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Long time control with applications

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A thesis submitted for the degree of

Doctor in Mathematics.

Supervised by:

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Madrid, 2020

To my parents.

This thesis has been mainly funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 694126-DyCon), and for an applied research secondment by the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 777822.

Acknowledgements

First of all, I would like to express my most sincere gratitude to my advisor, Enrique Zuazua Iriondo. He has been a mentor, who helped and motivated me all along these years of PhD studies. He has been a constant example of hard work, concentration and devotion. Furthermore, he gave me the opportunity to have a secondment in the company ‘‘Marposs S.p.A.’’. I am proud to have been his student. Eskerrik asko!!!

I thank professor Noboru Sakamoto, for being a guide during his visit to Bilbao. I am pleased to have the opportunity to work with him.

A special thank to professor Rafael Orive, who kindly accepted to be part of the PhD panel and lector. During these three years he has been a reference in the deportment of mathematics of Universidad Autónoma de Madrid. Thank you professor Fernando Soria and professor Eduardo Casas for accepting to be part of the PhD panel. I am deeply grateful to professor Emmanuel Trélat and professor Alessio Porretta for joining the PhD panel and for writing interesting reports on the present thesis. Alessio Porretta has been my supervisor of my master's thesis and the bridge to become PhD student of professor Enrique Zuazua.

I would like to express my most sincere gratitude to professor Matteo Gnuffi and engineer Francesco Ziprani, my supervisors during the secondment in the company ‘‘Marposs S.p.A.’’. I have learned a lot from them on mathematics and computer programming. I also thank the engineers Riccardo Cipriani, Stefano Fenara, Andrea Ferrari, Samuele Martelli and Alessandro Ruggeri for their contributions during the secondment.

A special thank to Miren Lourdes Arretxe for being a constant reference and guide for all members of the DyCon-ERC Advanced Grant. Thank you to all the workers of Universidad de Deusto for helping me in any kind of need during these three years. Among them, a special thank to Ondiz Zarraga, who also helped me in the translation of the present acknowledgments into Spanish.

It is now thank all the members of the Chair of Computational Mathematics in Universidad de Deusto: Alejandro B. Aceves, Omrane Abdennebi, Umberto Biccari, Darlis Bracho Tudares, Alejandro Cunillera, Carlos Esteve Yagüe, Borjan Geshkovski, Víctor Hernández-Santamaría, Aitziber Ibáñez, Dongnam Ko, Hicham Kouhkouh, Martin Lazar, Thibault Liard, José Vicente

Lorenzo, Jérôme Lohéac, Javier Martín, Debayan Maity, Jorge Mallo, Idriss Mazari, Sorin Micu, Césare Molinari, Azahar Monge, Salem Nafiri, Ana Navarro, Mohammed Nor Frioui, Miguel Nuñez, Leontine Nkamba, Héctor Olmedo, Jesús Oroya, Sergio Pascual, Charlotte Rodriguez, Domènec Ruiz-Balet, Alexandre Santos Francisco, Michael Schuster, Yacouba Simpore, Sebastian Zamorano, Yubiao Zhang and Yuhua Zhu. We have worked together these three years in a cooperative atmosphere I really appreciated. A special thank to Umberto Biccari and Víctor Hernández-Santamaría for being guides in the early stages of my PhD. Let me mention again “mis compañeros” Borjan Geshkovski and Domènec Ruiz-Balet, we are growing up together as mathematicians. I also thank again Carlos Esteve Yagüe and Jorge Mallo, who helped me in the translation of the abstract and the introduction of the present thesis.

Muchas gracias to UAM people. Among them, I remember professor Davide Barbieri who supported my secondment in the company “Marposs S.p.A.”, by the European project GHAIA. I thank professor Antonio Cuevas who gave me precious explanations concerning the PhD program in UAM. Thank you professor Eugenio Hernández, my neighbour in UAM. A special thank to all my collegues in UAM. In particular, I thank Alejandro Mas and Francisco Mengual. Their help and kindness during my visits in UAM have been really appreciated.

I am deeply grateful to my Judo mates in “Dojo Samurai” (Bilbao). A special thanks to the “Profe” Eduardo Pérez. Thank you Anaís, Guiomar and Gonzalo, my flatmates. Thank you Basque people for welcoming me with open arms. Eskerrik Asko !!!

A special thank to the Christian community I am proud to be part of in the quarter “San Ignacio”. In particular, I thank my reference group: Carlos, Christian, Itziar, Leire, Rebeca, Aitziber y Victor. Furthermore, I would like to thank father Rafael, a constant spiritual guide all along these three years.

Of course, I do not forget professor Anna Ceraso and professor Marisa Lucci, who made me enjoy Mathematics during the Secondary School.

A big building needs big pillars. My pillars in Rome (and in Zurich) are the components my family. There are no words to thank my parents Claudio and Antonella, who have supported my during all the the stages of my life. Their wise suggestions have guided me for the past and will guide me for the future. Their generosity have been immense, when they let me go thousands kilometers away to start my PhD. A special thought goes to my grandfathers Gino and Norina, who passed away and who have been an example of (concrete!) love. These three years abroad I always felt the help and support of my grandparents Nico and Vanda: avanti tutta !!! Let me now go back to young people like my nice Silvia and my nephew Elio, who taught me back tenderness. Grazie di cuore dalo zio!!! I include in the “young” list also my sister Claudia and my brother Daniele, who supported me with enthusiasm from the beginning in this adventure abroad. Let me mention my brothers in law Giovanni and Gözde, who gave a new flavor to our great family! Grazie mille!!!

I thank also my loyal friends in Ciampino (Rome), those I know from very long time and the most recent ones. Those who support the soccer team A.S. ROMA (like Matteo Pasanisi and Leonardo Vattermoli) and the ones who do not (like Matteo Mavilia). A special thank to my friends Alessandro Abbate, Letizia Calandrelli, Chiara Caracciolo, Paolo Girolami, Francesco Recupero and Alessandro Spadoni.

Agradecimientos

En primer lugar, me gustaría expresar mi más sincero agradecimiento a mi director de tesis, Enrique Zuazua Iriondo. Ha sido un mentor, que me ha ayudado y me ha motivado durante todos estos años de estudios de doctorado. Ha sido un ejemplo constante de trabajo, concentración y devoción. Además, me dio la oportunidad de realizar una estancia de investigación en la empresa "Marposs S.p.A.". Estoy orgulloso de haber sido su alumno. Eskerrik asko !!!

Agradezco al profesor Noboru Sakamoto el haber sido un guía durante su visita a Bilbao. Me complace haber tenido la oportunidad de trabajar con él.

Un agradecimiento especial al profesor Rafael Orive, quien amablemente aceptó ser parte del tribunal de doctorado y lector. Durante estos tres años ha sido un referente en el departamento de matemáticas de la Universidad Autónoma de Madrid (UAM). Gracias profesor Fernando Soria y profesor Eduardo Casas por aceptar ser parte del tribunal de doctorado. Estoy profundamente agradecido al profesor Emmanuel Trélat y al profesor Alessio Porretta por unirse al panel de doctorado y por escribir informes interesantes sobre la presente tesis. Alessio Porretta fue supervisor de mi tesis de maestría y el puente para ser estudiante de doctorado del profesor Enrique Zuazua.

Quisiera expresar mi más sincero agradecimiento al profesor Matteo Gnuffi y al ingeniero Francesco Ziprani, mis supervisores durante la estancia de investigación en la empresa "Marposs S.p.A.". He aprendido mucho de ellos sobre las matemáticas y programación de ordenadores. También agradezco a los ingenieros Riccardo Cipriani, Stefano Fenara, Andrea Ferrari, Samuele Martelli y Alessandro Ruggeri sus contribuciones durante la comisión de servicio.

Un agradecimiento especial a Miren Lourdes Arretxe por ser una referencia constante y una guía para todos los miembros del DyCon-ERC Advanced Grant. Gracias a todos los trabajadores de la Universidad de Deusto por ayudarme en cualquier tipo de necesidad durante estos tres años. Entre ellos, especialmente a Ondiz Zarraga, quien también me ayudó en la traducción de los reconocimientos actuales al español.

Me gustaría ahora mostrar mi agradecimiento a todos los miembros de la Cátedra de Matemática Computacional en la Universidad de Deusto: Alejandro B. Aceves, Omrane

Abdennebi, Umberto Biccari, Darlis Bracho Tudares, Alejandro Cunillera, Carlos Esteve Yagüe, Borjan Geshkovski, Víctor Hernández-Santamaría, Aitziber Ibáñez , Dongnam Ko, Hicham Kouhkouh, Martin Lazar, Thibault Liard, José Vicente Lorenzo, Jérôme Lohéac, Javier Martín, Debayan Maity, Jorge Mallo, Idriss Mazari, Sorin Micu, Césare Molinari, Azahar Monge, Salem Nafiri, Ana Navarro, Mohammed Nor Frioui, Leontine Nkamba, Miguel Nuñez, Héctor Olmedo, Jesús Oroya, Sergio Pascual, Charlotte Rodriguez, Domènec Ruiz-Balet, Alexandre Santos Francisco, Michael Schuster, Yacouba Simpore, Sebastian Zamorano, Yubiao Zhang y Yuhua Zhu. Hemos trabajado juntos estos tres años en una atmósfera cooperativa que realmente aprecié. Un agradecimiento especial a Umberto Biccari y Víctor Hernández-Santamaría por ser guías en las primeras etapas de mi doctorado. Permítanme mencionar nuevamente a “mis compañeros” Borjan Geshkovski y Domènec Ruiz-Balet, estamos creciendo juntos como matemáticos. También doy las gracias nuevamente a Carlos Esteve Yagüe y Jorge Mallo, quienes me ayudaron en la traducción del resumen y la introducción de la presente tesis.

Muchas gracias a toda la gente de la UAM. Entre ellos, recuerdo al profesor Davide Barbieri que apoyó mi comisión de servicio en la empresa “Marposs S.p.A.”, por el proyecto europeo GHAIA. Agradezco al profesor Antonio Cuevas que me dio preciosas explicaciones sobre el programa de doctorado en la UAM. Gracias profesor Eugenio Hernández, mi vecino de la UAM. Un agradecimiento especial a todos mis colegas en la UAM. En particular, doy las gracias a Alejandro Mas y Francisco Mengual. Su ayuda y amabilidad durante mis visitas en la UAM han sido realmente apreciadas.

Estoy profundamente agradecido a mis compañeros de judo en “Dojo Samurai” (Bilbao). Un agradecimiento especial al “Profe” Eduardo Pérez. Gracias Anaís, Guiomar y Gonzalo, mis compañeros de piso. Gracias pueblo vasco por darme la bienvenida con los brazos abiertos. Eskerrik asko !!!

Un agradecimiento especial a la comunidad cristiana de la que estoy orgulloso de ser parte en el barrio “San Ignacio”. En particular, doy las gracias a las personas de mi grupo de referencia: Carlos, Christian, Itziar, Leire, Rebeca, Aitziber y Victor. Ademas, quiero dar las gracias al cura Rafael por ser una costante referencia spiritual a lo largp de estos tres años.

Por supuesto, no me olvido de la profesora Anna Ceraso y de la profesora Marisa Lucci, quienes me hicieron disfrutar de las matemáticas durante la escuela secundaria.

Un gran edificio necesita grandes pilares. Mis pilares en Roma (y en Zurich) son los componentes de mi familia. No hay palabras para dar las gracias a mis padres Claudio y Antonella, quienes me han apoyado durante todas las etapas de mi vida. Sus sabias sugerencias me han guiado en el pasado y me guiarán en el futuro. Su generosidad ha sido inmensa, cuando me dejaron ir a miles de kilómetros para comenzar mi doctorado. Un pensamiento especial va para mi abuelos Gino y Norina, que fallecieron y que han sido un ejemplo de amor (concreto!). Estos tres años en el extranjero siempre sentí la ayuda y el apoyo de mis abuelos Nico y Vanda:

avanti tutta !!! Permítanme dirigirme ahora a los jóvenes como mi querida Silvia y mi sobrino Elio, que me enseñaron una otra vez la ternura. Grazie di cuore dallo zio!!! Incluyo en la lista de “jóvenes” también a mi hermana Claudia y mi hermano Daniele, quienes me apoyaron con entusiasmo desde el principio en esta aventura en el extranjero. Quiero mencionar también a mi cuñados Giovanni y Gözde que han dado un nuevo sabor a nuestra grande familia! Grazie mille!!!

También doy las gracias a mis fieles amigos en Roma, a los que conozco desde hace mucho tiempo y a los más recientes. A los que apoyan al equipo de fútbol A.S. ROMA (como Matteo Pasanisi y Leonardo Vattermoli) y a los que no (como Matteo Mavilia). Un agradecimiento especial a mis amigos Alessandro Abbate, Letizia Calandrelli, Chiara Caracciolo, Paolo Girolami, Francesco Recupero y Alessandro Spadoni.

Contents

Abstract

This thesis is concerned with the study of some control problems in a large time horizon.

The first part of the thesis is devoted to controllability of Partial Differential Equations under state and/or control constraints. In chapter 4, we address the controllability under positivity constraints of semilinear heat equations. We firstly obtain steady state controllability, by employing a "stair-case argument". Then, supposing dissipativity of the free dynamics, we extend our previous result to constrained controllability to trajectories. In any case, the targets must be defined by positive controls. We prove further the positivity of the minimal controllability time under positivity constraints, by applying a new method, based on the choice of a particular test function in the definition of weak solutions to evolution equations. Hence, despite the infinite velocity of propagation for parabolic equations, a waiting time phenomenon occurs in the constrained case. In chapter 5, controllability under positivity constraints is analyzed for wave equations. In this case, the zero state is reachable, by nonnegative controls. In chapter 6, we get a global turnpike result for an optimal control problem, governed by a semilinear heat equation. The running target in the cost functional is required to be small, whereas the initial datum for the evolution equation can be chosen arbitrarily. This is done by combining the available local results [116, 137], with an estimate of the L^∞ norm of the optima (uniform in the time horizon) and an estimate of the time needed to get close to the turnpike. If the target is large, we produce an example, where the steady problem admits (at least) two solutions (chapter 7). In chapter 8, we present an application of stabilization/turnpike theory to a problem of rotor balancing.

Resumen

Esta tesis concierne el estudio de algunos problemas de control en un largo horizonte temporal.

La primera parte de la tesis está dedicada a la controlabilidad de Ecuaciones en Derivadas Parciales bajo restricciones de estado y/o control. En el capítulo 4, abordamos la controlabilidad bajo restricciones de positividad para la ecuación del calor semilineal. En primer lugar, obtenemos la controlabilidad entre estados estacionarios, mediante el uso de un ‘‘stair-case argument’’. Luego, suponiendo disipatividad en la dinámica libre, extendemos nuestro resultado anterior a la controlabilidad bajo restricciones hacia trayectorias. En cualquier caso, los targets deben definirse mediante controles positivos. Ademas, probamos la positividad del tiempo mínimo de controlabilidad bajo restricciones de positividad, mediante la aplicación de un nuevo método, basado en la elección de una función test particular en la definición de solucione débil para la ecuación de evolución. Por lo tanto, a pesar de la velocidad infinita de propagación para las ecuaciones parabólicas, se produce un fenómeno de tiempo de espera en el caso restringido. En el capítulo 5, la controlabilidad bajo restricciones de positividad se analiza para la ecuación de ondas. En este caso, el estado cero es alcanzable por controles positivos. En el capítulo 6, obtenemos un resultado de turnpike global para un problema de control óptimo, sujeto a una ecuación del calor semilineal. En este caso, requerimos que el target en el funcional de coste sea pequeño, mientras que el dato inicial para la ecuación de evolución se puede elegir arbitrariamente. Esto se realiza combinando los resultados locales disponibles en [116, 137], con una estimación de la norma L^∞ para los óptimos (uniforme en el horizonte temporal) y una estimación del tiempo necesario para acercarse al turnpike. Para el caso de target grande, damos un ejemplo, donde el problema estacionario admite (al menos) dos soluciones (capítulo 7). En el capítulo 8, presentamos una aplicación de la teoría de estabilización/turnpike a un problema de equilibrio para un rotor.

Chapter 1

Introduction

A control system is a dynamical system whose behaviour may be influenced by an input parameter referred to as *a control*.

Early archetypes of control systems are the irrigation systems designed in Egypt and Mesopotamia around 8000 years ago [54, 124]. Later, Romans employed smart control strategies to keep the water level constant in their aqueducts. In the XVII century the Dutch mathematician and astronomer Christiaan Huygens addressed the problem of speed control while designing clocks. A similar work was carried out by the English physicist Robert Hooke [54, 134]. Later, a feedback-control mechanism was employed in the construction of flyball governors for windmills.

The outbreak of control theory was the industrial revolution starting from the XVIII century. An outstanding breakthrough was the steam engine (figure 1.1), invented by James Watt in 1769, the goal being to keep the velocity of rotation constant despite the variable load, by a flyball governor. When the velocity increases, two flyballs raises, activating some valves which let the vapour escape, thus slowing down the physical process. In the followings decades, a big challenge was to mathematically formulate the regulating system invented by Watt. Early attempts were made by the mathematician and astronomer George Airy and a complete analysis was carried out by the Scottish physicist James Clerk Maxwell, in 1868, to explain erratic behaviours of the steam engine and suggest solutions. A similar research was conducted by the mathematicians Adolf Hurwitz and Edward John Routh. These were the early stages of stability analysis in control theory.

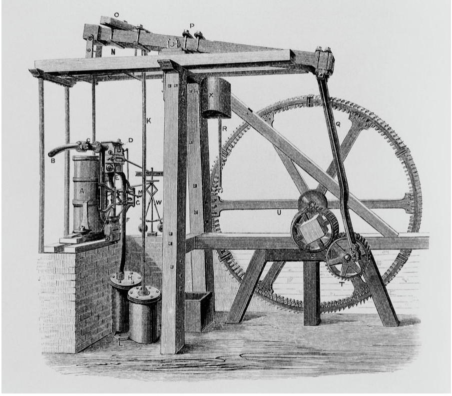

Figure 1.1: steam engine by James Watt
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Nowadays, Control Theory is a flourishing area of applied Mathematics and Engineering, enjoying fruitful interactions with several fields, including ODE and PDE analysis, numerical analysis, stability analysis, stochastic calculus, Fourier analysis, signal processing, computer science and "black-box" optimization. Applications can be found in several disciplines, including engineering, medicine, biology and economic sciences. The increasing performance of modern computers make control theory applicable to complex systems, like robots or traffic in a crowded city. Most probably, in the future control system will be ubiquitous in the real world, thanks to new interesting interactions between control and several disciplines like machine learning. For further details, see, for instance, [54] and the rich bibliography therein.

We give now a brief presentation of control theory. For further details, the reader is referred to chapter 3 and the references therein. Generally speaking, a control system is governed by a *state equation*

$$\frac{d}{dt}y(t) = Ay(t) + Bu(t), \tag{1.0.1}$$

where

- y is the *state*, the unknown of the system to be controlled. Typically, y is an element of a parabolic space $L^2(0, T; H)$, where H is known as state space;
- $A : D(A) \subset H \longrightarrow H$ is an operator generating a (linear or nonlinear) semigroup on the Hilbert space H. The operator A is the "model" of the system;
- u is the *control*, which belongs to a set of admissible controls $\mathscr{U}_{\text{ad}} \subseteq L^2(0, T; U)$, where U is a Banach space. The control is the free parameter at our disposal to influence the behaviour of the system;
- $B : U \longrightarrow X$ is the *control operator* which models the actuators, i.e. how the control acts on the system.

The set of admissible controls $\mathscr{U}_{\text{ad}}$ contains those controls which verify some conditions. Among the admissible controls, we call *optimal control* the best one according to some prescribed criteria (typically the minimizer of a given functional). The solution to the state equation (1.0.1), with the optimal control is named optimal state.

Some of the major challenges in control theory are

- well-posedeness of the state equation (1.0.1);
- existence of an admissible control;
- existence (and uniqueness) of an optimal control;
- first and second order optimality conditions characterizing the optimal control;
- structure and properties of optimal control and state;
- numerical methods to approximately solve the discretized control problem.

For further details, the interested reader is referred to chapter 3 and the references therein.

The object of the first part of the thesis is *controllability under constraints* of partial differential equations. We are given a state equation

$$\frac{d}{dt}y(t) = Ay(t) + Bu(t), \tag{1.0.2}$$

an initial datum y_0 and a final target y_1. Furthermore, some constraints are prescribed on the state and/or on the control

$$u(t) \in K_c \quad \text{and/or} \quad y(t) \in K_s, \quad \forall\, t \in [0, T], \tag{1.0.3}$$

where K_c and K_s are closed and convex subsets of U and H respectively. For instance, the control and the state can be required to be nonnegative. A control u is said to be admissible if the constraints (1.0.3) are satisfied, together with the terminal conditions

$$y(0) = y_0 \quad \text{and} \quad y(T) = y_1. \tag{1.0.4}$$

As we shall see, in presence of constraints, the control time T needs to be sufficiently large in order to fulfill the constraints reducing the amplitude of the oscillations of control and state. Large time is required for heat-like models as well, despite the infinite velocity of propagation.

The second part of the thesis is devoted to the validation of the *turnpike property* for optimal control problems like

$$\min_u J_T(u) = \int_0^T L(y, u)dt, \tag{1.0.5}$$

where

$$\begin{cases} \frac{d}{dt}y = A(y) + B(u) & \text{in } (0, T) \\ y(0) = y_0. \end{cases} \tag{1.0.6}$$

By dropping the time in the above problem, we get the following steady problem

$$\min_{u_s} J_s(u_s) = L(y_s, u_s), \quad \text{with the constraint} \quad A(y_s) + B(u_s) = 0.$$

We suppose the minimum is achieved for both problems. The optimal pair optimal control-optimal state for the time-evolution problem is denoted by (u^T, y^T), while the optimal pair for the steady problem is denoted by $(\overline{u}, \overline{y})$. We prove that, in large time, any optimal pair (u^T, y^T) for the time-evolution problem (1.0.6)-(1.0.5) is exponentially close to an optimal pair $(\overline{u}, \overline{y})$ of the steady problem.

1.1 Controllability of PDEs under constraints

Controllability of Partial Differential Equations is by now a classical topic in mathematical analysis. One of the pioneering works is [51], where H.O. Fattorini and D.L. Russell studied the controllability of the heat equation in one space dimension. Another milestone for controllability of linear PDEs is the SIAM Review article of 1988 by J.L. Lions [93], where the Hilbert Uniqueness Method (HUM) was introduced. Further references can be found in the following articles and books and the references therein: [153, 53, 92, 12, 87, 45, 77, 34, 140, 86].

On the one hand, many of these results and the corresponding numerical methods have been developed in absence of constraints. On the other hand, in practical applications constraints on the state and/or on the control are ubiquitous.

For instance, when controlling a heat-like phenomenon we frequently require the temperature to be above a lower threshold. In several biological, chemical and economical models, reaction-diffusion equations are solved by densities, which must be nonnegative for any time (see, e.g. the book by J.D. Murray [107, chapter 11], [74] or the celebrated paper by A.M. Turing [141]). In finance, under some assumptions, the fair value (solution to a parabolic PDE [71, Theorem 2.5, chapter 2]) must be kept above a lower threshold. Furthermore, in applications, the machine power is bounded, whence some constraints on the control needs to be imposed (see the earlier works [132] for the heat equation and [65] for the wave equation).

From a mathematical viewpoint, the introduction of constraints in controllability problems can be very challenging. For instance, considering the pure heat equation, norm-optimal controls in small time enjoy large oscillations in proximity of the final time. Indeed, they are restrictions of solutions of the adjoint system with a critical final datum. When the time horizon is too short, these oscillations prevent the control to fulfill any constraint. Then, despite the infinite velocity of propagation, the minimal controllability time under constraints is positive (see [95] and chapter 4).

On the other hand, in a large time horizon, we can construct small-amplitude controls in order to fulfill the constraints. Namely constrained controllability holds in large time, under suitable assumptions on the initial datum and the target. If the initial datum and the final target are steady states connected within the set of steady states, we implement a "stair-case argument" (see figure 1.2), consisting in moving in an iterative manner from one steady state to a neighbouring one using small amplitude controls. Iterating this procedure, one can drive the state to the final target by small amplitude controls. This ensures that the state remains in a tubular neighborhood of the path of steady states. We remark that this strategy relies on local controllability and on the continuous dependence of the state on the control. Hence, such strategy can be applied to a broad class of PDEs and it can be employed to fulfill both unilateral and bilateral constraints. This has been inspired by the seminal paper [36] by J.M. Coron and E. Trélat, where quasi-static deformations were employed to control semilinear heat equations. Note however that our approach differs from [36]. Indeed, the quasi-static deformations strategy is based on the following steps:

1. follow the given path of steady states at a small velocity obtaining "almost" a trajectory for the time-evolution control system;

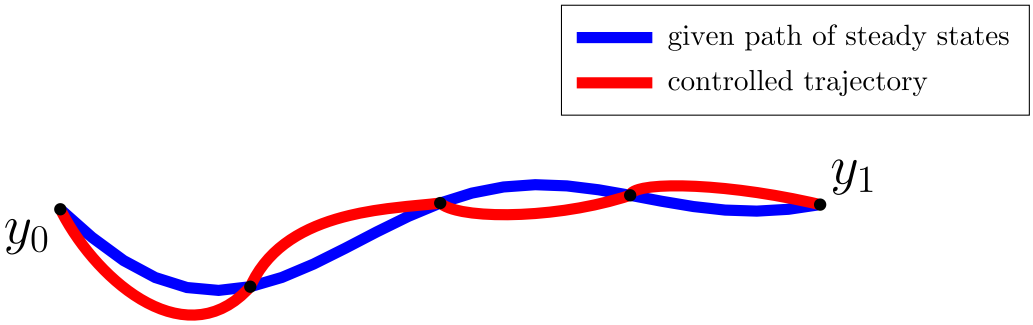


Figure 1.2: stair-case argument

2. determine stabilizing feedback to stabilize a trajectory of the control system along the obtained "almost"-trajectory. In this way, we can get arbitrarily close to the target;

3. local controllability to reach exactly the target.

Instead, as illustrated in figure 1.2, in the "stair-case argument"

1. we subdivide the given path of steady states into small arcs of steady states;

2. we employ local controllability in each small arc to link the steady states at the endpoints of the arc.

Controllability under positivity constraint has been addressed by J. Lohéac, E. Trélat and E. Zuazua in [95] for dissipative heat equations. The proof relies on the dissipativity of the system, which leads to an exponential decay of the observability constant. This allows one to show that, in large time intervals, the controls can be chosen to be small, which in turn implies constrained controllability. For controllability under positivity constraints of finite dimensional systems we refer to the recent paper [96]. Finally, the controllability problem under linear projection constraints have been analyzed by S. Ervedoza in [47].

We start illustrating our main results for heat-like equations. Hereafter, Ω is a connected bounded open set of $\mathbb{R}^n$, $n \geq 1$, with C^∞ boundary.

1.1.1 Controllability under positivity constraints of semilinear heat equations

Consider the semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0,T) \times \Omega \\ y = u & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{1.1.1}$$

where $y = y(t,x)$ is the state and while $u = u(t,x)$ is the control acting on the boundary $\partial\Omega$. The nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^1. Note that f can be of blowing-up type. The results we are going to present hold for more general operators, involving nonconstant diffusivity matrix, drift and localized control (see chapter 4).

Our analysis is inspired by [36], where J.M. Coron and E. Trélat control semilinear heat equations, by using quasi-static deformations. Our initial datum and final target are steady states, joined by a continuous path of steady states.

More precisely, we assume the existence of a continuous arc

$$\gamma : [0,1] \longrightarrow L^\infty(\Omega),$$

$$r \longmapsto \gamma_r,$$

such that $\gamma_0 = y_0$ and $\gamma_1 = y_1$ and for any $r \in [0,1]$, γ_r solves the steady problem

$$\begin{cases} -\Delta\gamma_r(x) + f(\gamma_r(x)) = 0 & x \in \Omega \\ \gamma_r(x) = \overline{u}_r(x) \geq \nu > 0 & x \in \partial\Omega. \end{cases}$$

where $\nu > 0$ is a constant. The construction of the path of steady state in some nonlinear models can be found in [117, 126, 101].

Given the path of steady states γ and time T large enough, the "stair-case argument" consists in linking neighbouring steady states along γ, by a control u remaining in a ν-neighborhood of $\overline{u}_r$

$$\|u - \overline{u}_r\|_{L^\infty} \leq \nu. \tag{1.1.2}$$

Now, since $\overline{u}_r \geq \nu > 0$, the control u fulfills the nonnegativity constraint

$$u = u - \overline{u}_r + \overline{u}_r \geq -\nu + \nu = 0, \quad \text{a.e. on } (0,T) \times \partial\Omega, \tag{1.1.3}$$

as desired. Note that, since the nonlinearity is of blowing-up type, by choosing an arbitrary control, the solution y to the state equation may blow-up. However, by choosing the above control, the solution to (1.1.1), with initial datum y_0 and control u remains in a bounded set, thus avoiding blow-up in finite time.

Theorem 1.1.1 (Steady state controllability)**.** *Under the above assumptions, let y_0 and y_1 be path connected bounded steady states, such that*

$$\overline{u}_r \geq \nu, \quad a.e. \text{ on } \Gamma \tag{1.1.4}$$

for any $r \in [0,1]$. Then, if T is large enough, there exists $u \in L^\infty((0,T) \times \partial\Omega)$, a control such that:

- *the problem* (1.1.1) *with initial datum y_0 and control u admits a unique solution y verifying $y(T,\cdot) = y_1$;*

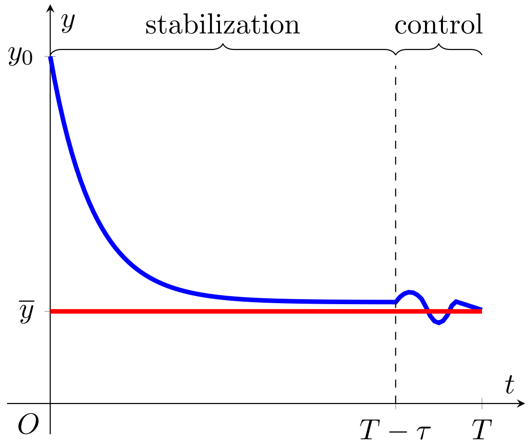


Figure 1.3: illustration of the proof of Theorem 4.1.2 in two steps: stabilization + control

- $u \geq 0$ *a.e. on* $(0,T)\times\Gamma$.

At this point, our purpose is to consider a wider set of initial data and final targets. To this end, we suppose hereafter f is increasing, which leads to dissipativity of the free dynamics and well posedeness of the state equation for any initial datum $y_0 \in L^2$ and control $u \in L^2$.

Take an initial datum $y_0 \in L^2(\Omega)$ and a target trajectory $\overline{y}$ solution to

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + f(\overline{y}) = 0 & \text{in } (0,T)\times\Omega \\ \overline{y} = \overline{u} \geq \nu > 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{1.1.5}$$

with bounded control $\overline{u} \geq \nu > 0$. Our goal is to find a nonnegative control u, such that the unique solution y to

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{in } (0,T)\times\Omega \\ y = u \geq 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{1.1.6}$$

satisfies the end condition $y(T,\cdot) = \overline{y}(T,\cdot)$, namely the controlled trajectory y matches the target trajectory $\overline{y}$ at time T.

Fix $\tau > 0$. The strategy we use to solve this control problem is the following:

- **stabilization** in $[0, T-\tau]$: for long time, we choose the steady control $u = \overline{u}$ to stabilize the system to the target trajectory $\overline{y}$. This is possible since f is increasing;

- **control** in $[T-\tau, T]$: we use local controllability to match exactly the target trajectory at time T.

Theorem 1.1.2 (Controllability of general initial data to trajectories)**.** *Assume f is increasing.*

Consider a target trajectory $\overline{y}$, solution to (1.1.6) *with initial datum $\overline{y}_0 \in L^2$ and control $\overline{u} \in L^\infty$, verifying the positivity condition:*

$$\overline{u} \geq \nu > 0, \quad a.e. \text{ on } (0,T)\times\Gamma. \tag{1.1.7}$$

Then, for any initial datum $y_0 \in L^2(\Omega)$, we can find, in a sufficiently large time, a bounded control $u \geq 0$ such that:

- *the unique solution y to (1.1.1) with initial datum y_0 and control u is such that $y(T, \cdot) = \overline{y}(T, \cdot)$;*
- *$u \geq 0$ a.e. on $(0, T) \times \Gamma$.*

Remark 1.1.1. *Now, suppose f is increasing and $f(0) = 0$. In the context of Theorem 1.1.2, assume in addition the initial datum $y_0 \geq 0$. Then, by the maximum principle, the controlled solution $y \geq 0$.*

So far, we have assumed the control time T to be large for constrained controllability. We show now that constrained controllability fails in time too small, namely the minimal controllability time (under constraints) is positive.

To fix ideas, let us consider the linear case

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ y = u & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0, & \text{in } \Omega. \end{cases} \tag{1.1.8}$$

with constant initial datum y_0 and target y_1. In chapter 4, the interested reader can find the general case with time-space dependent coefficients and nonlinear terms.

We define the concept of minimal controllability time as

$$T_{\min} := \inf \left\{ T > 0 \;\middle|\; \exists u \in L^\infty((0,T) \times \Gamma)^+, \ y(T, \cdot) = y_1 \right\}, \tag{1.1.9}$$

where we use the convention $\inf(\varnothing) = +\infty$.

Theorem 1.1.3 (Positivity of the minimal controllability time)**.** *Assume the initial datum y_0 differs from the target y_1. Suppose y_1 is defined by a boundary control $u_1 \geq \nu > 0$.*

Then,

1. *there exists $T_0 > 0$ such that, for any $T \in (0, T_0)$ and for any nonnegative control $u \in L^\infty((0,T) \times \Gamma)$ the solution y to (1.1.8) with initial datum y_0 and control u is such that $y(T, \cdot) \neq y_1$.*
2. *Consequently,*

$$T_{min} > 0.$$

We consider two paradigmatic examples: $y_0 > y_1$ and $y_0 < y_1$.

Case $y_0 > y_1$.

This is the most intuitive case. For any nonnegative control u, by comparison principle

$$y \geq z, \quad \text{a.e. in } (0,T) \times \Omega, \tag{1.1.10}$$

where y is the solution to (1.1.8), with initial datum y_0 and control $u \geq 0$ and z solves the homogeneous problem

$$\begin{cases} z_t - \Delta z = 0 & \text{in } (0,T)\times\Omega \\ z = 0 & \text{on } (0,T)\times\partial\Omega \\ z(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{1.1.11}$$

Let λ_1 be the first eigenvalue of the Dirichlet laplacian and let ϕ_1 be the corresponding eigenfunction, with $\phi_1 \geq 0$ and $\|\phi_1\|_{L^2} = 1$. We have

$$\int_\Omega y(t,x)\phi_1(x)dx \geq \int_\Omega z(t,x)\phi_1(x)dx = \exp(-\lambda_1 t)\int_\Omega y_0\phi_1 dx \tag{1.1.12}$$

Now,

$$\exp(-\lambda_1 t)\int_\Omega y_0\phi_1 dx > \int_\Omega y_1\phi_1 dx \tag{1.1.13}$$

if and only if

$$t < \frac{1}{\lambda_1}\ln\left[\frac{\int_\Omega y_0\phi_1 dx}{\int_\Omega y_1\phi_1 dx}\right], \tag{1.1.14}$$

whence

$$T_{\min} \geq \frac{1}{\lambda_1}\ln\left[\frac{\int_\Omega y_0\phi_1 dx}{\int_\Omega y_1\phi_1 dx}\right] > 0, \tag{1.1.15}$$

where the last inequality is justified by the assumption $y_0 > y_1$.

Case $y_0 < y_1$.

This is the most delicate case. To show the waiting-time phenomenon, we consider the adjoint problem

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x), & \text{in } \Omega \end{cases} \tag{1.1.16}$$

where φ^0 is a given final datum in $L^2(\Omega)$.

By definition, $y \in L^2((0,T)\times\Omega)\cap C^0([0,T];H^{-1}(\Omega))$ is said to be solution by transposition to (1.1.8), with initial datum y_0 and control u if

$$\langle y(T,\cdot),\varphi^0\rangle + \int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}d\sigma(x)dt = 0, \tag{1.1.17}$$

for any final datum $\varphi^0 \in L^2$ for the adjoint problem.

By contradiction, let us suppose for any time $T > 0$, there exists a nonnegative control u, such that $y(T,\cdot) = y_1$, whence

$$\int_\Omega y_1\varphi^0 dx + \int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}d\sigma(x)dt = 0, \tag{1.1.18}$$

for any final datum $\varphi^0 \in L^2$.

Now, to conclude, it suffices to construct a final datum φ^0 and $T_0 > 0$, such that, the solution φ of the adjoint system with final datum φ^0 satisfies:

$$\begin{cases} \dfrac{\partial \varphi}{\partial n} \leq 0 & \text{on } (0, T_0) \times \partial\Omega \\ \displaystyle\int_{\Omega} y_1 \varphi^0 dx < 0, & \forall T \in [0, T_0). \end{cases} \tag{1.1.19}$$

Indeed, if the above relation is satisfied, (1.1.18) fails for any $T \in (0, T_0)$ and final datum φ^0.

In the proof of Theorem 4.5.1 in chapter 4, we construct the final datum as in figure 1.4. The corresponding solution to the adjoint problem 1.1.16 is depicted in figure 1.5.

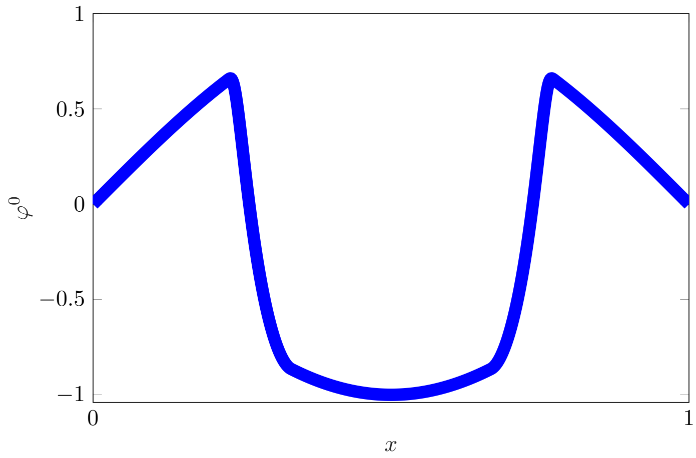


Figure 1.4: final datum for the adjoint system.

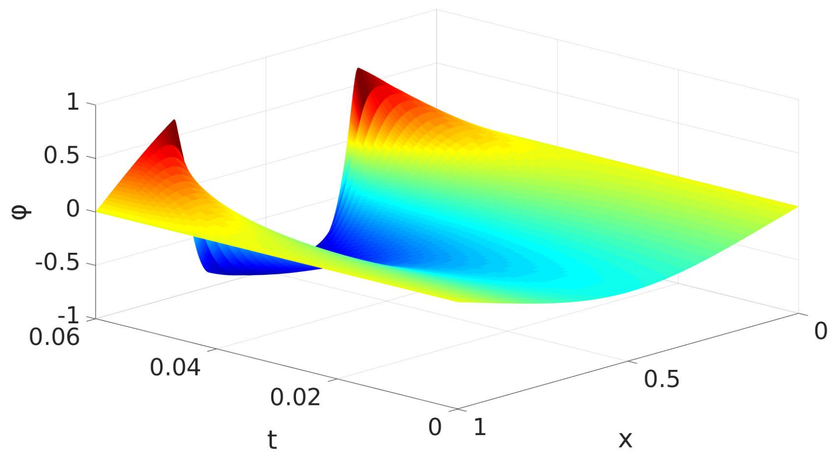


Figure 1.5: evolution of the adjoint heat equation with final datum φ^0.

By similar adjoint techniques, we prove that controllability holds in the minimal time by controls in the space of Radon measures (see Proposition 4.5.1 in chapter 4).

1.1.2 Controllability under positivity constraints of multi-d wave equations

As we anticipated, the "stair-case argument" is effective to control a broad class of PDEs. In this subsection, we will implement it for the wave equation. Furthermore, due to the time-reversibility of the wave equation, we are able to reach the zero state as final target by a nonnegative control. This null controllability by nonnegative controls is also a consequence of the absence of the comparison principle, which was an obstruction to reach zero for heat equations.

Note however that for heat like equations is much easier to fulfill nonnegative constraints on the state, provided that initial datum and final target are positive steady states. Indeed, due to the comparison principle, the nonnegativity of the control suffices to guarantee the non-negativity of the state. For the wave equation, one needs to be more carefully to avoid that oscillations push the state beyond the constraints.

To fix ideas, we consider the pure wave equation, controlled everywhere from the boundary. The initial datum y_0 and the final target y_1 are steady states.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ y = u & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x),\ y_t(0,x) = 0 & \text{in } \Omega \end{cases} \tag{1.1.20}$$

The reader can find in chapter 5 the more general case of the wave equation with potential and a localized control in the interior or on the boundary.

We assume the *Geometric Control Condition* on $(\Omega, \partial\Omega, T^*)$ which asserts that all generalized bicharacteristics touch the boundary $\partial\Omega$ at a non diffractive point in time smaller than some $T^* > 0$. By now, it is well known in the literature that this geometric condition is equivalent to (unconstrained) controllability [12, 17].

We have the following steady state controllability result, which corresponds to Theorem 5.1.5 in chapter 5. Note that we keep both the control and the state nonnegative along the control process.

Theorem 1.1.4. *Let y_i be nonnegative steady states solution to*

$$-\Delta y_i = 0 \qquad in\ \Omega. \tag{1.1.21}$$

Then, if the time-horizon T is large enough, there exists a control $u \in L^\infty$, such that

- *the unique solution (y, y_t) to (1.1.20) with initial datum $(y_0, 0)$ and control u verifies $(y(T,\cdot), y_t(T,\cdot)) = (y_1, 0)$;*
- *$u \geq 0$ on $(0,T) \times \partial\Omega$;*
- *$y \geq 0$ in $(0,T) \times \Omega$.*

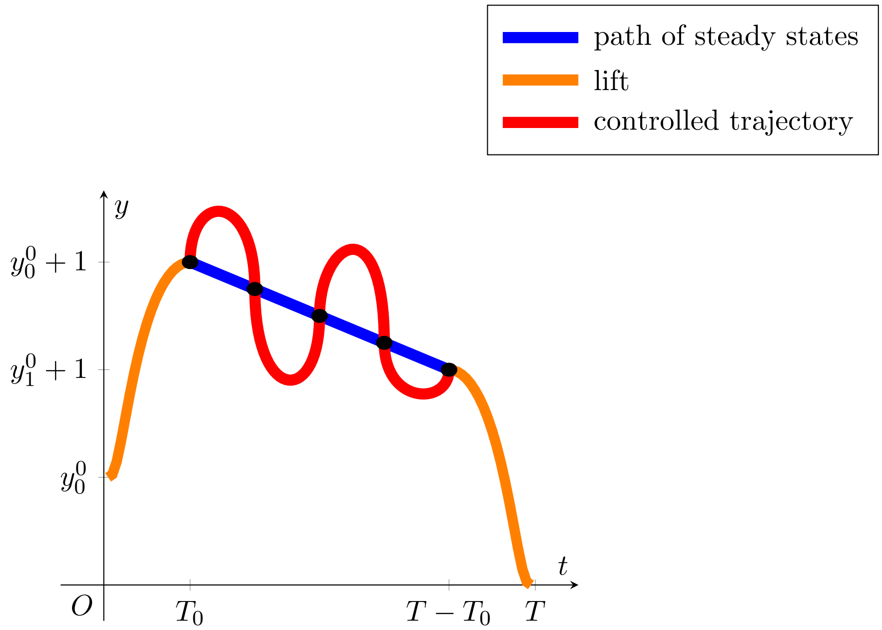


Figure 1.6: Control strategy to drive the solution of the wave equation to zero

The "stair-case argument" can be applied to satisfy both state and control constraint, by allowing only small oscillations around the path of steady states both at the level of the control and the state. However, the need of keeping both in a narrow tubular neighborhood of the path of steady states imposes a control time even larger than the case of control constraints only.

Note that we have required the steady states to be only non negative For heat equations zero was not reachable by nonnegative controls, because of the comparison principle. In the case of the wave equation, there is not such obstruction.

We are able to regain the room for oscillations needed to apply the "stair-case argument" even in case the final target $y_1^0 \equiv 0$, following the strategy (figure 1.6)

1. control the state (y, y_t) from $(y_0^0, 0)$ to $(y_0^0 + 1, 0)$ in time T_0;

2. employ the "stair-case method" in $[T_0, T - T_0]$ to link $(y_0 + 1, 0)$ and $(y_1^0 + 1, 0)$, taking T large enough;

3. drive the state (y, y_t) from $(y_1^0 + 1, 0)$ to $(y_1^0, 0)$ in $[T - T_0, T]$.

Part 3 can be accomplished, by defining the controlled solution

$$y(t, x) = y_1^0(x) + \tilde{y}(t + T - T_0, x), \tag{1.1.22}$$

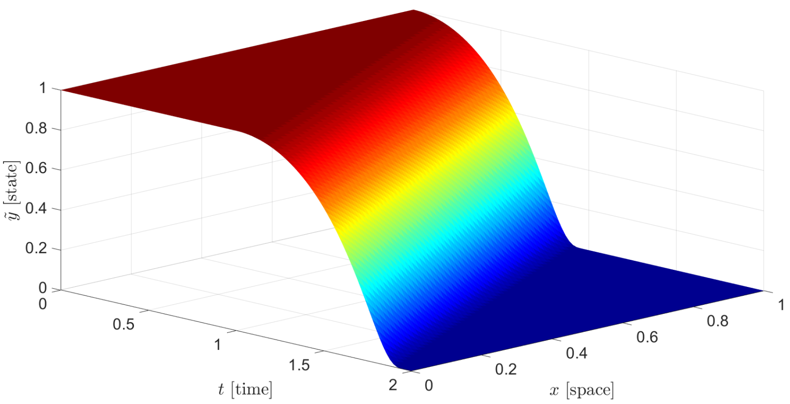


Figure 1.7: "lift"-solution to the wave equation joining the steady state $y_0^0 \equiv 1$, with the steady state $y_1^0 \equiv 0$ in time $T_0 = 2$. Both the state and the boundary control remain nonnegative along the control process.

where $\tilde{y}$ is a "lift"-solution (figure 1.7) of the problem

$$\begin{cases} \tilde{y}_{tt} - \Delta \tilde{y} = 0 & \text{in } (0, T_0) \times \Omega \\ \tilde{y} \geq 0 & \text{in } (0, T_0) \times \Omega \\ \tilde{y}(0, x) = 1, \ \tilde{y}_t(0, x) = 0 & \text{in } \Omega \\ \tilde{y}(T_0, x) = 0, \ \tilde{y}_t(T_0, x) = 0 & \text{in } \Omega, \end{cases} \tag{1.1.23}$$

with $T_0 > d$, where d is the diameter of Ω.

The "lift"-solution $\tilde{y}$ to (1.1.23) is of the form

$$\tilde{y}(t, x) = f(t + x_1), \tag{1.1.24}$$

where $f : \mathbb{R} \longmapsto \mathbb{R}$ is smooth and x_1 is the first component of $x \in \Omega$.
We illustrate how to build the profile f. By definition of diameter, there exists an interval $[a, b]$, with $|b - a| \leq d$ and

$$\left\{ x_1 \in \mathbb{R} \mid (x_1, x_2, \ldots, x_n) \in \overline{\Omega} \right\} \subseteq [a, b]. \tag{1.1.25}$$

Since $T_0 > d$, we have $a + T_0 > b$, whence there exists $f \in C^{\infty}(\mathbb{R}; [0, 1])$, such that

- $f(\xi) = 1$, for any $\xi \in [a, b]$;
- $f(\xi) = 0$, for any $\xi \in [a + T_0, b + T_0]$.

With the above f, $\tilde{y}$ defined in (1.1.24) is a solution to (1.1.23).

Part 1 can be handled likewise part 3, by using the time-reversibility of the wave equation.

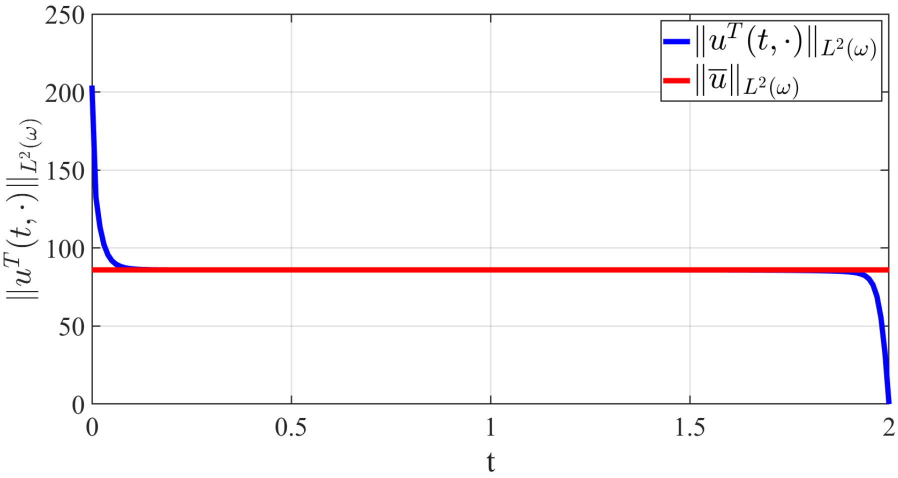


Figure 1.8: graph of the function $t \longrightarrow \|u^T(t)\|_{L^2(\omega)}$ (in blue) and $\|\overline{u}\|_{L^2(\omega)}$ (in red), where u^T denotes an optimal control for the time-evolution problem, whereas $\overline{u}$ stands for an optimal steady control.

1.2 The turnpike property in optimal control

The purpose of turnpike[1.1] theory is to establish a link between time-evolution control problems and its corresponding steady state version, as the time horizon $T \to +\infty$.

Generally speaking, we are given a time-evolution control problem $(OCP)_T$

$$\min_u J_T(u) = \int_0^T L(y,u)dt, \tag{1.2.1}$$

governed by the state equation:

$$\begin{cases} \frac{d}{dt}y = A(y) + B(u) & \text{in } (0,T) \\ y(0) = y_0. \end{cases} \tag{1.2.2}$$

The corresponding steady problem $(OCP)_s$ reads as

$$\min_{u_s} J_s(u_s) = L(y_s,u_s), \quad \text{with the constraint} \quad A(y_s) + B(u_s) = 0. \tag{1.2.3}$$

An optimal control for $(OCP)_T$ is denoted by u^T, while the optimal state is denoted by y^T. (u^T, y^T) is called the optimal pair for $(OCP)_T$. Besides, we indicate by $(\overline{u}, \overline{y})$ a minimizer of J_s. The pair $(\overline{u}, \overline{y})$ is known as the optimal pair for (1.2.3).

We assume the existence of an optimal pair (u^T, y^T) for $(OCP)_T$ as well as an optimal pair $(\overline{u}, \overline{y})$ for $(OCP)_s$.

[1.1] In American English, the word "turnpike" means "highway".

The "Turnpike Property" is verified if the time-evolution optima remain close to the steady optima up to some thin initial and final boundary layers (figure 1.8). More precisely, given any time-evolution optimal pair (u^T, y^T), we require the existence of a steady optimal pair $(\overline{u}, \overline{y})$ and a T-independent time $\tau \geq 0$ such that:

John Von Neumann

1. in the interval $[0, \tau]$, the optimal pair (u^T, y^T) moves approximately from $(u^T(0), y^T(0))$ to $(\overline{u}, \overline{y})$;
2. for a long time arc $[\tau, T - \tau]$, (u^T, y^T) remains close to $(\overline{u}, \overline{y})$;
3. for a final arc $[T - \tau, T]$, (u^T, y^T) moves approximately from $(\overline{u}, \overline{y})$ to $(u^T(T), y^T(T))$.

If the above property holds, the optimal pair $(\overline{u}, \overline{y})$ is called the *turnpike*. In econometric literature, the optimal state $\overline{y}$ is named *Von Neumann point*.

Note that in $(OCP)_T$ an initial condition for the state is imposed. Moreover, first order Optimality Conditions for $(OCP)_T$ leads to a final condition for the control. Hence, we cannot expect a proximity of (u^T, y^T) to $(\overline{u}, \overline{y})$, for any time $t \in [0, T]$. For example, if the initial datum y_0 is away from $\overline{y}$, in a thin time interval $[0, \tau]$, y^T must be far away from $\overline{y}$. Besides, if the norm of $\overline{u}$ is large, then u^T is away from $\overline{u}$ in an arc $[T - \tau, T]$.

This is a classical topic in mathematical control, econometrics and engineering. A pioneer on the topic has been John von Neumann [142]. The econometrician Paul Samuelson, Nobel Prize winner in 1970, introduced the concept of turnpike in the seminal book [44]:

...if we are planning long-run growth, no matter where we start and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

Paul Samuelson

An extensive literature is available on the topic. In econometrics the topic has been widely investigated by several scholars including P. Samuelson and L.W. McKenzie [131, 94, 102, 103, 22, 69]. Long time behaviour of optimal control problems have been studied by P. Kokotovic and collaborators in connection with Riccati theory and Hamilton-Jacobi equation [145, 6]. The same topic has been investigated in the Calculus of Variations by R.T. Rockafellar employing convex analysis

[123] and by A. Rapaport and P. Cartigny using Hamilton-Jacobi theory [120, 121]. A.J. Zaslavski wrote a book [148] on the topic. A turnpike-like asymptotic simplification have been obtained in the context of optimal design of the diffusivity matrix for the heat equation [4]. In the papers [38, 62, 61, 136], the concept of (measure) turnpike is related to the dissipativity of the control problem.

Recent papers on long time behaviour of Mean Field games [20, 21, 114] motivated new research on the topic. A special attention have been paid in providing an exponential estimate, like:

$$\|u^T(t)-\overline{u}\|_U+\|y^T(t)-\overline{y}\|_H\leq K\left[\exp\left(-\mu t\right)+\exp\left(-\mu(T-t)\right)\right], \qquad \forall t\in[0,T], \quad (1.2.4)$$

for some T-independent constants K and $\mu>0$. Note that $e^{-\mu t}$ is small far from $t=0$, whereas $e^{-\mu(T-t)}$ is small far from $t=T$. Then, if the above inequality is satisfied, (u^T,y^T) remains *exponentially* close to $(\overline{u},\overline{y})$, up to thin initial and final layers. Such estimates have been obtained by A. Porretta and E. Zuazua in [115] for linear quadratic control problems, governed by ODEs or PDEs. These results have later been extended in [138, 116, 147, 137, 64, 63] to control problems governed by a nonlinear state equation and applied to optimal control of the Lotka-Volterra system [76]. Recently turnpike property have been studied around nonsteady trajectories [137, 52]. In the reference [83], turnpike results have been connected to asymptotic properties of Hamilton-Jacobi equations.

Once we know a control system satisfy the turnpike property, we can construct quasi-optimal turnpike strategies as in figure 1.9:

1. in a short time interval $[0,\tau]$ drive the state from the initial configuration y_0 to the turnpike $\overline{y}$;
2. in a long time arc $[\tau,T-\tau]$, remain on $\overline{y}$;
3. in a short final arc $[T-\tau,T]$, use to control to match the required terminal condition at time $t=T$.

In general, the corresponding control and state are not optimal, being not smooth. However, they are easy to construct and, because of the turnpike effect, they are quasi-optimal.

1.2.1 The turnpike property in semilinear control

Turnpike theory is by now well understood for linear-quadratic problems, both in ODE control and in PDE control. When the governing state equation is nonlinear, available turnpike results are local [138, 116, 137].

The goal of chapter 6 is to develop global turnpike results for optimal control problems governed by a nonlinear equation. To fix ideas, we consider the semilinear control problem

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u)=\frac{1}{2}\int_0^T\int_\omega|u|^2dxdt+\frac{\beta}{2}\int_0^T\int_{\omega_0}|y-z|^2dxdt, \quad (1.2.5)$$

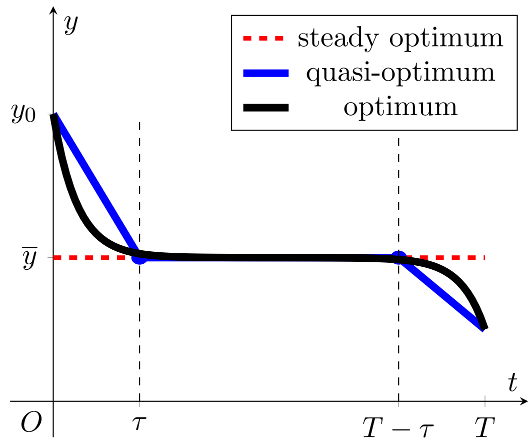


Figure 1.9: quasi-optimal turnpike strategies

where

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.2.6}$$

From now on, Ω is a regular bounded open subset of $\mathbb{R}^n$, with $n = 1, 2, 3$. The nonlinearity f is C^3 nondecreasing, with $f(0) = 0$, thus guaranteeing the existence of solutions to (1.2.6), globally time [10, chapter 5]. The control acts in the subdomain $\omega \subseteq \Omega$ and the state is observed in the subregion $\omega_0 \subseteq \Omega$. The target z is bounded. The weighting parameter $\beta \geq 0$ regulates the relevance of the state term in the cost functional (1.2.5).

The existence of an optimal control u^T for (1.2.5) follows from Proposition 3.1.1 in chapter 3. The corresponding optimal state is denoted by y^T.

The steady state version of the above problem reads as:

$$\min_{u_s\in L^2(\omega)} J_s(u_s) = \frac{1}{2}\int_\omega |u_s|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y_s - z|^2 dx, \tag{1.2.7}$$

where

$$\begin{cases} -\Delta y_s + f(y_s) = u_s\chi_\omega & \text{in } \Omega \\ y_s = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.8}$$

For any given control $u_s \in L^2(\omega)$, there exists a unique state $y_s \in H^2(\Omega)\cap H_0^1(\Omega)$ solution to (1.2.8).

The existence of an optimal control $\overline{u}$ for (1.2.7) follows from Proposition 3.1.1 in chapter 3. The corresponding optimal state is denoted by $\overline{y}$. As we shall see in chapter 7, the uniqueness of the minimizer fails for some large targets z. In case the nonuniqueness occurs, a question arises: if the turnpike property is satisfied, which minimizer for (1.2.8)-(1.2.7) attracts the optimal solutions to (1.2.6)-(1.2.5)?

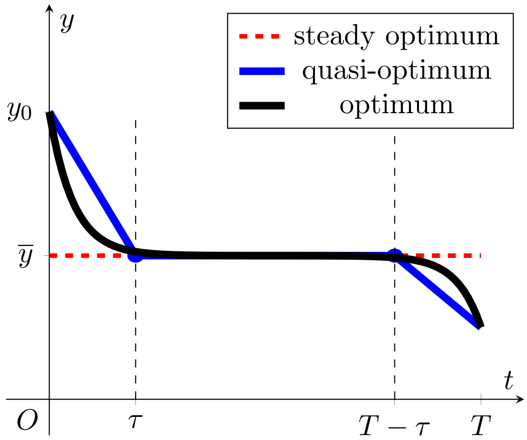


Figure 1.10: quasi-optimal turnpike strategies

The target is assumed to be small, whence the steady optimum is unique (see [116, subsection 3.2]). In case the target is large, as long as we know, the validity of the turnpike property is still an open problem. However as in figure 1.10, using controllability results, one could actually build quasi-optimal trajectories that simply use the steady optimum as a transition intermediate state, for very long time, to which one should add the first and the final controlled arcs.

