, \psi_h) &= 0, \quad \forall \psi_h \in Q_{h,r-1}. \end{aligned} \quad (12)$$

The following bound holds for $1 \leq l \leq r$ and $(u, p) \in (H^l(\Omega)^d \cap V) \times (H^{l-1}(\Omega) \cap Q)$, [18]:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \leq Ch^l (\|u\|_l + \|p\|_{l-1}). \quad (13)$$

And, for the pressure, [15]

$$\|p - q_h\|_0 \leq C_\beta h^{l-1} (\|u\|_l + \|p\|_{l-1}), \quad (14)$$

where C_β is a constant that depends on β in (10).

Following [7], one can also obtain the following bound for s_h

$$\|\nabla(u - s_h)\|_\infty \leq C \|\nabla u\|_\infty. \quad (15)$$

Let $H > h$ be the size of the coarse mesh. We will denote by $(u_H, p_H) \in (X_{H,r}, Q_{H,r-1})$ the time-continuous Galerkin approximation to the solution of (1) solving for $0 < t \leq T$, $\phi_H \in X_{H,r}$ and $\psi_H \in Q_{H,r-1}$

$$(\dot{u}_H, \phi_H) + \nu(\nabla u_H, \nabla \phi_H) + b(u_H, u_H, \phi_H) + (\nabla p_H, \phi_H) = (f, \phi_H), \quad (16)$$

$$(\nabla \cdot u_H, \psi_H) = 0, \quad (17)$$

where for the nonlinear term the following form will be used

$$b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\nabla \cdot u)v, w), \quad u, v, w \in H_0^1(\Omega)^d,$$

to keep the skew-symmetric property

$$b(u, v, w) = -b(u, w, v), \quad u, v, w \in H_0^1(\Omega)^d.$$

For the Galerkin approximation the following bounds for $2 \leq r \leq 4$ and $0 \leq t \leq T$ can be found in [17] and [18], see also [10]

$$\|u(t) - u_H(t)\|_0 + H\|u(t) - u_H(t)\|_1 \leq C \frac{H^r}{t^{(r-2)/2}}, \quad (18)$$

$$\|p(t) - p_H(t)\|_0 \leq C \frac{H^{r-1}}{t^{(r-2)/2}}. \quad (19)$$

2.1 Auxiliary results

In this subsection, we state and prove some auxiliary results that will be used for the error analysis of the method. In the next proposition we bound the nonlinear term using different norms. We do not include the proof that is simple and can be obtained case by case by using standard Hölder and Sobolev's inequalities.

Proposition 1 Let $a, b, c \in H_0^1(\Omega)^d$ (or $H^2(\Omega)^d \cap H_0^1(\Omega)^d$) when required. Then

$$|(a \cdot \nabla b, c)| \leq C \|a\|_\alpha \|b\|_\beta \|c\|_\gamma, \quad (20)$$

where $\alpha, \beta, \gamma = -1, 0, 1, 2$ and $\alpha + \beta + \gamma = 3$.

Let $\mu, \tau \in \mathbb{R}$ and let $(a, b, c)_{\mu, \tau}$ be the following trilinear form.

$$(a, b, c)_{\mu, \tau} := \mu(a \cdot \nabla b, c) + \tau(b \cdot \nabla a, c), \quad \forall a, b, c \in H_0^1(\Omega)^d$$

We will introduce the following notation for the most commons constants:

$$\begin{aligned} M_{\mu,\tau} &:= C(|\mu| + |\tau|), \\ M_{\nu,\lambda,\mu,\tau}(a) &:= \nu + \lambda + M_{\mu,\tau}\|a\|_1, \\ M_{\lambda,\mu,\tau}(a) &:= \lambda + M_{\mu,\tau}\|a\|_2. \end{aligned} \quad (21)$$

Proposition 2 Let $a, b, c \in H_0^1(\Omega)^d$ (or $H^2(\Omega)^d \cap H_0^1(\Omega)^d$) depending on the regularity required. Then

$$|(a, b, c)_{\mu,\tau}| \leq M_{\mu,\tau}\|a\|_\alpha\|b\|_\beta\|c\|_\gamma \quad (22)$$

where $\alpha, \beta, \gamma = -1, 0, 1, 2$ and $\alpha + \beta + \gamma = 3$ and $M_{\mu,\tau}$ is defined in (21).

Proof It is an immediate consequence of Proposition 1. \square

Now, we define a bilinear form depending on four parameters

$$(a; b, c)_{\nu,\lambda,\mu,\tau} := \nu(\nabla b, \nabla c) + \lambda(b, c) + (a, b, c)_{\mu,\tau}, \quad (23)$$

which is continuous on $H_0^1(\Omega)^d$ by applying Proposition 2 with $\alpha = \beta = \gamma = 1$:

$$|(a; b, c)_{\nu,\lambda,\mu,\tau}| \leq M_{\nu,\lambda,\mu,\tau}(a)\|b\|_1\|c\|_1. \quad (24)$$

Proposition 3 If a belongs to $H^2(\Omega)^d$ and b, c are regular enough such that all the appearing norms are bounded then

$$|(a; b, c)_{0,\lambda,\mu,\tau}| \leq M_{\lambda,\mu,\tau}(a)\|b\|_\alpha\|c\|_\beta, \quad (25)$$

where $\alpha, \beta = -1, 0, 1, 2$ and $\alpha + \beta = 1$ and $M_{\lambda,\mu,\tau}(a)$ is defined in (21).

Proof This is a consequence of Proposition 2. \square

Coercivity of $(a; b, c)_{\nu,\lambda,\mu,\tau}$ comes from the following bounds

$$\begin{aligned} |\mu(a \cdot \nabla b, b)| &= \left| -\frac{1}{2}\mu((\nabla \cdot a)b, b) \right| \leq \frac{d}{2}|\mu|\|\nabla a\|_\infty\|b\|_0^2 \\ |\tau((b \cdot \nabla)a, b)| &\leq |\tau|\|\nabla a\|_\infty\|b\|_0^2. \end{aligned}$$

Assuming that $\|\nabla a\|_\infty$ is finite and that

$$\lambda \geq \lambda_0 = \left(\frac{d}{2}|\mu| + |\tau| \right) \|\nabla a\|_\infty, \quad (26)$$

we get that $(a; b, c)_{\nu,\lambda,\mu,\tau}$ is coercive.

$$(a; b, b)_{\nu,\lambda,\mu,\tau} \geq \nu\|\nabla b\|_0^2 + \left(\lambda - \left(\frac{d}{2}|\mu| + |\tau| \right) \|\nabla a\|_\infty \right) \|b\|_0^2 \geq \nu\|\nabla b\|_0^2. \quad (27)$$

Remark 4 Let us observe that in general $\|\nabla a\|_\infty$ could be not finite. However, if $a = u$ is the solution of the NS equations, then by (5) $\|\nabla u\|_\infty \leq C\|u\|_{5/2+\alpha}$ for $\alpha > 0$ and $d = 2, 3$ and then by (9)

$$\|\nabla u\|_\infty \leq C\|u\|_3 \leq CM_3 t^{-1/2}. \quad (28)$$

Finally, we want to remark that since the postprocessed step is solved at a fixed time t strictly positive then $\|\nabla u(t)\|_\infty \leq CM_3 t^{-1/2}$ is bounded.

To finish this section we define the material derivative of $a \in H_0^1(\Omega)^d$ as $Da = a_t + a \cdot \nabla a$. We will use the following result that it is easy to obtain

$$(Da_1 - Da_2, c) = (D(a_1 - a_2), c) + (a_1 - a_2, a_2, c)_{1,1}. \quad (29)$$

Finally, taking into account that

$$(a_1, b_1, c)_{\mu,\tau} - (a_2, b_2, c)_{\mu,\tau} = (a_2, b_1 - b_2, c)_{\mu,\tau} + (a_1 - a_2, b_1, c)_{\mu,\tau}$$

we get

$$\begin{aligned} (a_1; b_1, c)_{\nu,\lambda,\mu,\tau} - (a_2; b_2, c)_{\nu,\lambda,\mu,\tau} &= (a_2; b_1 - b_2, c)_{\nu,\lambda,\mu,\tau} \\ &\quad + (a_1 - a_2, b_1, c)_{\mu,\tau}. \end{aligned} \quad (30)$$

3 The generalized linear problem

In this section we consider the steady generalized linear problem that will define the postprocessing step of our method, see (51). Also, we get error bounds of the error for this steady problem.

