# Singular integrals and boundary value problems for elliptic systems

Juan José Marín García

A thesis submitted for the degree of Doctor of Philosophy

Under the supervision of
**José María Martell** and **Marius Mitrea**

Madrid, 2019

# Acknowledgements

Many people made this dissertation possible in one way or another. First, I would like to express my gratitude to my advisors Chema Martell and Marius Mitrea. I am deeply thankful to Chema, who gave me this opportunity in the first place, for his patience in our countless meetings. I am also really grateful to Marius for our fruitful conversations and his great ideas.

I would like to thank Dorina Mitrea and Irina Mitrea for the generous amount of time and energy spent in our shared projects. I am also thankful to the HAPDEGMT group in ICMAT, especially to Cruz Prisuelos for her support. I acknowledge Olli Saari for the nice math conversations and for inviting me to visit him in Bonn. I have also enjoyed the hospitality of the staff of ICMAT, Universidad Autónoma de Madrid and University of Missouri, especially to deal with the paperwork.

En el ICMAT, el doctorado me ha permitido estar rodeado de personas cuyo apoyo ha sido esencial para llegar hasta el final de la tesis. Siempre dije que el despacho 412 era la segunda sala común del ICMAT. Gracias por ello a Jorge, Víctor y Tania, que me dieron tantos buenos momentos dentro y fuera del despacho. Estoy en deuda con muchos otros compañeros de la cuarta planta por convertir la rutina en algo menos rutinario: Cristina, Javi, Manuel, Dani y Ángel. Gracias al resto de compañeros del ICMAT que alguna vez tuvieron palabras de ánimo o complicidad: Diego, Mª Ángeles, Carlos, Ángel D., etc.

Fuera de las paredes del ICMAT, otras muchas personas merecen un agradecimiento. Gracias a los de aquí y a los de allí. A los que siempre estuvieron, a los que me llamaban matemático antes de serlo y a *los increíbles*, especialmente a Patricia, que me dejó escribir con ella y es la amiga que cualquiera querría tener.

Estoy tan agradecido con Ángela que cualquier cosa que escriba aquí no hará justicia con la realidad. Gracias por hacerme mejor persona, por aguantarme tanto tiempo y por sufrir conmigo los malos momentos de esta tesis, que en cierto modo también es tuya aunque no entiendas ni una palabra.

Finalmente, gracias a mi familia por valorarme y por su apoyo incondicional. A mi hermana, que siempre fue un referente para mí, y en especial a mis padres, a quien dedico esta tesis, por su confianza y por enseñarme el valor del esfuerzo y de casi todo lo demás.

# Contents

# Abstract and conclusions

This dissertation is devoted to the study of several problems lying at the intersection of harmonic analysis, partial differential equations and geometric measure theory. In general terms, it is focused on how the geometry of a domain in $\mathbb{R}^n$ influences the boundedness properties of certain operators defined in its boundary, and the applications of this in the realm of boundary value problems. More specifically, the behavior of the measure theoretic outward unit normal vector, which plays a central role in this work, is the key geometric feature that will allow us to bound singular integral operators (such as Riesz transforms or layer potentials) in certain function spaces. In turn, this is a fundamental step for the study of boundary value problems. In the opposite direction, we extract information from these operators about the geometry of a domain in terms of the behavior of its outward unit normal vector.

Some of the singular integral operators that will have a pivotal role in this work are harmonic double layer potentials, defined for each UR domain (cf. Definition 1.1.5) and each function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ according to

$$\mathcal{D}_\Delta f(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, d\sigma(y) \quad x \in \Omega,$$

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, d\sigma(y) \;\text{ for } \sigma\text{-a.e. } x \in \partial\Omega,$$

where $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^n$, $\nu$ is the geometric measure theoretic outward unit normal to $\Omega$ (cf. Section 1.1), and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ (where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$). If $\Delta$ denotes the Laplacian, one can show that $\Delta\mathcal{D}_\Delta f = 0$ and $\mathcal{D}_\Delta f\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (\frac{1}{2}I + K_\Delta)f$ $\sigma$-a.e. in $\partial\Omega$ for every $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$, where $I$ is the identity operator and the boundary trace is taken non-tangentially (cf. Section 1.1). The classical method of layer potentials uses the boundedness and invertibility properties of layer potential operators to study boundary value problems. For instance, if $\frac{1}{2}I + K_\Delta$ is invertible in some function space contained in

$L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ and $f$ belongs to the said space then $u := \mathcal{D}_\Delta\big((\frac{1}{2}I + K_\Delta)^{-1} f\big)$ satisfies

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_\Delta\big)\big(\tfrac{1}{2}I + K_\Delta\big)^{-1} f = f.$$

This scheme is used to show that the function $u$ above is a solution for a Dirichlet Problem. The case of the upper-half space, $\Omega = \mathbb{R}^n_+$, will be treated with a different approach, as in this case $K_\Delta \equiv 0$ (because $\nu(y)$ is perpendicular to $y - x$ whenever $x, y \in \partial\Omega$) and hence $2\,\mathcal{D}_\Delta f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f$. In fact, $2\,\mathcal{D}_\Delta f = P_t^\Delta * f$, where $P_t^\Delta$ is the harmonic Poisson kernel (cf. Theorem 1.2.4). This case highlights the importance of the behavior of the outward unit normal vector $\nu$.

More than 25 years ago, in [60, Problem 3.2.2, p. 117], C. Kenig asked to "*Prove that the layer potentials are invertible in appropriate* [...] *spaces in* [*suitable subclasses of uniformly rectifiable*] *domains*." Kenig's main motivation in this regard stems from the desire of establishing solvability results for boundary value problems formulated in a rather inclusive geometric setting. In the buildup to this open question on [60, p. 116], it is remarked that there are quite general classes of open sets $\Omega \subseteq \mathbb{R}^n$ with the property that the said layer potentials are bounded operators on $L^p(\partial\Omega, \sigma)$ for each exponent $p \in (1, \infty)$. Remarkably, this is the case whenever $\Omega \subseteq \mathbb{R}^n$ is an open set with a uniformly rectifiable boundary (cf. [33]).

The theory developed by S. Hofmann, M. Mitrea, and M. Taylor in [53] goes some way towards answering Kenig's open question. The stated goal of [53] was to "*find the optimal geometric measure theoretic context in which Fredholm theory can be successfully implemented, along the lines of its original development, for solving boundary value problems with $L^p$ data via the method of layer potentials* [*in domains with compact boundaries*]." In particular, [53] may be regarded as a sharp version of the fundamental work of E. Fabes, M. Jodeit, and N. Rivière in [39], dealing with the method of boundary layer potentials in bounded $\mathscr{C}^1$ domains.

However, the insistence on $\partial\Omega$ being a *compact* set is prevalent in [53]. In particular, the classical fact that the Dirichlet Problem (cf. (2.1.7)) is uniquely solvable in the case when $\Omega = \mathbb{R}^n_+$ does *not* fall under the tutelage of [53]. This leads one to speculate whether the treatment of layer potentials may be extended to a class of unbounded domains that includes the upper half-space. This is indeed the main goal of **Chapter 2**.

Specifically, we develop the theory of layer potentials to study boundary value problems in unbounded $\delta$-SKT domains (with SKT acronym for Semmes-Kenig-Toro), a class of domains whose key feature is that the BMO semi-norm (cf. (2.2.31)) of its outward unit normal $\nu$ is controlled by $\delta \in (0, 1)$, which is assumed to be small. The class of $\delta$-SKT domains emerged from the earlier work of S. Semmes [107], [108], and C. Kenig and T. Toro [61], [62], [63], and is related to a class of domains introduced in [53]. The latter was designed to work well when the domains in question have compact boundaries. By way of contrast, the fact that we are now demanding $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < 1$ has topological and metric implications for $\Omega$. Specifically, $\Omega$ is a connected unbounded open set, with a connected unbounded boundary and an unbounded connected complement. For example, in the two-dimensional setting we show that the class of $\delta$-SKT with

$\delta \in (0,1)$ small agrees with the category of chord-arc domains with small constant.

In this context, we prove that if $\delta$ is sufficiently small then the operator norm of Calderón-Zygmund singular integrals whose kernels exhibit a certain algebraic structure is $O(\delta)$ as $\delta \to 0^+$ as in the Theorem 2.4.4, which we state next.

**Theorem.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain satisfying a two-sided local John condition (cf. Definitions 1.1.2 and 1.1.10). Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$ (cf. (2.2.300)). Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$. Given a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, consider the maximal operator $T_*$ whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as*

$$T_*f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \quad \text{for each} \quad x \in \partial\Omega,$$

*where, for each $\varepsilon > 0$,*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \quad \text{for all} \quad x \in \partial\Omega.$$

*Then there exists some $C \in (0,\infty)$, which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,.$$

We also establish estimates in the opposite direction, quantifying the flatness of a "surface" by estimating the BMO semi-norm of its unit normal in terms of the operator norms of certain singular integrals associated with the given surface. Ultimately, this shows that the two-way bridge between geometry and analysis constructed here is in the nature of best possible.

Significantly, the operator norm estimates above permit us to invert the boundary double layer potentials associated with certain class of second-order homogeneous constant complex coefficient PDE. Fix $n \in \mathbb{N}$ with $n \geq 2$, along with $M \in \mathbb{N}$, and consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$

$$L = \big(a^{\alpha\beta}_{jk}\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq M},$$

where the summation convention over repeated indices is in effect (here and elsewhere in the manuscript). The weak ellipticity of the system $L$ amounts to demanding that

> the characteristic matrix $L(\xi) := -\big(a^{\alpha\beta}_{jk}\xi_j\xi_k\big)_{1\leq\alpha,\beta\leq M}$ is invertible for each vector $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n \setminus \{0\}$.

This should be contrasted with the more stringent strong (Legendre-Hadamard) ellipticity condition which asks for the existence of some $\kappa_0 > 0$ such that

$$-\,\mathrm{Re}\,\big\langle L(\xi)\zeta\,,\overline{\zeta}\,\big\rangle \geq \kappa_0\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad \zeta \in \mathbb{C}^M.$$

Examples of strongly (and hence weakly) elliptic operators include scalar operators, such as the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ or, more generally, operators of the form $\mathrm{div} A\nabla$ with $A = (a_{rs})_{1\leq r,s\leq n}$ an $n \times n$ matrix with complex entries satisfying the ellipticity condition

$$\inf_{\xi\in S^{n-1}} \mathrm{Re}\left[a_{rs}\xi_r\xi_s\right] > 0,$$

(where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$), as well as the complex version of the Lamé system of elasticity in $\mathbb{R}^n$,

$$L := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}.$$

Above, the constants $\lambda, \mu \in \mathbb{C}$ (typically called Lamé moduli), are assumed to satisfy

$$\mathrm{Re}\,\mu > 0 \quad \text{and} \quad \mathrm{Re}\,(2\mu + \lambda) > 0,$$

a condition equivalent to the demand that the Lamé system satisfies the strong ellipticity condition. While the Lamé system is symmetric, we stress that the results in this thesis require no symmetry for the systems involved.

The main result regarding invertibility of double layer potentials is the following (cf. Theorem 2.4.24).

**Theorem.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ (cf. (2.3.83)). Pick $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and consider the boundary-to-boundary double layer potential operators $K_A, K_A^{\#}$ associated with $\Omega$ and the coefficient tensor $A$ as in (2.3.4) and (2.3.5), respectively. Finally, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, \infty)$.*

*Then there exists some small threshold $\delta_0 \in (0, 1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $A$, $\varepsilon$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are invertible:*

$$zI + K_A : \left[L^p(\partial\Omega, w)\right]^M \longrightarrow \left[L^p(\partial\Omega, w)\right]^M,$$

$$zI + K_A : \left[L_1^p(\partial\Omega, w)\right]^M \longrightarrow \left[L_1^p(\partial\Omega, w)\right]^M,$$

$$zI + K_A^{\#} : \left[L^p(\partial\Omega, w)\right]^M \longrightarrow \left[L^p(\partial\Omega, w)\right]^M,$$

*where $L_1^p(\partial\Omega, w)$ is a certain brand of $L^p$-based weighted Sobolev space of order one on $\partial\Omega$ (cf. Section 2.2.6).*

The condition $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ above amounts to say that $L = (a_{jk}^{\alpha\beta}\partial_j\partial_k)_{1\leq\alpha,\beta\leq M}$ for some distinguished coefficient tensor $A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$, that is, a coefficient tensor $A$ for which the integral kernel of $K_A$ contains the inner product to $\nu(y)$ with the 'chord' $x-y$. This algebraic structure is neccesary to apply the operator norm estimates previously stated and obtain that $\|K_A\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$, from where one deduces the invertibility result for $zI+K_A$ in $[L^p(\partial\Omega,w)]^M$ if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is small enough.

Concisely put, in the previous theorem we are able to answer Kenig's open question (formulated above) pertaining to any given weakly elliptic homogeneous constant complex coefficient second-order system $L$ in $\mathbb{R}^n$ with $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$, in the setting of $\delta$-SKT domains $\Omega \subseteq \mathbb{R}^n$ with $\delta \in (0,1)$ small (relative to the original geometric characteristics of $\Omega$), for ordinary Lebesgue spaces, Muckenhoupt weighted Lebesgue spaces, as well as Sobolev spaces on $\partial\Omega$ suitably defined in relation to each of the aforementioned scales. Analogue results are proved for Lorentz spaces and Morrey spaces (see Remark 2.4.25, Theorem 2.4.29, Theorem 2.7.12, Theorem 2.7.13). As indicated in Remark 2.4.28, the smallness condition imposed on the parameter $\delta$ is actually in the nature of best possible as far as the aforementioned invertibility results are concerned.

The invertibility results in the previous theorem open the door for solving boundary value problems of Dirichlet, Regularity, Neumann, and Transmission type in the class of $\delta$-SKT domains with $\delta \in (0,1)$ small (relative to the original geometric characteristics of $\Omega$) for second-order weakly elliptic constant complex coefficient systems which (either themselves and/or their transposed) possess distinguished coefficient tensors.

For example, in such a setting, we succeed in establishing the well-posedness of the Muckenhoupt weighted Dirichlet Problem and the Muckenhoupt weighted Regularity Problem, formulated using the nontangential maximal operator introduced in (1.1.2), and nontangential boundary traces defined as in (1.1.5):

$$(D)_{p,w}\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[L^p(\partial\Omega,w)\right]^M, \end{cases} \qquad (R)_{p,w}\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[L_1^p(\partial\Omega,w)\right]^M, \end{cases}$$

for each integrability exponent $p \in (1,\infty)$ and each Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, under the assumption that both $L$ and $L^\top$ have a distinguished coefficient tensor (cf. Theorems 2.6.2 and 2.6.5). Moreover, we provide counterexamples which show that the well-posedness result just described may fail if these assumptions on the existence of distinguished coefficient tensors are simply dropped. Our results are therefore optimal in this regard. We also establish solvability for boundary value results with boundary data from Lorentz spaces, Morrey spaces, vanishing Morrey spaces, block spaces, and from Sobolev spaces naturally associated with these scales.

This extends previously known well-posedness results for boundary value problems in the upper half-space, which is the simplest example of unbounded SKT domain. Indeed, if $\Omega = \mathbb{R}^n_+$, then the Dirichlet Problem $(D)_{p,w}$ is uniquely solvable by taking the convolution of the boundary datum $f$ with the Poisson kernel associated with $L$ in the upper half-space (cf. [8], [42], [82], [115], [117]). Poisson kernels for elliptic boundary value problems in a half-space have been studied extensively in [1], [2], [69, §10.3], [112], [113], [114].

In this direction, in **Chapter 3** we establish a Fatou-type theorem and a naturally accompanying Poisson integral representation formula for null-solutions in the upper-half space. The main result is the following (cf. Theorem 3.1.1).

**Theorem.** *Consider a homogeneous, second-order, constant complex coefficient, strongly elliptic $M \times M$ system $L$ and fix some aperture parameter $\kappa > 0$. Assume that*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} (\mathcal{N}_\kappa u)(x')\,\frac{dx'}{1+|x'|^{n-1}} < \infty, \end{cases}$$

*where $\mathcal{N}_\kappa$ denotes the nontangential maximal operator (cf. (1.1.2)). Then,*

$$\begin{cases} u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ exists at } \mathscr{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1}, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ belongs to } \ \Big[L^1\Big(\mathbb{R}^{n-1}\,,\,\dfrac{dx'}{1+|x'|^{n-1}}\Big)\Big]^M, \\ u(x',t) = \Big(P^L_t * \big(u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\big)\Big)(x') \ \text{ for each } \ (x',t) \in \mathbb{R}^n_+, \end{cases}$$

*where $P^L = \big(P^L_{\beta\alpha}\big)_{1\le\beta,\alpha\le M}$ denotes the Poisson kernel for $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4 and $P^L_t(x') := t^{1-n}P^L(x'/t)$ for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.*

This refines [82, Theorem 6.1, p. 956], where a stronger integrability condition is assumed. We also wish to remark that even in the classical case when $L := \Delta$, the Laplacian in $\mathbb{R}^n$, the previous theorem is more general (in the sense that it allows for a larger class of functions) than the existing results in the literature. Indeed, the latter typically assume an $L^p$ integrability condition for the harmonic function which, in the range $p \in (1,\infty)$, implies our weighted $L^1$ integrability condition for the nontangential maximal function demanded above. In this vein see, e.g., [42, Theorems 4.8-4.9, pp. 174-175], [115, Corollary, p. 200], [116, Proposition 1, p. 119].

Moreover, this Fatou theorem has a natural associated uniqueness result which allows the study of very general results regarding the well-posedness of boundary value problems for elliptic systems (cf. Corollaries 3.1.3 and 3.1.4).

Going on with the study of boundary value problems in the upper-half space, in **Chapter 4** we study the Dirichlet Problem for elliptic systems with data in generalized Hölder spaces and generalized Morrey-Campanato spaces. Furthermore, via PDE-based techniques, we prove that these two function spaces are actually equivalent.

Generalized Hölder spaces, denoted by $\dot{\mathscr{C}}^\omega(\partial\Omega,\mathbb{C}^M)$, quantify continuity in terms of the modulus, or "growth" function, $\omega$. Specifically, given $U \subseteq \mathbb{R}^n$, $M \ge 1$, and a non-decreasing function $\omega : (0,\infty) \to (0,\infty)$ whose limit at the origin vanishes, the homogeneous space $\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$ is the collection of functions $u : U \to \mathbb{C}^M$ with

$$[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} := \sup_{\substack{x,y\in U \\ x\neq y}} \frac{|u(x)-u(y)|}{\omega(|x-y|)} < \infty.$$

Similarly, for $D \in (0,\infty]$ and a non-decreasing function $\omega : (0,D) \to (0,\infty)$ whose limit at the origin vanishes and which is bounded if $D < \infty$, the space $\mathscr{C}^\omega(U,\mathbb{C}^M)$ is defined by the norm

$$\|u\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} := \sup_U |u| + [u]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(U,\mathbb{C}^M)},$$

where $\widetilde{\omega}(t) := \omega(\min\{t,D\})$ for each $t \in (0,\infty)$.

Given a non-decreasing function $\omega : (0,\infty) \to (0,\infty)$ whose limit at the origin vanishes, along with some integrability exponent $p \in [1,\infty)$, we define the semi-norm

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))}\left(⨍_Q |f(x') - f_Q|^p\,dx'\right)^{1/p},$$

and we denote the associated function space by $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$, called the generalized Morrey-Campanato space in $\mathbb{R}^{n-1}$. The choice $\omega(t) := t^\alpha$ with $\alpha \in (0,1)$ corresponds to the classical Morrey-Campanato spaces, while the special case $\omega(t) := 1$ yields the usual space of functions of bounded mean oscillations (BMO). We also define, for every $u \in \left[\mathscr{C}^1(\mathbb{R}^n_+)\right]^M$ and $q \in (0,\infty)$,

$$\|u\|_{**}^{(\omega,q)} := \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))}\left(⨍_Q \left(\int_0^{\ell(Q)} |(\nabla u)(x',t)|^2\,t\,dt\right)^{q/2} dx'\right)^{1/q}.$$

We next state our main result on this subject, which is included in Theorem 4.1.2 and generalizes work in [85] (where the case $\omega(t) = t^\alpha$ for $\alpha \in (0,1)$ is studied) by allowing more flexible scales to measure regularity in Hölder spaces and Morrey-Campanato spaces.

**Theorem.** *Consider a homogeneous, second-order, constant complex coefficient, strongly elliptic $M \times M$ system $L$. Also, fix an aperture parameter $\kappa > 0$, $p \in [1,\infty)$ along with $q \in (0,\infty)$. Finally, let $\omega : (0,\infty) \to (0,\infty)$ be a non-decreasing function, whose limit at the origin vanishes, and satisfying*

$$\sup_{t>0}\left\{\frac{1}{\omega(t)}\left(\int_0^t \omega(s)\frac{ds}{s} + t\int_t^\infty \frac{\omega(s)}{s}\,\frac{ds}{s}\right)\right\} < +\infty.$$

*Then the following statements are true.*

(a) *The generalized Hölder Dirichlet Problem for the system $L$ in $\mathbb{R}^n_+$, i.e.,*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \\ Lu = 0 \quad \textit{in} \quad \mathbb{R}^n_+, \\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} < \infty, \\ u\big|_{\partial\mathbb{R}^n_+} = f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M) \quad \textit{on} \quad \mathbb{R}^{n-1}, \end{cases}$$

*is well-posed. More specifically, there is a unique solution which is given by*

$$u(x',t) = (P_t^L * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+,$$

*where $P^L$ denotes the Poisson kernel for $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4. In addition, $u$ belongs to the space $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$, satisfies $u|_{\partial\mathbb{R}^n_+}=f$, and there exists a finite constant $C=C(n,L,\omega)\geq 1$ such that*

$$C^{-1}[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}\leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}\leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}.$$

(b) *The generalized Morrey-Campanato Dirichlet Problem for $L$ in $\mathbb{R}^n_+$, formulated as*

$$\begin{cases} u\in\left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M,\\ Lu=0 \quad in \quad \mathbb{R}^n_+,\\ \|u\|_{**}^{(\omega,q)}<\infty,\\ u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}}=f\in\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \quad a.e. \ on \quad \mathbb{R}^{n-1}, \end{cases}$$

*is well-posed. More precisely, there is a unique solution which is given by*

$$u(x',t)=(P_t^L*f)(x'),\qquad \forall\,(x',t)\in\mathbb{R}^n_+,$$

*where $P^L$ denotes the Poisson kernel for $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4. In addition, $u$ belongs to $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$, satisfies $u|_{\partial\mathbb{R}^n_+}=f$ a.e. on $\mathbb{R}^{n-1}$, and there exists a finite constant $C=C(n,L,\omega,p,q)\geq 1$ such that*

$$C^{-1}\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\leq\|u\|_{**}^{(\omega,q)}\leq C\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}.$$

(c) *The following equality between vector spaces holds*

$$\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)=\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$$

*with equivalent norms, where the right-to-left inclusion is understood in the sense that for each $f\in\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ there exists a unique $\widetilde{f}\in\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$ with the property that $f=\widetilde{f}$ a.e. in $\mathbb{R}^{n-1}$.*

*As a result, the Hölder Dirichlet Problem in (a) and the Morrey-Campanato Dirichlet Problem in (b) are equivalent. Specifically, for any pair of boundary data which may be identified in the sense described in the previous paragraph, these problems have the same unique solution.*

We would like to notice that in Section 4.7 we are able to weaken the hypothesis on the growth function and still prove well-posedness for the two Dirichlet problems. The main difference is that in that case they are no longer equivalent (see Example 4.7.4).

The interplay between analysis and geometry described at the outset of this section allows us to give characterizations of certain class of domains based on purely analytic conditions. Specifically, in **Chapter 5** we characterize Lyapunov $\mathscr{C}^{1,\omega}$-domains. Lyapunov $\mathscr{C}^{1,\omega}$-domains are open sets of locally finite perimeter whose geometric measure theoretic outward unit normal $\nu$ belongs to $\mathscr{C}^\omega(\partial\Omega)$ (after possibly being redefined on a set of $\sigma$-measure zero). Here, to simplify the notation, we call $\mathscr{C}^\omega(U):=\mathscr{C}^\omega(U,\mathbb{C})$.

Borrowing ideas from [52], the class of $\mathscr{C}^{1,\omega}$-domains can also be described as the collection of all open subsets of $\mathbb{R}^n$ which locally coincide (up to a rigid transformation of the space) with the upper-graph of a real-valued continuously differentiable function defined in $\mathbb{R}^{n-1}$ whose first-order partial derivatives belong to $\mathscr{C}^\omega(\mathbb{R}^{n-1})$.

The characterizations of the class of Lyapunov domains that we prove are in terms of the boundedness properties of certain classes of singular integral operators acting on generalized Hölder spaces on the boundary of an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ with compact boundary (cf. Definition 1.1.2). The most important examples of such singular integral operators are the Riesz transforms $R_j$ (cf. (5.1.3)-(5.1.4)).

Our present work adds further credence to the heuristic principle that the action of the distributional Riesz transforms on the constant function 1 encapsulates much information, both of analytic and geometric flavor, about the underlying Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ (with compact boundary). At the most basic level, the main result of F. Nazarov, X. Tolsa, and A. Volberg in [101] states that

$$\partial\Omega \ \text{ is a UR set } \Longleftrightarrow R_j 1 \in \mathrm{BMO}(\partial\Omega,\sigma) \ \text{ for each } \ j \in \{1,\dots,n\},$$

and it has been proved in [96] that

$$\left.\begin{array}{r} \nu \in \mathrm{VMO}(\partial\Omega,\sigma) \\ \text{and } \partial\Omega \text{ is a UR set} \end{array}\right\} \Longleftrightarrow R_j 1 \in \mathrm{VMO}(\partial\Omega,\sigma) \ \text{ for all } \ j \in \{1,\dots,n\},$$

where $\mathrm{VMO}(\partial\Omega,\sigma)$ stands for the Sarason space of functions with vanishing mean oscillation on $\partial\Omega$, with respect to the measure $\sigma$. By further assigning additional regularity for the functionals $\{R_j 1\}_{1\le j\le n}$ yields the following result (proved in [96])

$$\left.\begin{array}{r} \Omega \text{ is a domain} \\ \text{of class } \mathscr{C}^{1+\alpha} \end{array}\right\} \Longleftrightarrow R_j 1 \in \mathscr{C}^\alpha(\partial\Omega) \ \text{ for all } \ j \in \{1,\dots,n\},$$

where $\alpha \in (0,1)$ and $\mathscr{C}^\alpha(\partial\Omega)$ is the classical Hölder space of order $\alpha$ on $\partial\Omega$. This is generalized by the following result, contained in Theorem 5.1.4, which allows us to consider more flexible scales of spaces measuring Hölder regularity (see the discussion in Example 1.3.4 in this regard).

**Theorem.** *Suppose $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain whose boundary is compact. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Finally, let $\omega : (0, \mathrm{diam}(\partial\Omega)) \to (0,\infty)$ be a bounded, non-decreasing function, whose limit at the origin vanishes, and satisfying*

$$\sup_{0<t<\mathrm{diam}(\partial\Omega)} \left\{ \frac{1}{\omega(t)} \left( \int_0^t \omega(s)\frac{ds}{s} + t\int_t^{\mathrm{diam}(\partial\Omega)} \frac{\omega(s)}{s}\,\frac{ds}{s} \right) \right\} < +\infty.$$

*Then the following statements are equivalent:*

(a) *After possibly being altered on a set of $\sigma$-measure zero, the outward unit normal $\nu$ to $\Omega$ belongs to the generalized Hölder space $\mathscr{C}^\omega(\partial\Omega)$.*

(*b*) *The Riesz transforms on* $\partial\Omega$ *satisfy*

$$R_j 1 \in \mathscr{C}^\omega(\partial\Omega) \ \ \textit{for each} \ \ j \in \{1, \dots, n\}.$$

(*c*) *The set* $\Omega$ *is a* UR *domain* (*in the sense of Definition 1.1.5*)*, and given any odd homogenous polynomial* $P$ *of degree* $\ell \geq 1$ *in* $\mathbb{R}^n$ *the singular integral operator acting on each function* $f \in \mathscr{C}^\omega(\partial\Omega)$ *according to*

$$(Tf)(x) := \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{P(x-y)}{|x-y|^{n-1+\ell}} f(y)\, d\sigma(y) \ \ \textit{for } \sigma\textit{-a.e.} \ \ x \in \partial\Omega$$

*is well-defined and maps the generalized Hölder space* $\mathscr{C}^\omega(\partial\Omega)$ *boundedly into itself.*

(*d*) *The set* $\Omega$ *is a* UR *domain, and the boundary-to-domain version of the Riesz transforms defined for each* $j \in \{1, \dots, n\}$ *and each* $f \in L^1(\partial\Omega, \sigma)$ *as*

$$(\mathscr{R}_j^{\pm} f)(x) := \frac{1}{\varpi_{n-1}} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_{\pm},$$

*satisfy*

$$\mathscr{R}_j^{\pm} 1 \in \mathscr{C}^\omega(\Omega_{\pm}) \ \ \textit{for each} \ \ j \in \{1, \dots, n\}.$$

(*e*) *The set* $\Omega$ *is a* UR *domain, and given any odd homogenous polynomial* $P$ *of degree* $\ell \geq 1$ *in* $\mathbb{R}^n$*, the integral operators acting on each function* $f \in \mathscr{C}^\omega(\partial\Omega)$ *according to*

$$\mathbb{T}_{\pm} f(x) := \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+\ell}} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_{\pm},$$

*map the generalized Hölder space* $\mathscr{C}^\omega(\partial\Omega)$ *continuously into* $\mathscr{C}^\omega(\Omega_{\pm})$*.*

This thesis has led to the following papers:

(a) *Singular integral operators, quantitative flatness, and boundary problems*, book manuscript, 2019 (joint work with J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea).

(b) *A Fatou theorem and Poisson's integral representation formula for elliptic systems in the upper-half space*, to appear in "Topics in Clifford Analysis", special volume in honor of Wolfgang Sprößig, Swanhild Bernstein editor, Birkhäuser, 2019 (joint work with J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea).

(c) *The generalized Hölder and Morrey-Campanato Dirichlet problems for elliptic systems in the upper-half space*, to appear in Potential Anal., 2019 (joint work with J.M. Martell and M. Mitrea).

(d) *Characterizations of Lyapunov domains in terms of Riesz transforms and generalized Hölder spaces*, preprint, 2019 (joint work with J.M. Martell and M. Mitrea).

The material in (*a*) is elaborated in Chapter 2, (*b*) is contained in Chapter 3, (*c*) is developed in Chapter 4, and (*d*) is discussed in Chapter 5. They correspond, respectively, to [77], [76], [79], and [78] in the bibliography.

# Resumen y conclusiones (Spanish)

Esta tesis está dedicada al estudio de varios problemas que se encuentran en la intersección del análisis armónico, las ecuaciones en derivadas parciales y la teoría geométrica de la medida. En términos generales, se centra en analizar cómo la geometría de un dominio en $\mathbb{R}^n$ influye en las propiedades de acotación de ciertos operadores definidos en su frontera y las aplicaciones que esto tiene en los problemas de valor en la frontera. Más específicamente, el comportamiento del vector normal unitario exterior, que juega un papel central en este trabajo, es la característica geométrica clave que nos permitirá acotar operadores integrales singulares (como las transformadas de Riesz o los potenciales de capa) en ciertos espacios de funciones. A su vez, este es un paso fundamental para el estudio de los problemas de valor en la frontera. En la dirección contraria, usando estos operadores extraemos información sobre la geometría del dominio, en función del comportamiento de su vector normal unitario exterior.

Algunos de los operadores integrales singulares que tendrán un papel crucial en este trabajo son los potenciales de doble capa armónicos, definidos para cada dominio UR (cf. Definición 1.1.5) y para cada función $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ como

$$\mathcal{D}_\Delta f(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle \nu(y), y-x \rangle}{|x-y|^n} f(y)\, d\sigma(y) \quad x \in \Omega,$$

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x \rangle}{|x-y|^n} f(y)\, d\sigma(y) \text{ para } \sigma\text{-c.t.p. } x \in \partial\Omega,$$

donde $\omega_{n-1}$ es la medida de superficie de la esfera unidad en $\mathbb{R}^n$, $\nu$ es el normal unitario exterior a $\Omega$ (cf. Sección 1.1) y $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ (donde $\mathcal{H}^{n-1}$ denota la medida de Hausdorff de dimensión $n-1$ en $\mathbb{R}^n$). Si $\Delta$ denota el laplaciano, se puede probar que $\Delta\mathcal{D}_\Delta f = 0$ y $\mathcal{D}_\Delta f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \left(\frac{1}{2}I + K_\Delta\right)f$ en $\sigma$-casi todo punto de $\partial\Omega$ para cada $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$, donde $I$ es el operador identidad y la traza a la frontera se toma no tangencialmente (cf. Sección 1.1). El método clásico de los potenciales de capa hace uso de las propiedades de acotación e invertibilidad de los operadores de potencial de capa para estudiar problemas de valor en la frontera. Por ejemplo, si $\frac{1}{2}I + K_\Delta$ es invertible en cierto espacio de funciones contenido en $L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ y $f$ pertenece a dicho espacio entonces $u := \mathcal{D}_\Delta((\frac{1}{2}I + K_\Delta)^{-1}f)$ satisface

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_\Delta\big)\big(\tfrac{1}{2}I + K_\Delta\big)^{-1}f = f.$$

Este esquema se usa para mostrar que la función $u$ es una solución del problema de Dirichlet. El caso del semiespacio superior, $\Omega = \mathbb{R}^n_+$, será tratado con un enfoque distinto, ya que en este caso $K_\Delta \equiv 0$ (porque $\nu(y)$ es perpendicular a $y - x$ siempre que $x, y \in \partial\Omega$) y por tanto $2\,\mathcal{D}_\Delta f\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f$. De hecho, $2\,\mathcal{D}_\Delta f = P_t^\Delta * f$, donde $P_t^\Delta$ es el núcleo de Poisson armónico (cf. Teorema 1.2.4). Esto pone de manifiesto la importancia del comportamiento del vector normal unitario exterior $\nu$.

Hace más de 25 años, en [60, Problema 3.2.2, p. 117], C. Kenig pidió probar que los potenciales de capa son invertibles en espacios apropiados en subclases adecuadas de dominios uniformemente rectificables. La motivación principal de Kenig a este respecto surge del deseo de establecer resultados para resolver problemas de valor en la frontera formulados en un escenario geométrico general. En el preámbulo de esta pregunta abierta, en [60, p. 116], se observa que hay clases generales de conjuntos abiertos $\Omega \subseteq \mathbb{R}^n$ con la propiedad de que los citados operadores de capa están acotados en $L^p(\partial\Omega, \sigma)$ para cada exponente $p \in (1, \infty)$. Notablemente, este es el caso siempre que $\Omega \subseteq \mathbb{R}^n$ sea un conjunto abierto con una frontera uniformemente rectificable (cf. [33]).

La teoría desarrollada por S. Hofmann, M. Mitrea y M. Taylor en [53] presenta un avance para responder a la pregunta abierta de Kenig. El objetivo de [53] era el de encontrar las condiciones óptimas en el contexto de la teoría geométrica de la medida para las cuales la teoría de Fredholm pueda ser implementada de manera satisfactoria, en las líneas de su desarrollo original, para resolver problemas de valor en la frontera con dato en $L^p$ mediante el método de los potenciales de capa en dominios con frontera compacta. En particular, [53] puede considerarse como una versión óptima del trabajo fundamental de E. Fabes, M. Jodeit y N. Rivière en [39] sobre el método de los potenciales de capa en dominios $\mathscr{C}^1$ acotados.

Sin embargo, la exigencia de que $\partial\Omega$ sea un conjunto *compacto* prevalece en [53]. En particular, el hecho clásico de que el problema de Dirichlet (cf. (2.1.7)) es únicamente resoluble en el caso en que $\Omega = \mathbb{R}^n_+$ no se encuentra dentro del alcance de [53]. Esto lleva a especular si el tratamiento de los potenciales de capa puede extenderse a una clase de dominios no acotados que incluya al semiespacio superior. Este es, de hecho, el objetivo principal del **Capítulo 2**.

Específicamente, desarrollamos la teoría de potenciales de capa para estudiar problemas de valor en la frontera en dominios $\delta$-SKT no acotados (donde SKT son las siglas de Semmes-Kenig-Toro), una clase de dominios cuya característica clave es que la seminorma BMO (cf. (2.2.31)) de su normal unitario exterior $\nu$ está controlada por $\delta \in (0, 1)$, un parámetro que se asume pequeño. La clase de dominios $\delta$-SKT emergió de trabajos de S. Semmes [107], [108] y C. Kenig y T. Toro [61], [62], [63] y está relacionada con la clase de dominios introducidos en [53]. Esta última fue diseñada para funcionar bien cuando los dominios en cuestión tienen fronteras compactas. En contraste, el hecho de que pidamos aquí que $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < 1$ tiene implicaciones topológicas y métricas sobre $\Omega$. Específicamente, $\Omega$ es un conjunto abierto, conexo, no acotado, con frontera conexa no acotada y con complementario conexo no acotado. Por ejemplo, en el contexto bidimensional, probamos que la clase de dominios $\delta$-SKT con $\delta \in (0,1)$ pequeña coincide con la familia de dominios cuerda-arco con constante pequeña.

En este contexto, probamos que si $\delta$ es suficientemente pequeña entonces la norma de los operadores integrales singulares de Calderón-Zygmund cuyo núcleo posee cierta estructura algebraica es $O(\delta)$ cuando $\delta \to 0^+$ como en el Teorema 2.4.4, que enunciamos a continuación.

**Teorema.** *Sea $\Omega \subseteq \mathbb{R}^n$ un dominio Ahlfors regular que satisface una condición two-sided local John (cf. Definiciones 1.1.2 y 1.1.10). Abreviamos $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ y denotamos por $\nu$ al normal unitario exterior a $\Omega$. Fijamos un exponente de integrabilidad $p \in (1,\infty)$, así como un peso de Muckenhoupt $w \in A_p(\partial\Omega,\sigma)$ (cf. (2.2.300)). Consideramos también un entero suficientemente grande $N = N(n) \in \mathbb{N}$. Dada una función $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ que toma valores completos y es par y positivamente homogénea de grado $-n$, consideramos el operador maximal $T_*$, que actúa en cada función $f \in L^p(\partial\Omega, w)$ de acuerdo a la fórmula*

$$T_* f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \quad \textit{para cada} \quad x \in \partial\Omega,$$

*donde, para cada $\varepsilon > 0$,*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \quad \textit{para todo} \quad x \in \partial\Omega.$$

*Entonces existe $C \in (0,\infty)$, que depende solo de $n$, $p$, $[w]_{A_p}$, las constantes local John de $\Omega$ y las constantes de regularidad Ahlfors de $\partial\Omega$, tal que*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,.$$

También establecemos estimaciones en la dirección contraria, cuantificando la planicidad de una “superficie” estimando la seminorma BMO de su normal unitario en función de las normas de ciertos operadores integrales singulares asociados con la superficie dada. En definitiva, esto muestra que el puente de doble sentido entre la geometría y el análisis que construimos aquí es el mejor posible.

Significativamente, las estimaciones para la norma de operadores enunciadas arriba nos permiten invertir los potenciales de doble capa en la frontera asociados con cierta clase de EDP de segundo orden, homogéneas y con coeficientes complejos constantes. Fijemos $n \in \mathbb{N}$ con $n \geq 2$, así como $M \in \mathbb{N}$, y consideremos un sistema $M \times M$ de segundo orden, homogéneo, con coeficientes complejos constantes y débilmente elíptico en $\mathbb{R}^n$,

$$L = \big(a^{\alpha\beta}_{jk}\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq M},$$

donde usamos (aquí y en el resto de esta memoria) el convenio de suma sobre índices repetidos. La condición de elipticidad débil del sistema $L$ equivale a imponer que

> la matriz característica $L(\xi) := -(a^{\alpha\beta}_{jk}\xi_j\xi_k)_{1\leq\alpha,\beta\leq M}$ es invertible para cada vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$.

Esta condición debe ser contrastada con la condición más estricta de elipticidad fuerte (Legendre-Hadamard), que requiere que exista $\kappa_0 > 0$ tal que

$$-\operatorname{Re}\left\langle L(\xi)\zeta\,,\overline{\zeta}\,\right\rangle \geq \kappa_0\,|\xi|^2\,|\zeta|^2 \quad \text{para todo} \quad \xi \in \mathbb{R}^n \quad \text{y} \quad \zeta \in \mathbb{C}^M.$$

Ejemplos de operadores fuertemente (y por tanto débilmente) elípticos incluyen operadores escalares, como el laplaciano $\Delta = \sum_{j=1}^{n} \partial_j^2$, o más en general, operadores de la forma $\operatorname{div} A\nabla$ con $A = (a_{rs})_{1\leq r,s\leq n}$ una matriz $n\times n$ con entradas complejas satisfaciendo la condición de elipticidad

$$\inf_{\xi\in S^{n-1}} \operatorname{Re}\left[a_{rs}\xi_r\xi_s\right] > 0,$$

(donde $S^{n-1}$ denota la esfera unidad en $\mathbb{R}^n$), así como la versión compleja del sistema de Lamé de elasticidad en $\mathbb{R}^n$,

$$L := \mu\Delta + (\lambda + \mu)\nabla\operatorname{div}.$$

Aquí, suponemos que las constantes $\lambda, \mu \in \mathbb{C}$ satisfacen

$$\operatorname{Re}\mu > 0 \quad \text{y} \quad \operatorname{Re}(2\mu + \lambda) > 0,$$

una condición equivalente al hecho de que el sistema de Lamé satisfaga la condición de elipticidad fuerte. Si bien el sistema de Lamé es simétrico, los resultados en esta tesis no requieren ninguna hipótesis de simetría.

El resultado principal sobre la invertibilidad de potenciales de doble capa es el siguiente (cf. Teorema 2.4.24).

**Teorema.** *Sea $\Omega \subseteq \mathbb{R}^n$ un conjunto abierto que satisface una condición two-sided local John y cuya frontera topológica es un conjunto Ahlfors regular. Abreviamos $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ y denotamos por $\nu$ al vector unitario exterior a $\Omega$. Además, sea $L$ un sistema $M\times M$ en $\mathbb{R}^n$ débilmente elíptico, de segundo orden, homogéneo, con coeficientes complejos constantes y para el cual $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$ (cf. (2.3.83)). Escogemos $A \in \mathfrak{A}^{\mathrm{dis}}_L$ y consideramos los operadores potenciales de doble capa en la frontera $K_A, K^{\#}_A$ asociados con $\Omega$ y con el tensor de coeficientes $A$ como en (2.3.4) y (2.3.5) respectivamente. Finalmente, fijamos un exponente de integrabilidad $p \in (1,\infty)$, un peso de Muckenhoupt $w \in A_p(\partial\Omega, \sigma)$ y un número $\varepsilon \in (0,\infty)$.*

*Entonces existe $\delta_0 \in (0,1)$, que depende solo de $n$, $p$, $[w]_{A_p}$, $A$, $\varepsilon$, las constantes local John de $\Omega$ y las constantes de regularidad Ahlfors de $\partial\Omega$, con la propiedad de que* *si* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ *entonces se sigue que para cada parámetro espectral* $z \in \mathbb{C}$ *con* $|z| \geq \varepsilon$ *los siguientes operadores son invertibles:*

$$zI + K_A : \left[L^p(\partial\Omega, w)\right]^M \longrightarrow \left[L^p(\partial\Omega, w)\right]^M,$$

$$zI + K_A : \left[L^p_1(\partial\Omega, w)\right]^M \longrightarrow \left[L^p_1(\partial\Omega, w)\right]^M,$$

$$zI + K^{\#}_A : \left[L^p(\partial\Omega, w)\right]^M \longrightarrow \left[L^p(\partial\Omega, w)\right]^M,$$

*donde* $L^p_1(\partial\Omega, w)$ *es un espacio de Sobolev en* $\partial\Omega$ *de orden uno basado en* $L^p$ *(cf. Sección 2.2.6).*

La condición $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$ de arriba quiere decir que $L = \left(a^{\alpha\beta}_{jk}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}$ para algún tensor de coeficientes distinguido $A = \left(a^{\alpha\beta}_{jk}\right)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$, es decir, un tensor de coeficientes $A$ para el cual el núcleo integral de $K_A$ contiene el producto escalar de $\nu(y)$ con la "cuerda" $x - y$. Esta estructura algebraica es necesaria para aplicar las estimaciones para la norma de los operadores enunciadas previamente y obtener que $\|K_A\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$, de lo cual se deducen los resultados de invertibilidad para $zI + K_A$ en $\left[L^p(\partial\Omega, w)\right]^M$ si $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ es suficientemente pequeño.

En términos concisos, en el teorema anterior somos capaces de contestar a la pregunta abierta de Kenig (formulada arriba) para sistemas $L$ en $\mathbb{R}^n$ débilmente elípticos, de segundo orden, homogéneos, con coeficientes complejos constantes y con $\mathfrak{A}^{\mathrm{dis}}_L \neq \varnothing$, en el contexto de dominios $\delta$-SKT $\Omega \subseteq \mathbb{R}^n$ con $\delta \in (0, 1)$ pequeña (con respecto a las características geométricas originales de $\Omega$), para espacios de Lebesgue ordinarios, espacios de Lebesgue con pesos de Muckenhoupt, así como para los espacios de Sobolev en $\partial\Omega$ adecuadamente definidos en relación a las escalas anteriores. Se prueban resultados análogos para espacios de Lorentz y espacios de Morrey (ver Observación 2.4.25, Teorema 2.4.29, Teorema 2.7.12, Teorema 2.7.13). Como se indica en la Observación 2.4.28, la condición de que el parámetro $\delta$ sea pequeño es de hecho la mejor posible para los resultados de invertibilidad antes mencionados.

Los resultados de invertibilidad en el teorema anterior abren la puerta a la resolución de problemas de valor en la frontera de tipo Dirichlet, Regularidad, Neumann y Transmisión en dominios $\delta$-SKT con $\delta \in (0, 1)$ pequeña (con respecto a las características geométricas originales de $\Omega$) para sistemas débilmente elípticos, de segundo orden, homogéneos, con coeficientes complejos constantes y que tienen (ellos y/o sus traspuestos) un tensor de coeficientes distinguido.

Por ejemplo, en este contexto, conseguimos establecer que los problemas de Dirichlet y de Regularidad en espacios con pesos de Muckenhoupt, formulados usando el operador maximal no tangencial que introducimos en (1.1.2) y las trazas a la frontera no tangenciales definidas en (1.1.5), están bien propuestos:

$$(D)_{p,w}\begin{cases} u\in\left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu=0 \;\text{ en }\; \Omega, \\ \mathcal{N}_\kappa u\in L^p(\partial\Omega,w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}=f\in\left[L^p(\partial\Omega,w)\right]^M, \end{cases} \qquad (R)_{p,w}\begin{cases} u\in\left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu=0 \;\text{ en }\; \Omega, \\ \mathcal{N}_\kappa u\in L^p(\partial\Omega,w), \\ \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega,w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}=f\in\left[L^p_1(\partial\Omega,w)\right]^M, \end{cases}$$

para cada exponente de integrabilidad $p\in(1,\infty)$ y cada peso de Muckenhoupt $w\in A_p(\partial\Omega,\sigma)$, ambos bajo la hipótesis de que $L$ y $L^\top$ tienen un tensor de coeficientes distinguido (cf. Teoremas 2.6.2 y 2.6.5). Además, proporcionamos contraejemplos que muestran que estos problemas pueden no estar bien propuestos si no asumimos la existencia de un tensor de coeficientes distinguido. Nuestros resultados son por tanto óptimos a este respecto. Establecemos también resultados análogos para problemas de valor en la frontera con dato en la frontera en espacios de Lorentz, espacios de Morrey, espacios vanishing Morrey, espacios block y en los espacios de Sobolev asociados de manera natural a estas escalas.

Esto extiende resultados previamente conocidos sobre problemas de valor en la frontera en el semiespacio superior, que es el ejemplo más sencillo de dominio SKT no acotado. En efecto, si $\Omega=\mathbb{R}^n_+$, entonces el problema de Dirichlet $(D)_{p,w}$ está bien propuesto y la solución viene dada como la convolución del dato en la frontera $f$ con el núcleo de Poisson asociado con $L$ en el semiespacio superior (cf. [8], [42], [82], [115], [117]). Los núcleos de Poisson para problemas de valor en la frontera elípticos en el semiespacio superior han sido estudiados en profundidad en [1], [2], [69, §10.3], [112], [113], [114].

En esta dirección, en el **Capítulo 3** establecemos un resultado de tipo Fatou y una fórmula de representación integral de Poisson para soluciones en el semiespacio superior. El resultado principal es el siguiente (cf. Teorema 3.1.1).

**Teorema.** *Sea $L$ un sistema $M\times M$ fuertemente elíptico, de segundo orden, homogéneo y con coeficientes complejos constantes y fijamos un parámetro de apertura $\kappa>0$. Asumimos que*

$$\begin{cases} u\in\left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \quad Lu=0 \;\text{ en }\; \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}}(\mathcal{N}_\kappa u)(x')\,\frac{dx'}{1+|x'|^{n-1}}<\infty, \end{cases}$$

*donde $\mathcal{N}_\kappa$ denota el operador maximal no tangencial (cf. (1.1.2)). Entonces,*

$$\begin{cases} u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} \;\text{ existe en } \mathscr{L}^{n-1}\text{-c.t.p. en } \mathbb{R}^{n-1}, \\ u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} \;\text{ pertenece a }\; \left[L^1\Big(\mathbb{R}^{n-1},\,\dfrac{dx'}{1+|x'|^{n-1}}\Big)\right]^M, \\ u(x',t)=\Big(P^L_t*\big(u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}}\big)\Big)(x') \;\text{ para cada }\; (x',t)\in\mathbb{R}^n_+, \end{cases}$$

*donde $P^L=\big(P^L_{\beta\alpha}\big)_{1\le\beta,\alpha\le M}$ denota el núcleo de Poisson para $L$ en $\mathbb{R}^n_+$ del Teorema 1.2.4 y $P^L_t(x'):=t^{1-n}P^L(x'/t)$ para cada $x'\in\mathbb{R}^{n-1}$ y $t>0$.*

Este resultado refina [82, Teorema 6.1, p. 956], donde se asume una condición de integrabilidad más restrictiva. Es conveniente notar que incluso en el caso clásico en que $L := \Delta$ es el laplaciano en $\mathbb{R}^n$, el teorema anterior es más general (en el sentido de que se puede aplicar sobre una clase más amplia de funciones) que los resultados que existen en la literatura. En efecto, normalmente se asume una condición de integrabilidad en $L^p$ para la función armónica que, en el rango $p \in (1, \infty)$, implica nuestra condición de integrabilidad sobre el espacio $L^1$ con peso sobre la función maximal no tangencial. En este sentido, véase, por ejemplo, [42, Teorema 4.8-4.9, pp. 174-175], [115, Corolario, p. 200], [116, Proposición 1, p. 119].

Además, este teorema de Fatou tiene resultados de unicidad asociados de manera natural, que permiten demostrar resultados muy generales que indican que ciertos problemas de valor en la frontera elípticos están bien propuestos (cf. Corolarios 3.1.3 y 3.1.4).

Continuando con el estudio de problemas de valor en la frontera en el semiespacio superior, en el **Capítulo 4** estudiamos el problema de Dirichlet para sistemas elípticos con dato en la frontera en espacios generalizados de Hölder y espacios generalizados de Morrey-Campanato. Además, mediante técnicas basadas en EDP, probamos que estos dos espacios de funciones son de hecho equivalentes.

Los espacios generalizados de Hölder, denotados por $\dot{\mathscr{C}}^\omega(\partial\Omega, \mathbb{C}^M)$, cuantifican la continuidad en términos de un módulo o función de crecimiento, $\omega$. Específicamente, dado $U \subseteq \mathbb{R}^n$, $M \geq 1$ y una función no decreciente $\omega : (0, \infty) \to (0, \infty)$ cuyo límite en el origen se anula, el espacio homogéneo $\dot{\mathscr{C}}^\omega(U, \mathbb{C}^M)$ es la colección de funciones $u : U \to \mathbb{C}^M$ tales que

$$[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} := \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{\omega(|x - y|)} < \infty.$$

De forma similar, para $D \in (0, \infty]$ y una función no decreciente $\omega : (0, D) \to (0, \infty)$ cuyo límite en el origen se anula y que es acotada si $D < \infty$, el espacio $\mathscr{C}^\omega(U, \mathbb{C}^M)$ viene definido por la norma

$$\|u\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} := \sup_U |u| + [u]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(U,\mathbb{C}^M)},$$

donde $\widetilde{\omega}(t) := \omega(\min\{t, D\})$ para cada $t \in (0, \infty)$.

Dada una función no decreciente $\omega : (0, \infty) \to (0, \infty)$ cuyo límite en el origen se anula, así como un exponente de integrabilidad $p \in [1, \infty)$, definimos la seminorma

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \sup_{Q \subseteq \mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left( \fint_Q |f(x') - f_Q|^p \, dx' \right)^{1/p},$$

y denotamos el espacio de funciones asociado por $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, llamado espacio generalizado de Morrey-Campanato en $\mathbb{R}^{n-1}$. La elección $\omega(t) := t^\alpha$ con $\alpha \in (0, 1)$ corresponde con los espacios clásicos de Morrey-Campanato, mientras que el caso especial $\omega(t) := 1$ produce el espacio usual de oscilación media acotada (BMO). También definimos, para cada $u \in [\mathscr{C}^1(\mathbb{R}^n_+)]^M$ y $q \in (0, \infty)$,

$$\|u\|_{**}^{(\omega,q)} := \sup_{Q \subseteq \mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))} \left( \fint_Q \left( \int_0^{\ell(Q)} |(\nabla u)(x', t)|^2 \, t \, dt \right)^{q/2} dx' \right)^{1/q}.$$

Enunciamos a continuación nuestro resultado principal a este respecto, que está incluido en el Teorema 4.1.2 y que generaliza resultados de [85] (donde se estudia el caso $\omega(t) = t^\alpha$ con $\alpha \in (0,1)$) permitiendo escalas más flexibles al medir la regularidad en los espacios de Hölder y Morrey-Campanato.

**Teorema.** *Sea $L$ un sistema $M \times M$ fuertemente elíptico, de segundo orden, homogéneo y con coeficientes complejos constantes. Fijamos además un parámetro de apertura $\kappa > 0$, $p \in [1,\infty)$, así como $q \in (0,\infty)$. Finalmente, sea $\omega : (0,\infty) \to (0,\infty)$ una función no decreciente cuyo límite en el origen se anula y que satisface*

$$\sup_{t>0} \left\{ \frac{1}{\omega(t)} \left( \int_0^t \omega(s) \frac{ds}{s} + t \int_t^\infty \frac{\omega(s)}{s} \frac{ds}{s} \right) \right\} < +\infty.$$

*Entonces las siguientes afirmaciones son ciertas.*

(a) *El problema de Dirichlet en el espacio generalizado de Hölder para el sistema $L$ en $\mathbb{R}^n_+$, es decir,*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \\ Lu = 0 \quad en \quad \mathbb{R}^n_+, \\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} < \infty, \\ u|_{\partial\mathbb{R}^n_+} = f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M) \quad en \quad \mathbb{R}^{n-1}, \end{cases}$$

*está bien propuesto. Más específicamente, existe una única solución, que viene dada por*

$$u(x',t) = (P_t^L * f)(x'), \qquad \forall\, (x',t) \in \mathbb{R}^n_+,$$

*donde $P^L$ denota el núcleo de Poisson para $L$ en $\mathbb{R}^n_+$ del Teorema 1.2.4. Además, $u$ pertenece al espacio $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$, satisface $u|_{\partial\mathbb{R}^n_+} = f$ y existe una constante finita $C = C(n,L,\omega) \geq 1$ tal que*

$$C^{-1}[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}.$$

(b) *El problema de Dirichlet en el espacio generalizado de Morrey-Campanato para el sistema $L$ en $\mathbb{R}^n_+$, formulado como*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \\ Lu = 0 \quad en \quad \mathbb{R}^n_+, \\ \|u\|_{**}^{(\omega,q)} < \infty, \\ u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} = f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \quad c.t.p. \ en \quad \mathbb{R}^{n-1}, \end{cases}$$

*está bien propuesto. Concretamente, existe una única solución, que viene dada por*

$$u(x',t) = (P_t^L * f)(x'), \qquad \forall\, (x',t) \in \mathbb{R}^n_+,$$

*donde $P^L$ denota el núcleo de Poisson para $L$ en $\mathbb{R}^n_+$ del Teorema 1.2.4. Además, $u$ pertenece a $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$, satisface $u|_{\partial\mathbb{R}^n_+} = f$ en casi todo punto de $\mathbb{R}^{n-1}$ y existe una constante finita $C = C(n,L,\omega,p,q) \geq 1$ tal que*

$$C^{-1}\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \|u\|_{**}^{(\omega,q)} \leq C\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}.$$

(c) *Se tiene la siguiente igualdad entre espacios vectoriales*

$$\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$$

*con normas equivalentes, donde la inclusión de derecha a izquierda se entiende en el sentido de que para cada* $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *existe una única* $\widetilde{f} \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *con la propiedad de que* $f = \widetilde{f}$ *en casi todo punto de* $\mathbb{R}^{n-1}$.

*Como resultado, el problema de Dirichlet en el espacio generalizado de Hölder de (a) y el problema de Dirichlet en el espacio generalizado de Morrey-Campanato de (b) son equivalentes. Específicamente, para cada par de datos en la frontera que puedan ser indentificados en el sentido descrito en el párrafo anterior, estos problemas tienen la misma solución única.*

Notemos que en la Sección 4.7 debilitamos las hipótesis sobre la función de crecimiento y probamos que los problemas de Dirichlet están bien propuestos. La diferencia principal es que en este caso no son equivalentes (ver Ejemplo 4.7.4).

La interacción entre el análisis y la geometría descrita al comienzo de esta sección nos permite dar una caracterización de ciertas clases de dominios basándonos en condiciones puramente analíticas. Específicamente, en el **Capítulo 5** caracterizamos dominios de Lyapunov $\mathscr{C}^{1,\omega}$. Los dominios de Lyapunov $\mathscr{C}^{1,\omega}$ son conjuntos abiertos con perímetro localmente finito cuyo normal unitario exterior $\nu$ pertenece a $\mathscr{C}^{\omega}(\partial\Omega)$ (después de, posiblemente, ser modificado en un conjunto de $\sigma$-medida cero). Aquí, para simplificar la notación, llamamos $\mathscr{C}^{\omega}(U) := \mathscr{C}^{\omega}(U, \mathbb{C})$.

Usando ideas de [52], la clase de dominios $\mathscr{C}^{1,\omega}$ puede ser descrita también como la colección de todos los subconjuntos abiertos de $\mathbb{R}^n$ que localmente coinciden (tras una transformación rígida del espacio) con la región sobre el grafo de una función continuamente diferenciable que toma valores reales, definida en $\mathbb{R}^{n-1}$, cuyas derivadas parciales de primer orden pertenecen a $\mathscr{C}^{\omega}(\mathbb{R}^{n-1})$.

Las caracterizaciones de la clase de dominios de Lyapunov que probamos vienen dadas en términos de las propiedades de acotación de ciertas clases de operadores integrales singulares actuando en espacios de Hölder generalizados en la frontera de un dominio Ahlfors regular $\Omega \subseteq \mathbb{R}^n$ con frontera compacta (cf. Definición 1.1.2). El ejemplo más importante de estos operadores integrales singulares son las transformadas de Riesz $R_j$ (cf. (5.1.3)-(5.1.4)).

Nuestro trabajo añade credibilidad al principio heurístico de que la acción de la transformada distribucional de Riesz sobre la función constante 1 encierra mucha información, de tipo analítico y también geométrico, sobre el dominio Ahlfors regular subyacente $\Omega \subseteq \mathbb{R}^n$ (con frontera compacta). Al nivel más básico, el resultado principal de F. Nazarov, X. Tolsa y A. Volberg en [101] establece que

$$\partial\Omega \ \text{ es un conjunto UR } \Longleftrightarrow R_j 1 \in \mathrm{BMO}(\partial\Omega, \sigma) \quad \text{para cada} \quad j \in \{1, \dots, n\}$$

y se ha probado en [96] que

$$\left.\begin{array}{r} \nu \in \mathrm{VMO}(\partial\Omega, \sigma) \\ \text{y } \partial\Omega \text{ es un conjunto UR} \end{array}\right\} \Longleftrightarrow R_j 1 \in \mathrm{VMO}(\partial\Omega, \sigma) \quad \text{para todo} \quad j \in \{1, \dots, n\},$$

donde $\mathrm{VMO}(\partial\Omega, \sigma)$ denota el espacio de Sarason de funciones en $\partial\Omega$ con oscilación media que se anula, con respecto a la medida $\sigma$. Añadiendo más regularidad a los funcionales $\{R_j 1\}_{1\leq j\leq n}$ obtenemos el siguiente resultado (probado en [96])

$$\left.\begin{array}{r}\Omega \text{ es un dominio}\\ \text{de clase } \mathscr{C}^{1+\alpha}\end{array}\right\} \Longleftrightarrow R_j 1 \in \mathscr{C}^{\alpha}(\partial\Omega) \ \text{ para todo } \ j \in \{1, \dots, n\},$$

donde $\alpha \in (0,1)$ y $\mathscr{C}^{\alpha}(\partial\Omega)$ es el espacio clásico de Hölder de orden $\alpha$ en $\partial\Omega$. Este resultado es generalizado por el siguiente, contenido en el Teorema 5.1.4, que nos permite considerar escalas más flexibles para medir la regularidad Hölder (ver la exposición en el Ejemplo 1.3.4 a este respecto).

**Teorema.** *Sea $\Omega \subset \mathbb{R}^n$ un dominio Ahlfors regular cuya frontera es compacta. Abreviamos $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ y denotamos por $\nu$ al vector unitario exterior a $\Omega$. Además, definimos $\Omega_+ := \Omega$ y $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Finalmente, sea $\omega : (0, \mathrm{diam}(\partial\Omega)) \to (0, \infty)$ una función acotada, no decreciente, cuyo límite en el origen se anula y que satisface*

$$\sup_{0<t<\mathrm{diam}(\partial\Omega)} \left\{ \frac{1}{\omega(t)} \left( \int_0^t \omega(s) \frac{ds}{s} + t \int_t^{\mathrm{diam}(\partial\Omega)} \frac{\omega(s)}{s} \frac{ds}{s} \right) \right\} < +\infty.$$

*Entonces las siguientes afirmaciones son equivalentes:*

(a) *Después de ser posiblemente modificado en un conjunto de $\sigma$-medida cero, el normal unitario exterior $\nu$ a $\Omega$ pertenece al espacio generalizado de Hölder $\mathscr{C}^{\omega}(\partial\Omega)$.*

(b) *Las transformadas de Riesz en $\partial\Omega$ satisfacen*

$$R_j 1 \in \mathscr{C}^{\omega}(\partial\Omega) \ \ \textit{para cada} \ \ j \in \{1, \dots, n\}.$$

(c) *El conjunto $\Omega$ es un dominio* UR *(en el sentido de la Definición 1.1.5), y dado un polinomio homogéneo e impar $P$ de grado $\ell \geq 1$ en $\mathbb{R}^n$ el operador integral singular que actúa en cada función $f \in \mathscr{C}^{\omega}(\partial\Omega)$ de acuerdo a la fórmula*

$$(Tf)(x) := \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{P(x-y)}{|x-y|^{n-1+\ell}} f(y)\, d\sigma(y) \ \ \textit{for } \sigma\textit{-c.t.p.} \ \ x \in \partial\Omega$$

*está bien definido y acotado del espacio generalizado de Hölder $\mathscr{C}^{\omega}(\partial\Omega)$ en sí mismo.*

(d) *El conjunto $\Omega$ es un dominio* UR *y la versión en la frontera de las transformadas de Riesz, definidas para cada $j \in \{1, \dots, n\}$ y cada $f \in L^1(\partial\Omega, \sigma)$ como*

$$(\mathscr{R}_j^{\pm} f)(x) := \frac{1}{\varpi_{n-1}} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_{\pm},$$

*satisfacen*

$$\mathscr{R}_j^{\pm} 1 \in \mathscr{C}^{\omega}(\Omega_{\pm}) \ \ \textit{para cada} \ \ j \in \{1, \dots, n\}.$$

(*e*) *El conjunto* $\Omega$ *es un dominio* UR *y dado un polinomio homogéneo e impar* $P$ *de grado* $\ell \geq 1$ *en* $\mathbb{R}^n$*, los operadores integrales que actúan en cada función* $f \in \mathscr{C}^\omega(\partial\Omega)$ *de acuerdo a la fórmula*

$$\mathbb{T}_\pm f(x) := \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+\ell}}\, f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_\pm,$$

*están acotados del espacio generalizado de Hölder* $\mathscr{C}^\omega(\partial\Omega)$ *al espacio* $\mathscr{C}^\omega(\Omega_\pm)$*.*

Esta tesis ha dado lugar a los siguientes artículos:

(a) *Singular integral operators, quantitative flatness, and boundary problems*, book manuscript, 2019 (trabajo conjunto con J.M. Martell, D. Mitrea, I. Mitrea y M. Mitrea).

(b) *A Fatou theorem and Poisson's integral representation formula for elliptic systems in the upper-half space*, to appear in "Topics in Clifford Analysis", special volume in honor of Wolfgang Sprößig, Swanhild Bernstein editor, Birkhäuser, 2019 (trabajo conjunto con J.M. Martell, D. Mitrea, I. Mitrea y M. Mitrea).

(c) *The generalized Hölder and Morrey-Campanato Dirichlet problems for elliptic systems in the upper-half space*, to appear in Potential Anal., 2019 (trabajo conjunto con J.M. Martell y M. Mitrea).

(d) *Characterizations of Lyapunov domains in terms of Riesz transforms and generalized Hölder spaces*, preprint, 2019 (trabajo conjunto con J.M. Martell y M. Mitrea).

El material en ($a$) está elaborado en el Capítulo 2, ($b$) está contenido en el Capítulo 3, ($c$) está desarrollado en el Capítulo 4 y ($d$) está expuesto en el Capítulo 5. Corresponden, respectivamente, a [77], [76], [79] y [78] en la bibliografía.

CHAPTER 1

# Preliminaries

## Contents



We begin with a quick review of notational conventions used in the dissertation. Throughout, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ with $n \geq 2$, and $\mathcal{L}^n$ stands for the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. For each $k \in \mathbb{N}$, we denote by $\mathbb{N}_0^k$ the collection of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in \mathbb{N}_0$ for $1 \leq j \leq k$. Also, we let $\mathcal{H}^{n-1}$ denote the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. For each set $E \subseteq \mathbb{R}^n$, we let $\mathbf{1}_E$ denote the characteristic function of $E$ (i.e., $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \in \mathbb{R}^n \setminus E$). Also, $\delta_{jk}$ is the Kronecker symbol (i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$). By $\{\mathbf{e}_j\}_{1\leq j\leq n}$ we shall denote the standard orthonormal basis in $\mathbb{R}^n$, i.e., $\mathbf{e}_j := (\delta_{jk})_{1\leq k\leq n}$ for each $j \in \{1, \ldots, n\}$. For each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ set $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The dot product of two vectors $u, v \in \mathbb{R}^n$ is denoted by $u \cdot v = \langle u, v\rangle$. Next, $\mathbb{R}^n_\pm := \{x \in \mathbb{R}^n : \pm\langle x, \mathbf{e}_n\rangle > 0\}$ denote, respectively, the upper-space and lower half-space in $\mathbb{R}^n$. For an arbitrary open set $\Omega \subseteq \mathbb{R}^n$ we shall let $\mathcal{D}'(\Omega)$ stand for the space of distributions in $\Omega$ and $\mathcal{E}'(\Omega)$ will denote the space of compactly supported distributions in $\Omega$. Given an integrability exponent $p \in [1, \infty]$ along with an integer $k \in \mathbb{N}$, we shall define the local $L^p$-based Sobolev space of order $k$ in $\Omega$ as $W^{k,p}_{\rm loc}(\Omega) := \{u \in \mathcal{D}'(\Omega) : \partial^\alpha u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n),\ |\alpha| \leq k\}$. Next, $S^{n-1} := \partial B(0, 1)$ denotes the unit sphere in $\mathbb{R}^n$, and $\varpi_{n-1} = \omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$ is the surface area of $S^{n-1}$. In addition, we shall let $\upsilon_{n-1}$ denote the volume of the unit ball in $\mathbb{R}^{n-1}$. Given any

$x, y \in \mathbb{R}^n$, by $[x, y]$ we shall denotes the line segment with endpoints $x, y$. We shall also need $\operatorname{dist}(x, E) := \inf\{|x-y| : y \in E\}$, the distance from a given point $x \in \mathbb{R}^n$ to a nonempty set $E \subseteq \mathbb{R}^n$. If $(X, \mu)$ is a given measure space, for each $p \in (0, \infty]$ we shall denote by $L^p(X, \mu)$ the Lebesgue space of $\mu$-measurable functions which are $p$-th power integrable on $X$ with respect to $\mu$. Also, by $L^{p,q}(X, \mu)$ with $p, q \in (0, \infty]$ we shall denote the scale of Lorentz spaces on $X$ with respect to the measure $\mu$. In the same setting, for each $\mu$-measurable set $E \subseteq X$ with $0 < \mu(E) < \infty$ and each function $f$ which is absolutely integrable on $E$ we set $⨍_E f\, d\mu := \mu(E)^{-1} \int_E f\, d\mu$.

Finally, we adopt the common convention of writing $A \approx B$ if there exists a constant $C \in (1, \infty)$ with the property that $A/C \leq B \leq CA$ for all values of the relevant parameters entering the definitions of $A, B$ (something that is self evident in each context we employ this notation).

## 1.1 Classes of Euclidean sets of locally finite perimeter

Given an open set $\Omega \subseteq \mathbb{R}^n$ and an aperture parameter $\kappa \in (0, \infty)$, define the non-tangential approach regions

$$\Gamma_\kappa(x) := \big\{y \in \Omega : \ |y - x| < (1+\kappa) \operatorname{dist}(y, \partial\Omega)\big\} \ \text{ for each } \ x \in \partial\Omega. \tag{1.1.1}$$

In turn, these are used to define the nontangential maximal operator $\mathcal{N}_\kappa$, acting on each $\mathcal{L}^n$-measurable function $u$ defined in $\Omega$ according to

$$(\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega, \tag{1.1.2}$$

with the convention that $(\mathcal{N}_\kappa u)(x) := 0$ whenever $x \in \partial\Omega$ is such that $\Gamma_\kappa(x) = \varnothing$. Note that, if we work (as one usually does) with equivalence classes, obtained by identifying functions which coincide $\mathcal{L}^n$-a.e., the nontangential maximal operator is independent of the specific choice of a representative in a given equivalence class. It turns out that $\mathcal{N}_\kappa u : \partial\Omega \to [0, +\infty]$ is a lower-semicontinuous function. Also, it is apparent from definitions that

$$\begin{gathered} \text{whenever } u \in \mathscr{C}^0(\Omega) \text{ one actually has} \\ (\mathcal{N}_\kappa u)(x) = \sup_{y \in \Gamma_\kappa(x)} |u(y)| \ \text{ for all } \ x \in \partial\Omega. \end{gathered} \tag{1.1.3}$$

More generally, if $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function and $E \subseteq \Omega$ is a $\mathcal{L}^n$-measurable set, we denote by $\mathcal{N}_\kappa^E u$ the non-tangential maximal function of $u$ restricted to $E$, i.e.,

$$\begin{gathered} \mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \ \text{ defined as} \\ (\mathcal{N}_\kappa^E u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x) \cap E, \mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega. \end{gathered} \tag{1.1.4}$$

Hence, $\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(u \cdot \mathbf{1}_E)$. Throughout, we agree to use the simpler notation $\mathcal{N}_\kappa^\delta$ in the case when $E = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$ for some $\delta \in (0, \infty)$.

Continue to assume that $\Omega$ is an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$ and suppose $u$ is some vector-valued $\mathcal{L}^n$-measurable function defined in $\Omega$. Also, fix an aperture parameter $\kappa > 0$ and consider a point $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ (i.e., $x$ is an accumulation point for the nontangential approach region $\Gamma_\kappa(x)$). In this context, we shall say that the nontangential limit of $u$ at $x$ from within $\Gamma_\kappa(x)$ exists, and its value is the vector $a \in \mathbb{C}^M$, provided

$$\begin{array}{l} \text{for every } \varepsilon > 0 \text{ there exists some } r > 0 \text{ such that } |u(y) - \\ a| < \varepsilon \text{ for } \mathcal{L}^n\text{-a.e. point } y \in \Gamma_\kappa(x) \cap B(x,r). \end{array} \tag{1.1.5}$$

Whenever the nontangential limit of $u$ at $x$ from within $\Gamma_\kappa(x)$ exists, we agree to denote its value by the symbol $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$.

Moving on, recall that an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter if its `measure theoretic boundary`, i.e.,

$$\partial_*\Omega := \Big\{x \in \partial\Omega :\ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap\Omega)}{r^n} > 0,\ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus\Omega)}{r^n} > 0\Big\}, \tag{1.1.6}$$

satisfies

$$\mathcal{H}^{n-1}(\partial_*\Omega \cap K) < +\infty \quad \text{for each compact} \quad K \subseteq \mathbb{R}^n \tag{1.1.7}$$

(cf. [38, Sections 5.7 and 5.11]). Alternatively, an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter if, with the gradient taken in the sense of distributions in $\mathbb{R}^n$,

$$\mu_\Omega := \nabla \mathbf{1}_\Omega \tag{1.1.8}$$

is an $\mathbb{R}^n$-valued Borel measure in $\mathbb{R}^n$ of locally finite total variation. Fundamental work of De Giorgi-Federer (cf., e.g., [38]) then gives the following Polar Decomposition of the Radon measure $\mu_\Omega$:

$$\mu_\Omega = \nabla \mathbf{1}_\Omega = -\nu\,|\nabla \mathbf{1}_\Omega| \tag{1.1.9}$$

where $|\nabla \mathbf{1}_\Omega|$, the total variation measure of the measure $\nabla \mathbf{1}_\Omega$, is given by

$$|\nabla \mathbf{1}_\Omega| = \mathcal{H}^{n-1}\lfloor\partial_*\Omega, \tag{1.1.10}$$

and where

$$\begin{array}{l} \nu \in \big[L^\infty(\partial_*\Omega, \mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function} \\ \text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial_*\Omega. \end{array} \tag{1.1.11}$$

We shall refer to $\nu$ above as the `geometric measure theoretic outward unit normal` to $\Omega$. Note here that by simply eliminating the distribution theory jargon implicit in the interpretation of (1.1.9) (and using a straightforward limiting argument involving a mollifier) one already arrives at the formula

$$\begin{array}{c} \displaystyle\int_\Omega \mathrm{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial\Omega}\big)\, d\mathcal{H}^{n-1} \\ \text{for each vector field } \vec{F} \in \big[\mathscr{C}^1_0(\mathbb{R}^n)\big]^n. \end{array} \tag{1.1.12}$$

For a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, we let $\partial^*\Omega$ denote the `reduced boundary` of $\Omega$, that is,

$$\begin{aligned}&\partial^*\Omega \text{ consists of all points } x \in \partial\Omega \text{ satisfying the following}\\&\text{properties: } 0 < \mathcal{H}^{n-1}(B(x,r) \cap \partial_*\Omega) < +\infty \text{ for each } r \in\\&(0,\infty), \text{ and } \lim_{r\to 0^+} \fint_{B(x,r)\cap\partial_*\Omega} \nu \, d\mathcal{H}^{n-1} = \nu(x) \in S^{n-1}.\end{aligned} \tag{1.1.13}$$

For any set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter we then have (cf. [38, p. 208])

$$\partial^*\Omega \subseteq \partial_*\Omega \subseteq \partial\Omega \quad \text{and} \quad \mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0. \tag{1.1.14}$$

**Definition 1.1.1.** A closed set $\Sigma \subseteq \mathbb{R}^n$ is called an `Ahlfors regular set` (or an Ahlfors-David regular set) if there exists a constant $C \in [1,\infty)$ such that

$$r^{n-1}/C \leq \mathcal{H}^{n-1}(B(x,r) \cap \Sigma) \leq C r^{n-1}, \quad \forall\, r \in (0\,,\,2\,\mathrm{diam}\,(\Sigma)), \quad \forall\, x \in \Sigma. \tag{1.1.15}$$

We say that $\Sigma$ is a lower Ahlfors regular set if it satisfies the first inequality in (1.1.15) and an upper Ahlfors regular set if it satisfies the second inequality in (1.1.15).

For a given closed set $\Sigma \subseteq \mathbb{R}^n$, being Ahlfors regular is not a regularity condition in a traditional analytic sense, but rather a property guaranteeing that, at all locations, $\Sigma$ behaves (in a quantitative, scale-invariant fashion) like an $(n-1)$-dimensional "surface," with respect to the Hausdorff measure $\mathcal{H}^{n-1}$. For example, the classical four-corner Cantor set in the plane is an Ahlfors regular set (cf., e.g., [94, Proposition 4.79, p. 238]).

**Definition 1.1.2.** An open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is called an `Ahlfors regular domain` provided $\partial\Omega$ is an Ahlfors regular set and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.

If $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain then the upper Ahlfors regularity condition satisfied by $\partial\Omega$ (i.e., the second inequality in (1.1.15) with $\Sigma := \partial\Omega$) together with (1.1.7) guarantee that $\Omega$ is a set of locally finite perimeter. Also, the fact that the measure theoretic boundary $\partial_*\Omega$ is presently assumed to have full measure (with respect to $\mathcal{H}^{n-1}$) in the topological boundary $\partial\Omega$, ensures that the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ (cf. (1.1.11)) is actually well defined at $\mathcal{H}^{n-1}$-a.e. point on $\partial\Omega$. Ultimately,

$$\begin{gathered}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an Ahlfors regular domain then}\\\nu \in \big[L^\infty(\partial\Omega, \mathcal{H}^{n-1})\big]^n \text{ is an } \mathbb{R}^n\text{-valued function}\\\text{satisfying } |\nu(x)| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial\Omega.\end{gathered} \tag{1.1.16}$$

From [53, Proposition 2.9, p. 2588] we also know that

$$\begin{aligned}&\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an Ahlfors regular domain, and if } \kappa > 0 \text{ is an}\\&\text{arbitrary aperture parameter, then } x \in \overline{\Gamma_\kappa(x)} \text{ (i.e., } x \text{ is an}\\&\text{accumulation point for the nontangential approach region } \Gamma_\kappa(x))\\&\text{for } \mathcal{H}^{n-1}\text{-a.e. point } x \text{ in the topological boundary } \partial\Omega.\end{aligned} \tag{1.1.17}$$

In particular, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and $u$ is an $\mathcal{L}^n$-measurable function $u$ defined in $\Omega$, then for any fixed aperture parameter $\kappa > 0$ it is meaningful to attempt to define the nontangential boundary trace $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial\Omega$. For future endeavors, it is also useful to remark that (see [93] for a proof)

$$\begin{array}{l}\text{if } \Omega \subset \mathbb{R}^n \text{ is an Ahlfors regular domain then } \Omega_- := \mathbb{R}^n \setminus \overline{\Omega} \text{ is also}\\ \text{an Ahlfors regular domain, whose topological boundary coincides}\\ \text{with that of } \Omega\text{, and whose geometric measure theoretic boundary}\\ \text{agrees with that of } \Omega\text{, i.e., } \partial(\Omega_-) = \partial\Omega \text{ and } \partial_*(\Omega_-) = \partial_*\Omega.\\ \text{Moreover, the geometric measure theoretic outward unit normal}\\ \text{to } \Omega_- \text{ is } -\nu \text{ at } \sigma\text{-a.e. point on } \partial\Omega.\end{array} \tag{1.1.18}$$

We continue by recalling the notion of countable rectifiability.

**Definition 1.1.3.** A closed set $E \subseteq \mathbb{R}^n$ is said to be `countably rectifiable` (of dimension $(n-1)$) provided

$$E = \left(\bigcup_{j=1}^{\infty} S_j\right) \cup N, \tag{1.1.19}$$

where $N$ is a null-set for $\mathcal{H}^{n-1}$ and each $S_j$ is the image of a compact subset of $\mathbb{R}^{n-1}$ under a Lipschitz map from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$.

The following definition is due to G. David and S. Semmes (cf. [34]).

**Definition 1.1.4.** A closed set $\Sigma \subseteq \mathbb{R}^n$ is said to be a `uniformly rectifiable` set (or simply a UR set) if $\Sigma$ is an Ahlfors regular set and there exist $\varepsilon, M \in (0,\infty)$ such that for each location $x \in \Sigma$ and each scale $R \in (0\,,\,2\,\mathrm{diam}\,(\Sigma))$ it is possible to find a Lipschitz map $\varphi : B^{n-1}_R \to \mathbb{R}^n$ (where $B^{n-1}_R$ is a ball of radius $R$ in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$ and such that

$$\mathcal{H}^{n-1}(\Sigma \cap B(x,R) \cap \varphi(B^{n-1}_R)) \geq \varepsilon R^{n-1}. \tag{1.1.20}$$

Uniformly rectifiability is a quantitative version of countable rectifiability. Any UR set is countably rectifiable. We also remark that any Ahlfors regular domain in $\mathbb{R}^n$ has a $(n-1)$-dimensional countably rectifiable boundary (cf. [96, Section 2]). Following [53] we also make the following definition.

**Definition 1.1.5.** An open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is called a `UR domain` (short for uniformly rectifiable domain) provided $\partial\Omega$ is a UR set (in the sense of Definition 1.1.4) and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.

By design, any UR domain is an Ahlfors regular domain. A basic subclass of UR domains has been identified by G. David and D. Jerison in [32]. To state (a version of) their result, we first recall the following definition.

**Definition 1.1.6.** Fix $R \in (0,\infty]$ and $c \in (0,1)$. A nonempty proper subset $\Omega$ of $\mathbb{R}^n$ is said to satisfy the $(R,c)$-`corkscrew condition` (or, simply, a `corkscrew condition` if the particular values of $R, c$ are not important) if for each location $x \in \partial\Omega$ and each scale $r \in (0, R)$ there exists a point $z \in \Omega$ (called a corkscrew point relative to $x$ and $r$) with the property that $B(z, c\,r) \subseteq B(x, r) \cap \Omega$.

Also, a nonempty proper subset $\Omega$ of $\mathbb{R}^n$ is said to satisfy the $(R, c)$-`two-sided corkscrew condition` provided both $\Omega$ and $\mathbb{R}^n \setminus \Omega$ satisfy the $(R, c)$-corkscrew condition (with the same convention regarding the omission of $R, c$).

It is then clear from definitions that we have

$$\partial_*\Omega = \partial\Omega \text{ for any } \mathcal{L}^n\text{-measurable set } \Omega \subseteq \mathbb{R}^n \text{ satisfying a two-sided corkscrew condition.} \tag{1.1.21}$$

Also, [32, Theorem 1, p. 840] implies that

if $\Omega$ is a nonempty proper open subset of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition and whose boundary is an Ahlfors regular set, then $\Omega$ is a UR domain. (1.1.22)

Following [57], we define the class of nontangentially accessible domains as those open sets satisfying a two-sided corkscrew conditions and the following Harnack chain condition.

**Definition 1.1.7.** Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open set $\Omega \subseteq \mathbb{R}^n$ is said to satisfy the $(R, N)$-`Harnack chain condition` (or, simply, a `Harnack chain condition` if the particular values of $R, N$ are irrelevant) provided whenever $\varepsilon > 0$, $k \in \mathbb{N}$, $z \in \partial\Omega$, $r \in (0, R)$, and $x, y \in B(z, r/4) \cap \Omega$ satisfy $|x - y| \leq 2^k\varepsilon$ and $\min\{\mathrm{dist}\,(x, \partial\Omega), \mathrm{dist}\,(y, \partial\Omega)\} \geq \varepsilon$, one may find a chain of balls $B_1, B_2, \dots, B_K$ with $K \leq Nk$, such that $x \in B_1$, $y \in B_K$, $B_i \cap B_{i+1} \neq \varnothing$ for every $i \in \{1, \dots, K-1\}$, and

$$N^{-1} \cdot \mathrm{diam}\,(B_i) \leq \mathrm{dist}\,(B_i, \partial\Omega) \leq N \cdot \mathrm{diam}\,(B_i), \tag{1.1.23}$$

$$\mathrm{diam}\,(B_i) \geq N^{-1} \cdot \min\{\mathrm{dist}\,(x, B_i), \mathrm{dist}\,(y, B_i)\}, \tag{1.1.24}$$

for every $i \in \{1, \dots, K\}$.

Following [57, pp. 93-94] (cf. also [64, Definition 2.1, p. 3]), we introduce the class of NTA domains.

**Definition 1.1.8.** Fix $R \in (0, \infty]$ and $N \in \mathbb{N}$. An open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is said to be a $(R, N)$-`nontangentially accessible domain` (or simply an NTA domain if the particular values of $R, N$ are not important) if $\Omega$ satisfies both the $(R, N^{-1})$-two-sided corkscrew condition and the $(R, N)$-Harnack chain condition. Finally, an open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is said to be a $(R, N)$-`two-sided nontangentially accessible domain` (or, simply, a `two-sided NTA domain` if the particular values of $R, N$ are not relevant) provided both $\Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ are $(R, N)$-nontangentially accessible domains.

There is also the related notion of $(R,N)$-one-sided NTA domain, i.e., an open set satisfying a $(R,N)$-Harnack chain condition and a $(R,N^{-1})$-corkscrew condition (once again, we agree to drop the parameters $R,N$ if theirs values are not relevant). For example, the complement of the classical four-corner Cantor set in the plane is a one-sided NTA domain with an Ahlfors regular boundary. We continue with the definition of uniform domains.

**Definition 1.1.9.** A nonempty, proper, open subset $\Omega$ of $\mathbb{R}^n$ is called a `uniform domain` if there exist $\varkappa \in [1,\infty)$ such that any two points $x,y \in \Omega$ may be joined in $\Omega$ by a rectifiable path $\gamma$ satisfying

$$\begin{aligned} &\mathrm{length}(\gamma) \leq \varkappa|x-y| \quad \text{and, for each } z \in \gamma, \\ &\min\big\{\mathrm{length}(\gamma_{x,z})\,,\,\mathrm{length}(\gamma_{y,z})\big\} \leq \varkappa\,\mathrm{dist}(z,\partial\Omega), \end{aligned} \tag{1.1.25}$$

where $\gamma_{x,z}$ and $\gamma_{y,z}$ are the sub-arcs of $\gamma$ joining $z$ with $x$ and $y$, respectively.

The following definition of yet another brand of local path connectivity condition first appeared in [53].

**Definition 1.1.10.** An open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is said to satisfy a `local John condition` if there exist $\theta \in (0,1)$ and $R > 0$ (with the requirement that $R = \infty$ if $\partial\Omega$ is unbounded) such that for every $x \in \partial\Omega$ and $r \in (0,R)$ one may find $x_r \in B(x,r)\cap\Omega$ such that $B(x_r,\theta r) \subseteq \Omega$ and with the property that for each $y \in B(x,r)\cap\partial\Omega$ there exists a rectifiable path $\gamma_y : [0,1] \to \overline{\Omega}$ whose length is $\leq \theta^{-1}r$ and such that

$$\gamma_y(0) = y, \quad \gamma_y(1) = x_r, \quad \mathrm{dist}\big(\gamma_y(t)\,,\,\partial\Omega\big) > \theta|\gamma_y(t) - y| \ \text{ for all } \ t \in (0,1]. \tag{1.1.26}$$

Finally, a nonempty open set $\Omega \subseteq \mathbb{R}^n$ which is not dense in $\mathbb{R}^n$ is said to satisfy a `two-sided local John condition` if both $\Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a local John condition.

It is clear from the definitions that

> any set satisfying a local John condition (respectively, a two-sided local John condition) also satisfies a corkscrew condition (respectively, a two-sided corkscrew condition). (1.1.27)

Moreover, given any $R \in (0,\infty]$ and $N \in \mathbb{N}$, from [53, Lemma 3.13, p. 2634] we know that

> any $(R,N)$-nontangentially accessible domain satisfies a local John condition, and any $(R,N)$-two-sided nontangentially accessible domain satisfies a two-sided local John condition (in all cases demanding that $R = \infty$ if the said domain has an unbounded boundary). (1.1.28)

## 1.2 Elliptic operators

Fix $n \in \mathbb{N}$ with $n \geq 2$ along with $M \in \mathbb{N}$, and denote by $\mathfrak{L}$ the collection of all homogeneous constant complex coefficient second-order $M \times M$ systems $L$ in $\mathbb{R}^n$. Hence, any element $L$ in $\mathfrak{L}$ may be written as a matrix of differential operators of the form $L = \left(a_{jk}^{\alpha\beta}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}$ for some complex numbers $a_{jk}^{\alpha\beta}$ (here and elsewhere, we shall use the usual convention of summation over repeated indices). In particular, the action of $L$ on any given vector-valued distribution $u = (u_\beta)_{1\leq\beta\leq M}$ may be described as

$$Lu = \left(a_{jk}^{\alpha\beta}\partial_j\partial_k u_\beta\right)_{1\leq\alpha\leq M}, \tag{1.2.1}$$

and we denote by $L^\top := \left(a_{kj}^{\beta\alpha}\partial_j\partial_k\right)_{1\leq\alpha,\beta\leq M}$ the (real) transposed of $L$. We also define the characteristic matrix of $L$ as

$$L(\xi) := -\left[(a_{jk}^{\alpha\beta}\xi_j\xi_k)_{1\leq\alpha,\beta\leq M}\right] \quad \text{for each} \quad \xi = (\xi_i)_{1\leq i\leq n} \in \mathbb{R}^n, \tag{1.2.2}$$

and introduce

$$\mathfrak{L}_* := \{L \in \mathfrak{L} : \det[L(\xi)] \neq 0 \ \text{ for each } \ \xi \in \mathbb{R}^n \setminus \{0\}\}. \tag{1.2.3}$$

We shall refer to a system $L \in \mathfrak{L}$ as being `weakly elliptic` if actually $L \in \mathfrak{L}_*$. This is in contrast with the more stringent condition of `Legendre-Hadamard (strong) ellipticity`, asks for the existence of some $\kappa_0 > 0$ such that

$$\begin{gathered}\operatorname{Re}\left[a_{jk}^{\alpha\beta}\xi_j\xi_k\overline{\zeta_\alpha}\zeta_\beta\right] \geq \kappa_0\,|\xi|^2\,|\zeta|^2 \quad \text{for all} \\ \xi = (\xi_j)_{1\leq j\leq n} \in \mathbb{R}^n \quad \text{and} \quad \zeta = (\zeta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M.\end{gathered} \tag{1.2.4}$$

Examples of strongly (and hence weakly) elliptic operators include scalar operators, such as the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ or, more generally, operators of the form $\mathrm{div}A\nabla$ with $A = (a_{rs})_{1\leq r,s\leq n}$ an $n\times n$ matrix with complex entries satisfying the ellipticity condition

$$\inf_{\xi\in S^{n-1}} \operatorname{Re}\left[a_{rs}\xi_r\xi_s\right] > 0, \tag{1.2.5}$$

(where $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$), as well as the complex version of the Lamé system of elasticity in $\mathbb{R}^n$,

$$L := \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}. \tag{1.2.6}$$

Above, the constants $\lambda, \mu \in \mathbb{C}$ (typically called Lamé moduli), are assumed to satisfy

$$\operatorname{Re}\mu > 0 \quad \text{and} \quad \operatorname{Re}(2\mu+\lambda) > 0, \tag{1.2.7}$$

a condition equivalent to the demand that the Lamé system (1.2.6) satisfies the Legendre-Hadamard ellipticity condition (1.2.4). While the Lamé system is symmetric, we stress that the results in this thesis require no symmetry for the systems involved.

Next, let us consider

$$\mathfrak{A} := \Big\{ A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} : \text{ each } a_{jk}^{\alpha\beta} \text{ belongs to } \mathbb{C} \Big\}, \tag{1.2.8}$$

the collection of coefficient tensors with constant complex entries. Adopting natural operations (i.e., componentwise addition and multiplication by scalars), this becomes a finite dimensional vector space (over $\mathbb{C}$) which we endow with the norm

$$\|A\| := \sum_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} |a_{jk}^{\alpha\beta}| \quad \text{for each} \quad A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}. \tag{1.2.9}$$

Hence, if the transposed of each given $A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}$ is the coefficient tensor $A^\top := (a_{kj}^{\beta\alpha})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}}$, then $\mathfrak{A} \ni A \mapsto A^\top \in \mathfrak{A}$ is an isometry. With each coefficient tensor $A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}$ associate the system $L_A \in \mathfrak{L}$ according to

$$L_A := \Big( a_{jk}^{\alpha\beta} \partial_j \partial_k \Big)_{1\leq\alpha,\beta\leq M}. \tag{1.2.10}$$

Then the map

$$\mathfrak{A} \ni A \longmapsto L_A \in \mathfrak{L} \tag{1.2.11}$$

is linear and surjective, though it fails to be injective. Specifically, if we introduce

$$\mathfrak{A}^{\rm ant} := \Big\{ B = (b_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A} : b_{jk}^{\alpha\beta} = -b_{kj}^{\alpha\beta} \quad \text{whenever} \quad 1\leq j,k\leq n \quad \text{and} \quad 1\leq\alpha,\beta\leq M \Big\}, \tag{1.2.12}$$

the collection of all coefficient tensors which are antisymmetric in the lower indices, then $\mathfrak{A}^{\rm ant}$ is a closed linear subspace of $\mathfrak{A}$ and for each $A, \widetilde{A} \in \mathfrak{A}$ we have

$$L_A = L_{\widetilde{A}} \Longleftrightarrow A - \widetilde{A} \in \mathfrak{A}^{\rm ant}. \tag{1.2.13}$$

If we now define

$$\mathfrak{A}_L := \big\{ A \in \mathfrak{A} : L = L_A \big\} \quad \text{for each} \quad L \in \mathfrak{L}, \tag{1.2.14}$$

and for each $L \in \mathfrak{L}$ we set (with the distance considered in the normed vector space $\mathfrak{A}$)

$$\|L\| := \operatorname{dist}\big(A, \mathfrak{A}^{\rm ant}\big) \quad \text{for each/some} \quad A \in \mathfrak{A}_L, \tag{1.2.15}$$

then $\mathfrak{L} \ni L \mapsto \|L\|$ is an unambiguously defined norm on the vector space $\mathfrak{L}$. In the topology induced by this norm, $\mathfrak{L}_*$ from (1.2.3) is an open subset of $\mathfrak{L}$, the mapping (1.2.11) is continuous, and $\mathfrak{L} \ni L \mapsto L^\top \in \mathfrak{L}$ is an isometry.

Finally, we denote by $\mathfrak{A}_{\rm WE}$ the collection of all coefficient tensors $A$ with the property that the $M \times M$ homogeneous second-order system $L_A$ associated with $A$ in $\mathbb{R}^n$ as in (1.2.10) is weakly elliptic, i.e.,

$$\mathfrak{A}_{\rm WE} := \big\{ A \in \mathfrak{A} : L_A \in \mathfrak{L}_* \big\}. \tag{1.2.16}$$

Then $\mathfrak{A}_{\text{WE}}$ is an open subset of $\mathfrak{A}$.

The following theorem, itself a special case of [92, Theorem 11.1, p. 393], summarizes some of the main properties of a certain type of fundamental solution canonically associated with any given homogeneous, constant complex coefficient, weakly elliptic second-order system in $\mathbb{R}^n$.

**Theorem 1.2.1.** *Let $L$ be a homogeneous, second-order, constant complex coefficient, $M \times M$ system in $\mathbb{R}^n$, which is weakly elliptic (cf. (1.2.3)). Then there exists an $M \times M$ matrix-valued function $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$, canonically associated with the given system $L$, such that the following properties are true.*

(a) *For each $\alpha, \beta \in \{1, \dots, M\}$ one has $E_{\alpha\beta} \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and $E_{\alpha\beta}(x) = E_{\alpha\beta}(-x)$ for every $x \in \mathbb{R}^n \setminus \{0\}$.*

(b) *For each fixed point $y \in \mathbb{R}^n$ one has $L[E(\cdot - y)] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, where $I_{M\times M}$ is the $M \times M$ identity matrix and $\delta_y$ denotes the Dirac distribution with mass at $y$ in $\mathbb{R}^n$. That is, using the standard Kronecker delta notation,*

$$a^{\alpha\beta}_{jk} \partial_{x_j} \partial_{x_k} [E_{\beta\gamma}(x - y)] = \delta_{\alpha\gamma} \delta_y(x), \qquad x \in \mathbb{R}^n, \tag{1.2.17}$$

*in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.*

(c) *The transposed of $E$, i.e., $E^\top = (E_{\beta\alpha})_{1\leq\alpha,\beta\leq M}$, is a fundamental solution for the transposed system $L^\top$. In other words, for each fixed point $y \in \mathbb{R}^n$ one has $L^\top[E^\top(\cdot - y)] = \delta_y I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$, i.e.,*

$$a^{\beta\alpha}_{kj} \partial_{x_j} \partial_{x_k} [E_{\gamma\beta}(x - y)] = \delta_{\alpha\gamma} \delta_y(x), \qquad x \in \mathbb{R}^n, \tag{1.2.18}$$

*in the sense of distributions, for every $\alpha, \gamma \in \{1, \dots, M\}$.*

(d) *For every multi-index $\alpha \in \mathbb{N}^n_0$ with $n + |\alpha| > 2$, the function $\partial^\alpha E$ is positive homogeneous of degree $2 - n - |\alpha|$ and there exists a constant $C_\alpha \in (0, \infty)$ with the property that*

$$\big|(\partial^\alpha E)(x)\big| \leq C_\alpha |x|^{2-n-|\alpha|} \quad \textit{for all} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.19}$$

*Finally, corresponding to $n = 2$ and $\alpha = (0, \dots, 0)$, there exists $C \in (0, \infty)$ such that $|E(x)| \leq C(1 + |\ln|x||)$ for every $x \in \mathbb{R}^2 \setminus \{0\}$.*

The following result is a particular case of more general interior estimates found in [91, Theorem 11.9, p. 364].

**Theorem 1.2.2.** *Let $L$ be a constant complex coefficient system as in (1.2.1) satisfying the weak ellipticity condition in (1.2.3). Then for every $p \in (0, \infty)$, $\lambda \in (0, 1)$, and $m \in \mathbb{N}_0$ there exists a finite constant $C = C(L, p, m, \lambda, n) > 0$ with the property that* *for every null-solution $u$ of $L$ in a ball $B(x,R)$, where $x\in\mathbb{R}^n$ and $R>0$, and every $r\in(0,R)$ one has*

$$\sup_{z\in B(x,\lambda r)}|(\nabla^m u)(z)|\leq\frac{C}{r^m}\left(\fint_{B(x,r)}|u(x)|^p\,dx\right)^{1/p}. \tag{1.2.20}$$

The fundamental solution of $L$ will be the matrix denoted by $E:=(E_{jk})_{1\leq j,k\leq M}$. The following theorem from [91, Theorem 11.1] summarizes the properties of $E$ in our context.

We now proceed to study the properties of solutions in the upper-half space $\mathbb{R}^n_+:=\{x=(x',t)\in\mathbb{R}^n:\,x'\in\mathbb{R}^{n-1},\,t>0\}$. First, introduce

$$\begin{aligned}W^{1,2}_{\rm bdd}(\mathbb{R}^n_+):=\{u\in L^2_{\rm loc}(\mathbb{R}^n_+):\,u,\partial_j u\in L^2(\mathbb{R}^n_+\cap B(0,r))\\ \text{for each }\ j\in\{1,\dots,n\}\ \text{ and }\ r\in(0,\infty)\},\end{aligned} \tag{1.2.21}$$

and define the Sobolev trace Tr, whenever meaningful, as

$$({\rm Tr}\,u)(x'):=\lim_{r\to0^+}\fint_{B((x',0),r)\cap\mathbb{R}^n_+}u(y)\,dy,\qquad x'\in\mathbb{R}^{n-1}. \tag{1.2.22}$$

The following result is taken from [90, Corollary 2.4].

**Proposition 1.2.3.** *Let $L$ be a constant complex coefficient system as in (1.2.1) satisfying (1.2.4), and suppose $u\in\big[W^{1,2}_{\rm bdd}(\mathbb{R}^n_+)\big]^M$ satisfies $Lu=0$ in $\mathbb{R}^n_+$ and ${\rm Tr}\,u=0$ on $\mathbb{R}^{n-1}$. Then $u\in\big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M$ and there exists a finite constant $C>0$, independent of $u$, such that for each $x\in\overline{\mathbb{R}^n_+}$ and each $r>0$,*

$$\sup_{\mathbb{R}^n_+\cap B(x,r)}|\nabla u|\leq\frac{C}{r}\sup_{\mathbb{R}^n_+\cap B(x,2r)}|u|. \tag{1.2.23}$$

When dealing with the upper-half space, we agree to denote the $(n-1)$-dimensional Lebesgue measure of given Lebesgue measurable set $E\subseteq\mathbb{R}^{n-1}$ by $|E|$. Also, by a cube $Q$ in $\mathbb{R}^{n-1}$ we shall understand a cube with sides parallel to the coordinate axes. Its side-length will be denoted by $\ell(Q)$, and for each $\lambda>0$ we shall denote by $\lambda\,Q$ the cube concentric with $Q$ whose side-length is $\lambda\,\ell(Q)$. For every function $h\in\big[L^1_{\rm loc}(\mathbb{R}^{n-1})\big]^M$ we write

$$h_Q:=\fint_Q h(x')\,dx':=\frac{1}{|Q|}\int_Q h(x')\,dx'\in\mathbb{C}^M, \tag{1.2.24}$$

with the integration performed componentwise. Moving on, for each given function $f\in\big[L^1_{\rm loc}(\mathbb{R}^{n-1})\big]^M$ define the $L^p$-based mean oscillation of $f$ at a scale $r\in(0,\infty)$ as

$${\rm osc}_p(f;r):=\sup_{\substack{Q\subseteq\mathbb{R}^{n-1}\\ \ell(Q)\leq r}}\left(\fint_Q|f(x')-f_Q|^p\,dx'\right)^{1/p}. \tag{1.2.25}$$

Poisson kernels for elliptic boundary value problems in a half-space have been studied extensively in [1], [2], [69, §10.3], [112], [113], [114]. The following theorem is contained in [85, Theorem 2.3 and Proposition 3.1], [82, Theorem 3.1, p. 934], and [2]. Here and elsewhere, the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

**Theorem 1.2.4.** *Suppose $L$ is a constant complex coefficient system as in (1.2.1), satisfying (1.2.4). Then the following statements are true.*

(a) *There exists a matrix-valued function $P^L = (P^L_{\alpha\beta})_{1\leq\alpha,\beta\leq M} : \mathbb{R}^{n-1} \to \mathbb{C}^{M\times M}$, called the Poisson kernel for $L$ in $\mathbb{R}^n_+$, such that $P^L \in \big[\mathscr{C}^\infty(\mathbb{R}^{n-1})\big]^{M\times M}$, there exists some finite constant $C > 0$ such that*
$$|P^L(x')| \leq \frac{C}{(1+|x'|^2)^{n/2}}, \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{1.2.26}$$
*and*
$$\int_{\mathbb{R}^{n-1}} P^L(x')\,dx' = I_{M\times M}, \tag{1.2.27}$$
*where $I_{M\times M}$ stands for the $M \times M$ identity matrix.*

(b) *If for every $x' \in \mathbb{R}^{n-1}$ and $t > 0$ one defines*
$$K^L(x',t) := P^L_t(x') := t^{1-n}P^L(x'/t), \tag{1.2.28}$$
*then $K^L \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon))\big]^{M\times M}$ for every $\varepsilon > 0$ and the function $K^L = (K^L_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ satisfies*
$$LK^L_{\cdot\beta} = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ for each } \ \beta \in \{1,\dots,M\}, \tag{1.2.29}$$
*where $K^L_{\cdot\beta} := (K^L_{\alpha\beta})_{1\leq\alpha\leq M}$ is the $\beta$-th column in $K^L$. Moreover, for each multi-index $\alpha \in \mathbb{N}^n_0$ there exists $C_\alpha \in (0,\infty)$ such that*
$$\big|(\partial^\alpha K^L)(x)\big| \leq C_\alpha\,|x|^{1-n-|\alpha|}, \quad \text{for every } \ x \in \overline{\mathbb{R}^n_+} \setminus \{0\}. \tag{1.2.30}$$

(c) *For each function $f = (f_\beta)_{1\leq\beta\leq M} \in \big[L^1(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n})\big]^M$ define, with $P^L$ as above,*
$$u(x',t) := (P^L_t * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+. \tag{1.2.31}$$
*Then $u$ is meaningfully defined, via an absolutely convergent integral, and for every aperture parameter $\kappa > 0$, it satisfies*
$$u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \qquad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \qquad u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \ \text{ a.e. on } \mathbb{R}^{n-1} \tag{1.2.32}$$
*(with the last identity valid in the set of Lebesgue points of $f$), and there exists a constant $C = C(n,L,\kappa) \in (0,\infty)$ with the property that*
$$(\mathcal{N}_\kappa u)(x') \leq C(Mf)(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1} \tag{1.2.33}$$
*Furthermore, there exists a finite constant $C > 0$ such that*
$$|(\nabla u)(x',t)| \leq \frac{C}{t}\int_1^\infty \operatorname{osc}_1(f;st)\frac{ds}{s^2}, \qquad \forall\,(x',t) \in \mathbb{R}^n_+, \tag{1.2.34}$$
*and, for each cube $Q \subseteq \mathbb{R}^{n-1}$,*
$$\left(\int_0^{\ell(Q)} ⨍_Q |(\nabla u)(x',t)|^2\,t\,dx'\,dt\right)^{1/2} \leq C\int_1^\infty \operatorname{osc}_1(f;s\ell(Q))\frac{ds}{s^2}. \tag{1.2.35}$$

## 1.3 Growth functions and generalized Hölder spaces

**Definition 1.3.1.** Given a number $D \in (0, \infty)$, a function $\omega : (0, D) \to (0, \infty)$ is called a `growth function on` $(0, D)$ if $\omega$ is non-decreasing and

$$\lim_{t \to 0^+} \omega(t) = 0, \qquad \omega(D) := \lim_{t \to D^-} \omega(t) < \infty. \tag{1.3.1}$$

Corresponding to the case when $D = \infty$, a function $\omega : (0, \infty) \to (0, \infty)$ is called a `growth function on` $(0, \infty)$ if $\omega$ is non-decreasing and

$$\lim_{t \to 0^+} \omega(t) = 0. \tag{1.3.2}$$

It turns out that any growth function on an interval extends to a growth function on $(0, \infty)$. For ease of reference, we state this formally in the remark below.

*Remark* 1.3.2. Let $D \in (0, \infty]$. If $\omega$ is a growth function on $(0, D)$ then

$$\widetilde{\omega}(t) := \omega\big(\min\{t, D\}\big) \quad \text{for each} \quad t \in (0, \infty) \tag{1.3.3}$$

is a growth function on $(0, \infty)$ with the property that $\widetilde{\omega} = \omega$ on $(0, D)$.

We shall frequently impose additional conditions on the growth functions employed in this work. First, if $\omega$ is a growth function on $(0, D)$, with $D \in (0, \infty]$, we shall call $\omega$ doubling provided

$$\sup_{0 < t < D/2} \frac{\omega(2t)}{\omega(t)} < \infty, \tag{1.3.4}$$

and refer to the supremum in the left hand-side of (1.3.4) as the doubling constant of $\omega$. It is then immediate from definitions that

$$\begin{array}{l} \text{if } \omega \text{ is a doubling growth function on } (0, D), \text{ with } D \in (0, \infty], \text{ then} \\ \widetilde{\omega} \text{ is a doubling growth function on the interval } (0, \infty). \end{array} \tag{1.3.5}$$

Another assumption (which plays a natural role in a variety of contexts) imposed on a given growth function $\omega : (0, D) \to (0, \infty)$, with $D \in (0, \infty]$, is that

$$C_\omega := \sup_{0 < t < D} \left\{ \frac{1}{\omega(t)} \Big( \int_0^t \omega(s) \frac{ds}{s} + t \int_t^D \frac{\omega(s)}{s} \frac{ds}{s} \Big) \right\} < +\infty. \tag{1.3.6}$$

Hence, whenever (1.3.6) holds it follows that $C_\omega \in (0, \infty)$ and

$$\int_0^t \omega(s) \frac{ds}{s} + t \int_t^D \frac{\omega(s)}{s} \frac{ds}{s} \leq C_\omega \cdot \omega(t) \quad \text{for each} \quad t \in (0, D), \tag{1.3.7}$$

with the added bonus that when $D < \infty$ this inequality also extends to $t = D$ if $\omega(D)$ is interpreted as in (1.3.1).

Let us note that for any growth function $\omega$ on $(0, D)$, with $D \in (0, \infty]$, the demand in (1.3.6) implies that $\omega$ is doubling. Indeed, for each $t \in (0, D/4)$ we may write

$$(\ln 2)\omega(2t) \leq \int_{2t}^{4t} \omega(s) \frac{ds}{s} \leq 4t \int_t^D \frac{\omega(s)}{s} \frac{ds}{s} \leq 4C_\omega \cdot \omega(t) \tag{1.3.8}$$

which already proves that $\omega$ is doubling in the case when $D = \infty$, whereas if $D < \infty$ then for each $t \in (D/4, D/2)$ we have $\omega(2t) \leq C\omega(t)$ with $C := \omega(D)/\omega(D/4)$.

*Remark* 1.3.3. Condition (1.3.6) is closely related to the dilation indices of Orlicz spaces studied in [10], [40] (which are useful in the theory of interpolation of Orlicz spaces). Given a growth function $\omega : (0, D) \to (0, \infty)$, where $D \in (0, \infty]$, we set

$$h_\omega(s) := \sup_{0<t<\min\{D, D/s\}} \frac{\omega(ts)}{\omega(t)} \quad \text{for each} \quad s \in (0, \infty), \tag{1.3.9}$$

and define the lower and upper dilation indices of $\omega$, respectively, as

$$i_\omega := \lim_{s\to 0^+} \frac{\ln h_\omega(s)}{\ln s} = \sup_{0<s<1} \frac{\ln h_\omega(s)}{\ln s}, \qquad I_\omega := \lim_{s\to\infty} \frac{\ln h_\omega(s)}{\ln s} = \inf_{s>1} \frac{\ln h_\omega(s)}{\ln s}. \tag{1.3.10}$$

Then one may check that (1.3.6) holds if $0 < i_\omega \leq I_\omega < 1$. Indeed, from (1.3.10) we see that whenever $0 < \varepsilon < \min\{i_\omega, 1 - I_\omega\}$ there exists some $C_\varepsilon \in (0, \infty)$ such that $h_\omega(s) \leq C_\varepsilon s^{i_\omega - \varepsilon}$ for every $s \in (0, 1)$, and $h_\omega(s) \leq C_\varepsilon s^{I_\omega + \varepsilon}$ for every $s \in (1, \infty)$. Hence, there there exists $C \in (0, \infty)$ such that if $0 < t < D$ then

$$i_\omega > 0 \Rightarrow \int_0^t \frac{\omega(s)}{\omega(t)} \frac{ds}{s} = \int_0^1 \frac{\omega(ts)}{\omega(t)} \frac{ds}{s} \leq C_\varepsilon \int_0^1 s^{i_\omega - \varepsilon} \frac{ds}{s} \leq C, \tag{1.3.11}$$

$$I_\omega < 1 \Rightarrow t \int_t^D \frac{\omega(s)}{s\omega(t)} \frac{ds}{s} = \int_1^{D/t} \frac{\omega(ts)}{s\omega(t)} \frac{ds}{s} \leq C_\varepsilon \int_1^\infty s^{I_\omega + \varepsilon - 1} \frac{ds}{s} \leq C. \tag{1.3.12}$$

It is of interest to provide relevant examples of growth functions satisfying (1.3.6).

**Example 1.3.4.** Fix $D \in (0, \infty]$. Given $\alpha \in (0, 1)$, if $\omega(t) := t^\alpha$ for each $t \in (0, D)$ then $i_\omega = I_\omega = \alpha$, hence (1.3.6) holds. The particular version of Theorem 5.1.4 corresponding to this scenario has been established in [96]. There are many examples of interest that are treated here for the first time. To elaborate, fix an arbitrary $\alpha \in (0, 1)$ along with $\theta \in \mathbb{R}$ and, for each $t \in (0, \infty)$, define $\log_+ t := \max\{0, \ln t\}$. Then such examples include

$$\omega(t) := t^\alpha \left(A + \log_+ t\right)^\theta \quad \text{for all} \quad t \in (0, D), \quad \text{where} \quad A := \max\{1, -\theta/\alpha\}, \tag{1.3.13}$$

and

$$\omega(t) := t^\alpha \left(B + \log_+(1/t)\right)^\theta \quad \text{for all} \quad t \in (0, D), \quad \text{where} \quad B := \max\{1, \theta/\alpha\}. \tag{1.3.14}$$

In these situations $i_\omega = I_\omega = \alpha$ which, as noted earlier, guarantees that (1.3.6) holds. Another relevant example is offered by

$$\omega(t) := \max\{t^\alpha, t^\beta\} \quad \text{for all} \quad t \in (0, D), \quad \text{where} \quad 0 < \alpha, \beta < 1. \tag{1.3.15}$$

Indeed, in such a case we have $i_\omega = \min\{\alpha, \beta\}$ and $I_\omega = \max\{\alpha, \beta\}$ if $D = \infty$, and $i_\omega = I_\omega = \min\{\alpha, \beta\}$ if $D < \infty$. Similarly, if

$$\omega(t) := \min\{t^\alpha, t^\beta\} \quad \text{for all} \quad t \in (0, D), \quad \text{where} \quad 0 < \alpha, \beta < 1, \tag{1.3.16}$$

then $i_\omega = \min\{\alpha, \beta\}$ and $I_\omega = \max\{\alpha, \beta\}$ if $D = \infty$ and $i_\omega = I_\omega = \max\{\alpha, \beta\}$ if $D < \infty$. Hence, once again, condition (1.3.6) is verified.

Next, we introduce generalized Hölder spaces, consisting of functions whose continuity is quantified using growth functions of the sort previously discussed.

**Definition 1.3.5.** Let $U \subseteq \mathbb{R}^n$ be an arbitrary set and $M \geq 1$.

(a) Given a growth function $\omega$ on $(0, \infty)$, for each vector-valued function $u : U \to \mathbb{C}^M$ consider $[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}$ to be zero if $U$ is a singleton, and

$$[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u(x)-u(y)|}{\omega(|x-y|)} \in [0,\infty] \tag{1.3.17}$$

if the cardinality of the set $U$ is at least two. Then the `homogeneous` $\omega$-`Hölder space` on $U$ is introduced as

$$\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M) := \{u : U \to \mathbb{C}^M : [u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} < \infty\}. \tag{1.3.18}$$

(b) If $D \in (0, \infty]$ and $\omega$ is a growth function on $(0, D)$, define the `inhomogeneous` $\omega$-`Hölder space` on $U$ as

$$\mathscr{C}^\omega(U,\mathbb{C}^M) := \{u \in \dot{\mathscr{C}}^{\widetilde{\omega}}(U,\mathbb{C}^M) : u \text{ bounded on } U\}, \tag{1.3.19}$$

and is equipped with the norm

$$\mathscr{C}^\omega(U,\mathbb{C}^M) \ni u \longmapsto \|u\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} := \sup_U |u| + [u]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(U,\mathbb{C}^M)}. \tag{1.3.20}$$

In relation to Definition 1.3.5 a few comments are in order. First, in the context of item (a), $[\cdot]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}$ is a semi-norm for the space $\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$. Second, the fact that $\omega(t) \to 0$ as $t \to 0^+$ implies that if $u \in \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$ then $u$ is a uniformly continuous function on $U$. Finally, we note that the choice $\omega(t) := t^\alpha$ for each $t \in (0,\infty)$, with $\alpha \in (0,1)$, yields the classical scale of Hölder spaces of order $\alpha$ on $U$.

Going further, it is clear from definitions that if $\omega$ is a growth function on $(0,\infty)$ and $U \subseteq \mathbb{R}^n$ is an arbitrary set, then

$$\begin{gathered} fg \in \mathscr{C}^\omega(U,\mathbb{C}^M) \ \text{ and } \ \|fg\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} \leq \|f\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)}\|g\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} \\ \text{for any two functions } f,g \in \mathscr{C}^\omega(U,\mathbb{C}^M). \end{gathered} \tag{1.3.21}$$

Also, for each subset $V$ of $U$, the restriction operators

$$\begin{gathered} \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M) \ni u \mapsto u\big|_V \in \dot{\mathscr{C}}^\omega(V,\mathbb{C}^M) \ \text{ and } \ \mathscr{C}^\omega(U,\mathbb{C}^M) \ni u \mapsto u\big|_V \in \mathscr{C}^\omega(V,\mathbb{C}^M) \\ \text{are well-defined, linear, and bounded.} \end{gathered} \tag{1.3.22}$$

In the opposite direction, we have the following extension result.

**Lemma 1.3.6.** *Let $U \subseteq \mathbb{R}^n$ be an arbitrary set, $M \geq 1$, and suppose $\omega$ is a doubling growth function on $(0,\infty)$. Then $\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$ and $\dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)$ coincide as vector spaces and have equivalent semi-norms. More specifically, the restriction map*

$$\dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M) \ni u \longmapsto u\big|_U \in \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M) \tag{1.3.23}$$

*is a linear bijection which, under the canonical identification of functions $u \in \dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)$ with their restrictions $u\big|_U \in \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$, satisfies*

$$\begin{gathered}[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)} \leq C[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)} \\ \text{for each function } \ u \in \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M),\end{gathered} \tag{1.3.24}$$

*where $C \in [1,\infty)$ is the doubling constant of $\omega$.*

*As a corollary of this, (1.3.5), and definitions, whenever $U \subseteq \mathbb{R}^n$ is an arbitrary set, and $\omega$ is a doubling growth function on $(0,D)$ for some $D \in (0,\infty]$, one may canonically identify $\mathscr{C}^\omega(U,\mathbb{C}^M) \equiv \mathscr{C}^\omega(\overline{U},\mathbb{C}^M)$ and there exists $C \in [1,\infty)$ such that*

$$\begin{gathered}\|u\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} \leq \|u\|_{\mathscr{C}^\omega(\overline{U},\mathbb{C}^M)} \leq C\|u\|_{\mathscr{C}^\omega(U,\mathbb{C}^M)} \\ \text{for each function } \ u \in \mathscr{C}^\omega(U,\mathbb{C}^M).\end{gathered} \tag{1.3.25}$$

*Proof.* Fix an arbitrary $u \in \dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)$. As noted earlier, this membership ensures that $u$ is uniformly continuous, hence $u$ extends uniquely to a continuous function $v$ on $\overline{U}$. To show that $v$ actually belongs to $\dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)$ pick two arbitrary distinct points $x,y \in \overline{U}$. Then there exist two sequences $\{x_j\}_{j\in\mathbb{N}}$, $\{y_j\}_{j\in\mathbb{N}}$ of points in $U$ such that $x_j \to x$ and $y_j \to y$ as $j \to \infty$. After discarding finitely many terms, there is no loss of generality in assuming that $0 < |x_j - y_j| < 2|x-y|$ for each $j \in \mathbb{N}$. Relying on the fact that $\omega$ is non-decreasing we may then write

$$\begin{aligned}|v(x)-v(y)| &= \lim_{j\to\infty}|u(x_j)-u(y_j)| \leq [u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}\limsup_{j\to\infty}\omega(|x_j-y_j|) \\ &\leq [u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}\,\omega(2|x-y|) \leq C[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}\,\omega(|x-y|),\end{aligned} \tag{1.3.26}$$

where $C \in [1,\infty)$ is the doubling constant of $\omega$. This ultimately proves that $v \in \dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)$ and $[v]_{\dot{\mathscr{C}}^\omega(\overline{U},\mathbb{C}^M)} \leq C[u]_{\dot{\mathscr{C}}^\omega(U,\mathbb{C}^M)}$. All desired conclusions now follow. □

To simplify the notation, we call $\dot{\mathscr{C}}^\omega(U) := \dot{\mathscr{C}}^\omega(U,\mathbb{C})$, and $\mathscr{C}^\omega(U) := \mathscr{C}^\omega(U,\mathbb{C})$. We continue by making the following definition.

**Definition 1.3.7.** Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and denote by $\mathscr{C}^1(\Omega)$ the space of continuously differentiable functions in $\Omega$. Given a growth function $\omega$ on $(0,D)$, with $D \in (0,\infty]$, for each $u \in \mathscr{C}^1(\Omega)$ define

$$\|u\|_{\mathscr{C}^{1,\omega}(\Omega)} := \sup_\Omega |u| + \sup_\Omega |\nabla u| + [\nabla u]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\Omega)} \in [0,\infty], \tag{1.3.27}$$

then introduce

$$\mathscr{C}^{1,\omega}(\Omega) := \{u \in \mathscr{C}^1(\Omega) :\ \|u\|_{\mathscr{C}^{1,\omega}(\Omega)} < \infty\}. \tag{1.3.28}$$

To close this section, we make the following convention. When simply speaking of a growth function, we shall understand a growth function $\omega$ defined on some interval $(0,D)$, with $D \in (0,\infty]$.

## 1.4 Clifford algebras

This section is a brief tutorial about Clifford algebras, which are a non-commutative higher-dimensional version of the field of complex numbers, where some of the magic cancellations and algebraic miracles typically associated with the complex plane still occur.

The Clifford algebra with $n$ imaginary units is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra $(\mathcal{C}\ell_n, +, \odot)$, which is not generated as an algebra by any proper subspace of $\mathbb{R}^n$ and such that

$$x \odot x = -|x|^2 \quad \text{for every} \quad x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{1.4.1}$$

In particular, with $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ denoting the standard orthonormal basis in $\mathbb{R}^n$, we have

$$\begin{gathered} \mathbf{e}_j \odot \mathbf{e}_j = -1 \quad \text{for all} \quad j \in \{1, \dots, n\} \quad \text{and} \\ \mathbf{e}_j \odot \mathbf{e}_k = -\mathbf{e}_k \odot \mathbf{e}_j \quad \text{for each distinct} \quad j, k \in \{1, \dots, n\}. \end{gathered} \tag{1.4.2}$$

This allows us define an embedding $\mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$ by identifying

$$\mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \sum_{j=1}^{n} x_j \mathbf{e}_j \in \mathcal{C}\ell_n. \tag{1.4.3}$$

In particular, $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ become $n$ imaginary units in $\mathcal{C}\ell_n$. Any element $u \in \mathcal{C}\ell_n$ has a unique representation of the form

$$u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \mathbf{e}_I, \quad u_I \in \mathbb{R}, \tag{1.4.4}$$

where $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices $I$, i.e., $I = (i_1, i_2, \dots, i_\ell)$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, and $\mathbf{e}_I$ denotes the Clifford algebra product $\mathbf{e}_I := \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_\ell}$. Write $\mathbf{e}_0 := \mathbf{e}_\varnothing := 1$ for the multiplicative unit in $\mathcal{C}\ell_n$. For each $u \in \mathcal{C}\ell_n$ represented as in (1.4.4) define the projection

$$u_{\text{proj}} := \sum_{j=1}^{n} u_j \mathbf{e}_j \in \mathbb{R}^n, \tag{1.4.5}$$

and denote by

$$u_{\text{scal}} := u_\varnothing \mathbf{e}_\varnothing = u_\varnothing \in \mathbb{R}, \ \text{ the scalar component of } \ u. \tag{1.4.6}$$

We endow $\mathcal{C}\ell_n$ with the natural Euclidean metric, hence

$$|u| := \Bigg( \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' |u_I|^2 \Bigg)^{1/2} \quad \text{for each} \quad u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \mathbf{e}_I \in \mathcal{C}\ell_n. \tag{1.4.7}$$

Next, define the conjugate of each $\mathbf{e}_I$ as the unique element $\overline{\mathbf{e}_I} \in \mathcal{C}\ell_n$ with the property that $\mathbf{e}_I \odot \overline{\mathbf{e}_I} = \overline{\mathbf{e}_I} \odot \mathbf{e}_I = 1$. Thus, if $I = (i_1, \dots, i_\ell)$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, then the conjugate of $\mathbf{e}_I$ is given by $\overline{\mathbf{e}_I} = (-1)^\ell \mathbf{e}_{i_\ell} \odot \cdots \odot \mathbf{e}_2 \odot \mathbf{e}_1$. More generally, for an arbitrary element $u \in \mathcal{C}\ell_n$ represented as in (1.4.4) we define

$$\overline{u} := \sum_{\ell=0}^{n} \sum_{|I|=\ell}{}' u_I \overline{\mathbf{e}_I}. \tag{1.4.8}$$

Note that $\overline{x} = -x$ for every $x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$, and $|u| = |\overline{u}|$ for every $u \in \mathcal{C}\ell_n$. One may also check that for any $u, v \in \mathcal{C}\ell_n$ we have

$$|u \odot v| \leq 2^{n/2}|u||v|, \qquad \overline{u \odot v} = \overline{v} \odot \overline{u}, \tag{1.4.9}$$

and, in fact,

$$\begin{gathered} |u \odot v| = |u||v| \quad \text{if either} \\ u \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n, \quad \text{or} \quad v \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n, \end{gathered} \tag{1.4.10}$$

For further details on Clifford algebras, the reader is referred to [99].

To study the boundedness of operators acting on Clifford algebra-valued functions, we need appropriate norms on the spaces to which the said functions belong. If $(X, \|\cdot\|_X)$ is a Banach space then $X \otimes \mathcal{C}\ell_n$ will denote the Banach space consisting of elements of the form

$$u = \sum_{\ell=0}^{n} \sum_{|I|=\ell}{}' u_I e_I, \qquad u_I \in X, \tag{1.4.11}$$

equipped with the norm

$$X \otimes \mathcal{C}\ell_n \ni u = \sum_{\ell=0}^{n} \sum_{|I|=\ell}{}' u_I e_I \mapsto \|u\|_{X \otimes \mathcal{C}\ell_n} := \sum_{\ell=0}^{n} \sum_{|I|=\ell}{}' \|u_I\|_X \,. \tag{1.4.12}$$

# CHAPTER 2

## Singular integral operators and quantitative flatness

We develop the theory of layer potentials in the context of $\delta$-SKT domains in $\mathbb{R}^n$ (with SKT acronym for Semmes-Kenig-Toro) where the parameter $\delta \in (0,1)$, regulating the size of the BMO semi-norm of the outward unit normal $\nu$ to $\Omega$, is assumed to be small. This category of two-sided NTA domains with Ahlfors regular boundaries, which emerged from the earlier work of S. Semmes, C. Kenig, and T. Toro, is related to a class of domains introduced in [53]. The latter was designed to work well when the domains in question have compact boundaries. By way of contrast, the fact that we are now demanding $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1$ (where $\sigma$ is the "surface measure" $\mathcal{H}^{n-1}\lfloor\partial\Omega$) has topological and metric implications for $\Omega$, namely $\Omega$ is a connected unbounded open set, with a connected unbounded boundary and an unbounded connected complement. For example, in the two-dimensional setting we show that the class of $\delta$-SKT with $\delta \in (0,1)$ small agrees with the category of chord-arc domains with small constant.

Assuming $\Omega \subseteq \mathbb{R}^n$ to be a $\delta$-SKT with $\delta \in (0,1)$ sufficiently small (relative to other geometric characteristics of $\Omega$) we prove that the operator norm of Calderón-Zygmund singular integrals whose kernels exhibit a certain algebraic structure (specifically, they contain the inner product to $\nu(y)$ with the "chord" $x - y$ as a factor) is $O(\delta)$ as $\delta \to 0^+$ in the context of Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, vanishing Morrey spaces, block spaces, as well as for the brands of Sobolev spaces naturally associated with these scales. Simply put, the problem that we completely solve is that of determining when singular integral operators of double layer type have small operator norms. We also establish estimates in the opposite direction, quantifying the flatness of a "surface" by estimating the BMO semi-norm of its unit normal in terms of the operator norms of certain singular integrals canonically associated with the given surface (such as the harmonic double layer, the family of Riesz transforms, and commutators between Riesz transforms and pointwise multiplication by the components of the unit normal). Ultimately, this goes to show that the two-way bridge between geometry and analysis constructed here is in the nature of best possible.

Significantly, the operator norm estimates described in the previous paragraph permit us to invert the boundary double layer potentials associated with certain classes of second-order PDE (such as the Laplacian, any scalar homogeneous constant complex coefficient second-order operator which is weakly elliptic when $n \geq 3$ or strongly elliptic in any dimension, the Lamé system of elasticity and, most generally, any weakly elliptic homogeneous constant complex coefficient second-order systems having a certain distinguished coefficient tensor), acting on a large variety of function spaces considered on the boundary of a sufficiently flat domain (specifically, a $\delta$-SKT domain with $\delta \in (0,1)$ suitably small relative to other geometric characteristics of the said domain). In particular, this portion of our work goes in the direction of answering the question posed by C. Kenig in [60, Problem 3.2.2, p. 117] asking to invert layer potentials in appropriate spaces on certain uniformly rectifiable sets.

In turn, these invertibility results allow us to establish solvability results for boundary value problems in the class of weakly elliptic second-order systems mentioned above, in a sufficiently flat $\delta$-SKT domain, with boundary data from Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Morrey spaces, vanishing Morrey spaces, block spaces, and from Sobolev spaces naturally associated with these scales.

In summary, a central theme in Geometric Measure Theory is understanding how geometric properties translate into analytical ones, and here we explore the implications of demanding that Gauss' map $\partial\Omega \ni x \mapsto \nu(x) \in S^{n-1}$ has small BMO semi-norm, in the realm of singular integral operators and boundary value problems. The sharp theory developed here complements the results of S. Hofmann, M. Mitrea, and M. Taylor obtained for bounded domains in [53], and extends previously known well-posedness results for elliptic PDE's in the upper half-space (which is a $\delta$-SKT domain for each $\delta \in (0,1)$).

The material in this chapter is based on joint work with J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea (cf. [77]).

## Contents

## 2.1 Introduction

More than 25 years ago, in [60, Problem 3.2.2, p. 117], C. Kenig asked to "*Prove that the layer potentials are invertible in appropriate* [...] *spaces in* [*suitable subclasses of uniformly rectifiable*] *domains*." Kenig's main motivation in this regard stems from the desire of establishing solvability results for boundary value problems formulated in a rather inclusive geometric setting. In the buildup to this open question on [60, p. 116], it is remarked that there are quite general classes of open sets $\Omega \subseteq \mathbb{R}^n$ with the property that if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ (where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$) then the said layer potentials are bounded operators on $L^p(\partial\Omega, \sigma)$ for each exponent $p \in (1, \infty)$. Remarkably, this is the case whenever $\Omega \subseteq \mathbb{R}^n$ is an open set with a uniformly rectifiable boundary (cf. [33]).

To further elaborate on this issue, recall the terminology introduced in Section 1.2. In this context, note that the given system $L$ in (1.2.1) does not determine uniquely the coefficient tensor

$$A := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\le j,k\le n\\ 1\le\alpha,\beta\le M}} \tag{2.1.1}$$

since employing $\widetilde{A} := \big(\widetilde{a}^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ in place of $A$ in the right-hand side of (1.2.1) yields the same system whenever the difference $a^{\alpha\beta}_{jk} - \widetilde{a}^{\alpha\beta}_{jk}$ is antisymmetric in the indices $j,k$ (for each $\alpha,\beta \in \{1,\dots,M\}$). Hence, there are a multitude of coefficient tensors $A$ which may be used to represent the given system $L$ as in (1.2.1). For each such coefficient tensor $A := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}}$ we shall associate a double layer potential operator $K_A$ on the boundary of a given uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ (see Definition 1.1.5). Specifically, if $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the "surface measure" on $\partial\Omega$ and if $\nu = (\nu_1,\dots,\nu_n)$ denotes the geometric measure theoretic outward unit normal to $\Omega$, then for each function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{2.1.2}$$

we define, at $\sigma$-a.e. point $x \in \partial\Omega$,

$$K_A f(x) := \Big(-\lim_{\varepsilon\to0^+}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(y)a^{\beta\alpha}_{jk}\,(\partial_j E_{\gamma\beta})\,(x-y)f_\alpha(y)\,d\sigma(y)\Big)_{1\leq\gamma\leq M}. \tag{2.1.3}$$

(Note that (2.1.2) is the most general environment in which each truncated integral in (2.1.3) is absolutely convergent.)

To offer a simple example, consider the case when $L = \Delta$, the Laplacian, in $\mathbb{R}^2$. Then $n = 2$ and $M = 1$. In this scalar case, we agree to drop the Greek superscripts labeling the entries of the coefficient tensor (2.1.1) used to express $L$ as in (1.2.1). Hence, we shall consider writings $\Delta = a_{jk}\partial_j\partial_k$ corresponding to various choices of the matrix $A = (a_{jk})_{1\leq j,k\leq 2} \in \mathbb{C}^{2\times2}$. Two such natural choices are

$$A_0 := \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}, \qquad A_1 := \begin{pmatrix} 1 & i\\ -i & 1\end{pmatrix}, \tag{2.1.4}$$

corresponding to which the recipe given in (2.1.3) yields

$$K_{A_0} f(x) = \lim_{\varepsilon\to0^+}\frac{1}{2\pi}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle\nu(y),y-x\rangle}{|x-y|^2} f(y)\,d\sigma(y) \ \text{ for } \sigma\text{-a.e. } x\in\partial\Omega, \tag{2.1.5}$$

i.e., the (two-dimensional) harmonic boundary-to-boundary double layer potential operator and, under the natural identification $\mathbb{R}^2 \equiv \mathbb{C}$,

$$K_{A_1} f(z) = \lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_{\partial\Omega\setminus\overline{B(z,\varepsilon)}} \frac{f(\zeta)}{\zeta - z}\,d\zeta \ \text{ for } \sigma\text{-a.e. } z\in\partial\Omega, \tag{2.1.6}$$

i.e., the boundary-to-boundary Cauchy integral operator, respectively.

Returning to the mainstream discussion in the general setting considered earlier, fundamental work in [33] guarantees that, if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, then for each coefficient tensor $A$ as in (2.1.1) which may be employed to write the given system $L$ as in (1.2.1), the boundary-to-boundary double layer potential $K_A$ from (2.1.3) is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega,\sigma)\big]^M$ for each $p \in (1,\infty)$. This property is particularly relevant in the treatment of the Dirichlet Problem for the system $L$ in the uniformly rectifiable domain $\Omega$ when the boundary data are selected from the space $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in (1,\infty)$, i.e.,

$$(D)_p \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = g \in \big[L^p(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{2.1.7}$$

where $\mathcal{N}_\kappa u$ is the nontangential maximal function, and $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ is the nontangential boundary trace, of the solution $u$ (see the body of the manuscript for precise definitions). Indeed, the essence of the boundary layer method is to consider as a candidate for the solution of the Dirichlet Problem (2.1.7) the $\mathbb{C}^M$-valued function $u$ defined at each point $x \in \Omega$ by

$$u(x) := \Big( - \int_{\partial\Omega} \nu_k(y) a^{\beta\alpha}_{jk} \left(\partial_j E_{\gamma\beta}\right)(x-y) f_\alpha(y)\, d\sigma(y) \Big)_{1\leq\gamma\leq M}, \tag{2.1.8}$$

for some yet-to-be-determined function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^p(\partial\Omega,\sigma)\big]^M$. In light of the special format of $u$ (in particular, thanks to the jump-formula (2.3.67)), this ultimately reduces the entire aforementioned Dirichlet Problem to the issue of solving the boundary integral equation

$$\left(\tfrac{1}{2}I + K_A\right) f = g \ \text{ on } \ \partial\Omega, \tag{2.1.9}$$

where $I$ is the identity operator (see Section 2.6 for the actual implementation of this approach). As such, having the operator $K_A$ well defined, linear, and bounded on $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in (1,\infty)$ opens the door for bringing in functional analytic techniques for inverting $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega,\sigma)\big]^M$ and eventually expressing the solution $f$ as $\left(\frac{1}{2}I + K_A\right)^{-1} g$.

A breakthrough in this regard has been registered by S. Hofmann, M. Mitrea, and M. Taylor in [53], where they have employed Fredholm theory in order to solve the boundary integral equation (2.1.9). To describe one of their main results, suppose $L = \Delta$, the Laplacian in $\mathbb{R}^n$, is written as $\Delta = a_{jk}\partial_j\partial_k$ for the identity matrix $A := (\delta_{jk})_{1\leq j,k\leq n}$. The blueprint provided in (2.1.3) then presently produces the classical harmonic double layer potential operator $K_\Delta$, acting on each function $f \in L^p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$ according to

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, d\sigma(y) \ \text{ for } \sigma\text{-a.e. } \ x \in \partial\Omega, \tag{2.1.10}$$

where $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^n$. In regard to this operator, S. Hofmann, M. Mitrea, and M. Taylor have proved in [53, Theorem 4.36, pp. 2728-2729] that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (which depends only on the said geometric characteristics of $\Omega$, $n$, $p$, and $\varepsilon$) such that

$$\operatorname{dist}\big(\nu\,,\, \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta \Longrightarrow \operatorname{dist}\big(K_\Delta\,,\, \mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \varepsilon. \tag{2.1.11}$$

The distance in the left-hand side of (2.1.11) is measured in the John-Nirenberg space $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of bounded mean oscillations on $\partial\Omega$ (with respect to the surface measure $\sigma$), from the unit vector $\nu \in \big[L^\infty(\partial\Omega,\sigma)\big]^n$ to the Sarason space $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$ of vector-valued functions of vanishing mean oscillations on $\partial\Omega$ (with respect to the surface measure $\sigma$), which is a closed subspace of $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$. The distance in the right-hand side of (2.1.11) is considered from $K_\Delta \in \mathrm{Bd}(L^p(\partial\Omega,\sigma))$, the Banach space of all linear and bounded operators on $L^p(\partial\Omega,\sigma)$ equipped with the operator norm, to $\mathrm{Cp}(L^p(\partial\Omega,\sigma))$ which is the closed linear subspace of $\mathrm{Bd}(L^p(\partial\Omega,\sigma))$ consisting of all compact operators on $L^p(\partial\Omega,\sigma)$. In particular, in the class of domains currently considered, $K_\Delta$ is a *compact* operator on $L^p(\partial\Omega,\sigma)$ whenever $\nu$ belongs to $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$. This is remarkable in as much that a purely geometric condition implies a functional analytic property of a singular integral operator. Most importantly, (2.1.11) ensures the existence of some small threshold $\delta > 0$ (which depends only on the said geometric characteristics of $\Omega$, $n$, and $p$) with the property that

$$\begin{aligned} &\operatorname{dist}\big(\nu\,,\,\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta \Longrightarrow \operatorname{dist}\big(K_\Delta\,,\,\mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) < \tfrac12 \\ &\qquad \Longrightarrow \tfrac12 I + K_\Delta \text{ Fredholm operator with index zero on } L^p(\partial\Omega,\sigma). \end{aligned} \tag{2.1.12}$$

This is the main step in establishing that $\frac12 I + K_\Delta$ is actually an invertible operator on $L^p(\partial\Omega,\sigma)$ in the said geometric setting, under the additional assumption that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected (see [53, Theorem 6.13, p. 2806]).

Another key result of a similar flavor to (2.1.11) proved in [53] pertains to the commutators $\big[M_{\nu_k}, R_j\big] := M_{\nu_k}R_j - R_j M_{\nu_k}$, where $j,k \in \{1,\dots,n\}$, between the operator $M_{\nu_k}$ of pointwise multiplication by $\nu_k$, the $k$-th scalar component of the geometric measure theoretic outward unit normal $\nu$ to $\Omega$, and $j$-th Riesz transform $R_j$ on $\partial\Omega$, acting on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon\to0^+} \frac{2}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{x_j - y_j}{|x-y|^n} f(y)\,d\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.1.13}$$

Specifically, [53, Theorem 2.19, p. 2608] states that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, and if some $p \in (1,\infty)$ has been fixed, then there exists some $C \in (0,\infty)$ (depending only on the aforementioned geometric characteristics of $\Omega$, $n$, and $p$) such that

$$\sum_{j,k=1}^n \operatorname{dist}\big(\big[M_{\nu_k}, R_j\big]\,,\,\mathrm{Cp}(L^p(\partial\Omega,\sigma))\big) \le C \operatorname{dist}\big(\nu\,,\,\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big). \tag{2.1.14}$$

Estimates of this type (with the Riesz transforms replaced by more general singular integral operators of the same nature) turned out to be a key ingredient in the proof of the fact that, if $\Omega$ is as above and $p \in (1,\infty)$, then for every threshold $\varepsilon > 0$ there exists some $\delta > 0$ (of the same nature as before) such that

$$\operatorname{dist}\big(\nu\,,\,\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta \Longrightarrow \operatorname{dist}\big(K_\Delta\,,\,\mathrm{Cp}(L^p_1(\partial\Omega,\sigma))\big) < \varepsilon, \tag{2.1.15}$$

where $L^p_1(\partial\Omega,\sigma)$ is a certain brand of $L^p$-based Sobolev space of order one on $\partial\Omega$, introduced in [53] (and further developed in [95], [93]).

These considerations have led to the development of a theory of boundary layer potentials in what was labeled in [53] as bounded $\delta$-SKT domains, a subclass of the family of bounded uniformly rectifiable domains inspired by work of S. Semmes [107], [108], and C. Kenig and T. Toro [61], [62], [63], whose trademark feature is the fact that the distance $\operatorname{dist}\big(\nu\,,\,\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big)$, measured in the John-Nirenberg space $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$, is $<\delta$. In turn, this was used in [53] to establish the well-posedness of the Dirichlet, Regularity, Neumann, and Transmission problems for the Laplacian in the class of bounded $\delta$-SKT domains with $\delta$ sufficiently small (relative to other geometric characteristics of $\Omega$). Quite recently, this theory has been extended in [80] to the case when the boundary data belong to Muckenhoupt weighted Lebesgue and Sobolev spaces.

In addition, the class of bounded $\delta$-SKT domains also turns out to be in the nature of best possible as far as the "close-to-compactness" results mentioned in (2.1.11) and (2.1.14) are concerned. Indeed, [53, Theorem 4.41, p. 2743] states that, if $\Omega\subseteq\mathbb{R}^n$ is a uniformly rectifiable domain with compact boundary and if some $p\in(1,\infty)$ has been fixed, then there exists some $C\in(0,\infty)$ (depending only on the uniform rectifiability character of $\Omega$, $n$, and $p$) such that

$$\begin{aligned}&\operatorname{dist}\big(\nu\,,\,\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n\big) &&(2.1.16)\\ &\quad\leq C\Big\{\operatorname{dist}\big(K_\Delta\,,\,\mathrm{Cp}(L^p(\partial\Omega,\sigma))\big)+\sum_{j,k=1}^{n}\operatorname{dist}\big([M_{\nu_k},R_j]\,,\,\mathrm{Cp}(L^p(\partial\Omega,\sigma))\big)\Big\}^{1/n}.\end{aligned}$$

In particular, if $K_\Delta$ and all commutators $\big[M_{\nu_k},R_j\big]$ are compact on $L^p(\partial\Omega,\sigma)$ then $\nu$ belongs to $\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n$.

The stated goal of [53] was to "*find the optimal geometric measure theoretic context in which Fredholm theory can be successfully implemented, along the lines of its original development, for solving boundary value problems with $L^p$ data via the method of layer potentials* [*in domains with compact boundaries*]." In particular, [53] may be regarded as a sharp version of the fundamental work of E. Fabes, M. Jodeit, and N. Rivière in [39], dealing with the method of boundary layer potentials in bounded $\mathscr{C}^1$ domains. As such, the theory developed in [53] goes some way towards answering Kenig's open question formulated at the beginning of this section.

However, the insistence on $\partial\Omega$ being a *compact* set is prevalent in this work. In particular, the classical fact that the Dirichlet Problem (2.1.7) is uniquely solvable in the case when $\Omega=\mathbb{R}^n_+$ (by taking the convolution of the boundary datum $g$ with the harmonic Poisson kernel in the upper half-space; cf. [8], [42], [115], [117]) does *not* fall under the tutelage of [53]. The issue is that once the uniformly rectifiable domain $\Omega$ is allowed to have an unbounded boundary then, generally speaking, singular integral operators like the harmonic double layer (2.1.10) are no longer (close to being) compact on $L^p(\partial\Omega,\sigma)$, though they remain well defined, linear, and bounded on this space, as long as $1<p<\infty$. The fact that the theory developed in [53] is not applicable in this scenario leads one to speculate whether the treatment of layer potentials may be extended to a class of unbounded domains that includes the upper half-space. In particular, it is natural to ask whether there is a parallel theory for unbounded domains $\Omega \subseteq \mathbb{R}^n$ in which we control the mean oscillations of its outward unit normal $\nu$ by suitably adapting the condition $\operatorname{dist}\big(\nu, \big[\operatorname{VMO}(\partial\Omega,\sigma)\big]^n\big) < \delta$ which is ubiquitous in [53]. This is indeed the main goal in the present monograph.

A seemingly peculiar aspect of the harmonic double layer operator (which, in hindsight turns out to be one of its salient features) is that, as visible from (2.1.10), if $\Omega = \mathbb{R}^n_+$ then $K_\Delta = 0$. Indeed, in such a case we have $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ and $\nu = (0, \dots, 0, -1)$, hence $\langle \nu(y), y - x\rangle = 0$ for all $x, y \in \partial\Omega$. This observation lends some credence to the conjecture loosely formulated as follows:

$$\begin{array}{l}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a uniformly rectifiable domain and } 1 < p < \infty, \\ \text{then the operator norm } \|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \text{ is small if } \Omega \text{ is} \\ \text{close to being a half-space in } \mathbb{R}^n.\end{array} \tag{2.1.17}$$

To make this precise, one needs to choose an appropriate way of quantifying the proximity of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$ to a half-space in $\mathbb{R}^n$. Since work in [52] gives that a uniformly rectifiable domain $\Omega \subsetneq \mathbb{R}^n$ actually is a half-space in $\mathbb{R}^n$ if and only if its geometric measure theoretic outward unit normal $\nu$ is a constant vector field, in which scenario $\|\nu\|_{[\operatorname{BMO}(\partial\Omega,\sigma)]^n} = 0$, it is natural to make the following conjecture (which is a precise, quantitative version of (2.1.17)):

$$\begin{array}{l}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an open set satisfying a two-sided local John} \\ \text{condition and whose boundary is Ahlfors regular, then for each} \\ p \in (1,\infty) \text{ there exists a constant } C \in (0,\infty) \text{ (which depends} \\ \text{only on the said geometric characteristics of } \Omega,\ n, \text{ and } p) \text{ such} \\ \text{that } \|K_\Delta\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \le C\,\|\nu\|_{[\operatorname{BMO}(\partial\Omega,\sigma)]^n}.\end{array} \tag{2.1.18}$$

We may go a step further and adopt a broader perspective, by replacing the Laplacian by a more general system of the sort discussed in (1.2.1). Specifically, consider a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system $L$ in $\mathbb{R}^n$ written as in (1.2.1) for some coefficient tensor $A$ as in (2.1.1). Also, suppose $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Finally, fix an integrability exponent $p \in (1,\infty)$. Then one may speculate whether there exists a constant $C \in (0,\infty)$ (which depends only on the said geometric characteristics of $\Omega$, $n$, $p$, and $A$) such that the double layer potential operator $K_A$ associated with the set $\Omega$ and the coefficient tensor $A$ as in (2.1.3) satisfies

$$\|K_A\|_{[L^p(\partial\Omega,\sigma)]^M \to [L^p(\partial\Omega,\sigma)]^M} \le C\,\|\nu\|_{[\operatorname{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.1.19}$$

It turns out that the choice of the coefficient tensor $A$ used to write the given system $L$ drastically affects the veracity of (2.1.19). Indeed, consider the case when $L := \Delta$ is the Laplacian in $\mathbb{R}^2$, and $\Omega := \mathbb{R}^2_+$. Observe that $\|\nu\|_{[\operatorname{BMO}(\partial\Omega,\sigma)]^2} = 0$ in this case, since $\nu$ is constant. From (2.1.4)-(2.1.5) we see that $K_{A_0} = 0$, which is in agreement with what (2.1.19) predicts in this case. On the other hand, the operator $K_{A_1}$ from (2.1.6) becomes (under the natural identification $\partial\Omega \equiv \mathbb{R}$)

$$K_{A_1} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{y-x}\,dy \quad \text{for } \mathcal{L}^1\text{-a.e. } \ x\in\mathbb{R}, \tag{2.1.20}$$

i.e., $K_{A_1} = (i/2)H$ where

$$Hf(x) := \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{f(y)}{x-y}\,dy \quad \text{for } \mathcal{L}^1\text{-a.e. } \ x\in\mathbb{R}, \tag{2.1.21}$$

is the classical Hilbert transform on the real line. In particular, since $H^2 = -I$ we have $(K_{A_1})^2 = 4^{-1}I$ which goes to show that $\|K_{A_1}\|_{[L^p(\partial\Omega,\sigma)]^M\to[L^p(\partial\Omega,\sigma)]^M} \geq 2^{-1}$, invalidating (2.1.19) in this case.