The starting point is the local analysis carried out by A. Porretta and E. Zuazua in [116], which leads to the existence of a solution to the Optimality System fulfilling the turnpike property, under smallness conditions on the initial datum y_0 and the target z. Our main purpose is to

1. prove that in fact the turnpike property is satisfied by the optima;

2. remove the smallness condition on the initial datum.

We state our main result.

Theorem 1.2.1. *Consider the control problem* (1.2.6)-(1.2.5)*. Suppose the nonlinearity f is C^3 nondecreasing, with $f(0) = 0$. Let u^T be a minimizer of* (1.2.5)*. There exists $\rho > 0$ such that for every $y_0 \in L^\infty(\Omega)$ and z verifying*

$$\|z\|_{L^\infty} \leq \rho, \tag{1.2.9}$$

we have

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t} + e^{-\mu(T-t)}\right], \qquad \forall t \in [0, T], \tag{1.2.10}$$

the constants K and $\mu > 0$ being independent of the time horizon T.

Our strategy to prove Theorem 1.2.1 is the following.

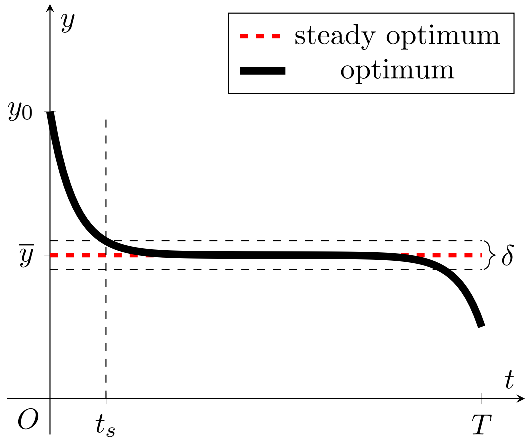


Figure 1.11: global-local argument

1. Deduce an L^∞ bound for the norm of the optimal control, uniform in the time horizon $T > 0$ (Lemma 6.2.1 in subsection 6.2.1);

2. Prove turnpike for *small data* and *small targets*. Note that, in [116, Theorem 1 subsection 3.1], the authors prove the existence of a solution to the Optimality System enjoying the turnpike property. In this preliminary step, for *small data* and *small targets*, we prove that any optimal control verifies the turnpike property (Lemma 6.2.2 in subsection 6.2.1);

3. For *small targets* and *any data*, show that $\|y^T(t)\|_{L^\infty(\Omega)}$ is small for t large (subsection 6.2.2). This is done by estimating the critical time t_s needed to approach the turnpike (figure 1.11);

4. Conclude by concatenating the two former steps (subsection 6.2.2).

Let us briefly sketch the proof of 3, the existence of τ upper bound for the minimal time needed to approach the turnpike t_s.
Suppose, by contradiction, that the critical time t_s to approach the turnpike is very large. Accordingly, the time-evolution optimal strategy obeys the following plan:

1. stay away from the turnpike for long time;

2. move close to the turnpike;

3. enjoy a final time-evolution performance, cheaper than the steady one.

We plug the steady optimum $\overline{u}$ in the time-evolving functional, getting the inequality

$$J_T\left(u^T\right) \leq J_T\left(\overline{u}\right). \tag{1.2.11}$$

Now, since the running target z is small, $\frac{1}{T}J_T(\overline{u})$ is also small. Then, in phase 1, with respect to the steady performance, an extra cost is generated, which should be regained in phase 3. At this point, we realize that this is prevented by validity of the local turnpike property. Indeed, once the time-evolution optima approach the turnpike at some time t_s, the optimal pair satisfies the turnpike property for larger times $t \geq t_s$. Hence, for $t \geq t_s$, the time-evolution performance cannot be significantly cheaper than the steady one. Accordingly, we cannot regain the extra-cost generated in phase 1, so obtaining a contradiction.

1.2.2 Non-uniqueness of minimizers for semilinear optimal control problems

In the previous section (referred to chapter 6), we have considered the semilinear control problem (1.2.8)-(1.2.7) for small targets. We have presented our contributions to turnpike. But as we emphasized, the results we have so far are limited to the case where the target is small. This smallness condition of the target assures that the optimal controls and controlled states are unique for the steady problem.

It has been recognized in the literature that one of the challenging problems in optimal control of elliptic and parabolic problems is the uniqueness, or the lack of, optimal control and controlled states, when the targets are large.

The main result of chapter 7 is the construction of an example of elliptic optimal control problem for which the uniqueness is lost when the target is large.

To fix ideas, in this introduction, we consider the case of cubic nonlinearity, leaving to chapter 7 the case of a more general increasing nonlinearity. We illustrate first the counterexample in boundary control and then the counterexample in internal control.

In this section, to simplify the notation, we have dropped the s subscript to denote steady controls/states.

1.2.2.1 Boundary control

We consider the optimal control problem

$$\min_{u\in L^2(\partial B(0,R))} J_s(u) = \frac{1}{2}\int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |y-z|^2 dx, \tag{1.2.12}$$

where

$$\begin{cases} -\Delta y + y^3 = 0 & \text{in } B(0,R) \\ y = u & \text{on } \partial B(0,R). \end{cases} \tag{1.2.13}$$

Here, $B(0,R)$ is the ball in $\mathbb{R}^n$, $n = 1,2,3$, of radius R, centred at the origin. The target z is of class $L^2(B(0,R))$ and the weighting parameter β is strictly positive.

Note that our result holds for any value of the radius R of the domain. By scaling the problem can be always be reduced to the case $R = 1$.

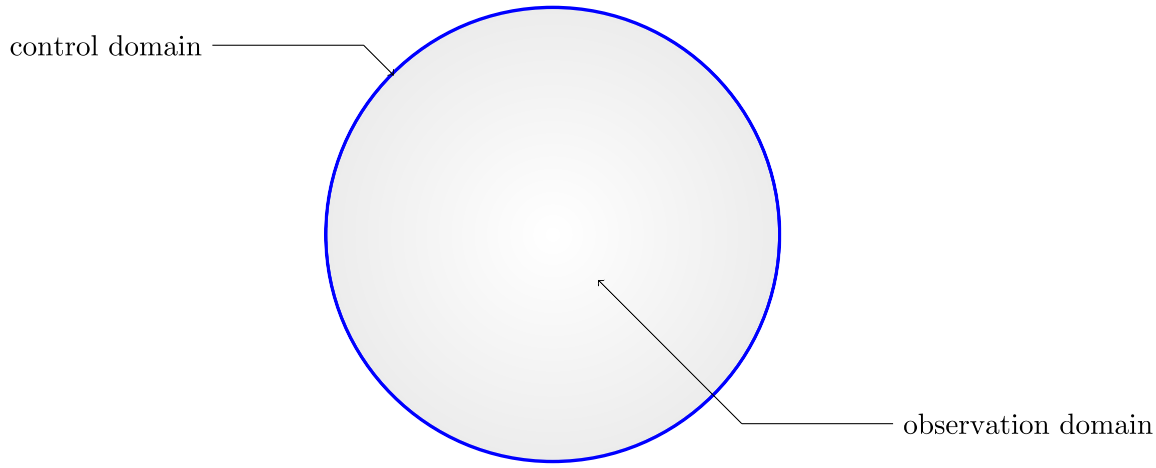


Figure 1.12: control and observation domains. The control domain is the blue boundary of the ball.

Theorem 1.2.2. *Consider the control problem* (1.2.13)-(1.2.12)*. For any* $R > 0$ *and* $\beta > 0$*, there exists a target* $z \in L^\infty(B(0,R))$ *such that the functional* J_s *defined in* (1.2.12) *admits (at least) two global minimizers.*

The main steps for the proof of our nonuniqueness result are the following:

Step 1 **reduction to constant controls**: by choosing radial targets and by using the rotational invariance of $B(0,R)$, we reduce to the case the control set consists of constant controls;

Step 2 **existence of two local minimizers**: we look for a target such that there exists two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the steady functional J_s;

Step 3 **existence of two global minimizers**: by the former step and a bisection argument, we prove the existence of a target z such that J_s admits two *global* minimizers.

The constructed target for nonuniqueness is a step function, as in figure 1.13.

1.2.2.2 Internal control

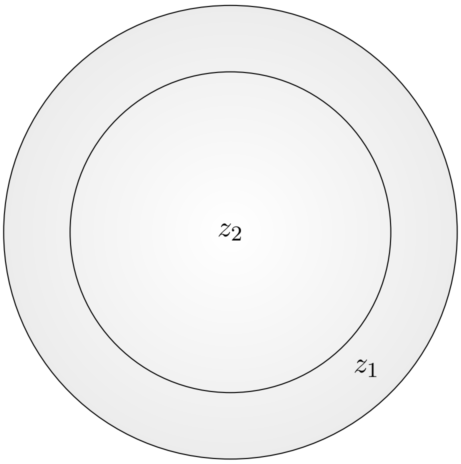
We consider the elliptic optimal control problem

$$\min_{u \in L^2(B(0,r))} J_s(u) = \frac{1}{2}\int_{B(0,r)} |u|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |y-z|^2 dx, \tag{1.2.14}$$

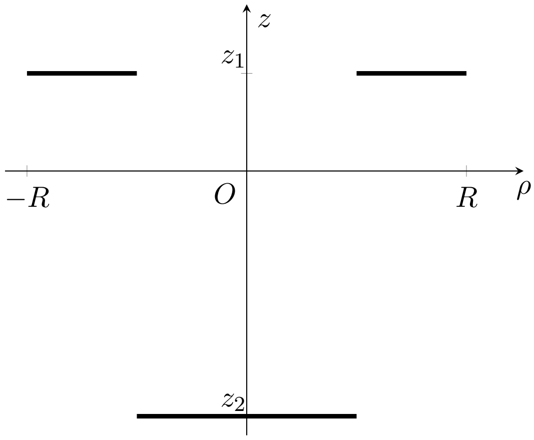
where

$$\begin{cases} -\Delta y + y^3 = u\chi_{B(0,r)} & \text{in } B(0,R) \\ y = 0 & \text{on } \partial B(0,R). \end{cases} \tag{1.2.15}$$

Here, $B(0,R)$ is a ball of $\mathbb{R}^n$, $n = 1,2,3$, centered at the origin of radius R. The control acts in the ball $B(0,r)$, with $r \in (0,R)$. The observation domain is $B(0,R) \setminus B(0,r)$ (see figure 1.14). The target z is of class $L^2(B(0,R) \setminus B(0,r))$ and the weighting parameter β is strictly positive.



(a) level curves of the constructed radial target



(b) constructed radial target versus the radius ρ

Figure 1.13: target yielding nonuniqueness in boundary control

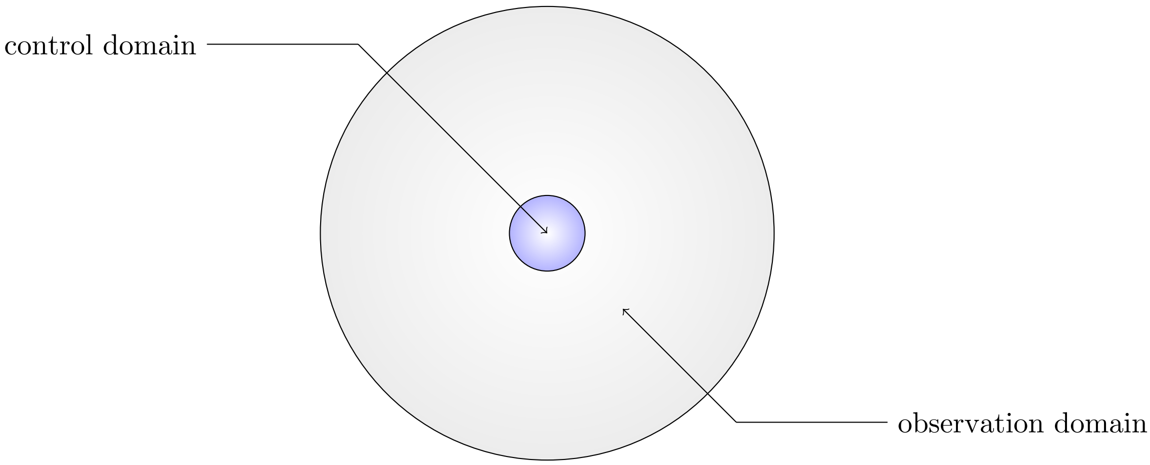


Figure 1.14: control and observation domains

Theorem 1.2.3. *Consider the control problem* (1.2.15)-(1.2.14). *For any* $0 < r < R$ *and* $\beta > 0$*, here exists a target* $z \in L^{\infty}(B(0,R) \setminus B(0,r))$ *such that the functional* J_s *defined in* (1.2.14) *admits (at least) two global minimizers.*

The proof is similar to the boundary control case, with the difference that in this case, by rotational invariance of control and observation domains, we reduce to radial instead of constant controls. See chapter 7 for further details.

1.2.3 Rotors imbalance suppression by optimal control

Chapter 8 illustrates the outcome of secondment in the company "Marposs S.p.A.". The content has been submitted [59]. During the secondment, we applied the turnpike/stabilization theory to a problem of vibrations suppression for an imbalanced rotor. We are given an imbalanced rotor together with two balancing heads. Each balancing head is made of two balancing masses. An initial configuration of the balancing masses is given. Our goal is to determine four angular trajectories steering the masses from their initial configuration to a steady configuration, where the balancing masses compensate the imbalance. Note that, differently from the classical wheel balancing machines, our balancing device rotates together with the rotor and the rotor is moving while the balancing procedure is accomplished. This motivates us to formulate the problem as a dynamic optimization problem so that transient responses are also taken into account.

In the spirit of turnpike/stabilization theory, the determined open-loop optimal trajectories stabilizes the system towards some optimal steady configuration, as time $t \to +\infty$. This is proved by Łojasiewicz inequality [99, Théorème 2 page 62]. In case the imbalance is below a computed threshold, the convergence occurs exponentially fast. This is shown by the stable manifold theorem applied to the Pontryagin optimality system.

Rotor balancing is a classical problem in engineering. Indeed, often times, the mass of the rotor is not symmetrically distributed around the axis, due to wear, damage and other reasons. This leads to dangerous vibrations, which seriously affect the performance of the rotor.

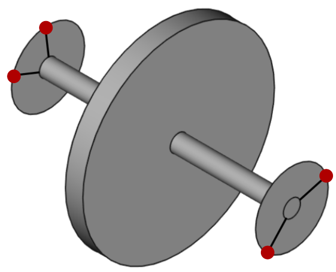


Figure 1.15: the rotor and the balancing device are represented. In the special case represented, the balancing heads are located at the endpoints of the spindle. The four balancing masses (two for each balancing head) are drawn in red.

Vibrations are a significant reason of concern in rotor dynamics. For instance, grinding machines often get deteriorated during their operational life-cycle. This leads to dangerous imbalance vibrations, which affects their performance while shaping objects (see, for instance, [68, 75, 149, 30]). Imbalance is a significant concern for wind turbines as well. In this case, the imbalance may affect the efficiency of power production and the life-cycle of the turbine. If the vibrations become too large, the turbine may collapse. For this reason, vibration detection and correction systems have been developed (see the U.S. patent [80]). Balancing devices have been developed to stabilize CD-ROM drives and washing machines (see [32, 119, 28, 29, 82]). Another classical topic in engineering is car's wheels balance. Indeed, easily the wheels can go out of alignment from encountering potholes and/or striking raised objects. Misalignment may cause irregular wear of the tyres and suspensions components may be damaged as well. For this reason, refined machines have been designed for wheel balancing (see, e.g., [46, chapter 44]). The classical engineering literature on imbalance suppression is concerned with imbalance detection and/or imbalance correction.

In chapter 8, the imbalance correction problem is addressed. (The imbalance is an input.) In our model, an imbalanced rotor rotates about a fixed axis at a constant angular velocity. The rotor is affected by dynamical imbalance. Namely, the imbalance exerts both a force and a torque on the rotation axle. Two balancing heads are mounted integrally with the rotor, thus rotating together with the rotor. In each balancing head, we have two balancing masses, which rotates in a plane orthogonal to the rotation axis (figure 1.15).
Given an initial configuration of the balancing masses, our goal is to determine four optimal trajectories for the four balancing masses to compensate the imbalance. As we mentioned, the balancing device rotates together with the rotor. Then, in view of minimizing the vibrations, we are interested in:

- drive the system to a balanced configuration in large time $t > T$;
- minimize the imbalance in the correction process, for $t \in [0, T]$.

For this reason the problem is formulated as a dynamical optimization problem, so that transient responses are also taken into account.

A control problem is formulated. We employ an open-loop control strategy to move the balancing heads from their initial configuration to a steady configuration, where they compensate the imbalance of the rotor. First of all, viewing the problem in the framework of the Calculus of Variations, the existence of the optimum is proved and the related Euler-Lagrange optimality conditions have been derived. By Łojasiewicz inequality, the stabilization of the optimal trajectories towards steady optima is proved in any condition. In case the imbalance is below a given threshold, we provide an exponential estimate of the stabilization. The estimate is obtained by seeing the problem as an optimal control problem, thus writing the Optimality Condition as a first order Pontryagin system. In this context, we prove the hyperbolicity of the Pontryagin system around steady optima in order to apply the stable manifold theorem (see [109, Corollary page 115] and [129]). Our conclusions fit in the general framework of Control Theory and in particular of stabilization, turnpike and controllability (see e.g. [54, 134, 153, 115, 138, 151]).

The position of the four balancing masses is given by the vector $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2) \in \mathbb{T}^4 = (\mathbb{S}^1)^4$.

We formulate the problem in the framework of the Calculus of Variation, with Lagrangian

$$L : \mathbb{T}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}, \quad L(\Phi, \psi) := \frac{1}{2}\left[\|\psi\|^2 + \beta \hat{G}(\Phi)\right],$$

where $\beta > 0$ is a parameter to be fixed and $\hat{G} = G - \inf G$, G being an imbalance indicator introduced in (8.2.6) (chapter 8). In the above definition, we have a trade-off between the cost of controlling the system to a stable regime and the velocity of the balancing masses, with respect to the rotor. If β is large, the primary concern for the optimal strategy is to minimize the cost of the control, while if β is small our priority is to minimize the velocities.

Now, let $\Phi_0 \in \mathbb{T}^4$ be an initial configuration. We introduce the space of admissible trajectories

$$\mathscr{A} := \left\{\Phi \in H^1_{loc}([0, +\infty); \mathbb{T}^4) \;\middle|\; \Phi(0) = \Phi_0, \quad \text{and} \quad L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)\right\},$$

where $\mathbb{T} := \mathbb{S}^1$ is the one dimensional sphere. Note that the requirement $L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)$ is equivalent to

$$\dot{\Phi} \in L^2(0, +\infty) \quad \text{and} \quad G(\Phi) - \inf G \in L^1(0, +\infty).$$

Our goal is to minimize the functional $J : \mathscr{A} \longrightarrow \mathbb{R}$

$$J(\Phi) := \frac{1}{2}\int_0^\infty \left[\|\dot{\Phi}\|^2 + \beta \hat{G}(\Phi)\right] dt. \tag{1.2.16}$$

Let F_1 and F_2 be the forces exerted by the imbalance in the two balancing planes. Let m_1 and m_2 be the mass of the two balancing masses and let r_1 and r_2 be their distance to the axle. We state now our main result.

Proposition 1.2.1. *Consider the functional* (1.2.16)*. For* $i = 1, 2$*, set*

$$c^i := \frac{1}{2 m_i r_i \omega^2}(F_{i,x}, F_{i,y}) \tag{1.2.17}$$

where ω is the velocity of rotation of the rotor. Then,

1. *there exists $\Phi \in \mathscr{A}$ minimizer of J;*

2. *$\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ is C^∞ smooth and, for $i = 1, 2$, the following Euler-Lagrange equations are satisfied, for $t > 0$*

$$\begin{cases} -\ddot{\alpha}_i = \beta \cos(\gamma_i)\left[-c_1^i \sin(\alpha_i) + c_2^i \cos(\alpha_i)\right] \\ -\ddot{\gamma}_i = -\beta \sin(\gamma_i)\left[c_1^i \cos(\alpha_i) + c_2^i \sin(\alpha_i) - \cos(\gamma_i)\right] \\ \alpha_i(0) = \alpha_{0,i}, \;\; \gamma_i(0) = \gamma_{0,i}, \;\; \dot{\Phi}(T) \underset{T \to +\infty}{\longrightarrow} 0. \end{cases} \tag{1.2.18}$$

(3) for any optimal trajectory Φ for (1.2.16)*, there exists $\overline{\Phi} \in \mathscr{S} = zeros\left(\widehat{G}\right)$ such that*

$$\Phi(t) \underset{t \to +\infty}{\longrightarrow} \overline{\Phi}, \tag{1.2.19}$$

$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{1.2.20}$$

and

$$\left|\hat{G}\left(\Phi(t)\right)\right| \underset{t\to+\infty}{\longrightarrow} 0. \tag{1.2.21}$$

If, in addition

$$m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \text{and} \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}, \tag{1.2.22}$$

we have the exponential estimate for any $t \geq 0$

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |G\left(\Phi(t)\right)| \leq C \exp\left(-\mu t\right), \tag{1.2.23}$$

with C, $\mu > 0$ *independent of* t.

We perform some numerical simulations, minimizing the discretized functional by the expert interior-point optimization routine `IpOpt` (see [143]), the modeling language being `AMPL` (see [55]).

In figures 1.16, 1.17, 1.18 and 1.19, we plot the computed optimal trajectory for (1.2.16), with initial datum $\Phi_0 = (\alpha_{0,1}, \gamma_{0,1}; \alpha_{0,2}, \gamma_{0,2}) \coloneqq (2.6, 0.6, 2.5, 1.5)$. We choose F, N and m_i, such that the condition (1.2.22) is fulfilled. The exponential stabilization proved in Proposition 1.2.1 emerges. In figure 1.20, we depict the imbalance indicator versus time along the computed trajectories. As expected, it decays to zero exponentially.

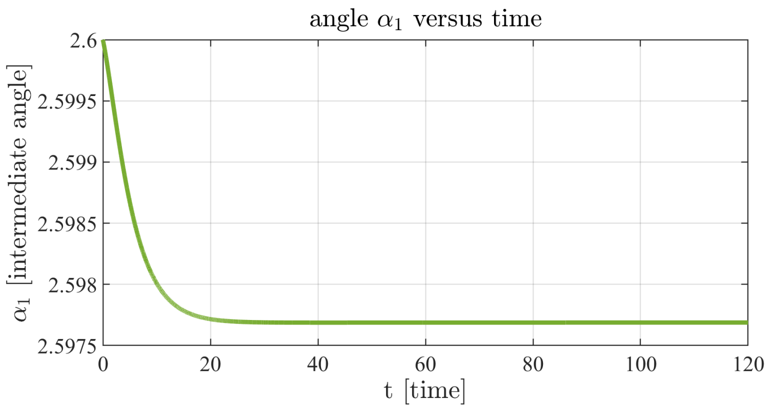


Figure 1.16: intermediate angle α_1 versus time

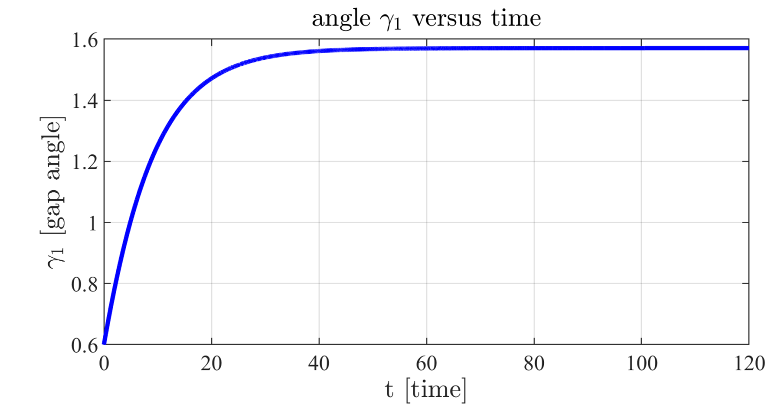


Figure 1.17: gap angle γ_1 versus time

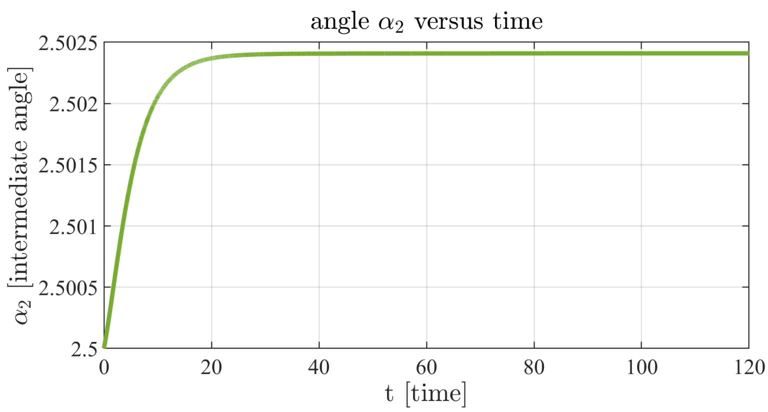


Figure 1.18: intermediate angle α_2 versus time

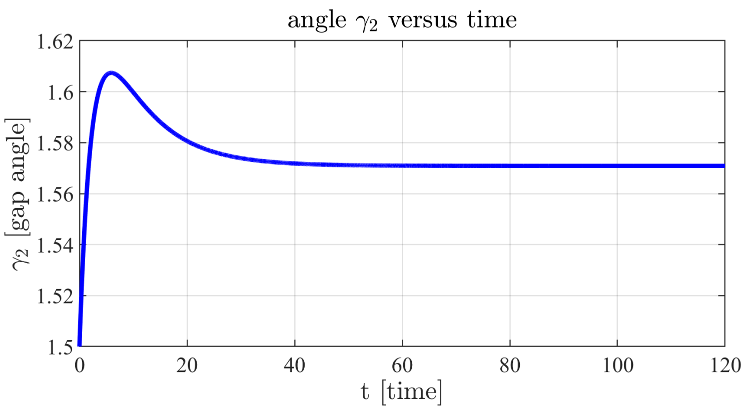


Figure 1.19: gap angle γ_2 versus time

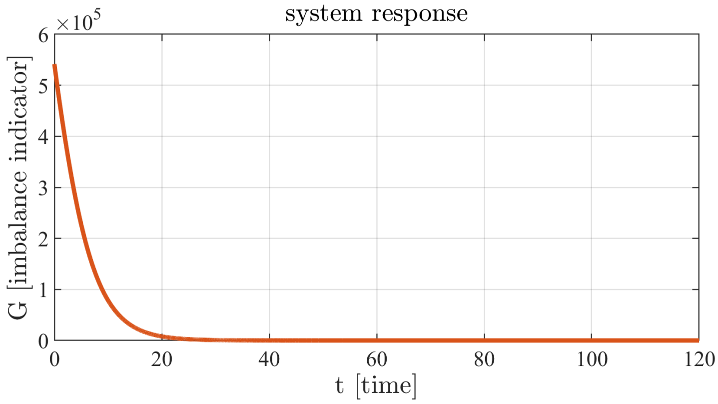


Figure 1.20: the imbalance indicator G along the computed trajectory versus time.

1.3 Conclusions

In this thesis we have analyzed the long time behaviour of some control problems. In chapter 4 (corresponding to [111]), we have given sufficient conditions for controllability under positivity constraints of semilinear heat equations. We have shown the positivity constraints lead to a waiting time phenomenon, namely, the minimal controllability time is strictly positive. In chapter 5 (corresponding to [112]), similar results have been proved for wave-like equations under positivity constraints, with the additional possibility of reaching the zero state at the final time. In chapter 6, we obtain global turnpike results for an optimal control problem governed by a semilinear heat equation. The running target is required to be small, but the initial data for the controlled state equation can be chosen arbitrarily large. If the target is large, nonuniqueness issues occur at the level of the steady problem, as shown in chapter 7. In chapter 8 (corresponding to [59]), turnpike/stabilization theory is applied to an industrial problem of rotor balancing.

1.3.1 Open problems

We formulate now some interesting open problems.

1.3.1.1 Controllability of the obstacle problem

A striking open problem is the boundary controllability of the obstacle problem. This is related to chapter 4, because the solution to the obstacle problem can be seen as the limit of solutions to a family of semilinear elliptic equations [56, 8]. The results in this chapter apply to the penalized problem, but passing to the limit to get relevant results for the obstacle problem is an open topic.

In the literature, we can find controllability-type results on related topics. For instance, [5] deals with the controllability of the one dimensional wave equation, with obstacle at right endpoint of the space interval. The control acts on the left endpoint of the space interval. The null controllability is shown, by combining D'Alembert's formula and a fixed point argument. Moreover, in [41] the approximate controllability of a parabolic variational inequality has been proved.

Furthermore, a rich literature is available on optimal control of the obstacle problem [31, 105, 9, 1, 104, 78, 79].

One of the difficulties in control problems with obstacle is the lack of differentiability of the control-to-state map. We show how differentiability fails in the following example.

Consider the obstacle $\psi(x) = x(1-x)$ and the convex set

$$\begin{aligned} K_{(a,b)} &:= \{y \in H^1(0,1) \mid y(0) = a,\ y(1) = b \\ &\quad \text{and } \ y(x) \geq x(1-x) \quad \forall x \in [0,1]\}, \end{aligned} \tag{1.3.1}$$

with a and b positive real numbers. For any $a \geq 0$ and $b \geq 0$, we consider the obstacle problem

$$\min_{y \in K_{(a,b)}} \int_0^1 |y_x|^2 dx. \tag{1.3.2}$$

The functional $J(y) = \int_0^1 |y_x|^2 dx$ is coercive, strictly convex and continuous. Then, (1.3.2) admits a unique solution $y_{(a,b)}$ [16, Theorem 5.6 section 5.3]. The solution can be computed explicitly. For instance, if both $0 < a < \frac{1}{4}$ and $0 < b < \frac{1}{4}$, we have

$$y_{(a,b)}(x) = \begin{cases} \left(-2\sqrt{a}+1\right)\left(x-\sqrt{a}\right) - a + \sqrt{a} & x \in [0, \sqrt{a}) \\ x(1-x) & x \in [\sqrt{a}, 1-\sqrt{b}) \\ \left(2\sqrt{b}-1\right)\left(x-1+\sqrt{b}\right) - b + \sqrt{b} & x \in [1-\sqrt{b}, 1], \end{cases} \tag{1.3.3}$$

while if both $a > \frac{1}{4}$ and $b > \frac{1}{4}$, the solution is a segment $y_{(a,b)}(x) = (b-a)x + a$.

The real numbers a and b are the controls and $y_{(a,b)}$ is the corresponding state. We introduce the control-to-state map

$$G : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow L^2(0,1), \quad (a,b) \longmapsto y_{(a,b)}. \tag{1.3.4}$$

Take $(a,b) = \left(\frac{1}{4}, \frac{1}{4}\right)$. We are going to show that G is not differentiable at $\left(\frac{1}{4}, \frac{1}{4}\right)$ in the direction $v = (1,1)$. Indeed, on the one hand, for any $x \in (0,1)$, the right derivative

$$\lim_{h \to 0^+} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} = 1. \tag{1.3.5}$$

On the other hand, the left derivative

$$\begin{aligned} \lim_{h \to 0^-} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} &= -2x+1, \quad 0 < x < \frac{1}{2} \\ \lim_{h \to 0^-} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} &= 2x-1, \quad \frac{1}{2} < x < 1. \end{aligned} \tag{1.3.6}$$

Hence, left and right derivatives differ, whence G is not differentiable at $\left(\frac{1}{4}, \frac{1}{4}\right)$ in the direction $v = (1,1)$.

1.3.1.2 Sharper estimates of the minimal controllability time by adjoint method

In chapter 4 we have introduced a new adjoint technique to prove the positivity of the minimal controllability time under constraints (Theorem 1.1.3). Our techniques rely on the existence of a special final datum for the adjoint problem

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x), & \text{in } \Omega \end{cases} \tag{1.3.7}$$

such that

$$\begin{cases} \dfrac{\partial\varphi}{\partial n} \leq 0 & \text{on } (0,T_0)\times\partial\Omega \\ \displaystyle\int_\Omega y_1\varphi^0 dx < 0, & \forall T \in [0,T_0), \end{cases} \tag{1.3.8}$$

provided that T_0 is small.

We constructed a specific final datum which satisfies the aforementioned requirements. An interesting open problem is to find the final datum φ^0 satisfying (1.3.8) and maximizing the time T_0 while the normal derivative remains nonpositive. This would lead to a sharper lower estimate of the minimal controllability time under constraints.

Namely, set

$$\mathscr{F} := \left\{ \varphi^0 \in L^2(\Omega) \;\middle|\; \int_\Omega y_1\varphi^0 dx < 0 \right\}. \tag{1.3.9}$$

For any final datum $\varphi^0 \in \mathscr{F}$, define

$$T_{\varphi^0} := \sup\left\{ T > 0 \;\middle|\; \frac{\partial\varphi}{\partial n} \leq 0 \;\text{ on } (0,T)\times\partial\Omega \right\}, \tag{1.3.10}$$

where φ is the solution to

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{1.3.11}$$

The problem is to maximize

$$\mathscr{T} : \mathscr{F} \longrightarrow \mathbb{R}^+, \quad \varphi^0 \longmapsto T_{\varphi^0}. \tag{1.3.12}$$

1.3.1.3 The turnpike property in semilinear control for large targets

In chapter 6, we have proved that turnpike property in semilinear control is valid for small targets and any initial data for the state equation. It would be interesting to explore the case of large targets.

The problem can be formulated as follows. Consider the time-evolution control problem

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u)=\frac{1}{2}\int_0^T\int_\omega |u|^2dxdt+\frac{\beta}{2}\int_0^T\int_{\omega_0}|y-z|^2dxdt, \tag{1.3.13}$$

where

$$\begin{cases} y_t-\Delta y+f(y)=u\chi_\omega & \text{in } (0,T)\times\Omega\\ y=0 & \text{on } (0,T)\times\partial\Omega\\ y(0,x)=y_0(x) & \text{in } \Omega.\end{cases} \tag{1.3.14}$$

The nonlinearity f is C^3 and nondecreasing. The assumptions on the state equation are the same of chapter 6.

The steady problem reads as

$$\min_{u_s\in L^2(\omega)} J_s(u_s)=\frac{1}{2}\int_\omega |u_s|^2dx+\frac{\beta}{2}\int_{\omega_0}|y_s-z|^2dx, \tag{1.3.15}$$

where

$$\begin{cases} -\Delta y_s+f(y_s)=u_s\chi_\omega & \text{in } \Omega\\ y_s=0 & \text{on } \partial\Omega.\end{cases} \tag{1.3.16}$$

We denote by $(\overline{u},\overline{y})$ an optimal pair, where $\overline{u}$ is an optimal control and $\overline{y}$ the corresponding optimal state.

Conjecture 1. *Consider the control problem* (1.3.14)-(1.3.13). *Take any initial datum* $y_0\in L^\infty(\Omega)$ *and any target* $z\in L^\infty(\omega_0)$. *Let* u^T *be a minimizer of* (1.3.13). *There exists an optimal pair* $(\overline{u},\overline{y})$ *for* (1.3.16)-(1.3.15) *such that*

$$\|u^T(t)-\overline{u}\|_{L^\infty(\omega)}+\|y^T(t)-\overline{y}\|_{L^\infty(\Omega)}\leq K\left[e^{-\mu t}+e^{-\mu(T-t)}\right],\qquad \forall t\in[0,T], \tag{1.3.17}$$

the constants K *and* $\mu>0$ *being independent of the time horizon* T.

In chapter 7 we construct special large targets z, such that the optimal control for the steady problem (1.3.16)-(1.3.15) is not unique. For those targets, a question arises: if the turnpike property is satisfied, which minimizer for (1.3.16)-(1.3.15) attracts the optimal solutions to (1.3.14)-(1.3.13)?

According to some numerical simulations we have performed, independently of the initial datum y_0, only one steady optimum is chosen as turnpike. From the perspective of quasi-optimal turnpike strategies this could be related to the cost of getting into the steady state and the cost of getting to the terminal condition for the adjoint state $p^T(T)=0$. But all this requires further investigation.

Generally speaking a further investigation is required for the linearized optimality system determined in [116, subsection 3.1]. We introduce the problem. As in [116], consider the optimality system for (1.3.14)-(1.3.13)

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T \chi_\omega & \text{in } (0,T)\times\Omega \\ y^T = 0 & \text{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \text{in } \Omega \\ -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q^T = 0 & \text{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{1.3.18}$$

Pick any optimal pair $(\overline{u},\overline{y})$ for (1.3.16)-(1.3.15). By the first order optimality conditions (see Proposition 3.1.7 in chapter 3), the steady optimal control reads as $\overline{u} = -\overline{q}\chi_\omega$, with

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3.19}$$

As in [116], we introduce the perturbation variables

$$\eta^T := y^T - \overline{y} \quad \text{and} \quad \varphi^T := q^T - \overline{q} \tag{1.3.20}$$

and we write down the linearized optimality system around $(\overline{u},\overline{y})$

$$\begin{cases} \eta_t^T - \Delta\eta^T + f'(\overline{y})\eta^T = -\varphi^T\chi_\omega & \text{in } (0,T)\times\Omega \\ \eta^T = 0 & \text{on } (0,T)\times\partial\Omega \\ \eta^T(0,x) = y_0(x) - \overline{y}(x) & \text{in } \Omega \\ -\varphi_t^T - \Delta\varphi^T + f'(\overline{y})\varphi^T = \left(\beta\chi_{\omega_0} - f''\left(\overline{y}\right)\overline{q}\right)\eta^T & \text{in } (0,T)\times\Omega \\ \varphi^T = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi^T(T,x) = -\overline{q}(x) & \text{in } \Omega. \end{cases} \tag{1.3.21}$$

As pointed out in [116, Theorem 1 in subsection 3.1], a key point is to check the validity of the turnpike property for the linearized optimality system (1.3.21). This is complicated because of the term $\beta\chi_{\omega_0} - f''\left(\overline{y}\right)\overline{q}$, whose sign is unknown for general large targets. Furthermore, in case of nonuniqueness of steady optimum, it would be interesting to compute the spectrum of the linearized system around any steady optima to check if among them one is a better attractor.

Capítulo 2

Introducción

Un sistema de control es un sistema dinámico cuyo comportamiento puede ser influenciado por un parámetro de entrada denominado *control*.

Los primeros arquetipos de sistemas de control fueron los sistemas de riego diseñados en Egipto y Mesopotamia hace aproximadamente 8000 años [54, 124]. Más tarde, los romanos emplearon estrategias de control inteligente para mantener constante el nivel del agua en sus acueductos. En el siglo XVII, el matemático y astrónomo holandés Christiaan Huygens abordó el problema del control de la velocidad para el diseño de relojes. Un trabajo similar fue llevado a cabo por el físico inglés Robert Hooke [54, 134]. Más tarde, se empleó un mecanismo de control de feedback para la construcción de reguladores centrífugos para molinos de viento.

La teoría de control nació en la revolución industrial a partir del siglo XVIII. Un avance notable fue la máquina de vapor (figura 2.1), inventada por James Watt en 1769, con el objetivo de mantener constante la velocidad de rotación a pesar de la carga variable por un regulador centrífugo. Cuando aumenta la velocidad, se levantan dos bolas activando algunas válvulas que permiten que el vapor escape, lo que ralentiza el proceso físico. En las décadas siguientes, un gran reto fue formular matemáticamente el sistema de regulación inventado por Watt. Después que el matemático y astrónomo George Airy hiciese los primeros intentos, el físico escocés James Clerk Maxwell realizó un análisis completo en 1868 para explicar el comportamiento errático de la máquina de vapor y sugerir soluciones. Una investigación similar fue realizada por los matemáticos Adolf Hurwitz y Edward John Routh. Estas fueron las primeras etapas del análisis de estabilidad en la teoría de control.

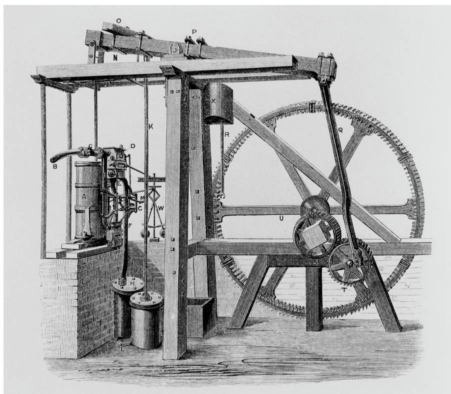


Figura 2.1: el motor a vapor de James Watt *Science Photo Library*

Hoy en día, la teoría de control es un área emergente en matemáticas aplicadas y ingeniería, que goza de interacciones fructíferas con varios campos, incluidos análisis de EDOs y EDPs, análisis numérico, análisis de estabilidad, cálculo estocástico, análisis de Fourier, procesamiento de señales, informática y ”black-box” optimization. Las aplicaciones se pueden encontrar en varias disciplinas, incluyendo ingeniería, medicina, biología y ciencias económicas. Gracias a las prestaciones crecientes de los ordenadores modernos, la teoría de control es aplicable a sistemas complejos, como robots o tráfico en una ciudad congestionada. Lo más probable es que en el futuro, los sistemas de control sean omnipresentes en el mundo real, gracias a nuevas interacciones interesantes entre el control y varias disciplinas como el machine learning. Para obtener más detalles, véase, por ejemplo, [54] y la rica bibliografía que contiene.

A continuación, damos una breve presentación de la teoría de control. Para más detalles, se remite el lector al capítulo 3 y a las referencias que ahí se dan. En términos generales, un sistema de control es definido por un *ecuación de estado*

$$\frac{d}{dt}y(t) = Ay(t) + Bu(t), \tag{2.0.1}$$

donde

- y es el *estado*, la incógnita del sistema a controlar. Típicamente, y es un elemento de un espacio parabólico $L^2(0,T;H)$, donde H es llamado espacio de estado;

- $A : D(A) \subset H \longrightarrow H$ es un operador que genera un semigrupo (lineal or no lineal) en el espacio de Hilbert H. El operador A es el “modelo” del sistema.

- u es el *control*, que pertenece a un conjunto de controles admisibles $\mathscr{U}_{\mathrm{ad}} \subseteq L^2(0,T;U)$, donde U es un espacio de Banach. El control es el parámetro libre a nuestra disposición para influir el comportamiento del sistema;

- $B : \mathscr{U}_{\mathrm{ad}} \longrightarrow X$ es el *operador de control* que modela los actuadores, es decir, cómo el control actúa en el sistema.

El conjunto de los controles admisibles $\mathscr{U}_{\mathrm{ad}}$ contiene aquellos controles que verifican algunas condiciones. Entre los controles admisibles, llamamos *control óptimo* al mejor según algunos criterios prescritos (generalmente, el minimizador de un funcional dado). La solución de la ecuación de estado (2.0.1), con el control óptimo se denomina estado óptimo.

Algunos de los principales retos en la teoría de control son

- existencia y unicidad de la ecuación de estado (2.0.1);

- existencia de un control admisible;

- existencia (y unicidad) de un control óptimo;

- condiciones de optimalidad de primer y segundo orden que caractericen el control óptimo;

- estructura y propiedades del control y estado óptimos;
- métodos numéricos para resolver aproximadamente el problema de control discretizado.

Para más detalles, nos remitimos al capítulo 3 y a las relativas referencias.

En la primera parte de la tesis estudiamos la *controlabilidad bajo restricciones* de ecuaciones en derivadas parciales. Consideremos la siguiente ecuación de estado

$$\frac{d}{dt}y(t) = Ay(t) + Bu(t), \tag{2.0.2}$$

un dato inicial y_0 y un objetivo y_1. Además, se prescriben algunas restricciones en el estado y/o en el control

$$u(t) \in K_c \quad \text{y/o} \quad y(t) \in K_s, \quad \forall\, t \in [0, T], \tag{2.0.3}$$

donde K_c y K_s son subconjuntos cerrados y convexos de U y H respectivamente. Por ejemplo, se puede requerir que el control y el estado sean positivos. Se dice que un control u es admisible si cumple las restricciones (2.0.3), junto con las condiciones terminales

$$y(0) = y_0 \quad \text{and} \quad y(T) = y_1. \tag{2.0.4}$$

Como veremos, en presencia de restricciones, el tiempo de control T debe ser lo suficientemente grande para cumplir las restricciones, reduciendo la amplitud de las oscilaciones del control y del estado. También se requiere que el tiempo sea grande para modelos similares al calor, a pesar de la velocidad infinita de propagación.

La segunda parte de la tesis está dedicada a la demostración de la *propiedad de turnpike* para problemas de control óptimo como

$$\min_u J_T(u) = \int_0^T L(y, u)dt, \tag{2.0.5}$$

donde

$$\begin{cases} \frac{d}{dt}y = A(y) + B(u) & \text{en } (0, T) \\ y(0) = y_0. \end{cases} \tag{2.0.6}$$

Si eliminamos el tiempo en el problema anterior, obtenemos el siguiente problema estacionario

$$\min_u J_s(u_s) = L(y_s, u_s), \quad \text{bajo la restricción} \quad A(y_s) + B(u_s) = 0.$$

Supongamos que existe un mínimo para ambos problemas. El par control óptimo-estado óptimo para el problema de evolución temporal se denota (u^T, y^T), mientras que el par óptimo para el problema estacionario se denota $(\overline{u}, \overline{y})$. Demostramos que, en tiempo T grande, cualquier par óptimo (u^T, y^T) para el problema de evolución temporal (2.0.6)-(2.0.5) está exponencialmente cerca de un par óptimo $(\overline{u}, \overline{y})$ del problema estacionario.

2.1 Controlabilidad de EDPs bajo restricciones

La controlabilidad de las ecuaciones en derivadas parciales es hoy en día un tema clásico en el análisis matemático. Uno de los trabajos pioneros es [51], donde H.O. Fattorini y D.L. Russell estudiaron la controlabilidad de la ecuación de calor en una dimensión espacial. Otro hito para la controlabilidad de las EDP lineales es el artículo SIAM Review de 1988 de J.L. Lions [93], donde se introdujo el Método de Unicidad de Hilbert (HUM). Se pueden encontrar más referencias en los siguientes artículos, así como en las referencias que ahí se dan: [153, 53, 92, 12, 87, 45, 77, 34, 140, 86].

Por un lado, muchos de estos resultados y los métodos numéricos correspondientes se han desarrollado en ausencia de restricciones. Por otro lado, en aplicaciones prácticas, las restricciones en el estado y/o en el control son ubicuas.

Por ejemplo, al controlar un fenómeno de difusión, frecuentemente requerimos que la temperatura se quede mayor que un umbral inferior. En varios modelos biológicos, químicos y económicos, las ecuaciones de reacción-difusión se resuelven mediante densidades, que no deben ser negativas en ningún momento (véase, por ejemplo, el libro de J.D. Murray [107, capítulo 11], [74] o el célebre artículo de A.M. Turing [141]). En finanzas, bajo algunos supuestos, el valor de mercado (solución de una EDP parabólica [71, Teorema 2.5, capítulo 2]) debe mantenerse mayor que un umbral inferior. Además, en las aplicaciones, la potencia de las máquinas está limitada, por lo que es necesario imponer algunas restricciones en el control (véase los trabajos anteriores [132] para la ecuación de calor y [65] para la ecuación de onda).

Desde un punto de vista matemático, cumplir las restricciones en los problemas de controlabilidad puede ser muy difícil. Por ejemplo, considerando la ecuación del calor pura, los controles de norma L^2 mínima en tiempo pequeño exhiben grandes oscilaciones cerca del tiempo final. De hecho, estos controles son restricciones de soluciones del sistema adjunto con un dato final crítico. Cuando el horizonte temporal es demasiado corto, estas oscilaciones impiden que el control cumpla con cualquier restricción. Entonces, a pesar de la velocidad infinita de propagación, el tiempo mínimo de control bajo restricciones es positivo (véase [95] y el capítulo 4).

Por otro lado, en un horizonte temporal grande, podemos construir controles de pequeña amplitud para cumplir con las restricciones. Es decir, la controlabilidad bajo restricciones es posible en tiempo grande, bajo hipótesis adecuadas sobre el dato inicial y el objetivo final. Por ejemplo, si el dato inicial y el objetivo final son estados estacionarios conectados por un camino de estados estacionarios dentro del conjunto de estados estacionarios, se puede implementar un argumento ‘‘paso a paso’’ para controlar el sistema bajo restricciones (ver figura 2.2). Este método consiste en pasar de un estado estacionario a uno vecino utilizando controles de pequeña amplitud. Iterando este procedimiento, uno puede conducir el estado al objetivo final preservando las restricciones impuestas a priori en el control. Este método

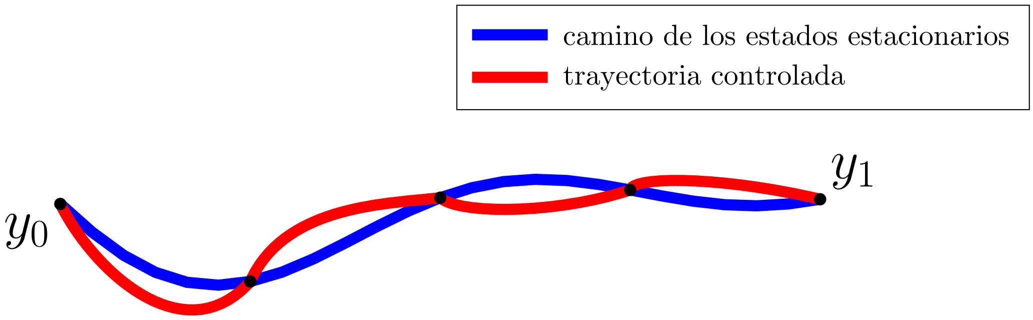


Figura 2.2: el argumento “paso a paso”

está inspirado en el articulo seminal [36] de J.M. Coron y E. Trélat, donde se emplearon deformaciones cuasiestacionarias para controlar las ecuaciones del calor semilineal. Sin embargo, queremos matizar que nuestro enfoque difiere de [36]. De hecho, la estrategia de deformaciones cuasiestacionarias se basa en los siguientes pasos:

1. seguimos el camino dado de estados estacionarios a una velocidad pequeña obteniendo una “casi”-trayectoria para el sistema de control de evolución;

2. determinamos un feedback estabilizador para estabilizar una trayectoria del sistema de control a lo largo de la “casi”-trayectoria obtenida. De esta manera, podemos acercarnos arbitrariamente al objetivo final;

3. controlabilidad local para alcanzar exactamente el objetivo final.

En cambio, como se ilustra en la figura 2.2, en el argumento “paso a paso”

1. subdividimos el camino de los estados estacionarios (dado por el problema) en pequeños arcos de estados estacionarios;

2. empleamos la controlabilidad local en cada arco pequeño para llevar el estado desde el extremo inicial de l'arco hasta el extremo final.

La controlabilidad bajo restricciones de positividad fué abordada por J. Lohéac, E. Trélat y E. Zuazua en [95] para ecuaciones disipativas. La prueba se basa en la disipatividad del sistema, lo que conduce a un decaimiento exponencial de la constante de observabilidad. Esto permite demostrar que, en intervalos grandes de tiempo, los controles se pueden elegir pequeños, lo que a su vez implica controlabilidad bajo restricciones. Para la controlabilidad bajo restricciones de positividad de sistemas de dimensión finita, nos referimos al reciente artículo [96]. Finalmente, el problema de controlabilidad bajo restricciones de proyección lineal ha sido analizado por S. Ervedoza en [47].

Para empezar, ilustramos resultados principales para ecuaciones similares al calor. De aquí en adelante, Ω será un conjunto abierto acotado conexo de $\mathbb{R}^n$, $n \geq 1$, con borde C^∞.

2.1.1 Controlabilidad bajo restricciones de positividad para la ecuación de calor semilineal

Consideramos la ecuación de calor semilineal

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{en } (0,T) \times \Omega \\ y = u & \text{sobre } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{en } \Omega \end{cases} \tag{2.1.1}$$

donde $y = y(t,x)$ es el estado y $u = u(t,x)$ es el control que actúa en el borde $\partial\Omega$. La no-linealidad $f : \mathbb{R} \longrightarrow \mathbb{R}$ es de la clase C^1. Téngase en cuenta que f puede ser de tipo blow-up. Los resultados que vamos a presentar son válidos para operadores más generales, que involucran una matriz de difusividad no constante, términos de convección y control localizado (véase capítulo 4).

Nuestro análisis está inspirado en [36], donde J.M. Coron y E. Trélat controlan la ecuacion de calor semilineal, mediante el uso de deformaciones cuasiestacionarias. Nuestro dato inicial y objetivo final son estados estacionarios, unidos por un camino continuo de estados estacionarios.

Más precisamente, asumimos la existencia de un arco continuo

$$\gamma : [0,1] \longrightarrow L^\infty(\Omega),$$

$$r \longmapsto \gamma_r,$$

tal que $\gamma_0 = y_0$ y $\gamma_1 = y_1$ y para cualquier $r \in [0,1]$, γ_r resuelve el problema elíptico

$$\begin{cases} -\Delta\gamma_r(x) + f(\gamma_r(x)) = 0 & x \in \ \Omega \\ \gamma_r(x) = \overline{u}_r(x) \geq \nu > 0 & x \in \ \partial\Omega. \end{cases}$$

donde $\nu > 0$ es una constante. La construcción del camino de estados estacionarios en algunos modelos no lineales se puede encontrar en [117, 126, 101].

Dado el camino de los estados estacionarios γ y el tiempo T suficientemente grande, el argumento ‘‘paso a paso’’ consiste en vincular estados estacionarios vecinos a lo largo de γ, mediante un control u que permanece en un vecindario ν de $\overline{u}_r$

$$\|u - \overline{u}_r\|_{L^\infty} \leq \nu. \tag{2.1.2}$$

Ahora vemos que, debido a que $\overline{u}_r \geq \nu > 0$, el control u cumple la restricción de no negatividad

$$u = u - \overline{u}_r + \overline{u}_r \geq -\nu + \nu = 0, \quad \text{casi por todo } (0,T) \times \partial\Omega, \tag{2.1.3}$$

como se deseaba. Téngase en cuenta que, dado que la no-linealidad es de tipo blow-up, al elegir un control arbitrario, la solución y de la ecuación de estado puede explotar. Sin embargo, al elegir el control anterior, la solución de (2.1.1), con el dato inicial y_0 y el control u permanece en un conjunto limitado, evitando así la explosión en tiempo finito.

Teorema 1 (Controlabilidad de estado estacionarios). *Bajo los supuestos anteriores, sean y_0 y y_1 dos estados estacionarios acotados conectados, de modo que*

$$\overline{u}_r \geq \nu, \quad \textit{casi por todo } \Gamma \tag{2.1.4}$$

para cualquier $r \in [0,1]$. Entonces, si T es lo suficientemente grande, existe $u \in L^\infty((0,T) \times \partial\Omega)$, un control tal que:

- *el problema* (2.1.1) *con dato inicial y_0 y el control u admite una solución única y que verifica $y(T,\cdot) = y_1$;*
- *$u \geq 0$ casi por todo $(0,T) \times \Gamma$.*

En este punto, nuestro propósito es considerar un conjunto más amplio de datos iniciales y objetivos finales. Suponemos de ahora en adelante que f es creciente, por lo cual la dinámica libre es disipativa y la ecuación de estado está bien puesta para cualquier dato inicial $y_0 \in L^2$ y control $u \in L^2$.

Sea $y_0 \in L^2(\Omega)$ un dato inicial e $\overline{y}$ una trayectoria objetivo, solución del problema

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + f(\overline{y}) = 0 & \text{en } (0,T) \times \Omega \\ \overline{y} = \overline{u} \geq \nu > 0 & \text{sobre } (0,T) \times \partial\Omega, \end{cases} \tag{2.1.5}$$

con control acotado $\overline{u} \geq \nu > 0$. Nuestro objetivo es encontrar un control no negativo u, tal que la solución (única) y de

$$\begin{cases} y_t - \Delta y + f(y) = 0 & \text{en } (0,T) \times \Omega \\ y = u \geq 0 & \text{sobre } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{en } \Omega \end{cases} \tag{2.1.6}$$

satisface la condición final $y(T,\cdot) = \overline{y}(T,\cdot)$, es decir, la trayectoria controlada y coincide con la trayectoria objetivo $\overline{y}$ en tiempo T.

Fijar $\tau > 0$. La estrategia que utilizamos para resolver este problema de control es la siguiente:

- **estabilización** en $[0, T-\tau]$: durante un largo intervalo de tiempo, elegimos el control estacionario $u = \overline{u}$ para estabilizar el sistema a la trayectoria objetivo $\overline{y}$. Esto es posible ya que f es creciente;
- **control** en $[T-\tau, T]$: utilizamos la controlabilidad local para coincidir exactamente con la trayectoria objetivo en el tiempo T.

Teorema 2 (Controlabilidad de datos iniciales generales a trayectorias). *Sea f una función creciente.*

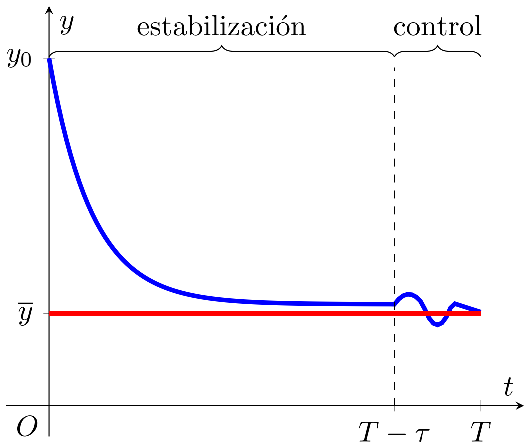


Figura 2.3: ilustración de la prueba del teorema 4.1.2 en dos pasos: estabilización + control

Considere una trayectoria objetivo $\overline{y}$*, solución de* (2.1.6) *con el dato inicial* $\overline{y}_0 \in L^2$ *y el control* $\overline{u} \in L^\infty$*, verificando la condición de positividad:*

$$\overline{u} \geq \nu > 0, \quad \textit{casi por todo } (0,T) \times \Gamma. \tag{2.1.7}$$

Entonces, para cualquier dato inicial $y_0 \in L^2(\Omega)$*, podemos encontrar, en un tiempo suficientemente largo, un control acotado* $u \geq 0$ *tal que:*

- *la solución única* y *de* (2.1.1) *con dato inicial* y_0 *y control* u *es tal que* $y(T,\cdot) = \overline{y}(T,\cdot)$*;*
- $u \geq 0$ *casi por todo* $(0,T) \times \Gamma$*.*

Observación 1. *Ahora, suponga que* f *está creciente y* $f(0) = 0$*. En el contexto del Teorema 1.1.2, asuma además el dato inicial* $y_0 \geq 0$*. Entonces, por el principio máximo, la solución controlada* $y \geq 0$*.*

Hasta ahora, hemos asumido que el tiempo de control T es largo para obtener la controlabilidad bajo restricciones. Mostramos ahora que la controlabilidad bajo restricciones falla en un tiempo demasiado pequeño, es decir, el tiempo de controlabilidad mínimo (bajo restricciones) es positivo.