Let us consider the following generalized problem where $\|\nabla a\|_\infty < \infty$ and $\varphi \in L^2(\Omega)^d$:

$$\left. \begin{aligned} -\nu \Delta v + \lambda v + \mu(a \cdot \nabla)v + \tau(v \cdot \nabla)a + \nabla j &= \varphi \\ \operatorname{div}(v) &= 0 \end{aligned} \right\} \text{ in } \Omega \quad (31)$$

$$v = 0, \quad \text{on } \partial\Omega.$$

The variational formulation of this problem is

$$\begin{aligned} (a; v, \phi)_{\nu,\lambda,\mu,\tau} + (\nabla j, \phi) &= (\varphi, \phi), \quad \forall \phi \in H_0^1(\Omega)^d, \\ (\nabla \cdot v, \psi) &= 0, \quad \forall \psi \in Q, \end{aligned}$$

where $(a; v, \phi)_{\nu,\lambda,\mu,\tau}$ is defined in (23). Assuming condition (26) holds existence and uniqueness of the generalized problem can be derived from the continuity and coercivity of the bilinear form over $V \subset H_0^1(\Omega)^d$ ((24) and (27)) applying the Lax-Milgram theorem and then the inf-sup condition (7), (see [15]). Following [14] we also have

$$\|v\|_2 + \|j\|_1 \leq C\|\varphi\|_0. \quad (32)$$

We define the dual problem of (31) as follows

$$\left. \begin{aligned} -\nu\Delta w + \lambda w - \mu(a \cdot \nabla)w - \mu(\nabla \cdot a)w + \tau(\nabla a)^T w + \nabla k &= \varphi, \\ \operatorname{div}(v) &= 0, \\ v &= 0, \end{aligned} \right\} \begin{array}{l} \text{in } \Omega \\ \\ \text{on } \partial\Omega. \end{array} \quad (33)$$

Let $(a; w, v)_{\nu, \lambda, \mu, \tau}^*$ be the bilinear form associated to the problem (33).

$$\begin{aligned} (a; w, v)_{\nu, \lambda, \mu, \tau}^* &:= \nu(\nabla w, \nabla v) + \lambda(w, v) - \mu((a \cdot \nabla)w, v) \\ &\quad - \mu((\nabla \cdot a)w, v) + \tau((\nabla a)^T w, v). \end{aligned}$$

By definition of $(a; w, v)_{\nu, \lambda, \mu, \tau}^*$ integrating by parts it is easy to obtain

$$(a; w, v)_{\nu, \lambda, \mu, \tau}^* = (a; v, w)_{\nu, \lambda, \mu, \tau}.$$

The dual form is also continuous and coercive and in the sequel we shall assume the regularity conditions for both the generalized problem and its dual one:

$$\|w\|_2 + \|k\|_1 \leq C\|\varphi\|_0. \quad (34)$$

3.1 Error bounds for the approximation to the linear generalized problem

Let u be the velocity of the Navier-Stokes equations (1) and assume that the solution (v, j) of (31) with $a = u(t)$, $t > 0$, is regular enough. We will denote by (s_h, q_h) its Stokes projection on $(X_{h,r}, Q_{h,r-1})$, that is defined by (see (12))

$$\begin{aligned} (u; s_h, \phi_h)_{\nu, 0, 0, 0} + (\nabla q_h, \phi_h) &= (u; v, \phi_h)_{\nu, 0, 0, 0} + (\nabla j, \phi_h) \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot s_h, \psi_h) &= 0, \quad \forall \psi_h \in Q_{h,r-1}. \end{aligned} \quad (35)$$

From (13) and (14) the following bounds hold:

$$\begin{aligned} \|v - s_h\|_0 + h\|v - s_h\|_1 &\leq Ch^m (\|v\|_m + \|j\|_{m-1}), \\ \|j - q_h\|_0 &\leq Ch^{m-1} (\|v\|_m + \|j\|_{m-1}). \end{aligned} \quad (36)$$

Let us now consider the mixed finite element approximation to (31) with $a = u(t)$, $t > 0$: $(v_h, j_h) \in (X_{h,r}, Q_{h,r-1})$ satisfying

$$\begin{aligned} (u; v_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla j_h, \phi_h) &= (\varphi, \phi_h), \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot v_h, \psi_h) &= 0, \quad \forall \psi_h \in Q_{h,r-1}. \end{aligned} \quad (37)$$

According to Remark 4, $\|\nabla u\|_\infty$ is finite for any $t > 0$. Assuming λ satisfies (26), the bilinear form associated to problem (37) is continuous and coercive over the discrete divergence free subspace $V_{h,r}$. From Lax-Milgram theorem and the discrete inf-sup condition (10) we can prove that (37) has a unique solution.

In the following lemma we bound the error between the solution of (31) and its mixed finite element approximation solving (37).

Lemma 1 *Let (v, j) be the solution of the linearized problem (31) with $a = u(t)$, $t > 0$, u being the velocity in the Navier-Stokes equations (1). Assume that λ satisfies (26). Then, the following bounds hold for $2 \leq m \leq r$*

$$\|v - v_h\|_0 + h\|v - v_h\|_1 \leq Ch^m (\|v\|_m + \|j\|_{m-1}), \quad (38)$$

$$\|j - j_h\|_0 \leq Ch^{m-1} (\|v\|_m + \|j\|_{m-1}). \quad (39)$$

Proof Consider the variational formulation of (31) on $(X_{h,r}, Q_{h,r-1})$.

$$\begin{aligned} (u; v, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla j, \phi_h) &= (\varphi, \phi_h), \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot v, \psi_h) &= 0, \quad \forall \psi_h \in Q_{h,r-1}, \end{aligned} \quad (40)$$

and subtract (37) from (40) to obtain the Galerkin orthogonality condition

$$\begin{aligned} (u, v - v_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla(j - j_h), \phi_h) &= 0, \quad \forall \phi_h \in X_{h,r}, \\ (\nabla \cdot (v - v_h), \psi_h) &= 0, \quad \forall \psi_h \in Q_{h,r-1}. \end{aligned} \quad (41)$$

The error can be decompose as

$$\begin{aligned} v - v_h &= (v - s_h) + (s_h - v_h), \\ j - j_h &= (j - q_h) + (q_h - j_h), \end{aligned} \quad (42)$$

where (s_h, q_h) is the Stokes projection of (v, j) , (35). We denote the velocity error by $e_h = s_h - v_h$ and by $r_h = q_h - j_h$ the pressure error. The first terms on the right-hand side of (42) are bounded by (36). For the second terms, we rewrite (41) to get

$$(u; e_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla r_h, \phi_h) = (u; s_h - v, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla(q_h - j), \phi_h),$$

and then, by (35) and (25) with $\alpha = 0$ and $\beta = 1$ we get

$$\begin{aligned} (u; e_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla r_h, \phi_h) &= (u; s_h - v, \phi_h)_{\nu, \lambda, \mu, \tau} - (u; s_h - v, \phi_h)_{\nu, 0, 0, 0} \\ &= (u; s_h - v, \phi_h)_{0, \lambda, \mu, \tau} \\ &\leq M_{\lambda, \mu, \tau}(u) \|s_h - v\|_0 \|\phi_h\|_1. \end{aligned} \quad (43)$$

We take now $\phi_h = e_h$ and apply the coercivity condition (27). Then

$$\begin{aligned} \|e_h\|_1 &\leq \frac{M_{\lambda, \mu, \tau}(u)}{\nu} \|s_h - v\|_0 \\ &\leq \frac{M_{\lambda, \mu, \tau}(u)}{\nu} Ch^m (\|v\|_m + \|j\|_{H^{m-1}/\mathbb{R}}), \end{aligned} \quad (44)$$

where in the last inequality we have applied (36).

To bound the error in the pressure, we get from (43) and (24), (25) and (44)

$$\begin{aligned}
|(\nabla r_h, \phi_h)| &\leq |(u; e_h, \phi_h)_{\nu, \lambda, \mu, \tau}| + |(u; s_h - v, \phi_h)_{0, \lambda, \mu, \tau}| \\
&\leq M_{\nu, \lambda, \mu, \tau}(u) \|e_h\|_1 \|\phi_h\|_1 \\
&\quad + M_{\lambda, \mu, \tau}(u) \|s_h - v\|_0 \|\phi_h\|_1 \\
&\leq M_{\nu, \lambda, \mu, \tau}(u) \frac{M_{\lambda, \mu, \tau}(u)}{\nu} \|s_h - v\|_0 \|\phi_h\|_1 \\
&\quad + M_{\lambda, \mu, \tau}(u) \|s_h - v\|_0 \|\phi_h\|_1 \\
&\leq \left(\frac{M_{\nu, \lambda, \mu, \tau}(u)}{\nu} + 1 \right) M_{\lambda, \mu, \tau}(u) \|s_h - v\|_0 \|\phi_h\|_1.
\end{aligned}$$

From the inf-sup condition (10) and applying (36) again we get

$$\|r_h\|_0 \leq \frac{\nu + M_{\nu, \lambda, \mu, \tau}(u)}{\nu} \frac{M_{\lambda, \mu, \tau}(u)}{\beta} Ch^m (\|v\|_m + \|j\|_{m-1}).$$

Going back to (42) and applying (36) we complete the proof of the bound for the H^1 norm in (38) and the bound (39).