This brings up the question of determining which of the many coefficient tensors $A$ that may be used in the representation of the given system $L$ as in (1.2.1) actually give rise to double layer potential operators $K_A$ (via the blueprint (2.1.3)) that have a chance of satisfying the estimate formulated in (2.1.19). This question is of an algebraic nature. To answer it, we find it convenient to adopt a more general point of view and consider the class of singular integral operators acting at $\sigma$-a.e. point $x\in\partial\Omega$ on functions $f$ as in (2.1.2) according to

$$T_\Theta f(x) := \left(\lim_{\varepsilon\to 0^+}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle \Theta_\gamma(x-y)\nu(y)\,,\,f(y)\rangle\,d\sigma(y)\right)_{1\leq\gamma\leq M} \tag{2.1.22}$$

where

$$\begin{gathered}\Theta = (\Theta_\gamma)_{1\leq\gamma\leq M} \ \text{ with each } \Theta_\gamma \in \big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^{M\times n}\\ \text{odd and positive homogeneous of degree } \ 1-n.\end{gathered} \tag{2.1.23}$$

Note that $K_A$ fits into this class, as it corresponds to (2.1.22) with $\Theta = (\Theta_\gamma)_{1\leq\gamma\leq M}$ given by $\Theta_\gamma := \big(a^{\beta\alpha}_{jk}\partial_j E_{\gamma\beta}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq k\leq n}}$ for each index $\gamma\in\{1,\dots,M\}$.

In this notation, the question is to find what additional condition should be imposed on $\Theta = (\Theta_\gamma)_{1\leq\gamma\leq M}$ so that the analogue of (2.1.19) holds with the operator $K_A$ replaced by $T_\Theta$. The latter inequality implies that $T_\Theta$ must vanish whenever $\Omega$ is a half-space in $\mathbb{R}^n$. Choosing $\Omega := \{z\in\mathbb{R}^n : \langle z,\omega\rangle > 0\}$ with $\omega\in S^{n-1}$ arbitrary then leads to the conclusion that for each index $\gamma\in\{1,\dots,M\}$ we have

$$\Theta_\gamma(x-y)\omega = 0 \ \text{ for each } \omega\in S^{n-1} \text{ and each } x,y\in\langle\omega\rangle^\perp \text{ with } x\neq y. \tag{2.1.24}$$

Specializing this to the case when $y=0$ and observing that $x\in\langle\omega\rangle^\perp$ is equivalent to having $\omega\in\langle x\rangle^\perp$, we arrive at

$$\Theta_\gamma(x)\omega = 0\in\mathbb{C}^M \ \text{ whenever } x\neq 0 \text{ and } \omega\in\langle x\rangle^\perp, \tag{2.1.25}$$

which is the same as saying that for each vector $x\in\mathbb{R}^n\setminus\{0\}$ the rows of the matrix $\Theta_\gamma(x)\in\mathbb{C}^{M\times n}$ are scalar multiples of $x$. Thus, there exists a family of scalar functions

$k_{\gamma,1},\dots,k_{\gamma,M}$ defined in $\mathbb{R}^n\setminus\{0\}$ such that the rows of $\Theta_\gamma(x)$ are $k_{\gamma,1}(x)x,\dots,k_{\gamma,M}(x)x$ for each $x\in\mathbb{R}^n\setminus\{0\}$. Ultimately, this implies that $k:=(k_{\gamma,\alpha})_{\substack{1\leq\gamma\leq M\\1\leq\alpha\leq M}}$ is a matrix-valued function belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^{M\times M}$ which is even, positive homogeneous of degree $-n$, and such that for each $\gamma\in\{1,\dots,M\}$ we have

$$\Theta_\gamma(x)\omega=\langle x,\omega\rangle k_{\gamma\cdot}(x)\quad\text{for each}\quad x\in\mathbb{R}^n\setminus\{0\}\quad\text{and}\quad\omega\in\mathbb{R}^n. \tag{2.1.26}$$

Consequently, $T_\Theta$ from (2.1.22) may be simply re-cast as

$$Tf(x)=\lim_{\varepsilon\to0^+}\int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}}\langle x-y,\nu(y)\rangle k(x-y)f(y)\,d\sigma(y)\quad\text{for }\sigma\text{-a.e. }\ x\in\partial\Omega. \tag{2.1.27}$$

In terms of the original double layer potential operator $K_A$, the above argument proves that

if (2.1.19) holds then the integral kernel of $K_A$ is necessarily of the form $\langle x-y,\nu(y)\rangle k(x-y)$ for some matrix-valued function $k\in\big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^{M\times M}$ which is even and positive homogeneous of degree $-n$. (2.1.28)

An algebraic condition, formulated solely in terms of $A$, guaranteeing that the integral kernel of $K_A$ has the distinguished structure singled out in (2.1.28) has been identified in [86] (see (2.3.74)-(2.3.75) where the said algebraic condition is recalled). Henceforth, we shall refer to such a coefficient tensor $A$ as being "distinguished", and we shall denote by $\mathfrak{A}_L^{\rm dis}$ the collection of all distinguished coefficient tensors which may be employed in the writing of a given system $L$.

Examples of weakly elliptic second-order homogeneous constant coefficient systems $L$ in $\mathbb{R}^n$ which possess distinguished coefficient tensors (i.e., for which $\mathfrak{A}_L^{\rm dis}\neq\varnothing$) include *all scalar operators* in dimension $n\geq3$. In particular, this is the case for the Laplacian $\Delta=\sum\limits_{j=1}^n\partial_j^2$ or, more generally, for operators of the form $L=\operatorname{div}A\nabla$ with the coefficient matrix $A=(a_{jk})_{1\leq j,k\leq n}\in\mathbb{C}^{n\times n}$ satisfying the weak ellipticity condition

$$\sum_{j,k=1}^n a_{jk}\xi_j\xi_k\neq0,\qquad\forall\,\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n\setminus\{0\}. \tag{2.1.29}$$

Other examples of weakly elliptic second-order homogeneous constant coefficient systems which possess distinguished coefficient tensors are obtained by considering the complex version of the Lamé system of elasticity in $\mathbb{R}^n$,

$$L_{\mu,\lambda}:=\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}, \tag{2.1.30}$$

where the Lamé moduli $\lambda,\mu\in\mathbb{C}$ are assumed to satisfy

$$\mu\neq0,\quad 2\mu+\lambda\neq0,\quad 3\mu+\lambda\neq0. \tag{2.1.31}$$

The first two requirements in (2.1.31) are equivalent to having the system $L_{\mu,\lambda}$ weakly elliptic (in the sense of (1.2.3)), while the last requirement in (2.1.31) ensures the existence of a distinguished coefficient tensor for $L_{\mu,\lambda}$. Recall from (1.2.7) that the (strong) Legendre-Hadamard ellipticity condition (1.2.4) holds for the complex Lamé system $L_{\mu,\lambda}$ if and only if

$$\operatorname{Re}\mu > 0 \quad \text{and} \quad \operatorname{Re}(2\mu+\lambda) > 0. \tag{2.1.32}$$

As such, our results apply to certain classes of weakly elliptic second-order systems which are not necessarily strongly elliptic (in the sense of Legendre-Hadamard). Also, while the Lamé system is symmetric, we stress that the results in this monograph require no symmetry for the systems involved.

One of the main results in this work asserts that if $L$ is a second-order, homogeneous, constant complex coefficient, weakly elliptic, $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, and if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, then for each $A \in \mathfrak{A}_L^{\rm dis}$ and each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$ (which depends only on the said geometric characteristics of $\Omega$, $n$, $p$, and $A$) such that estimate (2.1.19) actually holds (hence, in particular, the conjecture formulated in (2.1.18) is true). See Theorem 2.4.20 for a result of a more general flavor, formulated in terms of Muckenhoupt weighted Lebesgue spaces. Specifically, if the system $L$, the coefficient tensor $A$, and the set $\Omega$ are as just described, then for each Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$ with $1 < p < \infty$ there exists a constant $C \in (0,\infty)$ (which now also depends on $[w]_{A_p}$) with the property that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.1.33}$$

In turn, Theorem 2.4.20 is painlessly implied by the even more general result presented in Theorem 2.4.4 which, de facto, is the central point of this monograph. The proof of Theorem 2.4.4 uses a combination of tools of a purely geometric nature (such as Theorem 2.2.25 containing a versatile version of a decomposition result originally established by S. Semmes for smooth surfaces in [107] then subsequently strengthened as to apply to rough settings in [53], and the estimate from Proposition 2.2.24 controlling the inner product between the integral average of the outward unit normal and the "chord" in terms of the BMO semi-norm of the outward unit normal to a domain), and techniques of a purely harmonic analytic nature (like good-$\lambda$ inequalities, maximal operator estimates, stopping time arguments, and Muckenhoupt weight theory).

These considerations lead us to adopt (as we do in Definition 2.2.14) the following basic piece of terminology. An open, nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is said to be a $\delta$-`SKT domain` (for some $\delta > 0$) if $\Omega$ satisfies a two-sided local John condition, $\partial\Omega$ is an Ahlfors regular set and, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta. \tag{2.1.34}$$

Remarkably, demanding that $\delta$ in (2.1.34) is small has topological and metric implications for the underlying domain, namely $\Omega$ is a connected unbounded open set,

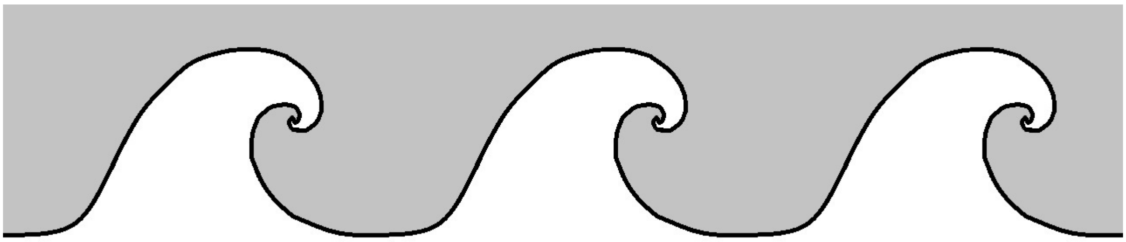

Figure 2.1: A prototype of an unbounded $\delta$-SKT domain for which $\delta > 0$ may be made as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of $\Omega$ (cf. (2.2.195), (2.2.197))

with a connected unbounded boundary and an unbounded connected complement (see Theorem 2.2.33). In the two-dimensional setting we actually show that the class of $\delta$-SKT with $\delta \in (0,1)$ small agrees with the category of chord-arc domains with small constant (see Theorem 2.2.38 for a precise statement). Most importantly, (2.1.33) shows that the oscillatory behavior of the outward unit normal is a key factor in determining the size of the operator norm for the double layer potential operator $K_A$.

Inspired by the format of a double layer operator (cf. (2.1.3)), so far we have been searching for singular integral operators fitting the general template in (2.1.22) for which it may be possible to control their operator norm in terms of $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$. While $\{T_\Theta : \Theta \text{ as in (2.1.23)}\}$ is a linear space, this is not stable under transposition (which is an isometric transformation and, hence, preserves the quality of having a small norm). This suggests that we cast a wider net and consider the class of singular integrals acting at $\sigma$-a.e. point $x \in \partial\Omega$ on functions $f$ as in (2.1.2) according to

$$T_{\Theta^1,\Theta^2} f(x) := \left( \lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \Big\langle \Theta^1_\gamma(x-y)\nu(y) - \Theta^2_\gamma(x-y)\nu(x), f(y) \Big\rangle \, d\sigma(y) \right)_{1\leq\gamma\leq M} \tag{2.1.35}$$

where $\Theta_1 = (\Theta^1_\gamma)_{1\leq\gamma\leq M}$ and $\Theta_2 = (\Theta^2_\gamma)_{1\leq\gamma\leq M}$ are as in (2.1.23). The latter condition ensures that $T_{\Theta^1,\Theta^2}$ is a well-defined, linear, and bounded operator on $\big[L^p(\partial\Omega, w)\big]^M$ (recall that we are assuming $\Omega$ to be a uniformly rectifiable domain). Consequently, $\{T_{\Theta^1,\Theta^2} : \Theta^1, \Theta^2 \text{ as in (2.1.23)}\}$ is a linear subspace of the space of linear and bounded operators on $\big[L^p(\partial\Omega, w)\big]^M$ which contains each double layer $K_A$ as in (2.1.3) as well as its formal transposed whose action on each function $f$ as in (2.1.2) at $\sigma$-a.e. $x \in \partial\Omega$ is given by

$$K_A^{\#} f(x) := \left( \lim_{\varepsilon\to 0^+} \int\limits_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \nu_k(x) a^{\beta\alpha}_{jk} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y) \, d\sigma(y) \right)_{1\leq\alpha\leq M}. \tag{2.1.36}$$

If an estimate like (2.1.33) holds for the operator (2.1.35), then we would necessarily have $T_{\Theta^1,\Theta^2} = 0$ whenever $\Omega \subseteq \mathbb{R}^n$ is a half-space. Taking $\Omega := \{z \in \mathbb{R}^n : \langle z, \omega\rangle > 0\}$ with

$\omega \in S^{n-1}$ arbitrary then forces that for each index $\gamma \in \{1, \dots, M\}$ we have

$$\begin{gathered}[\Theta^1_\gamma(x-y) - \Theta^2_\gamma(x-y)]\omega = 0 \quad \text{for each} \quad \omega \in S^{n-1} \\ \text{and each } x, y \in \langle\omega\rangle^\perp \text{ with } x \neq y. \end{gathered} \tag{2.1.37}$$

The same type of reasoning which, starting with (2.1.24), has produced (2.1.26) then shows that there exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M}$, which is even and positive homogeneous of degree $-n$, such that for each index $\gamma \in \{1, \dots, M\}$ we have

$$[\Theta^1_\gamma(z) - \Theta^2_\gamma(z)]\omega = \langle x, \omega\rangle k_{\gamma\cdot}(x) \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \omega \in \mathbb{R}^n. \tag{2.1.38}$$

In turn, this implies that (2.1.35) may be re-cast as

$$\begin{aligned} T_{\Theta^1,\Theta^2} f(x) = \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \\ + \Bigg( \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \langle \Theta^2_\gamma(x-y)(\nu(y) - \nu(x)), f(y)\rangle\, d\sigma(y)\Bigg)_{1\leq\gamma\leq M} \end{aligned} \tag{2.1.39}$$

for $\sigma$-a.e. $x \in \partial\Omega$. The first principal-value integral in (2.1.39) has been encountered earlier in (2.1.27), while the second one is of commutator type. Specifically, the second principal-value integral in (2.1.39) may be thought of as a finite linear combination of commutators between singular integral operators of convolution type with kernels which are odd and positive homogeneous of degree $1-n$ (like the entries in any of the matrices $\Theta^2_\gamma$) and operators $M_{\nu_j}$ of pointwise multiplication with the scalar components $\nu_j$, $1 \leq j \leq n$, of the outward unit normal $\nu$.

The ultimate conclusion is that, in addition to the family of operators described in (2.1.27), the class of commutators of the sort just described provides the only other viable candidates for operators whose norms become small when the ambient surface on which they are defined becomes flatter. That such an eventuality actually materializes is implied by [53, Theorem 2.16, p. 2603] which, in particular, gives (in the same setting as above)

$$\sum_{j,k=1}^{n} \big\| [M_{\nu_k}, R_j] \big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.1.40}$$

In the opposite direction, in Theorem 2.5.5 we prove that whenever $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain, $1 < p < \infty$, and $w \in A_p(\partial\Omega, \sigma)$, there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\ + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \Big\}^{1/(2n-1)}. \end{aligned} \tag{2.1.41}$$

This is done using the Clifford algebra machinery (cf. Section 1.4) and exploiting the relationship between the Cauchy-Clifford operator (cf. (2.5.1)) and the operators $K_\Delta$,

$[M_{\nu_k}, R_j]$ with $1 \leq j, k \leq n$, intervening in (2.1.41). Collectively, these results point to the optimality of the class of $\delta$-SKT domains with $\delta \in (0,1)$ small as the geometric environment in which $\|K_\Delta\|_{[L^p(\partial\Omega,w)]^M \to [L^p(\partial\Omega,w)]^M}$ and $\|[M_{\nu_k}, R_j]\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ for $1 \leq j, k \leq n$ can possibly be small (relative to $n$, $p$, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$).

We also succeed in characterizing flatness solely in terms of the behavior of the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ (defined in (2.1.13)). In one direction, in Theorem 2.5.7 we show that if $\Omega \subseteq \mathbb{R}^n$ is a uniformly rectifiable domain with an unbounded boundary and $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the uniform rectifiability character of $\partial\Omega$ with the property that

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{\Big\|I + \sum_{j=1}^n R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\ + \max_{1\leq j,k\leq n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}\Big\}^{1/(2n-1)}. \end{aligned} \tag{2.1.42}$$

Moreover, in the unweighted case (i.e., when $w \equiv 1$) the exponent $1/(2n-1)$ may be replaced by $1/n$. In the opposite direction, in Theorem 2.5.8 we prove that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set, then for each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ there exists some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\Big\|I + \sum_{j=1}^n R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.1.43}$$

and

$$\max_{1\leq j<k\leq n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.1.44}$$

Collectively, (2.1.42)-(2.1.44) give a fully satisfactory answer to the question of quantifying flatness of a given "surface" $\Sigma$ (thought of as the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$) in terms of the operator theoretic nature of the Riesz transforms on $\Sigma$. Informally, these estimates amount to saying that the flatter $\Sigma$ is, the closer $\{R_j\}_{1\leq j\leq n}$ are to satisfying the "usual" Riesz transform identities

$$\sum_{j=1}^n R_j^2 = -I \quad \text{and} \quad R_j R_k = R_k R_j \quad \text{for all} \quad j, k \in \{1, \dots, n\}, \tag{2.1.45}$$

when all operators are considered on Muckenhoupt weighted Lebesgue spaces on $\Sigma$, and vice versa. In the limit case when $\Sigma$ is genuinely flat (manifested through the vanishing of the BMO semi-norm of its unit normal), all formulas in (2.1.45) become genuine identities. The best know case is that when $\Sigma$ is the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ in $\mathbb{R}^n$, a scenario in which (2.1.45) may be readily checked when $p = 2$ and $w \equiv 1$ based on the fact that each $R_j$ is a Fourier multiplier corresponding to the symbol $i\xi_j/|\xi|$.

The insistence on Muckenhoupt weights is justified by the fact that the boundedness of the Riesz transforms on a weighted Lebesgue space $L^p$ with $p \in (1, \infty)$ actually forces the intervening weight to belong to the Muckenhoupt class $A_p$. See the discussion in Section 2.5.4 in this regard, where other related results may be found.

While estimate (2.1.33) is valid irrespective of whether $\partial\Omega$ is bounded or not, its usefulness is most apparent when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is sufficiently small (relative to the geometry of $\Omega$ and the weight $w$) since, in the context of (2.1.33),

$$\begin{gathered}\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1/(2C) \text{ implies that } \tfrac{1}{2}I + K_A \text{ is invertible} \\ \text{on } \big[L^p(\partial\Omega, w)\big]^M \text{ and } (\tfrac{1}{2}I + K_A)^{-1} \text{ may be expressed as the} \\ \text{Neumann series } 2^{-1}\textstyle\sum_{j=0}^{\infty}(-2K_A)^j, \text{ which is convergent in the} \\ \text{operator norm,}\end{gathered} \tag{2.1.46}$$

and one can actually show that having $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1$ forces $\partial\Omega$ to be unbounded. We may therefore recast (2.1.46) as saying that we may invert $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega, w)\big]^M$ whenever $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain for some $\delta \in (0, 1)$ sufficiently small (relative to the basic geometric features of $\Omega$ and the weight $w$), and the latter condition implies that $\partial\Omega$ is unbounded.

Estimate (2.1.33) then becomes a powerful tool in the proof of similar results on other function spaces. First, in concert with the homogeneous space version of the commutator theorem of Coifman et al., [27], proved in [53, Theorem 2.16, p. 2603], this implies an analogous estimate on Muckenhoupt weighted Sobolev spaces (see (2.2.349)). That is, retaining the assumptions on the domain $\Omega$ and the system $L$ made in the build-up to (2.1.33), whenever $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$ we have

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M \to [L_1^p(\partial\Omega,w)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.1.47}$$

for some constant $C \in (0, \infty)$ of the same nature as before. To elaborate on this crucial estimate, one should think of our Muckenhoupt weighted Sobolev space $L_1^p(\partial\Omega, w)$ as being naturally associated with a family $\{\partial_{\tau_{jk}}\}_{1 \leq j,k \leq n}$ of first-order "tangential" differential operators along $\partial\Omega$, which may loosely be described as $\partial_{\tau_{jk}} = \nu_j\partial_k - \nu_k\partial_j$ for each $j, k \in \{1, \dots, n\}$. Specifically, $L_1^p(\partial\Omega, w)$ is the linear space consists of functions $f \in L^p(\partial\Omega, w)$ with $\partial_{\tau_{jk}} f \in L^p(\partial\Omega, w)$ for each $j, k \in \{1, \dots, n\}$ (see the discussion in Section 2.2.6 in this regard). From this perspective it is then of paramount importance to understand the manner in which a double layer operator $K_A$ commutes with a generic tangential differential operators $\partial_{\tau_{jk}}$. It turns out that

$$\begin{gathered}\text{each commutator } [K_A, \partial_{\tau_{jk}}] \text{ acting on a function } f \text{ belonging to a} \\ \text{Muckenhoupt weighted Sobolev space may be expressed as a finite} \\ \text{linear combination of commutators of the form } [M_\nu, R] \text{ acting on} \\ \text{the components of } \nabla_{\tan} f, \text{ the tangential gradient of } f, \text{ where } M_\nu \\ \text{stands for the operator of pointwise multiplication by (generic} \\ \text{components of) the unit normal } \nu, \text{ and } R \text{ is a convolution-type} \\ \text{singular integral operator on } \partial\Omega \text{ of similar nature as the Riesz} \\ \text{transforms on } \partial\Omega \text{ (cf. (2.1.13)).}\end{gathered} \tag{2.1.48}$$

Based on this, (2.1.33), and a suitable analogue of (2.1.40), we then conclude that the key estimate stated in (2.1.47) holds. In turn, (2.1.47) permits us to invert $\frac{1}{2}I + K_A$ on the Muckenhoupt weighted Sobolev space $\big[L^p_1(\partial\Omega, w)\big]^M$, for each $w \in A_p(\partial\Omega, \sigma)$ with $1 < p < \infty$, via a Neumann series converging in the operator norm, whenever $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain for some $\delta \in (0,1)$ sufficiently small (a condition that renders $\partial\Omega$ unbounded) relative to the geometry of $\Omega$ and the weight $w$.

Second, we use the operator norm estimate on Muckenhoupt weighted Lebesgue spaces from (2.1.33) as a gateway to establishing similar estimates via extrapolation procedures. One of the best known embodiments of this principle is Rubio de Francia's celebrated extrapolation theorem, according to which estimates on Muckenhoupt weighted Lebesgue spaces for a fixed integrability exponent and all weights imply similar estimates for all integrability exponents (prompting Antonio Córdoba to famously declare that "there are no $L^p$ spaces, only weighted $L^2$ spaces"). Here we use (2.1.33) together with an extrapolation procedure from [93] (recalled in Proposition 2.7.5) to obtain norm estimates for double layer operators on the scale of Morrey spaces on the boundary of uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$, i.e.,

$$M^{p,\lambda}(\partial\Omega, \sigma) := \big\{ f \in L^1_{\rm loc}(\partial\Omega, \sigma) : \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty \big\} \tag{2.1.49}$$

with $p \in (1,\infty)$ and $\lambda \in (0, n-1)$, where

$$\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} := \sup_{\substack{x\in\partial\Omega \text{ and} \\ 0<R<2\,{\rm diam}(\partial\Omega)}} \left\{ R^{\frac{n-1-\lambda}{p}} \Big( ⨍_{\partial\Omega\cap B(x,R)} |f|^p \, d\sigma \Big)^{\frac{1}{p}} \right\}. \tag{2.1.50}$$

(Note that the scale of ordinary Lebesgue spaces on $\partial\Omega$ corresponds to the end-point case $\lambda = 0$, while the end-point $\lambda = n-1$ corresponds to the space of essentially bounded functions on $\partial\Omega$.) Retaining the same geometric context as before and assuming $A \in \mathfrak{A}^{\rm dis}_L$, the extrapolation procedure alluded to above yields

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.1.51}$$

for some constant $C \in (0,\infty)$ of the same nature as before (cf. Theorem 2.7.11 for this, and other related results). We may take this a step further and establish a similar operator norm estimate involving the Morrey-styled Sobolev space $M^{p,\lambda}_1(\partial\Omega,\sigma)$. These, in turn, allow us to us to invert $\frac{1}{2}I + K_A$ both on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and on the Morrey-based Sobolev space $\big[M^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$, under similar assumptions as before. See Theorem 2.7.12 where this and other invertibility results on related spaces are proved. In addition, (2.1.33) implies (via real interpolation) norm estimates and invertibility results for double layer potential operators on Lorentz spaces and Lorentz-based Sobolev spaces (cf. Remark 2.4.21 and Remark 2.4.25).

Concisely put, in this work we are able to answer Kenig's open question (formulated at the outset of this section) pertaining to any given weakly elliptic homogeneous constant complex coefficient second-order system $L$ in $\mathbb{R}^n$ with $\mathfrak{A}^{\rm dis}_L \neq \varnothing$, in the setting of $\delta$-SKT domains $\Omega \subseteq \mathbb{R}^n$ with $\delta \in (0,1)$ small (relative to the original geometric characteristics of $\Omega$), for ordinary Lebesgue spaces, Lorentz spaces, Muckenhoupt weighted Lebesgue, Morrey spaces, as well as Sobolev spaces on $\partial\Omega$ suitably defined in relation to each of the aforementioned scales (see Theorem 2.4.24, Remark 2.4.25, Theorem 2.4.29, Theorem 2.7.12, Theorem 2.7.13). As indicated in Remark 2.4.28, the smallness condition imposed on the parameter $\delta$ is actually in the nature of best possible as far as the aforementioned invertibility results are concerned.

In turn, the aforementioned invertibility results open the door for solving boundary value problems of Dirichlet, Regularity, Neumann, and Transmission type in the class of $\delta$-SKT domains with $\delta \in (0,1)$ small (relative to the original geometric characteristics of $\Omega$) for second-order weakly elliptic constant complex coefficient systems which (either themselves and/or their transposed) possess distinguished coefficient tensors.

For example, in such a setting, we succeed in establishing the well-posedness of the Muckenhoupt weighted Dirichlet Problem and the Muckenhoupt weighted Regularity Problem (formulated using the nontangential maximal operator introduced in (1.1.2), and nontangential boundary traces defined as in (1.1.5), for some fixed aperture parameter $\kappa > 0$):

$$(D)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \qquad (R)_{p,w} \begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^p_1(\partial\Omega, w)\big]^M, \end{cases} \tag{2.1.52}$$

for each integrability exponent $p \in (1,\infty)$ and each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, under the assumption that both $L$ and $L^\top$ have a distinguished coefficient tensor. Moreover, we provide counterexamples which show that the well-posedness result just described may fail if these assumptions on the existence of distinguished coefficient tensors are simply dropped. See Theorem 2.6.2 and Theorem 2.6.5 for more nuanced statements. Our results are therefore optimal in this regard. We wish to note that even in the scalar (i.e., $M = 1$), unweighted case (i.e., $w \equiv 1$), the well-posedness of the problems in (2.1.52) would still be new for such basic constant complex coefficient differential operators as

$$L = \partial_1^2 + \cdots + \partial_{n-1}^2 + i\partial_n^2. \tag{2.1.53}$$

Existence for $(D)_{p,w}$, $(R)_{p,w}$ is established by looking for a solution which is expressed as in (2.1.8), making use of the jump-formula (2.3.67), and the fact that $\frac{1}{2}I + K_A$ is invertible both on the Muckenhoupt weighted Lebesgue space $\big[L^p(\partial\Omega, w)\big]^M$ as well as on the Muckenhoupt weighted Sobolev space $\big[L^p_1(\partial\Omega, w)\big]^M$. The issue of uniqueness requires a new set of techniques, and may be challenging even in the classical setting of the upper half-space $\Omega := \mathbb{R}^n_+$. In the particular case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ has been treated at length in a number of monographs in the unweighted case (i.e., when $w = 1$), including [8], [42],

[115], [116], and [117]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle, or the Schwarz reflection principle for harmonic functions. Neither of these techniques may be adapted successfully to prove uniqueness in the case of general systems treated here. Subsequently, the Dirichlet boundary value problem $(D)_{p,w}$ in $\Omega := \mathbb{R}^n_+$ for a general strongly elliptic, second-order, homogeneous, constant complex coefficient, system $L$, and for an arbitrary Muckenhoupt weight $w$ has been treated in [82], where existence employs the Agmon-Douglis-Nirenberg Poisson kernel for $L$, while uniqueness relies on special properties of the Green function for $L$ in the upper half-space $\mathbb{R}^n_+$.

In the present setting, when $\Omega$ is merely a $\delta$-SKT domain with $\delta \in (0,1)$ small (relative to the original geometric characteristics of $\Omega$), in order to deal with the issue of uniqueness for the Muckenhoupt weighted Dirichlet Problem $(D)_{p,w}$ we construct a Green function $G$ for $L$ in $\Omega$ by correcting the fundamental solution $E$ of $L$ in $\mathbb{R}^n$ (as to ensure its boundary trace on $\partial\Omega$ vanishes) using the existence part for the Regularity Problem $(R)_{p',w'}$ (formulated for the transposed system $L^\top$, the conjugate exponent $p'$, and the dual weight $w'$) and then employ a rather general Poisson integral representation formula recently established in [93] (cf. Theorem 2.6.1 for a precise statement).

In the same geometric setting, of $\delta$-SKT domains, we also discuss the solvability of the Muckenhoupt weighted Neumann Problem (in Theorem 2.6.10) and the Muckenhoupt weighted Transmission Problem (in Theorem 2.6.14), i.e.,

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial^A_\nu u = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \qquad \begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \text{ in } \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w), \\ u^+\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = u^-\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \ \sigma\text{-a.e. on } \partial\Omega, \\ \partial^A_\nu u^+ - \mu \cdot \partial^A_\nu u^- = f \in \big[L^p(\partial\Omega, w)\big]^M, \end{cases} \tag{2.1.54}$$

(where $\partial^A_\nu$ is the conormal derivative operator associated with the coefficient tensor $A$ used to represent the given system $L$, and $\mu \in \mathbb{C} \setminus \{\pm 1\}$ is a transmission parameter), as well as variants of those boundary value problems involving Lorentz spaces. In all cases, we show that the boundary layer method may be successfully implemented for any second-order homogeneous constant complex coefficient weakly elliptic system $L$ in $\mathbb{R}^n$ whose transposed possesses a distinguished coefficient tensor, assuming $A \in \mathfrak{A}^{\text{dis}}_{L^\top}$. Moreover, in the two-dimensional setting we show that the Neumann and Transmission Problems (2.1.54) remain solvable for a larger spectrum of choices of the coefficient tensor for the Lamé system (see the results in Section 2.4.4, as well as Remark 2.6.12 and Remark 2.6.16, in this regard).

In [93], a robust Calderón-Zygmund theory for singular integral operators of boundary layer type associated with weakly elliptic systems and uniformly rectifiable domains has been developed. Here we use such a platform (consisting of results recalled in Proposition 2.7.6, Theorem 2.7.7, and Theorem 2.7.8) to prove solvability results for a variety of boundary value problems of Dirichlet, Regularity, Neumann, and Transmission type (akin those formulated in (2.1.52) and (2.1.54)) with data in Morrey spaces, vanishing Morrey spaces, and block spaces (cf. Theorem 2.7.20, Theorem 2.7.22, Theorem 2.7.24, Theorem 2.7.25).

Lastly, we develop a perturbation theory to the effect that, in all cases discussed so far in this narrative, solvability of a boundary value problem for a certain system $L_o$ implies solvability for any other system $L$ which is sufficiently close to $L_o$ (with proximity quantified using the norm introduced in (1.2.15)). For results of this nature, the reader is referred to Theorem 2.6.4, Theorem 2.6.9, Theorem 2.6.13, Theorem 2.6.18, Theorem 2.7.21.

## 2.2 Geometric measure theory

### 2.2.1 More on classes of Euclidean sets of locally finite perimeter

Recall the notion of Harnack chain condition in Definition 1.1.7. Note that, in the context of Definition 1.1.7, consecutive balls must have comparable radii. The "notangentiality" condition (1.1.23) further implies that

$$\lambda B_i \subseteq \Omega \quad \text{for each} \quad \lambda \in (0, 2N^{-1}+1] \quad \text{and} \quad i \in \{1, \dots, K\}. \tag{2.2.1}$$

The Harnack chain condition described in Definition 1.1.7 should be thought of as a quantitative local connectivity condition. In particular, any open set $\Omega \subseteq \mathbb{R}^n$ satisfying an $(\infty, N)$-Harnack chain condition (for some $N \in \mathbb{N}$) is pathwise connected in a quantitative fashion.

To elaborate on the latter aspect, we find it convenient to eliminate the parameter $\varepsilon > 0$ in Definition 1.1.7. Assuming $R = \infty$, this implies that for each $k \geq 2$ there exists $L_k \in \mathbb{N}$ (which is of the order of $N \cdot \log_2 k$) with the property that for each

$$x_1, x_2 \in \Omega \quad \text{with} \quad |x_1 - x_2| \leq k \cdot \min\left\{\operatorname{dist}(x_1, \partial\Omega), \operatorname{dist}(x_2, \partial\Omega)\right\} \tag{2.2.2}$$

one can find a sequence of balls

$$\begin{array}{l} \{B(y_j, r_j)\}_{1 \leq j \leq \ell} \text{ with } \ell \in \mathbb{N} \text{ satisfying } \ell \leq L_k, \text{ such} \\ \text{that } B(y_j, (2N^{-1}+1)r_j) \subseteq \Omega \text{ for every } j \in \{1, \dots, \ell\}, \\ x_1 \in B(y_1, r_1),\ x_2 \in B(y_\ell, r_\ell), \text{ and there exists some} \\ z_j \in B(y_j, r_j) \cap B(y_{j+1}, r_{j+1}) \text{ for all } j \in \{1, \dots, \ell-1\}. \end{array} \tag{2.2.3}$$

The fact that $L_k = O(\log_2 k)$ as $k \to \infty$ quantifies the intuitive idea that the closer to the boundary the points $x_1, x_2$ are, and the further apart for each other they happen to be, the larger the numbers of balls in the Harnack chain joining them. To proceed, we agree to abbreviate

$$\delta_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega) \quad \text{for each} \quad x \in \Omega. \tag{2.2.4}$$

Then the first property in (2.2.3) implies that for each $j \in \{1, \dots, \ell\}$ and $a, b \in B(y_j, r_j)$ we have $\delta_{\partial\Omega}(a) \geq 2N^{-1} r_j$ and $\delta_{\partial\Omega}(a) \leq (N+1) \cdot \delta_{\partial\Omega}(b)$. In particular, for each index

$j \in \{1, \dots, \ell-1\}$ we have

$$(N+1)^{-1} \cdot \delta_{\partial\Omega}(z_j) \leq \delta_{\partial\Omega}(z_{j+1}) \leq (N+1) \cdot \delta_{\partial\Omega}(z_j). \tag{2.2.5}$$

Joining $x_1, y_1, z_1, y_2, z_2, y_3, \dots, y_{\ell-1}, z_{\ell-1}, y_\ell, x_2$ with line segments yields a polygonal arc $\gamma$ joining $x_1$ with $x_2$ in $\Omega$, whose length may be estimated as follows:

$$\begin{aligned} \text{length}(\gamma) &\leq N \sum_{j=1}^{\ell} r_j \leq N \sum_{j=1}^{\ell} \delta_{\partial\Omega}(z_j) \leq N \sum_{j=1}^{L_k} (N+1)^j \cdot \delta_{\partial\Omega}(x_1) \\ &\leq (N+1)^{L_k+1} \cdot \delta_{\partial\Omega}(x_1). \end{aligned} \tag{2.2.6}$$

In a similar fashion, $\text{length}(\gamma) \leq (N+1)^{L_k+1} \cdot \delta_{\partial\Omega}(x_2)$ hence, ultimately,

$$\text{length}(\gamma) \leq (N+1)^{L_k+1} \cdot \min\{\delta_{\partial\Omega}(x_1)\,,\, \delta_{\partial\Omega}(x_2)\}. \tag{2.2.7}$$

In addition, for each $x \in \gamma$ there exists $j_x \in \{1, \dots, \ell\}$ such that $x \in B(y_{j_x}, r_{j_x})$ so

$$\delta_{\partial\Omega}(x) \geq (N+1)^{-j_x} \cdot \delta_{\partial\Omega}(x_1) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_1). \tag{2.2.8}$$

Analogously, $\delta_{\partial\Omega}(x) \geq (N+1)^{-L_k} \cdot \delta_{\partial\Omega}(x_2)$ which goes to show that

$$\delta_{\partial\Omega}(x) \geq (N+1)^{-L_k} \cdot \max\{\delta_{\partial\Omega}(x_1)\,,\, \delta_{\partial\Omega}(x_2)\} \ \text{ for each } \ x \in \gamma. \tag{2.2.9}$$

The existence of such a path $\gamma$ is going to be used in Lemma 2.2.1 and Lemma 2.2.2 which, in turn, play a significant role in the proof of Theorem 2.2.38.

Next, recall the notion of NTA domain in Definition 1.1.8. It turns out that from any point in a given one-sided NTA domain one may proceed along a path towards to the interior of said domain, which progressively distances itself from the boundary. This is made precise in the lemma below.

**Lemma 2.2.1.** *Let $\Omega \subset \mathbb{R}^n$ be an unbounded $(\infty, N)$-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For each location $x \in \Omega$ and each scale $r \in (0, \infty)$ there exist a point $x_* \in \Omega$ and a polygonal arc $\gamma$ joining $x$ with $x_*$ in $\Omega$ such that*

$$\begin{aligned} &|x - x_*| < 2r, \quad \delta_{\partial\Omega}(x_*) \geq r/N^2, \quad \text{length}(\gamma) \leq C_N \cdot r, \\ &\textit{and } \ \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \ \textit{ for each point } \ y \in \gamma, \end{aligned} \tag{2.2.10}$$

*where $\gamma_{x,y}$ is the sub-arc of $\gamma$ joining $x$ with $y$.*

*Proof.* Without loss of generality assume $N \geq 2$. If $\delta_{\partial\Omega}(x) \geq r/N$, simply take $x_* := x$ and $\gamma := \{x\}$. If $\delta_{\partial\Omega}(x) < r/N$, there exists $m \in \mathbb{N}$ such that $r/N^{m+1} \leq \delta_{\partial\Omega}(x) < r/N^m$. Pick some point $z \in \partial\Omega$ such that $\delta_{\partial\Omega}(x) = |x - z|$ and define $r_j := N^j \cdot \delta_{\partial\Omega}(x) \in (0, \infty)$ for each $j \in \{1, \dots, m\}$. The fact that $\Omega$ satisfies $(\infty, N^{-1})$-corkscrew condition guarantees that for each $j \in \{1, \dots, m\}$ there exists a corkscrew point $x_j \in \Omega$ relative to the location

$z$ and scale $r_j$. Hence, for each $j \in \{1, \dots, m\}$ we have $B(x_j, r_j/N) \subseteq B(z, r_j) \cap \Omega$ which entails

$$\begin{aligned} N^j \cdot \delta_{\partial\Omega}(x) = r_j > \delta_{\partial\Omega}(x_j) > r_j/N = N^{j-1} \cdot \delta_{\partial\Omega}(x) \\ \text{and} \quad |x_j - z| < r_j = N^j \cdot \delta_{\partial\Omega}(x) \quad \text{for each} \quad j \in \{1, \dots, m\}. \end{aligned} \tag{2.2.11}$$

Denote $x_0 := x$ and observe that for each $j \in \{1, \dots, m\}$ we have $x_{j-1}, x_j \in B(z, r_j)$. Together with (2.2.11), for each $j \in \{1, \dots, m\}$ this permits us to estimate

$$|x_{j-1} - x_j| < 2r_j = 2N^j \cdot \delta_{\partial\Omega}(x) \leq 2N^2 \cdot \min\{\delta_{\partial\Omega}(x_{j-1}), \delta_{\partial\Omega}(x_j)\}. \tag{2.2.12}$$

Hence, we are in the scenario described in (2.2.2) with $x_{j-1}$, $x_j$ playing the roles of $x_1$, $x_2$, and $k := 2N^2$. From (2.2.7)-(2.2.9) we then conclude that there exists $C_N \in (1, \infty)$ with the property that for each $j \in \{1, \dots, m\}$ we may find a polygonal arc $\gamma_j$ joining $x_{j-1}$ with $x_j$ in $\Omega$ such that

$$\text{length}(\gamma_j) \leq C_N \cdot \min\{\delta_{\partial\Omega}(x_{j-1}), \delta_{\partial\Omega}(x_j)\} \leq C_N \cdot N^j \cdot \delta_{\partial\Omega}(x), \tag{2.2.13}$$

and

$$C_N \cdot \delta_{\partial\Omega}(y) \geq \max\{\delta_{\partial\Omega}(x_{j-1}), \delta_{\partial\Omega}(x_j)\} \geq N^{j-1} \cdot \delta_{\partial\Omega}(x) \quad \text{for each} \quad y \in \gamma_j. \tag{2.2.14}$$

If we now define $x_* := x_m$ and take $\gamma := \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_m$ then $\gamma$ is a polygonal arc joining $x = x_0$ with $x_* = x_m$ in $\Omega$ whose length satisfies

$$\begin{aligned} \text{length}(\gamma) &= \sum_{j=1}^{m} \text{length}(\gamma_j) \leq \sum_{j=1}^{m} C_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\ &\leq \frac{N \cdot C_N}{N-1} N^m \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N \cdot C_N}{N-1}\Big) r, \end{aligned} \tag{2.2.15}$$

thanks to (2.2.13) and our choice of $m$. Also, for each $y \in \gamma$ there exists $j_y \in \{1, \dots, m\}$ such that $y \in \gamma_{j_y}$, hence we may use (2.2.14) to bound the length of the sub-arc $\gamma_{x,y}$ of $\gamma$ joining $x$ with $y$ by

$$\begin{aligned} \text{length}(\gamma_{x,y}) &\leq \sum_{j=1}^{j_y} \text{length}(\gamma_j) \leq \sum_{j=1}^{j_y} C_N \cdot N^j \cdot \delta_{\partial\Omega}(x) \\ &\leq \frac{N^2 \cdot C_N}{N-1} N^{j_y - 1} \cdot \delta_{\partial\Omega}(x) \leq \Big(\frac{N^2 \cdot C_N^2}{N-1}\Big) \delta_{\partial\Omega}(y). \end{aligned} \tag{2.2.16}$$

Our choice of $x_*$, the first line in (2.2.11), and our choice of $m$ also permit us to conclude that

$$\delta_{\partial\Omega}(x_*) = \delta_{\partial\Omega}(x_m) > N^{m-1} \cdot \delta_{\partial\Omega}(x) \geq r/N^2. \tag{2.2.17}$$

Finally, since $x, x_* \in B(z, r_m)$ it follows that $|x - x_*| < 2r_m = 2N^m \cdot \delta_{\partial\Omega}(x) < 2r$, so all properties claimed in (2.2.10) are verified. $\square$

Our next lemma shows that one-sided NTA domains satisfy a quantitative connectivity property of the sort considered by O. Martio and J. Sarvas in [87], where the class of uniform domains has been introduced.

**Lemma 2.2.2.** *Let $\Omega \subset \mathbb{R}^n$ be an unbounded $(\infty, N)$-one-sided* NTA *domain for some $N \in \mathbb{N}$. Then there exists a constant $C_N \in (1, \infty)$ with the following significance. For any two points $x, \widetilde{x} \in \Omega$ and any scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$ there exists a polygonal arc $\Gamma$ joining $x$ with $\widetilde{x}$ in $\Omega$ such that*

$$\begin{aligned} &\mathrm{length}(\Gamma) \leq C_N \cdot r, \quad \textit{and for each point} \;\; y \in \Gamma \\ &\min\{\mathrm{length}(\Gamma_{x,y}), \mathrm{length}(\Gamma_{y,\widetilde{x}})\} \leq C_N \cdot \delta_{\partial\Omega}(y), \end{aligned} \tag{2.2.18}$$

*where $\Gamma_{x,y}$ and $\Gamma_{y,\widetilde{x}}$ are the sub-arcs of $\Gamma$ joining $x$ with $y$ and, respectively, $y$ with $\widetilde{x}$.*

*Proof.* Fix two points $x, \widetilde{x} \in \Omega$ and pick a scale $r \in (0, \infty)$ with $r \geq |x - \widetilde{x}|$. If $\delta_{\partial\Omega}(x) > 2r$ then $\widetilde{x} \in \overline{B(x,r)} \subseteq B(x, 2r) \subseteq \Omega$. In such a scenario, take $\Gamma$ to be the line segment with end-points $x, \widetilde{x}$ and all desired properties follow. There remains to treat the case when

$$\delta_{\partial\Omega}(x) \leq 2r. \tag{2.2.19}$$

To proceed, let $x_*, \widetilde{x}_*$ be associated with the given points $x, \widetilde{x}$ as in Lemma 2.2.1, and denote by $\gamma, \widetilde{\gamma}$ the polygonal arcs joining $x$ with $x_*$ and $\widetilde{x}$ with $\widetilde{x}_*$ in $\Omega$, having the properties described in (2.2.10), for the current scale $r$. Specifically, for this choice of the scale, (2.2.10) gives

$$\begin{aligned} &|x - x_*| < 2r, \quad |\widetilde{x} - \widetilde{x}_*| < 2r, \\ &\delta_{\partial\Omega}(x_*) \geq r/N^2, \quad \delta_{\partial\Omega}(\widetilde{x}_*) \geq r/N^2, \\ &\mathrm{length}(\gamma) \leq C_N \cdot r, \quad \mathrm{length}(\widetilde{\gamma}) \leq C_N \cdot r, \\ &\mathrm{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \;\; \text{for each} \;\; y \in \gamma, \\ &\mathrm{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \;\; \text{for each} \;\; y \in \widetilde{\gamma}. \end{aligned} \tag{2.2.20}$$

Note that

$$|x_* - \widetilde{x}_*| \leq |x_* - x| + |x - \widetilde{x}| + |\widetilde{x} - \widetilde{x}_*| < 2r + r + 2r = 5r. \tag{2.2.21}$$

From (2.2.21) and the second line in (2.2.20) we then see that

$$|x_* - \widetilde{x}_*| < 5r \leq 5N^2 \cdot \min\{\delta_{\partial\Omega}(x_*), \delta_{\partial\Omega}(\widetilde{x}_*)\}. \tag{2.2.22}$$

Thus, we are in the scenario described in (2.2.2) with $x_1 := x_*$, $x_2 := \widetilde{x}_*$, and with $k := 5N^2$. From (2.2.7)-(2.2.9) we then conclude that there exist a constant $C_N \in (1, \infty)$ along with a polygonal arc $\widehat{\gamma}$ joining $x_*$ with $\widetilde{x}_*$ in $\Omega$ such that

$$\mathrm{length}(\widehat{\gamma}) \leq C_N \cdot \min\{\delta_{\partial\Omega}(x_*), \delta_{\partial\Omega}(\widetilde{x}_*)\} \leq 2C_N \cdot r, \tag{2.2.23}$$

where the last inequality comes from (2.2.19), and

$$C_N \cdot \delta_{\partial\Omega}(y) \geq \max\{\delta_{\partial\Omega}(x_*), \delta_{\partial\Omega}(\widetilde{x}_*)\} \geq r/N^2 \;\; \text{for each} \;\; y \in \widehat{\gamma}, \tag{2.2.24}$$

with the last inequality provided by the second line in (2.2.20).

If we now define

$$\Gamma := \gamma \cup \widehat{\gamma} \cup \widetilde{\gamma} \tag{2.2.25}$$

then $\Gamma$ is a polygonal arc joining $x$ with $\widetilde{x}$ in $\Omega$. Also, (2.2.20) and (2.2.23) allow us to estimate

$$\text{length}(\Gamma) = \text{length}(\gamma) + \text{length}(\widehat{\gamma}) + \text{length}(\widetilde{\gamma}) \leq C_N \cdot r, \tag{2.2.26}$$

proving the first estimate in (2.2.18). Fix now an arbitrary point $y \in \Gamma$. If $y$ belongs to $\gamma$, then $\Gamma_{x,y} = \gamma_{x,y}$ which further entails $\text{length}(\Gamma_{x,y}) = \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ by (2.2.20). Thus, the last estimate in (2.2.18) holds in this case. Similarly, if $y \in \widetilde{\gamma}$, then $\text{length}(\Gamma_{y,\widetilde{x}}) = \text{length}(\widetilde{\gamma}_{\widetilde{x},y}) \leq C_N \cdot \delta_{\partial\Omega}(y)$ again by (2.2.20), so the last estimate in (2.2.18) holds in this case as well. Finally, in the case when $y \in \widehat{\gamma}$ we may write

$$\min \left\{\text{length}(\Gamma_{x,y}), \text{length}(\Gamma_{y,\widetilde{x}})\right\} \leq \text{length}(\Gamma) \leq C_N \cdot r \leq C_N \cdot \delta_{\partial\Omega}(y), \tag{2.2.27}$$

by (2.2.26) and (2.2.24). □

When its end-points belong to a suitable neighborhood of infinity, the polygonal arc constructed in Lemma 2.2.2 may be made to avoid any given bounded set. This property, established in the next lemma, is going to be relevant later on, in the course of the proof of Theorem 2.2.38.

**Lemma 2.2.3.** *Let $\Omega \subset \mathbb{R}^n$ be an unbounded $(\infty, N)$-one-sided* NTA *domain for some $N \in \mathbb{N}$ such that $\mathbb{R}^n \setminus \overline{\Omega} \neq \varnothing$. Fix some point $z_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and some radius $R \in (0,\infty)$. Then there exist a large constant $C = C(N) \in (0,\infty)$ together with a small number $\varepsilon = \varepsilon(N) \in (0,1)$ with the property that for any two points $x, \widetilde{x} \in \Omega \setminus B(z_0,R)$ and any scale $r \in (0,\infty)$ with $r \geq \max\{|x-\widetilde{x}|, C \cdot R\}$ the polygonal arc $\Gamma$ joining $x$ with $\widetilde{x}$ in $\Omega$ as in Lemma 2.2.2 is disjoint from $B(z_0,\varepsilon R)$.*

*Proof.* Consider $\varepsilon \in (0,1)$ and $C \in (0,\infty)$ to be specified momentarily. Recall formula (2.2.25). Assume there exists a point $y \in \gamma \cap B(z_0,\varepsilon R)$. Then $y \in \gamma \subseteq \Omega$ so the line segment with end-points $y$ and $z_0$ intersects $\partial\Omega$. As such, $\delta_{\partial\Omega}(y) \leq \varepsilon R$. Also, $\gamma_{x,y}$ joins the point $x \in \mathbb{R}^n \setminus B(z_0,R)$ with the point $y \in B(z_0,\varepsilon R)$, which forces $\text{length}(\gamma_{x,y}) \geq (1-\varepsilon)R$. In concert with the last line in (2.2.10) this permits us to write

$$(1-\varepsilon)R \leq \text{length}(\gamma_{x,y}) \leq C_N \cdot \delta_{\partial\Omega}(y) \leq C_N \cdot \varepsilon R, \tag{2.2.28}$$

which leads to a contradiction if we choose $\varepsilon := 1/[2(C_N+1)]$. Thus, for this choice of $\varepsilon$ we have $\gamma \cap B(z_0,\varepsilon R) = \varnothing$. In a similar fashion, we also have $\widetilde{\gamma} \cap B(z_0,\varepsilon R) = \varnothing$. Finally, if there exists a point $y \in \widehat{\gamma} \cap B(z_0,\varepsilon R)$ then based on (2.2.24) and the nature of the scale $r$ we may estimate

$$\varepsilon R \geq \delta_{\partial\Omega}(y) \geq r/(N^2 \cdot C_N) \geq (C \cdot R)/(N^2 \cdot C_N) \tag{2.2.29}$$

which leads to a contradiction if $C = C(N) \in (0,\infty)$ is sufficiently large. □

To be able to define the class of Semmes-Kenig-Toro domains we have in mind we first need to formally introduce the John-Nirenberg space of functions of bounded mean oscillations on Ahlfors regular sets. Specifically, given a closed set $\Sigma \subseteq \mathbb{R}^n$, for each $x \in \Sigma$ and $r > 0$ define the surface ball $\Delta := \Delta(x,r) := B(x,r) \cap \Sigma$. For any constant $\lambda > 0$ we also agree to define $\lambda\Delta := \Delta(x, \lambda r) := B(x, \lambda r) \cap \Sigma$. Make the assumption that $\Sigma$ is Ahlfors regular and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. For each $f \in L^1_{\rm loc}(\Sigma, \sigma)$ introduce

$$f_\Delta := \fint_\Delta f\, d\sigma \quad \text{for each surface ball} \quad \Delta \subseteq \Sigma, \tag{2.2.30}$$

then consider the semi-norm

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} := \sup_{\Delta \subseteq \Sigma} \fint_\Delta |f - f_\Delta|\, d\sigma, \tag{2.2.31}$$

where the supremum in the right side of (2.2.31) is taken over all surface balls $\Delta \subseteq \Sigma$. We shall then denote by $\mathrm{BMO}(\Sigma,\sigma)$ the space of all functions $f \in L^1_{\rm loc}(\Sigma,\sigma)$ with the property that $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} < \infty$.

The above considerations may be naturally adapted to the case of vector-valued functions. Specifically, given $N \in \mathbb{N}$, for each $f : \Sigma \to \mathbb{C}^N$ with locally integrable scalar components, we define

$$\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} := \sup_{\Delta \subseteq \Sigma} \fint_\Delta |f - f_\Delta|\, d\sigma, \tag{2.2.32}$$

where the supremum in the right side of (2.2.32) is taken over all surface balls $\Delta \subseteq \Sigma$, the integral average $f_\Delta \in \mathbb{C}^N$ is taken componentwise, and $|\cdot|$ is the standard Euclidean norm in $\mathbb{C}^N$. In an analogous fashion, we then define $\big[\mathrm{BMO}(\Sigma,\sigma)\big]^N$ as the space of all $\mathbb{C}^N$-valued functions $f \in \big[L^1_{\rm loc}(\Sigma,\sigma)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\sigma)]^N} < \infty$.

A natural version of the classical John-Nirenberg inequality concerning exponential integrability of functions of bounded mean oscillations remains valid in this setting. Specifically, [75, Theorem 1.4, p. 2000] implies that there exist some small constant $c \in (0,\infty)$ along with some large constant $C \in (0,\infty)$, both of which depend only on the doubling character of $\sigma$, with the property that

$$\fint_\Delta \exp\Big\{\frac{c\,|f - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\, d\sigma \leq C \tag{2.2.33}$$

for each non-constant function $f \in \mathrm{BMO}(\Sigma,\sigma)$ and each surface ball $\Delta \subseteq \Sigma$. Note that for each surface ball $\Delta \subseteq \Sigma$ and each $\lambda \in (0,\infty)$ we have

$$\begin{gathered} 1 \leq \exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \cdot \exp\Big\{\tfrac{c\,|f(x) - f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\} \\ \text{for every} \quad x \in \Delta \quad \text{with} \quad |f(x) - f_\Delta| > \lambda. \end{gathered} \tag{2.2.34}$$

This shows that (2.2.33) implies the following level set estimate with exponential decay:

$$\begin{aligned}\sigma\Big(\{x\in\Delta:|f(x)-f_\Delta|>\lambda\}\Big) &\leq \exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\int_\Delta \exp\Big\{\frac{c\,|f-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\,d\sigma\\ &\leq C\cdot\exp\Big\{-\tfrac{c\lambda}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\sigma(\Delta)\end{aligned}\tag{2.2.35}$$

for each non-constant function $f\in\mathrm{BMO}(\Sigma,\sigma)$, each surface ball $\Delta\subseteq\Sigma$, and each $\lambda\in(0,\infty)$. Conversely, (2.2.35) implies an estimate like (2.2.33), namely

$$\fint_\Delta \exp\Big\{\frac{c_o\,|f-f_\Delta|}{\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}}\Big\}\,d\sigma\leq 1+\frac{C}{c/c_o-1},\tag{2.2.36}$$

for each non-constant function $f\in\mathrm{BMO}(\Sigma,\sigma)$ and each surface ball $\Delta\subseteq\Sigma$, as long as $c_o\in(0,c)$. See also [14, Theorem 3.15], [37, Theorem 3.1, p. 1397], [65, Lemma 2.4, p. 409], [88], and [118, Theorem 2, p. 33] in this regard. Here we wish to emphasize that only the doubling property of the underlying measure plays a role. In turn, the John-Nirenberg level set estimate (2.2.35) has many notable consequences. For one thing, (2.2.33) implies that $e^f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ if $f$ is a $\sigma$-measurable function on $\Sigma$ with $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}$ small enough (with $\ln|\cdot|$ a representative example of this local exponential integrability phenomenon). Second, (2.2.35) guarantees that

$$\mathrm{BMO}(\Sigma,\sigma)\subseteq L^p_{\mathrm{loc}}(\Sigma,\sigma)\ \text{ for each }\ p\in(0,\infty).\tag{2.2.37}$$

Third, (2.2.35) allows for more flexibility in describing the size of the BMO semi-norm. Specifically, for each $p\in[1,\infty)$ and $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ define

$$\|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)}:=\sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta|f-f_\Delta|^p\,d\sigma\Big)^{1/p},\tag{2.2.38}$$

where the supremum in (2.2.38) is taken over all surface balls $\Delta\subseteq\Sigma$. Then for each integrability exponent $p\in[1,\infty)$ there exists some constant $C_{\Sigma,p}\in(0,\infty)$ with the property that for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ we have

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\leq\|f\|_{\mathrm{BMO}_p(\Sigma,\sigma)}\leq C_{\Sigma,p}\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}.\tag{2.2.39}$$

Indeed, the first estimate in (2.2.39) is a direct consequence of definitions and Hölder's inequality, while the second estimate in (2.2.39) relies on the John-Nirenberg inequality (2.2.35). Parenthetically, we wish to note that when $\Sigma:=\mathbb{R}$ (hence $\sigma=\mathcal{L}^1$) and $p:=2$ the value of the optimal constant in (2.2.39) is known. Concretely, for each $f\in L^1_{\mathrm{loc}}(\mathbb{R},\mathcal{L}^1)$ we have

$$\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\leq\|f\|_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}\leq\tfrac{1}{2}e^{1+(2/e)}\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}.\tag{2.2.40}$$

The justification of the second estimate in (2.2.40) uses a sharp version of the one-dimensional version of the John-Nirenberg inequality (cf. [73]) according to which for each function $f \in \mathrm{BMO}(\mathbb{R},\mathcal{L}^1)$, each nonempty finite sub-interval $I \subset \mathbb{R}$, and each $\lambda \in (0,\infty)$ we have (with $f_I := \fint_I f\,d\mathcal{L}^1$)

$$\mathcal{L}^1\Big(\{t\in I:\,|f(t)-f_I|>\lambda\}\Big)\leq\frac{1}{2}e^{4/e}\mathcal{L}^1(I)\cdot\exp\Big\{-\frac{2\lambda/e}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\}. \tag{2.2.41}$$

Specifically, for each nonempty finite sub-interval $I\subset\mathbb{R}$ we may write

$$\begin{aligned}\fint_I|f(t)-f_I|^2\,dt&=\frac{1}{\mathcal{L}^1(I)}\int_0^\infty 2\lambda\cdot\mathcal{L}^1\Big(\{t\in I:\,|f(t)-f_I|>\lambda\}\Big)\,d\lambda\\&\leq e^{4/e}\int_0^\infty\lambda\cdot\exp\Big\{-\tfrac{2\lambda/e}{\|f\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}}\Big\}\,d\lambda\\&=e^{4/e}(e/2)^2\,\|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\int_0^\infty\lambda\cdot e^{-\lambda}\,d\lambda\\&=e^{4/e}(e/2)^2\,\|f\|^2_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)},\end{aligned} \tag{2.2.42}$$

thanks to (2.2.41) and some natural changes of variables, so the second estimate in (2.2.40) readily follows from (2.2.42) and (2.2.38).

Returning to the mainstream discussion, observe that (2.2.39) implies that for each integrability exponent $p\in[1,\infty)$ we have

$$\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}\approx\sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta|f-f_\Delta|^p\,d\sigma\Big)^{\frac{1}{p}}\approx\sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\Big(\fint_\Delta|f-c|^p\,d\sigma\Big)^{\frac{1}{p}}, \tag{2.2.43}$$

uniformly for $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$. For further use, let us also note here that if $\Delta$ and $\Delta'$ are two concentric surface balls in $\Sigma$ then for any $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ and any $q\in[1,\infty)$ we have

$$\Big(\fint_\Delta|f-f_{\Delta'}|^q\,d\sigma\Big)^{\frac{1}{q}}\leq C_{q,n}\Big[1+\Big(\frac{\sigma(\Delta\cup\Delta')}{\sigma(\Delta\cap\Delta')}\Big)^{\frac{1}{q}}\Big]\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}. \tag{2.2.44}$$

In particular, (2.2.44) readily implies that there exists some constant $C\in(0,\infty)$ which depends only on $n$ and the Ahlfors regular constant of $\Sigma$ with the property that for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$ and each surface ball $\Delta\subseteq\Sigma$ we have

$$|f_{2\Delta}-f_\Delta|\leq C\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}. \tag{2.2.45}$$

In turn, (2.2.45) may be used to estimate

$$|f_{2^j\Delta}-f_\Delta|\leq\sum_{k=1}^{j}|f_{2^k\Delta}-f_{2^{k-1}\Delta}|\leq Cj\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}, \tag{2.2.46}$$

for each function $f\in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, each surface ball $\Delta\subseteq\Sigma$, and each integer $j\in\mathbb{N}$.

More generally, suppose $\Sigma\subseteq\mathbb{R}^n$ is a closed set and assume $\mu$ is a doubling Borel measure on $\Sigma$. This means that there exists $C\in(0,\infty)$ with the property that for each surface ball $\Delta\subseteq\Sigma$ we have

$$0<\mu(2\Delta)\leq C\mu(\Delta)<+\infty. \tag{2.2.47}$$

In this setting, we shall denote by $\mathrm{BMO}(\Sigma,\mu)$ the space of all functions $f\in L^1_{\mathrm{loc}}(\Sigma,\mu)$ with the property that

$$\|f\|_{\mathrm{BMO}(\Sigma,\mu)}:=\sup_{\Delta\subseteq\Sigma}\fint_\Delta\Big|f-\fint_\Delta f\,d\mu\Big|\,d\mu<+\infty, \tag{2.2.48}$$

where the supremum is once again taken over all surface balls $\Delta\subseteq\Sigma$. Much as before, since the John-Nirenberg inequality holds for generic Borel doubling measures (as noted in the discussion pertaining to (2.2.33)-(2.2.35)), for each integrability exponent $p\in[1,\infty)$ we then have

$$\begin{aligned}\|f\|_{\mathrm{BMO}(\Sigma,\mu)}&\approx\|f\|_{\mathrm{BMO}_p(\Sigma,\mu)}\\&\approx\sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta\fint_\Delta|f(x)-f(y)|^p\,d\mu(x)d\mu(y)\Big)^{\frac1p}\\&\approx\sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\Big(\fint_\Delta|f-c|^p\,d\mu\Big)^{\frac1p},\end{aligned} \tag{2.2.49}$$

uniformly for $f\in L^1_{\mathrm{loc}}(\Sigma,\mu)$, where

$$\|f\|_{\mathrm{BMO}_p(\Sigma,\mu)}:=\sup_{\Delta\subseteq\Sigma}\Big(\fint_\Delta\Big|f-\fint_\Delta f\,d\mu\Big|^p\,d\mu\Big)^{1/p}, \tag{2.2.50}$$

with the supremum taken over all surface balls $\Delta\subseteq\Sigma$. As before, for any given integer $N\in\mathbb{N}$, we shall denote by $\big[\mathrm{BMO}(\Sigma,\mu)\big]^N$ the space of $\mathbb{C}^N$-valued functions $f\in\big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ with the property that $\|f\|_{[\mathrm{BMO}(\Sigma,\mu)]^N}<\infty$, where the semi-norm $\|\cdot\|_{[\mathrm{BMO}(\Sigma,\mu)]^N}$ is defined much as in (2.2.32). Finally, for each $f\in\big[L^1_{\mathrm{loc}}(\Sigma,\mu)\big]^N$ we agree to define $\|f\|_{[\mathrm{BMO}_p(\Sigma,\mu)]^N}$ as in (2.2.50), now interpreting $|\cdot|$ as the standard Euclidean norm in $\mathbb{C}^N$.

**Lemma 2.2.4.** *Let $(X,\mu)$ be a measure space with the property that $0<\mu(X)<\infty$. Also, fix an integer $N\in\mathbb{N}$ and suppose $f\in\big[L^1(X,\mu)\big]^N$. Then*

$$\fint_X\Big|f-\fint_X f\,d\mu\Big|^2\,d\mu=\fint_X|f|^2\,d\mu-\Big|\fint_X f\,d\mu\Big|^2. \tag{2.2.51}$$

*In particular,*

$$\begin{gathered}\textit{if } |f(x)|=1 \textit{ for } \mu\textit{-a.e. } x\in X \textit{ then}\\ \fint_X\Big|f-\fint_X f\,d\mu\Big|^2\,d\mu=1-\Big|\fint_X f\,d\mu\Big|^2\ \ \textit{and}\\ \Big(1-\Big|\fint_X f\,d\mu\Big|\Big)^2\le\fint_X\Big|f-\fint_X f\,d\mu\Big|^2\,d\mu\le2\Big(1-\Big|\fint_X f\,d\mu\Big|\Big),\\ 1-\Big|\fint_X f\,d\mu\Big|\le\fint_X\Big|f-\fint_X f\,d\mu\Big|\,d\mu\le\sqrt2\sqrt{1-\Big|\fint_X f\,d\mu\Big|}.\end{gathered} \tag{2.2.52}$$

*Proof.* Keeping in mind that $|Z-W|^2 = |Z|^2 - 2\mathrm{Re}(Z\cdot\overline{W}) + |W|^2$ for each $Z, W \in \mathbb{C}^N$, we may compute

$$\begin{aligned}
\fint_X \Big|f - \fint_X f\,d\mu\Big|^2\,d\mu &= \fint_X \Big(|f|^2 - 2\mathrm{Re}\Big[f\cdot\Big(\fint_X \overline{f}\,d\mu\Big)\Big] + \Big|\fint_X f\,d\mu\Big|^2\Big)\,d\mu \\
&= \fint_X |f|^2\,d\mu - 2\mathrm{Re}\fint_X f\cdot\Big(\fint_X \overline{f}\,d\mu\Big)\,d\mu + \Big|\fint_X f\,d\mu\Big|^2 \\
&= \fint_X |f|^2\,d\mu - \Big|\fint_X f\,d\mu\Big|^2, \qquad (2.2.53)
\end{aligned}$$

proving (2.2.51). Then (2.2.52) follows from this by observing that

$$1 - \Big|\fint_X f\,d\mu\Big|^2 = \Big(1 + \Big|\fint_X f\,d\mu\Big|\Big)\Big(1 - \Big|\fint_X f\,d\mu\Big|\Big) \leq 2\Big(1 - \Big|\fint_X f\,d\mu\Big|\Big) \qquad (2.2.54)$$

and

$$\begin{aligned}
1 - \Big|\fint_X f\,d\mu\Big| &= \fint_X |f|\,d\mu - \Big|\fint_X f\,d\mu\Big| \\
&\leq \fint_X \Big|f - \fint_X f\,d\mu\Big|\,d\mu \leq \Big(\fint_X \Big|f - \fint_X f\,d\mu\Big|^2\,d\mu\Big)^{1/2}, \qquad (2.2.55)
\end{aligned}$$

by the reverse triangle inequality and the Cauchy-Schwarz inequality. □

Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, Lemma 2.2.4 applies to the geometric measure theoretic outward unit normal $\nu$ to $\Omega$, taking $X := \Delta$, an arbitrary surface ball on $\partial\Omega$, and $\mu := \mathcal{H}^{n-1}\lfloor\Delta$. As indicated below, this yields a better bound for the BMO semi-norm of $\nu$ than directly estimating $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu\|_{[L^\infty(\partial\Omega,\sigma)]^n} = 2$.

**Lemma 2.2.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Then*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} \leq 1, \qquad (2.2.56)$$

*and*

$$1 - \inf_{\Delta\subseteq\partial\Omega}\Big|\fint_\Delta \nu\,d\sigma\Big| \leq \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq \sqrt{2}\sqrt{1 - \inf_{\Delta\subseteq\partial\Omega}\Big|\fint_\Delta \nu\,d\sigma\Big|}, \qquad (2.2.57)$$

*where the two infima are taken over all surface balls $\Delta \subseteq \partial\Omega$. Also,*

$$\text{if } \partial\Omega \text{ is bounded then } \quad \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} = 1. \qquad (2.2.58)$$

*As a consequence,*

$$\partial\Omega \text{ is unbounded whenever } \quad \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < 1. \qquad (2.2.59)$$

In relation to (2.2.58) we wish to note that, in the class of Ahlfors regular domains, having the BMO semi-norm of its geometric measure theoretic outward unit normal precisely 1 is not an exclusive attribute of bounded domains. For example, a straightforward computation shows that an infinite strip in $\mathbb{R}^n$ (i.e., the region in between two parallel hyperplanes in $\mathbb{R}^n$) is an unbounded Ahlfors regular domain with the property that the BMO semi-norm of its outward unit normal is equal to 1.

*Proof of Lemma 2.2.5.* Hölder's inequality and Lemma 2.2.4 imply that for each surface ball $\Delta \subseteq \partial\Omega$ we have

$$\Big(\fint_\Delta |\nu - \nu_\Delta|\, d\sigma\Big)^2 \leq \fint_\Delta |\nu - \nu_\Delta|^2\, d\sigma = 1 - \Big|\fint_\Delta \nu\, d\sigma\Big|^2 \leq 1, \tag{2.2.60}$$

from which (2.2.56) follows on account of (2.2.32), (2.2.38), and (2.2.39). Next, (2.2.57) follows from (2.2.52), used with $X := \Delta$, arbitrary surface ball on $\partial\Omega$, and $\mu := \mathcal{H}^{n-1}\lfloor\Delta$.

To justify the claim made in (2.2.58), assume first that the set $\Omega$ is bounded. In such a case, fix some point $x_0 \in \partial\Omega$ along with some real number $r_0 > \operatorname{diam}(\partial\Omega)$ and note that the latter choice entails $\Delta_0 := B(x_0, r_0) \cap \partial\Omega = \partial\Omega$. Also, since $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ (cf. Definition 1.1.2) the Divergence Formula (1.1.12) gives

$$\nu_{\Delta_0} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_{\partial\Omega} \nu \cdot \mathbf{e}_j\, d\sigma\Big)_{1\leq j\leq n} = \Big(\frac{1}{\sigma(\partial\Omega)}\int_\Omega \operatorname{div}\mathbf{e}_j\, d\mathcal{L}^n\Big)_{1\leq j\leq n} = 0. \tag{2.2.61}$$

In concert with (1.1.11) this implies (again, bearing in mind that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$)

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \sup_{\Delta\subseteq\partial\Omega} \fint_\Delta |\nu - \nu_\Delta|\, d\sigma \geq \fint_{\partial\Omega} |\nu - \nu_{\Delta_0}|\, d\sigma = 1. \tag{2.2.62}$$

In light of (2.2.56), we then conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = \|\nu\|_{[\mathrm{BMO}_2(\partial\Omega,\sigma)]^n} = 1$ in this case. When $\Omega$ is an unbounded Ahlfors regular domain with compact boundary in $\mathbb{R}^n$, having $n \geq 2$ implies that $\mathbb{R}^n \setminus \overline{\Omega}$ is a bounded Ahlfors regular domain whose topological boundary coincides with that of $\Omega$, whose geometric measure theoretic boundary agrees with that of $\Omega$, and whose geometric measure theoretic outward unit normal is $-\nu$ at $\sigma$-a.e. point on $\partial\Omega$ (cf. [93] for a proof). Granted these properties, we may run the same argument as in (2.2.61)-(2.2.62) with $\mathbb{R}^n \setminus \overline{\Omega}$ in place of $\Omega$ and conclude that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} = 1$ in this case as well. This finishes the proof of (2.2.58). $\square$

To close this section, recall for further use that $\mathrm{CMO}(\mathbb{R}^n, \mathcal{L}^n)$ is the closure of $\mathscr{C}_0^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n, \mathcal{L}^n)$. As may be seen with the help of [17, Théorème 7, p. 198], the space $\mathrm{CMO}(\mathbb{R}^n, \mathcal{L}^n)$ may be alternatively described as the linear subspace of $\mathrm{BMO}(\mathbb{R}^n, \mathcal{L}^n)$ consisting of functions $f$ satisfying the following three conditions:

$$\lim_{r\to 0^+}\Big[\sup_{x\in\mathbb{R}^n}\Big(\fint_{B(x,r)}\Big|f - \fint_{B(x,r)} f\, d\mathcal{L}^n\Big|\, d\mathcal{L}^n\Big)\Big] = 0, \tag{2.2.63}$$

$$\lim_{r\to\infty}\Big[\sup_{x\in\mathbb{R}^n}\Big(\fint_{B(x,r)}\Big|f - \fint_{B(x,r)} f\, d\mathcal{L}^n\Big|\, d\mathcal{L}^n\Big)\Big] = 0, \tag{2.2.64}$$

and

$$\lim_{|x|\to\infty}\Big[\sup_{r\in[R_0,R_1]}\Big(\fint_{B(x,r)}\Big|f - \fint_{B(x,r)} f\, d\mathcal{L}^n\Big|\, d\mathcal{L}^n\Big)\Big] = 0 \quad \text{for each } R_0, R_1 \in (0,\infty) \text{ with } R_0 < R_1. \tag{2.2.65}$$

This is going to be relevant later on, in Proposition 2.2.10.

### 2.2.2 Chord-arc curves in the plane

Shifting gears, in this section we shall work in the two-dimensional setting. We being by recalling some known results of topological flavor. First, for bounded sets, we know from [9, Corollary 1, p. 352] that

$$\text{an open bounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if } \mathbb{R}^2 \setminus \Omega \text{ is connected,} \tag{2.2.66}$$

and

$$\text{an open bounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if } \partial\Omega \text{ is connected.} \tag{2.2.67}$$

For unbounded sets, [9, Corollary 2, p. 352] gives

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component of } \mathbb{R}^2 \setminus \Omega \text{ is unbounded,} \tag{2.2.68}$$

and

$$\text{an open unbounded connected set } \Omega \subseteq \mathbb{R}^2 \text{ is simply connected if and only if every connected component } \Sigma \text{ of } \partial\Omega \text{ is unbounded.} \tag{2.2.69}$$

(Parenthetically, it is worth noting that the boundary of an open set $\Omega \subseteq \mathbb{R}^2$ which is both connected and simply connected is not necessarily connected: e.g., take $\Omega := \mathbb{R}^2 \setminus E$ where $E := [0, \infty) \times \{0, 1\}$.) Finally, according to [9, Corollary 3, p. 352],

$$\text{if } E \subseteq \mathbb{R}^2 \text{ is a closed set such that each connected component of } E \text{ is unbounded, then } \mathbb{R}^2 \setminus E \text{ is a simply connected set,} \tag{2.2.70}$$

and according to [105, Theorem 13.11, p. 274]

$$\text{an open connected set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ is simply connected if and only if } \widehat{\mathbb{C}} \setminus \Omega \text{ is connected, where } \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \text{ is the extended complex plane (i.e., the one-point compactification of } \mathbb{C}\text{, aka Riemann's sphere).} \tag{2.2.71}$$

Next, recall that a (compact) `curve` in the Euclidean plane $\mathbb{R}^2$ (canonically identified with $\mathbb{C}$) is a set of the form $\Sigma = \gamma([a, b])$, where $a, b \in \mathbb{R}$ are two numbers satisfying $a < b$, and $\gamma : [a, b] \to \mathbb{R}^2$ is a continuous function, called a parametrization of $\Sigma$. Call the curve $\Sigma$ `simple` if $\Sigma$ has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ whose restriction to $[a, b)$ is injective (hence, $\Sigma$ is simple if it is non self-intersecting). Call the curve $\Sigma$ `closed` if it has a parametrization $\gamma : [a, b] \to \mathbb{R}^2$ satisfying $\gamma(a) = \gamma(b)$. Also, call $\Sigma \subset \mathbb{C}$ a `Jordan curve`, if $\Sigma$ is a simple closed curve. Thus, a curve is Jordan if and only if it is the homeomorphic image of the unit circle $S^1$. The classical Jordan curve theorem asserts that

$$\text{the complement of a Jordan curve } \Sigma \subset \mathbb{C} \text{ consists precisely of two connected components, one bounded } \Omega_+\text{, and one unbounded } \Omega_-\text{, called the } \texttt{inner} \text{ and } \texttt{outer} \text{ domains of } \Sigma\text{, satisfying } \partial\Omega_\pm = \Sigma. \tag{2.2.72}$$

In light of (2.2.67), we also conclude that

$$\text{the inner domain } \Omega_+ \text{ of a Jordan curve } \Sigma \subset \mathbb{C} \text{ is simply connected.} \tag{2.2.73}$$

We are also going to be interested in Jordan curves `passing through infinity` in the plane. This class consists of sets of the form $\Sigma = \gamma(\mathbb{R})$, where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. For this class of curves a version of the Jordan separation theorem is also valid, namely

$$\text{if } \Sigma \text{ is a Jordan curve passing through infinity, then its complement in } \mathbb{C} \text{ consists precisely of two open connected components, called } \Omega_\pm, \text{ which satisfy } \partial\Omega_+ = \Sigma = \partial\Omega_-. \tag{2.2.74}$$

Once (2.2.74) has been established, we deduce from (2.2.69) that

$$\text{in the context of (2.2.74), the sets } \Omega_\pm \text{ are simply connected.} \tag{2.2.75}$$

To justify (2.2.74), let $\Sigma$ be a Jordan curve passing through infinity. From definitions, it follows that $\Sigma$ is a closed subset of $\mathbb{C}$. Fix an arbitrary point $z_o \in \mathbb{C}\setminus\Sigma$ and consider the homeomorphisms

$$\begin{aligned} &\Phi : \mathbb{C}\setminus\{z_o\} \longrightarrow \mathbb{C}\setminus\{0\}, \quad \Phi(z) := (z-z_o)^{-1} \text{ for all } z \in \mathbb{C}\setminus\{z_o\}, \\ &\Phi^{-1} : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}\setminus\{z_o\}, \quad \Phi^{-1}(\zeta) := z_o + \zeta^{-1} \text{ for all } \zeta \in \mathbb{C}\setminus\{0\}, \end{aligned} \tag{2.2.76}$$

which are inverse to each other. We then claim that

$$\widetilde{\Sigma} := \Phi(\Sigma) \cup \{0\} \tag{2.2.77}$$

is a simple closed curve which contains the origin in $\mathbb{C}$. To see that this is indeed the case, start by expressing $\Sigma = \gamma(\mathbb{R})$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with the property that $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. Then $\widetilde{\gamma} : [-\pi/2, \pi/2] \to \mathbb{C}$ defined for each $t \in [-\pi/2, \pi/2]$ as

$$\widetilde{\gamma}(t) := \begin{cases} \big(\gamma(\tan t) - z_o\big)^{-1} & \text{if } t \in (-\pi/2, \pi/2), \\ 0 & \text{if } t \in \{\pm\pi/2\}, \end{cases} \tag{2.2.78}$$

is a continuous function whose restriction to $[-\pi/2, \pi/2)$ is injective, and whose image is precisely $\widetilde{\Sigma}$. Also, $0 \in \widetilde{\Sigma}$ by design. Hence, as claimed, $\widetilde{\Sigma}$ is a simple closed curve passing through $0 \in \mathbb{C}$. The classical Jordan curve theorem recalled in (2.2.72) then ensures that $\mathbb{C}\setminus\widetilde{\Sigma}$ consists precisely of two open connected components, one bounded $\widetilde{\Omega}_+$, and one unbounded $\widetilde{\Omega}_-$, satisfying $\partial\widetilde{\Omega}_\pm = \widetilde{\Sigma}$. In particular,

$$\mathbb{C}\setminus\{0\} = \widetilde{\Omega}_+ \sqcup \big(\widetilde{\Sigma}\setminus\{0\}\big) \sqcup \widetilde{\Omega}_- \ \text{(disjoint unions)}. \tag{2.2.79}$$

Then $O_\pm := \Phi^{-1}(\widetilde{\Omega}_\pm)$ are open connected subsets of $\mathbb{C}\setminus\{z_o\}$, and applying the homeomorphism $\Phi^{-1}$ to (2.2.79) yields

$$\mathbb{C}\setminus\{z_o\} = O_+ \sqcup \Sigma \sqcup O_- \ \text{(disjoint unions)}. \tag{2.2.80}$$

Let us also observe that since $\widetilde{\Omega}_-$ is unbounded, there exists a sequence $\{\zeta_j\}_{j\in\mathbb{N}} \subseteq \widetilde{\Omega}_-$ with $|\zeta_j| \to \infty$ as $j \to \infty$. Consequently, the sequence $\{z_j\}_{j\in\mathbb{N}}$ defined for each $j \in \mathbb{N}$ as $z_j := \Phi^{-1}(\zeta_j) = z_o + \zeta_j^{-1}$ is contained in $\Phi^{-1}(\widetilde{\Omega}_-) = O_-$ and converges to $z_o$. This shows that

$$z_o \in \overline{O_-}. \tag{2.2.81}$$

Next, since $\Sigma$ is a closed set, the fact that $z_o \in \mathbb{C} \setminus \Sigma$ guarantees the existence of some $r > 0$ with the property that $B(z_o, r) \cap \Sigma = \varnothing$. In the context of (2.2.80) this shows that the connected set $B(z_o, r) \setminus \{z_o\}$ is covered by the open sets $O_\pm$. As such, $B(z_o, r) \setminus \{z_o\}$ is fully contained in either $O_+$, or $O_-$. In view of (2.2.81) we ultimately conclude that $B(z_o, r) \setminus \{z_o\} \subseteq O_-$. Then $\Omega_+ := O_+$ and $\Omega_- := O_- \cup \{z_o\}$ are open, connected, disjoint subsets of $\mathbb{C}$, with

$$\mathbb{C} = \Omega_+ \sqcup \Sigma \sqcup \Omega_- \text{ (disjoint unions)}, \tag{2.2.82}$$

and

$$\partial\Omega_\pm = \partial O_\pm \setminus \{z_o\} = \Phi^{-1}(\partial\widetilde{\Omega}_\pm \setminus \{0\}) = \Phi^{-1}(\widetilde{\Sigma} \setminus \{0\}) = \Sigma. \tag{2.2.83}$$

This finishes the proof of (2.2.74).

Moving on, the `length` $L \in [0, +\infty]$ of a given compact curve $\Sigma = \gamma([a, b])$ is defined as

$$L := \sup \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})|, \tag{2.2.84}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of the interval $[a, b]$. As is well-known (cf., e.g., [72, Theorem 4.38, p. 135]), the length $L$ of any simple compact curve $\Sigma$ may be expressed in terms of the Hausdorff measure by

$$L = \mathcal{H}^1(\Sigma), \tag{2.2.85}$$

and

$$\begin{aligned} &|z_1 - z_2| \leq \mathcal{H}^1(\Sigma) \text{ for any compact curve } \Sigma \\ &\text{in the plane with endpoints } z_1, z_2 \in \mathbb{C}. \end{aligned} \tag{2.2.86}$$

Call a curve $\Sigma$ `rectifiable` provided its length is finite (i.e., $L < +\infty$), and call $\Sigma$ `locally rectifiable` if each of its compact sub-curves is rectifiable. The latter condition is equivalent to demanding that $\gamma(I)$ is a rectifiable curve for each compact sub-interval $I$ of the domain of definition of some (or any) parametrization on $\Sigma$. In particular, a Jordan curve $\Sigma$ passing through infinity in the plane, with parametrization $\gamma : \mathbb{R} \to \Sigma$, is locally rectifiable if and only if $\gamma(I)$ is a rectifiable curve for any compact sub-interval $I$ of $\mathbb{R}$.

Suppose $\Sigma$ is a rectifiable, simple, compact curve in the plane, and denote by $L$ its length. Then there exists a unique parametrization $[0, L] \ni s \mapsto z(s) \in \Sigma$ of $\Sigma$, called the `arc-length parametrization` of $\Sigma$, with the property that for each $s_1, s_2 \in [0, L]$ with $s_1 < s_2$ the length of the curve with end-points at $z(s_1)$ and $z(s_2)$ is $s_2 - s_1$. It is well known (see, e.g., [72, Definition 4.21 and Theorem 4.22, pp. 128-129]) that the arch-length parametrization exists and satisfies

$$\begin{gathered} z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } [0, L] \\ \text{and } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in [0, L]. \end{gathered} \tag{2.2.87}$$

Also, (2.2.85)-(2.2.86) imply

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2|, \quad \forall\, s_1, s_2 \in [0, L]. \tag{2.2.88}$$

**Lemma 2.2.6.** *Let $\Sigma$ be a rectifiable, simple, compact curve in the plane. Denote by $L$ its length, and let $[0, L] \ni s \mapsto z(s) \in \Sigma$ be its arc-length parametrization. Given $s_1, s_2 \in [0, L]$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z'_I := ⨍_I z'(s)\, ds$. Then*

$$⨍_I |z'(s) - z'_I|^2\, ds = 1 - \Big|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\Big|^2. \tag{2.2.89}$$

*Proof.* Upon observing that

$$z'_I = ⨍_I z'(s)\, ds = \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} z'(s)\, ds = \frac{z(s_2) - z(s_1)}{s_2 - s_1}, \tag{2.2.90}$$

this is a direct consequence of the formula in the second line of (2.2.52). □

*Remark* 2.2.7. The arch-length parametrization of a locally rectifiable Jordan curve passing through infinity in the plane is defined similarly, with $\mathbb{R}$ now playing the role of the interval $[0, L]$, and satisfies properties analogous to (2.2.87), (2.2.88), and Lemma 2.2.6.

We continue by recalling an important category of curves, introduced in 1936 by Mikhail A. Lavrentiev in [71] (also known as the class of Lavrentiev curves).

**Definition 2.2.8.** Given some number $\varkappa \in [0, \infty)$, recall that a set $\Sigma \subset \mathbb{C}$ is said to be a $\varkappa$-`CAC`, or simply `CAC` (acronym for chord-arc curve) if the parameter $\varkappa$ is de-emphasized, provided $\Sigma$ is a locally rectifiable Jordan curve passing through infinity with the property that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \Sigma, \tag{2.2.91}$$

where $\ell(z_1, z_2)$ denotes the length of the sub-arc of $\Sigma$ joining $z_1$ and $z_2$.

In general, the presence of a cusp prevents a curve from being chord-arc. For example, $\Sigma := \{(x, \sqrt{|x|}) : x \in \mathbb{R}\}$ is a Jordan curve passing through infinity in $\mathbb{R}^2 \equiv \mathbb{C}$ which nonetheless fails to be chord-arc. Indeed, if for $x > 0$ we set $z_1 := x + i\sqrt{x} \in \Sigma$ and $z_2 := -x + i\sqrt{x} \in \Sigma$ then L'Hôspital's Rule gives

$$\lim_{x\to 0^+} \frac{\ell(z_1, z_2)}{|z_1 - z_2|} = \lim_{x\to 0^+} \frac{2\int_0^x \sqrt{1 + \frac{1}{4t}}\, dt}{2x} = \lim_{x\to 0^+} \sqrt{1 + \frac{1}{4x}} = +\infty, \tag{2.2.92}$$

which shows that condition (2.2.91) is violated for each $\varkappa \in [0, \infty)$.

There are fundamental links between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line. Such connections, along with other basic properties of chord-arc curves, are brought to the forefront in Proposition 2.2.9 below. To facilitate stating and proving it, we first wish to recall the following version for bi-Lipschitz maps of the classical Kirszbraun extension theorem proved in [68, Theorem 1.2] with a linear bound on the distortion:

> any function $f : \mathbb{R} \to \mathbb{C}$ with the property that there exist $C, C' \in (0, \infty)$ such that $C|t_1 - t_2| \leq |f(t_1) - f(t_2)| \leq C'|t_1 - t_2|$ for all $t_1, t_2 \in \mathbb{R}$ extends to a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ with $(C/120)|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq (2000C')|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. (2.2.93)

Results of this nature have also been proved in [120], [121], [58, Proposition 1.13, p. 227] (see also [104, Theorem 7.10, p. 166] and [29] in the case when the real line is replaced by the unit circle), though the quantitative aspect is less precise, or not explicitly mentioned, in these works.

Here is the proposition dealing with basic properties of chord-arc curves mentioned above.

**Proposition 2.2.9.** *Let $\Sigma \subset \mathbb{C}$ be a $\varkappa$-CAC in the plane, for some $\varkappa \in [0, \infty)$, and consider its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$. Then the following statements are true.*

*(i) For each $s_1, s_2 \in \mathbb{R}$ one has*

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2| \leq (1 + \varkappa)|z(s_1) - z(s_2)|, \tag{2.2.94}$$

*and*

$$\begin{gathered} z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}, \\ \text{with } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \end{gathered} \tag{2.2.95}$$

*(ii) For each $z_o \in \Sigma$ and $r \in (0, \infty)$ abbreviate $\Delta(z_o, r) := B(z_o, r) \cap \Sigma$. Then for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ one has*

$$(s_o - r, s_o + r) \subseteq z^{-1}(\Delta(z(s_o), r)) \subseteq (s_o - (1 + \varkappa)r\, ,\, s_o + (1 + \varkappa)r). \tag{2.2.96}$$

*(iii) For every Lebesgue measurable set $A \subseteq \mathbb{R}$ one has*

$$\mathcal{H}^1(z(A)) = \mathcal{L}^1(A), \tag{2.2.97}$$

*and for each $\mathcal{H}^1$-measurable set $E \subseteq \Sigma$ one has*

$$\mathcal{H}^1(E) = \mathcal{L}^1(z^{-1}(E)). \tag{2.2.98}$$

*(iv) With the arc-length measure $\sigma$ on $\Sigma$ defined as*

$$\sigma := \mathcal{H}^1 \lfloor \Sigma, \tag{2.2.99}$$

*for each $\sigma$-measurable set $E \subseteq \Sigma$ and each non-negative $\sigma$-measurable function $g$ on $E$ one has*

$$\int_E g\, d\sigma = \int_{z^{-1}(E)} g(z(s))\, ds. \tag{2.2.100}$$

*(v) Denote by $\Omega$ the region of the plane lying to the left of the curve $\Sigma$ (relative to the orientation $\Sigma$ inherits from its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then $\Omega$ is a set of locally finite perimeter and its geometric measure theoretic outward unit normal $\nu$ is given by*

$$\nu(z(s)) = -i\, z'(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } \ s \in \mathbb{R}. \tag{2.2.101}$$

*As a consequence, for $\mathcal{L}^1$-a.e. number $s \in \mathbb{R}$ the line $\{z(s) + t\, z'(s) : t \in \mathbb{R}\}$ is an approximate tangent line to $\Sigma$ at the point $z(s)$. Hence, $\Omega$ has an approximate tangent line at $\mathcal{H}^1$-almost every point on $\partial\Omega$.*

*(vi) The set $\Omega$ introduced in item (v) is a connected, simply connected, unbounded, two-sided* NTA *domain with an Ahlfors regular boundary (hence also an Ahlfors regular domain which satisfies a two-sided local John condition and, in particular, a* UR *domain) and whose topological boundary is precisely $\Sigma$, i.e., $\partial\Omega = \Sigma$. In fact,*

*there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $120^{-1}(1+\varkappa)^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$, and with the property that $\Omega = F(\mathbb{R}^2_+)$, $\mathbb{R}^2 \backslash \overline{\Omega} = F(\mathbb{R}^2_-)$, as well as $\partial\Omega = F(\mathbb{R} \times \{0\})$.* (2.2.102)

*(vii) With the piece of notation introduced in (2.2.38) one has*

$$\frac{1}{2(1+\varkappa)}\|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq \|z'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq \|z'\|_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \leq \frac{\sqrt{\varkappa(2+\varkappa)}}{1+\varkappa} < 1 \tag{2.2.103}$$

*and*

$$\frac{1}{2(1+\varkappa)}\|z'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq \|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.2.104}$$

*Moreover, $\Sigma$ is a $\varkappa_*$-*CAC *with $\varkappa_* \in [0, \varkappa]$ defined as*

$$\begin{aligned} \varkappa_* &:= \frac{1}{\sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}} - 1 \\ &= \frac{\|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}{\sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}\Big(1 + \sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}\Big)}. \end{aligned} \tag{2.2.105}$$

*Proof.* The claims in item *(i)* are seen from definitions and Remark 2.2.7, while the claim in item *(ii)* is an elementary consequence of (2.2.94). Next, in view of (2.2.95), the area formula (cf. [38, Theorem 1, p. 96]) gives (2.2.97), which may be equivalently recast as in (2.2.98). Also, the change of variable formula (cf. [38, Theorem 2, p. 99]) gives (2.2.100). This takes care of items *(iii)-(iv)*.

To proceed, from the version of the Jordan curve theorem recorded in (2.2.74) we conclude that

the complement of the curve $\Sigma$ in $\mathbb{C}$ consists precisely of two open connected components, namely $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{C} \setminus \overline{\Omega}$, satisfying $\partial\Omega_+ = \Sigma = \partial\Omega_-$. (2.2.106)

In addition, from (2.2.96) and (2.2.98) we see that for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ we have

$$\begin{aligned} \mathcal{H}^1(\Delta(z(s_o), r)) &= \mathcal{L}^1\Big(z^{-1}(\Delta(z(s_o), r))\Big) \\ &\leq \mathcal{L}^1\Big((s_o - (1+\varkappa)r\,,\ s_o + (1+\varkappa)r)\Big) \\ &= 2(1+\varkappa)r. \end{aligned} \tag{2.2.107}$$

Based on this and the criterion for finite perimeter from [38, Theorem 1, p. 222] we then conclude that $\Omega$ is a set of locally finite perimeter. Next, if $s_o \in \mathbb{R}$ is a point of differentiability for the complex-valued function $z(\cdot)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z(s_o + s) \in B\big(z(s_o) + s\, z\,'(s_o)\,,\, \varepsilon|s|\big) \quad \text{for each} \quad s \in (-\delta, \delta). \tag{2.2.108}$$

In turn, from this geometric property we deduce that for each angle $\theta \in (0, \pi)$ there exists a height $h = h(\theta) > 0$ such that if $\Gamma^{\pm}_{\theta,h}$ denote the open truncated plane sectors with common vertex at $z(s_o)$, common aperture $\theta$, common height $h$, and symmetry axes along the vectors $\pm i\, z'(s_o)$, then

$$\Gamma^{+}_{\theta,h} \subseteq \Omega = \Omega_{+} \quad \text{and} \quad \Gamma^{-}_{\theta,h} \subseteq \mathbb{C} \setminus \overline{\Omega} = \Omega_{-}. \tag{2.2.109}$$

To proceed, observe that that the measure-theoretic boundary of $\Omega$ (cf. (1.1.6)) may be presently described as

$$\partial_*\Omega = \Big\{ z \in \partial\Omega : \limsup_{r \to 0^+} \frac{\mathcal{L}^2\big(B(z,r) \cap \Omega_{\pm}\big)}{r^2} > 0 \Big\}. \tag{2.2.110}$$

Together, (2.2.109) and (2.2.110) imply that

$$\begin{aligned} &\mathcal{A} := \big\{ z(s_o) : \ s_o \in A \big\} \subseteq \partial_*\Omega, \ \text{where we have set} \\ &A := \big\{ s_o \in \mathbb{R} : \ s_o \text{ differentiability point for } z(\cdot) \big\}. \end{aligned} \tag{2.2.111}$$

Meanwhile, from (2.2.97) and the fact that $z(\cdot)$ is differentiable at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ we deduce (also using $\partial\Omega = \Sigma$) that

$$\mathcal{H}^1(\partial\Omega \setminus \mathcal{A}) = \mathcal{H}^1(\Sigma \setminus \mathcal{A}) = \mathcal{H}^1\big(z(\mathbb{R} \setminus A)\big) = \mathcal{L}^1(\mathbb{R} \setminus A) = 0. \tag{2.2.112}$$

With this in hand, formula

$$\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0 \tag{2.2.113}$$

follows by combining (2.2.111) with (2.2.112). As a consequence of (2.2.112)-(2.2.113) and (1.1.14) we then conclude that

$$\mathcal{A} \cap \partial^*\Omega \text{ has full } \mathcal{H}^1\text{-measure in } \partial\Omega. \tag{2.2.114}$$

Next, pick an arbitrary point $z_o \in \mathcal{A}$ and recall that (2.2.109) holds. From this and [52, Proposition 2.14, p. 606] it follows that if $\Gamma_{\pi-\theta}$ is the infinite open plane sector with vertex at $z_o$, aperture $\pi - \theta$, and symmetry axis along the vector $-i\, z\,'(s_o)$, then the geometric measure theoretic outward unit normal to $\Omega$ satisfies

$$\nu(z(s_o)) \in \Gamma_{\pi-\theta} \tag{2.2.115}$$

provided $\nu(z(s_o))$ exists, i.e., if $z(s_o) \in \partial^*\Omega$. The fact that the angle $\theta \in (0, \pi)$ may chosen arbitrarily close to $\pi$ then forces $\nu(z(s_o)) = -i\, z\,'(s_o)$ whenever $z(s_o) \in \partial^*\Omega$, i.e.

for $s_o \in z^{-1}(\mathcal{A} \cap \partial^*\Omega)$. Given that by (2.2.114) and (2.2.97) the latter set has full one-dimensional Lebesgue measure in $\mathbb{R}$, the claim in (2.2.101) is established. This finishes the treatment of item *(v)*.

Turning our attention to item *(vi)*, first observe that (2.2.94) implies

$$(1+\varkappa)^{-1}|s_1 - s_2| \leq |z(s_1) - z(s_2)| \leq |s_1 - s_2| \quad \text{for all} \quad s_1, s_2 \in \mathbb{R}, \tag{2.2.116}$$

hence $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is a bi-Lipschitz map. When used in conjunction with (2.2.116), the extension result recalled in (2.2.93) gives that

$$\begin{array}{l} \mathbb{R} \ni s \mapsto z(s) \in \Sigma \text{ extends to a bi-Lipschitz homeomorphism} \\ F : \mathbb{C} \to \mathbb{C} \text{ with the property that for any points } z_1, z_2 \in \mathbb{C} \text{ one} \\ \text{has } [120(1+\varkappa)]^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2|. \end{array} \tag{2.2.117}$$

As a consequence, work in [52] implies that $\Omega$ is a connected two-sided NTA domain with an Ahlfors regular boundary (hence also a connected Ahlfors regular domain which satisfies a two-sided local John condition; cf. (1.1.21) and (1.1.28)). As far as item *(vi)* is concerned, there remains to observe that $\partial\Omega = \Sigma$ has been noted earlier in (2.2.106).

Turning our attention to item *(vii)*, fix $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, abbreviate $I := [s_1, s_2]$ and set $z'_I := ⨍_I z'(s)\, ds$. We may then use Lemma 2.2.6 and (2.2.94) to estimate

$$⨍_I |z'(s) - z'_I|^2\, ds = 1 - \Big|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\Big|^2 \leq 1 - \Big(\frac{1}{1+\varkappa}\Big)^2 = \frac{\varkappa(2+\varkappa)}{(1+\varkappa)^2}. \tag{2.2.118}$$

In view of (2.2.38), this readily yields the penultimate inequality in (2.2.103). The second inequality in (2.2.103) is seen directly from the first inequality in (2.2.40).

To prove the very first inequality in (2.2.103), fix an arbitrary point $z_o \in \Sigma$ along with a radius $r \in (0, \infty)$, and set $\Delta := B(z_o, r) \cap \Sigma$. Then there exists a unique number $s_o \in \mathbb{R}$ such that $z_o = z(s_o) \in \Sigma$, and we abbreviate $\mathcal{I} := \big(s_o - (1+\varkappa)r\,,\, s_o + (1+\varkappa)r\big)$. In particular, (2.2.96) and (2.2.98) imply

$$\begin{aligned} \sigma(\Delta) &= \mathcal{H}^1(\Delta(z(s_o), r)) = \mathcal{L}^1\Big(z^{-1}(\Delta(z(s_o), r))\Big) \\ &\geq \mathcal{L}^1\Big((s_o - r\,,\, s_o + r)\Big) = 2r = (1+\varkappa)^{-1}\mathcal{L}^1(\mathcal{I}). \end{aligned} \tag{2.2.119}$$

With $c := -i ⨍_{\mathcal{I}} z'(s)\, ds \in \mathbb{C}$ we may then write

$$\begin{aligned} ⨍_\Delta |\nu - c|\, d\sigma &= \frac{1}{\sigma(\Delta)} \int_\Delta |\nu - c|\, d\sigma = \frac{1}{\sigma(\Delta)} \int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\, ds \\ &\leq \frac{1}{\sigma(\Delta)} \int_{\mathcal{I}} |\nu(z(s)) - c|\, ds = \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)} ⨍_{\mathcal{I}} |\nu(z(s)) - c|\, ds \\ &= \frac{\mathcal{L}^1(\mathcal{I})}{\sigma(\Delta)} ⨍_{\mathcal{I}} |z'(s) - ic|\, ds \leq (1+\varkappa) ⨍_{\mathcal{I}} |z'(s) - ic|\, ds \\ &\leq (1+\varkappa)\|z'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}, \end{aligned} \tag{2.2.120}$$

making use of (2.2.100), (2.2.96), (2.2.101), (2.2.119), and the choice of $c$. With (2.2.120) in hand, the first inequality in (2.2.103) readily follows. The last estimate in (2.2.104) is implicit in (2.2.103). To prove the first estimate in (2.2.104), retain notation introduced above and, now with the choice $c := ⨍_\Delta \nu\, d\sigma \in \mathbb{C}$, estimate

$$\begin{aligned}
⨍_{s_o-r}^{s_o+r} |z'(s) - ic|\, ds &= \frac{1}{2r}\int_{s_o-r}^{s_o+r} |z'(s) - ic|\, ds \le \frac{1}{2r}\int_{z^{-1}(\Delta)} |z'(s) - ic|\, ds \\
&= \frac{1}{2r}\int_{z^{-1}(\Delta)} |\nu(z(s)) - c|\, ds = \frac{1}{2r}\int_\Delta |\nu - c|\, d\sigma \\
&= \frac{\sigma(\Delta)}{2r} ⨍_\Delta |\nu - c|\, d\sigma \le (1+\varkappa) ⨍_\Delta |\nu - c|\, d\sigma \\
&\le (1+\varkappa)\|\nu\|_{\mathrm{BMO}\,(\Sigma,\sigma)},
\end{aligned} \tag{2.2.121}$$

thanks to (2.2.96), (2.2.101), (2.2.100), and (2.2.107). This readily yields the first estimate in (2.2.104).

To deal with the very last claim in item *(vii)*, fix some $s_1, s_2 \in \mathbb{R}$ with $s_1 < s_2$, set $I := [s_1, s_2]$ and abbreviate $z'_I := ⨍_I z'(s)\, ds$. Lemma 2.2.6 then permits us to estimate

$$\|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \ge ⨍_I |z'(s) - z'_I|^2\, ds = 1 - \left|\frac{z(s_2) - z(s_1)}{s_2 - s_1}\right|^2. \tag{2.2.122}$$

In turn, this implies

$$|s_1 - s_2| \le \frac{|z(s_1) - z(s_2)|}{\sqrt{1 - \|z'\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}} = (1+\varkappa_*)|z(s_1) - z(s_2)|, \tag{2.2.123}$$

provided $\varkappa_*$ is defined as in (2.2.105). This goes to show that, indeed, $\Sigma$ is a $\varkappa_*$-CAC. $\square$

Having discussed a number of basic properties of chord-arc curves in Proposition 2.2.9, we now wish to elaborate on the manner on which concrete examples of chord-arc curves may be produced. To set the stage for the subsequent discussion orserve that, when specialized to the one-dimensional setting, (2.2.63)-(2.2.65) imply that for each function $f \in \mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ we have

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ |s_1| + |s_2| \to \infty}} \Big( ⨍_{s_1}^{s_2} \Big| f - ⨍_{s_1}^{s_2} f\, d\mathcal{L}^1 \Big|\, d\mathcal{L}^1 \Big) = 0, \tag{2.2.124}$$

and

$$\lim_{\substack{-\infty < s_1 < s_2 < +\infty \\ s_2 - s_1 \to 0^+}} \Big( ⨍_{s_1}^{s_2} \Big| f - ⨍_{s_1}^{s_2} f\, d\mathcal{L}^1 \Big|\, d\mathcal{L}^1 \Big) = 0. \tag{2.2.125}$$

These properties are relevant in the context of the next proposition, describing a wealth of examples of chord-arc curves in the plane.

**Proposition 2.2.10.** *Suppose $b \in \mathrm{CMO}(\mathbb{R}, \mathcal{L}^1)$ is a real-valued function and consider the assignment*

$$\mathbb{R} \ni s \longmapsto z(s) := \int_0^s e^{ib(t)}\, dt \in \mathbb{C}. \tag{2.2.126}$$

*If the said assignment is injective then $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ is, in fact, the arc-length parametrization of a chord-arc curve (which, in particular, passes through infinity in the plane).*

*Proof.* Introduce

$$F(s_1,s_2):=\frac{z(s_1)-z(s_2)}{s_1-s_2}\quad\text{for each } s_1,s_2\in\mathbb{R} \text{ with } s_1\neq s_2. \tag{2.2.127}$$

Then, whenever $-\infty<s_1<s_2<+\infty$ and with $b_I$ abbreviating $\fint_{s_1}^{s_2} b(t)\,dt$, we may write

$$F(s_1,s_2)=\fint_{s_1}^{s_2} e^{ib(t)}\,dt=\fint_{s_1}^{s_2}\big(e^{ib(t)}-e^{ib_I}\big)\,dt+e^{ib_I}. \tag{2.2.128}$$

Recall that

$$|e^{i\theta}-1|=\Big|\int_0^\theta ie^{it}\,dt\Big|\leq\Big|\int_0^\theta |ie^{it}|\,dt\Big|=|\theta|\quad\text{for each}\quad\theta\in\mathbb{R}. \tag{2.2.129}$$

Then, since $b$ is real-valued, we may use (2.2.129) to estimate

$$\begin{aligned}\Big|\fint_{s_1}^{s_2}\big(e^{ib(t)}-e^{ib_I}\big)\,dt\Big|&=\Big|\fint_{s_1}^{s_2}\big(e^{i(b(t)-b_I)}-1\big)\,dt\Big|\\&\leq\fint_{s_1}^{s_2}\big|e^{i(b(t)-b_I)}-1\big|\,dt\leq\fint_{s_1}^{s_2}|b(t)-b_I|\,dt.\end{aligned} \tag{2.2.130}$$

According to (2.2.124)-(2.2.125) (written for $b$ in place of $f$), the last integral in (2.2.130) converges to zero as either $|s_1|+|s_2|\to\infty$, or $s_2-s_1\to 0^+$. Since $\big|e^{ib_I}\big|=1$, we conclude that

$$\lim_{\substack{-\infty<s_1\neq s_2<+\infty\\ |s_1|+|s_2|\to\infty}}|F(s_1,s_2)|=1\quad\text{and}\quad\lim_{\substack{-\infty<s_1\neq s_2<+\infty\\ |s_1-s_2|\to 0^+}}|F(s_1,s_2)|=1. \tag{2.2.131}$$

Given that, by assumption, the assignment $\mathbb{R}\ni s\mapsto z(s)\in\mathbb{C}$ is injective, we also have

$$F(s_1,s_2)\neq 0\quad\text{whenever}\quad-\infty<s_1\neq s_2<+\infty. \tag{2.2.132}$$

From (2.2.131), (2.2.132), and the fact that $F:\{(s_1,s_2)\in\mathbb{R}^2:\ s_1\neq s_2\}\to\mathbb{C}$ is continuous, we conclude that there exists $c\in(0,1)$ with the property that $|F(s_1,s_2)|\geq c$ for each $s_1,s_2\in\mathbb{R}$ with $s_1\neq s_2$. In view of (2.2.127), this implies

$$|s_1-s_2|\leq c^{-1}|z(s_1)-z(s_2)|\quad\text{for each}\quad s_1,s_2\in\mathbb{R}. \tag{2.2.133}$$

In particular, this entails $\lim_{s\to\pm\infty}|z(s)|=\infty$. In addition, the mapping $\mathbb{R}\ni s\mapsto z(s)\in\mathbb{C}$ is continuous, and it is assumed to be injective. Given that $|z'(s)|=\big|e^{ib(s)}\big|=1$ for $\mathcal{L}^1$-a.e. $s\in\mathbb{R}$, since $b$ is real-valued, it follows that $\mathbb{R}\ni s\mapsto z(s)\in\mathbb{C}$ is the arc-length parametrization of a Jordan curve in the plane which passes through infinity. $\square$

Here is a version of Proposition 2.2.10 in which the membership of $b$ to $\mathrm{CMO}(\mathbb{R},\mathcal{L}^1)$ is replaced by the demand that $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)}<\frac{\pi}{2}$. In an interesting twist, this forces the image of (2.2.126) to be a Lipschitz graph.

**Proposition 2.2.11.** *If $b\in L^\infty(\mathbb{R},\mathcal{L}^1)$ is a real-valued function with $\|b\|_{L^\infty(\mathbb{R},\mathcal{L}^1)}<\frac{\pi}{2}$ then the assignment* (2.2.126) *is actually the arc-length parametrization of a Lipschitz graph in the plane* (*hence, in particular, a chord-arc curve*).

*Proof.* Suppose there exists $\theta \in (0, \pi/2)$ such that $b(t) \in (-\theta, \theta)$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$. Since for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ we have $z'(t) = e^{ib(t)} = \cos(b(t)) + i\sin(b(t))$ given that $b$ is real-valued, it follows that

$$\operatorname{Re} z'(t) = \cos(b(t)) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } \ t \in \mathbb{R}. \tag{2.2.134}$$

Granted this, whenever $-\infty < s_1 < s_2 < +\infty$ we may estimate

$$\begin{aligned} |z(s_2) - z(s_1)| \geq \operatorname{Re}\big(z(s_2) - z(s_1)\big) = \operatorname{Re}\int_{s_1}^{s_2} z'(t)\,dt = \int_{s_1}^{s_2} \operatorname{Re} z'(t)\,dt \\ \geq \int_{s_1}^{s_2} \cos\theta\,dt = (\cos\theta)(s_2 - s_1), \end{aligned} \tag{2.2.135}$$

which, in light of Proposition 2.2.10, implies that the image of $z(\cdot)$ is a chord-arc curve $\Sigma$ in the plane. As such, Proposition 2.2.9 implies that if $\Omega$ denotes the region in $\mathbb{C}$ lying to the left of the curve $\Sigma$ (relative to the orientation $\Sigma$ inherits from its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$), then $\Omega$ is an Ahlfors regular domain whose topological boundary is $\Sigma$, and whose geometric measure theoretic outward unit normal $\nu$ is given at $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ by $\nu(z(s)) = -i\,z'(s)$. Consider next the constant vector field $h := (0, -1) \equiv -i$ in $\mathbb{C}$ and regard $\nu$ as a $\mathbb{C}^2$-valued function. Then, with $\langle \cdot, \cdot \rangle$ denoting the standard inner product in $\mathbb{R}^2$, we have

$$\begin{aligned} \big\langle \nu(z(s)), h(z(s)) \big\rangle &= \operatorname{Re}\big(i\nu(z(s))\big) \\ &= \operatorname{Re} z'(s) \geq \cos\theta > 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } \ s \in \mathbb{R}. \end{aligned} \tag{2.2.136}$$

This goes to show that there exists a constant vector field which is transversal to $\Omega$ and, as a consequence of work in [52], we conclude that $\Omega$ is the upper-graph of a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$. The desired conclusion now follows. $\square$

Another sub-category of chord-arc curves is offered by graphs of real-valued $\mathrm{BMO}_1$ functions defined on the real line.

**Proposition 2.2.12.** *Let $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ be such that $\varphi' \in \mathrm{BMO}\,(\mathbb{R}, \mathcal{L}^1)$ and consider its graph $\Sigma := \big\{(x, \varphi(x)) : x \in \mathbb{R}\big\} \subseteq \mathbb{R}^2$. Then $\Sigma$ is a $\varkappa$-CAC with $\varkappa = \|\varphi'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}$.*

*Proof.* Throughout, identify $\mathbb{R}^2$ with $\mathbb{C}$. Since functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ are locally absolutely continuous (cf., e.g., [72, Corollary 7.14, p. 223]), we conclude that $\Sigma$ is a curve in the plane, with parametrization $\mathbb{R} \ni x \mapsto x + i\varphi(x) \in \Sigma$. Hence, $\Sigma$ is a Jordan curve that passes through infinity in the plane. From [53, Proposition 2.25, p. 2616] we know that $\Sigma$ is an Ahlfors regular set which, in light of (2.2.85) implies that the curve $\Sigma$ is also locally rectifiable. Consider two arbitrary points $z_1, z_2 \in \Sigma$, say $z_1 := \big(a, \varphi(a)\big)$ and $z_2 := \big(b, \varphi(b)\big)$ for some $a, b \in \mathbb{R}$ with $a < b$, and denote by $\Sigma_{z_1,z_2}$ the sub-arc of $\Sigma$ with end-points $z_1, z_2$. From [53, Proposition 2.25, p. 2616] we also know that the arc-length measure $\sigma := \mathcal{H}^1\lfloor\Sigma$ on the curve $\Sigma$ satisfies

$$\ell(z_1, z_2) = \sigma(\Sigma_{z_1,z_2}) = \int_a^b \sqrt{1 + |\varphi'(x)|^2}\,dx. \tag{2.2.137}$$

Observe that the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(t) := \sqrt{1+t^2}$ for each $t \in \mathbb{R}$ is Lipschitz, with Lipschitz constant $\leq 1$, since $|F'(t)| = |t|/\sqrt{1+t^2} \leq 1$ for each $t \in \mathbb{R}$. Consequently, if we set

$$\varphi'_I := \fint_a^b \varphi' \, d\mathcal{L}^1 = \frac{\varphi(b)-\varphi(a)}{b-a}, \tag{2.2.138}$$

then

$$\begin{aligned}
\int_a^b \sqrt{1+|\varphi'(x)|^2}\, dx &= \int_a^b F(\varphi'(x))\, dx \\
&\leq \int_a^b |F(\varphi'(x)) - F(\varphi'_I)|\, dx + (b-a)F(\varphi'_I) \\
&\leq \int_a^b |\varphi'(x) - \varphi'_I|\, dx + (b-a)\sqrt{1+(\varphi'_I)^2} \\
&\leq (b-a)\|\varphi'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} + (b-a)\sqrt{1+\Big(\frac{\varphi(b)-\varphi(a)}{b-a}\Big)^2} \\
&\leq |z_1-z_2|\|\varphi'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} + |z_1-z_2| \\
&= \big(1+\|\varphi'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}\big)|z_1-z_2|. 
\end{aligned} \tag{2.2.139}$$

From (2.2.137) and (2.2.139) we conclude that (2.2.91) holds with $\varkappa = \|\varphi'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}$, and the desired conclusion follows. $\square$

Another basic link between chord-arc curves in the plane and the John-Nirenberg space BMO on the real line has been noted by R. Coifman and Y. Meyer. Specifically, [25] contains the following result: if $\Sigma \subseteq \mathbb{C}$ is a chord-arc curve then its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ satisfies $z'(s) = e^{ib(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ for some real-valued function $b \in \mathrm{BMO}(\mathbb{R},\mathcal{L}^1)$ and, in the converse direction, for any given real-function $b \in \mathrm{BMO}(\mathbb{R},\mathcal{L}^1)$ whose BMO semi-norm is sufficiently small, the function $\mathbb{R} \ni s \mapsto z(s) := \int_0^s e^{ib(t)}\, dt \in \mathbb{C}$ is the arc-length parametrization of a chord-arc curve (cf. also [26] for related results). Below we further elaborate on this last part of Coifman-Meyer's result. In particular, the analysis contained in our next proposition (which may be thought of as a quantitative version of Proposition 2.2.10) is going to be instrumental in producing a large variety of examples of $\delta$-SKT domains a little later (see Example 2.2.20).

**Proposition 2.2.13.** *Let $b \in \mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)$ be a real-valued function with*

$$\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} < 1 \tag{2.2.140}$$

*and introduce*

$$\varkappa := \frac{\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}{1-\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}} \in [0,\infty). \tag{2.2.141}$$

*Define $z : \mathbb{R} \to \mathbb{C}$ by setting*

$$z(s) := \int_0^s e^{i\,b(t)}\, dt \quad \textit{for each} \quad s \in \mathbb{R}. \tag{2.2.142}$$

*Finally, consider $\Sigma := z(\mathbb{R})$, the image of $\mathbb{R}$ under the mapping $z(\cdot)$. Then the following statements are true.*

*(i) The set $\Sigma$ is a $\varkappa$-`CAC` which contains the origin $0 \in \mathbb{C}$, and $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. In addition,*

$$\|z'\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} \leq 2\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}. \tag{2.2.143}$$

*(ii) Denote by $\Omega$ the region of the plane lying to the left of the curve $\Sigma$ (relative to the orientation $\Sigma$ inherits from its arc-length parametrization $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$). Then the set $\Omega$ is the image of the upper half-plane under a global bi-Lipschitz homeomorphism of $\mathbb{C}$, and*

*the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of $\Omega$ stay bounded as $\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} \longrightarrow 0^+$.* (2.2.144)

*Furthermore, the geometric measure theoretic outward unit normal $\nu$ satisfies*

$$\|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq 4\varkappa. \tag{2.2.145}$$

*(iii) With the piece of notation introduced in (2.2.38), if in place of (2.2.140) one now assumes*

$$\|b\|_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} < \sqrt{2} \tag{2.2.146}$$

*then $\Sigma$ is a $\varkappa_2$-CAC with*

$$\varkappa_2 := \frac{\|b\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}}{2 - \|b\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)}} \in [0, \infty). \tag{2.2.147}$$

*As a consequence of this and (2.2.104), in such a scenario one has*

$$\|\nu\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq 2\sqrt{\varkappa_2(2 + \varkappa_2)}. \tag{2.2.148}$$

*Proof.* The fact that $b$ is real-valued entails that $e^{i\,b(\cdot)} \in L^\infty(\mathbb{R}, \mathcal{L}^1)$. In turn, this membership guarantees that $z(\cdot)$ in (2.2.142) is a well-defined Lipschitz function on $\mathbb{R}$, with $z(0) = 0 \in \mathbb{C}$, and such that $z'(s) = e^{i\,b(s)}$ for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. In particular,

$$|z'(s)| = 1 \quad \text{for } \mathcal{L}^1\text{-a.e. } \ s \in \mathbb{R}. \tag{2.2.149}$$

We claim that the inequalities in (2.2.94) hold. To see this, for each $s_1, s_2 \in \mathbb{R}$ we write (keeping in mind that $b$ is real-valued)

$$\begin{aligned} |z(s_1) - z(s_2)| &= \Big|\int_0^{s_1} e^{i\,b(t)}\,dt - \int_0^{s_2} e^{i\,b(t)}\,dt\Big| = \Big|\int_{s_2}^{s_1} e^{i\,b(t)}\,dt\Big| \\ &\leq \Big|\int_{s_2}^{s_1} |e^{i\,b(t)}|\,dt\Big| = |s_1 - s_2|, \end{aligned} \tag{2.2.150}$$

justifying the first inequality in (2.2.94). To prove the second inequality in (2.2.94), for each finite, non-trivial, sub-interval $I$ of $\mathbb{R}$ introduce

$$b_I := \fint_I b(t)\,dt, \qquad m_I := e^{i\,b_I}, \tag{2.2.151}$$

and note that the fact that $b$ is real-valued implies $|m_I| = 1$. Also, $m_I^{-1} = e^{-i\,b_I}$. Assume $-\infty < s_1 < s_2 < +\infty$ and set $I := [s_1, s_2]$. We may then estimate

$$\begin{aligned}
\big|z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)\big| &= \Big|\int_{s_1}^{s_2} (z'(t) - m_I)\,dt\Big| \\
&= \Big|\int_{s_1}^{s_2} (z'(t)m_I^{-1} - 1)\,dt\Big| = \Big|\int_{s_1}^{s_2} (e^{i(b(t)-b_I)} - 1)\,dt\Big| \\
&\leq \int_{s_1}^{s_2} |e^{i(b(t)-b_I)} - 1|\,dt \leq \int_{s_1}^{s_2} |b(t) - b_I|\,dt \\
&= |s_1 - s_2| \fint_I |b(t) - b_I|\,dt \leq |s_1 - s_2| \|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)} \\
&= \Big(\frac{\varkappa}{1+\varkappa}\Big)|s_1 - s_2|,
\end{aligned} \tag{2.2.152}$$

where we have used the fact that Lipschitz functions are locally absolutely continuous (hence, the fundamental theorem of calculus applies), as well as the elementary inequality from (2.2.129). From (2.2.152), we obtain

$$\begin{aligned}
|s_1 - s_2| = |m_I \cdot (s_1 - s_2)| &\leq |z(s_1) - z(s_2)| + |z(s_1) - z(s_2) - m_I \cdot (s_1 - s_2)| \\
&\leq |z(s_1) - z(s_2)| + \Big(\frac{\varkappa}{1+\varkappa}\Big)|s_1 - s_2|,
\end{aligned} \tag{2.2.153}$$

which then readily yields the second estimate in (2.2.94). In particular, (2.2.94) implies that $\mathbb{R} \ni s \mapsto z(s) \in \Sigma$ is a bi-Lipschitz bijection. The argument so far shows that $\Sigma$ is a $\varkappa$-`CAC` passing through the origin $0 \in \mathbb{C}$, and $\mathbb{R} \ni s \longmapsto z(s) \in \Sigma$ is its arc-length parametrization. To finish the treatment of the claims in item *(i)*, there remains to justify (2.2.143). To this end, given any finite interval $I \subset \mathbb{R}$, set $b_I := \fint_I b(t)\,dt \in \mathbb{R}$ and $m_I := e^{i\,b_I} \in S^1$ (the two memberships a consequence of the fact that $b$ is real-valued). With $z'_I := \fint_I z'(s)\,ds \in \mathbb{C}$ we may then estimate (bearing in mind that $m_I^{-1} = e^{-i\,b_I}$ and the inequality in (2.2.129))

$$\begin{aligned}
\fint_I |z'(s) - z'_I|\,ds \leq 2\fint_I |z'(s) - m_I|\,ds &= 2\fint_I |z'(s)m_I^{-1} - 1|\,ds \\
&= 2\fint_I |e^{i(b(s)-b_I)} - 1|\,ds \leq 2\fint_I |b(s) - b_I|\,ds \\
&\leq 2\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)},
\end{aligned} \tag{2.2.154}$$

and (2.2.143) readily follows from this. Next, all but the last claim in item *(ii)* are consequences of (2.2.102). The estimate in (2.2.145) is obtain by combing the first inequality in (2.2.103) with (2.2.143) and (2.2.141).

To deal with the claims in item *(iii)*, make the assumption that (2.2.146) holds and define $\varkappa_2$ as in (2.2.147). Whenever $-\infty < s_1 < s_2 < +\infty$ and $I := [s_1, s_2]$ we may estimate

$$\begin{aligned}
s_2 - s_1 &\leq \sqrt{(s_2-s_1)^2 + \Big|\int_{s_1}^{s_2} (b(t) - b_I)\,dt\Big|^2} = \Big|(s_2 - s_1) + i\int_{s_1}^{s_2} (b(t) - b_I)\,dt\Big| \\
&= \Big|m_I \cdot (s_2 - s_1) + m_I \cdot \int_{s_1}^{s_2} i(b(t) - b_I)\,dt\Big| \\
&\leq |z(s_2) - z(s_1)| \\
&\qquad + \Big|z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} i(b(t) - b_I)\,dt\Big|. \qquad (2.2.155)
\end{aligned}$$

Note that the last term above may be written as

$$\begin{aligned}
&\Big|z(s_2) - z(s_1) - m_I \cdot (s_2 - s_1) - m_I \cdot \int_{s_1}^{s_2} i(b(t) - b_I)\,dt\Big| \\
&\qquad = \Big|\int_{s_1}^{s_2} \big(z'(t) - m_I - m_I \cdot i(b(t) - b_I)\big)\,dt\Big| \\
&\qquad = \Big|\int_{s_1}^{s_2} \big(z'(t)m_I^{-1} - 1 - i(b(t) - b_I)\big)\,dt\Big| \\
&\qquad = \Big|\int_{s_1}^{s_2} \big(e^{i(b(t)-b_I)} - 1 - i(b(t) - b_I)\big)\,dt\Big|. \qquad (2.2.156)
\end{aligned}$$

Also, for each $\theta \in \mathbb{R}$ we may use (2.2.129) to write

$$\begin{aligned}
|e^{i\theta} - 1 - i\theta| &= \Big|\int_0^\theta i(e^{it} - 1)\,dt\Big| \leq \Big|\int_0^\theta |i(e^{it} - 1)|\,dt\Big| \\
&\leq \Big|\int_0^\theta |t|\,dt\Big| = \theta^2/2. \qquad (2.2.157)
\end{aligned}$$

From (2.2.155), (2.2.156), (2.2.157), (2.2.38), and (2.2.147) we then conclude that

$$\begin{aligned}
s_2 - s_1 &\leq |z(s_2) - z(s_1)| + \frac{1}{2}\int_{s_1}^{s_2} |b(t) - b_I|^2\,dt \\
&\leq |z(s_2) - z(s_1)| + \tfrac{1}{2}(s_2 - s_1)\|b\|^2_{\mathrm{BMO}_2(\mathbb{R},\mathcal{L}^1)} \\
&= |z(s_2) - z(s_1)| + \Big(\frac{\varkappa_2}{1+\varkappa_2}\Big)(s_2 - s_1). \qquad (2.2.158)
\end{aligned}$$

From (2.2.158) we conclude that the version of (2.2.94) with $\varkappa$ replaced by $\varkappa_2$ holds. In particular, $\Sigma$ is a $\varkappa_2$-CAC. The proof of Proposition 2.2.13 is therefore complete. $\square$

### 2.2.3 The class of $\delta$-SKT domains

We begin making the following definition which is central for the present work. This should be compared with [53, Definitions 4.7-4.9, p. 2690] where related, yet distinct, variants have been considered (the said definitions in [53] are designed to work particularly well when dealing with domains with compact boundaries, as opposed to the preset endeavors where we shall consider domains with unbounded boundaries).

**Definition 2.2.14.** Consider a parameter $\delta > 0$. Call $\Omega$ a $\delta$-`SKT domain` if $\Omega$ is an open, nonempty, proper subset of $\mathbb{R}^n$, which satisfies a two-sided local John condition, whose topological boundary $\partial\Omega$ is an Ahlfors regular set, and whose geometric measure theoretic outward unit normal $\nu$ is such that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta, \tag{2.2.159}$$

where $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Whenever $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose topological boundary $\partial\Omega$ is an Ahlfors regular set, it follows from (1.1.27), (1.1.22), and (1.1.21) that $\Omega$ is a UR domain (hence, in particular, an Ahlfors regular domain). Since in this class of domains we always have $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$ (cf. (2.2.56)), condition (2.2.159) is redundant when $\delta > 1$. We will, however, be interested in the case when $\delta$ is small. In particular, when $\delta \in (0,1)$, Lemma 2.2.5 ensures that $\partial\Omega$ is an unbounded set.

Let us also note here that, as is visible from the first inequality in (2.2.57), whenever $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain with $\delta \in (0,1)$ then its geometric measure theoretic outward unit normal $\nu$ satisfies (with the infimum taken over all surface balls $\Delta \subseteq \partial\Omega$)

$$\inf_{\Delta\subseteq\partial\Omega} \left| ⨍_{\Delta} \nu \, d\sigma \right| > 1 - \delta. \tag{2.2.160}$$

Conversely, whenever $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose topological boundary $\partial\Omega$ is an Ahlfors regular set, it follows from the second inequality in (2.2.57) that $\Omega$ is a $\delta$-SKT domain for each

$$\delta > \sqrt{2}\sqrt{1 - \inf_{\Delta\subseteq\partial\Omega} \left| ⨍_{\Delta} \nu \, d\sigma \right|}, \tag{2.2.161}$$

where the infimum is taken over all surface balls $\Delta \subseteq \partial\Omega$.

Examples and counterexamples of $\delta$-SKT domains in $\mathbb{R}^n$ are as follows.

**Example 2.2.15.** The set $\Omega := \mathbb{R}^n_+$ is a $\delta$-SKT domain for each $\delta > 0$. Indeed, the outward unit normal $\nu = -\mathbf{e}_n = (0,\dots,0,-1)$ to $\Omega$ is constant, hence its BMO seminorm vanishes. More generally, any half-space in $\mathbb{R}^n$, i.e., any set of the form

$$\begin{gathered} \Omega_{x_o,\xi} := \{x \in \mathbb{R}^n : \langle x - x_o, \xi\rangle > 0\} \\ \text{with } \ x_o \in \mathbb{R}^n \ \text{ and } \ \xi \in S^{n-1}, \end{gathered} \tag{2.2.162}$$

is a $\delta$-SKT domain for each $\delta > 0$.

Consider next a sector of aperture $\theta \in (0,2\pi)$ in the two-dimensional space, i.e., a planar set of the form

$$\begin{gathered} \Omega_\theta := \Big\{x \in \mathbb{R}^2 \setminus \{x_o\} : \tfrac{x-x_o}{|x-x_o|} \cdot \xi > \cos(\theta/2)\Big\} \\ \text{with } \ x_o \in \mathbb{R}^2, \ \theta \in (0,2\pi), \ \text{ and } \ \xi \in S^1, \end{gathered} \tag{2.2.163}$$

and abbreviate $\sigma_\theta := \mathcal{H}^1\lfloor\partial\Omega_\theta$. Then a direct computation shows that the outward unit normal vector $\nu$ to $\Omega_\theta$, regarded as a complex-valued function, satisfies

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)} = |\cos(\theta/2)|. \tag{2.2.164}$$

Hence,

$$\Omega_\theta \text{ is a } \delta\text{-SKT domain if and only if } \ \delta > |\cos(\theta/2)|. \tag{2.2.165}$$

One last example in the same spirit is offered by the cone of aperture $\theta \in (0, 2\pi)$ in $\mathbb{R}^n$ with vertex at the origin and axis along $\mathbf{e}_n$, i.e.,

$$\begin{aligned}\Omega_\theta &:= \Big\{x \in \mathbb{R}^n \setminus \{0\} : \frac{x_n}{|x|} > \cos(\theta/2)\Big\} \\ &= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\},\end{aligned} \tag{2.2.166}$$

where $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is given by $\phi(x') := |x'| \cot(\theta/2)$ for each $x' \in \mathbb{R}^{n-1}$. If we abbreviate $\sigma_\theta := \mathcal{H}^{n-1}\lfloor\partial\Omega_\theta$, then a direct computation (cf. (2.2.169) below) shows that the outward unit normal vector $\nu$ to $\Omega_\theta$ satisfies

$$\begin{aligned}&\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^n} = |\cos(\theta/2)|, \quad \text{hence once again} \\ &\Omega_\theta \text{ is a } \delta\text{-SKT domain if and only if } \ \delta > |\cos(\theta/2)|.\end{aligned} \tag{2.2.167}$$

**Example 2.2.16.** If $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain for some $\delta > 0$, then $\mathbb{R}^n \setminus \overline{\Omega}$ is also a $\delta$-SKT domain (having the same topological and measure theoretic boundaries as $\Omega$, and whose geometric measure theoretic outward unit normal is the opposite of the one for $\Omega$). Also, any rigid transformation of $\mathbb{R}^n$ preserves the class of $\delta$-SKT domains. One may also check from definitions that there exists a dimensional constant $c_n \in (0, \infty)$ with the property that if $\Omega$ is a $\delta$-SKT domain in $\mathbb{R}^n$ for some $\delta > 0$ then $\Omega \times \mathbb{R}$ is a $(c_n\delta)$-SKT domain in $\mathbb{R}^{n+1}$.

**Example 2.2.17.** Given $\delta > 0$, the region $\Omega := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\}$ above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ whose Lipschitz constant is $< 2^{-3/2}\delta$ is a $\delta$-SKT domain. To see this is indeed the case, it is relevant to note that

$$\begin{aligned}&F : \mathbb{R}^n \to \mathbb{R}^n \ \text{ defined for all } x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \\ &\text{as } \ F(x', x_n) := x + \phi(x')\mathbf{e}_n = (x', x_n + \phi(x')),\end{aligned} \tag{2.2.168}$$

is a bijective function with inverse $F^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given at each point $y = (y', y_n)$ in $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ by $F^{-1}(y', y_n) = y - \phi(y')\mathbf{e}_n = (y', y_n - \phi(y'))$, and that both $F, F^{-1}$ are Lipschitz functions with constant $\leq 1 + \|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$. Hence, $\Omega$ is the image of the upper half-space $\mathbb{R}^n_+$ under the bi-Lipschitz homeomorphism $F$, which also maps $\mathbb{R}^n_-$ onto $\mathbb{R}^n \setminus \overline{\Omega}$ and $\mathbb{R}^{n-1} \times \{0\}$ onto $\partial\Omega$. This goes to show that $\Omega$ is a two-sided NTA domain with an Ahlfors regular boundary, hence also an Ahlfors regular domain satisfying a two-sided local John condition (cf. (1.1.21) and (1.1.28)). To conclude that $\Omega$ is a $\delta$-SKT domain we need to estimate the BMO semi-norm of its geometric measure theoretic outward unit normal. Since this satisfies

$$\nu(x', \phi(x')) = \frac{(\nabla\phi(x'), -1)}{\sqrt{1 + |\nabla\phi(x')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } \ x' \in \mathbb{R}^{n-1}, \tag{2.2.169}$$

it follows that for $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} \nu(x', \phi(x')) + \mathbf{e}_n &= \left( \frac{\nabla\phi(x')}{\sqrt{1+|\nabla\phi(x')|^2}}\, , \, 1 - \frac{1}{\sqrt{1+|\nabla\phi(x')|^2}} \right) \qquad (2.2.170) \\ &= \left( \frac{\nabla\phi(x')}{\sqrt{1+|\nabla\phi(x')|^2}}\, , \, \frac{|\nabla\phi(x')|^2}{\sqrt{1+|\nabla\phi(x')|^2}(1+\sqrt{1+|\nabla\phi(x')|^2})} \right). \end{aligned}$$

Therefore, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, we may estimate

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} &= \|\nu + \mathbf{e}_n\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 2\,\|\nu + \mathbf{e}_n\|_{[L^\infty(\partial\Omega,\sigma)]^n} \\ &= 2^{3/2} \left\| \frac{|\nabla\phi|}{(1+|\nabla\phi|^2)^{1/4}(1+\sqrt{1+|\nabla\phi|^2})^{1/2}} \right\|_{L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \\ &\leq 2^{3/2}\,\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \delta. \qquad (2.2.171) \end{aligned}$$

All things considered, the above analysis establishes that $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain, with $\delta = O\Big( \|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \Big)$ as $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \longrightarrow 0^+$. In addition, since the Lipschitz constants of the functions $F, F^{-1}$ stay bounded when the Lipschitz constant of $\phi$, i.e., $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$, stays bounded, we ultimately conclude that that

> by taking $\|\nabla\phi\|_{[L^\infty(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}$ sufficiently small, matters may be arranged so that the above set $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain with $\delta > 0$ as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of $\Omega$. (2.2.172)

**Example 2.2.18.** Given any $\delta > 0$, the region $\Omega := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \phi(x')\}$ above the graph of some $\mathrm{BMO}_1$ function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, (i.e., a function $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $\nabla\phi$ belonging to $\big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}$), satisfying (for some purely dimensional constant $C_n \in (1, \infty)$)

$$\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \min\{1, \delta/C_n\} \qquad (2.2.173)$$

is a $\delta$-SKT domain. Indeed, $\mathrm{BMO}_1$ domains are contained in the class of Zygmund domains (cf. [53, Proposition 3.15, p. 2637]) which, in turn, are NTA domains (cf. [57, Proposition 3.6, p. 94]). Since the set $\mathbb{R}^n \setminus \overline{\Omega}$ happens to be a reflection across the origin of $\widetilde{\Omega} := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \widetilde{\phi}(x')\}$ where $\widetilde{\phi}(x') := -\phi(-x')$ for each $x' \in \mathbb{R}^{n-1}$, we also conclude that $\mathbb{R}^n \setminus \overline{\Omega}$ is an NTA domain. Thus, $\Omega$ is a two-sided NTA domain. In particular, $\Omega$ satisfies a two-sided local John condition (cf. (1.1.28)). From [53, Corollary 2.26, p. 2622] we also know that $\partial\Omega$ is an Ahlfors regular set. Finally, [53, Proposition 2.27, p. 2622] guarantees the existence of a purely dimensional constant $C \in (0, \infty)$ such that

$$\begin{aligned} &\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \qquad (2.2.174) \\ &\quad \leq C\,\|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\,\mathcal{L}^{n-1})]^{n-1}} \Big( 1 + \|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\,\mathcal{L}^{n-1})]^{n-1}} \Big). \end{aligned}$$

Hence $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ if (2.2.173) is satisfied with $C_n := 2C$, proving that a $\Omega$ is indeed a $\delta$-SKT domain. In addition,

$$\text{choosing } \|\nabla\phi\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \text{ sufficiently small, ensures that the above set } \Omega \subseteq \mathbb{R}^n \text{ is a } \delta\text{-SKT domain with } \delta > 0 \text{ as small as wanted, relative to the Ahlfors regularity constant of } \partial\Omega \text{ and the local John constants of } \Omega. \tag{2.2.175}$$

To offer concrete, interesting examples and counterexamples pertaining to $\mathrm{BMO}_1$, work in the two-dimensional setting, i.e., when $n = 2$. For a fixed arbitrary number $\varepsilon \in (0, \infty)$ consider the continuous odd function $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\phi_\varepsilon(x) := \begin{cases} \varepsilon x\big(\ln|x| - 1\big) & \text{if } x \in \mathbb{R}\setminus\{0\}, \\ 0 & \text{if } x = 0, \end{cases} \qquad \text{for each } x \in \mathbb{R}. \tag{2.2.176}$$

Then from [92, Exercise 2.127, p. 89] we know that the distributional derivative of this function is $\phi_\varepsilon' = \varepsilon \ln|\cdot|$. Hence, for some absolute constant $C \in (0, \infty)$,

$$\|\phi_\varepsilon'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \leq C\varepsilon \tag{2.2.177}$$

so $\phi_\varepsilon$ is indeed in $\mathrm{BMO}_1$. This being said, $\phi_\varepsilon$ is not a Lipschitz function, so this example is outside the scope of item (c) above. Consequently, the planar region $\Omega_\varepsilon$ lying above the graph of $\phi_\varepsilon$ (cf. Figure 2.2) is a non-Lipschitz $\delta$-SKT domain in the plane with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ (as seen from (2.2.174) and (2.2.177)).

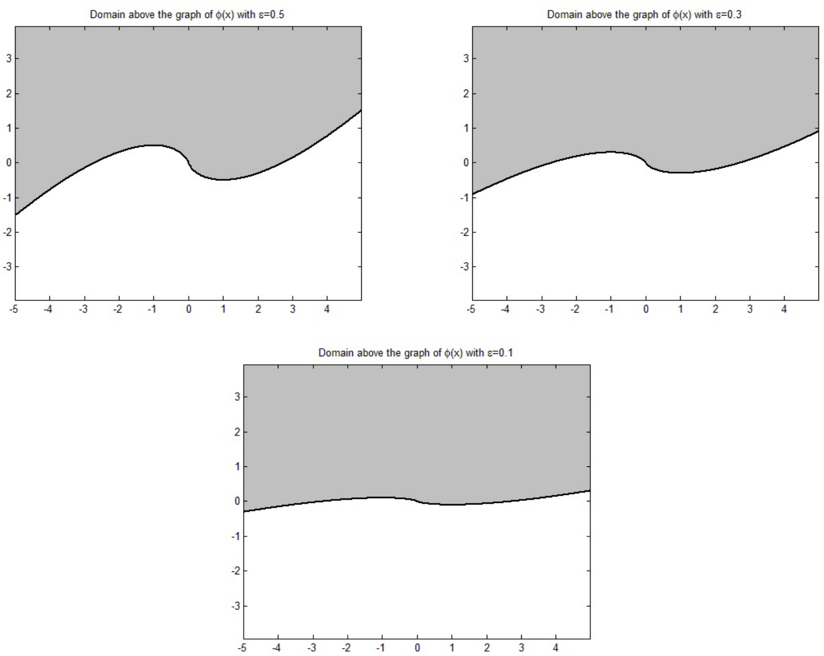


Figure 2.2: The prototype of a non-Lipschitz $\delta$-SKT domain $\Omega_\varepsilon$ for which $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$ and such that the Ahlfors regularity constant of $\partial\Omega_\varepsilon$ and the local John constants of $\Omega_\varepsilon$ are uniformly bounded in $\varepsilon$

On the other hand, the distributional derivative of the function $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined as

$$\psi_\varepsilon(x) := \begin{cases} \varepsilon x\big(\ln|x| - 1\big) & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0, \end{cases} \qquad \text{for each } \ x \in \mathbb{R}, \tag{2.2.178}$$

is $\psi'_\varepsilon = \varepsilon\big(\ln|\cdot|\big)\mathbf{1}_{(0,\infty)}$ which *fails* to be in $\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$ (recall that the latter space is not stable under multiplication by cutoff functions). Hence, $\psi_\varepsilon$ does *not* belong to $\mathrm{BMO}_1$. In this vein, we wish to note that while the planar region $\widetilde{\Omega}_\varepsilon$ lying above the graph of $\psi_\varepsilon$ continues to be an Ahlfors regular domain satisfying a two-sided local John condition for each $\varepsilon > 0$, its (complex-valued) geometric measure theoretic outward unit normal $\nu$ satisfies, due to the corner singularity at $0 \in \partial\widetilde{\Omega}_\varepsilon$ and (2.2.164) with $\theta = \pi/2$,

$$\|\nu\|_{\mathrm{BMO}(\partial\widetilde{\Omega}_\varepsilon, \widetilde{\sigma}_\varepsilon)} \geq \frac{1}{\sqrt{2}} \quad \text{for each } \ \varepsilon > 0, \tag{2.2.179}$$

where $\widetilde{\sigma}_\varepsilon := \mathcal{H}^1\lfloor\partial\widetilde{\Omega}_\varepsilon$. Consequently, as $\varepsilon \to 0^+$, the set $\widetilde{\Omega}_\varepsilon$ never becomes a $\delta$-SKT domain if $\delta \in (0, 1/\sqrt{2})$ (cf. Figure 2.3).

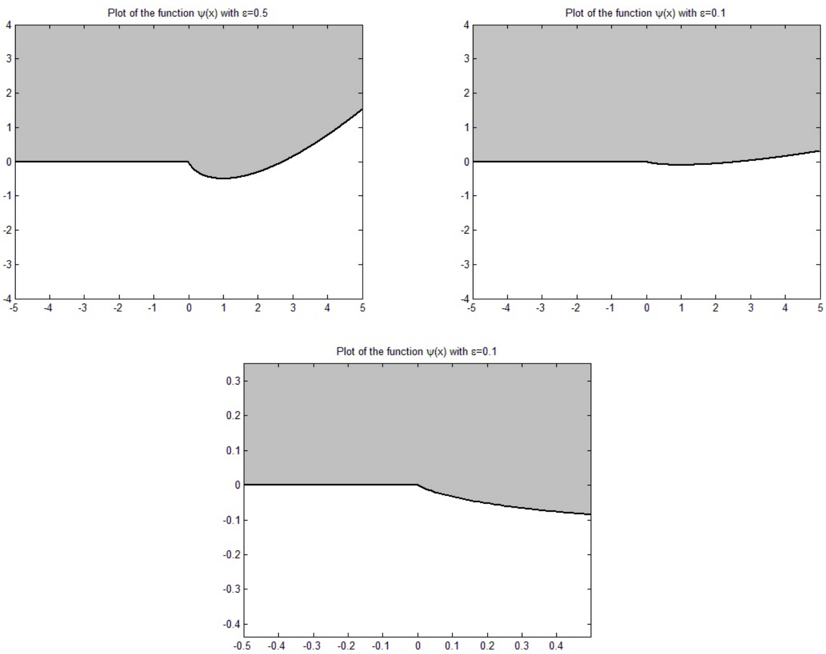


Figure 2.3: A family $\{\widetilde{\Omega}_\varepsilon\}_{\varepsilon>0}$ of Ahlfors regular domains satisfying a two-sided local John condition with uniform constants which does not contain a $\delta$-SKT domain with $\delta \in (0, 1/\sqrt{2})$

**Example 2.2.19.** From [61, Theorem 2.1, p. 515] and [61, Remark 2.2, pp. 514-515] we know that there exist two purely dimensional constants, $\delta_n \in (0,\infty)$ and $C_n \in (0,\infty)$, with the property that if $\Omega \subseteq \mathbb{R}^n$ is a $\delta_o$-Reifenberg flat domain, in the sense of [61, Definition 1.2, pp. 509-510] with $R = \infty$ and with $0 < \delta_o \leq \delta_n$, and if the surface measure

$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ satisfies

$$\begin{aligned}\sigma\big(B(x,r)\cap\partial\Omega\big) \leq (1+\delta_o)\upsilon_{n-1}r^{n-1}\\ \text{for each } x\in\partial\Omega \text{ and } r>0,\end{aligned} \tag{2.2.180}$$

(with $\upsilon_{n-1}$ denoting the volume of the unit ball in $\mathbb{R}^{n-1}$), then $\Omega$ is an Ahlfors regular domain whose geometric measure theoretic outward unit normal $\nu$ to $\Omega$ satisfies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C_n\sqrt{\delta_o}. \tag{2.2.181}$$

See also [20, p. 11] and [107] in this regard. Consequently, given any number $\delta>0$, any $\delta_o$-Reifenberg flat domain with $0<\delta_o<\min\{\delta_n\,,\,(\delta/C_n)^2\}$ which satisfies (2.2.180) as well as a two-sided local John condition is a $\delta$-SKT domain.

**Example 2.2.20.** Denote by $\Omega$ the region of the plane lying to one side of a $\varkappa$-CAC $\Sigma\subset\mathbb{C}$. Then Proposition 2.2.9 implies that $\Omega$ is a $\delta$-SKT domain for any $\delta>2\sqrt{\varkappa(2+\varkappa)}$.

To offer a concrete example, consider a real-valued function $b\in\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)$ with $\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}<1$ and define $z:\mathbb{R}\to\mathbb{C}$ by setting

$$z(s) := \int_0^s e^{i\,b(t)}\,dt \quad \text{for each} \quad s\in\mathbb{R}. \tag{2.2.182}$$

If $\Omega\subseteq\mathbb{C}\equiv\mathbb{R}^2$ is the region of the plane to one side of the curve $\Sigma:=z(\mathbb{R})$, then Proposition 2.2.9 implies that $\Omega$ is a connected Ahlfors regular domain which satisfies a two-sided local John condition, and $\partial\Omega=\Sigma$, and whose geometric measure theoretic outward unit normal $\nu$ to $\Omega$ is given by

$$\nu(z(s)) = -i\,e^{i\,b(s)} \quad \text{for } \mathcal{L}^1\text{-a.e.} \quad s\in\mathbb{R}. \tag{2.2.183}$$

In addition, if we set $\sigma:=\mathcal{H}^1\lfloor\partial\Omega$ then (2.2.145) gives

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq \frac{4\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}{1-\|b\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}}. \tag{2.2.184}$$

As a consequence, $\Omega$ is a $\delta$-SKT domain in $\mathbb{R}^2$ for each $\delta\in(0,\infty)$ bigger than the number in the right hand-side of (2.2.184).