Para simplificar la presentación, consideremos el caso lineal

$$\begin{cases} y_t - \Delta y = 0 & \text{en } (0,T) \times \Omega \\ y = u & \text{sobre } (0,T) \times \partial\Omega \\ y(0,x) = y_0, & \text{en } \Omega. \end{cases} \tag{2.1.8}$$

con dato inicial y_0 y objetivo y_1 estados estacionarios. En el capítulo 4, el lector interesado puede encontrar el caso general con coeficientes dependientes del espacio-tiempo y términos no lineales.

Definimos el concepto de tiempo mínimo de controlabilidad como

$$T_{\min} \stackrel{\text{def}}{=} \text{ínf} \left\{ T > 0 \;\middle|\; \exists u \in L^{\infty}((0,T)\times\Gamma)^{+},\; y(T,\cdot) = y_1 \right\}, \tag{2.1.9}$$

donde usamos la convención $\text{ínf}(\varnothing) = +\infty$.

Teorema 3 (Positividad del tiempo mínimo de controlabilidad)**.** *Supongamos que $y_0 \neq y_1$. Supongamos que y_1 está definido mediante un control acotado $u_1 \geq \nu > 0$.*

Entonces,

1. *existe $T_0 > 0$ tal que, para cualquier $T \in (0, T_0)$ y para cualquier control no negativo $u \in L^{\infty}((0,T)\times\Gamma)$ la solución y de (2.1.8) con dato inicial y_0 y control u es tal que $y(T,\cdot) \neq y_1$.*
2. *Consecuentemente,*

$$T_{min} > 0.$$

Consideramos dos ejemplos paradigmáticos: $y_0 > y_1$ y $y_0 < y_1$.

Caso $y_0 > y_1$.

Este es el caso más intuitivo. Para cualquier control no negativo u, por el principio de comparación

$$y \geq z, \quad \text{casi por todo } (0,T)\times\Omega, \tag{2.1.10}$$

donde y es la solución de (2.1.8), con el dato inicial y_0 y el control $u \geq 0$ y z resuelve el problema homogéneo

$$\begin{cases} z_t - \Delta z = 0 & \text{en } (0,T)\times\Omega \\ z = 0 & \text{sobre } (0,T)\times\partial\Omega \\ z(0,x) = y_0(x), & \text{en } \Omega. \end{cases} \tag{2.1.11}$$

Supongamos que λ_1 es el primer valor propio del laplaciano de Dirichlet y que ϕ_1 es la función propia correspondiente, con $\phi_1 \geq 0$ y $\|\phi_1\|_{L^2} = 1$. Tenemos

$$\int_{\Omega} y(t,x)\phi_1(x)dx \geq \int_{\Omega} z(t,x)\phi_1(x)dx = \exp(-\lambda_1 t)\int_{\Omega} y_0\phi_1 dx \tag{2.1.12}$$

Ahora,

$$\exp(-\lambda_1 t)\int_{\Omega} y_0\phi_1 dx > \int_{\Omega} y_1\phi_1 dx \tag{2.1.13}$$

si y solo si

$$t < \frac{1}{\lambda_1}\ln\left[\frac{\int_{\Omega} y_0\phi_1 dx}{\int_{\Omega} y_1\phi_1 dx}\right], \tag{2.1.14}$$

por lo cual

$$T_{\min} \geq \frac{1}{\lambda_1}\ln\left[\frac{\int_{\Omega} y_0\phi_1 dx}{\int_{\Omega} y_1\phi_1 dx}\right] > 0, \tag{2.1.15}$$

donde la última desigualdad se justifica por el supuesto $y_0 > y_1$.

Caso $y_0 < y_1$.

Este es el caso más delicado. Para mostrar el fenómeno del tiempo de espera, consideramos el problema adjunto

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{en } (0,T)\times\Omega \\ \varphi = 0 & \text{sobre } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x), & \text{en } \Omega \end{cases} \tag{2.1.16}$$

donde φ^0 es un dato final dado en $L^2(\Omega)$.

Por definición, se dice que $y \in L^2((0,T)\times\Omega)\cap C^0([0,T];H^{-1}(\Omega))$ es la solución por transposición de (2.1.8), con el dato inicial y_0 y el control u si

$$\langle y(T,\cdot),\varphi^0\rangle + \int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}d\sigma(x)dt = 0, \tag{2.1.17}$$

para cualquier dato final $\varphi^0 \in L^2(\Omega)$ para el problema adjunto.

Por contradicción, supongamos que en cualquier tiempo $T > 0$, existe un control no negativo u, tal que $y(T,\cdot) = y_1$, de donde

$$\int_\Omega y_1\varphi^0 dx + \int_0^T\int_{\partial\Omega} u\frac{\partial\varphi}{\partial n}d\sigma(x)dt = 0, \tag{2.1.18}$$

para cualquier dato final $\varphi^0 \in L^2(\Omega)$.

Ahora, para concluir, es suficiente construir un dato final φ^0 y $T_0 > 0$, tal que la solución φ del sistema adjunto con el dato final φ^0 satisface:

$$\begin{cases} \dfrac{\partial\varphi}{\partial n} \leq 0 & \text{sobre } (0,T_0)\times\partial\Omega \\ \displaystyle\int_\Omega y_1\varphi^0 dx < 0, & \forall T \in [0,T_0). \end{cases} \tag{2.1.19}$$

De hecho, si se cumple la relación anterior, (2.1.18) falla para cualquier $T \in (0,T_0)$ y dato final φ^0.

En la prueba del Teorema 4.5.1 en el capítulo 4, construímos el dato final como en la figura 2.4. La solución correspondiente al problema adjunto 2.1.16 se representa en la figura 2.5.

Mediante técnicas adjuntas similares, demostramos que la controlabilidad se mantiene valida en el tiempo mínimo mediante controles en el espacio de las medidas de Radón (véase Proposición 4.5.1 en el capítulo 4).

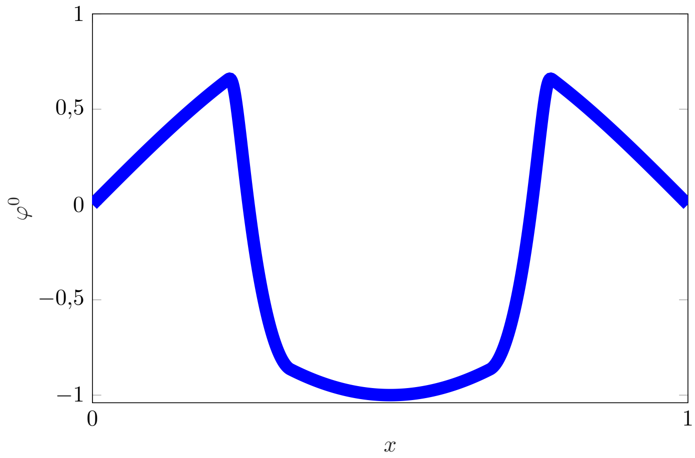


Figura 2.4: dato final para el sistema adjunto.

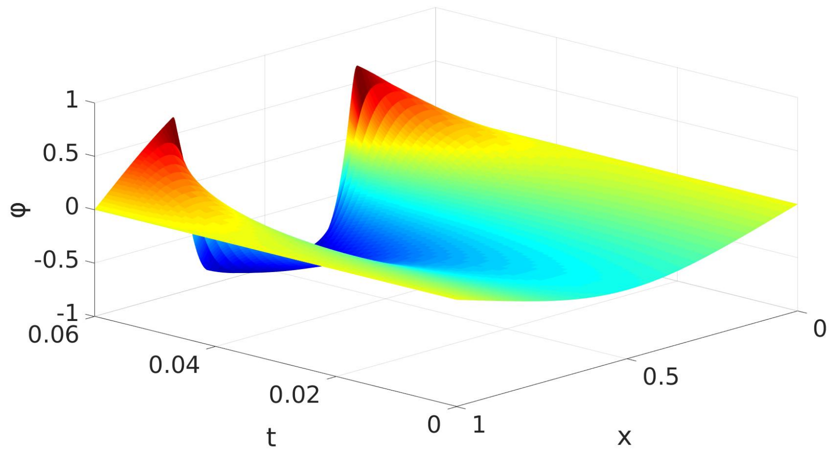


Figura 2.5: evolución de la ecuación de calor adjunta con dato final φ^0.

2.1.2 Controlabilidad bajo restricciones de positividad de la ecuación de las ondas multi-d

Como anticipamos, el argumento “paso a paso” es efectivo para controlar una amplia clase de EDPs. En esta subsección, lo implementaremos para la ecuación de ondas. Además, gracias a la reversibilidad en tiempo de la ecuación de onda, podemos lograr el estado cero como objetivo final mediante un control no negativo. Esta controlabilidad a cero por controles no negativos también es una consecuencia de la ausencia del principio de comparación, que fue una obstrucción para llegar a cero para la ecuación del calor.

Sin embargo, hay que tener en cuenta que para ecuaciones de difusión es mucho más fácil cumplir con restricciones no negativas sobre el estado, siempre que el dato inicial y el objetivo final sean estados estacionarios positivos. De hecho, debido al principio de comparación, la no negatividad del control es suficiente para garantizar la no negatividad del estado. Para la ecuación de ondas, hay que ser más cuidadoso para evitar que las oscilaciones empujen el estado más allá de las restricciones.

Para fijar las ideas, consideramos la ecuación de ondas pura, controlada en todo el borde del dominio Ω. El dato inicial y_0 y el objetivo final y_1 son estados estacionarios.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{en } (0,T)\times\Omega \\ y = u & \text{sobre } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x),\ y_t(0,x) = 0 & \text{en } \Omega \end{cases} \tag{2.1.20}$$

El lector puede encontrar en el capítulo 5 el caso más general de la ecuación de onda con potencial y un control localizado en el interior o en el borde del dominio Ω.

Asumimos la *Condición de Control Geométrico* en $(\Omega, \partial\Omega, T^*)$ que afirma que todas las bicaracterísticas generalizadas tocan el borde $\partial\Omega$ en un punto no difractivo en tiempo menor que T^*. Por ahora, es bien conocido en la literatura que esta condición geométrica es equivalente a la controlabilidad (sin restricciones) [12, 17].

Hemos obtenido el siguiente resultado de controlabilidad de estados estacionarios, que corresponde al Teorema 5.1.5 en el capítulo 5. La estrategia de control que hemos determinado nos permite de respectar tanto restricciones en el control como en el estado.

Teorema 4. *Sean y_i soluciones estáticas de*

$$-\Delta y_i = 0 \qquad en\,\Omega, \tag{2.1.21}$$

con $y_i \geq 0$, casi por todo Ω.

Entonces, si el horizonte temporal T es lo suficientemente largo, existe un control $u \in L^2$, tal que

- *la solución (y, y_t) del problema (2.1.20) con dato inicial $(y_0, 0)$ y control u verifica $(y(T,\cdot), y_t(T,\cdot)) = (y_1, 0)$;*

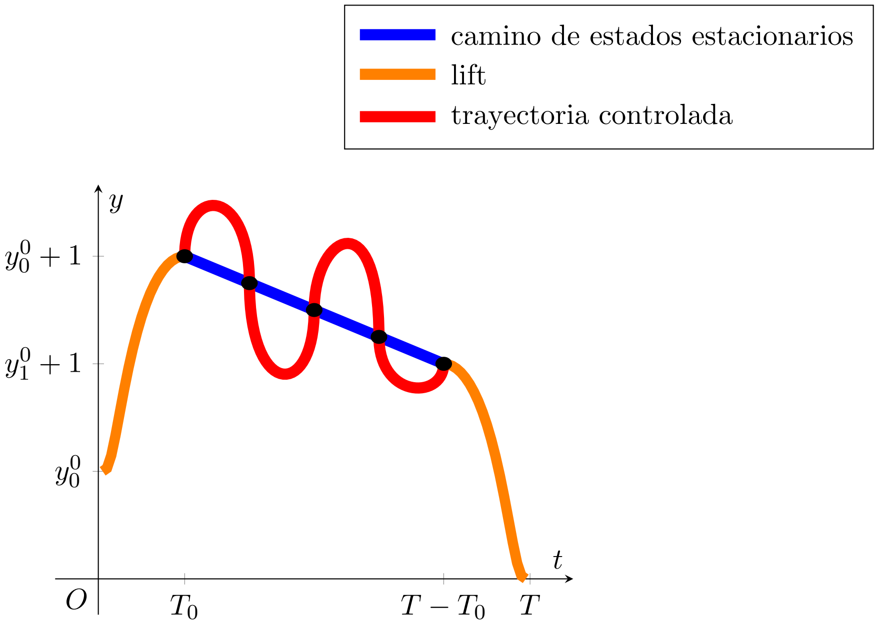


Figura 2.6: Estrategia de control para llevar la solución de la ecuación de onda a cero

- $u \geq 0$ *casi por todo* $(0, T) \times \partial\Omega$.

El argumento "paso a paso" se puede aplicar para satisfacer tanto el estado como la restricción de control, permitiendo solo pequeñas oscilaciones alrededor de la ruta de los estados estacionarios tanto a nivel del control como del estado. Sin embargo, la necesidad de mantener ambos en un estrecho vecindario tubular de la ruta de los estados estacionarios impone un tiempo de control aún mayor que el caso de las restricciones de control solamente.

Tenga en cuenta que hemos requerido que los estados estacionarios sean solo no negativos. Para la ecuacion del calor, los controles no negativos no podían alcanzar el cero, debido al principio de comparación parabólico. En el caso de la ecuación de onda, no existe tal obstrucción.

Podemos recuperar el espacio para las oscilaciones necesarias para aplicar el argumento "paso a paso" incluso en caso de que el objetivo final $y_1^0 \equiv 0$, siguiendo la estrategia (figura 2.6)

1. controlar el estado (y, y_t) desde $(y_0^0, 0)$ hasta $(y_0^0 + 1, 0)$ en tiempo T_0;

2. emplear el método "paso a paso" en $[T_0, T - T_0]$ para conectar $(y_0 + 1, 0)$ y $(y_1^0 + 1, 0)$, eligiendo T bastante grande;

3. portar el estado (y, y_t) desde $(y_1^0 + 1, 0)$ hasta $(y_1^0, 0)$ en $[T - T_0, T]$.

La parte 3 se puede lograr definiendo la solución controlada

$$y(t, x) = y_1^0(x) + \tilde{y}(t + T - T_0, x), \tag{2.1.22}$$

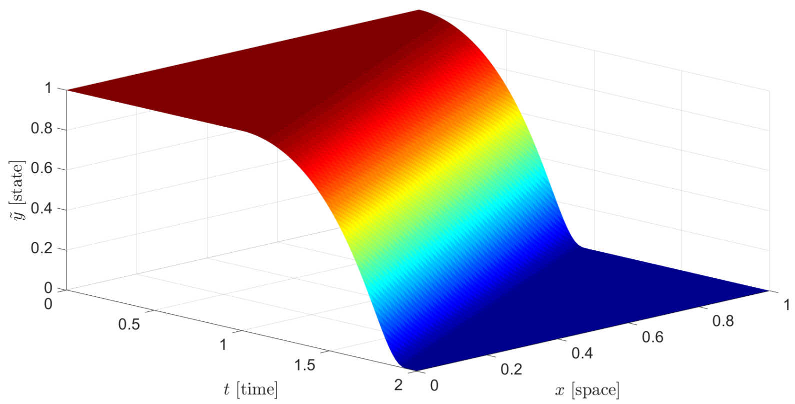


Figura 2.7: solución "lift" de la ecuación de onda que une el estado estacionario $y_0^0 \equiv 1$, con el estado estacionario $y_1^0 \equiv 0$ en tiempo $T_0 = 2$. Tanto el estado como el control de frontera permanecen no negativos a lo largo del proceso de control.

donde $\tilde{y}$ es la solución-"lift" (figura 2.7) del problema

$$\begin{cases} \tilde{y}_{tt} - \Delta \tilde{y} = 0 & \text{in } (0, T_0) \times \Omega \\ \tilde{y} \geq 0 & \text{in } (0, T_0) \times \Omega \\ \tilde{y}(0, x) = 1, \ \tilde{y}_t(0, x) = 0 & \text{in } \Omega \\ \tilde{y}(T_0, x) = 0, \ \tilde{y}_t(T_0, x) = 0 & \text{in } \Omega, \end{cases} \tag{2.1.23}$$

con $T_0 > d$, donde d es el diámetro de Ω.

La solución-"lift" $\tilde{y}$ de (2.1.23) es de la forma

$$\tilde{y}(t, x) = f(t + x_1), \tag{2.1.24}$$

donde $f : \mathbb{R} \longmapsto \mathbb{R}$ es suave y x_1 es la primer componente de $x \in \Omega$. Explicamos a continuación cómo construir el perfil f. Por definición de diámetro, existe un intervalo $[a, b]$, con $|b - a| \leq d$ y

$$\left\{ x_1 \in \mathbb{R} \mid \ (x_1, x_2, \dots, x_n) \in \overline{\Omega} \right\} \subseteq [a, b]. \tag{2.1.25}$$

Ya que $T_0 > d$, tenemos $a + T_0 > b$, de donde existe $f \in C^{\infty}(\mathbb{R}; [0, 1])$, tal que

- $f(\xi) = 1$, por cualquiera $\xi \in [a, b]$;
- $f(\xi) = 0$, por cualquiera $\xi \in [a + T_0, b + T_0]$.

Con la f anterior, $\tilde{y}$ definida en (2.1.24) es una solución de (2.1.23).

La parte 1 se puede manejar de la misma manera que la parte 3, utilizando la reversibilidad en el tiempo de la ecuación de onda.

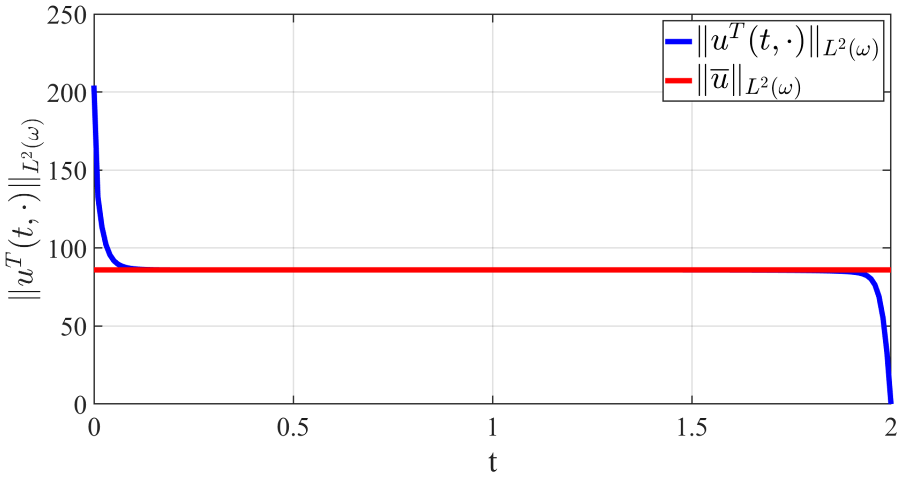


Figura 2.8: gráfico de la función $t \longrightarrow \|u^T(t)\|_{L^2(\omega)}$ (en azul) y $\|\overline{u}\|_{L^2(\omega)}$ (en rojo), donde u^T denota un control óptimo para el problema de evolución del tiempo, mientras que $\overline{u}$ representa un control estacionario óptimo.

2.2 La propiedad de turnpike en control óptimo

El propósito de la teoría de turnpike [2.1] es establecer una relación entre los problemas de control que dependen del tiempo y su correspondiente versión estática, cuando el horizonte temporal $T \to +\infty$.

En términos generales, se nos da un problema de control evolutivo $(OCP)_T$

$$\min_u J_T(u) = \int_0^T L(y,u)dt, \tag{2.2.1}$$

sujeto a la ecuación de estado:

$$\begin{cases} \dfrac{d}{dt}y = A(y) + B(u) & \text{en } (0,T) \\ y(0) = y_0. \end{cases} \tag{2.2.2}$$

El problema estacionario correspondiente $(OCP)_s$ seria

$$\min_u J_s(u_s) = L(y_s,u_s), \quad \text{con la restricción} \quad A(y_s) + B(u_s) = 0. \tag{2.2.3}$$

Un control óptimo para $(OCP)_T$ se denota con u^T, mientras que el estado óptimo se denota con y^T. El par (u^T, y^T) se llama el par óptimo para $(OCP)_T$. Además, denotamos por $(\overline{u}, \overline{y})$ un minimizador de J_s. El par $(\overline{u}, \overline{y})$ se conoce como el par óptimo para (2.2.3).

Suponemos la existencia de un par óptimo (u^T, y^T) para $(OCP)_T$, así como un par óptimo $(\overline{u}, \overline{y})$ para $(OCP)_s$.

[2.1] en inglés americano, la palabra "turnpike" significa "autopista".

La propiedad de " turnpike " se verifica si los óptimos del problema evolutivo permanecen cerca a los óptimos estacionarios excepto en dos pequeños intervalos temporales al principio y al final (figura 2.8). Más precisamente, dado cualquier par óptimo evolutivo (u^T, y^T), requerimos la existencia de un par óptimo estacionario $(\overline{u}, \overline{y})$ y un $\tau \geq 0$, independiente del horizonte temporal T, tal que:

John Von Neumann

1. en el intervalo $[0, \tau]$, el par óptimo (u^T, y^T) se mueve aproximadamente de $(u^T(0), y^T(0))$ a $(\overline{u}, \overline{y})$;

2. durante mucho tiempo $[\tau, T-\tau]$, el par (u^T, y^T) permanece cerca de $(\overline{u}, \overline{y})$;

3. en el intervalo final $[T-\tau, T]$, el par (u^T, y^T) se mueve aproximadamente de $(\overline{u}, \overline{y})$ a $(u^T(T), y^T(T))$.

Si se satisface la propiedad anterior, el par óptimo $(\overline{u}, \overline{y})$ se llama *turnpike*. En la literatura econométrica, el estado óptimo $\overline{y}$ se denomina *punto de Von Neumann*.

Téngase en cuenta que en $(OCP)_T$ se impone una condición inicial para el estado. Además, las condiciones de optimalidad de primer orden para $(OCP)_T$ conducen a una condición final para el control. Por lo tanto, no podemos esperar una proximidad de (u^T, y^T) a $(\overline{u}, \overline{y})$, en cualquier tiempo $t \in [0, T]$. Por ejemplo, si el dato inicial y_0 está lejos de $\overline{y}$, en un intervalo de tiempo reducido $[0, \tau]$, y^T va a estar muy lejos de $\overline{y}$. Además, si la norma de $\overline{u}$ es grande, entonces u^T está lejos de $\overline{u}$ en un arco $[T-\tau, T]$.

Este es un tema clásico en control matemático, econometría e ingeniería. Un pionero en el tema fue John von Neumann [142]. El econométrico Paul Samuelson, ganador del Premio Nobel en 1970, introdujo el concepto de turnpike en el libro seminal [44]:

. . . if we are planning long-run growth, no matter where we start and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.

Paul Samuelson

Existe una extensa literatura sobre este tema. En econometría, el tema ha sido ampliamente investigado por varios académicos, incluidos P. Samuelson y L.W. McKenzie [131, 94, 102, 103, 22, 69].

P. Kokotovic y sus colaboradores estudiaron el comportamiento a largo plazo de los problemas de control óptimo en relación con la teoría de Riccati y la ecuación de Hamilton-Jacobi [145, 6]. El mismo tema fue investigado en el cálculo de variaciones por R.T. Rockafellar empleando análisis convexo [123] y por A. Rapaport y P. Cartigny usando la teoría de Hamilton-Jacobi [120, 121]. A.J. Zaslavski escribió un libro [148] sobre el tema. Se ha obtenido una simplificación asintótica similar al turnpike en el contexto del diseño óptimo de la matriz de difusividad para la ecuación de calor [4]. En los trabajos [38, 62, 61, 136], el concepto de (medida) turnpike está relacionado con la disipatividad del problema de control.

Artículos recientes sobre el comportamiento a largo plazo de los mean field games [20, 21, 114] motivaron una nueva investigación sobre el tema. Se ha prestado especial atención a obtener una estimación exponencial:

$$\|u^T(t) - \overline{u}\|_U + \|y^T(t) - \overline{y}\|_H \leq K \left[\exp\left(-\mu t\right) + \exp\left(-\mu(T-t)\right)\right], \qquad \forall t \in [0,T], \quad (2.2.4)$$

para algunas constantes independientes de T, K y $\mu > 0$. Téngase en cuenta que $e^{-\mu t}$ es pequeño lejos de $t = 0$, mientras que $e^{-\mu(T-t)}$ es pequeño lejos de $t = T$. Entonces, si se satisface la desigualdad anterior, (u^T, y^T) permanece *exponencialmente* cerca de $(\overline{u}, \overline{y})$, excepto en un pequeño intervalo inicial y final. Tales estimaciones han sido obtenidas por A. Porretta y E. Zuazua en [115] para problemas de control cuadrático lineal, gobernados por EDOs o EDPs. Estos resultados se han extendido más tarde en [138, 116, 147, 137, 64, 63] para controlar problemas gobernados por una ecuación de estado no lineal y aplicados al control óptimo del sistema Lotka-Volterra [76]. Recientemente, la propiedad de turnpike se ha estudiado en torno a trayectorias no estáticas [137, 52]. En la referencia [83], los resultados de turnpike se han relacionado con las propiedades asintóticas de las ecuaciones de Hamilton-Jacobi.

Una vez que sabemos que un sistema de control satisface la propiedad de turnpike, podemos construir estrategias de turnpike casi óptimas como en la figura 2.9:

1. en un intervalo de tiempo corto $[0, \tau]$ se conduce el estado desde la configuración inicial y_0 al turnpike $\overline{y}$;

2. en un arco de tiempo largo $[\tau, T - \tau]$, permanece en $\overline{y}$;

3. en un arco final corto $[T - \tau, T]$, se usa un control para que y coincida con la condición terminal requerida en el tiempo $t = T$.

En general, el control y estado correspondientes no son óptimos, ya que no regulares. Sin embargo, son fáciles de construir y, dado que la propiedad de turnpike es verificada, son casi óptimos.

2.2.1 La propiedad de turnpike en control semilineal

La teoría de turnpike ha sido estudiada para problemas lineales-cuadráticos, tanto en el control de EDOs como en el control de EDPs. Cuando la ecuación de estado es no lineal, los resultados

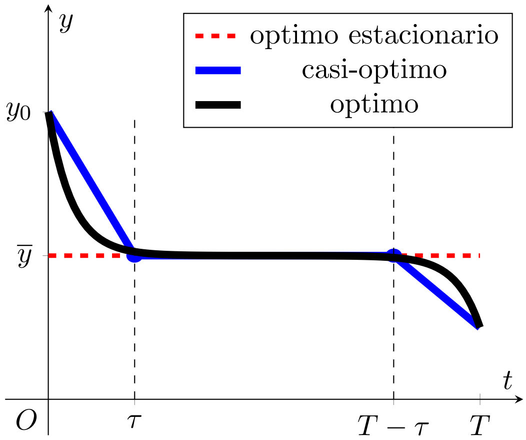


Figura 2.9: estrategias cuasi-óptimas de turnpike

de turnpike disponibles son locales [138, 116, 137].

El objetivo del capítulo 6 es desarrollar resultados de turnpike globales para problemas de control óptimo sujeto a una ecuación no lineal. Para fijar las ideas, consideramos el problema de control semilineal

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u) = \frac{1}{2}\int_0^T\int_\omega |u|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\omega_0} |y-z|^2 dxdt, \tag{2.2.5}$$

donde

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{en } (0,T)\times\Omega \\ y = 0 & \text{sobre } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{en } \Omega. \end{cases} \tag{2.2.6}$$

De ahora en adelante, Ω es un subconjunto abierto acotado regular de $\mathbb{R}^n$, con $n = 1,2,3$. La no-linealidad f es de clase C^3 y no decreciente, con $f(0) = 0$, garantizando así la existencia de soluciones para (2.2.6), en todo instante de tiempo [10, chapter 5]. El control actúa en el subdominio $\omega \subseteq \Omega$ y el estado se observa en la subregión $\omega_0 \subseteq \Omega$. El objetivo z está acotado. El parámetro de ponderación $\beta \geq 0$ regula la relevancia del término de estado en el funcional coste (2.2.5).

La existencia de un control óptimo u^T para (2.2.5) es una consecuencia de la Proposición 3.1.1 en el capítulo 3. El estado óptimo correspondiente se denota por y^T.

La versión estacionaria del problema anterior viene dada por:

$$\min_{u_s\in L^2(\omega)} J_s(u_s) = \frac{1}{2}\int_\omega |u_s|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y_s-z|^2 dx, \tag{2.2.7}$$

donde

$$\begin{cases} -\Delta y_s + f(y_s) = u_s\chi_\omega & \text{en } \Omega \\ y_s = 0 & \text{sobre } \partial\Omega. \end{cases} \tag{2.2.8}$$

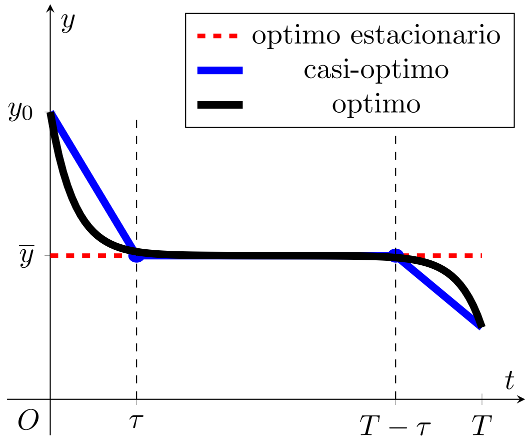


Figura 2.10: estrategias de turnpike casi-optimas

Para cualquier control dado $u_s \in L^2(\omega)$, existe un único estado $y_s \in H^2(\Omega) \cap H_0^1(\Omega)$ que es solución de (2.2.8) .

La existencia de un control óptimo $\overline{u}$ para (2.2.7) se sigue de la Proposición 3.1.1 en el capítulo 3. El estado óptimo correspondiente se denota por $\overline{y}$. Como veremos en el capítulo 7, la unicidad del minimizador se pierde para algunos objetivos grandes z. En caso de no unicidad, surge una pregunta: si la propiedad de turnpike se satisface, qué minimizador para (2.2.8)-(2.2.7) atrae las soluciones óptimas para (2.2.6)-(2.2.5)?

Se supone que el objetivo es pequeño, por lo que el control óptimo estacionario es único (véase [116, subsección 3.2]). En caso de que el objetivo sea grande, hasta donde sabemos, la validez de la propiedad de la turnpike sigue siendo un problema abierto. Sin embargo, como se muestra en la figura 2.10, utilizando la controlabilidad de la ecuación de estado, se podrían construir trayectorias cuasi-óptimas que simplemente utilicen el óptimo estacionario como un estado intermedio de transición, durante mucho tiempo, al cual se deben agregar arcos iniciales y finales controlados.

El punto de departida es el análisis local llevado a cabo por A. Porretta y E. Zuazua en [116], que conduce a la existencia de una solución para el sistema de optimalidad satisfaciendo la propiedad de turnpike, en condiciones de pequeñez en el dato inicial y_0 y en el objetivo z. Nos planteamos

1. probar que, de hecho, la propiedad de turnpike está satisfecha por el control óptimo;

2. eliminar la condición de pequeñez en el dato inicial.

Suponemos que el objetivo es pequeño, por lo que el control óptimo estacionario es único (véase [116, subsección 3.2]). Enunciamos nuestro resultado principal.

Teorema 5. *Considérese el problema de control* (2.2.6)-(2.2.5). *Sea* u^T *sea un minimizador*

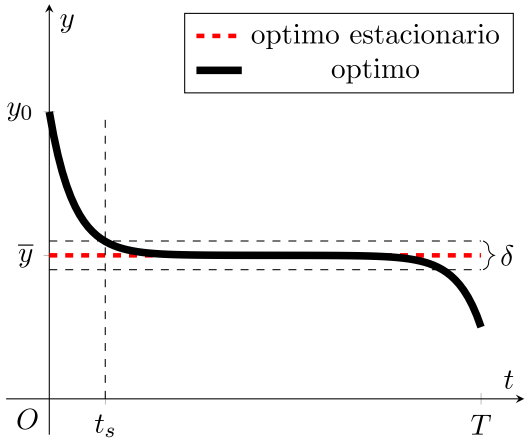


Figura 2.11: argumento global-local

de (2.2.5). *Existe* $\rho > 0$ *tal que, para cada* $y_0 \in L^\infty(\Omega)$ *y todo* z *verificando*

$$\|z\|_{L^\infty} \leq \rho, \tag{2.2.9}$$

se tiene:

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t} + e^{-\mu(T-t)}\right], \qquad \forall t \in [0,T], \tag{2.2.10}$$

las constantes K *y* $\mu > 0$ *son independientes del horizonte temporal* T.

Nuestra estrategia para demostrar el teorema 1.2.1 es la siguiente.

1. Primero, deducimos un cota L^∞ para la norma del control óptimo. Esta cota es uniforme en el horizonte temporal $T > 0$ (Lema 6.2.1 en la subsección 6.2.1);
2. Luego, probamos la propiedad de turnpike para *datos pequeños* y *objetivo pequeños*. Téngase en cuenta que, en [116, Teorema 1 subsección 3.1], los autores demuestran la existencia de una solución para el sistema de optimalidad que satisface la propiedad de turnpike. En este paso preliminar, para *datos pequeños* y *objetivos pequeños*, demostramos que cualquier control óptimo verifica la propiedad de turnpike (Lema 6.2.2 en la subsección 6.2.1);
3. Para *objetivos pequeños* y *cualquier dato*, demostramos que $\|y^T(t)\|_{L^\infty(\Omega)}$ es pequeño para t suficientemente grande (subsección 6.2.2). Esto se consigue estimando el tiempo crítico t_s necesario para acercarse al turnpike (figura 2.11);
4. Finalmente, concluímos juntando los dos pasos anteriores (subsección 6.2.2).

Describimos brevemente las ideas de la prueba de 3, la existencia de una cota superior τ para el tiempo mínimo necesario t_s para acercarse a la turnpike.
Supongamos, por contradicción, que el tiempo crítico t_s para acercarse al turnpike es muy grande. En consecuencia, la estrategia óptima de evolución temporal obedece al siguiente plan:

1. mantenerse alejado de la turnpike durante mucho tiempo;
2. moverse cerca del turnpike;
3. obtener un coste final del sistema evolutivo más barato que el estacionario.

Aplicando el funcional no estacionario al control optimo estacionario $\overline{u}$ obtenemos

$$J_T\left(u^T\right) \leq J_T\left(\overline{u}\right). \tag{2.2.11}$$

Ahora, dado que el objetivo z es pequeño, deducimos que $\frac{1}{T}J_T\left(\overline{u}\right)$ es pequeño también. Entonces, en la fase 1, con respecto al rendimiento estacionario, se genera un coste adicional, que debería recuperarse en la fase 3. En este punto, nos damos cuenta de que esto no puede ser por la propiedad de turnpike local. De hecho, una vez que los óptimos del funcional evolutivo J_T se acercan al turnpike en algún momento t_s, el par óptimo satisface la propiedad del turnpike por todos los tiempos más grandes $t \geq t_s$. Por lo tanto, para $t \geq t_s$, el rendimiento de el óptimo evolutivo no puede ser significativamente mejor que el estacionario. En consecuencia, no podemos recuperar el coste adicional generado en la fase 1, obteniendo así una contradicción.

2.2.2 No unicidad de minimizadores para problemas de control óptimos semilineales

En la sección anterior (referida al capítulo 6), hemos considerado el problema de control semilineal (2.2.8)-(2.2.7) para objetivos pequeños. Hemos presentado nuestras contribuciones a la teoría de turnpike. Pero como enfatizamos, los resultados que tenemos hasta ahora se limitan al caso en que el objetivo es pequeño. Esta condición sobre el objetivo asegura la unicidad de los controles óptimos y los estados controlados para el problema estacionario.

En la literatura, se ha mostrado que uno de los problemas más desafiantes en el control óptimo de problemas elípticos y parabólicos es la unicidad o la falta de unicidad del control óptimo y sus relativos estados controlados, cuando los objetivos son grandes.

En el capítulo 7 se construye un ejemplo de problema de control óptimo elíptico para el cual hay falta de unicidad debida a que el objetivo es grande.

Para fijar las ideas, en esta introducción consideramos el caso de un no-linealidad cúbica, dejando el caso de una no-linealidad creciente más general para el capítulo 7. Primero ilustramos el contraejemplo en un problema en el que el control desde la frontera y luego el contraejemplo en control interno.

En esta sección, para simplificar la notación, hemos eliminado el subíndice s para denotar controles/estados estables.

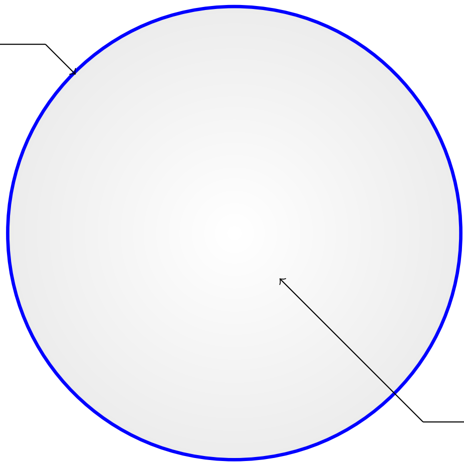


Figura 2.12: dominios de control y observación. El dominio de control es la frontera azul de la bola.

2.2.2.1 Control desde la frontera

Consideramos el problema de control óptimo

$$\min_{u \in L^2(\partial B(0,R))} J_s(u) = \frac{1}{2} \int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2} \int_{B(0,R)} |y - z|^2 dx, \tag{2.2.12}$$

donde

$$\begin{cases} -\Delta y + y^3 = 0 & \text{en } B(0,R) \\ y = u & \text{sobre } \partial B(0,R). \end{cases} \tag{2.2.13}$$

Aquí, $B(0,R)$ es la bola en $\mathbb{R}^n$, $n = 1,2,3$, de radio R, centrada en el origen. El objetivo z es un elemento de $L^2(B(0,R))$ y el parámetro de ponderación β es estrictamente positivo.

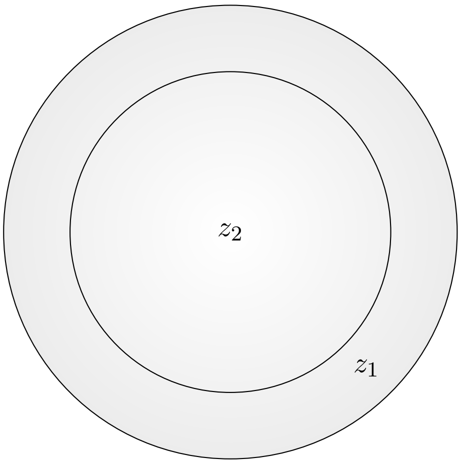
Téngase en cuenta que nuestro resultado es válido para cualquier valor del radio R del dominio. Ajustando el valor del parámetro β, el problema siempre se puede reducir al caso $R = 1$.

Teorema 6. *Considere el problema de control* (2.2.13) - (2.2.12). *Existe un objetivo $z \in L^\infty(B(0,R))$ tal que el funcional J_s definido en* (2.2.12) *admite (al menos) dos minimizadores globales.*

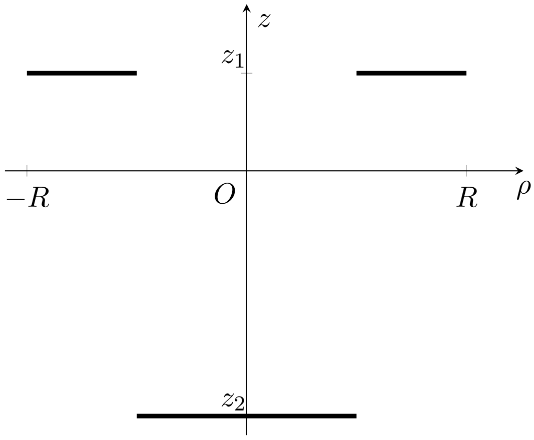
El esquema del la demostración de nuestro resultado de no unicidad es el siguiente:

Paso 1 **reducción al caso de controles constantes**: al elegir objetivos radiales y al usar la invariancia rotacional de $B(0,R)$, reducimos el problema al caso en el que al caso el conjunto de controles consiste en controles constantes;

Paso 2 **existencia de dos minimizadores locales**: buscamos un objetivo tal que existan dos minimizadores *locales* ($u_1 < 0$ y $u_2 > 0$) para el funcional estacionario J_s;



(a) curvas de nivel del objetivo radial construido



(b) objetivo radial construido en función del radio ρ

Figura 2.13: objetivo que produce falta de unicidad ene el control desde la frontera

Paso 3 **existencia de dos minimizadores globales**: mediante el primer paso y un argumento de bisección, demostramos la existencia de un objetivo z tal que J_s admite dos minimizadores *globales*.

El objetivo construido para demostrar la falta de unicidad es una función escalonada, como la que se muestra en la figura 2.13.

2.2.2.2 Control en el interior

Consideramos el problema de control óptimo elíptico

$$\min_{u \in L^2(B(0,r))} J_s(u) = \frac{1}{2} \int_{B(0,r)} |u|^2 dx + \frac{\beta}{2} \int_{B(0,R) \setminus B(0,r)} |y - z|^2 dx, \tag{2.2.14}$$

donde

$$\begin{cases} -\Delta y + y^3 = u\chi_{B(0,r)} & \text{en } B(0,R) \\ y = 0 & \text{sobre } \partial B(0,R). \end{cases} \tag{2.2.15}$$

Aquí, $B(0,R)$ es una bola de $\mathbb{R}^n$, $n = 1,2,3$, centrada en el origen y con radio R. El control actúa en la bola $B(0,r)$, con $r \in (0,R)$. El dominio de observación es $B(0,R) \setminus B(0,r)$ (véase figura 2.14). El objetivo z es un elemento de $L^2(B(0,R) \setminus B(0,r))$ y el parámetro de ponderación β es estrictamente positivo.

Teorema 7. *Considérese el problema de control* (2.2.15)-(2.2.14). *Existe un objetivo* $z \in L^\infty(B(0,R) \setminus B(0,r))$ *tal que el funcional* J_s *definido en* (2.2.14) *admite (al menos) dos minimizadores globales.*

La prueba es similar al caso en el que el control actúa desde la frontera, con la diferencia de que en este caso, por la invariancia rotacional de los dominios de control y observación,

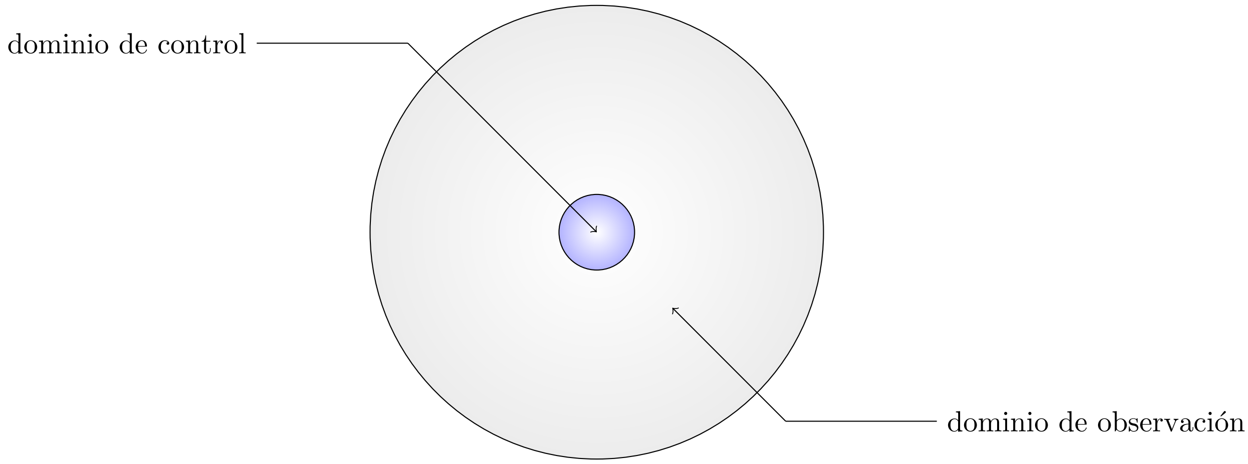


Figura 2.14: dominios de control y observación

podemos reducir el problema al caso de controles radiales en lugar de controles constantes. Véase el capítulo 7 para más detalles.

2.2.3 Supresión del desequilibrio de rotores mediante control óptimo

El capítulo 8 ilustra el resultado de una estancia de investigación en la empresa "Marposs S.p.A.". El contenido ha sido sometido [59]. Durante la estancia de investigación, aplicamos la teoría de la turnpike/estabilización a un problema de supresión de la vibración de un rotor fuera de equilibrio. Se da un rotor desequilibrado junto con dos cabezales de equilibrado. Cada cabeza de equilibrado está hecha de dos masas de equilibrado. Dada una configuración inicial de las mencionada masas, nuestro objetivo es determinar cuatro trayectorias angulares que dirigían las masas desde su configuración inicial a una configuración estable, donde compensen el desequilibrio. Téngase en cuenta que, a diferencia de las máquinas de balanceo de ruedas clásicas, nuestro dispositivo de balanceo gira junto con el rotor y el rotor se mueve mientras se realiza el procedimiento de balanceo. Todo esto nos sirve de motivación para formular el problema como un problema de optimización dinámica, de manera que también se tengan en cuenta las dinámica transitoria.

En el espíritu de la teoría de turnpike/estabilización, las trayectorias óptimas (de tipo open-loop) estabilizan el sistema hacia una configuración estable óptima. Esto es consecuencia de la desigualdad de Łojasiewicz [99, Théorème 2 página 62]. Si el desequilibrio está por debajo de un umbral calculado, la convergencia se produce exponencialmente rápido. Esto se demuestra mediante el teorema de la variedad estable aplicado al sistema de optimalidad de Pontryagin.

Equilibrar rotores es un problema clásico en ingeniería. En muchas ocasiones, la masa del rotor no está distribuida simétricamente alrededor del eje, debido al desgaste, daños estructurales y otras razones. Esto tiene como consecuencia la aparición de vibraciones, que afectan seriamente el rendimiento del rotor.

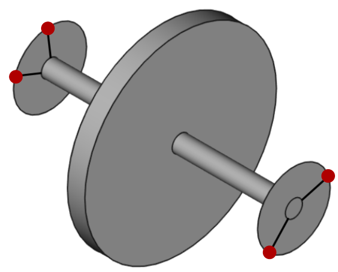


Figura 2.15: Representación del rotor y el dispositivo de equilibrado. En el caso especial representado, los cabezales de equilibrado se encuentran en los puntos terminales del huso. Las cuatro masas de equilibrado (dos para cada cabeza de equilibrado) se dibujan en rojo.

La supresión de vibraciones es un objecto de estudio clásico/típico de la dinámica de rotores. Por ejemplo, las máquinas rectificadoras a menudo se deterioran durante su ciclo de vida operativo. Esto conduce a peligrosas vibraciones de desequilibrio, lo que afecta su rendimiento al moldear objetos (véase, por ejemplo, [68, 75, 149, 30]). Este desequilibrio también es una problema para las turbinas eólicas. En este caso, el desequilibrio puede afectar la eficiencia de la producción de energía y la durabilidad de la turbina. Si las vibraciones se hacen demasiado grandes, la turbina puede colapsar. Debido a este y otros motivos, se han desarrollado sistemas de detección y corrección de vibraciones (véase la patente de EE. UU. [80]). Se han desarrollado dispositivos de equilibrado para estabilizar las unidades de CD-ROM y las lavadoras (véase [32, 119, 28, 29, 82]). Otro tema clásico en ingeniería es el equilibrio de las ruedas del automóvil. De hecho, fácilmente las ruedas pueden desalinearse cayendo en socavónes y/o golpeando objetos elevados. La desalineación puede causar un desgaste irregular de los neumáticos y los componentes de las suspensiones también pueden dañarse. Por esta razón, sofisticados dispositivos se han proyectado para equilibrar las ruedas (véase, por ejemplo, [46, capítulo 44]). La literatura de ingeniería clásica sobre la supresión del desequilibrio se refiere a la detección y / o corrección del desequilibrio.

En el capítulo 8, abordamos el problema de corrección del desequilibrio (el desequilibrio es un input). En nuestro modelo, un rotor desequilibrado gira a una velocidad angular constante alrededor de un eje fijo. Este rotor es afectado por un desequilibrio dinámico, es decir, el desequilibrio ejerce tanto una fuerza como un par en el eje de rotación. Dos cabezales de equilibrado están montados unidos rígidamente con el rotor, girando juntos a él. En cada cabeza de equilibrado tenemos dos masas de equilibrado, que giran en un plano ortogonal al eje de rotación (figura 2.15).
Dada una configuración inicial de las masas de equilibrado, nuestro objetivo es determinar cuatro trayectorias óptimas para las cuatro masas de equilibrado para compensar el desequilibrio. Como mencionamos, el dispositivo de equilibrado gira junto con el rotor. Luego, con la intención de minimizar las vibraciones, estamos interesados en:

- llevar el sistema a una configuración de equilibrio en un tiempo $t > T$;
- minimizar el desequilibrio que se produce al mientras se conduce el sistema al configuración estable, i.e. en $t \in [0, T]$.

El problema se formulará como un problema de optimización dinámica, de modo que las respuestas transitorias también se tengan en cuenta.

Empleamos una estrategia de control de open loop para mover los cabezales de equilibrado desde su configuración inicial a una configuración estable, donde compensan el desequilibrio del rotor. En primer lugar, al enunciar el problema en el marco del cálculo de variaciones, se demuestra la existencia de las trayectorias optimas óptimo y se derivan las correspondientes ecuaciones de Euler-Lagrange. Utilizando la desigualdad de-Lojasiewicz, se demuestra que en cualquier condición la estabilización de las trayectorias óptimas se estabilizan hacia óptimos estacionarios. En el caso de que el desequilibrio esté por debajo de un umbral dado, mostramos que la estabilización se produce exponencialmente rápida. Esta estimación se obtiene, utilizando el lenguaje del control optimo, mediante la condición de optimalidad de Pontryagin, que se escribe como un sistema de EDOs de primer orden en ele tiempo. Concretamente, demostramos la hiperbolicidad del sistema Pontryagin alrededor de los óptimos estacionarios para aplicar el teorema de la variedad estable (véase [109, Corollary page 115] y [129]). Nuestras conclusiones se ajustan al marco general de la teoría de control y, en particular, de la estabilización, la teoría de turnpike y la controlabilidad (véase, por ejemplo, [54, 134, 153, 115, 138, 151]).

La posición de las cuatro masas de equilibrado viene dada por el vector $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2) \in \mathbb{T}^4 = \left(\mathbb{S}^1\right)^4$.
Consideramos la lagrangiana

$$L : \mathbb{T}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}, \quad L(\Phi, \psi) := \frac{1}{2}\left[\|\psi\|^2 + \beta\hat{G}(\Phi)\right],$$

donde $\beta > 0$ es un parámetro de ponderación y $\hat{G} = G - \inf G$, siendo G una función introducida en (8.2.6) (capítulo 8) que indica el desequilibrio. En la definición anterior, tenemos una compensación entre el costo de controlar el sistema a un régimen estable y la velocidad de las masas de equilibrado, con respecto al rotor. Si β es grande, el principal objetivo de la estrategia óptima es minimizar el costo del control, mientras que, si β es pequeño, la prioridad es minimizar las velocidades.

Ahora, sea $\Phi_0 \in \mathbb{T}^4$ una configuración inicial. Introducimos el espacio de trayectorias admisibles:

$$\mathscr{A} := \left\{\Phi \in \bigcap_{T>0} H^1\left(0, T; \mathbb{T}^4\right) \;\middle|\; \Phi(0) = \Phi_0, \quad \text{y} \quad L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)\right\}.$$

Téngase en cuenta que la condición $L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)$ es equivalente a

$$\dot{\Phi} \in L^2(0, +\infty) \quad \text{y} \quad G(\Phi) - \inf G \in L^1(0, +\infty).$$

Nuestro objetivo es entonces minimizar el funcional $J : \mathscr{A} \longrightarrow \mathbb{R}$

$$J(\Phi) := \frac{1}{2}\int_0^{\infty} \left[\|\dot{\Phi}\|^2 + \beta\hat{G}(\Phi)\right] dt. \tag{2.2.16}$$

Sean F_1 y F_2 las fuerzas producidas por el desequilibrio en los dos planos de equilibrado. Sean m_1 y m_2 la masa de las dos masas de equilibrado y denotemos por r_1 y r_2 su distancia al eje respectivas. Enunciamos ahora nuestro resultado principal.

Proposición 1. *Considérese el funcional* (2.2.16). *Por $i = 1, 2$, sea*

$$c^i := \frac{1}{2m_i r_i \omega^2}\left(F_{i,x}, F_{i,y}\right), \tag{2.2.17}$$

donde ω es la velocidad de rotación del rotor. Entonces,

1. *existe un minimizador $\Phi \in \mathscr{A}$ de J;*

2. *cada minimizador $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ de J es C^∞ suave y, para $i = 1, 2$ y todo $t > 0$, se cumplen las siguientes ecuaciones de Euler-Lagrange*

$$\begin{cases} -\ddot{\alpha}_i = \beta \cos(\gamma_i) \left[-c_1^i \sin(\alpha_i) + c_2^i \cos(\alpha_i) \right] \\ -\ddot{\gamma}_i = -\beta \sin(\gamma_i) \left[c_1^i \cos(\alpha_i) + c_2^i \sin(\alpha_i) - \cos(\gamma_i) \right] \\ \alpha_i(0) = \alpha_{0,i}, \ \ \gamma_i(0) = \gamma_{0,i}, \ \ \dot{\Phi}(T) \underset{T\to+\infty}{\longrightarrow} 0. \end{cases} \tag{2.2.18}$$

(3) para cualquier trayectoria óptima Φ *para* (2.2.16)*, existe un estado estacionario optimo* $\overline{\Phi} \in \mathscr{S} := ceros\left(\widehat{G}\right)$ *tal que*

$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi}, \tag{2.2.19}$$

$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{2.2.20}$$

y

$$\left|\hat{G}\left(\Phi(t)\right)\right| \underset{t\to+\infty}{\longrightarrow} 0. \tag{2.2.21}$$

Si, además, se cumple que

$$m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad and \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}, \tag{2.2.22}$$

entonces tenemos la siguiente estimación exponencial para cualquier $t \geq 0$

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |G(\Phi(t))| \leq C \exp(-\mu t), \tag{2.2.23}$$

con C, $\mu > 0$ *independiente de* t.

Realizamos algunas simulaciones numéricas, minimizando el funcional discretizado por la rutina experta de optimización de “interior point” `IpOpt` (véase [143]), modelado en el lenguaje `AMPL` (véase [55]).

En las figuras 1.16, 1.17, 1.18 y 1.19, mostramos la trayectoria óptima calculada para (2.2.16), con dato inicial $\Phi_0 = (\alpha_{0,1}, \gamma_{0,1}; \alpha_{0,2}, \gamma_{0,2}) := (2{,}6, 0{,}6, 2{,}5, 1{,}5)$. Elegimos F, N y m_i, tales que se cumpla la condición (2.2.22), por lo cual se puede apreciar el decadimento exponencial demostrado en la Proposición 1.2.1. En la figura 2.20, representamos el indicador de desequilibrio G frente al tiempo a lo largo de las trayectorias calculadas. Como se esperaba, decae a cero exponencialmente.

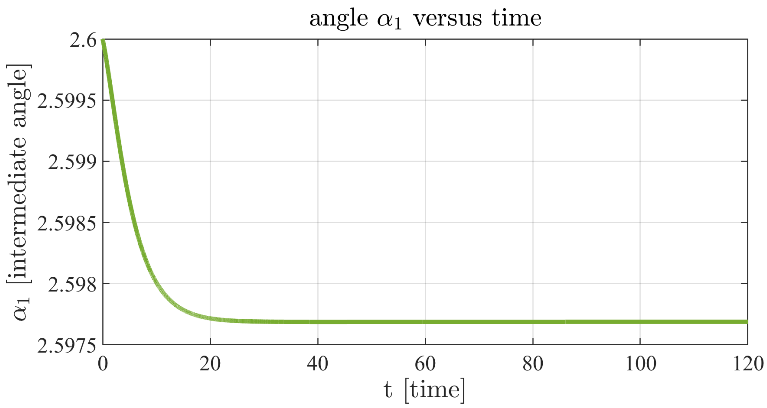


Figura 2.16: ángulo intermedio α_1 versus tiempo

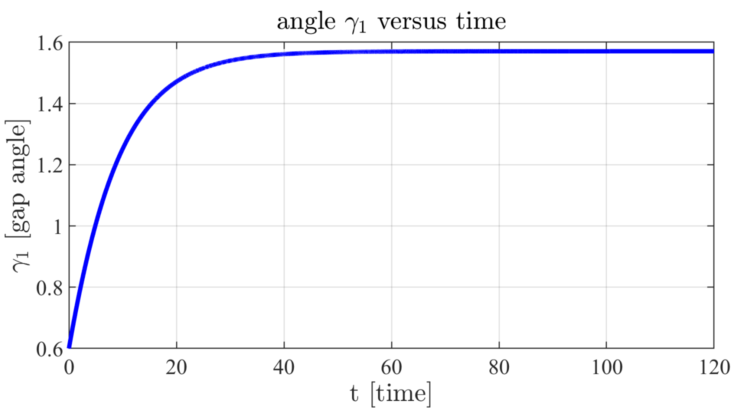


Figura 2.17: ángulo de separación γ_1 versus tempo

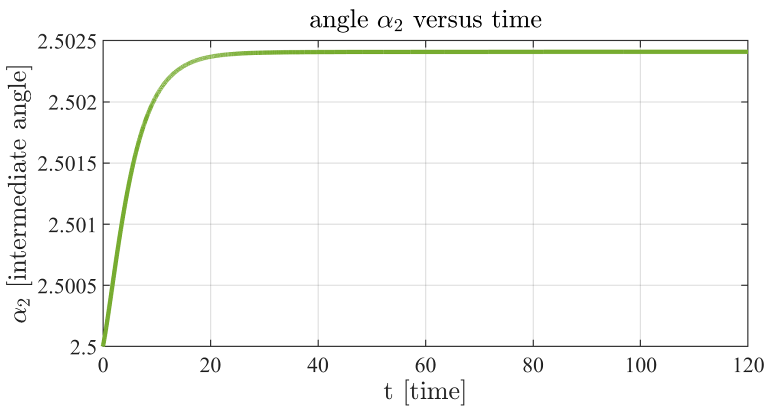


Figura 2.18: ángulo intermedio α_2 versus tempo

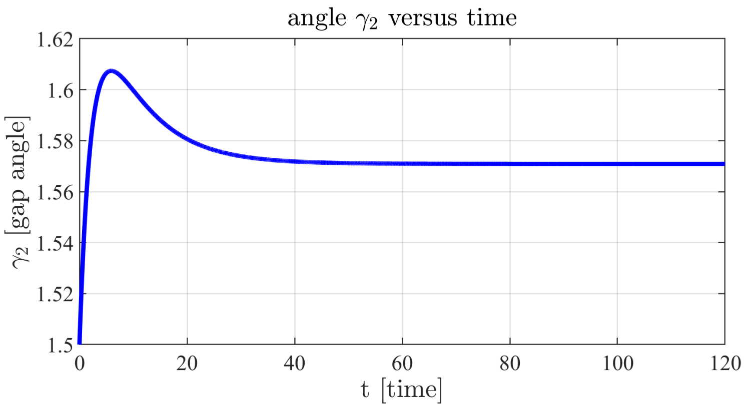


Figura 2.19: ángulo de separación γ_2 versus tempo

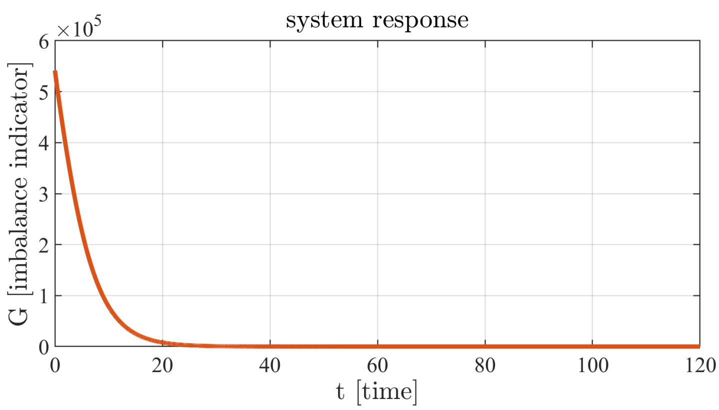


Figura 2.20: el indicador de desequilibrio G a lo largo de la trayectoria calculada versus tempo.