The convergence in the L^2 norm is deduced from the super-convergence of the H^1 norm:

$$\|e_h\|_0 \leq \|e_h\|_1 \leq \frac{M_{\lambda, \mu, \tau}(u)}{\nu} Ch^m (\|v\|_m + \|j\|_{m-1}),$$

so that the proof is finished. \square

Remark 5 We observe that we can remove the parameters in the error bound of $\|e_h\|_1$ taking in (44) $h < \nu/M_{\lambda, \mu, \tau}(u)$ so that

$$\|e_h\|_1 \leq Ch^{m-1} (\|v\|_m + \|j\|_{m-1}).$$

However, writing the bound in this way the dependence on the parameters is hidden in the requirement on the value of h .

4 The generalized two-grid method

In this section we get a bound for the error of the generalized two-grid method. We first state some results that will be used to prove that bound.

Let us consider $(u_H; v, w)_{\nu, \lambda, \mu, \tau}$ where $v, w \in H_0^1(\Omega)^d$ and u_H is the mixed finite-element approximation to u defined in (16)-(17). Continuity of $(u_H; v, w)_{\nu, \lambda, \mu, \tau}$ can be proved in the following way. By (24) we get

$$(u_H; v, w)_{\nu, \lambda, \mu, \tau} \leq M_{\nu, \lambda, \mu, \tau}(u_H) \|v\|_1 \|w\|_1. \quad (45)$$

Let us observe that $M_{\nu, \lambda, \mu, \tau}(u_H)$ is finite since $\|u_H\|_1$ is bounded. More precisely, applying (18) with $r = 2$ we get

$$\|u_H\|_1 \leq \|u_H - u\|_1 + \|u\|_1 \leq CH + \|u\|_1 \leq C.$$

We will show that $\|\nabla u_H(t)\|_\infty$ is also bounded for any $t > 0$. Let s_h be the Stokes projection of the velocity u of the Navier-Stokes equations, (12). Applying (11), (15) and (28), we get

$$\begin{aligned}\|\nabla u_H\|_\infty &\leq \|\nabla(u_H - s_H)\|_\infty + \|\nabla s_H\|_\infty \\ &\leq CH^{-d/2}\|\nabla(u_H - s_H)\|_0 + C\|\nabla u\|_\infty \\ &\leq CH^{-d/2}\|\nabla(u_H - s_H)\|_0 + CM_3t^{-1/2}.\end{aligned}$$

Taking into account the super-convergence between the Stokes projection and the Galerkin approximation to the velocity, proved in [10, Remark 4.2], we get

$$CH^{-d/2}\|\nabla(u_H - s_H)\|_0 \leq CH^{-d/2}H^2|\log(H)|^2 \leq C,$$

with C depending on M_2 in (8). And then $\|\nabla u_H(t)\|_\infty \leq C + CM_3t^{-1/2}$. Now, from (26) and (27) for $t > 0$ and

$$\lambda \geq \lambda_0(t) = \left(\frac{d}{2}|\mu| + |\tau|\right) \|\nabla u_H(t)\|_\infty, \quad (46)$$

we get the coercivity of the bilinear form

$$(u_H; v, v)_{\nu, \lambda, \mu, \tau} \geq \nu \|v\|_1^2, \quad \forall v \in H_0^1(\Omega)^d. \quad (47)$$

In the following lemma the time derivative of the Galerkin error (using different norms) and the H^{-1} norm of the Galerkin error are bounded. We refer the reader to [12, Lemma 4] for the case $r = 2$, [10, Lemma 5.1] for $r = 3, 4$ and to [10, p. 226].

Lemma 2 *Let (u, p) and u_H be the solution of (1) and its velocity approximation defined in (16)-(17), respectively. Then, the following bounds hold for $t \in (0, T]$*

$$\|u_t(t) - \dot{u}_H(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, \quad r = 2, 3, 4, \quad (48)$$

$$\|A^{-1}\Pi(u_t(t) - \dot{u}_H(t))\|_0 \leq \frac{C}{t^{(r-1)/2}} H^{r+1} |\log(H)|, \quad r = 3, 4, \quad (49)$$

$$\|u(t) - u_H(t)\|_{-1} \leq \frac{C}{t^{(r-1)/2}} H^{r+1} |\log(H)|, \quad r = 3, 4, \quad (50)$$

where in (48) $r' = 2$ in the case $r = 2$ and $r' = 1$ otherwise.

Next, we write the equations of the postprocessed method. As explained before, we have a two-level or two-grid method in which we have a coarse mesh of size H and a fine mesh of size h with $H > h$. The first step is to compute the Galerkin approximation (u_H, p_H) solving equations (16)-(17) up to time $t > 0$. Then, the second step is to postprocess the approximation. To this end, we solve the following linear generalized problem: find $(\tilde{u}_h(t), \tilde{p}_h(t)) \in (X_{h,r}, Q_{h,r-1})$, such that for all $\phi_h \in X_{h,r}$ and $\psi_h \in Q_{h,r-1}$ it holds

$$\begin{aligned}(u_H; \tilde{u}_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla \tilde{p}_h, \phi_h) &= (f - Du_H, \phi_h) + (u_H; u_H, \phi_h)_{0, \lambda, \mu, \tau} \\ (\nabla \cdot \tilde{u}_h, \psi_h) &= 0.\end{aligned} \quad (51)$$

We can also solve equations (51) over a higher order space over the same grid. For example, using the Hood-Taylor pair P_2/P_1 to postprocess the mini-element, the Hood-Taylor pair P_3/P_2 to postprocess the Hood-Taylor pair P_2/P_1 and so on. However, for simplicity, we consider in the following theorem only the case in which we refine the mesh at the second level. Also, for the numerical experiments of Section 5 we have chosen to refine the mesh since analogous results can be obtained increasing the degrees of the local polynomials. We refer the reader to reference [3] in which, for the Stokes postprocessing, numerical experiments increasing the degree of the polynomials at the postprocessing step can be found.

Next theorem proves an error bound for the generalized postprocessed approximation.

Theorem 1 *Let (u, p) be the solution of the Navier-Stokes equations (1) and let $(\tilde{u}_h, \tilde{p}_h)$ be its postprocessed approximation defined in (51). Then, for any positive time $0 < t \leq T$ and for λ big enough so that both (26) with $a = u(t)$ and (46) hold, the following bounds hold:*

$$\|u(t) - \tilde{u}_h(t)\|_1 + \|p(t) - \tilde{p}_h(t)\|_0 \leq \frac{C}{t^{(r-2)/2}} h^{r-1} \quad (52)$$

$$+ \frac{C}{t^{(r-1)/2}} H^r |\log(H)|^{r'}, \quad r = 2, 3, 4,$$

$$\|u(t) - \tilde{u}_h(t)\|_0 \leq \frac{C}{t^{(r-2)/2}} h^r \quad (53)$$

$$+ \frac{C}{t^{(r-1)/2}} H^{r+1} |\log(H)|, \quad r = 3, 4,$$

where $r' = 2$ in the case $r = 2$ and $r' = 1$ otherwise.

Proof We consider the auxiliary problem (31) with $a = u$ and $\varphi = f - Du + \lambda u + (\mu + \tau)(u \cdot \nabla)u$ (for simplicity we omit the dependence on t) and we observe that the solution (u, p) of the Navier-Stokes equations (1) is the unique solution of problem (31) (for any fixed time t), i.e. it holds:

$$\begin{aligned} (u; u, \phi)_{\nu, \lambda, \mu, \tau} + (\nabla j, \phi) &= (f - Du, \phi) + (u; u, \phi)_{0, \lambda, \mu, \tau} \\ (\nabla \cdot u, \psi) &= 0, \end{aligned}$$

for all $\phi \in H_0^1(\Omega)^d$ and $\psi \in Q$. We will denote by $(v_h, j_h) \in (X_{h,r}, Q_{h,r-1})$ its mixed finite-element approximation satisfying

$$\begin{aligned} (u; v_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla j_h, \phi_h) &= (f - Du, \phi_h) + (u; u, \phi_h)_{0, \lambda, \mu, \tau} \\ (\nabla \cdot v_h, \psi_h) &= 0, \end{aligned} \quad (54)$$

for all $(\phi_h, \psi_h) \in (X_{h,r}, Q_{h,r})$. We decompose

$$\begin{aligned} u - \tilde{u}_h &= (u - v_h) + (v_h - \tilde{u}_h) \\ p - \tilde{p}_h &= (p - j_h) + (j_h - \tilde{p}_h) \end{aligned} \quad (55)$$

and we denote by $\tilde{e}_h = v_h - \tilde{u}_h$ and by $\tilde{r}_h = j_h - \tilde{p}_h$. To bound the first terms on the right-hand side of (55) we can apply Lemma 1 (since λ satisfies (26) with