For instance, we may take $b$ to be a small multiple of the logarithm on the real line, i.e.,

$$\begin{aligned} b(s) := \varepsilon\ln|s| \quad \text{for each} \quad s\in\mathbb{R}\setminus\{0\},\\ \text{with} \quad 0<\varepsilon<\big\|\ln|\cdot|\big\|^{-1}_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}\end{aligned} \tag{2.2.185}$$

(e.g., the computation on [45, p. 520] shows that $\big\|\ln|\cdot|\big\|_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}\leq 3\ln(3/2)$, so taking $0<\varepsilon<[3\ln(3/2)]^{-1}\approx 0.8221$ will do). Such a choice makes $b$ a real-valued function with small BMO semi-norm which nonetheless maps $\mathbb{R}\setminus\{0\}$ onto $\mathbb{R}$. In view of the formula given in (2.2.183), this goes to show that Gauss' map $\Sigma\ni z\mapsto\nu(z)\in S^1$ is surjective, which may be interpreted as saying that the unit normal rotates arbitrarily much along the boundary. In particular, the chord-arc curve $\Sigma$ produced in this fashion, which is actually the topological boundary of a $\delta$-SKT domain $\Omega \subseteq \mathbb{R}^2$ (with $\delta > 0$ which can be made as small as one pleases by taking $\varepsilon > 0$ appropriately small), fails to be a rotation of the graph of a function (even locally, near the origin). This being said, from Proposition 2.2.9 we know that

the set $\Omega \subseteq \mathbb{R}^2$ is actually bi-Lipschitz homeomorphic to the upper half-plane. (2.2.186)

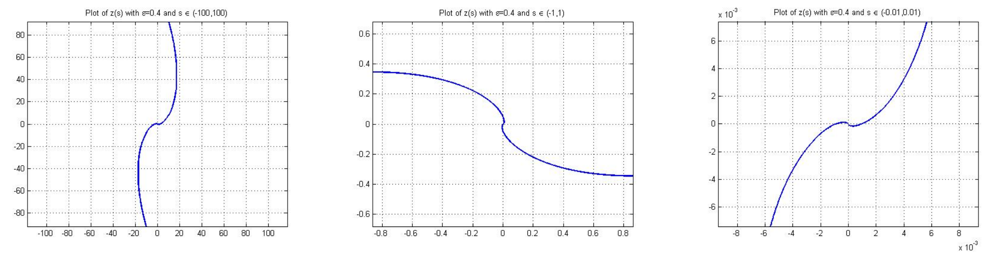


Figure 2.4: Zooming in the curve $s \mapsto z(s)$ at the point $0 \in \mathbb{C}$

The above pictures in Figure 2.4 depict an unbounded $\delta$-SKT domain $\Omega \subseteq \mathbb{R}^2$ which is not the upper-graph of a function (in any system of coordinates isometric to the standard one in the plane). The set $\Omega$ is the region lying to one side of the curve $\Sigma = z(\mathbb{R})$ with $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ defined by the formula given in (2.2.182) for the function $b$ as in (2.2.185) with $0 < \varepsilon < \|\ln|\cdot|\|^{-1}_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}$. As visible from (2.2.184), we have $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In the above pictures we have taken $\varepsilon = 0.4 < \frac{1}{2}\|\ln|\cdot|\|^{-1}_{\mathrm{BMO}\,(\mathbb{R},\mathcal{L}^1)}$ and progressively zoomed in at the point $0 \in \partial\Omega$. The boundary of the set $\Omega$ is the plot of the curve $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ with

$$z(s) = \int_0^s e^{i\varepsilon \ln|t|}\,dt = \begin{cases} (i\varepsilon+1)^{-1} s\, e^{i\varepsilon \ln|s|} & \text{if } s \in \mathbb{R}\setminus\{0\}, \\ 0 \in \mathbb{C} & \text{if } s = 0. \end{cases} \tag{2.2.187}$$

Here, $(i\varepsilon + 1)^{-1}$ is merely a complex constant, $s$ is the scaling factor that determines how far $z(s)$ is from the origin (specifically, $|z(s)| = |s|/\sqrt{\varepsilon^2+1}$), and $e^{i\varepsilon\ln|s|}$ is the factor that determines how the two spirals (making up $\partial\Omega\setminus\{0\}$, namely $z((-\infty,0))$ and $z((0,+\infty))$) spin about the point $0 \in \mathbb{C}$. Note that $|z(s)|$ growths linearly (with respect to $s$) which is very fast compared to the spinning rate (which is logarithmic) and this is why we have chosen to zoom in at the point $0 \in \mathbb{C}$ in several distinct frames to get a better understanding of how $\partial\Omega$ looks near 0. The fact that $\partial\Omega$ is symmetric with respect to the origin is a direct consequence of $\mathbb{R} \ni s \mapsto z(s) \in \mathbb{C}$ being odd. If $z(s) = re^{i\theta}$ is the polar representation of (2.2.187) for $s \in (0,\infty)$ then, with $\omega := 2\pi - \arccos(\frac{1}{\sqrt{\varepsilon^2+1}})$, we have $\theta = \omega + \varepsilon\ln|s|$ and $r = |z(s)| = (\varepsilon^2+1)^{-1/2}|s| = (\varepsilon^2+1)^{-1/2}e^{(\theta-\omega)/\varepsilon}$. Thus, in polar coordinates, the curve $\Sigma_+ := z((0,+\infty))$ has the equation $r = \alpha e^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}e^{-\omega/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which identifies it precisely as a logarithmic spiral. In a similar fashion, the polar equation of the curve $\Sigma_- := z((-\infty,0))$ is $r = \alpha e^{\beta\theta}$ with $\alpha := (\varepsilon^2+1)^{-1/2}e^{-(\omega+\pi)/\varepsilon} \in (0,\infty)$ and $\beta := \varepsilon^{-1} \in (0,\infty)$ which once again identifies it as a logarithmic spiral.

The MATLAB code that generated these pictures reads as follows:

```
s = [−100 : 0.001 : 100];
p = 0.4;
z = (1/(i ∗ p + 1.0)) ∗ s. ∗ exp(i ∗ p ∗ log(abs(s)));
plot(real(z), imag(z), 'LineWidth', 2), grid on, axis equal
```

Finally, we wish to elaborate on (2.2.186) and, in the process, get independent confirmation of (2.2.102) and (2.2.144). First, we observe that the $\delta$-SKT domain $\Omega \subseteq \mathbb{C}$ described above is the image of the upper half-plane $\mathbb{R}^2_+$ under map $F : \mathbb{C} \to \mathbb{C}$ defined for each $z \in \mathbb{C}$ by

$$F(z) := \begin{cases} (i\varepsilon + 1)^{-1} z\, e^{i\varepsilon \ln|z|} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ 0 \in \mathbb{C} & \text{if } z = 0. \end{cases} \tag{2.2.188}$$

Note that $F$ is a bijective, odd function, with inverse $F^{-1} : \mathbb{C} \to \mathbb{C}$ given at each $\zeta \in \mathbb{C}$ by

$$F^{-1}(\zeta) = \begin{cases} (i\varepsilon + 1)\zeta\, e^{-i\varepsilon \ln(|\zeta|\sqrt{\varepsilon^2+1})} & \text{if } \zeta \in \mathbb{C} \setminus \{0\}, \\ 0 \in \mathbb{C} & \text{if } \zeta = 0. \end{cases} \tag{2.2.189}$$

Also, whenever $z_1, z_2 \in \mathbb{C}$ are such that $|z_1| \geq |z_2| > 0$ we may estimate

$$|F(z_1) - F(z_2)| \leq \frac{1}{\sqrt{\varepsilon^2 + 1}} \Big\{ |z_1 - z_2| + |z_2| \big| e^{i\varepsilon \ln|z_1|} - e^{i\varepsilon \ln|z_2|} \big| \Big\} \tag{2.2.190}$$

and

$$\begin{aligned} \big| e^{i\varepsilon \ln|z_1|} - e^{i\varepsilon \ln|z_2|} \big| &= \big| e^{i\varepsilon(\ln|z_1| - \ln|z_2|)} - 1 \big| \leq \varepsilon \big| \ln|z_1| - \ln|z_2| \big| \\ &= \varepsilon \ln\big(\tfrac{|z_1|}{|z_2|}\big) \leq \varepsilon\big(\tfrac{|z_1|}{|z_2|} - 1\big) = \varepsilon\big(\tfrac{|z_1| - |z_2|}{|z_2|}\big) \\ &\leq \varepsilon \tfrac{|z_1 - z_2|}{|z_2|}, \end{aligned} \tag{2.2.191}$$

using the fact that $|e^{i\theta} - 1| \leq |\theta|$ for each $\theta \in \mathbb{R}$ and $0 \leq \ln x \leq x - 1$ for each $x \in [1, \infty)$. From this we then eventually deduce that

$$|F(z_1) - F(z_2)| \leq \frac{\varepsilon + 1}{\sqrt{\varepsilon^2 + 1}} |z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \mathbb{C}, \tag{2.2.192}$$

hence $F$ is Lipschitz. The same type of argument also shows that $F^{-1}$ is also Lipschitz, namely

$$|F^{-1}(\zeta_1) - F^{-1}(\zeta_2)| \leq (\varepsilon + 1)\sqrt{\varepsilon^2 + 1}\, |\zeta_1 - \zeta_2| \quad \text{for all} \quad \zeta_1, \zeta_2 \in \mathbb{C}, \tag{2.2.193}$$

so we ultimately conclude that $F : \mathbb{C} \to \mathbb{C}$ is an odd bi-Lipschitz homeomorphism of the complex plane. In summary,

> the $\delta$-SKT domain $\Omega \subseteq \mathbb{C}$ lying to the left of the curve $\mathbb{R} \ni s \longmapsto z(s) \in \mathbb{C}$ defined in (2.2.187) is in fact the image of the upper half-plane $\mathbb{R}^2_+$ under the odd bi-Lipschitz homeomorphism $F : \mathbb{C} \to \mathbb{C}$ from (2.2.188). (2.2.194)

Note that $F$ also maps the lower half-plane $\mathbb{R}^2_-$ onto $\mathbb{R}^2 \setminus \overline{\Omega}$, and $\mathbb{R} \times \{0\}$ onto $\partial\Omega$. This is in agreement with (2.2.102). Moreover, since the Lipschitz constants of $F, F^{-1}$ stay bounded uniformly in $\varepsilon \in (0,1)$ (as is clear from (2.2.192), (2.2.193)) while, as noted earlier, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$, we see that (as predicted in (2.2.144))

by taking $\varepsilon \in (0,1)$ sufficiently small, matters may be arranged so that the above set $\Omega \subseteq \mathbb{R}^2$ is a $\delta$-SKT domain with $\delta > 0$ as small as one wishes, relative to the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of $\Omega$. (2.2.195)

**Example 2.2.21.** We may also construct examples of $\delta$-SKT domains exhibiting *multiple* spiral points (cf. Figure 2.1). Specifically, suppose $-\infty < t_1 < t_2 < \cdots < t_{N-1} < t_N < +\infty$, for some $N \in \mathbb{N}$, and consider

$$b(t) := \varepsilon \sum_{j=1}^{N} \ln|t - t_j| \quad \text{for each} \quad t \in \mathbb{R} \setminus \{t_1, \dots, t_N\}, \tag{2.2.196}$$

for some sufficiently small $\varepsilon > 0$. Next, define $z : \mathbb{R} \to \mathbb{C}$ as in (2.2.182) for this choice of the function $b$. Then Proposition 2.2.13 and Proposition 2.2.9 imply that the region $\Omega$ in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is indeed a $\delta$-SKT domain and, in fact, $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. Moreover, from (2.2.183) and (2.2.196) we see that $\partial\Omega = \Sigma$ looks like a spiral at each of the points $z(t_1), \dots, z(t_N)$. Yet, once again, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Omega = F(\mathbb{R}^2_+)$, $\mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-)$, and $\partial\Omega = F(\mathbb{R} \times \{0\})$ (cf. (2.2.102)). Also, (2.2.144) presently entails

by choosing $\varepsilon \in (0,1)$ appropriately small, we may ensure that $\Omega$ is a $\delta$-SKT domain in $\mathbb{R}^2$ with $\delta > 0$ as small as desired, relative to the Ahlfors regularity constant of $\partial\Omega$ and the local John constants of $\Omega$. (2.2.197)

**Example 2.2.22.** We wish to note that the construction in Example 2.2.21 may be modified as to allow *infinitely many* spiral points. Specifically, assume $\{t_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}$ is a given sequence of real numbers and consider

$$0 < \lambda_j < 2^{-j} \min\Big\{1\,,\, \big\| \ln|\cdot - t_j| \big\|^{-1}_{L^1([-j,j],\mathcal{L}^1)}\Big\} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.2.198}$$

Also, suppose $0 < \varepsilon < \big\| \ln|\cdot| \big\|^{-1}_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ and define

$$b(t) := \varepsilon \sum_{j=1}^{\infty} \lambda_j \ln|t - t_j| \quad \text{for each} \quad t \in \mathbb{R} \setminus \{t_j\}_{j \in \mathbb{N}}. \tag{2.2.199}$$

The choice in (2.2.198) ensures that the above series converges absolutely in $L^1(K, \mathcal{L}^1)$ for any compact subset $K$ of $\mathbb{R}$. This has two notable consequences. First, the series in (2.2.199) converges absolutely in a pointwise sense $\mathcal{L}^1$-a.e. in $\mathbb{R}$; in particular, $b$ is well defined at $\mathcal{L}^1$-a.e. point in $\mathbb{R}$ and takes real values. Second,

$$\begin{aligned}\|b\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} &\leq \varepsilon \sum_{j=1}^{\infty} \lambda_j \big\| \ln|\cdot - t_j| \big\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \\ &= \varepsilon \big\| \ln|\cdot| \big\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} \sum_{j=1}^{\infty} \lambda_j < 1. \end{aligned} \tag{2.2.200}$$

Granted this, if we now define $z : \mathbb{R} \to \mathbb{C}$ as in (2.2.182) for this choice of the function $b$ then Proposition 2.2.13 and Proposition 2.2.9 imply that the region $\Omega$ in $\mathbb{R}^2$ lying to one side of the curve $\Sigma := z(\mathbb{R})$ is a $\delta$-SKT domain with $\delta = O(\varepsilon)$ as $\varepsilon \to 0^+$. In fact, there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ as in (2.2.102), and (2.2.144) holds. We claim that matters may be arranged so that $\partial\Omega = \Sigma$ develops a spiral at each of the points $\{z(t_j)\}_{j\in\mathbb{N}}$. To this end, start by making the assumption that the sequence $\{t_j\}_{j\in\mathbb{N}}$ does not have any finite accumulation points. Inductively, we may then select a sequence of small positive numbers $\{r_j\}_{j\in\mathbb{N}} \subseteq (0,1)$ with the property that the family of intervals $I_j := (t_j - r_j, t_j + r_j)$, $j \in \mathbb{N}$, are mutually disjoint. For each $j \in \mathbb{N}$ consider the non-empty compact set $K_j := [-j, j] \setminus I_j$ and, in addition to (2.2.198), impose the condition that

$$0 < \lambda_j < 2^{-j} \big\| \ln|\cdot - t_j| \big\|^{-1}_{L^\infty(K_j,\mathcal{L}^1)} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.2.201}$$

Pick now $j_o \in \mathbb{N}$ arbitrary. Then for each $t \in I_{j_o}$ decompose $b(t) = f(t) + g(t)$ where

$$f(t) := \varepsilon\lambda_{j_o} \ln|t - t_{j_o}| \quad \text{and} \quad g(t) := \varepsilon \sum_{j\in\mathbb{N}\setminus\{j_o\}} \lambda_j \ln|t - t_j|. \tag{2.2.202}$$

In view of (2.2.201), the series defining $g$ converges uniformly on $I_{j_o}$, hence $g$ is a continuous and bounded function on $I_{j_o}$. Since $f$ is continuous and unbounded from below on $(t_{j_o}, t_{j_o} + r_{j_o})$, it follows that the restriction of $b$ to $(t_{j_o}, t_{j_o} + r_{j_o})$ is continuous and unbounded from below. This implies that $b\big((t_{j_o}, t_{j_o} + r_{j_o})\big)$ contains an interval of the form $(-\infty, a_{j_o})$, for some $a_{j_o} \in \mathbb{R}$. Similarly, $b\big((t_{j_o} - r_{j_o}, t_{j_o})\big)$ contains an interval of the form $(-\infty, c_{j_o})$, for some $c_{j_o} \in \mathbb{R}$. Based on this and (2.2.183) we then conclude that the normal $\nu(z(t))$ completes infinitely many rotations on the unit circle as $t$ approaches $t_{j_o}$ either from the left or from the right. Hence, $\partial\Omega = \Sigma$ develops a spiral at the point $z(t_{j_o})$.

**Example 2.2.23.** All sets considered so far have been connected. In the class of disconnected sets in the plane consider a double sector of arbitrary aperture $\theta \in (0,\pi)$, i.e., a set of the form

$$\begin{gathered}\Omega := \Big\{ x \in \mathbb{R}^2 \setminus \{x_0\} : \big| \tfrac{x - x_0}{|x - x_0|} \cdot \xi \big| > \cos(\theta/2) \Big\} \\ \text{with} \quad x_0 \in \mathbb{R}^2, \ \ \theta \in (0,\pi), \quad \text{and} \quad \xi \in S^1, \end{gathered} \tag{2.2.203}$$

and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. Then simple symmetry considerations show that for each $r \in (0,\infty)$ the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ satisfies

$\fint_{B(x_o,r)\cap\partial\Omega} \nu\, d\sigma = 0$, hence

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} &\geq \fint_{B(x_o,r)\cap\partial\Omega} \Big|\nu - \fint_{B(x_o,r)\cap\partial\Omega} \nu\, d\sigma\Big|\, d\sigma \\ &= \fint_{B(x_o,r)\cap\partial\Omega} |\nu|\, d\sigma = 1. \end{aligned} \tag{2.2.204}$$

As a consequence,

the double sector $\Omega$ from (2.2.203) is a disconnected Ahlfors regular domain which satisfies a two-sided local John condition but fails to be a $\delta$-SKT domain for each $\delta \in (0,1]$. (2.2.205)

We may even arrange matters so that the set in question has a disconnected boundary. Specifically, given any two distinct points $x_0, x_1 \in \mathbb{R}^2$, along with an angle $\theta \in (0,\pi)$, and a direction vector $\xi \in S^1$, such that

$$\frac{x_1 - x_0}{|x_1 - x_0|} \cdot \xi < \cos(\theta/2), \tag{2.2.206}$$

consider

$$\begin{aligned} \Omega := &\Big\{x \in \mathbb{R}^2 \setminus \{x_0\} : \frac{x - x_0}{|x - x_0|} \cdot \xi > \cos(\theta/2)\Big\} \\ &\bigcup \Big\{x \in \mathbb{R}^2 \setminus \{x_1\} : \frac{x - x_1}{|x - x_1|} \cdot (-\xi) > \cos(\theta/2)\Big\}. \end{aligned} \tag{2.2.207}$$

This is the union of two planar sectors with vertices at $x_0$ and $x_1$, axes along $\xi$ and $-\xi$, and common aperture $\theta$. The condition in (2.2.206) ensures that the said sectors are disjoint, hence $\Omega$ is disconnected, with disconnected boundary. Note that if $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and $\nu$ stands for the geometric measure theoretic outward unit normal to $\Omega$ then

$$\lim_{r\to\infty} \fint_{B(x_o,r)\cap\partial\Omega} \nu\, d\sigma = 0 \tag{2.2.208}$$

which, much as in (2.2.204), once again implies that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} \geq 1$. Consequently,

the set $\Omega$ from (2.2.207) is an Ahlfors regular domain satisfying a two-sided local John condition which is disconnected and has a disconnected boundary, and which fails to be a $\delta$-SKT domain for each $\delta \in (0,1]$. (2.2.209)

Similar considerations apply virtually verbatim in $\mathbb{R}^n$ with $n \geq 2$ (working with cones in place of sectors).

These examples are particularly relevant in the context of Theorem 2.2.33.

Moving on, our next result, which slightly refines work in [53], identifies general geometric conditions on a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter so that the inner product between the integral average $\nu_\Delta$ of outward unit normal $\nu$ to $\Omega$ in any given surface ball $\Delta \subseteq \partial\Omega$ and the "chord" $x - y$ with $x, y \in \Delta$ may be controlled in terms of the radius of the said ball and the BMO semi-norm of the outward unit normal $\nu$.

**Proposition 2.2.24.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Then there exists $C \in (0,\infty)$ depending only on the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$ such that for each $\lambda \in [1,\infty)$ one has*

$$\sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(z,\lambda R)} R^{-1}|\langle x-y,\nu_{\Delta(z,R)}\rangle| \leq C\lambda(1+\log_2\lambda)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.2.210}$$

*Proof.* From [53, Corollary 4.15, pp. 2697-2698] we know that there exists some constant $C \in (0,\infty)$ depending only on the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\sup_{x\in\partial\Omega}\sup_{R>0}\sup_{y\in\Delta(x,2R)} R^{-1}|\langle x-y,\nu_{\Delta(x,R)}\rangle| \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.2.211}$$

Fix $\lambda \in [1,\infty)$ along with $x \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z,\lambda R)$. Then $|x-y| \leq 2\lambda R$, hence $y \in \Delta(x, 2\lambda R)$, so

$$\begin{aligned}
|\langle x-y,\nu_{\Delta(z,R)}\rangle| &\leq |\langle x-y,\nu_{\Delta(x,2\lambda R)}\rangle| + |x-y||\nu_{\Delta(x,2\lambda R)}-\nu_{\Delta(z,R)}| \\
&\leq C\lambda R\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} + 2\lambda R|\nu_{\Delta(x,2\lambda R)}-\nu_{\Delta(z,3\lambda R)}| \\
&\qquad\qquad + 2\lambda R|\nu_{\Delta(z,3\lambda R)}-\nu_{\Delta(z,R)}| \\
&\leq CR\lambda(1+\log_2\lambda)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},
\end{aligned} \tag{2.2.212}$$

by (2.2.211) and elementary estimates involving integral averages (cf. (2.2.44), (2.2.46)). After dividing the most extreme sides by $R$, then taking the supremum over all $z \in \partial\Omega$, $R > 0$, and $x, y \in \Delta(z,\lambda R)$, we arrive at (2.2.210). □

We continue by discussing a basic decomposition theorem. The general idea originated in [107, Proposition 5.1, p. 212] where such a decomposition result has been stated for surfaces of class $\mathscr{C}^2$, via a proof which makes essential use of smoothness, though the main quantitative aspects only depend on the rough character of the said surface. A formulation in which the $\mathscr{C}^2$ smoothness assumption is replaced by Reifenberg flatness appears in [62, Theorem 4.1, p. 398] (see also the comments on [20, p. 66]).

The most desirable version of such a decomposition result has been proved by S. Hofmann, M. Mitrea, and M. Taylor in [53, Theorem 4.16, p. 2701], starting with a different set of hypotheses which, a priori, do not specifically require the domain in question to be Reifenberg flat. Below we present a variant of this result which is well suited to the applications we have in mind.

**Theorem 2.2.25.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition whose boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Then there exists a threshold $\delta_* \in (0,1)$, depending only on the dimension $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that whenever*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < \delta_* \tag{2.2.213}$$

*there exist* $C_0 \in (0, \delta_*^{-1})$ *along with* $C_1, C_2, C_3 \in (0, \infty)$*, depending only on the dimension* $n$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the following significance. For every location* $x_0 \in \partial\Omega$ *and every scale* $r > 0$ *there exist a unit vector* $\vec{n}_{x_0,r} \in S^{n-1}$ *along with a Lipschitz function*

$$h : H(x_0, r) := \langle \vec{n}_{x_0,r} \rangle^{\perp} \to \mathbb{R} \quad \textit{with} \quad \sup_{\substack{y_1, y_2 \in H(x_0,r) \\ y_1 \neq y_2}} \frac{|h(y_1) - h(y_2)|}{|y_1 - y_2|} \leq C_0 \delta, \tag{2.2.214}$$

*whose graph*

$$\mathcal{G} := \{x = x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0, r),\ t = h(x')\} \tag{2.2.215}$$

(*in the coordinate system* $x = (x', t) \Leftrightarrow x = x_0 + x' + t\vec{n}_{x_0,r}$*,* $x' \in H(x_0, r)$*,* $t \in \mathbb{R}$) *is a good approximation of* $\partial\Omega$ *in the cylinder*

$$\mathcal{C}(x_0, r) := \{x_0 + x' + t\vec{n}_{x_0,r} : x' \in H(x_0, r),\ |x'| \leq r,\ |t| \leq r\} \tag{2.2.216}$$

*in the following sense. First, with* $\upsilon_{n-1}$ *denoting the volume of the unit ball in* $\mathbb{R}^{n-1}$*,*

$$\sigma\left(\mathcal{C}(x_0, r) \cap (\partial\Omega \,\triangle\, \mathcal{G})\right) \leq C_1 \upsilon_{n-1} r^{n-1} e^{-C_2/\delta}, \tag{2.2.217}$$

*where* $\triangle$ *denotes the symmetric set-theoretic difference. Second, there exist two disjoint* $\sigma$*-measurable sets* $G(x_0, r)$ *and* $E(x_0, r)$ *such that*

$$\mathcal{C}(x_0, r) \cap \partial\Omega = G(x_0, r) \cup E(x_0, r), \tag{2.2.218}$$

$$G(x_0, r) \subseteq \mathcal{G}, \quad \sigma(E(x_0, r)) \leq C_1 \upsilon_{n-1} r^{n-1} e^{-C_2/\delta}. \tag{2.2.219}$$

*Third, if* $\Pi : \mathbb{R}^n \to H(x_0, r)$ *is defined by* $\Pi(x) := x'$ *for each* $x = x_0 + x' + t\vec{n}_{x_0,r} \in \mathbb{R}^n$ *with* $x' \in H(x_0, r)$ *and* $t \in \mathbb{R}$*, then*

$$\begin{gathered} \big|x - (x_0 + \Pi(x) + h(\Pi(x))\vec{n}_{x_0,r})\big| \leq C_0 \delta \cdot \operatorname{dist}\left(\Pi(x), \Pi(G(x_0, r))\right) \\ \textit{for each point } \ x \in E(x_0, r), \end{gathered} \tag{2.2.220}$$

*and*

$$\mathcal{C}(x_0, r) \cap \partial\Omega \subseteq \{x_0 + x' + t\vec{n}_{x_0,r} : |t| \leq C_0 \delta r,\ x' \in H(x_0, r)\}, \tag{2.2.221}$$

$$\Pi\big(\mathcal{C}(x_0, r) \cap \partial\Omega\big) = \{x' \in H(x_0, r) : |x'| < r\}. \tag{2.2.222}$$

*Fourth, if*

$$\begin{aligned} \mathcal{C}^+(x_0, r) &:= \{x_0 + x' + t\,\vec{n}_{x_0,r} : x' \in H(x_0, r),\ |x'| \leq r,\ -r < t < -C_0 \delta r\}, \\ \mathcal{C}^-(x_0, r) &:= \{x_0 + x' + t\,\vec{n}_{x_0,r} : \ x' \in H(x_0, r),\ |x'| \leq r,\ C_0 \delta r < t < r\}, \end{aligned} \tag{2.2.223}$$

*then*

$$\mathcal{C}^+(x_0, r) \subseteq \Omega \quad \textit{and} \quad \mathcal{C}^-(x_0, r) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.2.224}$$

*Finally,*

$$(1 - C_3 \delta) \upsilon_{n-1} r^{n-1} \leq \sigma\big(\Delta(x_0, r)\big) \leq (1 + C_3 \delta) \upsilon_{n-1} r^{n-1}. \tag{2.2.225}$$

*Proof.* This is established by reasoning as in [53, pp. 2703-2709] with (2.2.213) replacing the small local BMO assumption (which, in particular, frees us from having to restrict $x_0$ to a compact subset of $\partial\Omega$). The key observation is that in the present context, the parameter $R_*$ from [53, Theorem 4.16, p. 2701] (which limits the size of the scale $r$) may be taken to be $+\infty$. In turn, this is seen by a careful inspection of the proof of [53, Theorem 4.16, p. 2701], in which we now rely on [53, Corollary 4.15, p. 2697] in place of [53, Theorem 4.14, p. 2697].

Finally, since the claim pertaining to (2.2.223)-(2.2.224) does not explicitly appear in the statement of [53, Theorem 4.16, p. 2701], we provide a proof here. From (2.2.221)-(2.2.222) it follows that the connected sets $\mathcal{C}^{\pm}(x_0, r)$ introduced in (2.2.223) do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $\mathcal{C}^{\pm}(x_0, r)$, hence

$$\begin{aligned}&\mathcal{C}^{+}(x_0, r) \text{ is fully contained in either } \Omega_+ \text{ or } \Omega_-, \text{ and}\\ &\text{also } \mathcal{C}^{-}(x_0, r) \text{ is fully contained in either } \Omega_+ \text{ or } \Omega_-.\end{aligned} \tag{2.2.226}$$

Denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location $x_0$ and scale $r$. In particular,

$$|x_r^{\pm} - x_0| < r \quad \text{and} \quad B(x_r^{\pm}, \theta r) \subseteq \Omega_{\pm} \tag{2.2.227}$$

where the parameter $\theta \in (0, 1)$ is as in Definition 1.1.10. Assume $0 < \delta < \theta/C_0$ to begin with. This makes it impossible to contain either of the balls $B(x_r^+, \theta r)$, $B(x_r^-, \theta r)$ in the strip $\{x_0 + x' + t\vec{n}_{x_0,r} : |t| \leq C_0\delta r,\ x' \in H(x_0, r)\}$. Since, as seen from (2.2.227), their centers $x_r^{\pm}$ belong to $B(x_0, r) \subset \mathcal{C}(x_0, r)$, in turn this forces one of the following four alternatives to be true:

$$B(x_r^+, \theta r) \cap \mathcal{C}^+(x_0, r) \neq \varnothing \quad \text{and} \quad B(x_r^-, \theta r) \cap \mathcal{C}^+(x_0, r) \neq \varnothing, \tag{2.2.228}$$

$$B(x_r^+, \theta r) \cap \mathcal{C}^-(x_0, r) \neq \varnothing \quad \text{and} \quad B(x_r^-, \theta r) \cap \mathcal{C}^-(x_0, r) \neq \varnothing, \tag{2.2.229}$$

$$B(x_r^+, \theta r) \cap \mathcal{C}^+(x_0, r) \neq \varnothing \quad \text{and} \quad B(x_r^-, \theta r) \cap \mathcal{C}^-(x_0, r) \neq \varnothing, \tag{2.2.230}$$

$$B(x_r^+, \theta r) \cap \mathcal{C}^-(x_0, r) \neq \varnothing \quad \text{and} \quad B(x_r^-, \theta r) \cap \mathcal{C}^+(x_0, r) \neq \varnothing. \tag{2.2.231}$$

Note that the alternative described in (2.2.228) cannot possibly hold. Indeed, the existence of two points $z_1 \in B(x_r^+, \theta r) \cap \mathcal{C}^+(x_0, r)$ and $z_2 \in B(x_r^-, \theta r) \cap \mathcal{C}^+(x_0, r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $\mathcal{C}^+(x_0, r)$, hence also either in $\Omega_+$ or in $\Omega_-$ by (2.2.226). However, the fact that $z_1 \in B(x_r^+, \theta r) \subseteq \Omega_+$ and $z_2 \in B(x_r^-, \theta r) \subseteq \Omega_-$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.2.228). Reasoning in a similar fashion we may rule out (2.2.229). When (2.2.230) holds, from (2.2.226) and the fact that $B(x_r^{\pm}, \theta r) \subseteq \Omega_{\pm}$ (cf. (2.2.227)) we conclude that the inclusions in (2.2.224) hold as stated. Finally, when (2.2.231) holds, from (2.2.226) and (2.2.227) we deduce that $\mathcal{C}^+(x_0, r) \subseteq \Omega_-$ and $\mathcal{C}^-(x_0, r) \subseteq \Omega_+$. In such a scenario, we may ensure that the inclusions in (2.2.224) are valid simply by re-denoting $\vec{n}_{x_0,r}$ as $-\vec{n}_{x_0,r}$ which amounts to reversing the roles of $\mathcal{C}^+(x_0, r)$ and $\mathcal{C}^-(x_0, r)$. This concludes the proof of (2.2.224). $\square$

In the final portion of this section we explore the implications of the quality of being a $\delta$-SKT domain with $\delta > 0$ small in terms of flatness (in the Reifenberg sense) and topology (cf. Theorem 2.2.33). To facilitate the subsequent discussion, the reader is reminded that the Hausdorff distance between two arbitrary nonempty sets $A, B \subset \mathbb{R}^n$ is defined as

$$\mathrm{Dist}[A,B] := \max\Big\{\sup\{\mathrm{dist}\,(a,B) : a \in A\}\,,\,\sup\{\mathrm{dist}\,(b,A) : b \in B\}\Big\}. \tag{2.2.232}$$

We also recall the following definitions from [61].

**Definition 2.2.26.** Fix $R \in (0,\infty]$ along with $\delta \in (0,\infty)$ and let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then $\Sigma$ is said to be a $(R,\delta)$-`Reifenberg flat set` if for each for each $x \in \Sigma$ and each $r \in (0,R)$ there exists an $(n-1)$-dimensional plane $\pi(x,r)$ in $\mathbb{R}^n$ which contains $x$ and satisfies

$$\mathrm{Dist}\big[\Sigma \cap B(x,r)\,,\,\pi(x,r) \cap B(x,r)\big] \leq \delta\, r. \tag{2.2.233}$$

For example, given $\delta > 0$, the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ with a sufficiently small Lipschitz constant is a $(\infty,\delta)$-Reifenberg flat set.

**Definition 2.2.27.** Fix $R \in (0,\infty]$ along with $\delta \in (0,\infty)$. A nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is said to satisfy the $(R,\delta)$-`separation property` if for each $x \in \partial\Omega$ and $r \in (0,R)$ there exist an $(n-1)$-dimensional plane $\widetilde{\pi}(x,r)$ in $\mathbb{R}^n$ passing through $x$ and a choice of unit normal vector $\vec{n}_{x,r}$ to $\widetilde{\pi}(x,r)$ such that

$$\begin{aligned} &\{y + t\,\vec{n}_{x,r} \in B(x,r) : y \in \widetilde{\pi}(x,r),\ t > 2\delta r\} \subset \Omega \ \text{ and} \\ &\{y + t\,\vec{n}_{x,r} \in B(x,r) : y \in \widetilde{\pi}(x,r),\ t < -2\delta r\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{2.2.234}$$

**Definition 2.2.28.** Fix $R \in (0,\infty]$ along with $\delta \in (0,\infty)$. A nonempty, proper subset $\Omega$ of $\mathbb{R}^n$ is called an $(R,\delta)$-`Reifenberg flat domain` (or simply a `Reifenberg flat domain` if the particular values of $R, \delta$ are not important) provided $\Omega$ satisfies the $(R,\delta)$-separation property and $\partial\Omega$ is an $(R,\delta)$-Reifenberg flat set.

Recall the two-sided corkscrew condition from Definition 1.1.6.

**Proposition 2.2.29.** *Let $\Omega$ be a nonempty proper subset of $\mathbb{R}^n$ satisfying the $(R,c)$-two-sided corkscrew condition for some $R \in (0,\infty]$ and $c \in (0,1)$. In addition, suppose $\partial\Omega$ is a $(R,\delta)$-Reifenberg flat set for some $\delta \in (0,c/2)$. Then $\Omega$ is an $(R,\delta)$-Reifenberg flat domain.*

*Proof.* Pick a location $x \in \partial\Omega$ and a scale $r \in (0,R)$. Definition 2.2.26 ensures the existence of of an $(n-1)$-dimensional plane $\pi(x,r)$ in $\mathbb{R}^n$ passing through $x$ which satisfies (2.2.233). Make a choice of a unit normal vector $\vec{n}_{x,r}$ to $\pi(x,r)$ and abbreviate

$$\begin{aligned} \mathcal{C}^+(x,r) &:= \{y + t\,\vec{n}_{x,r} \in B(x,r) : y \in \pi(x,r),\ t > 2\delta r\}, \\ \mathcal{C}^-(x,r) &:= \{y + t\,\vec{n}_{x,r} \in B(x,r) : y \in \pi(x,r),\ t < -2\delta r\}. \end{aligned} \tag{2.2.235}$$

We claim that matters may be arranged (by taking $\delta$ sufficiently small to begin with, and by making a judicious choice of the orientation of $\vec{n}_{x,r}$) so that

$$\mathcal{C}^+(x,r) \subset \Omega \ \text{ and } \ \mathcal{C}^-(x,r) \subset \mathbb{R}^n \setminus \Omega. \tag{2.2.236}$$

With this goal in mind, first observe that (2.2.233) guarantees that the connected sets $\mathcal{C}^{\pm}(x,r)$ do not intersect $\partial\Omega$. As such, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ form a disjoint, open cover of $\mathcal{C}^{\pm}(x,r)$, hence

$$\begin{array}{l}\mathcal{C}^{+}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-, \text{ and} \\ \text{also } \mathcal{C}^{-}(x,r) \text{ is entirely contained in either } \Omega_+ \text{ or } \Omega_-.\end{array} \tag{2.2.237}$$

To proceed, denote by $x_r^{\pm} \in \Omega_{\pm}$ the two corkscrew points corresponding to the location $x$ and scale $r$. In particular,

$$|x_r^{\pm} - x| < r \ \text{ and } \ B(x_r^{\pm}, cr) \subseteq \Omega_{\pm} \tag{2.2.238}$$

where the constant $c \in (0,1)$ is as in Definition 1.1.6. Hence, if we consider the balls $B(x_r^+, cr)$, $B(x_r^-, cr)$, their centers $x_r^{\pm}$ belong to $B(x,r)$. The fact that we are presently assuming $0 < \delta < c/2$ with $c \in (0,1)$ ensures that $\delta < (c/2)\sqrt{4-c^2}$ which, as some elementary geometry shows, forces each of the balls $B(x_r^+, cr)$, $B(x_r^-, cr)$ to intersect one of the sets $\mathcal{C}^+(x,r)$, $\mathcal{C}^-(x,r)$. As such, one of the following four alternatives is true:

$$B(x_r^+, cr) \cap \mathcal{C}^+(x,r) \neq \varnothing \ \text{ and } \ B(x_r^-, cr) \cap \mathcal{C}^+(x,r) \neq \varnothing, \tag{2.2.239}$$

$$B(x_r^+, cr) \cap \mathcal{C}^-(x,r) \neq \varnothing \ \text{ and } \ B(x_r^-, cr) \cap \mathcal{C}^-(x,r) \neq \varnothing, \tag{2.2.240}$$

$$B(x_r^+, cr) \cap \mathcal{C}^+(x,r) \neq \varnothing \ \text{ and } \ B(x_r^-, cr) \cap \mathcal{C}^-(x,r) \neq \varnothing, \tag{2.2.241}$$

$$B(x_r^+, cr) \cap \mathcal{C}^-(x,r) \neq \varnothing \ \text{ and } \ B(x_r^-, cr) \cap \mathcal{C}^+(x,r) \neq \varnothing. \tag{2.2.242}$$

Observe that the alternative described in (2.2.239) cannot possibly hold. Otherwise, the existence of two points $z_1 \in B(x_r^+, cr) \cap \mathcal{C}^+(x,r)$ and $z_2 \in B(x_r^-, cr) \cap \mathcal{C}^+(x,r)$ would imply that, on the one hand, the line segment $[z_1, z_2]$ lies in the convex set $\mathcal{C}^+(x,r)$, hence also either in $\Omega_+$ or in $\Omega_-$ by (2.2.237). This being said, the fact that $z_1 \in B(x_r^+, cr) \subseteq \Omega_+$ and $z_2 \in B(x_r^-, cr) \subseteq \Omega_-$ prevents either one of these eventualities form materializing. This contradiction therefore excludes (2.2.239). Reasoning in a similar fashion we may rule out (2.2.240). When (2.2.241) holds, from (2.2.237) and the fact that $B(x_r^{\pm}, cr) \subseteq \Omega_{\pm}$ (cf. (2.2.238)) we conclude that the inclusions in (2.2.236) hold as stated. Finally, when (2.2.242) holds, from (2.2.226) and (2.2.238) we deduce that $\mathcal{C}^+(x,r) \subseteq \Omega_-$ and $\mathcal{C}^-(x,r) \subseteq \Omega_+$. In such a scenario, we may ensure that the inclusions in (2.2.236) are valid simply by re-denoting $\vec{n}_{x,r}$ as $-\vec{n}_{x,r}$ which amounts to reversing the roles of $\mathcal{C}^+(x,r)$ and $\mathcal{C}^-(x,r)$. This concludes the proof of (2.2.236). In turn, from (2.2.236) and (2.2.235) we conclude that (2.2.234) holds with $\widetilde{\pi}(x,r) := \pi(x,r)$. Definition 2.2.27 then implies that $\Omega$ is, indeed, an $(R,\delta)$-Reifenberg flat domain. □

Any $\delta$-SKT domain with $\delta > 0$ sufficiently small is a Reifenberg flat domain. Specifically, we have the following result.

**Proposition 2.2.30.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$.*

*Then there exists some threshold $\delta_* \in (0,1)$ along with some constant $C \in (0,\infty)$, both depending only on the dimension $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ with $\delta \in (0,\delta_*)$ then $\Omega$ is an $(\infty, C\delta)$-Reifenberg flat domain.*

A version of this result continues to hold with the two-sided local John condition relaxed to a two-sided corkscrew condition, in which scenario one concludes that $\partial\Omega$ is a Reifenberg flat set but only at small scales (see [15, Theorem 2.1] for details).

*Proof of Proposition 2.2.30.* Choose the parameter $\delta_* \in (0,1)$ as in Theorem 2.2.25 and make the assumption that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ with $\delta \in (0,\delta_*)$. Then from (2.2.221) it follows that there exists some $C \in (0,\infty)$ which depends only on the dimension $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that $\partial\Omega$ is an $(\infty, C\delta)$-Reifenberg flat set. Also, (2.2.223)-(2.2.224) guarantee that, for a constant $C \in (0,\infty)$ of the same nature as before, $\Omega$ satisfies the $(\infty, C\delta)$-separation property. Alternatively, we may invoke Proposition 2.2.29 (keeping in mind that the two-sided local John condition implies the two-sided corkscrew condition). Granted these qualities, Definition 2.2.28 then implies that $\Omega$ is an $(\infty, C\delta)$-Reifenberg flat domain. □

It turns out that sufficiently flat Reifenberg domains are NTA domains. More specifically, from [61, Theorem 3.1, p. 524] and its proof we see that

> there exists a purely dimensional constant $\delta_n \in (0,\infty)$ with the property that for each $\delta \in (0,\delta_n)$ and $R \in (0,\infty]$ one may find a number $N = N(\delta, R) \in \mathbb{N}$ with the property that that any $(R,\delta)$-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^n$ is also an $(R,N)$-nontangentially accessible domain (in the sense of Definition 1.1.8). (2.2.243)

This has a number of useful consequences. For example, it allows us to conclude that any open set satisfying a two-sided corkscrew condition and whose topological boundary is a sufficiently flat Reifenberg set is actually an NTA domain.

**Proposition 2.2.31.** *Let $\Omega$ be a nonempty proper subset of $\mathbb{R}^n$ satisfying the $(R,c)$-two-sided corkscrew condition for some $R \in (0,\infty]$ and $c \in (0,1)$. In addition, suppose $\partial\Omega$ is a $(R,\delta)$-Reifenberg flat set with $0 < \delta < \min\{c/2, \delta_n\}$, where $\delta_n \in (0,\infty)$ is the purely dimensional constant from (2.2.243). Then there exist $N = N(\delta, R) \in \mathbb{N}$ with the property that $\Omega$ is an $(R,N)$-nontangentially accessible domain.*

*Proof.* The desired conclusion is a direct consequence of Proposition 2.2.29, (2.2.243), and Definition 1.1.8. □

In concert with Proposition 2.2.30, the result recalled in (2.2.243) also shows that any $\delta$-SKT domain with $\delta > 0$ sufficiently small is a two-sided NTA domain. Here is a precise statement.

**Proposition 2.2.32.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$.*

*Then there exists a threshold $\delta_0 \in (0,1)$ and a number $N \in \mathbb{N}$, both depending only on the dimension $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ then $\Omega$ is an $(\infty, N)$-two-sided nontangentially accessible domain* (*in the sense of Definition 1.1.8*).

In the converse direction, fix $\delta \in (0,\infty)$, $N \in \mathbb{N}$, $R \in (0,\infty]$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an $(R,N)$-two-sided nontangentially accessible domain (with the requirement that $R = \infty$ if $\partial\Omega$ is unbounded) whose boundary is an Ahlfors regular set. Then from (1.1.28) and definitions we see that $\Omega$ is a $\delta$-SKT domain whenever its geometric measure theoretic outward unit normal $\nu$ satisfies $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (where, as usual, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$).

*Proof of Proposition 2.2.32.* Let $\delta_* \in (0,1)$ and $C \in (0,\infty)$ be as in Proposition 2.2.30, and recall the purely dimensional constant $\delta_n \in (0,\infty)$ from (2.2.243). Take

$$\delta_0 := \min\{\delta_*\,,\,\delta_n/C\} \in (0,1) \tag{2.2.244}$$

and assume that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$. Also, fix some $\delta \in \big(\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,,\delta_0\big)$. Proposition 2.2.30 then guarantees that $\Omega$ is an $(\infty, C\delta)$-Reifenberg flat domain. Given that $C\delta < \delta_n$, from (2.2.243) we conclude that there exists some $N \in \mathbb{N}$ such that $\Omega$ is an $(\infty, N)$-nontangentially accessible domain in the sense of Definition 1.1.8.

Going further, observe that the set $\mathbb{R}^n\backslash\overline{\Omega}$ satisfies a two-sided local John condition and recall from Example 2.2.16 that its topological and measure theoretic boundaries coincide with those of $\Omega$. Also, the geometric measure theoretic outward unit normal to $\mathbb{R}^n\setminus\overline{\Omega}$ is $-\nu$ at $\sigma$-a.e. point on $\partial\Omega$. In particular, the BMO semi-norm of the geometric measure theoretic outward unit normal to $\mathbb{R}^n\setminus\overline{\Omega}$ is $< \delta$. As such, the argument in the first part of the proof applies and gives that $\mathbb{R}^n\setminus\overline{\Omega}$ is an $(\infty, N)$-nontangentially accessible domain as well. All together, this makes $\Omega$ an $(\infty, N)$-two-sided nontangentially accessible domain. □

We are now in a position to show that for a $\delta$-SKT domain $\Omega \subseteq \mathbb{R}^n$ the demand that the parameter $\delta \in (0,1)$ is suitably small relative to the geometry of $\Omega$ has a string of remarkable topological and metric consequences for the set $\Omega$. To set the stage, from [70, Theorem 2 in 49.VI, 57.I.9(i), 57.III.1] (cf. also [67, Lemma 4(1) and Lemma 5, p. 1702]) we first note that

$$\text{if } \mathcal{O} \subseteq \mathbb{R}^n \text{ is some arbitrary connected open set, then any connected component of } \mathbb{R}^n\setminus\overline{\mathcal{O}} \text{ has a connected boundary.} \tag{2.2.245}$$

**Theorem 2.2.33.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$.*

*Then there exists a threshold $\delta_0 \in (0,1)$ depending only on the ambient dimension $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that if*

$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ *it follows that* $\Omega$ *is an unbounded connected set,* $\overline{\Omega}$ *is an unbounded connected set,* $\partial\Omega$ *is an unbounded connected set,* $\mathbb{R}^n \setminus \overline{\Omega}$ *is an unbounded connected set,* $\mathbb{R}^n \setminus \Omega$ *is an unbounded connected set, and* $\partial(\overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$, $\partial(\mathbb{R}^n \setminus \Omega) = \partial\Omega$.

As is apparent from Example 2.2.23, the demand that the parameter $\delta > 0$ is sufficiently small cannot be dispense with in the context of Theorem 2.2.33.

*Proof of Theorem 2.2.33.* Bring in the threshold $\delta_0 \in (0,1)$ from Proposition 2.2.32 and assume that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$. From Proposition 2.2.32, Definition 1.1.8, and Definition 1.1.7 we then conclude that both $\Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ are pathwise connected open sets (hence, connected open sets). Having established this, from (2.2.245) we then see that $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial(\overline{\Omega})$ is connected. The fact that $\Omega$ satisfies an exterior corkscrew condition further implies $\partial(\overline{\Omega}) = \partial\Omega$. Since $\delta < 1$, Lemma 2.2.5 ensures that $\partial\Omega$ is unbounded, and this forces both $\Omega$ and $\mathbb{R}^n \setminus \overline{\Omega}$ to be unbounded (given that they have $\partial\Omega$ as their topological boundary). Also, the fact that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected implies that its closure is connected. However, $\overline{\mathbb{R}^n \setminus \overline{\Omega}} = \mathbb{R}^n \setminus \mathring{\overline{\Omega}}$ and

$$\mathring{\overline{\Omega}} = \overline{\Omega} \setminus \partial(\overline{\Omega}) = \overline{\Omega} \setminus \partial\Omega = \mathring{\Omega} = \Omega, \tag{2.2.246}$$

so $\mathbb{R}^n \setminus \Omega = \overline{\mathbb{R}^n \setminus \overline{\Omega}}$ is connected. □

In the two-dimensional setting, it turns out that having an outward unit normal with small BMO semi-norm implies (under certain background assumptions) that the domain in question is actually simply connected. This makes the object of Corollary 2.2.34, which augments Theorem 2.2.33.

**Corollary 2.2.34.** *Let* $\Omega \subseteq \mathbb{R}^2$ *be an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Abbreviate* $\sigma := \mathcal{H}^1\lfloor\partial\Omega$ *and denote by* $\nu$ *the geometric measure theoretic outward unit normal to* $\Omega$*. Then there exists a threshold* $\delta_0 \in (0,1)$*, depending only on the local John constants of* $\Omega$ *and the Ahlfors regularity constant of* $\partial\Omega$*, such that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta_0$ *it follows that* $\Omega$ *is an unbounded connected set which is simply connected,* $\partial\Omega$ *is an unbounded connected set,* $\mathbb{R}^2 \setminus \overline{\Omega}$ *is an unbounded connected set which is simply connected, and* $\partial(\mathbb{R}^2 \setminus \overline{\Omega}) = \partial\Omega$.

*Proof.* All claims are consequences of Theorem 2.2.33 and either (2.2.68), or (2.2.69), or (2.2.70). □

### 2.2.4 Chord-arc domains in the plane

In the two-dimensional setting, an important category of sets is the class of chord-arc domains, discussed next.

**Definition 2.2.35.** Given a nonempty, proper, open subset $\Omega$ of $\mathbb{R}^2$ and $\varkappa \in [0,\infty)$, one calls $\Omega$ a $\varkappa$-`CAD` (or simply `chord-arc domain`, if the value of $\varkappa$ is not important) provided $\partial\Omega$ is a locally rectifiable simple curve, which is either a closed curve or a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$, with the property that

$$\ell(z_1,z_2) \leq (1+\varkappa)|z_1 - z_2| \quad \text{for all} \quad z_1, z_2 \in \partial\Omega, \tag{2.2.247}$$

where $\ell(z_1, z_2)$ denotes the length of the shortest arc of $\partial\Omega$ joining $z_1$ and $z_2$.

For example, a planar sector $\Omega_\theta$ of aperture $\theta \in (0, 2\pi)$ (cf. (2.2.163)) is a $\varkappa$-CAD with constant $\varkappa := [\sin(\theta/2)]^{-1} - 1$. While Proposition 2.2.12 shows that the upper-graph of any real-valued $\mathrm{BMO}_1$ function defined on the real line is a chord-arc domain (hence, in particular, any Lipschitz domain in the plane is a chord-arc domain), from our earlier discussion (see, e.g., Example 2.2.20) we know that the boundaries of chord-arc domains may actually contain spiral points. There are also subtle connections between the quality of being a chord-arc domain and the behavior of the conformal mapping (see, e.g., [20] and the references therein).

Our next major goal is to establish, in the two-dimensional setting, the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of $\delta$-SKT domains with $\delta > 0$ small. This is accomplished in Theorem 2.2.38. For now recall the concept of UR domain from Definition 1.1.5.

**Proposition 2.2.36.** *Assume $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is a chord-arc domain. Then $\Omega$ is a connected* UR *domain, satisfying a two-sided local John condition. Moreover, $\partial\Omega = \partial(\overline{\Omega})$ and if either $\partial\Omega$ is unbounded, or $\Omega$ is bounded, then $\Omega$ is also simply connected.*

*Proof.* If $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C}$ then the desired conclusions follow from item *(vi)* of Proposition 2.2.9 and (2.2.69). If $\partial\Omega$ is bounded, then there exists a bi-Lipschitz homeomorphism $F$ of the complex plane onto itself such that $F(\partial B(0,1)) = \partial\Omega$ (cf. [104, Theorem 7.9, p. 165]). This implies that each of the connected sets $F(B(0,1))$, $F(\mathbb{C} \setminus \overline{B(0,1)})$ is contained in the disjoint union of $\Omega$ with $\mathbb{C} \setminus \overline{\Omega}$. Since $F$ is surjective, this forces that either

$$F(B(0,1)) = \Omega \quad \text{and} \quad F(\mathbb{C} \setminus \overline{B(0,1)}\,) = \mathbb{C} \setminus \overline{\Omega}, \tag{2.2.248}$$

or

$$F(B(0,1)) = \mathbb{C} \setminus \overline{\Omega} \quad \text{and} \quad F(\mathbb{C} \setminus \overline{B(0,1)}\,) = \Omega. \tag{2.2.249}$$

All desired conclusions readily follow from this and the transformational properties under bi-Lipschitz maps established in [52]. □

A chord-arc domain with a sufficiently small constant is necessarily unbounded (and, in fact, has an unbounded boundary).

**Proposition 2.2.37.** *If $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-*`CAD` *with $\varkappa \in \big[0\,,\,\sqrt{2}-1\big)$ then $\partial\Omega$ is unbounded.*

*Proof.* Seeking a contradiction, assume $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-`CAD` with $\varkappa \in \big[0\,,\,\sqrt{2}-1\big)$ and such that $\partial\Omega$ is a bounded set. In particular, $\partial\Omega$ is a rectifiable closed curve. Abbreviate $L := \mathcal{H}^1(\partial\Omega) \in (0, \infty)$ and let $[0, L] \ni s \mapsto z(s) \in \partial\Omega$ be the arc-length parametrization of $\partial\Omega$. Define $z_0 := z(0)$, $z_{1/4} := z(L/4)$, $z_{1/2} := z(L/2)$, $z_{3/4} := z(3L/4)$. Since

$$\begin{aligned} |z_0 - z_{1/4}| &\leq \ell(z_0, z_{1/4}) = L/4, & |z_{3/4} - z_0| &\leq \ell(z_{3/4}, z_0) = L/4, \\ |z_{1/2} - z_{3/4}| &\leq \ell(z_{1/2}, z_{3/4}) = L/4, & |z_{1/4} - z_{1/2}| &\leq \ell(z_{1/4}, z_{1/2}) = L/4, \end{aligned} \tag{2.2.250}$$

it follows that

$$z_{1/4}, z_{3/4} \in D := \overline{B(z_0, L/4)} \cap \overline{B(z_{1/2}, L/4)}, \tag{2.2.251}$$

hence

$$|z_{1/4} - z_{3/4}| \leq \operatorname{diam}(D). \tag{2.2.252}$$

On the one hand, with $R := |z_0 - z_{1/2}|$, elementary geometry gives that

$$\operatorname{diam}(D) = 2\sqrt{(L/4)^2 - (R/2)^2} = \sqrt{L^2/4 - R^2}. \tag{2.2.253}$$

On the other hand, $L/2 = \ell(z_0, z_{1/2}) \leq (1+\varkappa)|z_0 - z_{1/2}| = (1+\varkappa)R$ so

$$\operatorname{diam}(D) \leq \sqrt{L^2/4 - (L/(2+2\varkappa))^2} = \frac{L}{2}\sqrt{1 - \Big(\frac{1}{1+\varkappa}\Big)^2}. \tag{2.2.254}$$

Based on the chord-arc property, (2.2.252), and (2.2.254) we then conclude that

$$\begin{aligned} \frac{L}{2} &= \ell(z_{1/4}, z_{3/4}) \leq (1+\varkappa)|z_{1/4} - z_{3/4}| \\ &\leq (1+\varkappa)\operatorname{diam}(D) \leq \frac{L}{2}\sqrt{(1+\varkappa)^2 - 1}, \end{aligned} \tag{2.2.255}$$

which further implies that $\varkappa \geq \sqrt{2} - 1$, a contradiction. $\square$

By design, the boundary of any chord-arc domain is a simple curve, and this brings into focus the question: when is the boundary of an open, connected, simply connected planar set a Jordan curve? According to the classical Carathéodory theorem, this is the case if and only if some (or any) conformal mapping $\varphi : \mathbb{D} \to \Omega$ (where $\mathbb{D}$ is the unit disk in $\mathbb{C}$) extends to a homeomorphism $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$ (see, e.g., [43, Theorem 3.1, p. 13]). A characterization of bounded planar Jordan regions in terms of properties having no reference to their boundaries has been given by R.L. Moore in 1918. According to [100],

> given an open, bounded, connected, simply connected set $\Omega \subseteq \mathbb{R}^2$, in order for $\partial\Omega$ to be a simple closed curve it is necessary and sufficient that $\Omega$ is *uniformly connected im kleinen* (i.e., if for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset $\Gamma$ of $\Omega$ satisfying $|P - Q| < \varepsilon_o$ for each point $Q \in \Gamma$). (2.2.256)

A moment's reflection shows that the uniform connectivity condition (im kleinen) formulated above is equivalent to the demand that for every $\varepsilon_o > 0$ there exists $\delta_o > 0$ such that any two points $P, \widetilde{P} \in \Omega$ with $|P - \widetilde{P}| < \delta_o$ lie in a connected subset $\Gamma$ of $\Omega$ with $\operatorname{diam}(\Gamma) < \varepsilon_o$. This condition is meant to prevent the boundary of $\Omega$ to "branch out" (like in the case of a partially slit disk).

We are now in a position to establish the coincidence of the class of $\varkappa$-CAD domains with $\varkappa \geq 0$ small constant with that of $\delta$-SKT domains with $\delta > 0$ small, in the two-dimensional Euclidean setting.

**Theorem 2.2.38.** *If $\Omega \subset \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in [0, \sqrt{2}-1)$ then $\Omega$ is a $\delta$-SKT domain for any $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. In particular, $\Omega$ is a $\delta$-SKT domain for, say, $\delta := 4\sqrt{\varkappa(2+\varkappa)}$ a choice which satisfies $\delta = O(\sqrt{\varkappa})$ as $\varkappa \to 0^+$.*

*Conversely, given any $M \in (0, \infty)$ there exists $\delta_* \in (0,1)$ with the property that whenever $\delta \in (0, \delta_*)$ it follows that any $\delta$-SKT domain $\Omega \subset \mathbb{R}^2$ whose Ahlfors regularity constant as well as local John constants are $\leq M$ is a $\varkappa$-CAD with $\varkappa = O(\delta)$ as $\delta \to 0^+$.*

*Proof.* Suppose $\Omega \subset \mathbb{R}^2$ is a $\varkappa$-CAD with $\varkappa \in [0, \sqrt{2}-1)$. Proposition 2.2.37 then ensures that $\partial\Omega$ is an unbounded set. Keeping this in mind, from Definition 2.2.35 we then conclude that $\partial\Omega$ is a Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$. Granted (2.2.247), it follows that $\partial\Omega$ is a $\varkappa$-CAC. From Proposition 2.2.9 and (2.2.74) we then see that $\Omega$ satisfies a two-sided local John condition and has an Ahlfors regular boundary. Moreover, if $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and $\nu$ is the geometric measure theoretic outward unit normal to $\Omega$, from (2.2.103) we deduce that

$$\|\nu\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq 2\sqrt{\varkappa(2+\varkappa)}. \tag{2.2.257}$$

It follows from this Definition 2.2.14 that $\Omega$ is a $\delta$-SKT whenever $\delta > 2\sqrt{\varkappa(2+\varkappa)}$. This completes the proof of the claim in the first part of the statement of the theorem.

In the converse direction, let $\Omega \subseteq \mathbb{R}^2$ be a $\delta$-SKT domain with $\delta$ is sufficiently small relative to the Ahlfors regularity constant and the local John constants of $\Omega$. Then Proposition 2.2.32 implies that $\Omega$ is an $(\infty, N)$-two-sided nontangentially accessible domain (in the sense of Definition 1.1.8), for some $N \in \mathbb{N}$. From Corollary 2.2.34 we also know that $\Omega$ is an unbounded connected set which is simply connected, and whose topological boundary is an unbounded connected set.

The first order of business is to show that actually $\partial\Omega$ is a simple curve. To establish this, we intend to make use of Moore's criterion recalled in (2.2.256). Since this pertains to bounded sets, as a preliminary step we fix a point $z_0 \in \mathbb{C} \setminus \overline{\Omega}$ and consider

$$\widetilde{\Omega} := \Phi(\Omega) \subseteq \mathbb{C}, \tag{2.2.258}$$

where

$$\begin{gathered} \Phi : \mathbb{C} \setminus \{z_0\} \longrightarrow \mathbb{C} \setminus \{0\} \\ \Phi(z) := (z - z_0)^{-1} \ \text{ for each } \ z \in \mathbb{C} \setminus \{z_0\}. \end{gathered} \tag{2.2.259}$$

Note that, when restricted to $\Omega$, the function $\Phi$ satisfies a Lipschitz condition. Specifically, if $r_0 := \mathrm{dist}\,(z_0, \partial\Omega)$ then $r_0 \in (0, \infty)$ and we may estimate

$$\big|\Phi(z_1) - \Phi(z_2)\big| = \frac{|z_1 - z_2|}{|z_1 - z_0||z_2 - z_0|} \leq r_0^{-2}|z_1 - z_2| \ \text{ for all } \ z_1, z_2 \in \Omega. \tag{2.2.260}$$

Also, since $\Phi$ is a homeomorphism and $\Omega \subseteq \mathbb{C} \setminus \{z_0\}$ it follows that $\widetilde{\Omega} = \Phi(\Omega)$ is an open, connected, simply connected subset of $\mathbb{C} \setminus \{0\}$. Moreover, $\Omega \subseteq \mathbb{C} \setminus \overline{B(z_0, r_0)}$ and since $\Phi$ maps $\mathbb{C} \setminus \overline{B(z_0, r_0)}$ into $B(0, 1/r_0)$ it follows that $\widetilde{\Omega} \subseteq B(0, 1/r_0)$, hence $\widetilde{\Omega}$ is also bounded. The idea is then to check Moore's criterion (cf. (2.2.256)) for $\widetilde{\Omega}$, conclude that $\partial\widetilde{\Omega}$ is a simple curve, then use $\Phi^{-1}$ to reach a similar conclusion for $\partial\Omega$. Since $\Phi^{-1}$ is singular at $0 \in \partial\widetilde{\Omega}$, special care is required when checking the uniform connectivity condition (im kleinen) near the origin. This requires some preparations.

To proceed, fix some large number $R \in (0,\infty)$, to be specified later in the proof. Pick two points $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0,1/R)$ then define $x := \Phi^{-1}(P)$ and $\widetilde{x} := \Phi^{-1}(\widetilde{P})$. It follows that $x, \widetilde{x} \in \Omega \setminus \overline{B(z_0,R)}$. Bring in the polygonal arc $\Gamma$ joining $x$ with $\widetilde{x}$ in $\Omega$ as in Lemma 2.2.2. As noted in Lemma 2.2.3, there exists $\varepsilon = \varepsilon(N) \in (0,1)$ with the property that this curve is disjoint from $B(z_0, \varepsilon R)$. Next, abbreviate $L := \text{length}(\Gamma) \in (0,\infty)$ and let $[0,L] \ni s \mapsto \Gamma(s) \in \Gamma$ be the arc-length parametrization of $\Gamma$. In particular, $|\Gamma'(s)| = 1$ for $\mathcal{L}^1$-a.e. $s \in (0,L)$. If we define

$$\widetilde{\Gamma}(s) := \Phi(\Gamma(s)) = \frac{1}{\Gamma(s) - z_0} \quad \text{for each} \quad s \in [0,L], \tag{2.2.261}$$

then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining $P$ with $\widetilde{P}$ in $\widetilde{\Omega}$. In particular, this curve is a connected subset of $\widetilde{\Omega}$ containing $P, \widetilde{P}$ and, with (2.2.256) in mind, the immediate goal is to estimate the length of this curve. Retaining the symbol $\widetilde{\Gamma}$ for the said curve, we have

$$\begin{aligned} \text{length}(\widetilde{\Gamma}) &= \int_0^L |\widetilde{\Gamma}'(s)|\,ds = \int_0^L |\Phi'(\Gamma(s))| \cdot |\Gamma'(s)|\,ds \\ &= \int_0^L \frac{ds}{|\Gamma(s) - z_0|^2}. \end{aligned} \tag{2.2.262}$$

For each $s \in [0,L]$ we have $\Gamma(s) \in \Omega$. Given that $z_0 \notin \overline{\Omega}$, the line segment joining $\Gamma(s)$ with $z_0$ intersects $\partial\Omega$, hence $|\Gamma(s) - z_0| \geq \delta_{\partial\Omega}(\Gamma(s))$. On the other hand, for each $s \in [0,L]$ the last line in (2.2.18) implies that $C_N \cdot \delta_{\partial\Omega}(\Gamma(s)) \geq \min\{s, L-s\}$. All together, $C_N \cdot |\Gamma(s) - z_0| \geq \min\{s, L-s\}$ for each $s \in [0,L]$. Upon recalling that the polygonal arc $\Gamma$ is disjoint from $B(z_0, \varepsilon R)$, we also have $|\Gamma(s) - z_0| \geq \varepsilon R$ for each $s \in [0,L]$. Ultimately, this proves that there exists some $c_N \in (0,\infty)$ with the property that

$$|\Gamma(s) - z_0| \geq c_N \cdot (R + \min\{s, L-s\}) \quad \text{for each} \quad s \in [0,L]. \tag{2.2.263}$$

Combining (2.2.262) with (2.2.263) then gives

$$\begin{aligned} \text{length}(\widetilde{\Gamma}) &= \int_0^L \frac{ds}{|\Gamma(s) - z_0|^2} \leq C_N \int_0^L \frac{ds}{\big(R + \min\{s, L-s\}\big)^2} \\ &= C_N \int_0^{L/2} \frac{ds}{\big(R + \min\{s, L-s\}\big)^2} + C_N \int_{L/2}^L \frac{ds}{\big(R + \min\{s, L-s\}\big)^2} \\ &= 2C_N \int_0^{L/2} \frac{ds}{(R+s)^2} \leq 2C_N \int_0^\infty \frac{ds}{(R+s)^2} = \frac{2C_N}{R}. \end{aligned} \tag{2.2.264}$$

Armed with (2.2.264), we now proceed to check that the set $\widetilde{\Omega}$ is uniformly connected im kleinen (in the sense made precise in (2.2.256)). To get started, suppose some threshold $\varepsilon_o > 0$ has been given. Make the assumption that

$$R > \max\Big\{r_0\,,\, \frac{2C_N}{\varepsilon_o}\Big\} \quad \text{and pick} \quad \delta_o \in (0, 1/(2R)), \tag{2.2.265}$$

reserving the right to make further specifications regarding the size of $\delta_o$. Consider two points $P, \widetilde{P} \in \widetilde{\Omega}$ with $|P - \widetilde{P}| < \delta_o$. The goal is to find a connected subset of $\widetilde{\Omega}$ whose every point is at distance $\leq \varepsilon_o$ from $P$. To this end, we distinguish two cases.

`Case I`: *Assume* $P, \widetilde{P} \in \widetilde{\Omega} \cap B(0, 1/R)$. Then $\widetilde{\Gamma}$, the curve introduced in (2.2.261), is a connected subset of $\widetilde{\Omega}$ containing $P, \widetilde{P}$, and (2.2.264) implies (in view of (2.2.265)) that $\mathrm{length}(\widetilde{\Gamma}) < \varepsilon_o$. In particular, for any point $Q \in \widetilde{\Gamma}$ we have $|P - Q| \leq \mathrm{length}(\widetilde{\Gamma}) < \varepsilon_o$, as wanted.

`Case II`: *Assume either* $P \notin \widetilde{\Omega} \cap B(0, 1/R)$, *or* $\widetilde{P} \notin \widetilde{\Omega} \cap B(0, 1/R)$. Since we know that $|P - \widetilde{P}| < \delta_o < 1/(2R)$ to begin with, this forces $P, \widetilde{P} \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$. To proceed, observe that the restriction of $\Phi : \Omega \to \widetilde{\Omega}$ to $\Omega \cap B(z_0, 2R)$, i.e., the function

$$\begin{aligned} \widetilde{\Phi} : \Omega \cap B(z_0, 2R) &\longrightarrow \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \\ \widetilde{\Phi}(z) := (z - z_0)^{-1} \quad &\text{for each} \quad z \in \Omega \cap B(z_0, 2R), \end{aligned} \tag{2.2.266}$$

is a bijection, whose inverse

$$\begin{aligned} \widetilde{\Phi}^{-1} : \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))} &\longrightarrow \Omega \cap B(z_0, 2R), \\ \widetilde{\Phi}^{-1}(\zeta) := \zeta^{-1} + z_0 \quad &\text{for each} \quad \zeta \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}, \end{aligned} \tag{2.2.267}$$

is Lipschitz since for each $\zeta_1, \zeta_2 \in \widetilde{\Omega} \setminus \overline{B(0, 1/(2R))}$ we may estimate

$$|\widetilde{\Phi}^{-1}(\zeta_1) - \widetilde{\Phi}^{-1}(\zeta_2)| = \frac{|\zeta_1 - \zeta_2|}{|\zeta_1||\zeta_2|} \leq (2R)^2 |\zeta_1 - \zeta_2|. \tag{2.2.268}$$

In particular, if we set $x := \widetilde{\Phi}^{-1}(P) \in \Omega$ and $\widetilde{x} := \widetilde{\Phi}^{-1}(\widetilde{P}) \in \Omega$, it follows that

$$|x - \widetilde{x}| = |\widetilde{\Phi}^{-1}(P) - \widetilde{\Phi}^{-1}(\widetilde{P})| \leq (2R)^2 |P - \widetilde{P}| \leq (2R)^2 \delta_o. \tag{2.2.269}$$

Let $\Gamma$ be the polygonal arc joining $x$ with $\widetilde{x}$ in $\Omega$ as in Lemma 2.2.2 with the scale $r := |x - \widetilde{x}|$. Then the first inequality in (2.2.18) tells us that $\mathrm{length}(\Gamma) \leq C_N \cdot |x - \widetilde{x}|$, so $L := \mathrm{length}(\Gamma) \leq C_N \cdot (2R)^2 \delta_o$ by (2.2.269). Let $[a, b] \ni t \mapsto \gamma(t) \in \Gamma$ be a parametrization of the curve $\Gamma$ and define $\widetilde{\Gamma} := \Phi \circ \gamma$. Then the image of $\widetilde{\Gamma}$ is a rectifiable curve joining $P$ with $\widetilde{P}$ in $\widetilde{\Omega}$. Indeed, $\Phi(\Gamma) \subseteq \Phi(\Omega) = \widetilde{\Omega}$ and

$$\begin{aligned} \Phi(\gamma(a)) &= \Phi(x) = \Phi(\widetilde{\Phi}^{-1}(P)) = \widetilde{\Phi}(\widetilde{\Phi}^{-1}(P)) = P, \\ \Phi(\gamma(b)) &= \Phi(\widetilde{x}) = \Phi(\widetilde{\Phi}^{-1}(\widetilde{P})) = \widetilde{\Phi}(\widetilde{\Phi}^{-1}(\widetilde{P})) = \widetilde{P}, \end{aligned} \tag{2.2.270}$$

given that $\widetilde{\Phi}^{-1}(P), \widetilde{\Phi}^{-1}(\widetilde{P})$ belong to $\Omega \cap B(z_0, 2R)$ where $\Phi$ agrees with $\widetilde{\Phi}$. Retaining the symbol $\widetilde{\Gamma}$ for the the said curve, we may estimate

$$\mathrm{length}(\widetilde{\Gamma}) \leq r_0^{-2} \cdot \mathrm{length}(\Gamma) = L / r_0^2 \leq C_N \cdot (2R)^2 \delta_o / r_0^2, \tag{2.2.271}$$

where the first inequality is a consequence of (2.2.84) and the fact that $\Phi : \Omega \to \widetilde{\Omega}$ is a Lipschitz function with constant $\leq r_0^{-2}$ (cf. (2.2.260)). Choosing $\delta_o > 0$ sufficiently small, to begin with, so that $C_N \cdot (2R)^2 \delta_o / r_0^2 < \varepsilon_o$, we ultimately conclude that $\mathrm{length}(\widetilde{\Gamma}) < \varepsilon_o$.

Hence, once again, $\widetilde{\Gamma}$ is a connected subset of $\widetilde{\Omega}$ containing $P, \widetilde{P}$, and with the property that $|P-Q| \le \text{length}(\widetilde{\Gamma}) < \varepsilon_o$ for each point $Q \in \widetilde{\Gamma}$.

Let us summarize our progress. In view of (2.2.256), the proof so far gives that

$$\partial\widetilde{\Omega} \text{ a simple closed curve in the plane.} \tag{2.2.272}$$

Moreover, since $\Phi(\partial\Omega) \subseteq \partial\widetilde{\Omega}$, the origin $0 \in \mathbb{C}$ is an accumulation point for $\Phi(\partial\Omega)$ (as is visible from (2.2.259), keeping in mind that $\partial\Omega$ is unbounded), and $\partial\widetilde{\Omega}$ is a closed set, we conclude that $0 \in \partial\widetilde{\Omega}$. In turn, this implies that $\partial\widetilde{\Omega} \setminus \{0\}$ is a simple curve, and that the function (2.2.259) induces a homeomorphism $\Phi : \partial\Omega \to \partial\widetilde{\Omega} \setminus \{0\}$. As a consequence, $\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\})$ is a simple curve in the plane. In addition, the (upper) Ahlfors regularity property of $\partial\Omega$ ensures that the curve $\partial\Omega$ is locally rectifiable, hence

$$\partial\Omega = \Phi^{-1}(\partial\widetilde{\Omega} \setminus \{0\}) \text{ is a locally rectifiable simple curve in the plane.} \tag{2.2.273}$$

Next, if $\widetilde{\gamma} : \big[-\frac{\pi}{2}, \frac{\pi}{2}\big] \to \partial\widetilde{\Omega}$ is a parametrization of $\partial\widetilde{\Omega}$ with $\widetilde{\gamma}(\pm\pi/2) = 0$, then

$$\gamma : \mathbb{R} \to \partial\Omega, \quad \gamma(t) := \Phi^{-1}\big(\widetilde{\gamma}(\arctan t)\big) \text{ for each } t \in \mathbb{R}, \tag{2.2.274}$$

becomes a parametrization of the curve $\partial\Omega$. Given that $\lim\limits_{t\to\pm\infty} |\gamma(t)| = 0$, we ultimately conclude that

$$\partial\Omega \text{ is a Jordan curve passing through infinity in the plane.} \tag{2.2.275}$$

At this stage, there remains to prove that $\partial\Omega$ satisfies the chord-arc condition (2.2.247) with a constant $\varkappa = O(\delta)$ as $\delta \to 0^+$. In this regard, we note that Theorem 2.2.25 gives (cf. (2.2.225) with $n = 2$) that there exists a finite geometrical constant $C_o > 1$, independent of $\delta$, with the property that

$$\left|\frac{\mathcal{H}^1(B(z,r) \cap \partial\Omega)}{2r} - 1\right| \le C_o\,\delta, \qquad \forall\, z \in \partial\Omega, \quad \forall\, r \in (0,\infty). \tag{2.2.276}$$

Without loss of generality, henceforth assume $0 < \delta < 1/(4C_o)$. Consider now two points $z_1, z_2 \in \partial\Omega$. Abbreviate $r := \ell(z_1, z_2)$ and denote by $z_3$ the first exit point of the curve $\partial\Omega$ out of $B(z_1, r)$. Hence, $|z_1 - z_3| = r$ and the ordering $z_1, z_2, z_3$ conforms with the positive orientation of $\partial\Omega$. Moreover,

$$\text{the portion of } \partial\Omega \text{ between } z_1 \text{ and } z_3 \text{ is contained inside } B(z_1, r). \tag{2.2.277}$$

To proceed, introduce $\Delta := B(z_1, r) \cap \partial\Omega$ and decompose $\Delta = \Delta^+ \cup \Delta^-$ (disjoint union), where $\Delta^\pm$ denote the sets of points in $\Delta$ lying, respectively, to the left and to the right of $z_1$. Also, denote by $\ell(\Delta^\pm)$ the arc-lengths of $\Delta^\pm$. Then

$$\mathcal{H}^1(B(z_1, r) \cap \partial\Omega) = \ell(\Delta^-) + \ell(\Delta^+) \quad \text{and} \quad \ell(\Delta^\pm) \ge r. \tag{2.2.278}$$

Making use of (2.2.276) and (2.2.278) we may therefore estimate

$$\begin{aligned} C_o\,\delta &\ge \left|\frac{\mathcal{H}^1(B(z,r) \cap \partial\Omega)}{2r} - 1\right| = \left|\frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r}\right| \\ &= \frac{\ell(\Delta^-) - r}{2r} + \frac{\ell(\Delta^+) - r}{2r} \ge \frac{\ell(\Delta^+) - r}{2r}. \end{aligned} \tag{2.2.279}$$

Hence, by (2.2.277) and (2.2.279),

$$|z_2 - z_3| \leq \ell(\Delta^+) - r \leq 2rC_o\,\delta \tag{2.2.280}$$

which further implies

$$|z_1 - z_2| \geq |z_1 - z_3| - |z_2 - z_3| \geq r - 2rC_o\,\delta = (1 - 2C_o\,\delta)\ell(z_1, z_2). \tag{2.2.281}$$

This proves that

$$\ell(z_1, z_2) \leq (1 + \varkappa)|z_1 - z_2| \quad \text{with} \quad \varkappa := \frac{2C_o\,\delta}{1 - 2C_o\,\delta}, \tag{2.2.282}$$

which goes to show that $\partial\Omega$ is a chord-arc curve. Moreover, the fact that we have assumed $0 < \delta < 1/(4C_o)$ implies $0 < \varkappa < 4C_o\delta$, so in particular $\varkappa = O(\delta)$ as $\delta \to 0^+$. Hence, $\Omega$ is a $\varkappa$-CAD with $\varkappa = O(\delta)$ as $\delta \to 0^+$, finishing the proof of Theorem 2.2.38. $\square$

### 2.2.5 Dyadic grids and Muckenhoupt weights on Ahlfors regular sets

The following result, pertaining to the existence of a dyadic grid structure on a given Ahlfors regular set, is essentially due to M. Christ [22] (cf. also [33], [34]), with some refinements worked out in [51, Proposition 2.11, pp. 19-20].

**Proposition 2.2.39.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed, unbounded, Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then there are finite constants $a_1 \geq a_0 > 0$ such that for each $m \in \mathbb{Z}$ there exists a collection*

$$\mathbb{D}_m(\Sigma) := \{Q^m_\alpha\}_{\alpha \in I_m} \tag{2.2.283}$$

*of subsets of $\Sigma$ indexed by a nonempty, at most countable set of indices $I_m$, as well as a family $\{x^m_\alpha\}_{\alpha\in I_m}$ of points in $\Sigma$, for which the collection of all dyadic cubes in $\Sigma$, i.e.,*

$$\mathbb{D}(\Sigma) := \bigcup_{m\in\mathbb{Z}} \mathbb{D}_m(\Sigma), \tag{2.2.284}$$

*has the following properties:*

*(1)* [All dyadic cubes are open] *For each $m \in \mathbb{Z}$ and each $\alpha \in I_m$ the set $Q^m_\alpha$ is relatively open in $\Sigma$.*

*(2)* [Dyadic cubes are mutually disjoint within the same generation] *For each $m \in \mathbb{Z}$ and each $\alpha, \beta \in I_m$ with $\alpha \neq \beta$ there holds $Q^m_\alpha \cap Q^m_\beta = \varnothing$;*

*(3)* [No partial overlap across generations] *For each $m, \ell \in \mathbb{Z}$ with $\ell > m$ and each $\alpha \in I_m$, $\beta \in I_\ell$, either $Q^\ell_\beta \subseteq Q^m_\alpha$ or $Q^m_\alpha \cap Q^\ell_\beta = \varnothing$.*

*(4)* [Any dyadic cube has a unique ancestor in any earlier generation] *For each integers $m, \ell \in \mathbb{Z}$ with $m > \ell$ and each $\alpha \in I_m$ there is a unique $\beta \in I_\ell$ such that $Q^m_\alpha \subseteq Q^\ell_\beta$. In particular, for each $m \in \mathbb{Z}$ and each $\alpha \in I_m$ there exists a unique $\beta \in I_{m-1}$ such that $Q^m_\alpha \subseteq Q^{m-1}_\beta$ (a scenario in which $Q^{m-1}_\beta$ is referred to as the parent of $Q^m_\alpha$).*

*(5)* [The size is dyadically related to the generation] *For each $m \in \mathbb{Z}$ and each $\alpha \in I_m$ one has*

$$\Delta(x_\alpha^m, a_0 2^{-m}) \subseteq Q_\alpha^m \subseteq \Delta_{Q_\alpha^m} := \Delta(x_\alpha^m, a_1 2^{-m}). \tag{2.2.285}$$

*(6)* [Control of the number of children] *There exists an integer $M \in \mathbb{N}$ with the property that for each $m \in \mathbb{Z}$ and each $\alpha \in I_m$ one has*

$$\#\{\beta \in I_{m+1} : Q_\beta^{m+1} \subseteq Q_\alpha^m\} \leq M. \tag{2.2.286}$$

*Also, this integer may be chosen such that for each $m \in \mathbb{Z}$, each $x \in \Sigma$, and each $r \in (0, 2^{-m})$ the number of $Q$'s in $\mathbb{D}_m(\Sigma)$ that intersect $\Delta(x, r)$ is at most $M$.*

*(7)* [Each generation covers the space $\sigma$-a.e.] *For each $m \in \mathbb{Z}$ one has*

$$\sigma\Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) = 0. \tag{2.2.287}$$

*In particular,*

$$N := \bigcup_{m \in \mathbb{Z}} \Big(\Sigma \setminus \bigcup_{\alpha \in I_m} Q_\alpha^m\Big) \Longrightarrow \sigma(N) = 0, \tag{2.2.288}$$

*and for each $m \in \mathbb{Z}$ and each $\alpha \in I_m$ one has*

$$\sigma\Big(Q_\alpha^m \setminus \bigcup_{\beta \in I_{m+1},\, Q_\beta^{m+1} \subseteq Q_\alpha^m} Q_\beta^{m+1}\Big) = 0. \tag{2.2.289}$$

*(8)* [Dyadic cubes have thin boundaries] *There exist constants, some small $\vartheta \in (0,1)$ along with some large $C \in (0, \infty)$, such that for each $m \in \mathbb{Z}$, each $\alpha \in I_m$, and each $t > 0$ one has*

$$\sigma\Big(\{x \in Q_\alpha^m : \operatorname{dist}(x, \Sigma \setminus Q_\alpha^m) \leq t \cdot 2^{-k}\}\Big) \leq C t^\vartheta \cdot \sigma(Q_\alpha^m). \tag{2.2.290}$$

Moving on, assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. It has been noted in [93] that

> if $\mathcal{H}^{n-1}(K \cap \Sigma) < +\infty$ for each compact subset $K$ of $\mathbb{R}^n$ then $\sigma$ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on $\Sigma$, where the latter set is endowed with the topology canonically inherited from the ambient space. (2.2.291)

Let $w$ be a `weight` on $\Sigma$, i.e., a $\sigma$-measurable function satisfying $0 < w(x) < \infty$ for $\sigma$-a.e. point $x \in \Sigma$. We agree to also use the symbol $w$ for the weighted measure $w\,\sigma$, i.e., define

$$w(E) := \int_E w \, d\sigma \quad \text{for each } \sigma\text{-measurable set } \ E \subseteq \Sigma. \tag{2.2.292}$$

Then the measures $w$ and $\sigma$ have the same sigma-algebra of measurable sets, and are mutually absolutely continuous with each other. Recall that, for a generic measure space $(X, \mu)$, the measure $\mu$ is said to be `semi-finite` if for each $\mu$-measurable set $E \subseteq X$ with $\mu(E) = \infty$ there exists some $\mu$-measurable set $F \subseteq E$ such that $0 < \mu(F) < \infty$ (cf., e.g., [41, p. 25]).

**Lemma 2.2.40.** *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Let $w$ be an arbitrary weight on $\Sigma$ and pick an arbitrary $\sigma$-measurable set $\Delta \subseteq \Sigma$ with $\sigma(\Delta) < \infty$. Then the measure $w\lfloor\Delta$ is semi-finite and, whenever $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, it follows that*

$$\|w^{-1}\|_{L^{p'}(\Delta,w)} = \sup_{\substack{f \in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\, d\sigma. \tag{2.2.293}$$

*Proof.* Consider a $w$-measurable set $E \subseteq \Delta$ with $w(E) = \infty$. In particular, the set $E$ is $\sigma$-measurable. If for each $N \in \mathbb{N}$ we set $E_N := \{x \in E : w(x) < N\}$ then $E_N$ is a $\sigma$-measurable subset of $\Delta$ and $E_N \subseteq E_{N+1}$. Also, $\bigcup_{N\in\mathbb{N}} E_N = \{x \in E : w(x) < \infty\}$ hence $\sigma\big(E \setminus \bigcup_{N\in\mathbb{N}} E_N\big) = 0$. Consequently,

$$\lim_{N\to\infty} w(E_N) = \lim_{N\to\infty} \int_{E_N} w\, d\sigma = \int_E w\, d\sigma = w(E) = \infty, \tag{2.2.294}$$

by Lebesgue's Monotone Convergence Theorem. In turn, (2.2.294) implies that there exists $N_o \in \mathbb{N}$ such that $w(E_{N_o}) > 0$. Since we also have

$$w(E_{N_o}) = \int_{E_{N_o}} w\, d\sigma \leq N_o \cdot \sigma(E_{N_o}) \leq N_o \cdot \sigma(\Delta) < \infty, \tag{2.2.295}$$

we conclude that $E_{N_o}$ is a $w$-measurable subset of $E$ with $0 < w(E_{N_o}) < \infty$. This implies that $w\lfloor\Delta$ is indeed a semi-finite measure.

With an eye on (2.2.293), let $S_{\rm fin}(\Delta, w)$ be the vector space of all complex-valued functions defined on $\Delta$ which may be expressed as $f = \sum_{j=1}^N \lambda_j \mathbf{1}_{E_j}$ where $N \in \mathbb{N}$, each $\lambda_j$ is a complex number, the family $\{E_j\}_{1\leq j\leq N}$ consists of $w$-measurable mutually disjoint subset of $\Delta$, and $w\big(\bigcup_{j=1}^N E_j\big) < +\infty$. Note that each such function $f$ happens to be $\sigma$-measurable and, for each $q \in (0, \infty)$, satisfies $\int_\Delta |f|^q \leq \sum_{j=1}^N |\lambda_j|^q \cdot \sigma(\Delta) < \infty$. Hence,

$$S_{\rm fin}(\Delta, w) \subseteq \bigcap_{0<q<\infty} L^q(\Delta, \sigma) \tag{2.2.296}$$

and, in particular,

$$fw^{-1} \in L^1(\Delta, w) \quad \text{for each} \quad f \in S_{\rm fin}(\Delta, w). \tag{2.2.297}$$

Having picked $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$, we may then write

$$\begin{aligned} \|w^{-1}\|_{L^{p'}(\Delta,w)} &= \sup_{\substack{f \in S_{\rm fin}(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big|\int_\Delta f w^{-1}\, dw\Big| = \sup_{\substack{f \in S_{\rm fin}(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \Big|\int_\Delta f\, d\sigma\Big| \\ &\leq \sup_{\substack{f \in L^p(\Delta,w)\\ \|f\|_{L^p(\Delta,w)}=1}} \int_\Delta |f|\, d\sigma. \end{aligned} \tag{2.2.298}$$

The first equality above is a consequence of [41, Theorem 6.14, p. 189], whose applicability in the present setting is ensured by (2.2.297) and the fact that the measure $w\lfloor\Delta$ is semi-finite. The second equality in (2.2.298) is justified upon recalling that $dw = w d\sigma$, and the inequality in (2.2.298) is trivial. There remains to observe that for each $f \in L^p(\Delta, w)$ with $\|f\|_{L^p(\Delta,w)} = 1$ Hölder's inequality gives

$$\int_{\Delta} |f|\, d\sigma = \int_{\Delta} |f| w^{-1}\, dw \leq \|w^{-1}\|_{L^{p'}(\Delta,w)}. \tag{2.2.299}$$

At this stage, (2.2.293) becomes a consequence of (2.2.298) and (2.2.299). □

Next, assume that $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (1, \infty)$, we say that a weight $w$ on $\Sigma$ belongs to the Muckenhoupt class $A_p(\Sigma, \sigma)$ if

$$[w]_{A_p} := \sup_{\Delta \subseteq \Sigma} \Big( \fint_{\Delta} w(x)\, d\sigma(x) \Big) \Big( \fint_{\Delta} w(x)^{1-p'}\, d\sigma(x) \Big)^{p-1} < \infty, \tag{2.2.300}$$

where $p'$ is the Hölder conjugate exponent of $p$ (i.e., $p' \in (1, \infty)$ satisfies $1/p + 1/p' = 1$) and the supremum runs over all surface balls $\Delta$ in $\Sigma$. Corresponding to $p = 1$, we say that $w \in A_1(\Sigma, \sigma)$ if

$$[w]_{A_1} := \sup_{\Delta \subseteq \Sigma} \Big( \operatorname*{ess\,inf}_{x \in \Delta} w(x) \Big)^{-1} \Big( \fint_{\Delta} w\, d\sigma \Big) < \infty. \tag{2.2.301}$$

Recall that the (non-centered) Hardy-Littlewood maximal operator $M$ on $\Sigma$ acts on each $\sigma$-measurable function $f$ on $\Sigma$ according to

$$Mf(x) := \sup_{\Delta \ni x} \fint_{\Delta} |f|\, d\sigma, \qquad \forall\, x \in \Sigma, \tag{2.2.302}$$

where the supremum is taken over all surface balls $\Delta$ in $\Sigma$ which contain the point $x$. In particular, a weight $w$ on $\Sigma$ belongs to $A_1(\Sigma, \sigma)$ if and only if there exists a constant $C \in (0, \infty)$ with the property that $Mw(x) \leq Cw(x)$ at $\sigma$-a.e. point $x \in \Sigma$, and the best constant is actually $[w]_{A_1}$. Corresponding to the end-point $p = \infty$,

$$\begin{array}{l}\text{the class } A_\infty(\Sigma, \sigma) \text{ is defined as the} \\ \text{union of all } A_p(\Sigma, \sigma) \text{ with } p \in [1, \infty).\end{array} \tag{2.2.303}$$

**Lemma 2.2.41.** *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $p \in (1, \infty)$, each Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$, and each $\sigma$-measurable function $f$ on $\Sigma$ one has*

$$\begin{array}{c}\displaystyle \fint_{\Delta} |f|\, d\sigma \leq [w]_{A_p}^{1/p} \Big( \fint_{\Delta} |f|^p\, dw \Big)^{1/p}, \\ \text{for each surface ball } \Delta \subseteq \Sigma.\end{array} \tag{2.2.304}$$

*Conversely, if $p \in (1, \infty)$ and $w$ is a weight on $\Sigma$ with the property that there exists a constant $C \in (0, \infty)$ such that*

$$\begin{array}{c}\displaystyle \fint_{\Delta} |f|\, d\sigma \leq C \Big( \fint_{\Delta} |f|^p\, dw \Big)^{1/p} \text{ for each} \\ \text{function } f \in L^p_{\text{loc}}(\Sigma, w) \text{ and surface ball } \Delta \subseteq \Sigma,\end{array} \tag{2.2.305}$$

*then actually $w \in A_p(\Sigma, \sigma)$ and $[w]_{A_p} \leq C^p$.*

*Proof.* Let $p' \in (1,\infty)$ denote the Hölder conjugate exponent of $p$ and fix an arbitrary $\sigma$-measurable function $f$ on $\Sigma$. Then for each surface ball $\Delta \subseteq \Sigma$ we may estimate

$$
\begin{aligned}
\fint_{\Delta} |f|\, d\sigma &= \frac{1}{\sigma(\Delta)} \int_{\Delta} |f| w^{1/p} w^{-1/p}\, d\sigma \\
&\leq \frac{1}{\sigma(\Delta)} \Big( \int_{\Delta} |f|^p\, w\, d\sigma \Big)^{1/p} \Big( \int_{\Delta} w^{-p'/p}\, d\sigma \Big)^{1/p'} \\
&= \Big( \fint_{\Delta} w^{1-p'}\, d\sigma \Big)^{1/p'} \Big( \fint_{\Delta} w\, d\sigma \Big)^{1/p} \Big( \fint_{\Delta} |f|^p\, dw \Big)^{1/p} \\
&\leq [w]_{A_p}^{1/p} \Big( \fint_{\Delta} |f|^p\, dw \Big)^{1/p},
\end{aligned}
\tag{2.2.306}
$$

by Hölder's inequality and (2.2.300). This proves (2.2.304).

As for the converse, fix $p \in (1,\infty)$ and suppose $w$ is a generic weight function on $\Sigma$ for which there exists a constant $C \in (0,\infty)$ such that (2.2.305) holds. Once again, denote $p' \in (1,\infty)$ the Hölder conjugate exponent of $p$ and fix an arbitrary surface ball $\Delta \subseteq \Sigma$. Then, with tilde denoting the extension by zero of a function originally defined on $\Delta$ to the entire set $\Sigma$, we may write

$$
\begin{aligned}
\|w^{-1}\|_{L^{p'}(\Delta,w)} &= \sup_{\substack{f \in L^p(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \int_{\Delta} |f|\, d\sigma = \sigma(\Delta) \cdot \sup_{\substack{f \in L^p(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \fint_{\Delta} |\widetilde{f}\,|\, d\sigma \\
&\leq C\sigma(\Delta) \cdot \sup_{\substack{f \in L^p(\Delta,w) \\ \|f\|_{L^p(\Delta,w)}=1}} \Big( \fint_{\Delta} |\widetilde{f}\,|^p\, dw \Big)^{1/p} \leq C \frac{\sigma(\Delta)}{w(\Delta)^{1/p}},
\end{aligned}
\tag{2.2.307}
$$

where the first equality comes from Lemma 2.2.40, and the first inequality is implied by (2.2.305). This proves that $\|w^{-1}\|_{L^{p'}(\Delta,w)} \leq C \cdot \sigma(\Delta)/w(\Delta)^{1/p}$ which, after unraveling notation, yields

$$
\Big( \fint_{\Delta} w\, d\sigma \Big) \Big( \fint_{\Delta} w^{1-p'}\, d\sigma \Big)^{p-1} \leq C^p. \tag{2.2.308}
$$

Ultimately, in view of the arbitrariness of the surface ball $\Delta \subseteq \Sigma$, this implies that $w \in A_p(\Sigma,\sigma)$ and $[w]_{A_p} \leq C^p$. $\square$

In this work we are particularly interested in the scale of weighted Lebesgue space $L^p(\Sigma,w) := L^p(\Sigma,\, w\sigma)$ with $p \in (1,\infty)$ and $w \in A_p(\Sigma,\sigma)$. As in the Euclidean setting, given a weight $w$ on $\Sigma$ along with an integrability exponent $p \in (1,\infty)$, the Hardy-Littlewood maximal operator $M$ is bounded on $L^p(\Sigma,w)$ if and only if $w \in A_p(\Sigma,\sigma)$, in which case there exists $C = C(p,\Sigma) \in (0,\infty)$ with the property that

$$
\|Mf\|_{L^p(\Sigma,w)} \leq C [w]_{A_p}^{1/(p-1)} \|f\|_{L^p(\Sigma,w)} \quad \text{for each} \quad f \in L^p(\Sigma,w) \tag{2.2.309}
$$

(see, e.g., [55, Proposition 7.13]). Also, corresponding to $p=1$, the operator $M$ satisfies the weak-$(1,1)$ inequality

$$
\begin{gathered}
\sup\nolimits_{0<\lambda<\infty} \lambda \cdot w\big(\{x \in \Sigma :\, Mf(x) > \lambda\}\big) \leq C\|f\|_{L^1(\Sigma,w)} \\
\text{for all } f \in L^1(\Sigma,w), \text{ with } C \in (0,\infty) \text{ independent of } f,
\end{gathered}
\tag{2.2.310}
$$

if and only if $w \in A_1(\Sigma, \sigma)$. For the reader's convenience, other useful properties of Muckenhoupt weights are summarized in the proposition below (for a more extensive discussion pertaining to the theory of weights in the general context of spaces of homogeneous type the reader is referred to [4], [44], [56], [66], [118]).

**Proposition 2.2.42.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then the following properties hold.*

*(1) [Openness/Self-Improving] If $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ then there exist some $\tau \in (1, \infty)$ and some $\varepsilon \in (0, p-1)$ (both of which depend only on $p$, $[w]_{A_p}$, $n$, and the Ahlfors regularity constant of $\Sigma$) such that*

$$w^\tau \in A_p(\Sigma, \sigma) \quad \text{and} \quad w \in A_{p-\varepsilon}(\Sigma, \sigma). \tag{2.2.311}$$

*In addition, both $[w^\tau]_{A_p}$ and $[w]_{A_{p-\varepsilon}}$ are controlled in terms of $p$, $[w]_{A_p}$, $n$, and the Ahlfors regularity constant of $\Sigma$. In fact, matters may be arranged so that, in a quantitative fashion,*

$$w^\theta \in A_q(\Sigma, \sigma) \quad \text{for each} \quad \theta \in (\tau^{-1}, \tau) \quad \text{and} \quad q \in (p-\varepsilon, \infty). \tag{2.2.312}$$

*(2) [Monotonicity] If $1 \leq p \leq q \leq \infty$ then $A_p(\Sigma, \sigma) \subseteq A_q(\Sigma, \sigma)$ and $[w]_{A_q} \leq [w]_{A_p}$ for each $w \in A_p(\Sigma, \sigma)$.*

*(3) [Dual Weights] Given any $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$, it follows that $w^{1-p'}$ belongs to $A_{p'}(\Sigma, \sigma)$ and $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$, where $p' \in (1, \infty)$ is the Hölder conjugate exponent of $p$.*

*(4) [Products/Factorization] If $w_1, w_2 \in A_1(\Sigma, \sigma)$ then for every exponent $p \in (1, \infty)$ one has $w_1 \cdot w_2^{1-p} \in A_p(\Sigma, \sigma)$ and $[w_1 \cdot w_2^{1-p}]_{A_p} \leq [w_1]_{A_1} \cdot [w_2]_{A_1}^{p-1}$. Also, given any two weights $w_1, w_2 \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ along with some $\alpha \in [0, 1]$, it follows that $w_1^\alpha \cdot w_2^{1-\alpha} \in A_p(\Sigma, \sigma)$ and $[w_1^\alpha \cdot w_2^{1-\alpha}]_{A_p} \leq [w_1]_{A_p}^\alpha \cdot [w_2]_{A_p}^{1-\alpha}$.*

*(5) [Doubling] If $w \in A_p(\Sigma, \sigma)$ with $p \in (1, \infty)$ then for every surface ball $\Delta \subseteq \Sigma$ and every $\sigma$-measurable set $E \subseteq \Delta$ one has*

$$\left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^p \leq [w]_{A_p} \cdot \frac{w(E)}{w(\Delta)} \tag{2.2.313}$$

*In particular, the measure $w$ is doubling, that is, there exists some $C \in (0, \infty)$ which depends only on $p$, $n$, and the Ahlfors regularity constant of $\Sigma$, such that $w(2\Delta) \leq C[w]_{A_p} \cdot w(\Delta)$ for every surface ball $\Delta \subseteq \Sigma$. More generally, with the constant $C \in (0, \infty)$ of the same nature as above, one has the inequality $w(\lambda\Delta) \leq C[w]_{A_p} \cdot \lambda^{p(n-1)} \cdot w(\Delta)$ for each $\lambda \in (1, \infty)$ and each surface ball $\Delta \subseteq \Sigma$ (where, as in the past, $\lambda\Delta$ denotes the concentric dilate of $\Delta$ by a factor of $\lambda$).*

*(6) [Reverse Hölder Inequalities] For every $w \in A_\infty(\Sigma, \sigma)$ there exist some $q \in (1, \infty)$ and some $C \in (0, \infty)$ (which both depend only on $p$, $[w]_{A_p}$, $n$, and the Ahlfors*

*regularity constant of* $\Sigma$*, for some* $p \in (1, \infty)$ *for which* $w \in A_p(\Sigma, \sigma)$*) such that*

$$\left(⨍_\Delta w^q \, d\sigma\right)^{1/q} \leq C ⨍_\Delta w \, d\sigma, \tag{2.2.314}$$

*for every surface ball* $\Delta \subseteq \Sigma$*. Consequently, there exist some power* $\tau > 0$ *and some constant* $C \in (0, \infty)$ *(in fact,* $C$ *is the same as in* (2.2.314) *and* $\tau = 1/q'$ *where* $q'$ *is the Hölder conjugate of the exponent* $q$ *from* (2.2.314)*) such that*

$$\frac{w(E)}{w(\Delta)} \leq C \left(\frac{\sigma(E)}{\sigma(\Delta)}\right)^\tau \tag{2.2.315}$$

*for every surface ball* $\Delta \subseteq \Sigma$ *and every* $\sigma$*-measurable set* $E \subseteq \Delta$*. Another useful consequence of* (2.2.314) *and Hölder's inequality is that for each* $\sigma$*-measurable function* $f$ *on* $\Sigma$ *and each surface ball* $\Delta \subseteq \Sigma$ *one has*

$$⨍_\Delta |f| \, dw \leq C \left(⨍_\Delta |f|^{q'} \, d\sigma\right)^{1/q'} \tag{2.2.316}$$

*where* $q' \in (1, \infty)$ *is the Hölder conjugate exponent of* $q$*, and* $C \in (0, \infty)$ *is as in* (2.2.314)*.*

*(7)* *[Building* $A_1$ *Weights] There exists* $C \in (0, \infty)$ *which depends only on* $n$ *and* $\Sigma$*, with the property that if* $f \in L^1_{\rm loc}(\Sigma, \sigma)$ *is not identically zero and satisfies* $Mf < \infty$ *at* $\sigma$*-a.e. point on* $\Sigma$ *then for each* $\theta \in (0, 1)$ *one has* $(Mf)^\theta \in A_1(\Sigma, \sigma)$ *and* $\left[(Mf)^\theta\right]_{A_1} \leq C(1 - \theta)^{-1}$*.*

*(8)* *[*BMO *and Weights] For each* $p \in (1, \infty)$ *and* $w \in A_p(\Sigma, \sigma)$ *there exist some small* $\varepsilon = \varepsilon(\Sigma, p, [w]_{A_p}) > 0$ *and some large* $C = C(\Sigma, p, [w]_{A_p}) \in (0, \infty)$ *such that for each* $b \in \mathrm{BMO}(\Sigma, \sigma)$ *with* $\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} < \varepsilon$ *one has* $w \cdot e^b \in A_p(\Sigma, \sigma)$ *and* $\left[w \cdot e^b\right]_{A_p} \leq C$*. In particular, for each fixed integrability exponent* $p \in (1, \infty)$ *the set* $\mathcal{U}_p := \left\{b \in \mathrm{BMO}(\Sigma, \sigma) : e^b \in A_p(\Sigma, \sigma)\right\}$ *is open in* $\mathrm{BMO}(\Sigma, \sigma)$*. Also, for each weight* $w \in A_1(\Sigma, \sigma)$*, the function* $\log w$ *belongs to the space* $\mathrm{BMO}(\Sigma, \sigma)$ *and one has* $\|\log w\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq C(\Sigma, n, [w]_{A_1})$*. Finally, for each* $b \in \mathrm{BMO}(\Sigma, \sigma)$ *and each* $p \in (1, \infty)$ *it follows that the function* $\max\{1, |b|\}$ *belongs to* $A_p(\Sigma, \sigma)$ *and there exists some constant* $C_{\Sigma,p} \in (0, \infty)$ *which is independent of* $b$ *with the property that* $[\max\{1, |b|\}]_{A_p} \leq C_{\Sigma,p}(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)})$*.*

*(9)* *[Dyadic Cubes] If* $\Sigma$ *is unbounded, then properties* (2.2.313)*,* (2.2.314)*, and* (2.2.315) *also hold if surface balls* $\Delta$ *are replaced by dyadic "cubes", as described in Proposition 2.2.39.*

*Proof.* For the memberships in (2.2.311), (2.2.312) (including their quantitative aspects) see [56, Theorems 1.1-1.2], [16, Theorem 2.31, p. 58]. The claims in items *(2)-(4)* may be justified straight from definitions, much as in the Euclidean setting (cf., e.g., [42]). The estimate in (2.2.313) may be seen from Lemma 2.2.41, used here with $f := \mathbf{1}_E$. In concert with the Ahlfors regularity of $\Sigma$, this implies all subsequent claims in item *(5)*.

The reverse Hölder inequality claimed in (2.2.314) is contained in [56, Theorem 2.3], [118, Theorem 15, p. 9]. Moreover, if $q'$ is the Hölder conjugate of the exponent $q$ from (2.2.314) then for every surface ball $\Delta \subseteq \Sigma$ and every $\sigma$-measurable set $E \subseteq \Delta$ we may estimate

$$\begin{aligned} \frac{w(E)}{w(\Delta)} &= ⨍_{\Delta} \mathbf{1}_E \, dw = \frac{\sigma(\Delta)}{w(\Delta)} ⨍_{\Delta} \mathbf{1}_E \, w \, d\sigma \\ &\leq \frac{\sigma(\Delta)}{w(\Delta)} \Big( ⨍_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big( ⨍_{\Delta} w^q \, d\sigma \Big)^{1/q} \\ &\leq C \frac{\sigma(\Delta)}{w(\Delta)} \Big( ⨍_{\Delta} \mathbf{1}_E \, d\sigma \Big)^{1/q'} \Big( ⨍_{\Delta} w \, d\sigma \Big) = C \left( \frac{\sigma(E)}{\sigma(\Delta)} \right)^{1/q'}, \end{aligned} \tag{2.2.317}$$

thanks to Hölder's inequality and (2.2.314). This establishes (2.2.315) with $\tau := 1/q' > 0$ and $C \in (0, \infty)$ the same constant as in (2.2.314).

Consider next the claim made in item *(7)*. Suppose $f \in L^1_{\rm loc}(\Sigma, \sigma)$ is not identically zero and has the property that $Mf < \infty$ at $\sigma$-a.e. point on $\Sigma$. Fix a surface ball $\Delta \subseteq \Sigma$ and decompose $f = f_1 + f_2$ with $f_1 := f \mathbf{1}_{2\Delta}$ and $f_2 := f \mathbf{1}_{\Sigma \setminus 2\Delta}$. The fact that $Mf < \infty$ at $\sigma$-a.e. point on $\Sigma$ then entails $f_1 \in L^1(\Sigma, \sigma)$. Since $0 < \theta < 1$ and $0 \leq Mf \leq Mf_1 + Mf_2$, we conclude that

$$(Mf)^\theta \leq (Mf_1)^\theta + (Mf_2)^\theta \quad \text{on} \quad \Sigma. \tag{2.2.318}$$

Based on Kolmogorov's inequality, the fact that $M$ satisfies the weak-$(1,1)$ inequality, the membership of $f_1$ to $L^1(\Sigma, \sigma)$, and the fact that the measure $\sigma$ is doubling we may estimate

$$\begin{aligned} \Big( ⨍_{\Delta} |Mf_1|^\theta \, d\sigma \Big)^{1/\theta} &\leq \Big( \frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|Mf_1\|_{L^{1,\infty}(\Sigma,\sigma)} \\ &\leq C \Big( \frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \sigma(\Delta)^{-1} \|f_1\|_{L^1(\Sigma,\sigma)} \\ &\leq C \Big( \frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} ⨍_{2\Delta} |f| \, d\sigma \\ &\leq C \Big( \frac{1}{1-\theta} \Big)^{\frac{1}{\theta}} \inf_{x \in 2\Delta} (Mf)(x). \end{aligned} \tag{2.2.319}$$

Hence, on the one hand,

$$⨍_{\Delta} |Mf_1|^\theta \, d\sigma \leq \frac{C}{1-\theta} \Big( \inf_{x \in 2\Delta} (Mf)(x) \Big)^\theta. \tag{2.2.320}$$

On the other hand, the fact that

$$\begin{gathered} \text{for each surface ball } \Delta' \subseteq \Sigma \text{ such that } \Delta' \cap \Delta \neq \varnothing \\ \text{and } \Delta' \cap (\Sigma \setminus 2\Delta) \neq \varnothing \text{ it follows that } \Delta \subseteq 6\Delta' \end{gathered} \tag{2.2.321}$$

readily implies that there exists a geometric constant $C \in (0, \infty)$ with the property that

$$(Mf_2)(y) \leq C (Mf_2)(x) \quad \text{for each} \quad x, y \in \Delta. \tag{2.2.322}$$

In turn, this forces

$$\fint_{\Delta} |Mf_2|^{\theta}\,d\sigma \leq C\Big(\inf_{x\in\Delta}(Mf_2)(x)\Big)^{\theta} \leq C\Big(\inf_{x\in\Delta}(Mf)(x)\Big)^{\theta} \tag{2.2.323}$$

which, in concert with (2.2.320) and (2.2.318) proves that

$$\fint_{\Delta} |Mf|^{\theta}\,d\sigma \leq \frac{C}{1-\theta}\cdot\inf_{x\in\Delta}[(Mf)(x)]^{\theta}. \tag{2.2.324}$$

Since $0 < [(Mf)(x)]^{\theta} < \infty$ for $\sigma$-a.e. point $x \in \Sigma$, ultimately (2.2.324) goes to show that $(Mf)^{\theta} \in A_1(\Sigma,\sigma)$ and $\big[(Mf)^{\theta}\big]_{A_1} \leq C(1-\theta)^{-1}$.

For the first two claims in item *(8)* see [59, p. 33 and p. 60] for a proof in the Euclidean ambient which readily adapts to the present setting, given the availability of a John-Nirenberg inequality for doubling measures (see the discussion pertaining to (2.2.33)-(2.2.35)) and the results in the current items *(1)-(6)*. For the third claim in item *(8)* see [42, Theorem 3.3, p. 157] for a proof in the Euclidean space which goes through in the present setting as well. We may justify the very last claim in item *(8)* by arguing along the lines of the proof of [47, Lemma 1.12, p. 471]. Specifically, given $b \in \mathrm{BMO}(\Sigma,\sigma)$ set $w := \max\{1,|b|\}$ and fix some $p \in (1,\infty)$. Then for an arbitrary surface ball $\Delta$ in $\Sigma$ we may write

$$\begin{aligned}
&\Big(\fint_{\Delta} w\,d\sigma\Big)\Big(\fint_{\Delta} w^{-\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\qquad\leq \Big(\fint_{\Delta}[1+|b-b_{\Delta}|]\,d\sigma\Big)\Big(\fint_{\Delta}\Big(\frac{1}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\qquad\quad + |b_{\Delta}|\Big(\fint_{\Delta}\Big(\frac{1}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\qquad\leq 1+\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}+\Big(\fint_{\Delta}\Big(\frac{|b_{\Delta}|}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}.
\end{aligned} \tag{2.2.325}$$

Also, if $E_0 := \{x \in \Delta : |b(x)| > |b_{\Delta}|/2\}$ and $E_1 := \{x \in \Delta : |b(x)| \leq |b_{\Delta}|/2\}$, then for each $x \in E_0$ we have $|b_{\Delta}|/|b(x)| \leq 2$ while for each $x \in E_1$ we have $|b_{\Delta}| \leq 2|b(x) - b_{\Delta}|$. Consequently,

$$\begin{aligned}
\Big(\fint_{\Delta}\Big(\frac{|b_{\Delta}|}{\max\{1,|b|\}}\Big)^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1} &\leq \max\{1,2^{p-2}\}\cdot\Big(\frac{1}{\sigma(\Delta)}\int_{E_0}\Big(\frac{|b_{\Delta}|}{|b|}\Big)^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\quad + \max\{1,2^{p-2}\}\cdot\Big(\frac{1}{\sigma(\Delta)}\int_{E_1}|b_{\Delta}|^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\leq \max\{2,2^{p-1}\}\cdot\Big(\frac{\sigma(E_0)}{\sigma(\Delta)}\Big)^{p-1}\\
&\quad + \max\{2,2^{p-1}\}\cdot\Big(\fint_{\Delta}|b-b_{\Delta}|^{\frac{1}{p-1}}\,d\sigma\Big)^{p-1}\\
&\leq C_{\Sigma,p}\big(1+\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\big),
\end{aligned} \tag{2.2.326}$$

where the last step uses the John-Nirenberg inequality. In view of the arbitrariness of $\Delta$, from (2.2.325)-(2.2.326) we conclude that $w \in A_p(\Sigma, \sigma)$ and $[w]_{A_p} \leq C_{\Sigma,p}(1 + \|b\|_{\mathrm{BMO}(\Sigma,\sigma)})$ for some constant $C_{\Sigma,p} \in (0, \infty)$ which is independent of $b$. This takes care of the very last claim item *(8)*. Finally, the claim in item *(9)* is a consequence of (2.2.285) and doubling. $\square$

Given that the class of Muckenhoupt weights is going to play a prominent role in this work, it is appropriate to include some relevant concrete examples of interest.

**Example 2.2.43.** Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some $p \in (1, \infty)$ along with an arbitrary point $x_0 \in \Sigma$. Then for each power $a \in \big(1 - n, (p - 1)(n - 1)\big)$ the function

$$w : \Sigma \to [0, \infty], \quad w(x) := |x - x_0|^a \ \text{ for each } \ x \in \Sigma, \tag{2.2.327}$$

is a Muckenhoupt weight in the class $A_p(\Sigma, \sigma)$. Furthermore, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of $\Sigma$, $p$, and $a$.

See, for example, [44, Proposition 1.5.9, p. 42]. More generally, the following result appears in [93].

**Proposition 2.2.44.** *Assume $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $d \in [0, n - 1)$ and consider a $d$-set $E \subseteq \Sigma$, i.e., a closed subset $E$ of $\Sigma$ with the property that there exists some Borel outer-measure $\mu$ on $E$ satisfying*

$$\mu(B(x, r) \cap E) \approx r^d, \quad \textit{uniformly for } x \in E \textit{ and } r \in \big(0\,,\, 2\,\mathrm{diam}\,(E)\big). \tag{2.2.328}$$

*Then for each $p \in (1, \infty)$ and $a \in \big(d + 1 - n\,,\, (p - 1)(n - 1 - d)\big)$ the function $w := \big[\mathrm{dist}\,(\cdot, E)\big]^a$ is a Muckenhoupt weight in the class $A_p(\Sigma, \sigma)$. Moreover, $[w]_{A_p}$ depends only on the Ahlfors regularity constant of $\Sigma$, the proportionality constants in (2.2.328), $d$, $p$, and $a$.*

We continue to explore properties of Muckenhoupt weights in the context of Ahlfors regular sets which are relevant for this work.

**Lemma 2.2.45.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then for each $w \in A_\infty(\Sigma, \sigma)$ one has*

$$\mathrm{BMO}(\Sigma, \sigma) \subseteq L^1_{\mathrm{loc}}(\Sigma, w). \tag{2.2.329}$$

*Proof.* This is a direct consequence of (2.2.303), item *(2)* in Proposition 2.2.42, (2.2.316), and (2.2.37). $\square$

If $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, then for each weight function $w$ on $\Sigma$ we have $L^\infty(\Sigma, \sigma) = L^\infty(\Sigma, w)$, i.e., these vector spaces coincide and they have identical norms. Remarkably, whenever $w \in A_\infty(\Sigma, \sigma)$ it follows that the BMO spaces on $\Sigma$ with respect to $\sigma$ and $w$ are once again identical. Here is a formal statement of this fact.

**Lemma 2.2.46.** *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some weight $w \in A_\infty(\Sigma,\sigma)$ (hence, there exists some $p \in (1,\infty)$ for which $w \in A_p(\Sigma,\sigma)$). Then there exists a constant $C \in [1,\infty)$ which depends only on $p$, $[w]_{A_p}$, $n$, and the Ahlfors regularity constant of $\Sigma$ such that*

$$C^{-1}\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq \|f\|_{\mathrm{BMO}(\Sigma,w)} \leq C\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \tag{2.2.330}$$

*for each function $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)\cap L^1_{\mathrm{loc}}(\Sigma,w)$.*

*Moreover, for each $\sigma$-measurable function $f$ on $\Sigma$ one has the equivalence*

$$f \in \mathrm{BMO}(\Sigma,\sigma) \Longleftrightarrow f \in \mathrm{BMO}(\Sigma,w) \tag{2.2.331}$$

*and if either of these memberships materializes then $\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} \approx \|f\|_{\mathrm{BMO}(\Sigma,w)}$ where the implicit proportionality constants depend only on $p$, $[w]_{A_p}$, $n$, and the Ahlfors regularity constant of $\Sigma$. Succinctly put,*

$$\begin{gathered}\textit{the spaces } \mathrm{BMO}(\Sigma,\sigma) \textit{ and } \mathrm{BMO}(\Sigma,w) \textit{ coincide as sets}\\ \textit{and have equivalent semi-norms.}\end{gathered} \tag{2.2.332}$$

*Proof.* Pick a function $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)\cap L^1_{\mathrm{loc}}(\Sigma,w)$. To prove the first inequality in (2.2.330), start by writing (2.2.304) with $f$ replaced by $f - ⨍_\Delta f\,dw$ for some arbitrary surface ball $\Delta \subseteq \Sigma$, then invoke (2.2.43) to obtain

$$\begin{aligned}\|f\|_{\mathrm{BMO}(\Sigma,\sigma)} &\leq 2\sup_{\Delta\subseteq\Sigma}\inf_{c\in\mathbb{R}}\Big(⨍_\Delta |f-c|\,d\sigma\Big) \leq 2\sup_{\Delta\subseteq\Sigma}⨍_\Delta \Big|f - ⨍_\Delta f\,dw\Big|\,d\sigma\\ &\leq 2[w]_{A_p}^{1/p}\cdot\sup_{\Delta\subseteq\Sigma}\Big(⨍_\Delta \Big|f-⨍_\Delta f\,dw\Big|^p\,dw\Big)^{1/p}\\ &\leq C\,\|f\|_{\mathrm{BMO}(\Sigma,w)},\end{aligned} \tag{2.2.333}$$

for some constant $C \in (0,\infty)$ as in the statement. To prove the second inequality in (2.2.330), observe first that $w$ belongs to some Reverse Hölder class, say $w \in \mathrm{RH}_q(\Sigma,\sigma)$ for some $q \in (1,\infty)$. if $q' \in (1,\infty)$ denoting the Hölder conjugate exponent of $q$, then (2.2.316) allows to estimate

$$\begin{aligned}\inf_{c\in\mathbb{R}}\Big(⨍_\Delta |f-c|\,dw\Big) &\leq ⨍_\Delta\Big|f-⨍_\Delta f\,d\sigma\Big|\,dw\\ &\leq C\Big(⨍_\Delta\Big|f-⨍_\Delta f\,d\sigma\Big|^{q'}\,d\sigma\Big)^{1/q'},\end{aligned} \tag{2.2.334}$$

for some constant $C \in (0,\infty)$ of the same nature as before. Taking the supremum over all surface balls $\Delta \subseteq \Sigma$ and then using John-Nirenberg's inequality, we ultimately obtain $\|f\|_{\mathrm{BMO}(\Sigma,w)} \leq C\,\|f\|_{\mathrm{BMO}(\Sigma,\sigma)}$, as desired.

As regards the equivalence in (2.2.331), assume first that $f \in \mathrm{BMO}(\Sigma,\sigma)$. Then (2.2.329) implies that $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)\cap L^1_{\mathrm{loc}}(\Sigma,w)$, so (2.2.330) holds. Conversely, assume the function $f$ belongs to $\mathrm{BMO}(\Sigma,w)$. In particular, $f \in L^1_{\mathrm{loc}}(\Sigma,w)$ and the John-Nirenberg inequality (for the doubling measure $w$) guarantees that we also have $f \in L^p_{\mathrm{loc}}(\Sigma,w)$. In concert with (2.2.304) the latter membership implies that $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, hence once again (2.2.330) applies. $\square$

The doubling and self-improving properties of Muckenhoupt weights yield the following result (see [93] for a proof).

**Lemma 2.2.47.** *Suppose $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. In this setting, fix some $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$. Then*

$$\int_\Sigma \frac{w(x)}{(1+|x|^{n-1})^p}\, d\sigma(x) < +\infty. \tag{2.2.335}$$

*Also, there exists an exponent $p_o \in (1, p]$ with the property that*

$$\begin{gathered} L^p(\Sigma, w) \hookrightarrow L^q\Big(\Sigma\,,\, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \\ \textit{continuously, for each fixed } q \in (0, p_o). \end{gathered} \tag{2.2.336}$$

*As a consequence,*

$$L^p(\Sigma, w) \hookrightarrow L^1\Big(\Sigma\,,\, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \quad \textit{continuously}, \tag{2.2.337}$$

*and*

$$L^p(\Sigma, w) \subseteq \bigcup_{1<q<p} L^q_{\rm loc}(\Sigma, \sigma). \tag{2.2.338}$$

### 2.2.6 Sobolev spaces on Ahlfors regular sets

Consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, (2.2.291) implies that

$$\begin{gathered} \sigma \text{ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on } \partial\Omega, \text{ where the latter set is} \\ \text{endowed with the topology canonically inherited from } \mathbb{R}^n. \end{gathered} \tag{2.2.339}$$

Among other things, this implies (cf. [93]) that for every $f \in L^1_{\rm loc}(\partial\Omega, \sigma)$ we have

$$f = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega \iff \int_{\partial\Omega} f\phi\, d\sigma = 0 \text{ for every } \phi \in \mathscr{C}^\infty_0(\mathbb{R}^n). \tag{2.2.340}$$

In this context, define the family of first-order tangential derivative operators, $\partial_{\tau_{jk}}$ with $j, k \in \{1, \dots, n\}$, acting on functions $\varphi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ according to

$$\partial_{\tau_{jk}}\varphi := \nu_j(\partial_k\varphi)\big|_{\partial\Omega} - \nu_k(\partial_j\varphi)\big|_{\partial\Omega} \text{ for all } j, k \in \{1, \dots, n\}. \tag{2.2.341}$$

The starting point in the development of a brand of first-order Sobolev spaces on $\partial\Omega$ is the observation that for any two functions $\varphi, \psi \in \mathscr{C}^\infty_0(\mathbb{R}^n)$ and every pair of indices $j, k \in \{1, \dots, n\}$ one has the boundary integration by parts formula

$$\int_{\partial\Omega} (\partial_{\tau_{jk}}\varphi)\psi\, d\sigma = -\int_{\partial\Omega} \varphi(\partial_{\tau_{jk}}\psi)\, d\sigma. \tag{2.2.342}$$

Indeed, identity (2.2.342) is a consequence of the Divergence Formula (1.1.12) applied to a suitable vector field, namely $\vec{F} := \partial_k(\varphi\psi)e_j - \partial_j(\varphi\psi)e_k$ (where $\{e_i\}_{1\leq i\leq n}$ is the standard orthonormal basis in $\mathbb{R}^n$), which is smooth, compactly supported, divergence-free, and satisfies $\nu\cdot\vec{F} = (\partial_{\tau_{jk}}\varphi)\psi + \varphi(\partial_{\tau_{jk}}\psi)$ at $\sigma$-a.e. point on $\partial\Omega$.

Next, given a function $f \in L^1_{\rm loc}(\partial\Omega,\sigma)$ along with two indices $j,k\in\{1,\dots,n\}$, we shall say that $\partial_{\tau_{jk}}f$ exists in (or, belongs to) the space $L^1_{\rm loc}(\partial\Omega,\sigma)$ if there exists a function $f_{jk}\in L^1_{\rm loc}(\partial\Omega,\sigma)$ such that

$$\int_{\partial\Omega}(\partial_{\tau_{jk}}\varphi)f\,d\sigma = -\int_{\partial\Omega}\varphi f_{jk}\,d\sigma \quad\text{for all}\quad \varphi\in\mathscr{C}^\infty_0(\mathbb{R}^n). \tag{2.2.343}$$

In view of (2.2.340), we conclude that the function $f_{jk}$ is unambiguously defined by the demand in (2.2.343). Henceforth we shall favor the notation

$$\partial_{\tau_{jk}}f := f_{jk} \tag{2.2.344}$$

which, in particular, allows us to recast (2.2.343) more in line with (2.2.342), namely as

$$\int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi)\,d\sigma = -\int_{\partial\Omega}(\partial_{\tau_{jk}}f)\varphi\,d\sigma \quad\text{for all}\quad \varphi\in\mathscr{C}^\infty_0(\mathbb{R}^n). \tag{2.2.345}$$

In analogy with the classical flat, Euclidean case, it is natural to regard $\partial_{\tau_{jk}}f$ as a weak (tangential) derivative of the function $f$. The developments so far allow us to define a convenient functional analytic environment within which is possible to consider such weak (tangential) derivatives of functions in $L^1_{\rm loc}(\partial\Omega,\sigma)$. Specifically, for each $p\in[1,\infty]$ we introduce the local Sobolev space $L^p_{1,\rm loc}(\partial\Omega,\sigma)$ as

$$L^p_{1,\rm loc}(\partial\Omega,\sigma) := \{f\in L^p_{\rm loc}(\partial\Omega,\sigma):\ \partial_{\tau_{jk}}f\in L^p_{\rm loc}(\partial\Omega,\sigma),\ 1\leq j,k\leq n\}. \tag{2.2.346}$$

In such a context, we define the tangential gradient operator as

$$L^p_{1,\rm loc}(\partial\Omega,\sigma)\ni f\mapsto \nabla_{\tan}f := \left(\sum_{k=1}^n \nu_k\partial_{\tau_{kj}}f\right)_{1\leq j\leq n}. \tag{2.2.347}$$

If $\Omega$ is actually a UR domain, we may recover the weak tangential derivatives from the components of the tangential gradient operator via (cf. [93], [53, Lemma 3.40])

$$\begin{gathered}\partial_{\tau_{jk}}f = \nu_j(\nabla_{\tan}f)_k - \nu_k(\nabla_{\tan}f)_j, \quad 1\leq j,k\leq n,\\ \text{for every}\quad f\in L^p_{1,\rm loc}(\partial\Omega,\sigma)\quad\text{with}\quad p\in(1,\infty).\end{gathered} \tag{2.2.348}$$

Going further, having fixed an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$, define the (boundary) weighted Sobolev space

$$L^p_1(\partial\Omega,w) := \{f\in L^p(\partial\Omega,w):\ \partial_{\tau_{jk}}f\in L^p(\partial\Omega,w),\ 1\leq j,k\leq n\} \tag{2.2.349}$$

which is a Banach space when equipped with the norm

$$L^p_1(\partial\Omega,w)\ni f\mapsto \|f\|_{L^p_1(\partial\Omega,w)} := \|f\|_{L^p(\partial\Omega,w)} + \sum_{j,k=1}^n \left\|\partial_{\tau_{jk}}f\right\|_{L^p(\partial\Omega,w)}. \tag{2.2.350}$$

Since there exists $q \in (1,\infty)$ such that $L^p(\partial\Omega,w) \hookrightarrow L^q_{\rm loc}(\partial\Omega,\sigma)$ (cf. Lemma 2.2.47), we see that $L^p_1(\partial\Omega,w) \hookrightarrow L^q_{1,\rm loc}(\partial\Omega,\sigma)$ for such an exponent $q$. In particular, the equality in (2.2.348) holds for every function $f \in L^p_1(\partial\Omega,w)$.

In the same geometric setting, recall that $L^{p,q}(\partial\Omega,\sigma)$ with $p,q \in (0,\infty]$ stands for the scale of Lorentz spaces on $\partial\Omega$, with respect to the measure $\sigma$. These are quasi-Banach spaces which arise naturally as intermediate spaces for the real interpolation method used within the scale of ordinary Lebesgue spaces. In particular, this implies that

$$L^{p,q}(\partial\Omega,\sigma) \hookrightarrow L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap \Big(\textstyle\bigcap_{1<s<p} L^s_{\rm loc}(\partial\Omega,\sigma)\Big) \quad \text{whenever} \quad p \in (1,\infty) \quad \text{and} \quad q \in (0,\infty]. \tag{2.2.351}$$

In relation to this scale of spaces, it is also of interest to consider (boundary) Lorentz-based Sobolev spaces. Specifically, following [93], for each $p \in (1,\infty)$ and $q \in (0,\infty]$ we set

$$L^{p,q}_1(\partial\Omega,w) := \{f \in L^{p,q}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in L^{p,q}(\partial\Omega,\sigma),\ 1 \leq j,k \leq n\} \tag{2.2.352}$$

which is a quasi-Banach space when equipped with the quasi-norm

$$L^{p,q}_1(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{L^{p,q}_1(\partial\Omega,\sigma)} := \|f\|_{L^{p,q}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \Big\|\partial_{\tau_{jk}} f\Big\|_{L^{p,q}(\partial\Omega,\sigma)}. \tag{2.2.353}$$

In the proposition below, which refines [53, Lemma 3.36, p. 2678], we study the manner in which weak tangential derivatives interact with pointwise nontangential traces. See [93] for a proof.

**Proposition 2.2.48.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in [1,\infty]$ and an aperture parameter $\kappa \in (0,\infty)$. In this context, assume the function $u \in W^{1,1}_{\rm loc}(\Omega)$ satisfies*

$$\mathcal{N}_\kappa u \in L^p_{\rm loc}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa(\nabla u) \in L^p_{\rm loc}(\partial\Omega,\sigma), \tag{2.2.354}$$

*and the nontangential traces*

$$u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \ \ \textit{and}\ \ (\partial_j u)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \ \ \textit{for}\ j \in \{1,\dots,n\} \quad \textit{exist at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{2.2.355}$$

*Then $u\big|^{\kappa-\rm n.t.}_{\partial\Omega}$ belongs to $L^p_{1,\rm loc}(\partial\Omega,\sigma)$, the functions $(\partial_1 u)\big|^{\kappa-\rm n.t.}_{\partial\Omega},\dots,(\partial_n u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}$ belong to $L^p_{\rm loc}(\partial\Omega,\sigma)$ and, for each $j,k \in \{1,\dots,n\}$ and for $\sigma$-a.e. point on $\partial\Omega$, one has*

$$\partial_{\tau_{jk}}\Big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big) = \nu_j\Big((\partial_k u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big) - \nu_k\Big((\partial_j u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big). \tag{2.2.356}$$

*In particular, for each $j,k \in \{1,\dots,n\}$ one has*

$$\Big|\partial_{\tau_{jk}}\Big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)\Big| \leq 2\mathcal{N}_\kappa(\nabla u) \quad \textit{at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{2.2.357}$$

The following result from [93] may be regarded as a weighted counterpart of Proposition 2.2.48.

**Proposition 2.2.49.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa \in (0,\infty)$ and an integrability exponent $p \in (1,\infty)$. Also, assume $w : \partial\Omega \to [0,+\infty]$ is a $\sigma$-measurable function satisfying $0 < w(x) < +\infty$ for $\sigma$-a.e. $x \in \partial\Omega$ and $w^{-1/p} \in L^{p'}_{\rm loc}(\partial\Omega,\sigma)$, where $p' \in (1,\infty)$ denotes the Hölder conjugate exponent of $p$; in particular, $L^p(\partial\Omega, w\sigma) \hookrightarrow L^1_{\rm loc}(\partial\Omega,\sigma)$. In this setting, suppose that some complex-valued function $u \in W^{1,1}_{\rm loc}(\Omega)$ has been given which satisfies the following conditions:*

$$\begin{gathered} u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \;\; \textit{exists at } \sigma\textit{-a.e. point on } \partial\Omega \textit{ and} \\ \mathcal{N}_\kappa u \in L^1_{\rm loc}(\partial\Omega,\sigma), \;\; \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w\sigma). \end{gathered} \tag{2.2.358}$$

*Then the nontangential trace $u\big|^{\kappa-\rm n.t.}_{\partial\Omega}$ belongs to $L^1_{1,\rm loc}(\partial\Omega,\sigma)$ and satisfies*

$$\begin{aligned} &\partial_{\tau_{jk}}\Big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big) \in L^p(\partial\Omega, w\sigma) \;\; \textit{for each } \; j,k \in \{1,\dots,n\} \\ &\textit{and } \; \sum_{j,k=1}^n \Big\|\partial_{\tau_{jk}}\Big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)\Big\|_{L^p(\partial\Omega,w\sigma)} \le C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,w\sigma)} \end{aligned} \tag{2.2.359}$$

*for some constant $C \in (0,\infty)$ independent of $u$.*

## 2.3 Calderón-Zygmund theory for boundary layers in UR domains

In [18], A.P. Calderón has initiated a breakthrough by proving the $L^p$-boundedness of the principal-value Cauchy integral operator on Lipschitz curves with small Lipschitz constant. Subsequently, R. Coifman, A. McIntosh, and Y. Meyer have successfully extended Calderón's estimate on Cauchy integrals to general Lipschitz curves in [24], and used this to establish the boundedness of higher-dimensional singular integral operators (such as the harmonic double layer $K_\Delta$) on Lebesgue spaces $L^p(\Sigma,\mathcal{H}^{n-1})$ with $p \in (1,\infty)$, whenever $\Sigma$ is a strongly Lipschitz surface in $\mathbb{R}^n$. This gave the impetus for studying such singular integral operators on surfaces more general than the boundaries of Lipschitz domains. Works of G. David [30], [31], G. David and D. Jerison [32], G. David and S. Semmes [33], [34], and of S. Semmes [106] yield such boundedness when the $\Sigma \subseteq \mathbb{R}^n$ is a UR set, i.e., $\Sigma$ is a closed Ahlfors regular set which contains "big pieces" of Lipschitz images in a quantitative, uniform, scale-invariant fashion (cf. Definition 1.1.4).

This body of results, which interfaced tightly with geometric measure theory, has been applied to problems in PDE's for the first time by S. Hofmann, M. Mitrea, and M. Taylor in [53] (see also [95] for PDE's in the setting of Riemannian manifolds). Here we continue this line of work with two specific goals in mind. First, we consider singular integral operators (SIO's) acting on a larger variety of function spaces and, second, we seek finer bounds on the operator norm of the singular integrals of double layer type. We begin by discussing the general setup.

### 2.3.1 Boundary layer potentials: the setup

First, assume $\Omega \subseteq \mathbb{R}^n$ is a given UR domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. In addition, consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system $L$ in $\mathbb{R}^n$ and recall the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with $L$ as in Theorem 1.2.1. Finally, fix a coefficient tensor $A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$, and pick an arbitrary function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega\,,\,\tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M. \tag{2.3.1}$$

In this setting, define the action of the boundary-to-domain double layer potential operator $\mathcal{D}_A$ on $f$ as

$$\mathcal{D}_A f(x) := \Big(-\int_{\partial\Omega} \nu_k(y) a_{jk}^{\beta\alpha}\, (\partial_j E_{\gamma\beta})\,(x-y) f_\alpha(y)\, d\sigma(y)\Big)_{1\leq\gamma\leq M}, \tag{2.3.2}$$

at each point $x \in \Omega$. From (1.2.19) we see that (2.3.1) is the most general functional analytic setting in which the integral in (2.3.2) is absolutely convergent. The double layer operator $\mathcal{D}$ may be regarded as a mechanism for generating lots of null-solutions for the given system $L$ in $\Omega$ since, as is apparent from (2.3.2) and Theorem 1.2.1,

$$\begin{gathered}\text{for each function } f \text{ as in (2.3.1) we have}\\ \mathcal{D}_A f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \text{ and } \ L(\mathcal{D}_A f) = 0 \ \text{ in } \ \Omega.\end{gathered} \tag{2.3.3}$$

Going further, let us define the action of the boundary-to-boundary double layer potential operator $K_A$ on $f$ as in (2.3.1) by setting

$$K_A f(x) := \Big(-\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(y) a_{jk}^{\beta\alpha}\, (\partial_j E_{\gamma\beta})\,(x-y) f_\alpha(y)\, d\sigma(y)\Big)_{1\leq\gamma\leq M}, \tag{2.3.4}$$

at $\sigma$-a.e. point $x \in \partial\Omega$. Another singular integral operator which is closely related to (2.3.4) is the so-called "transposed" double layer operator $K_A^{\#}$ defined by setting

$$K_A^{\#} f(x) := \Big(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x) a_{jk}^{\beta\alpha} (\partial_j E_{\gamma\beta})(x-y) f_\gamma(y)\, d\sigma(y)\Big)_{1\leq\alpha\leq M} \tag{2.3.5}$$

at $\sigma$-a.e. $x \in \partial\Omega$, for each function $f$ as in (2.3.1). Since we are presently assuming that $\Omega$ is a UR domain, work in [93] guarantees that the above singular integral operators are well-defined in a $\sigma$-a.e. pointwise fashion for each function as in (2.3.1).

**Example 2.3.1.** The standard fundamental solution for the Laplacian in $\mathbb{R}^n$ is defined for $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)}\dfrac{1}{|x|^{n-2}}, & \text{if } \ n \geq 3,\\[2mm] \dfrac{1}{2\pi}\ln|x|, & \text{if } \ n = 2,\end{cases} \tag{2.3.6}$$

where, as usual, $\omega_{n-1}$ denotes the surface area of the unit sphere in $\mathbb{R}^n$ (cf. [92, Section 7.1]). Given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Set $a_{jk}^{\alpha\beta} := a_{jk} := \delta_{jk}$ in (1.2.1) so that $L = \Delta$, and refer to $\mathcal{D}_\Delta$, $K_\Delta$ (constructed as in (2.3.2), (2.3.4)) for this choice of coefficient tensor, i.e., for $A := I_{n\times n}$, the identity matrix) as being the (classical) harmonic double layer potentials. Concretely, for each function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ we have (writing, in this case, $\mathcal{D}_\Delta, K_\Delta, K_\Delta^\#$ in place of $\mathcal{D}_{I_{n\times n}}, K_{I_{n\times n}}, K^\#_{I_{n\times n}}$)

$$\mathcal{D}_\Delta f(x) = \frac{1}{\omega_{n-1}}\int_{\partial\Omega}\frac{\langle\nu(y), y-x\rangle}{|x-y|^n}f(y)\,d\sigma(y), \qquad \forall\, x\in\Omega, \tag{2.3.7}$$

and, at $\sigma$-a.e. point $x \in \partial\Omega$,

$$K_\Delta f(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle\nu(y), y-x\rangle}{|x-y|^n}f(y)\,d\sigma(y), \tag{2.3.8}$$

$$K_\Delta^\# f(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle\nu(x), x-y\rangle}{|x-y|^n}f(y)\,d\sigma(y). \tag{2.3.9}$$

Returning to the mainstream discussion, continue to assume that $\Omega \subseteq \mathbb{R}^n$ is a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that $L$ is a homogeneous constant complex coefficient weakly elliptic second-order $M\times M$ system in $\mathbb{R}^n$. Then, for each coefficient tensor $A \in \mathfrak{A}_L$, a basic identity relating the boundary-to-domain double layer potential operator $\mathcal{D}_A$ to the boundary-to-boundary double layer potential operator $K_A$ is the jump-formula (proved in [93]), to the effect that for each aperture parameter $\kappa > 0$ and each function $f \in \Big[L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})\Big]^M$ we have (with $I$ denoting the identity operator)

$$\mathcal{D}_A f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_A\big)f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.3.10}$$

Another fundamental property of the boundary-to-domain double layer potential operator is the ability of absorbing an arbitrary spacial derivative and eventually relocate it, via integration by parts on the boundary, all the way to the function on which this was applied to begin with. This is made precise in the following basic proposition, proved in [93].

**Proposition 2.3.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. Also, for some $M \in \mathbb{N}$, consider a weakly elliptic, homogeneous, constant (complex) coefficient, second-order, $M\times M$ system $L$ in $\mathbb{R}^n$, written as in (1.2.1) for some choice of a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$. Finally, associate with $A$ and $\Omega$ the double* *layer potential operator $\mathcal{D}_A$ as in (2.3.2), and consider a function*

$$
\begin{aligned}
&f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \textit{with the property that} \\
&\partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big) \ \textit{ for } 1\leq j,k\leq n \ \textit{ and } 1\leq\alpha\leq M.
\end{aligned}
\tag{2.3.11}
$$

*Then for each index $\ell \in \{1,\dots,n\}$ and each point $x\in\Omega$ one has*

$$
\partial_\ell(\mathcal{D}_A f)(x) = \left(\int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,d\sigma(y)\right)_{1\leq\gamma\leq M}. \tag{2.3.12}
$$

*In particular, for each aperture parameter $\kappa>0$, the nontangential boundary trace*

$$
(\nabla\mathcal{D}_A f)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \textit{exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{2.3.13}
$$

Once again, assume that $\Omega\subseteq\mathbb{R}^n$ is a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, as before, continue to assume that $L$ is a homogeneous constant complex coefficient weakly elliptic second-order $M\times M$ system in $\mathbb{R}^n$. In general, different choices of the coefficient tensor $A\in\mathfrak{A}_L$ yield different double layer potential operators, so it makes sense to use the subscript $A$ to highlight the dependence on the choice of the coefficient tensor $A$. One integral operator of layer potential variety which is intrinsically associated with the given system $L$ is the so-called single layer potential operator $\mathscr{S}$, whose integral kernel is the matrix-valued function $E(x-y)$, $x,y\in\partial\Omega$. In order to make sense of the action of such an operator on any function as in (2.3.1), it is necessary to alter the said integral kernel and consider the following modified single layer potential operator

$$
\begin{aligned}
&\mathscr{S}_{\rm mod} f(x) := \int_{\partial\Omega}\{E(x-y)-E_*(-y)\}f(y)\,d\sigma(y) \quad \text{for each} \quad x\in\Omega, \\
&\text{for each} \quad f\in\big[L^1(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}})\big]^M, \quad \text{where} \quad E_* := E\cdot\mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}.
\end{aligned}
\tag{2.3.14}
$$

In this regard, it is worth noting that for each $f\in\Big[L^1(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}})\Big]^M$ the function $\mathscr{S}_{\rm mod} f$ is well defined, belongs to the space $[\mathscr{C}^\infty(\Omega)]^M$, and for each multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\geq 1$ one has

$$
\partial^\alpha(\mathscr{S}_{\rm mod} f)(x) = \int_{\partial\Omega}(\partial^\alpha E)(x-y)f(y)\,d\sigma(y) \quad \text{for each} \quad x\in\Omega. \tag{2.3.15}
$$

In particular,

$$
L(\mathscr{S}_{\rm mod} f) = 0 \text{ in } \Omega \text{ for each } f\in\Big[L^1(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}})\Big]^M. \tag{2.3.16}
$$

Analogously to (2.3.14), we define the following modified version of the boundary-to-boundary single layer operator

$$
\begin{aligned}
&S_{\rm mod} f(x) := \int_{\partial\Omega}\{E(x-y)-E_*(-y)\}f(y)\,d\sigma(y) \quad \text{at } \sigma\text{-a.e.} \quad x\in\partial\Omega, \\
&\text{for each} \quad f\in\big[L^1(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^{n-1}})\big]^M, \quad \text{where} \quad E_* := E\cdot\mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}.
\end{aligned}
\tag{2.3.17}
$$

Then this operator is meaningfully defined, via an absolutely convergent integral, at $\sigma$-a.e. point in $\partial\Omega$, and

$$S_{\text{mod}} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M \longrightarrow \big[L^1_{\text{loc}}(\partial\Omega,\sigma)\big]^M \tag{2.3.18}$$

is a well-defined, linear and continuous mapping. Also, with the modified boundary-to-domain single layer operator $\mathscr{S}_{\text{mod}}$ as in (2.3.14), for each aperture parameter $\kappa > 0$ and each $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ one has

$$\Big((\mathscr{S}_{\text{mod}} f)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = (S_{\text{mod}} f)(x) \ \text{ at } \sigma\text{-a.e. point } \ x \in \partial\Omega. \tag{2.3.19}$$

See [93] for a more in-depth discussion on this topic.

To close this section, we define the conormal derivative operator associated with a given Ahlfors regular domain and a given coefficient tensor. Specifically, suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, $\Omega$ is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ is defined $\sigma$-a.e. on $\partial\Omega$. Also, fix a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ along with some aperture parameter $\kappa > 0$. In such a setting, define the `conormal derivative` of a given function $u = (u_\beta)_{1\le\beta\le M} \in \big[W^{1,1}_{\text{loc}}(\Omega)\big]^M$ as the $\mathbb{C}^M$-valued function

$$\partial^A_\nu u := \Big(\nu_r a^{\alpha\beta}_{rs}(\partial_s u_\beta)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)_{1\le\alpha\le M} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.3.20}$$

It has been proved in [93] that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then for each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$, the conormal derivative $\partial^A_\nu \mathscr{S}_{\text{mod}} f$ may be meaningfully considered in the sense of (2.3.20), and

$$\partial^A_\nu \mathscr{S}_{\text{mod}} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \ \text{ at } \sigma\text{-a.e. point in } \ \partial\Omega, \tag{2.3.21}$$

where $I$ is the identity, and $K^{\#}_{A^\top}$ is the operator associated as in (2.3.5) with the UR domain $\Omega$ and the transposed coefficient tensor $A^\top$.

### 2.3.2 SIO's on Muckenhoupt weighted Lebesgue and Sobolev spaces

We begin by considering garden variety Calderón-Zygmund singular integral operators (SIO's), i.e., operators of convolution-type with odd, homogeneous, sufficiently smooth kernels, which otherwise lack any particular algebraic characteristics. The goal is to obtain estimates in Muckenhoupt weighted Lebesgue spaces on UR sets in $\mathbb{R}^n$.

**Proposition 2.3.3.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed UR set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ which is odd and positive homogeneous of degree $1-n$. Also, fix an integrability exponent $p \in (1,\infty)$, along with a Muckenhoupt weight $w \in A_p(\Sigma,\sigma)$. In this setting, for each $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ define*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y) \ \ \textit{for each } \ x \in \Sigma, \tag{2.3.22}$$

$$T_*f(x) := \sup_{\varepsilon>0} |T_\varepsilon f(x)| \quad \textit{for each} \quad x \in \Sigma, \tag{2.3.23}$$

$$Tf(x) := \lim_{\varepsilon\to 0^+} T_\varepsilon f(x) \quad \textit{for } \sigma\textit{-a.e.} \quad x \in \Sigma. \tag{2.3.24}$$

*Then there exists a constant $C \in (0,\infty)$ which depends exclusively on $n$, $p$, $[w]_{A_p}$, and the* UR *constants of $\Sigma$ with the property that for each $f \in L^p(\Sigma, w)$ one has*

$$\|T_*f\|_{L^p(\Sigma,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\Sigma,w)}, \tag{2.3.25}$$

*In particular,*

$$\textit{the truncated integral operators } T_\varepsilon : L^p(\Sigma,w) \to L^p(\Sigma,w) \textit{ are well defined, linear, and bounded in a uniform fashion with respect to the truncation parameter } \varepsilon > 0. \tag{2.3.26}$$

*Moreover, for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ the limit defining $Tf(x)$ in (2.3.24) exists at $\sigma$-a.e. $x \in \Sigma$ and the operator*

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{2.3.27}$$

*is well defined, linear, and bounded. Let $p' \in (1,\infty)$ denote the Hölder conjugate exponent of $p$ and, with $w' := w^{1-p'} \in A_{p'}(\Sigma,\sigma)$, consider the natural identification*

$$\big(L^p(\Sigma, w)\big)^* = L^{p'}(\Sigma, w'). \tag{2.3.28}$$

*Then, under the canonical integral pairing $(f,g) \mapsto \int_\Sigma fg\,d\sigma$, it follows that*

$$\textit{the (real) transposed of the operator (2.3.27) is the operator } -T : L^{p'}(\Sigma,w') \to L^{p'}(\Sigma,w'). \tag{2.3.29}$$

*Finally, assume $\Omega \subseteq \mathbb{R}^n$ be an open set such that $\partial\Omega$ is a* UR *set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix some $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and pick an aperture parameter $\kappa > 0$. With the integral kernel $k$ as before, for each $f \in L^p(\partial\Omega, w)$ define*

$$\mathcal{T}f(x) := \int_{\partial\Omega} k(x-y)f(y)d\sigma(y) \quad \textit{for each} \quad x \in \Omega. \tag{2.3.30}$$

*Then there exists a constant $C \in (0,\infty)$ which depends exclusively on $n$, $p$, $[w]_{A_p}$, and the* UR *constants of $\partial\Omega$ with the property that for each $f \in L^p(\partial\Omega, w)$ one has*

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|f\|_{L^p(\partial\Omega,w)}. \tag{2.3.31}$$

The above proposition points to uniform rectifiability as being intimately connected with the boundedness of a large class of Calderón-Zygmund like operators on Muckenhoupt weighted Lebesgue spaces. From the work of G. David and S. Semmes (cf. [33], [34]) we know that UR sets make up the most general context in which Calderón-Zygmund like operators are bounded on ordinary Lebesgue spaces. David and Semmes have also proved that, under the background assumption of Ahlfors regularity, uniform rectifiability is implied by the simultaneous $L^2$ boundedness of all truncated integral convolution type operators on $\Sigma$ (uniformly with respect to the truncation), whose kernels are smooth, odd, and satisfy standard growth conditions, i.e., odd functions $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying

$$\sup_{x \in \mathbb{R}^n \setminus \{0\}} \Big[ |x|^{(n-1)+|\alpha|} \big|(\partial^\alpha k)(x)\big| \Big] < +\infty, \qquad \forall\, \alpha \in \mathbb{N}_0^n. \tag{2.3.32}$$

In fact, a remarkable result proved by F. Nazarov, X. Tolsa, and A. Volberg in [101] states that the $L^2$-boundedness of the truncated Riesz transforms on $\Sigma$ alone (uniformly with respect to the truncation) yields uniform rectifiability. The corresponding result in the plane was proved earlier in [89]. In light of (2.3.26), the above discussion highlights the optimality of demanding that $\Sigma$ is a UR set in the context of Proposition 2.3.3.