2.3 Conclusiones

En esta tesis hemos analizado el comportamiento de algunos problemas de control en intervalos de tiempo grandes. En el capítulo 4 (que corresponde a [111]), hemos dado condiciones suficientes para garantizar la controlabilidad de la ecuación de calor semilineal bajo restricciones de positividad. Hemos demostrado que las restricciones de positividad conducen a un fenómeno de tiempo de espera, i.e. el tiempo de control mínimo es estrictamente positivo. En el capítulo 5 (que corresponde a [112]), se han demostrado resultados similares para la ecuación de ondas bajo restricciones de positividad, con la posibilidad adicional de alcanzar cero como objetivo final. En el capítulo 6, obtenemos resultados globales de turnpike para un problema de control óptimo sujeto a una ecuación de calor semilineal. Se requiere que el objetivo sea pequeño, pero los datos iniciales pueden elegirse arbitrariamente grandes. Si el objetivo es grande, pueden existir problemas de falta de unicidad de los minimizadores en el caso estacionario. En el capítulo 7 damos un ejemplo de dicha situación. En el capítulo 8 (que corresponde [59]), la teoría de turnpike/estabilización se aplica al problema industrial de equilibrio de rotores.

2.3.1 Problemas abiertos

Formulamos ahora algunos problemas abiertos interesantes.

2.3.1.1 Controlabilidad del problema del obstáculos

Un problema abierto llamativo es demostrar la controlabilidad desde la frontera del problema del obstáculo. Esto está relacionado con el capítulo 4, porque la solución del problema del obstáculo puede verse como el límite de soluciones para una familia de ecuaciones elípticas semilineales [56, 8]. Los resultados de este capítulo se aplican al problema penalizado, pero pasar al límite para obtener resultados relevantes para el problema del obstáculo es un tema abierto.

En la literatura, podemos encontrar resultados de controlabilidad en temas relacionados. Por ejemplo, el articulo [5] se ocupa de la controlabilidad de la ecuación de ondas unidimensional, con un obstáculo en el extremo derecho del intervalo espacial. El control actúa en el extremo izquierdo del intervalo espacial. La controlabilidad a cero se muestra combinando la fórmula de D'Alembert y con un argumento de punto fijo. Además, en [41] se ha comprobado la controlabilidad aproximada de una desigualdad variacional parabólica.

Además, hay disponible una abundante literatura sobre el control óptimo del problema del obstáculo [31, 105, 9, 1, 104, 78, 79].

Una de las dificultades en los problemas de control con obstáculo es la falta de diferenciabilidad de la aplicación control-estado. Mostramos cómo falla la diferenciabilidad en el ejemplo siguiente.

Considere el obstáculo $\psi(x) = x(1-x)$ y el conjunto convexo

$$\begin{aligned} K_{(a,b)} &:= \{y \in H^1(0,1) \mid y(0) = a,\ y(1) = b \\ &\quad \text{y} \ \ y(x) \geq x(1-x) \quad \forall x \in [0,1]\}, \end{aligned} \tag{2.3.1}$$

con a y b números reales positivos. Para cualquier $a \geq 0$ y $b \geq 0$, nos consideramos el problema de obstáculo

$$\min_{y \in K_{(a,b)}} \int_0^1 |y_x|^2 dx. \tag{2.3.2}$$

El funcional $J(y) = \int_0^1 |y_x|^2 dx$ Es coercitivo, estrictamente convexo y continuo. Luego, (2.3.2) admite una solución única $y_{(a,b)}$ [16, Theorem 5.6 section 5.3]. La solución se puede calcular explícitamente. Por ejemplo, si ambos a y b satisfacen $0 < a < \frac{1}{4}$ y $0 < b < \frac{1}{4}$, entonces tenemos

$$y_{(a,b)}(x) = \begin{cases} \left(-2\sqrt{a}+1\right)\left(x-\sqrt{a}\right) - a + \sqrt{a} & x \in [0, \sqrt{a}) \\ x(1-x) & x \in [\sqrt{a}, 1-\sqrt{b}) \\ \left(2\sqrt{b}-1\right)\left(x-1+\sqrt{b}\right) - b + \sqrt{b} & x \in [1-\sqrt{b}, 1], \end{cases} \tag{2.3.3}$$

mientras que si ambos $a > \frac{1}{4}$ y $b > \frac{1}{4}$, la solución es un segmento $y_{(a,b)}(x) = (b-a)x + a$.

Los números reales a y b son los controles y $y_{(a,b)}$ es el estado correspondiente. Introducimos la aplicación control-estado

$$G : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow L^2(0,1), \quad (a,b) \longmapsto y_{(a,b)}. \tag{2.3.4}$$

Tome $(a,b) = \left(\frac{1}{4}, \frac{1}{4}\right)$. Vamos a mostrar que G no es diferenciable en $\left(\frac{1}{4}, \frac{1}{4}\right)$ en la dirección $v = (1,1)$. De hecho, por un lado, para cualquier $x \in (0,1)$, la derivada por la derecha

$$\lim_{h \to 0^+} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} = 1. \tag{2.3.5}$$

Por otro lado, la derivada por la izquierda

$$\begin{aligned} \lim_{h \to 0^-} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} &= -2x+1, \quad 0 < x < \frac{1}{2} \\ \lim_{h \to 0^-} \frac{G\left(\frac{1}{4}+h, \frac{1}{4}+h\right) - G\left(\frac{1}{4}, \frac{1}{4}\right)}{h} &= 2x-1, \quad \frac{1}{2} < x < 1. \end{aligned} \tag{2.3.6}$$

Por lo tanto, las derivadas por la izquierda y por la derecha difieren, por lo que G no es diferenciable en $\left(\frac{1}{4}, \frac{1}{4}\right)$ en la dirección $v = (1,1)$.

2.3.1.2 Estimaciones más precisas del tiempo mínimo de controlabilidad por método adjunto

En el capítulo 4 hemos introducido una nueva técnica adjunta para demostrar la positividad del tiempo mínimo de control bajo restricciones (Teorema 1.1.3). Nuestras técnicas se basan en la existencia de un dato final especial para el problema adjunto

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{en } (0,T)\times\Omega \\ \varphi = 0 & \text{sobre } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x), & \text{en } \Omega \end{cases} \tag{2.3.7}$$

tal que

$$\begin{cases} \dfrac{\partial\varphi}{\partial n} \leq 0 & \text{sobre } (0,T_0)\times\partial\Omega \\ \displaystyle\int_\Omega y_1\varphi^0 dx < 0, & \forall T \in [0,T_0), \end{cases} \tag{2.3.8}$$

siempre que T_0 sea pequeño.

Construímos un dato final específico que satisface los requisitos antes mencionados. Un problema abierto interesante es encontrar el dato final φ^0 que satisface (2.3.8) y maximizar el tiempo T_0 mientras la derivada normal permanece no positiva. Esto conduciría a una estimación más nítida del tiempo mínimo de control bajo restricciones.

Es decir, establecer

$$\mathscr{F} := \left\{ \varphi^0 \in L^2(\Omega) \;\middle|\; \int_\Omega y_1\varphi^0 dx < 0 \right\}. \tag{2.3.9}$$

Para cualquier dato final $\varphi^0 \in \mathscr{F}$, definir

$$T_{\varphi^0} := \sup\left\{ T > 0 \;\middle|\; \frac{\partial\varphi}{\partial n} \leq 0 \;\; \text{sobre } (0,T)\times\partial\Omega \right\}, \tag{2.3.10}$$

donde φ es la solución del problema

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{en } (0,T)\times\Omega \\ \varphi = 0 & \text{sobre } (0,T)\times\partial\Omega \\ \varphi(T,x) = \varphi^0(x) & \text{en } \Omega. \end{cases} \tag{2.3.11}$$

El problema es maximizar

$$\mathscr{T} : \mathscr{F} \longrightarrow \mathbb{R}^+, \quad \varphi^0 \longmapsto T_{\varphi^0}. \tag{2.3.12}$$

2.3.1.3 La propiedad de turnpike en control semilineal para objetivos grandes

En el capítulo 6, hemos demostrado que la propiedad de turnpike en el control semilineal es válida para objetivos pequeños y cualquier dato inicial para la ecuación de estado. Sería interesante explorar el caso de objetivos grandes.

El problema se puede formular de la siguiente manera. Considere el problema de control de la evolución del tiempo

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u) = \frac{1}{2}\int_0^T\int_\omega |u|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\omega_0} |y-z|^2 dxdt, \tag{2.3.13}$$

donde

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{en } (0,T) \times \Omega \\ y = 0 & \text{sobre } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x) & \text{en } \Omega. \end{cases} \tag{2.3.14}$$

La no-linealidad f es C^3 y no decreciente. Las hipótesis sobre la ecuación de estado son las mismas del capítulo 6.

El problema estacionario es el siguiente

$$\min_{u_s \in L^2(\omega)} J_s(u_s) = \frac{1}{2} \int_\omega |u_s|^2 dx + \frac{\beta}{2} \int_{\omega_0} |y_s - z|^2 dx, \tag{2.3.15}$$

donde

$$\begin{cases} -\Delta y_s + f(y_s) = u_s \chi_\omega & \text{en } \Omega \\ y_s = 0 & \text{sobre } \partial\Omega. \end{cases} \tag{2.3.16}$$

Denotamos por $(\overline{u}, \overline{y})$ un par óptimo, donde $\overline{u}$ es un control óptimo y $\overline{y}$ el estado óptimo correspondiente.

Conjetura 1. *Considere el problema de control* (2.3.14)-(2.3.13)*. Tome cualquier dato inicial* $y_0 \in L^\infty(\Omega)$ *y cualquier objetivo* $z \in L^\infty(\omega_0)$*. Sea* u^T *un minimizador de* (2.3.13)*. Existe un par óptimo* $(\overline{u}, \overline{y})$ *de* (2.3.16)-(2.3.15) *tal que*

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K \left[e^{-\mu t} + e^{-\mu(T-t)} \right], \quad \forall t \in [0,T], \tag{2.3.17}$$

las constantes K *y* $\mu > 0$ *son independientes del horizonte temporal* T.

En el capítulo 7 construímos objetivos especiales z, en los que el control óptimo de (2.3.16)-(2.3.15) no es único. Para esos objetivos, hay la siguiente pregunta: si la propiedad de turnpike se satisface, ?‘cuál es el minimizador de (2.3.16)-(2.3.15) que atrae los óptimos del problema evolutivo (2.3.14)-(2.3.13)?

Segundo algunas simulaciones numéricas que hemos realizado, independientemente del dato inicial y_0, sólo un estado estacionario es elegido por tener el rol de turnpike. Desde la perspectiva de las estrategias turnpike casi-optimas esto puede estar relacionado con el coste de llegar al estado estacionario y al coste de cumplir las condiciones final sobre el estado adjunto $p^T(T) = 0$. Sin embargo, el asunto requiere más investigación.

Más en general una investigación adicional seria necesaria para explorar el linearizado del sistema de optimalidad determinado en [116, subsection 3.1]. Procedemos a introducir el problema. Como en [116], se considera el sistema de optimalidad para (2.3.14)-(2.3.13)

$$
\begin{cases}
y_t^T - \Delta y^T + f(y^T) = -q^T \chi_\omega & \text{en } (0,T) \times \Omega \\
y^T = 0 & \text{sobre } (0,T) \times \partial\Omega \\
y^T(0,x) = y_0(x) & \text{en } \Omega \\
-q_t^T - \Delta q^T + f'(y^T) q^T = \beta(y^T - z)\chi_{\omega_0} & \text{en } (0,T) \times \Omega \\
q^T = 0 & \text{sobre } (0,T) \times \partial\Omega \\
q^T(T,x) = 0 & \text{en } \Omega.
\end{cases}
\tag{2.3.18}
$$

Se toma una par $(\overline{u}, \overline{y})$ optima para el problema estacionario (2.3.16)-(2.3.15). Por las condiciones de optimalidad de primer orden (véase la Proposición 3.1.7 en el capítulo 3), el control optimo estacionario se escribe $\overline{u} = -\overline{q}\chi_\omega$, con

$$
\begin{cases}
-\Delta \overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{en } \Omega \\
\overline{y} = 0 & \text{sobre } \partial\Omega \\
-\Delta \overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{en } \Omega \\
\overline{q} = 0 & \text{sobre } \partial\Omega.
\end{cases}
\tag{2.3.19}
$$

Como en [116], introducimos las variables de perturbación

$$
\eta^T := y^T - \overline{y} \quad \text{and} \quad \varphi^T := q^T - \overline{q}
\tag{2.3.20}
$$

y escribimos el sistema de optimalidad linearizado alrededor de $(\overline{u}, \overline{y})$

$$
\begin{cases}
\eta_t^T - \Delta \eta^T + f'(\overline{y})\eta^T = -\varphi^T \chi_\omega & \text{en } (0,T) \times \Omega \\
\eta^T = 0 & \text{sobre } (0,T) \times \partial\Omega \\
\eta^T(0,x) = y_0(x) - \overline{y}(x) & \text{en } \Omega \\
-\varphi_t^T - \Delta \varphi^T + f'(\overline{y})\varphi^T = \left(\beta\chi_{\omega_0} - f''(\overline{y})\,\overline{q}\right)\eta^T & \text{en } (0,T) \times \Omega \\
\varphi^T = 0 & \text{sobre } (0,T) \times \partial\Omega \\
\varphi^T(T,x) = -\overline{q}(x) & \text{en } \Omega.
\end{cases}
\tag{2.3.21}
$$

Como ha sido señalado en [116, Teorema 1 en subsección 3.1], un punto clave es comprobar la propiedad de turnpike por el linearizado del sistema de optimalidad (2.3.21). Esto es complicado por el termino $\beta\chi_{\omega_0} - f''(\overline{y})\,\overline{q}$, cuyo signo es desconocido para objetivos grandes generales. Además, en caso de falta de unicidad del óptimo estacionario, sería interesante calcular el espectro del sistema linealizado alrededor de cualquier óptimo estacionario para verificar si entre ellos hay un atractor mejor.

Chapter 3

Preliminaries

The purpose of this chapter is to introduce some notions of control theory. Our presentation is inspired by [153, 26, 106, 89, 140, 139, 33, 122, 13, 135] and we focus on deterministic systems. In a control system we have two actors: the *control* u and the *state* y, which are related by the state equation:

$$\Lambda(y) = B(u). \tag{3.0.1}$$

$\Lambda : D(\Lambda) \subset X \longrightarrow X$ is a (linear or nonlinear) operator describing the behaviour of the uncontrolled system and $B : U \longrightarrow X$ is the *control operator*, modelling the actuators. Typically, the state equation is a controlled ODE or PDE. The operator Λ can be steady or time-evolution as in (1.0.1), where $\Lambda := \frac{d}{dt} - A$, with A generator of an operator semigroup.

We furthermore introduce the observation of the state Cy, where $C : X \longrightarrow Z$ is called the *observation operator* and Z the *space of observations*. The idea is that, for any given control, we are able to observe Cy.

Hereafter, we deal with Banach spaces with real scalars. With appropriate modifications, the same results can be deduced in the complex case.

3.1 Optimal control theory

The goal of optimal control theory is to find an *optimal control*, minimizing a prescribed *cost functional*

$$J : \mathscr{U}_{\mathrm{ad}} \longrightarrow \mathbb{R}, \quad J(u) := \Phi(u, Cy). \tag{3.1.1}$$

Note that in the above functional the state y depends on the control u, through the state equation (3.0.1). Namely, the functional depends only on the control u.

The mathematical challenges in optimal control theory include:

1. well posedness of the state equation (3.0.1);

2. existence of an admissible control;

3. existence (and uniqueness) of an optimal control, minimizing the cost functional (3.1.1);
4. first and second order optimality conditions characterizing the optimal control;
5. structure and properties of optimal control and state;
6. numerical methods to approximately solve the discretized control problem.

The well posedness of the state equation can be accomplished by ODE or PDE analysis. The existence of an admissible control may require a deep analysis of the properties of the state equation. If the state equation depends on time and a final condition for the state is imposed, the existence of an admissible control is called a controllability problem.

3.1.1 Existence of the optimal control

The existence of an optimal control can be tackled by the *direct method in the calculus of variations* (DMCV), based on the proof of extreme value Weierstrass theorem. We will present these techniques in what follows. We start with the definition of lower semicontinuous function.

Definition 3.1.1. *Let U be a reflexive Banach space. A function $J : U \longmapsto \mathbb{R}$ is said to be weakly lower semicontinuous if for any $\hat{u} \in U$ and for any sequence $\{u_m\}_{m\in\mathbb{N}^*} \subset U$ weakly convergent to $\hat{u}$, we have*

$$J(\hat{u}) \leq \liminf_{m\to+\infty} J(u_m) = \lim_{M\to+\infty} \left[\inf_{m\geq M} J(u_m) \right].$$

We have the following theorem. Two main assumptions will be required:

1. lower semicontinuity of the function;
2. boundedness of a sublevel set.

Theorem 3.1.1. *Let U be a reflexive Banach space and let $K \subseteq U$ be a nonempty, closed and convex set. Consider $J : K \longmapsto \mathbb{R}$ and assume*
(H_1) J is weakly lower semicontinuous (in the sense of Definition 3.1.1);
(H_2) there exists $r > \inf_K J$ such that the sublevel set

$$\mathscr{S}_r := \{u \in K \mid J(u) \leq r\} \tag{3.1.2}$$

is bounded in the strong norm topology. Then, there exists $\overline{u} \in K$ minimizer of J.

Proof. Step 1 **Convergence of a minimizing sequence**
By definition of the infimum of a function, there exists a minimizing sequence $\{u_m\}_{m\in\mathbb{N}^*} \subset \mathscr{S}_r$, with

$$J(u_m) \underset{m\to+\infty}{\longrightarrow} \inf_K J. \tag{3.1.3}$$

By assumption (H_2) , $\mathscr{S}_r$ is bounded. Now, the Banach space U is reflexive. Then, by the Banach-Alaoglu theorem, there exists $\overline{u} \in U$ such that, up to subsequences,

$$u_m \underset{m\to+\infty}{\rightharpoonup} \overline{u},$$

weakly in U. Furthermore, K is weakly closed, being convex and (strongly) closed. Hence, $\overline{u} \in K$.

Step 2 **Conclusion by lower semicontinuity of J**

The definition of infimum, the lower semicontinuity of J (H_1) and (3.1.3) lead to

$$\inf_K J \leq J(\overline{u}) \leq \liminf_{m\to+\infty} J(u_m) = \inf_K J,$$

whence $\overline{u}$ is the required minimizer. This concludes the proof. □

By using the above theorem, we state and prove a result which yields existence of an optimal control for the control problems considered in this dissertation. To this end, we introduce the concept of weakly continuity.

Definition 3.1.2. *Let U and Y be reflexive Banach spaces. A map $G : U \longrightarrow Y$ is said to be weakly continuous if for any weakly converging sequence $\{u_m\}_{m\in\mathbb{N}^*}$*

$$u_m \underset{m\to+\infty}{\rightharpoonup} \overline{u}, \quad in U$$

the image of $\{u_m\}_{m\in\mathbb{N}^*}$ by G converges weakly

$$G(u_m) \underset{m\to+\infty}{\rightharpoonup} G(\overline{u}), \quad in Y.$$

Note that G may be linear or nonlinear. In our case, we will suppose the weak continuity the control-to-state map, which maps a control u to its corresponding state y, solution to (3.0.1) (see e.g. [26, page 15]).

Proposition 3.1.1. *Let U and Y be reflexive Banach spaces and let $K \subseteq U$ be a nonempty closed and convex set. Let*

$$G : U \longrightarrow Y$$

be weakly continuous. Set

$$J : K \longrightarrow \mathbb{R}, \quad J(u) := \|u\|_U^2 + \|G(u) - z\|_Y^2 , \tag{3.1.4}$$

with $z \in Y$. Then, there exists $\overline{u} \in K$ minimizer of J.

Proof. Step 1 **Weak lower semicontinuity of J**

For any sequence

$$u_m \underset{m\to+\infty}{\rightharpoonup} \overline{u},$$

by the compactness of G, we have the weak convergence in Y of the sequence $\{G(u_m) - z\}_{m\in\mathbb{N}^*}$

$$G(u_m) - z \underset{m\to+\infty}{\rightharpoonup} G(\overline{u}) - z.$$

Then, by the lower semi continuity of the norm with respect to the weak convergence, we have

$$J(\overline{u}) \leq \liminf_{m\to+\infty} J(u_m).$$

Step 2 **Boundedness of sublevel sets**

For any $r \geq 0$, consider the sublevel set

$$\mathscr{S}_r := \{u \in K \mid J(u) \leq r\}$$

By definition of $\mathscr{S}_r$ and (3.1.4), for any $u \in \mathscr{S}_r$ we have

$$r \geq J(u) \geq \|u\|_U^2 ,$$

whence the sublevel set $\mathscr{S}_r$ is bounded.

Step 3 **Conclusion**

The conclusion follows from Theorem 3.1.1. □

3.1.2 Characterization of optimal controls: first and second order optimality conditions

In case the functional J is smooth enough, optimal controls satisfy first and second order optimality conditions. As we shall see, these conditions play a key role when characterizing the optimal controls.

To this purpose, we introduce some notions on calculus in Banach spaces. For further details, the reader is referred to the book by H. Cartan [23]. First of all, we give the following definition.

Definition 3.1.3. *Let X and Y be Banach spaces and let $F : X \longrightarrow Y$ be some map. Take x_0 in X.*

- *If there exists a bounded linear operator $DF(x_0) \in \mathscr{L}(X, Y)$ such that for any direction v in X,*
$$\exists \lim_{h\to 0} \frac{F(x_0 + hv) - F(x_0)}{h} = DF(x_0)(v) \in Y,$$
then F is said to be Gâteaux differentiable at x_0 and $DF(x_0)$ is said to be the Gâteaux differential of F at x_0;

- *if, in addition,*
$$\exists \lim_{x\to x_0} \frac{\|F(x) - F(x_0) - DF(x_0)(x - x_0)\|_Y}{\|x - x_0\|_X} = 0,$$
we call F Fréchet differentiable at x_0 and $DF(x_0)$ the Fréchet differential of F at x_0.

We have now the well known Fermat necessary condition, which stipulates that the Gâteaux differential of a real valued function vanishes at a local minimizer. For completeness, let us define the concept of local minimizer.

Definition 3.1.4. *Let X be a Banach space and let $F : X \longrightarrow \mathbb{R}$. A point $\overline{x} \in X$ is a local minimizer for F if there exists $r > 0$ such that $F(\overline{x}) \leq F(x)$, for any $x \in X$, with $\|x - \overline{x}\|_X < r$,*

Proposition 3.1.2 (Fermat necessary condition)**.** *Let X be a Banach space and let $F : X \longrightarrow \mathbb{R}$ be Gâteaux differentiable function. Suppose $\overline{x} \in X$ is a local minimizer of F. Then, the Gâteaux differential vanishes*

$$DF(\overline{x}) = 0.$$

Proof of Proposition 3.1.2. Take an arbitrary direction $v \in X \setminus \{0\}$. By definition of local minimizer, $F(\overline{x} + hv) - F(\overline{x}) \geq 0$, for any h small enough. Hence, by definition of Gâteaux differential,

$$\langle DF(\overline{x}), v\rangle = \lim_{h\to 0} \frac{F(\overline{x} + hv) - F(\overline{x})}{h} \geq 0.$$

We can apply the same procedure to the direction $\tilde{v} = -v$, getting

$$-\langle DF(\overline{x}), v\rangle = \langle DF(\overline{x}), -v\rangle = \lim_{h\to 0} \frac{F(\overline{x} - hv) - F(\overline{x})}{h} \geq 0,$$

which yields

$$\langle DF(\overline{x}), v\rangle = 0,$$

as required. □

We formulate now second order conditions for optimality.

Proposition 3.1.3. *Let X be a Banach space and let $F : X \longrightarrow \mathbb{R}$ be twice Fréchet differentiable. Take $\overline{x} \in X$.*

1. *If $\overline{x}$ is a local minimizer for F, then the second Fréchet differential of F at $\overline{x}$ is positive semidefinite*

$$\langle D^2F(\overline{x})\, v, v\rangle \geq 0, \quad \forall\ v \in X. \tag{3.1.5}$$

2. *Conversely, assume $DF(\overline{x}) = 0$ and the existence of $\alpha > 0$ such that*

$$\langle D^2F(\overline{x})\, v, v\rangle \geq \alpha\|v\|^2, \quad \forall\ v \in X. \tag{3.1.6}$$

Then there exists $r > 0$ such that

$$F(x) \geq F(\overline{x}) + \frac{\alpha}{2}\|x - \overline{x}\|^2, \quad \forall\ x \in B(\overline{x}, r), \tag{3.1.7}$$

with $B(\overline{x}, r) := \{x \in X \mid \|x - \overline{x}\| < r\}$.
Consequently, $\overline{x}$ is a local minimizer.

Proof. *Step 1* **Proof of (1.)**
For any direction $v \in X$, set $f(h) := F(\overline{x} + hv)$. Now, f has a local minimizer at 0. Then, by second order optimality conditions in one dimension, $\langle D^2F(\overline{x})v, v\rangle = f''(0) \geq 0$.
Step 2 **Proof of (2.)**
By Taylor's theorem, we have, for any $x \in X$

$$F(x) = F(\overline{x}) + \langle D^2F(\overline{x})x - \overline{x}, x - \overline{x}\rangle + \Lambda(x - \overline{x}), \tag{3.1.8}$$

where

$$\frac{|\Lambda(x-\overline{x})|}{\|x-\overline{x}\|^2} \underset{\|y\|\to 0}{\longrightarrow} 0. \tag{3.1.9}$$

In particular, there exists $r > 0$ such that if $\|x - \overline{x}\| < r$, then

$$|\Lambda(x-\overline{x})| \leq \frac{\alpha}{2}\|x-\overline{x}\|^2, \tag{3.1.10}$$

whence by (3.1.8),

$$F(x) \geq F(\overline{x}) + \frac{\alpha}{2}\|x-\overline{x}\|^2, \tag{3.1.11}$$

as required. □

3.1.3 An example: optimal control of semilinear elliptic equations

Consider the optimal control problem

$$\min_{u\in L^2(\omega)} J_s(u) = \frac{1}{2}\int_\omega |u|^2dx + \frac{\beta}{2}\int_{\omega_0}|y-z|^2dx, \tag{3.1.12}$$

where:

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.13}$$

$\Omega \subset \mathbb{R}^n$ is a regular bounded open set. We suppose the space dimension $n = 1, 2, 3$. The nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is C^3 nondecreasing. The action of the control is localized by multiplication by χ_ω, characteristic function of the open subregion $\omega \subseteq \Omega$. The target z is of class $L^2(\omega_0)$ and the difference state-target is penalized over the observation domain $\omega_0 \subseteq \Omega$. The weighting parameter $\beta \geq 0$. As β increases, the distance between the optimal state and the target decreases. By elliptic regularity (see, e.g. [14]), for any given control $u \in L^2(\omega)$, there exists a unique state $y \in H^2(\Omega) \cap H_0^1(\Omega)$ solution to (3.1.13).

We start by verifying the existence of an optimal control for (3.1.13)-(3.1.12) by using Theorem 3.1.1.

To this end, set $U := L^2(\omega)$ and $Y := L^2(\omega_0)$ and the control-to-state map

$$G : L^2(\omega) \longrightarrow L^2(\omega_0) \tag{3.1.14}$$

$$u \longmapsto y,$$

where y is the solution to (3.1.13), with control u. We have the following result.

Proposition 3.1.4. *Under the above assumptions, there exists an optimal control $\overline{u}$ for the control problem* (3.1.13)-(3.1.12).

Proof. In the notation of Theorem 3.1.1, set $K = U := L^2(\omega)$ and $Y := L^2(\omega_0)$ and the control-to-state map G as in (3.1.14). To conclude, we show that G is weakly continuous. Take an arbitrary weakly converging sequence

$$u_m \underset{m\to+\infty}{\rightharpoonup} \overline{u}, \quad \text{in} U.$$

By elliptic PDE theory (see, e.g. [14]), the sequence $\{G(u_m)\}_{m\in\mathbb{N}}$ is bounded $H_0^1(\Omega)$. Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and the unique accumulation point of $\{G(u_m)\}_{m\in\mathbb{N}}$ is $G(u)$, we have

$$G(u_m) \underset{m\to+\infty}{\rightarrow} G(\overline{u})$$

strongly in $L^2(\omega_0)$, as required. □

We show now that the optimal control is given by the solution of a state-adjoint state optimality system. Moreover, we present necessary and sufficient second order optimality conditions. We follow the approach of [26]. We start computing the first Fréchet differential of the control-to-state map (3.1.14).

Proposition 3.1.5. *Let $G : L^2(\omega) \longrightarrow H^2(\Omega)$ be the control-to-state map defined in* (3.1.14). *Then,*

- *G is of class $C^2(L^2(\omega); H^2(\Omega))$;*
- *for any control $u \in L^2(\omega)$ and for any direction $v \in L^2(\omega)$, set $y = G(u)$ and $w := \langle DG(u), v\rangle$ the Fréchet differential of G at u along direction v. Then, w_v solves the linearized problem*
$$\begin{cases} -\Delta w + f'(y)w = v\chi_\omega & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega; \end{cases} \tag{3.1.15}$$
- *for any direction v_1 and v_2 in $L^2(\omega)$, $w_{v_1,v_2} := \langle D^2G(u)v_1, v_2\rangle$ satisfies*
$$\begin{cases} -\Delta w_{v_1,v_2} + f'(y)w_{v_1,v_2} + f''(y)w_{v_1}w_{v_2} = 0 & \text{in } \Omega \\ w_{v_1,v_2} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.16}$$

Proof. We introduce the differential operator

$$F : H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\omega) \longrightarrow L^2(\Omega)$$

$$F(y, u) := -\Delta y + f(y) - u\chi_\omega.$$

F is of class C^2 and the Fréchet differential with respect to y is given by

$$\langle D_yF(y, u), w\rangle = -\Delta w + f'(y)w.$$

Indeed, by the Lagrange mean value theorem and the continuous embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ (we have supposed the space dimension $n = 1, 2, 3$),

$$\begin{aligned}
\frac{\|F(\tilde{y}) - F(y) + \Delta(\tilde{y} - y) - f'(y)(\tilde{y} - y)\|_{L^2(\Omega)}}{\|\tilde{y} - y\|_{H^2(\Omega)}} &= \frac{\|f(\tilde{y}) - f(y) - f'(y)(\tilde{y} - y)\|_{L^2(\Omega)}}{\|\tilde{y} - y\|_{H^2(\Omega)}} \\
&= \frac{\|(f'(\xi(x)) - f'(y))(\tilde{y} - y)\|_{L^2(\Omega)}}{\|\tilde{y} - y\|_{H^2(\Omega)}} \\
&\leq K \left\|f'(\xi(x)) - f'(y)\right\|_{L^\infty(\Omega)} \frac{\|\tilde{y} - y\|_{L^\infty(\Omega)}}{\|\tilde{y} - y\|_{L^\infty(\Omega)}} \\
&\leq K \left\|f'(\xi(x)) - f'(y)\right\|_{L^\infty(\Omega)} \underset{\|\tilde{y}-y\|_{L^\infty} \to 0^+}{\longrightarrow} 0,
\end{aligned}$$

where the last limit is justified by the continuity of f'. Hence, the first differential $D_y F(y, u)$ is invertible. Then, by the implicit function theorem, $G \in C^2(L^2(\omega); H^2(\Omega))$ and $\langle DG(u), v\rangle$ solves (3.1.15), with control u and direction v. The last statement follows by differentiating twice with respect to u in the functional equation

$$-\Delta G(u) + f(G(u)) = u\chi_\omega.$$

This finishes the proof. □

We are now ready to compute the first Fréchet differential of J_s.

Proposition 3.1.6. *The functional $J_s : L^2(\omega) \longrightarrow \mathbb{R}$ defined in* (3.1.13) *is of class C^2. Take a control $u \in L^2(\omega)$, a direction $v \in L^2(\omega)$ and the state $y = G(u)$. The Fréchet differential of J_s at u along v reads as*

$$\langle DJ_s(u), v\rangle = \int_\omega (u + q)\, v dx,$$

where q is the adjoint state, solution to

$$\begin{cases} -\Delta q + f'(y)q = (y - z)\,\chi_{\omega_0} & \textit{in } \Omega \\ q = 0 & \textit{on } \partial\Omega. \end{cases} \tag{3.1.17}$$

Furthermore, the second Fréchet differential of J_s at u along v_1 and v_2

$$\langle d^2 J_s(v)v_1, v_2\rangle = \int_\omega v_1 v_2 dx + \beta \int_{\omega_0} \psi_{v_1}\psi_{v_2} dx - \beta \int_\Omega f''(y)\, q\psi_{v_1}\psi_{v_2} dx, \tag{3.1.18}$$

where ψ_{v_i} is the solution to the linearized problem

$$\begin{cases} -\Delta w + f'(y)w = v_i\chi_\omega & \textit{in } \Omega \\ w = 0 & \textit{on } \partial\Omega. \end{cases} \tag{3.1.19}$$

Proof. Step 1 **Computation of the first Fréchet differential**
By Proposition 3.1.5, the control-to-state map G is C^2. Then, by the chain rule, $J_s \in C^2(L^2(\omega); \mathbb{R})$ and we have

$$\langle DJ_s(u), v\rangle = \int_\omega uv dx + \beta \int_{\omega_0} (y - z)\, w dx, \tag{3.1.20}$$

where w solves the linearized problem

$$\begin{cases} -\Delta w + f'(y)w = v\chi_\omega & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiply the adjoint problem (3.1.17) by w, getting

$$\beta \int_{\omega_0} (y - z)\, w dx = \int_\Omega \left[\nabla q \cdot \nabla w + f'(y)qw\right] dx = \int_\omega qv dx.$$

By the above equality and (3.1.20), we have

$$\langle DJ_s(u), v\rangle = \int_\omega (u + q)\, v dx,$$

as required.

Step 2 **Computation of the second Fréchet differential**

By applying the chain rule, we have

$$\langle d^2 J_s(v)v_1, v_2\rangle = \int_\omega v_1 v_2 dx + \beta \int_{\omega_0} \psi_{v_1}\psi_{v_2} dx + \beta \int_{\omega_0} w_{v_1,v_2}\,(y - z)\, dx, \tag{3.1.21}$$

where w_{v_1,v_2} satisfies

$$\begin{cases} -\Delta w_{v_1,v_2} + f'(y) w_{v_1,v_2} + f''(y) w_{v_1} w_{v_2} = 0 & \text{in } \Omega \\ w_{v_1,v_2} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.22}$$

At this point, we multiply the adjoint equation (3.1.17) by w_{v_1,v_2}, obtaining

$$\int_{\omega_0} (y - z)\, w_{v_1,v_2} dx = \int_\Omega \left[\nabla q \cdot \nabla w_{v_1,v_2} + f'(y) q w_{v_1,v_2}\right] dx = -\int_\Omega f''(y) q w_{v_1} w_{v_2} dx, \tag{3.1.23}$$

where the last equality is given by (3.1.22). By (3.1.21) and the above equality, we get

$$\langle d^2 J_s(v)v_1, v_2\rangle = \int_\omega v_1 v_2 dx + \beta \int_{\omega_0} \psi_{v_1}\psi_{v_2} dx - \beta \int_\Omega f''(y)\, q \psi_{v_1}\psi_{v_2} dx, \tag{3.1.24}$$

as required. □

The Fermat necessary condition for (3.1.12)-(3.1.13) reads as system of elliptic PDEs.

Proposition 3.1.7 (First order optimality conditions)**.** *Let $\overline{u}$ be an optimal control for* (3.1.13)-(3.1.12)*. Then, $\overline{u} = -\overline{q}$, where the pair $(\overline{y}, \overline{q})$ satisfies the optimality system*

$$\begin{cases} -\Delta \overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta \overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.25}$$

Proof. By Proposition 3.1.6 and Proposition 3.1.2, for any direction $v \in L^2(\omega)$, we have

$$\int_\omega (\overline{u} + \overline{q})\, v dx = \langle DJ_s(\overline{u}), v\rangle = 0,$$

whence $\overline{u} = -\overline{q}\chi_\omega$, as required. □

In the following proposition, we show how second order necessary and sufficient conditions read for J_s. Note that the second order sufficient condition requires only $\langle D^2 J_s(\overline{u})\, v, v\rangle > 0$, for $v \neq 0$, instead of the stronger one $\langle D^2 J_s(\overline{u})\, v, v\rangle \geq \alpha \|v\|^2_{L^2(\omega)}$. Such conditions are equivalent in finite dimension. In general, the equivalence does not hold in infinite dimension. Despite of that, in our special case, we have the equivalence, because of:

- the compactness of the relation control-to-state;
- the strict convexity in u of the control term in the functional.

We present now a result obtained by E. Casas and F. Tröltzsch in [27].

Proposition 3.1.8 (Second order optimality conditions)**.** *Consider the functional J_s introduced in* (3.1.12)-(3.1.13)*.*

1. *If $\overline{u}$ is a local minimizer of J_s, we have*

$$\langle D^2 J_s(\overline{u})\, v, v\rangle \geq 0, \quad \forall\ v \in L^2(\omega); \tag{3.1.26}$$

2. *if $DJ_s(\overline{u}) = 0$ and*

$$\langle D^2 J_s(\overline{u})\, v, v\rangle > 0, \quad \forall\ v \in L^2(\omega) \setminus \{0\}, \tag{3.1.27}$$

then $\overline{u}$ is a local minimizer and there exists $\alpha > 0$ and $r > 0$ such that

$$J_s(u) \geq J_s(\overline{u}) + \frac{\alpha}{2}\|u - \overline{u}\|_{L^2(\omega)}, \quad \forall\ u \in B(\overline{u}, r), \tag{3.1.28}$$

with $B(\overline{u}, r) := \left\{u \in L^2(\omega) \mid \|u - \overline{u}\| < r\right\}$.

Proof. In view of Proposition 3.1.3, it suffices to prove that (3.1.27) yields

$$\langle D^2 J_s(\overline{u})\, v, v\rangle \geq \alpha \|v\|^2_{L^2(\omega)}, \quad \forall\ v \in B(\overline{u}, r), \tag{3.1.29}$$

for some $\alpha > 0$. Suppose, by contradiction, for any positive integer k, there exists a direction v_k such that

$$\langle D^2 J_s(\overline{u})\, v_k, v_k\rangle < \frac{1}{k}\|v_k\|^2_{L^2(\omega)}. \tag{3.1.30}$$

By the above inequality, $v_k \neq 0$. Set $\tilde{v}_k := \frac{v_k}{\|v_k\|_{L^2(\omega)}}$. By Banach-Alaoglu theorem, there exists $v \in L^2(\omega)$, such that, up to subsequences,

$$\tilde{v}_k \underset{k\to+\infty}{\rightharpoonup} v, \tag{3.1.31}$$

weakly in $L^2(\omega)$. Set $w_k := \langle DG(\overline{u}), \tilde{v}_k\rangle$ and $w := \langle DG(\overline{u}), v\rangle$. Now, $DG(\overline{u}) : L^2(\omega) \longrightarrow C^0(\overline{\Omega})$ is compact. Then,

$$w_k \underset{k\to+\infty}{\longrightarrow} w, \tag{3.1.32}$$

strongly in $C^0(\overline{\Omega})$. Set $\overline{y} = G(\overline{u})$ and $\overline{q}$ adjoint state solution to (3.1.17), with state $\overline{y}$. Then,

$$\langle D^2 J_s(\overline{u}) v, v\rangle \leq \liminf_{k\to+\infty} \langle D^2 J_s(\overline{u}) v_k, v_k\rangle \leq 0. \tag{3.1.33}$$

Hence, (3.1.27) leads to $v = 0$. Now,

$$\begin{aligned} 1 &= \|\tilde{v}_k\|^2_{L^2(\omega)} = \liminf_{k\to+\infty} \int_\omega \tilde{v}_k^2 dx \\ &= \liminf_{k\to+\infty} \left[\langle D^2 J_s(\overline{u})\tilde{v}_k, \tilde{v}_k\rangle - \beta\int_{\omega_0} \psi^2_{\tilde{v}_k} dx + \beta \int_\Omega f''(\overline{y})\, \overline{q}\psi^2_{\tilde{v}_k} dx\right] \\ &\leq 0, \end{aligned} \tag{3.1.34}$$

so obtaining a contradiction. □

3.2 Controllability of Partial Differential Equations

This section is devoted to the presentation of some existing results on controllability of PDEs. Our presentation has been inspired by [153, 92, 34, 140].

The exact controllability problem is one of the most challenging problems in control theory. Consider a controlled ODE or PDE

$$\frac{d}{dt} y = A(y) + B(u). \tag{3.2.1}$$

Roughly speaking, given an initial datum y_0 and a final target y_1, the equation (3.2.1) is said to be *exactly controllable* in time T from y_0 to y_1 if there exists a control u such that the following initial-final value problem admits a solution

$$\begin{cases} \frac{d}{dt} y = A(y) + B(u) & \text{in } (0,T) \\ y(0) = y_0, \; y(T) = y^1. \end{cases} \tag{3.2.2}$$

Control of Ordinary Differential Equations has been widely investigated. In the linear case, the controllability is equivalent to the Kalman rank condition ([135, chapter 2, section 2.2]). A rich literature is available on controllability of nonlinear ODEs ([34, 134]).

Controllability of Partial Differential Equations (both linear and nonlinear) is more delicate. By now an extensive literature is available in the topic. We shall give a general presentation on the topic. It is impossible to mention all the contributions. For further details, we refer to the following articles and books and the references therein: [153, 53, 51, 93, 92, 12, 87, 45, 77, 34, 140, 86].

We start by considering linear PDEs in the abstract framework introduced in [140]. Let H and U be two Hilbert spaces endowed with scalar products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_U$ respectively. H

is called the state space and U the control space. Let $A : D(A) \subset H \longrightarrow H$ be a generator of a C_0-semigroup $(\mathbb{T}_t)_{t\in\mathbb{R}^+}$, with $\mathbb{R}^+ = [0, +\infty)$. The domain of the generator $D(A)$ is endowed with the graph norm $\|x\|^2_{D(A)} = \|x\|^2_H + \|Ax\|^2_H$. We define H_{-1} as the completion of H with respect to the norm $\|\cdot\|_{-1} = \|(\beta I - A)^{-1}(\cdot)\|_H$, with real β such that $(\beta I - A)$ is invertible from H to H with continuous inverse. Adapting the techniques of [140, Proposition 2.10.2], one can check that the definition of H_{-1} is actually independent of the choice of β. By applying the techniques of [140, Proposition 2.10.3], we deduce that A admits a unique bounded extension A from H to H_{-1}. For simplicity, we still denote by A the extension. Hereafter, we write $\mathscr{L}(E, F)$ for the space of all bounded linear operators from a Banach space E to another Banach space F.

Our control system is governed by:

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t), \quad t \in (0, \infty), \\ y(0) = y_0, \end{cases} \tag{3.2.3}$$

where $y_0 \in H$, $u \in L^2_{loc}([0, +\infty), U)$ is a control function and the control operator $B \in \mathscr{L}(U, H_{-1})$ satisfies the admissibility condition in the following definition (see [140, Definition 4.2.1]).

Definition 3.2.1. *The control operator $B \in \mathscr{L}(U, H_{-1})$ is said to be admissible if for all $\tau > 0$ we have Range$(\Phi_\tau) \subset H$, where $\Phi_\tau : L^2((0, +\infty); U) \to H_{-1}$ is defined by:*

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - r} Bu(r) dr.$$

From now on, we will always assume the control operator to be admissible. One can check that for any $y_0 \in H$ and $u \in L^2_{loc}((0, +\infty); U)$ there exists a unique mild solution $y \in C^0([0, +\infty), H)$ to (5.2.1) (see, for instance, [140, Proposition 4.2.5]). We denote by $y(\cdot; y_0, u)$ the unique solution to (5.2.1) with initial datum y_0 and control u.

We now introduce the concepts of *approximate controllability*, *null controllability* and *exact controllability* in time $T > 0$.

Definition 3.2.2. *Let $T > 0$.*

- *the pair (A, B) is approximately controllable in time T if for any y_0 and y_1 in H and for any $\varepsilon > 0$, there exists a control $u \in L^2(0, T; U)$ such that $\|y(T; y_0, u) - \|_H < \varepsilon$;*
- *the pair (A, B) is null controllable in time T if for any $y_0 \in H$, there exists a control $u \in L^2(0, T; U)$ such that $y(T; y_0, u) = 0$;*
- *the pair (A, B) is exactly controllable in time T if for any y_0 and y_1 in H, there exists a control $u \in L^2(0, T; U)$ such that $y(T; y_0, u) = y_1$.*

By the linearity of the equation, *null controllability* is equivalent to *controllability to trajectories*.

Definition 3.2.3. *The pair (A, B) is controllable to trajectories in time T if for any y_0, $\overline{y}_0$ in H and $\overline{u} \in L^2(0, T; U)$, there exists a control $u \in L^2(0, T; U)$ such that $y(T; y_0, u) = y(T; \overline{y}_0, \overline{u})$.*

Remark 3.2.1. *The exact controllability of the pair (A, B) in time T implies both null controllability and approximate controllability. Moreover, if the Range$(\mathbb{T}_T)$ is dense in H, null controllability implies approximate controllability. Finally, if $\mathbb{T}$ is right invertible, then (A, B) is exactly controllable in time T if and only if (A, B) is null controllable in time T.*

Already in the linear case, we observe substantial differences between controllability of ODEs and PDEs:

- in the finite dimensional case, all the aforementioned concepts of controllability are equivalent to the Kalman rank condition ([135, chapter 2, section 2.2])

$$\text{rank}\left[B, AB, \ldots, A^{N-1}B\right] = N.$$

  There exist *null controllable* PDEs, which fail to be *exactly controllable* in the sense of definition 3.2.2. An example can be the heat equation controlled from the boundary, where the equation evolves. In this case, the regularizing effect prevents the state to reach exactly arbitrary targets in the state space L^2;

- in ODE control if controllability holds in some time T, then it is verified in any time. This is not the case, for instance, for the wave equation, due to the finite velocity of propagation.

The aforementioned controllability concepts are equivalent to corresponding observability concepts. This was pointed out by D. L. Russell in [128] and by J. L. Lions [93]. We follow the presentation of [140].

Let Y be a Hilbert space, hereafter called *observation space*. Let $C \in \mathscr{L}(D(A), Y)$ be a bounded linear operator named *observation operator*. We introduce the output operator.

Definition 3.2.4. *Let $T > 0$ be a time horizon. The output operator $\psi_T : D(A) \longrightarrow L^2(0, +\infty; Y)$ is defined as*

$$\psi_T \varphi_0 := \begin{cases} C\mathbb{T}_t\varphi_0 & t \in [0, T] \\ 0 & t > T \end{cases} \tag{3.2.4}$$

for any $\varphi_0 \in D(A)$.

The operator ψ_T belongs to $\mathscr{L}(D(A), L^2(0, +\infty; Y))$.

Definition 3.2.5. *The operator $C \in \mathscr{L}(D(A), Y)$ is said to be admissible observation operator for $\mathbb{T}$ if, for some $T > 0$, ψ_T has a continuous extension to H.*

By [140, Proposition 4.3.2], the admissibility concept is independent of T. We have the following observability concepts for the pair (A, C).

Definition 3.2.6. *Let* $T > 0$.

- *the pair* (A, C) *is approximatively observable in time* T *if* $\ker(\psi_T) = \{0\}$*;*
- *the pair* (A, C) *is final state observable in time* T *if there exists* $K_T > 0$ *such that*

$$\int_0^T \|C\mathbb{T}_t\varphi_0\|_Y^2 dt \geq K_T^2 \|\mathbb{T}_T\varphi_0\|_H^2, \quad \forall\ \varphi_0 \in X;$$

- *the pair* (A, C) *is exactly observable in time* T *if there exists* $K_T > 0$ *such that*

$$\int_0^T \|C\mathbb{T}_t\varphi_0\|_Y^2 dt \geq K_T^2 \|\varphi_0\|_H^2, \quad \forall\ \varphi_0 \in X;$$

First of all, the admissibility concepts are dual. Indeed, given an operator $B \in \mathscr{L}(U, H_{-1})$, B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for $\mathbb{T}^*$ [140, Theorem 4.4.3]. We have now the duality of observability and controllability (see [140, Theorem 11.2.1]).

Theorem 3.2.1. *Assume* $B \in \mathcal{L}(U, H_{-1})$ *is an admissible control operator for* $\mathbb{T}$*. Let* $T > 0$ *be a time horizon.*

1. *The pair* (A, B) *is approximatively controllable in time* T *if and only if* (A^*, B^*) *is approximately observable in time* T*.*
2. *The pair* (A, B) *is null controllable in time* T *if and only if* (A^*, B^*) *is final state observable in time* T*.*
3. *The pair* (A, B) *is exactly controllable in time* T *if and only if* (A^*, B^*) *is exactly observable in time* T*.*

We have reduced the proof of controllability to the proof of an observability inequality.

We now consider two paradigmatic examples: the linear heat and the wave equations.

Let Ω be a connected bounded open set of $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary, and consider the heat equation controlled from the interior

$$\begin{cases} y_t - \Delta y = u\chi_\omega & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0 \in L^2(\Omega), & \text{in } \Omega \end{cases} \tag{3.2.5}$$

where y is the state, while u is the control acting on a nonempty subdomain $\omega \subsetneq \Omega$. First of all, we observe that exact controllability to any target $y_1 \in L^2(\Omega)$ cannot be achieved. Indeed, by the regularizing effect of the heat equation, for any control initial datum y_0 and control u, the corresponding controlled solution $y \in C^\infty(\Omega \setminus \overline{\omega})$. However, the null controllability holds in any time $T > 0$. Even in time small, we can drive the system to zero by an highly oscillatory control. This is due to the infinite speed of propagation for the heat equation. In one space dimension, this has been proved by H. O. Fattorini and D. L. Russell in the seminal paper [51], by moment method. Alternative approaches for the one dimensional case are the flatness method (see [34, section 2.5] and [100]) and the characterization of the reachable set in the class of holomorphic functions (see [67, 85, 39, 81]). By now, different strategies are available for the multidimensional case as well: Lebeau-Robbiano strategy [87], Carleman estimates for the parabolic operators [45] and transmutation [49]. More recently, J. M. Coron and H. M. Nguyen proved controllability by means of a feedback control of the heat equation in one dimension using a backstepping approach [35].

Under the same assumptions, we consider the internal controllability problem for the wave equation

$$\begin{cases} y_{tt} - \Delta y + cy = u\chi_\omega & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0^0(x), \ y_t(0,x) = y_0^1(x) & \text{in } \Omega \end{cases} \tag{3.2.6}$$

where $y = y(t,x)$ is the state, while $u = u(t,x)$ is the control whose action is localized on the subdomain $\omega \subset \Omega$. The coefficient $c = c(x)$ is C^∞ smooth in $\overline{\Omega}$. It is well known in the literature (e.g. [50, section 7.2]) that, for any initial datum $(y_0^0, y_0^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and for any control $u \in L^2((0,T) \times \omega)$, the above problem admits an unique solution $(y, y_t) \in C^0([0,T]; H_0^1(\Omega) \times L^2(\Omega))$, with $y_{tt} \in L^2(0,T; H^{-1}(\Omega))$. Controllability of the wave equation in time T^* has been completely characterized in terms of the *Geometric Control Condition* (GCC) [12, 17], on (Ω, ω, T^*), which basically asserts that all bicharacteristic rays enter in the subdomain ω in time smaller than T^*. In our problem (3.2.6), the bicharacteristic rays read as:

$$(t(s), x(s)) = (s\tau, s\xi), \quad \text{with} \quad |\tau| = \|\xi\|.$$

We conclude this section with the study of the controllability of a semilinear heat equation

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{3.2.7}$$

where the nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is C^1 regular, with $f(0) = 0$.

If the nonlinearity f is globally Lipschitz, null controllability holds in any time $T > 0$. The proof is based on the controllability of the linearized problem and fixed-point argument (see, for instance, [77, Theorem 3.1]).

Consider a nonlinearity fulfilling the following conditions

$$|f'(y)| \leq C\left(1+|y|^{p}\right), \quad \text{a.e.}, \quad \text{with } p \leq 1+\frac{4}{n}$$

and

$$\frac{f(y)}{|y|\log^{3/2}(1+|y|)} \to 0, \quad \text{as } |y| \to +\infty.$$

By a fine estimate of the cost of controllability of the heat equation with a linear potential and a Kakutani fixed-point theorem, null controllability holds. In particular, the control can be employed to avoid blow up in finite time (E. Fernández-Cara and E. Zuazua [53]). Recently, these results has been extended to the blowing up nonlinearity $f(y) = -|y|\log^{\alpha}(1+|y|)$, with $\alpha \in [3/2, 2)$ [86].

We conclude considering the case, when f is nondecreasing. In this case, the free dynamics is exponential dissipative. Null controllability holds in large time. This can be seen by applying a strategy "stabilization+control", as in the proof of Theorem 4.1.2, chapter 4. Now, take f is strongly superlinear, namely satisfying one of the following condition

$$\begin{aligned} f(z) &\geq Cz\left(\log|z|\right)^{p}, \quad z > z_0 > 1, p > 2 \\ f(z) &\leq Cz\left(\log|z|\right)^{p}, \quad z < z_0 < -1, p > 2. \end{aligned}$$

Then, null controllability fails in time too small [7, Theorem 3], namely the controllability time is strictly positive. In case the nonlinearity is strongly superlinear, the approximate controllability is not verified (see [42, Theorem 3]). Indeed, strongly dissipative nonlinearities produce a damping the effect of the control as it expands from the controlled region into the uncontrolled region.

In case no growth or sign conditions are imposed on f, then semilinear heat can be controlled in between path connected steady states [36]. The proof is based on quasi-static deformations.

Chapter 4

Controllability under positivity constraints of semilinear heat equations

4.1 Introduction

This chapter corresponds to [111].

Controllability of partial differential equations has been widely investigated during the past decades (see, for instance, the following articles and books and the references therein: [153, 53, 92, 12, 87, 45, 77, 34, 140, 86]).

On the other hand, in many models of heat conduction in thermal diffusion, biology or population dynamics, some constraints on the state need to be imposed when facing practical applications (see, for instance, [107, 74, 141]). Furthermore, some constraints can be imposed at the control level, to model some prescribed bounds on the available power to control the system.

In this chapter we mainly focus on the controllability problem for semilinear heat equations under unilateral constraints. In other words, our aim is to analyse if the parabolic equation under consideration can be driven to a desired final target by means of the control action, but preserving some constraints on the control and/or the state. To fix ideas we focus on nonnegativity constraints.

As it is well known by now, a wide class of linear and semilinear parabolic systems, in the absence of constraints, is controllable in any positive time (see [87, 45]). And, often times, norm-optimal controls achieving the target at the final time are restrictions of solutions of the adjoint system. Accordingly these controls experience large oscillations in the proximity of the final time. In particular, when the time horizon is too short, these oscillations prevent the control to fulfill the positivity constraint.

Therefore, the question of controlling the system by means of nonnegative controls requires further investigation. This question has been addressed in [95] where, in the context of the linear heat equation, the constrained controllability in large time was proved, showing also the existence of a minimal controllability or waiting time.

The purpose of the present chapter is to extend the analysis in [95] to semilinear parabolic equations, considering the controllability problem under positivity constraints on the boundary control. As a consequence, employing the comparison or maximum principle for parabolic equations, we deduce the controllability under positivity constraints on the *state* too.

In [95] the constrained controllability property was proved using the dissipativity of the system, enabling to show the exponential decay of the observability constant. This allows to show that, in large time intervals, the controls can be chosen to be small, which in turn implies constrained controllability. In the present chapter, inspired by [36], we show that dissipativity is not needed for steady state constrained controllability, the aim being to control the system from one steady-state to another one, both belonging to the same connected component of the set of steady states. The method of proof, that uses a ''stair-case argument'', does not require any dissipativity assumption and is merely based on the controllability of the system without constraints. It consists in moving from one steady state to a neighbouring one, using small amplitude controls, in a recursive manner, so to reach the final target after a number of iterations and preserving the constraints on the control imposed a priori.

This iterative method, though, leads to constrained control results only when the time of control is long enough, and this time horizon increases when the distance between the initial and final steady states increases. On the other hand, the method cannot be applied to an arbitrary initial state. In fact, we give an example showing that, when the system is nondissipative, constrained controllability in large time may fail for general L^2-initial data. Achieving the constrained controllability result for general initial data and final target trajectories of the system requires to assume that the system to be dissipative and the control time long enough. Summarising, although dissipativity is not needed for steady state constrained controllability, it is crucial when considering general initial data.

Once the control property has been achieved with nonnegative controls, the classical comparison or maximum principle for parabolic equations allows proving that the same property holds under positivity constraints on the state.

But all previous techniques and results require the control time to be long enough. It is then natural to analyse whether constrained controllability can be achieved in an arbitrarily small time. In [95] it was shown, for the linear heat equation, that constrained controllability does not hold when the time horizon is too short. As we shall see, under some assumptions on the nonlinearity, the same occurs for semilinear parabolic equations so that, the minimal constrained controllability time, $T_{\min}$, is necessarily strictly positive, showing a *waiting time phenomenon*. Finally, in [95] it was also proved that, actually, constrained controllability of the heat equation holds in the minimal time with finite measures as controls. This result is also generalised in the present chapter.

4.1.1 Statement of the main results

Let Ω be a connected bounded open set of $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary, and let us consider the boundary control system:

$$\begin{cases} y_t - \operatorname{div}(A\nabla y) + b \cdot \nabla y + f(t,x,y) = 0 & \text{in } (0,T) \times \Omega \\ y = u\mathbf{1}_\Gamma & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{4.1.1}$$

where $y = y(t,x)$ is the state, while $u = u(t,x)$ is the control acting on a relatively open and non-empty subset Γ of the boundary $\partial\Omega$.

Moreover, $A \in W^{1,\infty}(\mathbb{R}^+ \times \Omega; \mathbb{R}^{n\times n})$ is a uniformly coercive symmetric matrix field, $b \in W^{1,\infty}(\mathbb{R}^+ \times \Omega; \mathbb{R}^n)$ and the nonlinearity $f : \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be of class C^1.