$a = u(t)$. For the second terms on the right-hand side of (55) we subtract (51) from (54) and apply (30) and (29) to obtain

$$\begin{aligned} (u_H; \tilde{e}_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (u - u_H, v_h, \phi_h)_{\mu, \tau} + (\nabla \tilde{r}_h, \phi_h) &= (D(u_H - u), \phi_h) \\ &+ (u_H - u, u, \phi_h)_{1,1} + (u_H; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} \\ &+ (u - u_H, u, \phi_h)_{\mu, \tau} \end{aligned} \quad (56)$$

We rewrite

$$\begin{aligned} (u_H - u, u, \phi_h)_{1,1} + (u_H; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} &= -(u, u - u_H, \phi_h)_{1,1} \\ &+ (u_H - u; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} + (u, u - u_H, \phi_h)_{\mu, \tau} \\ &= (u_H - u; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} + (u, u - u_H, \phi_h)_{\mu-1, \tau-1}. \end{aligned}$$

Inserting the above expression in (56) we get

$$\begin{aligned} (u_H; \tilde{e}_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla \tilde{r}_h, \phi_h) &= (D(u_H - u), \phi_h) \\ &+ (u - u_H, u - v_h, \phi_h)_{\mu, \tau} + (u_H - u; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} \\ &+ (u; u - u_H, \phi_h)_{\mu-1, \tau-1}. \end{aligned} \quad (57)$$

For the first term on the right-hand side of (57) we apply (20) with $\alpha = \beta = \gamma = 1$, (48) and (18) twice with $r = 2$ for one of the terms to get

$$\begin{aligned} (D(u_H - u), \phi_h) &\leq (\dot{u}_H - u_t, \phi_h) + ((u_H - u) \cdot \nabla (u_H - u), \phi_h) \\ &\leq \|u_t - \dot{u}_H\|_{-1} \|\phi_h\|_1 + C \|u - u_H\|_1^2 \|\phi_h\|_1 \\ &\leq \frac{CH^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1 + \frac{CH^r}{t^{(r-2)/2}} \|\phi_h\|_1. \end{aligned} \quad (58)$$

For the second term on the right-hand side of (57), (22) with $\alpha = \beta = \gamma = 1$, (18) and (38) with $r = 2$ leads to

$$\begin{aligned} (u - u_H, u - v_h, \phi_h)_{\mu, \tau} &\leq M_{\mu, \tau} \|u - u_H\|_1 \|u - v_h\|_1 \|\phi_h\|_1 \\ &\leq M_{\mu, \tau} \frac{CH^{r-1}}{t^{(r-2)/2}} Ch \|u\|_2 \|\phi_h\|_1. \end{aligned} \quad (59)$$

To bound the third term on the right-hand side of (57) we apply again (22) with $\alpha = \beta = \gamma = 1$ to obtain

$$\begin{aligned} (u_H - u; u - u_H, \phi_h)_{0, \lambda, \mu, \tau} &\leq \lambda \|u - u_H\|_{-1} \|\phi_h\|_1 + M_{\mu, \tau} \|u - u_H\|_1^2 \|\phi_h\|_1 \\ &\leq \lambda \frac{CH^r}{t^{(r-2)/2}} \|\phi_h\|_1 + M_{\mu, \tau} \frac{CH^r}{t^{(r-2)/2}} \|\phi_h\|_1, \end{aligned} \quad (60)$$

where we have bounded the $\|\cdot\|_{-1}$ norm by the $\|\cdot\|_0$ norm and used (18) and we have bounded the second term on the right-hand side as the second term on the right-hand side in (58).

Finally, for the last term on the right-hand side of (57) by (22) with $\alpha = 2$, $\beta = 0$ and $\gamma = 1$ and applying (18) again we obtain

$$\begin{aligned} (u, u - u_H, \phi_h)_{\mu-1, \tau-1} &\leq M_{\mu-1, \tau-1} \|u\|_2 \|u - u_H\|_0 \|\phi_h\|_1 \\ &\leq M_{\mu-1, \tau-1} \frac{CH^r}{t^{(r-2)/2}} \|\phi_h\|_1. \end{aligned} \quad (61)$$

Inserting (58), (59), (60) and (61) into (57) and bounding terms by the biggest one we get

$$(u_H; \tilde{e}_h, \phi_h)_{\nu, \lambda, \mu, \tau} + (\nabla \tilde{r}_h, \phi_h) \leq \frac{CK_{\mu, \tau} H^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1, \quad (62)$$

where

$$K_{\mu, \tau} = 1 + \lambda + M_{\mu, \tau} + M_{\mu-1, \tau-1}.$$

Taking $\phi_h = \tilde{e}_h$ in (62) and applying (47) we get

$$\|\tilde{e}_h\|_1 \leq C \frac{K_{\mu, \tau} H^r}{\nu t^{(r-1)/2}} |\log(H)|^{r'}, \quad r = 2, 3, 4. \quad (63)$$

For the pressure, rewriting (62) and applying (24) and (63) we obtain

$$\begin{aligned} |(\nabla \tilde{r}_h, \phi_h)| &= |(u_H; \tilde{e}_h, \phi_h)_{\nu, \lambda, \mu, \tau}| + \frac{CK_{\mu, \tau} H^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1 \\ &\leq M_{\nu, \lambda, \mu, \tau} (u_H) \|\tilde{e}_h\|_1 \|\phi_h\|_1 + \frac{CK_{\mu, \tau} H^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1 \\ &\leq \left(\frac{M_{\nu, \lambda, \mu, \tau} (u_H)}{\nu} + 1 \right) \frac{CK_{\mu, \tau} H^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1. \end{aligned}$$

Finally, applying inf-sup condition (10) we reach

$$\|\tilde{r}_h\|_0 \leq \frac{1}{\beta} \left(\frac{M_{\nu, \lambda, \mu, \tau} (u_H)}{\nu} + 1 \right) \frac{CK_{\mu, \tau} H^r}{t^{(r-1)/2}} |\log(H)|^{r'} \|\phi_h\|_1.$$

Applying the triangle inequality together with (38) and (39) we conclude (52).

In order to get a bound for the L^2 norm of the error in the velocity for $r = 3, 4$ we will use a duality argument. We have

$$\|\tilde{e}_h\|_0 = \sup_{\varphi^* \in L^2(\Omega)^d, \varphi^* \neq 0} \frac{|(\tilde{e}_h, \varphi^*)|}{\|\varphi^*\|_0}.$$

Now, we consider the dual problem:

$$\begin{aligned} -\nu \Delta v + \lambda v - \mu(u_H \cdot \nabla)v - \mu(\nabla \cdot u_H)v + \tau(\nabla u_H)^T v + \nabla j &= \varphi^*, \\ \operatorname{div}(v) &= 0, \\ v &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad \left. \vphantom{\begin{aligned} -\nu \Delta v + \lambda v - \mu(u_H \cdot \nabla)v - \mu(\nabla \cdot u_H)v + \tau(\nabla u_H)^T v + \nabla j \\ \operatorname{div}(v) \\ v \end{aligned}} \right\} \text{ in } \Omega$$

for $\varphi^* \in L^2(\Omega)^d$. By (34) we have

$$\|v\|_2 + \|j\|_1 \leq C \|\varphi^*\|_0. \quad (64)$$

For the rest of the proof we denote by (s_h, q_h) the Stokes projection of (v, j) , (12). Then,

$$\begin{aligned} (\tilde{e}_h, \varphi^*) &= (u_H; v, \tilde{e}_h)_{\nu, \lambda, \mu, \tau}^* + (\nabla j, \tilde{e}_h) \\ &= (u_H; \tilde{e}_h, v)_{\nu, \lambda, \mu, \tau} - (j - q_h, \nabla \cdot \tilde{e}_h) \\ &= (u_H; \tilde{e}_h, v - s_h)_{\nu, \lambda, \mu, \tau} - (j - q_h, \nabla \cdot \tilde{e}_h) + (u_H; \tilde{e}_h, s_h)_{\nu, \lambda, \mu, \tau}, \end{aligned}$$

and we apply (57) for the last term with $\phi_h = s_h$ to obtain

$$\begin{aligned} (\tilde{e}_h, \varphi^*) &= (u_H; \tilde{e}_h, v - s_h)_{\nu, \lambda, \mu, \tau} - (j - q_h, \nabla \cdot \tilde{e}_h) \\ &\quad + (D(u_H - u), s_h) + (u - u_H, u - v_h, s_h)_{\mu, \tau} \\ &\quad + (u_H - u; u - u_H, s_h)_{0, \lambda, \mu, \tau} + (u; u - u_H, s_h)_{\mu-1, \tau-1}. \end{aligned} \quad (65)$$