Results like Proposition 2.3.3 have been recently established in [93]. Here we present an alternative approach which makes essential use of the Fefferman-Stein sharp maximal function, considered in the setting of spaces of homogeneous type (for the Euclidean context, see [59, p. 52], [42, Theorem 3.6, p. 161]).

*Proof of Proposition 2.3.3.* To set the stage, recall the Fefferman-Stein sharp maximal operator $M^\#$ on $\Sigma$, acting on each function $f \in L^1_{\rm loc}(\Sigma,\sigma)$ according to

$$M^\# f(x) := \sup_{\Delta \ni x} \fint_\Delta \Big| f - \fint_\Delta f\, d\sigma \Big|\, d\sigma, \qquad \forall\, x \in \Sigma, \tag{2.3.33}$$

where the supremum is taken over all surface balls $\Delta \subseteq \Sigma$ containing the point $x \in \Sigma$. Clearly, for each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ and each $x \in \Sigma$ we have

$$\sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} \fint_\Delta |f - a|\, d\sigma \leq M^\# f(x) \leq 2 \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} \fint_\Delta |f - a|\, d\sigma. \tag{2.3.34}$$

Also, given $\alpha \in (0,1)$, for each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ set

$$M^\#_\alpha f(x) := M^\#(|f|^\alpha)(x)^{1/\alpha} \ \text{ for all } \ x \in \Sigma. \tag{2.3.35}$$

Since having $0 < \alpha < 1$ ensures that $|X^\alpha - Y^\alpha| \leq |X - Y|^\alpha$ for all $X, Y \in [0,\infty)$, from (2.3.35) and the last inequality in (2.3.34) one may readily check that

$$M^\#_\alpha f(x) \leq 2^{1/\alpha} \sup_{\Delta \ni x} \inf_{a \in \mathbb{C}} \Big( \fint_\Delta |f - a|^\alpha\, d\sigma \Big)^{1/\alpha} \tag{2.3.36}$$

for each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ and each $x \in \Sigma$. Finally, recall from (2.2.302) the (non-centered) Hardy-Littlewood maximal operator $M$ on $\Sigma$.

From (2.3.22)-(2.3.24) it is clear that the maximal operator $T_*$ and the principal-value singular integral operator $T$ depend in a homogeneous fashion on the kernel function $k$. In view of this observation, by working with $k/K$ (in the case when $k$ is not identically zero) where $K := \sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k|$, there is no loss of generality in assuming that

$$\sum_{|\alpha| \leq N} \sup_{S^{n-1}} |\partial^\alpha k| = 1. \tag{2.3.37}$$

The fact that for each function $f \in L^1(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}})$ the limit defining $Tf(x)$ in (2.3.24) exists at $\sigma$-a.e. $x \in \Sigma$ has been proved in [93]. To proceed, denote by $L^\infty_{\rm comp}(\Sigma, \sigma)$ the subspace of $L^\infty(\Sigma, \sigma)$ consisting of functions with compact support. Also, fix a power $\alpha \in (0, 1)$. We will first show that there exists a constant $C = C(\Sigma, n, \alpha) \in (0, \infty)$ such that

$$\begin{gathered} M^\#_\alpha(Tf)(x) \leq C \cdot Mf(x) \\ \text{for all } f \in L^\infty_{\rm comp}(\Sigma, \sigma) \text{ and } x \in \Sigma. \end{gathered} \tag{2.3.38}$$

To this end, fix a function $f \in L^\infty_{\rm comp}(\Sigma, \sigma)$ along with a point $x \in \Sigma$, and consider an arbitrary surface ball $\Delta = \Delta(x_0, r_0)$, with center at $x_0 \in \Sigma$ and radius $r_0 > 0$, containing the point $x$. Decompose $f = f_1 + f_2$, where $f_1 := f\mathbf{1}_{2\Delta}$ and $f_2 := f\mathbf{1}_{\Sigma\setminus 2\Delta}$. Then $|Tf_2(x_0)| < +\infty$ and we abbreviate $a := Tf_2(x_0) \in \mathbb{C}$. Note that

$$⨍_\Delta |Tf - a|^\alpha \, d\sigma \leq ⨍_\Delta |Tf_1|^\alpha \, d\sigma + ⨍_\Delta |Tf_2 - a|^\alpha \, d\sigma. \tag{2.3.39}$$

For the first term in the right-hand side of (2.3.39), using Kolmogorov's inequality, the fact that $T$ is bounded from $L^1(\Sigma, \sigma)$ to $L^{1,\infty}(\Sigma, \sigma)$ (cf. [93], [53, Proposition 3.19]) and the fact that $\Sigma$ is an Ahlfors regular set to write

$$\begin{aligned} ⨍_\Delta |Tf_1|^\alpha \, d\sigma &\leq \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|Tf_1\|^\alpha_{L^{1,\infty}(\Sigma,\sigma)} \leq \frac{C_\alpha}{\sigma(\Delta)^\alpha} \|f_1\|^\alpha_{L^1(\Sigma,\sigma)} \\ &\leq C_\alpha \Big( ⨍_{2\Delta} |f| \, d\sigma \Big)^\alpha \leq C_\alpha \cdot Mf(x)^\alpha. \end{aligned} \tag{2.3.40}$$

For the second term in the right-hand side of (2.3.39), note that the properties of $k$ and (2.3.37) entail

$$|(\nabla k)(z)| = \Big| (\nabla k)\Big( \frac{z}{|z|} |z| \Big) \Big| \leq |z|^{-n} \sup_{|\omega|=1} |(\nabla k)(\omega)| = C_n |z|^{-n}, \tag{2.3.41}$$

for each $z \in \mathbb{R}^n \setminus \{0\}$, where $C_n \in (0, \infty)$ is a purely dimensional constant. On account of (2.3.41) and the Mean Value Theorem, we see that there exists a dimensional constant $C_n \in (0, \infty)$ with the property that for each $y \in \Delta$ and $z \in \Sigma \setminus 2\Delta$ we have

$$|k(y - z) - k(x_0 - z)| \leq C_n \frac{|y - x_0|}{|x_0 - z|^n} \leq \frac{C_n r_0}{|x_0 - z|^n}. \tag{2.3.42}$$

Using this, for every $y \in \Delta$ we may write

$$\begin{aligned} |Tf_2(y) - a| &= |Tf_2(y) - Tf_2(x_0)| \\ &\leq \int_{\Sigma \setminus 2\Delta} |k(y - z) - k(x_0 - z)| |f(z)| \, d\sigma(z) \\ &\leq C r_0 \sum_{j=1}^\infty \int_{2^j r_0 \leq |x_0 - z| < 2^{j+1} r_0} \frac{|f(z)|}{|x_0 - z|^n} \, d\sigma(z) \\ &\leq C \sum_{j=1}^\infty 2^{-j} ⨍_{2^{j+1}\Delta} |f(z)| \, d\sigma(z) \\ &\leq C \cdot Mf(x), \end{aligned} \tag{2.3.43}$$

where $C \in (0,\infty)$ depends only on dimension and the Ahlfors regularity constant of $\Sigma$. At this stage, the claim in (2.3.38) follows by combining (2.3.36), (2.3.39), (2.3.40), and (2.3.43).

We shall now analyze two cases, depending on whether $\Sigma$ is bounded or not. Consider first the latter case, in which $\Sigma$ is unbounded. In such a setting, the $A_\infty$-weighted version of the Fefferman-Stein inequality for spaces of homogeneous type (cf., e.g. [7, Sections 3.2 and 5]) gives that for every $q \in (0,\infty)$ there exists a constant $C_w \in (0,\infty)$, which depends on the weight $w \in A_p(\Sigma,\sigma) \subseteq A_\infty(\Sigma,\sigma)$ only through its characteristic $[w]_{A_p}$ (indeed, it can be expressed as an increasing function of $[w]_{A_p}$), such that

$$\begin{aligned} \|Mg\|_{L^q(\Sigma,w)} \leq C_w \left\|M^{\#}g\right\|_{L^q(\Sigma,w)} \quad \text{for each} \\ g \in L^1_{\rm loc}(\Sigma,\sigma) \ \text{ such that } \ Mg \in L^q(\Sigma,w). \end{aligned} \tag{2.3.44}$$

To proceed, fix $\alpha \in (0,1)$ and $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$. Let us momentarily work under the additional assumption that the weight $w$ belongs to $L^\infty(\Sigma,\sigma)$. This permits us to estimate

$$\begin{aligned} \|M(|Tf|^\alpha)\|_{L^{p/\alpha}(\Sigma,w)} &\leq \|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)} \|M(|Tf|^\alpha)\|_{L^{p/\alpha}(\Sigma,\sigma)} \\ &\leq C\, \|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)} \|Tf\|^\alpha_{L^p(\Sigma,\sigma)} \\ &\leq C\, \|w\|^{\alpha/p}_{L^\infty(\Sigma,\sigma)} \|f\|^\alpha_{L^p(\Sigma,\sigma)} < +\infty, \end{aligned} \tag{2.3.45}$$

where we have used the boundedness of $M$ on $L^{p/\alpha}(\Sigma,\sigma)$ and the boundedness of $T$ on $L^p(\Sigma,\sigma)$ (cf. [53, Proposition 3.18]). This allows us to use (2.3.44) (with $g := |Tf|^\alpha$ and $q := p/\alpha$) to obtain, for some constant $C_w \in (0,\infty)$ (again, depending in an increasing fashion on $[w]_{A_p}$),

$$\begin{aligned} \|Tf\|_{L^p(\Sigma,w)} &\leq \left\|M(|Tf|^\alpha)^{1/\alpha}\right\|_{L^p(\Sigma,w)} = \|M(|Tf|^\alpha)\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} \\ &\leq C_w \left\|M^{\#}(|Tf|^\alpha)\right\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} = C_w \left\|M^{\#}_\alpha(Tf)\right\|_{L^p(\Sigma,w)} \\ &\leq C_w \|Mf\|_{L^p(\Sigma,w)} \leq C_w \|f\|_{L^p(\Sigma,w)}, \end{aligned} \tag{2.3.46}$$

where the first inequality follows from Lebesgue Differentiation Theorem (cf. [6]), the last equality is a consequence of (2.3.35), the penultimate inequality comes from (2.3.38), the last inequality is implied by the boundedness of the Hardy-Littlewood operator $M$ on $L^p(\Sigma,w)$.

To remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we proceed as follows. For each $j \in \mathbb{N}$, let $w_j := \min\{w,j\} \in L^\infty(\Sigma,\sigma)$. Moreover, as in [46, Ex. 9.1.9], we have

$$[w_j]_{A_p} \leq C_p(1+[w]_{A_p}) \tag{2.3.47}$$

for some $C_p \in (0,\infty)$ independent of $j \in \mathbb{N}$. As such, we may invoke (2.3.46) written for each $w_j$ (which now involves a constant whose dependence in $w_j$ may be expressed in terms of a non-decreasing function of $[w_j]_{A_p}$) to conclude that

$$\|Tf\|_{L^p(\Sigma,w_j)} \leq C\, \|f\|_{L^p(\Sigma,w_j)} \leq C\, \|f\|_{L^p(\Sigma,w)}, \tag{2.3.48}$$

for some constant $C \in (0, \infty)$ independent of $j \in \mathbb{N}$. Upon letting $j \to \infty$ and relying on Lebesgue's Monotone Convergence Theorem, we arrive at the conclusion that $\|Tf\|_{L^p(\Sigma,w)} \leq C \|f\|_{L^p(\Sigma,w)}$ for every $f \in L^\infty_{\mathrm{comp}}(\Sigma, \sigma)$. Given that $L^\infty_{\mathrm{comp}}(\Sigma, \sigma)$ is dense in $L^p(\Sigma, w)$, this ultimately establishes the boundedness of the operator $T$ in the context of (2.3.27) when $\Sigma$ is unbounded.

Let us now consider the case when $\Sigma$ is bounded. In this case, compared to (2.3.44), the $A_\infty$-weighted version of the Fefferman-Stein inequality includes an extra term, namely it now reads (cf. [7, Sections 3.2 and 5])

$$
\begin{aligned}
\|Mg\|_{L^q(\Sigma,w)} \leq C_w \left\|M^\# g\right\|_{L^q(\Sigma,w)} \\
+ C\sigma(\Sigma)^{-1}\Big(\int_\Sigma w\, d\sigma\Big)^{1/q} \|g\|_{L^1(\Sigma,\sigma)} \\
\text{for all } g \in L^1(\Sigma, \sigma) \text{ with } Mg \in L^q(\Sigma, w),
\end{aligned}
\tag{2.3.49}
$$

where $C_w \in (0, \infty)$ is as before and $C \in (0, \infty)$ is a purely geometric constant. Fix $\alpha \in (0, 1)$ and $f \in L^\infty_{\mathrm{comp}}(\Sigma, \sigma)$. Assume first that $w \in L^\infty(\Sigma, \sigma)$ and note that (2.3.45) holds in the same way. This permits us to invoke (2.3.49) (with $g := |Tf|^\alpha$ and $q := p/\alpha$), so in place of (2.3.46) we now get

$$
\begin{aligned}
\|Tf\|_{L^p(\Sigma,w)} &\leq \|M(|Tf|^\alpha)\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} \\
&\leq C_w \|M^\#(|Tf|^\alpha)\|^{1/\alpha}_{L^{p/\alpha}(\Sigma,w)} \\
&\qquad + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\, d\sigma\Big)^{1/p} \||Tf|^\alpha\|^{1/\alpha}_{L^1(\Sigma,\sigma)} \\
&\leq C_w \|f\|_{L^p(\Sigma,w)} + C\sigma(\Sigma)^{-1/\alpha}\Big(\int_\Sigma w\, d\sigma\Big)^{1/p} \|Tf\|_{L^\alpha(\Sigma,\sigma)},
\end{aligned}
\tag{2.3.50}
$$

where the first and last estimates follow as before. Here, the constant $C_w \in (0, \infty)$ depends on $w$ only through its characteristic $[w]_{A_p}$ (again, this may be expressed as an increasing function of $[w]_{A_p}$), while $C \in (0, \infty)$ depends just on $p$, $\alpha$, $n$ and the Ahlfors regularity constant of $\Sigma$.

It remains to estimate $\|Tf\|_{L^\alpha(\Sigma,\sigma)}$ in a satisfactory manner. Using Kolmogorov's inequality and the fact that $T$ is bounded from $L^1(\Sigma, \sigma)$ into $L^{1,\infty}(\Sigma, \sigma)$ (cf. [93], [53, Proposition 3.19]) and Hölder's inequality we obtain

$$
\begin{aligned}
\|Tf\|_{L^\alpha(\Sigma,\sigma)} &\leq (1-\alpha)^{-1/\alpha}\sigma(\Sigma)^{(1-\alpha)/\alpha} \|Tf\|_{L^{1,\infty}(\Sigma,\sigma)} \\
&\leq C\sigma(\Sigma)^{(1-\alpha)/\alpha} \|f\|_{L^1(\Sigma,\sigma)} \\
&= C\sigma(\Sigma)^{(1-\alpha)/\alpha}\Big(\int_\Sigma w^{1-p'}\, d\sigma\Big)^{1/p'} \|f\|_{L^p(\Sigma,w)}.
\end{aligned}
\tag{2.3.51}
$$

Let us record our progress. The argument so far proves that, if $\Sigma$ is bounded, then for each $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$ we have

$$\begin{aligned}\|Tf\|_{L^p(\Sigma,w)} &\leq \Big(C_w + C\,\sigma(\Sigma)^{-1}\Big(\int_\Sigma w\,d\sigma\Big)^{1/p}\Big(\int_\Sigma w^{1-p'}\,d\sigma\Big)^{1/p'}\Big)\,\|f\|_{L^p(\Sigma,w)}\\ &\leq \big(C_w + C\,[w]_{A_p}^{1/p}\big)\,\|f\|_{L^p(\Sigma,w)}\,, \end{aligned}\tag{2.3.52}$$

where $C_w \in (0,\infty)$ is as above. As before, to remove the restriction $w \in L^\infty(\Sigma,\sigma)$, we work with $w_j := \min\{w,j\}$ for $j \in \mathbb{N}$. Thanks to (2.3.47) the constant in the right-hand side of (2.3.52) may be controlled uniformly in $j$. After passing to limit $j \to \infty$ and once again relying on the density $L^\infty_{\rm comp}(\Sigma,\sigma)$ into $L^p(\Sigma,w)$, we eventually conclude that the operator $T$ is bounded in the context of (2.3.27) in this case as well. Moreover,

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C, \tag{2.3.53}$$

where $C \in (0,\infty)$ depends only on $n$, $p$, $[w]_{A_p}$, and the UR constants of $\Sigma$. This finishes the proof of (2.3.27).

Next, (2.3.25) follows from (2.3.27), Cotlar's inequality, to the effect that there exists some $C \in (0,\infty)$ which depends only on $n$, and the Ahlfors regularity constant of $\Sigma$, with the property that for every function $f \in L^\infty_{\rm comp}(\Sigma,\sigma)$ we have

$$(T_*f)(x) \leq C\cdot M(Tf)(x) + C\cdot Mf(x) \quad \text{for each} \quad x \in \Sigma, \tag{2.3.54}$$

the boundedness of the Hardy-Littlewood operator $M$ on $L^p(\Sigma,w)$, and a density argument. Going further, (2.3.29) may be justified by first establishing a similar claim for the truncated operators (2.3.22) using Fubini's theorem, then invoking Lebesgue's Dominated Convergence Theorem (whose applicability is guaranteed by (2.3.25)) to pass to limit $\varepsilon \to 0^+$.

Finally, consider the claim made in the very last part of the statement. It is apparent from (2.3.30) that the boundary-to-domain operator $\mathcal{T}$ depends in a homogeneous fashion on the kernel function $k$. Much as before, this permits us to work under the additional assumption that (2.3.37) holds. Granted this, the estimate claimed in (2.3.31) is a direct consequence of inequality (2.3.25) and the formula (cf. [53, eq. (3.2.22)])

$$\mathcal{N}_\kappa(\mathcal{T}f)(x) \leq C\cdot T_*f(x) + C\cdot Mf(x) \quad \text{for each} \quad x \in \Sigma, \tag{2.3.55}$$

where $C \in (0,\infty)$ depends only on $n$ and the Ahlfors regularity constant of $\Sigma$, and where the maximal operator $T_*$ and the Hardy-Littlewood maximal function $M$ are now associated with the UR set $\Sigma := \partial\Omega$. $\square$

The stage has been set for considering the action of the boundary layer potentials associated with a given weakly elliptic system $L$ and UR domain $\Omega$ in $\mathbb{R}^n$ as in (2.3.2)-(2.3.5) and (2.3.14) on Muckenhoupt weighted Lebesgue and Sobolev spaces on $\partial\Omega$. To state our main result in this regard, given any two Banach spaces $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$, denote

$$\mathrm{Bd}(X\to Y) := \{T : X \to Y : \ T \text{ linear and bounded}\}, \tag{2.3.56}$$

and equip this space with the standard operator norm $\mathrm{Bd}(X \to Y) \ni T \mapsto \|T\|_{X\to Y}$ (cf. (2.4.1)). Finally, corresponding to the case when $Y = X$, we agree to abbreviate

$$\mathrm{Bd}(X) := \mathrm{Bd}(X \to X). \tag{2.3.57}$$

**Proposition 2.3.4.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Pick $A \in \mathfrak{A}_L$ and consider the boundary layer potential operators $\mathcal{D}_A, K_A, K_A^{\#}$ associated with $\Omega$ and the coefficient tensor $A$ as in (2.3.2), (2.3.4), and (2.3.5). Also, recall the modified single layer potential operator $\mathscr{S}_{\rm mod}$ associated with $\Omega$ and $L$ as in (2.3.14). Finally, fix an integrability exponent $p \in (1,\infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and an aperture parameter $\kappa > 0$.*

*(a) The following operators are well defined, sub-linear, and bounded:*

$$\big[L^p(\partial\Omega,w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa(\mathcal{D}_A f) \in L^p(\partial\Omega,w), \tag{2.3.58}$$

$$\big[L^p_1(\partial\Omega,w)\big]^M \ni f \longmapsto \mathcal{N}_\kappa\,(\nabla\mathcal{D}_A f) \in L^p(\partial\Omega,w). \tag{2.3.59}$$

*Also,*

$$\begin{array}{c} \textit{for each } f \in \big[L^p_1(\partial\Omega,w)\big]^M \textit{ the nontangential trace} \\ (\nabla\mathcal{D}_A f)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \textit{exists (in } \mathbb{C}^{n\cdot M}\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega. \end{array} \tag{2.3.60}$$

*(b) For every $f \in \big[L^p(\partial\Omega,w)\big]^M$ the limits in (2.3.4) and (2.3.5) exist at $\sigma$-a.e. point on $\partial\Omega$. Moreover, the operators $K_A$ and $K_A^{\#}$ are well defined, linear, and bounded in the following contexts:*

$$K_A : \big[L^p(\partial\Omega,w)\big]^M \longrightarrow \big[L^p(\partial\Omega,w)\big]^M, \tag{2.3.61}$$

$$K_A : \big[L^p_1(\partial\Omega,w)\big]^M \longrightarrow \big[L^p_1(\partial\Omega,w)\big]^M, \tag{2.3.62}$$

$$K_A^{\#} : \big[L^p(\partial\Omega,w)\big]^M \longrightarrow \big[L^p(\partial\Omega,w)\big]^M. \tag{2.3.63}$$

*Additionally, the operators $K_A, K_A^{\#}$ in (2.3.61)-(2.3.63) depend continuously on the underlying coefficient tensor $A$. More specifically, with the piece of notation introduced in (1.2.16), the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,w)\big]^M\Big), \tag{2.3.64}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[L^p_1(\partial\Omega,w)\big]^M\Big), \tag{2.3.65}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A^{\#} \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,w)\big]^M\Big). \tag{2.3.66}$$

*Furthermore, the nontangential boundary trace of the boundary-to-domain double layer is related to the boundary-to-boundary double layer via a jump-formula, to the effect that for every $f \in \big[L^p(\partial\Omega,w)\big]^M$ and $\sigma$-a.e. in $\partial\Omega$ one has*

$$\mathcal{D}_A f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_A\big)f, \tag{2.3.67}$$

*where $I$ is the identity operator.*

(c) *For each $f \in [L^p(\partial\Omega, w)]^M$, one has*

$$\mathscr{S}_{\rm mod} f \in [\mathscr{C}^\infty(\Omega)]^M, \quad L(\mathscr{S}_{\rm mod} f) = 0 \;\text{ in }\; \Omega. \tag{2.3.68}$$

*In addition, the conormal derivative of the modified boundary-to-domain single layer satisfies the following jump-formula*

$$\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) f \;\text{ at } \sigma\text{-a.e. point in }\; \partial\Omega, \tag{2.3.69}$$

*where $I$ is the identity, and $K^{\#}_{A^\top}$ is the operator associated as in (2.3.5) with the* UR *domain $\Omega$ and the transposed coefficient tensor $A^\top$. Also, there exists some constant $C = C(\Omega, p, w, L, \kappa) \in (0, \infty)$ independent of $f$ such that*

$$\|\mathcal{N}_\kappa(\nabla \mathscr{S}_{\rm mod} f)\|_{L^p(\partial\Omega,w)} \leq C\|f\|_{[L^p(\partial\Omega,w)]^M}. \tag{2.3.70}$$

*Proof.* With the exception of (2.3.64)-(2.3.66), all the claims may be justified based on (2.3.2)-(2.3.16), Lemma 2.2.47, Proposition 2.3.2, Proposition 2.2.49, and Proposition 2.3.3. Finally, the continuity properties of the operator-valued maps in (2.3.64)-(2.3.66) have been proved in [93]. □

### 2.3.3 Distinguished coefficient tensors

To each weakly elliptic system $L$ we may canonically associate a fundamental solution $E$ as in Theorem 1.2.1. Having fixed a UR domain, this is then used to create a variety of double layer potential operators $K_A$, in relation to each choice of a coefficient tensor $A \in \mathfrak{A}_L$. While any such double layer $K_A$ has a rich Calderón-Zygmund theory (as discussed in Proposition 2.3.4), seeking more specialized properties requires placing additional demands on the coefficient tensor $A$. We begin by recording a result proved in [86] (see also [81]), describing the said demands phrased in several equivalent forms.

**Proposition 2.3.5.** *Let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and consider the matrix-valued function defined for each $\xi \in \mathbb{R}^n \setminus \{0\}$ as*

$$\big(\mathcal{E}_{\gamma\beta}(\xi)\big)_{1\leq\gamma,\beta\leq M} := -\big[L(\xi)\big]^{-1} \in \mathbb{C}^{M\times M} \tag{2.3.71}$$

*(recall that the characteristic matrix $L(\xi)$ of $L$ has been defined in (1.2.2)). Also, let $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the given system $L$ as in Theorem 1.2.1.*

*Then for each coefficient tensor $A = (a_{jk}^{\alpha\beta})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$ (cf. (1.2.14)) the following conditions are equivalent:*

(a) *For each $k, k' \in \{1, \dots, n\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ there holds*

$$(x_{k'} a_{jk}^{\beta\alpha} - x_k a_{jk'}^{\beta\alpha})(\partial_j E_{\gamma\beta})(x) = 0 \;\text{ for all }\; x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n \setminus \{0\}. \tag{2.3.72}$$

*(b) For each $s, s' \in \{1, \dots, n\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$, in the sense of tempered distributions in $\mathbb{R}^n$ one has*

$$\Big[a^{\beta\alpha}_{rs}\partial_{\xi_{s'}} - a^{\beta\alpha}_{rs'}\partial_{\xi_s}\Big]\big[\xi_r\mathcal{E}_{\gamma\beta}(\xi)\big] = 0. \tag{2.3.73}$$

*(c) For each $k, k' \in \{1, \dots, n\}$ and each $\alpha, \gamma \in \{1, \dots, M\}$ one has*

$$(a^{\beta\alpha}_{k'k} - a^{\beta\alpha}_{kk'} + \xi_j a^{\beta\alpha}_{jk}\partial_{\xi_{k'}} - \xi_j a^{\beta\alpha}_{jk'}\partial_{\xi_k})\mathcal{E}_{\gamma\beta}(\xi) = 0 \ \text{ for all } \ \xi \in \mathbb{R}^n \setminus \{0\} \tag{2.3.74}$$

*and also*

$$\int_{S^1}(a^{\beta\alpha}_{jk}\xi_{k'} - a^{\beta\alpha}_{jk'}\xi_k)\xi_j\,\mathcal{E}_{\gamma\beta}(\xi)\,d\mathcal{H}^1(\xi) = 0 \ \text{ if } \ n = 2. \tag{2.3.75}$$

*(d) One has*

$$\begin{gathered}\xi_r\xi_j\Big[a^{\beta\alpha}_{rs'}(a^{\lambda\mu}_{sj} + a^{\lambda\mu}_{js}) - a^{\beta\alpha}_{rs}(a^{\lambda\mu}_{s'j} + a^{\lambda\mu}_{js'})\Big]\mathcal{E}_{\mu\beta}(\xi) + a^{\lambda\alpha}_{s's} - a^{\lambda\alpha}_{ss'} = 0\\ \text{for all } \xi \in S^{n-1}, \text{ all } s, s' \in \{1, \dots, n\}, \text{ and all } \alpha, \lambda \in \{1, \dots, M\},\end{gathered} \tag{2.3.76}$$

*with the cancellation condition*

$$\begin{gathered}\int_{S^1}\Big(a^{\beta\alpha}_{rs}\xi_{s'} - a^{\beta\alpha}_{rs'}\xi_s\Big)\xi_r\mathcal{E}_{\lambda\beta}(\xi)\,d\mathcal{H}^1(\xi) = 0\\ \text{for all } s, s' \in \{1, \dots, n\} \text{ and } \alpha, \lambda \in \{1, \dots, M\},\end{gathered} \tag{2.3.77}$$

*additionally imposed in the case when $n = 2$.*

*(e) For each $\xi \in S^{n-1}$ and each $\alpha, \lambda \in \{1, \dots, M\}$,*

$$\begin{gathered}\text{the expression } \ (a^{\lambda\mu}_{sj} + a^{\lambda\mu}_{js})\mathcal{E}_{\mu\beta}(\xi)\xi_j\xi_r a^{\beta\alpha}_{rs'} + a^{\lambda\alpha}_{s's}\\ \text{is symmetric in the indices } s, s' \in \{1, \dots, n\},\end{gathered} \tag{2.3.78}$$

*with the condition that for each $\alpha, \lambda \in \{1, \dots, M\}$*

$$\begin{gathered}\text{the expression } \ \int_{S^1} a^{\beta\alpha}_{rs}\xi_{s'}\xi_r\mathcal{E}_{\lambda\beta}(\xi)\,d\mathcal{H}^1(\xi)\\ \text{is symmetric in the indices } s, s' \in \{1, 2\},\end{gathered} \tag{2.3.79}$$

*also imposed in the case when $n = 2$.*

*(f) There exists a matrix-valued function*

$$k = \big\{k_{\gamma\alpha}\big\}_{1\leq\gamma,\alpha\leq M} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}^{M\times M} \tag{2.3.80}$$

*with the property that for each $\gamma, \alpha \in \{1, \dots, M\}$ and $s \in \{1, \dots, n\}$ one has*

$$a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x) = x_s k_{\gamma\alpha}(x) \ \text{ for all } \ x \in \mathbb{R}^n \setminus \{0\}. \tag{2.3.81}$$

It is worth noting that the conditions in items *(a)-(f)* above are intrinsically formulated in terms of the given weakly elliptic system $L$.

**Definition 2.3.6.** Given a second-order, weakly elliptic, homogeneous, $M \times M$ system $L$ in $\mathbb{R}^n$, with constant complex coefficients, call

$$A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}} \in \mathfrak{A}_L \tag{2.3.82}$$

a `distinguished coefficient tensor` for the system $L$ provided any of the conditions *(a)-(f)* in Proposition 2.3.5 holds. Also, denote by $\mathfrak{A}_L^{\rm dis}$ the family of such distinguished coefficient tensors for $L$, say,

$$\begin{aligned} \mathfrak{A}_L^{\rm dis} := \Big\{ A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}} \in \mathfrak{A}_L :\ & \text{conditions (2.3.74)-(2.3.75)} \\ & \text{hold for each}\ \ k, k' \in \{1, \dots, n\}\ \ \text{and}\ \ \alpha, \gamma \in \{1, \dots, M\} \Big\}. \end{aligned} \tag{2.3.83}$$

Finally, introduce the class of weakly elliptic systems which posses a distinguished coefficient tensor, by setting

$$\mathfrak{L}^{\rm dis} := \big\{ L \in \mathfrak{L}_* :\ \mathfrak{A}_L^{\rm dis} \neq \varnothing \big\}. \tag{2.3.84}$$

The relevance of the distinguished coefficient tensors is most apparent from the following result proved in [86] (see also [81]).

**Proposition 2.3.7.** *Let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and suppose $A \in \mathfrak{A}_L$. Then the following statements are equivalent.*

*(i) The coefficient tensor $A$ belongs to $\mathfrak{A}_L^{\rm dis}$.*

*(ii) Whenever $\Omega$ is a half-space in $\mathbb{R}^n$, the boundary-to-boundary double layer potential $K_A$ associated with $A$ and $\Omega$ as in (2.3.4) is the zero operator.*

*(iii) There exists a matrix-valued function $k \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M \times M}$ which is even, positive homogeneous of degree $-n$, and with the property that for each* UR *domain $\Omega \subseteq \mathbb{R}^n$ the (matrix-valued) integral kernel of the double layer potential operator $K_A$ associated with $A$ and $\Omega$ as in (2.3.4) has the form*

$$\begin{gathered} \langle \nu(y), x - y \rangle k(x - y) \\ \textit{for each } x \in \partial\Omega \textit{ and } \mathcal{H}^{n-1}\textit{-a.e. } y \in \partial\Omega, \end{gathered} \tag{2.3.85}$$

*where $\nu$ is the geometric measure theoretic outward unit normal to $\Omega$.*

*(iv) There exists a matrix-valued function $k^{\#} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M \times M}$ which is even, positive homogeneous of degree $-n$, and with the property that for each* UR *domain $\Omega \subseteq \mathbb{R}^n$ the (matrix-valued) integral kernel of the "transposed" double layer potential operator $K_A^{\#}$ associated with $A$ and $\Omega$ as in (2.3.5) has the form*

$$\begin{gathered} \langle \nu(x), y - x \rangle k^{\#}(x - y) \\ \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \partial\Omega \textit{ and each } y \in \partial\Omega, \end{gathered} \tag{2.3.86}$$

*where $\nu$ is the geometric measure theoretic outward unit normal to $\Omega$.*

*Moreover, whenever either (hence all) of the above conditions materializes, the matrices $k, k^{\#}$ in items (iii), (iv) above are related to each other via $k^{\#} = k^{\top}$, where the superscript $\top$ indicates transposition.*

In light of Proposition 2.3.7 and (2.1.28), we are particularly interested in the class of weakly elliptic homogeneous constant complex coefficient second-order systems $L$ with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$. The following example shows that the latter condition is always satisfied by strongly elliptic scalar operators.

**Example 2.3.8.** Assume $L$ is a second-order, homogeneous, constant complex coefficient, scalar differential operator in $\mathbb{R}^n$ (i.e., as in (1.2.1) with $M = 1$), which is `strongly elliptic`. Specifically, suppose $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \dots, n\}$ having the property that there exists a constant $c \in (0, \infty)$ such that

$$\mathrm{Re}\Big[\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k\Big] \geq c|\xi|^2, \quad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \tag{2.3.87}$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$ then define

$$(\widetilde{a}_{jk})_{1\leq j,k\leq n} := \mathrm{sym}\, A := \frac{A + A^{\top}}{2}, \quad (b_{jk})_{1\leq j,k\leq n} := (\mathrm{sym}\, A)^{-1}. \tag{2.3.88}$$

In particular, $L = L_{\mathrm{sym}\, A} := \widetilde{a}_{jk}\partial_j\partial_k$, i.e., the coefficient matrix $\mathrm{sym}\, A$ may be used to represent the given differential operator $L$. In this case, it turns out that the fundamental solution $E$ canonically associated with the operator $L$ as in Theorem 1.2.1 may be explicitly identified (cf. [92, Theorem 7.68, pp. 314-315]) as the function $E \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ given at each point $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E(x) = \begin{cases} -\dfrac{1}{(n-2)\,\omega_{n-1}\sqrt{\det(\mathrm{sym}\, A)}}\langle(\mathrm{sym}\, A)^{-1}x, x\rangle^{-\frac{n-2}{2}} & \text{if } n \geq 3, \\ \dfrac{1}{4\pi\sqrt{\det(\mathrm{sym}\, A)}}\log(\langle(\mathrm{sym}\, A)^{-1}x, x\rangle) + c_A & \text{if } n = 2, \end{cases} \tag{2.3.89}$$

where log denotes the principal branch of the complex logarithm (defined for complex numbers $z \in \mathbb{C} \setminus (-\infty, 0]$ so that $z^a = e^{a\log z}$ for each $a \in \mathbb{R}$), and $c_A$ is a complex constant which depends solely on $A$. As both $\mathrm{sym}\, A$ and $B$ are symmetric matrices, for each index $j \in \{1, \dots, n\}$ and each point $x = (x_i)_{1\leq i\leq n} \in \mathbb{R}^n \setminus \{0\}$ we therefore have (in all dimensions $n \geq 2$)

$$\begin{aligned}(\partial_j E)(x) &= \frac{\langle(\mathrm{sym}\, A)^{-1}x, x\rangle^{-\frac{n}{2}}(\delta_{rj}b_{rs}x_s + \delta_{sj}b_{rs}x_r)}{2\omega_{n-1}\sqrt{\det(\mathrm{sym}\, A)}} \\ &= \frac{\langle(\mathrm{sym}\, A)^{-1}x, x\rangle^{-\frac{n}{2}}b_{rj}x_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\, A)}}.\end{aligned} \tag{2.3.90}$$

Thus, with $C_{A,n}$ abbreviating $\big(\omega_{n-1}\sqrt{\det(\mathrm{sym}\, A)}\big)^{-1} \in \mathbb{C}$, for each $k, k' \in \{1, \dots, n\}$ we may compute

$$
\begin{aligned}
(x_{k'}\widetilde{a}_{jk} - x_k\widetilde{a}_{jk'})(\partial_j E)(x) &= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}(x_{k'}\widetilde{a}_{kj} - x_k\widetilde{a}_{k'j})(b_{jr}x_r)\\
&= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}(x_{k'}\delta_{kr} - x_k\delta_{k'r})x_r\\
&= C_{A,n}\langle(\mathrm{sym}\,A)^{-1}x, x\rangle^{-\frac{n}{2}}(x_{k'}x_k - x_kx_{k'}) = 0. \qquad (2.3.91)
\end{aligned}
$$

This shows that condition (2.3.72) is presently verified for the choice of coefficient tensor $\mathrm{sym}\,A$ in the representation of the given differential operator $L$. Hence, $\mathrm{sym}\,A \in \mathfrak{A}_L^{\mathrm{dis}}$ which proves that, in the case when $M = 1$, we have

$\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ for every scalar, strongly elliptic, homogeneous, second-order, constant complex coefficient operator $L$ in $\mathbb{R}^n$. (2.3.92)

Consequently, Proposition 2.3.7 guarantees that for each UR domain $\Omega \subseteq \mathbb{R}^n$ the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ associated with $\mathrm{sym}\,A$ and $\Omega$ as in (2.3.4) has the form (2.3.85). This being said, it is actually of interest to identify the said integral kernel explicitly. Based on (2.3.88)-(2.3.90) and (2.3.4) we see that the kernel of if $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to $\Omega$ then the integral kernel of the double layer potential operator $K_{\mathrm{sym}\,A}$ is

$$
\begin{aligned}
-\nu_k(y)\widetilde{a}_{jk}\,(\partial_j E)\,(x-y) &= -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\nu_k(y)b_{rj}\widetilde{a}_{jk}(x-y)_r}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}\\
&= -\frac{\langle(\mathrm{sym}\,A)^{-1}(x-y), x-y\rangle^{-\frac{n}{2}}\langle\nu(y), x-y\rangle}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}} \qquad (2.3.93)\\
&\qquad \text{for each } x \in \partial\Omega \text{ and } \mathcal{H}^{n-1}\text{-a.e. } y \in \partial\Omega,
\end{aligned}
$$

which, as already anticipated, is of the form (2.3.85) with

$$
k(z) := -\frac{\langle(\mathrm{sym}\,A)^{-1}z, z\rangle^{-\frac{n}{2}}}{\omega_{n-1}\sqrt{\det(\mathrm{sym}\,A)}}, \qquad \forall\, z \in \mathbb{R}^n \setminus \{0\}. \qquad (2.3.94)
$$

Our next example shows that, for scalar operators in dimensions $n \geq 3$, weak ellipticity itself guarantees the existence of a distinguished coefficient tensor.

**Example 2.3.9.** Suppose $n \geq 3$, and consider an arbitrary second-order, homogeneous, constant complex coefficient, scalar differential operator $L$ in $\mathbb{R}^n$ (i.e., as in (1.2.1) with $M = 1$), which is merely *weakly elliptic*. Recall (cf. (1.2.3)) that this means that we may express $L = a_{jk}\partial_j\partial_k$ with $a_{jk} \in \mathbb{C}$ for $j, k \in \{1, \dots, n\}$ having the property that

$$
\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \neq 0, \qquad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \qquad (2.3.95)
$$

Introduce $A := (a_{jk})_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}$. It has been shown in [86] that (here is where $n \geq 3$ is used)

there exists an angle $\theta \in [0, 2\pi)$ such that if we set $A_\theta := e^{i\theta}A$ then the matrix $\mathrm{sym}A_\theta := (A_\theta + A_\theta^\top)/2 \in \mathbb{C}^{n\times n}$ is strongly elliptic, in the sense that there exists some $c \in (0, \infty)$ such that $\mathrm{Re}\,\langle(\mathrm{sym}\,A_\theta)\xi, \xi\rangle \geq c|\xi|^2$ for each $\xi \in \mathbb{R}^n$ (cf. (2.3.87)). (2.3.96)

From this and (2.3.89) we conclude that the fundamental solution $E \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ canonically associated as in Theorem 1.2.1 with the operator $L = e^{-i\theta}L_{A_\theta} = e^{-i\theta}L_{{\rm sym}A_\theta}$ presently may be expressed at each point $x \in \mathbb{R}^n \setminus \{0\}$ as

$$E(x) = -\frac{e^{i\theta}}{(n-2)\omega_{n-1}\sqrt{\det({\rm sym}\,A_\theta)}}\langle({\rm sym}\,A_\theta)^{-1}x, x\rangle^{\frac{2-n}{2}}. \tag{2.3.97}$$

In view of this formula and the fact that ${\rm sym}\,A := (A + A^\top)/2$ is related to ${\rm sym}\,A_\theta$ via ${\rm sym}\,A_\theta = e^{i\theta}{\rm sym}\,A$, we conclude from (2.3.89)-(2.3.91) that condition (2.3.72) currently holds for the choice of coefficient matrix ${\rm sym}\,A$ in the representation of the given differential operator $L$. Thus, ${\rm sym}\,A \in \mathfrak{A}^{\rm dis}_L$ which goes to show that, in the case when $n \geq 3$ and $M = 1$, we have

if $n \geq 3$ then $\mathfrak{A}^{\rm dis}_L \neq \varnothing$ for every scalar, weakly elliptic, homogeneous, second-order, constant complex coefficient operator $L$ in $\mathbb{R}^n$. (2.3.98)

Turning our attention to genuine systems, below we pay special attention to the Lamé system of elasticity.

**Example 2.3.10.** Consider the following complexified version of the Lamé system (originally arising in the study of linear elasticity), defined for any two parameters $\mu, \lambda \in \mathbb{C}$ (referred to as Lamé moduli) as

$$L := \mu\Delta + (\mu + \lambda)\nabla\,{\rm div}, \tag{2.3.99}$$

acting on vector fields $u = (u_\beta)_{1\leq\beta\leq n}$ defined in (subsets) of $\mathbb{R}^n$ (with the Laplacian applied componentwise). Hence, $L = L^\top$, and one may check (cf. [92, Proposition 10.14, p. 366]) that

the complex Lamé system (2.3.99) is weakly elliptic if and only if $\mu \neq 0$ and $2\mu + \lambda \neq 0$. (2.3.100)

We may express the complex Lamé system $L$ as in (1.2.1) (with $M := n$) using a variety of coefficient tensors, such as those belonging to the one-parameter family

$$\begin{gathered} A(\zeta) = \big(a^{\alpha\beta}_{jk}(\zeta)\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq n}} \quad \text{defined for each } \zeta \in \mathbb{C} \text{ according to} \\ a^{\alpha\beta}_{jk}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu + \lambda - \zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \quad 1 \leq j,k,\alpha,\beta \leq n. \end{gathered} \tag{2.3.101}$$

In other words, for each vector field $u = (u_\beta)_{1\leq\beta\leq n} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ and each parameter $\zeta \in \mathbb{C}$, the Lamé system (2.3.99) satisfies

$$Lu = \Big(a^{\alpha\beta}_{jk}(\zeta)\partial_j\partial_k u_\beta\Big)_{1\leq\alpha\leq n} \quad \text{in} \quad \big[\mathcal{D}'(\mathbb{R}^n)\big]^n. \tag{2.3.102}$$

In relation to the coefficient tensor (2.3.101) it turns out that for any given $\mu, \lambda, \zeta \in \mathbb{C}$ with $\mu \neq 0$ and $2\mu + \lambda \neq 0$ if $L$ is as in (2.3.99) then we have (cf. [53], [86] for specific details)

$$A(\zeta) \in \mathfrak{A}^{\rm dis}_L \iff 3\mu + \lambda \neq 0 \;\text{ and }\; \zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{2.3.103}$$

This ultimately shows that

$$\text{whenever the Lamé moduli } \mu, \lambda \in \mathbb{C} \text{ are such that } \mu \neq 0,\ 2\mu + \lambda \neq 0, \text{ and } 3\mu + \lambda \neq 0, \text{ the Lamé operator } L \text{ defined as in (2.3.102) has the property that } \mathfrak{A}_L^{\text{dis}} = \mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing. \tag{2.3.104}$$

It is of interest to concretely identity the format of the double layer potential operators associated with the complex Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda+\mu)\nabla\text{div}$ in $\mathbb{R}^n$, associated as in (2.1.30) to Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying

$$\mu \neq 0 \quad\text{and}\quad 2\mu + \lambda \neq 0 \tag{2.3.105}$$

(thus ensuring the weak ellipticity of $L_{\mu,\lambda}$; cf. (2.3.100)). For this system, the fundamental solution $E$ of $L_{\mu,\lambda}$ from Theorem 1.2.1 has the explicit form $E = (E_{jk})_{1\leq j,k\leq n}$, a matrix whose $(j,k)$ entry is defined at each point $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$ according to

$$E_{jk}(x) = \begin{cases} \dfrac{-(3\mu+\lambda)}{2\mu(2\mu+\lambda)\omega_{n-1}}\left[\dfrac{\delta_{jk}}{(n-2)|x|^{n-2}} + \dfrac{(\mu+\lambda)x_j x_k}{(3\mu+\lambda)|x|^n}\right] & \text{if } n \geq 3, \\[2ex] \dfrac{3\mu+\lambda}{4\pi\mu(2\mu+\lambda)}\left[\delta_{jk}\ln|x| - \dfrac{(\mu+\lambda)x_j x_k}{(3\mu+\lambda)|x|^2}\right] + c_{\mu,\lambda}\delta_{jk} & \text{if } n = 2, \end{cases} \tag{2.3.106}$$

for every $j,k \in \{1,\dots,n\}$, where $c_{\mu,\lambda} \in \mathbb{C}$ is the constant given by

$$c_{\mu,\lambda} := \frac{(1+\ln 4)(\lambda+\mu)}{8\pi\mu(\lambda+2\mu)} - \frac{\ln 2}{2\pi\mu}. \tag{2.3.107}$$

Let us now fix an arbitrary UR domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. In such a setting, with each choice of $\zeta \in \mathbb{C}$, associate a double layer potential operator $K_{A(\zeta)}$ as in (2.3.4). A direct computation based on (2.3.106), (2.3.101), and (2.3.4) then shows that the integral kernel $\Theta^\zeta(x,y)$ of the principal-value double layer potential operator $K_{A(\zeta)}$ is an $n\times n$ matrix whose $(j,k)$ entry, $1 \leq j,k \leq n$, is explicitly given by

$$\begin{aligned} \Theta^\zeta_{jk}(x,y) = -C_1(\zeta)&\frac{\delta_{jk}}{\omega_{n-1}}\frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} \\ &- (1 - C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}} \\ &- C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x_j-y_j)\nu_k(y) - (x_k-y_k)\nu_j(y)}{|x-y|^n}, \end{aligned} \tag{2.3.108}$$

for $\sigma$-a.e. $x, y \in \partial\Omega$, where the constants $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ are defined as

$$C_1(\zeta) := \frac{\mu(3\mu+\lambda) - \zeta(\mu+\lambda)}{2\mu(2\mu+\lambda)}, \qquad C_2(\zeta) := \frac{\mu(\mu+\lambda) - \zeta(3\mu+\lambda)}{2\mu(2\mu+\lambda)}. \tag{2.3.109}$$

Thus, if for any $a = (a_1,\dots,a_n) \in \mathbb{C}^n$ and $b = (b_1,\dots,b_n) \in \mathbb{C}^n$, we agree to define $a \otimes b$ to be the $n \times n$ matrix

$$a \otimes b := (a_j b_k)_{1\leq j,k\leq n} \in \mathbb{C}^{n\times n}, \tag{2.3.110}$$

then for each $\zeta \in \mathbb{C}$ the integral kernel $\Theta^\zeta(x,y)$ of $K_{A(\zeta)}$ may be recast as

$$\begin{aligned}\Theta^\zeta(x,y) = -C_1(\zeta)\frac{1}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle}{|x-y|^n}I_{n\times n}\\ -(1-C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle(x-y)\otimes(x-y)}{|x-y|^{n+2}}\\ -C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x-y)\otimes\nu(y)-\nu(y)\otimes(x-y)}{|x-y|^n},\end{aligned} \tag{2.3.111}$$

for $\sigma$-a.e. $x,y \in \partial\Omega$, where $I_{n\times n}$ is the $n\times n$ identity matrix. The penultimate term above suggests that for each function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ we define

$$\begin{aligned}Qf(x) &:= \lim_{\varepsilon\to 0^+}\frac{n}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle x-y,\nu(y)\rangle(x-y)\otimes(x-y)}{|x-y|^{n+2}}f(y)\,d\sigma(y)\\ &= \lim_{\varepsilon\to 0^+}\frac{n}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle x-y,\nu(y)\rangle\langle x-y,f(y)\rangle}{|x-y|^{n+2}}(x-y)\,d\sigma(y),\end{aligned} \tag{2.3.112}$$

at $\sigma$-a.e. point $x \in \partial\Omega$. Then, if

$$3\mu+\lambda \neq 0 \quad\text{and}\quad \zeta_* := \frac{\mu(\mu+\lambda)}{3\mu+\lambda}, \tag{2.3.113}$$

from (2.3.109) we see that $C_2(\zeta_*) = 0$, so the last term in (2.3.111) drops out and the principal-value double layer potential operator $K_{A(\zeta_*)}$ becomes

$$\begin{aligned}K_{A(\zeta_*)} &= C_1(\zeta_*)K_\Delta I_{n\times n} - (1-C_1(\zeta_*))Q\\ &= \frac{2\mu}{3\mu+\lambda}K_\Delta I_{n\times n} - \frac{\mu+\lambda}{3\mu+\lambda}Q,\end{aligned} \tag{2.3.114}$$

where $K_\Delta$ is the harmonic double layer potential operator (cf. (2.3.8)). In view of (2.3.8) and (2.3.112), this is in agreement with the prediction made in item *(iii)* of Proposition 2.3.7.

Traditionally, the singular integral operator $K_{A(\zeta_*)}$ from (2.3.114) has been called the (boundary-to-boundary) `pseudo-stress` double layer potential operator for the Lamé system, and the alternative notation $K_\Psi$ has been occasionally employed.

We conclude this series of examples by discussing a case of a second-order, homogeneous, real constant coefficient, weakly elliptic system which does *not* posses a distinguished coefficient tensor.

**Example 2.3.11.** Work in the plane $\mathbb{R}^2 \equiv \mathbb{C}$, and consider the second-order, homogeneous, real constant coefficient, $2\times 2$ system

$$L = \frac{1}{4}\begin{pmatrix}\partial_x^2-\partial_y^2 & -2\partial_x\partial_y\\ 2\partial_x\partial_y & \partial_x^2-\partial_y^2\end{pmatrix}. \tag{2.3.115}$$

An example of a coefficient tensor in $\mathfrak{A}_L$ is given by $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq 2\\ 1\leq\alpha,\beta\leq 2}}$ with

$$\begin{aligned} &a^{11}_{11} = a^{22}_{11} = \tfrac14, \quad a^{11}_{22} = a^{22}_{22} = -\tfrac14, \quad a^{11}_{12} = a^{11}_{21} = a^{22}_{12} = a^{22}_{21} = 0, \\ &a^{12}_{12} = a^{12}_{21} = -\tfrac14, \quad a^{21}_{12} = a^{21}_{21} = \tfrac14, \quad a^{21}_{11} = a^{21}_{22} = a^{12}_{22} = a^{12}_{11} = 0. \end{aligned} \tag{2.3.116}$$

The characteristic matrix of the system $L$ is given by

$$L(\xi) = -\frac{1}{4}\begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix} \ \text{ at each } \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{2.3.117}$$

Hence, at each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ we have

$$\det[L(\xi)] = \tfrac{1}{16}\big[(\xi_1^2 - \xi_2^2)^2 + (2\xi_1\xi_2)^2\big] = \tfrac{1}{16}(\xi_1^2 + \xi_2^2)^2 = \tfrac{1}{16}|\xi|^4 \neq 0, \tag{2.3.118}$$

which goes to show that

$$\text{the system } L \text{ from (2.3.115) is weakly elliptic.} \tag{2.3.119}$$

In particular, $L$ has a fundamental solution as in Theorem 1.2.1 which, once a UR domain in the plane has been fixed, may then be used to associate double layer potential operators $K_A$ with any coefficient tensor $A \in \mathfrak{A}_L$ as in (2.3.4), and all these singular integral operators enjoy the properties discussed in Proposition 2.3.4.

This being said, since with $\eta := (1, 0) \in \mathbb{C}^2$ we have

$$\langle L(\xi)\eta, \overline{\eta}\rangle = -\tfrac14(\xi_1^2 - \xi_2^2) \ \text{ for each } \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \tag{2.3.120}$$

and since the last expression above vanishes identically on the diagonal of $\mathbb{R}^2$, it follows that the system $L$ from (2.3.115) fails to satisfy the Legendre-Hadamard strong ellipticity condition.

To better understand this system, observe that its transposed is

$$L^\top = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix} \tag{2.3.121}$$

and, if $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ are the canonical coordinate projections, defined as

$$\pi_1(z_1, z_2) := z_1 \ \text{ and } \ \pi_2(z_1, z_2) = z_2 \ \text{ for each } \ (z_1, z_2) \in \mathbb{C}^2, \tag{2.3.122}$$

then

$$\begin{gathered} L(u_1, u_2) = \Big(\pi_1 L^\top(u_1, -u_2)\,,\, -\pi_2 L^\top(u_1, -u_2)\Big) \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \\ \text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega). \end{gathered} \tag{2.3.123}$$

As a consequence,

$$\begin{gathered} L(u_1, u_2) = 0 \Longleftrightarrow L^\top(u_1, -u_2) = 0 \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \\ \text{complex-valued function } \ U = u_1 + iu_2 \in \mathscr{C}^2(\Omega). \end{gathered} \tag{2.3.124}$$

Pressing on, recall the Cauchy-Riemann operator and its conjugate

$$\partial_{\bar z} := \tfrac{1}{2}(\partial_x + i\partial_y), \quad \partial_z := \tfrac{1}{2}(\partial_x - i\partial_y), \quad \text{where} \quad z = x + iy, \tag{2.3.125}$$

then bring in Bitsadze's operator (cf. [12], [13]), which is simply the square of $\partial_{\bar z}$, i.e,

$$\mathbb{L} := \partial_{\bar z}^2 = \tfrac{1}{4}\partial_x^2 + \tfrac{i}{2}\partial_x\partial_y - \tfrac{1}{4}\partial_y^2, \qquad z = x + iy. \tag{2.3.126}$$

To place things into a broader perspective, there are three basic prototypes of scalar, constant coefficient, second-order, elliptic operators in the plane: the Laplacian $4\partial_z\partial_{\bar z}$, plus Bitsadze's operator $\partial_{\bar z}^2$ and its complex conjugate $\partial_z^2$. With $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ the canonical coordinate projections from (2.3.122), the system $L$ introduced in (2.3.115) is related to Bitsadze's operator $\mathbb{L} = \partial_{\bar z}^2$ via

$$\begin{gathered} \mathbb{L}(u_1 + iu_2) = \pi_1 L(u_1, u_2) + i\pi_2 L(u_1, u_2) \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \\ \text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega). \end{gathered} \tag{2.3.127}$$

In particular,

$$\begin{gathered} L(\operatorname{Re} U\,,\, \operatorname{Im} U) = \big(\operatorname{Re}(\mathbb{L}U)\,,\, \operatorname{Im}(\mathbb{L}U)\big) \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \\ \text{complex-valued function } \ U \in \mathscr{C}^2(\Omega). \end{gathered} \tag{2.3.128}$$

On the other hand, given any open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ along with any complex-valued function $U \in \mathscr{C}^2(\Omega)$, we have $\partial_{\bar z}^2 U = 0$ if and only if $f := -\partial_{\bar z} U$ is holomorphic in $\Omega$, and the latter condition further implies that the function $g(z) := U(z) + \bar z f(z)$ for each $z \in \Omega$ is holomorphic in $\Omega$. As such, the general format of null-solution of $\partial_{\bar z}^2$ in an open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ is

$$\begin{gathered} U(z) = g(z) - \bar z f(z) \ \text{ for all } \ z \in \Omega, \ \text{ where} \\ f \text{ and } g \text{ are holomorphic functions in } \Omega. \end{gathered} \tag{2.3.129}$$

This is akin to the description of affine functions on the real line as null-solutions of the one-dimensional Laplacian $d^2/dx^2$, with the role of $d/dx$ now played by the Cauchy-Riemann operator $\partial_{\bar z}$, with $\bar z$ now playing the role of the variable $x$, and with holomorphic functions playing the role of constants.

Specializing the expression of $U$ in (2.3.129) to the case when $g(z) := zf(z)$ for each $z \in \Omega$, we obtain the following particular family of null-solutions for Bitsadze's operator $\mathbb{L}$ in any given open set $\Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C}$:

$$\begin{gathered} U(z) = (z - \bar z) f(z), \ \text{ where } f \\ \text{is any holomorphic function in } \Omega. \end{gathered} \tag{2.3.130}$$

From this and (2.3.128) we then conclude that

$$\begin{gathered} \text{given any holomorphic function } f \text{ in an open set } \Omega \subseteq \mathbb{C}, \\ \text{the vector-valued function } u = (u_1, u_2) \text{ with components} \\ u_1(z) := \operatorname{Re}\big[(z - \bar z) f(z)\big] \text{ and } u_2(z) := \operatorname{Im}\big[(z - \bar z) f(z)\big] \text{ for} \\ z \in \Omega \text{ is a null-solution of the system } L \text{ from (2.3.115).} \end{gathered} \tag{2.3.131}$$

In particular, by further specializing this property to the case when $\Omega := \mathbb{R}^2_+ \equiv \mathbb{C}_+$ and the holomorphic function $f(z) := (z+i)^{-m}$ for $z \in \mathbb{C}_+$, where $m \in \mathbb{N}$ is arbitrary, shows that the vector-valued function $u^{(m)} = (u_1^{(m)}, u_2^{(m)})$ with components defined at each $z \in \mathbb{C}_+$ as

$$u_1^{(m)}(z) := \operatorname{Re}\left[(z-\bar{z})(z+i)^{-m}\right] \quad \text{and} \quad u_2^{(m)}(z) := \operatorname{Im}\left[(z-\bar{z})(z+i)^{-m}\right] \tag{2.3.132}$$

is a null-solution of the system $L$ from (2.3.115). Note that each function $u^{(m)}$ belongs to $\big[\mathscr{C}^\infty(\overline{\mathbb{R}^2_+})\big]^2$, vanishes identically on $\partial\mathbb{R}^2_+ \equiv \mathbb{R}$ (since $z - \bar{z} = 0$ for each $z \in \mathbb{R}$), and for each multi-index $\alpha \in \mathbb{N}_0^2$ there exists some $C_\alpha \in (0,\infty)$ with the property that

$$\big|\partial^\alpha u^{(m)}(z)\big| \leq C_\alpha (1+|z|)^{1-m-|\alpha|} \quad \text{for all} \quad z \in \mathbb{R}^2_+. \tag{2.3.133}$$

The estimate above implies that, having fixed an aperture parameter $\kappa > 0$, for each multi-index $\alpha \in \mathbb{N}_0^2$ there exists some $C_\alpha \in (0,\infty)$ such that

$$\mathcal{N}_\kappa(\partial^\alpha u^{(m)})(x) \leq C_\alpha (1+|x|)^{1-m-|\alpha|} \quad \text{for all} \quad x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+. \tag{2.3.134}$$

As such, for any given $p \in (1,\infty)$, any Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and any multi-index $\alpha \in \mathbb{N}_0^2$, we have $\mathcal{N}_\kappa(\partial^\alpha u^{(m)}) \in L^p(\mathbb{R}, w)$ as long as either $m \geq 2$ or $|\alpha| > 0$. Ultimately, this proves that the null-space of the Infinite-Order Regularity Problem for the system $L$ in $\mathbb{R}^2_+$, i.e., the linear space of all vector-valued functions $u$ satisfying

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \\ \mathcal{N}_\kappa(\partial^\alpha u) \in L^p(\mathbb{R}, w) \ \text{ for all } \ \alpha \in \mathbb{N}_0^2, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}, \end{cases} \tag{2.3.135}$$

is infinite dimensional. In particular, the space of null-solutions of the corresponding Dirichlet Problem for the system $L$ in $\mathbb{R}^2_+$ (cf. (2.1.52)) is infinite dimensional. Since in item *(c)* of Theorem 2.6.2 we shall learn that this cannot happen if $\mathfrak{A}^{\text{dis}}_{L^\top} \neq \varnothing$, we then conclude that we necessarily have $\mathfrak{A}^{\text{dis}}_{L^\top} = \varnothing$ in this case. In other words, $L^\top$ from (2.3.121) is a weakly elliptic, second-order, homogeneous, real constant coefficient, $2 \times 2$ system in $\mathbb{R}^2$ which does not posses any distinguished coefficient tensor.

We may also run a variant of this argument, in which we now start with $L^\top$ instead of $L$. If

$$\overline{\mathbb{L}} = \partial_z^2 = \tfrac{1}{4}\partial_x^2 - \tfrac{i}{2}\partial_x\partial_y - \tfrac{1}{4}\partial_y^2 \tag{2.3.136}$$

is the complex conjugate of Bitsadze's operator $\mathbb{L}$ from (2.3.126), then in place of (2.3.127)-(2.3.128) we now have

$$\begin{gathered} \overline{\mathbb{L}}(u_1 + iu_2) = \pi_1 L^\top(u_1,u_2) + i\pi_2 L^\top(u_1,u_2) \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \\ \text{complex-valued functions } \ u_1, u_2 \in \mathscr{C}^2(\Omega), \end{gathered} \tag{2.3.137}$$

and, respectively,

$$\begin{aligned} &L^\top(\operatorname{Re}U\,,\,\operatorname{Im}U) = (\operatorname{Re}(\overline{\mathbb{L}}U)\,,\,\operatorname{Im}(\overline{\mathbb{L}}U)) \\ &\text{for any open set } \Omega\subseteq\mathbb{R}^2\equiv\mathbb{C} \text{ and any} \\ &\text{complex-valued function } \ U\in\mathscr{C}^2(\Omega). \end{aligned} \tag{2.3.138}$$

Keeping in mind that $U$ is a null-solution of $\overline{\mathbb{L}}$ if and only if $\overline{U}$ is a null-solution of $\mathbb{L}$ and reasoning as before, we conclude that, for each $m\in\mathbb{N}$, the vector-valued function $v^{(m)}=(v_1^{(m)},v_2^{(m)})$ with components defined at each $z\in\mathbb{C}_+$ as

$$v_1^{(m)}(z) := \operatorname{Re}\big[(\bar z - z)(\bar z - i)^{-m}\big] \ \text{ and } \ v_2^{(m)}(z) := \operatorname{Im}\big[(\bar z - z)(\bar z - i)^{-m}\big] \tag{2.3.139}$$

is a null-solution of the system $L^\top$ from (2.3.121). In turn, this goes to show that the null-space of the Infinite-Order Regularity Problem for the system $L^\top$ in $\mathbb{R}^2_+$ (formulated as in (2.3.135) with $L^\top$ now replacing $L$) is infinite dimensional. Once this has been established, from item *(c)* in Theorem 2.6.2 we then conclude that $\mathfrak{A}_L^{\rm dis}=\varnothing$. The bottom line is that

$$\begin{aligned} &L \text{ in (2.3.115) is an example of a weakly elliptic, second-order,} \\ &\text{homogeneous, real constant coefficient, } 2\times 2 \text{ system in } \mathbb{R}^2\text{, with} \\ &\text{the property that } \mathfrak{A}_L^{\rm dis}=\varnothing \text{ and } \mathfrak{A}_{L^\top}^{\rm dis}=\varnothing. \end{aligned} \tag{2.3.140}$$

In particular, this goes to show that not every weakly elliptic, second-order, homogeneous, real constant coefficient, system has a distinguished coefficient tensor.

In relation to the system $L$ from (2.3.115) it is also of interest to identify the space of boundary traces of its null-solutions whose nontangential maximal function belongs to a Muckenhoupt weighted Lebesgue space.

**Proposition 2.3.12.** *Fix an integrability index $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\mathbb{R},\mathcal{L}^1)$, and choose an aperture parameter $\kappa>0$. Also, recall the $2\times 2$ system $L$ defined in the plane as in (2.3.115).*

*Then if $u\in\big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function satisfying*

$$Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u\in L^p(\mathbb{R},w), \tag{2.3.141}$$

*and such that the nontangential boundary trace*

$$f := u\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^2_+} \ \text{ exists (in } \mathbb{C}^2) \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}, \tag{2.3.142}$$

*it follows that the function $f$ belongs to $\big[L^p(\mathbb{R},w)\big]^2$ and, if $f_1,f_2\in L^p(\mathbb{R},w)$ are the scalar components of $f$ (i.e., $f=(f_1,f_2)$), then with $H$ denoting the Hilbert transform on the real line (cf. (2.1.21)) one has*

$$Hf_1 = f_2 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \tag{2.3.143}$$

*In the converse direction, for any given $f\in L^p(\mathbb{R},w)$ there exists a vector-valued function $u\in\big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying*

$$\begin{aligned} &Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u\in L^p(\mathbb{R},w), \quad \text{and} \\ &u\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^2_+} = (f, Hf) \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \end{aligned} \tag{2.3.144}$$

*All together, the space of admissible boundary data for the Dirichlet Problem formulated in terms of Muckenhoupt weighted Lebesgue spaces for the system $L$ in the upper half-plane, i.e.,*

$$\Big\{u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \ in\ \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ and\ \ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\ \ exists\ at\ \mathcal{L}^1\text{-}a.e.\ point\ on\ \ \mathbb{R}\Big\}, \tag{2.3.145}$$

*is precisely*

$$\{(f, Hf) : f \in L^p(\mathbb{R}, w)\}. \tag{2.3.146}$$

*As a consequence of this and* (2.3.124)*, one also has*

$$\Big\{u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ L^\top u = 0 \ \ in\ \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ and\ \ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\ \ exists\ at\ \mathcal{L}^1\text{-}a.e.\ point\ on\ \ \mathbb{R}\Big\} \\ = \{(f, -Hf) : f \in L^p(\mathbb{R}, w)\}. \tag{2.3.147}$$

*Proof.* That the function $f$ belongs to $\big[L^p(\mathbb{R}, w)\big]^2$ is clear from $\big|u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big| \leq \mathcal{N}_\kappa u$, the fact that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ is $\mathcal{L}^1$-measurable (cf. [93]), and the last property in (2.3.141).

To proceed, fix a function $u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying (2.3.141)-(2.3.142) and denote by $u_1, u_2 \in \mathscr{C}^\infty(\mathbb{R}^2_+)$ its scalar components. Hence, $u = (u_1, u_2)$ in $\mathbb{R}^2_+$. Also, pick an arbitrary $\varepsilon > 0$ and define

$$U_\varepsilon(z) := u_1(z + \varepsilon i) + iu_2(z + \varepsilon i) \ \text{ for each } \ z \in (\mathbb{R}^2_+ - \varepsilon i). \tag{2.3.148}$$

Then $U_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^2_+ - \varepsilon i)$ and, as seen from (2.3.127), the fact that $Lu = 0$ in $\mathbb{R}^2_+$ translates into $\partial^2_{\bar z} U_\varepsilon = 0$ in $\mathbb{R}^2_+ - \varepsilon i$. Granted this, (2.3.129) then guarantees the existence of two holomorphic functions $f_\varepsilon, g_\varepsilon$ in $\mathbb{R}^2_+ - \varepsilon i$ with the property that

$$U_\varepsilon(z) = g_\varepsilon(z) - \bar z f_\varepsilon(z) \ \text{ for each } \ z \in (\mathbb{R}^2_+ - \varepsilon i). \tag{2.3.149}$$

More specifically, the unique holomorphic functions $f, g$ which do the job in (2.3.149) are

$$f_\varepsilon(z) := -\partial_{\bar z} U_\varepsilon(z) \ \text{ and } \ g_\varepsilon(z) := U_\varepsilon(z) + \bar z f_\varepsilon(z) \ \text{ for each } \ z \in (\mathbb{R}^2_+ - \varepsilon i). \tag{2.3.150}$$

Henceforth, we agree to restrict $U_\varepsilon, f_\varepsilon, g_\varepsilon$ to $\mathbb{R}^2_+$. With this interpretation, introduce

$$W_\varepsilon(z) := g_\varepsilon(z) - z f_\varepsilon(z) \ \text{ for each } \ z \in \mathbb{R}^2_+. \tag{2.3.151}$$

Hence, $W_\varepsilon$ is holomorphic in $\mathbb{R}^2_+$, extends continuously to $\overline{\mathbb{R}^2_+}$, and

$$U_\varepsilon(z) - W_\varepsilon(z) = 2iy f_\varepsilon(z) = -2iy(\partial_{\bar z} U_\varepsilon)(z) \\ \text{for each } \ z = x + iy \in \mathbb{R}^2_+. \tag{2.3.152}$$

From the fact that $\partial_{\bar z}^2 U_\varepsilon = 0$ in $\mathbb{R}^2_+$ we also conclude that $0 = \partial_z^2\partial_{\bar z}^2 U_\varepsilon = \frac{1}{16}\Delta^2 U_\varepsilon$, i.e., the function $U_\varepsilon$ is bi-harmonic in $\mathbb{R}^2_+$. Select $\theta\in(0,1)$ and $\widetilde\kappa\in(0,\kappa)$ both small so that

$$\frac{1+\theta+\widetilde\kappa}{1-\theta} < 1+\kappa. \tag{2.3.153}$$

Fix an arbitrary point $x\in\mathbb{R}\equiv\partial\mathbb{R}^2_+$ and pick some $z_o = x_o+iy_o\in\Gamma_{\widetilde\kappa}(x)$. The inequality demanded in (2.3.153) ensures that

$$B(z_o,\theta y_o)\subseteq\Gamma_\kappa(x). \tag{2.3.154}$$

Based on interior estimates for bi-harmonic functions (cf. [92, Theorem 11.12, p. 415]), (2.3.152), and (2.3.154) we may then estimate, for some constant $C = C(\theta)\in(0,\infty)$,

$$\begin{aligned} \big|U_\varepsilon(z_o)-W_\varepsilon(z_o)\big| &= 2y_o\big|(\partial_{\bar z}U_\varepsilon)(z_o)\big| \leq \sqrt{2}y_o\big|(\nabla U_\varepsilon)(z_o)\big| \\ &\leq C\fint_{B(z_o,\theta y_o)}|U_\varepsilon|\,d\mathcal{L}^2 \leq C(\mathcal{N}_\kappa U_\varepsilon)(x). \end{aligned} \tag{2.3.155}$$

Taking the supremum over all $z_o\in\Gamma_{\widetilde\kappa}(x)$ this ultimately yields

$$\big(\mathcal{N}_{\widetilde\kappa}(U_\varepsilon-W_\varepsilon)\big)(x)\leq C(\mathcal{N}_\kappa U_\varepsilon)(x) \quad\text{for each}\quad x\in\mathbb{R}\equiv\partial\mathbb{R}^2_+. \tag{2.3.156}$$

In turn, (2.3.156) implies

$$\begin{aligned} \mathcal{N}_{\widetilde\kappa}W_\varepsilon &\leq \mathcal{N}_{\widetilde\kappa}U_\varepsilon + \mathcal{N}_{\widetilde\kappa}(U_\varepsilon-W_\varepsilon)\leq\mathcal{N}_\kappa U_\varepsilon + C\mathcal{N}_\kappa U_\varepsilon \\ &= (1+C)\mathcal{N}_\kappa U_\varepsilon \leq (1+C)\mathcal{N}_\kappa u \quad\text{on}\quad \mathbb{R}\equiv\partial\mathbb{R}^2_+. \end{aligned} \tag{2.3.157}$$

Upon recalling that the nontangential maximal function $\mathcal{N}_{\widetilde\kappa}W_\varepsilon$ is non-negative and lower-semicontinuous, we then conclude from (2.3.157), the last property in (2.3.141), and (2.2.337) that

$$\mathcal{N}_{\widetilde\kappa}W_\varepsilon\in L^1\Big(\mathbb{R}\,,\,\tfrac{\mathcal{L}^1(x)}{1+|x|}\Big). \tag{2.3.158}$$

Let us record our progress. The argument so far shows that the function $W_\varepsilon$ is holomorphic in $\mathbb{R}^2_+$, extends continuously to $\overline{\mathbb{R}^2_+}$, and there exists some aperture parameter $\widetilde\kappa>0$ such that $\mathcal{N}_{\widetilde\kappa}W_\varepsilon$ belongs to $L^1\Big(\mathbb{R}\,,\,\frac{\mathcal{L}^1(x)}{1+|x|}\Big)$. These properties allow us to invoke the Cauchy reproducing formula (proved in [93] in much more general geometric settings) which asserts that

$$W_\varepsilon(z) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{(W_\varepsilon\big|_{\mathbb{R}})(y)}{y-z}\,dy, \quad\text{for each}\quad z\in\mathbb{R}^2_+. \tag{2.3.159}$$

Since $f_\varepsilon, g_\varepsilon$ extend continuously to $\overline{\mathbb{R}^2_+}$, from (2.3.149), (2.3.151), and the fact that $z=\bar z$ on $\mathbb{R}\equiv\partial\mathbb{R}^2_+$, we conclude that

$$W_\varepsilon\big|_{\mathbb{R}} = U_\varepsilon\big|_{\mathbb{R}} \quad\text{on}\quad \mathbb{R}\equiv\partial\mathbb{R}^2_+. \tag{2.3.160}$$

Hence, if we abbreviate

$$h_\varepsilon := U_\varepsilon\big|_{\mathbb{R}} \quad\text{on}\quad \mathbb{R}\equiv\partial\mathbb{R}^2_+, \tag{2.3.161}$$

after taking the nontangential boundary traces of both sides in (2.3.159) and using the Plemelj jump-formula for the Cauchy operator (which continues to be valid in this setting; see [93]) we arrive at

$$h_\varepsilon = \big(\tfrac{1}{2}I + \tfrac{i}{2}H\big)h_\varepsilon \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}, \tag{2.3.162}$$

where $I$ is the identity and $H$ is the Hilbert transform on $\mathbb{R}$. Hence, on the one hand, we may rewrite (2.3.162) simply as

$$Hh_\varepsilon = -ih_\varepsilon \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \tag{2.3.163}$$

On the other hand, from (2.3.161) and (2.3.148) we see that

$$h_\varepsilon(x) = u_1(x+\varepsilon i) + iu_2(x+\varepsilon i) \ \text{ for } \mathcal{L}^1\text{-a.e. } \ x \in \mathbb{R}. \tag{2.3.164}$$

In turn, this implies

$$|h_\varepsilon(x)| \le \sqrt{2}(\mathcal{N}_\kappa u)(x) \ \text{ for } \mathcal{L}^1\text{-a.e. } \ x \in \mathbb{R}, \tag{2.3.165}$$

and, when used in concert with (2.3.142), that

$$\begin{aligned}\lim_{\varepsilon\to 0^+} h_\varepsilon(x) &= \big(u_1\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) + i\big(u_2\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\big)(x) \\ &= f_1(x) + if_2(x) \ \text{ for } \mathcal{L}^1\text{-a.e. } \ x \in \mathbb{R}. \end{aligned} \tag{2.3.166}$$

Thanks to (2.3.165)-(2.3.166) and the last property in (2.3.141), we may now invoke Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + if_2 \ \text{ in } \ L^p(\mathbb{R}, w). \tag{2.3.167}$$

Having established this, on account of (2.3.163) and the continuity of the Hilbert transform $H$ on the Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}, w)$ we obtain

$$H(f_1 + if_2) = -i(f_1 + if_2) \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \tag{2.3.168}$$

The idea is now write $u = \operatorname{Re} u + i \operatorname{Im} u$ and observe that, since the coefficients of the system $L$ are real, $\operatorname{Re} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ and $\operatorname{Im} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ enjoy the same properties as the function $u$ in (2.3.141)-(2.3.142). Granted what we have proved already, it follows that if $\phi_1, \phi_2$ are the scalar components of $(\operatorname{Re} u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ and if $\psi_1, \psi_2$ are the scalar components of $(\operatorname{Im} u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ then $\phi_1, \phi_2, \psi_1, \psi_2$ are real-valued functions belonging to $L^p(\mathbb{R}, w)$, and the conclusion in (2.3.168) written separately for $\operatorname{Re} u$ and $\operatorname{Im} u$ gives

$$H(\phi_1 + i\phi_2) = -i(\phi_1 + i\phi_2) \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}, \tag{2.3.169}$$

and, respectively,

$$H(\psi_1 + i\psi_2) = -i(\psi_1 + i\psi_2) \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \tag{2.3.170}$$

In particular, taking the real parts in (2.3.169)-(2.3.170) (keeping in mind that $H$ maps real-valued functions into real-valued functions) leads to the conclusion that

$$H\phi_1 = \phi_2 \quad \text{and} \quad H\psi_1 = \psi_2. \tag{2.3.171}$$

Upon observing that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = (\operatorname{Re} u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} + i\,(\operatorname{Im} u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ implies $f_1 = \phi_1 + i\psi_1$ and $f_2 = \phi_2 + i\psi_2$, from (2.3.171) we readily obtain the formula claimed in (2.3.143).

In the converse direction, suppose first that the function $f \in L^p(\mathbb{R}, w)$ is real-valued. Then $Hf \in L^p(\mathbb{R}, w)$ and work in [93] then ensures that

$$U(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(f+iHf)(y)}{y-z}\,dy, \quad \text{for each} \quad z \in \mathbb{R}^2_+, \tag{2.3.172}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ and

$$U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = \big(\tfrac{1}{2}I + \tfrac{i}{2}H\big)(f+iHf) = f + iHf \quad \text{at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R}, \tag{2.3.173}$$

since the Hilbert transform satisfies $H^2 = -I$ on $L^p(\mathbb{R}, w)$. If we now set $u_1 := \operatorname{Re} U$ and $u_2 := \operatorname{Im} U$, then $u := (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components. Thanks to (2.3.128), we have

$$Lu = L(\operatorname{Re} U\,,\,\operatorname{Im} U) = \big(\operatorname{Re}(\partial^2_{\bar z} U)\,,\,\operatorname{Im}(\partial^2_{\bar z} U)\big) = 0 \in \mathbb{C}^2 \quad \text{in} \quad \mathbb{R}^2_+, \tag{2.3.174}$$

since $\partial_{\bar z} U = 0$ in $\mathbb{R}^2_+$ by the Cauchy-Riemann equations. Also, $\mathcal{N}_\kappa u = \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w)$ given that, by design, $|u| = |U|$. Finally, at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$ we have

$$u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = \Big(\operatorname{Re} U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\,,\,\operatorname{Im} U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = (f, Hf), \tag{2.3.175}$$

by virtue of (2.3.173) and the fact that $f$ is real-valued. Thus, $u$ satisfies all requirements in (2.3.144).

To deal with an arbitrary function $f \in L^p(\mathbb{R}, w)$ which is not necessarily real-valued, denote by $\phi, \psi$ the real and imaginary parts of $f$. In particular, $f = \phi + i\psi$. From what we have proved so far, there exist $v, w \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ as in (2.3.144) such that $v\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = (\phi, H\phi)$ and $w\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = (\psi, H\psi)$. Then the function $u := v + iw \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is as in (2.3.144) and satisfies $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = (f, Hf)$, as wanted. □

*Remark* 2.3.13. Suppose $w \in A_p(\mathbb{R}, \mathcal{L}^1)$ for some $p \in (1, \infty)$ and choose an aperture parameter $\kappa > 0$. Also, let $L$ be the $2\times 2$ system from (2.3.115) and assume $u : \mathbb{R}^2_+ \to \mathbb{C}^2$ is a function satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^2_+, \quad \mathcal{N}_\kappa u \in L^p(\mathbb{R}, w), \\ \text{and} \ \ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}. \end{gathered} \tag{2.3.176}$$

In particular, $u$ satisfies (2.3.141)-(2.3.142) with $f = (f_1, f_2) = (0, 0)$. Retaining notation from the proof of Proposition 2.3.12, from (2.3.159), (2.3.160), and (2.3.161) we see that

$$W_\varepsilon(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{h_\varepsilon(y)}{y-z}\,dy, \quad \text{for each} \quad z \in \mathbb{R}^2_+. \tag{2.3.177}$$

Let $U := u_1 + iu_2$, where $u_1, u_2$ are the two scalar components of the $\mathbb{C}^2$-valued function $u$. On the one hand, from (2.3.148)-(2.3.150) it is clear that

$$\lim_{\varepsilon\to 0^+} W_\varepsilon(z) = U(z) + (z-\bar{z})(\partial_{\bar{z}}U)(z) \quad \text{for fixed each} \quad z \in \mathbb{R}^2_+. \tag{2.3.178}$$

On the other hand, for each fixed $z \in \mathbb{R}^2_+$, on account of (2.3.167) and the fact that we currently have $f_1 + if_2 = 0$, we conclude that the limit as $\varepsilon \to 0^+$ of the integral in (2.3.177) is zero. Based on these observations and (2.3.177), we ultimately conclude that

if $u$ is as in (2.3.176) then the $\mathbb{C}$-valued function $U := u_1 + iu_2$ (where $u_1, u_2$ are the two scalar components of the $\mathbb{C}^2$-valued function $u$) satisfies $U(z) = (\bar{z} - z)(\partial_{\bar{z}}U)(z)$ for each $z \in \mathbb{R}^2_+$. (2.3.179)

The same type of argument also shows that

$$\left.\begin{array}{r} U \in \mathscr{C}^\infty(\mathbb{R}^2_+) \\ \partial^2_{\bar{z}} U = 0 \ \text{ in } \ \mathbb{R}^2_+ \\ \mathcal{N}_\kappa U \in L^p(\mathbb{R}, w) \\ U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = 0 \ \text{ on } \ \mathbb{R} \end{array}\right\} \Longrightarrow U(z) = (\bar{z} - z)(\partial_{\bar{z}}U)(z) \ \text{ for all } \ z \in \mathbb{R}^2_+. \tag{2.3.180}$$

Bearing in mind that for any $U$ as in the left side of (2.3.180) the function $f := -\partial_{\bar{z}}U$ is holomorphic in $\mathbb{R}^2_+$, we may recast the conclusion in (2.3.180) as saying that there exists some holomorphic function $f$ in $\mathbb{R}^2_+$ such that $U(z) = (z - \bar{z})f(z)$ for each $z \in \mathbb{R}^2_+$. In particular, this shows that that the choice $g(z) := zf(z)$ which has led to the conclusion in (2.3.130) is actually canonical in the case when $\Omega = \mathbb{R}^2_+$, the nontangential trace of $U$ vanishes, and the nontangential maximal function of $U$ belongs to a Muckenhoupt weighted Lebesgue space.

By further building on Proposition 2.3.12, below we identify the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system $L$ from (2.3.115) in the upper half-plane.

**Proposition 2.3.14.** *Fix an integrability index $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$, and choose an aperture parameter $\kappa > 0$. Also, recall the $2\times 2$ system $L$ defined in the plane as in (2.3.115). Then the space of admissible boundary data for the Muckenhoupt weighted version of the Regularity Problem for the system $L$ in the upper half-plane, i.e.,*

$$\Big\{ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ Lu = 0 \ \text{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w), \\ \text{and } \ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{exists at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R} \Big\}, \tag{2.3.181}$$

*coincides with*

$$\big\{ (f, Hf) : f \in L^p_1(\mathbb{R}, w) \big\}. \tag{2.3.182}$$

*As a consequence of this and (2.3.124), one also has*

$$\Big\{u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} : u \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2,\ L^\top u = 0 \ \text{ in } \ \mathbb{R}^2_+,\ \mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w),$$
$$\text{and } \ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \ \text{ exists at } \mathcal{L}^1\text{-a.e. point on } \ \mathbb{R}\Big\}$$
$$= \big\{(f, -Hf) : f \in L^p_1(\mathbb{R}, w)\big\}. \tag{2.3.183}$$

*Proof.* Consider some function $u = (u_1, u_2) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ satisfying $Lu = 0$ in $\mathbb{R}^2_+$, with $\mathcal{N}_\kappa u, \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$, and such that $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ exists at $\mathcal{L}^1$-a.e. point on $\mathbb{R}$. In particular, Proposition 2.3.12 guarantees that $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = (f, Hf)$ for some $f \in L^p(\mathbb{R}, w)$. Then actually $f = u_1\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \in L^p_1(\mathbb{R}, w)$, thanks to Proposition 2.2.49 (used with $u_1$ in place of $u$ and with $\Omega := \mathbb{R}^2_+$). This proves that the nontangential boundary trace $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}$ belongs to the space in (2.3.182).

Conversely, start with a function $f \in L^p_1(\mathbb{R}, w)$, which is first assumed to be real-valued. Work in [93] ensures that $Hf \in L^p_1(\mathbb{R}, w)$ and

$$U(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{(f + iHf)(y)}{y - z}\,dy, \quad \text{for each } \ z \in \mathbb{R}^2_+, \tag{2.3.184}$$

is a holomorphic function in $\mathbb{R}^2_+$ satisfying $\mathcal{N}_\kappa U$, $\mathcal{N}_\kappa(\nabla U) \in L^p(\mathbb{R}, w)$ and, much as in (2.3.173), $U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = f + iHf$. Then $u := (\operatorname{Re} U, \operatorname{Im} U) \in \big[\mathscr{C}^\infty(\mathbb{R}^2_+)\big]^2$ is a vector-valued function with real-valued scalar components, satisfying $\mathcal{N}_\kappa u$, $\mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, w)$ and

$$u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} = \Big(\operatorname{Re} U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}, \operatorname{Im} U\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+}\Big) = (f, Hf), \tag{2.3.185}$$

since $f$ is real-valued. Given that, much as in (2.3.174) we also have $Lu = 0$ in $\mathbb{R}^2_+$, it follows that $(f, Hf)$ belongs to the space in (2.3.181). Finally, the general case when $f \in L^p_1(\mathbb{R}, w)$ is not necessarily real-valued is dealt with based on what we have just proved, decomposing $f$ into its real and imaginary parts. This eventually shows that the space from (2.3.182) is contained in the space from (2.3.181). By double inclusion, we may therefore conclude that the said spaces are in fact equal. □

In the last part of this section we elaborate on connections with Poisson kernels, introduced in Theorem 1.2.4. We wish to augment Theorem 1.2.4 with the following result proved in [84], which identifies yet another scenario when a Poisson kernel exists.

**Proposition 2.3.15.** *Let $L$ be an $M \times M$ homogeneous constant complex coefficient second-order system in $\mathbb{R}^n$ which is weakly elliptic and satisfies $\mathfrak{A}^{\text{dis}}_L \neq \varnothing$. Fix a coefficient tensor $A \in \mathfrak{A}^{\text{dis}}_L$ and bring in the function $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}^{M\times M}$ from (2.3.80), associated with $A$ as in item (f) of Proposition 2.3.5. Then the matrix-valued function*

$$\begin{gathered} P : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M\times M} \ \text{ defined by} \\ P(x') := 2k(x', 1) \ \text{ for each } \ x' \in \mathbb{R}^{n-1} \end{gathered} \tag{2.3.186}$$

*is a Poisson kernel for $L$ in $\mathbb{R}^n_+$ (in the sense that it satisfies the properties in Theorem 1.2.4).*

Based on the mere knowledge that the system $L \in \mathfrak{L}_*$ has a Poisson kernel in $\mathbb{R}^n_+$, the Muckenhoupt weighted Dirichlet Problem for $L$ in the upper half-space (formulated as in (2.1.52) with $\Omega := \mathbb{R}^n_+$) has been shown in [82] to be always solvable. In concert with Proposition 2.3.12, this solvability result offers an alternative proof of (2.3.140).

## 2.4 Boundedness and invertibility of double layer potentials

The key result in this work is Theorem 2.4.4 which elaborates on the nature of the operator-norm of a singular integral operator $T$ defined on the boundary of a UR domain $\Omega$ whose integral kernel has a special algebraic format, through the presence of the inner product between the outward unit normal $\nu$ to $\Omega$ and the chord, as a factor. Proving this theorem requires extensive preparations and takes quite a bit of effort, but the redeeming feature of Theorem 2.4.4 is that the said operator-norm estimate involves the BMO semi-norm of $\nu$ as a factor. This trademark attribute (which is shared by the double layer operator $K_A$ associated with a distinguished coefficient tensor $A$) entails that the flatter $\partial\Omega$ is, the smaller $\|T\|$ is. In particular, having $\partial\Omega$ sufficiently flat ultimately allows us to invert $\frac{1}{2}I + K_A$ on Muckenhoupt weighted Lebesgue spaces via a Neumann series, and this is of paramount importance later on, when dealing with boundary value problems via the method of boundary layer potentials.

### 2.4.1 Estimates for Euclidean singular integral operators

We begin with a few generalities of functional analytic nature. Given two normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, consider a positively homogeneous mapping $T : X \to Y$, i.e., a function $T$ sending $X$ into $Y$ and satisfying $T(\lambda u) = \lambda T(u)$ for each $u \in X$ and each $\lambda \in (0, \infty)$ (note that taking $u := 0 \in X$ and $\lambda := 2$ implies $T(0) = 0 \in Y$). We shall denote by

$$\|T\|_{X\to Y} := \sup\big\{\|Tu\|_Y : \, u \in X, \, \|u\|_X = 1\big\} \in [0, \infty] \tag{2.4.1}$$

the operator norm of such a mapping $T$; in particular,

$$\|Tu\|_Y \leq \|T\|_{X\to Y}\|u\|_X \quad \text{for each} \quad u \in X. \tag{2.4.2}$$

It is then straightforward to check that a positively homogeneous mapping $T : X \to Y$ is continuous at $0 \in X$ if and only if $T$ is bounded (i.e., it maps bounded subsets of $X$ into bounded subsets of $Y$) if and only if $\|T\|_{X\to Y} < +\infty$.

Consider now the special case when $X, Y$ are Lebesgue spaces (associated with a generic measure space) and $T$ is a sub-linear mapping of $X$ into $Y$ (i.e., $T : X \to Y$ satisfies $T(\lambda u) = |\lambda| T(u)$ for each $u \in X$ and each scalar $\lambda$, as well as $T(u+w) \leq Tu + Tw$ at a.e. point, for each $u, w \in X$). Then, for each $u, w \in X$ we have $|Tu - Tw| \leq T(u - w)$ at a.e. point, hence $\|Tu - Tw\|_Y \leq \|T(u - w)\|_Y \leq \|T\|_{X\to Y}\|u - w\|_X$. Consequently,

$$\begin{gathered}\text{a sub-linear map } T : X \to Y \text{ is continuous}\\ \text{if and only if } \|T\|_{X\to Y} < +\infty.\end{gathered} \tag{2.4.3}$$

Let us now start in earnest. To facilitate dealing with Theorem 2.4.2 a little later, we first isolate a useful estimate in the lemma below.

**Lemma 2.4.1.** *Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^n,\mathcal{L}^n)$. Then there exists a constant $C \in (0,\infty)$ which only depends on $n$, $p$, and $[w]_{A_p}$, with the property that for each $x \in \mathbb{R}^n$, each $r \in (0,\infty)$, and real-valued function $A \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ with*

$$\nabla A \in \big[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)\big]^n \tag{2.4.4}$$

*one has*

$$\begin{aligned}\int\limits_{\substack{y\in\mathbb{R}^n\\ |x-y|>r}} &\frac{\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p}{|x-y|^{(n+1)p}}\,dw(y)\\ &\qquad\leq C r^p\, w\big(B(x,r)\big)\big\|\nabla A\big\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}.\end{aligned}\tag{2.4.5}$$

*Proof.* For starters, from Lemma 2.2.45 and (2.4.4) we see that

$$\nabla A \in \big[L^1_{\rm loc}(\mathbb{R}^n,w)\big]^n. \tag{2.4.6}$$

Next, recall from (2.2.311) that there exists $\varepsilon \in (0,p-1)$ which depends only on $p$, $n$, and $[w]_{A_p}$, such that

$$w \in A_{p-\varepsilon}(\mathbb{R}^n,\mathcal{L}^n). \tag{2.4.7}$$

Fix $x \in \mathbb{R}^n$, $r \in (0,\infty)$, and a function $A$ as in the statement of the lemma. By breaking up the integral dyadically, estimating the denominator, and using the doubling property of $w \in A_{p-\varepsilon}(\mathbb{R}^n,\mathcal{L}^n)$ (cf. item *(5)* of Proposition 2.2.42) we may dominate

$$\begin{aligned}\int\limits_{\substack{y\in\mathbb{R}^n\\ |x-y|>r}} &\frac{\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p}{|x-y|^{(n+1)p}}\,dw(y)\\ &\leq C_{n,p}\sum_{j=1}^{\infty}\frac{w(B(x,2^j r))}{2^{j(n+1)p}}\cdot \mathrm{I}_j \leq C_{n,p,w}\sum_{j=1}^{\infty}\frac{2^{jn(p-\varepsilon)}w(B(x,r))}{2^{j(n+1)p}}\cdot \mathrm{I}_j,\end{aligned}\tag{2.4.8}$$

where, for each $j \in \mathbb{N}$,

$$\mathrm{I}_j := \frac{1}{w(B(x,2^j r))}\int\limits_{2^{j-1}r<|x-y|\leq 2^j r}\big|A(x)-A(y)-\langle\nabla A(y),x-y\rangle\big|^p\,dw(y). \tag{2.4.9}$$

To proceed, for each $j \in \mathbb{N}$ introduce

$$A_j(z) := A(z) - \Big(\fint_{B(x,2^j r)}\nabla A\,dw\Big)\cdot z \quad\text{for each}\quad z\in\mathbb{R}^n \tag{2.4.10}$$

(making use of (2.4.6) to ensure that this is meaningful), and observe that $\mathrm{I}_j$, originally defined in (2.4.9), does not change if the function $A$ is replaced by $A_j$. Consequently, for each $j \in \mathbb{N}$ we have

$$\mathrm{I}_j \leq C_p\cdot \mathrm{II}_j + C_p\cdot \mathrm{III}_j, \tag{2.4.11}$$

where

$$\mathrm{II}_j := \frac{1}{w(B(x,2^j r))} \int_{2^{j-1}r<|x-y|\leq 2^j r} |A_j(x)-A_j(y)|^p\,dw(y), \tag{2.4.12}$$

and

$$\mathrm{III}_j := \frac{2^{jp}r^p}{w(B(x,2^j r))} \int_{2^{j-1}r<|x-y|\leq 2^j r} |\nabla A_j(y)|^p\,dw(y). \tag{2.4.13}$$

Fix an integrability exponent $q \in (n,\infty)$ and pick $j \in \mathbb{N}$ arbitrary. Then for each $y \in \mathbb{R}^n$ such that $2^{j-1}r < |x-y| \leq 2^j r$ we may estimate

$$\begin{aligned}
|A_j(x)-A_j(y)| &\leq C_{q,n}|x-y|\Big(⨍_{|x-z|\leq 2|x-y|} |\nabla A_j(z)|^q\,dz\Big)^{1/q}\\
&\leq C_{q,n,w}\cdot 2^j r\Big(⨍_{B(x,2|x-y|)} |\nabla A_j|^{pq}\,dw\Big)^{1/(pq)}\\
&\leq C_{q,n,w}\cdot 2^j r\Big(⨍_{B(x,2|x-y|)} \Big|\nabla A - ⨍_{B(x,2|x-y|)} \nabla A\,dw\Big|^{pq}\,dw\Big)^{1/(pq)}\\
&\quad + C_{q,n,w}\cdot 2^j r\Big|⨍_{B(x,2^j r)} \nabla A\,dw - ⨍_{B(x,2|x-y|)} \nabla A\,dw\Big|\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n}\\
&\leq C_{q,n,w}\cdot 2^j r\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}.
\end{aligned} \tag{2.4.14}$$

Above, the first estimate is provided by Mary Weiss' Lemma (cf. [19, Lemma 1.4, p. 144], or [47, Lemma 2.10, p. 477]), the second estimate uses $|x-y| \leq 2^j r$ and Lemma 2.2.41, the third estimate is implied by (2.4.10) which gives $\nabla A_j = \nabla A - ⨍_{B(x,2^j r)} \nabla A\,dw$, the penultimate estimate is a consequence of the John-Nirenberg inequality, (2.2.44) (written with $w$ in place of $\sigma$), and the doubling property of $w$, while the final estimate in (2.4.14) comes from Lemma 2.2.46. In turn, (2.4.12) and (2.4.14) yield

$$\mathrm{II}_j \leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n}. \tag{2.4.15}$$

By combining (2.4.13) and (2.4.10) we also see that

$$\begin{aligned}
\mathrm{III}_j &\leq 2^{jp}r^p ⨍_{B(x,2^j r)} \Big|\nabla A - ⨍_{B(x,2^j r)} \nabla A\,dw\Big|^p\,dw\\
&\leq C\cdot 2^{jp}r^p\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^n,w)]^n} \leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n},
\end{aligned} \tag{2.4.16}$$

where the last inequality is once again provided by Lemma 2.2.46. From (2.4.15)-(2.4.16) and (2.4.11) we then conclude that

$$\mathrm{I}_j \leq C\cdot 2^{jp}r^p\|\nabla A\|^p_{[\mathrm{BMO}(\mathbb{R}^n,\mathcal{L}^n)]^n} \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.4.17}$$

Using this back in (2.4.8) now readily yields (2.4.5), since $\sum_{j=1}^{\infty} 2^{-jn\varepsilon} < \infty$. ☐

The next result, dealing with boundedness for certain type of singular integral operators in the Euclidean context, refines work in [53, Theorem 4.34, p. 2725].

**Theorem 2.4.2.** *Pick an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. Recall the dual weight $w' := w^{1-p'} \in A_{p'}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ of $w$, where $p' \in (1, \infty)$ is the Hölder conjugate exponent of $p$. Next, fix three numbers $n, m, d \in \mathbb{N}$ with $n \geq 2$, and let $N = N(n, m) \in \mathbb{N}$ be a sufficiently large integer. Let $A \in W^{1,1}_{\rm loc}(\mathbb{R}^{n-1})$ be a complex-valued function with the property that*

$$\nabla A \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}. \tag{2.4.18}$$

*Also, for each $j \in \{1, \dots, m\}$ consider a real-valued function $B_j \in W^{1,1}_{\rm loc}(\mathbb{R}^{n-1})$ with the property that*

$$\nabla B_j \in \big[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}, \tag{2.4.19}$$

*and set $B := (B_1, \dots, B_m)$. In addition, consider a function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^d$ for which there exists $c \in (0, 1]$ such that*

$$c|x' - y'| \leq |\Phi(x') - \Phi(y')| \leq c^{-1}|x' - y'| \quad \textit{for all} \quad x', y' \in \mathbb{R}^{n-1}; \tag{2.4.20}$$

*hence, $\Phi$ is bi-Lipschitz. Going further, suppose $F \in \mathscr{C}^{N+2}(\mathbb{R}^m)$ is a complex-valued function which is even, has the property that $\partial^\alpha F$ belongs to $L^1(\mathbb{R}^m, \mathcal{L}^m)$ for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N + 2$, and*

$$\sup_{X \in \mathbb{R}^m} \big[(1 + |X|)|F(X)|\big] < +\infty. \tag{2.4.21}$$

*Finally, for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and each point $x' \in \mathbb{R}^{n-1}$ define*

$$\begin{aligned} T^{A,B}_{\Phi,*} g(x') := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x') - \Phi(y')| > \varepsilon}} & \frac{A(x') - A(y') - \langle \nabla A(y'), x' - y' \rangle}{|x' - y'|^n} \times \\ & \times F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y')\, dy' \Bigg|. \end{aligned} \tag{2.4.22}$$

*Then $T^{A,B}_{\Phi,*}$ is a well-defined, continuous, sub-linear mapping of the Muckenhoupt weighted Lebesgue space $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself, and there exists some constant $C(n, p, w) \in (0, \infty)$ which depends only on $n$, $p$, and $[w]_{A_p}$ with the property that*

$$\begin{aligned} & \Big\| T^{A,B}_{\Phi,*} \Big\|_{L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})} \\ & \quad \leq C(n, p, w) \cdot c^{-3n} \Big( \sum_{|\alpha| \leq N+2} \|\partial^\alpha F\|_{L^1(\mathbb{R}^m, \mathcal{L}^m)} + \sup_{X \in \mathbb{R}^m} (1 + |X|)|F(X)| \Big) \\ & \qquad \times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big(1 + \sum_{j=1}^m \|\nabla B_j\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \Big)^N. \end{aligned} \tag{2.4.23}$$

Theorem 2.4.2 is an intricate piece of machinery allowing us to estimate, in a rather detailed and specific manner, the maximal operator associated with integral kernels that exhibit a certain type of algebraic structure. We shall put this to good use in Lemma 2.4.3 which, in turn, is a basic ingredient in the proof of Theorem 2.4.4 (the main result in this section). This being said, Theorem 2.4.2 is useful for a variety of other purposes.

To give an example, work in the one-dimensional setting and recall the Hilbert transform $H$ on the real line from (2.1.21). Also, consider a complex-valued function $A \in W^{1,1}_{\rm loc}(\mathbb{R})$ with the property that $A' \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$. Let $M_A$ stand for the operator of pointwise multiplication by $A$, and denote by $D$ the one-dimensional derivative operator $f \mapsto df/dx$ on the real line. Finally, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\mathbb{R}, \mathcal{L}^1)$. Then the commutator $[H, M_A D]$, originally defined on $\mathscr{C}^\infty_0(\mathbb{R})$, extends to a bounded linear mapping on $L^p(\mathbb{R}, w)$ with operator norm $\leq C\,\|A'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}$ where $C \in (0, \infty)$ is an absolute constant. Indeed, given any $f \in \mathscr{C}^\infty_0(\mathbb{R})$, at $\mathcal{L}^1$-a.e. differentiability point $x \in \mathbb{R}$ for $A$ (hence, at $\mathcal{L}^1$-a.e. $x \in \mathbb{R}$) we may write (keeping in mind that, since the Hilbert transform is a multiplier, $H$ commutes with differentiation):

$$
\begin{aligned}
[H, M_A D]f(x) &= H(Af')(x) - A(x)\frac{d}{dx}(Hf(x)) = H(Af')(x) - A(x)(Hf')(x) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{A(y)-A(x)}{x-y} f'(y)\,dy \\
&= -\lim_{\varepsilon\to 0^+} \Big(\frac{A(y)-A(x)}{x-y} f(y)\Big|_{y=x-\varepsilon}^{y=x+\varepsilon}\Big) \\
&\quad - \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{d}{dy}\Big(\frac{A(y)-A(x)}{x-y}\Big) f(y)\,dy \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{A(x)-A(y)-A'(y)(x-y)}{(x-y)^2} f(y)\,dy. \qquad (2.4.24)
\end{aligned}
$$

(The fact that the limit in the third line of (2.4.24) vanishes is ensured by the differentiability of $A$ at $x$, and the continuity of $f$ at $x$). Granted this formula, Theorem 2.4.2 applies with $n = 2$, $m = 1$, $\Phi$ the identity, $B \equiv 0$, and taking $F \in \mathscr{C}^\infty_0(\mathbb{R})$ to be an even function with $F(0) = 1$. The desired conclusion then follows from (2.4.23).

To offer another example where Theorem 2.4.2 plays a decisive role, fix $\varkappa \in (0, \infty)$ and suppose $\Sigma$ is a $\varkappa$-CAC passing through infinity in $\mathbb{C}$. Recall the Cauchy integral operator on the chord-arc curve $\Sigma$ acts on each $f \in L^1\big(\Sigma, \frac{d\mathcal{H}^1(\zeta)}{1+|\zeta|}\big)$ according to

$$
(C_\Sigma f)(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi i} \int\limits_{\substack{\zeta\in\Sigma\\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\,d\zeta \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \Sigma. \qquad (2.4.25)
$$

Since from Proposition 2.2.9 we know that $\Sigma$ is the topological boundary of a UR domain, Proposition 2.3.3 guarantees that $C_\Sigma$ is a well-defined, linear, and bounded operator on $L^p(\Sigma,w)$ whenever $p\in(1,\infty)$ and $w\in A_p(\Sigma,\sigma)$, where $\sigma:=\mathcal{H}^1\lfloor\Sigma$. Let us indicate how Theorem 2.4.2 may be used to show that

the flatter the chord-arc curve $\Sigma$ becomes, the closer the corresponding Cauchy operator becomes (with proximity measured in the operator norm on Muckenhoupt weighted Lebesgue spaces) to the (suitably normalized) Hilbert transform on the real line. (2.4.26)

A brief discussion on this topic may be found in [23, pp. 138-139]. In order to facilitate a direct comparison between the two singular integral operators mentioned in (2.4.26), it is natural to consider the pull-back of $C_\Sigma$ to $\mathbb{R}$ under the arc-length parametrization $\mathbb{R}\ni s\mapsto z(s)\in\mathbb{C}$ of $\Sigma$. After natural adjustments in notation, this corresponds to the mapping sending each $f\in L^p(\mathbb{R},w)$ into

$$(C_{\mathbb{R}}f)(t):=\lim_{\varepsilon\to0^+}\frac{i}{2\pi}\int\limits_{\substack{s\in\mathbb{R}\\|z(t)-z(s)|>\varepsilon}}\frac{z'(s)}{z(t)-z(s)}f(s)\,ds\ \text{ for }\mathcal{L}^1\text{-a.e. }\ t\in\mathbb{R},\tag{2.4.27}$$

where $p\in(1,\infty)$ and $w\in A_p(\mathbb{R},\mathcal{L}^1)$. Recall from (2.2.94) that the function $z(\cdot)$ is bi-Lipschitz, specifically,

$$(1+\varkappa)^{-1}|t-s|\leq|z(t)-z(s)|\leq|t-s|\ \text{ for all }\ t,s\in\mathbb{R}.\tag{2.4.28}$$

Keeping this in mind, a suitable application[1] of [54, Proposition B.2] allows to change the truncation in (2.4.27) to

$$(C_{\mathbb{R}}f)(t)=\lim_{\varepsilon\to0^+}\frac{i}{2\pi}\int\limits_{\substack{s\in\mathbb{R}\\|t-s|>\varepsilon}}\frac{z'(s)}{z(t)-z(s)}f(s)\,ds\ \text{ for }\mathcal{L}^1\text{-a.e. }\ t\in\mathbb{R},\tag{2.4.29}$$

for each $f\in L^p(\mathbb{R},w)$ with $p\in(1,\infty)$ and $w\in A_p(\mathbb{R},\mathcal{L}^1)$. We wish to compare the operator written in this form with the (suitably normalized) Hilbert transform on the real line, acting on functions $f\in L^p(\mathbb{R},w)$, where $p\in(1,\infty)$ and $w\in A_p(\mathbb{R},\mathcal{L}^1)$, according to

$$(Hf)(t):=\lim_{\varepsilon\to0^+}\frac{1}{\pi}\int\limits_{\substack{s\in\mathbb{R}\\|t-s|>\varepsilon}}\frac{f(s)}{t-s}\,ds\ \text{ for }\mathcal{L}^1\text{-a.e. }\ t\in\mathbb{R}.\tag{2.4.30}$$

Fix $p\in(1,\infty)$, $w\in A_p(\mathbb{R},\mathcal{L}^1)$, and $f\in L^p(\mathbb{R},w)$. Then at $\mathcal{L}^1$-a.e. $t\in\mathbb{R}$ we may express

$$\begin{aligned}(C_{\mathbb{R}}-(i/2)H)f(t)&=\lim_{\varepsilon\to0^+}\frac{i}{2\pi}\int\limits_{\substack{s\in\mathbb{R}\\|t-s|>\varepsilon}}\Big(\frac{z'(s)}{z(t)-z(s)}-\frac{1}{t-s}\Big)f(s)\,ds\\&=\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int\limits_{\substack{s\in\mathbb{R}\\|t-s|>\varepsilon}}\frac{z(t)-z(s)-z'(s)(t-s)}{(z(t)-z(s))(t-s)}f(s)\,ds.\end{aligned}\tag{2.4.31}$$

[1]While [54, Proposition B.2] is stated for ordinary Lebesgue spaces, the same type of result holds in the class of Muckenhoupt weighted Lebesgue spaces (thanks to the fact that the phenomenon in question is local in nature, and (2.2.338)).

Pick an even function $\phi \in \mathscr{C}_0^\infty(\mathbb{C})$ satisfying (with $\varkappa$ as in (2.4.28))

$$\begin{gathered} 0 \leq \phi \leq 1, \quad \phi \equiv 0 \ \text{ near } \ 0 \in \mathbb{C}, \quad \operatorname{supp}\phi \subseteq B(0,2), \\ \phi \equiv 1 \ \text{ on } \ B(0,1) \setminus B(0,(1+\varkappa)^{-1}), \end{gathered} \tag{2.4.32}$$

along with a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ which is even and satisfies

$$0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subseteq [-4,4], \quad \text{and} \quad \psi \equiv 1 \ \text{ on } \ [-2,2] \setminus \big[-\tfrac{1}{2},\tfrac{1}{2}\big]. \tag{2.4.33}$$

We may then invoke Theorem 2.4.2 with $n := 2$, $m := 3$, and

$$\begin{gathered} \Phi(t) := t, \ \ A(t) := z(t), \ \ B(t) := \big(\operatorname{Re} z(t), \operatorname{Im} z(t), t\big) \ \text{ for all } \ t \in \mathbb{R}, \\ F(a,b,c) := \frac{c}{a+ib}\phi(a+ib)\psi(c) \ \text{ for all } \ (a,b,c) \in \mathbb{R}^3, \end{gathered} \tag{2.4.34}$$

and conclude from (2.4.23) and (2.2.103) that there exist some integer $\widetilde{N} \in \mathbb{N}$ and some constant $C_{p,w} \in (0,\infty)$ such that, with $\varkappa$ as in (2.4.28), we have

$$\big\|C_{\mathbb{R}} - (i/2)H\big\|_{L^p(\mathbb{R},w)\to L^p(\mathbb{R},w)} \leq C_{p,w}(1+\varkappa)^{\widetilde{N}}\sqrt{\varkappa}. \tag{2.4.35}$$

This lends credence to (2.4.26) since it implies

$$\big\|C_{\mathbb{R}} - (i/2)H\big\|_{L^p(\mathbb{R},w)\to L^p(\mathbb{R},w)} = O(\sqrt{\varkappa}\,) \ \text{ as } \ \varkappa \to 0^+. \tag{2.4.36}$$

After this preamble, we are ready to present the proof of Theorem 2.4.2.

*Proof of Theorem 2.4.2.* Throughout, let us abbreviate

$$K(x',y') := \frac{A(x') - A(y') - \langle \nabla A(y'), x'-y'\rangle}{|x'-y'|^n} F\Big(\frac{B(x')-B(y')}{|x'-y'|}\Big), \tag{2.4.37}$$

for each $x' \in \mathbb{R}^{n-1}$ and $\mathcal{L}^{n-1}$-a.e. $y' \in \mathbb{R}^{n-1}$. That, to begin with, $T_*^{A,B}g(x')$ in (2.4.56) is well-defined for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and each $x' \in \mathbb{R}^{n-1}$ is ensured by observing that

$$K(\cdot,\cdot) \text{ is an } \mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}\text{-measurable function on } \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \tag{2.4.38}$$

which is clear from (2.4.37), and that

$$\begin{gathered} \text{for each } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}),\ \varepsilon > 0,\ x' \in \mathbb{R}^{n-1}, \\ \text{one has} \quad \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} |K(x',y')||g(y')|\,dy' < +\infty. \end{gathered} \tag{2.4.39}$$

The finiteness property in (2.4.39) is a consequence of Hölder's inequality, (2.4.37), the fact that $F$ is bounded, and Lemma 2.4.1 (used with $n$ replaced by $n-1$, $p'$ in place of

$p$, and with $w'$ in place of $w$). In concert, these give that for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each $\varepsilon > 0$, and each $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned}\int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} |K(x',y')||g(y')|\,dy' \leq C\varepsilon\big[w'(B(x',\varepsilon))\big]^{1/p'}\Big(\sup_{X\in\mathbb{R}^m}|F(X)|\Big)\times \\ \times \|g\|_{L^p(\mathbb{R}^{n-1},w\mathcal{L}^{n-1})}\|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} < \infty.\end{aligned} \tag{2.4.40}$$

To proceed, for each function function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, each truncation parameter $\varepsilon > 0$, and each point $x' \in \mathbb{R}^{n-1}$ define

$$T^{A,B}_{\Phi,\varepsilon} g(x') := \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x')-\Phi(y')|>\varepsilon}} K(x',y')g(y')\,dy'. \tag{2.4.41}$$

Thanks to (2.4.20) and (2.4.38)-(2.4.39), the above integral is absolutely convergent, which means that $T^{A,B}_{\Phi,\varepsilon} g(x')$ is a well-defined number. If $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, we claim that for each function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$(T^{A,B}_{\Phi,*} g)(x') = \sup_{\varepsilon\in\mathbb{Q}_+} \big|(T^{A,B}_{\Phi,\varepsilon} g)(x')\big| \quad \text{for every} \quad x' \in \mathbb{R}^{n-1}. \tag{2.4.42}$$

To justify this, pick some $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. The idea is to show that if $x' \in \mathbb{R}^{n-1}$ is arbitrary and fixed then for each $\varepsilon \in (0,\infty)$ and each sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$ we have

$$\lim_{j\to\infty} (T^{A,B}_{\Phi,\varepsilon_j} g)(x') = (T^{A,B}_{\Phi,\varepsilon} g)(x'). \tag{2.4.43}$$

To justify (2.4.43) note that

$$\{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\} \nearrow \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\} \tag{2.4.44}$$

as $j \to \infty$, in the sense that

$$\{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\} = \bigcup_{j\in\mathbb{N}} \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\} \tag{2.4.45}$$

and

$$\{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_j\} \subseteq \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon_{j+1}\} \tag{2.4.46}$$

for every $j \in \mathbb{N}$. Then (2.4.43) follows from (2.4.44) and Lebesgue's Dominated Convergence Theorem (whose applicability is ensured by (2.4.38)-(2.4.39)). Having established this, (2.4.42) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

Moving on, we claim that

$$\begin{gathered}\text{for each fixed threshold } \varepsilon > 0, \text{ the function} \\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \ni (x',y') \longmapsto \big(\mathbf{1}_{\{y'\in\mathbb{R}^{n-1},\,|\Phi(x')-\Phi(y')|>\varepsilon\}}\big)(y') \in \mathbb{R} \\ \text{is lower-semicontinuous, hence } \mathcal{L}^{n-1}\otimes\mathcal{L}^{n-1}\text{-measurable.}\end{gathered} \tag{2.4.47}$$

To justify this claim, observe that for every number $\lambda \in \mathbb{R}$ the set of points in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ where the given function is $> \lambda$ may be described as

$$\begin{cases} \varnothing \ \text{ if } \ \lambda \geq 1, \\ \{(x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon\} \ \text{ if } \ \lambda \in [0, 1), \\ \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \ \text{ if } \ \lambda < 0. \end{cases} \tag{2.4.48}$$

Thanks to the fact that $\Phi$ is a continuous function, all sets appearing in (2.4.48) are open in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. This proves that the function (2.4.47) is indeed lower-semicontinuous.

We next claim that

$$\text{given any } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}), \text{ the function } T^{A,B}_{\Phi,*} g \text{ is } \mathcal{L}^{n-1}\text{-measurable.} \tag{2.4.49}$$

To see that this is the case, granted (2.4.42) and since the supremum of some countable family of $\mathcal{L}^{n-1}$-measurable functions is itself a $\mathcal{L}^{n-1}$-measurable function, it suffices to show that

$$\begin{gathered} T^{A,B}_{\Phi,\varepsilon} g \text{ is a } \mathcal{L}^{n-1}\text{-measurable function, for each fixed} \\ \varepsilon \in (0, \infty) \text{ and each fixed } g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}). \end{gathered} \tag{2.4.50}$$

With this goal in mind, fix $\varepsilon \in (0, \infty)$ along with $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$, and for each $j \in \mathbb{N}$ define

$$\begin{gathered} G_j : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \ \text{ given at every } (x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \text{ by} \\ G_j(x', y') := \big(\mathbf{1}_{B(0',j)}\big)(x') K(x', y') g(y') \big(\mathbf{1}_{\{y' \in \mathbb{R}^{n-1},\, |\Phi(x') - \Phi(y')| > \varepsilon\}}\big)(y'). \end{gathered} \tag{2.4.51}$$

Then, thanks to (2.4.38) and (2.4.47), it follows that $G_j$ is an $\mathcal{L}^{n-1} \otimes \mathcal{L}^{n-1}$-measurable function for each $j \in \mathbb{N}$. In addition, from (2.4.51), (2.4.39), and since balls have finite measure, we see that

$$\int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} |G_j(x', y')| \, dx' dy' < +\infty. \tag{2.4.52}$$

Granted these properties, Fubini's Theorem (whose applicability is ensured by the fact that $(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ is a sigma-finite measure space) then guarantees that

$$\begin{gathered} g_j : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g_j(x') := \int_{\mathbb{R}^{n-1}} G_j(x', y') \, dy', \quad \forall\, x' \in \mathbb{R}^{n-1}, \\ \text{is an } \mathcal{L}^{n-1}\text{-measurable function, for each integer } j \in \mathbb{N}. \end{gathered} \tag{2.4.53}$$

On the other hand, from (2.4.51), (2.4.53), and (2.4.41) it is apparent that for each $j \in \mathbb{N}$ we have

$$g_j = \mathbf{1}_{B(0',j)} \, T^{A,B}_{\Phi,\varepsilon} g \ \text{ everywhere in } \ \mathbb{R}^{n-1}. \tag{2.4.54}$$

In particular, this implies

$$\lim_{j \to \infty} g_j = T^{A,B}_{\Phi,\varepsilon} g \ \text{ pointwise everywhere in } \ \mathbb{R}^{n-1}. \tag{2.4.55}$$

At this stage, the fact that $T^{A,B}_{\Phi,\varepsilon}g$ is an $\mathcal{L}^{n-1}$-measurable function follows from (2.4.55) and (2.4.53). The claim in (2.4.49) is therefore established.

We next turn our attention to the main claim made in (2.4.23). The special case when $d := n-1$ and $\Phi(x') := x'$ for each $x' \in \mathbb{R}^{n-1}$ has been treated in [53], following basic work in [47]. Specifically, from [53, Theorem 4.34, p. 2725] we know that if for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we define

$$T^{A,B}_{*}g(x') := \sup_{\varepsilon>0}\Big|\int\limits_{\substack{y'\in\mathbb{R}^{n-1}\\ |x'-y'|>\varepsilon}} K(x',y')g(y')\,dy'\Big| \text{ at each } x' \in \mathbb{R}^{n-1}, \tag{2.4.56}$$

then

$$\begin{array}{l} T^{A,B}_{*} \text{ is a well-defined sub-linear operator from} \\ \text{the space } L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1}) \text{ into itself} \end{array} \tag{2.4.57}$$

and there exists a constant $C(n,p,w) \in (0,\infty)$ with the property that

$$\begin{aligned} &\big\|T^{A,B}_{*}\big\|_{L^p(\mathbb{R}^{n-1},w\mathcal{L}^{n-1})\to L^p(\mathbb{R}^{n-1},w\mathcal{L}^{n-1})} \\ &\qquad \le C(n,p,w)\Big(\sum_{|\alpha|\le N+2}\|\partial^\alpha F\|_{L^1(\mathbb{R}^m,\mathcal{L}^m)} + \sup_{X\in\mathbb{R}^m}(1+|X|)|F(X)|\Big) \\ &\qquad\qquad \times \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\Big(1+\sum_{j=1}^m\|\nabla B_j\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}}\Big)^N. \end{aligned} \tag{2.4.58}$$

To deal with the present case, in which the truncation is performed in the more general fashion described in (2.4.22), for each $\varepsilon > 0$ and each $x' \in \mathbb{R}^{n-1}$ abbreviate

$$\begin{aligned} D_\varepsilon(x') :=& \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| > \varepsilon \text{ and } |x'-y'| \le \varepsilon\} \\ &\bigcup \{y' \in \mathbb{R}^{n-1} : |\Phi(x') - \Phi(y')| \le \varepsilon \text{ and } |x'-y'| > \varepsilon\}. \end{aligned} \tag{2.4.59}$$

Fix an arbitrary $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ and define

$$\begin{aligned} Rg(x') := \sup_{\varepsilon>0}\int_{D_\varepsilon(x')}\Big|&\frac{A(x')-A(y')-\langle\nabla A(y'),x'-y'\rangle}{|x'-y'|^n}\times \\ &\times F\Big(\frac{B(x')-B(y')}{|x'-y'|}\Big)g(y')\Big|\,dy' \end{aligned} \tag{2.4.60}$$

at each point $x' \in \mathbb{R}^{n-1}$. These definitions imply that for each $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ we have

$$T^{A,B}_{\Phi,*}g(x') \le T^{A,B}_{*}g(x') + Rg(x') \text{ for every } x' \in \mathbb{R}^{n-1}. \tag{2.4.61}$$

To estimate the last term in appearing in the right-hand side of (2.4.61), pick some

$$\gamma \in (0, p-1) \text{ such that } w \in A_{p/(1+\gamma)}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{2.4.62}$$

fix an arbitrary point $x' \in \mathbb{R}^{n-1}$, consider an arbitrary threshold $\varepsilon > 0$, and select a function $g \in L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Also, abbreviate

$$Q := Q_{x',\varepsilon} := \{y' \in \mathbb{R}^{n-1} : |x' - y'| < \varepsilon\} \tag{2.4.63}$$

and introduce

$$A_Q(z') := A(z') - \Big(\fint_Q \nabla A \, d\mathcal{L}^{n-1}\Big) \cdot z' \quad \text{for each} \quad z' \in \mathbb{R}^{n-1}. \tag{2.4.64}$$

Observe that the number $Rg(x')$, originally defined in (2.4.60), does not change if the function $A$ is replaced by $A_Q$. Consequently,

$$Rg(x') \leq R_1 g(x') + R_2 g(x') \tag{2.4.65}$$

where

$$R_1 g(x') := \sup_{\varepsilon > 0} \int_{D_\varepsilon(x')} \left| \frac{A_Q(x') - A_Q(y')}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| dy' \tag{2.4.66}$$

and

$$R_2 g(x') := \sup_{\varepsilon > 0} \int_{D_\varepsilon(x')} \left| \frac{\langle \nabla A_Q(y'), x' - y' \rangle}{|x' - y'|^n} F\Big(\frac{B(x') - B(y')}{|x' - y'|}\Big) g(y') \right| dy'. \tag{2.4.67}$$

Note that, thanks to (2.4.20) and (2.4.59), we have

$$c\,\varepsilon \leq |x' - y'| \leq c^{-1}\varepsilon \quad \text{for each} \quad y' \in D_\varepsilon(x'). \tag{2.4.68}$$

Having fixed an integrability exponent $q \in (n-1, \infty)$, for each $y' \in D_\varepsilon(x')$ we may rely on Mary Weiss' Lemma (cf. [19, Lemma 1.4, p. 144]) in concert with (2.2.43), (2.2.44), (2.4.63), and (2.4.68) to estimate

$$\begin{aligned}
\frac{|A_Q(x') - A_Q(y')|}{|x' - y'|} &\leq C_{q,n} \Big( \fint_{|x'-z'| \leq 2|x'-y'|} |\nabla A_Q(z')|^q \, dz' \Big)^{1/q} \\
&\leq C_{q,n} \Big( \fint_{|x'-z'| \leq 2|x'-y'|} \Big| \nabla A(z') - \fint_{|x'-\zeta'| \leq 2|x'-y'|} \nabla A(\zeta') \, d\zeta' \Big|^q dz' \Big)^{1/q} \\
&\quad + C_{q,n} \Big| \fint_Q \nabla A \, d\mathcal{L}^{n-1} - \fint_{|x'-\zeta'| \leq 2|x'-y'|} \nabla A(\zeta') \, d\zeta' \Big| \\
&\leq C_{q,n} \cdot c^{-2(n-1)/q} \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}}.
\end{aligned} \tag{2.4.69}$$

Choosing $q := 2(n-1)$ it follows that there exists a constant $C_n \in (0, \infty)$, which depends only on $n$, such that

$$\begin{gathered}
|A_Q(x') - A_Q(y')| \leq (C_n/c)|x' - y'| \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})]^{n-1}} \\
\text{for each point} \quad y' \in D_\varepsilon(x').
\end{gathered} \tag{2.4.70}$$

In concert, (2.4.66), (2.4.68), and (2.4.70) allow us to conclude that

$$R_1 g(x') \leq C_n \cdot c^{1-2n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} \times \\ \times \sup_{\varepsilon>0} \Big( \fint_{|x'-y'|<c^{-1}\varepsilon} |g(y')|\, dy' \Big). \tag{2.4.71}$$

To estimate $R_2 g(x')$, bring in a brand of the Hardy-Littlewood maximal operator which associates to each $\mathcal{L}^{n-1}$-measurable function $f$ on $\mathbb{R}^{n-1}$ the function $M_\gamma f$ defined as

$$M_\gamma f(x') := \sup_{r>0} \Big( \fint_{|x'-y'|<r} |f(y')|^{1+\gamma}\, dy' \Big)^{1/(1+\gamma)} \quad \text{for each} \quad x' \in \mathbb{R}^{n-1}. \tag{2.4.72}$$

Then, using (2.4.67), (2.4.64), Hölder's inequality, and (2.2.44) we may write

$$\begin{aligned} R_2 g(x') &\leq C_n \cdot c^{2-2n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) \times \\ &\qquad \times \sup_{\varepsilon>0} \Big( \fint_{|x'-y'|<c^{-1}\varepsilon} \Big| \nabla A(y') - \fint_Q \nabla A \, d\mathcal{L}^{n-1} \Big| |g(y')|\, dy' \Big) \\ &\leq C_n \cdot c^{2-2n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) M_\gamma g(x') \times \\ &\qquad \times \sup_{\varepsilon>0} \Big( \fint_{|x'-y'|<c^{-1}\varepsilon} \Big| \nabla A(y') - \fint_Q \nabla A \, d\mathcal{L}^{n-1} \Big|^{(1+\gamma)/\gamma} dy' \Big)^{\gamma/(1+\gamma)} \\ &\leq C_{n,\gamma} \cdot c^{3-3n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} M_\gamma g(x'). \end{aligned} \tag{2.4.73}$$

Collectively, (2.4.65), (2.4.71), (2.4.73), and Hölder's inequality imply

$$Rg(x') \leq C_{n,\gamma} \cdot c^{-3n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} M_\gamma g(x'). \tag{2.4.74}$$

In turn, from (2.4.74) and (2.4.61) we conclude that for every $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} 0 \leq T^{A,B}_{\Phi,*} g(x') &\leq T^{A,B}_{*} g(x') \\ &\quad + C_{n,\gamma} \cdot c^{-3n} \big( \sup_{X \in \mathbb{R}^m} |F(X)| \big) \|\nabla A\|_{[\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})]^{n-1}} M_\gamma g(x'). \end{aligned} \tag{2.4.75}$$

Granted (2.4.62), the maximal operator $M_\gamma$ is a well-defined sub-linear bounded mapping from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. Bearing this in mind, from (2.4.75), (2.4.57), (2.4.58), (2.2.337), and the fact that the space $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ is a lattice, the estimate claimed in (2.4.23) now follows. As a consequence, $T^{A,B}_{\Phi,*}$ is a sub-linear mapping of finite operator norm on $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$. Hence, as remarked in (2.4.3), the operator $T^{A,B}_{\Phi,*}$ is continuous from $L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ into itself. □

The next step is to transfer the Euclidean result from Theorem 2.4.2 to singular integral operators on Lipschitz graphs, a task accomplished in the following lemma.

**Lemma 2.4.3.** *Given a unit vector $\vec{n} \in S^{n-1}$, consider the hyperplane $H := \langle \vec{n} \rangle^{\perp} \subseteq \mathbb{R}^{n-1}$ and suppose $h : H \to \mathbb{R}$ is a function satisfying*

$$M := \sup_{\substack{x,y \in H \\ x \neq y}} \frac{|h(x) - h(y)|}{|x - y|} < +\infty. \tag{2.4.76}$$

*Fix an arbitrary point $x_0 \in \mathbb{R}^n$ and let*

$$\mathcal{G} := \big\{ x_0 + x + h(x)\vec{n} : \, x \in H \big\} \subseteq \mathbb{R}^n \tag{2.4.77}$$

*denote the graph of $h$ in the coordinate system $X = (x,t) \Leftrightarrow X = x_0 + x + t\vec{n}$, with $x \in H$ and $t \in \mathbb{R}$. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \mathcal{G}$ and denote by $\nu$ the unique unit normal to $\mathcal{G}$ satisfying $\nu \cdot \vec{n} < 0$ at $\sigma$-a.e. point on $\mathcal{G}$. Also, fix some integrability exponent $p \in (1, \infty)$. Given a complex-valued function $k \in \mathscr{C}^{N+2}(\mathbb{R}^n \setminus \{0\})$, for some sufficiently large integer $N = N(n) \in \mathbb{N}$, which is even and positive homogeneous of degree $-n$, consider the maximal singular integral operator $T$ acting on each $f \in L^p(\mathcal{G}, \sigma)$ as*

$$T_* f(x) := \sup_{\varepsilon > 0} \Bigg| \int\limits_{\substack{y \in \mathcal{G} \\ |x-y| > \varepsilon}} \langle x - y, \nu(y) \rangle k(x-y) f(y) \, d\sigma(y) \Bigg|, \quad \forall\, x \in \mathcal{G}. \tag{2.4.78}$$

*Then $T_*$ is a well-defined continuous sub-linear mapping from the space $L^p(\mathcal{G}, \sigma)$ into itself and there exists a constant $C(n,p) \in (0, \infty)$, which depends only on $n, p$, with the property that*

$$\|T_*\|_{L^p(\mathcal{G},\sigma) \to L^p(\mathcal{G},\sigma)} \leq C(n,p) M (1+M)^{4n+N} \Big( \sum_{|\alpha| \leq N+2} \sup_{S^{n-1}} |\partial^\alpha k| \Big). \tag{2.4.79}$$

*Proof.* Recall that $\{\mathbf{e}_j\}_{1 \leq j \leq n}$ stands for the standard orthonormal basis in $\mathbb{R}^n$. Let us first treat the case when $x_0 = 0 \in \mathbb{R}^n$ and $\vec{n} := \mathbf{e}_n$, a scenario in which $H = \langle \mathbf{e}_n \rangle^{\perp}$ may be canonically identified with $\mathbb{R}^{n-1}$. Assume this is the case, and consider an even function $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that

$$\begin{gathered} 0 \leq \psi \leq 1, \quad \psi \text{ vanishes identically in } \mathbb{R}^n \setminus B\big(0, 2\sqrt{1+M^2}\,\big), \\ \psi \equiv 1 \text{ on } \overline{B(0, \sqrt{1+M^2})} \setminus B(0,1), \quad \psi \equiv 0 \text{ on } B(0,1/2), \\ \text{and for each } \alpha \in \mathbb{N}_0^n \text{ there exists } C_\alpha \in (0,\infty), \text{ depending only} \\ \text{on the given multi-index } \alpha, \text{ so that } \sup_{x \in \mathbb{R}^n} |(\partial^\alpha \psi)(x)| \leq C_\alpha. \end{gathered} \tag{2.4.80}$$

Then $F := \psi k$ is an even function belonging to $\mathscr{C}^{N+2}(\mathbb{R}^n)$, and satisfying

$$\begin{aligned} \sum_{|\alpha| \leq N+2} \|\partial^\alpha F\|_{L^1(\mathbb{R}^n, \mathcal{L}^n)} &+ \sup_{x \in \mathbb{R}^n} (1+|x|)|F(x)| \\ &\leq C_n (1+M)^n \Big( \sum_{|\alpha| \leq N+2} \sup_{S^{n-1}} |\partial^\alpha k| \Big), \end{aligned} \tag{2.4.81}$$

for some purely dimensional constant $C_n \in (0,\infty)$. Moreover, if for each point $x' \in \mathbb{R}^{n-1}$ we set $\Phi(x') := (x', h(x'))$ then $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a bi-Lipschitz function and (2.4.80) implies that

$$k\Big(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\Big) = F\Big(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\Big) \quad \text{for each } \ x', y' \in \mathbb{R}^{n-1} \ \text{ with } \ x' \neq y'. \tag{2.4.82}$$

To proceed, note that for each $\sigma$-measurable set $E \subseteq \mathcal{G}$ and each function $g \in L^1(E,\sigma)$ we have

$$\int_E g\, d\sigma = \int_{\{y' \in \mathbb{R}^{n-1}:\, (y', h(y')) \in E\}} g(y', h(y'))\sqrt{1 + |(\nabla h)(y')|^2}\, dy', \tag{2.4.83}$$

(cf., e.g., [119, Proposition 12.9, p. 164]) and

$$\nu(y', h(y')) = \frac{((\nabla h)(y'), -1)}{\sqrt{1 + |(\nabla h)(y')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } \ y' \in \mathbb{R}^{n-1}. \tag{2.4.84}$$

Also, fix $f \in L^p(\mathcal{G},\sigma)$ and define $\widetilde{f}(x') := f(x', h(x'))$ for each $x' \in \mathbb{R}^{n-1}$. In particular, from (2.4.83) we conclude that

$$\widetilde{f} \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \ \text{ and } \ \|\widetilde{f}\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq \|f\|_{L^p(\mathcal{G},\sigma)}. \tag{2.4.85}$$

Then based on (2.4.78), (2.4.83), (2.4.84), the homogeneity of $k$, and (2.4.82) we may write

$$\begin{aligned}
(T_* f)(x', h(x')) &= \sup_{\varepsilon>0}\Bigg|\int_{\substack{y' \in \mathbb{R}^{n-1} \text{ with} \\ \sqrt{|x'-y'|^2 + (h(x')-h(y'))^2} > \varepsilon}} \big(\langle \nabla h(y'), x'-y'\rangle + h(y') - h(x')\big)\times \\
&\qquad\qquad \times k(x'-y', h(x')-h(y'))\widetilde{f}(y')\, dy'\Bigg| \\
&= \sup_{\varepsilon>0}\Bigg|\int_{\substack{y' \in \mathbb{R}^{n-1} \\ |\Phi(x')-\Phi(y')|>\varepsilon}} \frac{h(x') - h(y') - \langle \nabla h(y'), x'-y'\rangle}{|x'-y'|^n}\times \\
&\qquad\qquad \times F\Big(\frac{\Phi(x')-\Phi(y')}{|x'-y'|}\Big)\widetilde{f}(y')\, dy'\Bigg|.
\end{aligned} \tag{2.4.86}$$

From (2.4.86), Theorem 2.4.2 (used with $m := n$, $d := n$, $A := h$, $B := \Phi$, and $w \equiv 1$), (2.4.81), and (2.4.83) we then conclude that (2.4.79) holds in this case.

To treat the case when $x_0 = 0$ but $\vec{n} \in S^{n-1}$ is arbitrary, pick an orthonormal basis $\{v_j\}_{1\leq j\leq n-1}$ in $H$ and consider the unitary transformation in $\mathbb{R}^n$ uniquely defined by the demand that $Uv_j = \mathbf{e}_j$ for $j \in \{1,\dots,n-1\}$ and $U\vec{n} = \mathbf{e}_n$. Then $\widetilde{\mathcal{G}} := U\mathcal{G}$ becomes the graph of $\widetilde{h} := h \circ U^{-1} : \mathbb{R}^{n-1} \to \mathbb{R}$, which is a Lipschitz function with the same Lipschitz constant $M$ as the original function $h$. Since the Hausdorff measure is rotation invariant, for each $g \in L^1(\mathcal{G},\sigma)$ we have

$$\int_{y\in\mathcal{G}} g(y)\, d\sigma(y) = \int_{\widetilde{y}\in\widetilde{\mathcal{G}}} (g \circ U^{-1})(\widetilde{y})\, d\widetilde{\sigma}(\widetilde{y}), \tag{2.4.87}$$

where $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\widetilde{\mathcal{G}}$. Moreover, the unique unit normal $\widetilde{\nu}$ to $\widetilde{\mathcal{G}}$ satisfying $\widetilde{\nu}\cdot\mathbf{e}_n < 0$ at $\mathcal{H}^{n-1}$-a.e. point on $\widetilde{\mathcal{G}}$ is $\widetilde{\nu} = U(\nu\circ U^{-1})$. Consider $\widetilde{k} := k\circ U^{-1}$ and note that this is a complex-valued function of class $\mathscr{C}^{N+2}(\mathbb{R}^n\setminus\{0\})$, which is even and positive homogeneous of degree $-n$. Finally, fix some function $f\in L^p(\mathcal{G},\sigma)$ and abbreviate $\widetilde{f} := f\circ U^{-1}$. Bearing in mind the fact that $U$ is a linear isometry satisfying $U^{-1} = U^{\top}$, from (2.4.78) and (2.4.87) we see that if $x\in\mathcal{G}$ and $\widetilde{x} := Ux$ then

$$T_*f(x) = \sup_{\varepsilon>0}\Bigg|\int\limits_{\substack{\widetilde{y}\in\widetilde{\mathcal{G}}\\ |\widetilde{x}-\widetilde{y}|>\varepsilon}} \langle\widetilde{x}-\widetilde{y},\widetilde{\nu}(\widetilde{y})\rangle\widetilde{k}(\widetilde{x}-\widetilde{y})\widetilde{f}(\widetilde{y})\,d\widetilde{\sigma}(\widetilde{y})\Bigg|. \tag{2.4.88}$$

Hence,

$$T_*f(x) = \widetilde{T}_*\widetilde{f}(\widetilde{x}) \quad\text{whenever}\quad x\in\mathcal{G} \quad\text{and}\quad \widetilde{x} = Ux, \tag{2.4.89}$$

where $\widetilde{T}_*$ is the maximal operator associated as in (2.4.78) with the Lipschitz graph $\widetilde{\mathcal{G}}$ and the kernel $\widetilde{k}$. In particular, given that (2.4.89) and (2.4.87) imply

$$\int_{\mathcal{G}}(T_*f)(x)^p\,d\sigma(x) = \int_{\widetilde{\mathcal{G}}}(\widetilde{T}_*\widetilde{f})(\widetilde{x})^p\,d\widetilde{\sigma}(\widetilde{x}), \tag{2.4.90}$$

the estimate claimed in (2.4.79) becomes a consequence of the corresponding estimate for the maximal operator $\widetilde{T}_*$ established in the first part of the current proof.

Finally, the case when both $x_0\in\mathbb{R}^n$ and $\vec{n}\in S^{n-1}$ are arbitrary follows from what we have proved so far using the natural invariance of the maximal operator (2.4.78) to translations. □

### 2.4.2 Estimates for certain classes of singular integrals on UR sets

The following theorem, which is central for the present work, is the main result regarding the size of the operator norm of certain maximal integral operators acting on Muckenhoupt weighted Lebesgue spaces on the boundary of Ahlfors regular domains satisfying a two-sided local John condition. In turn, this is going to be the key ingredient in obtain invertibility results for the brand of boundary double layer potential operators considered in this work. The proof is inspired by that of [53, Theorem 4.36, pp. 2728-2729].

**Theorem 2.4.4.** *Let $\Omega\subseteq\mathbb{R}^n$ be an Ahlfors regular domain satisfying a two-sided local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same same symbol $w$ for the measure associated with the given weight $w$ as in (2.2.292). Also, consider a sufficiently large integer $N = N(n)\in\mathbb{N}$. Given a complex-valued function $k\in\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ which is even and positive homogeneous of degree $-n$, consider the maximal operator $T_*$ whose action on each given function $f\in L^p(\partial\Omega,w)$ is defined as*

$$T_*f(x) := \sup_{\varepsilon>0}|T_\varepsilon f(x)| \quad\textit{for each}\quad x\in\partial\Omega, \tag{2.4.91}$$

*where, for each* $\varepsilon > 0$,

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \quad \textit{for all} \quad x \in \partial\Omega. \tag{2.4.92}$$

*Then there exists some* $C \in (0,\infty)$*, which depends only on* $n$*,* $p$*,* $[w]_{A_p}$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$*, such that*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.4.93}$$

Before presenting the proof of this theorem, several comments are in order.

*Remark* 2.4.5. In the context of Theorem 2.4.4, estimate (2.4.93) continues to hold with a fixed constant $C \in (0,\infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are allowed to vary with control. Specifically, an inspection of the proof of Theorem 2.4.4 given below shows that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on $n$, $I$, $W$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that (2.4.93) holds for each $p \in I$ and each $w \in A_p(\partial\Omega,\sigma)$ with $[w]_{A_p} \leq W$.

*Remark* 2.4.6. Since any Ahlfors regular domain satisfying a two-sided local John condition is a UR domain (cf. (1.1.22)), Proposition 2.3.3 applies and gives that $T_*$ is bounded on $L^p(\partial\Omega,w)$, with norm controlled in terms of $n$, $k$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. The crux of the matter here is the more refined version of the estimate of the operator norm of $T_*$ given in (2.4.93).

*Remark* 2.4.7. We focus on establishing the estimate claimed in (2.4.93) in the class of operators whose integral kernel factors as the product of $\langle x-y, \nu(y)\rangle$, i.e., the inner product between the unit normal $\nu(y)$ and the "chord" $x-y$, with some matrix-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, since it has been noted in (2.1.28) that this is the only type of kernel (in the class of double layer-like integral operators) for which the said estimate has a chance of materializing.

*Remark* 2.4.8. The class of domains to which Theorem 2.4.4 applies includes all two-sided NTA domains with an Ahlfors regular boundary (cf. (1.1.28)).

*Remark* 2.4.9. In the unweighted case, i.e., for $w \equiv 1$ (or, equivalently, when the measure $w$ coincides with $\sigma$), estimate (2.4.93) simply reads (for some $C \in (0,\infty)$ which now depends only on $n$, $p$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\|T_*\|_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.4.94}$$

It turns out that whenever (2.4.94) is available one may produce a weighted version of such an estimate via interpolation. Specifically, recall the interpolation theorem of Stein-Weiss (cf. [11, Theorem 5.4.1, p. 115]) according to which for any two $\sigma$-measurable functions $w_0, w_1 : \partial\Omega \to [0,\infty]$ and any $\theta \in (0,1)$ we have

$$\big(L^p(\partial\Omega, w_0\sigma)\,,\, L^p(\partial\Omega, w_1\sigma)\big)_{\theta,p} = L^p(\partial\Omega, \widetilde{w}\sigma) \quad \text{where} \quad \widetilde{w} := w_0^{1-\theta}\cdot w_1^{\theta}. \tag{2.4.95}$$

Now, given a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, from (2.2.311) we know that there exists some $\tau \in (1, \infty)$ (which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$) such that $w^\tau \in A_p(\partial\Omega, \sigma)$. Upon specializing (2.4.95) to the case when $\theta := 1 - \tau^{-1} \in (0, 1)$, $w_0 := w^\tau$, and $w_1 := 1$ we therefore obtain

$$\big(L^p(\partial\Omega, w^\tau\sigma)\,,\, L^p(\partial\Omega, \sigma)\big)_{\theta,p} = L^p(\partial\Omega, w). \tag{2.4.96}$$

Consequently, since $T_*$ is a sub-linear operator which is bounded both on $L^p(\partial\Omega, w^\tau\sigma)$ (given that $w^\tau \in A_p(\partial\Omega, \sigma)$), and on $L^p(\partial\Omega, \sigma)$ we may write

$$\begin{aligned} \|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} &\leq \|T_*\|^{1-\theta}_{L^p(\partial\Omega,w^\tau\sigma)\to L^p(\partial\Omega,w^\tau\sigma)}\, \|T_*\|^{\theta}_{L^p(\partial\Omega,\sigma)\to L^p(\partial\Omega,\sigma)} \\ &\leq C_{\Omega,n,p,k,[w]_{A_p}}\, \|\nu\|^{\theta}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \end{aligned} \tag{2.4.97}$$

with the last inequality provided by (2.4.94).

While the weighted norm inequality established in (2.4.97) is in the spirit of (2.4.93), the BMO semi-norm of the outward unit normal vector $\nu$ only picks up a small exponent $\theta \in (0, 1)$ in (2.4.97). This is in sharp contrast with (2.4.93) where the power of $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ is precisely 1. Hence, a two-step approach consisting first of proving the plain estimate (2.4.94) and, second, deriving a weighted version based on the procedure based on interpolation described above, only yields a weaker result than the one advertised in (2.4.93). Given this, in the proof of (2.4.93) presented below we shall devise an alternative approach, which deals with the weighted case directly, incorporating the weight in all relevant intermediary steps.

We are ready to proceed to the task of providing the proof of Theorem 2.4.4.

*Proof of Theorem 2.4.4.* As visible from (2.4.91)-(2.4.92), the maximal operator $T_*$ depends in a homogeneous fashion on the kernel function $k$. As such, by working with $k/K$ (in the case when $k$ is not identically zero) where $K := \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|$, matters are reduced to proving that whenever

$$\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \leq 1 \tag{2.4.98}$$

it is possible to find a constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.4.99}$$

Henceforth, assume (2.4.98).