Since f is not supposed to be globally Lipschitz, blow up phenomena in finite time may occur and the existence of solutions for bounded data can be guaranteed only locally in time. In fact, as it was shown in [53], the boundary controllability of semilinear parabolic equations can only be guaranteed for nonlinearities with a very mild superlinear growth. For the sake of completeness the well posedness of the above system is analyzed in the Appendix.

As mentioned above, in our analysis we distinguish the following two problems:

- *Steady state controllability*: In this case the system is not required to be dissipative, and the coefficients of (4.1.1) are assumed to be only space dependent;
- *Controllability of general initial data to trajectories*: The dissipativity of the system is needed in this case.

The main ingredients that our proofs require are as follows:

- local controllability;
- a recursive "stair-case" argument to get *global* steady state controllability;
- the dissipativity of the system to control general initial data to trajectories;
- the maximum or comparison principle to obtain a state constrained result.

We also prove the lack of constrained controllability when the control time horizon is too short. We do it employing different arguments:

- for the linear case, we use the definition of solution by transposition, and we choose specific solutions of the adjoint system as test functions;
- the same proof, with slight variations, applies when the nonlinearity is globally Lipschitz too;

- if the nonlinearity is "strongly" superlinear and nondecreasing, inspired by [70] and [7], a barrier function argument can also be applied.

Remark 4.1.1. *The conclusions of the present chapter can be deduced employing the same arguments for the internal control. In particular, the problem of local controllability of* (4.1.1) *can be addressed by using the techniques of [53].*

4.1.1.1 Steady state controllability

As announced, for steady state controllability, we do not assume the system to be dissipative but we ask the coefficients and the nonlinearity f to be time-independent, namely $A = A(x)$, $b = b(x)$ and $f = f(x, y)$. This allows to easily employ and exploit the concept of steady state and their very properties.

More precisely, let us first introduce the set of bounded steady states for (4.1.1).

Definition 4.1.1. *Let $\overline{u} \in L^\infty(\Gamma)$ be a steady boundary control. A function $\overline{y} \in L^\infty(\Omega)$ is said to be a bounded steady state for* (4.1.1) *if for any test function $\varphi \in C^\infty(\Omega)$ vanishing on $\partial\Omega$:*

$$\int_\Omega [-div(A\nabla\varphi) - div(\varphi b)]\, \overline{y} dx + \int_\Omega f(x, \overline{y})\varphi dx + \int_\Gamma \overline{u}\frac{\partial\varphi}{\partial n} d\sigma(x) = 0.$$

In the above equation, $n = A\hat{v}/\|A\hat{v}\|$, where $\hat{v}$ is the outward unit normal to Γ. We denote by $\mathscr{S}$ the set of bounded steady states endowed with the L^∞ distance.

In our first result, the initial and final data of the constrained control problem are steady states joined by a continuous arc within $\mathscr{S}$. This arc of steady states is then a datum of the problem, allowing to build the controlled path in an iterative manner, by the stair-case argument.

The existence of steady-state solutions with non-homogeneous boundary values (the control) can be analysed reducing it to the case of null Dirichlet conditions, splitting $y = z + w$, where z is the unique solution to the linear problem:

$$\begin{cases} -\text{div}(A\nabla z) + b \cdot \nabla z = 0 & \text{in } \Omega \\ z = u\mathbf{1}_\Gamma & \text{on } \partial\Omega \end{cases} \tag{4.1.2}$$

and w is a solution to:

$$\begin{cases} -\text{div}(A\nabla w) + b \cdot \nabla w + f(x, w + z) = 0 & \text{in } \Omega \\ w = 0. & \text{on } \partial\Omega \end{cases}$$

The first problem (4.1.2) can be treated by transposition techniques (see [91]), employing Fredholm Theory (see [57, Theorem 5.11 page 84]). But the second one needs the application of fixed point methods (see [57, Part II]).

As mentioned above, we assume the initial datum y_0 and the final target y_1 to be *path-connected* steady states, i.e. we suppose the existence of a continuous path:

$$\gamma : [0,1] \longrightarrow \mathscr{S},$$

$$r \longmapsto \gamma_r,$$

such that $\gamma_0 = y_0$ and $\gamma_1 = y_1$. Furthermore, we call $\overline{u}_r$ the boundary value of γ_r for each $r \in [0,1]$. In the linear case, two arbitrary steady states are linked by the convex combination $\gamma_r := (1-r)y_0 + ry_1$. The construction of the path of steady states in the nonlinear case is not trivial. However, in some nonlinear models the path has been constructed by viewing the steady equation as a dynamical system or by applying the implicit function theorem [117, 126, 101].

We have the following result, inspired by the methods in [36, Theorem 1.2].

Theorem 4.1.1 (Steady state controllability)**.** *Let y_0 and y_1 be path connected (in $\mathscr{S}$) bounded steady states. Assume there exists $\nu > 0$ such that*

$$\overline{u}_r \geq \nu, \quad a.e.\ on\ \Gamma \tag{4.1.3}$$

for any $r \in [0,1]$. Then, if T is large enough, there exists $u \in L^\infty((0,T) \times \Gamma)$, a control such that:

- *the problem* (4.1.1) *with initial datum y_0 and control u admits a unique solution y verifying $y(T,\cdot) = y_1$;*
- *$u \geq 0$ a.e. on $(0,T) \times \Gamma$.*

It is implicit in the statement of this result that the constructed trajectory does not blow up in $[0,T]$. The strategy of proof relies on keeping the trajectory in a narrow tubular neighborhood of the arc of steady states connecting the initial and the final ones. This result does not contradict the lack of controllability for blowing up semilinear systems (see [53, Theorem 1.1]), since we work in the special case where the initial and final data are bounded steady states connected within the set of steady states.

Remark 4.1.2. *Assume the target $y_1 \in C^\infty(\overline{\Omega})$. If T is large enough, by slightly changing our techniques, one can construct a C^∞-smooth nonnegative control u steering our system* (4.1.1) *from y_0 to y_1 in time T.*

4.1.1.2 Controllability of general initial data to trajectories

In this case, both the coefficients and the nonlinearity f can be time-dependent. We suppose the system to be dissipative, namely:

$$
\text{(D)} \begin{cases} \exists\, C_1 \in \mathbb{R}^+ \quad \text{such that} \quad f(t,x,y_2) - f(t,x,y_1) \geq -C_1(y_2 - y_1) \\ \text{a.e.}\, (t,x) \in \mathbb{R}^+ \times \Omega, \quad y_1 \leq y_2, \\ \\ \int_\Omega \left(\nabla(y_2 - y_1)\right)^T A \nabla(y_2 - y_1) dx + \int_\Omega (b \cdot \nabla(y_2 - y_1))\ (y_2 - y_1) dx \\ + \int_\Omega (f(t,x,y_2) - f(t,x,y_1))(y_2 - y_1) dx \geq \lambda \|y_2 - y_1\|^2_{H^1_0(\Omega)} \\ \text{a.e.}\, t \in \mathbb{R}^+ \ \text{ and } \ \forall (y_1, y_2) \in H^1_0(\Omega)^2, \end{cases}
$$

for some $\lambda > 0$.

These additional assumptions guarantee the global existence of solution for L^2 data in any time $T > 0$ (see [106] and [8]), i.e. for any $y_0 \in L^2(\Omega)$ and control $u \in L^2((0,T) \times \Gamma)$ the system (**??**) admits an unique solution:

$$
y \in L^2((0,T) \times \Omega) \cap C^0([0,T]; H^{-1}(\Omega)).
$$

As we shall see (proof of Theorem 4.1.2 and Lemma 4.3.1), this L^2-dissipativity property and the smoothing effect of the heat equation will allow also to control distances between differences of trajectories in L^∞.

In this context, we are able to extend Theorem 4.1.1 to more general initial data and final targets.

Theorem 4.1.2 (Controllability of general initial data to trajectories). *Assume that the dissipativity assumption (D) holds.*

Consider a target trajectory $\overline{y}$, solution to (**??**) *with initial datum $\overline{y}_0 \in L^2$ and control $\overline{u} \in L^\infty$, verifying the positivity condition:*

$$
\overline{u} \geq \nu, \quad \text{a.e. on } (0,T) \times \Gamma, \tag{4.1.4}
$$

with $\nu > 0$.

Then, for any initial datum $y_0 \in L^2(\Omega)$, in large time, we can find a control $u \in L^\infty((0,T) \times \Gamma)$ such that:

- *the unique solution y to* (4.1.1) *with initial datum y_0 and control u is such that $y(T,\cdot) = \overline{y}(T,\cdot)$;*
- *$u \geq 0$ a.e. on $(0,T) \times \Gamma$.*

Remark that the dissipativity property (D) is actually needed to control a general initial datum to trajectories. Indeed, even in the linear case, removing dissipativity, constrained controllability may fail in *any time* $T > 0$. This is the object of Proposition 4.4.1.

Remark 4.1.3. *As we have seen in Remark 4.1.2, if the target $\overline{y}$ is smooth, we can build a nonnegative control u smooth as well.*

4.1.1.3 State constraints

We assume that $f(t,x,0) = 0$ for any $(t,x) \in \mathbb{R}^+ \times \Omega$ so that $y \equiv 0$ is a steady-state.

We also assume that the dissipativity assumption (D) holds, so that the system (4.1.1) enjoys also the parabolic comparison or maximum principle (see [118, Theorem 2.2 page 187]). Then, the following state constrained controllability result is a consequence of the above one.

Under these conditions, in the framework of Theorem 4.1.2, if the initial datum $y_0 \geq 0$ a.e. in Ω, and in view of the fact that the control has been built to be nonnegative, then $y \geq 0$ a.e. in $(0,T) \times \Omega$, i.e. the full state satisfies the nonnegativity unilateral constraint too.

Again, in case the target $\overline{y}$ is smooth, we can construct a smooth control u as well, under state and control constrains.

Orientation

The rest of the chapter is organized as follows:

- Section 4.2: Proof of Theorem 4.1.1 by the stair-case method;
- Section 4.3: Proof of Theorem 4.1.2 using the dissipativity property;
- Section 4.4: Counterexample for general initial data in the nondissipative case;
- Section 4.5: Positivity of the minimal time;
- Section 4.6: Numerical simulations and experiments;
- Appendix: Proof of the well posedness and local null controllability for system (4.1.1).

4.2 Steady state controllability-The stair case method

In order to prove Theorem 4.1.1, we need the following two ingredients but we do not need/employ the dissipativity of the system:

1. Local null controllability with controls in L^∞;
2. The stair-case method to get the desired global result.

First of all, let us state the local controllability result. For the sake of simplicity, depending on the context, we denote by $\|\cdot\|_{L^\infty}$ the norm in $L^\infty(\Omega)$, $L^\infty((0,T)\times\Omega)$ or $L^\infty((0,T)\times\Gamma)$.

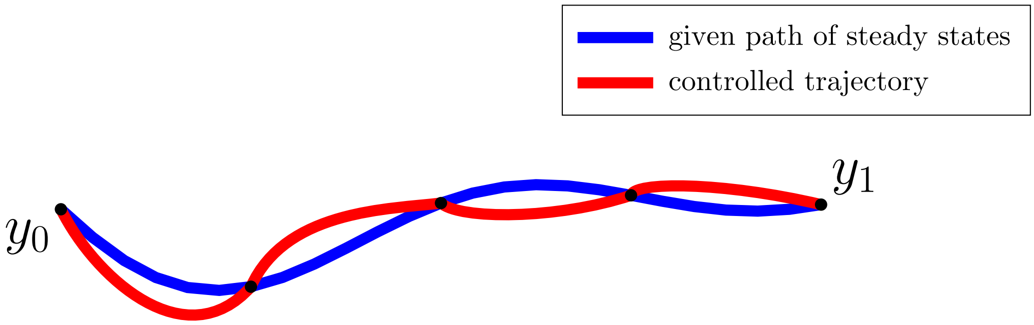


Figure 4.1: stair-case argument

Lemma 4.2.1. *Let $T > 0$ and $R > 0$. Then, there exist C and δ depending on R, and T such that, for all targets $\overline{y} \in L^\infty((0,T) \times \Omega)$ solutions to (4.1.1) with initial datum $\overline{y}_0$ and control $\overline{u}$ and for each initial data $y_0 \in L^\infty(\Omega)$ such that:*

$$\|y_0\|_{L^\infty} \leq R, \quad \|\overline{y}\|_{L^\infty} \leq R \quad \textit{and} \quad \|\overline{y}_0 - y_0\|_{L^\infty} < \delta, \tag{4.2.1}$$

we can find a control $u \in L^\infty((0,T) \times \Gamma)$ such that:

- *the problem (4.1.1) with initial datum y_0 and control u admits a unique solution y such that $y(T, \cdot) = \overline{y}(T, \cdot)$;*
- *the following estimate holds:*

$$\|u - \overline{u}\|_{L^\infty} \leq C \|\overline{y}_0 - y_0\|_{L^\infty}, \tag{4.2.2}$$

  where $\overline{u}$ is the control defining the target $\overline{y}$.

The proof of this lemma is presented in the Appendix.

We accomplish now Task 2, developing the stepwise iterative procedure, which enables us to employ the local controllability property to get the desired global result (see Figure 4.1).

Proof of Theorem 4.1.1. **Step 1.** Consequences of the null-controllability property.

First of all, we take $T = 1$ as time horizon. Let $R = \sup_{r\in[0,1]} \|\gamma_r\|_{L^\infty}$. By Lemma 4.2.1, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any pair of bounded steady states y_0 and y_1 lying on the arc γ such that:

$$\|y_0\|_{L^\infty} \leq R, \quad \|y_1\|_{L^\infty} \leq R \quad \text{and} \quad \|y_1 - y_0\|_{L^\infty} < \delta_\varepsilon \tag{4.2.3}$$

we can find a control $u \in L^\infty$ steering (4.1.1) from y_0 to y_1 and such that:

$$\|u - \overline{u}\|_{L^\infty((0,1)\times\Gamma)} < \varepsilon, \tag{4.2.4}$$

where $\overline{u}$ is the boundary value of y_1.

Step 2. Stepwise procedure and conclusion.

The initial datum y_0 and the final target y_1 to be controlled along a solution of the system, by hypothesis, are linked by a continuous arc γ. Then, let:

$$z_k = \gamma\left(\frac{k}{\overline{n}}\right), \quad k = 0, \dots, \overline{n}$$

be a finite sequence of bounded steady states. Let $\overline{u}_k$ be the boundary value of z_k. Right now, $\|z_k\|_{L^\infty} \leq R$. Moreover, by taking $\overline{n}$ sufficiently large,

$$\|z_k - z_{k-1}\|_{L^\infty(\Omega)} < \delta_\nu, \tag{4.2.5}$$

where δ_ν is given by the smallness condition (4.2.3) with $\varepsilon = \nu$. Then, for any $1 \leq k \leq \overline{n}$, we can find a control u_k joining the steady states z_{k-1} and z_k in time 1. Furthermore,

$$u_k = u_k - \overline{u}_k + \overline{u}_k \geq -\nu + \nu = 0, \quad \text{a.e. on } (0,1) \times \Gamma. \tag{4.2.6}$$

Finally, the control $u : (0, \overline{n}) \longrightarrow L^\infty(\Gamma)$ defined as $u(t) = u_k(t-k)$ for $t \in (k-1, k)$ is the desired one. This concludes the proof. □

By the implemented stepwise procedure, the time of control needed coincides with the number of steps we do. This is of course specific to the particular construction of the control we employ and this does not exclude the possibility of finding another nonnegative control driving (4.1.1) from y_0 to y_1 in a smaller time. Anyhow, the existence of a time, long enough, for control, allows defining the minimal constrained controllability time and, as we shall see in Theorem 4.5.2, under some conditions on the nonlinearity, this minimal time is positive, which exhibits a *waiting time* phenomenon for constrained control, as previously established in the linear case in [95].

Remark 4.2.1. *The stair-case method developed above can be employed to get an analogous state constrained controllability for the Neumann case as in [95, Theorem 4.1].*

Note however that in the Neumann case state constraints and controls constraints are not interlinked by the maximum principle.

4.3 Control of general initial data to trajectories for dissipative systems

As anticipated, in this case we assume that the system satisfies the dissipativity property (D). Then, for any $y_0 \in L^2(\Omega)$ and control $u \in L^2((0,T) \times \Gamma)$ the system (4.1.1) admits an unique solution $y \in L^2((0,T) \times \Omega) \cap C^0([0,T]; H^{-1}(\Omega))$ (see [106] and [8]).

The proof of Theorem 4.1.2 will need the following regularizing property.

Lemma 4.3.1. *Assume that the dissipativity property (D) holds. Let $y_0 \in L^2(\Omega)$ be an initial datum and $u \in L^\infty((0,T) \times \Gamma)$ be a control. Then, the unique solution y to* (4.1.1) *with initial* *datum y_0 and control u is such that $y(t,\cdot) \in L^\infty$ for any $t \in (0,T]$. Furthermore, there exists a constant $C = C(\Omega, A, b, f) > 0$ such that, for any t in $(0,T]$:*

$$\|y(t,\cdot)\|_{L^\infty} \le Ce^{CT} t^{-\frac{n}{4}} \left[\|y_0\|_{L^2} + \|u\|_{L^\infty} + \|f(\cdot,\cdot,0)\|_{L^\infty}\right]. \tag{4.3.1}$$

Proof. **Step 1.** Reduction to the linear case.

Let ψ be the solution to:

$$\begin{cases} \psi_t - \operatorname{div}(A\nabla\psi) + b\cdot\nabla\psi - C_1\psi = |f(\cdot,\cdot,0)| & \text{in } (0,T)\times\Omega \\ \psi = |u|\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ \psi(0,x) = |y_0|, & \text{in } \Omega \end{cases} \tag{4.3.2}$$

where C_1 is the constant appearing in assumptions (D). Then, by a comparison argument, for each $t \in [0,T]$:

$$|y(t,\cdot)| \le \psi(t,\cdot), \quad \text{a.e. in } \Omega. \tag{4.3.3}$$

Step 2. Regularization effect in the linear case.

First of all, we split $\psi = \xi + \chi$, where ξ solves:

$$\begin{cases} \xi_t - \operatorname{div}(A\nabla\xi) + b\cdot\nabla\xi - C_1\xi = |f(\cdot,\cdot,0)| & \text{in } (0,T)\times\Omega \\ \xi = 0 & \text{on } (0,T)\times\partial\Omega \\ \xi(0,x) = |y_0| & \text{in } \Omega \end{cases} \tag{4.3.4}$$

while χ satisfies:

$$\begin{cases} \chi_t - \operatorname{div}(A\nabla\chi) + b\cdot\nabla\chi - C_1\chi = 0 & \text{in } (0,T)\times\Omega \\ \chi = |u|\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ \chi(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{4.3.5}$$

By the maximum principle (see [118]), for each $t \in [0,T]$, $\chi(t,\cdot) \in L^\infty(\Omega)$ and there exists a constant $C = C(\Omega, A, b, f) > 0$ such that:

$$\|\chi(t,\cdot)\|_{L^\infty} \le e^{CT}\|u\|_{L^\infty}.$$

On the other hand, (4.3.4) enjoys a $L^2 - L^\infty$ regularization effect, namely $\xi(t,\cdot) \in L^\infty$ for any t in $(0,T]$. Moreover, there exists a constant $C = C(\Omega, A, b, f) > 0$ such that, for any t in $(0,T]$:

$$\|\xi(t,\cdot)\|_{L^\infty} \le Ce^{CT} t^{-\frac{n}{4}} \|y_0\|_{L^2}.$$

This can be proved using Moser-type techniques (see, for instance, [113, Theorem 1.7], [146] or [90]).

This yields the conclusion for ψ. The comparison argument (4.3.3) finishes the proof. □

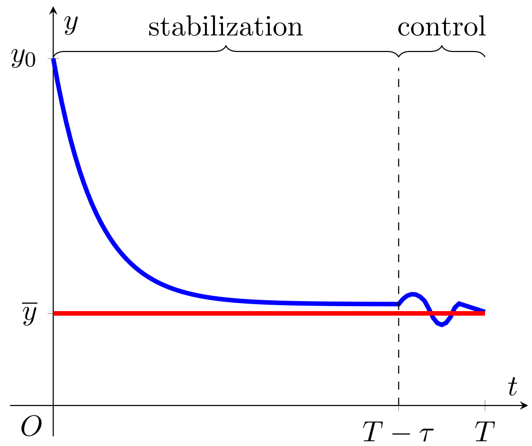


Figure 4.2: illustration of the proof of Theorem 4.1.2 in two steps: stabilization + control

We prove now Theorem 4.1.2, which is illustrated in Figure 4.2.

Proof of Theorem 4.1.2. **Step 1.** Stabilization.

Let $\tau > 0$ be fixed and $T > 2\tau$ be large enough. In the time interval $[0, T-\tau]$, we control y by means of $u = \overline{u}$ so to stabilize y towards $\overline{y}$ in norm L^∞. By the dissipativity property (D), we immediately have the stabilization property in L^2 in $[0, T-2\tau]$, namely:

$$\|y(t,\cdot) - \overline{y}(t,\cdot)\|_{L^2(\Omega)} \leq Ce^{-\frac{\lambda}{C_0^2}t}\|y_0 - \overline{y}_0\|_{L^2(\Omega)}, \quad \forall\, t \in [0, T-2\tau], \tag{4.3.6}$$

the constant C_0 being the Poincaré constant of the domain Ω (see [16, Corollary 9.19 page 290]). Then, we realize that $\eta = y - \overline{y}$ satisfies:

$$\begin{cases} \eta_t - \operatorname{div}(A\nabla\eta) + b\cdot\nabla\eta + \tilde{f}(t,x,\eta) = 0 & \text{in } (T-2\tau, T-\tau)\times\Omega \\ \eta = 0 & \text{on } (T-2\tau, T-\tau)\times\partial\Omega \\ \eta(T-2\tau, x) = y(T-2\tau,\cdot) - \overline{y}(T-2\tau,\cdot), & \text{in } \Omega \end{cases} \tag{4.3.7}$$

where $\tilde{f}(t,x,\eta) = f(t,x,\eta+\overline{y}(t,x)) - f(t,x,\overline{y}(t,x))$. Since the nonlinearity $\tilde{f}$, together with the coefficients A and b, fulfills the dissipative assumptions (D), we are in position to apply Lemma 4.3.1 to (4.3.7) with nonlinearity $\tilde{f}$, getting:

$$\begin{aligned} \|y(T-\tau,\cdot) - \overline{y}(T-\tau,\cdot)\|_{L^\infty(\Omega)} &\leq C(\tau)\|y(T-2\tau,\cdot) - \overline{y}(T-2\tau,\cdot)\|_{L^2(\Omega)} \\ &\leq C(\tau)e^{-\frac{\lambda}{C_0^2}(T-2\tau)}\|y_0 - \overline{y}_0\|_{L^2(\Omega)}, \end{aligned}$$

where in the last inequality we have used (4.3.6).

Step 2. Control. We conclude with an application of Lemma 4.2.1.

Let $\tilde{y}_0 = y(T-\tau,\cdot)$ be the new initial datum and $\overline{y}\restriction_{(T-\tau,T)\times\Omega}$ be the new target trajectory. By the above arguments, if T is large enough, they fulfill (4.2.1) with $R = \|\overline{y}\|_{L^\infty((T-\tau)\times\Omega)} + 1$.

Then, there exists a control $w \in L^\infty((T-\tau,T)\times\Gamma)$ driving (4.1.1) from $y(T-\tau,\cdot)$ to $\overline{y}(T,\cdot)$ in time τ. Furthermore,

$$\|w-\overline{u}\|_{L^\infty} \le C\|y(T-\tau,\cdot)-\overline{y}(T-\tau,\cdot)\|_{L^\infty} \le C(\tau)e^{-\frac{\lambda}{C_0^2}(T-2\tau)}\|y_0-\overline{y}_0\|_{L^\infty}.$$

Therefore, taking T sufficiently large, we have $\|w-\overline{u}\|_{L^\infty} < \nu$.

Step 3. Conclusion.

Finally, the control:

$$u = \begin{cases} \overline{u} & \text{in } (0,T-\tau) \\ w & \text{in } (T-\tau,T) \end{cases}$$

is the desired one. □

4.4 On the need of the dissipativity condition.

We now give an example showing that the result above does not hold in any time $T>0$ without imposing the dissipativity condition and the initial datum is not a steady-state.

Consider the linear system:

$$\begin{cases} y_t - \operatorname{div}(A(x)\nabla y) + b(x)\cdot\nabla y + c(x)y = 0 & \text{in } (0,T)\times\Omega \\ y = u\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{4.4.1}$$

Let $\mathcal{L} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ be the operator defined by:

$$\mathcal{L}(y) = -\operatorname{div}(A(x)\nabla y) + b(x)\cdot\nabla y + c(x)y.$$

According to [50, Theorem 3 page 361], there exists a real eigenvalue λ_1 for $\mathcal{L}$ such that if $\lambda \in \mathbb{C}$ is any other eigenvalue, then $\operatorname{Re}(\lambda) \ge \lambda_1$. Moreover, there exists a unique *nonnegative* eigenfunction $\phi_1 \in H_0^1(\Omega)$ such that $\|\phi_1\|_{L^2} = 1$. We suppose further that $\lambda_1 < 0$. By Fredholm Theory [57, Theorem 5.11 page 84], we can choose the coefficient c so that $\lambda_1 < 0$ *and* $\mathcal{L}$ is onto. For instance, one can consider the operator $\mathcal{L}(y) = -\Delta y - \lambda y$, with:

$$\lambda > \mu_1, \quad \lambda \ne \mu_k \ \forall k \in \mathbb{N}^*,$$

where $\{\mu_k\}$ is the set of eigenvalues of $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$.

Proposition 4.4.1. *In the framework above, with initial datum $y_0 = \phi_1$ and a steady-state final target $y_1 \in \mathscr{S}$ with boundary value*

$$\overline{u} \ge \nu > 0, \quad \textit{a.e. on } \Gamma, \tag{4.4.2}$$

the constrained controllability fails in any time $T>0$.

More precisely, for any time $T>0$ and nonnegative control $u \in L^\infty((0,T)\times\Gamma)$, the corresponding solution y to (4.4.1) is such that $y(T,\cdot) \ne y_1$.

Proof. Let $u \in L^2((0,T) \times \Gamma)$ be a nonnegative control and y be the solution of (4.4.1) with initial datum ϕ_1 and control u. Moreover, let z be the solution of the above system with initial datum ϕ_1 and null control. By the maximum principle for parabolic equations (see [118]), we have:

$$y \geq z = e^{-\lambda_1 t}\phi_1, \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega.$$

Hence, $\|y(T,\cdot)\|_{L^2} \geq e^{-\lambda_1 T} > \|y_1\|_{L^2(\Omega)}$ for T large enough since $\lambda_1 < 0$. Hence, constrained controllability in large time fails. Actually, since the final target y_1 is a steady state, constrained controllability can never be realized whatever $T > 0$ is. □

4.5 Positivity of the minimal controllability time.

First of all, we study the linear case to later address the semilinear one.

4.5.1 Linear case

We consider the linear system:

$$\begin{cases} y_t - \operatorname{div}(A\nabla y) + b \cdot \nabla y + cy = 0 & \text{in } (0,T) \times \Omega \\ y = u\mathbf{1}_\Gamma & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{4.5.1}$$

where, as usual, $A = A(t,x)$ and $b = b(t,x)$ are Lipschitz continuous, while $c = c(t,x)$ is bounded.

We take a target trajectory $\overline{y}$ solution to (4.5.1) with initial datum $\overline{y}_0 \in L^2(\Omega)$ and control $\overline{u} \in L^\infty((0,T) \times \Gamma)$ such that $\overline{u} \geq \nu$, with ν positive constant and an initial datum $y_0 \in L^2(\Omega)$.

We define the minimal controllability time (or waiting time) under positivity constraints:

$$T_{\min} := \inf \left\{ T > 0 \;\middle|\; \exists u \in L^\infty((0,T) \times \Gamma)^+,\ y(T,\cdot) = \overline{y}(T,\cdot) \right\}, \tag{4.5.2}$$

where we use the convention $\inf(\varnothing) = +\infty$.

Under the assumptions of Theorem 4.1.1 or Theorem 4.1.2, we know that constrained controllability holds in large time. This guarantees that this minimal time $T_{\min} < +\infty$.

On the other hand, when the system is not dissipative, we have shown that there exist initial data such that controllability fails in any time. In that case, the minimal time $T_{\min} = +\infty$.

The purpose of this section is to prove that, whenever the initial datum differs from the final target, constrained controllability fails in time too small, i.e. $T_{\min} \in (0, +\infty]$.

This result is natural and rather simple to prove if we impose bilateral bounds on the control, i.e. L^∞ bounds. Here, however, we prove it under the non-negativity constraint in which case the result is less obvious since, in principle we could expect, when the final target is larger than the initial datum, the minimal time to vanish, employing large positive controls. But this is not the case.

Before proving the positivity of the minimal time, we point out that, actually, the minimal time is independent of the L^p regularity of the controls, as already pointed out in [95, Proposition 2.1]. The above system admits an unique solution $y \in C^0([0,T];(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))')$, with $n+2 < p < +\infty$, for any $y_0 \in L^2(\Omega)$ and $u \in L^1((0,T)\times\Gamma)$, as it can be shown by transposition (see [91]). Thus the waiting time could also be defined with controls in $L^1((0,T)\times\Gamma)$. We have the following Lemma.

Lemma 4.5.1. *Let $T > 0$. We suppose there exists a nonnegative control $u \in L^1((0,T)\times\Gamma)$ such that $y(T,\cdot) = \overline{y}(T,\cdot)$. Then, for any $\tau > 0$, we can find a nonnegative control $\tilde{u} \in L^\infty((0,T+\tau)\times\Gamma)$ such that $y(T+\tau,\cdot) = \overline{y}(T+\tau,\cdot)$.*

Consequently, the minimal constrained controllability time T_{min} is independent of the L^p-regularity of controls.

Proof. **Step 1.** Regularization of the control.

First of all, by convolution, we construct a nonnegative regularized control $u^{\varepsilon,1} \in C^\infty([0,T]\times\partial\Omega)$ such that:

$$\|u - u^{\varepsilon,1}\|_{L^1((0,T)\times\Gamma)} < \varepsilon,$$

with $\varepsilon > 0$ to be specified later. By the well posedness of (4.5.1) with L^1 controls, we have that the unique solution $y^{\varepsilon,1}$ to (4.5.1) with initial datum y_0 and control $u^{\varepsilon,1}$ is such that:

$$\|y^{\varepsilon,1}(T,\cdot) - \overline{y}(T,\cdot)\|_{(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'} \le C\varepsilon,$$

where $n+2 < p < +\infty$. To conclude the proof, it remains to steer $y^{\varepsilon,1}(T,\cdot)$ to $\overline{y}(T+\tau,\cdot)$ by a small control. To this extent, we need first to regularize the difference $y^{\varepsilon,1}(T,\cdot) - \overline{y}(T,\cdot)$.

Step 2. Regularization of $y^{\varepsilon,1}(T,\cdot) - \overline{y}(T,\cdot)$.

Consider the unique solution $y^{\varepsilon,2}$ to:

$$\begin{cases} y_t - \operatorname{div}(A\nabla y) + b\cdot\nabla y + cy = 0 & \text{in } (T, T+\tau/2)\times\Omega \\ y = \overline{u}\mathbf{1}_\Gamma & \text{on } (T,T+\tau/2)\times\partial\Omega \\ y(T,x) = y^{\varepsilon,1}(T,x). & \text{in } \Omega \end{cases} \tag{4.5.3}$$

By the regularizing effect of parabolic equations, $y^{\varepsilon,2}(T+\tau/2,\cdot) - \overline{y}(T+\tau/2,\cdot) \in L^2(\Omega)$ and:

$$\|y^{\varepsilon,2}(T+\tau/2) - \overline{y}(T+\tau/2)\|_{L^2} \le C(\tau)\|y^{\varepsilon,1}(T) - \overline{y}(T)\|_{(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'} \le C(\tau)\varepsilon.$$

Step 3. Application of null controllability by small controls.

By Lemma 4.6.1, there exists a control $u^{\varepsilon,3} \in L^\infty((T+\tau/2, T+\tau)\times\Gamma)$ steering (4.5.1) from $y^{\varepsilon,2}(T+\tau/2,\cdot)$ to $\overline{y}(T+\tau,\cdot)$ such that:

$$\|u^{\varepsilon,3} - \overline{u}\|_{L^\infty} \le C(\tau)\|y^{\varepsilon,2}(T+\tau/2,\cdot) - \overline{y}(T+\tau/2,\cdot)\|_{L^2} \tag{4.5.4}$$

$$\leq C(\tau)\|y^{\varepsilon,1}(T,\cdot)-\overline{y}(T,\cdot)\|_{(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'}\leq C(\tau)\varepsilon.$$

Then, choosing ε sufficiently small, we have $\|u^{\varepsilon,3}-\overline{u}\|_{L^\infty}<\nu$, thus:

$$u^{\varepsilon,3}\geq 0,\quad \text{a.e. } (T+\tau/2,T+\tau)\times\Gamma.$$

Then,

$$\tilde{u}=\begin{cases}u^{\varepsilon,1}, & \text{in } (0,T)\\ \overline{u} & \text{in } (T,T+\tau/2)\\ u^{\varepsilon,3} & \text{in } (T+\tau/2,T+\tau)\end{cases} \tag{4.5.5}$$

is the desired control □

We are now ready to state the desired theorem on the waiting time, i.e. the positivity of $T_{\min}$.

Theorem 4.5.1 (Positivity of the minimal controllability time)**.** *Let $\overline{y}$ be the target trajectory solution to* (4.5.1) *with initial datum $\overline{y}_0\geq 0$ and boundary control $\overline{u}$ such that $\overline{u}\geq\nu>0$. Consider the initial datum $y_0\in L^2(\Omega)$ such that $y_0\neq\overline{y}_0$.*

Then,

1. *there exists $T_0>0$ such that, for any $T\in(0,T_0)$ and for any nonnegative control $u\in L^\infty((0,T)\times\Gamma)$ the solution y to* (4.5.1) *with initial datum y_0 and control u is such that $y(T,\cdot)\neq\overline{y}(T,\cdot)$.*

2. *Consequently,*

$$T_{min}>0.$$

The building block of the proof of above Theorem (at the end of this subsection) is the following lemma.

Lemma 4.5.2. *Let the diffusivity matrix $A=A(t,x)$ be uniformly coercive and Lipschitz continuous and let the drift coefficient $b=b(t,x)$ be Lipschitz continuous. Assume the potential coefficient $c=c(t,x)$ is bounded, with $\|c\|_{L^\infty((0,T)\times\Omega)}\leq L$, for some $L>0$. Let $\overline{y}$ be the target trajectory solution to* (4.5.1) *with initial datum $\overline{y}_0\geq 0$ and boundary control $\overline{u}\geq\nu>0$. Assume $\overline{y}_0\neq 0$. There exists $T_0=T_0(\Omega,\Gamma,A,b,M)>0$, such that for some final datum $\varphi^0\in\bigcap_{1<p<+\infty}W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ the solution to the adjoint problem*

$$\begin{cases}-\varphi_t-div(A\nabla\varphi)-div(\varphi b)+c\varphi=0 & in\ (0,T_0)\times\Omega\\ \varphi=0 & on\ (0,T_0)\times\partial\Omega\\ \varphi(T,x)=\varphi^0(x). & in\ \Omega\end{cases} \tag{4.5.6}$$

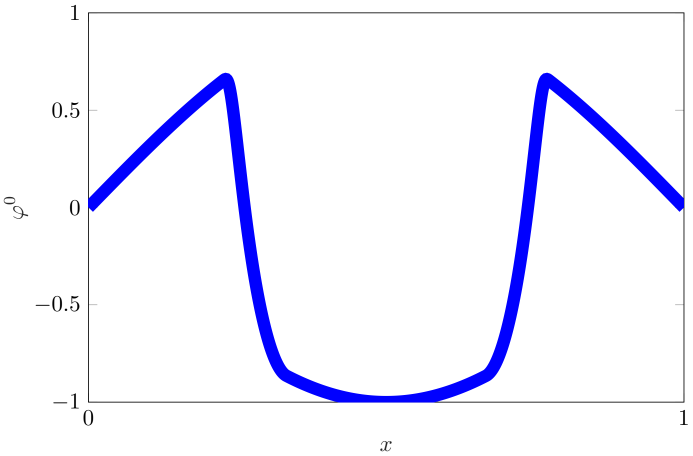


Figure 4.3: final datum for the adjoint system.

verifies the conditions

$$\begin{cases} \frac{\partial \varphi}{\partial n} \leq 0 & on\ (0,T_0)\times\partial\Omega \\ \langle \overline{y}(T,\cdot),\varphi^0\rangle < 0, & \forall T\in[0,T_0), \end{cases} \tag{4.5.7}$$

where $n = A\hat{v}/\|A\hat{v}\|$, *with* $\hat{v}$ *is the outward unit normal to* $\partial\Omega$.

Proof of Lemma 4.5.2. Step 1 **Definition of the candidate final datum**

Let ϕ_1 be the first eigenfunction of the Dirichlet laplacian in Ω, which is *strictly positive* in Ω ([50, Theorem 2 page 356]).

Let us also introduce a cut-off function $\zeta \in C^\infty(\overline{\Omega})$ such that:

- $\text{supp}(\zeta) \subset\subset \Omega$;
- $\zeta(x) = 1$ for any $x \in \Omega$ such that $\text{dist}(x,\partial\Omega) \geq \delta$,

with $\delta > 0$ to be made precise later.

We are now in position to define the final datum (see Figure 4.3):

$$\varphi^0 = -\phi_1 + 2(1-\zeta)\phi_1. \tag{4.5.8}$$

Step 3 **Condition at final time** T_0

We prove that $\langle \overline{y}(T,\cdot),\varphi^0\rangle < 0$, if δ is small enough. Indeed,

$$\int_\Omega \overline{y}_0\varphi^0 dx = \int_\Omega \overline{y}_0(-\phi_1 + 2(1-\zeta)\phi_1)dx.$$

On the one hand, since $\phi_1(x) > 0$ for any $x \in \Omega$, $\overline{y}_0 \geq 0$ and $\overline{y}_0 \neq 0$:

$$\int_\Omega \overline{y}_0(-\phi_1(x))dx \leq -\theta < 0,$$

with $\theta > 0$. On the other hand, taking $\delta > 0$ small enough,

$$\left| \int_\Omega \overline{y}_0 (1-\zeta)\phi_1 dx \right| \leq \|\overline{y}_0\|_{L^2} \|\phi_1\|_{L^\infty} \sqrt{|E_\delta|} < \frac{\theta}{4},$$

where $E_\delta := \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) < \delta\}$. Then,

$$\int_\Omega \overline{y}_0 \varphi^0(x) dx \leq 2\frac{\theta}{4} - \theta = -\frac{\theta}{2} < 0. \tag{4.5.9}$$

Finally, by transposition (see [91]), $\overline{y} \in C^0([0,T]; H^{-1}(\Omega))$. Hence, choosing T_0 small enough, we have:

$$\langle \overline{y}(T,\cdot), \varphi^0 \rangle < 0, \quad \forall T \in [0, T_0),$$

as required.

Step 3 **Condition on the normal derivative**
By the definition of ϕ_1 and ζ, $\varphi^0(x) < 0$ for any $x \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) \geq \delta$. On the other hand, $\varphi^0(x) > 0$ for any $x \in \Omega \setminus \mathrm{supp}(\zeta)$. This means that actually φ^0 is *negative* up to a small neighborhood of $\partial\Omega$.

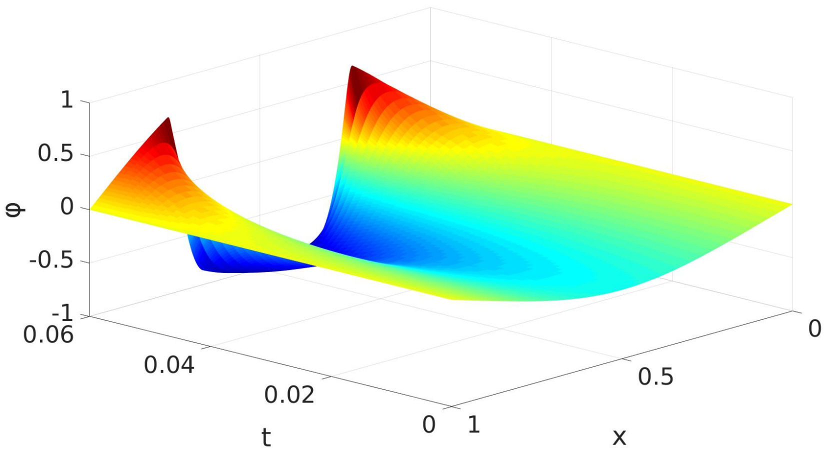


Figure 4.4: evolution of the adjoint heat equation with final datum φ^0.

Let us now check that $\frac{\partial\varphi}{\partial n} \leq 0$ on $(0,T) \times \partial\Omega$ (see Figure 4.4). To this purpose, we first observe that the final datum $\varphi^0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for any $1 < p < +\infty$, as a consequence of the well-known regularity properties of the first eigenfunction of the Laplacian ([16, Theorem 9.32 page 316] or [60, Theorem 2.4.2.5 page 124]).

In view of the regularity of φ^0 and Corollary 4.6.1, we have that

$$\frac{\partial\varphi}{\partial n} \in C^{0,\gamma}([0,+\infty) \times \overline{\Omega}) \quad \text{and} \quad \left\| \frac{\partial\varphi}{\partial n} \right\|_{C^{0,\gamma}} \leq K, \tag{4.5.10}$$

where both $0 < \gamma < 1$ and K are independent of the choice of the coefficient c verifying $\|c\|_{L^\infty} \leq L$. Then, since $\frac{\partial\varphi^0}{\partial n}(x) < 0$ for any $x \in \partial\Omega$, there exists $T_0 > 0$ such that:

$$\frac{\partial\varphi}{\partial n} < 0, \quad \forall\ (t,x) \in [0, T_0) \times \partial\Omega,$$

as desired. □

We are now in position to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. We distinguish two cases.

Case 1. $y_0 \leq \overline{y}_0$.

Step 1. Reduction to the case $y_0 = 0$.

We introduce z solution to (4.5.1) with initial datum y_0 and zero control. By *subtracting* z both to the target trajectory and to the controlled one, we justify the reduction. Indeed, let $\overline{\xi} = \overline{y} - z$ be the new target trajectory solution to (4.5.1) with initial datum $y_0 - \overline{y}_0$ and control $\overline{u}$, while $\xi = y - z$ is the solution to (4.5.1) with null initial datum and control u.

Now, if the result holds in the particular case where $y_0 = 0$, then for each time $0 < T < T_0$ and for any choice of the control $u \in L^\infty((0,T) \times \Gamma)^+$, we have $\xi(T, \cdot) \neq \overline{\xi}(T, \cdot)$. This, in turn, implies that for any $u \in L^\infty((0,T) \times \Gamma)^+$, we have $y(T, \cdot) \neq \overline{y}(T, \cdot)$, as desired.

Step 2. Weak solutions by transposition to (4.5.1).

The solution $y \in L^2((0,T) \times \Omega) \cap C^0([0,T]; H^{-1}(\Omega))$ of (4.5.1) with null initial datum and control $u \in L^\infty((0,T) \times \Gamma)$ is characterized by the duality identity

$$\langle y(T, \cdot), \varphi^0 \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt = 0, \tag{4.5.11}$$

where $y(T, \cdot) \in H^{-1}(\Omega)$ and φ is the solution to the adjoint problem:

$$\begin{cases} -\varphi_t - \operatorname{div}(A \nabla \varphi) - \operatorname{div}(\varphi b) + c\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T, x) = \varphi^0(x). & \text{in } \Omega \end{cases} \tag{4.5.12}$$

As usual $n = A\hat{v}/\|A\hat{v}\|$, where $\hat{v}$ is the outward unit normal to $\partial\Omega$.

Now, by Lemma 4.5.2, there exists $T_0 > 0$ and $\varphi^0 \in H_0^1(\Omega)$ such that, for any $T \in (0, T_0)$, the solution of the adjoint system (4.5.12) with final datum φ^0 satisfies:

$$\begin{cases} \frac{\partial \varphi}{\partial n} \leq 0 & \text{on } (0, T_0) \times \partial\Omega \\ \langle \overline{y}(T, \cdot), \varphi^0 \rangle < 0, & \forall T \in [0, T_0). \end{cases} \tag{4.5.13}$$

By contradiction, let us now suppose for some $T \in (0, T_0)$ we can find a nonnegative control u driving (4.5.1) from 0 to $\overline{y}(T, \cdot)$. By (4.5.11) and (4.5.13), we have:

$$0 = \langle \overline{y}(T, \cdot), \varphi^0 \rangle + \int_0^T \int_{\partial\Omega} u \frac{\partial \varphi}{\partial n} d\sigma(x) dt \leq \langle \overline{y}(T, \cdot), \varphi^0 \rangle < 0,$$

so obtaining a contradiction.

Case 2. $y_0 \not\leq \overline{y}_0$.

Since $y_0 \nleq \overline{y}_0$, $y_0 > \overline{y}_0$ on a set of positive measure. Then, there exists a nonnegative $\varphi \in H_0^1(\Omega) \setminus \{0\}$ such that $\int_\Omega (y_0 - \overline{y}_0)\varphi dx > 0$.

Let z be the unique solution to (4.5.1) with initial datum y_0 and zero control. Now,

$$\langle z(0,\cdot) - \overline{y}(0,\cdot), \varphi\rangle = \langle y_0 - \overline{y}_0, \varphi\rangle > 0.$$

Moreover, by transposition, z and $\overline{y}$ are of class $C^0([0,T]; H^{-1}(\Omega))$. Then, there exists $T_0 > 0$ such that:

$$\langle z(T,\cdot) - \overline{y}(T,\cdot), \varphi\rangle > 0, \quad \forall\, T \in [0, T_0). \tag{4.5.14}$$

On the other hand, the comparison principle (see [118]) yields $y \geq z$ in $(0,T)\times\Omega$. This, together with (4.5.14), implies that $\langle y(T,\cdot), \varphi\rangle > \langle \overline{y}(T,\cdot), \varphi\rangle$ for any $T \in [0, T_0)$, thus concluding the proof. □

4.5.2 Controllability in the minimal time for the linear case

We prove that controllability holds in the minimal time with measured valued controls. To this extent, let us define the solution to (4.5.1) with controls belonging to the space of Radon measures $\mathcal{M}([0,T] \times \partial\Omega)$ endowed with the norm:

$$\|\mu\|_{\mathcal{M}} = \sup\left\{ \int_{[0,T]\times\partial\Omega} \varphi(t,x) d\mu(t,x) \;\middle|\; \varphi \in C^0([0,T]\times\partial\Omega),\ \max_{[0,T]\times\partial\Omega} |\varphi| = 1 \right\}.$$

We firstly provide the notions of left and right limit of the solution of (4.5.1) by transposition (see [91]). Given $y_0 \in L^2(\Omega)$ and $u \in \mathcal{M}([0,T]\times\partial\Omega)$ with $n+2 < p < +\infty$, $y_l : [0,T] \longrightarrow (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'$ is the left limit of the solution to (4.5.1) if, for any $\varphi^0 \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$:

$$\langle y_l(t,\cdot), \varphi^0\rangle - \int_\Omega y_0(x)\varphi(0,x)dx + \int_{[0,t)\times\partial\Omega} \frac{\partial\varphi}{\partial n} du = 0, \tag{4.5.15}$$

where φ is the solution to the adjoint system:

$$\begin{cases} -\varphi_t - \operatorname{div}(A\nabla\varphi) - \operatorname{div}(\varphi b) + c\varphi = 0 & \text{in } (0,t)\times\Omega \\ \varphi = 0 & \text{on } (0,t)\times\partial\Omega \\ \varphi(t,x) = \varphi^0(x). & \text{in } \Omega. \end{cases} \tag{4.5.16}$$

Similarly, $y_r : [0,T] \longrightarrow (W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'$ is the right limit of the solution to (4.5.1) if, for any $\varphi^0 \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$:

$$\langle y_r(t,\cdot), \varphi^0\rangle - \int_\Omega y_0(x)\varphi(0,x)dx + \int_{[0,t]\times\partial\Omega} \frac{\partial\varphi}{\partial n} du = 0. \tag{4.5.17}$$

Note the difference between the left limit y_l and the right limit y_r is the domain of integration with respect to u. Indeed, for the left limit we integrate over $[0,t)\times\partial\Omega$, while for the right limit we integrate over $[0,t]\times\partial\Omega$. Actually, $y_l \neq y_r$ if, for instance, $u = \delta_{t_0}\otimes\delta_{x_0}$ for some

$t_0 \in [0,T]$ and $x_0 \in \partial\Omega$. On the other hand, $y_l = y_r$ as soon as u is absolutely continuous with respect to the Lebesgue measure on $[0,T] \times \partial\Omega$. This is the case, whenever $u \in L^1$.

We are now able to define the concept of (generalized) solution to (4.5.1) as:

$$y : [0,T] \longrightarrow \mathscr{P}((W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))')$$

$$t \longmapsto \{y_l(t), y_r(t)\},$$

where we have denoted as $\mathscr{P}((W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))')$ the power set of $(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))'$. Note that we have defined $y(t,\cdot)$ for any $t \in [0,T]$. Then, it makes sense the trace of y at time $t = T$.

The following holds:

Proposition 4.5.1 (Controllability in the minimal time)**.** *Let $\overline{y}$ be the target trajectory, solution to* (4.5.1) *with initial datum $\overline{y}_0 \in L^2$ and control $\overline{u} \in L^\infty$ such that $\overline{u} \geq \nu > 0$. Let $y_0 \in L^2(\Omega)$ be the initial datum to be controlled and suppose that $T_{min} < \infty$.*

Then, there exists a nonnegative control $\hat{u} \in \mathcal{M}([0,T_{min}] \times \overline{\Gamma})$ such that the right limit of the solution to (4.5.1) *with initial datum y_0 and control $\hat{u}$ verifies:*

$$y_r(T_{min},\cdot) = \overline{y}(T_{min},\cdot). \tag{4.5.18}$$

Proof. **Step 1.** L^1-bounds on the controls.

Let $T_k = T_{\min} + 1/k$. By definition of the minimal control time and the hypotheses, there exists a sequence of nonnegative controls $\{u^{T_k}\} \subset L^\infty$ such that u^{T_k} steers (4.5.1) from y_0 to $\overline{y}(T_k,\cdot)$ in time T_k. We extend these control by $\overline{u}$ on $(T_{\min} + \frac{1}{k}, T_{\min} + 1)$, getting a sequence $\{u^{T_k}\} \subset L^\infty((0,T_{\min}+1) \times \Gamma)$.

We want to prove that this sequence is bounded in $L^1((0,T_{\min}+1) \times \Gamma)$.

The arguments we employ resemble the ones employed in the proof of the positiveness of the minimal time and use the definition of solution to (4.5.1) by transposition.

Let ϕ_1 be the first eigenfunction of the Dirichlet laplacian in Ω. By applying the Corollary 4.6.1 to the adjoint system:

$$\begin{cases} -\varphi_t - \operatorname{div}(A\nabla\varphi) - \operatorname{div}(\varphi b) + c\varphi = 0 & \text{in } (0,T_k) \times \Omega \\ \varphi = 0 & \text{on } (0,T_k) \times \partial\Omega \\ \varphi(T_k,x) = \phi_1. & \text{in } \Omega \end{cases} \tag{4.5.19}$$

we get $\varphi \in C^0([0,T_k]; C^1(\overline{\Omega})) \cap W_p^{1,2}((0,T_k) \times \Omega)$ for any k.

By definition of the solution by transposition to (4.5.1), we have:

$$\langle \overline{y}(T_k,\cdot), \phi_1 \rangle - \int_\Omega y_0(x)\varphi(0,x)dx + \int_0^{T_k} \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} u^{T_k} dxdt = 0. \tag{4.5.20}$$

At this stage, we realize that:

$$\frac{\partial\varphi}{\partial n} \leq -\theta, \quad \forall\ (t,x) \in [0,T_k] \times \partial\Omega, \tag{4.5.21}$$

for some $\theta > 0$. Indeed, a strong maximum principle for (4.5.19) holds. The proof of this can be done in two steps. Firstly, by the transformation $\tilde{\varphi} = e^{-\lambda t}\varphi$ with $\lambda = \|c\|_{L^\infty} + \|\mathrm{div}(b)\|_{L^\infty}$, we reduce to the case $c - \mathrm{div}(b) \geq 0$. Then, we observe that proof of the Hopf Lemma (see [50]) works since $\varphi \in C^0([0,T_k]; C^1(\overline{\Omega})) \cap W_p^{1,2}((0,T_k)\times\Omega)$. This enables us to obtain (4.5.21). Finally, by (4.5.20), (4.5.21) and the positiveness of u^{T_k}, we have:

$$\theta\|u^{T_k}\|_{L^1} = \theta\int_0^{T_k}\int_{\partial\Omega} u^{T_k}dxdt \leq \int_0^{T_k}\int_{\partial\Omega} -\frac{\partial\varphi}{\partial n}u^{T_k}dxdt$$

$$= \langle\overline{y}(T_k,\cdot),\phi_1\rangle - \int_\Omega y_0(x)\varphi(0,x)dx \leq M,$$

where the last inequality is due to the continuous dependence for (4.5.1) and (4.5.19).

Step 2. Conclusion.

Since $\left\{u^{T_k}\right\}$ is bounded in $L^1((0,T_{\min}+1)\times\Gamma)$, there exists $\hat{u} \in \mathcal{M}([0,T_{\min}+1]\times\overline{\Gamma})$ such that, up to subsequences:

$$u^{T_k} \rightharpoonup \hat{u}$$

in the weak* sense. Clearly, $\hat{u}$ is a nonnegative measure. Finally, for any k large enough and $T_{\min} < T < T_{\min}+1$, by definition of u^{T_k}:

$$\langle\overline{y}(T,\cdot),\varphi^0\rangle - \int_\Omega y_0(x)\varphi(0,x)dx + \int_{[0,T]\times\partial\Omega}\frac{\partial\varphi}{\partial n}du^{T_k} = 0,$$

for any final datum for the adjoint system $\varphi^0 \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Right now, by definition of weak* limit, letting $k \to +\infty$:

$$\langle\overline{y}(T,\cdot),\varphi^0\rangle - \int_\Omega y_0(x)\varphi(0,x)dx + \int_{[0,T]\times\partial\Omega}\frac{\partial\varphi}{\partial n}d\hat{u} = 0, \tag{4.5.22}$$

which in turn implies that $y_r(T,\cdot) = \overline{y}(T,\cdot)$ in $(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))'$, where y_r is the right limit of the solution to (4.5.1) with initial datum y_0 and control $\hat{u}$. To show that actually $y_r(T_{\min},\cdot) = \overline{y}(T_{\min},\cdot)$, it remains to take the limit as $T \to T_{\min}$ in (4.5.22). This task can be accomplished by employing the regularity of the adjoint problem (Corollary 4.6.1) and $|\hat{u}|((T_{\min},T]\times\partial\Omega) = |\overline{u}|((T_{\min},T]\times\partial\Omega) \to 0$ as $T \to T_{\min}$. Then, we have $y_r(T_{\min},\cdot) = \overline{y}(T_{\min},\cdot)$, as required. □

4.5.3 Semilinear case

We now consider the semilinear system (4.1.1). We take as target trajectory $\overline{y}$ global solution to (4.1.1) with initial datum $\overline{y}_0 \in L^\infty$ and bounded control $\overline{u} \geq \nu$, where $\nu > 0$. Let $y_0 \in L^\infty(\Omega)$ be the initial datum and $T > 0$. We take a nonnegative control $u \in L^\infty((0,T)\times\Gamma)$ such that there exists y solution to (4.1.1) globally defined in $[0,T]$. We introduce the minimal controllability time:

$$T_{\min} := \inf\left\{T > 0 \;\middle|\; \exists u \in L^\infty((0,T)\times\Gamma)^+,\ \exists y(T,\cdot) = \overline{y}(T,\cdot)\right\}, \tag{4.5.23}$$

where, as usual, $\inf(\varnothing) = +\infty$. As before, our goal is to show the lack of controllability in small time, i.e. $T_{\min} \in (0, +\infty]$. Two different situations may occur:

- $y_0 \not\leq \overline{y}_0$, namely $y_0 > \overline{y}_0$ on a set of strictly positive measure. In this case the positivity of the waiting time may be proved without any additional assumption on the nonlinearity;
- $y_0 \leq \overline{y}_0$. We prove that $T_{\min} > 0$ under some assumptions on the nonlinearity. In particular, we assume either the nonlinearity to be globally Lipschitz:

$$|f(t,x,y_2) - f(t,x,y_1)| \leq L|y_2 - y_1|, \quad \forall\ (t,x,y_1,y_2) \in \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R}^2 \tag{4.5.24}$$

or "strongly" increasing, i.e. $y \longmapsto f(t,x,y)$ must be nondecreasing and

$$f(t,x,y_2) - f(t,x,y_1) \geq C(y_2 - y_1)\left(\ln(|y_2 - y_1|)\right)^p, \tag{4.5.25}$$

for any $(t,x) \in \mathbb{R}^+ \times \overline{\Omega}$ and $y_2 - y_1 > z_0$, with $z_0 > 1$ and $p > 2$.

These hypotheses correspond to two different situations which lead to the same result, i.e. the positivity of the waiting time. Hypothesis (4.5.24) imposes a lower and upper bound to the growth of $y \longmapsto f(t,x,y)$ which yields the lack of constrained controllability in time, with arguments similar to the linear case. On the other hand, the "strong" superlinear dissipativity condition (Hypothesis (4.5.25)) produces a damping effect on the action of the control (see [70] and [7]). As it is well known, this enables to prove that, actually, the minimal time is positive even in the *unconstrained* case.

We formulate now the result.

Theorem 4.5.2 (Positivity of the minimal time). *Let $y_0 \in L^\infty(\Omega)$ be an initial datum and $\overline{y}$ be a target trajectory solution to* (4.1.1) *with initial datum $\overline{y}_0$ and control $\overline{u} \in L^\infty((0,\overline{T}) \times \Gamma)$ with $\overline{u} \geq \nu > 0$, a.e.. We suppose $y_0 \neq \overline{y}_0$ and one of the following assumptions*

(H_1) $y_0 \not\leq \overline{y}_0$;

(H_2) f is globally Lipschitz

$$|f(t,x,y_2) - f(t,x,y_1)| \leq L|y_2 - y_1|, \quad \forall\ (t,x,y_1,y_2) \in \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R}^2; \tag{4.5.26}$$

(H_3) f "strong" superlinear dissipative (4.5.25)

$$f(t,x,y_2) - f(t,x,y_1) \geq C(y_2 - y_1)\left(\ln(|y_2 - y_1|)\right)^p, \tag{4.5.27}$$

for any $(t,x) \in \mathbb{R}^+ \times \overline{\Omega}$ and $y_2 - y_1 > z_0$, with $z_0 > 1$ and $p > 2$.

Then, $T_{min} > 0$.

First of all, we state the following remark.