For the first and second terms on the right-hand side of (65) by (45), (63), (36) with $r = 2$ and (64) we obtain

$$\begin{aligned} (u_H; \tilde{e}_h, v - s_h)_{\nu, \lambda, \mu, \tau} &\leq M_{\nu, \lambda, \mu, \tau}(u_H) \|\tilde{e}_h\|_1 \|v - s_h\|_1 \\ &\quad M_{\nu, \lambda, \mu, \tau}(u_H) \frac{K_{\mu, \tau} H^r}{\nu t^{(r-1)/2}} |\log(H)| Ch \|\varphi^*\|_0, \quad (66) \\ (j - q_h, \nabla \cdot \tilde{e}_h) &\leq \|j - q_h\|_{L^2/\mathbb{R}} \|\tilde{e}_h\|_1 \\ &\quad \frac{K_{\mu, \tau} H^r}{\nu t^{(r-1)/2}} |\log(H)| Ch \|\varphi^*\|_0. \quad (67) \end{aligned}$$

To bound the third term on the right-hand side of (65) we argue as in (58)

$$(D(u - u_H), s_h) \leq (u_t - \dot{u}_H, s_h) + C \|(u - u_H)_1^2\| s_h\|_1. \quad (68)$$

For the first term on the right-hand side of (68) we apply (48), (49), (36) and (64) to obtain

$$\begin{aligned} |(\dot{u}_H - u_t, s_h)| &= |(\dot{u}_H - u_t, s_h - v) + (\dot{u}_H - u_t, v)| \\ &\leq C \|u_t - \dot{u}_H\|_{-1} \|s_h - v\|_1 + C \|A^{-1} \Pi(\dot{u}_H - u_t)\|_0 \|Av\|_0 \\ &\leq C \|u_t - \dot{u}_H\|_{-1} h \|\varphi^*\|_0 + C \|A^{-1} \Pi(\dot{u}_H - u_t)\|_0 \|\varphi^*\|_0 \\ &\leq \frac{C}{t^{(r-1)/2}} H^{r+1} |\log(H)| \|\varphi^*\|_0 \end{aligned} \quad (69)$$

For the second term on the right-hand side of (68) we first observe that for $r = 3$ we get from (18) $\|u - u_H\|_1 \leq \frac{CH^2}{t^{1/2}}$ and then, applying (18) again we obtain for $r = 3, 4$

$$\|u - u_H\|_1^2 \leq \frac{CH^{r+1}}{t^{(r-1)/2}}. \quad (70)$$

Inserting (69) and (70) into (68) we finally get

$$(D(u - u_H), s_h) \leq \frac{CH^{r+1}}{t^{(r-1)/2}} |\log(H)| \|s_h\|_1 + \frac{CH^{r+1}}{t^{(r-1)/2}} \|s_h\|_1. \quad (71)$$

For the fourth term on the right-hand side of (65) taking into account that from (38) with $r = 3$ and (9) we get $\|u - v_h\|_1 \leq Ch^2 \|u\|_3 \leq CM_3 h^2 t^{-1/2}$ and then using (22) and (18) we obtain

$$\begin{aligned} (u - u_H, u - v_h, s_h)_{\mu, \tau} &\leq M_{\mu, \tau} \|u_H - u\|_1 \|u - v_h\|_1 \|s_h\|_1 \\ &\leq M_{\mu, \tau} \frac{CH^{r-1}}{t^{(r-1)/2}} Ch^2 \|s_h\|_1. \end{aligned} \quad (72)$$

For the fifth term on the right-hand side of (65) we can apply (50) which is valid for $r = 3, 4$. Then, (22) and (70) give

$$\begin{aligned} (u_H - u; u - u_H, s_h)_{0,\lambda,\mu,\tau} &\leq \lambda \|u - u_H\|_{-1} \|s_h\|_1 + M_{\mu,\tau} \|u - u_H\|_1^2 \|s_h\|_1 \quad (73) \\ &\leq \lambda \frac{CH^{r+1}}{t^{(r-1)/2}} |\log(H)| \|s_h\|_1 + M_{\mu,\tau} \frac{CH^{r+1}}{t^{(r-1)/2}} \|s_h\|_1. \end{aligned}$$

Finally, the last term on the right-hand side of (65) is bounded using (22) with $\alpha = 2$, $\beta = -1$ and $\gamma = 2$, (18), (50) and (64)

$$\begin{aligned} (u, u - u_H, s_h)_{\mu-1,\tau-1} &= (u, u - u_H, s_h - v)_{\mu-1,\tau-1} + (u, u - u_H, v)_{\mu-1,\tau-1} \\ &\leq M_{\mu-1,\tau-1} \|u\|_2 \|u - u_H\|_0 \|s_h - v\|_1 + M_{\mu-1,\tau-1} \|u - u_H\|_{-1} \|v\|_2 \\ &\leq M_{\mu-1,\tau-1} \frac{CH^r}{t^{(r-1)/2}} \|\varphi^*\|_0 \left(\frac{1}{t^{1/2}} Ch + H |\log(H)| \right). \quad (74) \end{aligned}$$

Taking into account that applying (64) again $\|s_h\|_1 \leq \|v\|_1 \leq \|v\|_2 \leq C \|\varphi^*\|_0$ and inserting (66), (67), (71), (72), (73) and (74) in (65) we obtain

$$\|\tilde{e}_h\|_0 \leq CK_{\nu,\lambda,\mu,\tau} t^{(r-1)/2} H^{r+1} |\log(H)|,$$

where

$$K_{\nu,\lambda,\mu,\tau} = M_{\nu,\lambda,\mu,\tau}(u_H) \frac{K_{\mu,\tau}}{\nu} + \frac{K_{\mu,\tau}}{\nu} + 1 + M_{\mu,\tau} + \lambda + M_{\mu-1,\tau-1}.$$

Applying triangle inequality together with (38) we conclude (53) \square

Remark 6 Let us observe that for $\mu, \tau = 1$ (Newton postprocessing) the last term in (57) and (65) is zero so that in some sense this selection of the parameters give an optimal method. This result is in agreement with the numerical experiments of [8] where, comparing with the Stokes and Oseen postprocessing, the Newton postprocessing produces the smaller errors in most of the examples.

5 Numerical experiments

In this section, we show some numerical experiments to check the rate of convergence we have proved for the generalized postprocessed method. Also, we compare the results obtained for different values of the parameters trying to find out which values work better in practice so that they can be recommended for users. For the experiments we consider a regular triangulation of $\Omega = [0, 1] \times [0, 1]$ with N intervals at each direction and $H = 1/N$ the size of the coarse mesh. We use the mini-element for the spatial discretization and the Crank-Nicolson scheme as time integrator with a semi-implicit version in which the spatial derivatives are treated implicitly. For the numerical experiments we want to show the spatial errors. To this end, the time step is chosen small enough so that the temporal errors are much smaller than the spatial errors and then they are negligible compared to them. For the errors, although the full linear plus bubble part is

computed in the Galerkin approximation to the velocity, only the linear part is postprocessed. We refer the reader to [13] where the idea of postprocessing only the linear part was proposed.

Although in Theorem 1 we need to assume λ is big enough so that both (26) with $a = u(t)$ and (46) hold, in practice we did not find problems computing the postprocessed approximations for all the values of λ we tried.

We compute the forcing term $f(x, t)$ so that the solution of the Navier-Stokes equations is the following:

$$\begin{aligned} u &= \pi t \begin{pmatrix} \sin^2(\pi x) \sin(2\pi y) \\ -\sin^2(2\pi y) \sin(2\pi x) \end{pmatrix}, \\ p &= 5tx^2y. \end{aligned}$$

In the experiments we will take different values of the diffusion parameter in a range that goes from $\nu = 1$ to $\nu = 0.001$. We postprocess the Galerkin approximation at time $T = 1$. The size $h < H$ of the finer mesh is chosen small enough so that in Theorem 1 the first terms depending on h on the error bounds are negligible. In this experiment we take for the coarse mesh $N = 40, 50, 60$ and 70 and for the fine mesh $n = 130, 175, n = 223$ and $n = 273$, respectively.