To proceed, recall the parameter $\delta_* > 0$ from Theorem 2.2.25. In the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \geq \min\{\delta_*, 1\}$, the estimate claimed in (2.4.99) follows directly from Proposition 2.3.3, which ensures that the maximal operator $T_*$ is bounded in $L^p(\partial\Omega, w)$. Therefore, there remains to consider the case when $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \min\{\delta_*, 1\}$. Assume this is the case and pick some $\delta$ such that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta < \min\{\delta_*, 1\}. \tag{2.4.100}$$

In such a scenario, Lemma 2.2.5 implies that the set $\partial\Omega$ is unbounded. We may also invoke Proposition 2.2.24 to conclude that there exists some $C_\Omega \in (0,\infty)$, which depends only on the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$, such that for each $\mu \in [1,\infty)$ we have

$$\sup_{z\in\partial\Omega}\sup_{R>0}\sup_{x,y\in\Delta(x,\mu R)} R^{-1}\big|\langle x-y,\nu_{\Delta(z,R)}\rangle\big| \leq C_\Omega\cdot\mu(1+\log_2\mu)\delta. \tag{2.4.101}$$

For reasons which are going to be clear momentarily, in addition to the truncated operators $T_\varepsilon$ from (2.4.92) we shall need a version in which the truncation is performed using a smooth cutoff function (rather than a characteristic function). Specifically, fix a function $\psi \in \mathscr{C}^\infty(\mathbb{R})$ satisfying $0 \leq \psi \leq 1$ on $\mathbb{R}$ and with the property that $\psi \equiv 0$ in $(-\infty,1]$ and $\psi \equiv 1$ in $[2,\infty)$. For each $\varepsilon > 0$ then define the action of the smoothly truncated operator $T_{(\varepsilon)}$ on each $f \in L^p(\partial\Omega,w)$ by setting

$$T_{(\varepsilon)}f(x) := \int_{\partial\Omega}\psi\Big(\frac{|x-y|}{\varepsilon}\Big)\langle x-y,\nu(y)\rangle\,k(x-y)f(y)\,d\sigma(y) \tag{2.4.102}$$

for each $x \in \partial\Omega$. Let us also define a smoothly truncated version of the maximal operator (2.4.91) by setting, for each $f \in L^p(\partial\Omega,w)$,

$$T_{(*)}f(x) := \sup_{\varepsilon>0}\big|T_{(\varepsilon)}f(x)\big| \text{ at every point } x \in \partial\Omega. \tag{2.4.103}$$

For the time being, the goal is compare roughly truncated singular integral operators with their smoothly truncated counterparts. To accomplish this task, for each fixed $\gamma \geq 0$ bring in a brand of Hardy-Littlewood maximal operator which associates to each $\sigma$-measurable function $f$ on $\partial\Omega$ the function $\mathcal{M}_\gamma f$ defined as

$$\mathcal{M}_\gamma f(x) := \sup_{\Delta\ni x}\Big(\fint_\Delta |f|^{1+\gamma}\,d\sigma\Big)^{1/(1+\gamma)} \text{ for each } x \in \partial\Omega, \tag{2.4.104}$$

where the supremum is taken over all surface balls $\Delta \subseteq \partial\Omega$ containing the point $x$. On to the task at hand, having fixed some $\varepsilon > 0$, for each $f \in L^p(\partial\Omega,w)$ and each $x \in \partial\Omega$

we may estimate

$$
\begin{aligned}
\big|(T_\varepsilon f - T_{(\varepsilon)} f)(x)\big| &\le \int_{\Delta(x,2\varepsilon)\setminus\overline{\Delta(x,\varepsilon)}} |\langle x-y,\nu(y)\rangle|\,|k(x-y)||f(y)|\,d\sigma(y) \\
&\le C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x-y,\nu(y)\rangle|\,|f(y)|\,d\sigma(y) \\
&\le C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x-y,\nu(y)-\nu_{\Delta(x,2\varepsilon)}\rangle||f(y)|\,d\sigma(y) \\
&\quad + C\varepsilon^{-1} \fint_{\Delta(x,2\varepsilon)} |\langle x-y,\nu_{\Delta(x,2\varepsilon)}\rangle||f(y)|\,d\sigma(y) \\
&\le C\left(\fint_{\Delta(x,2\varepsilon)} |\nu(y)-\nu_{\Delta(x,2\varepsilon)}|^{\frac{\gamma+1}{\gamma}}\,d\sigma(y)\right)^{\frac{\gamma}{1+\gamma}} \left(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}} \\
&\quad + C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}|\langle x-y,\nu_{\Delta(x,2\varepsilon)}\rangle|\Big)\left(\fint_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}} \\
&\le C\delta\cdot \inf_{\Delta(x,2\varepsilon)} \mathcal{M}_\gamma f,
\end{aligned}
\tag{2.4.105}
$$

using Hölder's inequality, (2.2.43), (2.4.100), (2.4.101), and (2.4.104). Then, (2.4.105) implies that there exists some $C \in (0,\infty)$, which depends only on $\gamma$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that for each function $f \in L^p(\partial\Omega, w)$ we have

$$\big|T_* f(x) - T_{(*)} f(x)\big| \le C\delta\cdot\mathcal{M}_\gamma f(x) \quad \text{for each} \quad x\in\partial\Omega. \tag{2.4.106}$$

Henceforth we agree to fix $\gamma \in (0, p-1)$, which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, such that $w \in A_{p/(1+\gamma)}(\partial\Omega,\sigma)$, with $[w]_{A_{p/(1+\gamma)}}$ controlled in terms of $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. From (2.2.311) we know that such a choice is possible.

To proceed, consider a dyadic grid $\mathbb{D}(\partial\Omega)$ on the Ahlfors regular set $\partial\Omega$ (as in Proposition 2.2.39, presently used with $\Sigma := \partial\Omega$). Also, choose a compactly supported function $f \in L^p(\partial\Omega, w)$. Note that for each $\varepsilon > 0$ the function $T_{(\varepsilon)} f$ is continuous on $\partial\Omega$, by Lebesgue's Dominated Convergence Theorem (whose applicability in the present setting is ensured by Lemma 2.2.47). Since the pointwise supremum of any collection of continuous functions is lower-semicontinuous, we conclude that for each $\lambda > 0$ the set

$$\{x\in\partial\Omega : T_{(*)} f(x) > \lambda\} \quad \text{is relatively open in} \quad \partial\Omega. \tag{2.4.107}$$

Next, fix a reference point $x_0 \in \partial\Omega$ and abbreviate $\Delta_0 := \Delta(x_0, 2^{-m})$ for some $m\in\mathbb{Z}$ chosen so that $\operatorname{supp} f \subseteq 2\Delta_0$. We emphasize that all subsequent constants are going to be independent of the function $f$, the point $x_0$, and the integer $m$. Upon recalling (2.2.283), define

$$\mathcal{Q}_0 := \{Q\in\mathbb{D}_m(\partial\Omega) : Q\cap 2\Delta_0 \ne \varnothing\} \tag{2.4.108}$$

then introduce

$$I_0 := \bigcup_{Q\in\mathcal{Q}_0} Q. \tag{2.4.109}$$

By design, $I_0$ is a relatively open subset of $\partial\Omega$. Recall the parameter $a_1 > 0$ appearing in (2.2.285) of Proposition 2.2.39. We claim that

$$I_0 \subseteq a\Delta_0 \quad \text{where} \quad a := 2(1+a_1) > 2. \tag{2.4.110}$$

Indeed, if $x \in I_0$ then $x \in Q$ for some $Q \in \mathcal{Q}_0$. In particular, $Q \cap 2\Delta_0 \neq \varnothing$ so we may pick some $y \in Q \cap 2\Delta_0$. Then $x, y \in Q \subseteq \Delta(x_Q, a_1 2^{-m})$ by (2.2.285), where $x_Q$ denotes the "center" of the dyadic cube $Q$. Consequently, $|x-y| < a_1 2^{-m+1}$ which, in turn, permits us to estimate $|x-x_0| \leq |x-y| + |y-x_0| < a_1 2^{-m+1} + 2^{-m+1} = a\cdot 2^{-m}$. Thus $x \in B(x_0, a\cdot 2^{-m}) \cap \partial\Omega = a\Delta_0$, proving the inclusion in (2.4.110).

We also claim that

$$\begin{array}{l}\text{there exists a } \sigma\text{-measurable set } N \subseteq \partial\Omega \text{ with the}\\ \text{property that } \sigma(N) = 0 \text{ and } 2\Delta_0 \setminus N \subseteq I_0.\end{array} \tag{2.4.111}$$

To justify this, recall from (2.2.287) that

$$\begin{array}{l}N := \partial\Omega \setminus \big(\bigcup_{Q\in\mathbb{D}_m(\partial\Omega)} Q\big) \text{ is a } \sigma\text{-measurable set}\\ \text{satisfying } \sigma(N) = 0 \text{ and } \partial\Omega \setminus N = \bigcup_{Q\in\mathbb{D}_m(\partial\Omega)} Q.\end{array} \tag{2.4.112}$$

Intersecting both sides of the last equality in (2.4.112) with $2\Delta_0$ while bearing in mind (2.4.108)-(2.4.109) then yields

$$2\Delta_0 \setminus N = \bigcup_{Q\in\mathbb{D}_m(\partial\Omega)} (Q \cap 2\Delta_0) = \bigcup_{Q\in\mathcal{Q}_0} (Q \cap 2\Delta_0) \subseteq \bigcup_{Q\in\mathcal{Q}_0} Q = I_0, \tag{2.4.113}$$

ultimately proving (2.4.111).

Since $w \in A_p(\partial\Omega, \sigma) \subseteq A_\infty(\partial\Omega, \sigma)$, there exists some small number $\tau > 0$ such that (2.2.315) holds. Let us define

$$A := \theta \cdot \delta^{-1} \in (0, \infty) \quad \text{for some small} \quad \theta \in (0,1). \tag{2.4.114}$$

At various stages in the proof we shall make specific demands on the size of $\theta$, though always in relation to the background geometric parameters and the weight, namely $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. We find it convenient to abbreviate

$$\eta(\theta, \delta) := C\Big(\theta^{1+\gamma} + \theta\delta^{-1} \cdot \exp\Big\{-\frac{c\,\gamma}{\delta(1+\gamma)}\Big\} + e^{-c/\delta}\Big), \tag{2.4.115}$$

where $C \in (0, \infty)$ is a constant which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, and where $c \in (0, \infty)$ is of a purely geometric nature (i.e., $c$ depends only the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$). We agree to retain the notation $\eta(\theta, \delta)$ even when $C, c \in (0, \infty)$ may occasionally change in size (while retaining the same nature, however).

Our long-term goal is to obtain the following type of good-$\lambda$ inequality: there exist $C, c \in (0, \infty)$ as above such that for each $\lambda > 0$ we have

$$\begin{aligned} w\Big(\{x \in I_0 : T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\}\Big) \\ \leq \eta(\theta, \delta)^\tau \cdot w\Big(\{x \in I_0 : T_{(*)} f(x) > \lambda\}\Big). \end{aligned} \tag{2.4.116}$$

Here and elsewhere, we employ our earlier convention of using the same same symbol $w$ for the measure associated with the given weight $w$ as in (2.2.292). The reader is also alerted to the fact that the maximal operator appearing in the right-hand side of (2.4.116) employs smooth truncations (as in (2.4.103)).

To prove (2.4.116), fix an arbitrary $\lambda > 0$ and abbreviate

$$\mathcal{F}_\lambda := \{x \in I_0 : T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\}. \tag{2.4.117}$$

Proposition 2.3.3 implies that $T_* f$ is a $\sigma$-measurable function. Since so is $\mathcal{M}_\gamma f$ (cf. [6] or [93] for a proof), it follows that $\mathcal{F}_\lambda$ is a $\sigma$-measurable set. From (2.4.107) and the fact that $I_0$ is a relatively open subset of $\partial\Omega$ we also conclude that $\{x \in I_0 : T_{(*)} f(x) > \lambda\}$ is a relatively open subset of $\partial\Omega$ (in particular, $\sigma$-measurable). As such, the good-$\lambda$ inequality is meaningfully formulated in (2.4.116).

Clearly, it is enough to consider the case $\mathcal{F}_\lambda \neq \varnothing$ since otherwise (2.4.116) is trivially satisfied by any choice of $C \in (0, \infty)$. For the remainder of the proof, assume this is the case. Since $\mathcal{F}_\lambda \subseteq I_0$ and $I_0 \subseteq a\Delta_0$, we conclude that

$$\mathcal{F}_\lambda \subseteq I_0 \subseteq a\Delta_0 \ \text{ and } \ \sup_{\mathcal{F}_\lambda} \mathcal{M}_\gamma f \leq A\lambda. \tag{2.4.118}$$

To proceed, decompose $I_0 = \mathcal{P}_\lambda \cup \mathcal{S}_\lambda$ (disjoint union) where, with the smoothly truncated maximal operator $T_{(*)}$ as in (2.4.103),

$$\mathcal{P}_\lambda := \{x \in I_0 : T_{(*)} f(x) \leq \lambda\} \ \text{ and } \ \mathcal{S}_\lambda := \{x \in I_0 : T_{(*)} f(x) > \lambda\}. \tag{2.4.119}$$

As a consequence of (2.4.107) and the fact that $I_0$ is a relatively open subset of $\partial\Omega$, the set $\mathcal{S}_\lambda$ is itself a relatively open subset of $\partial\Omega$. Moreover, using (2.4.106) and (2.4.118), for each point $x \in \mathcal{F}_\lambda$ we may estimate

$$\begin{aligned} 4\lambda < T_* f(x) &\leq T_{(*)} f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \leq T_{(*)} f(x) + C\delta A\lambda \\ &= T_{(*)} f(x) + C\theta\lambda < T_{(*)} f(x) + 3\lambda, \end{aligned} \tag{2.4.120}$$

by our choice of $A$ in (2.4.114) and by taking $\theta > 0$ small enough to begin with. From (2.4.120) we see that $T_{(*)} f(x) > \lambda$, hence $x \in \mathcal{S}_\lambda$ which ultimately goes to show that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. Thus,

$$\begin{array}{l} \mathcal{S}_\lambda \text{ is a nonempty relatively open subset of} \\ \partial\Omega, \text{ with the property that } \mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda \subseteq I_0. \end{array} \tag{2.4.121}$$

We first treat the case in which there exists $Q_0 \in \mathcal{Q}_0$ such that $\mathcal{P}_\lambda \cap Q_0 = \varnothing$ or, equivalently,

$$Q_0 \subseteq \mathcal{S}_\lambda. \tag{2.4.122}$$

Applying Theorem 2.2.25 to the surface ball $a\Delta_0$ permits us to decompose

$$a\Delta_0 = G \cup E, \tag{2.4.123}$$

where $G$ and $E$ are disjoint $\sigma$-measurable subsets of $\partial\Omega$ satisfying properties implied by (2.2.214)-(2.2.219) (relative to the location $x_0$ and the scale $r := a2^{-m}$) in the present setting. Specifically, there exists three constants $C_0, C_1, C_2 \in (0, \infty)$ of a purely geometric nature (i.e., depending only the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$) so that $G$ is contained in the graph $\mathcal{G} = \{x_0 + x + h(x)\vec{n} : x \in H\}$ of a Lipschitz function $h : H \to \mathbb{R}$ (where $\vec{n} \in S^{n-1}$ is a unit vector and $H = \langle\vec{n}\rangle^{\perp}$ is the hyperplane in $\mathbb{R}^n$ orthogonal to $\vec{n}$) such that

$$\sup_{\substack{x,y\in H\\ x\neq y}} \frac{|h(x) - h(y)|}{|x - y|} \leq C_0\delta, \tag{2.4.124}$$

whereas $E$ satisfies

$$\sigma(E) \leq C_1\, e^{-C_2/\delta}\sigma(a\Delta_0). \tag{2.4.125}$$

Since $\operatorname{supp} f \subseteq 2\Delta_0$ and $a > 2$ it follows that $f = f\mathbf{1}_{a\Delta_0}$. Based on this observation and the fact that $I_0 \subseteq a\Delta_0$ (cf. (2.4.118)), we may then estimate

$$\sigma(\mathcal{F}_\lambda) \leq \sigma\Big(\big\{x \in a\Delta_0 : T_*(f\mathbf{1}_{a\Delta_0})(x) > 4\lambda\big\}\Big). \tag{2.4.126}$$

By further decomposing $f\mathbf{1}_{a\Delta_0} = f\mathbf{1}_G + f\mathbf{1}_E$ (cf. (2.4.123)), then using the sub-linearity of $T_*$, as well as (2.4.123) and (2.4.125), we obtain

$$\begin{aligned}
\sigma\Big(\big\{x \in a\Delta_0 : &T_*(f\mathbf{1}_{a\Delta_0})(x) > 4\lambda\big\}\Big) \\
&\leq \sigma\Big(\big\{x \in G : T_*(f\mathbf{1}_G)(x) > 2\lambda\big\}\Big) \\
&\quad + \sigma\Big(\big\{x \in G : T_*(f\mathbf{1}_E)(x) > 2\lambda\big\}\Big) \\
&\quad + C_1\, e^{-C_2/\delta}\sigma(a\Delta_0).
\end{aligned} \tag{2.4.127}$$

To bound the first term in the right-hand side of (2.4.127), the idea is to use the fact that $G$ is contained in the graph $\mathcal{G}$ of the function $h$, then employ Lemma 2.4.3 while taking advantage of (2.4.124). Turning to specifics, denote by $\widetilde{\sigma}$ the surface measure on $\mathcal{G}$, and by $\widetilde{T}_*$ the maximal operator associated with $\mathcal{G}$ as in (2.4.78) (much as $T_*$ in (2.4.91)-(2.4.92) is associated with $\partial\Omega$). Also, fix a point $\widetilde{x} \in \mathcal{F}_\lambda$ (which, according to (2.4.118), also places $\widetilde{x}$ into $a\Delta_0$). Observe that the measures $\sigma$ and $\widetilde{\sigma}$ are compatible on $\partial\Omega \cap \mathcal{G}$ (as they are both induced by $\mathcal{H}^{n-1}$). This observation, Chebysheff's inequality, Lemma 2.4.3, (2.4.124), (2.4.123), (2.4.104), (2.4.118), and (2.4.114) then permit us to estimate

$$
\begin{aligned}
\sigma\Big(\{x\in G: T_*(f\mathbf{1}_G)(x)>2\lambda\}\Big)
&\le \widetilde{\sigma}\Big(\{x\in\mathcal{G}:\, \widetilde{T}_*(f\mathbf{1}_G)(x)>2\lambda\}\Big)\\
&\le \frac{1}{(2\lambda)^{1+\gamma}}\int_{\mathcal{G}}|\widetilde{T}_*(f\mathbf{1}_G)|^{1+\gamma}\,d\widetilde{\sigma}\le C\frac{\delta^{1+\gamma}}{\lambda^{1+\gamma}}\int_{\mathcal{G}}|f\mathbf{1}_G|^{1+\gamma}\,d\widetilde{\sigma}\\
&= C\frac{\delta^{1+\gamma}}{\lambda^{1+\gamma}}\int_G|f|^{1+\gamma}\,d\sigma\le C\delta^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\fint_{a\Delta_0}|f|^{1+\gamma}\,d\sigma\\
&\le C\delta^{1+\gamma}\,\frac{\sigma(a\Delta_0)}{\lambda^{1+\gamma}}\Big[\mathcal{M}_\gamma f(\widetilde{x})\Big]^{1+\gamma}\le C\,(A\delta)^{1+\gamma}\,\sigma(a\Delta_0)\\
&= C\,\theta^{1+\gamma}\,\sigma(a\Delta_0),
\end{aligned}
\tag{2.4.128}
$$

for some constant $C\in(0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and $C_0$. As for the second term in the right-hand side of (2.4.127), once again fix a point $\widetilde{x}\in\mathcal{F}_\lambda$ (which then also belongs to $a\Delta_0$). We may then use the fact that $T_*$ is bounded from $L^1(\partial\Omega,\sigma)$ into the weak Lebesgue space $L^{1,\infty}(\partial\Omega,\sigma)$ (cf. [53, Proposition 3.19]), (2.4.123), Hölder's inequality, (2.4.125), (2.4.104), (2.4.118), and (2.4.114) to obtain

$$
\begin{aligned}
\sigma\Big(\{x\in G:\, T_*(f\mathbf{1}_E)(x)>2\lambda\}\Big)&\le\sigma\Big(\{x\in\partial\Omega:\, T_*(f\mathbf{1}_E)(x)>2\lambda\}\Big)\\
&\le\frac{C}{\lambda}\int_{\partial\Omega}|f|\,\mathbf{1}_E\,d\sigma=\frac{C}{\lambda}\int_{a\Delta_0}|f|\,\mathbf{1}_E\,d\sigma\\
&\le\frac{C}{\lambda}\,\sigma(E)^{\frac{\gamma}{1+\gamma}}\left(\int_{a\Delta_0}|f|^{1+\gamma}\,d\sigma\right)^{\frac{1}{1+\gamma}}\\
&=\frac{C}{\lambda}\Big(\frac{\sigma(E)}{\sigma(a\Delta_0)}\Big)^{\frac{\gamma}{1+\gamma}}\left(\fint_{a\Delta_0}|f|^{1+\gamma}\,d\sigma\right)^{\frac{1}{1+\gamma}}\sigma(a\Delta_0)\\
&\le\frac{C}{\lambda}\exp\Big\{-\frac{C_2\,\gamma}{\delta(1+\gamma)}\Big\}\,\mathcal{M}_\gamma f(\widetilde{x})\,\sigma(a\Delta_0)\\
&\le CA\cdot\exp\Big\{-\frac{C_2\,\gamma}{\delta(1+\gamma)}\Big\}\,\sigma(a\Delta_0)\\
&=C\theta\delta^{-1}\cdot\exp\Big\{-\frac{C_2\,\gamma}{\delta(1+\gamma)}\Big\}\,\sigma(a\Delta_0),
\end{aligned}
\tag{2.4.129}
$$

where $C\in(0,\infty)$ depends only $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. Gathering (2.4.127), (2.4.128), and (2.4.129) then yields

$$
\begin{aligned}
\sigma\Big(\{x\in a\Delta_0: T_*(f\mathbf{1}_{a\Delta_0})(x)>4\lambda\}\Big)
&\le C\Big(\theta^{1+\gamma}+\theta\delta^{-1}\cdot\exp\Big\{-\frac{C_2\,\gamma}{\delta(1+\gamma)}\Big\}+e^{-C_2/\delta}\Big)\,\sigma(a\Delta_0)\\
&=\eta(\theta,\delta)\sigma(a\Delta_0),
\end{aligned}
\tag{2.4.130}
$$

where $\eta(\theta,\delta)\in(0,\infty)$ is as in (2.4.115). Finally, from (2.4.130) and (2.4.126) we see that

$$\sigma(\mathcal{F}_\lambda)\leq\eta(\theta,\delta)\sigma(a\Delta_0), \tag{2.4.131}$$

where $\eta(\theta,\delta)\in(0,\infty)$ is as in (2.4.115).

Moving on, observe that (2.2.285) implies that there exists a point $x_{Q_0}\in\partial\Omega$ with the property that

$$\Delta(x_{Q_0},a_0 2^{-m})\subseteq Q_0\subseteq\Delta(x_{Q_0},a_1 2^{-m}). \tag{2.4.132}$$

From this and (2.4.108) we then conclude that there exists some constant $c>0$, which only depends on the Ahlfors regularity constant of $\partial\Omega$, with the property that $a\Delta_0\subseteq c\Delta(x_{Q_0},a_1 2^{-m})$. As a consequence of this inclusion we may write (for some $C\in(0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$)

$$w(a\Delta_0)\leq w(c\Delta(x_{Q_0},a_1 2^{-m}))\leq C\,w(\Delta(x_{Q_0},a_0 2^{-m}))\leq Cw(Q_0) \tag{2.4.133}$$

where we have also used the fact that $w$ is a doubling measure (cf. (2.2.313)) and (2.4.132). With this in hand, we may now estimate

$$\begin{aligned} w(\mathcal{F}_\lambda) &\leq \eta(\theta,\delta)^\tau\cdot w(a\Delta_0)\leq\eta(\theta,\delta)^\tau\cdot w(Q_0)\\ &\leq\eta(\theta,\delta)^\tau\cdot w(\mathcal{S}_\lambda), \end{aligned} \tag{2.4.134}$$

where the first inequality uses (2.2.315), the fact that $\mathcal{F}_\lambda\subseteq a\Delta_0$ (cf. (2.4.118)), and (2.4.131), the second inequality is based on (2.4.133), while the last inequality is a consequence of (2.4.122). Therefore (2.4.116) holds whenever there exists $Q_0\in\mathcal{Q}_0$ such that $\mathcal{P}_\lambda\cap Q_0=\varnothing$.

To complete the proof of (2.4.116), it remains to consider the case when $\mathcal{P}_\lambda\cap Q\neq\varnothing$ for each $Q\in\mathcal{Q}_0$. In this scenario, take an arbitrary dyadic cube $Q\in\mathcal{Q}_0$. From (2.4.109) we know that $Q\subseteq I_0$. Subdivide $Q$ dyadically and stop when $\mathcal{P}_\lambda\cap Q'=\varnothing$. This produces a family of pairwise disjoint (stopping time) dyadic cubes $\{Q^j\}_{j\in J_Q}\subset\mathbb{D}(\partial\Omega)$ such that $Q^j\subseteq Q$, $Q^j\cap\mathcal{P}_\lambda=\varnothing$, and $Q'\cap\mathcal{P}_\lambda\neq\varnothing$ for all $Q'\in\mathbb{D}(\partial\Omega)$ such that $Q^j\subsetneq Q'\subseteq Q$. In particular $Q^j\subsetneq Q$ for every $j\in J_Q$ and $\widetilde{Q}^j$, the dyadic parent of $Q^j$, satisfies $\widetilde{Q}^j\subseteq Q$. With the $\sigma$-nullset $N$ as in (2.2.288), we now claim that

$$\bigcup_{j\in J_Q}Q^j\subseteq\mathcal{S}_\lambda\cap Q\subseteq\Big(\bigcup_{j\in J_Q}Q^j\Big)\cup N. \tag{2.4.135}$$

To justify the first inclusion above, observe that if $j\in J_Q$ then $Q^j\subseteq\mathcal{S}_\lambda\cap Q$, since $Q^j\subseteq Q\subseteq I_0$ and $Q^j\cap\mathcal{P}_\lambda=\varnothing$ imply that $Q_j\subseteq Q\setminus\mathcal{P}_\lambda=Q\cap\mathcal{S}_\lambda$. This establishes the first inclusion in (2.4.135). As regards the second inclusion claimed in (2.4.135), consider an arbitrary point $x\in(\mathcal{S}_\lambda\cap Q)\setminus N$. Then $T_{(*)}f(x)>\lambda$ which, in view of (2.4.107), ensures that we may find a surface ball $\Delta_x:=\Delta(x,r_x)$ such that $T_{(*)}f(y)>\lambda$ for every $y\in\Delta_x$. Thanks to (2.2.285) and (2.2.287) we may then choose a dyadic cube $Q_x\in\mathbb{D}(\partial\Omega)$ such that $x\in Q_x$ and $Q_x\subseteq\Delta_x\cap Q\subseteq I_0$. This forces $Q_x\subseteq\mathcal{S}_\lambda\cap Q$, hence $Q_x\cap\mathcal{P}_\lambda=\varnothing$. By the maximality of the family chosen above, $Q_x\subseteq Q^j$ for some $j\in J_Q$ which goes to show that $x\in Q^j$. Ultimately, this proves the second inclusion in (2.4.135).

Going further, the idea is to carry out the stopping-time argument just described for each dyadic cube $Q \in \mathcal{Q}_0$. For ease of reference, organize the resulting collection of dyadic cubes $\{Q^j : Q \in \mathcal{Q}_0 \text{ and } j \in J_Q\}$ (which is an at most countable set) as a single-index family $\{Q_\ell\}_{\ell \in \mathcal{I}}$ of mutually disjoint dyadic cubes; in particular,

$$\bigcup_{Q \in \mathcal{Q}_0} \bigcup_{j \in J_Q} Q^j = \bigcup_{\ell \in \mathcal{I}} Q_\ell, \tag{2.4.136}$$

with the latter union comprised of pairwise disjoint dyadic cubes in $\partial\Omega$. Note that $\mathcal{S}_\lambda \cap Q$ might be empty for some $Q \in \mathcal{Q}_0$ and in this case $J_Q = \varnothing$ (i.e., the family of cubes $\{Q^j\}_{j \in J_Q}$ is empty, since there are no stopping time dyadic cubes produced in this case). However, (2.4.109) and (2.4.121) imply that $\mathcal{S}_\lambda \cap Q$ cannot be empty for every $Q \in \mathcal{Q}_0$ and, as a consequence, $\mathcal{I} \neq \varnothing$. Going further, using (2.4.109) and the fact that $\mathcal{S}_\lambda \subseteq I_0$ (cf. (2.4.119)) we may write

$$\bigcup_{Q \in \mathcal{Q}_0} (\mathcal{S}_\lambda \cap Q) = \mathcal{S}_\lambda \tag{2.4.137}$$

which further entails, on account of (2.4.136) and (2.4.135), that

$$\bigcup_{\ell \in \mathcal{I}} Q_\ell \subseteq \mathcal{S}_\lambda \subseteq \Big(\bigcup_{\ell \in \mathcal{I}} Q_\ell\Big) \cup N. \tag{2.4.138}$$

By construction, for each index $\ell \in \mathcal{I}$ there exists a point $x_\ell^*$ such that

$$x_\ell^* \in \widetilde{Q}_\ell \cap \mathcal{P}_\lambda = \widetilde{Q}_\ell \cap (I_0 \setminus \mathcal{S}_\lambda), \tag{2.4.139}$$

where $\widetilde{Q}_\ell$ denotes the dyadic parent of $Q_\ell$ (cf. item *(4)* in Proposition 2.2.39). For each $\ell \in \mathcal{I}$ we let $\Delta_\ell := \Delta_{Q_\ell}$ and $\widetilde{\Delta}_\ell := \Delta_{\widetilde{Q}_\ell}$ be as in (2.2.285). Pressing on, split the collection $\{\Delta_\ell\}_{\ell \in \mathcal{I}}$ into two sub-classes. Specifically, bring in

$$\begin{gathered} \mathcal{I}_1 := \{\ell \in \mathcal{I} : \text{ there exists } x_\ell^{**} \in \Delta_\ell \text{ such that } \mathcal{M}_\gamma f(x_\ell^{**}) \leq A\lambda\} \\ \text{and } \ \mathcal{I}_2 := \mathcal{I} \setminus \mathcal{I}_1. \end{gathered} \tag{2.4.140}$$

Hence, by design, $\mathcal{F}_\lambda \cap \Delta_\ell = \varnothing$ for each $\ell \in \mathcal{I}_2$. Recall now from (2.4.121) that $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$. From this, (2.4.138), and (2.2.285) we then obtain (bearing in mind that $\sigma(N) = 0$; cf. (2.2.288))

$$w(\mathcal{F}_\lambda) = \sum_{\ell \in \mathcal{I}} w(\mathcal{F}_\lambda \cap Q_\ell) \leq \sum_{\ell \in \mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell). \tag{2.4.141}$$

Let us also consider

$$F_\ell := \{x \in \Delta_\ell : T_* f(x) > 4\lambda\} \ \text{ for each } \ \ell \in \mathcal{I}_1, \tag{2.4.142}$$

and observe that this entails

$$\mathcal{F}_\lambda \cap \Delta_\ell \subseteq F_\ell \ \text{ for each } \ \ell \in \mathcal{I}_1, \tag{2.4.143}$$

Our next goal is to prove that

$$\sigma(F_\ell) \leq \eta(\theta, \delta) \cdot \sigma(\Delta_\ell) \ \text{ for each } \ \ell \in \mathcal{I}_1. \tag{2.4.144}$$

Granted this, using (2.2.315) it would follow that

$$w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot w(\Delta_\ell) \text{ for each } \ell \in \mathcal{I}_1 \tag{2.4.145}$$

which, in concert with (2.4.141), (2.4.143), (2.2.285) plus the fact that $w$ is a doubling measure, and (2.4.138), would then imply

$$\begin{aligned} w(\mathcal{F}_\lambda) &\leq \sum_{\ell\in\mathcal{I}_1} w(\mathcal{F}_\lambda \cap \Delta_\ell) \leq \sum_{\ell\in\mathcal{I}_1} w(F_\ell) \leq \eta(\theta,\delta)^\tau \cdot \sum_{\ell\in\mathcal{I}_1} w(\Delta_\ell) \\ &\leq \eta(\theta,\delta)^\tau \cdot \sum_{\ell\in\mathcal{I}_1} w(Q_\ell) \leq \eta(\theta,\delta)^\tau \cdot \sum_{\ell\in\mathcal{I}} w(Q_\ell) \\ &= \eta(\theta,\delta)^\tau \cdot w(\mathcal{S}_\lambda), \end{aligned} \tag{2.4.146}$$

finishing the justification of (2.4.116).

We now turn to the proof of (2.4.144). Fix $\ell \in \mathcal{I}_1$ and, in order to lighten notation, in the sequel we agree to suppress the dependence of $\Delta_\ell$, $\widetilde{\Delta}_\ell$, $F_\ell$, $x^*_\ell$, and $x^{**}_\ell$ on the index $\ell$, and simply write $\Delta$, $\widetilde{\Delta}$, $F$, $x^*$, and $x^{**}$, respectively. With this convention in mind, observe first that

$$\Delta \subseteq 2\widetilde{\Delta}. \tag{2.4.147}$$

To justify this inclusion, recall from (2.2.285) that we may write $\Delta = B(x_Q, r_Q) \cap \partial\Omega$ and $\widetilde{\Delta} = B(x_{\widetilde{Q}}, r_{\widetilde{Q}}) \cap \partial\Omega$; moreover, since $\widetilde{Q}$ is the parent of $Q$, we have $r_{\widetilde{Q}} = 2r_Q$. Then for each $x \in \Delta$ we have $|x - x_{\widetilde{Q}}| \leq |x - x_Q| + |x_Q - x_{\widetilde{Q}}| < r_Q + r_{\widetilde{Q}} = (3/2)r_{\widetilde{Q}} < 2r_{\widetilde{Q}}$ which ultimately proves (2.4.147). Going forward, let us also denote by $\Delta^*$ the surface ball of center $x^*$ and radius $R := \Lambda \cdot r_Q$, for a sufficiently large constant $\Lambda \in (2,\infty)$ (depending only on the implicit constants in the dyadic grid construction, which in turn depend only on the Ahlfors regularity constant of $\partial\Omega$) chosen so that

$$2\widetilde{\Delta} \subseteq \Delta^*. \tag{2.4.148}$$

We then decompose

$$f = f_1 + f_2 \text{ where } f_1 := f\mathbf{1}_{\overline{2\Delta^*}} \text{ and } f_2 := f\mathbf{1}_{\partial\Omega\setminus\overline{2\Delta^*}}. \tag{2.4.149}$$

By virtue of the sub-linearity of $T_*$ and the fact that $\Delta \subseteq \Delta^* \subseteq 4\Delta^*$ (cf. (2.4.147)-(2.4.148)) this implies

$$\begin{aligned} \sigma(F) &\leq \sigma\Big(\{x \in \Delta : T_* f_1(x) > 2\lambda\}\Big) + \sigma\Big(\{x \in \Delta : T_* f_2(x) > 2\lambda\}\Big) \\ &\leq \sigma\Big(\{x \in 4\Delta^* : T_* f_1(x) > 2\lambda\}\Big) + \sigma\Big(\{x \in \Delta : T_* f_2(x) > 2\lambda\}\Big). \end{aligned} \tag{2.4.150}$$

The contribution from $f_1$ in the last line above is handled as in (2.4.123)-(2.4.125), (2.4.127)-(2.4.130) by performing a decomposition of $4\Delta^*$ as in Theorem 2.2.25. Indeed, $a\Delta_0$, $\widetilde{x}$, $f$, and $\lambda$ are replaced by $4\Delta^*$, $x^{**}$, $f_1$, and $\frac{1}{2}\lambda$, respectively, and we use that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (2.4.140)), $\operatorname{supp} f_1 \subseteq \overline{2\Delta^*} \subseteq 4\Delta^*$ (cf. (2.4.149)), and $\sigma(4\Delta^*) \leq c \cdot \sigma(\Delta)$ for some $c \in (0,\infty)$ depending only the Ahlfors regularity constant of $\partial\Omega$ (since

$\partial\Omega$ is Ahlfors regular and the surface balls $4\Delta^*$, $\Delta$ have comparable radii) to run the same proof as before. The conclusion is that

$$\sigma\Big(\{x \in 4\Delta^* : T_* f_1(x) > 2\lambda\}\Big) \leq \eta(\theta,\delta)\cdot\sigma(\Delta). \tag{2.4.151}$$

In view of the conclusion we seek (cf. (2.4.144)), this suits our purposes.

As for $f_2$, recall that $R$ is the radius of the surface ball $\Delta^*$, and for each given $\varepsilon > 0$ set $\varepsilon' := \max\{\varepsilon, 2R\}$. Based on this choice of $\varepsilon'$, the definition of the truncated singular integral operators in (2.4.92), the truncation in the definition of the function $f_2$, the estimate in (2.4.105) (presently used with $x^*$ in place of $x$ and $\varepsilon'$ in place of $\varepsilon$), the fact that $x^{**} \in \Delta \subseteq \Delta^* \subseteq \Delta(x^*, 2\varepsilon')$ (cf. (2.4.140) and (2.4.147)-(2.4.148)), the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (2.4.140)), the definition of $T_{(*)}f(x^*)$ (cf. (2.4.103)), the membership of $x^*$ to $\mathcal{P}_\lambda$ (cf. (2.4.139)), and the first formula in (2.4.119) we may write

$$\begin{aligned}
|T_\varepsilon f_2(x^*)| &= |T_{\varepsilon'} f(x^*)| \leq |T_{\varepsilon'} f(x^*) - T_{(\varepsilon')} f(x^*)| + |T_{(\varepsilon')} f(x^*)| \\
&\leq C\delta \cdot \mathcal{M}_\gamma f(x^{**}) + T_{(*)} f(x^*) \leq C\delta A\lambda + \lambda \\
&= C\theta\lambda + \lambda \leq \tfrac{3}{2}\lambda,
\end{aligned} \tag{2.4.152}$$

with the last line a consequence of our choice of $A$ in (2.4.114) and taking $\theta > 0$ small enough to begin with. With $\varepsilon > 0$ momentarily fixed, consider now an arbitrary point $x \in \Delta$ and bound

$$|T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*)| \leq \mathrm{I} + \mathrm{II} + \mathrm{III}, \tag{2.4.153}$$

where

$$\begin{aligned}
\mathrm{I} &:= \int_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\, |x^*-y|>\varepsilon}} \Big|\langle x-y, \nu(y)\rangle k(x-y) - \langle x^*-y, \nu(y)\rangle k(x^*-y)\Big| |f(y)|\, d\sigma(y), \\
\mathrm{II} &:= \int_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x-y|>\varepsilon,\, |x^*-y|\leq\varepsilon}} |\langle x-y, \nu(y)\rangle||k(x-y)||f(y)|\, d\sigma(y), \\
\mathrm{III} &:= \int_{\substack{y\in\partial\Omega\setminus\overline{2\Delta^*} \\ |x^*-y|>\varepsilon,\, |x-y|\leq\varepsilon}} |\langle x^*-y, \nu(y)\rangle||k(x^*-y)||f(y)|\, d\sigma(y).
\end{aligned} \tag{2.4.154}$$

In preparation for estimating the term I, we will first analyze the difference between I and a similar expression in which $\nu(y)$ has been replaced by the integral average $\nu_{\Delta^*} := ⨍_{\Delta^*} \nu\, d\sigma$. To set the stage, for each fixed $y \in \partial\Omega \setminus \overline{2\Delta^*}$ consider the function

$$F_y(z) := \langle z-y, \nu(y) - \nu_{\Delta^*}\rangle k(z-y) \quad \text{for each} \quad z \in B(x^*, R). \tag{2.4.155}$$

Then

$$|(\nabla F_y)(z)| \leq \Big(\sum_{|\alpha|\leq 1} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \frac{|\nu(y) - \nu_{\Delta^*}|}{|z-y|^n} \quad \text{for each} \quad z \in B(x^*, R). \tag{2.4.156}$$

Keeping in mind that $x \in \Delta \subseteq \Delta^* = B(x^*, R) \cap \partial\Omega$ (cf. (2.4.147)-(2.4.148)), we have

$$|x - x^*| < R. \tag{2.4.157}$$

Also, (recall that $[x, x^*]$ denotes the line segment with endpoints $x, x^*$),

$$|x^* - y| \leq 2|\xi - y| \quad \text{for each} \quad y \in \partial\Omega \setminus \overline{2\Delta^*} \quad \text{and each} \quad \xi \in [x, x^*]. \tag{2.4.158}$$

Hence, by (2.4.155)-(2.4.156), the Mean Value Theorem (bearing in mind (2.4.98)), (2.4.157)-(2.4.158), and Hölder's inequality it follows that

$$\begin{aligned}
&\int_{\partial\Omega\setminus\overline{2\Delta^*}} \Big|\langle x-y, \nu(y)-\nu_{\Delta^*}\rangle k(x-y) - \langle x^*-y, \nu(y)-\nu_{\Delta^*}\rangle k(x^*-y)\Big| |f(y)|\, d\sigma(y) \\
&= \int_{\partial\Omega\setminus\overline{2\Delta^*}} |F_y(x) - F_y(x^*)||f(y)|\, d\sigma(y) \\
&\leq \int_{\partial\Omega\setminus\overline{2\Delta^*}} |x-x^*| \cdot \sup_{\xi\in[x,x^*]} |(\nabla F_y)(\xi)||f(y)|\, d\sigma(y) \\
&\leq C \int_{\partial\Omega\setminus 2\Delta^*} \frac{R}{|x^*-y|^n} |\nu(y)-\nu_{\Delta^*}||f(y)|\, d\sigma(y) \\
&\leq C \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta^*\setminus 2^j\Delta^*} |\nu(y)-\nu_{\Delta^*}||f(y)|\, d\sigma(y) \\
&\leq C \sum_{j=1}^{\infty} 2^{-j} \Big( \fint_{2^{j+1}\Delta^*} \big(|\nu(y)-\nu_{2^{j+1}\Delta^*}| + |\nu_{2^{j+1}\Delta^*} - \nu_{\Delta^*}|\big)^{\frac{1+\gamma}{\gamma}}\, d\sigma(y) \Big)^{\frac{\gamma}{1+\gamma}} \times \\
&\qquad\qquad \times \Big( \fint_{2^{j+1}\Delta^*} |f(y)|^{1+\gamma}\, d\sigma(y) \Big)^{\frac{1}{1+\gamma}} \\
&\leq C \Big( \sum_{j=1}^{\infty} (j+2)\, 2^{-j} \Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \mathcal{M}_\gamma f(x^{**}) \\
&\leq C A\, \delta\lambda,
\end{aligned} \tag{2.4.159}$$

for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Above, the fifth inequality relies on (2.2.43) and the fact that

$$|\nu_{2^{j+1}\Delta^*} - \nu_{\Delta^*}| \leq C\,(j+1)\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \quad \text{for each} \quad j \in \mathbb{N} \tag{2.4.160}$$

for some $C \in (0, \infty)$ depending only on $n$ and the Ahlfors regular constant of $\partial\Omega$, which is a direct consequence of (2.2.46). The fifth inequality in (2.4.159) also uses the fact that $x^{**} \in \Delta \subseteq \Delta^* \subseteq 2^{j+1}\Delta^*$ for each integer $j \in \mathbb{N}$. The last inequality in (2.4.159) is a consequence of the fact that $\mathcal{M}_\gamma f(x^{**}) \leq A\lambda$ (cf. (2.4.140)).

On the other hand, from the properties of the kernel $k$ and the Mean Value Theorem we obtain

$$
\begin{aligned}
\int_{\partial\Omega\setminus\overline{2\Delta^*}} & \Big|\langle x-y,\nu_{\Delta^*}\rangle k(x-y)-\langle x^*-y,\nu_{\Delta^*}\rangle k(x^*-y)\Big||f(y)|\,d\sigma(y)\\
&=\int_{\partial\Omega\setminus\overline{2\Delta^*}}\Big|\left(\langle x-y,\nu_{\Delta^*}\rangle-\langle x^*-y,\nu_{\Delta^*}\rangle\right)k(x^*-y)\\
&\qquad+\langle x-y,\nu_{\Delta^*}\rangle\left(k(x-y)-k(x^*-y)\right)\Big||f(y)|\,d\sigma(y)\\
&\le C_n\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\Big(\frac{|\langle x-x^*,\nu_{\Delta^*}\rangle|}{|x^*-y|^n}+R\frac{|\langle x-y,\nu_{\Delta^*}\rangle|}{|x^*-y|^{n+1}}\Big)|f(y)|\,d\sigma(y)\\
&\le C_n\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle x-x^*,\nu_{\Delta^*}\rangle|}{|x^*-y|^n}|f(y)|\,d\sigma(y)\\
&\quad+C_nR\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle x-y,\nu_{\Delta^*}-\nu_{2^{j+1}\Delta^*}\rangle|}{|x^*-y|^{n+1}}|f(y)|\,d\sigma(y)\\
&\quad+C_nR\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle x-y,\nu_{2^{j+1}\Delta^*}\rangle|}{|x^*-y|^{n+1}}|f(y)|\,d\sigma(y)\\
&=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}\tag{2.4.161}
$$

To estimate $\mathrm{I}_1$, write

$$
\begin{aligned}
\mathrm{I}_1&\le C_nR^{-1}|\langle x-x^*,\nu_{\Delta^*}\rangle|\sum_{j=1}^{\infty}2^{-j}⨍_{2^{j+1}\Delta^*}|f(y)|\,d\sigma(y)\\
&\le C\delta\sum_{j=1}^{\infty}2^{-j}\mathcal{M}_\gamma f(x^{**})\le C\delta\mathcal{M}_\gamma f(x^{**})\\
&\le CA\,\delta\lambda,
\end{aligned}\tag{2.4.162}
$$

where $C\in(0,\infty)$ depends only on $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. The second inequality above is a consequence of (2.4.101) used here with $z:=x^*$, $y:=x^*$, $\mu:=2$ (a valid choice given that $x\in\Delta(x^*,2R)$ since, as seen from (2.4.147)-(2.4.148), we have $x\in\Delta\subseteq\Delta^*=\Delta(x^*,R)$), as well as the fact that $x^{**}\in\Delta\subseteq\Delta^*\subseteq 2^{j+1}\Delta^*$ for each $j\in\mathbb{N}$. The last inequality (2.4.162) uses $\mathcal{M}_\gamma f(x^{**})\le A\lambda$ (cf. (2.4.140)).

To treat $\mathrm{I}_2$, we write (for some $C\in(0,\infty)$ which depends only on $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$),

$$
\begin{aligned}
\mathrm{I}_2&\le CR\sum_{j=1}^{\infty}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\nu_{\Delta^*}-\nu_{2^{j+1}\Delta^*}|}{|x^*-y|^n}|f(y)|\,d\sigma(y)\\
&\le C\sum_{j=1}^{\infty}(j+1)\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,2^{-j}⨍_{2^{j+1}\Delta^*}|f(y)|\,d\sigma(y)\\
&\le C\delta\,\mathcal{M}_\gamma f(x^{**})\le CA\,\delta\lambda,
\end{aligned}\tag{2.4.163}
$$

where the first inequality uses the definition of $\mathrm{I}_2$ and the estimate $|x-y| \leq (3/2)|x^*-y|$ for each $y \in \partial\Omega \setminus 2\Delta^*$, the second inequality takes into account (2.4.160) and the Ahlfors regularity of $\partial\Omega$, while the remaining inequalities are justified as in (2.4.162).

As regards $\mathrm{I}_3$, write (again, with $C \in (0,\infty)$ depending only on $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\begin{aligned}\mathrm{I}_3 &\leq C\sum_{j=1}^{\infty} 2^{-j} ⨍_{2^{j+1}\Delta^*} \frac{|\langle x-y, \nu_{2^{j+1}\Delta^*}\rangle|}{2^{j+1}R}|f(y)|\,d\sigma(y) \\ &\leq C\delta \sum_{j=1}^{\infty} 2^{-j} ⨍_{2^{j+1}\Delta^*} |f(y)|\,d\sigma(y) \leq C\delta\mathcal{M}_\gamma f(x^{**}) \\ &\leq CA\,\delta\lambda. \end{aligned} \tag{2.4.164}$$

The second inequality in (2.4.164) is based on (2.4.101) used with $z := x^*$ and $R$ replaced by $2^{j+1}R$. The remaining inequalities in (2.4.164) are then justified much as in (2.4.162).

At this stage, by combining (2.4.159) and (2.4.161)-(2.4.164) we conclude that there exists some $C \in (0,\infty)$ which depends only on $n$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{I} \leq CA\,\delta\lambda. \tag{2.4.165}$$

To bound II in (2.4.154), recall that $x, x^{**} \in \Delta$ and assume $y \in \partial\Omega \setminus \overline{2\Delta^*}$ is such that $|x^*-y| \leq \varepsilon$ and $|x-y| > \varepsilon$. Then, $2R < |x^*-y| \leq \varepsilon$ and since $x, x^{**} \in \Delta \subseteq B(x_Q, r_Q)$ (where $x_Q$ and $r_Q$ are, respectively, the center and radius of the surface ball $\Delta$) and $R = \Lambda \cdot r_Q$ with $\Lambda > 2$, we have $|x - x^{**}| < 2r_Q < R < \varepsilon/2$. Hence, the point $x^{**}$ belongs to the surface ball $\Delta(x, \varepsilon/2)$. Moreover, on account of (2.4.157) we may write $|x-y| \leq |x-x^*| + |x^*-y| < R + \varepsilon < (3/2)\varepsilon$ which, in particular, guarantees that $y \in \Delta(x, 2\varepsilon)$. Consequently, $\varepsilon < |x-y| < 2\varepsilon$ hence $|k(x-y)| \leq \varepsilon^{-n}$ and (for some $C \in (0,\infty)$ which depends only on depends only on $n$ and the Ahlfors regularity constant of $\partial\Omega$),

$$\begin{aligned}\mathrm{II} &\leq C\varepsilon^{-1} ⨍_{\Delta(x,2\varepsilon)} |\langle x-y, \nu(y)\rangle|\,|f(y)|\,d\sigma(y) \\ &\leq C\varepsilon^{-1} ⨍_{\Delta(x,2\varepsilon)} |\langle x-y, \nu(y) - \nu_{\Delta(x,2\varepsilon)}\rangle|\,|f(y)|\,d\sigma(y) \\ &\quad + C\varepsilon^{-1} ⨍_{\Delta(x,2\varepsilon)} |\langle x-y, \nu_{\Delta(x,2\varepsilon)}\rangle|\,|f(y)|\,d\sigma(y) \\ &=: \mathrm{II}_1 + \mathrm{II}_2. \end{aligned} \tag{2.4.166}$$

Using Hölder's inequality, (2.2.43), (2.4.140), and (2.4.100) we obtain that there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\begin{aligned}\mathrm{II}_1 &\leq C\left(⨍_{\Delta(x,2\varepsilon)} |\nu(y)-\nu_{\Delta(x,2\varepsilon)}|^{\frac{1+\gamma}{\gamma}}\,d\sigma(y)\right)^{\frac{\gamma}{1+\gamma}}\left(⨍_{\Delta(x,2\varepsilon)} |f(y)|^{1+\gamma}\,d\sigma(y)\right)^{\frac{1}{1+\gamma}}\\ &\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,\mathcal{M}_\gamma f(x^{**}) \leq CA\,\delta\lambda, \end{aligned}\tag{2.4.167}$$

since $x^{**}$ is contained in $\Delta(x,\varepsilon/2)\subseteq\Delta(x,2\varepsilon)$ and $\mathcal{M}_\gamma f(x^{**})\leq CA\lambda$, as already noted earlier. As for $\mathrm{II}_2$, invoking (2.4.101), Hölder's inequality, and (2.4.140), it follows that (with $C\in(0,\infty)$ as above)

$$\begin{aligned}\mathrm{II}_2 &\leq C\Big(\sup_{y\in\Delta(x,2\varepsilon)} \varepsilon^{-1}|\langle x-y,\nu_{\Delta(x,2\varepsilon)}\rangle|\Big)⨍_{\Delta(x,2\varepsilon)}|f(y)|\,d\sigma(y)\\ &\leq C\delta\Big(⨍_{\Delta(x,2\varepsilon)}|f(y)|^{1+\gamma}\,d\sigma(y)\Big)^{\frac{1}{1+\gamma}}\\ &\leq C\delta\cdot\mathcal{M}_\gamma f(x^{**})\leq CA\,\delta\lambda.\end{aligned}\tag{2.4.168}$$

From (2.4.166)-(2.4.168) we see that there exists $C\in(0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\mathrm{II}\leq CA\,\delta\lambda.\tag{2.4.169}$$

Turning our attention to III, recall that $x,x^{**}\in\Delta$ and suppose $y\in\partial\Omega\setminus\overline{2\Delta^*}$ is such that $|x^*-y|>\varepsilon$ and $|x-y|\leq\varepsilon$. Then $|x^*-y|>2R>R+|x-x^*|$ by (2.4.157) which further entails $\varepsilon\geq|x-y|\geq|x^*-y|-|x-x^*|>R$. In particular, $R<\varepsilon$. If we now abbreviate $\widetilde{R}:=R+\varepsilon$ then, on the one hand, $|x^*-y|\leq|x^*-x|+|x-y|<R+\varepsilon=\widetilde{R}$, while on the other hand having $|x^*-y|>\varepsilon$ and $|x^*-y|>2R$ implies $|x^*-y|>R+(\varepsilon/2)>\frac{1}{2}\widetilde{R}$. As such, $|k(x^*-y)|\leq\widetilde{R}^{-n}$ and

$$\mathrm{III}\leq C_n\widetilde{R}^{-1}⨍_{\Delta(x^*,\widetilde{R})}|\langle x^*-y,\nu(y)\rangle|\,|f(y)|\,d\sigma(y).\tag{2.4.170}$$

Granted this, the same type of argument which, starting with the first line in (2.4.166) has produced (2.4.169) (reasoning with $\widetilde{R}/2$ replacing $\varepsilon$ and with $x^*$ replacing $x$) will now yield (for some $C\in(0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$)

$$\mathrm{III}\leq CA\,\delta\lambda,\tag{2.4.171}$$

as soon as we show that $x^{**}\in\Delta(x^*,\widetilde{R})$. To justify this membership, start by recalling that $|x-x^{**}|<2r_Q<R$ and then use (2.4.157), the triangle inequality, and the fact that $R<\varepsilon$ to estimate $|x^*-x^{**}|\leq|x-x^*|+|x-x^{**}|<2R<\widetilde{R}$. The proof of (2.4.171) is therefore complete.

Let us summarize our progress. From (2.4.153), (2.4.165), (2.4.169), and (2.4.171) we conclude that there exists some $C\in(0,\infty)$, which depends only on $\theta$, $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$\Big|T_\varepsilon f_2(x)-T_\varepsilon f_2(x^*)\Big|\leq C\,A\,\delta\lambda,\qquad\forall\,x\in\Delta,\quad\forall\,\varepsilon>0.\tag{2.4.172}$$

In view of the fact that $A = \theta \cdot \delta^{-1}$ this entails

$$\Big| T_\varepsilon f_2(x) - T_\varepsilon f_2(x^*) \Big| \leq \tfrac{1}{2}\lambda, \qquad \forall\, x \in \Delta, \quad \forall\, \varepsilon > 0, \tag{2.4.173}$$

provided we pick $\theta > 0$ small to begin with. From (2.4.152), (2.4.173), and (2.4.91) we then obtain

$$T_* f_2(x) \leq 2\lambda \ \text{ for all } \ x \in \Delta, \tag{2.4.174}$$

whenever $\theta > 0$ is small enough. Therefore, for this choice of $\theta$, we conclude that

$$\sigma\Big(\{x \in \Delta :\, T_* f_2(x) > 2\lambda\}\Big) = 0 \tag{2.4.175}$$

which, in concert with (2.4.150) and (2.4.151), establishes (2.4.144). This finishes the proof of the good-$\lambda$ inequality (2.4.116).

Once (2.4.116) has been established, we proceed to prove (2.4.99). First, using (2.4.106), by our definition of $A$, and by possibly choosing a smaller $\theta > 0$, for each point $x \in I_0$ with $T_{(*)} f(x) > \lambda$ and $\mathcal{M}_\gamma f(x) \leq A\lambda$ we may write

$$\begin{aligned} \lambda < T_{(*)} f(x) &\leq T_* f(x) + C\delta \cdot \mathcal{M}_\gamma f(x) \\ &\leq T_* f(x) + C\delta A\lambda = T_* f(x) + C\theta\lambda \\ &< T_* f(x) + \tfrac{1}{2}\lambda. \end{aligned} \tag{2.4.176}$$

Hence, for such a choice of $\theta$ we have

$$\begin{array}{l} \tfrac{1}{2}\lambda < T_* f(x) \text{ whenever the point } x \in I_0 \text{ is such} \\ \text{that } T_{(*)} f(x) > \lambda \text{ and } \mathcal{M}_\gamma f(x) \leq A\lambda. \end{array} \tag{2.4.177}$$

Consequently,

$$\begin{aligned} \{x \in I_0 :\, T_{(*)} f(x) > \lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\} & \\ \subseteq \{x \in I_0 :\, T_* f(x) > \tfrac{\lambda}{2}\} & \end{aligned} \tag{2.4.178}$$

which, in turn, permits us to estimate

$$\begin{aligned} w\Big(\{x \in I_0 :\, T_{(*)} f(x) > \lambda\}\Big) &\leq w\Big(\{x \in I_0 :\, T_{(*)} f(x) > \lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\}\Big) \\ &\qquad + w\Big(\{x \in I_0 :\, \mathcal{M}_\gamma f(x) > A\lambda\}\Big) \\ &\leq w\Big(\{x \in I_0 :\, T_* f(x) > \tfrac{\lambda}{2}\}\Big) \\ &\qquad + w\Big(\{x \in I_0 :\, \mathcal{M}_\gamma f(x) > A\lambda\}\Big). \end{aligned} \tag{2.4.179}$$

From (2.4.115) it is clear that

$$\lim_{\theta \to 0^+} \lim_{\delta \to 0^+} \eta(\theta, \delta) = 0 \tag{2.4.180}$$

so it is possible to choose the threshold $\delta_0 > 0$ and the coefficient $\theta > 0$ small enough depending only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, so that

$$\eta(\theta, \delta)^\tau < (2 \cdot 8^p)^{-1}. \tag{2.4.181}$$

This is the last demand imposed on $\delta, \theta$, and the totality of the such size specifications imply that the final choice of the said parameters ultimately depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. Combining (2.4.179) with (2.4.116) and keeping (2.4.181) in mind we then get

$$\begin{aligned}
w\Big(\{x \in I_0 : T_* f(x) > 4\lambda\}\Big)& \\
&\leq w\Big(\{x \in I_0 :\, T_* f(x) > 4\lambda \ \text{ and } \ \mathcal{M}_\gamma f(x) \leq A\lambda\}\Big) \\
&\qquad + w\Big(\{x \in I_0 :\, \mathcal{M}_\gamma f(x) > A\lambda\}\Big) \\
&\leq \eta(\theta, \delta)^\tau \cdot w\Big(\{x \in I_0 :\, T_{(*)} f(x) > \lambda\}\Big) \\
&\qquad + w\Big(\{x \in I_0 :\, \mathcal{M}_\gamma f(x) > A\lambda\}\Big) \\
&< (2 \cdot 8^p)^{-1}\, w\Big(\{x \in I_0 :\, T_* f(x) > \tfrac{\lambda}{2}\}\Big) \\
&\qquad + \big(1 + (2 \cdot 8^p)^{-1}\big)\, w\Big(\{x \in I_0 :\, \mathcal{M}_\gamma f(x) > A\lambda\}\Big).
\end{aligned} \tag{2.4.182}$$

Recall that $\gamma \in (0, p-1)$ has been chosen so that $w \in A_{p/(1+\gamma)}(\partial\Omega, \sigma)$, hence $\mathcal{M}_\gamma$ is bounded on $L^p(\partial\Omega, w)$. Multiply the most extreme sides of (2.4.182) by $p\lambda^{p-1}$ and integrate over $\lambda \in (0, \infty)$. Bearing in mind that $A = \theta \cdot \delta^{-1}$, after three natural changes of variables (namely, $\widetilde{\lambda} := 4\lambda$ in the first integral, $\widetilde{\lambda} := \frac{1}{2}\lambda$ in the second integral, and $\widetilde{\lambda} := \theta\delta^{-1}\lambda$ in the third integral) we therefore obtain

$$\begin{aligned}
\int_{I_0} |T_* f|^p\, w\, d\sigma &\leq \frac{1}{2}\int_{I_0} |T_* f|^p\, w\, d\sigma + \delta^p\theta^{-p}(2^{2p} + 2^{-p-1})\int_{I_0} (\mathcal{M}_\gamma f)^p\, w\, d\sigma \\
&\leq \frac{1}{2}\int_{I_0} |T_* f|^p\, w\, d\sigma + C\,\delta^p \int_{\partial\Omega} |f|^p\, w\, d\sigma,
\end{aligned} \tag{2.4.183}$$

for some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ (hence, in particular, independent of the function $f$, the quantity $\delta$, as well as the parameters $x_0, m$ defining the set $I_0$). Since $f \in L^p(\partial\Omega, w)$ and the operator $T_*$ maps the space $L^p(\partial\Omega, w)$ into itself (cf. Proposition 2.3.3), it follows that $\int_{I_0} |T_* f|^p\, w\, d\sigma \leq \|T_* f\|^p_{L^p(\partial\Omega, w)} < \infty$. Hence, the first integral in the right-most side of (2.4.183) may be absorbed in the left-most side. By also taking into account (2.4.111), we therefore obtain

$$\int_{2\Delta_0} |T_* f|^p\, w\, d\sigma \leq \int_{I_0} |T_* f|^p\, w\, d\sigma \leq C\delta^p \int_{\partial\Omega} |f|^p\, w\, d\sigma. \tag{2.4.184}$$

Recall that $2\Delta_0 = \Delta(x_0, 2^{-m+1})$ and the only constraint on the integer $m \in \mathbb{Z}$ has been that $\operatorname{supp} f \subseteq 2\Delta_0$. Upon letting $m \to -\infty$ and invoking Lebesgue's Monotone

Convergence Theorem we arrive at the conclusion that, for some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$, we have the estimate

$$\int_{\partial\Omega} |T_* f|^p \, w \, d\sigma \leq C\delta^p \int_{\partial\Omega} |f|^p \, w \, d\sigma, \qquad \text{for every } f \in L^p(\partial\Omega, w) \text{ with compact support.} \tag{2.4.185}$$

To treat the case when the function $f \in L^p(\partial\Omega, w)$ is now arbitrary, for each $j \in \mathbb{N}$ define $f_j := \mathbf{1}_{\Delta(x_0,j)} f$. Then Lebesgue's Dominated Convergence Theorem implies that $f_j \to f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$, and since $T_*$ is continuous on $L^p(\partial\Omega, w)$ we also have $T_* f_j \to T_* f$ in $L^p(\partial\Omega, w)$ as $j \to \infty$. Writing the estimate in (2.4.185) for $f_j$ in place of $f$ and passing to limit $j \to \infty$ then yields

$$\int_{\partial\Omega} |T_* f|^p \, w \, d\sigma \leq C\delta^p \int_{\partial\Omega} |f|^p \, w \, d\sigma \quad \text{for each} \quad f \in L^p(\partial\Omega, w), \tag{2.4.186}$$

where $C \in (0,\infty)$ depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Finally, sending $\delta \searrow \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ (cf. (2.4.100)), this finishes the proof of (2.4.99). □

Recall the notion of chord-arc domain introduced, in the two-dimensional setting, in Definition 2.2.35.

**Corollary 2.4.10.** *Fix $\varkappa_* \in (0,\infty)$ and let $\Omega \subseteq \mathbb{R}^2$ be a $\varkappa$-CAD for some $\varkappa \in [0, \varkappa_*)$. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^2 \setminus \{0\})$, for a sufficiently large integer $N \in \mathbb{N}$, which is even and positive homogeneous of degree $-2$, and define the maximal operator $T_*$ acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$T_* f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon f(x)\big| \quad \textit{for each} \quad x \in \partial\Omega, \tag{2.4.187}$$

*where, for each $\varepsilon > 0$,*

$$T_\varepsilon f(x) := \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y) \, d\sigma(y) \quad \textit{for all} \quad x \in \partial\Omega. \tag{2.4.188}$$

*Then there exists some $C \in (0,\infty)$, which depends only on $\varkappa_*$, $p$, $[w]_{A_p}$ such that*

$$\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^1} |\partial^\alpha k|\Big)\sqrt{\varkappa}. \tag{2.4.189}$$

Of course, the crux of the matter is the presence of $\sqrt{\varkappa}$ as a multiplicative factor in the right hand-side of (2.4.189). As a consequence, $\|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}$ is small if $\Omega \subseteq \mathbb{R}^2$ is a $\varkappa$-CAD whose constant $\varkappa > 0$ is sufficiently small (relative to the integral exponent $p$, the characteristic $[w]_{A_p}$ of the Muckenhoupt weight, and the integral kernel $k$).

*Proof of Corollary 2.4.10.* From Proposition 2.2.36 and Proposition 2.3.3 it follows that $T_*$ is bounded on $L^p(\partial\Omega, w)$, with norm controlled in terms of $\sum_{|\alpha|\leq N} \sup_{S^1} |\partial^\alpha k|$, and $\varkappa_*$, $p$, $[w]_{A_p}$. This trivially implies (2.4.189) when $\varkappa$ stays away from zero, say when $\varkappa \geq \sqrt{2}-1$, simply by adjusting constants. When $\varkappa \in [0\,, \sqrt{2}-1)$ the estimate claimed in (2.4.189) is implied by (2.4.93) and (2.2.257). □

Theorem 2.4.4 then readily implies similar operator norm estimates for principal-value singular integral operators whose integral kernel has a special algebraic format, in that it involves the inner product between the outward unit normal and the chord, as a factor. This is made precise in the corollary below. In turn, for a given second-order, homogeneous, constant complex coefficient system $L$ with $\mathfrak{A}_L^{\rm dis} \neq \varnothing$, and a given Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ satisfying a two-sided local John condition, Corollary 2.4.11 will be used for $T$ being either the boundary-to-boundary double layer potential operator $K_A$ associated with a coefficient tensor $A \in \mathfrak{A}_L^{\rm dis}$, or its "transposed" version $K_A^{\#}$, acting on Muckenhoupt weighted Lebesgue spaces on $\partial\Omega$.

**Corollary 2.4.11.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain satisfying a two-sided local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same same symbol $w$ for the measure associated with the given weight $w$ as in (2.2.292). Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$ and suppose $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ is a complex-valued function which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators $T, T^{\#}$ acting on each function $f \in L^p(\partial\Omega, w)$ according to*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \tag{2.4.190}$$

*and*

$$T^{\#}f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\, d\sigma(y), \tag{2.4.191}$$

*at $\sigma$-a.e. point $x \in \partial\Omega$. Then there exists a constant $C \in (0,\infty)$, which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that*

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{2.4.192}$$

*and*

$$\big\|T^{\#}\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.4.193}$$

*Furthermore, with $p' \in (1,\infty)$ denoting the Hölder conjugate exponent of $p$ and with $w' := w^{1-p'} \in A_{p'}(\partial\Omega,\sigma)$, it follows that*

$$\begin{array}{l}\text{the (real) transposed of } T : L^p(\partial\Omega,w) \to L^p(\partial\Omega,w) \text{ is} \\ \text{the operator } T^{\#} : L^{p'}(\partial\Omega,w') \to L^{p'}(\partial\Omega,w').\end{array} \tag{2.4.194}$$

*Proof.* In view of the fact that

$$\|T\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq \|T_*\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)}, \tag{2.4.195}$$

the estimate claimed in (2.4.192) follows directly from (2.4.93). As regards the claim made in (2.4.193), first observe that (2.4.194) holds, thanks to (2.4.190)-(2.4.191) and (2.3.29). As such,

$$\begin{aligned}\left\|T^{\#}\right\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} &= \|T\|_{L^{p'}(\partial\Omega,w')\to L^{p'}(\partial\Omega,w')} \\ &\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},\end{aligned} \tag{2.4.196}$$

thanks to (2.4.192) used with $p', w'$ in place of $p, w$. □

*Remark* 2.4.12. Of course, in the special case when $w \equiv 1$, Theorem 2.4.4 and Corollary 2.4.11 yield estimates on ordinary Lebesgue spaces, $L^p(\partial\Omega,\sigma)$ with $p \in (1,\infty)$. Via real interpolation, these further imply similar estimates on the scale of Lorentz spaces on $\partial\Omega$. Specifically, from (2.4.93), (2.4.192)-(2.4.193), and real interpolation (for sub-linear operators) we conclude that for each $p \in (1,\infty)$ and $q \in (0,\infty]$ there exists a constant $C \in (0,\infty)$, which depends only on $n$, $p$, $q$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$\|T_*\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.4.197}$$

$$\|T\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.4.198}$$

and

$$\left\|T^{\#}\right\|_{L^{p,q}(\partial\Omega,\sigma)\to L^{p,q}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.4.199}$$

*Remark* 2.4.13. In the context of Corollary 2.4.11, estimates (2.4.192)-(2.4.193) remain valid with a fixed constant $C \in (0,\infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are allowed to vary while retaining control. Concretely, Remark 2.4.5 implies that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on $n$, $I$, $W$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that (2.4.192)-(2.4.193) hold for each $p \in I$ and each $w \in A_p(\partial\Omega,\sigma)$ with $[w]_{A_p} \leq W$.

Similar considerations apply to the estimates in (2.4.197)-(2.4.199).

### 2.4.3 Norm estimates and invertibility results for double layers

We first recall a result (cf. [53, Theorem 2.16, p. 2603]) which is a combination of the extrapolation theorem of Rubio de Francia with the commutator theorem of Coifman et al., [27], suitably adapted to setting of spaces of homogeneous type.

**Theorem 2.4.14.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1,\infty)$ along with some non-decreasing function $\Phi : (0,\infty) \to (0,\infty)$ and let $T$ be a linear operator which is bounded on $L^{p_0}(\Sigma, w)$ for every $w \in A_{p_0}(\Sigma,\sigma)$, with operator norm $\leq \Phi([w]_{A_{p_0}})$.*

*Then for each integrability exponent $p \in (1,\infty)$ there exist $C_1, C_2 \in (0,\infty)$ which depend exclusively on the dimension $n$, the exponents $p_0, p$, and the Ahlfors regularity constant of $\Sigma$, with the property that for any Muckenhoupt weight $w \in A_p(\Sigma,\sigma)$ the operator*

$$T : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{2.4.200}$$

*is well defined, linear, and bounded, with operator norm*

$$\|T\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{\max\{1,(p_0-1)/(p-1)\}}\Big). \tag{2.4.201}$$

*In addition, given any $p \in (1,\infty)$ along with some $w \in A_p(\Sigma,\sigma)$, there exists a constant $C = C(\Sigma, n, p_0, p, [w]_{A_p}) \in (0,\infty)$ with the property that for every complex-valued function $b \in L^\infty(\Sigma,\sigma)$ one has (with $C_1$ as before)*

$$\|[M_b, T]\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,, \tag{2.4.202}$$

*where $[M_b, T]$ is the commutator of $T$ considered as in (2.4.200) and the operator $M_b$ of pointwise multiplication on $L^p(\Sigma, w)$ by the function $b$, i.e.,*

$$[M_b, T]f := bT(f) - T(bf) \quad \text{for each} \quad f \in L^p(\Sigma, w). \tag{2.4.203}$$

*In particular, from (2.4.202) with $w \equiv 1$ and real interpolation it follows that, for any given $p \in (1,\infty)$ and $q \in (0,\infty]$, there exists a constant $C = C(\Sigma, n, p, q) \in (0,\infty)$ with the property that for every complex-valued function $b \in L^\infty(\Sigma,\sigma)$ one has*

$$\|[M_b, T]\|_{L^{p,q}(\Sigma,\sigma)\to L^{p,q}(\Sigma,\sigma)} \leq C_1 \cdot \Phi(C)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.4.204}$$

Theorem 2.4.14 is a particular case of a more general result proved in Theorem 2.4.16, stated just after the following remark.

*Remark* 2.4.15. Even though Theorem 2.4.14 suffices for the purposes we have in mind, it is worth noting that there is a version of (2.4.202) in which the pointwise multiplier $b$ is allowed to belong to the larger space $\mathrm{BMO}(\Sigma,\sigma)$. The price to pay is that we now no longer may regard $[M_b, T]$ as in (2.4.203) and, instead, have to interpret this as an abstract extension (by density) of a genuine commutator. Specifically, given a real-valued function $b \in \mathrm{BMO}(\Sigma,\sigma)$, for each $N \in \mathbb{N}$ define

$$b_N := \min\Big\{\max\{b, -N\}\,,\, N\Big\} = \max\Big\{\min\{b, N\}\,,\, -N\Big\}, \tag{2.4.205}$$

and note that there exists $C \in (0,\infty)$ such that

$$
\begin{aligned}
& b_N \in L^\infty(\Sigma,\sigma), \ \text{ thus } \ b_N \in \mathrm{BMO}(\Sigma,\sigma), \ \text{ and} \\
& \|b_N\|_{\mathrm{BMO}(\Sigma,\sigma)} \leq 2\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} \ \text{ for all } \ N \in \mathbb{N}, \\
& |b_N(x)| \leq |b(x)| \ \text{ for all } \ x \in \Sigma \ \text{ and } \ N \in \mathbb{N}, \\
& \lim_{N\to\infty} b_N(x) = b(x) \ \text{ for each } x \text{ belonging to } \Sigma.
\end{aligned}
\tag{2.4.206}
$$

Keeping in mind that $\mathrm{BMO}(\Sigma,\sigma) \subseteq L^{p_0}_{\mathrm{loc}}(\Sigma,\sigma)$, Lebesgue's Dominated Convergence Theorem then implies $b_N \to b$ in $L^{p_0}_{\mathrm{loc}}(\Sigma,\sigma)$ as $N \to \infty$, hence $b_N f \to bf$ in $L^{p_0}(\Sigma,\sigma)$ as $N \to \infty$ for any $f$ in $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ (the space of essentially bounded functions with compact support in $\Sigma$). For each such function $f$ the hypotheses on $T$ then imply that $T(b_N f) \to T(bf)$ in $L^{p_0}(\Sigma,\sigma)$ as $N \to \infty$. Since we also have $b_N T(f) \to bT(f)$ at each point in $\Sigma$ as $N \to \infty$, we ultimately conclude that

$$
\begin{array}{l}
\text{for each function } f \in L^\infty_{\mathrm{comp}}(\Sigma,\sigma) \text{ there exists a strictly} \\
\text{increasing sequence } \{N_j\}_{j\in\mathbb{N}} \subseteq \mathbb{N} \text{ with the property that} \\
[M_{b_{N_j}},T]f \to [M_b,T]f \text{ at } \sigma\text{-a.e. point in } \Sigma \text{ as } j \to \infty.
\end{array}
\tag{2.4.207}
$$

Having fix $p \in (1,\infty)$ along with $w \in A_p(\Sigma,\sigma)$, for each such function $f$ in $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ we may now write

$$
\begin{aligned}
\int_\Sigma \big|[M_b,T]f\big|^p\,dw &= \int_\Sigma \liminf_{j\to\infty} \Big|[M_{b_{N_j}},T]f\Big|^p\,dw \\
&\leq \liminf_{j\to\infty} \int_\Sigma \Big|[M_{b_{N_j}},T]f\Big|^p\,dw \\
&\leq \liminf_{j\to\infty} \Big(C_1 \cdot \Phi(C)\,\big\|b_{N_j}\big\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p \|f\|_{L^p(\Sigma,w)} \\
&\leq \Big(C_1 \cdot \Phi(C)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\Big)^p \|f\|_{L^p(\Sigma,w)},
\end{aligned}
\tag{2.4.208}
$$

where the equality comes from (2.4.207), the first inequality is implies by Fatou's Lemma, the second inequality is a consequence of (2.4.202) (bearing in mind the first property in (2.4.206)), and the last inequality follows from the second line of (2.4.206). In turn, (2.4.208) proves that $[M_b,T]$ maps $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$, regarded as a subspace of $L^p(\Sigma,w)$, boundedly into $L^p(\Sigma,w)$. Given that $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$ is dense in $L^p(\Sigma,w)$, we finally conclude that $[M_b,T]$, acting as a commutator on $L^\infty_{\mathrm{comp}}(\Sigma,\sigma)$, extends by density to a linear and bounded mapping on $L^p(\Sigma,w)$.

Here is a generalization of Theorem 2.4.14, involving the "maximal commutator" associated with a given family of linear and bounded operators.

**Theorem 2.4.16.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0 \in (1,\infty)$ and let $\{T_j\}_{j\in\mathbb{N}}$ be a family of linear operators which are bounded on $L^{p_0}(\Sigma,w)$ for every $w \in A_{p_0}(\Sigma,\sigma)$. Define the action of the maximal operator associated with this family on each function $f \in L^{p_0}(\Sigma,w)$ with $w \in A_{p_0}(\Sigma,\sigma)$ as*

$$
T_{\max} f(x) := \sup_{j\in\mathbb{N}} |T_j f(x)| \quad \textit{for each} \quad x \in \Sigma. \tag{2.4.209}
$$

*Assume that for each $w \in A_{p_0}(\Sigma, \sigma)$ the sub-linear operator $T_{\max}$ maps $L^{p_0}(\Sigma, w)$ into itself, and that there exists some non-decreasing function $\Phi : (0, \infty) \to (0, \infty)$ with the property that*

$$\|T_{\max}\|_{L^{p_0}(\Sigma,w)\to L^{p_0}(\Sigma,w)} \leq \Phi\big([w]_{A_{p_0}}\big) \ \text{ for each } \ w \in A_{p_0}(\Sigma, \sigma). \tag{2.4.210}$$

*Then the following statements are true.*

*(i) For each integrability exponent $p \in (1, \infty)$ there exist $C_1, C_2 \in (0, \infty)$ which depend exclusively on the dimension $n$, the exponents $p_0, p$, and the Ahlfors regularity constant of $\Sigma$, with the property that for any Muckenhoupt weight $w \in A_p(\Sigma, \sigma)$ the operator*

$$T_{\max} : L^p(\Sigma, w) \longrightarrow L^p(\Sigma, w) \tag{2.4.211}$$

*is well defined, sub-linear, and bounded, with operator norm*

$$\|T_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi\Big(C_2 \cdot [w]_{A_p}^{\max\{1,(p_0-1)/(p-1)\}}\Big). \tag{2.4.212}$$

*In particular, for each $j \in \mathbb{N}$ the operator $T_j$ is a well-defined, linear, and bounded mapping on $L^p(\Sigma, w)$ with operator norm satisfying a similar estimate to (2.4.212).*

*(ii) Pick an arbitrary $p \in (1, \infty)$ along with some $w \in A_p(\Sigma, \sigma)$, and fix an arbitrary complex-valued function $b \in L^\infty(\Sigma, \sigma)$. Define the action of the "maximal commutator" (associated with the given function $b$ and the family $\{T_j\}_{j\in\mathbb{N}}$) on each function $f \in L^p(\Sigma, w)$ as*

$$C_{\max} f(x) := \sup_{j\in\mathbb{N}} \big|[M_b, T_j]f(x)\big| \ \text{ for each } \ x \in \Sigma, \tag{2.4.213}$$

*where, as in the past, $M_b$ denotes the operator of pointwise multiplication by the function $b$. Then there exist two constants $C_i = C_i(\Sigma, n, p_0, p, [w]_{A_p}) \in (0, \infty)$, $i \in \{1, 2\}$, independent of the function $b$ and the family $\{T_j\}_{j\in\mathbb{N}}$, with the property that*

$$\|C_{\max}\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)} \leq C_1 \cdot \Phi(C_2)\, \|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.4.214}$$

The particular case when all operators in the family $\{T_j\}_{j\in\mathbb{N}}$ are identical to one another corresponds to Theorem 2.4.14.

*Proof of Theorem 2.4.16.* The fact that for each $p \in (1, \infty)$ and $w \in A_p(\Sigma, \sigma)$ the sub-linear operator $T_{\max}$ induces a bounded mapping on $L^p(\Sigma, w)$ whose operator norm may be estimated as in (2.4.212) follows from Rubio de Francia's extrapolation theorem. The specific format of the constant in (2.4.212) is seen from a straightforward adaptation to the setting of measure metric spaces of the Euclidean argument in [35, Theorem 3.2], [28, Theorem 3.22, p.40] (making use of (2.2.309)). This takes care of item *(i)*.

To deal with item *(ii)*, we shall adapt the argument in [27], [59], [53]. First, from simple linearity and homogeneity considerations, there is no loss of generality in assuming that $b \in L^\infty(\Sigma,\sigma)$ is actually real-valued and satisfies $\|b\|_{\mathrm{BMO}(\Sigma,\sigma)} = 1$ (the case when $b$ is constant is trivial). Fix now $p \in (1,\infty)$ and $w \in A_p(\Sigma,\sigma)$. From item *(8)* of Proposition 2.2.42 we know that there exists some small $\varepsilon = \varepsilon(\Sigma, p, [w]_{A_p}) > 0$ with the property that for each complex number $z$ with $|z| \leq \varepsilon$ we have

$$w \cdot e^{(\mathrm{Re}\, z)b} \in A_p(\Sigma,\sigma) \ \text{ with } \ \big[w \cdot e^{(\mathrm{Re}\, z)b}\big]_{A_p} \leq C, \tag{2.4.215}$$

where the constant $C = C(\Sigma, p, [w]_{A_p}) \in (0,\infty)$ is independent of $z$.

To proceed, denote by $\mathscr{L}(L^p_w)$ the space of all linear and bounded operators from $L^p(\Sigma,w)$ into itself, equipped with the operator norm $\|\cdot\|_{L^p(\Sigma,w)\to L^p(\Sigma,w)}$. The idea is now to observe that, for each $j \in \mathbb{N}$,

$$\begin{gathered}\Phi_j : \{z \in \mathbb{C} : \ |z| < \varepsilon/2\} \longrightarrow \mathscr{L}(L^p_w) \ \text{ defined as} \\ \Phi_j(z) := M_{e^{zb}} T_j M_{e^{-zb}} \ \text{ for each } \ z \in \mathbb{C} \ \text{ with } \ |z| < \varepsilon/2,\end{gathered} \tag{2.4.216}$$

is an analytic map which, for each $z \in \mathbb{C}$ with $|z| < \varepsilon/2$ and each $f \in L^p(\Sigma,w)$, satisfies

$$\begin{aligned}\int_\Sigma & \sup_{j\in\mathbb{N}} \big|\Phi_j(z)f(x)\big|^p \, w(x)\, d\sigma(x) \\ &= \int_\Sigma \sup_{j\in\mathbb{N}} \big|T_j(e^{-zb}f)(x)\big|^p \, w(x)\cdot e^{(\mathrm{Re}\, z)b(x)}\, d\sigma(x) \\ &= \int_\Sigma \big|T_{\max}(e^{-zb}f)(x)\big|^p \, w(x)\cdot e^{(\mathrm{Re}\, z)b(x)}\, d\sigma(x) \\ &\leq \|T_{\max}\|^p_{L^p(\Sigma, w\cdot e^{(\mathrm{Re}\, z)b})\to L^p(\Sigma, w\cdot e^{(\mathrm{Re}\, z)b})} \times \\ &\qquad \times \int_\Sigma \big|e^{-zb(x)}f(x)\big|^p \, w(x)\cdot e^{(\mathrm{Re}\, z)b(x)}\, d\sigma(x) \\ &\leq C_1^p \cdot \Phi\Big(C_2 \cdot C^{\max\{1,(p_0-1)/(p-1)\}}\Big)^p \|f\|^p_{L^p(\Sigma,w)},\end{aligned} \tag{2.4.217}$$

thanks to (2.4.216), (2.4.209), (2.4.215), and (2.4.212). In addition, from (2.4.216) and Cauchy's reproducing formula for analytic functions we see that for each $j \in \mathbb{N}$ we have

$$[M_b, T_j] = \Phi_j'(0) = \frac{1}{2\pi i} \int_{|z|=\varepsilon/4} \frac{\Phi_j(z)}{z^2}\, dz. \tag{2.4.218}$$

Consequently, for each $f \in L^p(\Sigma,w)$ and $x \in \Sigma$, we have

$$C_{\max} f(x) = \sup_{j\in\mathbb{N}} \big|[M_b, T_j]f(x)\big| \leq \frac{8}{\pi\varepsilon^2} \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z)f(x)\big|\, d\mathcal{H}^1(z), \tag{2.4.219}$$

hence

$$\big|C_{\max} f(x)\big|^p \leq \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} \big|\Phi_j(z)f(x)\big|^p\, d\mathcal{H}^1(z). \tag{2.4.220}$$

From the last property in item *(i)* and (2.4.213) we see that for each $f \in L^p(\Sigma,w)$ the function $C_{\max} f$ is $\sigma$-measurable. In concert with (2.4.220) and (2.4.217), this permits us to estimate

$$
\begin{aligned}
&\int_{\Sigma} |C_{\max} f(x)|^p \, dw(x) \\
&\qquad \leq \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{\Sigma} \Bigg( \int_{|z|=\varepsilon/4} \sup_{j\in\mathbb{N}} |\Phi_j(z) f(x)|^p \, d\mathcal{H}^1(z) \Bigg) \, dw(x) \\
&\qquad = \Big(\frac{8}{\pi\varepsilon^2}\Big)^p \int_{|z|=\varepsilon/4} \Bigg( \int_{\Sigma} \sup_{j\in\mathbb{N}} |\Phi_j(z) f(x)|^p \, dw(x) \Bigg) \, d\mathcal{H}^1(z) \\
&\qquad \leq \Big(\frac{2^{3p-1}}{\pi^{p-1}\varepsilon^{2p-1}}\Big) C_1^p \cdot \Phi\Big(C_2 \cdot C^{\max\{1,(p_0-1)/(p-1)\}}\Big)^p \|f\|^p_{L^p(\Sigma,w)},
\end{aligned}
\tag{2.4.221}
$$

and (2.4.214) readily follows from this. □

We next discuss a companion result to Theorem 2.4.4, the novelty being the consideration of a maximal "transposed" operator as defined below in (2.4.222).

**Theorem 2.4.17.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain satisfying a two-sided local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and recall the earlier convention of using the same same symbol $w$ for the measure associated with the given weight $w$ as in (2.2.292). Also, consider a sufficiently large integer $N = N(n) \in \mathbb{N}$. Given a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, consider the maximal operator $T_*^{\#}$ whose action on each given function $f \in L^p(\partial\Omega, w)$ is defined as*

$$
T_*^{\#} f(x) := \sup_{\varepsilon>0} \big|T_\varepsilon^{\#} f(x)\big| \quad \text{for } \sigma\text{-a.e. } \ x \in \partial\Omega, \tag{2.4.222}
$$

*where, for each $\varepsilon > 0$,*

$$
T_\varepsilon^{\#} f(x) := \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y) \, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } \ x \in \partial\Omega. \tag{2.4.223}
$$

*Then there exists some $C \in (0,\infty)$, which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$
\Big\|T_*^{\#}\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.4.224}
$$

In particular, Theorem 2.4.17 may be used to give a direct proof of (2.4.193), without having to rely on duality.

*Proof of Theorem 2.4.17.* To get started, we make the observation that if $\mathbb{Q}_+$ denotes the collection of all positive rational numbers, then for each function $f \in L^1\big(\partial\Omega\,,\frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$
(T_*^{\#} f)(x) = \sup_{\varepsilon\in\mathbb{Q}_+} \big|(T_\varepsilon^{\#} f)(x)\big| \quad \text{for every } \ x \in \partial^*\Omega. \tag{2.4.225}
$$

To justify this, pick some $f \in L^1\big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\big)$. We claim that if $x \in \partial^*\Omega$ is arbitrary and fixed then for each $\varepsilon \in (0,\infty)$ and each sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ such that $\varepsilon_j \searrow \varepsilon$ as $j \to \infty$ we have

$$\begin{aligned}\lim_{j\to\infty} \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon_j}} &\langle y-x,\nu(x)\rangle k(x-y)f(y)\,d\sigma(y)\\ &= \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} \langle y-x,\nu(x)\rangle k(x-y)f(y)\,d\sigma(y).\end{aligned} \tag{2.4.226}$$

To justify (2.4.226) note that

$$\{y\in\partial\Omega:\,|x-y|>\varepsilon_j\} \nearrow \{y\in\partial\Omega:\,|x-y|>\varepsilon\} \text{ as } j\to\infty, \tag{2.4.227}$$

in the sense that

$$\begin{aligned}\{y\in\partial\Omega:\,|x-y|>\varepsilon\} = \bigcup_{j\in\mathbb{N}}\{y\in\partial\Omega:\,|x-y|>\varepsilon_j\} \text{ and}\\ \{y\in\partial\Omega:\,|x-y|>\varepsilon_j\} \subseteq \{y\in\partial\Omega:\,|x-y|>\varepsilon_{j+1}\} \text{ for every } j\in\mathbb{N}.\end{aligned} \tag{2.4.228}$$

Then (2.4.226) follows from (2.4.227), the properties of $k$, and Lebesgue's Dominated Convergence Theorem. What we have just proved amounts to saying that for every function $f \in L^1\big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\big)$ we have

$$\lim_{j\to\infty}(T^{\#}_{\varepsilon_j}f)(x) = (T^{\#}_{\varepsilon}f)(x) \text{ for every } x\in\partial^*\Omega, \tag{2.4.229}$$

whenever $\varepsilon \in (0,\infty)$ and $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty)$ are such that $\varepsilon_j \searrow \varepsilon$ as $j\to\infty$. Having established this, (2.4.225) readily follows on account of the density of $\mathbb{Q}_+$ in $(0,\infty)$.

To proceed, let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}_+$. Also, bring back the operators (2.4.92) and observe that for each $j\in\mathbb{N}$, each $f\in L^p(\partial\Omega,w)$, and each $x\in\partial^*\Omega$ we have

$$T^{\#}_{\varepsilon_j}f(x) + T_{\varepsilon_j}f(x) = \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon_j}} \langle y-x,\nu(x)-\nu(y)\rangle k(x-y)f(y)\,d\sigma(y). \tag{2.4.230}$$

Write $(\nu_i)_{1\le i\le n}$ for the scalar components of the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ and, for every $i\in\{1,\dots,n\}$, every $j\in\mathbb{N}$, and every $f\in L^p(\partial\Omega,w)$ set

$$\mathbb{T}^{(i)}_j f(x) := \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon_j}} (y_i-x_i)k(x-y)f(y)\,d\sigma(y) \text{ for each } x\in\partial\Omega. \tag{2.4.231}$$

Then, for each $j\in\mathbb{N}$ and each $f\in L^p(\partial\Omega,w)$ we may recast (2.4.230) as

$$T^{\#}_{\varepsilon_j}f(x) + T_{\varepsilon_j}f(x) = \sum_{i=1}^{n}\big[M_{\nu_i}\,,\,\mathbb{T}^{(i)}_j\big]f(x) \text{ for each } x\in\partial^*\Omega. \tag{2.4.232}$$

If for each $i \in \{1, \dots, n\}$ and each $f \in L^p(\partial\Omega, w)$ we now define

$$C^{(i)}_{\max} f(x) := \sup_{j\in\mathbb{N}} \Big| [M_{\nu_i}, \mathbb{T}^{(i)}_j] f(x) \Big| \quad \text{for each} \quad x \in \partial^*\Omega, \tag{2.4.233}$$

then, thanks to Proposition 2.3.3, for each $i \in \{1, \dots, n\}$ we may invoke Theorem 2.4.16 for the family $\{\mathbb{T}^{(i)}_j\}_{j\in\mathbb{N}}$ to conclude that

$$\Big\| C^{(i)}_{\max} \Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C \Big( \sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k| \Big) \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.4.234}$$

where $C \in (0, \infty)$, which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. Also, from (2.4.232), (2.4.233), (2.4.225), and (2.4.91) we deduce that for each $f \in L^p(\partial\Omega, w)$ we have

$$T^{\#}_* f(x) \leq T_* f(x) + \sum_{i=1}^{n} C^{(i)}_{\max} f(x) \quad \text{for each} \quad x \in \partial^*\Omega. \tag{2.4.235}$$

At this stage, the estimate claimed in (2.4.224) becomes a consequence of (2.4.235), (2.4.93), and (2.4.234), keeping in mind that, as is apparent from (2.4.225), the function $T^{\#}_* f$ is $\sigma$-measurable, and that we currently have $\sigma(\partial\Omega \setminus \partial^*\Omega) = 0$ (cf. Definition 1.1.2 and (1.1.14)). □

To discuss a significant application of Theorem 2.4.14 let us first formally introduce the family of Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on the boundary a UR domain $\Omega \subseteq \mathbb{R}^n$. Specifically, with $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, for each $j \in \{1, \dots, n\}$ the $j$-th Riesz transform $R_j$ acts on any given function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y) \tag{2.4.236}$$

at $\sigma$-a.e. point $x \in \partial\Omega$.

**Theorem 2.4.18.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from (2.3.8), the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ from (2.4.236), and for each $k \in \{1, \dots, n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by the $k$-th scalar component of $\nu$.*

*Then there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\begin{aligned} &\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \\ &\qquad + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \end{aligned} \tag{2.4.237}$$

*Proof.* The estimate claimed in (2.4.237) is implied by (2.3.8), Corollary 2.4.11, (2.4.236), Proposition 2.3.3, and Theorem 2.4.14. $\square$

We shall, once again, see Theorem 2.4.14 in action shortly, in the proof of Theorem 2.4.20 stated a little later. As a preamble, we recall the following lemma from [93], which identifies the commutator between the double layer potential operator $K_A$ from (2.3.4) and the first-order tangential derivative operators $\partial_{\tau_{jk}}$ from (2.2.344) as being yet another commutator, of the sort considered in Theorem 2.4.14 (with the function $b$ a scalar component of the outward unit normal $\nu$).

**Lemma 2.4.19.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. Let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ and consider the matrix-valued fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with $L$ as in Theorem 1.2.1. Also, pick a coefficient tensor $A = (a^{\alpha\beta}_{jk})_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq j,k\leq n}} \in \mathfrak{A}_L$ and bring in $K_A$, the boundary-to-boundary double layer potential operator associated with $\Omega$ and $A$ as in (2.3.4). In addition, for each $j,k \in \{1,\dots,n\}$ define the singular integral operator $U_{jk}$ acting on each given matrix-valued function $F = (F_{\alpha s})_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}$ with entries belonging to $L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})$ as $U_{jk}F = \big((U_{jk}F)_\gamma\big)_{1\leq\gamma\leq M}$ where, for each index $\gamma \in \{1,\dots,M\}$,*

$$
\begin{aligned}
(U_{jk}F)_\gamma(x) := &- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(x)-\nu_k(y)]\nu_j(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) F_{\alpha s}(y)\, d\sigma(y)\\
&+ \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(x)-\nu_j(y)]\nu_k(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) F_{\alpha s}(y)\, d\sigma(y)\\
&+ \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(y)-\nu_k(x)]\nu_s(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) F_{\alpha j}(y)\, d\sigma(y)\\
&- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(y)-\nu_j(x)]\nu_s(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) F_{\alpha k}(y)\, d\sigma(y)
\end{aligned}
\tag{2.4.238}
$$

*at $\sigma$-a.e. point $x \in \partial\Omega$. Finally, fix some integrability exponents $p,q \in (1,\infty]$ and consider a function*

$$
\begin{aligned}
&f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M \quad \textit{with the property that}\\
&\partial_{\tau_{jk}} f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^q_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M \quad \textit{for all } j,k \in \{1,\dots,n\}.
\end{aligned}
\tag{2.4.239}
$$

*Then for each $j,k, \in \{1,\dots,n\}$ one has*

$$
\partial_{\tau_{jk}}(K_A f) = K_A(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \quad \textit{at } \sigma\textit{-a.e. point on } \partial\Omega, \tag{2.4.240}
$$

*where* $\nabla_{\tan} f$ *is regarded as the* $M \times n$ *matrix-valued function* $F = (F_{\alpha s})_{\substack{1\leq\alpha\leq M\\1\leq s\leq n}}$ *whose entry* $F_{\alpha s}$ *is the* $s$*-th component of* $\nabla_{\tan} f_\alpha$*.*

In Theorem 2.4.20 below the focus is obtaining operator norm estimates for double layer potentials associated with distinguished coefficient tensors on Muckenhoupt weighted Lebesgue and Sobolev spaces, involving the BMO semi-norm of the unit normal to the boundary of the underlying domain as a factor.

**Theorem 2.4.20.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu$ *the geometric measure theoretic outward unit normal to* $\Omega$*. Also, let* $L$ *be a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$ *for which* $\mathfrak{A}_L^{\rm dis} \neq \varnothing$*. Pick* $A \in \mathfrak{A}_L^{\rm dis}$ *and consider the boundary-to-boundary double layer potential operators* $K_A, K_A^{\#}$ *associated with* $\Omega$ *and the coefficient tensor* $A$ *as in* (2.3.4) *and* (2.3.5)*, respectively. Finally, fix an integrability exponent* $p \in (1,\infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$*.*

*Then there exists some* $C \in (0,\infty)$ *which depends only on* $n$*,* $A$*,* $p$*,* $[w]_{A_p}$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$*, such that*

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.4.241}$$

$$\|K_A\|_{[L_1^p(\partial\Omega,w)]^M\to[L_1^p(\partial\Omega,w)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.4.242}$$

*and*

$$\left\|K_A^{\#}\right\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.4.243}$$

Note that the estimate in (2.4.241) implies that the operator $K_A$ becomes identically zero whenever $\Omega$ is a half-space in $\mathbb{R}^n$. From *(i)*$\Leftrightarrow$*(ii)* in Proposition 2.3.7 we know that this may only occur if $A \in \mathfrak{A}_L^{\rm dis}$. Hence, the assumption $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ is actually *necessary* in light of the conclusion in Theorem 2.4.20.

*Proof of Theorem 2.4.20.* The estimates claimed in (2.4.241) and (2.4.243) are direct consequences of Corollary 2.4.11 and Proposition 2.3.7, bearing in mind (2.3.4) and (2.3.5).

Turning to the task of proving (2.4.242), it is apparent from (2.4.238) that each $U_{jk}$ is a sum of operators of commutator type. Then, given any $f \in \big[L_1^p(\partial\Omega,w)\big]^M$, based on

(2.4.240), (2.4.241), and Theorem 2.4.14 we may write

$$
\begin{aligned}
\|K_A f\|_{[L^p_1(\partial\Omega,w)]^M} &= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}}(K_A f)\big\|_{[L^p(\partial\Omega,w)]^M} \\
&= \|K_A f\|_{[L^p(\partial\Omega,w)]^M} + \sum_{j,k=1}^{n} \Big(\big\|K_A(\partial_{\tau_{jk}} f)\big\|_{[L^p(\partial\Omega,w)]^M} + \|U_{jk}(\nabla_{\tan} f)\|_{[L^p(\partial\Omega,w)]^M}\Big) \\
&\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,\|f\|_{[L^p(\partial\Omega,w)]^M} + C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{[L^p(\partial\Omega,w)]^M} \\
&\quad + C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,\|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^M} \\
&\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,\|f\|_{[L^p_1(\partial\Omega,w)]^M}\,,
\end{aligned}
\tag{2.4.244}
$$

which finishes the proof. □

*Remark* 2.4.21. The unweighted case (i.e., the scenario in which $w \equiv 1$) of Theorem 2.4.20 gives norm estimates for the double layer operator and its formal transposed on ordinary Lebesgue and Sobolev spaces. By relying on (2.4.198)-(2.4.199), Lemma 2.4.19, (2.4.204), and (2.2.351) we may also obtain similar estimates on Lorentz spaces and Lorentz-based Sobolev spaces (cf. (2.2.352)-(2.2.353)). Specifically, in the same setting as Theorem 2.4.20, the aforementioned result imply that for each $p \in (1,\infty)$ and $q \in (0,\infty]$ there exists some $C \in (0,\infty)$ which depends only on $n$, $A$, $p$, $q$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that

$$
\|K_A\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.4.245}
$$

$$
\|K_A\|_{[L^{p,q}_1(\partial\Omega,\sigma)]^M \to [L^{p,q}_1(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.4.246}
$$

and

$$
\big\|K_A^{\#}\big\|_{[L^{p,q}(\partial\Omega,\sigma)]^M \to [L^{p,q}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.4.247}
$$

*Remark* 2.4.22. By reasoning much as in the proof of Theorem 2.4.20, we may also obtain operator norm estimates for the double layer $K_A$ with $A \in \mathfrak{A}_L^{\mathrm{dis}}$ on *off-diagonal* weighted Sobolev spaces, i.e., when the integrability exponents and the weights for the Lebesgue spaces to which the actual function and its tangential derivatives belong to are allowed to be different. Specifically, given two integrability exponents $p_1, p_2 \in (1,\infty)$ along with two Muckenhoupt weights $w_1 \in A_{p_1}(\partial\Omega,\sigma)$ and $w_2 \in A_{p_2}(\partial\Omega,\sigma)$, define the off-diagonal weighted Sobolev space

$$
L_1^{p_1;p_2}(\partial\Omega, w_1; w_2) := \big\{f \in L^{p_1}(\partial\Omega,w_1) :\, \partial_{\tau_{jk}} f \in L^{p_2}(\partial\Omega,w_2),\ 1 \leq j,k \leq n\big\}, \tag{2.4.248}
$$

equipped with the natural norm defined for each $f \in L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)$ as

$$
\|f\|_{L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)} := \|f\|_{L^{p_1}(\partial\Omega,w_1)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{L^{p_2}(\partial\Omega,w_2)}\,. \tag{2.4.249}
$$

Then much as in (2.4.244) we now obtain

$$\|K_A\|_{[L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M\to[L_1^{p_1;p_2}(\partial\Omega,w_1;w_2)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.4.250}$$

for some $C \in (0,\infty)$ which depends only on $n$, $A$, $p_1$, $p_2$, $[w_1]_{A_{p_1}}$, $[w_2]_{A_{p_2}}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$.

*Remark* 2.4.23. In the setting of Theorem 2.4.20, estimates (2.4.241)-(2.4.243) continue to hold with a fixed constant $C \in (0,\infty)$ when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary with control. Specifically, from Remark 2.4.13 and the proof of Theorem 2.4.20 we see that for each compact interval $I \subset (0,\infty)$ and each number $W \in (0,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on $n$, $I$, $W$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that (2.4.241)-(2.4.243) hold for each $p \in I$ and each $w \in A_p(\partial\Omega,\sigma)$ with $[w]_{A_p} \leq W$.

Having proved Theorem 2.4.20, we may now establish invertibility results for certain types of boundary layer potentials assuming $\Omega$ is a $\delta$-SKT domain with $\delta$ suitably small relative to the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$. As explained a little later, in Remark 2.4.28, the latter smallness condition is actually in the nature of best possible as far as the invertibility results from Theorem 2.4.24 are concerned.

**Theorem 2.4.24.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and consider the boundary-to-boundary double layer potential operators $K_A, K_A^{\#}$ associated with $\Omega$ and the coefficient tensor $A$ as in (2.3.4) and (2.3.5), respectively. Finally, fix an integrability exponent $p \in (1,\infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$, and some number $\varepsilon \in (0,\infty)$.*

*Then there exists some small threshold $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $A$, $\varepsilon$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ it follows that for each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ the following operators are invertible:*

$$zI + K_A : \big[L^p(\partial\Omega,w)\big]^M \longrightarrow \big[L^p(\partial\Omega,w)\big]^M, \tag{2.4.251}$$

$$zI + K_A : \big[L_1^p(\partial\Omega,w)\big]^M \longrightarrow \big[L_1^p(\partial\Omega,w)\big]^M, \tag{2.4.252}$$

$$zI + K_A^{\#} : \big[L^p(\partial\Omega,w)\big]^M \longrightarrow \big[L^p(\partial\Omega,w)\big]^M, \tag{2.4.253}$$

*Proof.* If we pick $\delta_0 := \min\{1\,,\,\varepsilon/C\}$ where $C$ is the constant appearing in the estimates

(2.4.241)-(2.4.243), then Theorem 2.4.20 ensures that

$$\|K_A\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M} < \varepsilon, \tag{2.4.254}$$

$$\|K_A\|_{[L^p_1(\partial\Omega,w)]^M\to[L^p_1(\partial\Omega,w)]^M} < \varepsilon, \tag{2.4.255}$$

$$\left\|K_A^{\#}\right\|_{[L^p(\partial\Omega,w)]^M\to[L^p(\partial\Omega,w)]^M} < \varepsilon. \tag{2.4.256}$$

In particular, the operators in (2.4.251)-(2.4.253) are invertible for each given $z \in \mathbb{C}$ with $|z| \geq \varepsilon$ using a Neumann series, i.e.,

$$\big(zI + K_A\big)^{-1} = z^{-1}\sum_{m=0}^{\infty}\big(-z^{-1}K_A\big)^m \tag{2.4.257}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega,w)\big]^M$ as well as on $\big[L^p_1(\partial\Omega,w)\big]^M$, and

$$\big(zI + K_A^{\#}\big)^{-1} = z^{-1}\sum_{m=0}^{\infty}\big(-z^{-1}K_A^{\#}\big)^m \tag{2.4.258}$$

with convergence in the space of linear and bounded operators on $\big[L^p(\partial\Omega,w)\big]^M$. □

*Remark* 2.4.25. In view of (2.4.245)-(2.4.247), and (2.4.250), invertibility results which are similar to those proved in Theorem 2.4.24 may be established on Lorentz spaces and Lorentz-based Sobolev spaces, as well as on the brand of off-diagonal Muckenhoupt weighted Sobolev spaces defined as in (2.4.248)-(2.4.249).

*Remark* 2.4.26. It is of interest to contrast Theorem 2.4.24 with the precise invertibility results known in the particular case when $\Omega$ is an infinite sector in the plane, with opening angle $\theta \in (0, 2\pi)$ and when $L = \Delta$ (the two-dimensional Laplacian). In such a setting, it is known (cf. [109, Theorem 5, p. 192]) that

given $p \in (1,\infty)$, the operators $\pm\frac{1}{2}I + K_\Delta$ are invertible on $L^p(\partial\Omega,\sigma)$ if and only if $p \neq 1 + |\pi - \theta|/\pi$ (which amounts to saying that $p \neq \frac{2\pi-\theta}{\pi}$ for $\theta \in (0,\pi)$ and $p \neq \frac{\theta}{\pi}$ for $\theta \in (\pi, 2\pi)$). (2.4.259)

When $\theta = \pi$ (i.e., when $\Omega$ is a half-plane) then, of course, any $p \in (1,\infty)$ will do. In this vein, see also [97, Lemma 4.5, p. 2042]. Consider next the case of the two-dimensional Lamé system in an infinite sector of aperture $\theta \in (0, 2\pi)$, and recall from the discussion at the end of Example 2.3.10 that pseudo-stress double layer potential operator for the Lamé system is denoted by $K_\Psi$. Then there are two critical values of the integrability exponent $p \in (1,\infty)$, which depend on $\theta$ and a specific combination of the Lamé moduli, for which the invertibility of the operators $\pm\frac{1}{2}I + K_\Psi$ on $\big[L^p(\partial\Omega,\sigma)\big]^2$ fails. See [98, Theorem 1.1(A.2) on pp. 153-154, and Theorem 1.3 on pp. 157-158] for more precise information in this regard (including the location of these critical values, which are no longer as explicit as in the case of the Laplacian, and certain monotonicity properties with respect to the angle $\theta$ and the Lamé moduli). We shall revisit the case of the two-dimensional Lamé system in Section 2.4.4.

*Remark* 2.4.27. In the context of Theorem 2.4.24, the operators in (2.4.251)-(2.4.253) continue to be invertible when the integrability exponent and the corresponding Muckenhoupt weight are permitted to vary while retaining control. More specifically, from Remark 2.4.23 and the proof of Theorem 2.4.24 it follows that for each compact interval $I \subset (0, \infty)$ and each number $W \in (0, \infty)$ there exists a threshold $\delta_0 \in (0, 1)$, which depends only on $n$, $I$, $W$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0 \tag{2.4.260}$$

then the operators (2.4.251)-(2.4.253) are invertible for each $p \in I$ and each $w \in A_p(\partial\Omega, \sigma)$ with $[w]_{A_p} \leq W$.

*Remark* 2.4.28. The more general version of Theorem 2.4.24 from Remark 2.4.27 is in the nature of best possible, in the sense that the simultaneous invertibility result described in Remark 2.4.27 forces $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}$ to be small (relative to the other geometric characteristics of $\Omega$). To illustrate this, consider the case when $\Omega = \Omega_\theta$, an infinite sector in the plane with opening angle $\theta \in (0, 2\pi)$ (cf. (2.2.163)), and when $L = \Delta$, the two-dimensional Laplacian. We are interested in the geometric implications of having the operators $\pm\frac{1}{2}I + K_\Delta$ are invertible on $L^p(\partial\Omega_\theta, \sigma_\theta)$ (where $\sigma_\theta := \mathcal{H}^1 \lfloor \partial\Omega_\theta$) for all $p$'s belonging to a compact sub-interval of $(1, \infty)$.

Specifically, suppose the said operators are invertible whenever $p \in I_\eta := [1 + \eta, 2]$ for some fixed $\eta \in (0, 1)$. From (2.4.259) we see that this forces $\theta \neq \pi(2 - p)$ if $\theta \in (0, \pi)$ and $\theta \neq \pi p$ if $\theta \in (\pi, 2\pi)$. As $p$ swipes the interval $[1 + \eta, 2]$, the set of prohibited values for the aperture $\theta$ becomes $\big(0, (1 - \eta)\pi\big] \cup \big[(1 + \eta)\pi, 2\pi\big)$. Hence, we necessarily have $\theta \in \big((1 - \eta)\pi, (1 + \eta)\pi\big)$ which further entails

$$-\sin\big(\eta\tfrac{\pi}{2}\big) = \cos\big((1 + \eta)\tfrac{\pi}{2}\big) < \cos(\theta/2) < \cos\big((1 - \eta)\tfrac{\pi}{2}\big) = \sin\big(\eta\tfrac{\pi}{2}\big). \tag{2.4.261}$$

If $\nu$ denotes the outward unit normal vector to $\Omega_\theta$, then from (2.4.261) and (2.2.164) we conclude that

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega_\theta,\sigma_\theta)]^2} = |\cos(\theta/2)| < \sin\big(\eta\tfrac{\pi}{2}\big) \longrightarrow 0^+ \quad \text{as} \quad \eta \to 0^+. \tag{2.4.262}$$

This goes to show that, in general, the smallness of the BMO semi-norm of the geometric measure theoretic outward unit normal stipulated in (2.4.260) *cannot* be dispensed with, as far as the invertibility of the operator in (2.4.251) (in this case, with $z \in \{\pm\frac{1}{2}\}$, $L = \Delta$, $A$ the identity matrix, $M = 1$, and $w \equiv 1$) for each $p \in I_\eta$ is concerned.

The invertibility results from Theorem 2.4.24 may be further enhanced by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Concretely, by combining Theorem 2.4.20 with the continuity of the operator-valued assignments in (2.3.64)-(2.3.66), we obtain the following result.

**Theorem 2.4.29.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.4.24 and, as before, fix an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight*

*$w \in A_p(\partial\Omega, \sigma)$, and some number $\varepsilon \in (0, \infty)$. Consider $L \in \mathfrak{L}^{\rm dis}$ (cf. (2.3.84)) and pick an arbitrary $A_o \in \mathfrak{A}_L^{\rm dis}$. Then there exist some small threshold $\delta_0 \in (0,1)$ along with some open neighborhood $\mathcal{O}$ of $A_o$ in $\mathfrak{A}_{\rm WE}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $A_o$, $\varepsilon$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ then for each $A \in \mathcal{O}$ and each spectral parameter $z \in \mathbb{C}$ with $|z| \geq \varepsilon$, the operators (2.4.251)-(2.4.253) are invertible.*

### 2.4.4 Another look at double layers for the two-dimensional Lamé system

Throughout this section, we shall work in the two-dimensional case, i.e., when $n = 2$. As a preamble, we introduce a singular integral operator which is going to be relevant shortly. To set the stage, suppose $\Omega \subseteq \mathbb{R}^2$ is a UR domain, abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$, and denote by $\nu = (\nu_1, \nu_2)$ the geometric measure theoretic outward unit normal to $\Omega$. Then for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ define

$$R_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\nu_1(y)(y_2 - x_2) - \nu_2(y)(y_1 - x_1)}{|x-y|^2} f(y)\, d\sigma(y), \tag{2.4.263}$$

at $\sigma$-a.e. point $x \in \partial\Omega$. Let us fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. It has been proved in [93] that the singular integral operator $R_\Delta$ introduced in (2.4.263) is bounded on $L^p(\partial\Omega, w)$ and satisfies

$$(R_\Delta)^2 = \big(\tfrac{1}{2}I + K_\Delta\big)\big(-\tfrac{1}{2}I + K_\Delta\big) \ \text{ on } \ L^p(\partial\Omega, w), \tag{2.4.264}$$

$$K_\Delta R_\Delta + R_\Delta K_\Delta = 0 \ \text{ on } \ L^p(\partial\Omega, w), \tag{2.4.265}$$

where $K_\Delta$ is the harmonic double layer potential operator in this setting (i.e., $K_\Delta$ is as in (2.3.8) with $n := 2$).

Our main result in this section is Theorem 2.4.30 below, which elaborates on the spectra of double layer potential operators, associated with the two-dimensional complex Lamé system, when acting on Muckenhoupt weighted Lebesgue and Sobolev spaces on the boundary of a $\delta$-SKT unbounded domains in the plane.

**Theorem 2.4.30.** *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix two Lamé moduli $\mu, \lambda \in \mathbb{C}$ satisfying*

$$\mu \neq 0, \qquad 2\mu + \lambda \neq 0, \tag{2.4.266}$$

*and bring back the one-parameter family coefficient tensors from (2.3.101) (with $n = 2$), i.e.,*

$$\begin{gathered} A(\zeta) = \big(a_{jk}^{\alpha\beta}(\zeta)\big)_{\substack{1\leq j,k\leq 2 \\ 1\leq\alpha,\beta\leq 2}} \ \textit{ defined for each } \zeta \in \mathbb{C} \textit{ according to} \\ a_{jk}^{\alpha\beta}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu + \lambda - \zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \quad 1 \leq j,k,\alpha,\beta \leq 2, \end{gathered} \tag{2.4.267}$$

*which allows to represent the $2 \times 2$ Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ in $\mathbb{R}^2$ as*

$$L_{\mu,\lambda} = \Big(a_{jk}^{\alpha\beta}(\zeta)\partial_j\partial_k\Big)_{1\leq\alpha,\beta\leq 2} \quad \textit{for each} \quad \zeta \in \mathbb{C}. \tag{2.4.268}$$

*Fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, suppose $z, \zeta \in \mathbb{C}$ are such that*

$$z \neq \pm\frac{\mu(\mu+\lambda) - \zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}, \tag{2.4.269}$$

*and associate the double layer potential operator $K_{A(\zeta)}$ with the coefficient tensor $A(\zeta)$ and the domain $\Omega$ as in (2.3.4).*

*Then there exists some small threshold $\delta \in (0,1)$ which depends only on $\mu$, $\lambda$, $p$, $[w]_{A_p}$, $z$, $\zeta$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta$ it follows that*

$$\begin{gathered} \textit{the operator} \quad zI_{2\times 2} + K_{A(\zeta)} \quad \textit{is invertible} \\ \textit{both on} \quad \big[L^p(\partial\Omega,w)\big]^2 \quad \textit{and on} \quad \big[L^p_1(\partial\Omega,w)\big]^2. \end{gathered} \tag{2.4.270}$$

Before presenting the proof of this theorem, a few clarifications are in order. From (2.4.251)-(2.4.252) in Theorem 2.4.24 and (2.3.103)-(2.3.104) we already know that, under suitable geometric assumptions, the conclusion in (2.4.270) holds (and this is true in all dimensions $n \geq 2$) when

$$3\mu + \lambda \neq 0 \quad \text{and} \quad \zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{2.4.271}$$

The point of Theorem 2.4.30 is that, for the two-dimensional Lamé system, the invertibility results from (2.4.251)-(2.4.252) holds with $A = A(\zeta)$ as in (2.3.101) for *a much larger range of* $\zeta$'s than the singleton in (2.4.271). (Parenthetically we wish to note that what is special about the scenario described in (2.4.271) is that this makes $\pm\frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{4\mu(2\mu+\lambda)}$ zero, so (2.4.269) simply reads $z \in \mathbb{C}\setminus\{0\}$ in this case, as was assumed in Theorem 2.4.24.) It should be also remarked that, in the setting on Theorem 2.4.30, the double layer $K_{A(\zeta)}$ does *not* necessarily have small operator norm, and this is in stark contrast with the case of the double layer operators considered in Theorem 2.4.24.

We are now ready to present the proof of Theorem 2.4.30.

*Proof of Theorem 2.4.30.* Recall the numbers $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ associated with $\zeta, \mu, \lambda$ as in (2.3.109). From (2.3.8), (2.3.111), (2.3.112), and (2.4.263) we see that for each $\zeta \in \mathbb{C}$ we have

$$K_{A(\zeta)} = C_1(\zeta)K_\Delta I_{2\times 2} - (1 - C_1(\zeta))Q + C_2(\zeta)\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix} \tag{2.4.272}$$

as operators on $\big[L^p(\partial\Omega,w)\big]^2$. Note that (2.4.264) implies

$$\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0 \end{pmatrix}^2 = (\tfrac{1}{4}I - (K_\Delta)^2)I_{2\times 2} \quad \text{on} \quad \big[L^p(\partial\Omega,w)\big]^2. \tag{2.4.273}$$

Staring with (2.4.272) and then using (2.4.273), (2.4.265) we may write, with all operators acting on the space $\big[L^p(\partial\Omega,w)\big]^2$,

$$
\begin{aligned}
(zI_{2\times 2}+K_{A(\zeta)})\big(-zI_{2\times 2}+K_{A(\zeta)}\big) &= (K_{A(\zeta)})^2 - z^2 I_{2\times 2} \\
&= \big[\tfrac14 C_2(\zeta)^2 - z^2\big] I_{2\times 2} + T_\zeta,
\end{aligned}
\tag{2.4.274}
$$

for all $z,\zeta\in\mathbb{C}$, where $T_\zeta$ is the operator

$$
\begin{aligned}
T_\zeta &= (C_1(\zeta)^2 - C_2(\zeta)^2)K_\Delta^2 I_{2\times 2} + (1-C_1(\zeta))^2 Q^2 \\
&\quad - C_1(\zeta)(1-C_1(\zeta))(K_\Delta I_{2\times 2})Q - C_1(\zeta)(1-C_1(\zeta))Q(K_\Delta I_{2\times 2}) \\
&\quad - C_2(\zeta)(1-C_1(\zeta))Q\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0\end{pmatrix} - C_2(\zeta)(1-C_1(\zeta))\begin{pmatrix} 0 & R_\Delta \\ -R_\Delta & 0\end{pmatrix} Q.
\end{aligned}
\tag{2.4.275}
$$

Fix now $\zeta\in\mathbb{C}$ along with $\varepsilon>0$ arbitrary. Note that $T_\zeta$ in (2.4.275) is a finite linear combination of compositions of pairs of singular integral operators such that, in each case, at least one of them falls under the scope of Corollary 2.4.11. As a consequence of this and Proposition 2.3.3, it follows that there exists $\delta\in(0,1)$ small enough (relative to $\mu$, $\lambda$, $\zeta$, $\varepsilon$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$), matters may be arranged so that, under the additional assumption that

$$
\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2} < \delta,
\tag{2.4.276}
$$

we have

$$
\|T_\zeta\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \le \varepsilon^2/2.
\tag{2.4.277}
$$

Consider now

$$
z\in\mathbb{C}\setminus\Big\{B\big(\tfrac12 C_2(\zeta)\,,\,\varepsilon\big)\cup B\big(-\tfrac12 C_2(\zeta)\,,\,\varepsilon\big)\Big\},
\tag{2.4.278}
$$

which entails

$$
\big|\tfrac14 C_2(\zeta)^2 - z^2\big| = \big|\tfrac12 C_2(\zeta) - z\big|\big|\tfrac12 C_2(\zeta)+z\big| \ge \varepsilon^2.
\tag{2.4.279}
$$

Then from (2.4.279), (2.4.277) it follows that

$$
\begin{gathered}
\big[\tfrac14 C_2(\zeta)^2 - z^2\big] I_{2\times 2} + T_\zeta \ \text{ is invertible on } \ \big[L^p(\partial\Omega,w)\big]^2 \\
\text{for each } z \text{ as in (2.4.278)},
\end{gathered}
\tag{2.4.280}
$$

and

$$
\begin{gathered}
\Big\|\big(\big[\tfrac14 C_2(\zeta)^2 - z^2\big] I_{2\times 2} + T_\zeta\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \le (\varepsilon^2/2)^{-1} \\
\text{for each } z \text{ as in (2.4.278)}.
\end{gathered}
\tag{2.4.281}
$$

Since the operators $zI_{2\times 2}+K_{A(\zeta)}$ and $-zI_{2\times 2}+K_{A(\zeta)}$ commute with one another, from (2.4.274) and (2.4.280) we ultimately conclude that

$$zI_{2\times 2}+K_{A(\zeta)} \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \text{ for each } z \text{ as in (2.4.278).} \tag{2.4.282}$$

In relation to (2.4.282) we also claim that there exists some small number

$$c := c(\Omega,\varepsilon,\zeta,p,w) \in (0,1] \tag{2.4.283}$$

where the dependence of $c$ on $\Omega$ manifests itself only through the local John constants of $\Omega$ and the Ahlfors regularity constant of $\partial\Omega$, with the property that

$$\begin{gathered} c\|f\|_{[L^p(\partial\Omega,w)]^2} \leq \big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)f\big\|_{[L^p(\partial\Omega,w)]^2} \\ \text{for each } z \text{ as in (2.4.278) and each } f\in\big[L^p(\partial\Omega,w)\big]^2. \end{gathered} \tag{2.4.284}$$

To prove this, first observe that

$$\begin{gathered} \text{whenever } |z|>1+\big\|K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \text{ then} \\ zI_{2\times 2}+K_{A(\zeta)} \text{ is invertible on } \big[L^p(\partial\Omega,w)\big]^2 \text{ and} \\ \Big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}<1. \end{gathered} \tag{2.4.285}$$

Hence, as long as $|z|>1+\big\|K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}$, the estimate in (2.4.284) is true for any choice of $c\in(0,1]$. As such, there remains to study the case in which

$$\begin{gathered} z \text{ is as in (2.4.278) and also satisfies} \\ |z|\leq 1+\big\|K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}. \end{gathered} \tag{2.4.286}$$

Henceforth assume $z$ is as in (2.4.286). From (2.4.274) and (2.4.281) we know that

$$\Big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\big(-zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}\leq(\varepsilon^2/2)^{-1}. \tag{2.4.287}$$

Write $\big(zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}$ as

$$\Big[\big(zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\big(-zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\Big]\big(-zI_{2\times 2}+K_{A(\zeta)}\big), \tag{2.4.288}$$

then use this formula and (2.4.287) to estimate

$$\begin{aligned} &\Big\|\big(zI_{2\times 2}+K_{A(\zeta)}\big)^{-1}\Big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \\ &\qquad\leq(\varepsilon^2/2)^{-1}\big\|-zI_{2\times 2}+K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2} \\ &\qquad\leq(\varepsilon^2/2)^{-1}\Big(|z|+\big\|K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}\Big) \\ &\qquad\leq C(\Omega,\varepsilon,\zeta,p,w), \end{aligned} \tag{2.4.289}$$

where the last inequality comes from (2.4.286), and

$$C(\Omega,\varepsilon,\zeta,p,w):=2\varepsilon^{-2}+4\varepsilon^{-2}\big\|K_{A(\zeta)}\big\|_{[L^p(\partial\Omega,w)]^2\to[L^p(\partial\Omega,w)]^2}. \tag{2.4.290}$$

Hence, if we define

$$c := c(\Omega, \varepsilon, \zeta, p, w) := \min\Big\{1\,,\ \big[C(\Omega, \varepsilon, \zeta, p, w)\big]^{-1}\Big\} \in (0, 1], \tag{2.4.291}$$

we may rely on (2.4.289) to write

$$c\|f\|_{[L^p(\partial\Omega,w)]^2} \le \big\|(zI_{2\times 2} + K_{A(\zeta)})f\big\|_{[L^p(\partial\Omega,w)]^2}, \quad \forall\, f \in \big[L^p(\partial\Omega, w)\big]^2, \tag{2.4.292}$$

finishing the proof of (2.4.284).

We next claim that, if the threshold $\delta \in (0, 1)$ appearing in (2.4.276) is taken sufficiently small to begin with, we also have

$$\begin{gathered} zI_{2\times 2} + K_{A(\zeta)} \ \text{ invertible on } \ \big[L^p_1(\partial\Omega, w)\big]^2 \\ \text{for each } z \text{ as in (2.4.278).} \end{gathered} \tag{2.4.293}$$

For starters, observe that for each $z \in \mathbb{C}$, and each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$, Lemma 2.4.19 gives

$$\partial_{\tau_{12}}\big[(zI_{2\times 2} + K_{A(\zeta)})f\big] = (zI_{2\times 2} + K_{A(\zeta)})(\partial_{\tau_{12}}f) + U^{\zeta}_{12}(\nabla_{\tan}f) \tag{2.4.294}$$

where the commutator $U^{\zeta}_{12}$ is defined as in (2.4.238) with $n = 2$, $j = 1$, $k = 2$, and the coefficient tensor $A(\zeta)$ as in (2.4.267). If $z$ as in (2.4.278) then, on account of (2.4.294), (2.4.284), and Theorem 2.4.14 (also keeping in mind Proposition 2.3.3) for each $f \in \big[L^p_1(\partial\Omega, w)\big]^2$ we may estimate

$$\begin{aligned} c\|\partial_{\tau_{12}}f\|_{[L^p(\partial\Omega,w)]^2} &\le \big\|(zI_{2\times 2} + K_{A(\zeta)})(\partial_{\tau_{12}}f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\le \big\|\partial_{\tau_{12}}\big[(zI_{2\times 2} + K_{A(\zeta)})f\big]\big\|_{[L^p(\partial\Omega,w)]^2} + \big\|U^{\zeta}_{12}(\nabla_{\tan}f)\big\|_{[L^p(\partial\Omega,w)]^2} \\ &\le \big\|(zI_{2\times 2} + K_{A(\zeta)})f\big\|_{[L^p_1(\partial\Omega,w)]^2} + C\delta\|\partial_{\tau_{12}}f\|_{[L^p(\partial\Omega,w)]^2}, \end{aligned} \tag{2.4.295}$$

(since we presently have $\partial_{\tau_{11}} = \partial_{\tau_{22}} = 0$ and $\partial_{\tau_{12}} = -\partial_{\tau_{21}}$), where $C \in (0, \infty)$ depends only on $\mu$, $\lambda$, $\zeta$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. Assuming $\delta < c/(2C)$ to begin with, the very last term above may be absorbed in the left-most side of (2.4.295). By combing the resulting inequality with (2.4.284) we therefore arrive at the conclusion that if $\delta$ in (2.4.276) is small enough then we may find some small $\eta > 0$ with the property that

$$\begin{gathered} \eta\|f\|_{[L^p_1(\partial\Omega,w)]^2} \le \big\|(zI_{2\times 2} + K_{A(\zeta)})f\big\|_{[L^p_1(\partial\Omega,w)]^2} \\ \text{for each } z \text{ as in (2.4.278) and each } \ f \in \big[L^p_1(\partial\Omega, w)\big]^2. \end{gathered} \tag{2.4.296}$$

In this scenario, (2.4.296) implies that the operator $zI_{2\times 2}+K_{A(\zeta)}$ acting on $\big[L^p_1(\partial\Omega, w)\big]^2$ is injective and has closed range for each $z$ as in (2.4.278). Consequently, the operator $zI_{2\times 2} + K_{A(\zeta)}$ acting on $\big[L^p_1(\partial\Omega, w)\big]^2$ is semi-Fredholm for each $z$ as in (2.4.278). Since this depends continuously on $z$, the homotopic invariance of the index on connected sets then ensures that the index of $zI_{2\times 2} + K_{A(\zeta)}$ on $\big[L^p_1(\partial\Omega, w)\big]^2$ is independent of $z$ in the said range. Given that, via a Neumann series argument,

$$\begin{gathered} zI_{2\times 2} + K_{A(\zeta)} \ \text{ is invertible on } \ \big[L^p_1(\partial\Omega, w)\big]^2 \\ \text{if } \ |z| > \big\|K_{A(\zeta)}\big\|_{[L^p_1(\partial\Omega,w)]^2\to[L^p_1(\partial\Omega,w)]^2}, \end{gathered} \tag{2.4.297}$$

we may therefore conclude that the index of $zI_{2\times2}+K_{A(\zeta)}$ on $\big[L^p_1(\partial\Omega,w)\big]^2$ is zero for each $z$ as in (2.4.278). In view of the fact that, as already noted from (2.4.296), the operator $zI_{2\times2}+K_{A(\zeta)}$ is injective on $\big[L^p_1(\partial\Omega,w)\big]^2$ for each $z$ as in (2.4.278), this ultimately proves that $zI_{2\times2}+K_{A(\zeta)}$ is invertible on $\big[L^p_1(\partial\Omega,w)\big]^2$ for each $z$ as in (2.4.278). Hence, the claim made in (2.4.293) is true. At this stage, the claim made in (2.4.270) readily follows from (2.4.282) and (2.4.293). □

It is of interest to single out the case $z=\pm\frac{1}{2}$ in (2.4.270), and in Corollary 2.4.31 stated next we do just that.

**Corollary 2.4.31.** *Let $\Omega\subseteq\mathbb{R}^2$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma:=\mathcal{H}^1\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix two Lamé moduli $\mu,\lambda\in\mathbb{C}$ satisfying*

$$\mu\neq0,\qquad 2\mu+\lambda\neq0,\qquad 3\mu+\lambda\neq0,\tag{2.4.298}$$

*and recall the one-parameter family coefficient tensors $A(\zeta)$ defined for each $\zeta\in\mathbb{C}$ as in (2.4.267). Fix an integrability exponent $p\in(1,\infty)$ along with a Muckenhoupt weight $w\in A_p(\partial\Omega,\sigma)$. Finally, pick some*

$$\zeta\in\mathbb{C}\setminus\Big\{-\mu\,,\,\tfrac{\mu(5\mu+3\lambda)}{3\mu+\lambda}\Big\}\tag{2.4.299}$$

*and associate double layer potential operator $K_{A(\zeta)}$ with the coefficient tensor $A(\zeta)$ and the domain $\Omega$ as in (2.3.4).*

*Then there exists some small threshold $\delta\in(0,1)$ which depends only on $\mu$, $\lambda$, $p$, $[w]_{A_p}$, $\zeta$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^2}<\delta$ it follows that*

$$\begin{array}{c}\textit{the operators}\quad \pm\frac{1}{2}I_{2\times2}+K_{A(\zeta)}\quad\textit{are invertible}\\ \textit{both on}\quad \big[L^p(\partial\Omega,w)\big]^2\quad\textit{and on}\quad \big[L^p_1(\partial\Omega,w)\big]^2,\end{array}\tag{2.4.300}$$

*and*

$$\textit{the operators}\quad \pm\tfrac{1}{2}I_{2\times2}+K^{\#}_{A(\zeta)}\quad\textit{are invertible on}\quad \big[L^p(\partial\Omega,w)\big]^2.\tag{2.4.301}$$

As seen from (2.4.299) (also keeping in mind (2.4.298)), under the additional assumption that $\mu+\lambda\neq0$ the value $\zeta:=\mu$ becomes acceptable in the formulation of the conclusions in (2.4.300)-(2.4.301). This special choice leads to the conclusion that, if $\Omega$ is sufficiently flat (relative to $\mu$, $\lambda$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$) then the operators

$$\pm\tfrac{1}{2}I_{2\times2}+K_{A(\mu)}:\big[L^p(\partial\Omega,w)\big]^2\longrightarrow\big[L^p(\partial\Omega,w)\big]^2,\tag{2.4.302}$$

$$\pm\tfrac{1}{2}I_{2\times2}+K_{A(\mu)}:\big[L^p_1(\partial\Omega,w)\big]^2\longrightarrow\big[L^p_1(\partial\Omega,w)\big]^2,\tag{2.4.303}$$

$$\pm\tfrac{1}{2}I_{2\times2}+K^{\#}_{A(\mu)}:\big[L^p(\partial\Omega,w)\big]^2\longrightarrow\big[L^p(\partial\Omega,w)\big]^2,\tag{2.4.304}$$

are all invertible whenever

$$\mu \neq 0, \qquad \mu + \lambda \neq 0, \qquad 2\mu + \lambda \neq 0, \qquad 3\mu + \lambda \neq 0. \tag{2.4.305}$$

This is relevant in the context of Remark 2.6.12.

*Proof of Corollary 2.4.31.* The claim in (2.4.300) follows at once from Theorem 2.4.30, upon observing that when $z = \pm 1/2$ the demand in (2.4.269) becomes equivalent to the condition stipulated in (2.4.299). The claim in (2.4.301) then follows from (2.4.300) and duality. $\square$

## 2.5 Controlling the BMO semi-norm of the unit normal

In the previous chapter we have succeeded in estimating the size of a certain brand of singular integrals operators (which includes the harmonic double layer operator; cf. Theorem 2.4.20) in terms of the geometry of the underlying "surface." A key characteristic of these estimates (originating with Theorem 2.4.4) is the presence of the BMO semi-norm of the unit normal to the surface as a factor in the right side. In particular, the flatter the said surface, the smaller the norm of the singular integral operators in question. Similar results are also valid for a specific type of commutators, of the sort described in Theorem 2.4.14.

By way of contrast, the principal goal in this chapter is to proceed in the opposite direction, and control geometry in terms of analysis. More specifically, we seek to quantify flatness of a given "surface" (by estimating the BMO semi-norm of its unit normal) in terms of analytic entities, such as the operator norms of the harmonic double layer and the commutators of Riesz transforms with the operator of pointwise multiplication by the (scalar components of the) unit normals, or various natural algebraic combinations of the said Riesz transforms (where all singular integral operators just mentioned are intrinsically defined on the given "surface").

In this endeavor, the catalyst is the language of Clifford algebras which allows us to glue together singular integral operators of the sort described above into a single, Cauchy-like, singular integral which exhibits excellent non-degeneracy properties (i.e., up to normalization, such a Cauchy-Clifford operator is its own inverse; cf. (2.5.9)). We therefore begin with a brief review about Cauchy-Clifford operators. This chapter ends with Section 2.5.4 which contains results characterizing Muckenhoupt weights in terms of the boundedness Riesz transforms. The Clifford algebra formalism turns out to be useful in this regard, both as tool and as a mean to bring into play other types of operators, like the Cauchy-Clifford singular integral operator alluded to above.

### 2.5.1 Cauchy-Clifford operators

Recall the Clifford algebras machinery introduced in Section 1.4 and consider an arbitrary UR domain $\Omega \subseteq \mathbb{R}^n$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ its geometric measure theoretic outward unit normal. For the goals we have in mind, it is natural to identify $\nu$ with the Clifford algebra-valued function $\nu = \nu_1\mathbf{e}_1 + \cdots + \nu_n\mathbf{e}_n$. Bearing this identification in mind, we then recall the action of the boundary-to-boundary Cauchy-Clifford operator of any given $\mathcal{C}\!\ell_n$-valued function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}) \otimes \mathcal{C}\!\ell_n$ as

$$\mathbf{C}f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y), \tag{2.5.1}$$

for $\sigma$-a.e. point $x \in \partial\Omega$ (cf. (5.4.21)). In particular, with Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ defined as in (2.4.236), for each function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}) \otimes \mathcal{C}\!\ell_n$ we have

$$\mathbf{C}f = \frac{1}{2} \sum_{1\leq j,k\leq n} \mathbf{e}_j \odot \mathbf{e}_k \odot R_j(\nu_k f) \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.5.2}$$

Another closely related integral operator which is of interest to us acts on each given function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}) \otimes \mathcal{C}\!\ell_n$ according to

$$\mathbf{C}^{\#}f(x) := -\lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu(x) \odot \frac{x-y}{|x-y|^n} \odot f(y)\, d\sigma(y) \tag{2.5.3}$$

for $\sigma$-a.e. $x \in \partial\Omega$. Analogously to (2.5.2), for each function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}) \otimes \mathcal{C}\!\ell_n$ we have

$$\mathbf{C}^{\#}f = -\frac{1}{2} \sum_{1\leq j,k\leq n} \mathbf{e}_k \odot \mathbf{e}_j \odot \nu_k R_j f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.5.4}$$

As is apparent from (2.5.2), (2.5.4), both $\mathbf{C}$ and $\mathbf{C}^{\#}$ are amenable to Proposition 2.3.3. Hence, whenever $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$,

$$\mathbf{C} : L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n \longrightarrow L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n \tag{2.5.5}$$

and

$$\mathbf{C}^{\#} : L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n \longrightarrow L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n \tag{2.5.6}$$

are well defined, linear and bounded operators, with

$$\begin{array}{l} \|\mathbf{C}\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n},\ \|\mathbf{C}^{\#}\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n} \\ \text{controlled in terms of } n,\ p,\ [w]_{A_p}, \text{ and the UR character of } \partial\Omega. \end{array} \tag{2.5.7}$$

In fact (see [53, Sections 4.6-4.7] and [93]),

> the transposed of $\mathbf{C}$ from (2.5.5) is the operator $\mathbf{C}^{\#}$ acting in the context of (2.5.6) with the exponent $p$ replaced by its Hölder conjugate $p' \in (1,\infty)$ and the weight $w$ replaced by $w^{1-p'} \in A_{p'}(\partial\Omega,\sigma)$. (2.5.8)

For this reason, it is natural to refer to $\mathbf{C}^{\#}$ as the "transposed" Cauchy-Clifford operator. Moreover, with $I$ denoting the identity operator, we have

$$\mathbf{C}^2 = \tfrac{1}{4}I \quad \text{and} \quad \big(\mathbf{C}^{\#}\big)^2 = \tfrac{1}{4}I, \tag{2.5.9}$$

on $L^p(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$ with $p \in (1, \infty)$ (cf. [53, Sections 4.6-4.7]). In view of (2.5.5)-(2.5.7), a standard density argument then shows that these formulas remain valid on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$ whenever $p \in (1, \infty)$ and $w \in A_p(\partial\Omega, \sigma)$.

Here we are interested in the difference $\mathbf{C} - \mathbf{C}^{\#}$ which, up to multiplication by $2^{-1}$, may be thought of as the antisymmetric part of the Cauchy-Clifford operator $\mathbf{C}$. The following lemma elaborates on the relationship between the antisymmetric part of the Cauchy-Clifford operator, i.e., $\mathbf{C} - \mathbf{C}^{\#}$, and the harmonic boundary double layer potential (cf. (2.3.8)) together with commutators between Riesz transforms (cf. (2.4.236)) and operators of pointwise multiplication by scalar components of the unit vector. For a proof see [53, Lemma 4.45].

**Lemma 2.5.1.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. For each index $j \in \{1, \dots, n\}$, denote by $M_{\nu_j}$ the operator of pointwise multiplication by $\nu_j$. Also, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from* (2.3.8) *and the family of Riesz transforms $\{R_j\}_{1\le j\le n}$ from* (2.4.236)*. Then*

$$\begin{aligned}(\mathbf{C} - \mathbf{C}^{\#})f = &-2\sum_{\ell=0}^{n} \sum_{|I|=\ell}{}^{'} (K_\Delta f_I)\mathbf{e}_I \\ &- \sum_{\ell=0}^{n} \sum_{|I|=\ell}{}^{'} \sum_{j,k=1}^{n} \Big([M_{\nu_j}, R_k] f_I\Big)\, \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_I\end{aligned} \tag{2.5.10}$$

*for each $\mathcal{C}\!\ell_n$-valued function $f = \sum_{\ell=0}^{n} \sum'_{|I|=\ell} f_I \odot \mathbf{e}_I$ belonging to the weighted Lebesgue space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes \mathcal{C}\!\ell_n$.*

In turn, the structural result from Lemma 2.5.1 is a basic ingredient in the proof of the following corollary.

**Corollary 2.5.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then there exists some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\big\|\mathbf{C} - \mathbf{C}^{\#}\big\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes\mathcal{C}\!\ell_n} \le C\, \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.5.11}$$

*Proof.* This is a consequence of Lemma 2.5.1, Lemma 2.2.47, (2.3.8), Corollary 2.4.11, (2.4.236), Proposition 2.3.3, and Theorem 2.4.14. ∎

### 2.5.2 Estimating the BMO semi-norm of the unit normal

The next goal is to establish a bound from below for the operator norm of $\mathbf{C} - \mathbf{C}^{\#}$ on Muckenhoupt weighted Lebesgue spaces on the boundary of a UR domain in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to the said domain.

**Proposition 2.5.3.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Then there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\|\mathbf{C} - \mathbf{C}^{\#}\right\|^{1/(2n-1)}_{L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n}. \tag{2.5.12}$$

A couple of comments are in order. First, as a consequence of (2.5.12) and work in [52] we see that

> given a UR domain $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is unbounded, and given $p \in (1,\infty)$ together with $w \in A_p(\partial\Omega,\sigma)$, we have $\mathbf{C} = \mathbf{C}^{\#}$ as operators on $L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n$ if and only if $\Omega$ is a half-space. (2.5.13)

Second, estimate (2.5.12) may fail without the assumption that $\partial\Omega$ is unbounded. Indeed, from (2.5.1)-(2.5.3) one may easily check that $\mathbf{C} = \mathbf{C}^{\#}$ if $\Omega$ is an open ball, or the complement of a closed ball, in $\mathbb{R}^n$ and yet $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} > 0$ in either case. In fact, open balls, complements of closed balls, and half-spaces in $\mathbb{R}^n$ are the only UR domains for which $\mathbf{C} = \mathbf{C}^{\#}$ (see [48] for more on this).

*Proof of Proposition 2.5.3.* If $\left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n} \geq 4^{1-2n}$ then (2.2.56) implies

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1 \leq 4 \left\|\mathbf{C} - \mathbf{C}^{\#}\right\|^{1/(2n-1)}_{L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n}, \tag{2.5.14}$$

so the estimate claimed in (2.5.12) holds in this case if $C \geq 4$. Hence, there remains to consider the scenario in which $\left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n} < 4^{1-2n}$. Assume this is the case and pick a real number $\eta > 0$ such that

$$\left\|\mathbf{C} - \mathbf{C}^{\#}\right\|_{L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,w)\otimes \mathcal{C}\!\ell_n} < \eta < 4^{1-2n}. \tag{2.5.15}$$

Fix a location $x_0 \in \partial\Omega$ along with a scale $R > 0$, and define $r := R \cdot \eta^{-1/(2n-1)} > 0$. If $C \in [1,\infty)$ is the Ahlfors regularity constant of $\partial\Omega$ (cf. (1.1.15)) then choosing some $\lambda \geq 1$ sufficiently large relative to $C$ it follows that (here we make use of the fact that no smallness condition on $r$ is necessary since $\partial\Omega$ is unbounded)

$$\begin{aligned}\sigma\big(\Delta(x_0,\lambda r) \setminus \Delta(x_0,r)\big) &= \sigma\big(\Delta(x_0,\lambda r)\big) - \sigma\big(\Delta(x_0,r)\big) \\ &\geq \Big(\frac{1}{C}\lambda^{n-1} - C\Big) r^{n-1} > 0.\end{aligned} \tag{2.5.16}$$

In turn, this guarantees that $\Delta(x_0, \lambda r) \setminus \Delta(x_0, r) \neq \varnothing$, hence we may choose some point $y_0 \in \Delta(x_0, \lambda r) \setminus \Delta(x_0, r)$. Define

$$\delta := |x_0 - y_0| \in [r, \lambda r) \tag{2.5.17}$$

then set

$$r_* := \delta\, \eta^{1/(2n-1)}. \tag{2.5.18}$$

In particular, the last estimate in (2.5.15) implies

$$r_* < \delta/4. \tag{2.5.19}$$

Also,

$$R = r\, \eta^{1/(2n-1)} \leq \delta\, \eta^{1/(2n-1)} = r_* < \lambda r \eta^{1/(2n-1)} = \lambda R. \tag{2.5.20}$$

Next, define

$$A := \delta^{n-2} \left( ⨍_{\Delta(y_0, r_*)} \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y)\, d\sigma(y) \right) \odot (x_0 - y_0) \in \mathcal{C}\!\ell_n. \tag{2.5.21}$$

In relation to this we claim that

$$\Big( ⨍_{\Delta(x_0, r_*)} |\nu(y) - A|^p\, dw(y) \Big)^{1/p} \leq C\, \eta^{1/(2n-1)} \tag{2.5.22}$$

for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Assuming this momentarily, we use (1.4.5), (2.5.20), and the fact that $w$ is a doubling measure to conclude that

$$\begin{aligned} \Big( ⨍_{\Delta(x_0, R)} |\nu(y) - A_{\rm proj}|^p\, dw(y) \Big)^{1/p} &\leq \Big( ⨍_{\Delta(x_0, R)} |\nu(y) - A|^p\, dw(y) \Big)^{1/p} \\ &\leq C \Big( ⨍_{\Delta(x_0, r_*)} |\nu(y) - A|^p\, dw(y) \Big)^{1/p} \\ &\leq C\, \eta^{1/(2n-1)}, \end{aligned} \tag{2.5.23}$$

for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. As $x_0 \in \partial\Omega$ and $R > 0$ are arbitrary, the estimate claimed in (2.5.12) is obtained from (2.5.23) by recalling (2.2.49), Lemma 2.2.46, and passing to limit

$$\eta \searrow \Big\| \mathbf{C} - \mathbf{C}^{\#} \Big\|_{L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n}. \tag{2.5.24}$$

It remains to prove (2.5.22). To this end, observe that for each point $x \in \Delta(x_0, r_*)$

we have

$$
\begin{aligned}
&\int_{\Delta(y_0,r_*)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y_0}{|x_0-y_0|^n}\Big\}\,d\sigma(y)\\
&\qquad=\int_{\Delta(y_0,r_*)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)-\frac{x-y}{|x-y|^n}\odot\nu(y)\Big\}\,d\sigma(y)\\
&\qquad\quad+\int_{\Delta(y_0,r_*)}\Big\{\frac{x-y}{|x-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x-y}{|x-y|^n}\Big\}\,d\sigma(y)\\
&\qquad\quad-\int_{\Delta(y_0,r_*)}\Big\{\nu(x)\odot\frac{x-y}{|x-y|^n}-\nu(x)\odot\frac{x_0-y}{|x_0-y|^n}\Big\}\,d\sigma(y)\\
&\qquad\quad-\int_{\Delta(y_0,r_*)}\Big\{\nu(x)\odot\frac{x_0-y}{|x_0-y|^n}-\nu(x)\odot\frac{x_0-y_0}{|x_0-y_0|^n}\Big\}\,d\sigma(y)\\
&\qquad=:\mathrm{I}+\mathrm{II}-\mathrm{III}-\mathrm{IV}.
\end{aligned}
\tag{2.5.25}
$$

Based on definitions, the second term above may be re-cast as

$$
\mathrm{II}=\omega_{n-1}\,(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,r_*)}(x).
\tag{2.5.26}
$$

For the remaining terms in (2.5.25) we use (2.5.19) and the Mean Value Theorem to estimate

$$
|\mathrm{I}|+|\mathrm{III}|+|\mathrm{IV}|\leq C\frac{r_*}{\delta^n}\,r_*^{n-1}=C\Big(\frac{r_*}{\delta}\Big)^n,
\tag{2.5.27}
$$

for some $C\in(0,\infty)$ which depends only on the Ahlfors regularity constant of $\partial\Omega$. On account of (2.5.25)-(2.5.27), (2.2.313), (2.5.19), and (2.5.18) we conclude that

$$
\begin{aligned}
&⨍_{\Delta(x_0,r_*)}\Big|\int_{\Delta(y_0,r_*)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y_0}{|x_0-y_0|^n}\Big\}\,d\sigma(y)\Big|^p\,dw(x)\\
&\qquad\leq C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}⨍_{\Delta(x_0,r_*)}|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,r_*)}(x)|^p\,dw(x)\\
&\qquad< C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\cdot w\big(\Delta(x_0,r_*)\big)^{-1}\cdot\eta^p\cdot\big\|\mathbf{1}_{\Delta(y_0,r_*)}\big\|^p_{L^p(\partial\Omega,w)}\\
&\qquad= C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\cdot\eta^p\cdot\frac{w\big(\Delta(y_0,r_*)\big)}{w\big(\Delta(x_0,r_*)\big)}\\
&\qquad\leq C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\cdot\eta^p\cdot\frac{w\big(\Delta(x_0,r_*+\delta)\big)}{w\big(\Delta(x_0,r_*)\big)}\\
&\qquad\leq C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p,w}\cdot\eta^p\cdot\bigg(\frac{\sigma\big(\Delta(x_0,r_*+\delta)\big)}{\sigma\big(\Delta(x_0,r_*)\big)}\bigg)^p\\
&\qquad\leq C\Big(\frac{r_*}{\delta}\Big)^{np}+C\,\eta^p\Big(\frac{\delta}{r_*}\Big)^{(n-1)p}=C\,\eta^{(np)/(2n-1)},
\end{aligned}
\tag{2.5.28}
$$

for some $C\in(0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. Based on (1.4.1), (2.5.17), (1.4.10), (2.5.21), (2.5.28), and (2.5.18) we may then write

$$\begin{aligned}
&\fint_{\Delta(x_0,r_*)} |\nu(x) - A|^p \, dw(x) \\
&= \delta^{(n-2)p} \fint_{\Delta(x_0,r_*)} \left| (\nu(x) - A) \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} \odot (x_0 - y_0) \right|^p dw(x) \\
&= \delta^{(n-2)p} \fint_{\Delta(x_0,r_*)} \left| (\nu(x) - A) \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} \right|^p |x_0 - y_0|^p \, dw(x) \\
&= \delta^{(n-1)p} \fint_{\Delta(x_0,r_*)} \left| \nu(x) \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} - A \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} \right|^p dw(x) \\
&= \delta^{(n-1)p} \fint_{\Delta(x_0,r_*)} \left| \fint_{\Delta(y_0,r_*)} \Big\{ \nu(x) \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} \right. \\
&\qquad\qquad\qquad \left. + \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \Big\} \, d\sigma(y) \right|^p dw(x) \\
&\leq C \Big( \frac{\delta}{r_*} \Big)^{(n-1)p} \fint_{\Delta(x_0,r_*)} \left| \int_{\Delta(y_0,r_*)} \Big\{ \nu(x) \odot \frac{x_0 - y_0}{|x_0 - y_0|^n} \right. \\
&\qquad\qquad\qquad \left. + \frac{x_0 - y}{|x_0 - y|^n} \odot \nu(y) \Big\} \, d\sigma(y) \right|^p dw(x) \\
&\leq C \eta^{-(np-p)/(2n-1)} \cdot \eta^{(np)/(2n-1)} = C\, \eta^{p/(2n-1)},
\end{aligned} \tag{2.5.29}$$

for some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$. From this (2.5.22) follows, completing the proof of the proposition. $\square$

*Remark* 2.5.4. In the unweighted case (i.e., for $w \equiv 1$ or, equivalently, when the measure $w$ coincides with $\sigma$) a slight variant of the above proof gives that, in the geometric context of Proposition 2.5.3, one actually has

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|^{1/n}_{L^p(\partial\Omega,\sigma)\otimes \mathcal{C}\!\ell_n \to L^p(\partial\Omega,\sigma)\otimes \mathcal{C}\!\ell_n}. \tag{2.5.30}$$

Specifically, the idea is to take $r_* := \delta\, \eta^{1/n}$ in place of (2.5.18) and run the same argument as above bearing in mind that we now have $w(\Delta(y_0, r_*))/w(\Delta(x_0, r_*)) \leq C$ in the fourth line of (2.5.28).