Remark 4.5.1. *The lack of unconstrained controllability in small time under the assumption of "strong" superlinear dissipativity (H_3) is well known in the literature (see [70] and [7])). One can check this by adapting the techniques developed in [7, Lemma 7].*

We prove now the Theorem 4.5.2 in the remaining cases. We proceed as follows:

- proof in case $y_0 \not\leq \overline{y}_0$ for general nonlinearities (under assumption (H_1));
- proof in case $y_0 \geq \overline{y}_0$ under assumption (H_2).

Proof of Theorem 4.5.2 in case $y_0 \not\leq \overline{y}_0$ for general nonlinearities (assumption (H_1)).
Case 1. $y_0 \not\leq \overline{y}_0$. By Proposition 4.6.1, taking $\overline{T} > 0$ sufficiently small, (4.1.1) admits an unique solution z defined in $[0, \overline{T}]$ with initial datum y_0 and null control. By assumptions, $y_0 > \overline{y}_0$ in a set of positive measure. Then, there exists a nonnegative $\varphi \in H_0^1(\Omega) \setminus \{0\}$ such that $\int_\Omega (y_0 - \overline{y}_0)\varphi dx > 0$. Now, since $z - \overline{y} \in C^0([0, \overline{T}]; H^{-1}(\Omega))$ and $\langle z(0, \cdot) - \overline{y}(0, \cdot), \varphi\rangle > 0$, we have:

$$\langle z(T, \cdot), \varphi\rangle > \langle \overline{y}(T, \cdot), \varphi\rangle, \quad \forall\, T \in [0, T_0), \tag{4.5.28}$$

with $T_0 \in (0, \overline{T})$ small enough.

At this point, we are going to show that $T_{\min} \geq T_0$. Indeed, let $T \in (0, T_0)$ and $u \in L^\infty((0, T) \times \Gamma)$ be a nonnegative control such that (4.1.1) admits a global solution y with initial datum y_0 and control u. Then, by the comparison principle, we have $y \geq z$. Indeed, one realizes that the difference $y - z$ satisfies the linear system:

$$\begin{cases} \xi_t - \mathrm{div}(A\nabla\xi) + b \cdot \nabla\xi + c\xi = 0 & \text{in } (0, T) \times \Omega \\ \xi = u\mathbf{1}_\Gamma & \text{on } (0, T) \times \partial\Omega \\ \xi(0, x) = 0, & \text{in } \Omega \end{cases}$$

where:

$$c(t, x) = \begin{cases} \frac{f(t,x,\xi(t,x)+z(t,x)) - f(t,x,z(t,x))}{\xi(t,x)} & \xi(t, x) \neq 0 \\ \frac{\partial f}{\partial y}(t, x, z(t, x)) & \xi(t, x) = 0. \end{cases}$$

Since $f \in C^1$ and both y and z are bounded, c is bounded too. Then, we apply the comparison principle to 4.5.3 (see [118]), getting $y \geq z$. This, together with (4.5.28), yields:

$$\langle y(T, \cdot), \varphi\rangle \geq \langle z(T, \cdot), \varphi\rangle > \langle \overline{y}(T, \cdot), \varphi\rangle.$$

Hence, $y(T, \cdot) \neq \overline{y}(T, \cdot)$, as desired. □

We now prove Theorem 4.5.2 in case $y_0 \leq \overline{y}_0$ under assumptions (4.5.24).

Proof of Theorem 4.5.2 in case $y_0 \leq \overline{y}_0$ assuming f globally Lipschitz (H_2). First of all, we notice that, since the nonlinearity is globally Lipschitz, finite time blow up never occurs and the corresponding solutions are global in time.

We proceed in several steps.

Step 1. Reduction to the case $y_0 = 0$.

We take z unique solution to (4.1.1) with initial datum y_0 and null control. Then, $\xi = y - z$ solves:

$$\begin{cases} \xi_t - \operatorname{div}(A\nabla\xi) + b\cdot\nabla\xi + \tilde{f}(t,x,\xi(t,x)) = 0 & \text{in } (0,\overline{T})\times\Omega \\ \xi = u\mathbf{1}_\Gamma & \text{on } (0,\overline{T})\times\partial\Omega \\ \xi(0,x) = 0, & \text{in } \Omega \end{cases} \tag{4.5.29}$$

where $\tilde{f}(t,x,\xi) = f(t,x,\xi + z(t,x)) - f(t,x,z(t,x))$. Besides, $\overline{\xi} = \overline{y} - z$ solves (4.5.29) with initial datum $\overline{y}_0 - y_0$ and control $\overline{u}$. Then, the problem is reduced to prove the existence of $T_0 \in (0,\overline{T})$ such that, for any $T \in (0,T_0)$ and for any nonnegative $u \in L^\infty((0,T)\times\Gamma)$, $\xi(T,\cdot) \neq \overline{\xi}(T,\cdot)$.

Step 2. Reduction to the linear case.

Let $T > 0$, $u \in L^\infty((0,T)\times\Gamma)$ be a nonnegative control and ξ be the unique solution to (4.5.29) with null initial datum and control u. We set:

$$c_u(t,x) = \begin{cases} \frac{\tilde{f}(t,x,\xi(t,x))}{\xi(t,x)} & \xi(t,x) \neq 0 \\ \frac{\partial\tilde{f}}{\partial\xi}(t,x,0) & \xi(t,x) = 0. \end{cases}$$

By (4.5.24), $c_u \in L^\infty$ and $\|c_u\|_{L^\infty} \leq L$. Therefore, ξ solves (4.5.1) with potential coefficient c_u, null initial datum and control u. Hence, to conclude, it suffices to show that there exist $T_0 \in (0,\overline{T})$ such that, whenever $\|c\|_{L^\infty} \leq L$ and $T < T_0$, the unique solution ξ to (4.5.1) with null initial datum and control u is such that $\xi(T,\cdot) \neq \overline{\xi}(T,\cdot)$.

Step 3. Conclusion.

Reasoning as in the proof of Theorem 4.5.1, it remains to apply Lemma 4.5.2, to have the existence of $T_0 \in (0,\overline{T})$, valid for all coefficients verifying $\|c\|_{L^\infty} \leq L$, such that:

$$\begin{cases} \frac{\partial\varphi}{\partial n} \leq 0 & \text{on } (0,T_0)\times\partial\Omega \\ \langle\overline{\xi}(T,\cdot),\varphi^0\rangle < 0, & \forall T \in [0,T_0), \end{cases} \tag{4.5.30}$$

where φ is the solution to the adjoint system (4.5.12) with final datum φ^0. This finishes the proof under assumptions (4.5.24). □

4.6 Numerical experiments and simulations.

This section is devoted to numerical experiments and simulations. We begin providing explicit lower bounds for the minimal time.

4.6.1 The linear case

Let us show that the dual method developed for the proof of Theorem 4.5.1 provides explicit lower bounds for the minimal controllability time. To simplify the presentation, we discuss the linear case, but a similar analysis can be done for the globally Lipschitz semilinear case. As usual, $\overline{y}$ stands for the target trajectory, while y_0 is the initial datum to be controlled.

Firstly, recall that the idea is to find $T_0 > 0$ and $\varphi^0 \in H_0^1(\Omega)$ such that, for any $T \in (0, T_0)$, the solution to the adjoint system (4.5.12) with final datum φ^0 satisfies:

$$\begin{cases} \frac{\partial\varphi}{\partial n} \leq 0 & \text{on } (0,T_0)\times\partial\Omega \\ \langle \overline{y}(T,\cdot) - z(T,\cdot), \varphi^0\rangle < 0, & \forall T \in [0,T_0). \end{cases} \tag{4.6.1}$$

where z is the solution to (4.5.1) with initial datum y_0 and null control.

The specific final datum φ^0 in the proof of Theorem 4.5.1 is valid and leads to a lower bound for $T_{\min}$ for the waiting time. But this lower bound can eventually be improved by a better choice of φ^0.

For example, let us consider the problem of driving the heat problem:

$$\begin{cases} y_t - y_{xx} = 0 & (t,x) \in (0,T)\times(0,1) \\ y(t,0) = u_1(t) & t \in (0,T) \\ y(t,1) = u_2(t), & t \in (0,T) \end{cases}$$

from $y_0 \in \mathbb{R}$ to $\overline{y} \equiv y_1 \in \mathbb{R}$, with $0 < y_0 < y_1$.

Let:

$$\varphi^0 = -\alpha\sin(\pi x) + \beta\sin(3\pi x),$$

where α and β are nonnegative parameters to be made precise later. We firstly check the second relation in (4.6.1).

First of all, when the initial datum $y_0 > 0$ is constant, we develop z the solution to the free heat problem :

$$z(t,x) = 4y_0\sum_{p=0}^{\infty}\frac{e^{-\pi^2(2p+1)^2 t}}{(2p+1)\pi}\sin((2p+1)\pi x),$$

which yields:

$$\int_0^1 z(T,x)\varphi^0(x)dx = y_0\left[-\frac{2}{\pi}\alpha e^{-\pi^2 T} + \frac{2}{3\pi}\beta e^{-9\pi^2 T}\right].$$

Hence,

$$\langle \overline{y}(T,\cdot) - z(T,\cdot), \varphi^0\rangle = y_1\left[-\frac{2\alpha}{\pi} + \frac{2\beta}{3\pi}\right] + y_0\left[\frac{2}{\pi}\alpha e^{-\pi^2 T} - \frac{2}{3\pi}\beta e^{-9\pi^2 T}\right].$$

The above quantity is strictly negative for any $T > 0$ whenever $0 < \frac{\beta}{\alpha} < \frac{3(y_1-y_0)}{y_1}$.

Finally, let us check the first relation in (4.6.1). By the Fourier expansion of φ, we have that:

$$\frac{\partial\varphi}{\partial n}(t,0) = \frac{\partial\varphi}{\partial n}(t,1) = \alpha\pi e^{-\pi^2(T-t)} - 3\pi\beta e^{-9\pi^2(T-t)}.$$

Then, $\frac{\partial\varphi}{\partial n}(t,0) = \frac{\partial\varphi}{\partial n}(t,1) \leq 0$ for any $0 \leq t \leq T$ if and only if:

$$T \leq \frac{1}{8\pi^2} \log\left(\frac{3\beta}{\alpha}\right).$$

Finally, optimising within the range $0 < \frac{\beta}{\alpha} < \frac{3(y_1-y_0)}{y_1}$, we have:

$$T_{\min} \geq \frac{1}{8\pi^2} \log\left(\frac{9(y_1-y_0)}{y_1}\right).$$

In case $y_0 \equiv 1$ and $y_1 \equiv 5$, this leads to the bound:

$$T_{\min} \geq \frac{1}{8\pi^2} \log\left(\frac{36}{5}\right) \cong 0.0250020.$$

One can also compute a numerical approximation of the minimal time employing `IpOpt` (as in [95]). This leads to $T_{\min} \cong 0.0498$, which is compatible lower bound.

The gap between the analytical estimate and the value of the numerical approximation is still significant and leaves for the improvement of the estimates above by means of the use of suitable adjoint solutions as test functions.

On the other hand, when y_0 and y_1 are constant and such that $y_0 > y_1$, by comparison (see [95]) we have

$$T_{\min} \geq \frac{1}{\pi^2} \log\left(\frac{y_0}{y_1}\right).$$

This can also be proved y employing the adjoint technique, choosing as final datum for the adjoint state $\varphi^0 = \sin(\pi x)$. For instance, when $y_0 \equiv 5$ and $y_1 \equiv 1$, it turns out:

$$T_{\min} \geq \frac{1}{\pi^2} \log\left(\frac{y_0}{y_1}\right) = \frac{1}{\pi^2} \log(5) \cong 0.1630702.$$

4.6.2 The semilinear case

So far the problem of constrained controllability in the minimal time has not been analysed in the semilinear case. In the particular case of globally Lipschitz nonlinearities the linear arguments apply, and allow showing that there is a measure value control obtained as limit of controls when the time of control T tends to the minimal one $T_{\min}$. This is so since, mainly, globally Lipschitz nonlinearities can be handled by fixed point techniques out of uniform estimates for linear equations with bounded potentials, the bound on the potential being uniform. But the analysis of the actual controllability properties that the system experiences in the minimal time for the limit nonnegative measure control need to be further explored. This is so because of the very weak regularity properties of solutions that make difficult their definition in the nonlinear case.

In this section we run some numerical simulations in the case of a sinusoidal nonlinearity showing that, in fact, one may expect similar properties as those encountered for the linear problem, namely:

- the positivity of the minimal controllability time in agreement with Theorem 4.5.2;
- the sparse structure of the controls in the minimal time.

We consider the nonlinearity $f(y) = sin(\pi y)$ in $1d$. The problem under consideration is then:

$$\begin{cases} y_t - y_{xx} + sin(\pi y) = 0 & (t,x) \in (0,T) \times (0,1) \\ y(t,0) = u_1(t) & t \in (0,T) \\ y(t,1) = u_2(t). & t \in (0,T) \\ y(0,x) = 1 & x \in (0,1) \end{cases} \tag{4.6.2}$$

with final target $y_1 \equiv 2$. Note that both the initial datum and the final target are steady states for the system under consideration. The nonlinearity appearing in the above system is globally Lipschitz. Then, we can apply Theorem 4.5.2 getting $T_{\min} > 0$. We are now interested in determining numerically $T_{\min} > 0$ and the control in the minimal time.

We perform the simulation by using `IpOpt`. As in [95], we employ a finite-difference discretisation scheme combining the explicit Euler discretisation in time and 3-point finite differences in space.

The time-space grid is uniform $\left\{(i\frac{T}{N_t}, \frac{j}{N_x})\right\}$, with indexes $i = 0, \ldots, N_t$ and $j = 0, \ldots, N_x$. We choose $N_t = 200$ and $N_x = 20$ so that they satisfy the Courant-Friedrich-Lewy condition $2\Delta t \leq (\Delta x)^2$, where $\Delta t = T/N_t$ and $\Delta x = 1/N_x$. The discretized space is then a matrix space of dimension $(N_t + 1) \times (N_x + 1)$.

We denote by Y the discretized state and by U^0 and U^1 the discretized boundary controls.

The problem of numerically approximating the minimal control time under constraints is addressed by computing the minimum number of time iterations for the discrete dynamics. In other words, the discretized state Y and the discretized controls U^i are subjected to the discrete dynamics, the boundary and terminal conditions both at the initial and final time and so that the positivity constraint is fulfilled.

We thus solve numerically:

$$\min T$$

under the constraints:

$$\begin{cases} \frac{Y_{i+1,j} - Y_{i,j}}{\Delta t} = \frac{Y_{i,j+1} - Y_{i,j} + Y_{i,j-1}}{\Delta x} + \sin(\pi j) & i = 0, \ldots, N_t - 1, \ j = 1, \ldots N_x - 1 \\ Y_{i,0} = U_i^0, \quad Y_{i,N_x} = U_i^1 & i = 0, \ldots, N_t, \\ U_i^0 \geq 0, \qquad U_i^1 \geq 0 & i = 0, \ldots, N_t, \\ Y_{0,j} = y_0, \quad Y_{N_t,j} = y_1 & j = 0, \ldots, N_x, \end{cases}$$

Numerical simulations are done employing the expert interior-point optimization routine `IpOpt` (see [143]), the modeling language being `AMPL` (see [55]).

We observe:

- the computed minimal time $T_{\min} = 0.045197$ is positive in agreement with Theorem 4.5.2;
- controllability holds in the minimal time, thus suggesting that the conclusions of Proposition 4.5.1 may hold even in the nonlinear context (see figure 4.5);
- the control in the minimal time exhibits the sparse structure described by figure 4.5. In particular, it seems that the nonnegative control driving (4.6.2) from $y_0 \equiv 1$ to $y_1 \equiv 2$ in the minimal time is a sum of Dirac masses.

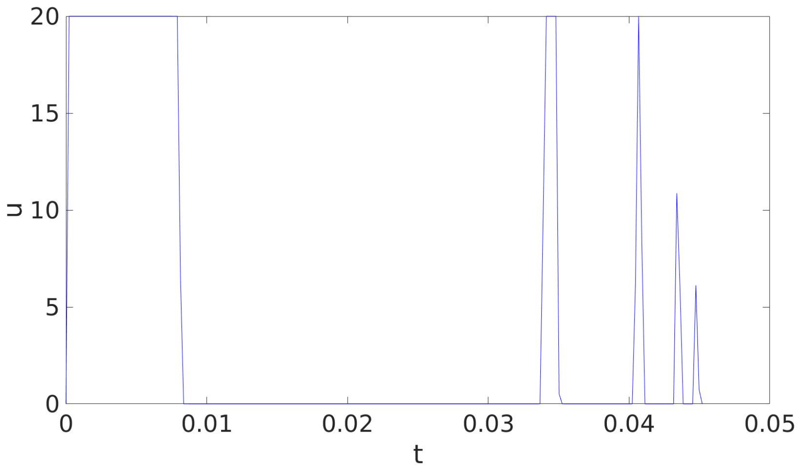


Figure 4.5: graph of the control in the minimal time.

Appendix

Regularity for linear parabolic equations.

We begin the Appendix stating a known result for the linear problem

$$\begin{cases} y_t - \operatorname{div}(A\nabla y) + b\cdot\nabla y + cy = h & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{4.6.3}$$

Let $1 \leq p \leq +\infty$ and consider the anisotropic Sobolev Space:

$$W_p^{1,2}((0,T)\times\Omega) = \left\{ y \in L^p((0,T);W^{2,p}(\Omega)) \mid y_t \in L^p((0,T)\times\Omega) \right\}$$

endowed with the norm:

$$\|y\|_{W_p^{1,2}} = \|u\|_{L^p} + \|\nabla_x y\|_{L^p} + \|D_x^2 y\|_{L^p} + \|y_t\|_{L^p}.$$

The following holds:

Theorem 4.6.1 (Parabolic regularity). *Let Ω be a bounded open set with $\partial\Omega \in C^2$. Assume that $A \in W^{1,\infty}((0,T) \times \Omega; \mathbb{R}^{n\times n})$, $b \in L^\infty((0,T) \times \Omega; \mathbb{R}^n)$ and $c \in L^\infty((0,T) \times \Omega)$. Let $1 < p < +\infty$. Then, for any $y_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $h \in L^p((0,T) \times \Omega)$,*

1. *there exists a unique solution $y \in W_p^{1,2}((0,T) \times \Omega)$ and the following estimate holds:*

$$\|y\|_{W_p^{1,2}} \leq C\left[\|y_0\|_{W^{2,p}} + \|h\|_{L^p}\right]; \tag{4.6.4}$$

2. *if $p > n+2$, $y \in C^{0,\gamma}([0,T]; C^{1,\gamma}(\overline{\Omega}))$ for some $\gamma \in (0,1)$ and:*

$$\|y\|_{C^{0,\gamma}([0,T];C^{1,\gamma}(\overline{\Omega}))} \leq C\left[\|y_0\|_{W^{2,p}} + \|h\|_{L^p}\right]. \tag{4.6.5}$$

The constants C and γ depend only on Ω, A, b, $\|c\|_{L^\infty}$ and T.

The proof of the first part of the Theorem can be found in [90, Theorem 7.32 page 182]. In this reference the considered parabolic operator is in a non divergence form. The Lipschitz assumption on the diffusion A we impose suffices to transform the operator under consideration from the divergence to the non divergence form, keeping the Lipschitz continuity on the diffusion matrix and the boundedness on the other coefficients. For the proof of the first part, see also [84, Theorem 9.1 page 341] and [146, Theorem 9.2.5 page 275].

The second part is due to the existence of the continuous embedding:

$$i : W_p^{1,2}((0,T) \times \Omega) \hookrightarrow C^{0,\gamma}([0,T]; C^{1,\gamma}(\overline{\Omega})),$$

provided that $p > n+2$ (see [84, Lemma 3.3 page 81] or [133]).

Finally, assuming further b Lipschitz continuous, we can get the same regularity result for the adjoint problem:

$$\begin{cases} -\varphi_t - \operatorname{div}(A\nabla\varphi) - \operatorname{div}(\varphi b) + c\varphi = h & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T,x) = \varphi^0(x). & \text{in } \Omega \end{cases} \tag{4.6.6}$$

Corollary 4.6.1. *We suppose the same hypotheses of Theorem 4.6.1 and we assume further that the drift term $b \in W^{1,\infty}((0,T) \times \Omega)$. Then, the same conclusions of Theorem 4.6.1 hold for the adjoint problem* (4.6.6).

This corollary can be proved employing the time inversion $t \to T - t$ and applying the above theorem. Expanding $-div(\varphi b) = -b \cdot \nabla\varphi - \operatorname{div}(b)\varphi$, one realizes that the Lipschitz condition on b guarantees the boundedness of the coefficient $-\operatorname{div}(b)$.

Well-posedness of the state equation.

Let us define the notion of (weak) solution of (4.1.1).

First of all, we introduce the class of test functions:

$$\mathscr{T} := \{\varphi \in C^\infty([0,T]\times\overline{\Omega}) \text{ such that}$$
$$\varphi(T,x)=0 \quad \forall x\in\Omega \qquad \text{and} \qquad \varphi(t,x)=0 \quad \forall (t,x)\in[0,T]\times\partial\Omega\}\,.$$

Definition 4.6.1. *Let $y_0 \in L^\infty(\Omega)$ be the initial datum and $u \in L^\infty((0,T)\times\Gamma)$ be the boundary control. Then, $y \in L^\infty((0,T)\times\Omega)\cap C^0([0,T];H^{-1}(\Omega))$ is said to be a solution to* (4.1.1) *if for any test function $\varphi \in \mathscr{T}$:*

$$\int_0^T\int_\Omega [-\varphi_t - div(A\nabla\varphi) - div(\varphi b)]\, y \; dxdt + \int_0^T\int_\Omega f(t,x,y)\varphi \; dxdt =$$
$$= \int_\Omega \varphi(x,0)y_0(x)\; dx - \int_0^T\int_\Gamma u\frac{\partial\varphi}{\partial n}\; d\sigma(x)dt.$$

In the above equation, $n = A\hat{v}/\|A\hat{v}\|$, where $\hat{v}$ is the outward unit normal to Γ.

The following local existence and uniqueness result holds.

Proposition 4.6.1. *Let $R>0$. Then, there exists $T_R>0$ such that for each time horizon $T\in(0,T_R)$ and for any initial datum $y_0\in L^\infty(\Omega)$ and boundary control $u\in L^\infty((0,T)\times\Gamma)$ fulfilling the smallness conditions:*

$$\|y_0\|_{L^\infty}\le R, \quad \|u\|_{L^\infty}\le R, \tag{4.6.7}$$

problem (4.1.1) *admits an unique solution $y\in L^\infty((0,T)\times\Omega)\cap C^0([0,T];H^{-1}(\Omega))$. Furthermore,*

$$\|y\|_{L^\infty}\le C_R[\|y_0\|_{L^\infty}+\|u\|_{L^\infty}], \tag{4.6.8}$$

the constant C_R depending only on R. Furthermore, both T_R and C_R can be chosen uniformly over the nonlinearities:

$$\tilde{f}_{\overline{y}}(t,x,y) = f(t,x,y+\overline{y}(t,x)) - f(t,x,\overline{y}(t,x)), \tag{4.6.9}$$

where $f\in C^1$, $\overline{y}\in L^\infty$ and $\|\overline{y}\|_{L^\infty}\le R$.

The uniqueness for (4.1.1) can be proved by energy estimates.

Existence can be addressed splitting the solution $y=z+w$, where z is the solution to:

$$\begin{cases} z_t - \text{div}(A\nabla z) + b\cdot\nabla z = 0 & \text{in } (0,T)\times\Omega \\ z = u\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ z(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{4.6.10}$$

and w solves:

$$\begin{cases} w_t - \mathrm{div}(A\nabla w) + b \cdot \nabla w + f(t,x,w+z) = 0 & \text{in } (0,T)\times\Omega \\ w = 0 & \text{on } (0,T)\times\partial\Omega \\ w(0,x) = 0. & \text{in } \Omega \end{cases} \tag{4.6.11}$$

The global existence for (4.6.10) holds by transposition by adapting [106, page 202] to the present case, while the local existence for (4.6.11) can be proved by fixed point as follows.

First of all, by the transformation $\hat{w} = e^{-\lambda t} w$, the linear part of the operator can be made to be dissipative. Then, for any $\eta \in L^\infty((0,T)\times\Omega)$, we consider $\phi(\eta)$ the unique solution to:

$$\begin{cases} w_t - \mathrm{div}(A\nabla w) + b \cdot \nabla w + \lambda w + f(t,x,\eta+z) = 0 & \text{in } (0,T)\times\Omega \\ w = 0 & \text{on } (0,T)\times\partial\Omega \\ w(0,x) = 0. & \text{in } \Omega \end{cases} \tag{4.6.12}$$

This actually defines the map:

$$\phi : \overline{B^{L^\infty}(0,2R)} \longrightarrow \overline{B^{L^\infty}(0,2R)}; \quad \eta \longmapsto \phi(\eta),$$

where $\overline{B^{L^\infty}(0,2R)}$ stands for the closed ball of radius $2R$ centered at 0 in L^∞.

For the sake of simplicity, let us consider the case where the coefficients of the linear part do not depend on time. In this case, $\phi(\eta)$ can be represented by the variation of constants formula as:

$$\phi(\eta) = \int_0^t S(t-s) f(s,x,\eta+z)\, ds, \tag{4.6.13}$$

where $\{S(t)\}$ is the semigroup associated to the linear part of our system. Then, by choosing T small enough, ϕ can be shown to be contractive in $\overline{B^{L^\infty}(0,2R)}$. We conclude applying the Banach fixed point theorem.

This argument can be applied uniformly over the set of nonlinearities (4.6.9) since the above fixed point argument can be accomplished uniformly, while z is independent of the nonlinearity.

Local controllability

In this section we prove the local controllability of the semilinear system. We will proceed as follows:

- proof of global controllability of (4.1.1) by controls in L^∞, under global Lipschitz assumptions on the nonlinearity. We employ an extension-restriction technique;
- proof of the local controllability of (4.1.1) in the general case (Lemma 4.2.1).

We make use of an extension-restriction argument so to avoid some technical difficulties that arise when dealing directly with the boundary control problem.

It is worth noticing that, by classical extension results (see Whitney extension theorem and [50, Theorem 1 page 268]), we can suppose the coefficients $A \in W^{1,\infty}((0,T) \times \mathbb{R}^n; \mathbb{R}^{n\times n})$, $b \in W^{1,\infty}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^n)$ and the nonlinearity $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R})$. Combining existing results of interior controllability of (4.1.1) (see [77]) and an extension-restriction argument one can prove a global controllability result for (4.1.1) with globally Lipschitz nonlinearity and general data in L^2.

Lemma 4.6.1. *Suppose that $f = f(t,x,y)$ is globally Lipschitz in y uniformly in (t,x), i.e.:*

$$|f(t,x,y_2) - f(t,x,y_1)| \leq L|y_2 - y_1|, \quad \forall\ (t,x,y_1,y_2) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2. \tag{4.6.14}$$

Let $T > 0$. Then, we consider a target trajectory $\overline{y}$ solution to (4.1.1) *with initial datum $\overline{y}_0 \in L^2$ and control $\overline{u} \in L^\infty$. Finally, we take an initial datum $y_0 \in L^2$.*

Then, there exists a control $u \in L^\infty((0,T) \times \Gamma)$ such that the unique solution y to (4.1.1) *with initial datum y_0 and control u verifies:*

$$y(T,\cdot) = \overline{y}(T,\cdot), \quad \text{in } \Omega.$$

Furthermore,

$$\|u - \overline{u}\|_{L^\infty} \leq C\|y_0 - \overline{y}_0\|_{L^2}, \tag{4.6.15}$$

the constant C being independent of y_0 and $\overline{y}$.

Note that, thanks to the regularizing effect of the heat equation, we are able to estimate the L^∞ norm of the control, with the L^2 norm of the initial data.

Proof of Lemma 4.6.1. **Step 1.** Reduction to null controllability.

Taking $\eta = y - \overline{y}$, the problem is reduced to prove the null controllability of the system:

$$\begin{cases} \eta_t - \operatorname{div}(A\nabla\eta) + b\cdot\nabla\eta + \tilde{f}(t,x,\eta) = 0 & \text{in } (0,T)\times\Omega \\ \eta = v\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ \eta(0,x) = y_0 - \overline{y}_0, & \text{in } \Omega \end{cases} \tag{4.6.16}$$

where $\tilde{f}(t,x,\eta) = f(t,x,\eta + \overline{y}(t,x)) - f(t,x,\overline{y}(t,x))$.

Step 2. Regularization of the initial datum.

Let $0 < \tau < T$. We firstly let the system evolve with zero boundary control in $[0,\tau]$ to regularize the initial datum. Indeed, by Moser-type techniques (see, for instance, [113, Theorem 1.7], [146] or [90]), the solution η to (4.6.16) with null control in $[0,\tau]$ is such that $\eta_1 = \eta(\tau,\cdot) \in C^0(\overline{\Omega})$ and $\eta_1(x) = 0$ for any $x \in \partial\Omega$, with estimate:

$$\|\eta_1\|_{C^0} \leq C\|y_0 - \overline{y}_0\|_{L^2}. \tag{4.6.17}$$

Step 3. Extension.

We extend our domain Ω around Γ getting an extended domain $\widehat{\Omega}$ such that:

- $\Omega \subset \widehat{\Omega}$;
- $\partial\Omega \setminus \Gamma \subset \partial\widehat{\Omega}$;
- there exists a ball ω such that $\overline{\omega} \subset \widehat{\Omega} \setminus \overline{\Omega}$;
- $\partial\widehat{\Omega} \in C^2$.

Then, we introduce $\widehat{\eta}_1 = \eta_1 \mathbf{1}_\Omega$ the extension by 0 of the regularized initial datum.

Step 4. Interior Controllabilty.

By [77, Theorem 3.1], there exists a control $h \in L^2((\tau,T)\times\omega)$ such that the unique solution $\widehat{\eta}$ to:

$$\begin{cases} \eta_t - \operatorname{div}(A\nabla\eta) + b\cdot\nabla\eta + \tilde{f}(t,x,\eta) = h\mathbf{1}_\omega & \text{in } (\tau,T)\times\widehat{\Omega} \\ \eta = 0 & \text{on } (\tau,T)\times\partial\widehat{\Omega} \\ \eta(\tau,x) = \widehat{\eta}_1 & \text{in } \widehat{\Omega} \end{cases} \tag{4.6.18}$$

verifies the final condition $\widehat{\eta}(T,\cdot) = 0$, with

$$\|h\|_{L^2((\tau,T)\times\omega)} \leq C\,\|\eta_1\|_{L^2(\Omega)} \leq C\,\|y_0 - \overline{y}_0\|_{L^2(\Omega)},$$

the constant C being independent of y_0 and $\overline{y}_0$. Now, since $\overline{\omega} \subset \widehat{\Omega} \setminus \overline{\Omega}$, by the regularization effect of parabolic equations, we have $\widehat{\eta} \in C^0([\tau,T]\times\overline{\Omega})$ and:

$$\|\widehat{\eta}\|_{C^0} \leq C[\|h\|_{L^2} + \|\widehat{\eta}_1\|_{C^0}] \leq C\|y_0 - \overline{y}_0\|_{L^2}. \tag{4.6.19}$$

Step 5. Restriction.

The boundary control

$$v = \begin{cases} 0 & \text{in } (0,\tau) \\ \widehat{\eta}\restriction_{(\tau,T)\times\Gamma} & \text{in } (T-\tau,T) \end{cases}$$

steers (4.6.16) from $y_0 - \overline{y}_0$ to 0. Hence, $u = v + \overline{u}$ drives (4.1.1) from y_0 to $\overline{y}(T,\cdot)$. Finally, (4.6.15) is a consequence of (4.6.19). □

Now, we are ready to prove the announced local controllability result (Lemma 4.2.1).

Proof of Lemma 4.2.1. **Step 1.** Controllability of the truncated system.

Let $M = 2R$. We introduce the cut-off function $\zeta \in C^\infty(\mathbb{R})$ such that:

- $supp(\zeta) \subseteq [-2M, 2M]$;
- $\zeta\restriction_{[-M,M]} \equiv 1$.

We are now in position to define the truncated nonlinearity $f_L(t,x,y) = f(t,x,\zeta(y)y)$. Note that f_L is globally Lipschitz in y uniformly in (t,x), i.e.:

$$|f_L(t,x,y_2) - f_L(t,x,y_1)| \le L|y_2 - y_1|, \quad \forall\ (t,x,y_1,y_2) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2. \tag{4.6.20}$$

Then, by Lemma 4.6.1, we can find a control $u \in L^\infty((0,T)\times\Gamma)$ such that the unique solution y_L to:

$$\begin{cases} y_t - \operatorname{div}(A\nabla y) + b\cdot\nabla y + f_L(t,x,y) = 0 & \text{in } (0,T)\times\Omega \\ y = u\mathbf{1}_\Gamma & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x), & \text{in } \Omega \end{cases} \tag{4.6.21}$$

verifies $y_L(T,\cdot) = \overline{y}(T,\cdot)$. Moreover, by (4.6.15),

$$\|u - \overline{u}\|_{L^\infty((0,T)\times\Gamma)} \le C\|y_0 - \overline{y}_0\|_{L^\infty(\Omega)}. \tag{4.6.22}$$

Therefore, choosing $\delta > 0$ small enough, whenever $\|\overline{y}_0 - y_0\|_{L^\infty} < \delta$, we have:

$$\|u - \overline{u}\|_{L^\infty} \le R. \tag{4.6.23}$$

Step 2. Conclusion with the original nonlinearity.

The final target is a trajectory for the system. Hence, in the notation of Proposition 4.6.1, we can suppose $T < T_R$. Right now, let y be the solution to (4.1.1) with the original nonlinearity f, initial datum y_0 and control u. Proposition 4.6.1 together with (4.2.1), (4.6.23) and (4.6.22) yields:

$$\begin{aligned} \|y - \overline{y}\|_{L^\infty} &\le C_R\left[\|y_0 - \overline{y}_0\|_{L^\infty} + \|u - \overline{u}\|_{L^\infty}\right] \\ &\le C_R\|y_0 - \overline{y}_0\|_{L^\infty} \le R, \end{aligned}$$

taking δ small enough. Hence, $\|y - \overline{y}\|_{L^\infty} \le R$. This in turn implies $\|y\|_{L^\infty} \le 2R = M$. Finally, by the definition of f_L, we have $y = y_L$ thus finishing the proof. □

Chapter 5

Controllability under positivity constraints of multi-d wave equations

5.1 Introduction

This chapter corresponds to [112].

We shall study the controllability properties of the wave equation, under *positivity* (or nonnegativity) constraints on the control.

We address both the *interior* controllability and the *boundary* controllability problem.

The controllability of the wave equation has been exhaustively considered in the unconstrained case but very little is known in the presence of constraints on the control, an issue of primary importance in applications, since whatever the applied context under consideration is, the available controls are always limited. For some of the basic literature on the unconstrained controllability of wave-like equations the reader is referred to: [12, 17, 19, 18, 40, 48, 93, 127, 128, 152, 150].

Concerning constrained controllability, in [65], authors analysed controllability of the one dimensional wave equation, under the more classical bilateral constraints on the control. Our work is, as far as we know, the first one considering unilateral constraints for wave-like equations.

The developments in this chapter are motivated by our earlier works on the constrained controllability of heat-like equations ([95, 111]). In that context, due to the well-known comparison principle for parabolic equations, control and state constraints are interlinked. In particular, for the heat equation, nonnegative controls imply that the solution is nonnegative too, provided that the initial configuration is nonnegative. Therefore, imposing non-negativity constraints on the control ensures that the state satisfies the non-negativity constraint too.

This is no longer true for wave-like equations in which the sign of the control does not determine that of solutions. However, as mentioned above, from a practical viewpoint, it is very natural to consider the problem of imposing control constraints. In this chapter, to fix ideas, we focus in the particular case of nonnegative controls.

First we address the problem of steady state controllability in which one aims at controlling the solution from a steady configuration to another one. In absence of constraints on the control, this problem was addressed in [37] for semilinear wave equations. Our main contribution here is to control the system by preserving some constraints on the controls given a priori. And, as we shall see, when the initial and final steady states are associated with positive time-independent control functions, the constrained controllability can be guaranteed to hold if the time-horizon is long enough.

The proof is developed by a step-wise procedure presented in [111] (which differs from the one in [37] and [95]), the so-called "stair-case argument", along an arc of steady-states linking the starting and final one. The proof consists on moving recursively from one steady state to the other by means of successive small amplitude controlled trajectories linking successive steady-states. This "stair-case argument" and the corresponding controllability result are presented in a general semigroup setting and they are applicable to any control system for which controllability holds by means of L^∞ controls.

The same recursive approach enables us to prove a state constrained result, under additional dissipativity assumptions. But the time needed for this to hold is even larger than before.

The problem of steady-state controllability is a particular instance of the more general trajectory control problem, in which, given two controlled trajectories of the system, both obtained from nonnegative controls, and one state in each of them (possibly corresponding to two different time-instances) one aims at driving one state to the other one by means of nonnegative constrained controls. Controllability between trajectories can also be proved by a similar iterative procedure, but under the added assumption that the system is conservative and its energy coercive so that uncontrolled trajectories are globally bounded.

Even in the unconstrained context, the control of waves requires a long-enough time horizon, determined by the finite velocity of propagation and the so-called Geometric Control Condition (GCC) [12, 17]. The implemented recursive procedure is based on the construction of small amplitude control, thus requiring a large control time, much beyond the time needed in the unconstrained context. It is then meaningful to define the minimal controllability time under control and/or state constraints.

There is plenty to be done to understand how these constrained minimal times depends on the data to be controlled. In this chapter, we give an answer to this important issue for constant steady states in one space dimension. Employing d'Alembert's formula for the one dimensional wave equation, we compute both minimal times for constant steady states, showing that they coincide with the unconstrained one. In that case we also show that the property of constrained controllability holds in the minimal time too.

Controllability under constraints has already been studied for finite-dimensional models and heat-like equations [95, 111]. In both cases it was also proved that controllability by nonnegative controls fails if time is too short, when the initial datum differs from the final target. This fact exhibits a big difference with respect to the unconstrained control problem for those systems, where controllability holds in arbitrary small time in both cases. In the wave-like context addressed in this chapter, the waiting phenomenon, according to which there is a minimal control time for the constrained problem, is less surprising. On the other hand, in some sense, the fact that constraints can be imposed on controls and state seems more striking too.

We start presenting our main results in the context of internal control, to later deal with boundary control.

5.1.1 Internal control

Let Ω be a connected bounded open set of $\mathbb{R}^n$, $n \geq 1$, with C^∞ boundary, and let ω and ω_0 be subdomains of Ω such that $\overline{\omega_0} \subset \omega$.

Let $\chi \in C^\infty(\mathbb{R}^n)$ be a smooth function supported in ω such that $\text{Range}(\chi) \subseteq [0,1]$, $\chi\restriction_{\omega_0} \equiv 1$.

We assume further that all derivatives of χ vanish on the boundary of Ω. We will discuss further this assumption in subsection 5.3.3.

We consider the wave equation controlled from the interior

$$\begin{cases} y_{tt} - \Delta y + cy = u\chi & \text{in } (0,T) \times \Omega \\ y = 0 & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0^0(x),\ y_t(0,x) = y_0^1(x) & \text{in } \Omega \end{cases} \tag{5.1.1}$$

where $y = y(t,x)$ is the state, while $u = u(t,x)$ is the control whose action is localized on ω by means of multiplication with the smooth cut-off function χ. The coefficient $c = c(x)$ is C^∞ smooth in $\overline{\Omega}$.

We assume the *Geometric Control Condition* on (Ω, ω_0, T^*), which basically asserts that all bicharacteristic rays enter in the subdomain ω_0 in time smaller than T^*. This geometric condition is actually equivalent to the property of (unconstrained) controllability of the system [12, 17].

It is well known in the literature (e.g. [50, section 7.2]) that, for any initial datum $(y_0^0, y_0^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and for any control $u \in L^2((0,T)\times\omega)$, the above problem admits a unique solution $(y, y_t) \in C^0([0,T]; H_0^1(\Omega) \times L^2(\Omega))$, with $y_{tt} \in L^2(0,T; H^{-1}(\Omega))$.

5.1.1.1 Steady state controllability

The purpose of our first result is to show that, in large time, we can drive (5.1.1) from one steady state to another by a *nonnegative* control, assuming the uniform *positivity* of the control defining the steady states.

Recall that a steady state is a solution to

$$\begin{cases} -\Delta\overline{y} + c\overline{y} = \overline{u}\chi & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.2}$$

where $\overline{u} \in L^2(\omega)$ and $\overline{y} \in H^2(\Omega) \cap H_0^1(\Omega)$. Note that, as a consequence of Fredholm Alternative (see [57, Theorem 5.11 page 84]), the existence and uniqueness of the solution to (5.1.2) can be guaranteed whenever zero is not an eigenvalue of $-\Delta + cI : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$.

The following result holds:

Theorem 5.1.1 (Controllability between steady states)**.** *Take $\overline{y}_0$ and $\overline{y}_1$ in $H^2(\Omega) \cap H_0^1(\Omega)$ steady states associated with L^2-controls $\overline{u}^1$ and $\overline{u}^2$, respectively. Assume further that there exists $\sigma > 0$ such that*

$$\overline{u}^i \geq \sigma, \quad a.e. \ in\ \omega. \tag{5.1.3}$$

Then, if T is large enough, there exists a control $u \in L^2((0,T) \times \omega)$, such that

- *the unique solution (y, y_t) to the problem (5.1.1) with initial datum $(\overline{y}_0, 0)$ and control u verifies $(y(T,\cdot), y_t(T,\cdot)) = (\overline{y}_1, 0)$;*
- *$u \geq 0$ a.e. on $(0,T) \times \omega$.*

Theorem 5.1.1 is proved in subsection 5.3.1. Inspired by [37], we implement a recursive "stair-case" argument to keep the control in a narrow tubular neighborhood of the segment connecting the controls defining the initial and final data. This will guarantee the actual positivity of the control obtained.

5.1.1.2 Controllability between trajectories

The purpose of this section is to extend Theorem 5.1.1 to more controllability between trajectories, under the additional assumption $c(x) > -\lambda_1$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω. This guarantees that the energy of the system defines a norm

$$\|(y^0, y^1)\|_E^2 = \int_\Omega \left[\|\nabla y^0\|^2 + c\big(y^0\big)^2\right] dx + \int_\Omega (y^1)^2 dx$$

on $H_0^1(\Omega) \times L^2(\Omega)$. Thus, by conservation of the energy, uncontrolled solutions are uniformly bounded for all t.

We assume that both the initial datum (y_0^0, y_0^1) and the final target (y_1^0, y_1^1) belong to controlled trajectories (see figure 5.1)

$$(y_i^0, y_i^1) \in \{(\overline{y}_i(\tau,\cdot), (\overline{y}_i)_t(\tau,\cdot) \mid \tau \in \mathbb{R}\}, \tag{5.1.4}$$

where $(\overline{y}_i, (\overline{y}_i)_t)$ solve (5.1.1) with *nonnegative* controls. We suppose that these trajectories are smooth enough, namely

$$(\overline{y}_i, (\overline{y}_i)_t) \in C^{s(n)}(\mathbb{R}; H_0^1(\Omega) \times L^2(\Omega)),$$

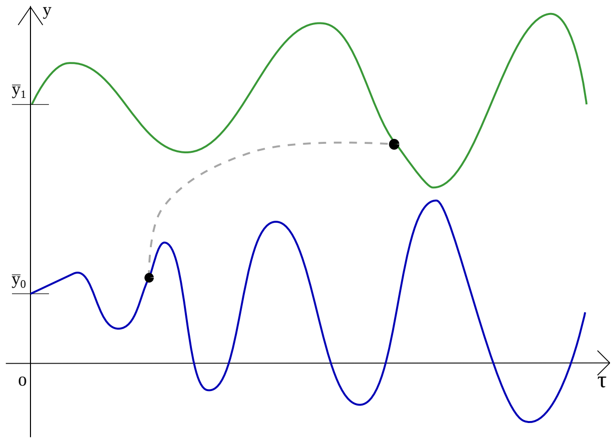


Figure 5.1: controllability between data lying on trajectories.

with $s(n) = \lfloor n/2 \rfloor + 1$. Hereafter, we denote by $(\overline{y}_0, (\overline{y}_0)_t)$ the initial trajectory, while $(\overline{y}_1, (\overline{y}_1)_t)$ stands for the target one.

Note that the regularity is assumed only in time and not in space. This allows to consider weak steady-state solutions.

We can in particular choose as final target the null state $(y_1^0, y_1^1) = (0, 0)$. It is important to highlight that this is something specific to the wave equation. In the parabolic case [95, 111], this is prevented by the comparison principle, since the zero target cannot be reached in finite time with non-negative controls. But, for the wave equation, the maximum principle does not hold and this obstruction does not apply.

The following result holds

Theorem 5.1.2 (Controllability between trajectories)**.** *Suppose $c(x) > -\lambda_1$, for any $x \in \overline{\Omega}$. Let $(\overline{y}_i, (\overline{y}_i)_t) \in C^{s(n)}(\mathbb{R}; H_0^1(\Omega) \times L^2(\Omega))$ be solutions to (5.1.1) associated with controls $\overline{u}^i \geq 0$ a.e. in $(0, T) \times \omega$, $i = 0, 1$. Take $(y_0^0, y_0^1) = (\overline{y}_0(\tau_0, \cdot), (\overline{y}_0)_t(\tau_0, \cdot))$ and $(y_1^0, y_1^1) = (\overline{y}_1(\tau_1, \cdot), (\overline{y}_1)_t(\tau_1, \cdot))$ for arbitrary values of τ_0 and τ_1. Then, in time $T > 0$ large enough, there exists a control $u \in L^2((0, T) \times \omega)$ such that*

- *the unique solution (y, y_t) to (5.1.1) with initial datum (y_0^0, y_0^1) verifies the terminal condition $(y(T, \cdot), y_t(T, \cdot)) = (y_1^0, y_1^1)$;*
- *$u \geq 0$ a.e. in $(0, T) \times \omega$.*

Remark 5.1.1. *This result is more general than Theorem 5.1.1 for two reasons*

1. *it enables us to link more general data, with nonzero velocity, and not only steady states;*
2. *the control defining the initial and target trajectories is assumed to be only nonnegative. This assumption is weaker than the uniform positivity one required in Theorem 5.1.1.*

On the other hand, the present result requires the condition $c(x) > -\lambda_1$ *on the potential* $c = c(x)$.

We give the proof of Theorem 5.1.2 in subsection 5.3.2.

5.1.2 Boundary control

Let Ω be a connected bounded open set of $\mathbb{R}^n$, $n \geq 1$, with C^∞ boundary, and let Γ_0 and Γ be open subsets of $\partial\Omega$ such that $\overline{\Gamma_0} \subset \Gamma$.

Let $\chi \in C^\infty(\partial\Omega)$ be a smooth function such that $\text{Range}(\chi) \subseteq [0,1]$, $\text{supp}(\chi) \subset \Gamma$ and $\chi\restriction_{\Gamma_0} \equiv 1$.

We now consider the wave equation controlled on the *boundary*

$$\begin{cases} y_{tt} - \Delta y + cy = 0 & \text{in } (0,T) \times \Omega \\ y = \chi u & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0^0(x), \ y_t(0,x) = y_0^1(x) & \text{in } \Omega \end{cases} \tag{5.1.5}$$

where $y = y(t,x)$ is the state, while $u = u(t,x)$ is the boundary control localized on Γ by the cut-off function χ. As before, the space-dependent coefficient c is supposed to be C^∞ regular in $\overline{\Omega}$.

We assume the *Geometric Control Condition* on (Ω, Γ_0, T^*) which asserts that all generalized bicharacteristics touch the sub-boundary Γ_0 at a non diffractive point in time smaller than T^*. By now, it is well known in the literature that this geometric condition is equivalent to (unconstrained) controllability [12, 17].

By transposition (see [93]) , one can realize that for any initial datum $(y_0^0, y_0^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and control $u \in L^2((0,T) \times \Gamma)$, the above problem admits a unique solution $(y, y_t) \in C^0([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$.

5.1.2.1 Steady state controllability

[12, 17]

As in the context of internal control, our first goal is to show that, in large time, we can drive (5.1.5) from one steady state to another, assuming the uniform positivity of the controls defining these steady states.

In boundary control a steady state is a time independent solution to (5.1.5), namely a solution to

$$\begin{cases} -\Delta \overline{y} + c\overline{y} = 0 & \text{in } \Omega \\ \overline{y} = \chi \overline{u} & \text{on } \partial\Omega. \end{cases} \tag{5.1.6}$$

In the present setting, $\overline{u} \in L^2(\partial\Omega)$ and $\overline{y} \in L^2(\Omega)$ solves (5.1.6) in the sense of transposition (see [106, chapter II, section 4.2] and [91]).

As in the context of internal control, if 0 is not an eigenvalue of $-\Delta+cI : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$, for any boundary control $\overline{u} \in L^2(\partial\Omega)$, there exists a unique $\overline{y} \in L^2(\Omega)$ solution to (5.1.6) with boundary control $\overline{u}$. This can be proved combining Fredholm Alternative (see [57, Theorem 5.11 page 84]) and transposition techniques [106, Theorem 4.1 page 73].

We prove the following result

Theorem 5.1.3 (Steady state controllability)**.** *Let $\overline{y}_i$ be steady states defined by controls $\overline{u}^i$, $i = 0, 1$, so that*

$$\overline{u}^i \geq \sigma, \quad \text{on } \Gamma, \tag{5.1.7}$$

with $\sigma > 0$.

Then, if T is large enough, there exists a control $u \in L^2([0, T] \times \Gamma)$, such that

- *the unique solution (y, y_t) to* (5.1.5) *with initial datum $(\overline{y}_0, 0)$ and control u verifies $(y(T, \cdot), y_t(T, \cdot)) = (\overline{y}_1, 0)$;*
- *$u \geq 0$ on $(0, T) \times \Gamma$.*

The proof of the above result can be found in subsection 5.4.1. The structure of the proof resembles the one of Theorem 5.1.1, with some technical differences due to the different nature of the control.

5.1.2.2 Controllability between trajectories

As in the internal control case, we suppose $c(x) > -\lambda_1$, where λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω. Then, the generator of the free dynamics is skew-adjoint (see [140, Proposition 3.7.6]), thus generating an unitary group of operators $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ on $L^2(\Omega) \times H^{-1}(\Omega)$.

Both the initial datum and final target (y_i^0, y_i^1) belong to a smooth trajectory, namely

$$(y_i^0, y_i^1) \in \{(\overline{y}_i(\tau, \cdot), (\overline{y}_i)_t(\tau, \cdot)) \mid \tau \in \mathbb{R}\}. \tag{5.1.8}$$

We assume the nonnegativity of the controls $\overline{u}^i$ defining $(\overline{y}_i, (\overline{y}_i)_t)$, for $i = 0, 1$. Hereafter, in the context of boundary control, we take trajectories of class $C^{s(n)}(\mathbb{R}; L^2(\Omega) \times H^{-1}(\Omega))$, with $s(n) = \lfloor n/2 \rfloor + 1$. We set $(\overline{y}_0, (\overline{y}_0)_t)$ to be the initial trajectory and $(\overline{y}_1, (\overline{y}_1)_t)$ be the target one.

Note that, with respect to Theorem 5.1.3, we have relaxed the assumptions on the sign of the controls $\overline{u}^i$. Now, they are required to be only *nonnegative* and not uniformly strictly positive.

Theorem 5.1.4 (Controllability between trajectories)**.** *Assume $c(x) > -\lambda_1$, for any $x \in \overline{\Omega}$. Let $(\overline{y}_i, (\overline{y}_i)_t)$ be solutions to* (5.1.5) *with non-negative controls $\overline{u}^i$ respectively. Suppose the trajectories $(\overline{y}_i, (\overline{y}_i)_t) \in C^{s(n)}([0, T]; L^2(\Omega) \times H^{-1}(\Omega))$. Pick $(y_0^0, y_0^1) = (\overline{y}_0(\tau_0, \cdot), (\overline{y}_0)_t(\tau_0, \cdot))$ and $(y_1^0, y_1^1) = (\overline{y}_1(\tau_1, \cdot), (\overline{y}_1)_t(\tau_1, \cdot))$. Then, in large time, we can find a control $u \in L^2((0, T) \times \Gamma)$ such that*

- *the solution* (y, y_t) *to* (5.1.5) *with initial datum* (y_0^0, y_0^1) *fulfills the final condition* $(y(T,\cdot), y_t(T,\cdot)) = (y_1^0, y_1^1)$;

- $u \geq 0$ *a.e. in* $(0,T) \times \Gamma$.

Theorem (5.1.4) is proved in subsection 5.4.2. Furthermore, in section 5.5, we show how this result applies in the one dimensional case, providing further information about the minimal time to control and the possibility of controlling the system in the minimal time.

5.1.2.3 State constraints

We impose now constraints both on the control and on the state. Namely both the *control* and the *state* are required to be nonnegative.

In the parabolic case [95, 111], one can employ the comparison principle to get a state constrained result from a control constrained one. But, now, as we have explained before, the comparison principle is not valid in general for the wave equation. Hence, we cannot rely on it to deduce our state constrained result from the control constrained one.

We shall rather apply the "stair-case argument" developed to prove steady state controllability, paying attention to the added need of preserving state constraints as well.

To fix ideas, we consider the pure wave equation, i.e. in (5.1.5) we take $c \equiv 0$. Furthermore, we suppose the control acts everywhere on the boundary. Given two steady states $y_0^0 \geq 0$ and $y_1^0 \geq 0$, we wish to solve in time T large the following controllability problem

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } (0,T) \times \Omega \\ y = u & \text{on } (0,T) \times \partial\Omega \\ y(0,x) = y_0^0(x), \; y_t(0,x) = 0 & \text{in } \Omega \\ y(T,x) = y_1^0(x), \; y_t(T,x) = 0 & \text{in } \Omega \end{cases} \tag{5.1.9}$$

under the constraints

$$\begin{aligned} u &\geq 0, \quad \text{a.e. on } (0,T) \times \partial\Omega \\ y &\geq 0, \quad \text{a.e. in } (0,T) \times \Omega. \end{aligned} \tag{5.1.10}$$

Note that both the initial datum and the final target are not required to be strictly positive. In particular, they can be identically zero.

Our strategy is the following

- reduce to the case of initial datum and target $y_i^0 \geq 1$ by constructing a "lift"-solution;

- employ the "stair-case argument" used to prove steady state controllability, to keep the control in a narrow tubular neighborhood of the segment connecting $\overline{u}^0$ and $\overline{u}^1$. This can be done by taking the time of control large enough. Since $\overline{u}^i \geq 1 > 0$, this guarantees the positivity of the control;

- by the continuous dependence of the solution on the data, the controlled trajectory remains also in a narrow neighborhood of the convex combination joining initial and final data. On the other hand, by construction, we have that $y_i^0 \geq 1$ in Ω, for $i = 0, 1$. In this way the state y can be assured to remain nonnegative.

Theorem 5.1.5 (State constraints)**.** *Let y_0^0 and y_1^0 be solutions to the steady problem*

$$-\Delta y = 0 \qquad in\, \Omega \tag{5.1.11}$$

with $y_i^0 \geq 0$ a.e. in Ω. We assume $y_i^0 \in H^{s(n)}(\Omega)$, with $s(n) = \lfloor n/2 \rfloor + 1$. Then, there exists $\overline{T} > 0$ such that for any $T > \overline{T}$ there exists a control $u \in L^\infty((0,T) \times \partial\Omega)$ and a corresponding state y such that

$$\begin{cases} y_{tt} - \Delta y = 0 & in\ (0,T) \times \Omega \\ y = u & on\ (0,T) \times \partial\Omega \\ y(0,x) = y_0^0(x),\ y_t(0,x) = 0 & in\ \Omega \\ y(T,x) = y_1^0(x),\ y_t(T,x) = 0 & in\ \Omega \end{cases} \tag{5.1.12}$$

under the constraints

$$\begin{aligned} u &\geq 0, \quad a.e.\ on\ (0,T) \times \partial\Omega \\ y &\geq 0, \quad a.e.\ in\ (0,T) \times \Omega. \end{aligned} \tag{5.1.13}$$

The proof of Theorem 5.1.5 can be found in subsection 5.4.3.

Note that the time needed to control the system keeping both the control and the state nonnegative is greater (or equal) than the corresponding one with no constraints on the state.

5.1.3 Orientation

The rest of the chapter is organized as follows:

- Section 5.2: Abstract results concerning constrained controllability of (linear) operator semigroups;
- Section 5.3: Internal Control: Proof of Theorem 5.1.1 and Theorem 5.1.2;
- Section 5.4: Boundary control: Proof of Theorem 5.1.3, Theorem 5.1.4 and Theorem 5.1.5;
- Section 5.5: The one dimensional case;
- Appendix.

5.2 Abstract results

The goal of this section is to provide some results on constrained controllability for some abstract control systems. We apply these results in the context of internal control and boundary control of the wave equation (see section 5.1).

We begin by introducing the abstract control system. Let H and U be two Hilbert spaces endowed with norms $\|\cdot\|_H$ and $\|\cdot\|_U$ respectively. H will be called the state space and U the control space. Let $A : D(A) \subset H \longrightarrow H$ be a generator of a C_0-semigroup $(\mathbb{T}_t)_{t\in\mathbb{R}^+}$, with $\mathbb{R}^+ = [0, +\infty)$. The domain of the generator $D(A)$ is endowed with the graph norm $\|x\|^2_{D(A)} = \|x\|^2_H + \|Ax\|^2_H$. We define H_{-1} as the completion of H with respect to the norm $\|\cdot\|_{-1} = \|(\beta I - A)^{-1}(\cdot)\|_H$, with real β such that $(\beta I - A)$ is invertible with continuous inverse from H to H. Adapting the techniques of [140, Proposition 2.10.2], one can check that the definition of H_{-1} is actually independent of the choice of β. Moreover, by applying the techniques of [140, Proposition 2.10.3], we deduce that A admits a unique bounded extension A from H to H_{-1}. For simplicity, we still denote by A the extension. Hereafter, we write $\mathscr{L}(E, F)$ for the space of all bounded linear operators from a Banach space E to another Banach space F.

Our control system is governed by:

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + Bu(t), \quad t \in (0, \infty), \\ y(0) = y_0, \end{cases} \tag{5.2.1}$$

where $y_0 \in H$, $u \in L^2_{loc}([0, +\infty), U)$ is a control function and the control operator $B \in \mathscr{L}(U, H_{-1})$ satisfies the admissibility condition in the following definition (see [140, Definition 4.2.1]).

Definition 5.2.1. *The control operator $B \in \mathscr{L}(U, H_{-1})$ is said to be admissible if for all $\tau > 0$ we have Range$(\Phi_\tau) \subset H$, where $\Phi_\tau : L^2((0, +\infty); U) \to H_{-1}$ is defined by:*

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - r} Bu(r) dr.$$

From now on, we will always assume the control operator to be admissible. One can check that for any $y_0 \in H$ and $u \in L^2_{loc}((0, +\infty); U)$ there exists a unique mild solution $y \in C^0([0, +\infty), H)$ to (5.2.1) (see, for instance, [140, Proposition 4.2.5]). We denote by $y(\cdot; y_0, u)$ the unique solution to (5.2.1) with initial datum y_0 and control u.