In the first experiment, we study the rate of convergence of two postprocessed solutions. The values defining the methods are $\mu = \tau = 0.5$ and $\mu = 0, \tau = 1$. The postprocess with $\mu = \tau = 0.5$ shares with the Oseen postprocess the right-hand side of the problem being solved (see (2)) and with the Newton postprocess the condition $\mu = \tau$. The postprocess with $\mu = 0, \tau = 1$ is a purely reactive postprocess, let us observe that we have on the left hand side of (4) the term: $\mu(u_H \cdot \nabla)\tilde{u} + \tau(\tilde{u} \cdot \nabla)u_H$. In principle, there is no reason for choosing these values for the parameters since the method does not include a convective term and then $\nabla\tilde{u}$ is not re-computed at the postprocessed step. Moreover, in general ∇u_H has a bigger error than u_H and in this method we give the biggest weight to the term including ∇u_H . Figures 1, 2 and 3 represent the rate of convergence of these two methods for $H = 40, 50, 60, 70$ and $\nu = 0.1$ for the L^2 errors of the velocity, H^1 errors of the velocity and the L^2 errors of the pressure. In the figures the errors of the Galerkin method are plotted in black, and the postprocessed methods with $\lambda = 0, 2$ and 5 are plotted in red, blue and green, respectively.

Figures 4, 5 and 6 show the rate of convergence of the same methods for $\nu = 0.001$. The results are in agreement with the theory. As it was predicted for the mini-element ($r=2$), the rate of convergence of the postprocessed methods is equal to the Galerkin one for the L^2 norm of the velocity and increases in one order the rate of the Galerkin method in the H^1 norm of the velocity and the L^2 norm of the pressure. In this experiment the rate of convergence of the pressure in the Galerkin method is bigger than expected.

For $\nu = 0.1$ the postprocessed errors are smaller than the Galerkin errors for all the values shown in the figures and the method with $\mu = \tau = 0.5$ works better than the method with $\mu = 0$ and $\tau = 1$, as expected. Comparing the two postprocessed methods, although the choice $\mu = \tau = 0.5$ is better, specially for the H^1 errors of the velocity, there is not a big difference between the two

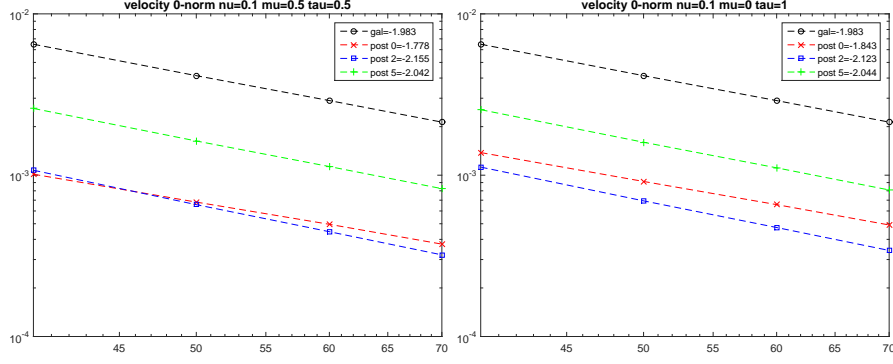


Figure 1: Velocity L^2 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.1$. On the left, postprocess with $\mu = \tau = 0.5$. On the right, postprocess with $\mu = 0, \tau = 1$.

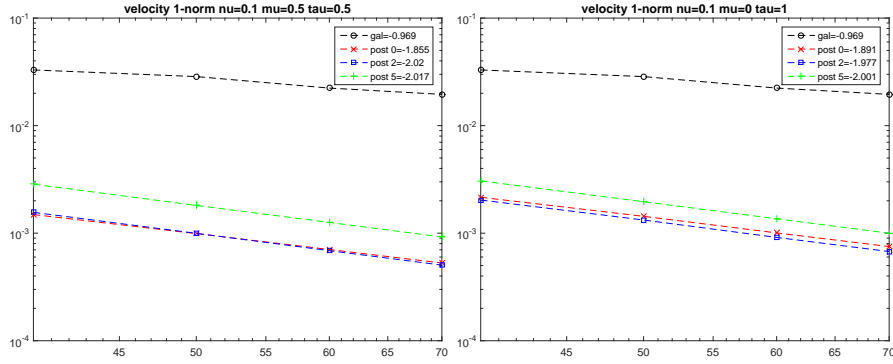


Figure 2: Velocity H^1 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.1$. On the left, postprocess with $\mu = \tau = 0.5$, on the right postprocess with $\mu = 0, \tau = 1$.

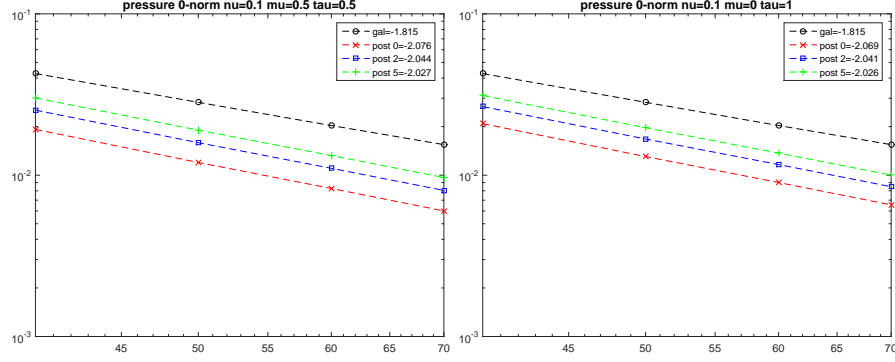


Figure 3: Pressure L^2 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.1$. On the left, postprocess with $\mu = \tau = 0.5$, on the right, postprocess with $\mu = 0, \tau = 1$.

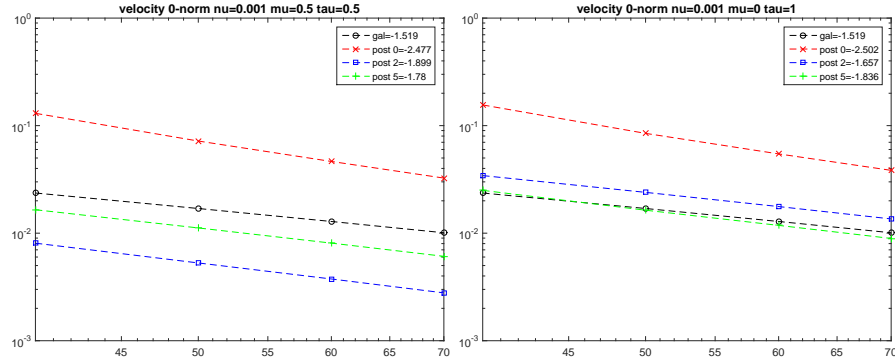


Figure 4: Velocity L^2 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.001$. On the left, postprocess with $\mu = \tau = 0.5$, on the right, postprocess with $\mu = 0, \tau = 1$.

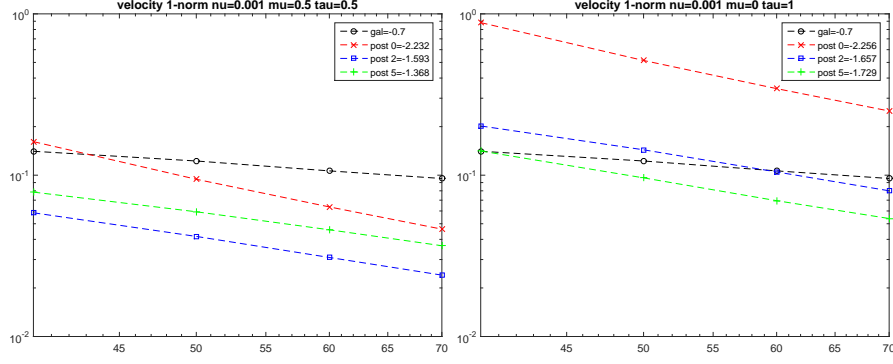


Figure 5: Velocity H^1 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.001$. On the left, postprocess with $\mu = \tau = 0.5$, on the right, postprocess $\mu = 0, \tau = 1$.

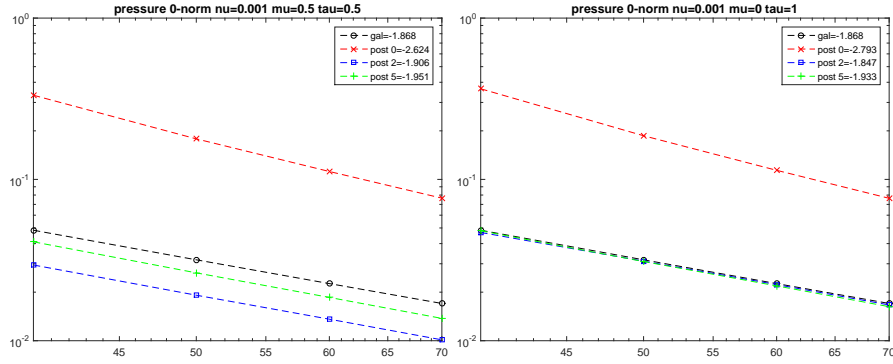


Figure 6: Pressure L^2 errors of the Galerkin approximation and the postprocessed solutions for $\lambda = 0, 2, 5$ and $\nu = 0.001$. On the left, postprocess with $\mu = \tau = 0.5$, on the right, postprocess $\mu = 0, \tau = 1$.

methods. Respect to the values of λ , the best errors correspond to $\lambda = 0$, which means that when the diffusion is not too small there is no need to add the term $\lambda \tilde{u}$ in equation (4). On the contrary, when the diffusion decreases, as we can observe in the experiments for $\nu = 0.001$, it is convenient to take a positive value of λ . In this case, $\lambda = 2$ is the best choice in most of the cases. We can also observe that for small values of the viscosity, since the errors in ∇u_H are bigger (compare the errors of the Galerkin method in Figures 2 and 5), there is a big difference between the postprocess with $\mu = \tau = 0.5$ and $\mu = 0, \tau = 1$, being the first one much better, as expected.