Our next result contains estimates in the opposite direction to those given in Theorem 2.4.18.

**Theorem 2.5.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by $\nu = (\nu_k)_{1 \leq k \leq n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from (2.3.8), the Riesz transforms $\{R_j\}_{1 \leq j \leq n}$ on $\partial\Omega$ from (2.4.236), and for each index $k \in \{1, \dots, n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by the $k$-th scalar component of $\nu$.*

*Then there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \Big\}^{1/(2n-1)}. \tag{2.5.31}$$

*Moreover, in the unweighted case the exponent $1/(2n-1)$ may be replaced by $1/n$.*

*Proof.* If $\partial\Omega$ is unbounded, then the estimate claimed in (2.5.31) is a direct consequence of Proposition 2.5.3 and Lemma 2.5.1 (also bearing in mind Lemma 2.2.47). In the case when $\partial\Omega$ is bounded, we have $K_\Delta 1 = \pm\frac{1}{2}$ (cf. [93]) with the sign plus if $\Omega$ is bounded, and the sign minus if $\Omega$ is unbounded, hence $\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \geq \frac{1}{2}$ in such a scenario. Since from (2.2.56) we know that we always have $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$, the estimate claimed in (2.5.31) holds in this case if we take $C \geq 2^{1/(2n-1)}$. Finally, that in the unweighted case the exponent $1/(2n-1)$ may be replaced by $1/n$ is seen from Remark 2.5.4. $\square$

We conclude this section by presenting a characterization of $\delta$-SKT domains in terms of the size of the operator norms of the classical harmonic double layer and commutators of Riesz transforms with pointwise multiplication by the scalar components of the unit normal.

**Corollary 2.5.6.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq j\leq n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ on $\partial\Omega$ from (2.3.8), the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (2.4.236), and for each $k \in \{1,\dots,n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by the $k$-th scalar component of $\nu$.*

*Then there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if*

$$\|K_\Delta\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} < \delta \tag{2.5.32}$$

*then $\Omega$ is a $(C\delta^{1/(2n-1)})$-SKT domain. Moreover, in the unweighted case the exponent $1/(2n-1)$ may be replaced by $1/n$.*

*Proof.* Since the current hypotheses imply that $\Omega$ is a UR domain, all desired conclusions follow from Theorem 2.5.5 and Definition 2.2.14. $\square$

### 2.5.3 Using Riesz transforms to quantify flatness

Recall from (2.1.13) that for each $j \in \{1,\dots,n\}$ the $j$-th Riesz transform $R_j$ associated with a UR domain $\Omega \subseteq \mathbb{R}^n$ is the formal convolution operator on $\partial\Omega$ with the kernel

$k_j(x) := \frac{2}{\omega_{n-1}} \frac{x_j}{|x|^n}$ for $x \in \mathbb{R}^n \setminus \{0\}$. From Proposition 2.3.3 we know that these are bounded operators on $L^p(\partial\Omega, w)$ for each $p \in (1, \infty)$ and $w \in A_p(\partial\Omega, \sigma)$. The most familiar setting is when $\Omega = \mathbb{R}^n_+$, in which case it is well known that

$$\sum_{j=1}^{n} R_j^2 = -I \quad \text{and} \quad R_j R_k = R_k R_j \quad \text{for all} \quad j, k \in \{1, \ldots, n\}, \tag{2.5.33}$$

when all operators are considered on Muckenhoupt weighted Lebesgue spaces. Indeed, in such a setting, for the integrability exponent $p = 2$ and the weight $w = 1$ these are immediate consequences of the fact that each $R_j$ is a Fourier multiplier in $\partial\Omega \equiv \mathbb{R}^{n-1}$ corresponding to the symbol $i\xi_j/|\xi|$, then the said identities extend to $L^p(\partial\Omega, w)$ via a density argument. For ease of reference, we shall refer to the formulas in (2.5.33) as being URTI, i.e., the `usual Riesz transform identities`.

Remarkably, Theorem 2.5.7 below provides a stability result to the affect that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI are "almost" true in the context of a Muckenhoupt weighted Lebesgue space, then $\partial\Omega$ is "almost" flat, in that the BMO semi-norm of the outward unit normal to $\Omega$ is small.

**Theorem 2.5.7.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain with an unbounded boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and recall the Riesz transforms $\{R_j\}_{1 \le j \le n}$ on $\partial\Omega$ from (2.4.236). Then there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, and the* UR *character of $\partial\Omega$ with the property that*

$$\begin{aligned} \|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \le C\Big\{ \Big\| I + \sum_{j=1}^{n} R_j^2 \Big\|_{L^p(\partial\Omega,w) \to L^p(\partial\Omega,w)} \\ + \max_{1 \le j,k \le n} \big\| [R_j, R_k] \big\|_{L^p(\partial\Omega,w) \to L^p(\partial\Omega,w)} \Big\}^{1/(2n-1)}. \end{aligned} \tag{2.5.34}$$

*Moreover, in the unweighted case the exponent $1/(2n-1)$ may be replaced by $1/n$.*

It is perhaps surprising (but nonetheless true; cf. [48]) that URTI are also valid in the context of Muckenhoupt weighted Lebesgue spaces when $\Omega$ is an open ball, or the complement of a closed ball in $\mathbb{R}^n$. This shows that, in the context of Theorem 2.5.7, our assumption that $\partial\Omega$ is unbounded is warranted, since otherwise (2.5.34) may fail.

*Proof of Theorem 2.5.7.* Formula [53, (4.6.46), p. 2752] (which is valid in any UR domain, irrespective of whether its boundary is compact or not) tells us that for each $f \in L^p(\partial\Omega, \sigma) \otimes \mathcal{C}\!\ell_n$ we have

$$(\mathbf{C} - \mathbf{C}^{\#})f = \mathbf{C}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f + \sum_{1 \le j < k \le n} \mathbf{C}\big(([R_j, R_k]f)\mathbf{e}_j \odot \mathbf{e}_k\big). \tag{2.5.35}$$

Since $\big(L^p(\partial\Omega, \sigma) \cap L^p(\partial\Omega, w)\big) \otimes \mathcal{C}\!\ell_n$ is a dense subspace of $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$ and since all operators involved are continuous on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$, we conclude that formula (2.5.35) continues to hold for each $f \in L^p(\partial\Omega, w) \otimes \mathcal{C}\ell_n$. From this version of (2.5.35) we then conclude that

$$\mathbf{C} - \mathbf{C}^{\#} = \mathbf{C}\Big(I + \sum_{j=1}^{n} R_j^2\Big) + \sum_{1\leq j<k\leq n} \mathbf{C}[R_j, R_k]\mathbf{e}_j \odot \mathbf{e}_k \tag{2.5.36}$$

as operators on $L^p(\partial\Omega, w) \otimes \mathcal{C}\ell_n$. In concert with (2.5.7), this implies

$$\begin{aligned} \|\mathbf{C} - \mathbf{C}^{\#}\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n \to L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n} & \\ \leq C\Big\| I + \sum_{j=1}^{n} R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} & \\ + C \sum_{1\leq j<k\leq n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} & \end{aligned} \tag{2.5.37}$$

Then (2.5.34) becomes a consequence of (2.5.37) and Proposition 2.5.3. Finally, the very last claim in the statement of the theorem follows from Remark 2.5.4. □

Our next result contains estimates in the opposite direction to those from Theorem 2.5.7. Collectively, Theorem 2.5.8 and Theorem 2.5.7 amount to saying that, under natural background geometric assumptions on the set $\Omega$, the URTI are "almost" true in the context of a Muckenhoupt weighted Lebesgue space if and only if $\partial\Omega$ is "almost" flat (in that the BMO semi-norm of the outward unit normal to $\Omega$ is small).

**Theorem 2.5.8.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and recall the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (2.4.236).*

*Then there exists some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\Big\| I + \sum_{j=1}^{n} R_j^2\Big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.5.38}$$

*and*

$$\max_{1\leq j<k\leq n} \big\|[R_j, R_k]\big\|_{L^p(\partial\Omega,w)\to L^p(\partial\Omega,w)} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.5.39}$$

*Proof.* From the Muckenhoupt version of (2.5.9) and (2.5.36) we see that for each $f \in L^p(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ we have

$$\mathbf{C}(\mathbf{C}^{\#} - \mathbf{C})f = -\tfrac{1}{4}\Big(I + \sum_{j=1}^{n} R_j^2\Big)f - \tfrac{1}{4}\sum_{1\leq j<k\leq n} ([R_j, R_k]f)\mathbf{e}_j \odot \mathbf{e}_k. \tag{2.5.40}$$

Fix an arbitrary scalar function $f \in L^p(\partial\Omega, w)$ normalized so that $\|f\|_{L^p(\partial\Omega,w)} = 1$. In particular, this entails $f \in L^p(\partial\Omega, w) \otimes \mathcal{C}\ell_n$ and $\|f\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n} = 1$. Bearing this in mind, we may then write

$$\begin{aligned}
&\max\Big\{\Big\|\tfrac14\Big(I+\sum_{j=1}^n R_j^2\Big)f\Big\|_{L^p(\partial\Omega,w)},\ \max_{1\le j<k\le n}\Big\|\tfrac14[R_j,R_k]f\Big\|_{L^p(\partial\Omega,w)}\Big\}\\
&\quad\le\Big\|\Big\{\Big|\tfrac14\Big(I+\sum_{j=1}^n R_j^2\Big)f\Big|^2+\sum_{1\le j<k\le n}\Big|\tfrac14[R_j,R_k]f\Big|^2\Big\}^{1/2}\Big\|_{L^p(\partial\Omega,w)}\\
&\quad=\Big\|\tfrac14\Big(I+\sum_{j=1}^n R_j^2\Big)f+\tfrac14\sum_{1\le j<k\le n}([R_j,R_k]f)\mathbf{e}_j\odot\mathbf{e}_k\Big\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n}\\
&\quad=\|\mathbf{C}(\mathbf{C}^\#-\mathbf{C})f\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n}\\
&\quad\le\|\mathbf{C}\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n\to L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n}\Big\|\mathbf{C}-\mathbf{C}^\#\Big\|_{L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n\to L^p(\partial\Omega,w)\otimes\mathcal{C}\ell_n}\\
&\quad\le C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n},
\end{aligned}\tag{2.5.41}$$

where the first inequality is trivial, the subsequent equality is implied by (1.4.7), the second equality is seen from formula (2.5.40), the penultimate estimate uses the normalization of $f$, while the last inequality is provided by (2.5.7) and (2.5.11). With (2.5.41) in hand, the claims in (2.5.38)-(2.5.39) readily follow (in view of the arbitrariness of the scalar-valued function $f\in L^p(\partial\Omega,w)$ with $\|f\|_{L^p(\partial\Omega,w)}=1$). $\square$

### 2.5.4 Using Riesz transforms to characterize Muckenhoupt weights

Assume $\Sigma\subseteq\mathbb{R}^n$, where $n\in\mathbb{N}$ with $n\ge 2$, is a closed UR set and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. For $j\in\{1,\dots,n\}$, the $j$-th Riesz transform $R_j$ on $\Sigma$ is defined as the operator acting on each $f\in L^1\big(\Sigma,\frac{\sigma(x)}{1+|x|^{n-1}}\big)$ according to

$$R_jf(x):=\lim_{\varepsilon\to0^+}\frac{2}{\omega_{n-1}}\int\limits_{\substack{y\in\Sigma\\|x-y|>\varepsilon}}\frac{x_j-y_j}{|x-y|^n}f(y)\,d\sigma(y)\ \text{ for }\sigma\text{-a.e. }x\in\Sigma.\tag{2.5.42}$$

From Proposition 2.3.3 we know that these Riesz transforms are well-defined in this context, and that for each integrability exponent $p\in(1,\infty)$ and Muckenhoupt weight $w\in A_p(\Sigma,\sigma)$ they induce linear and bounded mappings on $L^p(\Sigma,w)$. The goal in this section is to show that the class of Muckenhoupt weights is the largest class of weights for which the latter boundedness properties hold.

As a preamble, we note that for a variety of purposes it is convenient to glue together all Riesz transforms $\{R_j\}_{1\le j\le n}$ from (2.5.42) into a unique operator now acting on Clifford algebra-valued functions $f\in L^1\big(\Sigma,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\otimes\mathcal{C}\ell_n$ according to

$$\begin{aligned}
Rf(x)&:=\lim_{\varepsilon\to0^+}\frac{2}{\omega_{n-1}}\int\limits_{\substack{y\in\Sigma\\|x-y|>\varepsilon}}\frac{x-y}{|x-y|^n}\odot f(y)\,d\sigma(y)\\
&=\mathbf{e}_1\odot R_1f(x)+\cdots+\mathbf{e}_n\odot R_nf(x)\ \text{ for }\sigma\text{-a.e. }x\in\Sigma.
\end{aligned}\tag{2.5.43}$$

**Theorem 2.5.9.** *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a closed* UR *set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p \in (1,\infty)$ and consider a weight $w$ on $\Sigma$ which belongs to $L^1_{\rm loc}(\Sigma,\sigma)$ and has the property that, for $j \in \{1,\dots,n\}$, the $j$-th Riesz transform $R_j$ on $\Sigma$ originally defined as in (2.5.42) extends to a linear and bounded operator on $L^p(\Sigma,w)$. Then necessarily $w \in A_p(\Sigma,\sigma)$.*

From assumptions and (2.2.291) we know that $\sigma$ is a complete, locally finite (hence also sigma-finite), separable, Borel-regular measure on $\Sigma$. Since the weight $w$ belongs to $L^1_{\rm loc}(\Sigma,\sigma)$, it follows that

$$\text{the measure } dw := w\,d\sigma \text{ is complete, locally finite (hence also sigma-finite), separable, and Borel-regular on } \Sigma. \tag{2.5.44}$$

Granted this, results in [6], [93] then guarantee that the natural inclusion

$$\mathscr{X} := \big\{\phi\big|_\Sigma : \phi \in \mathscr{C}^\infty_0(\mathbb{R}^n)\big\} \hookrightarrow L^p(\Sigma,w) \ \text{ has dense range.} \tag{2.5.45}$$

From the preamble to Theorem 2.5.9 we know that the Riesz transforms (2.5.42) act in a meaningful fashion on $\mathscr{X}$, and this is the manner in which the $R_j$'s are originally considered in the context of Theorem 2.5.9. The point of the latter theorem is that if the $R_j$'s originally defined on $\mathscr{X}$ extend via density (cf. (2.5.45)) to linear and bounded operators on $L^p(\Sigma,w)$ then necessarily $w \in A_p(\Sigma,\sigma)$.

*Proof of Theorem 2.5.9.* The fact that all Riesz transforms on $\Sigma$ originally defined as in (2.5.42) on functions $f \in \mathscr{X} := \big\{\phi\big|_\Sigma : \phi \in \mathscr{C}^\infty_0(\mathbb{R}^n)\big\}$ induce (via density; cf. (2.5.45)) linear and bounded mappings on $L^p(\Sigma,w)$, implies that the operator $R$ from (2.5.43), originally defined on functions $f \in \mathscr{X} \otimes \mathcal{C}\!\ell_n$ induces (via density) a linear and bounded mapping on $L^p(\Sigma,w) \otimes \mathcal{C}\!\ell_n$.

To proceed in earnest, fix a number $\lambda \in (1,\infty)$ which is sufficiently large relative to the Ahlfors regularity constant of $\Sigma$. Much as in (2.5.16), this may be done as to ensure that

$$\Delta(x,\lambda R) \setminus \Delta(x,R) \neq \varnothing \quad \text{for each} \quad x \in \Sigma \quad \text{and} \quad R \in \big(0\,,\,{\rm diam}\,(\Sigma)/\lambda\big). \tag{2.5.46}$$

Fix $r \in \big(0\,,\,{\rm diam}\,(\Sigma)/(10\lambda)\big)$ and suppose $x_1, x_2 \in \Sigma$ are such that

$$10\,r < |x_1 - x_2| < 10\lambda r. \tag{2.5.47}$$

Abbreviate

$$\Delta_1 := \Delta(x_1,r) \quad \text{and} \quad \Delta_2 := \Delta(x_2,r). \tag{2.5.48}$$

Next, select a real-valued function $f \in \mathscr{X}$ and set $f_\pm := \max\{\pm f, 0\}$. We then have $0 \leq f_\pm \leq |f| = f_+ + f_-$ on $\Sigma$, and $f_\pm \in L^p(\Sigma,w)$ since $\mathscr{X} \subseteq L^p(\Sigma,w)$. To proceed, define

$$g_\pm(y) := \begin{cases} -\dfrac{x_2 - y}{|x_2 - y|} f_\pm(y) & \text{for } \ y \in \Delta_1, \\ \qquad 0 & \text{for } \ y \in \Sigma \setminus \Delta_1, \end{cases} \tag{2.5.49}$$

so $g_\pm$ belong to $L^p(\Sigma, w) \otimes \mathcal{C}\ell_n$ and are supported in $\Delta_1$. Consequently,

$$Rg_\pm(x) = \frac{2}{\omega_{n-1}} \int_{\Delta_1} \frac{x-y}{|x-y|^n} \odot \frac{-(x_2-y)}{|x_2-y|} f_\pm(y)\, d\sigma(y) \text{ for each } x \in \Delta_2. \tag{2.5.50}$$

Recall that the scalar component $u_{\rm scal}$ of a Clifford algebra element $u \in \mathcal{C}\ell_n$ is defined as in (1.4.6). For each $x \in \Delta_2$ and $y \in \Delta_1$ we may use (1.4.1), (1.4.7), (1.4.10), as well as (2.5.47) to compute

$$\begin{aligned}
\Big(\frac{x-y}{|x-y|^n} \odot \frac{-(x_2-y)}{|x_2-y|}\Big)_{\rm scal} &= \Big(\frac{x-y}{|x-y|^n} \odot \frac{-(x-y)}{|x_2-y|}\Big)_{\rm scal} \\
&\quad + \Big(\frac{x-y}{|x-y|^n} \odot \frac{x-x_2}{|x_2-y|}\Big)_{\rm scal} \\
&= \frac{1}{|x-y|^{n-2}\cdot|x_2-y|} + \Big(\frac{x-y}{|x-y|^n} \odot \frac{x-x_2}{|x_2-y|}\Big)_{\rm scal} \\
&\geq \frac{1}{|x-y|^{n-2}\cdot|x_2-y|} - \frac{|x-x_2|}{|x-y|^{n-1}\cdot|x_2-y|} \\
&= \frac{|x-y|-|x-x_2|}{|x-y|^{n-1}\cdot|x_2-y|} \\
&\geq \frac{7r}{(10\lambda r+2r)^{n-1}(10\lambda r+r)} = c_{n,\lambda}\cdot r^{1-n}.
\end{aligned} \tag{2.5.51}$$

Based on (2.5.50), (2.5.51), and the Ahlfors regularity of $\Sigma$ we conclude that

$$|Rg_\pm| \geq (Rg_\pm)_{\rm scal} \geq c_{n,\lambda} ⨍_{\Delta_1} f_\pm\, d\sigma \quad\text{on}\quad \Delta_2. \tag{2.5.52}$$

In concert with the boundedness of $R$ on $L^p(\Sigma, w) \otimes \mathcal{C}\ell_n$ (mentioned in the first part of the proof), this permits us to estimate

$$\begin{aligned}
c_{n,\lambda}^p\Big(⨍_{\Delta_1} f_\pm\, d\sigma\Big)^p &\leq \frac{1}{w(\Delta_2)}\int_{\Delta_2} |Rg_\pm|^p\, dw \leq \frac{1}{w(\Delta_2)}\int_\Sigma |Rg_\pm|^p\, dw \\
&\leq \frac{C}{w(\Delta_2)}\int_\Sigma |g_\pm|^p\, dw \leq \frac{C}{w(\Delta_2)}\int_{\Delta_1} |f|^p\, dw,
\end{aligned} \tag{2.5.53}$$

for some constant $C \in (0, \infty)$ independent of $f$, $x_1$, $x_2$, and $r$. Combining the two versions of (2.5.53), corresponding to $f_+$ and $f_-$, yields

$$c_{n,\lambda}^p\Big(⨍_{\Delta_1} |f|\, d\sigma\Big)^p \leq \frac{C}{w(\Delta_2)}\int_{\Delta_1} |f|^p\, dw, \tag{2.5.54}$$

for some constant $C \in (0, \infty)$ independent of $f$, $x_1$, $x_2$, and $r$. Specializing (2.5.54) to the case when the real-valued function $f \in \mathscr{X}$ is chosen such that $f \equiv 1$ on $\Delta_1 \cup \Delta_2$ then yields

$$c_{n,\lambda}^p \leq C\frac{w(\Delta_1)}{w(\Delta_2)}. \tag{2.5.55}$$

Running the same type of argument as above but with the roles of $x_1$ and $x_2$ (which are interchangeable) reversed then produces, in place of (2.5.55),

$$c^p_{n,\lambda} \leq C\frac{w(\Delta_2)}{w(\Delta_1)}. \tag{2.5.56}$$

From (2.5.56) and (2.5.54) we then conclude that for each real-valued function $f \in \mathscr{X}$ we have

$$⨍_{\Delta_1} |f|\, d\sigma \leq C\Big(⨍_{\Delta_1} |f|^p\, dw\Big)^{1/p} \tag{2.5.57}$$

for some constant $C \in (0,\infty)$ independent of $f$, $x_1$, and $r$.

Consider now an arbitrary function $h \in L^p_{\rm loc}(\Sigma, w)$. In particular, the extension of $h\big|_{\Delta_1}$ by zero to the rest of $\Sigma$ belongs to $L^p(\Sigma, w)$. Granted this, (2.5.45) guarantees the existence of a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathscr{X}$ such that

$$f_j\big|_{\Delta_1} \to h\big|_{\Delta_1} \quad \text{in} \quad L^p(\Delta_1, w) \quad \text{as} \quad j \to \infty. \tag{2.5.58}$$

By eventually passing to sub-sequences there is no loss of generality in also assuming that $\lim\limits_{j\to\infty} f_j(x) = h(x)$ for $\sigma$-a.e. $x \in \Delta_1$. Based on this, Fatou's lemma, and (2.5.57) we may then write

$$\begin{aligned} ⨍_{\Delta_1} |h|\, d\sigma &\leq \liminf_{j\to\infty} ⨍_{\Delta_1} |f_j|\, d\sigma \leq C \cdot \liminf_{j\to\infty} \Big(⨍_{\Delta_1} |f_j|^p\, dw\Big)^{1/p} \\ &\leq C\Big(⨍_{\Delta_1} |h|^p\, dw\Big)^{1/p}. \end{aligned} \tag{2.5.59}$$

Ultimately, this goes to show that for each $h \in L^p_{\rm loc}(\Sigma, w)$ we have

$$⨍_{\Delta_1} |h|\, d\sigma \leq C\Big(⨍_{\Delta_1} |h|^p\, dw\Big)^{1/p}, \tag{2.5.60}$$

with $C \in (0,\infty)$ independent of $h$, $x_1$, and $r$.

Start now with an arbitrary point $x \in \Sigma$, and continue to assume that the scale $r$ belongs to $\big(0\,,\, {\rm diam}\,(\Sigma)/(10\lambda)\big)$. We may then employ (2.5.46) with $R := 10\,r$ to conclude that there exists some $\widetilde{x} \in \Delta(x, 10\lambda r) \setminus \Delta(x, 10\,r)$. We then have $10\,r < |x - \widetilde{x}| < 10\lambda r$ which, in light of (2.5.47), shows that we may run the argument so far with $x_1 := x$ and $x_2 := \widetilde{x}$. In place of (2.5.60) we then arrive at the conclusion that there exists $C \in (0,\infty)$ with the property that

$$\begin{gathered} ⨍_{\Delta(x,r)} |h|\, d\sigma \leq C\Big(⨍_{\Delta(x,r)} |h|^p\, dw\Big)^{1/p} \quad \text{for each} \\ h \in L^p_{\rm loc}(\Sigma, w),\ \ x \in \Sigma,\ \ r \in \big(0\,,\, {\rm diam}\,(\Sigma)/(10\lambda)\big). \end{gathered} \tag{2.5.61}$$

In the case when $\Sigma$ is unbounded, from (2.5.61) (which now hold with no restriction on the size of the scale $r$ since ${\rm diam}\,(\Sigma) = \infty$) and the second part of Lemma 2.2.41 we conclude that $w \in A_p(\Sigma, \sigma)$ and $[w]_{A_p} \leq C^p$.

As such, there remains to treat the case when $\Sigma$ is bounded. When is the case, starting with (2.5.61), the argument in the proof of Lemma 2.2.41 that has led to (2.2.308) presently gives (with $p'$ denoting the Hölder conjugate exponent of $p$)

$$\Big(\fint_{\Delta(x,r)} w\,d\sigma\Big)\Big(\fint_{\Delta(x,r)} w^{1-p'}\,d\sigma\Big)^{p-1}\leq C^p \qquad (2.5.62)$$
$$\text{for each}\quad x\in\Sigma\quad\text{and}\quad r\in(0\,,\,\mathrm{diam}\,(\Sigma)/(10\lambda)).$$

To treat the case when

$$\mathrm{diam}\,(\Sigma)/(10\lambda)\leq r\leq \mathrm{diam}\,(\Sigma), \qquad (2.5.63)$$

observe that for each $x\in\Sigma$ we may estimate, using the Ahlfors regularity of $\Sigma$ and the fact that $r$ is comparable with $\mathrm{diam}\,(\Sigma)$,

$$\Big(\fint_{\Delta(x,r)} w\,d\sigma\Big)\Big(\fint_{\Delta(x,r)} w^{1-p'}\,d\sigma\Big)^{p-1}$$
$$\leq C_{\Sigma,p}\Big(\fint_{\Sigma} w\,d\sigma\Big)\Big(\fint_{\Sigma} w^{1-p'}\,d\sigma\Big)^{p-1}, \qquad (2.5.64)$$

for some constant $C_{\Sigma,p}\in(0,\infty)$ which depends only on $\Sigma$ and $p$. At this stage, there remains to show that the right hand-side of (2.5.64) is finite. To this end, introduce $r_0:=\mathrm{diam}\,(\Sigma)/(20\lambda)$ and cover the compact set $\Sigma$ with finitely surface balls of radius $r_0$, say $\Sigma\subseteq\bigcup_{j=1}^N\Delta(x_j,r_0)$ for some $N\in\mathbb{N}$ and $\{x_j\}_{1\leq j\leq N}\subseteq\Sigma$. Also, define

$$c_*:=\inf_{1\leq j\leq N}\frac{w(\Delta(x_j,r_0))}{w(\Sigma)}>0. \qquad (2.5.65)$$

From (2.5.65) and (2.5.61) used with $r:=r_0\in(0\,,\,\mathrm{diam}\,(\Sigma)/(10\lambda))$ we then obtain that for each $h\in L^p_{\rm loc}(\Sigma,w)$ we have

$$\fint_{\Delta(x_j,r_0)}|h|\,d\sigma\leq C\cdot c_*^{-1/p}\Big(\fint_{\Sigma}|h|^p\,dw\Big)^{1/p}\quad\text{for}\quad j\in\{1,\dots,N\}. \qquad (2.5.66)$$

Using the Ahlfors regularity of $\Sigma$ and summing up in $j$ further yields

$$\fint_{\Sigma}|h|\,d\sigma\leq C_\Sigma\cdot C\cdot c_*^{-1/p}\Big(\fint_{\Sigma}|h|^p\,dw\Big)^{1/p}\quad\text{for each}\quad h\in L^p_{\rm loc}(\Sigma,w), \qquad (2.5.67)$$

where $C_\Sigma\in(0,\infty)$ depends only on $\Sigma$. Having established (2.5.67), the argument in the proof of Lemma 2.2.41 that has produced (2.2.308) then currently gives

$$\Big(\fint_{\Sigma} w\,d\sigma\Big)\Big(\fint_{\Sigma} w^{1-p'}\,d\sigma\Big)^{p-1}\leq (C_\Sigma\cdot C)^p/c_*. \qquad (2.5.68)$$

In concert with (2.5.64) this finally proves that $w\in A_p(\Sigma,\sigma)$. $\square$

In concert with earlier results, Theorem 2.5.9 yields the following remarkable characterization of Muckenhoupt weights.

**Theorem 2.5.10.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix a function $w \in L^1_{\rm loc}(\partial\Omega, \sigma)$ which is strictly positive $\sigma$-a.e. on $\partial\Omega$, along with an integrability exponent $p \in (1, \infty)$. Then the following statements are equivalent.*

*(1) The weight $w$ belongs to the Muckenhoupt class $A_p(\partial\Omega, \sigma)$.*

*(2) For each $j \in \{1, \dots, n\}$, the $j$-th Riesz transform $R_j$ on $\partial\Omega$ (cf. (2.4.236)) induces a linear and bounded operator on $L^p(\partial\Omega, w)$.*

*(3) The Cauchy-Clifford operator $\mathbf{C}$ from (2.5.1) induces a linear and bounded mapping on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$.*

*(4) The "transposed" Cauchy-Clifford operator $\mathbf{C}^{\#}$ from (2.5.3) induces a linear and bounded mapping on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$.*

*(5) For each complex-valued function $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ which is odd and positive homogeneous of degree $1-n$, the integral operator originally defined for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ as*

$$Tf(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} k(x-y) f(y)\, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{2.5.69}$$

*induces a linear and bounded mapping on $L^p(\partial\Omega, w)$.*

*Proof.* The implications *(1)⇒(2)* and *(1)⇒(5)* are direct consequences of Proposition 2.3.3 and (2.4.236). From (2.4.236) it is also clear that *(5)⇒(2)*. To proceed, let $\nu = (\nu_1, \dots, \nu_n)$ denote the geometric measure theoretic outward unit normal to $\Omega$. Then (2.5.2) and (2.5.4) imply that the Cauchy-Clifford operator $\mathbf{C}$ from (2.5.1) as well as the "transposed" Cauchy-Clifford operator $\mathbf{C}^{\#}$ from (2.5.3) induce linear and bounded mappings on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$ whenever all Riesz transforms on $\partial\Omega$, i.e., $R_j$ as in (2.4.236) with $1 \leq j \leq n$, induce linear and bounded operators on $L^p(\partial\Omega, w)$. This takes care of the implications *(2)⇒(3)* and *(2)⇒(4)*.

Going further, bring in the integral operator $R$ defined as in (2.5.43) for $\Sigma := \partial\Omega$, i.e., $Rf = \mathbf{e}_1 \odot R_1 f + \cdots + \mathbf{e}_n \odot R_n f$ for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes \mathcal{C}\!\ell_n$, where $\{R_j\}_{1 \leq j \leq n}$ are Riesz transforms on $\partial\Omega$ defined in (2.4.236). From definitions and the fact that $\nu \odot \nu = -1$ at $\sigma$-a.e. point of $\partial\Omega$ (cf. (1.4.1)) we then see that for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes \mathcal{C}\!\ell_n$ we have

$$\begin{aligned} &\nu \odot \mathbf{C}^{\#} f = \tfrac{1}{2} Rf, \quad -\mathbf{C}(\nu \odot f) = \tfrac{1}{2} Rf, \quad \mathbf{C} f = \nu \odot \mathbf{C}^{\#}(\nu \odot f), \\ &\mathbf{C}^{\#} f = -\tfrac{1}{2} \nu \odot Rf, \quad \mathbf{C} f = \tfrac{1}{2} R(\nu \odot f), \quad \mathbf{C}^{\#} f = \nu \odot \mathbf{C}(\nu \odot f). \end{aligned} \tag{2.5.70}$$

It is also clear that the statement in item *(2)* is equivalent to the demand that $R$ induces a linear and bounded operator on $L^p(\partial\Omega, w) \otimes \mathcal{C}\!\ell_n$. On account of this and (2.5.70) we then conclude that the implications *(3)⇒(2)* and *(4)⇒(2)* are valid. Finally, Theorem 2.5.9 gives the implication *(2)⇒(1)*. The proof of Theorem 2.5.10 is therefore complete. □

## 2.6 Boundary value problems in Muckenhoupt weighted spaces

This section is devoted to studying the Dirichlet, Regularity, Neumann, and Transmission boundary value problems in unbounded $\delta$-SKT domains with boundary data in Muckenhoupt weighted Lebesgue and Sobolev spaces. The technology that we bring to bear on such problems also allows us to deal with similar boundary value problems formulated in terms of ordinary Lorentz spaces and Lorentz-based Sobolev spaces.

As a preamble, in Theorem 2.6.1 below we recall from [93] a Poisson integral representation formula for solutions of the Dirichlet Problem for a given weakly elliptic second-order system $L$, in domains of a very general geometric nature, which involves the conormal derivative of the Green function for the transposed system $L^\top$ as integral kernel. Stating this requires that we review a definition and a couple of related results. Specifically, following [93] we shall say that a set $\Omega$ is `globally pathwise nontangentially accessible` provided $\Omega$ is an open nonempty proper subset of $\mathbb{R}^n$ such that:

$$\begin{array}{l}\text{given any } \kappa > 0 \text{ there exist } \widetilde{\kappa} \geq \kappa \text{ along with } c \in [1,\infty) \\ \text{such that } \sigma\text{-a.e. point } x \in \partial\Omega \text{ has the property that any} \\ y \in \Gamma_\kappa(x) \text{ may be joined by a rectifiable curve } \gamma_{x,y} \text{ such that} \\ \gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x) \text{ and whose length is } \leq c|x-y|.\end{array} \tag{2.6.1}$$

It has been noted in [93] that

$$\begin{array}{l}\text{any one-sided NTA domain with unbounded boundary is} \\ \text{a globally pathwise nontangentially accessible set,}\end{array} \tag{2.6.2}$$

and that

$$\begin{array}{l}\text{any semi-uniform set (in the sense of Aikawa-Hirata; cf. [3])} \\ \text{is a globally pathwise nontangentially accessible set.}\end{array} \tag{2.6.3}$$

We are now ready to state the Poisson integral representation formula advertised earlier (for a proof see [93]).

**Theorem 2.6.1.** *Let $\Omega$ be an open nonempty proper subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) which is globally pathwise nontangentially accessible (in the sense of (2.6.1)), and such that $\partial\Omega$ is unbounded and Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to $\Omega$. Next, suppose $L$ is a weakly elliptic, homogeneous, constant (complex) coefficient, second-order, $M \times M$ system in $\mathbb{R}^n$. Fix an aperture parameter $\kappa \in (0,\infty)$, along with an arbitrary point $x_0 \in \Omega$, and suppose $0 < \rho < \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)$. Finally, define $K := \overline{B(x_0,\rho)}$.*

*Then there exists some $\widetilde{\kappa} > 0$, which depends only on $\Omega$ and $\kappa$, with the following* *significance. Assume $G$ is a matrix-valued function satisfying*

$$
\begin{cases}
G = (G_{\alpha\beta})_{1\leq\alpha,\beta\leq M} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^{M\times M}, \\
\big(L^\top G_{\bullet\beta}\big)_\alpha = -\delta_{x_0}\delta_{\alpha\beta} \;\; in \;\; \big[\mathcal{D}'(\Omega)\big]^M \;\; for\ all \;\; \alpha,\beta\in\{1,\dots,M\}, \\
(\nabla G)\big|_{\partial\Omega}^{\widetilde{\kappa}-{\rm n.t.}} \;\; exists\ (in\ \mathbb{C}^{n\cdot M^2}) \;\; at\ \sigma\text{-}a.e.\ point\ on \;\; \partial\Omega, \\
G\big|_{\partial\Omega}^{\widetilde{\kappa}-{\rm n.t.}} = 0 \in \mathbb{C}^{M\times M} \;\; at\ \sigma\text{-}a.e.\ point\ on \;\; \partial\Omega, \\
\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) < +\infty \;\; at\ \sigma\text{-}a.e.\ point\ on \;\; \partial\Omega,
\end{cases}
\tag{2.6.4}
$$

*and assume $u = (u_\beta)_{1\leq\beta\leq M}$ is a $\mathbb{C}^M$-valued function in $\Omega$ satisfying*

$$
\begin{cases}
u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \;\; in \;\; \Omega, \\
u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \;\; exists\ at\ \sigma\text{-}a.e.\ point\ on \;\; \partial\Omega, \\
\mathcal{N}_\kappa u < +\infty \;\; at\ \sigma\text{-}a.e.\ point\ on \;\; \partial\Omega, \\
\displaystyle\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\, d\sigma < +\infty.
\end{cases}
\tag{2.6.5}
$$

*Then for any choice of a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}} \in \mathfrak{A}_L$ one has the Poisson integral representation formula*

$$
u_\beta(x_0) = -\int_{\partial_*\Omega} \big\langle u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}, \partial_\nu^{A^\top} G_{\bullet\beta}\big\rangle\, d\sigma, \qquad \forall\, \beta\in\{1,\dots,M\},
\tag{2.6.6}
$$

*where $\partial_\nu^{A^\top}$ stands for the conormal derivative associated with $A^\top$, acting on the columns of the matrix-valued function $G$ according to (compare with (2.3.20))*

$$
\partial_\nu^{A^\top} G_{\bullet\beta} := \Big(\nu_r a_{sr}^{\gamma\alpha} (\partial_s G_{\gamma\beta})\big|_{\partial\Omega}^{\widetilde{\kappa}-{\rm n.t.}}\Big)_{1\leq\alpha\leq M} \;\; at\ \sigma\text{-}a.e.\ point\ on \;\; \partial_*\Omega,
\tag{2.6.7}
$$

*for each $\beta\in\{1,\dots,M\}$.*

One remarkable feature of this result is that the only quantitative aspect of the hypotheses made in its statement is the finiteness condition in the fourth line of (2.6.5). Not only is this is most natural (in view of the conclusion in (2.6.6)), but avoiding to specify separate memberships of $\mathcal{N}_\kappa u$ and $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)$ to concrete dual function spaces on $\partial\Omega$ gives Theorem 2.6.1 a wide range of applicability. In particular, the various Poisson integral representation formulas this provides in various contexts permits us to derive, rather painlessly, uniqueness results for the Dirichlet Problem.

### 2.6.1 The Dirichlet Problem in weighted Lebesgue spaces

Theorem 2.6.2 below describes solvability, regularity, and well-posedness results for the Dirichlet Problem in $\delta$-SKT domains $\Omega\subseteq\mathbb{R}^n$ with boundary data in Muckenhoupt weighted Lebesgue spaces for weakly elliptic second-order homogeneous constant coefficient systems $L$ in $\mathbb{R}^n$ with the property that $\mathfrak{A}_L^{\rm dis}\neq\varnothing$ and/or $\mathfrak{A}_{L^\top}^{\rm dis}\neq\varnothing$. Examples of such systems include the Laplacian, all scalar weakly elliptic operators when $n \geq 3$, as well as the complex Lamé system $L_{\mu,\lambda} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ with $\mu \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{-2\mu, -3\mu\}$. In particular, the well-posedness result described in item *(d)* of Theorem 2.6.2 holds in all these cases. Furthermore, we provide counterexamples showing that our results are optimal, in the sense that the aforementioned assumptions on the existence of distinguished coefficient tensors cannot be dispensed with.

**Theorem 2.6.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1, \infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system $L$ in $\mathbb{R}^n$, consider the Dirichlet Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \quad \text{in} \quad \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[L^p(\partial\Omega, w)\big]^M. \end{cases} \tag{2.6.8}$$

*The following claims are true:*

*(a)* [Existence and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (a scenario which ensures that $\Omega$ is a $\delta$-SKT domain; cf. Definition 2.2.14) for some $\delta \in (0, \delta_0)$ then $\frac{1}{2}I + K_A$ is an invertible operator on the weighted Lebesgue space $\big[L^p(\partial\Omega, w)\big]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \Big(\mathcal{D}_A \Big(\tfrac{1}{2}I + K_A\Big)^{-1} f\Big)(x) \quad \text{for all} \quad x \in \Omega, \tag{2.6.9}$$

*is a solution of the Dirichlet Problem (2.6.8). Moreover, there exists some constant $C \in (0, \infty)$ independent of $f$ with the property that*

$$\|f\|_{[L^p(\partial\Omega,w)]^M} \leq \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} \leq C\,\|f\|_{[L^p(\partial\Omega,w)]^M}\,. \tag{2.6.10}$$

*(b)* [Regularity] *Under the background assumptions made in item (a), for the solution $u$ of the Dirichlet Problem (2.6.8) defined in (2.6.9), one has the following regularity result. For any given $q \in (1, \infty)$ and $\omega \in A_q(\partial\Omega, \sigma)$, further decreasing $\delta_0 \in (0, 1)$ (relative to $q$ and $[\omega]_{A_q}$), one has*

$$\mathcal{N}_\kappa(\nabla u) \in L^q(\partial\Omega, \omega) \Longleftrightarrow \partial_{\tau_{jk}} f \in \big[L^q(\partial\Omega, \omega)\big]^M, \ 1 \leq j, k \leq n, \tag{2.6.11}$$

*and if either of these conditions holds then*

$$\begin{gathered} (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \quad \text{exists (in } \mathbb{C}^{n\cdot M}\text{) at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{and} \quad \|\mathcal{N}_\kappa(\nabla u)\|_{L^q(\partial\Omega,\omega)} \approx \|\nabla_{\tan} f\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}}\,. \end{gathered} \tag{2.6.12}$$

*In particular, corresponding to $q := p$ and $\omega := w$, if $\delta_0 \in (0,1)$ is sufficiently small to begin with then*

$$\begin{array}{l} \mathcal{N}_\kappa(\nabla u) \text{ belongs to } L^p(\partial\Omega, w) \text{ if and only if } f \text{ belongs to} \\ [L^p_1(\partial\Omega, w)]^M, \text{ and if either of these conditions holds then} \\ \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} + \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p_1(\partial\Omega,w)]^M}. \end{array} \tag{2.6.13}$$

*(c)* [Uniqueness] *Whenever $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$, there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0,\delta_0)$ then the Dirichlet Problem* (2.6.8) *has at most one solution.*

*(d)* [Well-Posedness] *If $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ and $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (in other words, if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0,\delta_0)$ then the Dirichlet Problem* (2.6.8) *is well posed (i.e., it is uniquely solvable and the solution satisfies the naturally accompanying estimate formulated in* (2.6.10)*).*

*(e)* [Sharpness] *If $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ then the Dirichlet Problem* (2.6.8) *may not be solvable. Also, if $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ then the Dirichlet Problem* (2.6.8) *may have more than one solution. In fact, there exists a homogeneous, second-order, constant real coefficient, weakly elliptic $2\times 2$ system $L$ in $\mathbb{R}^2$ with $\mathfrak{A}^{\rm dis}_{L} = \mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ and which satisfies the following two properties: (i) the Dirichlet Problem formulated for this system as in* (2.6.8) *with $\Omega := \mathbb{R}^2_+$ fails to have a solution for each non-zero boundary datum belonging to an infinite-dimensional linear subspace of $\big[L^p(\partial\Omega,w)\big]^2$, and (ii) the linear space of null-solutions for the Dirichlet Problem formulated for the system $L$ as in* (2.6.8) *with $\Omega := \mathbb{R}^2_+$ is actually infinite dimensional.*

From Example 2.2.43 we know that, once a point $x_0 \in \partial\Omega$ has been fixed, then for each power $a \in \big(1-n, (p-1)(n-1)\big)$ the function

$$w : \partial\Omega \to [0,\infty], \quad w(x) := |x - x_0|^a \text{ for each } x \in \partial\Omega, \tag{2.6.14}$$

is a Muckenhoupt weight in the class $A_p(\partial\Omega,\sigma)$. Boundary value problems for a real constant coefficient system $L$ satisfying the Legendre-Hadamard strong ellipticity condition in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with boundary data in weighted (Lebesgue and Sobolev) spaces on $\partial\Omega$ for a weight of the form (2.6.14) have been considered in [111].

More generally, Proposition 2.2.44 tells us that, for each $d$-set $E \subseteq \partial\Omega$ with $d \in [0, n-1)$ and each power $a \in \big(d+1-n\,,\,(p-1)(n-1-d)\big)$, the function $w := \big[\mathrm{dist}\,(\cdot, E)\big]^a$ is a Muckenhoupt weight in the class $A_p(\partial\Omega,\sigma)$. Theorem 2.6.2 may therefore be specialized to this type of weights. A natural choice corresponds to the case when $E$ is a subset of the set of singularities of the "surface" $\partial\Omega$. Weighted boundary value problems in which the weight is a power of the distance to the singular set (of the boundary) have been studied extensively in the setting of conical and polyhedral domains, for which there is a vast amount of literature (see, e.g., [69] and the references therein).

Finally, we wish to mention that, in the class of systems considered in Theorem 2.6.2, the ensuing solvability, regularity, uniqueness, and well-posedness results are new even in the standard case when $\Omega = \mathbb{R}^n_+$.

Here is the proof of Theorem 2.6.2.

*Proof of Theorem 2.6.2.* To deal with the claims made in item *(a)* assume $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L^{\rm dis}$. Then Theorem 2.4.24 guarantees the existence of some threshold $\delta_0 \in (0,1)$, whose nature is as specified in the statement of the theorem, such that if $\|\nu\|_{[{\rm BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-SKT domain) for some $\delta \in (0,\delta_0)$ then the operator $\frac{1}{2}I + K_A$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. Granted this, from (2.3.3) and Proposition 2.3.4 (also keeping in mind (2.2.337)) we conclude that the function $u$ defined as in (2.6.9) solves the Dirichlet Problem (2.6.8) and satisfies (2.6.10).

Let us now prove the claims made in item *(b)* pertaining to the regularity of the solution $u$ just constructed. Retain the background assumptions made in item *(a)* and fix some $q \in (1,\infty)$ along with some $\omega \in A_q(\partial\Omega,\sigma)$. As regards the equivalence claimed in (2.6.11), assume first that $f \in \big[L^p(\partial\Omega, w)\big]^M$ is such that $\partial_{\tau_{jk}} f \in \big[L^q(\partial\Omega,\omega)\big]^M$ for each $j,k \in \{1,\dots,n\}$. Set $g := \big(\frac{1}{2}I + K_A\big)^{-1} f \in \big[L^p(\partial\Omega, w)\big]^M$ where the inverse is considered in the space $\big[L^p(\partial\Omega, w)\big]^M$. As noted in Remark 2.4.25 (assuming $\delta_0$ is sufficiently small), the operator $\frac{1}{2}I + K_A$ is also invertible on the off-diagonal Muckenhoupt weighted Sobolev space $\big[L^{p;q}_1(\partial\Omega, w;\omega)\big]^M$ (cf. (2.4.248)-(2.4.249)). Moreover, since the latter is a subspace of $\big[L^p(\partial\Omega, w)\big]^M$, it follows that the inverse of $\frac{1}{2}I + K_A$ on $\big[L^{p;q}_1(\partial\Omega, w;\omega)\big]^M$ is compatible with the inverse of $\frac{1}{2}I + K_A$ on $\big[L^p(\partial\Omega, w)\big]^M$. In particular, since we are currently assuming that $f \in \big[L^{p;q}_1(\partial\Omega, w;\omega)\big]^M$, we conclude that $g \in \big[L^{p;q}_1(\partial\Omega, w;\omega)\big]^M$. As a consequence of this membership and (2.2.337), we have

$$
\begin{aligned}
& g = (g_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \text{and} \\
& \partial_{\tau_{jk}} g \in \Big[L^1\Big(\partial\Omega\,,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \quad \text{for all } j,k \in \{1,\dots,n\}.
\end{aligned}
\tag{2.6.15}
$$

Granted these, we may invoke Proposition 2.3.2 and from (2.3.13) we conclude that the nontangential boundary trace $(\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = (\nabla \mathcal{D}_A g)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists (in $\mathbb{C}^{n\cdot M}$) at $\sigma$-a.e. point on $\partial\Omega$ (hence, the first property listed in (2.6.12) holds). Also, formula (2.3.12) gives that for each index $\ell \in \{1,\dots,n\}$ and each point $x \in \Omega$ we have

$$
\begin{aligned}
(\partial_\ell u)(x) &= \partial_\ell(\mathcal{D}_A g)(x) \\
&= \bigg(\int_{\partial\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} g_\alpha)(y)\,d\sigma(y)\bigg)_{1\leq\gamma\leq M}
\end{aligned}
\tag{2.6.16}
$$

if the coefficient tensor $A$ is expressed as $\big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}}$, and if the fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ is as in Theorem 1.2.1. In concert with (2.3.31) and (2.2.348), this proves that

$$
\begin{aligned}
& \|\mathcal{N}_\kappa(\nabla u)\|_{L^q(\partial\Omega,\omega)} \leq C\,\|\nabla_{\tan} g\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}} \\
& \text{for some constant } C \in (0,\infty) \text{ independent of } g.
\end{aligned}
\tag{2.6.17}
$$

In particular, $\mathcal{N}_\kappa(\nabla u)$ belongs to $L^q(\partial\Omega,\omega)$, which finishes the justification of the right-to-left implication in (2.6.11). Also, in light of (2.6.17), to justify the left-pointing inequality in the equivalence claimed in (2.6.12), there remains to show that, for some constant $C\in(0,\infty)$ independent of $f$,

$$\|\nabla_{\tan}g\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}}\leq C\,\|\nabla_{\tan}f\|_{[L^q(\partial\Omega,\omega)]^{n\cdot M}}. \tag{2.6.18}$$

To this end, use (2.4.240) to write, for each $j,k\in\{1,\dots,n\}$,

$$\begin{aligned}\partial_{\tau_{jk}}f&=\partial_{\tau_{jk}}\big[(\tfrac12 I+K_A)g\big]=(\tfrac12 I+K_A)(\partial_{\tau_{jk}}g)+U_{jk}(\nabla_{\tan}g)\\&=(\tfrac12 I+K_A)(\partial_{\tau_{jk}}g)+U_{jk}\Big((\nu_r\partial_{\tau_{rs}}g_\alpha)_{\substack{1\leq\alpha\leq M\\1\leq s\leq n}}\Big)\end{aligned} \tag{2.6.19}$$

at $\sigma$-a.e. point on $\partial\Omega$, where $\nu=(\nu_r)_{1\leq r\leq n}$ is the geometric measure theoretic outward unit normal to $\Omega$. Using the abbreviations

$$\nabla_\tau f:=\big(\partial_{\tau_{jk}}f\big)_{1\leq j,k\leq n},\quad \nabla_\tau g:=\big(\partial_{\tau_{jk}}g\big)_{1\leq j,k\leq n}, \tag{2.6.20}$$

we find it convenient to recast the collection of all formulas as in (2.6.19), corresponding to all indices $j,k\in\{1,\dots,n\}$, simply as

$$\nabla_\tau f=(\tfrac12 I+R)(\nabla_\tau g), \tag{2.6.21}$$

where $I$ is the identity and $R$ is the operator acting from $\big[L^q(\partial\Omega,\omega)\big]^{M\cdot n^2}$ into itself according to

$$R:=K_A+\Big(U_{jk}\circ(M_{\nu_r}\circ\pi^\alpha_{rs})_{\substack{1\leq\alpha\leq M\\1\leq s\leq n}}\Big)_{1\leq j,k\leq n}. \tag{2.6.22}$$

Above, we let $K_A$ act on each $F=(F^\alpha_{rs})_{\substack{1\leq\alpha\leq M\\1\leq r,s\leq n}}\in\big[L^q(\partial\Omega,\omega)\big]^{M\cdot n^2}$ by setting

$$K_AF:=\Big(K_A(F^\alpha_{rs})_{1\leq\alpha\leq M}\Big)_{1\leq r,s\leq n}. \tag{2.6.23}$$

Also, recall that each $M_{\nu_r}$ denotes the operator of pointwise multiplication by $\nu_r$, the $r$-th scalar component of $\nu$. Finally, in (2.6.22) we let each $\pi^\alpha_{rs}$ be the "coordinate-projection" operator which acts as $\pi^\alpha_{rs}(X):=X^\alpha_{rs}$ for every $X=(X^\alpha_{rs})_{\substack{1\leq\alpha\leq M\\1\leq r,s\leq n}}\in\mathbb{C}^{M\cdot n^2}$. From (2.6.22), (2.4.241), (2.4.238), Theorem 2.4.14, and (2.3.27), we then conclude that

$$\|R\|_{[L^q(\partial\Omega,\omega)]^{M\cdot n^2}\to[L^q(\partial\Omega,\omega)]^{M\cdot n^2}}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \tag{2.6.24}$$

for some $C\in(0,\infty)$ which depends only on $n$, $A$, $q$, $[\omega]_{A_q}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. As a consequence of this, if we assume $\delta_0>0$ to be sufficiently small to begin with, a Neumann series argument gives that

$$\tfrac12 I+R\ \text{ is invertible on }\ \big[L^q(\partial\Omega,\omega)\big]^{M\cdot n^2}, \tag{2.6.25}$$

and provides an estimate for the norm on the inverse. At this stage, the estimate claimed in (2.6.18) follows from (2.6.21), (2.6.25), (2.6.20), and (2.2.347)-(2.2.348). As noted earlier, this concludes the proof of the left-pointing inequality in the equivalence claimed in (2.6.12). To complete the treatment of item *(b)*, there remains to observe that the right-pointing implication in (2.6.11) together with the right-pointing inequality in the equivalence claimed in (2.6.12) are consequences of Proposition 2.2.49 (bearing in mind (2.2.347)).

Consider next the uniqueness result claimed in item *(c)*. Suppose $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$. Also, denote by $p' \in (1,\infty)$ the Hölder conjugate exponent of $p$, and set $w' := w^{1-p'} \in A_{p'}(\partial\Omega,\sigma)$. From Theorem 2.4.24, presently used with $L$ replaced by $L^\top$, $p'$ in place of $p$, and $w'$ in place of $w$, we know that there exists $\delta_0 \in (0,1)$, which depends only on $n$, $p$, $[w]_{A_p}$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\Omega$ is a $\delta$-SKT domain with $0<\delta<\delta_0$ then

$$\tfrac{1}{2}I + K_{A^\top} : \big[L^{p'}_1(\partial\Omega,w')\big]^M \longrightarrow \big[L^{p'}_1(\partial\Omega,w')\big]^M \tag{2.6.26}$$

is an invertible operator.

By eventually decreasing the value of $\delta_0 \in (0,1)$ if necessary, we may ensure that $\Omega$ is an NTA domain with unbounded boundary (cf. Proposition 2.2.32 and Lemma 2.2.5). In such a case, (2.6.2) guarantees that $\Omega$ is globally pathwise nontangentially accessible.

To proceed, let $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the system $L$ as in Theorem 1.2.1. Fix $x_\star \in \mathbb{R}^n \setminus \overline{\Omega}$ along with $x_0 \in \Omega$, arbitrary. Also, pick $\rho \in \big(0\,,\,\frac{1}{4}\,{\rm dist}\,(x_0,\partial\Omega)\big)$ and define $K := \overline{B(x_0,\rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa} > 0$ associated with $\Omega$ and $\kappa$ as in Theorem 2.6.1. Next, for each fixed $\beta \in \{1,\dots,M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x-x_0) - E_{\beta\alpha}(x-x_\star)\big)_{1\leq\alpha\leq M}, \qquad \forall\, x \in \partial\Omega. \tag{2.6.27}$$

From (2.6.27), (2.2.349), (2.2.341), (2.2.335), (1.2.19), and the Mean Value Theorem we then conclude that

$$f^{(\beta)} \in \big[L^{p'}_1(\partial\Omega,w')\big]^M. \tag{2.6.28}$$

As a consequence, with $\big(\frac{1}{2}I + K_{A^\top}\big)^{-1}$ denoting the inverse of the operator in (2.6.26),

$$v_\beta := (v_{\beta\alpha})_{1\leq\alpha\leq M} := \mathcal{D}_{A^\top}\Big(\big(\tfrac{1}{2}I + K_{A^\top}\big)^{-1} f^{(\beta)}\Big) \tag{2.6.29}$$

is a well-defined $\mathbb{C}^M$-valued function in $\Omega$ which, thanks to Proposition 2.3.4, satisfies

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in L^{p'}(\partial\Omega,w'), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in L^{p'}(\partial\Omega,w'), \\ \text{and } \ v_\beta\Big|^{\widetilde{\kappa}-\rm n.t.}_{\partial\Omega} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{2.6.30}$$

Moreover, from (2.6.28)-(2.6.29) and (2.3.60) we see that

$$(\nabla v_\beta)\Big|^{\widetilde{\kappa}-\rm n.t.}_{\partial\Omega} \ \text{ exists (in } \mathbb{C}^{n\cdot M}) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.6.31}$$

Subsequently, for each pair of indices $\alpha, \beta \in \{1, \dots, M\}$ define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big), \qquad \forall\, x \in \Omega \setminus \{x_0\}. \tag{2.6.32}$$

If we now consider $G := (G_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in $\Omega$, then from (2.6.32) and Theorem 1.2.1 we see that $G \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^{M\times M}$. Also, by design,

$$\begin{gathered} L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and} \\ G\big|^{\widetilde{\kappa}-{\rm n.t.}}_{\partial\Omega} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ (\nabla G)\big|^{\widetilde{\kappa}-{\rm n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{gathered} \tag{2.6.33}$$

while if $v := (v_{\beta\alpha})_{1\leq\alpha,\beta\leq M}$ then from (1.1.4), (1.2.19), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{gathered} (\mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}} G)(x) \leq (\mathcal{N}_{\widetilde{\kappa}} v)(x) + C_{x_0,\rho}(1 + |x|)^{1-n} \ \text{ and} \\ (\mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}}(\nabla G))(x) \leq (\mathcal{N}_{\widetilde{\kappa}}(\nabla v))(x) + C_{x_0,\rho}(1 + |x|)^{-n}, \end{gathered} \tag{2.6.34}$$

where $C_{x_0,\rho} \in (0, \infty)$ is independent of $x$. In view of (2.6.30), (2.6.34), and (2.2.335) we see that the conditions listed in (2.6.4) are presently satisfied and, in fact,

$$\mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}}(\nabla G) \in L^{p'}(\partial\Omega, w') = \big(L^p(\partial\Omega, w)\big)^*. \tag{2.6.35}$$

Suppose now that $u = (u_\beta)_{1\leq\beta\leq M}$ is a $\mathbb{C}^M$-valued function in $\Omega$ satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa u \ \text{ belongs to the space } \ L^p(\partial\Omega, w). \end{gathered} \tag{2.6.36}$$

Since (2.6.35) implies

$$\int_{\partial\Omega} \mathcal{N}_\kappa u \cdot \mathcal{N}^{\Omega\setminus K}_{\widetilde{\kappa}}(\nabla G)\, d\sigma < +\infty, \tag{2.6.37}$$

we may then invoke Theorem 2.6.1 to conclude that the Poisson integral representation formula (2.6.6) holds. In particular, this proves that whenever $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = 0$ at $\sigma$-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that $x_0$ has been arbitrarily chosen in $\Omega$, this ultimately shows such a function $u$ is actually identically zero in $\Omega$. This finishes the proof of the claim made in item *(c)*.

Next, the well-posedness claim in item *(d)* is a consequence of what we have proved in items *(a)* and *(c)*. Finally, the two optimality results formulated in item *(e)* are seen from Proposition 2.3.12, and from Example 2.3.11, respectively. □

*Remark* 2.6.3. The approach used to prove Theorem 2.6.2 relies on mapping properties and invertibility results for boundary layer potentials on Muckenhoupt weighted Lebesgue and Sobolev spaces. Given that analogous of these results are also valid on Lorentz spaces and Lorentz-based Sobolev spaces (cf. Remark 2.4.25, and the Lorentz space version of (2.3.31) obtained via real interpolation), the type of argument used to establish Theorem 2.6.2 produces similar results for the Dirichlet Problem with data in *Lorentz spaces*, i.e., for

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p,q}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \left[L^{p,q}(\partial\Omega, \sigma)\right]^M. \end{cases} \tag{2.6.38}$$

More specifically, for this boundary problem existence holds in the setting of item *(a)* of Theorem 2.6.2 whenever $p \in (1, \infty)$ and $q \in (0, \infty]$, whereas uniqueness holds in the setting of item *(c)* of Theorem 2.6.2 provided $p \in (1, \infty)$ and $q \in (0, \infty]$ (see [45, Theorem 1.4.17, p. 52] for duality results for Lorentz spaces).

In particular, corresponding to $q = \infty$, whenever $\mathfrak{A}_L^{\text{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\text{dis}} \neq \varnothing$ it follows that for each $p \in (1, \infty)$ the weak-$L^p$ Dirichlet Problem

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \left[L^{p,\infty}(\partial\Omega, \sigma)\right]^M, \end{cases} \tag{2.6.39}$$

is well posed assuming $\Omega$ is a $\delta$-SKT domain for a sufficiently small $\delta > 0$, relative to $n$, $p$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$. As in the proof of Theorem 2.6.2, uniqueness is obtained relying on the Poisson integral representation formula from Theorem 2.6.1. This requires checking that the Green function with components as in (2.6.32) is well defined and satisfies $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in L^{p',1}(\partial\Omega, \sigma)$, where $p'$ is the Hölder conjugate exponents of $p$. Once this task is accomplished, the fact that $\mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma) = \big(L^{p',1}(\partial\Omega, \sigma)\big)^*$ (cf. [45, Theorem 1.4.17(v), p. 52]) guarantees that the finiteness condition (2.6.37) presently holds, and the desired conclusion follows. In turn, the membership of $\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)$ to $L^{p',1}(\partial\Omega, \sigma)$ is seen from (2.6.34) and (2.6.29), keeping in mind that the operator $\frac{1}{2}I + K_{A^\top}$ (where $A \in \mathfrak{A}_L$ is such that $A^\top \in \mathfrak{A}_{L^\top}^{\text{dis}}$) is invertible on the Lorentz-based Sobolev space $\big[L_1^{p',1}(\partial\Omega, \sigma)\big]^M$ and, as seen from standard real interpolation inclusions, $(1 + |x|)^{-N} \in L^{p,q}(\partial\Omega, \sigma)$ whenever $N \geq n - 1$, $p \in (1, \infty)$, and $q \in (0, \infty]$.

To offer an example, assume $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain and fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$. Set $p := (n-1)/a \in (1, \infty)$. Then, if $\delta \in (0, 1)$ is sufficiently small (relative to $n$, $a$, the Ahlfors regularity constant of $\partial\Omega$, and the local John constants of $\Omega$), it follows that for each point $x_o \in \partial\Omega$ the Dirichlet Problem

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \ \text{ in } \ \Omega, \quad \mathcal{N}_\kappa u \in L^{p,\infty}(\partial\Omega, \sigma), \\ \Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = |x - x_o|^{-a} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{cases} \tag{2.6.40}$$

is uniquely solvable. In addition, there exists a constant $C(\Omega, n, \kappa, a) \in (0, \infty)$ with the property that if $u_{x_o}$ denotes the unique solution of (2.6.40) then we have the estimate $\|\mathcal{N}_\kappa u_{x_o}\|_{L^{p,\infty}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a)$ for each $x_o \in \partial\Omega$. Indeed, since the function $f_{x_o}(x) := |x - x_o|^{-a}$ for $\sigma$-a.e. point $x \in \partial\Omega$ belongs to the Lorentz space $L^{p,\infty}(\partial\Omega, \sigma)$ and $\sup_{x_o \in \partial\Omega} \|f_{x_o}\|_{L^{p,\infty}(\partial\Omega,\sigma)} < \infty$, the solvability result in Remark 2.6.3 applies. This example is particularly relevant in view of the fact that the boundary datum $|\cdot - x_o|^{-a}$ does *not* belong to any ordinary Lebesgue space on $\partial\Omega$ with respect to the "surface measure" $\sigma$. In addition, since for each $j, k \in \{1, \dots, n\}$ the boundary datum $f_{x_o}$ satisfies

$$\begin{gathered}\partial_{\tau_{jk}} f_{x_o} \in L^{q,\infty}(\partial\Omega, \sigma) \quad \text{and} \quad \sup_{x_o \in \partial\Omega} \|\partial_{\tau_{jk}} f_{x_o}\|_{L^{q,\infty}(\partial\Omega,\sigma)} < \infty, \\ \text{where} \quad q := (n-1)/(a+1) \in (1, \infty),\end{gathered} \tag{2.6.41}$$

given that, if $(\nu_i)_{1 \leq i \leq n}$ are the components of the geometric outward unit normal vector to $\Omega$,

$$(\partial_{\tau_{jk}} f_{x_o})(x) = a \frac{(x - x_o)_j \nu_k(x) - (x - x_o)_k \nu_j(x)}{|x - x_o|^{a+2}} \quad \text{for } \sigma\text{-a.e.} \quad x \in \partial\Omega, \tag{2.6.42}$$

then the analogues of (2.6.11)-(2.6.12) in the current setting imply that the unique solution $u_{x_o}$ of the Dirichlet Problem (2.6.40) enjoys additional regularity. Specifically, if $\delta \in (0, 1)$ is sufficiently small to begin with, then

$$\begin{gathered}\text{for each } x_o \in \partial\Omega, \text{ the nontangential boundary trace} \\ (\nabla u_{x_o})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{exists (in } \mathbb{R}^n) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{and} \quad \sup_{x_o \in \partial\Omega} \|\mathcal{N}_\kappa(\nabla u_{x_o})\|_{L^{q,\infty}(\partial\Omega,\sigma)} < +\infty \quad \text{if } q := \tfrac{n-1}{a+1}.\end{gathered} \tag{2.6.43}$$

In relation to the Dirichlet Problem with data in weak-Lebesgue spaces formulated in (2.6.39), we also wish to note that, in contrast to the well-posedness result in the range $p \in (1, \infty)$, uniqueness no longer holds in the limiting case when $p = 1$. Indeed, if we take $\Omega := \mathbb{R}^n_+$ and $u(x) := x_n/|x|^n$ for each $x = (x_1, \dots, x_n) \in \Omega$ then, since under the identification $\partial\Omega \equiv \mathbb{R}^{n-1}$ we have $(\mathcal{N}_\kappa u)(x') \approx |x'|^{1-n}$ uniformly for $x' \in \mathbb{R}^{n-1} \setminus \{0\}$, wee see that

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \\ \Delta u = 0 \quad \text{in} \quad \Omega, \\ \mathcal{N}_\kappa u \in L^{1,\infty}(\partial\Omega, \sigma), \\ u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0 \quad \text{at } \sigma\text{-a.e. point} \quad x \in \partial\Omega, \end{cases} \tag{2.6.44}$$

and yet, obviously, $u \not\equiv 0$ in $\Omega$.

Moving on, it is remarkable that the solvability results described in Theorem 2.6.2 are also stable under small perturbations. This is made precise in the theorem below.

**Theorem 2.6.4.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.6.2 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then the following statements are true.*

*(a)* [Existence] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ (cf.* (2.3.84)) *there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0,\delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem* (2.6.8) *formulated for $L$ is solvable.*

*(b)* [Uniqueness] *For each given system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0,\delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem* (2.6.8) *formulated for $L$ has at most one solution.*

*(c)* [Well-Posedness] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0,\delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem* (2.6.8) *formulated for $L$ is well posed.*

*Proof.* To deal with the claim made in item *(a)*, start by observing that the assumption $L_o \in \mathfrak{L}^{\rm dis}$ guarantees the existence of some $A_o \in \mathfrak{A}^{\rm dis}_{L_o}$. By Theorem 2.4.29 (used with, say, $\varepsilon := 1/4$) then ensures the existence of some small threshold $\delta_0 \in (0,1)$ along with some open neighborhood $\mathcal{O}$ of $A_o$ in $\mathfrak{A}_{\rm WE}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $A_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ then for each $\widetilde{A} \in \mathcal{O}$ the operator $\frac{1}{2}I + K_{\widetilde{A}}$ is invertible on $\big[L^p(\partial\Omega,w)\big]^M$. Pick $\varepsilon > 0$ such that $\{A \in \mathfrak{A} : \|A - A_o\| < \varepsilon\} \subseteq \mathcal{O}$, and define $\mathcal{U} := \{L \in \mathfrak{L} : \|L - L_o\| < \varepsilon\}$. Choose now an arbitrary system $L \in \mathcal{U}$. By design, there exists $A \in \mathfrak{A}_L$ and $B \in \mathfrak{A}^{\rm ant}$ such that $\|A - A_o - B\| < \varepsilon$. Hence, if we now introduce $\widetilde{A} := A - B$, then $\widetilde{A} \in \mathfrak{A}_L$ and the fact that $\|\widetilde{A} - A_o\| < \varepsilon$ implies that $\widetilde{A} \in \mathcal{O}$. In particular, the latter property permits us to conclude (in light of our earlier discussion) that the operator $\frac{1}{2}I + K_{\widetilde{A}}$ is invertible on $\big[L^p(\partial\Omega,w)\big]^M$. Given that we also have $\widetilde{A} \in \mathfrak{A}_L$, it follows (much as in the proof of Theorem 2.6.2) that the function $u : \Omega \to \mathbb{C}^M$ defined as

$$u(x) := \Big(\mathcal{D}_{\widetilde{A}}\Big(\tfrac{1}{2}I + K_{\widetilde{A}}\Big)^{-1} f\Big)(x) \quad \text{for all} \quad x \in \Omega, \tag{2.6.45}$$

is a solution of the Dirichlet Problem (2.6.8) formulated for the current system $L$. This finishes the proof of the claim made in item *(a)*.

On to the claim in item *(b)*, pick some $A_o \in \mathfrak{A}_{L_o}$ with $A_o^\top \in \mathfrak{A}^{\rm dis}_{L_o^\top}$. Running the same argument as above (with $L_o^\top$ playing the role of $L_o$, $A_o^\top$ playing the role of $A_o$, and keeping in mind that transposition is an isometry) yields some small threshold $\delta_0 \in (0,1)$ along with some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$,

$[w]_{A_p}$, $A_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ then for each system $L \in \mathcal{U}$ we may find a coefficient tensor $\widetilde{A} \in \mathfrak{A}_L$ with the property that the operator $\frac{1}{2}I + K_{(\widetilde{A})^\top}$ is invertible on the Muckenhoupt weighted Sobolev space $\big[L_1^{p'}(\partial\Omega, w')\big]^M$. This is a perturbation of the invertibility result in (2.6.26) and, once this has been established, the same argument as in the proof of item *(b)* of Theorem 2.6.2 applies and gives the conclusion we presently seek.

Finally, the claim in item *(c)* is a direct consequence of what we have proved in items *(a)*-*(b)*. □

### 2.6.2 The Regularity Problem in weighted Sobolev spaces

Traditionally, the label "Regularity Problem" is intended for a version of the Dirichlet Problem in which both the boundary datum and the solution sought are more "regular" than in the standard formulation of the Dirichlet Problem. For us here, this means that we shall now select boundary data from Muckenhoupt weighted Sobolev spaces and also demand control of the nontangential maximal operator of the gradient of the solution. This being said, the specific manner in which we have formulated the solvability result for the Dirichlet Problem in Theorem 2.6.2, in particular, having already elaborated on how extra regularity of the boundary datum affects the regularity of the solution (cf. (2.6.11)), renders the Regularity Problem a "sub-problem" of the Dirichlet Problem. As seen below, this makes light work of the treatment of the Regularity Problem.

**Theorem 2.6.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1,\infty)$ and a Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system $L$ in $\mathbb{R}^n$, consider the Regularity Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \quad \text{in} \quad \Omega, \\ \mathcal{N}_\kappa u,\, \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \big[L_1^p(\partial\Omega, w)\big]^M. \end{cases} \tag{2.6.46}$$

*The following statements are true:*

*(a)* [Existence and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $[w]_{A_p}$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $\frac{1}{2}I + K_A$ is an invertible operator on the Muckenhoupt weighted Sobolev space $\big[L_1^p(\partial\Omega, w)\big]^M$ and the function*

$$u(x) := \Big(\mathcal{D}_A\left(\tfrac{1}{2}I + K_A\right)^{-1} f\Big)(x), \qquad \forall\, x \in \Omega, \tag{2.6.47}$$

*is a solution of the Regularity Problem* (2.6.46)*. In addition,*

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p(\partial\Omega,w)]^M}\,, \tag{2.6.48}$$

*and*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|\nabla_{\tan} f\|_{[L^p(\partial\Omega,w)]^{n\cdot M}}\,. \tag{2.6.49}$$

*In particular,*

$$\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,w)} + \|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \approx \|f\|_{[L^p_1(\partial\Omega,w)]^M}\,. \tag{2.6.50}$$

*(b)* [Uniqueness] *Whenever* $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$*, there exists* $\delta_0 \in (0,1)$ *which depends only on* $n$*,* $p$*,* $[w]_{A_p}$*,* $L$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\Omega$ *is a* $\delta$-SKT *domain with* $0 < \delta < \delta_0$ *then the Regularity Problem* (2.6.46) *has at most one solution.*

*(c)* [Well-Posedness] *If* $\mathfrak{A}^{\rm dis}_{L} \neq \varnothing$ *and* $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$ *then there exists* $\delta_0 \in (0,1)$ *which depends only on* $n$*,* $p$*,* $[w]_{A_p}$*,* $L$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\Omega$ *is a* $\delta$-SKT *domain with* $0 < \delta < \delta_0$ *then the Regularity Problem* (2.6.46) *is uniquely solvable and the solution satisfies* (2.6.48)*-*(2.6.50)*.*

*(d)* [Sharpness] *If* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ *the Regularity Problem* (2.6.46) *may fail to be solvable, and if* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ *the Regularity Problem* (2.6.46) *may posses more than one solution. In particular, if either* $\mathfrak{A}^{\rm dis}_{L} = \varnothing$*, or* $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$*, then the Regularity Problem* (2.6.46) *may fail to be well posed.*

*Proof.* All claims in items *(a)-(c)* are direct consequences of Theorem 2.4.24 and Theorem 2.6.2. As regards the sharpness results formulated in item *(d)*, the fact that the Regularity Problem (2.6.46) may fail to be solvable when $\mathfrak{A}^{\rm dis}_{L} = \varnothing$ is seen from Proposition 2.3.14 and (2.3.140). Finally, that the Regularity Problem (2.6.46) for $L$ may have more than one solution if $\mathfrak{A}^{\rm dis}_{L^\top} = \varnothing$ is seen from Example 2.3.11. □

*Remark* 2.6.6. From Remark 2.6.3 we see that the Regularity Problem with data in Lorentz-based Sobolev spaces, i.e.,

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u,\, \mathcal{N}_\kappa(\nabla u) \in L^{p,q}(\partial\Omega,\sigma), \\ u\big|^{\kappa-\rm n.t.}_{\partial\Omega} = f \in \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M, \end{cases} \tag{2.6.51}$$

enjoys similar solvability and well-posedness results to those described in Theorem 2.6.5. Concretely, for this boundary problem we have existence in the setting of item *(a)* of Theorem 2.6.5 whenever $p \in (1,\infty)$ and $q \in (0,\infty]$, and we have uniqueness in the setting of item *(b)* of Theorem 2.6.5 whenever $p, q \in (1,\infty)$.

*Remark* 2.6.7. An inspection of the proof of Theorem 2.6.5 reveals that similar solvability and well-posedness results are valid in the case when the boundary data belong to the off-diagonal Muckenhoupt weighted Sobolev spaces discussed in (2.4.248)-(2.4.249). More specifically, given two integrability exponents $p_1, p_2 \in (1, \infty)$ along with two Muckenhoupt weights $w_1 \in A_{p_1}(\partial\Omega, \sigma)$ and $w_2 \in A_{p_2}(\partial\Omega, \sigma)$, the off-diagonal Regularity Problem

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa u \in L^{p_1}(\partial\Omega, w_1), \\ \mathcal{N}_\kappa(\nabla u) \in L^{p_2}(\partial\Omega, w_2), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \in \big[L^{p_1;p_2}_1(\partial\Omega, w_1; w_2)\big]^M, \end{cases} \tag{2.6.52}$$

continues to enjoy similar solvability and well-posedness results to those described in Theorem 2.6.5. Of course, this time, the a priori estimates (2.6.48)-(2.6.49) read

$$\|\mathcal{N}_\kappa u\|_{L^{p_1}(\partial\Omega, w_1)} \approx \|f\|_{[L^{p_1}(\partial\Omega, w_1)]^M}, \tag{2.6.53}$$

and

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^{p_2}(\partial\Omega, w_2)} \approx \|\nabla_{\tan} f\|_{[L^{p_2}(\partial\Omega, w_2)]^{n\cdot M}}. \tag{2.6.54}$$

*Remark* 2.6.8. Once again, in the class of systems considered in Theorem 2.6.5, the solvability, uniqueness, and well-posedness results for the Regularity Problem (2.6.46) are new even in the standard case when $\Omega = \mathbb{R}^n_+$.

As in the case of the Dirichlet Problem, it turns out that the solvability results presented in Theorem 2.6.5 are stable under small perturbations, of the sort described below.

**Theorem 2.6.9.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.6.5 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then the following statements are true.*

*(a)* [Existence] *Given any system $L_o \in \mathfrak{L}^{\rm dis}$ (cf. (2.3.84)), there exist a threshold $\delta_0 \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-SKT domain) for some $\delta \in (0, \delta_0)$ then for each system $L \in \mathcal{U}$ the Regularity Problem (2.6.46) formulated for $L$ is solvable.*

*(b)* [Uniqueness] *Given any system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist a threshold $\delta_0 \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-SKT domain) for some $\delta \in (0, \delta_0)$ then for each system $L \in \mathcal{U}$ the Regularity Problem (2.6.46) formulated for $L$ has at most one solution.*

*(c)* [Well-Posedness] *Given any system* $L_o \in \mathfrak{L}^{\rm dis}$ *with* $L_o^\top \in \mathfrak{L}^{\rm dis}$ *there exist a threshold* $\delta_0 \in (0,1)$ *and an open neighborhood* $\mathcal{U}$ *of* $L_o$ *in* $\mathfrak{L}$, *both of which depend only on* $n$, $p$, $[w]_{A_p}$, $L_o$, *the local John constants of* $\Omega$, *and the Ahlfors regularity constant of* $\partial\Omega$, *with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ *(i.e., if* $\Omega$ *is a* $\delta$-SKT *domain) for some* $\delta \in (0,\delta_0)$ *then for each system* $L \in \mathcal{U}$ *the Regularity Problem* (2.6.46) *formulated for* $L$ *is well posed.*

*Proof.* The same type of argument used in the proof of Theorem 2.6.4 continues to work in this setting. ☐

### 2.6.3 The Neumann Problem in weighted Lebesgue spaces

To set the stage, recall the definition of the conormal derivative operator from (2.3.20).

**Theorem 2.6.10.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Denote by* $\nu$ *the geometric measure theoretic outward unit normal* $\nu$ *to* $\Omega$, *abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, *and fix an aperture parameter* $\kappa > 0$. *Also, pick an integrability exponent* $p \in (1,\infty)$ *and a Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$.

*Suppose* $L$ *is a homogeneous, second-order, constant complex coefficient, weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$, *with the property that* $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. *Select* $A \in \mathfrak{A}_L$ *such that* $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ *and consider the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \quad in \quad \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial^A_\nu u = f \in \big[L^p(\partial\Omega,w)\big]^M. \end{cases} \tag{2.6.55}$$

*Then there exists* $\delta_0 \in (0,1)$ *which depends only on* $n$, $p$, $[w]_{A_p}$, $A$, *the local John constants of* $\Omega$, *and the Ahlfors regularity constant of* $\partial\Omega$ *such that if* $\Omega$ *is a* $\delta$-SKT *domain with* $0 < \delta < \delta_0$ *then* $-\frac{1}{2}I + K^{\#}_{A^\top}$ *is an invertible operator on the Muckenhoupt weighted Lebesgue space* $\big[L^p(\partial\Omega,w)\big]^M$ *and the function* $u : \Omega \to \mathbb{C}^M$ *defined as*

$$u(x) := \Big(\mathscr{S}_{\rm mod}\Big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\Big)^{-1} f\Big)(x) \quad for\ all \quad x \in \Omega, \tag{2.6.56}$$

*is a solution of the Neumann Problem* (2.6.55) *which satisfies*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,w)} \leq C\,\|f\|_{[L^p(\partial\Omega,w)]^M}\,. \tag{2.6.57}$$

*for some constant* $C \in (0,\infty)$ *independent of* $f$.

*Proof.* From the current assumptions and Theorem 2.4.24 we know that there exists some threshold $\delta_0 \in (0,1)$, whose nature is as specified in the statement of the theorem, such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the operator $-\frac{1}{2}I + K^{\#}_{A^\top}$ is invertible on $\big[L^p(\partial\Omega,w)\big]^M$. Granted this, item *(c)* in Proposition 2.3.4 then guarantees that the function (2.6.56) solves the Neumann Problem (2.6.55) and satisfies (2.6.57). ☐

*Remark* 2.6.11. For similar reasons as in past situations, a solvability result which is analogous to the one described in Theorem 2.6.10 also holds for the Neumann Problem with data in Lorentz spaces, i.e., for

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^{p,q}(\partial\Omega, \sigma), \\ \partial_\nu^A u = f \in \big[L^{p,q}(\partial\Omega, \sigma)\big]^M, \end{cases} \tag{2.6.58}$$

with $p \in (1, \infty)$ and $q \in (0, \infty]$.

*Remark* 2.6.12. In light of the remarks made in (2.3.103)-(2.3.104), Theorem 2.6.10 applies in the case of the Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$ with $n \geq 2$, assuming $\mu \neq 0$, $2\mu + \lambda \neq 0$, and $3\mu + \lambda \neq 0$. Specifically, if $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain, and $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$, then if $\delta \in (0,1)$ sufficiently small (relative to $\mu$, $\lambda$, $p$, $[w]_{A_p}$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$) the Neumann Problem (2.6.55), which in this case reads

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \\ \mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, w), \\ \partial_\nu^{A(\zeta)} u = \Big[\mu(\nabla u)^\top + \zeta(\nabla u)\Big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu + (\mu+\lambda-\zeta)(\mathrm{div}u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu = f \end{cases} \tag{2.6.59}$$

is solvable (in the manner described in (2.6.56)) for each given function $f \in \big[L^p(\partial\Omega, w)\big]^n$, provided

$$\zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}. \tag{2.6.60}$$

By way of contrast, in the two-dimensional case, Corollary 2.4.31 ensures that the Neumann Problem (2.6.59) is actually solvable (again, in the manner described in (2.6.56)) for each given function $f \in \big[L^p(\partial\Omega, w)\big]^2$, in the much larger range

$$\zeta \in \mathbb{C} \setminus \Big\{ -\mu\,,\ \tfrac{\mu(5\mu+3\lambda)}{3\mu+\lambda} \Big\}. \tag{2.6.61}$$

In particular, if we also demand that $\mu + \lambda \neq 0$ then $\zeta := \mu$ becomes an admissible value, as far as (2.6.61) is concerned, and from (2.4.304), (2.6.56) we see that the Neumann Problem (2.6.59) with $\zeta := \mu$ is solvable for each given function $f \in \big[L^p(\partial\Omega, w)\big]^2$. This is of interest since the said problem involves the so-called `traction conormal derivative`, i.e.,

$$\partial_\nu^{A(\mu)} u = \mu\Big[(\nabla u)^\top + (\nabla u)\Big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu + \lambda(\mathrm{div}u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \nu, \tag{2.6.62}$$

which is particularly relevant in physics and engineering.

It is also of interest to note that the solvability result from Theorem 2.6.10 is stable under small perturbations. Specifically, by reasoning similarly as in the proof of Theorem 2.6.4 yields the following theorem.

**Theorem 2.6.13.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.6.10 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Also, consider a system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ (cf. (2.3.84)). Then for any $A_o \in \mathfrak{A}_{L_o}$ with $A_o^\top \in \mathfrak{A}^{\rm dis}_{L_o^\top}$ there exist a threshold $\delta_0 \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of $A_o$ in $\mathfrak{A}$, both of which depend only on $n$, $p$, $[w]_{A_p}$, $A_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-SKT domain) for some $\delta \in (0, \delta_0)$ then for each coefficient tensor $A \in \mathcal{U}$ the Neumann Problem (2.6.55) formulated for the system $L_A$ (cf. (1.2.10)) and the conormal derivative associated with $A$ (cf. (2.3.20)) is actually solvable.*

### 2.6.4 The Transmission Problem in weighted Lebesgue spaces

The trademark characteristic of a Transmission problem is the fact that one now seeks two functions, defined on either side of an interface, whose traces and conormal derivatives couple in a specific fashion along the common interface.

**Theorem 2.6.14.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.6.63}$$

*Also, pick an integrability exponent $p \in (1, \infty)$, a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, an aperture parameter $\kappa > 0$, and a transmission parameter $\eta \in \mathbb{C} \setminus \{\pm 1\}$.*

*Assume $L$ is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Select $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ and consider the Transmission Problem*

$$\begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \ \text{in} \ \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega, w), \\ u^+\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = u^-\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \ \ \text{at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \partial^A_\nu u^+ - \eta \cdot \partial^A_\nu u^- = f \in \big[L^p(\partial\Omega, w)\big]^M. \end{cases} \tag{2.6.64}$$

*Then there exists $\delta_0 \in (0, 1)$ which depends only on $n$, $\eta$, $p$, $[w]_{A_p}$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $\frac{\eta+1}{2(\eta-1)}I + K^\#_{A^\top}$ is an invertible operator on the Muckenhoupt weighted Lebesgue space $\big[L^p(\partial\Omega, w)\big]^M$ and the functions $u^\pm : \Omega_\pm \to \mathbb{C}^M$ defined as*

$$u^\pm(x) := (1 - \eta)^{-1}\Big(\mathscr{S}_{\rm mod}\Big(\tfrac{\eta+1}{2(\eta-1)}I + K^\#_{A^\top}\Big)^{-1} f\Big)(x) \ \ \text{for all} \ \ x \in \Omega_\pm \tag{2.6.65}$$

*solve the Transmission Problem (2.6.64) and satisfy*

$$\|\mathcal{N}_\kappa(\nabla u^\pm)\|_{L^p(\partial\Omega,w)} \le C\,\|f\|_{[L^p(\partial\Omega,w)]^M}\,. \tag{2.6.66}$$

*for some constant $C \in (0, \infty)$ independent of $f$.*

A few clarifications are in order here. First, work in [93] shows that

$$\begin{array}{l}\Omega_- \text{ is an open set in } \mathbb{R}^n \text{ satisfying a two-sided local John condi-}\\ \text{tion, whose topological boundary is Ahlfors regular and actually}\\ \text{coincides } \partial\Omega\text{, and whose geometric measure theoretic boundary}\\ \text{agrees with that of } \Omega \text{ (hence, } \partial(\Omega_-) = \partial\Omega \text{ and } \partial_*(\Omega_-) = \partial_*\Omega\text{); in}\\ \text{addition, the geometric measure theoretic outward unit normal to}\\ \Omega_- \text{ is } -\nu \text{ at } \sigma\text{-a.e. point on } \partial\Omega.\end{array} \tag{2.6.67}$$

In particular, this makes it meaningful to talk about the nontangential boundary trace $u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$, here understood as $u^-\big|_{\partial(\Omega_-)}^{\kappa-\mathrm{n.t.}}$. Second, the existence of $u^\pm\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ at $\sigma$-a.e. point on $\partial\Omega$ is an implicit demand in the formulation of the Transmission Problem (2.6.64). Third, the conormal derivative $\partial_\nu^A u^+$ is defined as in (2.3.20), while in light of the last property in (2.6.67) we take $\partial_\nu^A u^-$ to be the opposite of (i.e.,$-1$ times) the conormal derivative operator from (2.3.20) for the domain $\Omega_-$ acting on the function $u^-$, i.e.,

$$\partial_\nu^A u^- := -\partial_{(-\nu)}^A u^-. \tag{2.6.68}$$

We now turn to the task of giving the proof of Theorem 2.6.14.

*Proof of Theorem 2.6.14.* The present assumptions and Theorem 2.4.24 (currently used for the spectral parameter $z := \frac{\eta+1}{2(\eta-1)} \in \mathbb{C}\setminus\{0\}$) ensure the existence of some threshold $\delta_0 \in (0,1)$, whose nature is as specified in the statement of the theorem, such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the operator $\frac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}$ is invertible on $\big[L^p(\partial\Omega, w)\big]^M$. In particular, it is meaningful to define $u^\pm$ as in (2.6.65). In view of (2.6.67) and item *(c)* in Proposition 2.3.4 (used both for $\Omega_+$ and $\Omega_-$), these functions satisfy the first three conditions in (2.6.64), the estimates claimed in (2.6.66), and we have (keeping (2.6.68) and (2.6.67) in mind)

$$\begin{aligned}\partial_\nu^A u^+ - \eta\cdot\partial_\nu^A u^- &= (1-\eta)^{-1}\big(-\tfrac12 I + K^{\#}_{A^\top}\big)\Big(\tfrac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}\Big)^{-1} f\\ &\qquad - \eta(1-\eta)^{-1}(-1)\big(-\tfrac12 I - K^{\#}_{A^\top}\big)\Big(\tfrac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}\Big)^{-1} f\\ &= \Big(\tfrac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}\Big)\Big(\tfrac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}\Big)^{-1} f\\ &= f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega.\end{aligned} \tag{2.6.69}$$

Finally, thanks to (2.3.17)-(2.3.19), (2.2.337), and (2.6.67), we see that

$$\begin{aligned}u^+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = (1-\eta)^{-1} S_{\mathrm{mod}}\Big(\tfrac{\eta+1}{2(\eta-1)}I + K^{\#}_{A^\top}\Big)^{-1} f = u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\\ \text{at } \sigma\text{-a.e. point on } \ \partial\Omega.\end{aligned} \tag{2.6.70}$$

Hence, the functions $u^\pm$ defined as in (2.6.65) solve the Transmission Problem (2.6.64) and satisfy the estimates demanded in (2.6.66). $\square$

*Remark* 2.6.15. Once again, for familiar reasons, a similar solvability result to the one established in Theorem 2.6.14 turns out to be true for the Transmission Problem with data in Lorentz spaces, i.e., for

$$\begin{cases} u^{\pm} \in \big[\mathscr{C}^{\infty}(\Omega_{\pm})\big]^{M}, \\ Lu^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}_{\kappa}(\nabla u^{\pm}) \in L^{p,q}(\partial\Omega, \sigma), \\ u^{+}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = u^{-}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \partial_{\nu}^{A} u^{+} - \eta \cdot \partial_{\nu}^{A} u^{-} = f \in \big[L^{p,q}(\partial\Omega, \sigma)\big]^{M}, \end{cases} \tag{2.6.71}$$

with $p \in (1, \infty)$ and $q \in (0, \infty]$.

*Remark* 2.6.16. Thanks to (2.3.103)-(2.3.104), Theorem 2.6.14 is applicable to the Lamé system $L_{\mu,\lambda} = \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$ in $\mathbb{R}^n$ with $n \geq 2$, assuming $\mu \neq 0$, $2\mu + \lambda \neq 0$, $3\mu + \lambda \neq 0$, provided we work with the coefficient tensor $A(\zeta)$ defined as in (2.3.101) for the choice $\zeta = \frac{\mu(\mu+\lambda)}{3\mu+\lambda}$. In addition, when $n = 2$, we may rely on the invertibility result from Theorem 2.4.30 (and duality) to conclude that the transmission boundary problem for the two-dimensional Lamé system in sufficiently flat $\delta$-SKT domains in the plane is solvable when formulated in a similar fashion to (2.6.64) with $A := A(\zeta)$, for a much larger range of $\zeta$'s, namely

$$\zeta \in \mathbb{C} \setminus \Big\{ \pm \frac{\eta + 1}{\eta - 1}\Big[\frac{2\mu(2\mu + \lambda)}{3\mu + \lambda}\Big] + \frac{\mu(\mu + \lambda)}{3\mu + \lambda} \Big\}. \tag{2.6.72}$$

*Remark* 2.6.17. In the two-dimensional setting, for $L = \Delta$ the Laplacian and $\Omega$ an infinite sector in the plane, counterexamples to the well-posedness of the Transmission Problem (2.6.64) for certain values of $p$ (related to the aperture of $\Omega$ and the transmission parameter appearing in the formulation of the problem) have been given in [97].

Finally, it is possible to enhance the solvability result from Theorem 2.6.14 via perturbations, and our next theorem elaborates on this aspect.

**Theorem 2.6.18.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.6.14 and, as before, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ and a transmission parameter $\eta \in \mathbb{C}\setminus\{\pm 1\}$. Also, consider a system $L_o \in \mathfrak{L}$ with $L_o^{\top} \in \mathfrak{L}^{\mathrm{dis}}$ (cf. (2.3.84)). Then for any $A_o \in \mathfrak{A}_{L_o}$ with $A_o^{\top} \in \mathfrak{A}^{\mathrm{dis}}_{L_o^{\top}}$ there exist a threshold $\delta_0 \in (0, 1)$ and an open neighborhood $\mathcal{U}$ of $A_o$ in $\mathfrak{A}$, both of which depend only on $n$, $\eta$, $p$, $[w]_{A_p}$, $A_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-SKT domain) for some $\delta \in (0, \delta_0)$ then for each coefficient tensor $A \in \mathcal{U}$ the Transmission Problem (2.6.64) formulated for the system $L_A$ (cf. (1.2.10)) and the conormal derivative associated with $A$ (cf. (2.3.20)) is actually solvable.*

*Proof.* This is seen by reasoning as in the proofs of Theorem 2.6.4 and Theorem 2.6.14. □

## 2.7 Singular integrals and boundary problems in Morrey and block spaces

The spaces which bear the name of Morrey have been introduced by C. Morrey in 1930's in relation to regularity problems for solutions to partial differential equations in the Euclidean setting. Membership of a function to a Morrey space amounts to a bound on the size of the $L^p$-integral average of the said function over an arbitrary ball in terms of a fixed power of its radius. Since these are all measure-metric considerations, this brand of space naturally adapts to the more general setting of spaces of homogeneous type. Here we are concerned with the space of Morrey spaces when the ambient is the boundary of a uniformly rectifiable domain $\Omega \subseteq \mathbb{R}^n$. We make use of the Calderón-Zygmund theory for singular integral operators acting on Morrey spaces in such a setting as a platform that allows us to build in the direction of solving boundary value problems for weakly elliptic systems in $\delta$-SKT domains with boundary data in Morrey spaces (and their pre-duals).

### 2.7.1 Boundary layer potentials on Morrey and block spaces

The material in this section closely follows [93]. We begin by discussing the scale of Morrey spaces on Ahlfors regular sets. To set the stage, assume $\Sigma \subseteq \mathbb{R}^n$ (where, as in the past, $n \in \mathbb{N}$ with $n \geq 2$) is a closed Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Given $p \in (0,\infty)$ and $\lambda \in (0, n-1)$, we then define the `Morrey space` $M^{p,\lambda}(\Sigma,\sigma)$ as

$$M^{p,\lambda}(\Sigma,\sigma) := \Big\{ f : \Sigma \to \mathbb{C} : \ f \text{ is } \sigma\text{-measurable and } \ \|f\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty \Big\} \tag{2.7.1}$$

where, for each $\sigma$-measurable function $f$ on $\Sigma$, we have set

$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ 0<R<2\,\mathrm{diam}(\Sigma)}} \Big\{ R^{\frac{n-1-\lambda}{p}} \Big( ⨍_{\Sigma\cap B(x,R)} |f|^p \, d\sigma \Big)^{\frac{1}{p}} \Big\}. \tag{2.7.2}$$

The space $M^{p,\lambda}(\Sigma,\sigma)$ is complete, hence Banach (though not separable) when equipped with the norm (2.7.2), and (cf. [93] for a proof)

$$\begin{gathered} M^{p,\lambda}(\Sigma,\sigma) \hookrightarrow L^p_{\mathrm{loc}}(\Sigma,\sigma) \cap L^1\Big(\Sigma\,,\, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \\ \text{whenever } p \in [1,\infty) \text{ and } \lambda \in (0, n-1). \end{gathered} \tag{2.7.3}$$

As may be seen from (2.7.1)-(2.7.2), we also have

$$L^s(\Sigma,\sigma) \hookrightarrow M^{p,\lambda}(\Sigma,\sigma) \ \text{ continuously, with } \ s := \tfrac{p(n-1)}{n-1-\lambda} \in (p,\infty). \tag{2.7.4}$$

In particular, there exists some $C \in (0,\infty)$ which depends only on $n, p, \lambda$, and the Ahlfors regularity constant of $\Sigma$, with the property that for each $\sigma$-measurable set $E \subseteq \Sigma$ we have

$$\|\mathbf{1}_E\|_{M^{p,\lambda}(\Sigma,\sigma)} \leq C\|\mathbf{1}_E\|_{L^s(\Sigma,\sigma)} = C \cdot \sigma(E)^{(n-1-\lambda)/[p(n-1)]}. \tag{2.7.5}$$

As a consequence, $\mathbf{1}_E$ belongs to $M^{p,\lambda}(\Sigma,\sigma)$ whenever $E \subseteq \Sigma$ is a $\sigma$-measurable set with $\sigma(E) < +\infty$. Other examples of functions belonging to Morrey spaces are presented below (see [93]).

**Example 2.7.1.** Let $\Sigma, \sigma$ be as above, and for each fixed point $x_o \in \Sigma$ consider the function $f_{x_o} : \Sigma \to \mathbb{R}$ defined for each $x \in \Sigma \setminus \{x_o\}$ as $f_{x_o}(x) := |x - x_o|^{-(n-1-\lambda)/p}$. Then each $f_{x_o}$ belongs to the Morrey space $M^{p,\lambda}(\Sigma, \sigma)$ and, in fact,

$$\sup_{x_o \in \Sigma} \|f_{x_o}\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty. \tag{2.7.6}$$

This being said, each $f_{x_o}$ fails to be in $L^s(\Sigma, \sigma)$ with $s := \frac{p(n-1)}{n-1-\lambda}$, so the inclusion in (2.7.4) is strict.

In view of (2.7.4) it is of interest to define the space

$$\mathring{M}^{p,\lambda}(\Sigma, \sigma) := \text{the closure of } L^s(\Sigma, \sigma) \text{ with } s := \tfrac{p(n-1)}{n-1-\lambda} \text{ in } M^{p,\lambda}(\Sigma, \sigma). \tag{2.7.7}$$

Hence, by design,

$$\begin{array}{l} \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ is a closed linear subspace of } M^{p,\lambda}(\Sigma,\sigma) \text{ with} \\ \text{the property that } L^s(\Sigma,\sigma) \hookrightarrow \mathring{M}^{p,\lambda}(\Sigma,\sigma) \text{ continuously} \\ \text{and densely.} \end{array} \tag{2.7.8}$$

Thus, when equipped with the norm inherited from the larger ambient $M^{p,\lambda}(\Sigma, \sigma)$, the space $\mathring{M}^{p,\lambda}(\Sigma, \sigma)$ is complete (hence Banach). As a consequence of (2.7.8) and (2.2.291) we also see that

$$\text{the space } \mathring{M}^{p,\lambda}(\Sigma, \sigma) \text{ is separable.} \tag{2.7.9}$$

As noted in [93],

$$\begin{array}{l} \text{the operator of pointwise multiplication by a function } b \in \\ L^\infty(\Sigma,\sigma) \text{ is a bounded mapping from the space } \mathring{M}^{p,\lambda}(\Sigma,\sigma) \\ \text{into itself, with operator norm} \leq \|b\|_{L^\infty(\Sigma,\sigma)}. \end{array} \tag{2.7.10}$$

and

$$\begin{array}{l} \text{if } f, g : \Sigma \to \mathbb{C} \text{ are two } \sigma\text{-measurable functions with} \\ \text{the property that } |g| \leq |f| \text{ at } \sigma\text{-a.e. point on } \Sigma \text{ and} \\ f \in \mathring{M}^{p,\lambda}(\Sigma,\sigma), \text{ then } g \text{ also belongs to } \mathring{M}^{p,\lambda}(\Sigma,\sigma). \end{array} \tag{2.7.11}$$

In relation to the space introduced in (2.7.7), we also wish to remark that since $\mathrm{Lip}_{\mathrm{comp}}(\Sigma)$ (the space of Lipschitz functions with compact support on $\Sigma$) is dense in $L^s(\Sigma, \sigma)$ and since, according to (2.7.4), the latter space embeds continuously into $M^{p,\lambda}(\Sigma, \sigma)$, we have

$$\mathring{M}^{p,\lambda}(\Sigma, \sigma) = \text{the closure of } \mathrm{Lip}_{\mathrm{comp}}(\Sigma) \text{ in } M^{p,\lambda}(\Sigma, \sigma). \tag{2.7.12}$$

An immediate corollary of the latter description of the space $\mathring{M}^{p,\lambda}(\Sigma, \sigma)$ worth mentioning is that functions $f$ belonging to $\mathring{M}^{p,\lambda}(\Sigma, \sigma)$ enjoy the "vanishing" property

$$\lim_{\rho \to 0^+} \sup_{\substack{x \in \Sigma \text{ and} \\ R \in (0,\rho)}} \left\{ R^{\frac{n-1-\lambda}{p}} \Big( ⨍_{\Sigma \cap B(x,R)} |f|^p \, d\sigma \Big)^{\frac{1}{p}} \right\} = 0. \tag{2.7.13}$$

As such, it is natural to refer to $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as being a `vanishing Morrey space`.

The topic addressed next pertains to the pre-duals of Morrey spaces, and the duals of vanishing Morrey spaces. Continue to assume that $\Sigma\subseteq\mathbb{R}^n$ is a closed Ahlfors regular set and define $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. To set the stage, given an integrability exponent $q\in(1,\infty)$ and a parameter $\lambda\in(0,n-1)$, a function $b\in L^q(\Sigma,\sigma)$ is said to be a $\mathcal{B}^{q,\lambda}$-`block` on $\Sigma$ (or, simply, a block) provided there exist some point $x_o\in\Sigma$ and some radius $R\in\big(0,2\,\mathrm{diam}\,(\Sigma)\big)$ such that

$$\operatorname{supp} b\subseteq B(x_o,R)\cap\Sigma \quad\text{and}\quad \|b\|_{L^q(\Sigma,\sigma)}\leq R^{\lambda\left(\frac{1}{q}-1\right)}. \tag{2.7.14}$$

With $r:=\frac{q(n-1)}{n-1+\lambda(q-1)}\in(1,q)$ we then define the `block space`

$$\begin{aligned}\mathcal{B}^{q,\lambda}(\Sigma,\sigma):=\Big\{f\in L^r(\Sigma,\sigma):&\text{there exist a numerical sequence }\ \{\lambda_j\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})\\ &\text{and a family }\{b_j\}_{j\in\mathbb{N}}\text{ of }\mathcal{B}^{q,\lambda}\text{-blocks on }\Sigma\text{ with}\\ &f=\sum_{j=1}^{\infty}\lambda_j b_j\ \text{ in }\ L^r(\Sigma,\sigma)\Big\},\end{aligned} \tag{2.7.15}$$

and for each $f\in\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ define

$$\begin{aligned}\|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}:=\inf\Big\{\sum_{j=1}^{\infty}|\lambda_j|:\ f=\sum_{j=1}^{\infty}\lambda_j b_j\ \text{ in }\ L^r(\Sigma,\sigma)\ \text{ with}\\ \{\lambda_j\}_{j\in\mathbb{N}}\in\ell^1(\mathbb{N})\ \text{ and each }b_j\text{ a }\mathcal{B}^{q,\lambda}\text{-block on }\Sigma\Big\}.\end{aligned} \tag{2.7.16}$$

Work in [93] gives that

$$\begin{aligned}&\Big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\,,\,\|\cdot\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\Big)\ \text{ is a separable Banach space,}\\ &\text{and }\ \mathcal{B}^{q,\lambda}(\Sigma,\sigma)\hookrightarrow L^r(\Sigma,\sigma)\ \text{ with }\ r:=\tfrac{q(n-1)}{n-1+\lambda(q-1)}\in(1,q).\end{aligned} \tag{2.7.17}$$

and

the operator of pointwise multiplication by a function $b\in L^\infty(\Sigma,\sigma)$ is a bounded mapping from the space $\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ into itself, with operator norm $\leq\|b\|_{L^\infty(\Sigma,\sigma)}$. (2.7.18)

Note that the latter property further implies that

if $f,g:\Sigma\longrightarrow\mathbb{C}$ are two $\sigma$-measurable functions such that $|g|\leq|f|$ at $\sigma$-a.e. point on $\Sigma$ and $f\in\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$, then we have $g\in\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ and $\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\leq\|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}$. (2.7.19)

Examples of functions in the block space (2.7.15) may be produced using the following result from [93].

**Proposition 2.7.2.** *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix $q\in(1,\infty)$ along with $\lambda\in(0,n-1)$. Then for each $a>\lambda$ one has the continuous and dense embedding*

$$L^q\Big(\Sigma\,,\,(1+|x|)^{a(q-1)}\sigma(x)\Big)\hookrightarrow\mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{2.7.20}$$

*In particular,*

$$\begin{aligned}&\text{if } N > \tfrac{\lambda(q-1)+n-1}{q} \text{ and } f_N(x) := (1+|x|)^{-N} \text{ for } x \in \Sigma,\\ &\text{then the function } f_N \text{ belongs to the space } \mathcal{B}^{q,\lambda}(\Sigma,\sigma).\end{aligned} \tag{2.7.21}$$

Our primary interest in the space (2.7.15) stems from the fact that this turns out to be the pre-dual of a Morrey space. In turn, vanishing Morrey spaces are pre-duals of block spaces. Specifically, we have the following result proved in [93].

**Proposition 2.7.3.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two integrability exponents $p, q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. Then there exists $C \in (0,\infty)$ which depends only on the Ahlfors regularity constant of $\Sigma$, $n$, $p$, and $\lambda$, with the property that*

$$\begin{aligned}&\int_\Sigma |f||g|\,d\sigma \leq C\|f\|_{M^{p,\lambda}(\Sigma,\sigma)}\|g\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\\ &\text{for all } \ f \in M^{p,\lambda}(\Sigma,\sigma) \ \text{ and } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma).\end{aligned} \tag{2.7.22}$$

*In addition, the mapping*

$$\begin{aligned}&M^{p,\lambda}(\Sigma,\sigma) \ni f \longmapsto \Lambda_f \in \big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by}\\ &\Lambda_f(g) := \int_\Sigma fg\,d\sigma \ \text{ for each } \ g \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma)\end{aligned} \tag{2.7.23}$$

*is a well-defined, linear, bounded isomorphism, with bounded inverse. Simply put, the integral paring yields the quantitative identification*

$$\big(\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\big)^* = M^{p,\lambda}(\Sigma,\sigma). \tag{2.7.24}$$

*Furthermore, regarding $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$ as a Banach space equipped with the norm inherited from $M^{p,\lambda}(\Sigma,\sigma)$, the mapping*

$$\begin{aligned}&\mathcal{B}^{q,\lambda}(\Sigma,\sigma) \ni g \longmapsto \Lambda_g \in \big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* \ \text{ given by}\\ &\Lambda_g(f) := \int_\Sigma fg\,d\sigma \ \text{ for each } \ f \in \mathring{M}^{p,\lambda}(\Sigma,\sigma)\end{aligned} \tag{2.7.25}$$

*is a well-defined, linear, bounded isomorphism, with bounded inverse. As such, the integral paring yields the identification*

$$\big(\mathring{M}^{p,\lambda}(\Sigma,\sigma)\big)^* = \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{2.7.26}$$

In the setting of Proposition 2.7.3, from (2.7.24), (2.7.17), and the Sequential Banach-Alaoglu Theorem we conclude that any bounded sequence in $M^{p,\lambda}(\Sigma,\sigma)$ has a subsequence which is weak-$*$ convergent. A result in this spirit in which a stronger conclusion is reached, provided one assumes more than mere boundedness for the said sequence, has been proved in [93].

**Proposition 2.7.4.** *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix two integrability exponents $p, q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, along with a parameter $\lambda \in (0, n-1)$. In this setting, suppose $\{f_j\}_{j\in\mathbb{N}} \subseteq M^{p,\lambda}(\Sigma,\sigma)$ is a sequence of functions with the property that*

$$\begin{gathered} f(x) := \lim_{j\to\infty} f_j(x) \quad \text{exists for } \sigma\text{-a.e. } x \in \Sigma, \quad \text{and} \\ \text{there exists some } g \in M^{p,\lambda}(\Sigma,\sigma) \text{ such that for each} \\ j \in \mathbb{N} \quad \text{one has} \quad |f_j(x)| \leq |g(x)| \quad \text{for } \sigma\text{-a.e. } x \in \Sigma. \end{gathered} \tag{2.7.27}$$

*Then $f \in M^{p,\lambda}(\Sigma,\sigma)$ and $f_j \to f$ as $j \to \infty$ weak-$*$ in $M^{p,\lambda}(\Sigma,\sigma)$, i.e.,*

$$\lim_{j\to\infty} \int_\Sigma f_j h\, d\sigma = \int_\Sigma f h\, d\sigma \quad \text{for each} \quad h \in \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{2.7.28}$$

Remarkably, certain types of estimates on Muckenhoupt weighted Lebesgue space imply estimates on Morrey spaces. Here is a basic result of this flavor from [93] (cf. also [36] for related results in the Euclidean setting).

**Proposition 2.7.5.** *Let $\Sigma$ be a closed Ahlfors regular set in $\mathbb{R}^n$ and define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix an integrability exponent $p \in (1,\infty)$ along with some parameter $\lambda \in (0, n-1)$, and assume that some assignment (not necessarily linear) $f \mapsto \Theta(f)$, mapping functions $f \in L^1(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}})$ into $\sigma$-measurable functions on $\Sigma$ has been given, with the property that*

$$\begin{gathered} \text{for each Muckenhoupt weight } w \in A_1(\Sigma,\sigma) \text{ one may find} \\ \text{some constant } C_w = C([w]_{A_1}) \in (0,\infty) \text{ (which depends in a} \\ \text{non-decreasing fashion on the characteristic } [w]_{A_1}) \text{ such that} \\ \|\Theta(f)\|_{L^p(\Sigma,w)} \leq C_w \|f\|_{L^p(\Sigma,w)} \text{ for each function } f \text{ belonging to} \\ \text{the weighted Lebesgue space } L^p(\Sigma,w). \end{gathered} \tag{2.7.29}$$

*Then there exist two constants $C_{\Sigma,p} \in (0,\infty)$ (which depends only on the Ahlfors regularity constant of $\Sigma$ and $p$), and $W_{n,\lambda} \in (0,\infty)$ (which depends only on $n$ and $\lambda$) with the property that if one abbreviates*

$$C_\Theta := C_{\Sigma,p} \cdot \sup_{\substack{w\in A_1(\Sigma,\sigma) \\ [w]_{A_1} \leq W_{n,\lambda}}} C_w \tag{2.7.30}$$

*then*

$$\begin{gathered} \|\Theta(f)\|_{M^{p,\lambda}(\Sigma,\sigma)} \leq C_\Theta \|f\|_{M^{p,\lambda}(\Sigma,\sigma)} \text{ for each } f \\ \text{belonging to the Morrey space } M^{p,\lambda}(\Sigma,\sigma). \end{gathered} \tag{2.7.31}$$

Based on Proposition 2.7.5, Proposition 2.3.3, Proposition 2.7.3 (as well as Coltar's inequality and boundedness results for the Hardy-Littlewood maximal operator on Morrey and block spaces), the following result has been established in [93].

**Proposition 2.7.6.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $\partial\Omega$ is a* UR *set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is odd and positive homogeneous of* *degree* $1-n$. *Also, fix two integrability exponents* $p,q\in(1,\infty)$ *with* $1/p+1/q=1$, *along with a parameter* $\lambda\in(0,n-1)$, *and pick an aperture parameter* $\kappa>0$. *In this setting, for each* $f$ *belonging to either* $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *define*

$$T_\varepsilon f(x):=\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y)\quad\text{for each}\quad x\in\partial\Omega, \tag{2.7.32}$$

$$T_*f(x):=\sup_{\varepsilon>0}|T_\varepsilon f(x)|\quad\text{for each}\quad x\in\partial\Omega, \tag{2.7.33}$$

$$Tf(x):=\lim_{\varepsilon\to0^+}T_\varepsilon f(x)\quad\text{for }\sigma\text{-a.e.}\quad x\in\partial\Omega, \tag{2.7.34}$$

$$\mathcal{T}f(x):=\int_{\partial\Omega}k(x-y)f(y)d\sigma(y)\quad\text{for each}\quad x\in\Omega. \tag{2.7.35}$$

*Then there exists a constant* $C\in(0,\infty)$ *which depends exclusively on* $n$, $p$, $\lambda$, *and the* UR *constants of* $\partial\Omega$ *with the property that for each* $f\in M^{p,\lambda}(\partial\Omega,\sigma)$ *one has*

$$\|T_*f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{2.7.36}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{2.7.37}$$

*for each* $f\in\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ *one has*

$$\|T_*f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{2.7.38}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}, \tag{2.7.39}$$

*and for each* $f\in\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *one has*

$$\|T_*f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}, \tag{2.7.40}$$

$$\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{2.7.41}$$

*Furthermore, for each function* $f$ *belonging to either* $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, *or* $M^{p,\lambda}(\partial\Omega,\sigma)$ *the limit defining* $Tf(x)$ *in* (2.7.34) *exists at* $\sigma$*-a.e.* $x\in\partial\Omega$ *and the operators*

$$T:M^{p,\lambda}(\partial\Omega,\sigma)\longrightarrow M^{p,\lambda}(\partial\Omega,\sigma), \tag{2.7.42}$$

$$T:\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma), \tag{2.7.43}$$

$$T:\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \tag{2.7.44}$$

*are well defined, linear, and bounded. In addition,*

*the (real) transposed of the operator* (2.7.43) *is the operator* $-T$ *with* $T$ *as in* (2.7.44), *and the (real) transposed of the operator* (2.7.44) *is the operator* $-T$ *with* $T$ *as in* (2.7.42). (2.7.45)

In particular, the results from Proposition 2.7.6 are directly applicable to the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ defined as in (2.4.236) on the boundary of a UR domain $\Omega\subseteq\mathbb{R}^n$. This proves that, in such a setting, for each $p,q\in(1,\infty)$ and $\lambda\in(0,n-1)$

the operators $\{R_j\}_{1\leq j\leq n}$ are well defined, linear, and bounded on $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, and $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$. (2.7.46)

In concert with Theorem 2.4.14, (2.7.7), and duality (cf. Proposition 2.7.3), Proposition 2.7.5 also yields the following version of the commutator theorem from [27], in Morrey and block spaces.

**Theorem 2.7.7.** *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed Ahlfors regular set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p_0\in(1,\infty)$ along with some non-decreasing function $\Phi:(0,\infty)\to(0,\infty)$ and let $T$ be a linear operator which is bounded on $L^{p_0}(\Sigma,w)$ for every $w\in A_{p_0}(\Sigma,\sigma)$, with operator norm $\leq\Phi([w]_{A_{p_0}})$.*

*Then for each integrability exponent $p\in(1,\infty)$ and each parameter $\lambda\in(0,n-1)$ the operator $T$ induces well-defined, linear, and bounded mappings in the contexts*

$$T: M^{p,\lambda}(\Sigma,\sigma)\longrightarrow M^{p,\lambda}(\Sigma,\sigma), \tag{2.7.47}$$

$$T: \mathring{M}^{p,\lambda}(\Sigma,\sigma)\longrightarrow \mathring{M}^{p,\lambda}(\Sigma,\sigma). \tag{2.7.48}$$

*In addition, given any $p\in(1,\infty)$ along with some $\lambda\in(0,n-1)$, there exist two constants, $C_1=C_1(\Sigma,n,p_0,p,\lambda)\in(0,\infty)$ and $C_2=C_2(\Sigma,n,p_0,p)\in(0,\infty)$, with the property that for every complex-valued function $b\in L^\infty(\Sigma,\sigma)$ one has*

$$\begin{aligned}\|[M_b,T]\|_{\mathring{M}^{p,\lambda}(\Sigma,\sigma)\to\mathring{M}^{p,\lambda}(\Sigma,\sigma)} &\leq \|[M_b,T]\|_{M^{p,\lambda}(\Sigma,\sigma)\to M^{p,\lambda}(\Sigma,\sigma)} \\ &\leq C_1\Phi(C_2)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,,\end{aligned} \tag{2.7.49}$$

*where $[M_b,T]:=bT(\cdot)-T(b\,\cdot)$ is the commutator of $T$ (considered either as in (2.7.47), or as in (2.7.48)) and the operator $M_b$ of pointwise multiplication (either on $M^{p,\lambda}(\Sigma,\sigma)$, or on $\mathring{M}^{p,\lambda}(\Sigma,\sigma)$) by the function $b$.*

*Moreover, if $T^\top$ is the (real) transposed of the original operator $T$, then for each $q\in(1,\infty)$ and $\lambda\in(0,n-1)$ the operator $T^\top$ induces a well-defined, linear, and bounded mapping*

$$T^\top: \mathcal{B}^{q,\lambda}(\Sigma,\sigma)\longrightarrow \mathcal{B}^{q,\lambda}(\Sigma,\sigma). \tag{2.7.50}$$

*Finally, for each $q\in(1,\infty)$ and $\lambda\in(0,n-1)$ there exist two positive finite constants, $C_1=C_1(\Sigma,n,p_0,q,\lambda)$ and $C_2=C_2(\Sigma,n,p_0,q)$, with the property that for every complex-valued function $b\in L^\infty(\Sigma,\sigma)$ one has*

$$\Big\|[M_b,T^\top]\Big\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)\to\mathcal{B}^{q,\lambda}(\Sigma,\sigma)}\leq C_1\Phi(C_2)\,\|b\|_{\mathrm{BMO}(\Sigma,\sigma)}\,. \tag{2.7.51}$$

For example, if $\Omega\subseteq\mathbb{R}^n$ is a UR domain then, for each complex-valued function $k\in\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ (where $N=N(n)\in\mathbb{N}$ is sufficiently large) which is odd and positive homogeneous of degree $1-n$, Theorem 2.7.7 applies with $\Sigma := \partial\Omega$ and $T$ as in (2.7.34). In such a scenario, from (2.7.51) and (2.7.45) we see that for each $b \in L^\infty(\partial\Omega,\sigma)$, $q \in (1,\infty)$, and $\lambda \in (0, n-1)$, the following commutator estimate holds:

$$\|[M_b,T]\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|b\|_{\mathrm{BMO}(\partial\Omega,\sigma)}, \tag{2.7.52}$$

where $C \in (0,\infty)$ depends exclusively on $n$, $q$, $\lambda$, and the UR character of $\partial\Omega$.

Following [93], we may also consider `Morrey-based Sobolev spaces` on the boundaries of Ahlfors regular domains. Specifically, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then for each $p \in (1,\infty)$ and $\lambda \in (0, n-1)$ we define

$$\begin{aligned} M_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in M^{p,\lambda}(\partial\Omega,\sigma) \cap L^1_{1,\mathrm{loc}}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega,\sigma) \\ \text{for each } \ j,k \in \{1,\dots,n\}\Big\}, \end{aligned} \tag{2.7.53}$$

equipped with the natural norm

$$M_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^n \|\partial_{\tau_{jk}} f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{2.7.54}$$

A significant closed subspace of $M_1^{p,\lambda}(\partial\Omega,\sigma)$ is the `vanishing Morrey-based Sobolev space`

$$\begin{aligned} \mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \\ \text{one has } \ \partial_{\tau_{jk}} f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\Big\}. \end{aligned} \tag{2.7.55}$$

In the same vein, for each $q \in (1,\infty)$ let us also define the `block-based Sobolev space`

$$\begin{aligned} \mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) : \text{ for each } \ j,k \in \{1,\dots,n\} \\ \text{one has } \ \partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\Big\}, \end{aligned} \tag{2.7.56}$$

and endowed with the norm

$$\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^n \|\partial_{\tau_{jk}} f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{2.7.57}$$

It has been noted in [93] that by combining Proposition 2.7.5 with Proposition 2.3.4 (while also keeping in mind Lemma 2.4.19, (2.7.3), (2.7.8), (2.7.17), (2.7.18), (2.7.52)) one obtains the following result pertaining to the action of boundary layer potentials associated with weakly elliptic second-order systems in UR domains, on the scales of spaces discussed earlier.

**Theorem 2.7.8.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Let $L$ be a homogeneous, weakly elliptic, constant complex coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$) and recall the modified single layer potential operator $\mathscr{S}_{\mathrm{mod}}$ associated with $L$*

*and $\Omega$ as in (2.3.14). Also, pick a coefficient tensor $A \in \mathfrak{A}_L$ and consider the double layer potential operators $\mathcal{D}_A$, $K_A$, $K_A^{\#}$ associated with the coefficient tensor $A$ and the set $\Omega$ as in (2.3.2), (2.3.4), and (2.3.5), respectively. Finally, select $p \in (1, \infty)$ along with $\lambda \in (0, n-1)$ and some aperture parameter $\kappa > 0$.*

*Then the operators*

$$K_A, K_A^{\#} : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \tag{2.7.58}$$

*are well defined, linear, and bounded. Additionally, the operators $K_A, K_A^{\#}$ in the context of (2.7.58) depend continuously on the underlying coefficient tensor $A$. Specifically, with the piece of notation introduced in (1.2.16), the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A \in \mathrm{Bd}\Big(\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M\Big), \tag{2.7.59}$$

$$\mathfrak{A}_{\rm WE} \ni A \longmapsto K_A^{\#} \in \mathrm{Bd}\Big(\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M\Big). \tag{2.7.60}$$

*Furthermore, there exists a constant $C \in (0, \infty)$, depending only on the* UR *character of $\partial\Omega$, $L$, $n$, $\kappa$, $p$, and $\lambda$, with the property that*

$$\begin{gathered} \|\mathcal{N}_\kappa(\mathcal{D}_A f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla(\mathscr{S}_{\rm mod} f))\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \\ \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M. \end{gathered} \tag{2.7.61}$$

*Moreover, for each given function $f$ in the Morrey space $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ the following nontangential boundary trace formulas hold (with $I$ denoting the identity operator)*

$$\mathcal{D}_A f\Big|_{\partial\Omega}^{\kappa-\rm n.t.} = \big(\tfrac{1}{2}I + K_A\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.7.62}$$

*and*

$$\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K_{A^\top}^{\#}\big)f \quad \text{at } \sigma\text{-a.e. point in } \partial\Omega, \tag{2.7.63}$$

*where $K_{A^\top}^{\#}$ is the singular integral operator associated as in (2.3.5) with the set $\Omega$ and the transposed coefficient tensor $A^\top$.*

*In addition, for each function $f$ in the Morrey-based Sobolev space $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ it follows that*

$$\begin{gathered} \text{the nontangential boundary trace } (\partial_\ell \mathcal{D}_A f)\big|_{\partial\Omega}^{\kappa-\rm n.t.} \text{ exists (in} \\ \mathbb{C}^M) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each } \ell \in \{1, \dots, n\}, \end{gathered} \tag{2.7.64}$$

*and there exits some finite constant $C > 0$, depending only on $\partial\Omega$, $L$, $n$, $\kappa$, $p$, $\lambda$, such that*

$$\|\mathcal{N}_\kappa(\mathcal{D}_A f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla\mathcal{D}_A f)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{2.7.65}$$

*In fact, similar results are valid with the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ replaced throughout by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ (defined as in (2.7.7) with $\Sigma := \partial\Omega$), or by the block space $\mathcal{B}^{q,\lambda}(\Sigma, \sigma)$ with $q \in (1, \infty)$ (defined as in (2.7.15)-(2.7.16) with $\Sigma := \partial\Omega$).*

*Finally, the operators*

$$K_A : \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{2.7.66}$$

$$K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{2.7.67}$$

*are well defined, linear, bounded and, for each $q \in (1,\infty)$, so is*

$$K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{2.7.68}$$

*Also, much as in* (2.7.59)-(2.7.60)*, the operator $K_A$ in the context of* (2.7.66)-(2.7.68) *depends in a continuous fashion on the underlying coefficient tensor $A$.*

### 2.7.2 Inverting double layer operators on Morrey and block spaces

The starting point is deriving estimates for the operator norms of singular integral operators whose integral kernels contain, as a factor, the crucial inner product between the unit normal and the "chord" (cf. (2.7.69), (2.7.70)), of the sort obtained earlier in Theorem 2.4.4 and Corollary 2.4.11 in the context of Muckenhoupt weighted Lebesgue spaces, but now working in the framework of Morrey spaces, vanishing Morrey spaces, and block spaces. We carry out this task in Theorem 2.7.9 below.

**Theorem 2.7.9.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain satisfying a two-sided local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Fix an integrability exponent $p \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Also, consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ (for a sufficiently large integer $N = N(n) \in \mathbb{N}$) which is even and positive homogeneous of degree $-n$. In this setting consider the principal-value singular integral operators $T, T^{\#}$ acting on each function $f \in M^{p,\lambda}(\partial\Omega,\sigma)$ according to*

$$Tf(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \tag{2.7.69}$$

*and*

$$T^{\#}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle y-x, \nu(x)\rangle k(x-y) f(y)\, d\sigma(y), \tag{2.7.70}$$

*at $\sigma$-a.e. point $x \in \partial\Omega$. Also, define the action of the maximal operator $T_*$ on each given function $f \in M^{p,\lambda}(\partial\Omega,\sigma)$ as*

$$T_*f(x) := \sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y) \Bigg| \quad \textit{for each} \;\; x \in \partial\Omega. \tag{2.7.71}$$

*Then the following are well-defined, bounded operators*

$$T_*, T, T^{\#} : M^{p,\lambda}(\partial\Omega,\sigma) \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma), \tag{2.7.72}$$

$$T_*, T, T^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma), \tag{2.7.73}$$

*and there exists some $C \in (0,\infty)$, which depends only on $n$, $p$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\max\Big\{ \|T_*\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\,,\, \|T_*\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \Big\} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.7.74}$$

$$\max\Big\{ \|T\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\,,\, \|T\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \Big\} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,, \tag{2.7.75}$$

$$\max\Big\{ \big\|T^{\#}\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\,,\, \big\|T^{\#}\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \Big\} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.7.76}$$

*Furthermore, for each $q \in (1,\infty)$ the operators*

$$T, T^{\#} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \tag{2.7.77}$$

*are well defined, linear, bounded, and there exists some $C \in (0,\infty)$, which depends only on $n$, $q$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\max\Big\{ \|T\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\,,\, \big\|T^{\#}\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \Big\} \leq C\Big(\sum_{|\alpha|\leq N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,. \tag{2.7.78}$$

*Proof.* The claims made in (2.7.72)-(2.7.76) follow from Theorem 2.4.4, Corollary 2.4.11, and Proposition 2.7.5 (also keeping in mind (2.7.3) and (2.7.7)). Then the claims in (2.7.77)-(2.7.78) become consequences of what we have just proved and duality (cf. Proposition 2.7.3 and (2.7.45)). □

In concert with the commutator estimates discussed earlier (cf. Theorem 2.7.7), Theorem 2.7.9 implies the following result, which is the Morrey space (respectively, vanishing Morrey space, and block space) counterpart of Theorem 2.4.18.

**Corollary 2.7.10.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_k)_{1\leq k\leq n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix two integrability exponents $p, q \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from (2.3.8), the*

*Riesz transforms $\{R_j\}_{1\leq j\leq n}$ from (2.4.236), and for each $k\in\{1,\dots,n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by the $k$-th scalar component of $\nu$.*

*Then there exists some $C\in(0,\infty)$ which depends only on $n$, $p$, $q$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\begin{aligned}\|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}&\\ +\max_{1\leq j,k\leq n}\|[M_{\nu_k},R_j]\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}&\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \end{aligned}\tag{2.7.79}$$

$$\begin{aligned}\|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}&\\ +\max_{1\leq j,k\leq n}\|[M_{\nu_k},R_j]\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)}&\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \end{aligned}\tag{2.7.80}$$

*and*

$$\begin{aligned}\|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}&\\ +\max_{1\leq j,k\leq n}\|[M_{\nu_k},R_j]\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}&\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \end{aligned}\tag{2.7.81}$$

*Proof.* The estimates claimed in (2.7.79)-(2.7.81) are implied by (2.3.8), Theorem 2.7.9, (2.4.236), Proposition 2.3.3, and Theorem 2.7.7. □

We shall revisit Corollary 2.7.10 later, in Theorem 2.7.17, which contains estimates in the opposite direction to those obtained in (2.7.79)-(2.7.81).

For the time being, we take up the task of establishing estimates akin to those obtained in Theorem 2.4.20 for Muckenhoupt weighted Lebesgue and Sobolev spaces, now working in the setting of Morrey spaces, vanishing Morrey spaces, block spaces, as well as the brands of Sobolev spaces naturally associated with these scales.

**Theorem 2.7.11.** *Let $\Omega\subseteq\mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\mathrm{dis}}\neq\varnothing$. Pick $A\in\mathfrak{A}_L^{\mathrm{dis}}$ and consider the boundary-to-boundary double layer potential operators $K_A, K_A^\#$ associated with $\Omega$ and the coefficient tensor $A$ as in (2.3.4) and (2.3.5), respectively. Finally, fix two integrability exponents $p,q\in(1,\infty)$ and a parameter $\lambda\in(0,n-1)$.*

*Then there exists some constant $C\in(0,\infty)$ which depends only on $n$, $A$, $p$, $q$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\|K_A\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M\to[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.82}$$

$$\|K_A\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M\to[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.83}$$

$$\|K_A\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M\to[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.84}$$

$$\|K_A\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.85}$$

$$\|K_A\|_{[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.86}$$

$$\|K_A\|_{[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.87}$$

*as well as*

$$\left\|K_A^{\#}\right\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M \to [M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.88}$$

$$\left\|K_A^{\#}\right\|_{[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M \to [\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.89}$$

$$\left\|K_A^{\#}\right\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M \to [\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.7.90}$$

*Proof.* All claims are justified as in the proof of Theorem 2.4.20, now making use of Theorem 2.7.9, Lemma 2.4.19, Theorem 2.7.7, (2.7.53)-(2.7.54), (2.7.55), (2.7.56)-(2.7.57), as well as (2.7.3), (2.7.8), (2.7.10), (2.7.17), (2.7.18). □

The stage is now set for obtaining invertibility results for certain types of double layer potential operators acting on Morrey spaces, vanishing Morrey spaces, block spaces, as well as on the brands of Sobolev spaces naturally associated with these scales.

**Theorem 2.7.12.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, let $L$ be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ for which $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$. Pick $A \in \mathfrak{A}_L^{\mathrm{dis}}$ and consider the boundary-to-boundary double layer potential operators $K_A, K_A^{\#}$ associated with $\Omega$ and the coefficient tensor $A$ as in (2.3.4) and (2.3.5), respectively. Finally, fix two integrability exponents $p, q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$, and some number $\varepsilon \in (0,\infty)$.*

*Then there exists some small threshold $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $q$, $\lambda$, $A$, $\varepsilon$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ it follows that for each spectral parameter $z \in \mathbb{C}$*

*with* $|z| \geq \varepsilon$ *the following operators are invertible:*

$$zI + K_A : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.91}$$

$$zI + K_A : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.92}$$

$$zI + K_A : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.93}$$

$$zI + K_A : \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.94}$$

$$zI + K_A : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.95}$$

$$zI + K_A : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.96}$$

$$zI + K_A^{\#} : \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.97}$$

$$zI + K_A^{\#} : \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M, \tag{2.7.98}$$

$$zI + K_A^{\#} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M. \tag{2.7.99}$$

*Proof.* This is a direct consequence of Theorem 2.7.11, reasoning as in the proof of Theorem 2.4.24. □

We may be further enhance the invertibility results from Theorem 2.7.12 by allowing the coefficient tensor to be a small perturbation of any distinguished coefficient tensor of the given system. Specifically, Theorem 2.7.11 in concert with the continuity of the operator-valued assignments $\mathfrak{A}_{\mathrm{WE}} \ni A \mapsto K_A$ and $\mathfrak{A}_{\mathrm{WE}} \ni A \mapsto K_A^{\#}$, considered in all contexts discussed in Theorem 2.7.8, yield the following result.

**Theorem 2.7.13.** *Retain the original background assumptions on the set* $\Omega$ *from Theorem 2.7.12 and, as before, fix some integrability exponents* $p, q \in (1, \infty)$*, a parameter* $\lambda \in (0, n-1)$*, and some number* $\varepsilon \in (0, \infty)$*. Consider* $L \in \mathfrak{L}^{\mathrm{dis}}$ *(cf. (2.3.84)) and pick an arbitrary* $A_o \in \mathfrak{A}_L^{\mathrm{dis}}$*. Then there exist some small threshold* $\delta_0 \in (0, 1)$ *along with some open neighborhood* $\mathcal{O}$ *of* $A_o$ *in* $\mathfrak{A}_{\mathrm{WE}}$*, both of which depend only on* $n$*,* $p$*,* $q$*,* $\lambda$*,* $A_o$*,* $\varepsilon$*, the local John constants of* $\Omega$*, and the Ahlfors regularity constant of* $\partial\Omega$*, with the property that if* $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta_0$ *then for each* $A \in \mathcal{O}$ *and each spectral parameter* $z \in \mathbb{C}$ *with* $|z| \geq \varepsilon$*, the operators* (2.7.91)*-*(2.7.99) *are invertible.*

We close this section with the following remark.

*Remark* 2.7.14. In the two-dimensional setting, more can be said about the Lamé system. Specifically, the versions of Theorem 2.4.30 and Corollary 2.4.31 naturally formulated in terms of Morrey spaces, vanishing Morrey spaces, block spaces, as well as their associated Sobolev spaces, continue to hold, virtually with the same proofs (now making use of Proposition 2.7.6, Theorem 2.7.7, Theorem 2.7.8, and Theorem 2.7.9).

### 2.7.3 Characterizing flatness in terms of Morrey and block spaces

How do the quantitative aspects of the analysis of a certain geometric environment affect the very geometric features of the said environment? Here we address a specific aspect of this general question by characterizing the flatness of a "surface" in terms of the size of the norms of certain singular integral operators acting on Morrey and block spaces considered on this surface.

In order be able to elaborate on this topic, we need some notation. Given a UR domain $\Omega \subseteq \mathbb{R}^n$, denote by $\nu$ its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. From Proposition 2.7.5 (more specifically, from a version of it using the formalism of pairs of functions as in [28, p. 30]) and (2.5.5)-(2.5.7) we then conclude that whenever $p \in (1,\infty)$ and $\lambda \in (0, n-1)$, the operators

$$\mathbf{C} : M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{2.7.100}$$

$$\mathbf{C} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{2.7.101}$$

and

$$\mathbf{C}^{\#} : M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{2.7.102}$$

$$\mathbf{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{2.7.103}$$

are all well defined, linear, and continuous, with

$$\begin{gathered}\|\mathbf{C}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\ \|\mathbf{C}^{\#}\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\\ \|\mathbf{C}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\ \|\mathbf{C}^{\#}\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\\ \text{bounded exclusively in terms of } n, p, \lambda, \text{ and the UR character of } \partial\Omega.\end{gathered} \tag{2.7.104}$$

Granted these, via duality (cf. (2.5.8) and Proposition 2.7.3) we also obtain that for each $q \in (1,\infty)$ and $\lambda \in (0, n-1)$ the operators

$$\mathbf{C} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n, \tag{2.7.105}$$

$$\mathbf{C}^{\#} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{p,\lambda}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \tag{2.7.106}$$

are all well defined, linear, and bounded, with

$$\begin{gathered}\|\mathbf{C}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n},\ \|\mathbf{C}^{\#}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}\\ \text{controlled only in terms of } n, q, \lambda, \text{ and the UR character of } \partial\Omega.\end{gathered} \tag{2.7.107}$$

In addition, from (2.5.9) and duality (cf. (2.5.8) and Proposition 2.7.3) we conclude that, for each $p, q \in (1,\infty)$ and $\lambda \in (0, n-1)$,

$$\begin{gathered}\text{the operator identities } \mathbf{C}^2 = \tfrac{1}{4}I \text{ and } \big(\mathbf{C}^{\#}\big)^2 = \tfrac{1}{4}I\\ \text{are valid on either of the spaces } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n, \text{ and } \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n.\end{gathered} \tag{2.7.108}$$

More delicate estimates than (2.7.104), (2.7.107) turn out to hold for the antisymmetric part of the Cauchy-Clifford operator, i.e., for the difference $\mathbf{C} - \mathbf{C}^{\#}$, of the sort described in the proposition below.

**Proposition 2.7.15.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix two integrability exponents $p, q \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Then there exists some constant $C \in (0, \infty)$ which depends only on $n$, $p$, $q$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n} \leq C \left\| \nu \right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.109}$$

$$\left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n} \leq C \left\| \nu \right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}, \tag{2.7.110}$$

$$\left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n} \leq C \left\| \nu \right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}. \tag{2.7.111}$$

*Proof.* This is implied by the structural result from Lemma 2.5.1 (bearing in mind (2.7.3), (2.7.8), (2.7.17)), together with Theorem 2.7.7, Theorem 2.7.9, and (2.3.8). □

Remarkably, it is also possible to establish bounds from below for the operator norm of $\mathbf{C} - \mathbf{C}^{\#}$ on Morrey spaces and their pre-duals, considered on the boundary of a UR domain, in terms of the BMO semi-norm of the geometric measure theoretic outward unit normal vector to the said domain.

**Proposition 2.7.16.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain such that $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an integrability exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Then there exists some $C \in (0, \infty)$ which depends only on $n$, $p$, $\lambda$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\begin{aligned} \left\| \nu \right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} &\leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|^{1/n}_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n} \\ &\leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|^{1/n}_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n}. \end{aligned} \tag{2.7.112}$$

*Furthermore, for each $q \in (1, \infty)$ and $\lambda \in (0, n-1)$ there exists some $C \in (0, \infty)$ which depends only on $n$, $q$, $\lambda$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\left\| \nu \right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C \left\| \mathbf{C} - \mathbf{C}^{\#} \right\|^{1/n}_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n}. \tag{2.7.113}$$

*Proof.* The argument largely follows the proof of the unweighted version of Proposition 2.5.3 (i.e., when $w \equiv 1$), so we will only indicate the main changes. First, in place of (2.5.15) we now consider

$$\left\| \mathbf{C} - \mathbf{C}^{\#} \right\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n} < \eta < 4^{-n}. \tag{2.7.114}$$

Second, we now take $r_* := \delta\, \eta^{1/n}$ in place of (2.5.18). Lastly, in place of (2.5.28) we now write (making use of (2.5.25)-(2.5.27), (2.7.2), (2.7.114), the fact that $\mathbf{1}_{\Delta(y_0,r_*)} \in$

$\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, and (2.7.5))

$$\begin{aligned}
&\fint_{\Delta(x_0,r_*)}\Big|\int_{\Delta(y_0,r_*)}\Big\{\frac{x_0-y}{|x_0-y|^n}\odot\nu(y)+\nu(x)\odot\frac{x_0-y_0}{|x_0-y_0|^n}\Big\}\,d\sigma(y)\Big|^p\,d\sigma(x)\\
&\qquad\le C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\fint_{\Delta(x_0,r_*)}|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,r_*)}(x)|^p\,d\sigma(x)\\
&\qquad\le C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\cdot r_*^{-(n-1-\lambda)}\|(\mathbf{C}-\mathbf{C}^{\#})\mathbf{1}_{\Delta(y_0,r_*)}\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)}\\
&\qquad< C\Big(\frac{r_*}{\delta}\Big)^{np}+C_{n,p}\cdot\eta^p\cdot r_*^{-(n-1-\lambda)}\,\big\|\mathbf{1}_{\Delta(y_0,r_*)}\big\|^p_{M^{p,\lambda}(\partial\Omega,\sigma)}\\
&\qquad\le C\Big(\frac{r_*}{\delta}\Big)^{np}+C\cdot\eta^p\cdot r_*^{-(n-1-\lambda)}\sigma\big(\Delta(y_0,r_*)\big)^{(n-1-\lambda)/(n-1)}\\
&\qquad\le C\Big(\frac{r_*}{\delta}\Big)^{np}+C\cdot\eta^p=C\cdot\eta^p,
\end{aligned}\tag{2.7.115}$$

for some $C\in(0,\infty)$ which depends only on $n$, $p$, $\lambda$, and the Ahlfors regularity constant of $\partial\Omega$. Having established this, after running the same argument as in (2.5.29), in place of (2.5.22) we presently arrive at the conclusion that

$$\Big(\fint_{\Delta(x_0,r_*)}|\nu(y)-A|^p\,d\sigma(y)\Big)^{1/p}\le C\cdot\eta^{1/n},\tag{2.7.116}$$

for some $C\in(0,\infty)$ which depends only on $n$, $p$, $\lambda$, and the Ahlfors regularity constant of $\partial\Omega$. With this in hand, the first estimate claimed in (2.7.112) now follows upon passing to limit $\eta\searrow\big\|\mathbf{C}-\mathbf{C}^{\#}\big\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\to\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n}$ and reasoning as in (2.5.23). The second estimate in (2.7.112) is a direct consequence of (2.7.8).

Finally, the estimate claimed in (2.7.113) follows from the first inequality in (2.7.112), plus the fact that whenever $p,q\in(1,\infty)$ are such that $1/p+1/q=1$ then the (real) transposed of

$$\mathbf{C}-\mathbf{C}^{\#}:\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\longrightarrow\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\tag{2.7.117}$$

is the operator

$$\mathbf{C}^{\#}-\mathbf{C}:\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n\longrightarrow\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n.\tag{2.7.118}$$

See (2.5.8) and Proposition 2.7.3 in this regard. □

The next result contains estimates in the opposite direction to those presented in Corollary 2.7.10.

**Theorem 2.7.17.** *Let $\Omega\subseteq\mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu=(\nu_k)_{1\le j\le n}$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix two integrability exponent $p,q\in(1,\infty)$ along with a parameter $\lambda\in(0,n-1)$. Finally, recall the boundary-to-boundary harmonic double layer potential operator $K_\Delta$ from (2.3.8), the Riesz transforms $\{R_j\}_{1\le j\le n}$ on $\partial\Omega$ from (2.4.236), and for each index $k\in\{1,\dots,n\}$ denote by $M_{\nu_k}$ the operator of pointwise multiplication by the $k$-th scalar component of*

*$\nu$. Then there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $q$, $\lambda$, and the Ahlfors regularity constant of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \|K_\Delta\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \Big\}^{1/n}, \tag{2.7.119}$$

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \|K_\Delta\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\to \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)} \Big\}^{1/n}, \tag{2.7.120}$$

*and*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \|K_\Delta\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \|[M_{\nu_k}, R_j]\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\to \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \Big\}^{1/n}. \tag{2.7.121}$$

*Proof.* If $\partial\Omega$ is unbounded then all estimates are implied by Proposition 2.7.16 and the structural result from Lemma 2.5.1 (keeping in mind (2.7.3), (2.7.8), (2.7.17)). When $\partial\Omega$ is bounded, we have $K_\Delta 1 = \pm\frac{1}{2}$ (cf. [93]) with the sign plus if $\Omega$ is bounded, and the sign minus if $\Omega$ is unbounded, hence the norm of $K_\Delta$ on either $M^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, or $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ is $\geq \frac{1}{2}$ in such a case. Given that $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq 1$ (cf. (2.2.56)), the estimates claimed in (2.7.119)-(2.7.121) are valid in this case if we take $C \geq 2^{1/n}$. □

In turn, the results established in Theorem 2.7.17 may be used to characterize the class of $\delta$-SKT domains in $\mathbb{R}^n$, in the spirit of Corollary 2.5.6, using Morrey spaces and their pre-duals.

By way of contrast, Theorem 2.7.18 discussed next is a stability result stating that if $\Omega \subseteq \mathbb{R}^n$ is a UR domain with an unbounded boundary for which the URTI (cf. (2.5.33)) are "almost" true in the context of either Morrey or block spaces, then $\partial\Omega$ is "almost" flat, in that the BMO semi-norm of the outward unit normal to $\Omega$ is small.

**Theorem 2.7.18.** *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain with an unbounded boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix $p,q \in (1,\infty)$ along with $\lambda \in (0,n-1)$, and recall the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (2.4.236). Then there exists some $C \in (0,\infty)$ which depends only on $n$, $p$, $q$, $\lambda$, and the* UR *character of $\partial\Omega$ with the property that*

$$\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} \leq C\Big\{ \Big\| I + \sum_{j=1}^n R_j^2 \Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} + \max_{1\leq j,k\leq n} \|[R_j, R_k]\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)} \Big\}^{1/n}, \tag{2.7.122}$$

*plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$.*

*Proof.* A key ingredient is the fact that we have the operator identities

$$\begin{gathered}\mathbf{C}-\mathbf{C}^{\#}=\mathbf{C}\Big(I+\sum_{j=1}^{n}R_j^2\Big)+\sum_{1\leq j<k\leq n}\mathbf{C}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k\\ \text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n.\end{gathered}\tag{2.7.123}$$

These are proved much like formula [53, (4.6.46), p. 2752], now making use of (2.7.108). Once (2.7.123) has been established, Proposition 2.7.16 and (2.7.100)-(2.7.107) to conclude (much as in the proof of Theorem 2.5.7) that the estimate claimed in (2.7.122) as well as its related versions on vanishing Morrey spaces and block spaces are all true. □

The last result in this section contains estimates in the opposite direction to those from Theorem 2.7.18. Together, Theorem 2.7.19 and Theorem 2.7.18 amount to saying that, under natural background geometric assumptions on the set $\Omega$, the URTI are "almost" true on Morrey spaces or block spaces if and only if $\partial\Omega$ is "almost" flat (in that the BMO semi-norm of the outward unit normal to $\Omega$ is small).

**Theorem 2.7.19.** *Let $\Omega\subseteq\mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is an Ahlfors regular set. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix $p,q\in(1,\infty)$ along with $\lambda\in(0,n-1)$, and recall the Riesz transforms $\{R_j\}_{1\leq j\leq n}$ on $\partial\Omega$ from (2.4.236).*

*Then there exists some constant $C\in(0,\infty)$ which depends only on $n$, $p$, $q$, $\lambda$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that*

$$\Big\|I+\sum_{j=1}^{n}R_j^2\Big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,,\tag{2.7.124}$$

$$\max_{1\leq j<k\leq n}\big\|[R_j,R_k]\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)\to M^{p,\lambda}(\partial\Omega,\sigma)}\leq C\,\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n}\,,\tag{2.7.125}$$

*plus similar estimates with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$.*

*Proof.* The starting point is to observe that we have the operator identities

$$\begin{gathered}\mathbf{C}(\mathbf{C}^{\#}-\mathbf{C})=-\tfrac{1}{4}\Big(I+\sum_{j=1}^{n}R_j^2\Big)-\tfrac{1}{4}\sum_{1\leq j<k\leq n}[R_j,R_k]\mathbf{e}_j\odot\mathbf{e}_k,\\ \text{on } M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n,\end{gathered}\tag{2.7.126}$$

which are themselves consequences of (2.7.123) and (2.7.108). With (2.7.126) in hand, the estimates claimed in the statement of the theorem may then be justified via an estimate similar in spirit to (2.5.41), and also invoking Proposition 2.7.15 (as well as (2.7.104), (2.7.107)) in the process. □

### 2.7.4 Boundary value problems in Morrey and block spaces

We begin by discussing the Dirichlet Problem for weakly elliptic systems in $\delta$-SKT domains with boundary data in ordinary Morrey spaces, vanishing Morrey spaces, and block spaces.