Now, we introduce the following constrained controllability problem

> *Let $\mathscr{U}_{ad}$ be a nonempty subset of U. Find a subset E of H so that for each y_0, $y_1 \in E$, there exists $T > 0$ and a control $u \in L^\infty(0, T; U)$ with $u(t) \in \mathscr{U}_{ad}$ for a.e. $t \in (0, T)$, so that $y(T; y_0, u) = y_1$.*

We address this controllability problem in the next two subsections, under different assumptions on $\mathscr{U}_{\mathrm{ad}}$ and (A, B). In subsection 5.2.1, we study the above controllability problem, when the initial and final data are steady states, i.e. solutions to the steady equation:

$$Ay + Bu = 0 \quad \text{for some} \quad u \in U. \tag{5.2.2}$$

In subsection 5.2.2, we take initial and final data on two different trajectories of (5.2.1).

To study (5.2.2), we need two ingredients, which play a key role in the proofs of subsection 5.2.1 and subsection 5.2.2. The first one is the notion of smooth controllability. Before introducing this concept, we fix $s \in \mathbb{N}$ and a Hilbert space V so that

$$V \hookrightarrow U, \tag{5.2.3}$$

where $\hookrightarrow$ denotes the continuous embedding. Note that throughout the remainder of the section, s and V remain fixed.

The notation $y(\cdot; y_0, u)$ stands for the solution of the abstract controlled equation (5.2.1) with control u and initial data y_0. The concept of smooth controllability is given in the following definition.

Definition 5.2.2. *The control system* (5.2.1) *is said to be smoothly controllable in time* $T_0 > 0$ *if for any* $y_0 \in D(A^s)$*, there exists a control function* $v \in L^\infty((0, T_0); V)$ *such that*

$$y(T_0; y_0, v) = 0$$

and

$$\|v\|_{L^\infty((0,T_0);V)} \leq C \|y_0\|_{D(A^s)}, \tag{5.2.4}$$

the constant C *being independent of* y_0.

Remark 5.2.1. *The following facts are worth noticing. (i) The system is smoothly controllable in time* T_0 *if for each (regular) initial datum* $y_0 \in D(A^s)$*, there exists a* L^∞*-control* u *with values in the regular space* V *steering our control system to rest at time* T_0.

(ii) The smooth controllability in time T_0 *of system* (5.2.1) *is a consequence of the following observability inequality: there exists a constant* $C > 0$ *such that for any* $z \in D(A^*)$

$$\|\mathbb{T}_{T_0}^* z\|_{D(A^s)^*} \leq C \int_0^{T_0} \|i^* B^* \mathbb{T}_{T_0 - t}^* z\|_{V^*} dt,$$

where $D(A^s)^*$ *is the dual of* $D(A^s)$ *and* $i : V \hookrightarrow U$ *is the inclusion. This inequality, that can often be proved out of classical observability inequalities employing the regularizing properties of the system, provides a way to prove the smooth controllability for system* (5.2.1)*. This occurs for parabolic problem enjoying smoothing properties.*

(iii) Besides, for some systems (A, B)*, even if they do not enjoy smoothing effects, there is an alternative way to prove the aforementioned smooth controllability property exploiting the ellipticity properties of the control operator (see [48]).*

Under suitable assumptions, the wave system is smoothly controllable (see Lemma 5.3.1 and Lemma 5.4.1).

The second ingredient is the following lemma, which concerns the regularity of the inhomogeneous problem.

Lemma 5.2.1. *Fix $k \in \mathbb{N}$ and take $f \in H^k((0,T);H)$ such that*

$$\begin{cases} \frac{d^j}{dt^j} f(0) = 0, & \forall\, j \in \{0, \dots, k\} \\ f(t) = 0, & a.e.\ t \in (\tau, T), \end{cases} \tag{5.2.5}$$

with $0 < \tau < T$. Consider y solution to the problem

$$\begin{cases} \frac{d}{dt} y = Ay + f & t \in (0,T) \\ y(0) = 0. \end{cases} \tag{5.2.6}$$

Then, $y \in \cap_{j=0}^{k} C^j([\tau, T]; D(A^{k-j}))$ and

$$\sum_{j=0}^{k} \|y\|_{C^j([\tau,T];D(A^{k-j}))} \leq C \|f\|_{H^k((0,T);H)},$$

the constant C depending only on k.

Remark 5.2.2. *Note that the maximal regularity of the solution is only assured for $t \geq \tau$, after the right hand side term f vanishes.*

The proof of this lemma is given in an Appendix at the end of this chapter.

5.2.1 Steady state controllability

In this subsection, we study the constrained controllability for some steady states. Our results in this part will be based on two fundamental assumptions:

(H_1) the system (5.2.1) is smoothly controllable in time T_0 for some $T_0 > 0$.

(H_2) $\mathscr{U}_{\mathrm{ad}}$ is a closed and convex cone with vertex at 0 and $\mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) \neq \varnothing$, where int^V denotes the interior set in the topology of V.

Recall that s and V are given by (5.2.3).

Furthermore, we define the following subset

$$\mathscr{W} = \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) + \mathscr{U}_{\mathrm{ad}}. \tag{5.2.7}$$

(Note that, since $\mathscr{U}_{\mathrm{ad}}$ is a convex cone, then $\mathscr{W} \subset \mathscr{U}_{\mathrm{ad}}$.) In what follows, $y(\cdot; y_0, u)$ will denote the solution to (5.2.1) with initial datum y_0 and control u. The main result of this subsection is the following.

Theorem 5.2.1 (Steady state controllability). *Assume* (H_1) *and* (H_2) *hold. Let* $\left\{(y_i, \overline{u}^i)\right\}_{i=0}^{1} \subset H \times \mathscr{W}$ *satisfying*

$$Ay_i + B\overline{u}^i = 0, \qquad i = 0, 1.$$

Then there exists $T > T_0$ *and* $u \in L^2(0,T;U)$ *such that*

- $u(t) \in \mathscr{U}_{ad}$ *a.e. in* $(0,T)$;
- $y(T; y_0, u) = y_1$.

Remark 5.2.3. *As we shall see, in the application to the wave equation with positivity constraints:*

- *for internal control,* $U = L^2(\omega)$ *and* $V = H^{s(n)}(\omega)$, *with* $s = s(n) = \lfloor n/2 \rfloor + 1$;
- *for boundary control,* $U = L^2(\Gamma)$ *and* $V = H^{s(n)-\frac{1}{2}}(\Gamma)$, *where* $s(n) = \lfloor n/2 \rfloor + 1$.

$\mathscr{U}_{ad}$ *is the set of nonnegative controls in* U. *In both cases,* $\mathscr{W}$ *is nonempty and contains controls* u *in* $L^2(\omega)$ *(resp.* $L^2(\Gamma)$*) such that* $u \geq \sigma$, *for some* $\sigma > 0$. *For this to happen, it is essential that* $H^{s(n)}(\omega) \hookrightarrow C^0(\overline{\omega})$ *(resp.* $H^{s(n)-\frac{1}{2}}(\Gamma) \hookrightarrow C^0(\overline{\Gamma})$*). This is guaranteed by our special choice of* $s = s(n)$. *Furthermore, in these special cases:*

$$\overline{\mathscr{W}}^U = \mathscr{U}_{ad},$$

where $\overline{\mathscr{W}}^U$ *is the closure of* $\mathscr{W}$ *in the space* U.

In the remainder of the present subsection we prove Theorem 5.2.1. The following lemma is essential for the proof of Theorem 5.2.1. Fix $\rho \in C^\infty(\mathbb{R})$ such that

$$\mathrm{Range}(\rho) \subseteq [0,1], \quad \rho \equiv 1 \quad \text{over } (-\infty, 0] \quad \text{and} \quad \mathrm{supp}(\rho) \subset\subset (-\infty, 1/2). \tag{5.2.8}$$

Lemma 5.2.2. *Assume that the system* (5.2.1) *is smoothly controllable in time* T_0, *for some* $T_0 > 0$. *Let* $(\eta_0, \overline{v}^0) \in H \times U$ *be a steady state, i.e. solution to* (5.2.2) *with control* $\overline{v}^0$. *Then, there exists* $w \in L^\infty((1, T_0+1); V)$ *such that the control*

$$v(t) = \begin{cases} \rho(t)\overline{v}^0 & \text{in } (0,1) \\ w & \text{in } (1, T_0+1) \end{cases} \tag{5.2.9}$$

drives (5.2.1) *from* η_0 *to* 0 *in time* $T_0 + 1$. *Furthermore,*

$$\|w\|_{L^\infty((1,T_0+1);V)} \leq C\|\eta_0\|_H. \tag{5.2.10}$$

The proof of Lemma 5.2.2 can be found in the Appendix.

We prove now Theorem 5.2.1, by developing a "stair-case argument" (see figure 5.2).

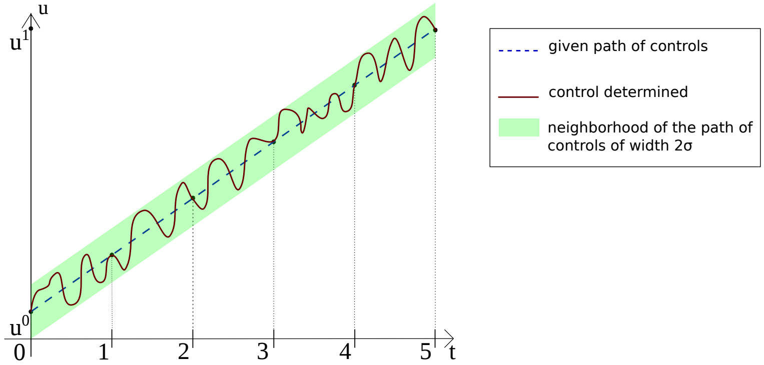


Figure 5.2: stepwise procedure

Proof of Theorem 5.2.1. Let $\left\{(y_i, \overline{u}^i)\right\}_{i=0}^{1}$ satisfy

$$Ay_i + B\overline{u}^i = 0 \quad \forall\, i \in \{0,1\}\,. \tag{5.2.11}$$

By the definition of $\mathscr{W}$, there exists $\left\{(q^i, z^i)\right\}_{i=0}^{1} \subset \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) \times \mathscr{U}_{\mathrm{ad}}$ such that

$$\overline{u}^i = q^i + z^i \quad i = 0, 1. \tag{5.2.12}$$

Define the segment joining y_0 and y_1

$$\gamma(s) = (1-s)y_0 + sy_1 \quad \forall\, s \in [0,1].$$

For each $s \in [0,1]$, $\gamma(s)$ solves

$$A\gamma(s) + B(q(s) + z(s)) = 0 \quad \forall\, i \in \{0,1\}\,.$$

where $(q(s), z(s)) \in \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) \times \mathscr{U}_{\mathrm{ad}}$ are defined by:

$$q(s) = (1-s)q^0 + sq^1 \quad \text{and} \quad z(s) = (1-s)z^0 + sz^1 \quad \forall\, s \in [0,1].$$

The rest of the proof is divided into two steps.
Step 1 **Show that there exists $\delta > 0$, such that for each $s \in [0,1]$, $q(s) + B^V(0,\delta) \subset \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V)$, where $B^V(0,\delta)$ denotes the closed ball in V, centered at 0 and of radius δ.**

Define

$$f(s) = \inf_{y \in V \setminus \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V)} \|q(s) - y\|_V, \quad s \in [0,1]. \tag{5.2.13}$$

One can check that f is Lipschitz continuous over the compact interval $[0,1]$. Then, by Weierstrass' theorem, we have that

$$\min_{s \in [0,1]} f(s) > 0.$$

Choose $0 < \delta < \min_{s\in[0,1]} f(s)$. Hence, by (5.2.13), it follows that, for each $s \in [0,1]$,

$$q(s) + B^V(0,\delta) \subset \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V),$$

as required.

Step 2 **Conclusion.**

Let $C > 0$ be given by Lemma 5.2.2. Let $\delta > 0$ be given by Step 1. Choose $N_0 \in \mathbb{N} \setminus \{0\}$ such that

$$N_0 > \frac{2C\|y_0 - y_1\|_H}{\delta}. \tag{5.2.14}$$

For each $k \in \{0, \ldots, N_0\}$, define:

$$y_k = \left(1 - \frac{k}{N_0}\right) y_0 + \frac{k}{N_0} y_1 \quad \text{and} \quad u_k = \left(1 - \frac{k}{N_0}\right) \overline{u}^0 + \frac{k}{N_0}\overline{u}^1. \tag{5.2.15}$$

It is clear that, by (5.2.12), for each $k \in \{0, \ldots, N_0 - 1\}$,

$$\|y_k - y_{k+1}\|_H = \frac{1}{N_0}\|y_0 - y_1\|_H \quad \text{and} \quad u_k - q\left(\frac{k}{N_0}\right) \in \mathscr{U}_{\mathrm{ad}}. \tag{5.2.16}$$

Fix arbitrarily $k \in \{0, \ldots, N_0 - 1\}$ and take $\eta_0 = y_k - y_{k+1}$ and $\overline{v}^0 = u_k - u_{k+1}$. Then, we can apply Lemma 5.2.2, to get a control $w_k \in L^\infty(1, T_0 + 1; V)$ such that

$$y(T_0 + 1; y_k - y_{k+1}, \hat{v}_k) = 0 \tag{5.2.17}$$

and

$$\|w_k\|_{L^\infty(1,T_0+1;V)} \leq C\|y_k - y_{k+1}\|_H, \tag{5.2.18}$$

where

$$\hat{v}_k(t) = \begin{cases} \rho(t)(u_k - u_{k+1}) & t \in (0,1] \\ w_k(t) & t \in (1, T_0 + 1). \end{cases} \tag{5.2.19}$$

Define

$$v_k(t) = \begin{cases} \rho(t)(u_k - u_{k+1}) + u_{k+1} & t \in (0,1] \\ w_k(t) + u_{k+1} & t \in (1, T_0 + 1). \end{cases} \tag{5.2.20}$$

By (5.2.11) and (5.2.15), we have

$$Ay^{k+1} + Bu_{k+1} = 0 \quad \text{and} \quad y(T_0 + 1; y_{k+1}, u_{k+1}) = y_{k+1}.$$

This, together with (5.2.17), (5.2.19) and (5.2.20), yields

$$\begin{aligned} y(T_0 + 1; y_k, v_k) &= y(T_0 + 1; y_k - y_{k+1}, \hat{v}_k) + y(T_0 + 1; y_{k+1}, u_{k+1}) \\ &= y_{k+1}. \end{aligned} \tag{5.2.21}$$

Next, we claim that

$$v_k(t) \in \mathscr{U}_{\mathrm{ad}} \quad \text{for a.e. } t \in (0, T_0 + 1). \tag{5.2.22}$$

Indeed, by (5.2.7) and since $\mathscr{U}_{\rm ad}$ is a convex cone, we have

$$\mathscr{W} \quad \text{is convex and} \quad \mathscr{W} \subset \mathscr{U}_{\rm ad}. \tag{5.2.23}$$

By (5.2.8), $0 \leq \rho(t) \leq 1$ for all $t \in \mathbb{R}$. Then, by (5.2.20) an (5.2.23), it follows that, for a.e $t \in (0,1)$,

$$v_k(t) = \rho(t)u_k + (1-\rho(t))u_{k+1} \in \rho(t)\mathscr{W} + (1-\rho(t))\mathscr{W} \subset \mathscr{W} \subset \mathscr{U}_{\rm ad}.$$

At this stage, to show (5.2.22), it remains to prove that

$$v_k(t) \in \mathscr{U}_{\rm ad} \quad \text{for a.e. } t \in (1, T_0+1). \tag{5.2.24}$$

Take $t \in (1, T_0+1)$. By (5.2.18), (5.2.16) and (5.2.14), we have

$$\|w_k(t)\|_V \leq \frac{C}{N_0}\|y_0 - y_1\|_H \leq \delta/2.$$

From this and Step 1, it follows

$$w_k(t) + q\left(\frac{k+1}{N_0}\right) \in \text{int}^V(\mathscr{U}_{\rm ad} \cap V).$$

By this, (5.2.16), (5.2.20) and (5.2.7), we get, for a.e. t in $(1, T_0+1)$,

$$\begin{aligned} v_k(t) &= w_k(t) + u_{k+1} \\ &= w_k(t) + q\left(\frac{k+1}{N_0}\right) + \left(u^{k+1} - q\left(\frac{k+1}{N_0}\right)\right) \\ &\in \text{int}^V(\mathscr{U}_{\rm ad} \cap V) + \mathscr{U}_{\rm ad} \\ &= \mathscr{W}. \end{aligned}$$

From this and (5.2.23), we are led to (5.2.24). Therefore, the claim (5.2.22) is true.

Finally, define

$$u(t) = v_k(t - k(T_0+1)), \qquad \forall\, t \in [k(T_0+1), (k+1)(T_0+1)), \qquad k \in \{0, \ldots, N_0 - 1\}.$$

Then, from (5.2.21) and (5.2.22), the conclusion of this theorem follows. □

In subsections 5.3.1 and 5.4.1, we apply Theorem 5.2.1 to prove Theorem 5.1.1 and Theorem 5.1.3 respectively. In particular:

- for internal control,

$$\mathscr{U}_{\rm ad} = \left\{u \in L^2(\omega) \mid u \geq 0, \text{ a.e. } \omega\right\};$$

- for boundary control,

$$\mathscr{U}_{\rm ad} = \left\{u \in L^2(\Gamma) \mid u \geq 0, \text{ a.e. } \Gamma\right\}.$$

Then, in both cases, $\mathscr{U}_{\rm ad}$ is closed convex cone with vertex at 0.

The above techniques can be adapted in a wide variety of contexts including parabolic and hyperbolic problems.

5.2.2 Controllability between trajectories

In this subsection, we study the constrained controllability for some general states lying on trajectories of the system with possibly nonzero time derivative. Our results in this subsection will rely on two fundamental assumptions:

(H_1') the system (5.2.1) is smoothly controllable in time T_0 for some $T_0 > 0$.

(H_2') the set $\mathscr{U}_{\text{ad}}$ is a closed and convex and $\text{int}^V(\mathscr{U}_{\text{ad}} \cap V) \neq \varnothing$, where int^V denotes the interior set in the topology of V;

(H_3') the operator A generates a C_0-group $\{\mathbb{T}_t\}_{t\in\mathbb{R}}$ over H and $\|\mathbb{T}_t\|_{\mathscr{L}(H,H)} = 1$ for all $t \in \mathbb{R}$. Furthermore, A is invertible from $D(A)$ to H, with continuous inverse.

Recall that s and V are given by (5.2.3).

The main result of this subsection is the following. The notation $y(\cdot; y_0, u)$ stands for the solution of the abstract controlled equation (5.2.1) with control u and initial data y_0.

Theorem 5.2.2. *Assume (H_1'), (H_2') and (H_3') hold. Let $\overline{y}_i \in C^s(\mathbb{R}; H)$ be solutions to (5.2.1) with controls $\overline{u}^i \in L^2_{loc}(\mathbb{R}; U)$ for $i = 0, 1$. Assume $\overline{u}^i(t) \in \mathscr{U}_{ad}$ for a.e. $t \in \mathbb{R}$. Let τ_0, $\tau_1 \in \mathbb{R}$. Then, there exists $T > 0$ and $u \in L^2(0, T; U)$ such that*

- *$y(T; \overline{y}_0(\tau_0), u) = \overline{y}_1(\tau_1)$;*
- *$u(t) \in \mathscr{U}_{ad}$ for a.e. $t \in (0, T)$.*

Remark 5.2.4. *(i) Roughly, Theorem 5.2.2 addresses the constrained controllability for all initial data y_0 and final target y_1, with y_0, $y_1 \in E$, where*

$$E = \Big\{ y(\tau) \;\Big|\; \tau \in \mathbb{R},\ y \in C^s(\mathbb{R}; H) \quad \text{and} \quad \exists\ u \in L^2_{loc}(\mathbb{R}; U),$$

$$\text{with} \quad u(t) \in \mathscr{U}_{ad} \quad \text{a.e. } t \in \mathbb{R} \quad \text{s.t.} \quad \frac{d}{dt} y(t) = Ay(t) + Bu(t), \quad t \in \mathbb{R} \Big\}.$$

By Lemma 5.2.1, one can check that

$$\Big\{ y(\tau; 0, u) \;\Big|\; \tau \in \mathbb{R},\ u \in C^s(\mathbb{R}, \mathscr{U}_{ad}), \frac{d^j}{dt^j} u(0) = 0,\ j = 0, \ldots, s \Big\} \subset E.$$

Furthermore, we observe that such set E includes some non-steady states.

(ii) There are at least two differences between Theorem 5.2.1 and Theorem 5.2.2. First of all, Theorem 5.2.1 studies constrained controllability for some steady states, whereas Theorem 5.2.2 can deal with constrained controllability for some non-steady states (see (i) of this remark). Secondly, in Theorem 5.2.2 the controls $\overline{u}^i$ $(i = 0, 1)$ defining the initial datum $\overline{y}^0(\tau_0)$ and final target $\overline{y}^1(\tau_1)$ are required to fulfill the constraint

$$\overline{u}^i(t) \in \mathscr{U}_{ad}, \quad \text{a.e. } t \in \mathbb{R},\ i = 0, 1,$$

while $\overline{u}^i$ in Theorem 5.2.1 is required to be in $\mathscr{W} \subsetneq \mathscr{U}_{ad}$. (Then, in Theorem 5.2.2 we have weakened the constraints on $\overline{u}^i$. In particular, we are able to apply Theorem 5.2.2 to the wave system with nonnegative controls with final target $\overline{y}^1 \equiv 0$.)

Before proving Theorem 5.2.2, we show a preliminary lemma. Note that such lemma works with any contractive semigroup. In particular, it holds both for wave-like and heat-like systems. A similar result was proved in [110, 97].

Lemma 5.2.3 (Null Controllability by small controls)**.** *Assume that A generates a contractive C_0-semigroup $(\mathbb{T}_t)_{t\in\mathbb{R}^+}$ over H. Suppose that (H_1') holds. Let $\varepsilon > 0$ and $\eta_0 \in D(A^s)$. Then, there exists $\overline{T} = \overline{T}(\varepsilon, \|\eta_0\|_{D(A^s)}) > 0$ such that, for any $T \geq \overline{T}$, there exists a control $v \in L^\infty((0,T);V)$ such that*

- $y(T;\eta_0,v) = 0$;
- $\|v\|_{L^\infty(\mathbb{R}^+;V)} \leq \varepsilon$.

The proof of the lemma above is given in the Appendix.

We are now ready to prove Theorem 5.2.2.

With respect to Theorem 5.1.5 we have weakened the constraints on the controls defining the initial and final trajectories. Then, a priori, we have lost the room for oscillations needed in the proof of that theorem. We shall see how to recover this by modifying the initial and final trajectories away from the initial and final data (see figure 5.3, figure 5.4 and figure 5.5).

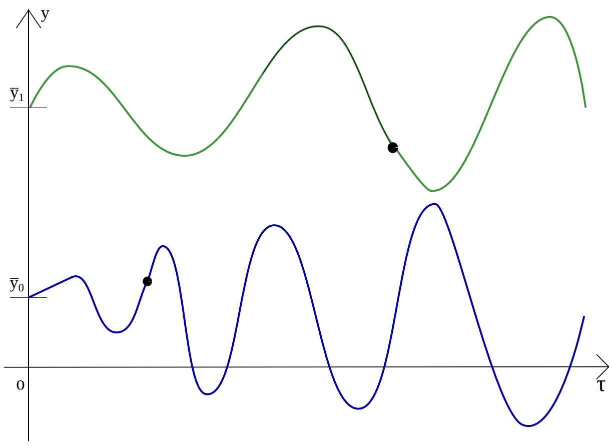


Figure 5.3: the two original trajectories. The time τ parameterizing the trajectories is just a parameter independent of the control time t.

Proof of Theorem 5.2.2. The main strategy of proof is the following:

(i) we reduce the constrained controllability problem (with initial data $\overline{y}_0(\tau_0)$ and final target $\overline{y}_1(\tau_1)$) to another controllability problem (with initial datum $\hat{y}_0$ and final target 0);

(ii) we solve the latter controllability problem by constructing two controls. The first control is used to improve the regularity of the solution. The second control is small in a regular space and steers the system to rest.

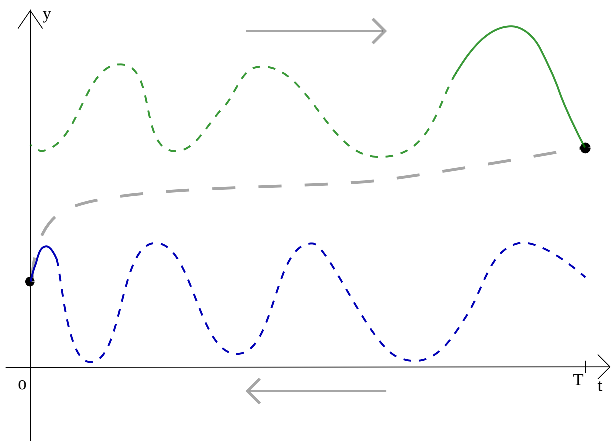


Figure 5.4: the new trajectories to be linked, now synchronized with the control time t. Note that 1) we have translated the time parameter defining the trajectories and 2) we have modified them away from the initial and the final data, to apply Lemma 5.2.3. The new initial trajectory is represented in blue, while the new final trajectory is drawn in green. The modified part is dashed. Following the notation of the proof of Theorem 5.2.2, the new initial trajectory is $y(\cdot; \hat{u}^0, \overline{y}_0(\tau_0))$, while the new final trajectory is φ_T.

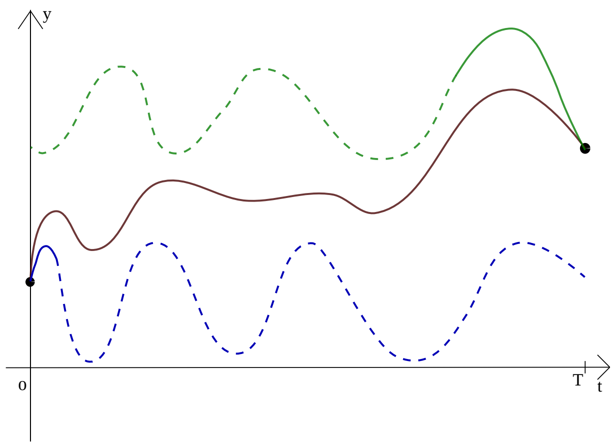


Figure 5.5: the new trajectories linked by the controlled trajectory y, pictured in red. As in figure 5.4, the new initial trajectory is drawn in blue, while the new final trajectory is represented in green.

Step 1 **The part (i) of the above strategy.**
For each $T > 0$, we aim to define a new trajectory with the final state $\overline{y}_1(\tau_1)$ as value at time $t = T$. Choose a smooth function $\zeta \in C^\infty(\mathbb{R})$ such that

$$\zeta \equiv 1 \text{ over } \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad \mathrm{supp}(\zeta) \subset\subset (-1, 1). \tag{5.2.25}$$

Take $\sigma \in \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V)$. Fix $T > 1$ and define a control

$$\hat{u}_T^1(t) = \zeta(t - T)\overline{u}^1(t - T + \tau_1) + (1 - \zeta(t - T))\sigma. \tag{5.2.26}$$

We denote by φ_T the unique solution to the problem

$$\begin{cases} \dfrac{d}{dt}\varphi(t) = A\varphi(t) + B\hat{u}_T^1(t) & t \in \mathbb{R} \\ \varphi(T) = \overline{y}_1(\tau_1). \end{cases} \tag{5.2.27}$$

In what follows, we will construct two controls which sends $\overline{y}_0(\tau_0) - \varphi_T(0)$ to 0 in time T, which is part (ii) of our strategy. Recall that ρ is given by (5.2.8). We define

$$\hat{u}^0(t) = \rho(t)\overline{u}^0(t + \tau_0) + (1 - \rho(t))\sigma \quad t \in \mathbb{R}.$$

Step 2 **Estimate of** $\|y(1; \overline{y}_0(\tau_0) - \varphi_T(0), \hat{u}^0 - \hat{u}_T^1)\|_{D(A^s)}$
We take the control $(\hat{u}^0 - \hat{u}_T^1)\restriction_{(0,1)}$ to be the first control mentioned in part (ii) of our strategy. In this step, we aim to prove the following regularity estimate associated with this control: there exists a constant $C > 0$ independent of T and σ such that

$$\begin{aligned} &\|y(1; \overline{y}_0(\tau_0) - \varphi_T(0), \hat{u}^0 - \hat{u}_T^1)\|_{D(A^s)} \\ &\leq C \left[\|\overline{y}_0\|_{C^s([\tau_0,\tau_0+1];H)} + \|\overline{y}_1\|_{C^s([\tau_1-1,\tau_1];H)} + \|\sigma\|_U\right]. \end{aligned} \tag{5.2.28}$$

To begin, we introduce ψ the solution to

$$A\psi + B\sigma = 0. \tag{5.2.29}$$

We have that

$$\begin{aligned} & y(1; \overline{y}(\tau_0) - \varphi_T(0), \hat{u}^0 - \hat{u}_T^1) \\ = \;& y(1; \overline{y}(\tau_0), \hat{u}^0) - y(1; \varphi_T(0), \hat{u}^1) \\ = \;& [y(1; \overline{y}(\tau_0), \hat{u}^0) - \psi] - [y(1; \varphi_T(0), \hat{u}_T^1) - \psi] \\ = \;& y(1; \overline{y}(\tau_0) - \psi, \hat{u}^0 - \sigma) - y(1; \varphi_T(0) - \psi, \hat{u}_T^1 - \sigma). \end{aligned} \tag{5.2.30}$$

To estimate (5.2.28), we need to compute the norms of the last two terms in (5.2.30), in the space $D(A^s)$. We claim that there exists $C_1 > 0$ (independent of T and σ) such that

$$\|y(1; \overline{y}(\tau_0) - \psi, \hat{u}^0 - \sigma)\|_{D(A^s)} \leq C_1 \left(\|\overline{y}_0\|_{C^s([\tau_0,\tau_0+1];H)} + \|\sigma\|_U\right). \tag{5.2.31}$$

To this end, we show that

$$y(t;\overline{y}(\tau_0)-\psi,\hat{u}^0-\sigma)=\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t),\quad t\in\mathbb{R},\tag{5.2.32}$$

where η_2 solves

$$\begin{cases}\dfrac{d}{dt}\eta_2(t)=A\eta_2(t)-\rho'(\overline{y}(t+\tau_0)-\psi)\quad t\in\mathbb{R}\\ \eta_2(0)=0.\end{cases}\tag{5.2.33}$$

Indeed,

$$\begin{aligned}&\frac{d}{dt}\left[\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t)\right]\\ =\;&\rho(t)(A\overline{y}^0(t+\tau_0)+B\overline{u}^0(t+\tau_0))+\rho'(t)(\overline{y}^0(t+\tau_0)-\psi)\\ &+A\eta_2(t)-\rho'(t)(\overline{y}^0(t+\tau_0)-\psi)\\ =\;&A(\rho(t)\overline{y}^0(t+\tau_0)+\eta_2(t))+B\left(\rho(t)\overline{u}^0(t+\tau_0)\right)\\ =\;&A(\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t))+\rho(t)A\psi+B\left(\rho(t)\overline{u}^0(t+\tau_0)\right)\\ =\;&A(\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t))+B\left(\rho(t)(\overline{u}^0(t+\tau_0)-\sigma)\right)\\ =\;&A(\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t))+B(\hat{u}^0(t)-\sigma).\end{aligned}\tag{5.2.34}$$

At the same time, since $\rho(0)=1$, from (5.2.33), it follows that

$$\rho(t)(\overline{y}^0(t+\tau_0)-\psi)+\eta_2(t)\restriction_{t=0}=\overline{y}^0(\tau_0)-\psi.$$

From this and (5.2.34), we are led to (5.2.32).

Next, we will use (5.2.32) and (5.2.33) to prove (5.2.31). To this end, since we assumed $\overline{y}^0\in C^s(\mathbb{R};H)$ and ψ is independent of t, we get that

$$\overline{y}^0(\cdot+\tau_0)-\psi\in C^s(\mathbb{R};H).$$

By this, we apply Lemma 5.2.1 obtaining the existence of $\hat{C}_1>0$ (independent of T and σ) such that

$$\|\eta_2(1)\|_{D(A^s)}\leq\hat{C}_1\left(\|\overline{y}^0\|_{C^s([\tau_0,\tau_0+1];H)}+\|\psi\|_H\right).\tag{5.2.35}$$

Now, since $\rho(1)=0$ (see (5.2.8)), by (5.2.32), we have that

$$y(1;\overline{y}(T_0)-\psi,\hat{u}^0-\sigma)=\eta_2(1).$$

This, together with (5.2.35) and (5.2.29), yields (5.2.31).

At this point, we estimate the norm of the second term in (5.2.30) in the space $D(A^s)$. Namely we prove the existence of $C_2>0$ (independent of T and σ) such that

$$\|y(1;\varphi_T(0)-\psi,\hat{u}_T^1-\sigma)\|_{D(A^s)}\leq C_2\left[\|\overline{y}^1\|_{C^s([\tau_1-1,\tau_1];H)}+\|\sigma\|_U\right].\tag{5.2.36}$$

To this end, as in the proof of (5.2.28), we get that

$$y(t;\varphi_T(0)-\psi,\hat{u}_T^1-\sigma)=\zeta(t-T)(\overline{y}^1(t-T+\tau_1)-\psi)+\tilde{\eta}_2(t),\quad t\in\mathbb{R},\tag{5.2.37}$$

where $\tilde{\eta}_2$ solves

$$\begin{cases} \dfrac{d}{dt}\tilde{\eta}_2(t) = A\tilde{\eta}_2(t) - \zeta'(t-T)(\overline{y}^1(t-T+\tau_1) - \psi) & t \in \mathbb{R} \\ \tilde{\eta}_2(T) = 0. \end{cases} \tag{5.2.38}$$

We will use (5.2.37) and (5.2.38) to prove (5.2.36). Indeed, set

$$\hat{\eta}(t) = \tilde{\eta}_2(T-t).$$

By definition of $\hat{\eta}$, we have

$$\begin{cases} \dfrac{d}{dt}\hat{\eta}(t) = -A\hat{\eta}(t) + \zeta'(-t)(\overline{y}^1(\tau_1 - t) - \psi) & t \in \mathbb{R} \\ \hat{\eta}(0) = 0. \end{cases} \tag{5.2.39}$$

Since we have assumed $\overline{y}^1 \in C^s(\mathbb{R}, H)$ and ψ is independent of t (see (5.2.29)), we have

$$\overline{y}^1 - \psi \in C^s(\mathbb{R}; H).$$

Recall that $\zeta(t) \equiv 1$ in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (see (5.2.25)). Then, $\zeta'(t) = 0$, for each $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Now, by hypothesis (H_3'), A generates a group of operators. Hence, we can apply Lemma 5.2.1 to (5.2.39) getting the existence of $\tilde{C}_2 > 0$ (independent of T and σ) such that

$$\|\hat{\eta}(1)\|_{D(A^s)} \leq \tilde{C}_2 \left(\|\overline{y}^1\|_{C^s([\tau_1-1,\tau_1];H)} + \|\psi\|_H\right),$$

whence

$$\|\tilde{\eta}_2(T-1)\|_{D(A^s)} \leq \tilde{C}_2 \left(\|\overline{y}^1\|_{C^s([\tau_1-1,\tau_1];H)} + \|\psi\|_H\right). \tag{5.2.40}$$

At the same time, by (H_3') and some computations, we have that

$$\|\mathbb{T}_t\|_{\mathscr{L}(D(A^s),D(A^s))} = 1, \quad \text{for each } t \in \mathbb{R}.$$

Since $\zeta(t-T) = 0$, for each $t \in [0, T-1]$ (see (5.2.25)), the above, together with (5.2.37) and (5.2.38), yields

$$\|y(1; \varphi_T(0) - \psi, \hat{u}_T^1 - \sigma)\|_{D(A^s)} = \|\tilde{\eta}_2(1)\|_{D(A^s)} = \|\tilde{\eta}_2(T-1)\|_{D(A^s)}.$$

This, together with (5.2.40) and (5.2.29), leads to (5.2.36).

Step 3 **Conclusion.**

In this step, we will first construct the second control mentioned in part (ii) of our strategy. Then we put together the first and second controls (mentioned in part (ii)) to get the conclusion.

By (5.2.36),

$$\|y(1; \varphi_T(0) - \psi, \hat{u}_T^1 - \sigma)\|_{D(A^s)} \leq C_2 \left[\|\overline{y}^1\|_{C^s([\tau_1-1,\tau_1];H)} + \|\sigma\|_U\right].$$

Notice that the above estimate is independent of T. Then for each $T > 0$, by Lemma 5.2.3, there exists

$$\overline{T} = \overline{T}(\sigma, \|\overline{y}^0\|_{C^s([\tau_0,\tau_0+1];H)}, \|\overline{y}^1\|_{C^s([\tau_1-1,\tau_1];H)}) > 0$$

and $w_T \in L^\infty(\mathbb{R}^+; V)$ such that

$$\begin{cases} \dfrac{d}{dt}z(t) = Az(t) + Bw_T(t) & t \in (1, \overline{T}) \\ z(1) = y(1; \overline{y}(\tau_0) - \varphi_T(0), \hat{u}^0 - \hat{u}^1_T), \quad z(\overline{T}) = 0 \end{cases} \tag{5.2.41}$$

and

$$\|w_T\|_{L^\infty(1,\overline{T};V)} \leq \frac{1}{2} \inf_{y \in V \setminus \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V)} \|\sigma - y\|_V. \tag{5.2.42}$$

Note that the last constant is positive, because σ is taken from $\mathrm{int}^V(\mathscr{U}_{\mathrm{ad}})$. Choose $T = \overline{T} + 1$. Define a control:

$$v = \begin{cases} \hat{u}^0(t) & t \in \ (0,1) \\ w_T(t) + \hat{u}^1_T(t) & t \in \ (1, \overline{T}) \\ \hat{u}^1_T(t) & t \in \ (\overline{T}, \overline{T}+1). \end{cases} \tag{5.2.43}$$

We aim to show that

$$y(\overline{T}+1; \overline{y}^0(\tau_0), v) = \overline{y}^0(\tau_1) \quad \text{and} \quad v(t) \in \mathscr{U}_{\mathrm{ad}} \text{ a.e. } t \in \ (1, \overline{T}+1). \tag{5.2.44}$$

To this end, by (5.2.43), (5.2.41) and (5.2.27), we get that

$$\begin{aligned} y(\overline{T}+1; \overline{y}^0(\tau_0), v) &= y(\overline{T}+1; \overline{y}^0(\tau_0) - \varphi_T(0), v - \hat{u}^1_T) + y(\overline{T}+1; \varphi_T(0), \hat{u}^1_T) \\ &= \mathbb{T}_1(z_T(\overline{T})) + \varphi_T(\overline{T}+1) \\ &= \overline{y}^1(\tau_1). \end{aligned}$$

This leads to the first conclusion of (5.2.44). It remains to show the second condition in (5.2.44). Arbitrarily fix $t \in (0,1)$. By (5.2.43) and (5.2.36), we have

$$v(t) = \rho(t)\overline{u}^0(t+\tau_0) + (1-\rho(t))\sigma \in \rho(t)\mathscr{U}_{\mathrm{ad}} + (1-\rho(t))\mathscr{U}_{\mathrm{ad}} \subset \mathscr{U}_{\mathrm{ad}}.$$

Choose also an arbitrary $s \in (1, \overline{T})$. By (5.2.43), (5.2.42) and (5.2.26), we obtain

$$\begin{aligned} v(s) &= w(s) + (1 - \zeta(s - \overline{T} - 1))\sigma + \zeta(s - \overline{T} - 1)\overline{u}^1(s - \overline{T} - 1 + \tau_1) \\ &= w(s) + \sigma \in \mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) \subset \mathscr{U}_{\mathrm{ad}}. \end{aligned}$$

Take any $t \in (\overline{T}, \overline{T}+1)$. We find from (5.2.43) and (5.2.26) that

$$\begin{aligned} v(t) &= \zeta(t - \overline{T} - 1)\overline{u}^1(t - \overline{T} - 1 + \tau_1) + (1 - \zeta(t - \overline{T} - 1))\sigma \\ &\in \zeta(t - \overline{T} - 1)\mathscr{U}_{\mathrm{ad}} + (1 - \zeta(t - \overline{T} - 1))\mathscr{U}_{\mathrm{ad}} \\ &\subset \mathscr{U}_{\mathrm{ad}}. \end{aligned}$$

Therefore, we are led to the second conclusion of (5.2.44). This ends the proof.

□

5.3 Internal Control: Proof of Theorem 5.1.1 and Theorem 5.1.2

The present section is organized as follows:

- Subsection 5.3.1: proof of Lemma 5.3.1 and Theorem 5.1.1;
- Subsection 5.3.2: proof of Theorem 5.1.2;
- Subsection 5.3.3: discussion of the issues related to the internal control touching the boundary.

5.3.1 Proof of Theorem 5.1.1

We now prove Theorem 5.1.1 by employing Theorem 5.2.1.

Firstly, we place our control system in the abstract framework introduced in section 5.2 and we prove that our control system is smoothly controllable (see Definition 5.2.2).

The free dynamics is generated by $A : D(A) \subset H \longrightarrow H$, where

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad \begin{cases} H = H_0^1(\Omega) \times L^2(\Omega) \\ D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega). \end{cases} \tag{5.3.1}$$

and $A_0 = -\Delta + cI : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$. The control operator

$$B(v) = \begin{pmatrix} 0 \\ \chi v. \end{pmatrix}$$

defined from $U = L^2(\omega)$ to $H = H_0^1(\Omega) \times L^2(\Omega)$ is bounded, then admissible.

Lemma 5.3.1. *In the above framework take $V = H^{s(n)}(\omega)$ and $s = s(n) = \lfloor n/2 \rfloor + 1$. Assume further that (Ω, ω_0, T^*) fulfills the Geometric Control Condition. Then, the control system* (5.1.1) *is smoothly controllable in any time $T_0 > T^*$.*

The proof of this lemma can be found in the reference [48, Theorem 5.1].

We are now ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. We choose as set of admissible controls

$$\mathscr{U}_{\text{ad}} = \left\{ u \in L^2(\omega) \mid u \geq 0, \text{ a.e. } \omega \right\}.$$

Then,

$$\bigcup_{\sigma > 0} \left\{ u \in L^2(\omega) \mid u \geq \sigma, \text{ a.e. } \omega \right\} \subset \mathscr{W}. \tag{5.3.2}$$

We highlight that, to prove (5.3.2), we need $H^{s(n)}(\omega) \hookrightarrow C^0(\overline{\omega})$. For this reason, we have chosen $s(n) = \lfloor n/2 \rfloor + 1$.

By Lemma 5.3.1, we have that the system is *Smoothly Controllable* with $s = s(n) = \lfloor n/2 \rfloor + 1$ and $V = H^{s(n)}(\omega)$. Then, by Theorem 5.2.1, we conclude. $\square$

5.3.2 Proof of Theorem 5.1.2

We prove now Theorem 5.1.2

Proof of Theorem 5.1.2. As we have seen, our system fits in the abstract framework. Moreover, we have checked in Lemma 5.3.1 that the system is *Smoothly Controllable* with $s(n) = \lfloor n/2 \rfloor + 1$ and $V = H^{s(n)}(\omega)$. Furthermore, $\mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V) \neq \varnothing$. Indeed, any constant $\sigma > 0$ belongs to $\mathrm{int}^V(\mathscr{U}_{\mathrm{ad}} \cap V)$, since $H^{s(n)}(\omega) \hookrightarrow C^0(\overline{\omega})$. This is guaranteed by our choice of $s(n) = \lfloor n/2 \rfloor + 1$.

Therefore, we are in position to apply Theorem 5.2.2 and finish the proof.

□

5.3.3 Internal controllability from a neighborhood of the boundary

So far, we have assumed that the control is localized by means of a smooth cut-off function χ so that all its derivatives vanish on the boundary of Ω. This implies that χ must be constant on any connected component of the boundary. This prevents us to localize the internal control in a region touching the boundary only on a subregion, as in figure 5.6.

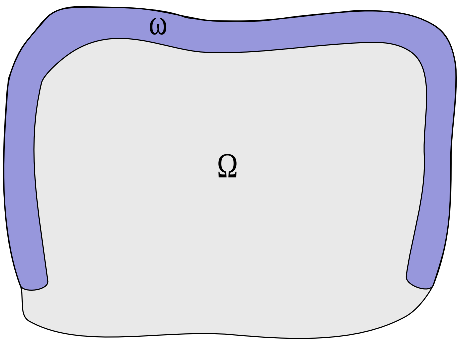


Figure 5.6: controlling from the interior touching the boundary.

In this case, as already pointed out in [40], some difficulties in finding regular controls may arise. Indeed, as indicated both in [40] and in [48] a crucial property needs to be verified in order to have controls in $C^0([0,T]; H^s(\omega))$, namely

$$BB^*(D(A^*)^k) \subset D(A^k) \tag{5.3.3}$$

for $k = 0, \ldots, s$, where we have used the notation of the proof of Theorem 5.1.1.

Now, for any $k \in \mathbb{N}$ we have

$$D(A^k) = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{array}{lll} \psi_1 \in H^{k+1}(\Omega), & \Delta^j \psi_1 = 0 \text{ on } \partial\Omega, & 0 \leq j \leq \lfloor k/2 \rfloor \\ \psi_2 \in H^{k}(\Omega), & \Delta^j \psi_2 = 0 \text{ on } \partial\Omega, & 0 \leq j \leq \lfloor (k+1)/2 \rfloor - 1 \end{array} \right\},$$

while

$$D((A^*)^k) = \left\{ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{array}{lll} \psi_1 \in H^{k}(\Omega), & \Delta^j \psi_1 = 0 \text{ on } \partial\Omega, & 0 \leq j \leq \lfloor (k-1)/2 \rfloor \\ \psi_2 \in H^{k-1}(\Omega), & \Delta^j \psi_2 = 0 \text{ on } \partial\Omega, & 0 \leq j \leq \lfloor k/2 \rfloor - 1 \end{array} \right\}. \tag{5.3.4}$$

Furthermore,

$$BB^* = \begin{pmatrix} 0 & 0 \\ \chi^2 & 0 \end{pmatrix}$$

Then, (5.3.3) is verified if and only if for any $\psi \in H^s(\Omega)$ such that

$$\Delta^j(\psi) = 0, \quad 0 \le j \le \lfloor (s-1)/2 \rfloor, \quad \text{a.e. on } \partial\Omega$$

the following hold

$$\Delta^j(\chi^2\psi) = 0, \quad 0 \le j \le \lfloor (s-1)/2 \rfloor, \quad \text{a.e. on } \partial\Omega. \tag{5.3.5}$$

Choosing χ so that all its normal derivatives vanish on $\partial\Omega$

- in case $s < 5$, we are able to prove (5.3.3). Then, by adapting the techniques of [48, Theorem 5.1], we have that our system is *Smoothly Controllable* (Definition 5.2.2), with $s(n) = \lfloor n/2 \rfloor + 1$. This enables us to prove Theorem 5.1.1 in space dimension $n < 8$.

- in case $s \ge 5$, in (5.3.5) the biharmonic operator Δ^2 enters into play. By computing it in normal coordinates on the boundary, some terms appear involving the curvature and $\frac{\partial}{\partial \xi_k}\chi \frac{\partial}{\partial v}\psi$, where $(\xi_1, \dots, \xi_{n-1})$ are tangent coordinates, while v is the normal coordinate. In general, these terms do not vanish, unless $\partial\Omega$ is flat. Then, for $n \ge 8$, we are unable to deduce a constrained controllability result in case the internal control is localized along a subregion of $\partial\Omega$.

5.4 Boundary control: proof of Theorem 5.1.3, Theorem 5.1.4 and Theorem 5.1.5

This section is devoted to boundary control and is organized as follows:

- Subsection 5.4.1: proof of Lemma 5.4.1 and Theorem 5.1.3;

- Subsection 5.4.2: proof of Theorem 5.1.4;

- Subsection 5.4.3: proof of Theorem 5.1.5.

5.4.1 Proof of Theorem 5.1.3

We start by explaining how our boundary control system fits in the abstract semigroup setting described in section 5.2. The generator of the free dynamics is:

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \begin{cases} H = L^2(\Omega) \times H^{-1}(\Omega) \\ D(A) = H^1_0(\Omega) \times L^2(\Omega), \end{cases} \tag{5.4.1}$$

where $A_0 = -\Delta + cI : H_0^1(\Omega) \subset H^{-1}(\Omega) \longrightarrow H^{-1}(\Omega)$. The definition of the control operator is subtler than in the internal control case. Let Δ_0 be the Dirichlet Laplacian. Then, the control operator reads as

$$B(v) = \begin{pmatrix} 0 \\ -\Delta_0 \tilde{z} \end{pmatrix}, \quad \text{where} \begin{cases} -\Delta \tilde{z} = 0 & \text{in } \Omega \\ \tilde{z} = \chi v(\cdot, t) & \text{on } \partial\Omega. \end{cases}$$

defined from $L^2(\Gamma)$ to $H^{-\frac{3}{2}}(\Omega)$. In this case, B is unbounded but admissible (see [93] or [140, Proposition 10.9.1, page 349]).

Lemma 5.4.1. *In the above framework, set* $V = H^{s(n)-\frac{1}{2}}(\Gamma)$ *and* $s = s(n)$, *with* $s(n) = \lfloor n/2 \rfloor + 1$. *Suppose (GCC) holds for* (Ω, Γ_0, T^*). *Then, in any time* $T_0 > T^*$, *the control system* (5.1.5) *is smoothly controllable in time* T_0.

One can prove Lemma (5.4.1), by employing [48, Theorem 5.4].

Proof of Theorem 5.1.3. We prove our Theorem, by choosing the set of admissible controls:

$$\mathscr{U}_{\text{ad}} = \left\{ u \in L^2(\Gamma) \mid u \geq 0, \text{ a.e. } \Gamma \right\}.$$

Hence,

$$\bigcup_{\sigma > 0} \left\{ u \in L^2(\Gamma) \mid u \geq \sigma, \text{ a.e. } \Gamma \right\} \subset \mathscr{W}. \tag{5.4.2}$$

Note that, in order to show (5.4.2), it is essential that the embedding $H^{s(n)-\frac{1}{2}}(\Gamma) \hookrightarrow C^0(\overline{\Gamma})$ is continuous. This is guaranteed by the choice $s(n) = \lfloor n/2 \rfloor + 1$.

By Lemma 5.4.1, we conclude that smooth controllability holds. At this point, it suffices to apply Theorem 5.2.1 to conclude. □

5.4.2 Proof of Theorem 5.1.4

We prove now Theorem 5.1.4.

Proof of Theorem 5.1.4. We have explained above how our control system (5.1.5) fits in the abstract framework presented in section 5.2. Furthermore, by Lemma 5.4.1, the system is *Smoothly Controllable* with $s(n) = \lfloor n/2 \rfloor + 1$ and $V = H^{s(n)-\frac{1}{2}}(\Gamma)$. Moreover, the set $\text{int}^V(\mathscr{U}_{\text{ad}} \cap V)$ is non empty, since all constants $\sigma > 0$ belong to it. This is a consequence of the continuity of $H^{s(n)-\frac{1}{2}}(\Gamma) \hookrightarrow C^0(\overline{\Gamma})$, valid for $s(n) = \lfloor n/2 \rfloor + 1$. The result holds as a consequence of Theorem 5.2.2.

□

5.4.3 State Constraints. Proof of Theorem 5.1.5

We conclude this section proving Theorem 5.1.5 about state constraints. The following result is needed.

Lemma 5.4.2. *Let $s \in \mathbb{N}^*$ and $T > T^*$. Take a steady state solution η_0 associated with the control $v^0 \in H^{s-\frac{1}{2}}(\Gamma)$. Then, there exists $v \in \cap_{j=0}^{s} C^j([0,T]; H^{s-\frac{1}{2}-j}(\Gamma))$ such that the unique solution (η, η_t) to (5.1.5) with initial datum $(\eta_0, 0)$ and control v is such that $(\eta(T,\cdot), \eta_t(T,\cdot)) = (0,0)$. Furthermore,*

$$\sum_{j=0}^{s} \|v\|_{C^j([0,T];H^{s-\frac{1}{2}-j}(\Gamma))} \leq C(T) \left\|v^0\right\|_{H^{s-\frac{1}{2}}(\Gamma)}, \tag{5.4.3}$$

the constant C being independent of η_0 and v^0. Finally, if $s = s(n) = \lfloor n/2 \rfloor + 1$, then the control $v \in C^0([0,T] \times \overline{\Gamma})$ and

$$\|v\|_{C^0([0,T]\times\overline{\Gamma})} \leq C(T) \left\|v^0\right\|_{H^{s(n)-\frac{1}{2}}(\Gamma)}. \tag{5.4.4}$$

Lemma 5.4.2 can be proved by using the techniques of Lemma 5.2.2. We now prove our theorem about state constraints.

Proof of Theorem 5.1.5. Step 1 **Reduction to the case $y_0^0 \geq 1$ and $y_1^0 \geq 1$ in Ω.**

In this step, we provide an explicit control strategies to drive the state (y, y_t) either from $(y_0^0, 0)$ to $(y_0^0 + 1, 0)$ or from $(y_1^0 + 1, 0)$ to $(y_1^0, 0)$ in some time $T_0 > d$, where $d := \sup\left\{\|x^1 - x^2\| \mid x^1,\ x^2 \in \Omega\right\}$ is the diameter of Ω. We start by constructing an explicit smooth solution $\tilde{y}$ solving the problem

$$\begin{cases} \tilde{y}_{tt} - \Delta\tilde{y} = 0 & \text{in } (0,T_0)\times\Omega \\ \tilde{y} \geq 0 & \text{in } (0,T_0)\times\Omega \\ \tilde{y}(0,x) = 1,\ \tilde{y}_t(0,x) = 0 & \text{in } \Omega \\ \tilde{y}(T_0,x) = 0,\ \tilde{y}_t(T_0,x) = 0 & \text{in } \Omega. \end{cases} \tag{5.4.5}$$

The solution $\tilde{y}$ to the above problem (figure 5.7) will be of the form

$$\tilde{y}(t,x) = f(t + x_1), \tag{5.4.6}$$

where $f : \mathbb{R} \longmapsto \mathbb{R}$ is smooth and x_1 is the first component of $x \in \Omega$. We firstly realize that, by definition of diameter, there exists an interval $[a,b]$, such that $|b-a| \leq d$ and

$$\left\{x_1 \in \mathbb{R} \mid\ (x_1, x_2, \ldots, x_n) \in \overline{\Omega}\right\} \subseteq [a,b]. \tag{5.4.7}$$

Since $T_0 > d$, we have $a + T_0 > b$, whence there exists $f \in C^\infty(\mathbb{R}; [0,1])$, such that

- $f(\xi) = 1$, for any $\xi \in [a,b]$;

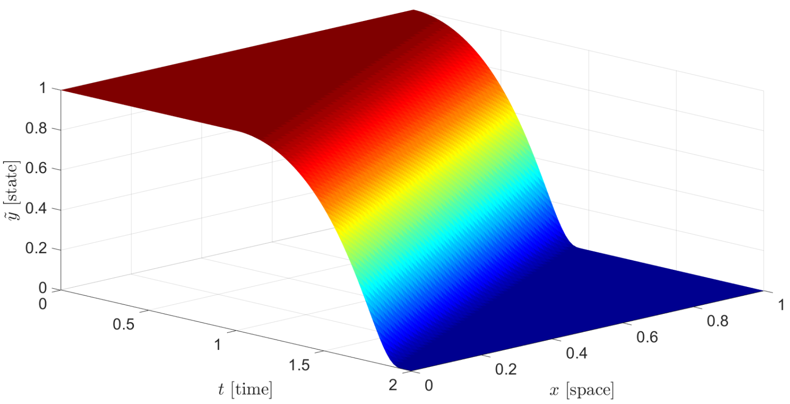


Figure 5.7: "lift"-solution to the wave equation joining the steady state $y_0^0 \equiv 1$, with the steady state $y_1^0 \equiv 0$ in time $T_0 = 2$. Both the state and the boundary control remain nonnegative along the control process.

- $f(\xi) = 0$, for any $\xi \in [a + T_0, b + T_0]$.

With the above f, $\tilde{y}$ defined in (5.4.6) is a solution to (5.4.5). Now,

1. the nonnegative trajectory $y(t,x) := y_0^0 + \tilde{y}(T_0 - t, x)$ is a solution to the wave equation, with terminal conditions $(y(0,\cdot), y_t(0,\cdot)) = (y_0^0, 0)$ and $(y(T_0,\cdot), y_t(T_0,\cdot)) = (y_0^0 + 1, 0)$;

2. the nonnegative trajectory $y(t,x) := y_1^0 + \tilde{y}(t,x)$ is a solution to the wave equation, with terminal conditions $(y(0,\cdot), y_t(0,\cdot)) = (y_1^0 + 1, 0)$ and $(y(T_0,\cdot), y_t(T_0,\cdot)) = (y_0^0, 0)$.

Then, if we are given two nonnengative steady states y_0^0 and y_1^0 we can reduce to the case $y_0^0 \geq 1$ and $y_1^0 \geq 1$, following the strategy (figure 5.8)

- control the state (y, y_t) from $(y_0^0, 0)$ to $(y_0^0 + 1, 0)$ in time T_0;
- apply the "stair-case argument" described after in $[T_0, T - T_0]$ to link $(y_0 + 1, 0)$ and $(y_1^0 + 1, 0)$, with T large enough;
- finally drive the state (y, y_t) from $(y_1^0 + 1, 0)$ to $(y_1^0, 0)$.

Hereafter, we will assume $y_0^0 \geq 1$ and $y_1^0 \geq 1$ in Ω.

Step 2 **Consequences of Lemma 5.4.2.**

Let $T_0 > T^*$, T^* being the critical time given by the *Geometric Control Condition*. By Lemma 5.4.2, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for any pair of steady states y_0 and y_1 defined by regular controls $\overline{u}^i \in H^{s(n)-\frac{1}{2}}(\Gamma)$ such that

$$\|\overline{u}^1 - \overline{u}^0\|_{H^{s(n)-\frac{1}{2}}(\Gamma)} < \delta_\varepsilon \tag{5.4.8}$$

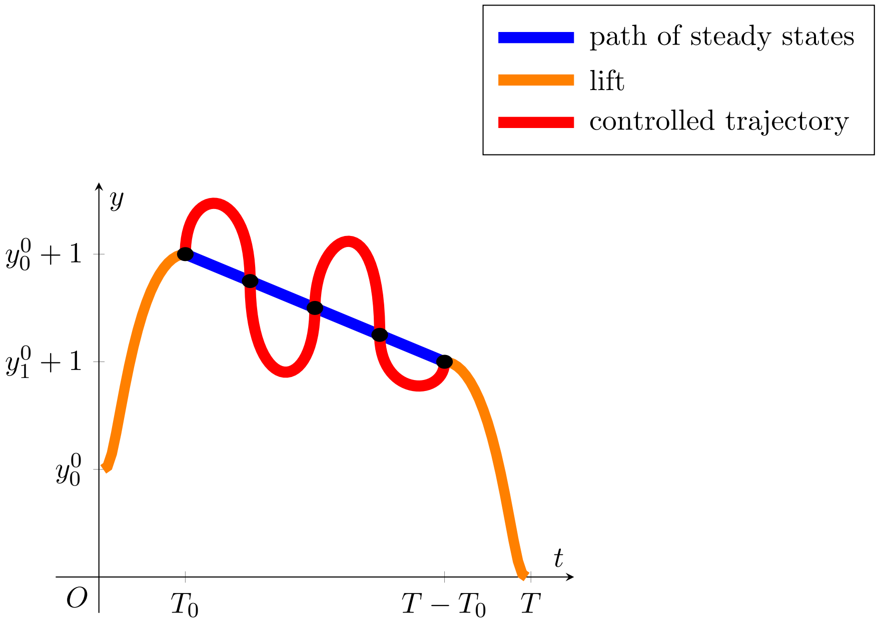


Figure 5.8: Control strategy to drive the solution of the wave equation to zero

we can find a control u driving (5.2.1) from y_0 to y_1 in time T_0 and verifying

$$\sum_{j=0}^{s(n)} \|u-\overline{u}^1\|_{C^j([0,T_0];H^{s(n)-\frac{1}{2}-j}(\Gamma))} < \varepsilon, \tag{5.4.9}$$

where $\overline{u}^1$ is the control defining y_1. Moreover, if (y, y_t) is the unique solution to (5.1.5) with initial datum $(y_0, 0)$ and control u, we have

$$\begin{aligned}\|y-y_1\|_{C^0([0,T_0]\times\overline{\Omega})} &\leq C\|y-y_1\|_{C^0([0,T_0];H^{s(n)}(\Omega))}\\ &\leq C\sum_{j=0}^{s(n)} \|u-\overline{u}^1\|_{C^j([0,T_0];H^{s(n)-\frac{1}{2}-j}(\Gamma))} \leq C\varepsilon,\end{aligned}$$

where we have used the boundedness of the inclusion $H^{s(n)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and the continuous dependence of the data .