In the second experiment, we revise previous results about the comparison of known postprocessed methods, see the numerical experiments in [8], since we consider the case of adding the term $\lambda \tilde{u}$ with $\lambda > 0$ in the Stokes and Oseen postprocessed methods. Although this term was introduced because the Newton postprocess is not coercive, the term can also be added in the other methods. Let us observe that Theorem 1 can be applied for any values of the parameters and then, in particular, it covers the analysis of the Stokes and Oseen postprocessing with λ different from 0 (the case $\lambda = 0$ was analyzed in previous references, [10], [13]). Then, Theorem 1 proves that adding this term does not change the rate of convergence of the Stokes and Oseen postprocessed methods. Moreover, the term $\lambda \tilde{u}$ improves in some sense the coercivity of the bilinear form associated to the postprocessed methods since $\lambda(\tilde{u}, \tilde{u}) = \lambda \|\tilde{u}\|_0^2$ is always a positive term, although it does not improve the value of the coercivity constant in H^1 . We have checked in our numerical experiments that adding this term is advisable, especially for moderate to small values of the viscosity. To illustrate this fact, in Figures 7, 8 and 9 we show the results we have obtained for $\nu = 0.001$. We compare the errors of four postprocessed methods: Stokes, Oseen, Newton and the postprocessed method with $\mu = \tau = 0.5$. As before, the red colour is for postprocessing with $\lambda = 0$, blue for $\lambda = 2$ and green for $\lambda = 5$. In Figure 7 we represent the L^2 errors of the velocity, in Figure 8 the H^1 errors of the velocity and in Figure 9 the L^2 errors of the pressure. We can observe that adding the extra term increases a lot the accuracy of all the postprocessed methods and that $\lambda = 2$ is the best choice for this experiment. Moreover, for $\lambda = 0$ all the postprocessed methods produce bigger errors than the Galerkin method in the L^2 norm of the velocity and all but the Stokes postprocessing also in the L^2 norm of the pressure. The situation is the opposite in the H^1 norm of the velocity, as we can observe in Figure 8, where for $\lambda = 0$ the Stokes method is the only one obtaining worse errors than the Galerkin method. On the other hand, if we compare the different postprocessing procedures for the best choice $\lambda = 2$, we can observe that the Newton postprocessing and the postprocess with $\mu = \tau = 0.5$ work equally well for the velocity errors and better than the Stokes and Oseen postprocessing. For the pressure errors, Figure 9, these two methods are still better than the others but they do not produce the same errors. It is the Newton postprocessing the one producing smaller errors in the pressure.

As we know, the generalized two grid postprocessing method (4) has three parameters λ , μ and τ where λ is a nonnegative real parameter. The theory does not restrict the values of μ and τ and we assume that they are real. Once

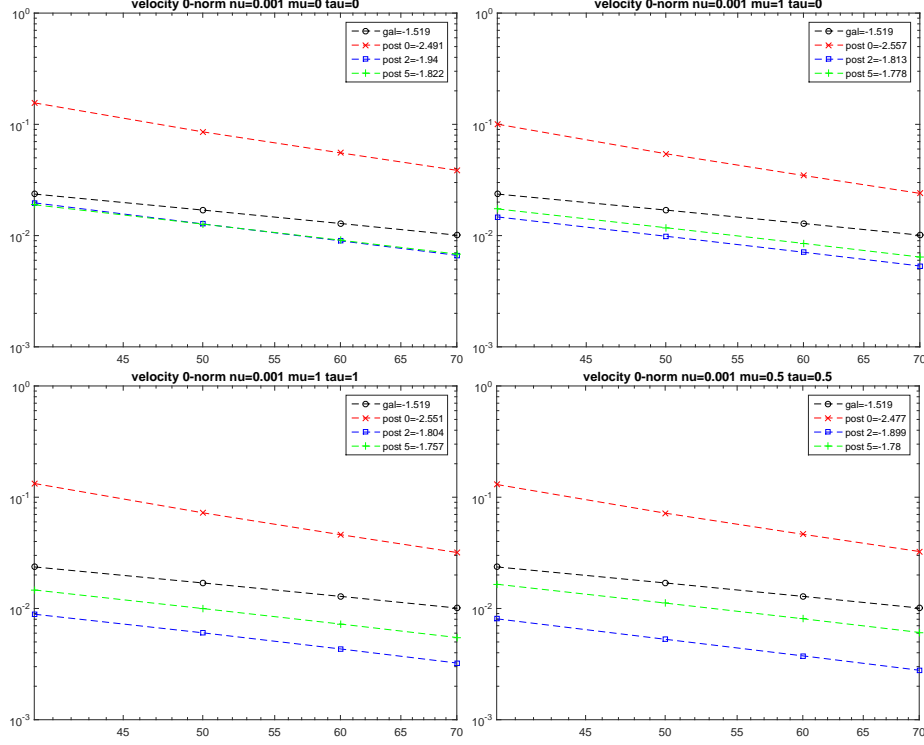


Figure 7: Velocity L^2 errors of the Galerkin approximation and the postprocessed solutions for $\nu = 0.001$ and $\lambda = 0, 2, 5$.

the parameter λ has been fixed, it is possible to calculate the postprocessed approximation for different values of μ and τ . Then, it is possible to build a function which assigns to each pair of values (μ, τ) , the error of the postprocessed approximation. This error function can be represented as a surface where Stokes, Oseen and Newton postprocess appear as notable points on it: those with coordinates $(0, 0)$, $(1, 0)$ and $(1, 1)$.

In Figures 10, 11 and 12 we represent the postprocess errors surface (μ, τ) for $\nu = 0.1$ and $\nu = 0.001$ and $\lambda = 2$, and for the L^2 errors of the velocity, the H^1 errors of the velocity and the L^2 errors of the pressure, respectively. The post-process surface is represented in the square $[0, 2] \times [0, 2]$ in which Stokes, Oseen and Newton postprocesses are marked by an asterisk. We take 10 equidistant values for both μ and τ in the interval $[0, 2]$ which gives a uniform mesh of size 0.2 at each direction. For $\nu = 0.1$ we have a regular surface in which we can observe that the Newton postprocessing is the best method and gives essentially the minimum error between the different possible choices of parameters. For $\nu = 0.001$ the surface is not so regular and the errors are much bigger. We can see that the errors increase as τ increases, in agreement with the previous results and comments for the method with $\mu = 0$ and $\tau = 1$ since for small values

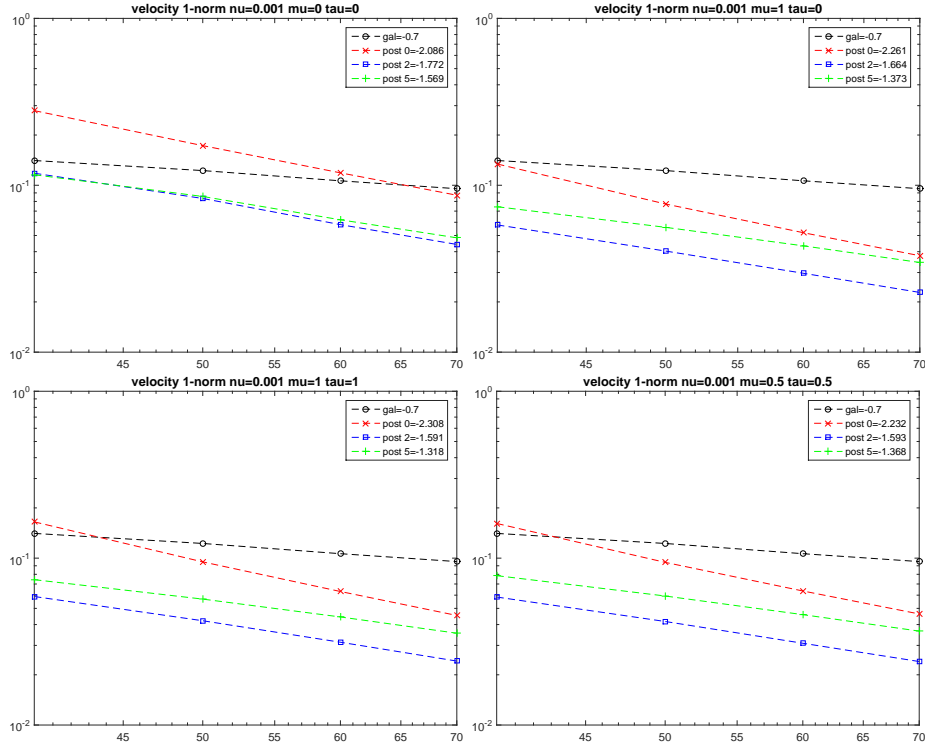


Figure 8: Velocity H^1 errors of the Galerkin approximation and the postprocessed solutions for $\nu = 0.001$ and $\lambda = 0, 2, 5$.