**Theorem 2.7.20.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1, \infty)$ and a parameter $\lambda \in (0, n-1)$. Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system $L$ in $\mathbb{R}^n$, consider the Dirichlet Problem*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \quad \text{in} \quad \Omega, \\ \mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M. \end{cases} \tag{2.7.127}$$

*The following claims are true:*

*(a)* [Existence, Regularity, and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists $\delta_0 \in (0, 1)$ which depends only on $n$, $p$, $\lambda$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $\frac{1}{2}I + K_A$ is an invertible operator on the Morrey space $\left[M^{p,\lambda}(\partial\Omega, \sigma)\right]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \left(\mathcal{D}_A\left(\tfrac{1}{2}I + K_A\right)^{-1} f\right)(x) \quad \text{for all} \quad x \in \Omega, \tag{2.7.128}$$

*is a solution of the Dirichlet Problem (2.7.127). Moreover,*

$$\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{2.7.129}$$

*Furthermore, the function $u$ defined in (2.7.128) satisfies the following regularity result*

$$\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma) \Longleftrightarrow f \in \left[M_1^{p,\lambda}(\partial\Omega, \sigma)\right]^M, \tag{2.7.130}$$

*and if either of these conditions holds then*

$$\begin{gathered} (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \quad \text{exists (in } \mathbb{C}^{n\cdot M}) \text{ at } \sigma\text{-a.e. point on } \partial\Omega \text{ and} \\ \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \end{gathered} \tag{2.7.131}$$

*(b)* [Uniqueness] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, there exists $\delta_0 \in (0, 1)$ which depends only on $n$, $p$, $\lambda$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the Dirichlet Problem (2.7.127) has at most one solution.*

*(c)* [Well-Posedness] *If $\mathfrak{A}_L^{\rm dis} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\rm dis} \neq \varnothing$ then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $\lambda$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the Dirichlet Problem* (2.7.127) *is uniquely solvable and the solution satisfies* (2.7.129)*.*

*(d)* [Sharpness] *If $\mathfrak{A}_L^{\rm dis} = \varnothing$ then the Dirichlet Problem* (2.7.127) *may fail to be solvable (actually for boundary data belonging to an infinite dimensional subspace of the corresponding Morrey space). Also, if $\mathfrak{A}_{L^\top}^{\rm dis} = \varnothing$ then the Dirichlet Problem* (2.7.127) *may have more than one solution (in fact, the linear space of null-solutions may actually be infinite dimensional).*

*(e)* [Other Spaces of Boundary Data] *Similar results to those described in items (a)-(d) above hold with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ with $q \in (1,\infty)$.*

To give an example, suppose $\Omega \subseteq \mathbb{R}^n$ is a $\delta$-SKT domain and fix an arbitrary aperture parameter $\kappa > 0$ along with some power $a \in (0, n-1)$. In addition, choose a number $\lambda \in (0, n-1-a)$ and define $p := (n-1-\lambda)/a \in (1,\infty)$. Then, if $\delta > 0$ is sufficiently small (relative to $n$, $a$, $\lambda$, the Ahlfors regularity constant of $\partial\Omega$, and the local John constants of $\Omega$), it follows that for each point $x_o \in \partial\Omega$ the Dirichlet Problem

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \ \text{ in } \ \Omega, \quad \mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma), \\ \Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = |x - x_o|^{-a} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{cases} \tag{2.7.132}$$

has a unique solution. Moreover, there exists a constant $C(\Omega, n, \kappa, a, \lambda) \in (0,\infty)$ with the property that the said solution satisfies $\|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, \lambda)$. The reason is that, as seen from Example 2.7.1, the function $f_{x_o}(x) := |x - x_o|^{-a}$ for $\sigma$-a.e. point $x \in \partial\Omega$ belongs to the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ and we have $\sup_{x_o \in \partial\Omega} \|f_{x_o}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty$. As such, the result in item *(c)* of Theorem 2.7.20 applies and yields the desired conclusion.

In addition, there is a naturally accompanying regularity result. To formulate it, assume $q \in (1,\infty)$ and $\mu \in (0, n-1)$ are such that $a + 1 = (n-1-\mu)/q$. Starting from the realization that the boundary datum $f_{x_o}$ actually actually belongs to a suitably defined off-diagonal Morrey-based Sobolev space on $\partial\Omega$, from (2.6.42) and Example 2.7.1 we see that actually there exists $C(\Omega, n, \kappa, a, q, \mu) \in (0,\infty)$ independent of $x_o \in \partial\Omega$ such that, if $\delta > 0$ is sufficiently small to begin with, then the unique solution of the Dirichlet Problem (2.7.132) satisfies the following additional regularity properties

$$\begin{aligned} &(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists (in } \mathbb{R}^n) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ &\text{and } \ \|\mathcal{N}_\kappa(\nabla u)\|_{M^{q,\mu}(\partial\Omega,\sigma)} \leq C(\Omega, n, \kappa, a, q, \mu). \end{aligned} \tag{2.7.133}$$

In closing, let us also mention that boundary value problems in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with boundary data with components in the Morrey spaces $M^{2,\lambda}(\partial\Omega,\sigma)$ (with $\lambda$ belonging to a certain sub-interval of $(0, n-1)$) for symmetric, homogeneous, second-order, systems with constant real coefficients satisfying the Legendre-Hadamard strong ellipticity condition have been considered in [110].

After this digression we turn to the task of giving the proof of Theorem 2.7.20.

*Proof of Theorem 2.7.20.* The argument parallels the proof of Theorem 2.6.2. First, Theorem 2.7.12 shows that there exists some $\delta_0 \in (0,1)$, whose nature is as specified in the statement of the theorem, with the property that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the operator $\frac{1}{2}I + K_A$ is invertible on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Hence, the function $u$ in (2.7.128) is meaningfully defined, and according to (2.3.3), (2.7.3), and Theorem 2.7.8, we have $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, $Lu = 0$ in $\Omega$, $\mathcal{N}_\kappa u \in M^{p,\lambda}(\partial\Omega,\sigma)$, and (2.7.129) holds. Concerning the equivalence claimed in (2.7.130), if $f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ then Theorem 2.7.12 gives (assuming $\delta_0$ is sufficiently small) that $\big(\frac{1}{2}I + K_A\big)^{-1} f \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. With this in hand, (2.7.64)-(2.7.65) then imply that the function $u$ defined as in (2.7.128) satisfies $\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma)$, the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists $\sigma$-a.e. on $\partial\Omega$, and the left-pointing inequality in the equivalence claimed in (2.7.131) holds. In particular, this justifies the left-pointing implication in (2.7.130). The right-pointing implication in (2.7.130) together with the right-pointing inequality in the equivalence claimed in (2.7.131) are consequences of (2.7.3) and Proposition 2.2.48.

Turning our attention to the uniqueness result claimed in item *(b)*, suppose $\mathfrak{A}^{\mathrm{dis}}_{L^\top} \neq \varnothing$ and pick some $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\mathrm{dis}}_{L^\top}$. Also, denote by $q \in (1,\infty)$ the Hölder conjugate exponent of $p$. From Theorem 2.7.12, presently used with $L$ replaced by $L^\top$, we know that there exists $\delta_0 \in (0,1)$, which depends only on $n$, $p$, $\lambda$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the following operator is invertible:

$$\tfrac{1}{2}I + K_{A^\top} : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{2.7.134}$$

Also, decreasing the value of $\delta_0 \in (0,1)$ if necessary guarantees that $\Omega$ is an NTA domain with unbounded boundary (cf. Proposition 2.2.32 and Lemma 2.2.5). In such a case, (2.6.2) ensures that $\Omega$ is globally pathwise nontangentially accessible.

Moving on, recall the fundamental solution $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with the system $L$ as in Theorem 1.2.1. Pick $x_\star \in \mathbb{R}^n \setminus \overline{\Omega}$ along with $x_0 \in \Omega$, arbitrary. Also, fix $\rho \in \big(0\,,\,\frac{1}{4}\,\mathrm{dist}\,(x_0,\partial\Omega)\big)$ and define $K := \overline{B(x_0,\rho)}$. Finally, recall the aperture parameter $\widetilde{\kappa} > 0$ associated with $\Omega$ and $\kappa$ as in Theorem 2.6.1. To proceed, for each fixed index $\beta \in \{1,\dots,M\}$, consider the $\mathbb{C}^M$-valued function

$$f^{(\beta)}(x) := \big(E_{\beta\alpha}(x-x_0) - E_{\beta\alpha}(x-x_\star)\big)_{1\leq\alpha\leq M}, \qquad \forall\, x \in \partial\Omega. \tag{2.7.135}$$

Based on (2.7.19), (2.7.135), (2.7.56), (2.2.341), (2.7.21), (1.2.19), and the Mean Value Theorem we then conclude that

$$f^{(\beta)} \in \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{2.7.136}$$

Consequently, with $\big(\frac{1}{2}I + K_{A^\top}\big)^{-1}$ denoting the inverse of the operator in (2.7.134),

$$v_\beta := (v_{\beta\alpha})_{1\leq\alpha\leq M} := \mathcal{D}_{A^\top}\Big(\big(\tfrac{1}{2}I + K_{A^\top}\big)^{-1} f^{(\beta)}\Big) \tag{2.7.137}$$

is a well-defined $\mathbb{C}^M$-valued function in $\Omega$ which, by virtue of Theorem 2.7.8, satisfies

$$\begin{gathered} v_\beta \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L^\top v_\beta = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_{\widetilde{\kappa}} v_\beta \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \quad \mathcal{N}_{\widetilde{\kappa}}(\nabla v_\beta) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma), \\ \text{and } \ v_\beta\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = f^{(\beta)} \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{gathered} \tag{2.7.138}$$

In addition, from (2.7.136)-(2.7.137) and (2.7.64) we see that

$$(\nabla v_\beta)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists (in } \mathbb{C}^{n\cdot M}) \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.7.139}$$

For each pair of indices $\alpha, \beta \in \{1, \dots, M\}$ let us now define

$$G_{\alpha\beta}(x) := v_{\beta\alpha}(x) - \big(E_{\beta\alpha}(x - x_0) - E_{\beta\alpha}(x - x_\star)\big), \qquad \forall\, x \in \Omega \setminus \{x_0\}. \tag{2.7.140}$$

Regarding $G := \big(G_{\alpha\beta}\big)_{1\le\alpha,\beta\le M}$ as a $\mathbb{C}^{M\times M}$-valued function defined $\mathcal{L}^n$-a.e. in $\Omega$, from (2.7.140) and Theorem 1.2.1 we then see that $G \in \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{M\times M}$. Furthermore, by design,

$$\begin{gathered} L^\top G = -\delta_{x_0} I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^{M\times M} \ \text{ and} \\ G\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ (\nabla G)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \end{gathered} \tag{2.7.141}$$

while if $v := \big(v_{\beta\alpha}\big)_{1\le\alpha,\beta\le M}$ then from (1.1.4), (1.2.19), and the Mean Value Theorem it follows that at each point $x \in \partial\Omega$ we have

$$\begin{gathered} \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K} G\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}} v\big)(x) + C_{x_0,\rho}(1 + |x|)^{1-n} \ \text{ and} \\ \big(\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\big)(x) \le \big(\mathcal{N}_{\widetilde{\kappa}}(\nabla v)\big)(x) + C_{x_0,\rho}(1 + |x|)^{-n}, \end{gathered} \tag{2.7.142}$$

where $C_{x_0,\rho} \in (0, \infty)$ is independent of $x$. From (2.7.138), (2.7.142), (2.7.21), and (2.7.19) we see that the conditions listed in (2.6.4) are presently satisfied and, in fact,

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma). \tag{2.7.143}$$

Assume now that $u = (u_\beta)_{1\le\beta\le M}$ is a $\mathbb{C}^M$-valued function in $\Omega$ satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \text{and } \ \mathcal{N}_\kappa u \ \text{ belongs to the space } \ M^{p,\lambda}(\partial\Omega, \sigma). \end{gathered} \tag{2.7.144}$$

Since (2.7.143) and (2.7.22) imply

$$\begin{aligned} \int_{\partial\Omega} & \mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\, d\sigma \\ & \le C \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} < \infty, \end{aligned} \tag{2.7.145}$$

we may rely on Theorem 2.6.1 to conclude that the Poisson integral representation formula (2.6.6) holds. In particular, the said formula proves that whenever $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0$ at $\sigma$-a.e. point on $\partial\Omega$ we necessarily have $u(x_0) = 0$. Given that $x_0$ has been arbitrarily chosen in $\Omega$, this ultimately shows such a function $u$ is actually identically zero in $\Omega$. This finishes the proof of the uniqueness claim made in item *(b)*. The well-posedness claim in item *(c)* is a consequence of what we have already proved in items *(a)*-*(b)*.

Going further, the first claim in item *(d)*, regarding the potential failure of solvability of the Dirichlet Problem (2.7.127), is a consequence of Proposition 2.3.12 formulated for Morrey spaces. Its proof goes through virtually unchanged, with one caveat. Specifically, to justify (2.3.167), instead of Lebesgue's Dominated Convergence Theorem on Muckenhoupt weighted Lebesgue spaces we now use the weak-$*$ convergence on Morrey spaces from Proposition 2.7.4 (bearing in mind the continuity and skew-symmetry of the Hilbert transform on Morrey and block spaces on the real line). Also, the second claim in item *(d)*, regarding the potential failure of uniqueness for the Dirichlet Problem (2.7.127), is a consequence of Example 2.3.11 (keeping in mind (2.3.134) and (2.7.4)).

Consider next the claim made in item *(e)*. When the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, virtually the same proof goes through, given that matters may be arranged (by taking $\delta_0$ sufficiently small) so that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and $\big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 2.7.12). Finally, in the scenario in which the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is replaced by the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ for some given $q \in (1,\infty)$, the same line of reasoning applies, with a few notable changes. First, if $p$ is the Hölder conjugate exponent of $q$, then taking $\delta_0$ sufficiently small we may ensure that the operator $\frac{1}{2}I + K_A$ is invertible on $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, and $\big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 2.7.12). Second, with $f^{(\beta)}$ as in (2.7.135), thanks to (2.7.4) in place of (2.7.136) we now have

$$f^{(\beta)} \in \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{2.7.146}$$

In place of (2.7.143), this eventually implies

$$\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G) \in M^{p,\lambda}(\partial\Omega,\sigma), \tag{2.7.147}$$

so in place of (2.7.145) we now have (again, thanks to (2.7.22))

$$\begin{aligned}\int_{\partial\Omega} &\mathcal{N}_\kappa u \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\,d\sigma \\ &\le C\|\mathcal{N}_\kappa u\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \|\mathcal{N}_{\widetilde{\kappa}}^{\Omega\setminus K}(\nabla G)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} < \infty.\end{aligned} \tag{2.7.148}$$

As before, this allows us to invoke Theorem 2.6.1 to conclude that the Poisson integral representation formula (2.6.6) holds. Ultimately, this readily implies the uniqueness result we presently seek. Finally, the versions of the claims in item *(d)* for vanishing Morrey spaces and block spaces are dealt with much as before (for the former scale, use (2.7.8); in the case of block spaces, it is useful to observe that (2.7.17) and Lebesgue's Dominated Convergence Theorem yield, in place of (2.3.167), that $\lim_{\varepsilon\to 0^+} h_\varepsilon = f_1 + i f_2$ in $L^r(\mathbb{R},\mathcal{L}^1)$ where $r$ is as in (2.7.17), and this suffices to conclude that (2.3.168) holds in this case). The proof of Theorem 2.7.20 is therefore complete. $\square$

It turns out that the solvability results established in Theorem 2.7.20 may be further enhanced, via perturbation arguments, as described in our next theorem.

**Theorem 2.7.21.** *Retain the original background assumptions on the set $\Omega$ from Theorem 2.7.20 and, as before, fix two integrability exponents $p, q \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$. Then the following statements are true.*

*(a)* [Existence] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ (cf. (2.3.84)) there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $q$, $\lambda$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0, \delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem (2.7.127), along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, are all solvable.*

*(a)* [Uniqueness] *For each given system $L_o \in \mathfrak{L}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $q$, $\lambda$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT* domain) for some $\delta \in (0, \delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem (2.7.127) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, have at most one solution.*

*(c)* [Well-Posedness] *For each given system $L_o \in \mathfrak{L}^{\rm dis}$ with $L_o^\top \in \mathfrak{L}^{\rm dis}$ there exist some small threshold $\delta_0 \in (0,1)$ and some open neighborhood $\mathcal{U}$ of $L_o$ in $\mathfrak{L}$, both of which depend only on $n$, $p$, $q$, $\lambda$, $L_o$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$, with the property that if $\|\nu\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta$ (i.e., if $\Omega$ is a $\delta$-*SKT *domain) for some $\delta \in (0, \delta_0)$ then for each system $L \in \mathcal{U}$ the Dirichlet Problem (2.7.127) along with its versions in which the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ is replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$, are all well posed.*

*Proof.* This may be justified by reasoning as in the proof of Theorem 2.6.4, now making use of the invertibility results from Theorem 2.7.13. □

We continue by discussing the Regularity Problem for weakly elliptic systems in $\delta$-SKT domains with boundary data in Morrey-based Sobolev spaces, vanishing Morrey-based Sobolev spaces, as well as block-based Sobolev spaces.

**Theorem 2.7.22.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1, \infty)$ and a parameter*

$\lambda \in (0, n-1)$. *Given a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system $L$ in $\mathbb{R}^n$, consider the Regularity Problem*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \\ Lu = 0 \quad in \quad \Omega, \\ \mathcal{N}_\kappa u,\, \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \in \left[M_1^{p,\lambda}(\partial\Omega,\sigma)\right]^M. \end{cases} \tag{2.7.149}$$

*The following statements are true:*

*(a)* [Existence and Estimates] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $A \in \mathfrak{A}_L^{\mathrm{dis}}$, then there exists some $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $\lambda$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $\frac{1}{2}I + K_A$ is an invertible operator on the Morrey-based Sobolev space $\left[M_1^{p,\lambda}(\partial\Omega,\sigma)\right]^M$ and the function*

$$u(x) := \Big(\mathcal{D}_A\left(\tfrac{1}{2}I + K_A\right)^{-1} f\Big)(x), \qquad \forall\, x \in \Omega, \tag{2.7.150}$$

*is a solution of the Regularity Problem* (2.7.149). *In addition,*

$$\begin{aligned} \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \quad and \\ \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|f\|_{[M_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \end{aligned} \tag{2.7.151}$$

*(b)* [Uniqueness] *Whenever $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$, there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $\lambda$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the Regularity Problem* (2.7.149) *has at most one solution.*

*(c)* [Well-Posedness] *If $\mathfrak{A}_L^{\mathrm{dis}} \neq \varnothing$ and $\mathfrak{A}_{L^\top}^{\mathrm{dis}} \neq \varnothing$ then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $\lambda$, $L$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the Regularity Problem* (2.7.149) *is uniquely solvable and the solution satisfies* (2.7.151).

*(d)* [Other Spaces of Boundary Data] *Analogous results to those described in items (a)-(c) above are also valid for the Regularity Problem formulated with boundary data in the vanishing Morrey-based Sobolev space $\left[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\right]^M$, or the block-based Sobolev space $\left[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\right]^M$ with $q \in (1,\infty)$.*

*(e)* [Perturbation Results] *In each of the cases considered in items (a)-(d), there are naturally accompanying perturbation results of the sort described in Theorem 2.7.21.*

*(f)* [Sharpness] *If $\mathfrak{A}_L^{\mathrm{dis}} = \varnothing$ the Regularity Problem* (2.7.149) *(and its variants involving vanishing Morrey-based Sobolev spaces, or block-based Sobolev spaces) may fail to be solvable, and if $\mathfrak{A}_{L^\top}^{\mathrm{dis}} = \varnothing$ the Regularity Problem* (2.7.149) *(along with its aforementioned variants) may posses more than one solution.*

*Proof.* The claims in items *(a)-(d)* are implied by Theorem 2.7.12 and Theorem 2.7.20, while the claim in item *(e)* may be justified by reasoning as in the proof of Theorem 2.6.4, now making use of the invertibility results from Theorem 2.7.13. Finally, the claims in item *(f)* are consequences of the versions of Example 2.3.11 and Proposition 2.3.14 formulated for Morrey spaces, as well as vanishing Morrey spaces and block spaces (whose proofs naturally adapt to these spaces; see the discussion in the proof of item *(d)* in Theorem 2.7.20). □

*Remark* 2.7.23. Much as in discussed in Remark 2.6.7, similar solvability and well-posedness results as in Theorem 2.7.22 hold for the versions of the Regularity Problem (2.7.149) formulated with boundary data belonging to suitably defined off-diagonal Morrey-based Sobolev spaces (as well as off-diagonal vanishing Morrey-based Sobolev spaces, and off-diagonal block-based Sobolev spaces).

We next treat the Neumann Problem for weakly elliptic systems in $\delta$-SKT domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces.

**Theorem 2.7.24.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Denote by $\nu$ the geometric measure theoretic outward unit normal $\nu$ to $\Omega$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa > 0$. Also, pick an integrability exponent $p \in (1,\infty)$ and a parameter $\lambda \in (0, n-1)$.*

*Suppose $L$ is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Select $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ and consider the Neumann Problem*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \\ Lu = 0 \ \ \text{in} \ \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma), \\ \partial^A_\nu u = f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{2.7.152}$$

*Then there exists $\delta_0 \in (0,1)$ which depends only on $n$, $p$, $\lambda$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $-\frac{1}{2}I + K^{\#}_{A^\top}$ is an invertible operator on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and the function $u : \Omega \to \mathbb{C}^M$ defined as*

$$u(x) := \Big(\mathscr{S}_{\rm mod}\Big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\Big)^{-1} f\Big)(x) \ \ \text{for all} \ \ x \in \Omega, \tag{2.7.153}$$

*is a solution of the Neumann Problem (2.7.152) which satisfies*

$$\|\mathcal{N}_\kappa(\nabla u)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{2.7.154}$$

*for some constant $C \in (0,\infty)$ independent of $f$.*

*Moreover, similar results are valid with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ with $q \in (1,\infty)$.*

*Finally, in each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 2.7.21.*

*Proof.* Theorem 2.7.12 guarantees the existence of some threshold $\delta_0 \in (0,1)$, whose nature is as specified in the statement of the theorem, with the property that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the operator $-\frac{1}{2}I + K^{\#}_{A^\top}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ (assuming $q \in (1,\infty)$ has been fixed to begin with). Granted this, all desired conclusions follow from Theorem 2.7.8 and Theorem 2.7.13. $\square$

In relation to Theorem 2.7.24, we wish to note that in the formulation of the Neumann Problem (2.7.152) for the two-dimensional Lamé system we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (2.4.267) for any $\zeta$ as in (2.6.61) (see Remark 2.7.14 and Remark 2.6.12 in this regard).

Finally, we formulate and solve the Transmission Problem for weakly elliptic systems in $\delta$-SKT domains with boundary data in Morrey spaces, vanishing Morrey spaces, and block spaces. In the formulation on this problem, the clarifications made right after the statement of Theorem 2.6.14 continue to remain relevant.

**Theorem 2.7.25.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a two-sided local John condition and whose topological boundary is Ahlfors regular. Denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and set*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.7.155}$$

*Also, pick an integrability exponent $p \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$, an aperture parameter $\kappa > 0$, and a transmission parameter $\mu \in \mathbb{C} \setminus \{\pm 1\}$.*

*Assume $L$ is a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, with the property that $\mathfrak{A}^{\rm dis}_{L^\top} \neq \varnothing$. Select $A \in \mathfrak{A}_L$ such that $A^\top \in \mathfrak{A}^{\rm dis}_{L^\top}$ and consider the Transmission Problem*

$$\begin{cases} u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \\ Lu^\pm = 0 \ \ \text{in} \ \ \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u^\pm) \in M^{p,\lambda}(\partial\Omega,\sigma), \\ u^+\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = u^-\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \ \text{at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \partial^A_\nu u^+ - \mu\cdot\partial^A_\nu u^- = f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \end{cases} \tag{2.7.156}$$

*Then there exists some number $\delta_0 \in (0,1)$ which depends only on $n$, $\mu$, $p$, $\lambda$, $A$, the local John constants of $\Omega$, and the Ahlfors regularity constant of $\partial\Omega$ such that if the set $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then $\frac{\mu+1}{2(\mu-1)}I + K^{\#}_{A^\top}$ is an invertible operator on the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and the functions $u^\pm : \Omega_\pm \to \mathbb{C}^M$ defined as*

$$u^\pm(x) := (1-\mu)^{-1}\Big(\mathscr{S}_{\rm mod}\Big(\tfrac{\mu+1}{2(\mu-1)}I + K^{\#}_{A^\top}\Big)^{-1} f\Big)(x) \ \ \text{for all} \ \ x \in \Omega_\pm \tag{2.7.157}$$

*solve the Transmission Problem (2.7.156) and satisfy*

$$\|\mathcal{N}_\kappa(\nabla u^\pm)\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\,\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{2.7.158}$$

*for some constant $C \in (0, \infty)$ independent of $f$.*

*Furthermore, analogous results hold with the Morrey space $M^{p,\lambda}(\partial\Omega, \sigma)$ replaced by the vanishing Morrey space $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$, or the block space $\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)$ with $q \in (1, \infty)$. In addition, in each of these cases there are naturally accompanying perturbation results of the sort described in Theorem 2.7.21.*

*Proof.* From the current assumptions and Theorem 2.7.12 (used for the spectral parameter $z := \frac{\mu+1}{2(\mu-1)} \in \mathbb{C} \setminus \{0\}$) we conclude that there exists some threshold $\delta_0 \in (0, 1)$, whose nature is as specified in the statement of the theorem, with the property that if $\Omega$ is a $\delta$-SKT domain with $0 < \delta < \delta_0$ then the operator $\frac{\mu+1}{2(\mu-1)} I + K^{\#}_{A^\top}$ is invertible on $\big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, $\big[\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, and $\big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ (assuming $q \in (1, \infty)$ has been fixed to begin with). With this in hand, the same type of argument as in the proof of Theorem 2.6.14 (which now relies on Theorem 2.7.8) and the proof of Theorem 2.6.4 (which now makes use of Theorem 2.7.13) yields all desired conclusions. □

We close by noting that, in the formulation of the Transmission Problem (2.7.156) for the two-dimensional Lamé system, we may allow conormal derivatives associated with coefficient tensors of the form $A = A(\zeta)$ as in (2.4.267) for any $\zeta$ as in (2.6.72) (see Remark 2.7.14 and Remark 2.6.16 in this regard).

CHAPTER 3

# A Fatou theorem and Poisson's integral representation formula in the upper-half space

Let $L$ be a second-order, homogeneous, constant (complex) coefficient elliptic system in $\mathbb{R}^n$. The goal of this chapter is to prove a Fatou-type result, regarding the a.e. existence of the nontangential boundary limits of any null-solution $u$ of $L$ in the upper-half space, whose nontangential maximal function satisfies an integrability condition with respect to the weighted Lebesgue measure $(1+|x'|^{n-1})^{-1}dx'$ in $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$. This is the best result of its kind in the literature. In addition, we establish a naturally accompanying integral representation formula involving the Agmon-Douglis-Nirenberg Poisson kernel for the system $L$. Finally, we use this machinery to derive well-posedness results for the Dirichlet boundary value problem for $L$ in $\mathbb{R}^n_+$ formulated in a manner which allows for the simultaneous treatment of a variety of function spaces.

The material in this chapter is based on joint work with J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea (cf. [76]).

### Contents



## 3.1 Introduction

As is known from the classical work of S. Agmon, A. Douglis, and L. Nirenberg in [1] and [2], every operator $L$ as in (1.2.1) satisfying (1.2.4) has a Poisson kernel, denoted by $P^L$ (an object whose properties mirror the most basic characteristics of the classical harmonic Poisson kernel). For details, see Theorem 1.2.4 above.

The main goal of this chapter is to establish a Fatou-type theorem and a naturally accompanying Poisson integral representation formula for null-solutions of an elliptic system $L$ in the upper-half space. Among other things, this is going to yield versatile well-posedness results for the Dirichlet Problem in $\mathbb{R}^n_+$ for such systems.

**Theorem 3.1.1** (A Fatou-Type Theorem and Poisson's Integral Formula)**.** *Let $L$ be an $M \times M$ system with constant complex coefficients as in* (1.2.1) *satisfying* (1.2.4)*, and fix some aperture parameter $\kappa > 0$. Then*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} (\mathcal{N}_\kappa u)(x') \, \frac{dx'}{1+|x'|^{n-1}} < \infty, \end{cases} \tag{3.1.1}$$

*implies that*

$$\begin{cases} u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} \ \ \textit{exists at } \mathcal{L}^{n-1}\textit{-a.e. point in } \ \mathbb{R}^{n-1}, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} \ \ \textit{belongs to} \ \ \Big[L^1\Big(\mathbb{R}^{n-1}, \dfrac{dx'}{1+|x'|^{n-1}}\Big)\Big]^M, \\ u(x',t) = \Big(P^L_t * \big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\big)\Big)(x') \ \ \textit{for each} \ \ (x',t) \in \mathbb{R}^n_+, \end{cases} \tag{3.1.2}$$

*where $P^L = (P^L_{\beta\alpha})_{1 \le \beta,\alpha \le M}$ is the Agmon-Douglis-Nirenberg Poisson kernel for the system $L$ in $\mathbb{R}^n_+$ and $P^L_t(x') := t^{1-n} P^L(x'/t)$ for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.*

This refines [82, Theorem 6.1, p. 956]. We also wish to remark that even in the classical case when $L := \Delta$, the Laplacian in $\mathbb{R}^n$, Theorem 3.1.1 is more general (in the sense that it allows for a larger class of functions) than the existing results in the literature. Indeed, the latter typically assume an $L^p$ integrability condition for the harmonic function which, in the range $p \in (1,\infty)$, implies our weighted $L^1$ integrability condition for the nontangential maximal function demanded in (3.1.1). In this vein see, e.g., [42, Theorems 4.8-4.9, pp. 174-175], [115, Corollary, p. 200], [116, Proposition 1, p. 119].

A special case of Theorem 3.1.1 worth singling out is as follows. Recall the Agmon-Douglis-Nirenberg kernel function

$$\begin{aligned} &K^L \in \textstyle\bigcap_{\varepsilon>0} \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon)\big)\big]^{M\times M}, \\ &K^L(x) := P^L_t(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+, \end{aligned} \tag{3.1.3}$$

associated with the elliptic system $L$ as in Theorem 1.2.4. Fix some $t_o > 0$ and define

$$u(x) := K^L(x', t+t_o) = P^L_{t+t_o}(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+. \tag{3.1.4}$$

Then

$$u \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+}\big)\big]^{M\times M}, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad u\big|_{\partial\mathbb{R}^n_+} = P^L_{t_o} \ \text{ on } \ \mathbb{R}^{n-1}. \tag{3.1.5}$$

In addition, (1.2.30) ensures that there exists a finite constant $C_{t_o} > 0$ with the property that $|u(x)| \leq C_{t_o}(1+|x|)^{1-n}$ for each $x \in \mathbb{R}^n_+$. For each fixed $\kappa > 0$ this readily entails

$$(\mathcal{N}_\kappa u)(x') \leq \frac{C}{1+|x'|^{n-1}}, \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{3.1.6}$$

This, in turn, guarantees that the finiteness condition demanded in (3.1.6) is presently satisfied. Having verified all hypotheses of Theorem 3.1.1, from the Poisson integral representation formula in the last line of (3.1.2) and (3.1.4)-(3.1.5) we conclude that

$$P^L_{t+t_o}(x') = u(x',t) = (P^L_t * P^L_{t_o})(x') \text{ for all } (x',t) \in \mathbb{R}^n_+, \tag{3.1.7}$$

where the convolution between the two matrix-valued functions in (3.1.7) is understood in a natural fashion, taking into account the algebraic multiplication of matrices. Ultimately, this provides an elegant proof of the following result (first established in [82, Theorem 5.1] via a conceptually different argument):

$$\begin{array}{l}\text{the Agmon-Douglis-Nirenberg Poisson kernel } P^L \text{ associated}\\ \text{with any given elliptic system } L \text{ as in Theorem 1.2.4 satisfies}\\ \text{the semi-group property } P^L_{t_0+t_1} = P^L_{t_0} * P^L_{t_1} \text{ for all } t_0, t_1 > 0.\end{array} \tag{3.1.8}$$

Here is another important corollary of Theorem 3.1.1, which refines [82, Theorem 3.2, p. 935].

**Corollary 3.1.2** (A General Uniqueness Result)**.** *Let $L$ be an $M \times M$ system with constant complex coefficients as in* (1.2.1) *satisfying* (1.2.4)*, and fix an aperture parameter $\kappa > 0$. Then*

$$\left.\begin{array}{l} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+,\\[4pt] \displaystyle\int_{\mathbb{R}^{n-1}} (\mathcal{N}_\kappa u)(x')\,\frac{dx'}{1+|x'|^{n-1}} < +\infty,\\[4pt] u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \ \textit{at } \mathcal{L}^{n-1}\textit{-a.e. point on} \ \ \mathbb{R}^{n-1}, \end{array}\right\} \Longrightarrow u = 0 \ \ \textit{in} \ \ \mathbb{R}^n_+. \tag{3.1.9}$$

Theorem 3.1.1 also interfaces tightly with the topic of boundary value problems. To elaborate on this aspect, we need more notation. Denote by $\mathbb{M}$ the collection of all (equivalence classes of) Lebesgue measurable functions $f : \mathbb{R}^{n-1} \to [-\infty, \infty]$ such that $|f| < \infty$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Also, call a subset $\mathbb{Y}$ of $\mathbb{M}$ a `function lattice` if the following properties hold:

(i) whenever $f, g \in \mathbb{M}$ satisfy $0 \leq f \leq g$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$ and $g \in \mathbb{Y}$ then necessarily $f \in \mathbb{Y}$;

(ii) $0 \leq f \in \mathbb{Y}$ implies $\lambda f \in \mathbb{Y}$ for every $\lambda \in (0, \infty)$;

(iii) $0 \leq f, g \in \mathbb{Y}$ implies $\max\{f, g\} \in \mathbb{Y}$.

In passing, note that, granted (i), one may replace (ii)-(iii) above by the condition: $0 \leq f, g \in \mathbb{Y}$ implies $f + g \in \mathbb{Y}$. As usual, we set $\log_+ t := \max\{0\,, \ln t\}$ for each $t \in (0, \infty)$. Also, the symbol $M$ is reserved for the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$; see (3.2.1).

We are now in a position to discuss the following refinement of [82, Theorem 1.1, p. 915].

**Corollary 3.1.3** (A Template for the Dirichlet Problem)**.** *Let $L$ be an $M \times M$ system with constant complex coefficients as in (1.2.1) satisfying (1.2.4), and fix an aperture parameter $\kappa > 0$. Also, assume that*

$$\mathbb{Y} \subseteq L^1\Big(\mathbb{R}^{n-1}\,, \frac{dx'}{1+|x'|^{n-1}}\Big), \quad \mathbb{Y} \textit{ is a function lattice,} \tag{3.1.10}$$

*and that*

$$\begin{array}{l}\mathbb{X} \textit{ is a collection of } \mathbb{C}^M\textit{-valued measurable} \\ \textit{functions on } \mathbb{R}^{n-1} \textit{ satisfying } M\mathbb{X} \subseteq \mathbb{Y}.\end{array} \tag{3.1.11}$$

*Then the $(\mathbb{X}, \mathbb{Y})$-Dirichlet boundary value problem for the system $L$ in the upper-half space, formulated as*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \\ Lu = 0 \quad \textit{in} \;\; \mathbb{R}^n_+, \\ \mathcal{N}_\kappa u \in \mathbb{Y}, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathbb{X}, \end{cases} \tag{3.1.12}$$

*has a unique solution. Moreover, the solution $u$ of (3.1.12) is given by*

$$u(x) = (P^L_t * f)(x') \quad \textit{for all} \quad x = (x', t) \in \mathbb{R}^{n-1} \times (0, \infty) = \mathbb{R}^n_+, \tag{3.1.13}$$

*where $P^L$ is the Poisson kernel for $L$ in $\mathbb{R}^n_+$, and satisfies*

$$(\mathcal{N}_\kappa u)(x') \leq C\, Mf(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{3.1.14}$$

*for some constant $C \in (0, \infty)$ that depends only on $L$, $n$, and $\kappa$.*

Corollary 3.1.3 contains as particular cases a multitude of well-posedness results for elliptic systems in the upper-half space. For example, one may take Muckenhoupt weighted Lebesgue spaces $\mathbb{X} := \big[L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})\big]^M$ and $\mathbb{Y} := L^p(\mathbb{R}^{n-1}, w\mathcal{L}^{n-1})$ with $p \in (1, \infty)$ and $w \in A_p$, or Morrey spaces in $\mathbb{R}^{n-1}$; for more on this, as well as other examples, see [82].

Here we wish to identify the most inclusive setting in which Corollary 3.1.3 yields a well-posedness result. Specifically, in view of the assumptions made in (3.1.10)-(3.1.11) it is natural to consider the linear space

$$\begin{aligned}\mathscr{X} &:= \Big\{f \in \big[L^1(\mathbb{R}^{n-1}\,, \tfrac{dx'}{1+|x'|^{n-1}})\big]^M : \; Mf \in L^1(\mathbb{R}^{n-1}\,, \tfrac{dx'}{1+|x'|^{n-1}})\Big\} \\ &= \Big\{f : \mathbb{R}^{n-1} \to \mathbb{C}^M : \;\; \text{measurable and} \;\; Mf \in L^1(\mathbb{R}^{n-1}\,, \tfrac{dx'}{1+|x'|^{n-1}})\Big\}\end{aligned} \tag{3.1.15}$$

(recall that $M$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$) equipped with the norm

$$\begin{aligned}\|f\|_{\mathscr{Z}} &:= \|f\|_{[L^1(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}})]^M} + \|Mf\|_{L^1\left(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}}\right)} \\ &\approx \|Mf\|_{L^1\left(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}}\right)}, \qquad \forall\, f \in \mathscr{Z}. \end{aligned} \tag{3.1.16}$$

Then, Corollary 3.1.3 applied with $\mathbb{X} := \mathscr{Z}$ and $\mathbb{Y} := L^1(\mathbb{R}^{n-1}, \frac{dx'|}{1+|x'|^{n-1}})$ yields the following result.

**Corollary 3.1.4** (The Most Inclusive Well-Posedness Result)**.** *Let $L$ be an $M \times M$ system with constant complex coefficients as in* (1.2.1) *satisfying* (1.2.4)*, and fix an aperture parameter $\kappa > 0$. Then the following boundary-value problem is well-posed:*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \quad Lu = 0 \ \ in \ \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} (\mathcal{N}_\kappa u)(x') \, \frac{dx'}{1+|x'|^{n-1}} < \infty, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{Z}. \end{cases} \tag{3.1.17}$$

The relevance of the fact that (3.1.1) implies (3.1.2) in the context of all the aforementioned boundary value problems (cf. (3.1.12), (3.1.17)) is that the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ is guaranteed to exist by the other conditions imposed on the function $u$ in the formulation of the said problems, and that the solution may be recovered from the boundary datum via convolution with the Poisson kernel canonically associated with the system $L$.

The type of boundary value problems treated here, in which the size of the solution is measured in terms of its nontangential maximal function and its trace is taken in a nontangential pointwise sense, has been dealt with in the particular case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$, in a number of monographs, including [8], [42], [115], [116], and [117]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle, or the Schwarz reflection principle for harmonic functions. Neither of the latter techniques may be adapted successfully to prove uniqueness in the case of general systems treated here, and our approach is more in line with the work in [82] (which involves Green function estimates and a sharp version of the Divergence Theorem), with some significant refinements. A remarkable aspect is that our approach works for the entire class of elliptic systems $L$ as in (1.2.1) satisfying (1.2.4).

## 3.2 Preliminary matters

Recall the notation related to the upper-half space introduced in Section 4.1. Additionally, the origin in $\mathbb{R}^{n-1}$ is denoted by $0'$ and we let $B_{n-1}(x', r)$ stand for the $(n-1)$-dimensional Euclidean ball of radius $r$ centered at $x' \in \mathbb{R}^{n-1}$. With this terminology, the action of the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$ on any Lebesgue measurable function $f$ defined in $\mathbb{R}^{n-1}$ is given by

$$(Mf)(x') := \sup_{r>0} \fint_{B_{n-1}(x',r)} |f| \, d\mathcal{L}^{n-1}, \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{3.2.1}$$

where, as usual, the barred integral denotes mean average (for functions which are $\mathbb{C}^M$-valued the average is taken componentwise).

We next recall a useful weak compactness result from [82, Lemma 6.2, p. 956]. To state it, denote by $\mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$ the space of continuous functions in $\mathbb{R}^{n-1}$ vanishing at infinity.

**Lemma 3.2.1.** *Let $v : \mathbb{R}^{n-1} \to (0,\infty)$ be a Lebesgue measurable function and consider a sequence $\{f_j\}_{j\in\mathbb{N}}$ in the weighted Lebesgue space $L^1(\mathbb{R}^{n-1}, v\mathcal{L}^{n-1})$ such that*

$$F := \sup_{j\in\mathbb{N}} |f_j| \in L^1(\mathbb{R}^{n-1}, v\mathcal{L}^{n-1}). \tag{3.2.2}$$

*Then there exists a subsequence $\{f_{j_k}\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ and $f \in L^1(\mathbb{R}^{n-1}, v\mathcal{L}^{n-1})$ with the property that*

$$\int_{\mathbb{R}^{n-1}} f_{j_k}(x')\varphi(x')v(x')\, dx' \longrightarrow \int_{\mathbb{R}^{n-1}} f(x')\varphi(x')v(x')\, dx' \quad \text{as} \quad k \to \infty, \tag{3.2.3}$$

*for every $\varphi \in \mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$.*

A key ingredient in the proof of our main result is understanding the nature of the Green function associated with a given elliptic system. While we elaborate on this topic in Theorem 3.2.3 below, we begin by providing a suitable definition for the said Green function (which, in particular, is going to ensure its uniqueness). To set the stage, denote by $\mathcal{D}'(\mathbb{R}^n_+)$ the space of distributions in $\mathbb{R}^n_+$.

**Definition 3.2.2.** Let $L$ be an $M \times M$ system with constant complex coefficients as in (1.2.1) satisfying (1.2.4). Call $G^L(\cdot,\cdot) : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus {\rm diag} \to \mathbb{C}^{M\times M}$ a `Green function` for $L$ in $\mathbb{R}^n_+$ provided for each $y = (y', y_n) \in \mathbb{R}^n_+$ the following properties hold (for some aperture parameter $\kappa > 0$):

$$G^L(\cdot\,, y) \in \big[L^1_{\rm loc}(\mathbb{R}^n_+)\big]^{M\times M}, \tag{3.2.4}$$

$$G^L(\cdot\,, y)\Big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathcal{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+, \tag{3.2.5}$$

$$\int_{\mathbb{R}^{n-1}} \Big(\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus \overline{B(y,y_n/2)}}\, G^L(\cdot\,, y)\Big)(x') \frac{dx'}{1+|x'|^{n-1}} < \infty, \tag{3.2.6}$$

$$L\big[G^L(\cdot\,, y)\big] = -\delta_y\, I_{M\times M} \ \text{ in } \ \big[\mathcal{D}'(\mathbb{R}^n_+)\big]^{M\times M}, \tag{3.2.7}$$

where the $M \times M$ system $L$ acts in the "dot" variable on the columns of $G$.

The existence and basic properties of the Green function just defined are discussed in our next theorem (a proof of which may be found in [83]). Before stating it, we make two conventions regarding notation. First, we agree to abbreviate ${\rm diag} := \{(x,x) : x \in \mathbb{R}^n_+\}$

for the diagonal in the Cartesian product $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. Second, given a function $G(\cdot,\cdot)$ of two vector variables, $(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag}$, for each $k \in \{1,\dots,n\}$ we agree to write $\partial_{X_k} G$ and $\partial_{Y_k} G$, respectively, for the partial derivative of $G$ with respect to $x_k$, and $y_k$. This convention may be iterated, lending a natural meaning to $\partial_X^\alpha \partial_Y^\beta G$, for each pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. We are now ready to present the result alluded to above.

**Theorem 3.2.3.** *Assume that $L$ is an $M \times M$ system with constant complex coefficient as in (1.2.1) satisfying (1.2.4). Then there exists a unique Green function $G^L(\cdot,\cdot)$ for $L$ in $\mathbb{R}^n_+$, in the sense of Definition 3.2.2. Moreover, this Green function also satisfies the following additional properties:*

*(1) Given $\kappa > 0$, for each $y \in \mathbb{R}^n_+$ and each compact neighborhood $K$ of $y$ in $\mathbb{R}^n_+$ there exists a finite constant $C_y = C(n, L, \kappa, K, y) > 0$ such that for every $x' \in \mathbb{R}^{n-1}$ there holds*

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K}(G^L(\cdot,y))(x') \le C_y \frac{1+\log_+|x'|}{1+|x'|^{n-1}}. \tag{3.2.8}$$

*Moreover, for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha|+|\beta| > 0$, there exists some constant $C_y = C(n, L, \kappa, \alpha, \beta, K, y) \in (0,\infty)$ such that*

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K}((\partial_X^\alpha \partial_Y^\beta G^L)(\cdot,y))(x') \le \frac{C_y}{1+|x'|^{n-2+|\alpha|+|\beta|}}. \tag{3.2.9}$$

*(2) For each fixed $y \in \mathbb{R}^n_+$, there holds*

$$G^L(\cdot\,,y) \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus B(y,\varepsilon))\big]^{M\times M} \quad \text{for every} \quad \varepsilon > 0. \tag{3.2.10}$$

*As a consequence of (3.2.10) and (3.2.5), for each fixed $y \in \mathbb{R}^n_+$ one has*

$$G^L(\cdot,y)\Big|_{\partial\mathbb{R}^n_+} = 0 \quad \text{everywhere on} \quad \mathbb{R}^{n-1}. \tag{3.2.11}$$

*(3) For each $\alpha, \beta \in \mathbb{N}_0^n$ the function $\partial_X^\alpha \partial_Y^\beta G^L$ is translation invariant in the tangential variables, in the sense that*

$$\begin{gathered}(\partial_X^\alpha \partial_Y^\beta G^L)(x-(z',0), y-(z',0)) = (\partial_X^\alpha \partial_Y^\beta G^L)(x,y) \\ \text{for each} \quad (x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag} \quad \text{and} \quad z' \in \mathbb{R}^{n-1},\end{gathered} \tag{3.2.12}$$

*and is positive homogeneous, in the sense that*

$$\begin{gathered}(\partial_X^\alpha \partial_Y^\beta G^L)(\lambda x, \lambda y) = \lambda^{2-n-|\alpha|-|\beta|}(\partial_X^\alpha \partial_Y^\beta G^L)(x,y) \\ \text{for each} \quad x, y \in \mathbb{R}^n_+ \quad \text{with} \quad x \neq y \quad \text{and} \quad \lambda \in (0,\infty), \\ \text{provided either } n \geq 3, \text{ or } |\alpha|+|\beta| > 0.\end{gathered} \tag{3.2.13}$$

*(4) If $G^{L^\top}(\cdot,\cdot)$ denotes the (unique, by the first part of the statement) Green function for $L^\top$ (the transposed of $L$) in $\mathbb{R}^n_+$, then*

$$G^L(x,y) = \Big[G^{L^\top}(y,x)\Big]^\top, \qquad \forall\,(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag}. \tag{3.2.14}$$

*Hence, as a consequence of* (3.2.14), (3.2.5), *and* (3.2.10), *for each fixed* $x \in \mathbb{R}^n_+$ *and* $\varepsilon > 0$,

$$G^L(x,\cdot) \in \left[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus B(x,\varepsilon))\right]^{M\times M} \quad \textit{and} \quad G^L(x,\cdot)\Big|_{\partial\mathbb{R}^n_+} = 0 \quad \textit{on} \quad \mathbb{R}^{n-1}. \tag{3.2.15}$$

(5) *For any multi-indices* $\alpha, \beta \in \mathbb{N}^n_0$ *there exists a finite constant* $C_{\alpha\beta} > 0$ *such that*

$$\begin{gathered} \left|(\partial^\alpha_X \partial^\beta_Y G^L)(x,y)\right| \leq C_{\alpha\beta}|x-y|^{2-n-|\alpha|-|\beta|}, \\ \forall\,(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \mathrm{diag},\ \textit{if either } n \geq 3,\ \textit{or } |\alpha|+|\beta|>0, \end{gathered} \tag{3.2.16}$$

*and, corresponding to* $|\alpha| = |\beta| = 0$ *and* $n = 2$, *there exists* $C \in (0,\infty)$ *such that*

$$\left|G^L(x,y)\right| \leq C + C\left|\ln|x-\overline{y}|\right|, \quad \forall\,(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \setminus \mathrm{diag}, \tag{3.2.17}$$

*where* $\overline{y} := (y', -y_n) \in \mathbb{R}^n$ *is the reflexion of* $y = (y', y_n) \in \mathbb{R}^n_+$ *across the boundary of the upper-half space.*

(6) *The Agmon-Douglis-Nirenberg Poisson kernel* $P^L = (P^L_{\gamma\alpha})_{1\leq\gamma,\alpha\leq M}$ *for* $L$ *in* $\mathbb{R}^n_+$ *from Theorem 1.2.4 is related to the Green function* $G^L$ *for* $L$ *in* $\mathbb{R}^n_+$ *according to the formula*

$$\begin{gathered} P^L_{\gamma\alpha}(z') = a^{\beta\alpha}_{nn}(\partial_{Y_n} G^L_{\gamma\beta})((z',1),0), \quad \forall\, z' \in \mathbb{R}^{n-1}, \\ \textit{for each} \ \ \alpha, \gamma \in \{1, \dots, M\}. \end{gathered} \tag{3.2.18}$$

We conclude by recording a suitable version of the divergence theorem recently obtained in [93]. To state it requires a few preliminaries which we dispense with first. Recall first that we write $\mathcal{E}'(\mathbb{R}^n_+)$ for the subspace of $\mathcal{D}'(\mathbb{R}^n_+)$ consisting of those distributions which are compactly supported. Hence,

$$\mathcal{E}'(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+) \quad \text{and} \quad L^1_{\mathrm{loc}}(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+). \tag{3.2.19}$$

For each compact set $K \subset \mathbb{R}^n_+$, define $\mathcal{E}'_K(\mathbb{R}^n_+) := \{u \in \mathcal{E}'(\mathbb{R}^n_+) : \operatorname{supp} u \subset K\}$ and consider

$$\begin{aligned} \mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) := \{u \in \mathcal{D}'(\mathbb{R}^n_+) :\ & \exists\, v_1 \in \mathcal{E}'_K(\mathbb{R}^n_+) \text{ and } \exists\, v_2 \in L^1(\mathbb{R}^n_+) \\ & \text{such that } u = v_1 + v_2 \text{ in } \mathcal{D}'(\mathbb{R}^n_+)\}. \end{aligned} \tag{3.2.20}$$

Also, introduce $\mathscr{C}^\infty_b(\mathbb{R}^n_+) := \mathscr{C}^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ and let $\left(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\right)^*$ denote its algebraic dual. Moreover, we let ${}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}(\cdot\,,\cdot)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)}$ denote the natural duality pairing between these spaces. It is useful to observe that for every compact set $K \subset \mathbb{R}^n_+$ one has

$$\mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) \subset \left(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\right)^*. \tag{3.2.21}$$

**Theorem 3.2.4** ([93]). *Assume that* $K \subset \mathbb{R}^n_+$ *is a compact set and that* $\vec{F} \in \left[L^1_{\mathrm{loc}}(\mathbb{R}^n_+)\right]^n$ *is a vector field satisfying the following conditions (for some aperture parameter* $\kappa > 0$*):*

*(a)* $\operatorname{div}\vec{F} \in \mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+)$, *where the divergence is taken in the sense of distributions;*

*(b) the nontangential maximal function* $\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus K}\vec{F}$ *belongs to* $L^1(\mathbb{R}^{n-1})$*;*

*(c) the nontangential boundary trace* $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}$ *exists (in* $\mathbb{C}^n$*) at* $\mathcal{L}^{n-1}$*-a.e. point in* $\mathbb{R}^{n-1}$*.*

*Then, with* $e_n := (0,\dots,0,1) \in \mathbb{R}^n$ *and "dot" denoting the standard inner product in* $\mathbb{R}^n$*,*

$$ {}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\big)\,d\mathcal{L}^{n-1}. \tag{3.2.22}$$

## 3.3 Proofs of main results

We take on the task of presenting the proof of Theorem 3.1.1.

*Proof of Theorem 3.1.1.* Fix an arbitrary point $x^\star \in \mathbb{R}^n_+$ and bring in $G^{L^\top}(\cdot, x^\star)$, the Green function with pole at $x^\star$ for $L^\top$, the transposed of the operator $L$ (cf. Definition 3.2.2 and Theorem 3.2.3 for details on this matter). For ease of notation, abbreviate

$$G(\cdot) := G^{L^\top}(\cdot, x^\star) \quad\text{in}\quad \mathbb{R}^n_+ \setminus \{x^\star\}. \tag{3.3.1}$$

By design, this is a matrix-valued function, say $G = (G_{\alpha\gamma})_{1\le\alpha,\gamma\le M}$. We shall apply Theorem 3.2.4 to a suitably chosen vector field. To set the stage, consider the compact set

$$K_\star := \overline{B(x^\star, r)} \subset \mathbb{R}^n_+, \qquad\text{where}\quad r := \operatorname{dist}(x^\star, \partial\mathbb{R}^n_+)\cdot \tfrac{\kappa}{2\sqrt{4+\kappa^2}}. \tag{3.3.2}$$

For each $\varepsilon > 0$ consider the function $u^\varepsilon : \overline{\mathbb{R}^n_+} \to \mathbb{C}^M$ given by

$$u^\varepsilon(x) := u(x', x_n + \varepsilon) \quad\text{for all}\quad x = (x', x_n) \in \overline{\mathbb{R}^n_+}. \tag{3.3.3}$$

Then

$$u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M, \quad Lu^\varepsilon = 0 \;\text{ in }\; \mathbb{R}^n_+, \;\text{ and }\; \mathcal{N}_\kappa u^\varepsilon \le \mathcal{N}_\kappa u \;\text{ on }\; \mathbb{R}^{n-1}. \tag{3.3.4}$$

Fix $\varepsilon > 0$ along with some $\beta \in \{1,\dots,M\}$ and, using the summation convention over repeated indices, define the vector field

$$\vec{F} := \Big(u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\,\partial_k G_{\gamma\beta} - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\,\partial_k u^\varepsilon_\gamma\Big)_{1\le j\le n} \quad\text{at } \mathcal{L}^n\text{-a.e. point in }\; \mathbb{R}^n_+. \tag{3.3.5}$$

From (3.3.5), Theorem 3.2.3, and the fact that $u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M$ it follows that

$$\vec{F} \in \big[L^1_{\text{loc}}(\mathbb{R}^n_+)\big]^n \cap \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+}\setminus K_\star)\big]^n \tag{3.3.6}$$

and, on account of (3.2.11) (used for $L^\top$ in place of $L$), we have

$$\vec{F}\Big|_{\partial\mathbb{R}^n_+} = \Big(\big(u^\varepsilon_\alpha\big|_{\partial\mathbb{R}^n_+}\big)a^{\gamma\alpha}_{kj}\big(\partial_k G_{\gamma\beta}\big)\big|_{\partial\mathbb{R}^n_+}\Big)_{1\le j\le n}. \tag{3.3.7}$$

Next, in the sense of distributions in $\mathbb{R}^n_+$, we may compute

$$\begin{aligned}\operatorname{div}\vec{F} &= (\partial_j u^\varepsilon_\alpha)\, a^{\gamma\alpha}_{kj}\, (\partial_k G_{\gamma\beta}) + u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\, (\partial_j\partial_k G_{\gamma\beta}) \\ &\quad - (\partial_j G_{\alpha\beta})\, a^{\alpha\gamma}_{jk}\, (\partial_k u^\varepsilon_\gamma) - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\, (\partial_j\partial_k u^\varepsilon_\gamma) \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.3.8}$$

where the last equality defines the $I_i$'s. Changing variables $j' = k$, $k' = j$, $\alpha' = \gamma$, and $\gamma' = \alpha$ in $I_3$ yields

$$I_3 = -(\partial_{k'} G_{\gamma'\beta})\, a^{\gamma'\alpha'}_{k'j'}\, (\partial_{j'} u^\varepsilon_{\alpha'}) = -I_1. \tag{3.3.9}$$

As regards $I_4$, we have

$$I_4 = -G_{\alpha\beta}\, (Lu^\varepsilon)_\alpha = 0, \tag{3.3.10}$$

by (3.3.4). Finally,

$$\begin{aligned} I_2 &= u^\varepsilon_\alpha (L_{A^\top} G_{\cdot\beta})_\alpha = u^\varepsilon_\alpha (L^\top G_{\cdot\beta})_\alpha \\ &= -u^\varepsilon_\alpha \delta_{\alpha\beta}\delta_{x^\star} = -u^\varepsilon_\beta\, \delta_{x^\star} = -u^\varepsilon_\beta(x^\star)\, \delta_{x^\star}. \end{aligned} \tag{3.3.11}$$

Collectively, these equalities permit us to conclude that, in the sense of distributions in $\mathbb{R}^n_+$,

$$\operatorname{div}\vec{F} = -u^\varepsilon_\beta(x^\star)\, \delta_{x^\star} \in \mathcal{E}'(\mathbb{R}^n_+). \tag{3.3.12}$$

In particular,

$$\operatorname{div}\vec{F} \in \mathcal{D}'(\mathbb{R}^n_+) \ \text{ induces a continuous functional in } \ \big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*. \tag{3.3.13}$$

Moving on, fix $x' \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$ and pick an arbitrary point

$$y = (y', y_n) \in \Gamma_{\kappa/2}(x') \setminus K_\star. \tag{3.3.14}$$

Choose a rectifiable path $\gamma : [0,1] \to \overline{\mathbb{R}^n_+}$ joining $(x', 0)$ with $y$ in $\Gamma_{\kappa/2}(x') \setminus K_\star$ and whose length is $\leq C y_n$. Then, for some constant $C \in (0,\infty)$ independent of $x'$ and $y$, we may estimate

$$\begin{aligned} |G(y)| &= |G(y) - G(x',0)| = \Big| \int_0^1 \frac{d}{dt}[G(\gamma(t))]\, dt \Big| \\ &= \Big| \int_0^1 (\nabla G)(\gamma(t)) \cdot \gamma'(t)\, dt \Big| \leq \Big( \sup_{\xi \in \gamma((0,1))} |(\nabla G)(\xi)| \Big) \int_0^1 |\gamma'(t)|\, dt \\ &\leq C y_n \cdot \mathcal{N}^{\mathbb{R}^n_+ \setminus K_\star}_{\kappa/2}(\nabla G)(x'), \end{aligned} \tag{3.3.15}$$

using the fact that $G$ vanishes on $\partial\mathbb{R}^n_+$, the Fundamental Theorem of Calculus, Chain Rule, and (1.1.4). Next, define

$$a := \frac{\kappa}{2(\kappa+1)} \in (0, \tfrac{1}{2}) \tag{3.3.16}$$

and write, using interior estimates (cf. Theorem 1.2.2) for the function $u^\varepsilon$,

$$
\begin{aligned}
|(\nabla u^\varepsilon)(y)| &\leq \frac{C}{y_n} \fint_{B(y,a\cdot y_n)} |u^\varepsilon(z)|\,dz \\
&\leq C y_n^{-1} \cdot \sup_{z\in\Gamma_\kappa(x')} |u^\varepsilon(z)| \leq C y_n^{-1} \cdot (\mathcal{N}_\kappa u^\varepsilon)(x'),
\end{aligned}
\tag{3.3.17}
$$

since having $z = (z', z_n) \in B(y, a \cdot y_n)$ entails

$$
y_n \leq z_n + |z - y| < z_n + a \cdot y_n \Longrightarrow y_n < (1-a)^{-1} z_n,
\tag{3.3.18}
$$

which, bearing in mind that $y$ is as in (3.3.14), permits us to conclude that

$$
\begin{aligned}
|z' - x'| &\leq |z' - y'| + |y' - x'| \leq |z - y| + (\kappa/2) y_n < a \cdot y_n + (\kappa/2) y_n \\
&= (\kappa/2 + a) y_n < \frac{\kappa/2 + a}{1-a} z_n = \kappa z_n, \quad \text{hence} \quad z \in \Gamma_\kappa(x').
\end{aligned}
\tag{3.3.19}
$$

Then combining (3.3.15) with (3.3.17) gives, on account of (3.2.9),

$$
\begin{aligned}
\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(|G||\nabla u^\varepsilon|)(x') &\leq C(\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(\nabla G))(x')(\mathcal{N}_\kappa u^\varepsilon)(x') \\
&\leq C(\mathcal{N}_\kappa u)(x') \frac{1}{1 + |x'|^{n-1}} \quad \text{at each point} \quad x' \in \mathbb{R}^{n-1}.
\end{aligned}
\tag{3.3.20}
$$

Since we also have

$$
\begin{aligned}
\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(|\nabla G||u^\varepsilon|)(x') &\leq (\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(\nabla G))(x')(\mathcal{N}_\kappa u^\varepsilon)(x') \\
&\leq C(\mathcal{N}_\kappa u)(x') \frac{1}{1 + |x'|^{n-1}} \quad \text{at each point} \quad x' \in \mathbb{R}^{n-1},
\end{aligned}
\tag{3.3.21}
$$

we conclude from (3.3.5), (3.3.20), (3.3.21), and the second line in (3.1.1) that

$$
\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star} \vec{F} \in L^1(\mathbb{R}^{n-1}).
\tag{3.3.22}
$$

Having established (3.3.6), (3.3.7), (3.3.13), and (3.3.22), Theorem 3.2.4 applies. To write the Divergence Formula (3.2.22) in this case, express $x^\star$ as $(x', t) \in \mathbb{R}^{n-1} \times (0, \infty)$.

Then, in view of (3.3.12) and (3.3.7) we may write

$$
\begin{aligned}
u_\beta(x^\star + \varepsilon e_n) = u^\varepsilon_\beta(x^\star) &= -{}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}(\operatorname{div}\vec{F}\,,\,1)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} \\
&= \int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\big|_{\partial\mathbb{R}^n_+}\big)\, d\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{kn} (\partial_k G_{\gamma\beta})(y',0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn} (\partial_n G_{\gamma\beta})(y',0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn} (\partial_{X_n} G^{L^\top}_{\gamma\beta})((y',0),x^\star)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn} (\partial_{Y_n} G^{L}_{\beta\gamma})(x^\star,(y',0))\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) a^{\gamma\alpha}_{nn} (\partial_{Y_n} G^{L}_{\beta\gamma})((x'-y',t),0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) t^{1-n} a^{\gamma\,\alpha}_{nn} (\partial_{Y_n} G^{L}_{\beta\gamma})(((x'-y')/t,1),0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y',\varepsilon) (P^L_{\beta\alpha})_t(x'-y')\, dy',
\end{aligned}
\tag{3.3.23}
$$

where the fifth equality uses the observation that $(\partial_k G)(y',0) = 0$ whenever $k < n$ since $G$ vanishes (in a smooth fashion) on $\mathbb{R}^{n-1} \times \{0\}$, the sixth equality is a consequence of (3.3.1), the seventh equality is implied by (3.2.14), the eighth equality makes use of (3.2.12) (bearing in mind that $x^\star = (x',t)$), the ninth equality is seen from (3.2.13), and the last equality comes from (3.2.18).

Since $\beta \in \{1,\dots,M\}$ and $x^\star = (x',t) \in \mathbb{R}^n_+$ have been arbitrarily chosen, the argument so far shows that

$$
u(x',t+\varepsilon) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y') f_\varepsilon(y')\, dy' \quad \text{for each} \quad x = (x',t) \in \mathbb{R}^n_+, \tag{3.3.24}
$$

where we have abbreviated

$$
f_\varepsilon := u(\cdot,\varepsilon) : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^M \quad \text{for each} \quad \varepsilon > 0. \tag{3.3.25}
$$

If we also consider the weight $v : \mathbb{R}^{n-1} \to (0,\infty)$ defined as $v(x') := (1 + |x'|^{n-1})^{-1}$ for each $x' \in \mathbb{R}^{n-1}$, then the last condition in (3.1.1) entails

$$
\sup_{\varepsilon>0} |f_\varepsilon| \le \mathcal{N}_\kappa u \in L^1(\mathbb{R}^{n-1}\,,\, v\,\mathcal{L}^{n-1}). \tag{3.3.26}
$$

Granted this, the weak-$*$ convergence result in Lemma 3.2.1 may be used for the sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbb{R}^{n-1}\,,\, v\,\mathcal{L}^{n-1})$ to conclude that there is some $f \in L^1(\mathbb{R}^{n-1}\,,\, v\,\mathcal{L}^{n-1})$ and some sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ which converges to zero with the property that

$$
\lim_{j\to\infty} \int_{\mathbb{R}^{n-1}} \varphi(y') f_{\varepsilon_j}(y') \frac{dy'}{1+|y'|^{n-1}} = \int_{\mathbb{R}^{n-1}} \varphi(y') f(y') \frac{dy'}{1+|y'|^{n-1}} \tag{3.3.27}
$$

for every continuous function $\varphi \in \mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$. The fact that there exists a constant $C \in (0,\infty)$ for which

$$|P^L(z')| \leq \frac{C}{(1+|z'|^2)^{n/2}} \quad \text{for each} \quad z' \in \mathbb{R}^{n-1} \tag{3.3.28}$$

(see (1.2.26)) ensures for each fixed point $(x',t) \in \mathbb{R}^n_+$ the assignment

$$\begin{gathered} \mathbb{R}^{n-1} \ni y' \mapsto \varphi(y') := (1+|y'|^{n-1})P^L_t(x'-y') \in \mathbb{C}^{M\times M} \\ \text{is a continuous function which vanishes at infinity.} \end{gathered} \tag{3.3.29}$$

At this stage, from (3.3.24) and (3.3.27) used for the function $\varphi$ defined in (3.3.29) we obtain (bearing in mind that $u$ is continuous in $\mathbb{R}^n_+$) that

$$u(x',t) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y')f(y')\,dy' \quad \text{for each} \quad x=(x',t) \in \mathbb{R}^n_+. \tag{3.3.30}$$

With this in hand, and since $L^1(\mathbb{R}^{n-1}, v\,\mathcal{L}^{n-1}) \subseteq L^1\Big(\mathbb{R}^{n-1}, \dfrac{dx'}{1+|x'|^n}\Big)$, we may invoke Theorem 1.2.4(c) to conclude that

$$u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists and equals } f \text{ at } \mathcal{L}^{n-1}\text{-a.e. point in } \mathbb{R}^{n-1}. \tag{3.3.31}$$

Once this has been established, all conclusions in (3.1.2) are implied by (3.3.30)-(3.3.31). $\square$

We close by presenting the proof of Corollary 3.1.3.

*Proof of Corollary 3.1.3.* As a preamble, let us first show that

$$\mathbb{X} \subseteq \Big[L^1\Big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\Big)\Big]^M. \tag{3.3.32}$$

To justify this, pick some arbitrary $f \in \mathbb{X}$. Then the inclusion in (3.1.11) gives that $\mathcal{M}f \in \mathbb{Y}$, hence $\mathcal{M}f$ is not identically $+\infty$. This implies that $f \in [L^1_{\rm loc}(\mathbb{R}^{n-1})]^M$ which, in concert with Lebesgue's Differentiation Theorem, implies that $|f| \leq \mathcal{M}f$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since $\mathbb{Y}$ is a function lattice, it follows that $|f| \in \mathbb{Y}$. Thus, ultimately, (3.3.32) holds by virtue of the inclusion in (3.1.10).

To prove the existence of a solution for (3.1.12), given any $f \in \mathbb{X}$ define $u$ as in (3.1.13). Note that (3.3.32) ensures that Theorem 1.2.4(c) is applicable. In turn, this guarantees that $u$ is a well-defined null-solution of $L$ belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$, satisfying the boundary condition $u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$, as well as the pointwise estimate in (3.1.14). The latter property, together with the last conditions imposed in (3.1.11) and (3.1.10), guarantees $\mathcal{N}_\kappa u \in \mathbb{Y}$. Thus, $u$ is indeed a solution for (3.1.12).

At this stage, there remains to establish that the boundary value problem (3.1.12) can have at most one solution. To this end, assume that both $u_1$ and $u_2$ solve (3.1.12) for the same datum $f \in \mathbb{X}$ and set $u := u_1 - u_2 \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$. Then $Lu = 0$ in $\mathbb{R}^n_+$

and $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = 0$ at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since we also have $\mathcal{N}_\kappa u_1, \mathcal{N}_\kappa u_2 \in \mathbb{Y}$, the pointwise estimate

$$0 \le \mathcal{N}_\kappa u \le \mathcal{N}_\kappa u_1 + \mathcal{N}_\kappa u_2 \le 2\max\{\mathcal{N}_\kappa u_1\,,\,\mathcal{N}_\kappa u_2\} \quad \text{on} \quad \mathbb{R}^{n-1} \tag{3.3.33}$$

forces $\mathcal{N}_\kappa u \in \mathbb{Y}$ by the properties of the function lattice $\mathbb{Y}$. Granted this, Corollary 3.1.2 applies (thanks to the first condition in (3.1.10)) and gives that $u \equiv 0$ in $\mathbb{R}^n_+$. Hence $u_1 = u_2$, as wanted. $\square$

# CHAPTER 4

# The generalized Hölder and Morrey-Campanato Dirichlet problems for elliptic systems in the upper-half space

We prove well-posedness results for the Dirichlet Problem in the upper-half space $\mathbb{R}^n_+$ for homogeneous, second-order, constant complex coefficient elliptic systems with boundary data in generalized Hölder spaces $\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and in generalized Morrey-Campanato spaces $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ under certain assumptions on the growth function $\omega$. We also identify a class of growth functions $\omega$ for which $\mathscr{C}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M) = \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and for which the aforementioned well-posedness results are equivalent, in the sense that they have the same unique solution, satisfying natural regularity properties and estimates.

The material in this chapter is based on joint work with J.M. Martell and M. Mitrea (cf. [79]).

## Contents

## 4.1 Introduction

This chapter is devoted to studying the Dirichlet Problem for elliptic systems in the upper-half space with data in generalized Hölder and generalized Morrey-Campanato spaces. As a byproduct of the PDE-based techniques developed here, we are able to establish the equivalence of these function spaces. To be more specific requires introducing some notation.

With each system $L$ as in (1.2.1) satisfying (1.2.4) one may associate a Poisson kernel, $P^L$, which is a $\mathbb{C}^{M\times M}$-valued function defined in $\mathbb{R}^{n-1}$ described in detail in Theorem 1.2.4. This Poisson kernel has played a pivotal role in the treatment of the Dirichlet Problem with data in $L^p$, BMO, VMO and Hölder spaces (see [82, 85]). For now, we make the observation that the Poisson kernel gives rise to a nice approximation to the identity in $\mathbb{R}^{n-1}$ by setting $P_t^L(x') = t^{1-n}P^L(x'/t)$ for every $x' \in \mathbb{R}^{n-1}$ and $t > 0$.

For every point $x \in \mathbb{R}^n$ write $x = (x', t)$, where $x' \in \mathbb{R}^{n-1}$ corresponds to the first $n-1$ coordinates of $x$, and $t \in \mathbb{R}$ is the last coordinate of $x$. As is customary, we shall let $\mathbb{R}^n_+ := \{x = (x', t) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, t > 0\}$ denote the upper-half space in $\mathbb{R}^n$, and typically identify its boundary with $(n-1)$-dimensional Euclidean space, via $\partial\mathbb{R}^n_+ \ni (x', 0) \equiv x' \in \mathbb{R}^{n-1}$. In this case, the nontangential approach regions defined in (1.1.1) are cones and we agree to denote the cone with vertex at $x' \in \mathbb{R}^{n-1}$ and aperture $\kappa > 0$ as

$$\Gamma_\kappa(x') := \{y = (y', t) \in \mathbb{R}^n_+ : |x' - y'| < \kappa t\}. \tag{4.1.1}$$

The nontangential limit of $u$ at $x' \in \mathbb{R}^{n-1}$ (that is, the limit from within the cone $\Gamma_\kappa(x')$ defined as in (4.1.1)) is denoted by $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x')$ following the terminology introduced in Section 1.1. The unrestricted pointwise trace of a vector-valued function $u$ defined in $\mathbb{R}^n_+$ at each $x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ is taken to be

$$(u|_{\partial\mathbb{R}^n_+})(x') := \lim_{\mathbb{R}^n_+ \ni y \to (x',0)} u(y), \quad x' \in \mathbb{R}^{n-1}, \tag{4.1.2}$$

whenever such a limit exists exists.

Remember the definitions of growth function and generalized Hölder space in Definitions 1.3.1 and 1.3.5. Growth functions are always defined on $(0, \infty)$ throughout this chapter.

**Definition 4.1.1.** Given a growth function $\omega$ along with some integrability exponent $p \in [1, \infty)$, the associated `generalized Morrey-Campanato space` in $\mathbb{R}^{n-1}$ is defined as

$$\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) := \{f \in \big[L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})\big]^M : \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} < \infty\}, \tag{4.1.3}$$

where $\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}$ stands for the semi-norm

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} := \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))}\Big(\fint_Q |f(x') - f_Q|^p \, dx'\Big)^{1/p}. \tag{4.1.4}$$

The choice $\omega(t) := t^\alpha$ with $\alpha \in (0,1)$ corresponds to the classical Morrey-Campanato spaces, while the special case $\omega(t) := 1$ yields the usual space of functions of bounded mean oscillations (BMO). We also define, for every $u \in \big[\mathscr{C}^1(\mathbb{R}^n_+)\big]^M$ and $q \in (0,\infty)$,

$$\|u\|_{**}^{(\omega,q)} := \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))}\bigg(⨍_Q \Big(\int_0^{\ell(Q)} |(\nabla u)(x',t)|^2\, t\, dt\Big)^{q/2} dx'\bigg)^{1/q}. \tag{4.1.5}$$

As far as this semi-norm is concerned, there are two reasonable candidates for the end-point $q=\infty$ (see Proposition 4.3.1 and Lemma 4.4.1). First, we may consider

$$\|u\|_{**}^{(\omega,\exp)} := \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(Q))}\bigg\|\Big(\int_0^{\ell(Q)} |(\nabla u)(\cdot,t)|^2\, t\, dt\Big)^{1/2}\bigg\|_{\exp L,Q} \tag{4.1.6}$$

where $\|\cdot\|_{\exp L,Q}$ is the version of the norm in the Orlicz space $\exp L$ localized and normalized relative to $Q$, i.e.,

$$\|f\|_{\exp L,Q} := \inf\bigg\{t>0 : ⨍_Q \Big(e^{\frac{|f(x')|}{t}}-1\Big)\,dx' \le 1\bigg\}. \tag{4.1.7}$$

Second, corresponding to the limiting case $q=\infty$ we may consider

$$\|u\|_{**}^{(\omega,\infty)} := \sup_{(x',t)\in\mathbb{R}^n_+} \frac{t}{\omega(t)}|(\nabla u)(x',t)|. \tag{4.1.8}$$

We are ready to describe our main result concerning the Dirichlet problems with data in generalized Hölder and generalized Morrey-Campanato spaces for homogeneous second-order strongly elliptic systems of differential operators with constant complex coefficients (cf. (1.2.1) and (1.2.4)). In Section 4.7 (cf. Theorems 4.7.1-4.7.2), we weaken the condition (4.1.9) and still prove well-posedness for the two Dirichlet problems. The main difference is that in that case they are no longer equivalent as (4.1.15) might fail (see Example 4.7.4).

**Theorem 4.1.2.** *Consider a strongly elliptic constant complex coefficient second-order $M\times M$ system $L$, as in* (1.2.1) *satisfying* (1.2.4). *Also, fix an aperture parameter $\kappa>0$, $p\in[1,\infty)$ along with $q\in(0,\infty]$, and let $\omega$ be a growth function satisfying, for some finite constant $C_\omega \ge 1$,*

$$\int_0^t \omega(s)\frac{ds}{s} + t\int_t^\infty \frac{\omega(s)}{s}\frac{ds}{s} \le C_\omega\,\omega(t) \quad \textit{for each} \quad t\in(0,\infty). \tag{4.1.9}$$

*Then the following statements are true.*

(a) *The generalized Hölder Dirichlet Problem for the system $L$ in $\mathbb{R}^n_+$, i.e.,*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \\ Lu = 0 \quad \textit{in} \quad \mathbb{R}^n_+, \\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} < \infty, \\ u\big|_{\partial\mathbb{R}^n_+} = f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M) \quad \textit{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{4.1.10}$$

*is well-posed. More specifically, there is a unique solution which is given by*

$$u(x',t) = (P^L_t * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+, \tag{4.1.11}$$

*where $P^L$ denotes the Poisson kernel for $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4. In addition, $u$ belongs to the space $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, satisfies $u|_{\partial\mathbb{R}^n_+} = f$, and there exists a finite constant $C = C(n, L, \omega) \geq 1$ such that*

$$C^{-1}[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{4.1.12}$$

(b) *The generalized Morrey-Campanato Dirichlet Problem for $L$ in $\mathbb{R}^n_+$, formulated as*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \\ Lu = 0 \ \ in \ \ \mathbb{R}^n_+, \\ \|u\|^{(\omega,q)}_{**} < \infty, \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \ \ a.e. \ on \ \ \mathbb{R}^{n-1}, \end{cases} \tag{4.1.13}$$

*is well-posed. More precisely, there is a unique solution (4.1.13) which is given by (4.1.11). In addition, $u$ belongs to $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, satisfies $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$, and there exists a finite constant $C = C(n, L, \omega, p, q) \geq 1$ such that*

$$C^{-1}\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \|u\|^{(\omega,q)}_{**} \leq C\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\,. \tag{4.1.14}$$

*Furthermore, all these properties remain true if $\|\cdot\|^{(\omega,q)}_{**}$ is replaced everywhere by $\|\cdot\|^{(\omega,\exp)}_{**}$.*

(c) *The following equality between vector spaces holds*

$$\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M) = \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \tag{4.1.15}$$

*with equivalent norms, where the right-to-left inclusion is understood in the sense that for each $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ there exists a unique $\widetilde{f} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$ with the property that $f = \widetilde{f}$ a.e. in $\mathbb{R}^{n-1}$.*

*As a result, the Dirichlet problems (4.1.10) and (4.1.13) are equivalent. Specifically, for any pair of boundary data which may be identified in the sense of (4.1.15) these problems have the same unique solution (given by (4.1.11)).*

A few comments regarding the previous result. In Lemma 4.2.1 we shall prove that, for growth functions as in (4.1.9), each $u \in \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)$ extends uniquely to a function $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$. Hence, the ordinary restriction $u|_{\partial\mathbb{R}^n_+}$ is well-defined in the context of item (a) of Theorem 4.1.2. In item (b) the situation is slightly different. One can first show that $u$ extends to a continuous function up to, and including, the boundary. Hence, the non-tangential pointwise trace agrees with the restriction to the boundary everywhere. However, since functions in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ are canonically identified whenever they agree outside of a set of zero Lebesgue measure, the boundary condition in (4.1.13) is most naturally formulated by asking that the non-tangential boundary trace agrees with the boundary datum almost everywhere. The same type of issue arises when interpreting (4.1.15). Specifically, while the left-to-right inclusion has a clear meaning, the converse inclusion should be interpreted as saying that each equivalence class in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ (induced by the aforementioned identification) has a unique representative from $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$.

The following result, providing a characterization of the generalized Hölder and generalized Morrey-Campanato spaces in terms of the boundary traces of solutions, is a byproduct of the proof of the above theorem.

**Corollary 4.1.3.** *Let $L$ be a strongly elliptic, constant complex coefficient, second-order $M\times M$ system in $\mathbb{R}^n$. Fix $p\in[1,\infty)$ along with $q\in(0,\infty)$, and let $\omega$ be a growth function for which (4.1.9) holds. Then for every function $u\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$ satisfying $Lu=0$ in $\mathbb{R}^n_+$ one has*

$$\|u\|_{**}^{(\omega,q)}\approx\|u\|_{**}^{(\omega,\exp)}\approx\|u\|_{**}^{(\omega,\infty)}\approx[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}\tag{4.1.16}$$

*where the implicit proportionality constants depend only on $L$, $n$, $q$, and the constant $C_\omega$ in (4.1.9). Moreover,*

$$\begin{aligned}\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)&=\Big\{\,u\big|_{\partial\mathbb{R}^n_+}:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M,\ Lu=0\ \text{in}\ \mathbb{R}^n_+,\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}<\infty\Big\}\\&=\Big\{\,u\big|_{\partial\mathbb{R}^n_+}:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M,\ Lu=0\ \text{in}\ \mathbb{R}^n_+,\ \|u\|_{**}^{(\omega,q)}<\infty\Big\}\\&=\Big\{\,u\big|_{\partial\mathbb{R}^n_+}:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M,\ Lu=0\ \text{in}\ \mathbb{R}^n_+,\ \|u\|_{**}^{(\omega,\exp)}<\infty\Big\}\\&=\Big\{\,u\big|_{\partial\mathbb{R}^n_+}:\,u\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M,\ Lu=0\ \text{in}\ \mathbb{R}^n_+,\ \|u\|_{**}^{(\omega,\infty)}<\infty\Big\}.\end{aligned}\tag{4.1.17}$$

The plan of the chapter is as follows. In Section 4.2 we present some properties of the growth functions and study some of the features of the generalized Hölder and Morrey-Campanato spaces which are relevant to this work. Section 4.3 is reserved for collecting some known results for elliptic systems, and for giving the proof of Proposition 4.3.1, where some a priori estimates for the null-solutions of such systems are established. In turn, these estimates allow us to compare the semi-norm $\|\cdot\|_{**}^{(\omega,q)}$ (corresponding to various values of $q$) with $[\cdot]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}$. In Section 4.4 we present a John-Nirenberg type inequality of real-variable nature, generalizing some results in [50, 49] by allowing more flexibility due to the involvement of growth functions. This is interesting and useful in its own right. In addition, we are able to show exponential decay for the measure of the associated level sets which, in turn, permits deriving estimates not only in arbitrary $L^q$ spaces but also in the space $\exp L$. Our approach for deriving such results is different from [50, 49], and uses some ideas which go back to a proof of the classical John-Nirenberg exponential integrability for BMO functions due to Calderón. As a matter of fact, our abstract method yields easily Calderón's classical result. In Section 4.5 we prove the existence of solutions for the Dirichlet problems with boundary data in $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$ and $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$. Section 4.6 contains a Fatou-type result for null-solutions of a strongly elliptic system $L$ belonging to the space $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, which will be a key ingredient when establishing uniqueness for the boundary value problems formulated in Theorem 4.1.2. Finally, in Section 4.7, combining the main results of the previous two sections yields two well-posedness results under different assumptions on the growth function: one for boundary data in $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and solutions in $\dot{\mathscr{C}}^{\omega}(\mathbb{R}^n_+, \mathbb{C}^M)$, and another one for boundary data in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and solutions satisfying $\|u\|_{**}^{(\omega,q)} < \infty$ for some $0 < q \leq \infty$, or even in the case where $q$ is replaced by exp. In concert, these two results cover all claims of Theorem 4.1.2.

## 4.2 Growth functions, generalized Hölder and Morrey-Campanato spaces

We begin by studying some basic properties of growth functions. As explained in the introduction, we ultimately wish to work with growth functions satisfying conditions weaker than (4.1.9). Indeed, the two mains conditions that we will consider are

$$\int_0^1 \omega(s) \frac{ds}{s} < \infty, \tag{4.2.1}$$

and

$$t \int_t^\infty \frac{\omega(s)}{s} \frac{ds}{s} \leq C'_\omega\, \omega(t), \qquad \forall\, t \in (0, \infty), \tag{4.2.2}$$

for some finite constant $C'_\omega \geq 1$. In what follows, $C'_\omega$ will always denote the constant in (4.2.2). Clearly, if $\omega$ satisfies it satisfies (4.1.9) then both (4.2.1) and (4.2.2) hold but the reverse implication is not true in general (see Example 4.7.4 in this regard).

Later on, we will need the auxiliary function $W$ defined as

$$W(t) := \int_0^t \omega(s) \frac{ds}{s} \quad \text{for each} \quad t \in (0, \infty). \tag{4.2.3}$$

Note that (4.2.1) gives that $W(t) < \infty$ for every $t > 0$. Then (4.1.9) holds if and only if (4.2.2) holds and there exists $C \in (0, \infty)$ such that $W(t) \leq C\, \omega(t)$ for each $t \in (0, \infty)$.

The following lemma gathers some useful properties on growth functions satisfying condition (4.2.2).

**Lemma 4.2.1.** *Given a growth function $\omega$ satisfying (4.2.2), the following statements are true.*

(a) *Whenever $0 < t_1 \leq t_2 < \infty$, one has*

$$\frac{\omega(t_2)}{t_2} \leq C'_\omega \frac{\omega(t_1)}{t_1}. \tag{4.2.4}$$

(b) *For every $t \in (0, \infty)$ one has*

$$\omega(2t) \leq 2C'_\omega\, \omega(t). \tag{4.2.5}$$

(c) *One has* $\lim_{t\to\infty} \omega(t)/t = 0$.

(d) *For each set* $E \subseteq \mathbb{R}^n$ *one has* $\dot{\mathscr{C}}^\omega(E, \mathbb{C}^M) = \dot{\mathscr{C}}^\omega(\overline{E}, \mathbb{C}^M)$, *with equivalent norms. More specifically, the restriction map*

$$\dot{\mathscr{C}}^\omega(\overline{E}, \mathbb{C}^M) \ni u \longmapsto u\big|_E \in \dot{\mathscr{C}}^\omega(E, \mathbb{C}^M) \tag{4.2.6}$$

*is a linear isomorphism which is continuous in the precise sense that, under the canonically identification of functions* $u \in \dot{\mathscr{C}}^\omega(\overline{E}, \mathbb{C}^M)$ *with* $u\big|_E \in \dot{\mathscr{C}}^\omega(E, \mathbb{C}^M)$, *one has*

$$[u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\overline{E},\mathbb{C}^M)} \leq 2C'_\omega [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \tag{4.2.7}$$

*for each* $u \in \dot{\mathscr{C}}^\omega(E, \mathbb{C}^M)$.

*Proof.* We start observing that for every $t > 0$

$$\frac{\omega(t)}{t} \leq \int_t^\infty \frac{\omega(s)}{s}\frac{ds}{s} \leq C'_\omega \frac{\omega(t)}{t}. \tag{4.2.8}$$

The first inequality uses that $\omega$ is non-decreasing and the second is just (4.2.2). Then, given $t_1 \leq t_2$, we may write

$$\frac{\omega(t_2)}{t_2} \leq \int_{t_2}^\infty \frac{\omega(s)}{s}\frac{ds}{s} \leq \int_{t_1}^\infty \frac{\omega(s)}{s}\frac{ds}{s} \leq C'_\omega \frac{\omega(t_1)}{t_1}, \tag{4.2.9}$$

proving (a). The doubling property in (b) follows at once from (a) by taking $t_2 := 2t_1$ in (4.2.4). Next, the claim in (c) is justified by passing to limit $t \to \infty$ in the first inequality in (4.2.8) and using Lebesgue's Dominated Convergence Theorem.

Turning our attention to (d), fix an arbitrary $u \in \dot{\mathscr{C}}^\omega(E, \mathbb{C}^M)$. As noted earlier, this membership ensures that $u$ is uniformly continuous, hence $u$ extends uniquely to a continuous function $v$ on $\overline{E}$. To show that $v$ belongs to $\dot{\mathscr{C}}^\omega(\overline{E}, \mathbb{C}^M)$ pick two arbitrary distinct points $y, z \in \overline{E}$ and choose two sequences $\{y_k\}_{k\in\mathbb{N}}$, $\{z_k\}_{k\in\mathbb{N}}$ of points in $E$ such that $y_k \to x$ and $z_k \to z$ as $k \to \infty$. By discarding finitely many terms, there is no loss of generality in assuming that $|y_k - z_k| < 2|y - z|$ for each $k \in \mathbb{N}$. Relying on the fact that $\omega$ is non-decreasing and (4.2.5), we may then write

$$\begin{aligned} |v(y) - v(z)| &= \lim_{k\to\infty} |u(y_k) - u(z_k)| \leq [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)} \limsup_{k\to\infty} \omega(|y_k - z_k|) \\ &\leq [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)}\, \omega(2|y - z|) \leq 2C'_\omega [u]_{\dot{\mathscr{C}}^\omega(E,\mathbb{C}^M)}\, \omega(|y - z|). \end{aligned} \tag{4.2.10}$$

From this, all claims in (d) follow, completing the proof of the lemma. $\square$

In the following lemma we treat $W$ defined in (4.2.3) as a growth function depending on the original $\omega$.

**Lemma 4.2.2.** *Let* $\omega$ *be a growth function satisfying* (4.2.1) *and* (4.2.2), *and let* $W(t)$ *be defined as in* (4.2.3). *Then* $W : (0, \infty) \to (0, \infty)$ *is a growth function satisfying* (4.2.2) *with*

$$C'_W \leq 1 + (C'_\omega)^2. \tag{4.2.11}$$

*Moreover,*

$$\omega(t) \leq C'_\omega W(t) \quad \textit{for each} \quad t \in (0, \infty). \tag{4.2.12}$$

*Proof.* By design, $W$ is a non-decreasing function and, thanks to Lebesgue's Dominated Convergence Theorem and (4.2.1) we have $W(t) \to 0$ as $t \to 0^+$. Also, on account of (4.2.4), for each $t \in (0, \infty)$ we may write

$$\omega(t) = \int_0^t \frac{\omega(t)}{t}\, ds \leq C'_\omega \int_0^t \frac{\omega(s)}{s}\, ds = C'_\omega W(t), \tag{4.2.13}$$

proving (4.2.12). In turn, Fubini's Theorem, (4.2.2), and (4.2.12) permit us to estimate

$$\begin{aligned} t\int_t^\infty \frac{W(s)}{s}\frac{ds}{s} &= t\int_t^\infty \left(\int_0^s \omega(\lambda)\frac{d\lambda}{\lambda}\right)\frac{ds}{s^2} \\ &= t\int_0^t \left(\int_t^\infty \frac{ds}{s^2}\right)\omega(\lambda)\frac{d\lambda}{\lambda} + t\int_t^\infty \left(\int_\lambda^\infty \frac{ds}{s^2}\right)\omega(\lambda)\frac{d\lambda}{\lambda} \\ &= t\int_0^t \frac{1}{t}\omega(\lambda)\frac{d\lambda}{\lambda} + t\int_t^\infty \frac{\omega(\lambda)}{\lambda}\frac{d\lambda}{\lambda} \\ &\leq W(t) + C'_\omega\, \omega(t) \\ &\leq \Big(1 + (C'_\omega)^2\Big)\, W(t), \end{aligned} \tag{4.2.14}$$

for each $t \in (0, \infty)$. This shows that $W$ satisfies (4.2.2) with constant $C'_W \leq 1 + (C'_\omega)^2$. $\square$

Remember the definition of $L^p$-based mean oscillation in (1.2.25). The following lemma gathers some results from [85, Lemmas 2.1 and 2.2].

**Lemma 4.2.3.** *Let $f \in \big[L^1_{\rm loc}(\mathbb{R}^{n-1})\big]^M$.*

(a) *For every $p, q \in [1, \infty)$ there exists some finite $C = C(p, q, n) > 1$ such that*

$$C^{-1} \operatorname{osc}_p(f; r) \leq \operatorname{osc}_q(f; r) \leq C \operatorname{osc}_p(f; r), \quad \forall\, r > 0. \tag{4.2.15}$$

(b) *For every $\varepsilon > 0$,*

$$\int_1^\infty \operatorname{osc}_1(f; s)\frac{ds}{s^{1+\varepsilon}} < \infty \Longrightarrow f \in \left[L^1\Big(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^{n-1+\varepsilon}}\Big)\right]^M. \tag{4.2.16}$$

We augment Lemma 4.2.3 with similar results which involve generalized Morrey-Campanato spaces and generalized Hölder spaces.

**Lemma 4.2.4.** *Let $\omega$ be a growth function and fix $p \in [1, \infty)$. Then the following properties are valid.*

(a) *If $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, then*

$$\operatorname{osc}_p(f; r) \leq \omega(r)\, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \quad \textit{for each} \quad r \in (0, \infty). \tag{4.2.17}$$

(b) *If $\omega$ satisfies (4.2.2), then for each $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ one has*

$$\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \sqrt{n-1}\, C'_\omega [f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \tag{4.2.18}$$

*and*

$$\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1},\mathbb{C}^M) \subseteq \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \subseteq \Big[L^1\Big(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^n}\Big)\Big]^M. \tag{4.2.19}$$

*Proof.* Note that given any $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ and $r>0$, based on (1.2.25), the fact that $\omega$ is non-decreasing, and (4.1.4) we may write

$$\begin{aligned}\operatorname{osc}_p(f;r) &= \sup_{\substack{Q\subseteq\mathbb{R}^{n-1}\\ \ell(Q)\le r}} \omega(\ell(Q))\frac{1}{\omega(\ell(Q))}\Big(⨍_Q |f(x')-f_Q|^p\,dx'\Big)^{1/p}\\ &\le \omega(r)\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\,,\end{aligned}\tag{4.2.20}$$

proving (a). Consider next the claims in (b). Given any $f \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1},\mathbb{C}^M)$, a combination of (4.1.4), (1.3.17), and (4.2.4) yields

$$\begin{aligned}\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} &\le \sup_{Q\subseteq\mathbb{R}^{n-1}} \Big(⨍_Q⨍_Q\Big(\frac{|f(x')-f(y')|}{\omega(\ell(Q))}\Big)^p\,dx'\,dy'\Big)^{1/p}\\ &\le \sup_{Q\subseteq\mathbb{R}^{n-1}} \frac{\omega(\sqrt{n-1}\ell(Q))}{\omega(\ell(Q))}\,[f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1},\mathbb{C}^M)}\\ &\le \sqrt{n-1}C'_\omega[f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1},\mathbb{C}^M)}.\end{aligned}\tag{4.2.21}$$

This establishes (4.2.18), hence also the first inclusion in (4.2.19). For the second inclusion in (4.2.19), using Jensen's inequality, (4.2.17), and (4.2.2) we may write

$$\begin{aligned}\int_1^\infty \operatorname{osc}_1(f;s)\frac{ds}{s^2} &\le \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\int_1^\infty \omega(s)\frac{ds}{s^2}\\ &\le C'_\omega\,\omega(1)\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} < \infty.\end{aligned}\tag{4.2.22}$$

The desired inclusion now follows from this and (4.2.16) with $\varepsilon := 1$. □

## 4.3 Properties of elliptic systems and their solutions

Our next proposition contains a number of a priori estimates comparing $\|u\|_{**}^{(\omega,q)}$, corresponding to different values of $q$, for solutions of $Lu=0$ in $\mathbb{R}^n_+$. To set the stage, we first state some simple estimates which are true for any function $u \in \big[\mathscr{C}^1(\mathbb{R}^n_+)\big]^M$:

$$\|u\|_{**}^{(\omega,p)} \le \|u\|_{**}^{(\omega,q)} \le C\,\|u\|_{**}^{(\omega,\exp)}\,, \qquad 0<p\le q<\infty, \tag{4.3.1}$$

where $C=C(q)\ge 1$. Indeed, the first estimate follows at once from Jensen's inequality. The second estimate is a consequence of the fact that $t^{\max\{1,q\}} \le C(e^t-1)$ (with $C>0$ depending on $\max\{1,q\}$) for each $t\in(0,\infty)$ and the definition of $\|\cdot\|_{\exp L,Q}$ (cf. (4.1.6)).

**Proposition 4.3.1.** *Let $L$ be a constant complex coefficient system as in* (1.2.1) *satisfying the strong ellipticity condition* (1.2.4)*, and let $u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$ be such that $Lu=0$ in $\mathbb{R}^n_+$. Then the following statements hold.*

(a) *For every* $q \in (0,\infty)$ *there there exists a finite constant* $C = C(L,n,q) \geq 1$ *such that for each* $(x',t) \in \mathbb{R}^n_+$ *one has*

$$t\,|(\nabla u)(x',t)| \leq C\Big(\fint_{|x'-y'|<\frac{t}{2}}\Big(\int_{t/2}^{3t/2}|(\nabla u)(y',s)|^2\,s\,ds\Big)^{q/2}dy'\Big)^{1/q}. \tag{4.3.2}$$

(b) *There exists a finite constant* $C = C(L,n) \geq 1$ *such that for each cube* $Q \subseteq \mathbb{R}^{n-1}$ *and each* $x' \in \mathbb{R}^{n-1}$ *one has*

$$\Big(\int_0^{\ell(Q)}|(\nabla u)(x',t)|^2\,t\,dt\Big)^{1/2} \leq C\Big(\int_0^{2\ell(Q)}\int_{|x'-y'|<s}|(\nabla u)(y',s)\,s|^2\,dy'\frac{ds}{s^n}\Big)^{1/2}. \tag{4.3.3}$$

*Furthermore, whenever* $2 \leq q < \infty$ *there exists a finite constant* $C = C(L,n,q) \geq 1$ *such that for each cube* $Q \subseteq \mathbb{R}^{n-1}$ *and each* $x' \in \mathbb{R}^{n-1}$ *one has*

$$\begin{aligned}\Big(\fint_Q\Big(\int_0^{\ell(Q)}\int_{|x'-y'|<s}|(\nabla u)(y',s)\,s|^2\,dy'\frac{ds}{s^n}\Big)^{q/2}dx'\Big)^{1/q}&\\ \leq C\Big(\fint_{3Q}\Big(\int_0^{3\ell(Q)}|(\nabla u)(x',t)|^2\,t\,dt\Big)^{q/2}dx'\Big)^{1/q}&.\end{aligned} \tag{4.3.4}$$

(c) *There exists a finite constant* $C = C(L,n) \geq 1$ *such that for each growth function* $\omega$ *one has*

$$\|u\|_{**}^{(\omega,\infty)} \leq C[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}. \tag{4.3.5}$$

(d) *For every* $q \in (0,\infty)$ *there exists a finite constant* $C = C(L,n,q) \geq 1$ *such that for each growth function* $\omega$ *satisfying* (4.2.2) *one has*

$$\|u\|_{**}^{(\omega,\infty)} \leq C\,C'_\omega\,\|u\|_{**}^{(\omega,q)}. \tag{4.3.6}$$

(e) *There exists a finite constant* $C = C(L,n) \geq 1$ *such that for each growth function* $\omega$ *satisfying* (4.2.2) *one has*

$$\|u\|_{**}^{(\omega,\exp)} \leq C(C'_\omega)^2\,\|u\|_{**}^{(\omega,2)}. \tag{4.3.7}$$

(f) *Let* $\omega$ *be a growth function satisfying* (4.2.1) *as well as* (4.2.2), *and define* $W(t)$ *as in* (4.2.3). *Then*

$$[u]_{\dot{\mathscr{C}}^W(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C'_\omega(2+C'_\omega)\,\|u\|_{**}^{(\omega,\infty)}, \tag{4.3.8}$$

*and, if the latter quantity is finite,* $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ *in the sense described in Lemma 4.2.1(d).*

(g) *Let* $\omega$ *be a growth function satisfying*

$$\int_0^t \omega(s)\frac{ds}{s} \leq C''_\omega\,\omega(t), \qquad \forall\, t \in (0,\infty), \tag{4.3.9}$$

*for some finite constant* $C''_\omega > 1$. *Then*

$$\|u\|_{**}^{(\omega,\exp)} \leq (C''_\omega)^{1/2}\,\|u\|_{**}^{(\omega,\infty)}. \tag{4.3.10}$$

(h) *Let $\omega$ be a growth function satisfying (4.1.9). Then for every $q \in (0, \infty)$*

$$\|u\|_{**}^{(\omega,q)} \approx \|u\|_{**}^{(\omega,\exp)} \approx \|u\|_{**}^{(\omega,\infty)} \approx [u]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^n_+,\mathbb{C}^M)} \tag{4.3.11}$$

*where the implicit constants depend only on $L$, $n$, $q$, and the constant $C_\omega$ in (4.1.9). In particular, if $\|u\|_{**}^{(\omega,q)} < \infty$ for some $q \in (0, \infty]$, or $\|u\|_{**}^{(\omega,\exp)} < \infty$, then $u \in \dot{\mathscr{C}}^{\omega}(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ in the sense of Lemma 4.2.1(d).*

*Proof.* We start by proving (a). Fix $(x', t) \in \mathbb{R}^n_+$ and let $Q_{x',t}$ be the cube in $\mathbb{R}^{n-1}$ centered at $x'$ with side-length $t$. Then from Theorem 1.2.2 (presently used with $m := 0$ and $p := \min\{q, 2\}$) and Jensen's inequality we obtain

$$\begin{aligned}
|(\nabla u)(x', t)| &\leq C \left( \fint_{|(y',s)-(x',t)|<\frac{t}{2}} |(\nabla u)(y', s)|^p \, dy' \, ds \right)^{1/p} \\
&\leq C \left( \fint_{|x'-y'|<\frac{t}{2}} \left( \fint_{(t/2,3t/2)} |(\nabla u)(y', s)|^2 \, ds \right)^{p/2} dy' \right)^{1/p} \\
&\leq C \Big( \fint_{|x'-y'|<\frac{t}{2}} \left( \fint_{(t/2,3t/2)} |(\nabla u)(y', s)|^2 \, ds \right)^{q/2} dy' \Big)^{1/q} \\
&= C t^{-1} \Big( \fint_{|x'-y'|<\frac{t}{2}} \left( \int_{t/2}^{3t/2} |(\nabla u)(y', s)|^2 \, s \, ds \right)^{q/2} dy' \Big)^{1/q},
\end{aligned} \tag{4.3.12}$$

proving (4.3.2). Turning our attention to (b), fix a cube $Q \subseteq \mathbb{R}^{n-1}$ along with a point $x' \in \mathbb{R}^{n-1}$. First, integrating (4.3.2) written for $q := 2$ yields

$$\begin{aligned}
\int_0^{\ell(Q)} |(\nabla u)(x', t)|^2 \, t \, dt &\leq C \int_0^{\ell(Q)} \frac{1}{t^{n+1}} \int_{t/2}^{3t/2} \int_{|x'-y'|<s} |(\nabla u)(y', s)|^2 \, s \, dy' \, ds \, t \, dt \\
&\leq C \int_0^{2\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y', s)|^2 \int_{2s/3}^{2s} t^{-n} \, dt \, dy' \, s \, ds \\
&= C \int_0^{2\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y', s) \, s|^2 \, dy' \, \frac{ds}{s^n},
\end{aligned} \tag{4.3.13}$$

and this readily leads to the estimate in (4.3.3). To justify (4.3.4), observe that for each nonnegative function $h \in L^1_{\text{loc}}(\mathbb{R}^{n-1})$ we have

$$\begin{aligned}
&\fint_Q \Big( \int_0^{\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y', s) \, s|^2 \, dy' \frac{ds}{s^n} \Big) h(x') \, dx' \\
&\leq 3^n \fint_{3Q} \int_0^{\ell(Q)} \Big( \frac{1}{s^{n-1}} \int_{|y'-x'|<s} h(x') \, dx' \Big) |(\nabla u)(y', s)|^2 \, s \, ds \, dy' \\
&\leq C_n \fint_{3Q} \Big( \int_0^{3\ell(Q)} |(\nabla u)(y', s)|^2 \, s \, ds \Big) (Mh)(x') \, dx',
\end{aligned} \tag{4.3.14}$$

where $M$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$. Note that if $q = 2$ then (4.3.14) gives at once (4.3.4) by taking $h = 1$ in $Q$ and using that $Mh \leq 1$. On the other hand, if $q > 2$, we impose the normalization condition $\|h\|_{L^{(q/2)'}(Q,dx'/|Q|)} = 1$ and then rely on (4.3.14) and Hölder's inequality to write

$$\begin{aligned}
&\fint_Q \Big( \int_0^{\ell(Q)} \int_{|x'-y'|<s} |(\nabla u)(y',s)\,s|^2\,dy'\frac{ds}{s^n}\Big) h(x')\,dx' \\
&\quad \le C_n \Big( \fint_{3Q} \Big( \int_0^{3\ell(Q)} |(\nabla u)(y',s)|^2\,s\,ds\Big)^{q/2}\,dx'\Big)^{2/q} \|Mh\|_{L^{(q/2)'}(Q,dx'/|Q|)} \\
&\quad \le C \Big( \fint_{3Q} \Big( \int_0^{3\ell(Q)} |(\nabla u)(y',s)|^2\,s\,ds\Big)^{q/2}\,dx'\Big)^{2/q}, \qquad (4.3.15)
\end{aligned}$$

bearing in mind that $M$ is bounded in $L^{(q/2)'}(\mathbb{R}^{n-1})$, given that $q > 2$. Taking now the supremum over all such functions $h$ yields (4.3.4) on account of Riesz' duality theorem.

As regards (c), fix $(x',t) \in \mathbb{R}^n_+$ and use Theorem 1.2.2 together with the fact that $\omega$ is a non-decreasing function to write

$$\begin{aligned}
|(\nabla u)(x',t)| &= \big|\nabla(u(\cdot) - u(x',t))(x',t)\big| \\
&\le \frac{C}{t} \fint_{|(y',s)-(x',t)|<t/2} |u(y',s) - u(x',t)|\,dy'\,ds \\
&\le C[u]_{\mathscr{C}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \frac{\omega(t)}{t}. \qquad (4.3.16)
\end{aligned}$$

In view of (4.1.8), this readily establishes (4.3.5).

The claim in (d) is proved by combining (4.3.2) and (4.2.5), which permit us to estimate (recall that $Q_{x',t}$ denotes the cube in $\mathbb{R}^{n-1}$ centered at $x'$ with side-length $t$)

$$\begin{aligned}
\|u\|_{**}^{(\omega,\infty)} &\le C \sup_{(x',t)\in\mathbb{R}^n_+} \frac{1}{\omega(t)} \Big( \fint_{(3/2)Q_{x',t}} \Big( \int_0^{3t/2} |(\nabla u)(y',s)|^2\,s\,ds\Big)^{q/2} dy'\Big)^{1/q} \\
&\le C\,C'_\omega\,\|u\|_{**}^{(\omega,q)}. \qquad (4.3.17)
\end{aligned}$$

Going further, consider the claim in (e). For starters, observe that the convexity of the function $t \mapsto e^t - 1$ readily implies that $2^{n-1}(e^t - 1) \le e^{2^{n-1}t} - 1$ for every $t > 0$ which, in view of (4.1.7), allows us to write

$$\|f\|_{\exp L,Q} \le 2^{n-1}\,\|f\|_{\exp L,2Q} \qquad (4.3.18)$$

for each cube $Q$ in $\mathbb{R}^{n-1}$ and each Lebesgue measurable function $f$ on $Q$.

Turning to the proof of (4.3.7) in earnest, by homogeneity we may assume that $\|u\|_{**}^{(\omega,2)} = 1$ to begin with. We are going to use Lemma 4.4.1. As a prelude, define

$$F(y',s) := |(\nabla u)(y',s)\,s|, \qquad \forall\,(y',s) \in \mathbb{R}^n_+, \qquad (4.3.19)$$

and, for each cube $Q$ in $\mathbb{R}^{n-1}$ and each threshold $N \in (0,\infty)$, consider the set

$$E_{N,Q} := \Big\{ x' \in Q : \frac{1}{\omega(\ell(Q))} \Big( \int_0^{\ell(Q)} \int_{|x'-y'|<\kappa s} |F(y',s)|^2\,dy'\frac{ds}{s^n}\Big)^{1/2} > N \Big\}. \qquad (4.3.20)$$

where $\kappa := 1 + 2\sqrt{n-1}$. Denoting $Q^* := (2\kappa+1)Q = (3+4\sqrt{n-1})Q$, then using Chebytcheff's inequality, and (4.2.4), for each cube $Q$ in $\mathbb{R}^{n-1}$ and each $N > 0$ we may write

$$
\begin{aligned}
|E_{N,Q}| &\leq \frac{1}{N^2}\frac{1}{\omega(\ell(Q))^2}\int_Q\int_0^{\ell(Q)}\int_{|x'-y'|<\kappa s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\,dx' \\
&\leq \frac{1}{N^2}\frac{1}{\omega(\ell(Q))^2}\int_{Q^*}\int_0^{\ell(Q)}\Big(\int_{|y'-x'|<\kappa s}dx'\Big)\,|F(y',s)|^2\frac{ds}{s^n}\,dy' \\
&\leq C\frac{1}{N^2}\frac{1}{\omega(\ell(Q))^2}\int_{Q^*}\int_0^{\ell(Q^*)}|(\nabla u)(y',s)\,s|^2\frac{ds}{s}\,dy' \\
&\leq C\frac{|Q^*|}{N^2}\frac{\omega(\ell(Q^*))^2}{\omega(\ell(Q))^2}\big(\,\|u\|_{**}^{(\omega,2)}\big)^2 = C\frac{|Q^*|}{N^2}\Big[\frac{\omega(\ell(Q^*))}{\omega(\ell(Q))}\Big]^2 \\
&\leq C_0(C'_\omega)^2\frac{1}{N^2}|Q|,
\end{aligned}
\tag{4.3.21}
$$

for some finite constant $C_0 > 0$. Therefore, taking $N := \sqrt{2C_0}C'_\omega > 0$, we conclude that

$$
|E_{N,Q}| \leq \frac{1}{2}|Q|. \tag{4.3.22}
$$

This allows us to invoke Lemma 4.4.1 with $\varphi := \omega$, which together with (4.3.3), (4.2.5), and (4.3.18), gives

$$
\begin{aligned}
\|u\|_{**}^{(\omega,\exp)} &\leq C\sup_{Q\subseteq\mathbb{R}^{n-1}}\frac{1}{\omega(\ell(Q))}\Big\|\Big(\int_0^{\ell(2Q)}\int_{|\,\cdot\,-y'|<s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\Big)^{1/2}\Big\|_{\exp L,Q} \\
&\leq C(C'_\omega)^2.
\end{aligned}
\tag{4.3.23}
$$

This completes the proof of (e).

Turning our attention to (f), fix $x = (x',t)$ and $y = (y',s)$ in $\mathbb{R}^n_+$, and abbreviate $r := |x-y|$. Then,

$$
\begin{aligned}
\frac{|u(x)-u(y)|}{W(|x-y|)} &\leq \frac{1}{W(r)}|u(x',t)-u(x',t+r)| + \frac{1}{W(r)}|u(x',t+r)-u(y',s+r)| \\
&\quad + \frac{1}{W(r)}|u(y',s+r)-u(y',s)| \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
\tag{4.3.24}
$$

To bound I, we use the Fundamental Theorem of Calculus, (4.2.4), and (4.2.3) and obtain

$$
\begin{aligned}
\mathrm{I} = \frac{1}{W(r)}\Big|\int_0^r(\partial_n u)(x',t+\xi)\,d\xi\Big| &\leq \|u\|_{**}^{(\omega,\infty)}\frac{1}{W(r)}\int_0^r\frac{\omega(t+\xi)}{t+\xi}\,d\xi \\
&\leq C'_\omega\,\|u\|_{**}^{(\omega,\infty)}\frac{1}{W(r)}\int_0^r\frac{\omega(\xi)}{\xi}\,d\xi = C'_\omega\,\|u\|_{**}^{(\omega,\infty)}.
\end{aligned}
\tag{4.3.25}
$$

Note that III is bounded analogously replacing $x'$ by $y'$ and $t$ by $s$. For II, we use again the Fundamental Theorem of Calculus, together with (4.2.4) and (4.2.12), to write

$$\begin{aligned}
\mathrm{II} &= \frac{1}{W(r)}\left|\int_0^1 \frac{d}{d\theta}[u(\theta(x',t+r)+(1-\theta)(y',s+r))]\,d\theta\right| \\
&= \frac{1}{W(r)}\left|\int_0^1 (x'-y',t-s)\cdot(\nabla u)(\theta(x',t+r)+(1-\theta)(y',s+r))\,d\theta\right| \\
&\le \|u\|_{**}^{(\omega,\infty)}\,\frac{r}{W(r)}\int_0^1 \frac{\omega((1-\theta)s+\theta t+r)}{(1-\theta)s+\theta t+r}\,d\theta \\
&\le C'_\omega\,\|u\|_{**}^{(\omega,\infty)}\,\frac{r}{W(r)}\int_0^1 \frac{\omega(r)}{r}\,d\theta \\
&\le (C'_\omega)^2\,\|u\|_{**}^{(\omega,\infty)}. \qquad (4.3.26)
\end{aligned}$$

As $x$ and $y$ were chosen arbitrarily, (4.3.24), (4.3.25), and (4.3.26) collectively justify (4.3.8).

To justify (g), observe that since $\omega$ is non-decreasing and satisfies (4.3.9) we may write

$$\begin{aligned}
\left(\fint_Q\left(\int_0^{\ell(Q)}|(\nabla u)(x',t)|^2\,t\,dt\right)^{q/2}dx'\right)^{1/q} &\le \left(\int_0^{\ell(Q)}\omega(t)^2\frac{dt}{t}\right)^{1/2}\|u\|_{**}^{(\omega,\infty)} \\
&\le (C''_\omega)^{1/2}\omega(\ell(Q))\,\|u\|_{**}^{(\omega,\infty)}, \qquad (4.3.27)
\end{aligned}$$

which readily leads to the desired inequality.

As regards (h), the idea is to combine (4.3.1), (g), and (d) for the first three equivalences. In concert, (c), the fact that (4.1.9) gives $W \le C_\omega\,\omega$, and (f) also give the last equivalence in (h). The proof of Proposition 4.3.1 is therefore complete. □

## 4.4 John-Nirenberg's inequality adapted to growth functions

In what follows we assume that all cubes are half-open, that is, they can be written in the form $Q = [a_1, a_1+\ell(Q)) \times \cdots \times [a_{n-1}, a_{n-1}+\ell(Q))$ with $a_i \in \mathbb{R}^{n-1}$ and $\ell(Q) > 0$. Notice that since $\partial Q$ has Lebesgue measure zero the half-open assumption is harmless. Subdividing dydically yields the collection of (half-open) dyadic-subcubes of a given cube $Q$, which we shall denote by $\mathbb{D}_Q$. For the following statement, and with the aim of considering global results, it is also convenient to consider the case $Q = \mathbb{R}^{n-1}$ in which scenario we take $\mathbb{D}_Q$ to be the classical dyadic grid generated by $[0,1)^{n-1}$, or any other dyadic grid. Let us also recall the definition of the dyadic Hardy-Littlewood maximal function localized to a given cube $Q$, i.e.,

$$(M_Q^d f)(x) := \sup_{x\in Q'\in\mathbb{D}_Q}\fint_{Q'}|f(y')|\,dy', \qquad x \in Q, \qquad (4.4.1)$$

for each $f \in L^1(Q)$. The following result is an extension of the John-Nirenberg inequality obtained in [50], [49] (when $\omega \equiv 1$) adapted to our growth function.

**Lemma 4.4.1.** *Let $F \in L^2_{\rm loc}(\mathbb{R}^n_+)$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a non-decreasing function. Let $Q_0 \subseteq \mathbb{R}^{n-1}$ be an arbitrary half-open cube, or $Q_0 = \mathbb{R}^{n-1}$. Assume that there are numbers $\alpha \in (0,1)$ and $N \in (0,\infty)$ such that*

$$\left|\left\{x' \in Q : \frac{1}{\varphi(\ell(Q))}\left(\int_0^{\ell(Q)}\int_{|x'-y'|<\kappa s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\right)^{1/2} > N\right\}\right| \leq \alpha|Q| \tag{4.4.2}$$

*for every cube $Q \in \mathbb{D}_{Q_0}$ and $\kappa := 1 + 2\sqrt{n-1}$. Then, for every $t > 0$*

$$\sup_{Q\in\mathbb{D}_{Q_0}}\frac{1}{|Q|}\left|\left\{x' \in Q : \frac{1}{\varphi(\ell(Q))}\left(\int_0^{\ell(Q)}\int_{|x'-y'|<\kappa s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\right)^{1/2} > t\right\}\right| \\ \leq \frac{1}{\alpha}\,e^{-\frac{\log(\alpha^{-1})}{N}t}. \tag{4.4.3}$$

*Hence, for each $q \in (0,\infty)$ there exists a finite constant $C = C(\alpha,q) \geq 1$ such that*

$$\sup_{Q\in\mathbb{D}_{Q_0}}\frac{1}{\varphi(\ell(Q))}\left(\fint_Q\left(\int_0^{\ell(Q)}\int_{|x'-y'|<s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\right)^{q/2}dx'\right)^{1/q} \leq CN. \tag{4.4.4}$$

*Moreover, there exists some finite $C = C(\alpha) \geq 1$ such that*

$$\sup_{Q\in\mathbb{D}_{Q_0}}\frac{1}{\varphi(\ell(Q))}\left\|\left(\int_0^{\ell(Q)}\int_{|\cdot-y'|<s}|F(y',s)|^2\,dy'\frac{ds}{s^n}\right)^{1/2}\right\|_{\exp L,Q} \leq CN. \tag{4.4.5}$$

The previous result can be proved using the arguments in [50], [49] with appropriate modifications. Here we present an alternative abstract argument based on ideas that go back to Calderón, as presented in [102] (see also [103], [74]). This also contains as a particular case the classical John-Nirenberg result concerning the exponential integrability of BMO functions.

**Proposition 4.4.2.** *Let $Q_0 \subseteq \mathbb{R}^{n-1}$ be an arbitrary half-open cube, or $Q_0 = \mathbb{R}^{n-1}$. For every $Q \in \mathbb{D}_{Q_0}$ assume that there exist two non-negative functions $G_Q$, $H_Q \in L^1_{\rm loc}(\mathbb{R}^{n-1})$ such that*

$$G_Q(x') \leq H_Q(x') \quad \text{for almost every} \quad x' \in Q, \tag{4.4.6}$$

*and, for every $Q' \in \mathbb{D}_Q \setminus \{Q\}$,*

$$G_Q(x') \leq G_{Q'}(x') + H_Q(y') \quad \text{for a.e. } x \in Q' \text{ and for a.e. } y' \in \widehat{Q'}, \tag{4.4.7}$$

*where $\widehat{Q'}$ is the dyadic parent of $Q'$. For each $\alpha \in (0,1)$ define*

$$\mathfrak{m}_\alpha := \sup_{Q\in\mathbb{D}_{Q_0}}\inf\{\lambda > 0 : |\{x' \in Q : H_Q(x') > \lambda\}| \leq \alpha|Q|\}. \tag{4.4.8}$$

*Then, for every $\alpha \in (0,1)$ one has*

$$\sup_{Q\in\mathbb{D}_{Q_0}}\frac{|\{x \in Q : G_Q(x') > t\}|}{|Q|} \leq \frac{1}{\alpha}\,e^{-\log(\alpha^{-1})\frac{t}{\mathfrak{m}_\alpha}}, \qquad \forall\, t > 0. \tag{4.4.9}$$

*As a consequence,*

$$\sup_{Q\in\mathbb{D}_{Q_0}} \|G_Q\|_{\exp L,Q} \le \frac{1+\alpha^{-1}}{\log(\alpha^{-1})}\,\mathfrak{m}_\alpha \tag{4.4.10}$$

*and for every $q \in (0,\infty)$ there exists a finite constant $C = C(q) > 0$ such that*

$$\sup_{Q\in\mathbb{D}_{Q_0}} \Big(\fint_Q G_Q(x')^q\,dx'\Big)^{1/q} \le C_q \frac{1}{\alpha^{1/q}\log(\alpha^{-1})}\,\mathfrak{m}_\alpha. \tag{4.4.11}$$

Before proving this result and Lemma 4.4.1, let us illustrate how Proposition 4.4.2 yields the classical John-Nirenberg result regarding the exponential integrability of BMO functions. Concretely, pick $f \in \mathrm{BMO}(\mathbb{R}^{n-1})$. Fix an arbitrary cube $Q_0$ and for every $Q \in \mathbb{D}_{Q_0}$ define $G_Q := |f - f_Q|$ and $H_Q := 2^{n-1}M^d_Q(|f-f_Q|)$ (cf. (4.4.1)). Clearly, (4.4.6) holds by Lebesgue's Differentiation Theorem. Moreover, for every $Q' \in \mathbb{D}_Q \setminus Q$, $x' \in Q'$, and $y' \in \widehat{Q'}$ we have

$$\begin{aligned} G_Q(x') &\le |f(x') - f_{Q'}| + |f'_Q - f_Q| \le G_{Q'}(x') + \fint_{Q'} |f(z') - f_Q|\,dz' \\ &\le G_{Q'}(x') + 2^{n-1}\fint_{\widehat{Q'}} |f(z') - f_Q|\,dz' \le G_{Q'}(x') + H_{Q'}(y'), \end{aligned} \tag{4.4.12}$$

and (4.4.7) follows. Going further, by the weak-type $(1,1)$ of the dyadic Hardy-Littlewood maximal function, for every $\lambda > 0$ we may write

$$|\{x' \in Q :\, H_Q(x') > \lambda\}| \le \frac{2^{n-1}}{\lambda}\int_Q |f(y') - f_Q|\,dy' \le \frac{2^{n-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}{\lambda}|Q|. \tag{4.4.13}$$

In particular, choosing for instance $\alpha := e^{-1}$, if we use the previous estimate with $\lambda := 2^{n-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}/\alpha$ we obtain $\mathfrak{m}_\alpha \le 2^{n-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}/\alpha$. Thus, (4.4.9) yields

$$\begin{aligned} \frac{|\{x \in Q_0 :\, |f(x') - f_{Q_0}| > t\}|}{|Q_0|} &\le \frac{1}{\alpha}\, e^{-\frac{\alpha\log(\alpha^{-1})}{2^{n-1}}\frac{t}{\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}} \\ &= e\cdot e^{-\frac{1}{2^{n-1}e}\frac{t}{\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})}}} \end{aligned} \tag{4.4.14}$$

while (4.4.10) gives

$$\|f - f_{Q_0}\|_{\exp L, Q_0} \le (1+e)\,e\,2^{n-1}\|f\|_{\mathrm{BMO}(\mathbb{R}^{n-1})} \tag{4.4.15}$$

which are the well-known John-Nirenberg inequalities.

We now turn to the proof of Lemma 4.4.1.

*Proof of Lemma 4.4.1.* Let $F$, $\alpha$, and $N$ be fixed as in the statement of the lemma. For every $Q \in \mathbb{D}_{Q_0}$ and $x' \in \mathbb{R}^{n-1}$, define

$$G_Q(x') := \frac{1}{\varphi(\ell(Q))}\Big(\int_0^{\ell(Q)}\int_{|x'-z'|<s} |F(z',s)|^2\,dz'\frac{ds}{s^n}\Big)^{1/2} \tag{4.4.16}$$

and

$$H_Q(x') = \frac{1}{\varphi(\ell(Q))}\Big(\int_0^{\ell(Q)}\int_{|x'-z'|<\kappa s}|F(z',s)|^2\,dz'\frac{ds}{s^n}\Big)^{1/2}. \tag{4.4.17}$$

Note that (4.4.6) is trivially verified since $\kappa > 1$. To proceed, fix $Q' \in \mathbb{D}_Q$ along with $x' \in Q'$ and $y' \in \widehat{Q'}$. If $|x'-z'| < s$ with $\ell(Q') \le s \le \ell(Q)$ then

$$|y'-z'| \le |y'-x'| + |x'-z'| < 2\sqrt{n-1}\,\ell(Q') + s \le \kappa s. \tag{4.4.18}$$

Therefore, since $\varphi$ is non-decreasing,

$$\begin{aligned} G_Q(x') &\le \frac{\varphi(\ell(Q'))}{\varphi(\ell(Q))}G_{Q'}(x') + \frac{1}{\varphi(\ell(Q))}\Big(\int_{\ell(Q')}^{\ell(Q)}\int_{|x'-z'|<s}|F(z',s)|^2\,dz'\frac{ds}{s^n}\Big)^{1/2} \\ &\le G_{Q'}(x') + H_Q(y'), \end{aligned} \tag{4.4.19}$$

establishing (4.4.7). Moreover, (4.4.2) gives immediately that $\mathfrak{m}_\alpha \le N$. Granted this, (4.4.9), (4.4.11), and (4.4.10), (with $\alpha \in (0,1)$ given by (4.4.2)) prove, respectively (4.4.3), (4.4.4), and (4.4.5). $\square$

Finally, we give the proof of Proposition 4.4.2.

*Proof of Proposition 4.4.2.* We start by introducing some notation. Set

$$\Xi(t) := \sup_{Q\in\mathbb{D}_{Q_0}}\frac{|E_Q(t)|}{|Q|} := \sup_{Q\in\mathbb{D}_{Q_0}}\frac{|\{x'\in Q : G_Q(x') > t\}|}{|Q|}, \qquad 0 < t < \infty. \tag{4.4.20}$$

Fix $\alpha \in (0,1)$, let $\varepsilon > 0$ be arbitrary, and write $\lambda_\varepsilon = \mathfrak{m}_\alpha + \varepsilon$. From (4.4.8) it follows that

$$|F_{Q,\varepsilon}| := |\{x' \in Q :\ H_Q(x') > \lambda_\varepsilon\}| \le \alpha|Q|, \qquad \forall\, Q \in \mathbb{D}_{Q_0}. \tag{4.4.21}$$

Fix now $Q \in \mathbb{D}_{Q_0}$, $\beta \in (\alpha, 1)$ (we will eventually let $\beta \to 1^+$) and set

$$\Omega_Q := \{x' \in Q :\ M_Q^d(1_{F_{Q,\varepsilon}})(x') > \beta\}. \tag{4.4.22}$$

Note that (4.4.21) ensures that

$$\fint_Q 1_{F_{Q,\varepsilon}}(y')\,dy' = \frac{|F_{Q,\varepsilon}|}{|Q|} \le \alpha < \beta, \tag{4.4.23}$$

hence we can extract a family of pairwise disjoint stopping-time cubes $\{Q_j\}_j \subseteq \mathbb{D}_Q \setminus \{Q\}$ so that $\Omega_Q = \cup_j Q_j$ and for every $j$

$$\frac{|F_{Q,\varepsilon}\cap Q_j|}{|Q_j|} > \beta, \qquad \frac{|F_{Q,\varepsilon}\cap Q'|}{|Q'|} \le \beta, \qquad Q_j \subsetneq Q' \in \mathbb{D}_Q. \tag{4.4.24}$$

Let $t > \lambda_\varepsilon$ and note that (4.4.6) gives

$$\lambda_\varepsilon < t < G_Q(x') \le H_Q(x') \ \text{ for a.e. } \ x' \in E_Q(t). \tag{4.4.25}$$

which implies that

$$\beta < 1 = \mathbf{1}_{F_{Q,\varepsilon}}(x') \le M_Q^d(\mathbf{1}_{F_{Q,\varepsilon}})(x') \ \text{ for a.e. } \ x' \in E_Q(t). \tag{4.4.26}$$

Hence,

$$|E_Q(t)| = |E_Q(t) \cap \Omega_Q| = \sum_j |E_Q(t) \cap Q_j|. \tag{4.4.27}$$

For every $j$, by the second estimate in (4.4.24) applied to $\widehat{Q}_j$, the dyadic parent of $Q_j$, we have $|F_{Q,\varepsilon} \cap \widehat{Q}_j|/|\widehat{Q}_j| \leq \beta < 1$, therefore $|\widehat{Q}_j \setminus F_{Q,\varepsilon}|/|\widehat{Q}_j| > 1 - \beta > 0$. In particular, (4.4.7) guarantees that we can find $\widehat{x}'_j \in \widehat{Q}_j \setminus F_{Q,\varepsilon}$, such that for a.e. $x' \in Q_j$ we have

$$G_Q(x') \leq G_{Q_j}(x') + H_Q(\widehat{x}'_j) \leq G_{Q_j}(x') + \lambda_\varepsilon. \tag{4.4.28}$$

Consequently, $G_{Q_j}(x') > t - \lambda_\varepsilon$ for a.e. $x' \in E_Q(t) \cap Q_j$ which further implies

$$|E_Q(t) \cap Q_j| \leq |\{x' \in Q_j : G_{Q_j}(x') > t - \lambda_\varepsilon\}| \leq \Xi(t - \lambda_\varepsilon)|Q_j|. \tag{4.4.29}$$

In turn, this permits us to estimate

$$\begin{aligned} |E_Q(t)| = \sum_j |E_Q(t) \cap Q_j| &\leq \Xi(t - \lambda_\varepsilon) \sum_j |Q_j| \leq \Xi(t - \lambda_\varepsilon) \frac{1}{\beta} \sum_j |F_{Q,\varepsilon} \cap Q_j| \\ &\leq \Xi(t - \lambda_\varepsilon) \frac{1}{\beta} |F_{Q,\varepsilon}| \leq \Xi(t - \lambda_\varepsilon) \frac{\alpha}{\beta} |Q|, \end{aligned} \tag{4.4.30}$$

where we have used (4.4.24), that the cubes $\{Q_j\}_j$ are pairwise disjoint and, finally, (4.4.21). Dividing by $|Q|$ and taking the supremum over all $Q \in \mathbb{D}_{Q_0}$ we arrive at

$$\Xi(t) \leq \frac{\alpha}{\beta} \Xi(t - \lambda_\varepsilon), \qquad t > \lambda_\varepsilon. \tag{4.4.31}$$

Since this is valid for all $\beta \in (\alpha, 1)$, we can now let $\beta \to 1^+$, iterate the previous expression, and use the fact that $\Xi(t) \leq 1$ to conclude that

$$\Xi(t) \leq \frac{1}{\alpha} \alpha^{\frac{t}{\lambda_\varepsilon}} = \frac{1}{\alpha} e^{-\log(\alpha^{-1}) \frac{t}{\lambda_\varepsilon}}, \qquad t > 0. \tag{4.4.32}$$

Recalling that $\lambda_\varepsilon = \mathfrak{m}_\alpha + \varepsilon$ and letting $\varepsilon \to 0^+$ establishes (4.4.9).

We shall next indicate how (4.4.9) implies (4.4.10). Concretely, if we take $t := \frac{1+\alpha^{-1}}{\log(\alpha^{-1})} \mathfrak{m}_\alpha$ we see that (4.4.9) gives

$$\begin{aligned} \fint_Q \left( e^{\frac{G_Q(x')}{t}} - 1 \right) dx' &= \int_0^\infty \frac{|\{x' \in Q : G_Q(x')/t > \lambda\}|}{|Q|} e^\lambda \, d\lambda \\ &\leq \frac{1}{\alpha} \int_0^\infty e^{-\log(\alpha^{-1}) \frac{\lambda t}{\mathfrak{m}_\alpha}} e^\lambda \, d\lambda = \frac{1}{\alpha} \int_0^\infty e^{-\alpha^{-1}\lambda} \, d\lambda = 1. \end{aligned} \tag{4.4.33}$$

With this in hand, (4.4.10) follows with the help of (4.1.7).

At this stage, there remains to justify (4.4.11). This can be done invoking again (4.4.9):

$$\begin{aligned} \fint_Q G_Q(x')^q \, dx' &= \int_0^\infty \frac{|\{x' \in Q : G_Q(x') > \lambda\}|}{|Q|} q \lambda^q \frac{d\lambda}{\lambda} \leq \frac{1}{\alpha} \int_0^\infty e^{-\log(\alpha^{-1}) \frac{\lambda}{\mathfrak{m}_\alpha}} q \lambda^q \frac{d\lambda}{\lambda} \\ &= \frac{1}{\alpha} \left( \frac{\mathfrak{m}_\alpha}{\log(\alpha^{-1})} \right)^q \int_0^\infty e^{-\lambda} q \lambda^q \frac{d\lambda}{\lambda} = C_q \frac{1}{\alpha} \left( \frac{\mathfrak{m}_\alpha}{\log(\alpha^{-1})} \right)^q. \end{aligned} \tag{4.4.34}$$

This completes the proof of Proposition 4.4.2. $\square$

## 4.5 Existence results

In this section we develop the main tools used to establish the existence of solutions for the boundary value problems formulated in the statement of Theorem 4.1.2. We start with the generalized Hölder Dirichlet Problem.

**Proposition 4.5.1.** *Let $L$ be a constant complex coefficient system as in (1.2.1) satisfying the strong ellipticity condition formulated in (1.2.4), fix an aperture parameter $\kappa > 0$ and let $\omega$ be a growth function satisfying (4.2.2). Given $f \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, define $u(x', t) := (P_t^L * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. Then $u$ is meaningfully defined via an absolutely convergent integral and satisfies*

$$u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \qquad Lu = 0 \ \ in \ \ \mathbb{R}^n_+, \qquad u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} = f \ \ a.e. \ \ on \ \ \mathbb{R}^{n-1}. \tag{4.5.1}$$

*Moreover, there exists a finite constant $C = C(L, n) > 0$ such that*

$$[u]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^n_+, \mathbb{C}^M)} \leq C\, C'_\omega (1 + C'_\omega) [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \tag{4.5.2}$$

*and $u \in \dot{\mathscr{C}}^{\omega}(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ with $u\big|_{\partial\mathbb{R}^n_+} = f$.*

*Proof.* Let $f \in \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and define $u(x', t) := (P_t^L * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. By (4.2.19) and Theorem 1.2.4(c), $u$ satisfies all properties listed in (4.5.1). To prove the estimate in (4.5.2), we first notice that for any $(x', t) \in \mathbb{R}^n_+$, we can write

$$\begin{aligned}(P_t^L * f)(x') &= \int_{\mathbb{R}^{n-1}} P_t^L(x' - y') f(y')\, dy' = \int_{\mathbb{R}^{n-1}} t^{1-n} P^L\left(\frac{x' - y'}{t}\right) f(y')\, dy' \\ &= \int_{\mathbb{R}^{n-1}} P^L(z') f(x' - tz')\, dz'.\end{aligned} \tag{4.5.3}$$

Fix now $x = (x', t)$ and $y = (y', s)$ arbitrary in $\mathbb{R}^n_+$, and set $r := |x - y|$. By (1.2.26) and the fact that $\omega$ is non-decreasing we obtain

$$\begin{aligned}|u(x', t) - u(y', s)| &= |(P_t^L * f)(x') - (P_s^L * f)(y')| \\ &\leq C \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z'|^2)^{n/2}} |f(x' - tz') - f(y' - sz')|\, dz' \\ &\leq C [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z'|^2)^{n/2}}\, \omega((1 + |z'|) r)\, dz' \\ &\leq C [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \int_0^\infty \frac{1}{(1 + \lambda^2)^{n/2}}\, \omega((1 + \lambda) r)\, \lambda^{n-1} \frac{d\lambda}{\lambda} \\ &\leq C [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \left(\int_0^1 \omega(2r)\, \lambda^{n-1} \frac{d\lambda}{\lambda} + \int_1^\infty \frac{\omega(2\lambda r)}{\lambda} \frac{d\lambda}{\lambda}\right) \\ &= C [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \left(\omega(2r) + 2r \int_{2r}^\infty \frac{\omega(\lambda)}{\lambda} \frac{d\lambda}{\lambda}\right) \\ &\leq C\, C'_\omega (1 + C'_\omega) \omega(r) [f]_{\dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1}, \mathbb{C}^M)},\end{aligned} \tag{4.5.4}$$

where in the last inequality we have used (4.2.2) and (4.2.5). Hence, (4.5.2) holds. In particular, $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$ by Lemma 4.2.1(d). This and the fact that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f$ a.e. in $\mathbb{R}^{n-1}$ with $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ then prove that, indeed, $u|_{\partial\mathbb{R}^n_+} = f$. $\square$

The result below is the main tool in the proof of existence of solutions for the generalized Morrey-Campanato Dirichlet Problem.

**Proposition 4.5.2.** *Let $L$ be a constant complex coefficient system as in (1.2.1) satisfying the strong ellipticity condition stated in (1.2.4), fix an aperture parameter $\kappa > 0$, and let $\omega$ be a growth function satisfying (4.2.2). Given $1 \leq p < \infty$, let $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and define $u(x', t) := (P^L_t * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$. Then $u$ is meaningfully defined via an absolutely convergent integral and satisfies*

$$u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \qquad Lu = 0 \;\; \text{in} \;\; \mathbb{R}^n_+, \qquad u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \;\; \text{a.e. on} \;\; \mathbb{R}^{n-1}. \tag{4.5.5}$$

*Moreover, for every $q \in (0, \infty]$ there exists a finite constant $C = C(L, n, p, q) > 0$ such that*

$$\|u\|^{(\omega,q)}_{**} \leq C(C'_\omega)^4 \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} . \tag{4.5.6}$$

*Furthermore, the same is true if $\|\cdot\|^{(\omega,q)}_{**}$ is replaced by $\|\cdot\|^{(\omega,\exp)}_{**}$.*

*Proof.* Given $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$, if $u(x', t) := (P^L_t * f)(x')$ for every $(x', t) \in \mathbb{R}^n_+$, from (4.2.19) and Theorem 1.2.4(c) we see that $u$ satisfies all properties listed in (4.5.5).

Next, having fixed an arbitrary exponent $q \in (0, \infty)$, based on Proposition 4.3.1(d), (4.3.1), Proposition 4.3.1(e), (1.2.35), (4.2.15), (4.2.17) and (4.2.2) we may write

$$\begin{aligned} \|u\|^{(\omega,\infty)}_{**} &\leq C\, C'_\omega \, \|u\|^{(\omega,q)}_{**} \leq C\, C'_\omega \, \|u\|^{(\omega,\exp)}_{**} \\ &\leq C(C'_\omega)^3 \, \|u\|^{(\omega,2)}_{**} \leq C(C'_\omega)^3 \sup_{t>0} \frac{1}{\omega(t)} \int_1^\infty \operatorname{osc}_1(f, st) \frac{ds}{s^2} \\ &= C(C'_\omega)^3 \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \sup_{t>0} \frac{t}{\omega(t)} \int_t^\infty \omega(s) \frac{ds}{s^2} \\ &\leq C(C'_\omega)^4 \, \|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} , \end{aligned} \tag{4.5.7}$$

which proves (4.5.6) and the corresponding estimate for $\|u\|^{(\omega,\exp)}_{**}$. $\square$

## 4.6 A Fatou-type result and uniqueness of solutions

We shall now prove a Fatou-type result which is going to be the main ingredient in establishing the uniqueness of solutions for the boundary value problems we are presently considering. More precisely, the following result establishes that any solution in $\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ can be obtained as a convolution of its trace with the associated Poisson kernel.

**Proposition 4.6.1.** *Let $L$ be a constant complex coefficient system as in (1.2.1) satisfying the strong ellipticity condition stated in (1.2.4), and let $\omega$ be a growth function satisfying*

(4.2.2). *If* $u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M \cap \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ *is a function satisfying* $Lu = 0$ *in* $\mathbb{R}^n_+$*, then* $u\big|_{\partial\mathbb{R}^n_+} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ *and*

$$u(x', t) = \Big(P_t^L * \big(u\big|_{\partial\mathbb{R}^n_+}\big)\Big)(x'), \qquad \forall\,(x', t) \in \mathbb{R}^n_+. \tag{4.6.1}$$

*Proof.* Let $u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M \cap \dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)$ satisfy $Lu = 0$ in $\mathbb{R}^n_+$. By Lemma 4.2.1(d), it follows that $u$ can be continuously extended to a function (which we call again $u$) $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$. In particular, the trace $u\big|_{\partial\mathbb{R}^n_+}$ is well-defined and belongs to the space $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$. To proceed, fix an arbitrary $\varepsilon > 0$ and define $u_\varepsilon = u(\cdot + \varepsilon e_n)$ in $\mathbb{R}^n_+$, where $e_n = (0, \dots, 0, 1) \in \mathbb{R}^n$. Then, by design, $u_\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, $Lu_\varepsilon = 0$ in $\mathbb{R}^n_+$, and $[u_\varepsilon]_{\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)}$. Moreover, using Proposition 4.3.1(c) and (4.2.4) we obtain

$$\begin{aligned}
\sup_{(x',t)\in\mathbb{R}^n_+} |(\nabla u_\varepsilon)(x', t)| &= \sup_{(x',t)\in\mathbb{R}^n_+} |(\nabla u)(x', t+\varepsilon)| \\
&\leq C[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \sup_{(x',t)\in\mathbb{R}^n_+} \frac{\omega(t+\varepsilon)}{t+\varepsilon} \\
&\leq C\,C'_\omega [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \frac{\omega(\varepsilon)}{\varepsilon}.
\end{aligned} \tag{4.6.2}$$

This implies that $\nabla u_\varepsilon$ is bounded in $\mathbb{R}^n_+$, hence $u_\varepsilon \in \big[W^{1,2}_{\rm bdd}(\mathbb{R}^n_+)\big]^M$.

Define next $f_\varepsilon(x') := u(x', \varepsilon) \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1}, \mathbb{C}^M)$ and $w_\varepsilon(x', t) := (P_t^L * f_\varepsilon)(x')$ for each $(x', t) \in \mathbb{R}^n_+$. Then, Proposition 4.5.1 implies that that $w_\varepsilon \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M \cap \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+}, \mathbb{C}^M)$, $Lw_\varepsilon = 0$ in $\mathbb{R}^n_+$ and $w_\varepsilon\big|_{\partial\mathbb{R}^n_+} = f_\varepsilon$. Moreover, for every pair of points $x', y' \in \mathbb{R}^{n-1}$ we have, on the one hand,

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)| \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)}\, \omega(|x' - y'|), \tag{4.6.3}$$

and, on the other hand, using the Mean Value Theorem and Proposition 4.3.1(c),

$$\begin{aligned}
|f_\varepsilon(x') - f_\varepsilon(y')| &= |u(x', \varepsilon) - u(y', \varepsilon)| \\
&\leq |x' - y'| \sup_{z'\in[x',y']} |(\nabla u)(z', \varepsilon)| \\
&\leq C\,|x' - y'|\,[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+, \mathbb{C}^M)} \frac{\omega(\varepsilon)}{\varepsilon}.
\end{aligned} \tag{4.6.4}$$

Therefore, we conclude that $f_\varepsilon \in \dot{\mathscr{C}}^\Psi(\mathbb{R}^{n-1}, \mathbb{C}^M)$, with norm depending (unfavorably) on the parameter $\varepsilon$, where the growth function $\Psi$ is given by

$$\Psi(t) := \min\left\{t, \frac{\omega(t)}{\omega(1)}\right\} = \begin{cases} t & \text{if } t \leq 1, \\ \omega(t)/\omega(1) & \text{if } t > 1. \end{cases} \tag{4.6.5}$$

For every $R > 1$ and $x = (x', t)$, let us now invoke (1.2.34), (4.2.15) and (4.2.17), with $\Psi$

in place of $\omega$, to write

$$\begin{aligned}\int_{B(0,R)\cap\mathbb{R}^n_+}|(\nabla w_\varepsilon)(x)|^2\,dx &\le \int_{B(0,R)\cap\mathbb{R}^n_+}\Big(\frac{C}{t}\int_1^\infty \operatorname{osc}_1(f_\varepsilon;st)\frac{ds}{s^2}\Big)^2\,dx\\ &\le C\,\|f_\varepsilon\|_{\mathscr{E}^{\Psi,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\int_{B(0,R)\cap\mathbb{R}^n_+}\Big(\int_1^\infty\frac{\Psi(st)}{st}\frac{ds}{s}\Big)^2\,dx\\ &\le C\,\|f_\varepsilon\|_{\mathscr{E}^{\Psi,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\,R^{n-1}\int_0^R\Big(\int_t^\infty\frac{\Psi(s)}{s}\frac{ds}{s}\Big)^2\,dt, \qquad (4.6.6)\end{aligned}$$

and then use (4.2.2) to observe that

$$\begin{aligned}\int_0^R\Big(\int_t^\infty\frac{\Psi(s)}{s}\frac{ds}{s}\Big)^2\,dt &\le \int_0^1\Big(\int_t^1\frac{ds}{s}+\frac{1}{\omega(1)}\int_1^\infty\frac{\omega(s)}{s}\frac{ds}{s}\Big)^2\,dt\\ &\quad+\int_1^R\Big(\frac{1}{\omega(1)}\int_1^\infty\frac{\omega(s)}{s}\frac{ds}{s}\Big)^2\,dt\\ &\le \int_0^1\big(\log(1/t)+C'_\omega\big)^2\,dt+(R-1)(C'_\omega)^2<\infty, \qquad (4.6.7)\end{aligned}$$

Collectively, (4.6.6) and (4.6.7) show that $w_\varepsilon\in\big[W^{1,2}_{\rm bdd}(\mathbb{R}^n_+)\big]^M$.

We now consider $v_\varepsilon:=u_\varepsilon-w_\varepsilon\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M\cap\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)\cap\big[W^{1,2}_{\rm bdd}(\mathbb{R}^n_+)\big]^M$, which satisfies $Lv_\varepsilon=0$ in $\mathbb{R}^n_+$ and $v_\varepsilon\big|_{\partial\mathbb{R}^n_+}=0$. Hence, $\operatorname{Tr} v_\varepsilon=0$ on $\mathbb{R}^{n-1}$ (see (1.2.22)) and for each $x\in\mathbb{R}^n$ we have

$$\begin{aligned}|v_\varepsilon(x)|&\le|v_\varepsilon(x)-v_\varepsilon(0)|+|v_\varepsilon(0)|\\ &\le\max\big\{[v_\varepsilon]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}\,,\,|v_\varepsilon(0)|\big\}(1+\omega(|x|)). \qquad (4.6.8)\end{aligned}$$

From this and Proposition 1.2.3 we then conclude that

$$\sup_{\mathbb{R}^n_+\cap B(0,r)}|\nabla v_\varepsilon|\le\frac{C}{r}\sup_{\mathbb{R}^n_+\cap B(0,2r)}|v_\varepsilon|\le C_\varepsilon\frac{1+\omega(2r)}{r}, \qquad (4.6.9)$$

and from Lemma 4.2.1(c) we see that the right side of (4.6.9) tends to 0 as $r\to\infty$. This forces $\nabla v_\varepsilon\equiv0$, and since $v_\varepsilon\in\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M\cap\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ with $v_\varepsilon\big|_{\partial\mathbb{R}^n_+}=0$ we ultimately conclude that $v_\varepsilon\equiv0$. Consequently,

$$u(x',t+\varepsilon)=(P^L_t*f_\varepsilon)(x'),\qquad\forall\,(x',t)\in\mathbb{R}^n_+. \qquad (4.6.10)$$

Since, as noted earlier, $u\big|_{\partial\mathbb{R}^n_+}\in\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$, for every $x'\in\mathbb{R}^{n-1}$ and $\varepsilon>0$ we may now write

$$\begin{aligned}\big|u(x',t+\varepsilon)-(P^L_t*(u\big|_{\partial\mathbb{R}^n_+}))(x')\big|&=\big|(P^L_t*(f_\varepsilon-u\big|_{\partial\mathbb{R}^n_+}))(x')\big|\\ &\le\|P^L_t\|_{L^1(\mathbb{R}^{n-1})}\sup_{y'\in\mathbb{R}^{n-1}}\big|f_\varepsilon(y')-u\big|_{\partial\mathbb{R}^n_+}(y')\big|\\ &=\|P^L\|_{L^1(\mathbb{R}^{n-1})}\sup_{y'\in\mathbb{R}^{n-1}}\big|u(y',\varepsilon)-u\big|_{\partial\mathbb{R}^n_+}(y')\big|\\ &\le\|P^L\|_{L^1(\mathbb{R}^{n-1})}[u]_{\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)}\,\omega(\varepsilon). \qquad (4.6.11)\end{aligned}$$

From (1.2.26) we know that $\|P^L\|_{L^1(\mathbb{R}^{n-1})} < \infty$. Upon letting $\varepsilon \to 0^+$ and using that $\omega$ vanishes in the limit at the origin, we see that (4.6.11) implies (4.6.1). This finishes the proof of Proposition 4.6.1. $\square$

## 4.7 Well-posedness results

We are now ready to prove well-posedness results. We first consider the case in which the boundary data belong to generalized Hölder spaces and we note that, in such a scenario, the only requirement on the growth function is (4.2.2).

**Theorem 4.7.1.** *Let $L$ be a constant complex coefficient $M \times M$ system as in (1.2.1) satisfying the strong ellipticity condition (1.2.4). Also, let $\omega$ be a growth function satisfying (4.2.2). Then the generalized Hölder Dirichlet Problem for $L$ in $\mathbb{R}^n_+$, formulated as*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \\ Lu = 0 \quad \textit{in} \quad \mathbb{R}^n_+, \\ [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} < \infty, \\ u\big|_{\partial\mathbb{R}^n_+} = f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M) \quad \textit{on} \quad \mathbb{R}^{n-1}, \end{cases} \tag{4.7.1}$$

*is well-posed. More specifically, there exists a unique solution which is given by*

$$u(x',t) = (P^L_t * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+, \tag{4.7.2}$$

*where $P^L$ denotes the Poisson kernel for the system $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4. In addition, $u$ extends to a function in $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ with $u\big|_{\partial\mathbb{R}^n_+} = f$, and there exists a finite constant $C = C(n,L,\omega) \geq 1$ such that*

$$C^{-1}[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)}. \tag{4.7.3}$$

*Proof.* Given $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$, define $u$ as in (4.7.2). Proposition 4.5.1 then implies that $u$ satisfies all conditions in (4.7.1). Also, $u$ extends to a function in $\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ with $u\big|_{\partial\mathbb{R}^n_+} = f$, and the second inequality in (4.7.3) holds. Moreover, (4.2.7) yields

$$[f]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} = [u|_{\partial\mathbb{R}^n_+}]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)} \leq 2C'_\omega[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)}, \tag{4.7.4}$$

so that the first inequality in (4.7.3) follows.