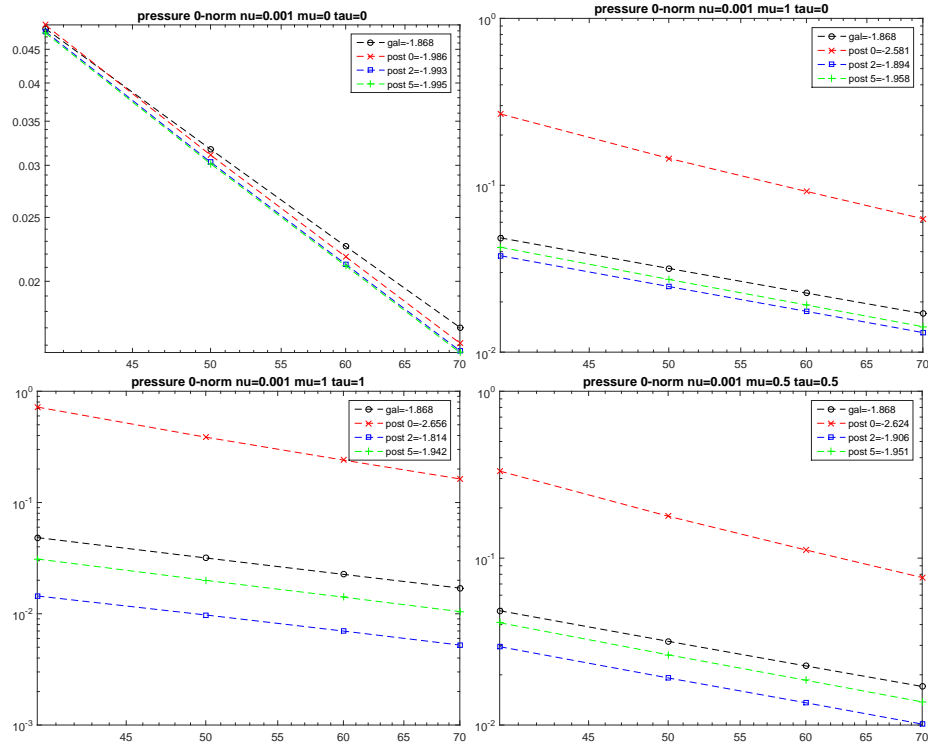


Figure 9: Pressure L^2 errors of the Galerkin approximation and the postprocessed solutions for $\nu = 0.001$ and $\lambda = 0, 2, 5$.

of ν the gradient of u_H , ∇u_H , has a big error and, as a consequence, giving a bigger weight to the reactive term $\tau(\tilde{u} \cdot \nabla)u_H$ in (4) is the wrong strategy. In the surfaces we can see that the method with $\mu = 0.4$ and $\tau = 2$ produces the worst results in this experiment for both the velocity and the pressure errors.

6 Conclusions

We have proposed and analyzed a generalized postprocessed method depending on several parameters (see (4)) in which previous postprocessed methods are included as particular cases. We prove that the generalized method has a rate of convergence one unit bigger than the rate of convergence of the plain Galerkin method, for any of the values of the different parameters. In particular, for the (old) Stokes and Oseen postprocessed methods we prove that adding a term $\lambda \tilde{u}$ on the left-hand side of the postprocessing equation (with λ positive) while adding also for consistency λu_H on the right-hand side, turns out in a method with the same rate of convergence than the case $\lambda = 0$ analyzed in [10], [13]. However, as shown in the numerical experiments of Section 5, a positive λ improves considerably the accuracy of these two postprocessed methods for small values of the diffusion. In particular, while for both postprocessing procedures most of the errors are worse than those of the plain Galerkin method for moderate resolution in space, adding a positive λ makes the postprocessed methods always more accurate than the Galerkin one, for all the errors we have computed.

With the new method analyzed in this paper, we open the possibility of postprocessing with different values of the parameters than those corresponding to the already known Stokes, Oseen and Newton postprocessing. According with the numerical experiments of [8] the Newton postprocessing seems to be one of the best choices. However, we show in Section 5 that taking $\mu = \tau = 0.5$ works also pretty well. Anyway, with the theory developed in this paper users could apply postprocessing with different values of parameters being aware of having always a method with a rate of convergence one unit bigger than Galerkin.

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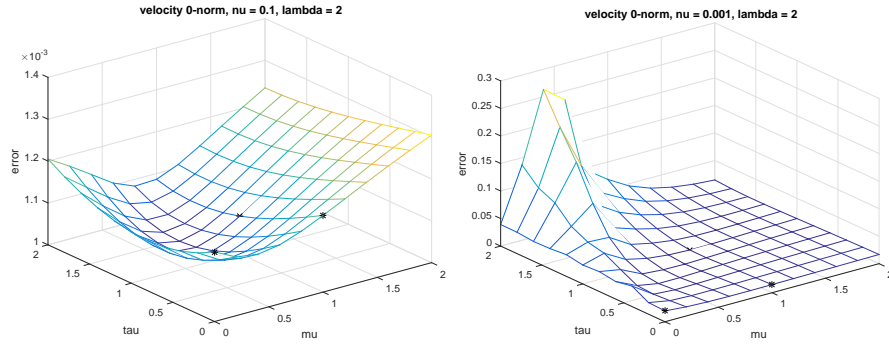


Figure 10: Velocity L^2 postprocess errors surface $\lambda = 2$. On the left, $\nu = 0.1$, on the right $\nu = 0.001$.

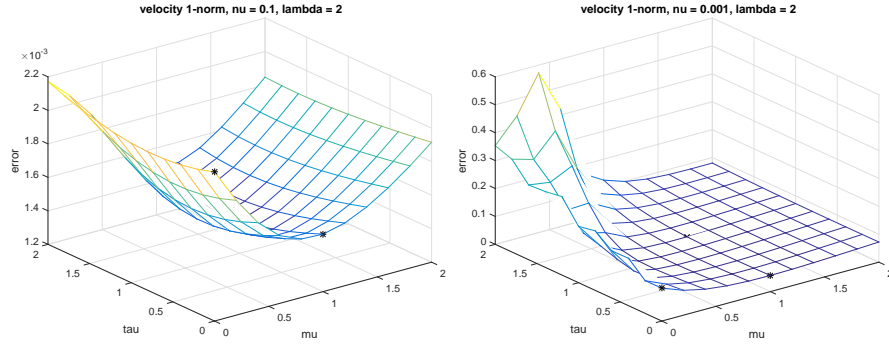


Figure 11: Velocity H^1 postprocess errors surface $\lambda = 2$. On the left, $\nu = 0.1$, on the right $\nu = 0.001$.

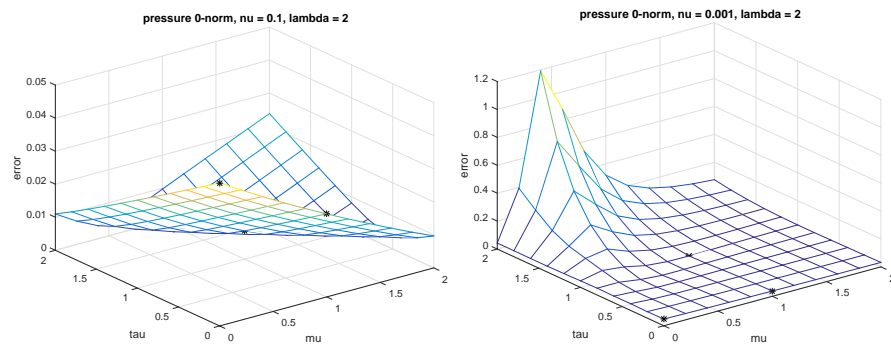


Figure 12: Pressure L^2 postprocess errors surface $\lambda = 2$. On the left, $\nu = 0.1$, on the right $\nu = 0.001$.