It remains to prove that the solution is unique. However, this follows at once from Proposition 4.6.1. Indeed, the first three conditions in (4.7.1) imply (4.6.1) and since $u\big|_{\partial\mathbb{R}^n_+} = f$ we conclude that necessarily $u(x',t) = \left(P^L_t * f\right)(x')$ for every $(x',t) \in \mathbb{R}^n_+$. $\square$

Here is the well-posedness for the generalized Morrey-Campanato Dirichlet Problem. In this case, the growth function is assumed to satisfy both (4.2.1) and (4.2.2).

**Theorem 4.7.2.** *Let $L$ be a constant complex coefficient $M \times M$ system as in (1.2.1) satisfying the strong ellipticity condition (1.2.4). Fix an aperture parameter $\kappa > 0$,*

*$p \in [1,\infty)$ along with $q \in (0,\infty]$, and let $\omega$ be a growth function satisfying (4.2.1) and (4.2.2). Then the generalized Morrey-Campanato Dirichlet Problem for $L$ in $\mathbb{R}^n_+$, namely*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\mathbb{R}^n_+)\right]^M, \\ Lu = 0 \quad in \ \ \mathbb{R}^n_+, \\ \|u\|_{**}^{(\omega,q)} < \infty, \\ u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} = f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) \quad a.e. \ on \ \ \mathbb{R}^{n-1}, \end{cases} \tag{4.7.5}$$

*is well-posed. More specifically, there exists a unique solution which is given by*

$$u(x',t) = (P_t^L * f)(x'), \qquad \forall\,(x',t) \in \mathbb{R}^n_+, \tag{4.7.6}$$

*where $P^L$ denotes the Poisson kernel for the system $L$ in $\mathbb{R}^n_+$ from Theorem 1.2.4. Moreover, with $W$ defined as in (4.2.3), the solution $u$ extends to a function in $\dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$, and there exists a finite constant $C = C(n,L,\omega,p,q) \geq 1$ for which*

$$C^{-1}\,\|f\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \|u\|_{**}^{(\omega,q)} \leq C\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\,. \tag{4.7.7}$$

*Furthermore, all results remain valid if $\|\cdot\|_{**}^{(\omega,q)}$ is replaced everywhere by $\|\cdot\|_{**}^{(\omega,\exp)}$.*

*Proof.* Having fixed $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$, if $u$ is defined as in (4.7.6) then Proposition 4.5.2 implies the validity of all conditions in (4.7.5) and also of the second inequality in (4.7.7) (even replacing $q$ by exp). In the case $q=\infty$ we invoke Proposition 4.3.1(f) to obtain that $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ in the sense of Lemma 4.2.1(d). Note that we also have

$$[u|_{\partial\mathbb{R}^n_+}]_{\dot{\mathscr{C}}^W(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq [u]_{\dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)} \leq 2C'_W[u]_{\dot{\mathscr{C}}^W(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C(C'_\omega)^4\,\|u\|_{**}^{(\omega,\infty)} \tag{4.7.8}$$

thanks to (4.2.7) (for the growth function $W$), Lemma 4.2.2, and (4.3.8).

Given that, on the one hand, $u|_{\partial\mathbb{R}^n_+} = u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}}$ everywhere in $\mathbb{R}^{n-1}$ because $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$, and that, on the other hand, $u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} = f$ a.e. in $\mathbb{R}^{n-1}$, we conclude that $u|_{\partial\mathbb{R}^n_+} = f$ a.e. in $\mathbb{R}^{n-1}$. In addition, (4.2.18) (applied to $W$), Lemma 4.2.2, and (4.7.8) permit us to estimate

$$\begin{aligned} \|f\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} &= \|\,u|_{\partial\mathbb{R}^n_+}\|_{\mathscr{E}^{W,p}(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq \sqrt{n-1}\,C'_W[u|_{\partial\mathbb{R}^n_+}]_{\dot{\mathscr{C}}^{\mathscr{W}}(\mathbb{R}^{n-1},\mathbb{C}^M)} \\ &\leq C(C'_\omega)^6\,\|u\|_{**}^{(\omega,\infty)} \leq C(C'_\omega)^7\,\|u\|_{**}^{(\omega,q)} \leq C(C'_\omega)^7\,\|u\|_{**}^{(\omega,\exp)}, \end{aligned} \tag{4.7.9}$$

where $0<q<\infty$ and where we have also used Proposition 4.3.1(d) and (4.3.1).

To prove that the solution is unique, we note that having $\|u\|_{**}^{(\omega,q)} < \infty$ for a given $q \in (0,\infty]$, or even $\|u\|_{**}^{(\omega,\exp)} < \infty$, implies that $\|u\|_{**}^{(\omega,\infty)} < \infty$ by Proposition 4.3.1(d) and (4.3.1). Having established this, Proposition 4.3.1(f) applies and yields that $u \in \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$. Consequently, $u|_{\partial\mathbb{R}^n_+} = u\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}}$ everywhere in $\mathbb{R}^{n-1}$, and if we also take into account the boundary condition from (4.7.5), we conclude that $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on

$\mathbb{R}^{n-1}$. Moreover, since Lemma 4.2.2 ensures that $W$ is a growth function satisfying (4.2.2), we may invoke Proposition 4.6.1 to write

$$u(x',t) = \Big(P_t^L * (u|_{\partial\mathbb{R}^n_+})\Big)(x') = \Big(P_t^L * f\Big)(x'), \qquad \forall\,(x',t)\in\mathbb{R}^n_+. \tag{4.7.10}$$

The proof of the theorem is therefore finished. □

*Remark* 4.7.3. Theorems 4.7.1 and 4.7.2 are closely related. To elaborate in this, fix a growth function $\omega$ satisfying (4.2.1) and (4.2.2). From (4.2.18) and Proposition 4.5.2 it follows that, given any $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$, the unique solution of the boundary value problem (4.7.1), i.e., $u(x',t) = (P_t^L * f)(x')$ for $(x',t)\in\mathbb{R}^n_+$, also solves (4.7.5), regarding now $f$ as a function in $\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ (cf. (4.2.19)) with $p\in[1,\infty)$ and $q\in(0,\infty]$ arbitrary (and even with $\|\cdot\|_{**}^{(\omega,q)}$ replaced by $\|\cdot\|_{**}^{(\omega,\exp)}$). As such, $u$ satisfies (4.7.7) whenever (4.2.1) holds.

This being said, the fact that $f\in\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ does not guarantee, in general, that the corresponding solution satisfies $u\in\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)$, even though we have established above that the solution to the boundary value problem (4.7.5) belongs to $\dot{\mathscr{C}}^W(\mathbb{R}^n_+,\mathbb{C}^M)$. Note that, as seen from (1.3.17)-(1.3.18) and (4.2.12), the space $\dot{\mathscr{C}}^W(\mathbb{R}^n_+,\mathbb{C}^M)$ contains $\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)$.

This aspect is fully clarified with the help of Example 4.7.4 discussed further below, where we construct some growth function $\omega$ satisfying (4.2.1), (4.2.2), and for which the space $\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1},\mathbb{C})$ is strictly bigger than $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C})$. Its relevance for the issue at hand is as follows. Consider the boundary problem (4.7.5) formulated with $L$ being the Laplacian in $\mathbb{R}^n$ and with $f\in\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1},\mathbb{C})\setminus\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C})$ as boundary datum. Its solution $u$ then necessarily satisfies $u\notin\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C})$, for otherwise Lemma 4.2.1(d) would imply $u\in\dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ and since $u|_{\partial\mathbb{R}^n_+} = u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$ and $f$ is continuous in $\mathbb{R}^{n-1}$ we would conclude that $f$ coincides everywhere with $u|_{\partial\mathbb{R}^n_+}\in\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C})$, a contradiction.

In spite of the previous remark, Theorem 4.1.2 states that the boundary problems (4.7.1) and (4.7.5) are actually equivalent under the stronger assumption (4.1.9) on the growth function. Here is the proof of Theorem 4.1.2.

*Proof of Theorem 4.1.2.* We start with the observation that (4.1.9) and Lemma 4.2.2 yield $C_\omega^{-1}W(t)\le\omega(t)\le C_\omega W(t)$ for each $t\in(0,\infty)$. Therefore,

$$\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M) = \dot{\mathscr{C}}^W(\mathbb{R}^n_+,\mathbb{C}^M), \quad \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M) = \dot{\mathscr{C}}^W(\overline{\mathbb{R}^n_+},\mathbb{C}^M), \tag{4.7.11}$$

and

$$\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M) = \mathscr{E}^{W,p}(\mathbb{R}^{n-1},\mathbb{C}^M), \tag{4.7.12}$$

as vector spaces, with equivalent norms.

Having made these identifications, we now proceed to observe that (a) follows directly from Theorem 4.7.1, while (b) is implied by Theorem 4.7.2 with the help of (4.7.11) and (4.7.12). To deal with (c), we first observe that the left-to-right inclusion follows from Lemma 4.2.4(b), whereas (4.2.18) provides the accompanying estimate for the norms.

For the converse inclusion, fix $f \in \mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)$ and set $u(x',t) := (P_t^L * f)(x')$ for every $(x',t) \in \mathbb{R}^n_+$. Theorem 4.7.2 and (4.7.11) then imply that $u \in \dot{\mathscr{C}}^\omega(\overline{\mathbb{R}^n_+},\mathbb{C}^M)$ with $u|_{\partial\mathbb{R}^n_+} = f$ a.e. on $\mathbb{R}^{n-1}$. Introduce $\widetilde{f} := u|_{\partial\mathbb{R}^n_+}$ and note that $\widetilde{f} \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$ with $\widetilde{f} = f$ a.e. on $\mathbb{R}^{n-1}$. Then $u(x',t) = (P_t^L * \widetilde{f})(x')$ and, thanks to (4.7.3), (4.3.11), and (4.7.7), we have

$$[\widetilde{f}\,]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)} \leq C[u]_{\dot{\mathscr{C}}^\omega(\mathbb{R}^n_+,\mathbb{C}^M)} \leq C\,\|f\|_{\mathscr{E}^{\omega,p}(\mathbb{R}^{n-1},\mathbb{C}^M)}\,. \tag{4.7.13}$$

This completes the treatment of (c), and finishes the proof of Theorem 4.1.2. □

We are now in a position to give the proof of Corollary 4.1.3.

*Proof of Corollary 4.1.3.* We start by observing that (4.1.16) is a direct consequence of Proposition 4.3.1(h). In particular, the last three equalities in (4.1.17) follow at once. Also, the fact that the second set in the first line of (4.1.17) is contained in $\dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$ is a consequence of Lemma 4.2.1(d). Finally, given any $f \in \dot{\mathscr{C}}^\omega(\mathbb{R}^{n-1},\mathbb{C}^M)$, if $u$ is the solution of (4.1.10) corresponding to this choice of boundary datum, then $u|_{\partial\mathbb{R}^n_+} = f$ and $u$ also satisfies the required conditions to be an element in the second set displayed in (4.1.17). □

The following example shows that conditions (4.2.1) and (4.2.2) do not imply (4.1.15).

**Example 4.7.4.** Fix two real numbers $\alpha,\beta \in (0,1)$ and consider the growth function $\omega : (0,\infty) \to (0,\infty)$ defined for each $t > 0$ as

$$\omega(t) := \begin{cases} t^\alpha, & \text{if } t \leq 1, \\ 1 + (\log t)^\beta, & \text{if } t > 1. \end{cases} \tag{4.7.14}$$

Clearly, $\omega$ satisfies (4.2.1), and we also claim that $\omega$ satisfies (4.2.2). Indeed, for $t \leq 1$,

$$\int_t^\infty \frac{\omega(s)}{s}\frac{ds}{s} = \int_t^1 s^{\alpha-1}\frac{ds}{s} + \int_1^\infty \frac{1+(\log s)^\beta}{s^2}\,ds \leq C(t^{\alpha-1}+1) \leq 2C\,t^{\alpha-1}. \tag{4.7.15}$$

For $t \in [1,\infty)$, define

$$F(t) := \frac{t\displaystyle\int_t^\infty \frac{\omega(s)}{s}\frac{ds}{s}}{\omega(t)} = \frac{\displaystyle\int_t^\infty \frac{1+(\log s)^\beta}{s^2\,ds}}{\dfrac{1+(\log t)^\beta}{t}}, \tag{4.7.16}$$

which is a continuous function in $[1,\infty)$ and satisfies $F(1) < \infty$. Moreover, using L'Hôpital's rule,

$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{-(1+(\log t)^\beta)}{\beta(\log t)^{\beta-1} - (1+(\log t)^\beta)} = 1. \tag{4.7.17}$$

Hence, $F$ is bounded, which amounts to having $\omega$ satisfy (4.2.2). The function $W$, defined as in (4.2.3), is currently given by

$$W(t) = \begin{cases} \dfrac{1}{\alpha}t^\alpha, & \text{if } t \leq 1, \\ \dfrac{1}{\alpha} + \dfrac{1}{\beta+1}(\log t)^{\beta+1} + \log t, & \text{if } t > 1. \end{cases} \tag{4.7.18}$$

Since (4.1.9) would imply $W(t) \leq C\omega(t)$ which is not the case for $t$ sufficiently large, we conclude that the growth function $\omega$ satisfies (4.2.1) and (4.2.2) but it does not satisfy (4.1.9).

For this choice of $\omega$, we now proceed to check that $\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1},\mathbb{C}) \neq \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1})$. To this end, consider the function

$$f(x) := \log_+ |x_1|, \qquad \forall\, x = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, \tag{4.7.19}$$

where $\log_+ t := \max\{0, \log t\}$. With $e_1 =: (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$ we then have

$$\sup_{x\neq y} \frac{|f(x) - f(y)|}{\omega(|x-y|)} \geq \lim_{x_1\to\infty} \frac{|f(x_1e_1) - f(e_1)|}{\omega(|x_1e_1 - e_1|)} = \lim_{x_1\to\infty} \frac{\log x_1}{1 + (\log(x_1 - 1))^\beta} = \infty, \tag{4.7.20}$$

since $\beta < 1$. This means that $f \notin \dot{\mathscr{C}}^{\omega}(\mathbb{R}^{n-1})$. To prove that $f \in \mathscr{E}^{\omega,1}(\mathbb{R}^{n-1},\mathbb{C})$, consider $\widetilde{Q} := (a,b) \times Q \subseteq \mathbb{R}^{n-1}$, where $Q$ is an arbitrary cube in $\mathbb{R}^{n-2}$ and $a, b \in \mathbb{R}$ are arbitrary numbers satisfying $a < b$. Then,

$$\begin{aligned} \|f\|_{\mathscr{E}^{\omega,1}(\mathbb{R}^{n-1},\mathbb{C})} &\leq \sup_{\widetilde{Q}\subseteq\mathbb{R}^{n-1}} \frac{1}{\omega(\ell(\widetilde{Q}))} ⨍_{\widetilde{Q}} ⨍_{\widetilde{Q}} |f(x) - f(y)|\, dx\, dy \\ &\leq \sup_{a<b} \frac{1}{\omega(b-a)} H(a,b), \end{aligned} \tag{4.7.21}$$

where

$$H(a,b) := ⨍_a^b ⨍_a^b \big| \log_+ |x_1| - \log_+ |y_1| \big|\, dx_1\, dy_1. \tag{4.7.22}$$

We shall now prove that the right-hand side of (4.7.21) is finite considering several different cases.

**Case I: $1 \leq a < b$.** In this scenario, define

$$G(\lambda) := 1 + 2\lambda - 2\lambda(\lambda+1)\log\left(1 + \frac{1}{\lambda}\right), \quad \forall\, \lambda > 0. \tag{4.7.23}$$

Note that $G$ is continuous in $(0,\infty)$, $G(0) = 1$, and by L'Hôpital's rule, $\lim_{\lambda\to\infty} G(\lambda) = 0$, hence $G$ is bounded. Also,

$$H(a,b) = \frac{b^2 - a^2 - 2ab\log(b/a)}{(b-a)^2} = G(a/(b-a)). \tag{4.7.24}$$

Consequently, whenever $b - a \geq 1$ we have

$$H(a,b) = G(a/(b-a)) \leq C \leq C\big(1 + (\log(b-a))^\beta\big) = C\omega(b-a). \tag{4.7.25}$$

Again by L'Hôpital's rule, $\lim_{\lambda\to\infty} \lambda^\alpha G(\lambda) = 0$, hence $\lambda^\alpha G(\lambda) \leq C$ for every $\lambda > 0$. Therefore, whenever $0 < b - a < 1$ we may write

$$H(a,b) = G(a/(b-a)) \leq C\left(\frac{b-a}{a}\right)^\alpha \leq C(b-a)^\alpha = C\omega(b-a). \tag{4.7.26}$$

All these show that $H(a,b) \leq C\omega(b-a)$ in this case.

**Case II: $a < b \le -1$.** This case is analogous to the previous one by symmetry.

**Case III: $-1 \le a < b \le 1$.** This case is straightforward since $H(a,b) = 0$, given that $\log_+ |x_1| = \log_+ |y_1| = 0$ whenever $a < x_1, y_1 < b$.

**Case IV: $-1 < a < 1 < b$.** In this case we obtain

$$\begin{aligned} H(a,b) &= \frac{1}{(b-a)^2}\int_1^b\int_1^b |\log x_1 - \log y_1|\, dx_1\, dy_1 \\ &\quad + \frac{1}{(b-a)^2}\int_a^1\int_1^b \log x_1\, dx_1\, dy_1 + \frac{1}{(b-a)^2}\int_1^b\int_a^1 \log y_1\, dx_1\, dy_1 \\ &\le \frac{(b-1)^2}{(b-a)^2}H(1,b) + 2\frac{(1-a)(b\log b - b + 1)}{(b-a)^2}. \end{aligned} \tag{4.7.27}$$

For the first term in the right-hand side of (4.7.27), we use (4.7.25) and (4.7.26) (written with $a := 1$) and obtain, keeping in mind that in this case $a < 1$,

$$\frac{(b-1)^2}{(b-a)^2}H(1,b) \le C\omega(b-1)\left(\frac{b-1}{b-a}\right)^2 \le C\omega(b-a). \tag{4.7.28}$$

To bound the second term in the right-hand side of (4.7.27), we first use the fact that $1 - a < 2$ and $\log t \le t - 1$ for every $t \ge 1$ to obtain

$$\begin{aligned} \frac{(1-a)(b\log b - b + 1)}{(b-a)^2} &\le 2\frac{b(b-1) - b + 1}{(b-a)^2} \le 2\left(\frac{b-1}{b-a}\right)^2 \le 2 \\ &\le 2(1 + (\log(b-a))^\beta) = 2\omega(b-a), \end{aligned} \tag{4.7.29}$$

whenever $b - a \ge 1$. To study the case when $b - a < 1$, bring in the auxiliary function

$$\widetilde{G}(\lambda) := \frac{\lambda\log\lambda - \lambda + 1}{(\lambda-1)^{1+\alpha}}, \qquad \forall \lambda > 1. \tag{4.7.30}$$

By L'Hôpital's rule, $\lim_{\lambda\to 1^+}\widetilde{G}(\lambda) = 0$, hence $\widetilde{G}(\lambda) \le C$ for each $\lambda \in (1,2]$. If $b - a < 1$, we clearly have $1 < b \le 2$ which, in turn, permits us to estimate

$$\begin{aligned} \frac{(1-a)(b\log b - b + 1)}{(b-a)^2} &= \frac{(1-a)(b-1)^{1+\alpha}G(b)}{(b-a)^2} \\ &\le \frac{C(b-1)^{1+\alpha}}{b-a} \le C(b-a)^\alpha = C\omega(b-a). \end{aligned} \tag{4.7.31}$$

Consequently, we have obtained that $H(a,b) \le C\omega(b-a)$ in this case as well.

**Case V: $a < -1 < b < 1$.** This is analogue to Case IV, again by symmetry.

**Case VI: $a < -1,\ b > 1$.** We break the interval $(a,b)$ into two intervals $(a,0)$ and $(0,b)$ to obtain

$$H(a,b) \le \frac{1}{(b-a)^2}(\mathrm{I} + \mathrm{II}), \tag{4.7.32}$$

where, using Case IV and Case V,

$$\begin{aligned}
\mathrm{I} &:= \int_a^0 \int_a^0 \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1 + \int_0^b \int_0^b | \log_+ x_1 - \log_+ y_1 | \, dx_1 \, dy_1 \\
&= H(a,0)(0-a)^2 + H(0,b)(b-0)^2 \leq C|a|^2 \omega(|a|) + C\, b^2 \omega(b) \\
&\leq 2C(b-a)^2 \omega(b-a). \qquad (4.7.33)
\end{aligned}$$

Similarly, by Case IV,

$$\begin{aligned}
\mathrm{II} &:= \int_a^0 \int_0^b \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1 + \int_0^b \int_a^0 \big| \log_+ |x_1| - \log_+ |y_1| \big| \, dx_1 \, dy_1 \\
&\leq 2 \big( \max\{|a|, b\} - 0 \big)^2 H\big( \max\{|a|, b\}, 0 \big) \\
&\leq 2C \max\{|a|, b\}^2 \, \omega(\max\{|a|, b\}) \\
&\leq 2C(b-a)^2 \, \omega(b-a). \qquad (4.7.34)
\end{aligned}$$

Thus, $H(a,b) \leq C\omega(b-a)$ in this case also.

Collectively, the results in Cases I-VI prove that $f \in \mathscr{E}^{\omega,1}(\mathbb{R}^{n-1}, \mathbb{C})$.

CHAPTER 5

# Characterizations of Lyapunov domains in terms of Riesz transforms and generalized Hölder spaces

We prove several characterizations of $\mathscr{C}^{1,\omega}$-domains (aka Lyapunov domains), where $\omega$ is a growth function satisfying natural assumptions. For example, given an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, we show that the modulus of continuity of the geometric measure theoretic outward unit normal $\nu$ to $\Omega$ is dominated by (a multiple of) $\omega$ if and only if the action of each Riesz transform $R_j$ associated with $\partial\Omega$ on the constant function 1 has a modulus of continuity dominated by (a multiple of) $\omega$. We also establish estimates for certain classes of singular integral operators on generalized Hölder space on Lyapunov domains.

The material in this chapter is based on joint work with J.M. Martell and M. Mitrea (cf. [78]).

## Contents



## 5.1 Introduction

The principal aim of this chapter is to characterize Lyapunov $\mathscr{C}^{1,\omega}$-domains in $\mathbb{R}^n$, $n \geq 2$. One way to think of such a domain $\Omega \subseteq \mathbb{R}^n$ is as an open set of locally finite perimeter whose geometric measure theoretic outward unit normal $\nu$, after possibly being redefined on a set of $\sigma$-measure zero, belongs to $\mathscr{C}^\omega(\partial\Omega)$. Here, $\mathscr{C}^\omega(\partial\Omega)$ is a generalized Hölder space, quantifying continuity in terms of the modulus, or "growth" function, $\omega$ (cf. Definition 1.3.5), and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the "surface measure" on $\partial\Omega$ (with $\mathcal{H}^{n-1}$ denoting the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$). Thanks to work in [52], the above class of domains may be equivalently described as the collection of all open subsets of $\mathbb{R}^n$ which locally coincide (up to a rigid transformation of the space) with the upper-graph of a real-valued continuously differentiable function defined in $\mathbb{R}^{n-1}$ whose first-order partial derivatives belong to $\mathscr{C}^\omega(\mathbb{R}^{n-1})$.

**Definition 5.1.1.** Let $\omega$ be a growth function. A nonempty, proper, open subset $\Omega$ of $\mathbb{R}^n$ is called a `Lyapunov` $\mathscr{C}^{1,\omega}$`-domain` (or simply a $\mathscr{C}^{1,\omega}$`-domain`) if there exist $r, h > 0$ such that for every point $x_0 \in \partial\Omega$ there exists a coordinate system $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ which is isometric to the canonical one and has $x_0$ as its origin, and a function $\varphi \in \mathscr{C}^{1,\omega}(\mathbb{R}^{n-1})$ such that

$$\Omega \cap C(r,h) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < r,\ \varphi(x') < x_n < h\}, \tag{5.1.1}$$

where $C(r,h)$ is the cylinder defined as

$$C(r,h) = \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < r,\ -h < x_n < h\}. \tag{5.1.2}$$

*Remark* 5.1.2. Any $\mathscr{C}^{1,\omega}$-domain with compact boundary is, in particular, simultaneously a UR domain and a uniform domain.

*Remark* 5.1.3. Analogously to the characterization of $\mathscr{C}^1$ domains given in [52], one can prove that $\mathscr{C}^{1,\omega}$-domain are those open sets of locally finite perimeter with the property that the geometric measure theoretic outward unit normal $\nu$ to $\Omega$, after possibly being altered on a set of $\sigma$-measure zero, belongs to $\mathscr{C}^\omega(\partial\Omega)$. See [52, Theorem 2.19] for the proof in the case $\omega(t) = t^\alpha$ for each $t \in (0, \infty)$ with $\alpha \in (0,1)$, which is easily adapted to our scenario.

The characterizations of the class of Lyapunov domains we presently seek are in terms of the boundedness properties of certain classes of singular integral operators acting on generalized Hölder spaces. The most prominent examples of such singular integral operators are offered by the Riesz transforms on the boundary of an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ with compact boundary (cf. Definition 1.1.2). Specifically, for each $j \in \{1, \dots, n\}$ we define the $j$-th distributional Riesz transform on $\partial\Omega$ as the operator

$$\begin{gathered} R_j : \mathscr{C}^\omega(\partial\Omega) \longrightarrow \big(\mathscr{C}^\omega(\partial\Omega)\big)^* \text{ satisfying, for every } f, g \in \mathscr{C}^\omega(\partial\Omega), \\ \langle R_j f, g\rangle = \frac{1}{2\varpi_{n-1}} \int_{\partial\Omega}\int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\big(f(y)g(x) - f(x)g(y)\big)\, d\sigma(y)\, d\sigma(x), \end{gathered} \tag{5.1.3}$$

where $\varpi_{n-1}$ stands for the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ denotes the natural duality pairing between $\big(\mathscr{C}^\omega(\partial\Omega)\big)^*$ and $\mathscr{C}^\omega(\partial\Omega)$. From the $T(1)$ theorem for spaces of homogeneous type (cf., e.g., [21]) we know that for each $j \in \{1, \dots, n\}$ the operator $R_j$ extends to a bounded linear mapping on $L^2(\partial\Omega,\sigma)$ if and only if $R_j 1$ belongs to $\mathrm{BMO}(\partial\Omega,\sigma)$, the John-Nirenberg space of functions with bounded mean oscillation on $\partial\Omega$ (with respect to the measure $\sigma$). Moreover, having $R_j 1 \in \mathrm{BMO}(\partial\Omega,\sigma)$ is actually equivalent to $\partial\Omega$ being uniformly rectifiable (cf. [101]). Ultimately this goes to show that the aforementioned extension of $R_j$ from (5.1.3) to a bounded mapping on $L^2(\partial\Omega,\sigma)$ is given by the principal-value integral operator acting on each $f \in L^2(\partial\Omega,\sigma)$ according to

$$(R_j f)(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{x_j-y_j}{|x-y|^n} f(y)\,d\sigma(y) \quad \text{for } \sigma\text{-a.e. } \ x\in\partial\Omega. \tag{5.1.4}$$

In this chapter, we find it convenient to make two terminology changes with respect to the previous chapters: the area of the unit sphere is denoted by $\varpi_{n-1}$ (in place of $\omega_{n-1}$), and the definition of the Riesz transform $R_j f$ above differs from the Riesz transform defined in Chapter 2 by a constant).

Our main result in this regard is the following theorem. For all relevant definitions, the reader is referred to Sections 1.1, 1.3, and 5.2.

**Theorem 5.1.4.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain whose boundary is compact. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Finally, let $\omega : (0, \mathrm{diam}(\partial\Omega)) \to (0,\infty)$ be a bounded, non-decreasing function, whose limit at the origin vanishes, and satisfying*

$$\sup_{0<t<\mathrm{diam}(\partial\Omega)} \left\{ \frac{1}{\omega(t)} \Big( \int_0^t \omega(s)\frac{ds}{s} + t \int_t^{\mathrm{diam}(\partial\Omega)} \frac{\omega(s)}{s}\frac{ds}{s} \Big) \right\} < +\infty. \tag{5.1.5}$$

*Then the following statements are equivalent:*

(a) *After possibly being altered on a set of $\sigma$-measure zero, the outward unit normal $\nu$ to $\Omega$ belongs to the generalized Hölder space $\mathscr{C}^\omega(\partial\Omega)$ (i.e. the set $\Omega$ is a $\mathscr{C}^{1,\omega}$-domain; cf. Remark 5.1.3).*

(b) *The Riesz transforms on $\partial\Omega$ satisfy*

$$R_j 1 \in \mathscr{C}^\omega(\partial\Omega) \ \ \textit{for each} \ \ j \in \{1,\dots,n\}. \tag{5.1.6}$$

(c) *The set $\Omega$ is a* UR *domain (in the sense of Definition 1.1.5), and given any odd homogenous polynomial $P$ of degree $\ell \geq 1$ in $\mathbb{R}^n$ the singular integral operator acting on each function $f \in \mathscr{C}^\omega(\partial\Omega)$ according to*

$$(Tf)(x) := \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{P(x-y)}{|x-y|^{n-1+\ell}} f(y)\,d\sigma(y) \quad \textit{for } \sigma\textit{-a.e. } \ x\in\partial\Omega \tag{5.1.7}$$

*is well-defined and maps the generalized Hölder space $\mathscr{C}^\omega(\partial\Omega)$ boundedly into itself.*

(*d*) *The set* $\Omega$ *is a* UR *domain, and the boundary-to-domain version of the Riesz transforms defined for each* $j \in \{1, \dots, n\}$ *and each* $f \in L^1(\partial\Omega, \sigma)$ *as*

$$(\mathscr{R}_j^{\pm} f)(x) := \frac{1}{\varpi_{n-1}} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_{\pm}, \tag{5.1.8}$$

*satisfy*

$$\mathscr{R}_j^{\pm} 1 \in \mathscr{C}^{\omega}(\Omega_{\pm}) \ \text{ for each } \ j \in \{1, \dots, n\}. \tag{5.1.9}$$

(*e*) *The set* $\Omega$ *is a* UR *domain, and given any odd homogenous polynomial* $P$ *of degree* $\ell \geq 1$ *in* $\mathbb{R}^n$*, the integral operators acting on each function* $f \in \mathscr{C}^{\omega}(\partial\Omega)$ *according to*

$$\mathbb{T}_{\pm} f(x) := \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+\ell}} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega_{\pm}, \tag{5.1.10}$$

*map the generalized Hölder space* $\mathscr{C}^{\omega}(\partial\Omega)$ *continuously into* $\mathscr{C}^{\omega}(\Omega_{\pm})$*.*

*In addition, if* $\Omega$ *is a* $\mathscr{C}^{1,\omega}$*-domain, there exists a constant* $C \in (0, \infty)$*, depending only on* $n$*,* $\omega$*, and* $\Omega$*, with the property that*

$$\|\mathbb{T}_{\pm} f\|_{\mathscr{C}^{\omega}(\Omega_{\pm})} \leq C^{\ell}\, 2^{\ell^2} \|P\|_{L^2(S^{n-1}, \mathcal{H}^{n-1})} \|f\|_{\mathscr{C}^{\omega}(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^{\omega}(\partial\Omega), \tag{5.1.11}$$

*and*

$$\|Tf\|_{\mathscr{C}^{\omega}(\partial\Omega)} \leq C^{\ell}\, 2^{\ell^2} \|P\|_{L^2(S^{n-1}, \mathcal{H}^{n-1})} \|f\|_{\mathscr{C}^{\omega}(\partial\Omega)}, \qquad \forall\, f \in \mathscr{C}^{\omega}(\partial\Omega). \tag{5.1.12}$$

This generalizes earlier work in [96] where similar characterizations for domains of class $\mathscr{C}^{1+\alpha}$, with $\alpha \in (0,1)$, have been obtained. The latter scenario presently corresponds to the particular choice $\omega(t) := t^{\alpha}$ for each $t > 0$ in Theorem 5.1.4. Our present work adds further credence to the heuristic principle that the action of the distributional Riesz transforms (5.1.3) on the constant function 1 encapsulates much information, both of analytic and geometric flavor, about the underlying Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ (with compact boundary). At the most basic level, the main result of F. Nazarov, X. Tolsa, and A. Volberg in [101] states that

$$\partial\Omega \ \text{ is a UR set} \iff R_j 1 \in \mathrm{BMO}(\partial\Omega, \sigma) \ \text{ for each } \ j \in \{1, \dots, n\}, \tag{5.1.13}$$

and it has been noted in [96] that

$$\left.\begin{array}{r} \nu \in \mathrm{VMO}(\partial\Omega, \sigma) \\ \text{and } \partial\Omega \text{ is a UR set} \end{array}\right\} \iff R_j 1 \in \mathrm{VMO}(\partial\Omega, \sigma) \ \text{ for all } \ j \in \{1, \dots, n\}, \tag{5.1.14}$$

where $\mathrm{VMO}(\partial\Omega, \sigma)$ stands for the Sarason space of functions with vanishing mean oscillation on $\partial\Omega$, with respect to the measure $\sigma$. By further assigning additional regularity for the functionals $\{R_j 1\}_{1 \leq j \leq n}$ yields the following result (proved in [96])

$$\left.\begin{array}{r} \Omega \text{ is a domain} \\ \text{of class } \mathscr{C}^{1+\alpha} \end{array}\right\} \iff R_j 1 \in \mathscr{C}^{\alpha}(\partial\Omega) \ \text{ for all } \ j \in \{1, \dots, n\}, \tag{5.1.15}$$

where $\alpha \in (0,1)$ and $\mathscr{C}^\alpha(\partial\Omega)$ is the classical Hölder space of order $\alpha$ on $\partial\Omega$.

Theorem 5.1.4 provides a satisfactory generalization of (5.1.15) by allowing considerably more flexible scales of spaces measuring Hölder regularity (see the discussion in Example 1.3.4 in this regard). To place this in perspective, observe that the operators described in (5.1.7) may be thought of as generalized Riesz transforms since they correspond to (5.1.4) in the case when

$$P(x) := x_j/\varpi_{n-1} \quad \text{for} \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad j \in \{1, \dots, n\}. \tag{5.1.16}$$

As such, our result may be interpreted as roughly saying that *the classical Riesz transforms are bounded on a generalized Hölder space if and only if all generalized Riesz transforms are bounded on a generalized Hölder space if and only if the underlying domain is Lyapunov.*

Due to their significant role in Partial Differential Equations, Lyapunov domains have received a good deal of attention in the literature, and a number of alternative characterizations have been discovered. For example, Theorem 5.1.4 should be compared with the following purely geometric characterization of Lyapunov domains obtained in [5], which amounts to the ability of threading the boundary of the said domain in between the two rounded components of an "hour-glass" shaped solid.

**Proposition 5.1.5.** *Fix $D \in (0,\infty)$ and let $\omega : (0,D] \to [0,\infty)$ be a continuous, strictly increasing function, with the property that*

$$\lim_{\lambda \to 0^+} \Big( \sup_{t \in (0, \min\{D, D/\lambda\}]} \frac{\omega(\lambda t)}{\omega(t)} \Big) = 0. \tag{5.1.17}$$

*Define the pseudo-ball associated with $\omega$ having apex at a point $x \in \mathbb{R}^n$, axis of symmetry along some vector $h \in S^{n-1}$, height $b > 0$, and aperture $a > 0$, as the set*

$$\mathscr{G}^\omega_{a,b}(x,h) := \{ y \in B(x,D) \setminus \{x\} : a|y-x|\,\omega(|y-x|) < h \cdot (y-x) < b \}. \tag{5.1.18}$$

*Then a nonempty, open, proper subset $\Omega$ of $\mathbb{R}^n$, with compact boundary is a $\mathscr{C}^{1,\omega}$-domain if and only if there exist $a > 0$, $b > 0$ and a function $h : \partial\Omega \to S^{n-1}$ with the property that*

$$\mathscr{G}^\omega_{a,b}(x,h(x)) \subseteq \Omega \quad \text{and} \quad \mathscr{G}^\omega_{a,b}(x,-h(x)) \subseteq \mathbb{R}^n \setminus \Omega \quad \text{for each} \quad x \in \partial\Omega. \tag{5.1.19}$$

In [5] this was used to prove a sharper version of the Hopf-Oleinik Boundary Point Principle in domains satisfying an "pseudo-ball condition" (in place of the classical interior ball condition).

The layout of the chapter is as follows. In Section 5.2 we study growth functions and generalized Hölder spaces, extending the results in Section 1.3. We then proceed to study singular integrals on generalized Hölder spaces in Section 5.3. Following [96], our approach relies on Clifford algebra techniques. In Section 5.4 we prove some basic estimates for the Cauchy-Clifford operators acting on generalized Hölder spaces. Finally, Section 5.5 contains the proof of our main result, Theorem 5.1.4, which proceeds by induction on the degree $\ell$ of the polynomial $P$ involved in (5.1.7) and (5.1.10).

## 5.2 More on growth functions

Given a growth function $\omega$, in the lemma below we study some of the implications of (1.3.6) for the function $\widetilde{\omega}$ associated with $\omega$ as in Remark 1.3.2.

**Lemma 5.2.1.** *Suppose $D \in (0,\infty]$ and let $\omega$ be a growth function on $(0,D)$ satisfying (1.3.6). If $\widetilde{\omega}$ is associated with $\omega$ as in Remark 1.3.2, then the following statements are true.*

*(a) For each $N \in [1,\infty)$ one has*

$$\int_0^t \widetilde{\omega}(s)\frac{ds}{s} \leq C_1\widetilde{\omega}(t) \quad \textit{for all} \quad t \in (0, ND), \tag{5.2.1}$$

*with $C_1 := C_\omega + \ln N$ if $D < \infty$ and $C_1 := C_\omega$ if $D = \infty$.*

*(b) One has*

$$t\int_t^\infty \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} \leq C_2\widetilde{\omega}(t) \quad \textit{for all} \quad t \in (0,\infty), \tag{5.2.2}$$

*with $C_2 := C_\omega + \max\{1, C_\omega\}\frac{\omega(D)}{\omega(D/2)}$ if $D < \infty$ and $C_2 := C_\omega$ if $D = \infty$.*

*(c) As a consequence of Remark 1.3.2 and (a)-(b), one concludes that $\widetilde{\omega}$ is growth function on $(0,\infty)$ which satisfies (1.3.6) with*

$$C_{\widetilde{\omega}} \leq C_1 + C_2, \tag{5.2.3}$$

*where $C_1, C_2 \in (0,\infty)$ are as above.*

*(d) Whenever $0 < t_1 \leq t_2 < \infty$ one has*

$$\frac{\widetilde{\omega}(t_2)}{t_2} \leq C_2\frac{\widetilde{\omega}(t_1)}{t_1}, \tag{5.2.4}$$

*with $C_2$ as in part (b). In particular, $\widetilde{\omega}$ is doubling with constant*

$$\sup_{0<t<\infty} \frac{\widetilde{\omega}(2t)}{\widetilde{\omega}(t)} \leq 2C_2. \tag{5.2.5}$$

*Proof.* If $D = \infty$, then the claims in $(a)$ and $(b)$ are direct consequences of (1.3.7), since $\widetilde{\omega} = \omega$ on $(0,\infty)$ in this case. Henceforth assume $D < \infty$. In such a scenario, if $t \in (0,D)$ then the claim in $(a)$ follows at once from (1.3.7) since $\widetilde{\omega} = \omega$ on $(0,D)$. On the other hand, if $t \in [D, ND)$ for some $N \in [1,\infty)$, then

$$\begin{aligned}\int_0^t \widetilde{\omega}(s)\frac{ds}{s} &= \int_0^D \omega(s)\frac{ds}{s} + \int_D^{ND} \omega(D)\frac{ds}{s} \leq C_\omega\omega(D) + \omega(D)\ln N \\ &= (C_\omega + \ln N)\widetilde{\omega}(t),\end{aligned} \tag{5.2.6}$$

keeping in mind that, as noted earlier, (1.3.7) extends to $t = D$ in this case.

As regards item $(b)$, we first claim that

$$t \leq \frac{\max\{1, C_\omega\}\, D}{\omega(D/2)}\,\omega(t) \quad \text{for each} \quad t \in (0, D). \tag{5.2.7}$$

Indeed, if $t \in (0, D/2]$ we may estimate

$$t\frac{\omega(D/2)}{D} \leq t\int_{D/2}^{D} \frac{\omega(s)}{s}\frac{ds}{s} \leq t\int_{t}^{D} \frac{\omega(s)}{s}\frac{ds}{s} \leq C_\omega \omega(t), \tag{5.2.8}$$

which suits our purposes. If $t \in (D/2, D)$, the fact hat $\omega$ is a non-decreasing function entails $\omega(D/2)/D \leq \omega(t)/t$, finishing the proof of (5.2.7). In turn, on account of (5.2.7) we see that for each $t \in (0, D)$ we have

$$\begin{aligned} t\int_{t}^{\infty} \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} &= t\int_{t}^{D} \frac{\omega(s)}{s}\frac{ds}{s} + t\int_{D}^{\infty} \frac{\omega(D)}{s}\frac{ds}{s} \leq C_\omega \omega(t) + t\frac{\omega(D)}{D} \\ &\leq \Big(C_\omega + \max\{1, C_\omega\}\frac{\omega(D)}{\omega(D/2)}\Big)\omega(t). \end{aligned} \tag{5.2.9}$$

To finish the proof of (5.2.2) there remains to observe that, if $t \geq D$,

$$t\int_{t}^{\infty} \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} = \omega(D) = \widetilde{\omega}(t). \tag{5.2.10}$$

Next, the claims in item $(c)$ are direct consequences of Remark 1.3.2 and parts *(a)-(b)*. Turning our attention to item $(d)$, if $0 < t_1 \leq t_2 < \infty$ then (5.2.2) implies

$$\frac{\widetilde{\omega}(t_2)}{t_2} \leq \int_{t_2}^{\infty} \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} \leq \int_{t_1}^{\infty} \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} \leq C_2\frac{\widetilde{\omega}(t_1)}{t_1}, \tag{5.2.11}$$

and the assertion in (5.2.5) follows by specializing this to the case when $t_1 := t$ and $t_2 := 2t$. $\square$

In the classical case when, for some $\alpha \in (0, 1)$, the growth function is defined as $\omega(t) := t^\alpha$ for each $t \in (0, \infty)$, the function $V(t) := t^{\alpha-1}$ for each $t \in (0, \infty)$ plays a significant role in ensuing analysis. Below we identify the general format of the latter function associated with general growth functions.

**Lemma 5.2.2.** *Given $D \in (0, \infty]$, let $\omega$ be a growth function on $(0, D)$ and recall the function $\widetilde{\omega}$ from Remark 1.3.2, defined as $\widetilde{\omega}(t) := \omega(\min\{t, D\})$ for each $t \in (0, \infty)$. Set*

$$V(t) := \int_{t}^{\infty} \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} \quad \textit{for each} \quad t \in (0, \infty). \tag{5.2.12}$$

*Then $V : (0, \infty) \to (0, \infty]$ is a non-increasing function which satisfies*

$$\frac{\widetilde{\omega}(t)}{t} \leq V(t) \quad \textit{for each} \quad t \in (0, \infty). \tag{5.2.13}$$

*Moreover, if $D < \infty$ then $V$ takes only finite values.*

*Proof.* By design, $V$ is a non-increasing function. Bearing in mind that $\widetilde{\omega}$ is non-decreasing, for every $t \in (0,\infty)$ we may write

$$\frac{\widetilde{\omega}(t)}{t} = \int_t^\infty \frac{\widetilde{\omega}(t)}{s}\frac{ds}{s} \le \int_t^\infty \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} = V(t), \tag{5.2.14}$$

proving (5.2.13). Also, when $D \in (0,\infty)$, for each $t \in (0,D)$ we have

$$\begin{aligned} V(t) &= \int_D^\infty \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} + \int_t^D \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} = \frac{\omega(D)}{D} + \int_t^D \frac{\widetilde{\omega}(s)}{s}\frac{ds}{s} \\ &\le \frac{\omega(D)}{D} + \omega(D)\Big(\frac{1}{t} - \frac{1}{D}\Big). \end{aligned} \tag{5.2.15}$$

As $V(t)$ is non-increasing, this proves that $V$ only takes finite values. $\square$

We close this section by proving some useful integral estimates on upper Ahlfors regular sets involving growth functions. Here and elsewhere, for each number $a \in \mathbb{R}$ we agree to abbreviate $(a)_+ := \max\{a,0\}$.

**Lemma 5.2.3.** *Suppose $\Sigma$ is closed subset of $\mathbb{R}^n$ satisfying an upper Ahlfors regularity condition with constant $C \in (0,\infty)$. Define $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, assume $\omega$ is a growth function on $(0,\infty)$. Then for each $x \in \Sigma$, $r \in \big(0\,,\operatorname{diam}(\Sigma)\big)$, and $d \in \mathbb{R}$ one has*

$$\int_{\Sigma\cap B(x,r)} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) \le C\frac{2^{n-1+d}\cdot 2^{d_+}\cdot 2^{(1-n-d)_+}}{\ln 2}\int_0^{2\,r} \frac{\omega(s)}{s^d}\frac{ds}{s}, \tag{5.2.16}$$

$$\int_{\Sigma\setminus B(x,r)} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) \le C\frac{2^{n-1+d}\cdot 2^{d_+}\cdot 2^{(1-n-d)_+}}{\ln 2}\int_{2\,r}^{4\operatorname{diam}(\Sigma)} \frac{\omega(s)}{s^d}\frac{ds}{s}. \tag{5.2.17}$$

*Proof.* Fix $x \in \Sigma$ and $r \in \big(0\,,\operatorname{diam}(\Sigma)\big)$ then abbreviate $B := B(x,r)$. Using the fact that $\omega$ is non-decreasing and the upper Ahlfors regularity of $\Sigma$ we may estimate

$$\begin{aligned} \int_{\Sigma\cap B} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) &= \sum_{k=0}^\infty \int_{\Sigma\cap(2^{-k}B\setminus 2^{-k-1}B)} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) \\ &\le \sum_{k=0}^\infty 2^{(1-n-d)_+}\frac{\omega(2^{-k}r)}{(2^{-k-1}\,r)^{n-1+d}}\,\sigma(\Sigma\cap 2^{-k}B) \\ &\le C\frac{2^{n-1+d}\cdot 2^{(d)_+}\cdot 2^{(1-n-d)_+}}{\ln 2}\sum_{k=0}^\infty \int_{2^{-k}\,r}^{2^{-k+1}\,r} \frac{\omega(s)}{s^d}\frac{ds}{s} \\ &= C\frac{2^{n-1+d}\cdot 2^{d_+}\cdot 2^{(1-n-d)_+}}{\ln 2}\int_0^{2\,r} \frac{\omega(s)}{s^d}\frac{ds}{s}, \end{aligned} \tag{5.2.18}$$

proving (5.2.16).

Let us now turn to (5.2.17). We may assume that $\Sigma \setminus B \neq \varnothing$, otherwise there is nothing to prove. Set $N := [\log_2(\operatorname{diam}(\Sigma)/r)] \in \mathbb{N}_0 \cup \{\infty\}$, so that $\partial\Omega \setminus 2^k B = \varnothing$

for every integer $k > N$. Then, using that $\omega$ is non-decreasing and the upper Ahlfors regularity of $\Sigma$ we may write

$$
\begin{aligned}
\int_{\Sigma\setminus B} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) &\leq \sum_{k=0}^{N} \int_{\Sigma\cap(2^{k+1}B\setminus 2^k B)} \frac{\omega(|x-y|)}{|x-y|^{n-1+d}}\,d\sigma(y) \\
&\leq \sum_{k=0}^{N} 2^{(1-n-d)_+} \frac{\omega(2^{k+1}\,r)}{(2^k\,r)^{n-1+d}}\,\sigma(\Sigma\cap 2^{k+1}B) \\
&\leq C\frac{2^{n-1+d}\cdot 2^{(d)_+}\cdot 2^{(1-n-d)_+}}{\ln 2} \sum_{k=0}^{N} \int_{2^{k+1}\,r}^{2^{k+2}\,r} \frac{\omega(s)}{s^d}\frac{ds}{s} \\
&\leq C\frac{2^{n-1+d}\cdot 2^{(d)_+}\cdot 2^{(1-n-d)_+}}{\ln 2} \int_{2\,r}^{4\,\mathrm{diam}(\Sigma)} \frac{\omega(s)}{s^d}\frac{ds}{s},
\end{aligned}
\tag{5.2.19}
$$

proving (5.2.17). □

## 5.3 Singular integrals on generalized Hölder spaces

We first recall a basic result pertaining to the behavior of singular integral operators on UR sets, which is a direct consequence of Proposition 2.3.3 with $w \equiv 1$ (except for the formula in (5.3.3), which is proved in [93]).

**Theorem 5.3.1.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set such that $\partial\Omega$ is a UR set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, assume $N = N(n) \in \mathbb{N}$ is a sufficiently large integer and consider a complex-valued function $k \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ which is odd and positive homogeneous of degree $1-n$. Finally, fix an aperture parameter $\kappa > 0$. In this setting, for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ define*

$$
\mathcal{T}f(x) := \int_{\partial\Omega} k(x-y)f(y)\,d\sigma(y) \quad \textit{for each} \quad x \in \Omega, \tag{5.3.1}
$$

$$
Tf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k(x-y)f(y)\,d\sigma(y) \quad \textit{for } \sigma\textit{-a.e.} \quad x \in \partial\Omega. \tag{5.3.2}
$$

*Then for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ the limit in (5.3.2) exists $\sigma$-a.e. and one has the jump-formula*

$$
\big(\mathcal{T}f\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) = \frac{1}{2i}\widehat{k}(\nu(x))f(x) + Tf(x), \tag{5.3.3}
$$

*for $\sigma$-a.e. $x \in \partial_*\Omega$, where $\widehat{k}$ denotes the Fourier transform of $k$ in $\mathbb{R}^n$ and $i := \sqrt{-1} \in \mathbb{C}$.*

*Also, for each integrability exponent $p \in (1,\infty)$ there exists a finite constant $C > 0$ such that for each function $f \in L^p(\partial\Omega,\sigma)$ one has*

$$
\|\mathcal{N}_\kappa(\mathcal{T}f)\|_{L^p(\partial\Omega,\sigma)} \leq C\,\|f\|_{L^p(\partial\Omega,\sigma)}\,, \tag{5.3.4}
$$

$$
\|Tf\|_{L^p(\partial\Omega,\sigma)} \leq C\,\|f\|_{L^p(\partial\Omega,\sigma)}\,. \tag{5.3.5}
$$

We now turn to the task of estimating singular integral operators acting on generalized Hölder spaces.

**Lemma 5.3.2.** *Suppose $\Omega$ is a nonempty, proper, open subset of $\mathbb{R}^n$ with compact boundary, satisfying an upper Ahlfors regularity condition with constant $C \in (0,\infty)$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a function $k : \Omega \times \partial\Omega \to \mathbb{R}$ with the property that $k(x,\cdot)$ is a $\sigma$-measurable function for each fixed point $x \in \Omega$, and there exists some finite constant $C_0 > 0$ such that*

$$|k(x,y)| \leq \frac{C_0}{|x-y|^{n-1}} \quad \textit{for each } x \in \Omega \textit{ and } \sigma\textit{-a.e. } \ y \in \partial\Omega. \tag{5.3.6}$$

*Define an integral operator acting on each $f \in L^1(\partial\Omega,\sigma)$ according to*

$$\mathscr{T}f(x) := \int_{\partial\Omega} k(x,y)f(y)\,d\sigma(y) \quad \textit{for all} \ \ x \in \Omega, \tag{5.3.7}$$

*and assume*

$$C_1 := \sup_{x\in\Omega} |\mathscr{T}1(x)| < +\infty. \tag{5.3.8}$$

*Finally, with $D := \operatorname{diam}(\partial\Omega)$, let $\omega$ be a growth function on $(0,D)$ for which*

$$C_2 := \int_0^D \omega(s)\,\frac{ds}{s} < \infty. \tag{5.3.9}$$

*Then for every $f \in \mathscr{C}^\omega(\partial\Omega)$ one has*

$$\begin{aligned}\sup_{x\in\Omega}|\mathscr{T}f(x)| \leq C\,C_0\,2^{n-1}\Big[(1+2^n)\,\omega(D) + \frac{2^n}{\ln 4}\,C_2\Big]\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)}& \\ + \max\{C_1\,,\,C\,C_0\}\cdot\sup_{\partial\Omega}|f|.&\end{aligned} \tag{5.3.10}$$

*Proof.* As in Remark 1.3.2, set $\widetilde{\omega}(t) := \omega(\min\{t,D\})$ for each $t \in (0,\infty)$. Pick an arbitrary function $f \in \mathscr{C}^\omega(\partial\Omega) \subseteq L^1(\partial\Omega,\sigma)$ and fix some point $x \in \Omega$. If $\operatorname{dist}(x,\partial\Omega) \geq D$ use (5.3.6) to estimate

$$|\mathscr{T}f(x)| \leq \frac{C_0}{D^{n-1}}\sigma(\partial\Omega)\cdot\sup_{\partial\Omega}|f| \leq C\,C_0\cdot\sup_{\partial\Omega}|f|. \tag{5.3.11}$$

Consider next the case when $\operatorname{dist}(x,\partial\Omega) < D$. Pick $x_* \in \partial\Omega$ such that

$$|x-x_*| = \operatorname{dist}(x,\partial\Omega) =: r \in (0,D), \tag{5.3.12}$$

and abbreviate $B_* := B(x_*,r)$. Then we may decompose $\mathscr{T}f(x) = \mathrm{I}+\mathrm{II}+\mathrm{III}$, where

$$\mathrm{I} := \int_{\partial\Omega\cap 2B_*} k(x,y)\,\big[f(y)-f(x_*)\big]\,d\sigma(y), \tag{5.3.13}$$

$$\mathrm{II} := \int_{\partial\Omega\setminus 2B_*} k(x,y)\,\big[f(y)-f(x_*)\big]\,d\sigma(y), \tag{5.3.14}$$

$$\mathrm{III} := (\mathscr{T}1)(x)\,f(x_*). \tag{5.3.15}$$

Start by estimating the first term above:

$$
\begin{aligned}
|\mathrm{I}| &\leq C_0\,[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\cap 2B_*} \frac{\widetilde{\omega}(|y-x_*|)}{|x-y|^{n-1}}\,d\sigma(y)\\
&\leq C_0\,[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)} \frac{\widetilde{\omega}(D)}{r^{n-1}}\,\sigma(\partial\Omega\cap 2\,B_*)\\
&\leq C\,C_0\,2^{n-1}\,\omega(D)\,[f]_{\mathscr{C}^{\omega}(\partial\Omega)}.
\end{aligned}
\tag{5.3.16}
$$

where we have used that $\omega$ is non-decreasing, that $r = \operatorname{dist}(x,\partial\Omega) \leq |x-y|$ for every $y \in \partial\Omega \cap 2B_*$, and the upper Ahlfors regularity of $\partial\Omega$.

Let us now estimate II. When $\partial\Omega \setminus 2B_* = \varnothing$ we have $\mathrm{II} = 0$. When $\partial\Omega \setminus 2B_* \neq \varnothing$ it follows that $2r < D$, hence (5.2.17) may be employed. Since for each $y \in \partial\Omega \setminus 2B_*$ we have $|y - x_*| \leq 2\,|y-x|$, this permits us to estimate

$$
\begin{aligned}
|\mathrm{II}| &\leq C_0\,[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\setminus 2B_*} \frac{\widetilde{\omega}(|y-x_*|)}{|x-y|^{n-1}}\,d\sigma(y)\\
&\leq 2^{n-1}\,C_0\,[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\setminus 2B_*} \frac{\widetilde{\omega}(|y-x_*|)}{|y-x_*|^{n-1}}\,d\sigma(y)\\
&\leq \frac{2^{2\,(n-1)}}{\ln 2}\,C\,C_0[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)} \int_{4\,r}^{4\,D} \widetilde{\omega}(s)\frac{ds}{s},\\
&\leq \frac{2^{2\,(n-1)}}{\ln 2}\,C\,C_0\,(C_2 + \omega(D)\,\ln 4)\,[f]_{\mathscr{C}^{\widetilde{\omega}}(\partial\Omega)}.
\end{aligned}
\tag{5.3.17}
$$

Finally,

$$
|\mathrm{III}| \leq C_1 \sup_{\partial\Omega} |f|, \tag{5.3.18}
$$

so (5.3.10) follows from (5.3.11) and (5.3.16)-(5.3.18). □

Here is a companion result to Lemma 5.3.2, for integral operators whose kernel exhibits a stronger singularity at the boundary (compared to (5.3.6)).

**Lemma 5.3.3.** *Assume $\Omega$ is a nonempty, proper, open subset of $\mathbb{R}^n$ with a compact boundary satisfying an upper Ahlfors regularity condition with constant $C \in (0,\infty)$, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a function $q : \Omega \times \partial\Omega \to \mathbb{R}$ with the property that there exists some finite constant $C_3 > 0$ such that*

$$
|q(x,y)| \leq \frac{C_3}{|x-y|^n} \quad \text{for each } x \in \Omega \text{ and } \sigma\text{-a.e. } \ y \in \partial\Omega, \tag{5.3.19}
$$

*and such that $q(x,\cdot)$ is a $\sigma$-measurable function for each fixed point $x \in \Omega$. Use this to define an integral operator acting on each $f \in L^1(\partial\Omega,\sigma)$ according to*

$$
\mathscr{Q}f(x) := \int_{\partial\Omega} q(x,y)f(y)\,d\sigma(y) \quad \text{for each } \ x \in \Omega. \tag{5.3.20}
$$

*Going further, let $\omega$ be a growth function on $(0,\operatorname{diam}(\partial\Omega))$. Associate with it the function $V : (0,\infty) \to (0,\infty)$ defined as in (5.2.12), and also the growth function $\widetilde{\omega}$ as in*

*Remark 1.3.2. Finally, set* $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ *for every* $x \in \Omega$*, and make the assumption that*

$$C_4 := \sup_{x\in\Omega} \frac{|\mathscr{Q}1(x)|}{V(\rho(x))} < +\infty. \tag{5.3.21}$$

*Then for each function* $f \in \mathscr{C}^\omega(\partial\Omega)$ *one has*

$$\sup_{x\in\Omega} \frac{|\mathscr{Q}f(x)|}{V(\rho(x))} \leq 2^n\, C\, C_3 \Big(1 + \frac{2^{n+1}}{\ln 2}\Big) [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} + C_4 \cdot \sup_{\partial\Omega} |f|. \tag{5.3.22}$$

*Proof.* Choose an arbitrary function $f \in \mathscr{C}^\omega(\partial\Omega)$ and fix some point $x \in \Omega$. Next, pick $x_* \in \partial\Omega$ such that

$$|x - x_*| = \operatorname{dist}(x, \partial\Omega) = \rho(x) =: r, \tag{5.3.23}$$

and abbreviate $B_* := B(x_*, r)$. Then we may decompose $\mathscr{Q}f(x) = \mathrm{I} + \mathrm{II} + \mathrm{III}$, where

$$\mathrm{I} := \int_{\partial\Omega\cap 2B_*} q(x,y)\,[f(y) - f(x_*)]\,d\sigma(y), \tag{5.3.24}$$

$$\mathrm{II} := \int_{\partial\Omega\setminus 2B_*} q(x,y)\,[f(y) - f(x_*)]\,d\sigma(y), \tag{5.3.25}$$

$$\mathrm{III} := (\mathscr{Q}1)(x)\, f(x_*). \tag{5.3.26}$$

Set $D := \operatorname{diam}(\partial\Omega)$ and recall from Remark 1.3.2 the growth function $\widetilde{\omega}$, extending $\omega$ to $(0,\infty)$ according to $\widetilde{\omega}(t) = \omega(\min\{t, D\})$ for each $t \in (0,\infty)$. We may then estimate

$$\begin{aligned}
|\mathrm{I}| &\leq C_3\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\cap 2B_*} \frac{\omega(|y - x_*|)}{|x-y|^n}\, d\sigma(y) \\
&\leq C_3\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \frac{\widetilde{\omega}(2r)}{r^n}\, \sigma(\partial\Omega \cap 2B_*) \\
&\leq 2^{n-1} C\, C_3\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \frac{\widetilde{\omega}(2r)}{r} \\
&\leq 2^n\, C\, C_3\, V(r)\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)},
\end{aligned} \tag{5.3.27}$$

using that $\omega$ is non-decreasing, that $r = \operatorname{dist}(x, \partial\Omega) \leq |x - y|$ for every $y \in \partial\Omega$, the upper Ahlfors regularity of $\partial\Omega$, (5.2.13), and the fact that $V$ is non-increasing.

Next, note that $\mathrm{II} = 0$ if $\partial\Omega \setminus 2B_* = \varnothing$. Consider the case when $\partial\Omega \setminus 2B_* \neq \varnothing$. Since in this scenario $2\,r \leq D$ and $|y - x_*| \leq 2\,|x - y|$ for each $y \in \partial\Omega \setminus 2B_*$, we may estimate

$$\begin{aligned}
|\mathrm{II}| &\leq C_3\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\setminus 2B_*} \frac{\widetilde{\omega}(|y - x_*|)}{|x-y|^n}\, d\sigma(y) \\
&\leq 2^n\, C_3\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \int_{\partial\Omega\setminus 2B_*} \frac{\widetilde{\omega}(|y - x_*|)}{|y-x_*|^n}\, d\sigma(y) \\
&\leq \frac{2^{2n+1}}{\ln 2}\, C\, C_3\, V(r)\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)},
\end{aligned} \tag{5.3.28}$$

where we have used (5.2.17) with $d := 1$, and the fact that $V$ is non-increasing. Finally, (5.3.21) gives

$$|\mathrm{III}| \leq C_4\, V(r) \cdot \sup_{\partial\Omega} |f|. \tag{5.3.29}$$

Gathering (5.3.27), (5.3.28), (5.3.29) then yields (5.3.22). □

From Lemmas 5.3.2-5.3.3 we then immediately derive the following result.

**Lemma 5.3.4.** *Let $\Omega$ be a nonempty, proper, open subset of $\mathbb{R}^n$ with a compact boundary satisfying an upper Ahlfors regularity condition with constant $C \in (0, \infty)$, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider function $K : \Omega \times \partial\Omega \to \mathbb{R}$, which is continuously differentiable in the first argument, with the property that there exists some finite constant $A_0 > 0$ such that*

$$|K(x,y)| + |x-y|\,|\nabla_x K(x,y)| \leq A_0|x-y|^{1-n} \quad \text{for each } x \in \Omega \text{ and } \sigma\text{-a.e. } \ y \in \partial\Omega, \tag{5.3.30}$$

*and such that $K(x,\cdot)$ is a $\sigma$-measurable function for each fixed point $x \in \Omega$. Define an integral operator acting on each function $f \in L^1(\partial\Omega, \sigma)$ according to*

$$\mathcal{T}f(x) := \int_{\partial\Omega} K(x,y)f(y)\,d\sigma(y) \quad \text{for each} \quad x \in \Omega. \tag{5.3.31}$$

*With $D := \operatorname{diam}(\partial\Omega)$, let $\omega$ be a growth function on $(0, D)$. Bring in the function $V$ associated with $\omega$ as in (5.2.12), and the growth function $\widetilde{\omega}$ associated with $\omega$ as in Remark 1.3.2. In relation to these, assume*

$$A_1 := \int_0^D \omega(s)\,\frac{ds}{s} < +\infty, \tag{5.3.32}$$

*and, if $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ for each $x \in \Omega$,*

$$A_2 := \sup_{x\in\Omega} |(\mathcal{T}1)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathcal{T}1)(x)|}{V(\rho(x))} < +\infty. \tag{5.3.33}$$

*Then for every $f \in \mathscr{C}^\omega(\partial\Omega)$ one has*

$$\begin{aligned} &\sup_{x\in\Omega} |(\mathcal{T}f)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathcal{T}f)(x)|}{V(\rho(x))} \\ &\qquad \leq C\,A_0\,2^{n-1}\Big[(1+2^n)\,\omega(D) + \frac{2^n}{\ln 4}\,A_1 + 2 + \frac{2^{n+2}}{\ln 2}\Big]\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)} \\ &\qquad\quad + (2\,A_2 + C\,A_0)\sup_{\partial\Omega}|f|. \end{aligned} \tag{5.3.34}$$

*As a consequence, there exists a finite constant $C_{\Omega,n,\omega} > 0$ with the property that for each function $f \in \mathscr{C}^\omega(\partial\Omega)$ one has*

$$\sup_{x\in\Omega} |(\mathcal{T}f)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathcal{T}f)(x)|}{V(\rho(x))} \leq C_{\Omega,n,\omega}\,(A_0\,(1+A_1) + A_2)\,\|f\|_{\mathscr{C}^\omega(\partial\Omega)}. \tag{5.3.35}$$

We conclude this section by estimating the generalized Hölder norm of a $\mathscr{C}^1$ function defined in a uniform domain (cf. Definition 1.1.9).

**Lemma 5.3.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform domain. Recall the constant $\varkappa \in [1, \infty)$ appearing in (1.1.25) and abbreviate $D := \operatorname{diam}(\partial\Omega) \in (0, \infty]$. Also, set $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ for each $x \in \Omega$. In addition, consider a growth function $\omega$ on $(0, D)$ satisfying (1.3.6) and associate with it the function $V$ as in (5.2.12).*

*Then there exists a finite constant $C = C(C_\omega, D) > 0$ with the property that for each function $u \in \mathscr{C}^1(\Omega)$ one has*

$$\|u\|_{\mathscr{C}^\omega(\Omega)} \leq C\Big(\sup_{x\in\Omega}|u(x)| + \varkappa \sup_{x\in\Omega}\frac{|(\nabla u)(x)|}{V(\rho(x))}\Big). \tag{5.3.36}$$

*Proof.* Fix $x, y \in \Omega$ and assume first that $|x-y| \leq D$. Since $\Omega$ is a uniform domain with constant $\varkappa \in [1,\infty)$, there exists a rectifiable curve $\gamma : [0, L] \to \Omega$ (in the arc-length parametrization) joining $x$ with $y$ and satisfying $L = \text{length}(\gamma) \leq \varkappa|x-y|$ as well as

$$\min\{s, L-s\} \leq \varkappa\, \text{dist}(\gamma(s), \partial\Omega) = \varkappa\, \rho(\gamma(s)), \qquad 0 \leq s \leq L. \tag{5.3.37}$$

Since $\gamma$ is a.e. differentiable and $|d\gamma/ds| = 1$ for almost every $s \in (0, L)$, for any given function $u \in \mathscr{C}^1(\Omega)$ we may write

$$\begin{aligned}|u(x) - u(y)| &= \left|\int_0^L \frac{d}{ds}[u(\gamma(s))]\,ds\right| \leq \int_0^L |(\nabla u)(\gamma(s))|\,ds\\ &\leq \sup_{x\in\Omega}\frac{|\nabla u(x)|}{V(\rho(x))}\int_0^L V\big(\rho(\gamma(s))\big)\,ds.\end{aligned} \tag{5.3.38}$$

On the other hand, since $V$ is non-increasing and $L \leq \varkappa|x-y|$, on account of (5.3.37) we may write

$$\begin{aligned}&\int_0^L V\big(\rho(\gamma(s))\big)\,ds \leq \int_0^L V\big(\varkappa^{-1}\min\{s, L-s\}\big)\,ds\\ &\quad = 2\int_0^{\frac{L}{2}} V(\varkappa^{-1}s)\,ds = 2\,\varkappa\int_0^{\frac{L}{2\varkappa}} V(s)\,ds \leq 2\,\varkappa\int_0^{|x-y|} V(s)\,ds.\end{aligned} \tag{5.3.39}$$

As in Remark 1.3.2, set $\widetilde{\omega}(t) := \omega(\min\{t, D\})$ for each $t \in (0,\infty)$. Then

$$\begin{aligned}\int_0^{|x-y|} V(s)\,ds &= \int_0^{|x-y|}\Big(\int_s^{|x-y|}\frac{\widetilde{\omega}(t)}{t}\frac{dt}{t}\Big)\,ds + \int_0^{|x-y|}\Big(\int_{|x-y|}^\infty \frac{\widetilde{\omega}(t)}{t}\frac{dt}{t}\Big)\,ds\\ &= \int_0^{|x-y|}\widetilde{\omega}(t)\,\frac{dt}{t} + |x-y|\int_{|x-y|}^\infty \frac{\widetilde{\omega}(t)}{t}\frac{dt}{t}\\ &\leq C\,\widetilde{\omega}(|x-y|),\end{aligned} \tag{5.3.40}$$

for some finite constant $C = C(C_\omega, D) > 0$, where in the last estimate we have used (1.3.7) and Lemma 5.2.1. Gathering (5.3.38), (5.3.39), and (5.3.40) we conclude that

$$|u(x) - u(y)| \leq 2\,\varkappa\, C\,\Big(\sup_{x\in\Omega}\frac{|\nabla u(x)|}{V(\rho(x))}\Big)\,\omega(|x-y|) \tag{5.3.41}$$

for every $x, y \in \Omega$ such that $|x-y| \leq D$. If $\text{diam}(\Omega) > D$, then for any $x, y \in \Omega$ satisfying $|x-y| > D$ we obtain

$$\frac{|u(x) - u(y)|}{\widetilde{\omega}(|x-y|)} \leq \frac{2}{\omega(D)}\sup_{x\in\Omega}|u(x)|. \tag{5.3.42}$$

From (5.3.41) and (5.3.42) the desired estimate now follows. □

## 5.4 Cauchy-Clifford operators on $\mathscr{C}^\omega(\partial\Omega)\otimes \mathcal{C}\!\ell_n$

The last part of our auxiliary results deals with Clifford algebras and the properties of Cauchy-Clifford operators. This complements the preliminary results and terminology introduced in Section 1.4. We recall the definition of the Dirac operator $D := \sum_{j=1}^n e_j\partial_j$. Given an open set $\Omega \subseteq \mathbb{R}^n$, we shall denote by $D_L$ and $D_R$ the action of $D$ on functions $u \in \mathscr{C}^1(\Omega)\otimes \mathcal{C}\!\ell_n$ from the left and from the right, respectively. More precisely, if $u$ is written as in (1.4.4) then

$$D_L u := \sum_{\ell=0}^n {\sum_{|I|=\ell}}' \sum_{j=1}^n \frac{\partial u_I}{\partial x_j}\, e_j \odot e_I, \quad D_R u := \sum_{\ell=0}^n {\sum_{|I|=\ell}}' \sum_{j=1}^n \frac{\partial u_I}{\partial x_j}\, e_I \odot e_j. \tag{5.4.1}$$

If $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and $\nu = (\nu_1,\dots,\nu_n)$ is the geometric measure theoretic outward unit normal to $\Omega$, we agree to identify the latter vector field with the Clifford algebra valued function defined at $\sigma$-a.e. point $x\in\partial\Omega$ as $\nu(x) = \nu_1(x)e_1+\dots+\nu_n(x)e_n \in \mathcal{C}\!\ell_n$. In this context, for any $u,v\in \mathscr{C}^1_0(\mathbb{R}^n)\otimes\mathcal{C}\!\ell_n$ formula (1.1.12) implies the following integration by parts holds:

$$\int_{\partial\Omega} u(x)\odot\nu(x)\odot v(x)\,d\sigma(x) = \int_\Omega \Big((D_R u)(x)\odot v(x) + u(x)\odot(D_L v)(x)\Big)\,dx. \tag{5.4.2}$$

The following lemma is proved in [96, Lemma 4.2]. To state it, we set

$$[x]_s := x_s \ \text{ for each } \ x=(x_1,\dots,x_n)\in\mathbb{R}^n \ \text{ and } \ s\in\{1,\dots,n\}. \tag{5.4.3}$$

**Lemma 5.4.1.** *Fix $n\geq 2$ and consider an odd, harmonic, homogeneous polynomial $P(x)$, with $x\in\mathbb{R}^n$, of degree $\ell\geq 3$. Associate with it the family $P_{rs}(x)$ with $r,s\in\{1,\dots,n\}$ of harmonic homogeneous polynomials of degree $\ell-2$ given by the formula*

$$P_{rs}(x) := \frac{1}{\ell(\ell-1)}(\partial_r\partial_s P)(x), \quad x\in\mathbb{R}^n. \tag{5.4.4}$$

*Then there exists a family of odd, $\mathscr{C}^\infty$ functions*

$$k_{rs} : \mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n \hookrightarrow \mathcal{C}\!\ell_n, \qquad r,s\in\{1,\dots,n\}, \tag{5.4.5}$$

*which are homogeneous of degree $1-n$, such that for each $r,s\in\{1,\dots,n\}$ and each $x\in\mathbb{R}\setminus\{0\}$ one has*

$$\frac{P(x)}{|x|^{n-1+\ell}} = \sum_{r,s=1}^n \big[k_{rs}(x)\big]_s \ \ (\textit{cf.}\ (5.4.3)), \tag{5.4.6}$$

$$(D_R k_{rs})(x) = \frac{\ell-1}{n+\ell-3}\frac{\partial}{\partial x_r}\left(\frac{P_{rs}(x)}{|x|^{n+\ell-3}}\right). \tag{5.4.7}$$

*Moreover, there exists a finite dimensional constant $c_n>0$ such that*

$$\max_{1\leq r,s\leq n}\sup_{S^{n-1}}|k_{rs}| + \max_{1\leq r,s\leq n}\sup_{S^{n-1}}|\nabla k_{rs}| \leq c_n 2^\ell \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}. \tag{5.4.8}$$

*Remark* 5.4.2. Let $P$ be an odd, harmonic homogeneous polynomial of degree $\ell \geq 1$. Then for each multi-index $\gamma \in \mathbb{N}_0^n$ there exists a finite constant $c_{n,\gamma} > 0$ such that

$$\begin{aligned} \|\partial^\gamma P\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} &\leq c_{n,\gamma} \int_{B(0,2)} |P(x)|\, dx \\ &= c_{n,\gamma} \int_{S^{n-1}} |P(\upsilon)| \Big( \int_0^2 r^{n-1+\ell}\, dr \Big)\, d\mathcal{H}^{n-1}(\upsilon) \\ &= c_{n,\gamma} \frac{2^\ell}{n+\ell} \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}, \end{aligned} \tag{5.4.9}$$

where we have used interior estimates for the harmonic function $P$ and its homogeneity of degree $\ell$.

We continue to assume that $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. As before, we identify the geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ to $\Omega$ with the Clifford algebra valued function defined at $\sigma$-a.e. point $x \in \partial\Omega$ as $\nu(x) = \nu_1(x)e_1 + \cdots + \nu_n(x)e_n \in \mathcal{C}\ell_n$. With $\varpi_{n-1}$ denoting the surface area of the unit sphere in $\mathbb{R}^n$, define the action of the boundary-to-domain Cauchy-Clifford operator on any function $f \in L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}) \otimes \mathcal{C}\ell_n$ as

$$(\mathcal{C}f)(x) := \frac{1}{\varpi_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y) \quad \text{for each} \quad x \in \Omega. \tag{5.4.10}$$

There is a remarkable Cauchy Reproducing Formula involving the above integral operator. Specifically, with $\Omega$ as above, if the function $u \in \mathscr{C}^\infty(\Omega) \otimes \mathcal{C}\ell_n$ satisfies (for some fixed aperture parameter $\kappa > 0$)

$$\begin{gathered} Du = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa u \in L^1(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}), \quad \text{and} \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \end{gathered} \tag{5.4.11}$$

then (cf. [93])

$$u = \mathcal{C}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \text{ in } \Omega. \tag{5.4.12}$$

Before proving boundedness properties for the Cauchy-Clifford integral operator introduced above, we need two lemmas. Their proofs may be found in [96, Lemmas 2.5, 5.2, and 5.3].

**Lemma 5.4.3.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain whose boundary is compact. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an aperture parameter $\kappa > 0$. Then for each $x \in \partial^*\Omega$ there exists a Lebesgue measurable set $\mathcal{O}_x \subseteq (0,1)$ satisfying*

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^1(\mathcal{O}_x \cap (0,\varepsilon))}{\varepsilon} = 1, \tag{5.4.13}$$

*with the property that*

$$\lim_{\mathcal{O}_x \ni \varepsilon \to 0^+} \lim_{\Gamma_\kappa(x) \ni z \to x} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|<\varepsilon}} \frac{z-y}{|z-y|^n} \odot \nu(y)\, d\sigma(y)$$

$$= \lim_{\mathcal{O}_x \ni \varepsilon \to 0^+} \frac{\mathcal{H}^{n-1}(\partial B(x,\varepsilon) \setminus \Omega)}{\varpi_{n-1}\varepsilon^{n-1}} = \frac{1}{2}. \tag{5.4.14}$$

Here is the second lemma from [96] alluded to above.

**Lemma 5.4.4.** *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter whose boundary is compact and upper Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Then there exists some finite constant $C_{\partial\Omega,n} > 0$, depending only on the dimension $n$ and the upper Ahlfors regularity constant of $\partial\Omega$, such that*

$$\left| \int_{\partial_*\Omega \setminus B(x,r)} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) \right| \leq C_{\partial\Omega,n} \tag{5.4.15}$$

*for every $x \in \mathbb{R}^n$ and $r > 0$. Moreover, if $\Omega$ is bounded,*

$$\int_{\partial_*\Omega \setminus B(x,r)} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) = \frac{\mathcal{H}^{n-1}(\Omega \cap \partial B(x,r))}{r^{n-1}} \tag{5.4.16}$$

*for every $x \in \mathbb{R}^n$ and $\mathcal{L}^1$-a.e. $r > 0$.*

*As a consequence, if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain whose boundary is compact, then for each $x \in \mathbb{R}^n$ one has*

$$\mathcal{C}1(x) = \begin{cases} 1 & \textit{if } \Omega \textit{ is bounded,} \\ 0 & \textit{if } \Omega \textit{ is unbounded.} \end{cases} \tag{5.4.17}$$

At this point, we have enough foundation material to conclude the following result.

**Proposition 5.4.5.** *Let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain whose boundary is compact. Suppose $\omega$ is a growth function on $(0, \operatorname{diam}(\partial\Omega))$ satisfying (1.3.6), and for each $x \in \Omega$ set $\rho(x) := \operatorname{dist}(x, \partial\Omega)$. Then there exists some finite constant $C_{\Omega,n,\omega} > 0$ with the property that*

$$\sup_{x\in\Omega} |(\mathcal{C}f)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathcal{C}f)(x)|}{V(\rho(x))} \leq C_{\Omega,n,\omega} \, \|f\|_{\mathscr{C}^\omega(\partial\Omega)} \tag{5.4.18}$$

*for each function $f \in \mathscr{C}^\omega(\partial\Omega)$.*

*Proof.* It is a consequence of Lemma 5.3.4, whose applicability is ensured by (5.4.17) and (1.3.6). □

**Proposition 5.4.6.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is simultaneously a uniform domain and an Ahlfors regular domain with compact boundary. Let $\omega$ be a growth function on $(0, \operatorname{diam}(\partial\Omega))$ satisfying (1.3.6). Then the boundary-to-domain Cauchy-Clifford operator defined in (5.4.10) is well-defined, and bounded in the context*

$$\mathcal{C} : \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\!\ell_n \longrightarrow \mathscr{C}^\omega(\overline{\Omega}) \otimes \mathcal{C}\!\ell_n, \tag{5.4.19}$$

*with operator norm controlled in terms of $n$, $\omega$, and $\Omega$.*

*Proof.* For each $x\in\Omega$ set $\rho(x):=\mathrm{dist}(x,\partial\Omega)$. Also, recall the constant $\varkappa\in[1,\infty)$ associated with the uniform domain $\Omega$ as in (1.1.25). Then from (5.4.18) and Lemma 5.3.5, whose applicability is ensured by (1.3.6), we obtain

$$\begin{aligned}\|\mathcal{C}f\|_{\mathscr{C}^\omega(\Omega)\otimes C\!\ell_n} &\leq C\Big(\sup_{x\in\Omega}|\mathcal{C}f(x)|+\varkappa\sup_{x\in\Omega}\frac{|\nabla(\mathcal{C}f)(x)|}{V(\rho(x))}\Big)\\ &\leq C_{\Omega,n,\omega}\,\|f\|_{\mathscr{C}^\omega(\partial\Omega)\otimes C\!\ell_n},\end{aligned}\tag{5.4.20}$$

and the desired result follows on account of Lemma 1.3.6. $\square$

The following result can be found in [96, Proposition 5.1]

**Proposition 5.4.7.** *Let $\Omega\subseteq\mathbb{R}^n$ be a* UR *domain. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. Also, fix an aperture parameter $\kappa>0$. Then for each function $f\in L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\otimes C\!\ell_n$ the limit*

$$\mathbf{C}f(x):=\lim_{\varepsilon\to0^+}\frac{1}{\varpi_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y)\tag{5.4.21}$$

*exists exists at $\sigma$-a.e. point $x\in\partial\Omega$ and, with $I$ denoting the identity operator, the following jump-formula holds*

$$\Big(\mathcal{C}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x)=\Big(\tfrac12 I+\mathbf{C}\Big)f(x)\quad\text{for }\sigma\text{-a.e. }\ x\in\partial\Omega.\tag{5.4.22}$$

*Moreover, for each $p\in(1,\infty)$ there exists a finite constant $C>0$ such that*

$$\begin{gathered}\|\mathcal{N}_\kappa(\mathcal{C}f)\|_{L^p(\partial\Omega,\sigma)}\leq C\,\|f\|_{L^p(\partial\Omega,\sigma)\otimes C\!\ell_n}\\ \text{for each }\ f\in L^p(\partial\Omega,\sigma)\otimes C\!\ell_n,\end{gathered}\tag{5.4.23}$$

*the operator $\mathbf{C}$ is well-defined and bounded on $L^p(\partial\Omega,\sigma)\otimes C\!\ell_n$, and*

$$\mathbf{C}^2=\tfrac14 I\quad\text{on}\quad L^p(\partial\Omega,\sigma)\otimes C\!\ell_n.\tag{5.4.24}$$

We next arrive at a central result in this section, regarding the action of the Cauchy-Clifford operator on generalized Hölder spaces in Ahlfors regular domains with compact boundaries.

**Theorem 5.4.8.** *Let $\Omega\subseteq\mathbb{R}^n$ be an Ahlfors regular domain whose boundary is compact. Set $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa>0$. Also, suppose $\omega$ is a growth function on $(0,\mathrm{diam}(\partial\Omega))$ satisfying (1.3.6).*

*Then for every function $f\in\mathscr{C}^\omega(\partial\Omega)\otimes C\!\ell_n$ the limit in (5.4.21) exists for $\sigma$-a.e. $x\in\partial\Omega$, and the operator $\mathbf{C}$ thus defined induces a well-defined, linear, and bounded mapping*

$$\mathbf{C}:\mathscr{C}^\omega(\partial\Omega)\otimes C\!\ell_n\longrightarrow\mathscr{C}^\omega(\partial\Omega)\otimes C\!\ell_n,\tag{5.4.25}$$

*with operator norm controlled in terms of $n$, $\omega$, $\mathrm{diam}(\partial\Omega)$, and the upper Ahlfors regularity constant of $\partial\Omega$.*

*Furthermore, for each $f \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\!\ell_n$ the following jump-formula holds*

$$\Big(\mathcal{C}f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = \Big(\tfrac{1}{2}I + \mathbf{C}\Big)f(x) \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.4.26}$$

*where $I$ denotes the identity operator. Finally, one has*

$$\mathbf{C}^2 = \tfrac{1}{4}I \quad \text{on } \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\!\ell_n. \tag{5.4.27}$$

*Proof.* Denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. From [96, Lemma 5.5] we know that, at $\sigma$-a.e. $x \in \partial\Omega$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y)\, d\sigma(y) = \begin{cases} +\frac{1}{2} & \text{if } \Omega \text{ is bounded,} \\ -\frac{1}{2} & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{5.4.28}$$

Set $D := \mathrm{diam}(\partial\Omega) \in (0,\infty)$ and let $\widetilde{\omega}$ be associated with $\omega$ as in Remark 1.3.2. Also, fix some $f \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\!\ell_n$. Then from (1.4.9), (5.2.17), (1.3.7), and Lemma 5.2.1 we see that there exists some $C \in (0,\infty)$, which depends only on the upper Ahlfors regularity constant of $\partial\Omega$ and $\omega$, such that for every $x \in \partial\Omega$ we have

$$\begin{aligned}
\int_{\partial\Omega} &\left| \frac{x-y}{|x-y|^n} \odot \nu(y) \odot (f(y) - f(x)) \right| d\sigma(y) \\
&\le 2^{n/2}\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n} \int_{\partial\Omega} \frac{\widetilde{\omega}(|x-y|)}{|x-y|^{n-1}}\, d\sigma(y) \\
&\le \frac{2^{\frac{3n}{2}-1}}{\ln 2}\, C\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n} \int_0^{4D} \widetilde{\omega}(s)\, \frac{ds}{s} \\
&\le \frac{2^{\frac{3n}{2}-1}}{\ln 2}\, C\, \widetilde{\omega}(4D)\, [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n} < +\infty.
\end{aligned} \tag{5.4.29}$$

Granted this, for every $x \in \partial\Omega$ we may write (based on (5.4.28) and Lebesgue's Dominated Convergence Theorem)

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y), \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot (f(y) - f(x))\, d\sigma(y) \pm \frac{1}{2} f(x) \\
&= \frac{1}{\varpi_{n-1}} \int\limits_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot (f(y) - f(x))\, d\sigma(y) \pm \frac{1}{2} f(x),
\end{aligned} \tag{5.4.30}$$

where the sign of $\frac{1}{2} f(x)$ is plus if $\Omega$ is bounded and minus if $\Omega$ is unbounded. This implies that the limit in the second line of (5.4.30) exists and thus for every $x \in \partial\Omega$,

$$\mathbf{C}f(x) = \pm\frac{1}{2} f(x) + \frac{1}{\varpi_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot (f(y) - f(x))\, d\sigma(y), \tag{5.4.31}$$

where $\pm$ depends on $\Omega$ being bounded or unbounded, as explained above.

Our next goal is to show that the operator (5.4.31) is well-defined and bounded on $\mathscr{C}^{\omega}(\partial\Omega)\otimes\mathcal{C}\!\ell_n$. To this end, fix two distinct points $x_1, x_2 \in \partial\Omega$ and use (5.4.31) to decompose

$$\mathbf{C}f(x_1) - \mathbf{C}f(x_2) = \mathrm{I} + \mathrm{II}, \tag{5.4.32}$$

where

$$\mathrm{I} := \pm\frac{1}{2}\big(f(x_1) - f(x_2)\big), \tag{5.4.33}$$

and

$$\begin{aligned}\mathrm{II} := \frac{1}{\varpi_{n-1}}\int_{\partial\Omega}\Big(&\frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\odot\big(f(y)-f(x_1)\big)\\ &-\frac{x_2-y}{|x_2-y|^n}\odot\nu(y)\odot\big(f(y)-f(x_2)\big)\Big)\,d\sigma(y).\end{aligned} \tag{5.4.34}$$

Set $r := |x_1 - x_2| \in (0, D)$ and note that this entails

$$|\mathrm{I}| \leq \frac{1}{2}\,[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(r). \tag{5.4.35}$$

Also, we may bound

$$|\mathrm{II}| \leq \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{III}_3, \tag{5.4.36}$$

where

$$\begin{aligned}\mathrm{II}_1 := \frac{1}{\varpi_{n-1}}\Bigg|\int\limits_{\substack{y\in\partial\Omega\\ |x_1-y|\geq 2r}}\Big(&\frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\odot\big(f(y)-f(x_1)\big)\\ &-\frac{x_2-y}{|x_2-y|^n}\odot\nu(y)\odot\big(f(y)-f(x_2)\big)\Big)\,d\sigma(y)\Bigg|,\end{aligned} \tag{5.4.37}$$

$$\mathrm{II}_2 := \frac{1}{\varpi_{n-1}}\,[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\int\limits_{\substack{y\in\partial\Omega\\ |x_1-y|<2r}}\frac{\widetilde{\omega}(|x_1-y|)}{|x_1-y|^{n-1}}\,d\sigma(y), \tag{5.4.38}$$

and

$$\begin{aligned}\mathrm{II}_3 &:= \frac{1}{\varpi_{n-1}}\,[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\int\limits_{\substack{y\in\partial\Omega\\ |x_1-y|<2r}}\frac{\widetilde{\omega}(|x_2-y|)}{|x_2-y|^{n-1}}\,d\sigma(y)\\ &\leq \frac{1}{\varpi_{n-1}}\,[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\int\limits_{\substack{y\in\partial\Omega\\ |x_2-y|<3r}}\frac{\widetilde{\omega}(|x_2-y|)}{|x_2-y|^{n-1}}\,d\sigma(y),\end{aligned} \tag{5.4.39}$$

with the last inequality a consequence of the fact that $|x_2-y| \leq |x_2-x_1| + |x_1-y| < 3\,r$ whenever $|x_1-y| < 2r$. We may then use (5.2.16), (5.2.1), and (5.2.4) to conclude that

$$\begin{aligned}\mathrm{II}_2 + \mathrm{II}_3 &\leq C\frac{2^n}{\varpi_{n-1}\ln 2}[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\int_0^{6r}\widetilde{\omega}(s)\frac{ds}{s}\\ &\leq C\frac{6\cdot 2^n}{\varpi_{n-1}\ln 2}[f]_{\widetilde{\mathscr{C}^{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(r).\end{aligned} \tag{5.4.40}$$

for $C \in (0,\infty)$ which depends only on $C_\omega$, $D$, and the upper Ahlfors regularity constant of $\partial\Omega$. Let us turn our attention to $\mathrm{II}_1$ which we further decompose as

$$\mathrm{II}_1 \leq \mathrm{II}_{11} + \mathrm{II}_{12}, \tag{5.4.41}$$

with

$$\begin{aligned} \mathrm{II}_{11} &:= \frac{1}{\varpi_{n-1}}\Big|\int_{\partial\Omega\setminus B(x_1,2r)} \frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\odot\big(f(x_2)-f(x_1)\big)\,d\sigma(y)\Big| \\ &\leq \frac{2^{n/2}}{\varpi_{n-1}}\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(r)\,\Big|\int_{\partial\Omega\setminus B(x_1,2r)} \frac{x_1-y}{|x_1-y|^n}\odot\nu(y)\,d\sigma(y)\Big| \\ &\leq C\frac{2^{n/2}}{\varpi_{n-1}}[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(r), \end{aligned} \tag{5.4.42}$$

for some $C \in (0,\infty)$ which depends only on $n$ and the upper Ahlfors regularity constant of $\partial\Omega$ (here we have used (1.4.9) and (5.4.15)), and

$$\begin{aligned} \mathrm{II}_{12} &:= \frac{1}{\varpi_{n-1}}\Big|\int_{\partial\Omega\setminus B(x_1,2r)}\Big(\frac{x_1-y}{|x_1-y|^n}-\frac{x_2-y}{|x_2-y|^n}\Big)\odot\nu(y)\odot\big(f(y)-f(x_2)\big)\,d\sigma(y)\Big| \\ &\leq \frac{2^{n/2}}{\varpi_{n-1}}\int_{\partial\Omega\setminus B(x_1,2r)}\Big|\frac{x_1-y}{|x_1-y|^n}-\frac{x_2-y}{|x_2-y|^n}\Big|\,\big|f(y)-f(x_2)\big|\,d\sigma(y) \\ &\leq C\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,r\int_{\partial\Omega\setminus B(x_2,r)}\frac{\widetilde{\omega}(|x_2-y|)}{|x_2-y|^n}\,d\sigma(y) \\ &\leq C\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,r\int_{2r}^{\infty}\frac{\widetilde{\omega}(s)}{s}\,\frac{ds}{s} \\ &\leq C\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(r), \end{aligned} \tag{5.4.43}$$

for some $C \in (0,\infty)$, which depends only on $n$, the upper Ahlfors regularity constant of $\partial\Omega$, $\omega$, and $D$ (here we have used (1.4.9), the Mean Value Theorem, the fact that $r \leq |y-x_1|/2 \leq |y-x_2| \leq 3\,|y-x_1|/2$ for every $y \in \partial\Omega\setminus B(x_1,2r)$, (5.2.17), (1.3.7), and Lemma 5.2.1).

After gathering (5.4.32), (5.4.35), (5.4.36), (5.4.40), (5.4.41), (5.4.42), and (5.4.43), and recalling that $r = |x_1-x_2|$, we may now conclude that there exists some $C \in (0,\infty)$, depending only on $n$, $\omega$, $D$, and the upper regularity constant of $\partial\Omega$ with the property that

$$\begin{gathered} \big|\mathbf{C}f(x_1)-\mathbf{C}f(x_2)\big| \leq C\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\,\widetilde{\omega}(|x_1-x_2|) \\ \text{for all distinct points } \ x_1,x_2\in\partial\Omega. \end{gathered} \tag{5.4.44}$$

Since (5.4.31) and (5.4.29) also imply

$$\sup_{\partial\Omega}|\mathbf{C}f| \leq \frac{2^{\frac{3n}{2}-1}}{\ln 2}\,C\,\widetilde{\omega}(4D)\,[f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n} + \frac{1}{2}\sup_{\partial\Omega}|f|, \tag{5.4.45}$$

we ultimately obtain

$$\|\mathbf{C}f\|_{\mathscr{C}^{\omega}(\partial\Omega)\otimes\mathcal{C}\!\ell_n} \leq C\big(\sup_{\partial\Omega}|f| + [f]_{\dot{\mathscr{C}}^{\widetilde{\omega}}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\big) = C\|f\|_{\mathscr{C}^{\omega}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}, \tag{5.4.46}$$

where $C\in(0,\infty)$ depends on $n$, $\omega$, $D$, and the upper regularity constant of $\partial\Omega$. This finishes the proof of (5.4.25).

We continue by observing that our current assumptions guarantee that (5.4.18) holds and therefore $\mathcal{C}f$ is a well-defined function for each $f\in\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\!\ell_n$. Also (1.1.17) ensures that it is meaningful to consider the limit in (5.4.26) given that $\Omega$ is an Ahlfors regular domain. Assume now that $f\in\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\!\ell_n$ is given and fix $x\in\partial^*\Omega$. Let $\mathcal{O}_x$ be the associated with the point $x$ as in Lemma 5.4.3. In particular, (5.4.14) holds. For every $\varepsilon>0$ we may then write

$$\begin{aligned}
\Big(\mathcal{C}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) &= \lim_{\Gamma_\kappa(x)\ni z\to x}\mathcal{C}f(z)\\
&= \lim_{\Gamma_\kappa(x)\ni z\to x}\frac{1}{\varpi_{n-1}}\int_{\partial\Omega\setminus B(x,\varepsilon)}\frac{z-y}{|z-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y)\\
&\quad+\lim_{\Gamma_\kappa(x)\ni z\to x}\frac{1}{\varpi_{n-1}}\int_{\partial\Omega\cap B(x,\varepsilon)}\frac{z-y}{|z-y|^n}\odot\nu(y)\odot(f(y)-f(x))\,d\sigma(y)\\
&\quad+\Big(\lim_{\Gamma_\kappa(x)\ni z\to x}\frac{1}{\varpi_{n-1}}\int_{\partial\Omega\cap B(x,\varepsilon)}\frac{z-y}{|z-y|^n}\odot\nu(y)\,d\sigma(y)\Big)\odot f(x)\\
&=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.
\end{aligned}\tag{5.4.47}$$

For each fixed $\varepsilon>0$, Lebesgue's Dominated Convergence Theorem applies to the limit $\Gamma_\kappa(x)\ni z\to x$ in $\mathrm{I}_1$. This allows us to compute

$$\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\mathrm{I}_1=\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\frac{1}{\varpi_{n-1}}\int_{\partial\Omega\setminus B(x,\varepsilon)}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,d\sigma(y)=\mathbf{C}f(x),\tag{5.4.48}$$

given that that the latter limit exists in view of (5.4.30). To handle $\mathrm{I}_2$, first notice that for every $x,y\in\partial\Omega$ and $z\in\Gamma_\kappa(x)$

$$|x-y|\le|z-y|+|z-x|\le|z-y|+(1+\kappa)\,\rho(z)\le(2+\kappa)\,|z-y|,\tag{5.4.49}$$

therefore

$$\Big|\frac{z-y}{|z-y|^n}\odot\nu(y)\Big|\,|f(y)-f(x)|\le(2+\kappa)^{n-1}\,[f]_{\widetilde{\mathscr{C}^\omega}(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\frac{\widetilde{\omega}(|x-y|)}{|x-y|^{n-1}}.\tag{5.4.50}$$

This, (5.2.16), (5.2.17), (1.3.7) and Lemma 5.2.1 then permit us to employ Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\mathrm{I}_2=0.\tag{5.4.51}$$

Finally, from (5.4.14),

$$\lim_{\mathcal{O}_x\ni\varepsilon\to0^+}\mathrm{I}_3=\frac12 f(x).\tag{5.4.52}$$

All together, the above argument implies that (5.4.26) holds for every $x\in\partial^*\Omega$, hence also for $\sigma$-a.e. point $x\in\partial\Omega$, by (1.1.14) and the fact that $\Omega$ is an Ahlfors regular domain.

To complete the proof of the theorem there remains to justify (5.4.27). Given any $f \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$, define $u := \mathcal{C}f$. By design, $u \in \mathscr{C}^\infty(\Omega) \otimes \mathcal{C}\ell_n$ and $D_L u = 0$ in $\Omega$. Also, (5.4.18) yields

$$\sup_{x\in\Omega} |u(x)| \leq C_{\Omega,n,\omega} \|f\|_{\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\ell_n} < +\infty. \tag{5.4.53}$$

Keeping in mind that $\mathcal{N}_\kappa u$ is a lower-semicontinuous function, from (5.4.53) we conclude that $\mathcal{N}_\kappa u \in L^\infty(\partial\Omega,\sigma) \subseteq L^1(\partial\Omega,\sigma)$ since $\partial\Omega$ is compact and therefore has finite measure. Also, the jump-formula (5.4.26) gives

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\mathcal{C}f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) = \big(\tfrac{1}{2}I + \mathbf{C}\big)f \quad \text{at } \sigma\text{-a.e. point in } \ \partial\Omega. \tag{5.4.54}$$

From this and (5.4.11)-(5.4.12) we then obtain

$$u = \mathcal{C}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) = \mathcal{C}\big((\tfrac{1}{2}I + \mathbf{C})f\big) \quad \text{in } \ \Omega. \tag{5.4.55}$$

Notice that $(\frac{1}{2}I + \mathbf{C})f \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$ by (5.4.25). As such, we may once again employ the jump-formula (5.4.26) for this function. Together with (5.4.54) and (5.4.55) this yields

$$\big(\tfrac{1}{2}I + \mathbf{C}\big)f = u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \mathcal{C}\Big(\big(\tfrac{1}{2}I + \mathbf{C}\big)f\Big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + \mathbf{C}\big)\big(\tfrac{1}{2}I + \mathbf{C}\big)f, \tag{5.4.56}$$

at $\sigma$-a.e. point on $\partial\Omega$. Now (5.4.27) follows from this and simple algebra, completing the proof of Theorem 5.4.8. □

In the last portion of this section we briefly review the harmonic single potential operator and highlight its relationship to the Cauchy-Clifford integral operator. To set the stage, assume $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain whose boundary is compact. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu$ the geometric measure theoretic outward unit normal to $\Omega$. In this setting, define action of the harmonic single layer operator on each $f \in L^1(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n$ as

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x-y)f(y)\,d\sigma(y) \quad \text{for each } \ x \in \Omega, \tag{5.4.57}$$

where $E$ denotes the standard fundamental solution for the Laplacian (cf. (2.3.6)). From definitions and the fact that $\nu \odot \nu = -1$ at $\sigma$-a.e. point on $\partial\Omega$, it follows that for each $f \in L^1(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n$ we have

$$D_L\mathcal{S}f = -\mathcal{C}(\nu \odot f) \quad \text{in } \ \Omega. \tag{5.4.58}$$

In particular, in light of the identification in (1.4.3),

$$\nabla(\mathcal{S}1) \equiv D_L\mathcal{S}1 = -\mathcal{C}\nu \quad \text{in } \ \Omega. \tag{5.4.59}$$

## 5.5 Proof of Theorem 5.1.4

In broad outline, the proof proceeds along the original approach in [96], where the authors have dealt with the classical growth function $\omega(t) := t^\alpha$ for each $t \in (0, \infty)$.

*Proof of Theorem 5.1.4.* Throughout, we set $D := \mathrm{diam}(\partial\Omega)$ and extend $\omega$ to $\widetilde{\omega}$, defined as in Remark 1.3.2, i.e., $\widetilde{\omega}(t) := \omega(\min\{t, D\})$ for each $t \in (0, \infty)$. Also, abbreviate $\rho(x) := \mathrm{dist}(x, \partial\Omega)$ for each $x \in \Omega$.

*Proof of (a) ⇒ (e), and of estimate (5.1.11).* Work under the assumption that $\Omega \subseteq \mathbb{R}^n$ is a $\mathscr{C}^{1,\omega}$-domain with compact boundary (hence also a UR domain and a uniform domain, as observed in Remark 5.1.2). Let $P$ be an odd homogeneous polynomial of degree $\ell \geq 1$ and suppose $\mathbb{T} := \mathbb{T}_+$ is associated with $P$ as in (5.1.10). For now, make the additional assumption that $P$ is harmonic. The goal is to prove that for every function $f \in \mathscr{C}^\omega(\partial\Omega)$ we have

$$\sup_{x\in\Omega} |\mathbb{T}f(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}f)(x)|}{V(\rho(x))} \leq C^\ell 2^{\ell^2} \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \, \|f\|_{\mathscr{C}^\omega(\partial\Omega)} \,, \tag{5.5.1}$$

where $C \in (0, \infty)$ depends only on the dimension $n$, $\omega$, $D$, $\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}$, and the upper Ahlfors regularity constant of $\partial\Omega$.

The strategy is to prove (5.5.1) by induction on the degree $\ell \in 2\mathbb{N} - 1$. If $\ell = 1$ then $P(x) = \sum_{j=1}^n a_j x_j$ for every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Hence, from (5.4.9),

$$\max_{1\leq j\leq n} |a_j| \leq \|P\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} \leq c_n \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \,. \tag{5.5.2}$$

The idea is to use Lemma 5.3.4 to deal with the case $\ell = 1$ of (5.5.1). To check that Lemma 5.3.4 is indeed applicable, note that $\nu$ is currently known to belong to $\mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$ hence, from (5.4.18),

$$\sup_{x\in\Omega} |(\mathcal{C}\nu)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathcal{C}\nu)(x)|}{V(\rho(x))} < +\infty. \tag{5.5.3}$$

Upon observing from (5.4.57) that in the present case $\mathbb{T}$ may be expressed as

$$\mathbb{T} = \varpi_{n-1} \sum_{j=1}^n a_j \partial_j \mathcal{S}, \tag{5.5.4}$$

and using (5.4.59) together with (5.5.3), we conclude that (5.3.33) holds for $\mathbb{T}$ (and the constant $A_2$ in (5.3.33) depends linearly on $\|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}$ by (5.5.2)). The other hypotheses in Lemma 5.3.4 are guaranteed by the structure of $\mathbb{T}$ and (1.3.6). Hence, (5.5.1) holds for $\ell = 1$ by Lemma 5.3.4.

Next, fix $\ell \geq 3$ and assume (5.5.1) is satisfied when the polynomial entering the definition of $\mathbb{T}$ has degree $\leq \ell - 2$. Pick an arbitrary odd harmonic homogeneous polynomial $P$ of degree $\ell$ and associate with it the operator $\mathbb{T}$ as in (5.1.10). Also, for $r, s \in \{1, ..., n\}$, let $P_{rs}(x)$ and $k_{rs}$ be as in Lemma 5.4.1 for this choice of $P$. For each $r, s \in \{1, \dots, n\}$ set

$$k^{rs}(x) := \frac{P_{rs}(x)}{|x|^{n+\ell-3}}, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{5.5.5}$$

then use this as a kernel to define an integral operator acting on every $g \in \mathscr{C}^\omega(\partial\Omega)\otimes \mathcal{C}\!\ell_n$, say $g = \sum_{\ell=0}^n \sum_{|I|=\ell}' g_I e_I$ with each $g_I$ in $\mathscr{C}^\omega(\partial\Omega)$, according to

$$\begin{aligned}\mathbb{T}^{rs} g(x) &:= \int_{\partial\Omega} k^{rs}(x-y) g(y)\, d\sigma(y)\\ &= \sum_{\ell=0}^n \sum_{|I|=\ell}{}' \Big(\int_{\partial\Omega} k^{rs}(x-y) g_I(y)\, d\sigma(y)\Big) e_I, \quad x \in \Omega.\end{aligned} \tag{5.5.6}$$

Based on the induction hypothesis (used component-wise), (1.4.9), (5.4.4), and (5.4.9) we may estimate

$$\begin{aligned}\sup_{x\in\Omega} |(\mathbb{T}^{rs} g)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}^{rs} g)(x)|}{V(\rho(x))}&\\ \leq 2^{n/2} C^{\ell-2} 2^{(\ell-2)^2} &\|P_{rs}\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|g\|_{\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\!\ell_n}\\ \leq c_n C^{\ell-2} 2^{(\ell-2)^2} 2^\ell &\|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|g\|_{\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\!\ell_n}.\end{aligned} \tag{5.5.7}$$

For each $r, s \in \{1, \dots, n\}$ and $g \in \mathscr{C}^\omega(\partial\Omega)\otimes \mathcal{C}\!\ell_n$ let us also define

$$(\mathbb{T}_{rs} g)(x) := \int_{\partial\Omega} k_{rs}(x-y) \odot g(y)\, d\sigma(y), \qquad \forall\, x \in \Omega. \tag{5.5.8}$$

By (5.4.6), for any real-valued function $f \in \mathscr{C}^\omega(\partial\Omega) \hookrightarrow \mathscr{C}^\omega(\partial\Omega)\otimes \mathcal{C}\!\ell_n$ we then have

$$(\mathbb{T} f)(x) = \sum_{r,s=1}^n [\mathbb{T}_{rs} f(x)]_s \ \text{ for each } \ x \in \Omega, \tag{5.5.9}$$

where, as usual, $[\cdot]_s$ singles out the $s$-th component of a vector in $\mathbb{R}^n$. If the set $\Omega$ is unbounded, fix $x \in \Omega$ along with $R_1 \in (0, \operatorname{dist}(x, \partial\Omega))$ and $R_2 > \operatorname{dist}(x,\partial\Omega) + D$, then define $\Omega_{R_1,R_2} := (B(x,R_2) \setminus \overline{B(x,R_1)}\,) \cap \Omega$, which is a bounded $\mathscr{C}^{1,\omega}$-domain such that $\partial\Omega_{R_1,R_2} = \partial B(x,R_2) \cup \partial B(x,R_1) \cup \partial\Omega$. Then for each $r, s \in \{1, \dots, n\}$ we have

$$\begin{aligned}\int_{\partial\Omega_{R_1,R_2}} k_{rs}(x-y) \odot \nu(y)\, d\sigma(y) &= -\int_{\Omega_{R_1,R_2}} (D_R k_{rs})(x-y)\, dy\\ &= \frac{\ell-1}{n+\ell-3} \int_{\Omega_{R_1,R_2}} \frac{\partial}{\partial y_r} \Big(\frac{P_{rs}(x-y)}{|x-y|^{n+\ell-3}}\Big)\, dy\\ &= \frac{\ell-1}{n+\ell-3} \int_{\partial\Omega_{R_1,R_2}} k^{rs}(x-y) \nu_r(y)\, d\sigma(y),\end{aligned} \tag{5.5.10}$$

where the first equality follows from (5.4.2), the second from (5.4.7), and the third from

(5.5.5) and the Divergence Theorem. Therefore, for each $r, s \in \{1, \dots, n\}$ we may write

$$
\begin{aligned}
(\mathbb{T}_{rs}\nu)(x) &= \int_{\partial\Omega} k_{rs}(x-y) \odot \nu(y)\, d\sigma(y) \\
&= \int_{\partial\Omega_{R_1,R_2}} k_{rs}(x-y) \odot \nu(y)\, d\mathcal{H}^{n-1}(y) - \int_{\partial B(x,R_1)} k_{rs}(x-y) \odot \frac{x-y}{|x-y|}\, d\mathcal{H}^{n-1}(y) \\
&\quad + \int_{\partial B(x,R_2)} k_{rs}(x-y) \odot \frac{x-y}{|x-y|}\, d\mathcal{H}^{n-1}(y) \\
&= \frac{\ell-1}{n+\ell-3} \int_{\partial\Omega_{R_1,R_2}} k^{rs}(x-y)\nu_r(y)\, d\mathcal{H}^{n-1}(y) - \int_{S^{n-1}} k_{rs}(\upsilon) \odot \upsilon\, d\mathcal{H}^{n-1}(\upsilon) \\
&\quad + \int_{S^{n-1}} k_{rs}(\upsilon) \odot \upsilon\, d\mathcal{H}^{n-1}(\upsilon) \\
&= \frac{\ell-1}{n+\ell-3} \Big( \int_{\partial\Omega} k^{rs}(x-y)\nu_r(y)\, d\sigma(y) - \int_{\partial B(x,R_1)} k^{rs}(x-y)\frac{x_r-y_r}{|x-y|}\, d\mathcal{H}^{n-1}(y) \\
&\quad + \int_{\partial B(x,R_2)} k^{rs}(x-y)\frac{x_r-y_r}{|x-y|}\, d\mathcal{H}^{n-1}(y) \Big) \\
&= \frac{\ell-1}{n+\ell-3} \Big( (\mathbb{T}^{rs}\nu_r)(x) - \int_{S^{n-1}} k^{rs}(\upsilon)\upsilon_r\, d\mathcal{H}^{n-1}(\upsilon) + \int_{S^{n-1}} k^{rs}(\upsilon)\upsilon_r\, d\mathcal{H}^{n-1}(\upsilon) \Big) \\
&= \frac{\ell-1}{n+\ell-3} (\mathbb{T}^{rs}\nu_r)(x). \qquad (5.5.11)
\end{aligned}
$$

From (5.5.11) and (5.5.7) used with $f := \nu_r \in \mathscr{C}^\omega(\partial\Omega)$, we obtain, for $r, s \in \{1, ..., n\}$,

$$
\begin{aligned}
&\sup_{x\in\Omega} |(\mathbb{T}_{rs}\nu)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}_{rs}\nu)(x)|}{V(\rho(x))} \\
&\qquad \le \sup_{x\in\Omega} |(\mathbb{T}^{rs}\nu_r)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}^{rs}\nu_r)(x)|}{V(\rho(x))} \\
&\qquad \le c_n C^{\ell-2} 2^{(\ell-2)^2} 2^\ell \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} . \qquad (5.5.12)
\end{aligned}
$$

Assume now that $\Omega$ is bounded. Fix a point $x \in \Omega$ and define $\Omega_{R_1} := \Omega \setminus \overline{B(x, R_1)}$ with $R_1$ as before, so that $\partial\Omega_{R_1} = \partial B(x, R_1) \cup \partial\Omega$. Proceeding as in the past, in place of (5.5.11) we now obtain

$$
\begin{aligned}
(\mathbb{T}_{rs}\nu)(x) = \frac{\ell-1}{n+\ell-3} (\mathbb{T}^{rs}\nu_r)(x) &- \frac{\ell-1}{n+\ell-3} \int_{S^{n-1}} k^{rs}(\upsilon)\upsilon_r\, d\mathcal{H}^{n-1}(\upsilon) \\
&- \int_{S^{n-1}} k_{rs}(\upsilon) \odot \upsilon\, d\mathcal{H}^{n-1}(\upsilon), \qquad (5.5.13)
\end{aligned}
$$

for each $r, s \in \{1, \dots, n\}$. From (5.4.8), (5.4.9), (5.4.4), and (5.5.5) we see that

$$
\|k^{rs}\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} + \|k_{rs}\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} \le c_n 2^\ell \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} . \qquad (5.5.14)
$$

In turn, from (5.5.13), (5.5.14), and the fact that $\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \ge \sup_{\partial\Omega} |\nu| = 1$ , we conclude that (5.5.12) also holds (with a possibly different dimensional constant $c_n \in (0, \infty)$) in the case when $\Omega$ is bounded.

Pressing on, fix $r, s \in \{1, \dots, n\}$ arbitrary and, for each $g \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$, set

$$\widetilde{\mathbb{T}}_{rs} g(x) := \int_{\partial\Omega} k_{rs}(x-y) \odot \nu(y) \odot g(y)\, d\sigma(y), \qquad \forall\, x \in \Omega. \tag{5.5.15}$$

Note that $\widetilde{\mathbb{T}}_{rs} 1 = \mathbb{T}_{rs}\nu$, hence (5.5.12) yields

$$\begin{aligned} \sup_{x\in\Omega} |(\widetilde{\mathbb{T}}_{rs}1)(x)| &+ \sup_{x\in\Omega} \frac{|\nabla(\widetilde{\mathbb{T}}_{rs}1)(x)|}{V(\rho(x))} \\ &\leq c_n C^{\ell-2} 2^{(\ell-2)^2} 2^\ell \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} . \end{aligned} \tag{5.5.16}$$

Thanks to the homogeneity properties of $k_{rs}$ and (5.4.8) we have

$$|k_{rs}(x-y) \odot \nu(y)| \leq c_n \frac{\|k_{rs}\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})}}{|x-y|^{n-1}} \leq \frac{c_n 2^\ell \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}}{|x-y|^{n-1}}, \tag{5.5.17}$$

$$|\nabla_x (k_{rs}(x-y) \odot \nu(y))| \leq c_n \frac{\|\nabla k_{rs}\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})}}{|x-y|^{n}} \leq \frac{c_n 2^\ell \, \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}}{|x-y|^{n}}, \tag{5.5.18}$$

so that we may invoke Lemma 5.3.4 to obtain that, for each $g \in \mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$,

$$\begin{aligned} &\sup_{x\in\Omega} |(\widetilde{\mathbb{T}}_{rs}g)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\widetilde{\mathbb{T}}_{rs}g)(x)|}{V(\rho(x))} \\ &\quad \leq C_{\Omega,n,\omega} 2^\ell \Big(1 + C_\omega\, \omega(D) + C^{\ell-2} 2^{(\ell-2)^2} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\Big) \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|g\|_{\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\ell_n} \\ &\quad \leq C_{\Omega,n,\omega} 2^\ell \Big(C^{\ell-2} 2^{(\ell-2)^2} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} + 1\Big) \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|g\|_{\mathscr{C}^\omega(\partial\Omega)\otimes\mathcal{C}\ell_n} . \end{aligned} \tag{5.5.19}$$

By specializing this to the case when $g := \nu \odot f$ with $f \in \mathscr{C}^\omega(\partial\Omega)$ arbitrary, and keeping in mind that $\nu \odot \nu = -1$, we arrive at the conclusion that

$$\begin{aligned} &\sup_{x\in\Omega} |(\mathbb{T}_{rs}f)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}_{rs}f)(x)|}{V(\rho(x))} \\ &\quad \leq C_{\Omega,n,\omega} 2^\ell (C^{\ell-2} 2^{(\ell-2)^2} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} + 1) \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \|f\|_{\mathscr{C}^\omega(\partial\Omega)} \end{aligned} \tag{5.5.20}$$

for every $f \in \mathscr{C}^\omega(\partial\Omega)$. In turn, from (5.5.20) and (5.5.9) we obtain

$$\begin{aligned} &\sup_{x\in\Omega} |(\mathbb{T}f)(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}f)(x)|}{V(\rho(x))} \\ &\quad \leq C_{\Omega,n,\omega} 2^\ell (C^{\ell-2} 2^{(\ell-2)^2} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} + 1) \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \|f\|_{\mathscr{C}^\omega(\partial\Omega)} , \end{aligned} \tag{5.5.21}$$

for every $f \in \mathscr{C}^\omega(\partial\Omega)$. The current working hypothesis is that $\ell \in \mathbb{N}$ is odd and $\ell \geq 3$, hence $2^{(\ell-2)^2} 2^\ell \leq 2^{\ell^2}$ and $2^\ell \leq C^{\ell-2} 2^{\ell^2}$ if $C \geq 1$. Therefore, in such a scenario, there exists some $C(n,\omega,\Omega) \in (0,\infty)$ such that

$$C_{\Omega,n,\omega} 2^\ell (C^{\ell-2} 2^{(\ell-2)^2} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} + 1) \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \leq C(n,\omega,\Omega) C^{\ell-2} 2^{\ell^2}, \tag{5.5.22}$$

where $C_{n,\omega,\Omega}$ is the constant appearing in (5.5.21). This suggests that, to begin with, we take $C \geq \max\{1, \sqrt{C(n,\omega,\Omega)}\}$ which then ensures both that $C \geq 1$ and that $C(n,\omega,\Omega)C^{\ell-2}2^{\ell^2} \leq C^{\ell}2^{\ell^2}$. Ultimately, this choice, together with (5.5.21) and (5.5.22), proves (5.5.1) and therefore the induction is complete.

Moving on, we claim that the additional assumption that $P$ is harmonic may be eliminated. The starting point in the justification of this claim is the observation that any homogeneous polynomial $P$ in $\mathbb{R}^n$ may be written as $P(x) = P_0(x) + |x|^2 Q_0(x)$ for every $x \in \mathbb{R}^n$, where $P_0$ and $Q_0$ are homogeneous polynomials in $\mathbb{R}^n$, and $P_0$ is harmonic (cf. [115, p. 69]). If $P$ has degree $\ell = 2N+1$ for some $N \in \mathbb{N}_0$, by iterating this process finitely many steps we conclude that for each $j \in \{0,1,\dots,N\}$ there exists a harmonic homogeneous polynomial $P_j$ of degree $\ell - 2j$ such that

$$P(x) = \sum_{j=0}^{N} |x|^{2j} P_j(x), \qquad \forall\, x \in \mathbb{R}^n. \tag{5.5.23}$$

Since the restrictions to the unit sphere of any two homogeneous harmonic polynomials of different degrees are orthogonal in $L^2(S^{n-1}, \mathcal{H}^{n-1})$ (cf. [115, p. 69]), we have

$$\|P\|^2_{L^2(S^{n-1},\mathcal{H}^{n-1})} = \sum_{j=0}^{N} \|P_j\|^2_{L^2(S^{n-1},\mathcal{H}^{n-1})}. \tag{5.5.24}$$

Thus, for every $j \in \{0,1,\dots,N\}$,

$$\|P_j\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \leq c_n \|P_j\|_{L^2(S^{n-1},\mathcal{H}^{n-1})} \leq c_n \|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})}. \tag{5.5.25}$$

The upshot of (5.5.23) is that we may now express the action of the operator $\mathbb{T}$ on any function $f \in \mathscr{C}^{\omega}(\partial\Omega)$ as

$$\mathbb{T}f(x) = \sum_{j=0}^{N} \int_{\partial\Omega} \frac{P_j(x-y)}{|x-y|^{n-1+(\ell-2j)}} f(y)\, d\sigma(y), \qquad \forall\, x \in \Omega. \tag{5.5.26}$$

and an estimate like (5.5.1) is valid for each integral operator in the above sum since each polynomial $P_j$ is odd, homogeneous, and harmonic. From this and (5.5.25) we then conclude that, in the current general case, for each function $f \in \mathscr{C}^{\omega}(\partial\Omega)$ we have

$$\sup_{x\in\Omega} |\mathbb{T}f(x)| + \sup_{x\in\Omega} \frac{|\nabla(\mathbb{T}f)(x)|}{V(\rho(x))} \leq c_n \ell C^{\ell} 2^{\ell^2} \|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})} \|f\|_{\mathscr{C}^{\omega}(\partial\Omega)}. \tag{5.5.27}$$

By choosing again $C$ big enough to begin with, matters may be arranged so that $c_n\ell \leq C^{\ell}$ for $\ell \geq 1$. Assume this is the case and rename $C^2$ as $C$. From (5.5.27) and Lemma 5.3.5 we then conclude that the operator $\mathbb{T}_+$ from (5.1.10) maps $\mathscr{C}^{\omega}(\partial\Omega)$ continuously into $\mathscr{C}^{\omega}(\Omega_+)$, and that the corresponding estimate claimed in (5.1.11) holds. Finally, $\Omega_-$ is also a $\mathscr{C}^{1,\omega}$-domain with compact boundary, so the same argument applies for the operator $\mathbb{T}_-$.

*Proof of (e) ⇒ (d).* This is a direct consequence of the observation that the operators $\mathscr{R}_j^{\pm}$ considered in item $(d)$ are particular cases of those considered in item $(e)$.

*Proof of (d) ⇒ (a).* Since $\Omega$ is a UR domain, the jump-formulas from Theorem 5.3.1 hold. In view of (1.1.18) this implies that for each $f \in L^1(\partial\Omega, \sigma)$, each $j \in \{1, \dots, n\}$, and $\sigma$-a.e. $x \in \partial\Omega$, we have

$$\Big(\mathscr{R}_j^{\pm} f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\pm}\Big)(x) = \mp\frac{1}{2}\nu_j(x)f(x) + \lim_{\varepsilon\to 0^+}\int_{\partial\Omega\setminus B(x,\varepsilon)} (\partial_j E)(x-y)f(y)\,d\sigma(y), \tag{5.5.28}$$

where $E$ is the fundamental solution for the Laplacian defined in (2.3.6). From this, (5.1.9), Lemma 1.3.6, and (1.3.22) we then see that for $j \in \{1, \dots, n\}$ we have

$$\nu_j = \mathscr{R}_j^{-} 1\big|_{\partial\Omega_-} - \mathscr{R}_j^{+} 1\big|_{\partial\Omega_+} \in \mathscr{C}^\omega(\partial\Omega), \tag{5.5.29}$$

where the first equality holds $\sigma$-a.e. on $\partial\Omega$. Thus, $\nu \in \mathscr{C}^\omega(\partial\Omega)$ which ultimately goes to show that $\Omega$ is a $\mathscr{C}^{1,\omega}$-domain.

*Proof of (a) ⇒ (c), and of estimate (5.1.12).* Suppose $\Omega \subseteq \mathbb{R}^n$ is a $\mathscr{C}^{1,\omega}$-domain with compact boundary and fix an odd homogenous polynomial $P$ of degree $\ell \geq 1$ in $\mathbb{R}^n$. Under these hypotheses, we have already proved (in the implication (a) ⇒ (e)) that the integral operators associated with $P$ as in (5.1.10) map $\mathscr{C}^\omega(\partial\Omega)$ continuously into $\mathscr{C}^\omega(\Omega_\pm)$ and that the estimates in (5.1.11) hold. Assume first that $P$ is harmonic and define

$$k(x) := \frac{P(x)}{|x|^{n-1+\ell}}, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}. \tag{5.5.30}$$

From [115, p. 73] we know that its Fourier transform is given by

$$\widehat{k}(\xi) = \gamma_{n,\ell}\frac{P(\xi)}{|\xi|^{\ell+1}}, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}, \tag{5.5.31}$$

where $\gamma_{n,\ell} = O(\ell^{-(n-2)/2})$ if $n$ is even, and $\gamma_{n,\ell} = O(\ell^{-(n-4)/2})$ if $n$ is odd as $\ell \to \infty$.

Fix two arbitrary distinct points $x, y \in \partial\Omega$. In the case when $|\nu(x) - \nu(y)| \geq 1/2$ we have $\omega(|x-y|) \geq (2\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)})^{-1}$, hence

$$\begin{aligned}
\frac{|P(\nu(x)) - P(\nu(y))|}{\omega(|x-y|)} &\leq 4\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\,\|P\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} \\
&\leq c_n 2^\ell\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\,\|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}\,,
\end{aligned} \tag{5.5.32}$$

where the last inequality above is a consequence of (5.4.9). Consider next the case when $|\nu(x) - \nu(y)| \leq 1/2$. In such a scenario, the line segment $[\nu(x), \nu(y)]$ is contained in the annulus $\overline{B(0,1)} \setminus B(0,1/2)$. Based on this, the Mean Value Theorem, the homogeneity of $P$, and (5.4.9), we obtain

$$\begin{aligned}
\frac{|P(\nu(x)) - P(\nu(y))|}{\omega(|x-y|)} &\leq \Big(\sup_{z\in[\nu(x),\nu(y)]} |(\nabla P)(z)|\Big)\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \\
&\leq \|\nabla P\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})}\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \\
&\leq c_n 2^\ell\,\|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}\,\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\,.
\end{aligned} \tag{5.5.33}$$

Moreover, using (5.4.9) and the fact that $\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \geq \sup_{\partial\Omega}|\nu| = 1$,

$$\begin{aligned}\sup_{x\in\partial\Omega}|P(\nu(x))| &\leq \|P\|_{L^\infty(S^{n-1},\mathcal{H}^{n-1})} \leq c_n 2^\ell \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \\ &\leq c_n 2^\ell \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})} \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}. \end{aligned} \tag{5.5.34}$$

From (5.5.32), (5.5.33), (5.5.34), and the fact that $\widehat{k}(\nu(x)) = \gamma_{n,\ell}P(\nu(x))$ for each $x \in \partial\Omega$ (as seen from (5.5.31)), we can conclude that the mapping $\partial\Omega \ni x \mapsto \widehat{k}(\nu(x)) \in \mathbb{C}$ belongs to $\mathscr{C}^\omega(\partial\Omega)$ and

$$\|\widehat{k}(\nu(\cdot))\|_{\mathscr{C}^\omega(\partial\Omega)} = \gamma_{n,\ell}\|P(\nu(\cdot))\|_{\mathscr{C}^\omega(\partial\Omega)} \leq c_n 2^\ell \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)} \|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}, \tag{5.5.35}$$

where the last inequality is based on the decay properties of $\gamma_{n,\ell}$ in the parameter $\ell$. Given that $\Omega$ is a UR domain (cf. Remark 5.1.2), Theorem 5.3.1 is applicable. Fix an aperture parameter $\kappa > 0$. Based the jump-formula (5.3.3), (5.1.11), Lemma 1.3.6, (1.3.22), (5.5.35), and (1.3.21) we may then estimate

$$\begin{aligned}\|Tf\|_{\mathscr{C}^\omega(\partial\Omega)} &\leq \Big\|\tfrac{1}{2i}\widehat{k}(\nu(\cdot))f + Tf\Big\|_{\mathscr{C}^\omega(\partial\Omega)} + \Big\|\tfrac{1}{2i}\widehat{k}(\nu(\cdot))f\Big\|_{\mathscr{C}^\omega(\partial\Omega)} \\ &\leq \Big\|\mathbb{T}f\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{\mathscr{C}^\omega(\partial\Omega)} + \tfrac{1}{2}\|\widehat{k}(\nu(\cdot))\|_{\mathscr{C}^\omega(\partial\Omega)}\|f\|_{\mathscr{C}^\omega(\partial\Omega)} \\ &\leq \|\mathbb{T}f\|_{\mathscr{C}^\omega(\overline{\Omega})} + c_n 2^\ell \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\|P\|_{L^1(S^{n-1},\mathcal{H}^{n-1})}\|f\|_{\mathscr{C}^\omega(\partial\Omega)} \\ &\leq \|\mathbb{T}f\|_{\mathscr{C}^\omega(\overline{\Omega})} + c_n 2^\ell \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})}\|f\|_{\mathscr{C}^\omega(\partial\Omega)} \\ &\leq \Big(C^\ell 2^{\ell^2} + c_n 2^\ell \|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\Big)\|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})}\|f\|_{\mathscr{C}^\omega(\partial\Omega)} \\ &\leq (2C)^\ell 2^{\ell^2}\|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})}\|f\|_{\mathscr{C}^\omega(\partial\Omega)}, \end{aligned} \tag{5.5.36}$$

assuming, without loss of generality, that $C \geq \max\{1\,,\,c_n\|\nu\|_{\mathscr{C}^\omega(\partial\Omega)}\}$. This ultimately proves that the singular integral operator $T$ associated with the polynomial $P$ as in (5.1.7) maps $\mathscr{C}^\omega(\partial\Omega)$ boundedly into itself in the case when $P$ is harmonic.

In the case when the polynomial $P$ is not necessarily harmonic, we decompose $P$ as in (5.5.23) and for each $f \in \mathscr{C}^\omega(\partial\Omega)$ write

$$Tf(x) = \sum_{j=0}^{N}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\frac{P_j(x-y)}{|x-y|^{n-1+(\ell-2j)}}f(y)\,d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x\in\partial\Omega. \tag{5.5.37}$$

Each term in the above sum may be regarded as the action of an integral operator on the function $f$, of the sort described in (5.1.7), though now associated with a harmonic homogeneous odd polynomial. As such, what we have proved up to this point applies and, on account of (5.5.25), gives that

$$\|Tf\|_{\mathscr{C}^\omega(\partial\Omega)} \leq \ell(2C)^\ell 2^{\ell^2}\|P\|_{L^2(S^{n-1},\mathcal{H}^{n-1})}\|f\|_{\mathscr{C}^\omega(\partial\Omega)}. \tag{5.5.38}$$

By choosing again $C$ big enough so that $\ell(2C)^\ell \leq C^\ell(2C)^\ell$ for $\ell \geq 1$ and renaming $2C^2$ as $C$, the claim in item $(c)$ and the estimate in (5.1.12) follow.

<u>*Proof of* $(c) \Rightarrow (b)$</u>. Since $\Omega$ is a UR domain and $1 \in L^2(\partial\Omega, \sigma)$ since $\partial\Omega$ is bounded, the functional $R_j 1 \in \big(\mathscr{C}^\omega(\partial\Omega)\big)^*$, originally defined as in (5.1.3), is given by the principal-value integral (5.1.4) with $f = 1$, for each $j \in \{1, \dots, n\}$. Indeed, for each $g \in \mathscr{C}^\omega(\partial\Omega)$ we may write

$$
\begin{aligned}
\langle R_j 1, g\rangle &= \frac{1}{2\varpi_{n-1}} \int_{\partial\Omega}\int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\big(g(x) - g(y)\big)\,d\sigma(y)\,d\sigma(x) \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{2\varpi_{n-1}} \iint\limits_{\substack{(x,y)\in\partial\Omega\times\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n}\big(g(x) - g(y)\big)\,d\sigma(y)\,d\sigma(x) \\
&= \Big\langle \lim_{\varepsilon\to 0^+} \frac{1}{\varpi_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |\cdot-y|>\varepsilon}} \frac{(\cdot - y)_j}{|\cdot - y|^n}\,d\sigma(y)\,,\, g\Big\rangle,
\end{aligned}
\tag{5.5.39}
$$

as wanted. As such, (5.1.6) is a direct consequence of item $(c)$.

<u>*Proof of* $(b) \Rightarrow (a)$</u>. Assume $R_j 1 \in \mathscr{C}^\omega(\partial\Omega)$ for every $j \in \{1, \dots, n\}$. Since we trivially have $\mathscr{C}^\omega(\partial\Omega) \subseteq L^\infty(\partial\Omega, \sigma) \subseteq \mathrm{BMO}(\partial\Omega, \sigma)$, from the discussion in Section 5.1 it follows that each distributional Riesz transform $R_j$ extends to a bounded linear operator on $L^2(\partial\Omega, \sigma)$, given by (5.1.4). Then at $\sigma$-a.e. point on $\partial\Omega$ we may write

$$
\frac{1}{4}\nu = \mathbf{C}(\mathbf{C}\nu) = -\mathbf{C}\Big(\sum_{j=1}^{n}(R_j 1)e_j\Big). \tag{5.5.40}
$$

The first equality above uses (5.4.24). The second equality in (5.5.40) uses the fact that at $\sigma$-a.e. point on $\partial\Omega$ we have $\mathbf{C}\nu = -\sum_{j=1}^{n}(R_j 1)e_j$, itself a consequence of the definition of $\mathbf{C}$ plus the fact that $\nu \odot \nu = -1$ and $x - y = \sum_{j=1}^{n}(x_j - y_j)e_j$ for each $x, y \in \mathbb{R}^n$. Thanks to (5.1.6) and Theorem 5.4.8, the function in the right hand-side of (5.5.40) belongs to $\mathscr{C}^\omega(\partial\Omega) \otimes \mathcal{C}\ell_n$, so ultimately $\nu \in \mathscr{C}^\omega(\partial\Omega)$ (hence $\Omega$ is a $\mathscr{C}^{1,\omega}$-domain, by Remark 5.1.3).

This concludes the proof of Theorem 5.1.4. $\square$

# Bibliography


[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] ———, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. **17** (1964), 35–92.

[3] H. Aikawa and K. Hirata, *Doubling conditions for harmonic measure in John domains*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 2, 429–445.

[4] H. Aimar, A. Bernardis, and B. Iaffei, *Comparison of Hardy-Littlewood and dyadic maximal functions on spaces of homogeneous type*, J. Math. Anal. Appl. **312** (2005), no. 1, 105–120.

[5] R. Alvarado, D. Brigham, V. Maz'ya, M. Mitrea, and E. Ziadé, *On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf-Oleinik boundary point principle*, J. Math. Sci. **176** (2011), no. 3, 281–360.

[6] R. Alvarado and M. Mitrea, *Hardy spaces on Ahlfors-regular quasi metric spaces. A sharp theory*, Lecture Notes in Mathematics, vol. 2142, Springer, 2015.

[7] P. Auscher and J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225–276.

[8] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, second ed., Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001.

[9] R.E. Basye, *Simply connected sets*, Trans. Amer. Math. Soc. **38** (1935), no. 2, 341–356.

[10] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press Inc., Boston, MA, 1988.

[11] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976.

[12] A.V. Bitsadze, *Ob edinstvennosti resheniya zadachi dirichlet dlya ellipticheskikh uravnenii s chastnymi proizvodnymi*, Uspekhi Matem. Nauk **3** (1948), no. 6, 211–212.

[13] ______, *Boundary value problems for second order elliptic equations*, North-Holland Publishing Co., Amsterdam, 1968.

[14] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, vol. 17, European Mathematical Society (EMS), Zürich, 2011.

[15] S. Bortz and M. Engelstein, *Reifenberg flatness and oscillation of the unit normal vector*, preprint, arXiv:1708.05331, 2017.

[16] A. Böttcher and Yu.I. Karlovich, *Carleson curves, Muckenhoupt weights, and Toeplitz operators*, Progress in Mathematics, vol. 154, Birkhäuser Verlag, Basel, 1997.

[17] G. Bourdaud, *Remarques sur certains sous-espaces de* $\mathrm{BMO}(\mathbb{R}^n)$ *et de* $\mathrm{bmo}(\mathbb{R}^n)$, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 4, 1187–1218.

[18] A.P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 4, 1324–1327.

[19] C.P. Calderón, *On commutators of singular integrals*, Studia Math. **53** (1975), no. 2, 139–174.

[20] L. Capogna, C. Kenig, and L. Lanzani, *Harmonic measure. Geometric and analytic points of view*, University Lecture Series, vol. 35, American Mathematical Society, Providence, RI, 2005.

[21] M. Christ, *Lectures on singular integral operators*, CBMS Regional Conference Series in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 1990.

[22] ______, *A* $T(b)$ *theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), no. 2, 601–628.

[23] R.R. Coifman, G. David, Y. Meyer, and S.W. Semmes, *$\omega$-Calderón-Zygmund operators*, Harmonic analysis and partial differential equations (El Escorial, 1987), Lecture Notes in Math., vol. 1384, Springer, Berlin, 1989, pp. 132–145.

[24] R.R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur $L^2$ pour les courbes lipschitziennes*, Ann. of Math. (2) **116** (1982), no. 2, 361–387.

[25] R.R. Coifman and Y. Meyer, *Le théorème de Calderón par les "méthodes de variable réelle"*, C. R. Acad. Sci. Paris Sér. A-B **289** (1979), no. 7, A425–A428.

[26] ______, *Une généralisation du théorème de Calderón sur l'intégrale de Cauchy*, Fourier analysis (Proc. Sem., El Escorial, 1979), vol. 1, Asoc. Mat. Española, Madrid, 1980, pp. 87–116.

[27] R.R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635.

[28] D. Cruz-Uribe, J.M. Martell, and C. Pérez, *Weights, extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, Birkhäuser/Springer Basel AG, Basel, 2011.

[29] S. Daneri and A. Pratelli, *A planar bi-Lipschitz extension theorem*, Adv. Calc. Var. **8** (2015), no. 3, 221–266.

[30] G. David, *Opérateurs d'intégrale singulière sur les surfaces régulières*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 2, 225–258.

[31] ______, *Wavelets and singular integrals on curves and surfaces*, Lecture Notes in Mathematics, vol. 1465, Springer-Verlag, Berlin, 1991.

[32] G. David and D. Jerison, *Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals*, Indiana Univ. Math. J. **39** (1990), no. 3, 831–845.

[33] G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbf{R}^n$: Beyond Lipschitz graphs*, Astérisque (1991), no. 193, 152.

[34] ______, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993.

[35] J. Duoandikoetxea, *Forty years of Muckenhoupt weights*, "Function Spaces and Inequalities," Lecture Notes Paseky nad Jizerou 2013, J. Lukes, L. Pick ed., Matfyzpress, Praga, 2013, pp. 23–75.

[36] J. Duoandikoetxea and M. Rosenthal, *Extension and boundedness of operators on Morrey spaces from extrapolation techniques and embeddings*, J. Geom. Anal. **28** (2018), no. 4, 3081–3108.

[37] X.T. Duong and L. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Comm. Pure Appl. Math. **58** (2005), no. 10, 1375–1420.

[38] L. Evans and R. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[39] E.B. Fabes, M. Jodeit Jr., and N.M. Rivière, *Potential techniques for boundary value problems on $C^1$-domains*, Acta Math. **141** (1978), no. 3-4, 165–186.

[40] A. Fiorenza and M. Krbec, *Indices of Orlicz spaces and some applications*, Comment. Math. Univ. Carolin. **38** (1997), no. 3, 433–451.

[41] G.B. Folland, *Real analysis. Modern techniques and their applications*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999.

[42] J. García-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985.

[43] J.B. Garnett and D.E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.

[44] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 92, Longman, Harlow, 1998.

[45] L. Grafakos, *Classical and modern Fourier analysis*, Pearson Education, Inc., Upper Saddle River, NJ, 2004.

[46] ———, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics 250, Springer New York, 2009.

[47] S. Hofmann, *On singular integrals of Calderón-type in $\mathbf{R}^n$, and BMO*, Rev. Mat. Iberoamericana **10** (1994), no. 3, 467–505.

[48] S. Hofmann, E. Marmolejo-Olea, M. Mitrea, S. Pérez-Esteva, and M. Taylor, *Hardy spaces, singular integrals and the geometry of Euclidean domains of locally finite perimeter*, Geom. Funct. Anal. **19** (2009), no. 3, 842–882.

[49] S. Hofmann, J.M. Martell, and S. Mayboroda, *Transference of scale-invariant estimates from lipschitz to non-tangentially accessible to uniformly rectifiable domains*, preprint, 2016.

[50] S. Hofmann and S. Mayboroda, *Hardy and* BMO *spaces associated to divergence form elliptic operators*, Math. Ann. **344** (2009), no. 1, 37–116.

[51] S. Hofmann, D. Mitrea, M. Mitrea, and A. Morris, *$L^p$-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets*, Mem. Amer. Math. Soc. **245** (2017), no. 1159.

[52] S. Hofmann, M. Mitrea, and M. Taylor, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, J. Geom. Anal. **17** (2007), no. 4, 593–647.

[53] ———, *Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains*, Int. Math. Res. Not. IMRN (2010), no. 14, 2567–2865.

[54] ———, *Symbol calculus for operators of layer potential type on Lipschitz surfaces with VMO normals, and related pseudodifferential operator calculus*, Anal. PDE **8** (2015), no. 1, 115–181.

[55] T. Hytönen and A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. **126** (2012), no. 1, 1–33.

[56] T. Hytönen, C. Pérez, and E. Rela, *Sharp reverse Hölder property for $A_\infty$ weights on spaces of homogeneous type*, J. Funct. Anal. **263** (2012), no. 12, 3883–3899.

[57] D. Jerison and C. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147.

[58] ———, *Hardy spaces, $A_\infty$, and singular integrals on chord-arc domains*, Math. Scand. **50** (1982), no. 2, 221–247.

[59] J.-L. Journé, *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics, vol. 994, Springer-Verlag, Berlin, 1983.

[60] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, vol. 83, AMS, Providence, RI, 1994.

[61] C. Kenig and T. Toro, *Harmonic measure on locally flat domains*, Duke Math. J. **87** (1997), no. 3, 509–551.

[62] ———, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454.

[63] ———, *Poisson kernel characterization of Reifenberg flat chord arc domains*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 3, 323–401.

[64] ———, *Free boundary regularity below the continuous threshold: 2-phase problems*, J. Reine Angew. Math. **596** (2006), 1–44.

[65] J. Kinnunen, R. Korte, N. Marola, and N. Shanmugalingam, *A characterization of BMO self-maps of a metric measure space*, Collect. Math. **66** (2015), no. 3, 405–421.

[66] J. Kinnunen and P. Shukla, *The structure of reverse Hölder classes on metric measure spaces*, Nonlinear Anal. **95** (2014), 666–675.

[67] J. Kolář and J. Kristensen, *Gradient ranges of bumps on the plane*, Proc. Amer. Math. Soc. **133** (2005), no. 6, 1699–1706.

[68] L.V. Kovalev, *Sharp distortion growth for bilipschitz extension of planar maps*, Conform. Geom. Dyn. **16** (2012), 124–131.

[69] V. Kozlov, V. Maz′ya, and J. Rossmann, *Spectral problems associated with corner singularities of solutions to elliptic equations*, Mathematical Surveys and Monographs, vol. 85, American Mathematical Society, Providence, RI, 2001.

[70] K. Kuratowski, *Topology. Vol. II*, Academic Press and PWN-Polish Scientific Publishers, 1966.

[71] M.A. Lavrentiev, *Boundary problems in the theory of univalent functions*, Math. Sbornik, N.S. **1(43)** (1936), no. 6, 815–845.

[72] G. Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009.

[73] A.K. Lerner, *The John-Nirenberg inequality with sharp constants*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 11-12, 463–466.

[74] A.K. Lerner and C. Pérez, *Self-improving properties of generalized Poincaré type inequalities through rearrangements*, Math. Scand. **97** (2005), no. 2, 217–234.

[75] P. MacManus and C. Pérez, *Trudinger inequalities without derivatives*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1997–2012.

[76] J.J. Marín, J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *A Fatou theorem and Poisson's integral representation formula for elliptic systems in the upper-half space*, to appear in "Topics in Clifford Analysis", special volume in honor of Wolfgang Sprößig, Swanhild Bernstein editor, Birkhäuser, 2019.

[77] ———, *Singular integral operators, quantitative flatness, and boundary problems*, book manuscript, 2019.

[78] J.J. Marín, J.M. Martell, and M. Mitrea, *Characterizations of Lyapunov domains in terms of Riesz transforms and generalized Hölder spaces*, preprint, 2019.

[79] ———, *The generalized Hölder and Morrey-Campanato Dirichlet problems for elliptic systems in the upper-half space*, to appear in Potential Anal., 2019.

[80] ———, *The weighted theory for layer potentials and elliptic boundary value problems on Semmes-Kenig-Toro domains*, preprint, 2019.

[81] J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *The higher order regularity Dirichlet problem for elliptic systems in the upper-half space*, Harmonic analysis and partial differential equations, Contemp. Math., vol. 612, Amer. Math. Soc., Providence, RI, 2014, pp. 123–141.

[82] ———, *The Dirichlet problem for elliptic systems with data in Köthe function spaces*, Rev. Mat. Iberoam. **32** (2016), no. 3, 913–970.

[83] ———, *The Green function for elliptic systems in the upper-half space*, preprint, 2018.

[84] ———, *Poisson kernels and boundary problems for elliptic systems in the upper-half space*, book manuscript, 2018.

[85] ———, *The BMO-Dirichlet problem for elliptic systems in the upper half-space and quantitative characterizations of VMO*, Anal. PDE **12** (2019), no. 3, 605–720.

[86] J.M. Martell, D. Mitrea, and M. Mitrea, *Higher order regularity for elliptic boundary value problems*, preprint, 2018.

[87] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), no. 2, 383–401.

[88] J. Mateu, P. Mattila, A. Nicolau, and J. Orobitg, *BMO for nondoubling measures*, Duke Math. J. **102** (2000), no. 3, 533–565.

[89] P. Mattila, M.S. Melnikov, and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. (2) **144** (1996), no. 1, 127–136.

[90] V. Maz′ya, M. Mitrea, and T. Shaposhnikova, *The Dirichlet problem in Lipschitz domains with boundary data in Besov spaces for higher order elliptic systems with rough coefficients*, Journal d'Analyse Mathématique **110** (2010), no. 1, 167–239.

[91] D. Mitrea, *Distributions, partial differential equations, and harmonic analysis*, Universitext, Springer, New York, NY, 2013.

[92] ———, *Distributions, partial differential equations, and harmonic analysis*, Universitext, Springer, New York, NY, 2019, second edition.

[93] D. Mitrea, I. Mitrea, and M. Mitrea, *A sharp divergence theorem with non-tangential pointwise traces and applications to singular integral operators*, book manuscript, 2018.

[94] D. Mitrea, I. Mitrea, M. Mitrea, and S. Monniaux, *Groupoid metrization theory with applications to analysis on quasi-metric spaces and functional analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2013.

[95] D. Mitrea, I. Mitrea, M. Mitrea, and M. Taylor, *The Hodge-Laplacian: Boundary value problems on riemannian manifolds*, Studies in Mathematics, vol. 64, De Gruyter, Berlin, 2016.

[96] D. Mitrea, M. Mitrea, and J. Verdera, *Characterizing regularity of domains via the Riesz transforms on their boundaries*, Anal. PDE **9** (2016), no. 4, 955–1018.

[97] I. Mitrea and K. Ott, *Counterexamples to the well-posedness of $L^p$ transmission boundary value problems for the Laplacian*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2037–2043.

[98] I. Mitrea, K. Ott, and W. Tucker, *Invertibility properties of singular integral operators associated with the Lamé and Stokes systems on infinite sectors in two dimensions*, Integr. Equ. Oper. Theory **89** (2017), no. 2, 151–207.

[99] M. Mitrea, *Clifford wavelets, singular integrals, and Hardy spaces*, Lecture Notes in Mathematics, vol. 1575, Springer-Verlag, Berlin, 1994.

[100] R.L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proc. Nat. Acad. Sci. USA. **4** (1918), no. 12, 364–370.

[101] F. Nazarov, X. Tolsa, and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1*, Acta Math. **213** (2014), no. 2, 237–321.

[102] U. Neri, *Some properties of functions with bounded mean oscillation*, Studia Math. **61** (1977), no. 1, 63–75.

[103] C. Pérez, *Calderón-Zygmund theory related to Poincaré-Sobolev inequalities, fractional integrals and singular integral operators*, Function Spaces, Nonlinear Analysis and Applications, Lecture notes of the Spring Lectures in analysis. Editors: J. Lukes and L. Pick, Charles University and Academy of Sciences, 1999, pp. 31–94. Available at http://grupo.us.es/anaresba/trabajos/carlosperez/31.pdf.

[104] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[105] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987.

[106] S.W. Semmes, *A criterion for the boundedness of singular integrals on hypersurfaces*, Trans. Amer. Math. Soc. **311** (1989), no. 2, 501–513.

[107] ———, *Chord-arc surfaces with small constant. I*, Adv. Math. **85** (1991), no. 2, 198–223.

[108] ———, *Chord-arc surfaces with small constant. II. Good parameterizations*, Adv. Math. **88** (1991), no. 2, 170–199.

[109] V.Yu. Shelepov, *On the index and spectrum of integral operators of potential type along Radon curves*, Math. USSR Sbornik **70** (1991), no. 1, 175–203.

[110] Z. Shen, *Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains*, Amer. J. Math. **125** (2003), no. 5, 1079–1115.

[111] ———, *Weighted estimates for elliptic systems on Lipschitz domains*, Indiana Univ. Math. J. **55** (2006), no. 3, 1135–1154.

[112] V. A. Solonnikov, *Estimates for solutions of general boundary value problems for elliptic systems*, Doklady Akad. Nauk. SSSR **151** (1963), 783–785 (Russian), English translation in Soviet Math., 4 (1963), 1089–1091.

[113] ———, *General boundary value problems for systems elliptic in the sense of A. Douglis and L. Nirenberg. I*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 665–706 (Russian).

[114] ———, *General boundary value problems for systems elliptic in the sense of A. Douglis and L. Nirenberg. II*, Trudy Mat. Inst. Steklov. **92** (1966), 233–297 (Russian).

[115] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[116] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[117] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32.

[118] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics, vol. 1381, Springer-Verlag, Berlin, 1989.

[119] M. Taylor, *Measure theory and integration*, Graduate Studies in Mathematics, vol. 76, American Mathematical Society, Providence, RI, 2006.

[120] P. Tukia, *The planar Schönflies theorem for Lipschitz maps*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), no. 1, 49–72.

[121] ———, *Extension of quasisymmetric and Lipschitz embeddings of the real line into the plane*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), no. 1, 89–94.