Step 3 **Stepwise procedure and conclusion.**

We consider the convex combination $\gamma(s) = (1-s)y_0 + sy_1$. Then, let

$$z_k = \gamma\left(\frac{k}{\overline{n}}\right), \quad k = 0, \dots, \overline{n}$$

be a finite sequence of steady states defined by the control $\overline{u}_k = \frac{\overline{n}-k}{\overline{n}}\overline{u}^0 + \frac{k}{\overline{n}}\overline{u}^1$. Let $\delta > 0$. By taking $\overline{n}$ sufficiently large,

$$\|\overline{u}_k - \overline{u}_{k-1}\|_{H^{s(n)-\frac{1}{2}}(\Gamma)} < \delta. \tag{5.4.10}$$

By the above reasonings, choosing δ small enough, for any $1 \leq k \leq \overline{n}$, we can find a control u^k joining the steady states z_{k-1} and z_k in time T_0, with

$$\|y^k - z_k\|_{C^0([0,T_0]\times\overline{\Omega})} \leq 1,$$

where $(y^k, (y^k)_t)$ is the solution to (5.1.5) with initial datum z_{k-1} and control u^k. Hence,

$$y^k = y^k - z_k + z_k \geq -1 + 1 = 0, \quad \text{on } (0, T_0) \times \Omega, \tag{5.4.11}$$

where we have used $z^k \geq 1$.

By taking the traces in (5.4.11), we have $u^k \geq 0$ for $1 \leq k \leq \overline{n}$.

In conclusion, the control $u : (0, \overline{n}T_0) \longrightarrow H^{s(n)-\frac{1}{2}}(\Gamma)$ defined as $u(t) = u_k(t - (k-1)T_0)$ for $t \in ((k-1)T_0, kT_0)$ is the required one. This finishes the proof. □

5.5 The one dimensional wave equation

We consider the one dimensional wave equation, controlled from the boundary

$$\begin{cases} y_{tt} - y_{xx} = 0 & (t,x) \in (0,T) \times (0,1) \\ y(t,0) = u_0(t),\ y(t,1) = u_1(t) & t \in (0,T) \\ y(0,x) = y_0^0(x),\ y_t(0,x) = y_0^1(x). & x \in (0,1) \end{cases} \tag{5.5.1}$$

As in the general case, by transposition (see [93]), for any initial datum $(y_0^0, y_0^1) \in L^2(0,1) \times H^{-1}(0,1)$ and controls $u_i \in L^2(0,T)$, the above problem admits a unique solution $(y, y_t) \in C^0([0,T]; L^2(0,1) \times H^{-1}(0,1))$.

We show how Theorem 5.1.4 reads in this one-dimensional setting, in the special case where both the initial trajectory $(\overline{y}_0, (\overline{y}_0)_t)$ and the final one $(\overline{y}_1, (\overline{y}_1)_t)$ are constant (independent of x) steady states.

We determine explicitly a pair of *nonnegative* controls steering (5.5.1) from one positive constant to the other. The controlled solution remains *nonnegative*.

In this special case, we show further that

- the minimal controllability time is the same, regardless whether we impose the positivity constraint on the control or not;
- constrained controllability holds in the minimal time.

The minimal controllability time for (5.5.1) is defined as follows.

Let $(y_0^0, y_0^1) \in L^2(0,1) \times H^{-1}(0,1)$ be an initial datum and $(y_1^0, y_1^1) \in L^2(0,1) \times H^{-1}(0,1)$ be a final target. Then the minimal controllability time without constraints is defined as follows:

$$T_{\min} \stackrel{\text{def}}{=} \inf \left\{ T > 0 \;\middle|\; \exists u_i \in L^2(0,T),\ (y(T,\cdot), y_t(T,\cdot)) = (y_1^0, y_1^1) \right\}. \tag{5.5.2}$$

Similarly, the minimal time under positivity constraints on the control is defined as:

$$T_{\min}^{c} \stackrel{\text{def}}{=} \inf \left\{ T > 0 \;\middle|\; \exists u_i \in L^2(0,T)^+,\ (y(T,\cdot), y_t(T,\cdot)) = (y_1^0, y_1^1) \right\}. \tag{5.5.3}$$

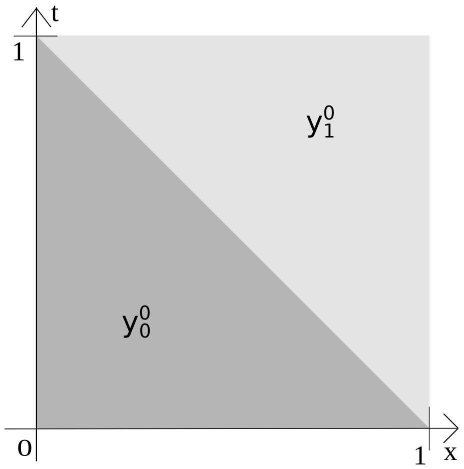


Figure 5.9: level sets of the solution to (5.5.1) with initial datum $(y_0^0, 0)$ and controls $\hat{u}^i$. In the darker region the solution takes value y_0^0, while in the complement it coincides with y_1^0.

Finally, we introduce the minimal time with constraints on the state and and the control:

$$T^s_{\min} \stackrel{\text{def}}{=} \inf\left\{T > 0 \mid \exists u_i \in L^2(0,T)^+,\ (y(T,\cdot), y_t(T,\cdot)) = (y_1^0, y_1^1),\ y \geq 0\right\}. \tag{5.5.4}$$

The problem of controllability of the one-dimensional wave equation under bilateral constraints on the control has been studied in [65]. In the next proposition, we concentrate on unilateral constraints and we compute explicitly the minimal time for the specific data considered.

Proposition 5.5.1. *Let $(y_0^0, 0)$ be the initial datum and $(y_1^0, 0)$ be the final target, with $y_0^0 \in \mathbb{R}^+$ and $y_1^0 \in \mathbb{R}^+$. Then,*

1. *for any time $T > 1$, there exists two nonnegative controls*

$$u_0(t) = \begin{cases} y_0^0 & t \in [0,1) \\ (y_1^0 - y_0^0)\frac{t-1}{T-1} + y_0^0 & t \in (1,T] \end{cases} \tag{5.5.5}$$

$$u_1(t) = \begin{cases} (y_1^0 - y_0^0)\frac{t}{T-1} + y_0^0 & t \in [0,T-1) \\ y_1^0 & t \in [T-1,T] \end{cases} \tag{5.5.6}$$

driving (5.5.1) from $(y_0^0, 0)$ to $(y_1^0, 0)$ in time T. Moreover, the corresponding solution remains nonnegative, i.e.

$$y(t,x) \geq 0, \quad \forall (t,x) \in [0,T] \times [0,1].$$

2. $T^s_{min} = T^c_{min} = T_{min} = 1$;

3. *the nonnegative controls $\hat{u}_0 \equiv y_0^0$ and $\hat{u}_1 \equiv y_1^0$ in $L^2(0,1)$ steer (5.5.1) from $(y_0^0, 0)$ to $(y_1^0, 0)$ in the minimal time. Furthermore, the corresponding solution satisfies $y \geq 0$ a.e. in $(0,1) \times (0,1)$;*

4. *the controls in the minimal time are not unique. In particular, for any* $\lambda \in [0,1]$, $\hat{u}^0_\lambda = (1-\lambda)y^0_0 + \lambda y^0_1$ *and* $\hat{u}^1_\lambda = (1-\lambda)y^0_1 + \lambda y^0_0$ *drive* (5.5.1) *from* $(y^0_0, 0)$ *to* $(y^0_1, 0)$ *in the minimal time.*

Proof. We proceed in several steps.

Step 1. **Proof of the constrained controllability in time $T > 1$.**
By D'Alembert's formula, the solution (y, y_t) to (5.5.1) with initial datum $(y^0_0, 0)$ and controls u_i defined in (5.5.5) and (5.5.6), reads as

$$y(t,x) = f(x+t), \quad (t,x) \in [0,T] \times [0,1],$$

where

$$f(\xi) = \begin{cases} y^0_0 & \xi \in [0,1) \\ (y^0_1 - y^0_0)\frac{\xi-1}{T-1} + y^0_0 & \xi \in [1,T) \\ y^0_1. & \xi \in [T, T+1]. \end{cases}$$

This finishes the proof of (1.).
Step 2 **Computation of the minimal time.**
In any time $T > 1$, controllability under state and control constraints holds. Then, $T_{\min} \leq T^c_{\min} \leq T^s_{\min} \leq 1$.

It remains to prove that $T_{\min} \geq 1$. This can be obtained by adapting the techniques of [98, Proposition 4.1].
Step 3 **Controllability in the minimal time.**
One can check (see figure 5.9) that the unique solution to (5.5.1) with initial datum $(y^0_0, 0)$ and controls $\hat{u}^i$ is

$$y(t,x) = \begin{cases} y^0_0 & t+x < 1 \\ y^0_1 & t+x > 1 \end{cases} \tag{5.5.7}$$

This concludes the argument.

□

Appendix

Regularity results

In what follows, H is a real Hilbert space and $A : D(A) \subset H \longrightarrow H$ is a generator of a C^0-semigroup.

Lemma 5.5.1. *Let* $k \in \mathbb{N}$. *Take* $y \in C^k([0,T]; H) \cap H^{k+1}((0,T); H_{-1})$ *solution to the homogeneous equation:*

$$\frac{d}{dt}y = Ay, \quad t \in (0,T). \tag{5.5.8}$$

Then, $y \in \cap_{j=0}^{k} C^j([0,T]; D(A^{k-j}))$ *and*

$$\sum_{j=0}^{k} \|y\|_{C^j([0,T];D(A^{k-j}))} \leq C(k)\|y\|_{C^k([0,T];H)},$$

the constant $C(k)$ *depending only on* k*.*

The proof of the Lemma 5.5.1 can be done by using the equation (5.5.8) (see [16]).

We prove now Lemma 5.2.1.

Proof of Lemma 5.2.1. Step 1 **Time regularity**
By induction on $j = 0, \dots, k$, we prove that $y \in C^j([0,T]; H)$ and

$$\|y\|_{C^j([0,T];H)} \leq C\|f\|_{H^j((0,T);H)}.$$

For $j = 0$, the validity of the assertion is a consequence of classical semigroup theory (e.g. [140, Proposition 4.2.5] with control space $U = H$ and control operator $B = Id_H$). Assume now that the result hold up to $j - 1$. Then, let w solution to

$$\begin{cases} \frac{d}{dt}w = Aw + f' & t \in (0,T) \\ w(0) = 0. \end{cases} \tag{5.5.9}$$

By induction assumption, $w \in C^{j-1}([0,T]; H)$ and the corresponding estimate holds. Then, $\tilde{y}(t) = \int_0^t w(\sigma)d\sigma \in C^j([0,T]; H)$ and

$$\|\tilde{y}\|_{C^j([0,T];H)} \leq C\|f\|_{H^j((0,T);H)}.$$

Then, it remains to show that $y = \tilde{y}$. Now, for any $t \in [0,T]$

$$\begin{aligned} \tilde{y}(t) - \tilde{y}(0) &= \int_0^t [w(\sigma) - w(0)]d\sigma = \int_0^t \int_0^\sigma [Aw(\xi) + f'(\xi)]d\xi d\sigma \\ &= \int_0^t [A\tilde{y}(\sigma) + f(\sigma)]d\sigma. \end{aligned}$$

By the uniqueness of solution to (5.2.6), we have $y = \tilde{y}$. This finishes the first step.
Step 2 **Conclusion**
We start observing that y solves

$$y_t = Ay, \qquad t \in (\tau, T).$$

Then, by classical semigroup arguments (see [16, Chapter 7]), we conclude.

□

Proof of Lemma 5.2.2

We give the proof of Lemma 5.2.2.

Proof of Lemma 5.2.2. Let v be given by (5.2.9). The proof is made of two steps.
Step 1 **Show that $y(1;\eta_0,v)\in D(A^s)$, with s given by** (5.2.3)
We apply Lemma 5.2.1 with $y=y(\cdot;\eta_0,\rho\overline{v}^0)-\rho\eta_0$ and $f=\rho'\eta_0$, getting

$$y(1;\eta_0,\rho\overline{v}^0)-\rho\eta_0\in D(A^s).$$

Since $\rho\eta_0=0$ over $(\delta,1)$, for some $\delta\in(0,1)$, we have that

$$y(1;\eta_0,\rho\overline{v}^0)\in D(A^s).$$

Step 2 **Conclusion**
Since $y(1;\eta_0,\rho\overline{v}^0)\in D(A^s)$, we are in position to apply the smooth controllability (see Definition 5.2.2) and determine $w\in L^\infty((1,T_0+1);V)$ steering the solution to (5.2.1) from $y(1;\eta_0,v)$ at time $t=1$ to 0 at time $t=T_0+1$.
Hence, the desired control v reads as (5.2.9).

Finally, by similar reasonings the estimate (5.2.10) follows. This ends the proof of this lemma.

□

Proof of Lemma 5.2.3

We prove now Lemma 5.2.3.

Proof of Lemma 5.2.3. We split the proof in two steps.
Step 1 **Proof of the inequality $\|\mathbb{T}_t\|_{\mathscr{L}(D(A^s),D(A^s))}\leq 1$ with $t\in\mathbb{R}^+$**
Recall that

$$\|x\|^2_{D(A^s)}=\sum_{j=0}^{s}\|A^jx\|^2_H\quad\forall\,x\in D(A^s).$$

Now, for any $x\in D(A^s)$ and $t\in\mathbb{R}^+$, we have

$$\|A^j\mathbb{T}_tx\|_H=\|\mathbb{T}_tA^jx\|_H\leq\|A^jx\|_H\quad\forall\,j=0,\dots,s.$$

This yields $\|\mathbb{T}_t\|_{\mathscr{L}(D(A^s),D(A^s))}\leq 1$ for any $t\in\mathbb{R}^+$.
Step 2 **Conclusion.**
Let $C>0$ be given by (5.2.2). Take

$$N>\frac{C\|\eta_0\|_{D(A^s)}}{\varepsilon}. \tag{5.5.10}$$

Arbitrarily fix $k \in \{0, \ldots, N-1\}$. Consider the following equation

$$\begin{cases} \dfrac{d}{dt} y(t) = Ay(t) + B\chi_{(kT_0,(k+1)T_0)}(t)u_k(t) \quad t \in \mathbb{R}^+ \\ y(0) = \frac{1}{N}\eta_0, \end{cases} \tag{5.5.11}$$

where $\chi_{(kT_0,(k+1)T_0)}$ is the characteristic function of the set $(kT_0, (k+1)T_0)$ and $u_k \in L^2(\mathbb{R}^+, V)$. From step 1 and (5.5.10), we have that

$$\|y(kT_0; (1/N)\eta_0, 0)\|_{D(A^s)} \leq (1/N)\|\eta_0\|_{D(A^s)} \leq \varepsilon. \tag{5.5.12}$$

Then, we apply smooth controllability (given by (H_1')) to find some control $\hat{u}_k \in L^\infty(\mathbb{R}^+; V)$ so that the solution to (5.5.11) with control $u_k = \hat{u}_k$ satisfies

$$y((k+1)T_0; (1/T_0)\eta_0, \chi_{(kT_0,(k+1)T_0)}\hat{u}_k) = 0 \quad \text{and} \quad \|\hat{u}_k\|_{L^\infty((kT_0,(k+1)T_0);V)} \leq \varepsilon. \tag{5.5.13}$$

Now, we define:

$$v(t) = \sum_{k=0}^{N-1} \chi_{(kT_0,(k+1)T_0)}(t)u_k(t) \quad t \in \mathbb{R}^+. \tag{5.5.14}$$

Then, from (5.5.13) and (5.5.14), we know

$$y(NT_0; \eta_0, v) = 0 \quad \text{and} \quad \|v\|_{L^\infty((0,NT_0);V)} \leq \varepsilon.$$

This leads to the conclusion where $\overline{T} = NT_0$. □

Chapter 6

The turnpike property in semilinear control

6.1 Introduction

In this chapter, the long time behaviour of semilinear optimal control problems as the time-horizon tends to infinity is analyzed.

Our results do not need smallness assumption on the initial datum for the governing state equation.

In [116], Alessio Porretta and Enrique Zuazua studied turnpike property for control problems governed by a semilinear heat equation, with dissipative nonlinearity. In particular, [116, Theorem 1] yields the existence of a solution to the optimality system fulfilling the turnpike property, under smallness conditions on the initial datum and the target. Our goal is to

1. prove that in fact the turnpike property is satisfied by the optimal control and state;

2. remove the smallness assumption on the initial datum.

We keep the smallness assumption on the target. This leads to the smallness and uniqueness of the steady optima (see [116, subsection 3.2]), whence existence and uniqueness of the turnpike follows.

First of all, in Lemma 6.2.1, for any initial datum and target, we estimate the L^∞ norm of the optima in terms of the norms of the corresponding initial data, uniformly in the time-horizon. This is employed to show that, in case both the initial datum and the target are small, the time-evolution functional to be minimized is strictly convex around the optimum. Then, the solution to the optimality system fulfilling the turnpike property determined in [116, Theorem 1] is in fact the unique optimum.

Concerning large initial data, it is well known that the smallness of the steady optima yields the smallness of the time-evolution optima in an averaged sense [116]. Accordingly, the time-evolution optimal pair approaches the turnpike at some time t_s. The proof of the validity of the turnpike property requires an upper bound τ for the critical time t_s, so to avoid

t_s to increase as the time horizon increase. Namely, we need to ensure that the time-evolution optima approach the turnpike, in time smaller than some τ independent of the time horizon.

6.1.1 Statement of the main result

We consider the semilinear optimal control problem:

$$\min_{u\in L^2((0,T)\times\omega)} J_T(u) = \frac{1}{2}\int_0^T\int_\omega |u|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\omega_0} |y-z|^2 dxdt, \tag{6.1.1}$$

where:

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{6.1.2}$$

As usual, Ω is a regular bounded open subset of $\mathbb{R}^n$, with $n = 1, 2, 3$. The nonlinearity f is C^3 nondecreasing. The action of the control is localized by multiplication by χ_ω, characteristic function of the open subregion $\omega \subseteq \Omega$. The target z is assumed to be bounded. Since that the nonlinearity is nondecreasing, the semilinear problem (6.1.2) is well-posed [10, chapter 5] . Namely, given an initial datum $y_0 \in L^2(\Omega)$ and a control $u \in L^2((0,T)\times\omega)$, there exists a unique solution

$$y \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

$\beta \geq 0$ is a penalization parameter. As β increases, the distance between the optimal state and the target decreases.

By the direct method in the calculus of variations (see Proposition 3.1.1 in chapter 3), there exists a global minimizer of (6.1.1). As we shall see, uniqueness can be guaranteed, provided that the initial datum and the target are small enough in the uniform norm.

Taking the Gâteaux differential of the functional (6.1.1) and imposing the Fermat stationary condition, we realize that any optimal control reads as $u^T = -q^T\chi_\omega$, where (y^T, q^T) solves

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T\chi_\omega & \text{in } (0,T)\times\Omega \\ y^T = 0 & \text{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \text{in } \Omega \\ -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q^T = 0 & \text{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{6.1.3}$$

In order to study the turnpike, we need to study the steady version of (6.1.2)-(6.1.1):

$$\min_{u_s\in L^2(\omega)} J_s(u_s) = \frac{1}{2}\int_\omega |u_s|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y_s - z|^2 dx, \tag{6.1.4}$$

where:

$$\begin{cases} -\Delta y_s + f(y_s) = u_s\chi_\omega & \text{in } \Omega \\ y_s = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1.5}$$

Under the same assumptions required for the problem (6.1.2)-(6.1.1), for any given control $u_s \in L^2(\omega)$, there exists a unique state $y_s \in H^2(\Omega) \cap H_0^1(\Omega)$ solution to (6.1.5).

The analysis of (6.1.5)-(6.1.4) is accomplished in chapter 3, section 3.1.3. By Proposition 3.1.4, we have the existence of a global minimizer $\overline{u}$ for (6.1.4). The corresponding optimal state is denoted by $\overline{y}$. If the target is sufficiently small in the uniform norm, the optimal control is unique (see [116, subsection 3.2]).

By the Fermat stationary condition (Proposition 3.1.7), any optimal control $\overline{u} = -\overline{q}\chi_\omega$, where

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1.6}$$

The analysis in [116, section 3], leads to the following local result.

Theorem 6.1.1 (Porretta-Zuazua). *Consider the control problem* (6.1.5)-(6.1.4). *There exists* $\delta > 0$ *such that if the **initial datum** and the target fulfil the **smallness condition***

$$\|y_0\|_{L^\infty} \leq \delta \quad \textit{and} \quad \|z\|_{L^\infty} \leq \delta,$$

there exists a solution (y^T, q^T) *to the Optimality System*

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T\chi_\omega & \textit{in } (0,T)\times\Omega \\ y^T = 0 & \textit{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \textit{in } \Omega \\ -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \textit{in } (0,T)\times\Omega \\ q^T = 0 & \textit{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \textit{in } \Omega \end{cases}$$

satisfying for any $t \in [0,T]$

$$\|q^T(t) - \overline{q}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t} + e^{-\mu(T-t)}\right],$$

where K *and* μ *are* T*-independent.*

We observe that the turnpike property is satisfied by one solution to the optimality system. Since our problem may be not convex, we cannot directly assert that such solution of the optimality system is the unique minimizer (optimal control) for (6.1.5)-(6.1.4).

The main result of this chapter is the following.

Theorem 6.1.2. *Consider the control problem* (6.1.2)-(6.1.1)*. Let* u^T *be a minimizer of* (6.1.1)*. There exists* $\rho > 0$ *such that for every initial datum* $y_0 \in L^\infty(\Omega)$ *and target* z *verifying*

$$\|z\|_{L^\infty} \leq \rho, \tag{6.1.7}$$

we have

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t} + e^{-\mu(T-t)}\right], \qquad \forall t \in [0, T], \tag{6.1.8}$$

the constants K *and* $\mu > 0$ *being independent of the time horizon* T*.*

Note that ρ is smaller than the smallness parameter δ in Theorem 6.1.1.

The main ingredients our proofs require are:

- prove a L^∞ bound of the norm of the optimal control, uniform in the time horizon $T > 0$ (Lemma 6.2.1 in subsection 6.2.1);
- proof of the turnpike property for *small data* and *small targets*. Note that, in Theorem 6.1.1, the authors prove the existence of a solution to the optimality system enjoying the turnpike property. In this preliminary step, for *small data* and *small targets*, we prove that any optimal control verifies the turnpike property (Lemma 6.2.2 in subsection 6.2.1);
- for *small targets* and *any data*, proof of the smallness of $\|y^T(t)\|_{L^\infty(\Omega)}$ in time t large (subsection 6.2.2). This is done by estimating the critical time t_s needed to approach the turnpike;
- conclude concatenating the two former steps (subsection 6.2.2).

Theorem 6.1.2 ensures that the conclusion of Theorem 6.1.1 holds for the optimal pair.

Remark 6.1.1. *The above method fails when the target is large. As a matter of fact, a key point in our method is the smallness of steady optima* $(\overline{u}, \overline{y})$*, for small targets. Then, by triangle inequality, we can replace the smallness condition*

$$\|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq \delta,$$

with the simpler one

$$\|y^T(t)\|_{L^\infty(\Omega)} \leq \delta', \qquad \text{with } \delta' := \frac{\delta}{2}.$$

This cannot be done if the target is large. Indeed, in that case the steady optima may be large as well.

The rest of the chapter is organized as follows:

- Section 6.2: Proof of Theorem 6.1.2;
- Section 6.3: Numerical simulations;
- Appendix.

6.2 Proof of the main result

6.2.1 Preliminary Lemmas

As announced, we firstly exhibit an upper bound of the norms of the optima in terms of the data. Note that the Lemma below yields an uniform bound for large targets as well.

Lemma 6.2.1. *Consider the control problem* (6.1.2)-(6.1.1)*. Let* $R > 0$*,* $y_0 \in L^\infty(\Omega)$ *and* $z \in L^\infty(\omega_0)$*, satisfying* $\|y_0\|_{L^\infty(\Omega)} \leq R$ *and* $\|z\|_{L^\infty(\omega_0)} \leq R$*. Let* u^T *be an optimal control for* (6.1.2)-(6.1.1)*. Then,* u^T *and* y^T *are bounded and*

$$\|u^T\|_{L^\infty((0,T)\times\omega)} + \|y^T\|_{L^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\omega_0)}\right], \tag{6.2.1}$$

where the constant K *is independent of the time horizon* T*, but it depends on* R*.*

The proof is postponed to the Appendix.

The second ingredient for the proof of Theorem 6.1.2 is the following Lemma.

Lemma 6.2.2. *Consider the control problem* (6.1.2)-(6.1.1)*. Let* $y_0 \in L^\infty(\Omega)$ *and* $z \in L^\infty(\omega_0)$*. There exists* $\delta > 0$ *such that, if*

$$\|z\|_{L^\infty} \leq \delta \qquad \text{and} \qquad \|y_0\|_{L^\infty} \leq \delta, \tag{6.2.2}$$

the functional (6.1.1) *admits a unique global minimizer* u^T*. Furthermore, for every* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that, if*

$$\|z\|_{L^\infty} \leq \delta_\varepsilon \qquad \text{and} \qquad \|y_0\|_{L^\infty} \leq \delta_\varepsilon, \tag{6.2.3}$$

the functional (6.1.1) *admits a unique global minimizer* u^T*. Furthermore, we have*

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq \varepsilon\left[e^{-\mu t} + e^{-\mu(T-t)}\right], \qquad \forall t \in [0,T], \tag{6.2.4}$$

$(\overline{u}, \overline{y})$ *being the optimal pair for* (6.1.4)*. The constants* δ_ε *and* $\mu > 0$ *are independent of the time horizon and* μ *is given by*

$$\begin{aligned} \left\|\mathscr{E}(t) - \widehat{E}\right\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} &\leq C\exp(-\mu t), \\ \|\exp(-tM)\|_{\mathscr{L}(L^2(\Omega),L^2(\Omega))} &\leq \exp(-\mu t), \quad M := -\Delta + f'(\overline{y}) + \widehat{E}\chi_\omega. \end{aligned} \tag{6.2.5}$$

where $\mathscr{E}$ *and* $\widehat{E}$ *denote respectively the differential and algebraic Riccati operators (see [116, equation (22)]).*

Proof of Lemma 6.2.2. We introduce the critical ball

$$B := \left\{u \in L^\infty((0,T)\times\omega) \;\middle|\; \|u\|_{L^\infty} \leq K\left[\|y_0\|_{L^\infty} + \|z\|_{L^\infty}\right]\right\}, \tag{6.2.6}$$

where K is the constant appearing in (6.2.1).

Step 1 **Strict convexity in B for small data**

By [27, section 5] or [26], the second order Gâteaux differential of J reads as

$$\langle d^2 J_T(u)w, w\rangle = \int_0^T \int_\omega w^2 dxdt + \int_0^T \int_{\omega_0} |\psi_w|^2 dxdt - \int_0^T \int_\Omega f'(y) q |\psi_w|^2 dxdt,$$

where y solves (6.1.2) with control u and initial datum y_0, ψ_w solves the linearized problem

$$\begin{cases} (\psi_w)_t - \Delta\psi_w + f'(y)\psi_w = w\chi_\omega & \text{in } (0,T)\times\Omega \\ \psi_w = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi_w(0,x) = 0 & \text{in } \Omega \end{cases} \tag{6.2.7}$$

and

$$\begin{cases} -q_t - \Delta q + f'(y)q = (y-z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q = 0 & \text{on } (0,T)\times\partial\Omega \\ q(T,x) = 0 & \text{in } \Omega. \end{cases} \tag{6.2.8}$$

Since $f'(y) \geq 0$,

$$\|\psi_w\|_{L^2((0,T)\times\Omega)} \leq K\|w\|_{L^2((0,T)\times\omega)}.$$

Let $u \in B$. By applying a comparison argument to (6.1.2) and (6.2.8),

$$\|y\|_{L^\infty} + \|q\|_{L^\infty} \leq K\left[\|y_0\|_{L^\infty} + \|z\|_{L^\infty}\right].$$

Hence,

$$\langle d^2 J_T(u)w, w\rangle \geq \int_0^T \int_{\omega_0} |\psi_w|^2 dxdt + \{1 - K[\|y_0\|_{L^\infty} + \|z\|_{L^\infty}]\} \int_0^T \int_\omega |w|^2 dxdt,$$

If $\|y_0\|_{L^\infty}$ and $\|z\|_{L^\infty}$ are small enough, we have

$$\langle d^2 J_T(u)w, w\rangle \geq \frac{1}{2}\int_0^T \int_\omega |w|^2 dxdt,$$

whence the strict convexity of J in the critical ball B. Now, by (6.2.1) and (6.2.6), if $\|y_0\|_{L^\infty}$ and $\|z\|_{L^\infty}$ are small enough, any optimal control u^T belongs to B. Then, there exists a unique solution to the optimality system, with control in the critical ball B and such control coincides with u^T the unique global minimizer of (6.1.1).

Step 2 **Conclusion**

Let $\varepsilon > 0$. By following the fixed-point argument developed in the proof of [116, Theorem 1 subsection 3.1] and in [116, subsection 3.2], we can find $\delta_\varepsilon > 0$ such that, if

$$\|z\|_{L^\infty} \leq \delta_\varepsilon \qquad \text{and} \qquad \|y_0\|_{L^\infty} \leq \delta_\varepsilon,$$

there exists a solution (y^T, q^T) to the optimality system such that

$$\|u^T\|_{L^\infty} < \varepsilon$$

and

$$\|u^T(t)-\overline{u}\|_{L^\infty(\omega)}+\|y^T(t)-\overline{y}\|_{L^\infty(\Omega)}\leq K\left[e^{-\mu t}+e^{-\mu(T-t)}\right],\qquad \forall t\in[0,T].$$

By Step 1, if ε is small enough, $u^T:=-q^T\chi_\omega$ is a strict global minimizer for J_T. Then, being strict, it is the unique one. This finishes the proof.

□

In the following Lemma, we compare the value of the time evolution functional (6.1.1) at a control u, with the value of the steady functional (6.1.4) at control $\overline{u}$, supposing that u and $\overline{u}$ satisfy a turnpike-like estimate.

Lemma 6.2.3. *Consider the time-evolution control problem* (6.1.2)-(6.1.1) *and its steady version* (6.1.5)-(6.1.4)*. Fix $y_0\in L^2(\Omega)$ an initial datum and $z\in L^2(\omega_0)$ a target. Let $\overline{u}\in L^\infty(\omega)$ be a control and let $\overline{y}$ be the corresponding solution to* (6.1.5)*. Let $u\in L^\infty((0,T)\times\omega)$ be a control and y the solution to* (6.1.2)*, with control u. Assume*

$$\|u(t)-\overline{u}\|_{L^\infty(\omega)}+\|y(t)-\overline{y}\|_{L^\infty(\Omega)}\leq K\left[e^{-\mu t}+e^{-\mu(T-t)}\right],\qquad \forall t\in[0,T],\tag{6.2.9}$$

with $K=K(\Omega,\beta,y_0)$ and $\mu=\mu(\Omega,\beta)$. Then,

$$|J_T(u)-TJ_s(\overline{u})|\leq C\left[1+\|\overline{u}\|_{L^\infty(\omega)}+\|z\|_{L^\infty}\right],\tag{6.2.10}$$

the constant C depending only on the above constant K and μ.

Proof. We estimate

$$\begin{aligned}
&|J_T(u)-TJ_s(\overline{u})|\\
=\ &\left|\frac{1}{2}\|u\|^2_{L^2((0,T)\times\omega)}+\frac{\beta}{2}\|y-z\|^2_{L^2((0,T)\times\omega_0)}-T\left[\frac{1}{2}\|\overline{u}\|^2_{L^2(\omega)}+\frac{\beta}{2}\|\overline{y}-z\|^2_{L^2(\omega_0)}\right]\right|\\
=\ &\left|\frac{1}{2}\|u-\overline{u}\|^2_{L^2((0,T)\times\omega)}+\frac{\beta}{2}\|y-\overline{y}\|^2_{L^2((0,T)\times\omega_0)}\right.\\
&\left.+\int_0^T\int_\omega(u-\overline{u})\overline{u}\,dxdt+\beta\int_0^T\int_{\omega_0}(y-\overline{y})(\overline{y}-z)\,dxdt\right|\\
\leq\ &C\left[1+\|\overline{u}\|_{L^\infty(\omega)}+\|z\|_{L^\infty(\omega_0)}\right]\left\{\int_0^T\left[\|u-\overline{u}\|^2_{L^\infty(\omega)}+\|u-\overline{u}\|_{L^\infty(\omega)}\right]dt\right.\\
&\left.+\int_0^T\left[\|y-\overline{y}\|^2_{L^\infty(\omega_0)}+\|y-\overline{y}\|_{L^\infty(\omega_0)}\right]dt\right\}\\
\leq\ &C\left[1+\|\overline{u}\|_{L^\infty(\omega_0)}+\|z\|_{L^\infty(\omega_0)}\right],
\end{aligned}$$

where the last inequality follows from (6.2.9). □

The following Lemma plays a key role in the proof of Theorem 6.1.2.
Let u^T be an optimal control for (6.1.2)-(6.1.1). Let y^T be the corresponding optimal state. For any $\varepsilon>0$, let δ_ε be given by (6.2.3). Set

$$t_s:=\inf\left\{t\in[0,T]\mid\|y^T(t)\|_{L^\infty(\Omega)}\leq\delta_\varepsilon\right\},$$

where we use the convention $\inf(\varnothing) = T$.

Lemma 6.2.4 (Global attractor property). *Consider the control problem* (6.1.2)-(6.1.1). *Let* $y_0 \in L^\infty(\Omega)$ *and* $z \in L^\infty(\omega_0)$. *Let* u^T *be an optimal control for* (6.1.2)-(6.1.1) *and let* y^T *be the corresponding optimal state. For any* $\varepsilon > 0$. *there exist* $\rho_\varepsilon = \rho_\varepsilon(\Omega, \beta, \varepsilon)$ *and* $\tau_\varepsilon = \tau_\varepsilon(\Omega, \beta, y_0, \varepsilon)$, *such that if* $\|z\|_{L^\infty} \leq \rho_\varepsilon$ *and* $T \geq \tau_\varepsilon$,

$$\|y^T(t_s)\|_{L^\infty(\Omega)} \leq \delta_\varepsilon, \qquad \text{with the upper bound } t_s \leq \tau_\varepsilon \tag{6.2.11}$$

and

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq \varepsilon \left[e^{-\mu(t-t_s)} + e^{-\mu(T-(t-t_s))}\right], \qquad \forall t \in [t_s, T]. \tag{6.2.12}$$

The constant μ *is given by* (6.2.5) *and* δ_ε *is given by* (6.2.3).

Proof of Lemma 6.2.4. Throughout the proof, constant $K_1 = K_1(\Omega, \beta)$ is chosen as small as needed, whereas constant $K_2 = K_2(\Omega, \beta, y_0)$ is chosen as large as needed.

Step 1 **Estimate of the L^∞ norm of steady optimal controls**

Let $\overline{u} \in L^2(\omega)$ be an optimal control for (6.1.5)-(6.1.4). By definition of minimizer (optimal control),

$$\frac{1}{2}\|\overline{u}\|_{L^2}^2 \leq J_s(\overline{u}) \leq J_s(0) = \frac{\beta}{2}\|z\|_{L^2}^2 \leq \frac{\beta \mu_{leb}(\omega_0)}{2}\|z\|_{L^\infty(\omega_0)}^2.$$

Now, any optimal control is of the form $\overline{u} = -\overline{q}\chi_\omega$, where the pair $(\overline{y}, \overline{q})$ satisfies the optimality system (6.1.6). Since $n = 1, 2, 3$, by elliptic regularity (see, e.g. [50, Theorem 4 subsection 6.3.2]) and Sobolev embeddings (see e.g. [50, Theorem 6 subsection 5.6.3]), $\overline{q} \in C^0(\overline{\Omega})$ and $\|\overline{q}\|_{L^\infty} \leq K\|z\|_{L^\infty}$, where $K = K(\Omega)$. This yields $\overline{u} \in C^0(\overline{\omega})$ and

$$\|\overline{u}\|_{L^\infty} \leq K\|z\|_{L^\infty}. \tag{6.2.13}$$

Step 2 **There exist $\rho_\varepsilon = \rho_\varepsilon(\Omega, \beta, \varepsilon)$ and $\tau_\varepsilon = \tau_\varepsilon(\Omega, \beta, y_0, \varepsilon)$, such that if $\|z\|_{L^\infty} \leq \rho_\varepsilon$, then the critical time satisfies $t_s \leq \tau_\varepsilon$**

Let $\overline{u}$ be an optimal control for the steady problem. Then, by definition of minimizer (optimal control),

$$J_T(u^T) \leq J_T(\overline{u}) \tag{6.2.14}$$

and, by Lemma 6.2.3,

$$J_T(\overline{u}) \leq T \inf_{L^2(\omega)} J_s + K_2. \tag{6.2.15}$$

Now, we split the integrals in J_T into $[0, t_s]$ and $(t_s, T]$

$$\begin{aligned} J_T(u^T) &= \frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \\ &+ \frac{1}{2}\int_{t_s}^T\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_{t_s}^T\int_{\omega_0} |y^T - z|^2 dxdt. \end{aligned} \tag{6.2.16}$$

Set:

$$c_y(t,x) := \begin{cases} \frac{f(y^T(t,x))}{y^T(t,x)} & y^T(t,x) \neq 0 \\ f'(0) & y^T(t,x) = 0. \end{cases}$$

Since f is nondecreasing and $f(0) = 0$, we have $c_y \geq 0$. Then, Lemma 6.3.1 (with potential c_y and source term $h := u^T \chi_\omega$) yields

$$\frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \geq K_1 \int_0^{t_s} \|y^T(t)\|^2_{L^\infty(\Omega)} dt - K_2.$$

Furthermore, by definition of t_s, for any $t \in [0, t_s]$, $\|y^T(t)\|_{L^\infty(\Omega)} \geq \delta_\varepsilon$. Then,

$$\frac{1}{2}\int_0^{t_s}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_0^{t_s}\int_{\omega_0} |y^T - z|^2 dxdt \geq K_1 t_s \delta_\varepsilon^2 - K_2. \tag{6.2.17}$$

Once again, by definition of t_s,

$$\|y^T(t_s)\|_{L^\infty(\Omega)} = \delta_\varepsilon \quad \text{and} \quad \|z\|_{L^\infty} \leq \delta_\varepsilon,$$

where δ_ε is given by (6.2.3). Therefore, by Lemma 6.2.2, the turnpike estimate (6.2.4) is satisfied in $[t_s, T]$. Lemma 6.2.3 applied in $[t_s, T]$ gives

$$\begin{aligned} & \frac{1}{2}\int_{t_s}^{T}\int_\omega |u^T|^2 dt + \frac{\beta}{2}\int_{t_s}^{T}\int_{\omega_0} |y^T - z|^2 dxdt \\ \geq\ & (T - t_s) \inf_{L^2(\omega)} J_s - K_2 \left[1 + \|\overline{u}\|_{L^\infty(\omega)} + \|z\|_{L^\infty}\right] \\ \geq\ & (T - t_s) \inf_{L^2(\omega)} J_s - K_2, \end{aligned} \tag{6.2.18}$$

where the last inequality is due to (6.2.13) and $\|z\|_{L^\infty} \leq \delta_\varepsilon$.

At this point, by (6.2.16), (6.2.17) and (6.2.18)

$$J_T(u^T) \geq K_1 t_s \delta_\varepsilon^2 + (T - t_s) \inf_{L^2(\omega)} J_s - K_2. \tag{6.2.19}$$

Therefore, by (6.2.19), (6.2.14) and (6.2.15)

$$K_1 t_s \delta_\varepsilon^2 + (T - t_s) \inf_{L^2(\omega)} J_s - K_2 \leq T \inf_{L^2(\omega)} J_s + K_2,$$

whence

$$t_s \left[K_1 \delta_\varepsilon^2 - \inf_{L^2(\omega)} J_s\right] \leq K_2. \tag{6.2.20}$$

Now, by (6.2.13), there exists $\rho_\varepsilon = \rho_\varepsilon(\Omega, \beta, \varepsilon) \leq \delta_\varepsilon$ such that, if the target $\|z\|_{L^\infty(\omega_0)} \leq \rho_\varepsilon$, then $\inf_{L^2(\omega)} J_s \leq \frac{K_1 \delta_\varepsilon^2}{2}$. This, together with (6.2.20), yields

$$t_s \frac{K_1 \delta_\varepsilon^2}{2} \leq K_2,$$

whence

$$t_s \leq \frac{K_2}{\delta_\varepsilon^2}.$$

Set

$$\tau_{\varepsilon} := \frac{K_2}{\delta_{\varepsilon}^2} + 1.$$

This finishes this step.

Step 3 **Conclusion**

By Step 2, for any $T \geq \tau_{\varepsilon}$, there exists $t_s \leq \tau_{\varepsilon}$ such that

$$\|y^T(t_s)\|_{L^{\infty}(\Omega)} \leq \delta_{\varepsilon}, \tag{6.2.21}$$

where δ_{ε} is given by (6.2.4). Now, by Bellman's Principle of Optimality, $u^T\restriction_{(t_s,T)}$ is optimal for (6.1.2)-(6.1.1), with initial datum $y^T(t_s)$ and target z. We took $\rho_{\varepsilon} \leq \delta_{\varepsilon}$, Then, we also have

$$\|z\|_{L^{\infty}} \leq \rho_{\varepsilon} \leq \delta_{\varepsilon}. \tag{6.2.22}$$

Then, we can apply Lemma 6.2.2, getting (6.2.12). This completes the proof. □

6.2.2 Proof of Theorem 6.1.2

We now prove Theorem 6.1.2.

The proof consists of two steps:

- *Step 1* **From global to local.** By employing Lemma 6.2.4, we prove that the optimal state $\|y^T(t)\|_{L^{\infty}(\Omega)}$ is small in large time;
- *Step 2* **Local turnpike property.** In the spirit of [116], we conclude applying the local turnpike property (Lemma 6.2.2).

Proof of Theorem 6.1.2. Step 1 **Smallness of $\|y^T(t)\|_{L^{\infty}(\Omega)}$ in large time**

Arbitrarily fix $\varepsilon > 0$. By Lemma 6.2.4, there exist $\rho_{\varepsilon} = \rho_{\varepsilon}(\Omega, \beta, \varepsilon)$ and $\tau_{\varepsilon} = \tau_{\varepsilon}(\Omega, \beta, y_0, \varepsilon)$, such that if $\|z\|_{L^{\infty}} \leq \rho_{\varepsilon}$ and $T \geq \tau_{\varepsilon}$,

$$\|y^T(t_s)\|_{L^{\infty}(\Omega)} \leq \delta_{\varepsilon}, \qquad \text{with the upper bound } t_s \leq \tau_{\varepsilon}. \tag{6.2.23}$$

Step 2 **Conclusion**

By Lemma 6.2.4, there exists $\rho_{\varepsilon}(\Omega, \beta, \varepsilon) > 0$ such that if

$$\|z\|_{L^{\infty}} \leq \rho_{\varepsilon} \qquad , \tag{6.2.24}$$

the unique optimal control satisfies the turnpike estimate

$$\|u^T(t) - \overline{u}\|_{L^{\infty}(\omega)} + \|y^T(t) - \overline{y}\|_{L^{\infty}(\Omega)} \leq \varepsilon \left[e^{-\mu(t-t_s)} + e^{-\mu(T-(t-t_s))}\right], \qquad \forall t \in [t_s, T]. \tag{6.2.25}$$

Set

$$K_0 := \exp(\mu\tau) K \left[1 + \|y_0\|_{L^{\infty}} + \delta\right],$$

with $\mu > 0$ the exponential rate defined in (6.2.5) and K is given by (6.2.1). Note that $K_0 = K_0(\Omega, \beta, y_0)$ and, in particular, it is independent of the time horizon. By the above definition, for every $T > 0$ and for each $t \in [0, \tau_\varepsilon] \cap [0, T]$

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K_0 \exp(-\mu\tau) \leq K_0 \exp(-\mu t). \tag{6.2.26}$$

On the other hand, for $t \geq t_s$, (6.2.25) holds. Then, (6.1.8) follows. □

6.3 Numerical simulations

This section is devoted to a numerical illustration of Theorem 6.1.2.

Our goal is to check that turnpike property is fulfilled for small target, regardless of the size of the initial datum.

We deal with the optimal control problem

$$\min_{u \in L^2((0,T)\times(0,\frac{1}{2}))} J_T(u) = \frac{1}{2}\int_0^T \int_0^{\frac{1}{2}} |u|^2 dxdt + \frac{\beta}{2}\int_0^T \int_0^1 |y - z|^2 dxdt,$$

where:

$$\begin{cases} y_t - y_{xx} + y^3 = u\chi_{(0,\frac{1}{2})} & (t,x) \in (0,T) \times (0,1) \\ y(t,0) = y(t,1) = 0 & t \in (0,T) \\ y(0,x) = y_0(x) & x \in (0,1). \end{cases}$$

We choose as initial datum $y_0 \equiv 10$ and as target $z \equiv 1$.

We solve the above semilinear heat equation by using the semi-implicit method:

$$\begin{cases} \frac{Y_{i+1} - Y_i}{\Delta t} - \Delta Y_{i+1} + Y_i^3 = U_i \chi_{(0,\frac{1}{2})} & i = 0, \ldots, N_t - 1 \\ Y_0 = y_0, \end{cases}$$

where Y_i and U_i denote resp. a time discretization of the state and the control.

The optimal control is determined by a Gradient Descent method, with constant stepsize. The optimal state is depicted in figure 6.1.

Appendix

The first part of the Appendix is devoted to some regularity results for parabolic equations. The second part is devoted to the proof of Lemma 6.2.1.

Parabolic regularity results

One of the key tool to carry on the proof of Lemma 6.2.1 is the following regularity result.

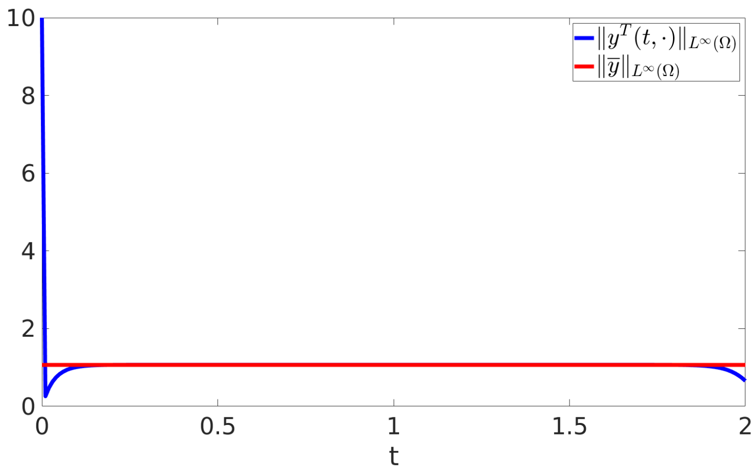


Figure 6.1: graph of the function $t \longrightarrow \|y^T(t)\|_{L^\infty(\Omega)}$ (in blue) and $\|\overline{y}\|_{L^\infty(\Omega)}$ (in red), where y^T denotes an optimal state, whereas $\overline{y}$ stands for an optimal steady state.

Lemma 6.3.1. *Let Ω be a bounded open set of $\mathbb{R}^n$, $n \in \{1,2,3\}$, with C^2 boundary. Let $c \in L^\infty((0,T)\times\Omega)$ be nonnegative. Let $y_0 \in L^\infty(\Omega)$ an initial datum and $h \in L^\infty((0,T)\times\Omega)$ a source term. Let y be the solution to*

$$\begin{cases} y_t - \Delta y + cy = h & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

Then, $y \in L^2((0,T);L^\infty(\Omega))$ and we have

$$\|y\|_{L^2((0,T);L^\infty(\Omega))} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \|h\|_{L^2((0,T)\times\Omega)}\right], \tag{6.3.1}$$

where K is independent of the potential $c \geq 0$, the time horizon T and the initial datum y_0.

Proof of Lemma 6.3.1. Step 1 **Comparison**

Let ψ be the solution to:

$$\begin{cases} \psi_t - \Delta\psi = |h| & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi(0,x) = |y_0|. & \text{in } \Omega \end{cases} \tag{6.3.2}$$

Since $c \geq 0$, a.e. in $(0,T)\times\Omega$, by a comparison argument, for each $t \in [0,T]$:

$$|y(t,x)| \leq \psi(t,x), \quad \text{a.e. } x \in \Omega. \tag{6.3.3}$$

Now, since y_0 and h are bounded, again by comparison principle applied to (6.3.2), ψ is bounded. Hence, by (6.3.3), y is bounded as well and

$$\int_0^T \|y(t)\|^2_{L^\infty(\Omega)}dt \leq \int_0^T \|\psi(t)\|^2_{L^\infty(\Omega)}dt. \tag{6.3.4}$$

Then, to conclude it suffices to show

$$\|\psi\|_{L^2(0,T;L^\infty(\Omega))} \leq K\left[\|y_0\|_{L^\infty} + \|h\|_{L^2((0,T)\times\Omega)}\right],$$

the constant K being independent of T.

Step 2 **Splitting**

Split $\psi = \xi + \chi$, where ξ solves:

$$\begin{cases} \xi_t - \Delta\xi = 0 & \text{in } (0,T)\times\Omega \\ \xi = 0 & \text{on } (0,T)\times\partial\Omega \\ \xi(0,x) = |y_0| & \text{in } \Omega \end{cases} \tag{6.3.5}$$

while χ satisfies:

$$\begin{cases} \chi_t - \Delta\chi = |h| & \text{in } (0,T)\times\Omega \\ \chi = 0 & \text{on } (0,T)\times\partial\Omega \\ \chi(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{6.3.6}$$

First of all, we prove an estimate like (6.3.1) for ξ. We start by employing maximum principle (see [118]) to (6.3.4), getting

$$\|\xi\|_{L^\infty} \leq \|y_0\|_{L^\infty}. \tag{6.3.7}$$

Now, if $T \geq 1$, by the regularizing effect and the exponential stability of the heat equation, for any $t \in [1,T]$, we have

$$\|\xi(t)\|_{L^\infty} \leq K\|\xi(t-1)\|_{L^2} \leq Ke^{-\lambda_1(t-1)}\|y_0\|_{L^2}, \tag{6.3.8}$$

the constant K depending only on the domain Ω. Then, by (6.3.7) and (6.3.8), for any $T > 0$, for every $t \in [0,T]$,

$$\|\xi(t,\dot{})\|_{L^\infty(\Omega)} \leq K\min\left\{1, e^{-\lambda_1(t-1)}\right\}\|y_0\|_{L^\infty}, \tag{6.3.9}$$

with $K = K(\Omega)$.

Now, we focus on (6.3.6). By parabolic regularity (see e.g. [50, Theorem 5 subsection 7.1.3]), $\chi \in L^2(0,T;H^2(\Omega))$, with $\chi_t \in L^2((0,T)\times\Omega)$. Then, by multiplying (6.3.6) by $-\Delta\chi$ and integrating over $[0,T]\times\Omega$, we obtain

$$\frac{1}{2}\|\nabla\chi(T)\|_{L^2}^2 + \int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \|h\|_{L^2}\|\Delta\chi\|_{L^2}.$$

By Young's Inequality,

$$\int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \frac{1}{2}\|h\|_{L^2}^2 + \frac{1}{2}\|\Delta\chi\|_{L^2}^2,$$

which leads to

$$\int_0^T\int_\Omega |\Delta\chi|^2 dxdt \leq \|h\|_{L^2}^2.$$

Now, by [50, Theorem 6 subsection 5.6.3] and [50, Theorem 4 subsection 6.3.2],

$$\int_0^T \|\chi\|_{L^\infty(\Omega)}^2 dt \leq K \int_0^T \|\chi\|_{H^2}^2 dt \leq K \int_0^T \int_\Omega |\Delta\chi|^2 dxdt \leq K\|h\|_{L^2}^2. \tag{6.3.10}$$

Finally, by (6.3.4), (6.3.9) and (6.3.10),

$$\int_0^T \|y\|_{L^\infty(\omega_0)}^2 dt \leq 2\int_0^T \|\xi\|_{L^\infty(\omega_0)}^2 dt + 2 \leq K \int_0^T \|\chi\|_{L^\infty(\omega_0)}^2 dt \leq K\left[\|y_0\|_{L^\infty(\omega_0)}^2 + \|h\|_{L^2}^2\right],$$

as required. □

The following regularity result is employed in the proof of Lemma 6.2.1.

Lemma 6.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, with $\partial\Omega \in C^\infty$. Let $c \in L^\infty((0,T)\times\Omega)$ be nonnegative. Let $y_0 \in L^\infty(\Omega)$ an initial datum and $h \in L^\infty((0,T)\times\Omega)$ a source term. Let $\overline{T} \in (0,T)$ and set $N := \lfloor T/\overline{T}\rfloor$. Let y be the solution to*

$$\begin{cases} y_t - \Delta y + cy = h & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

Then, $y \in L^\infty((0,T)\times\Omega)$ and we have

$$\|y\|_{L^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \max_{i=1,\dots,N} \|h\|_{L^2(((i-1)\overline{T},i\overline{T});L^\infty(\Omega))} + \|h\|_{L^2(N\overline{T},T;L^\infty(\Omega))}\right], \tag{6.3.11}$$

where K is independent of the potential $c \geq 0$ and the time horizon T.

Proof of Lemma 6.3.2. Step 1 **Comparison argument**
Let ψ be the solution to:

$$\begin{cases} \psi_t - \Delta\psi = |h| & \text{in } (0,T)\times\Omega \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega \\ \psi(0,x) = |y_0|. & \text{in } \Omega \end{cases} \tag{6.3.12}$$

Since $c \geq 0$, a.e. in $(0,T)\times\Omega$, by a comparison argument, for each $t \in [0,T]$:

$$|y(t,x)| \leq \psi(t,x), \quad \text{a.e. } x \in \Omega. \tag{6.3.13}$$

Now, since y_0 and h are bounded, again by comparison principle applied to (6.3.12), ψ is bounded. Hence, by (6.3.13), y is bounded as well and

$$\|y\|_{L^\infty((0,T)\times\Omega)} \leq \|\psi\|_{L^\infty((0,T)\times\Omega)}. \tag{6.3.14}$$

Then, to conclude it suffices to show

$$\|\psi\|_{L^\infty((0,T)\times\Omega)} \leq K\left[\|y_0\|_{L^\infty(\Omega)} + \max_{i=1,\dots,N} \|h\|_{L^2(((i-1)\overline{T},i\overline{T});L^\infty(\Omega))} + \|h\|_{L^2(N\overline{T},T;L^\infty(\Omega))}\right],$$

the constant K being independent of T.

Step 2 **Conclusion**

Now, fix $\varepsilon \in (0, \overline{T})$. By the regularizing effect of the heat equation (see, e.g. [16, Theorem 10.1, section 10.1]), for any $t \geq \varepsilon$,

$$\|S(t)y_0\|_{L^\infty(\Omega)} \leq K \exp(-\mu(t-\varepsilon))\|y_0\|_{L^2(\Omega)} \leq K \exp(-\mu(t-\varepsilon))\|y_0\|_{L^\infty(\Omega)}. \tag{6.3.15}$$

For $t \in [0, \varepsilon]$, by comparison principle, we have

$$\|S(t)y_0\|_{L^\infty(\Omega)} \leq K\|y_0\|_{L^\infty(\Omega)} \leq K \exp(-\mu(t-\varepsilon))\|y_0\|_{L^\infty(\Omega)},$$

being $\exp(-\mu(t-\varepsilon)) \geq 1$. Hence, for any $t \geq 0$,

$$\|S(t)y_0\|_{L^\infty(\Omega)} \leq K \exp(-\mu(t-\varepsilon))\|y_0\|_{L^\infty(\Omega)}. \tag{6.3.16}$$

Let $\{S(t)\}_{t\in\mathbb{R}^+}$ be the heat semigroup on Ω. Then, by the Duhamel formula, for any $t \in [0, T]$, we have

$$\psi(t) = S(t)\left(|y_0|\right) + \int_0^t S(t-s)\left(|h(s)|\right) ds. \tag{6.3.17}$$

Now, by (6.3.16), for any $t \geq 0$,

$$\|S(t)\left(|y_0|\right)\|_{L^\infty(\Omega)} \leq K \exp\left(-\mu(t-\delta)\right)\|y_0\|_{L^\infty(\Omega)}. \tag{6.3.18}$$

Besides, by applying (6.3.16) to the integral term $\eta(t) := \int_0^t S(t-s)\left(|h(s)|\right) ds$ in (6.3.17), we obtain

$$
\begin{aligned}
\|\eta(t)\|_{L^\infty} &\leq \int_0^t \|S(t-s)\left(|h(s)|\right)\|_{L^\infty} ds \\
&\leq K\int_0^t \exp(-\mu(t-s-\varepsilon))\|h(s)\|_{L^\infty} ds \\
&\leq K\left[\sum_{i=1}^{\lfloor \frac{t}{T}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\int_{(i-1)\overline{T}}^{i\overline{T}} \exp(-\mu(i\overline{T}-s))\|h(s)\|_{L^\infty} ds\right. \\
&\quad \left. +K\int_{(\lfloor \frac{t}{T}\rfloor-1)\overline{T}}^{t} \exp(-\mu(t-s-\varepsilon))\|h(s)\|_{L^\infty} ds\right] \\
&\leq K\left\{\sum_{i=1}^{\lfloor \frac{t}{T}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\left[\int_{(i-1)\overline{T}}^{i\overline{T}} \exp(-2\mu(i\overline{T}-s))ds\right]^{\frac{1}{2}}\left[\int_{(i-1)\overline{T}}^{i\overline{T}} \|h(s)\|_{L^\infty}^2 ds\right]^{\frac{1}{2}}\right. \\
&\quad \left. +K\left[\int_{(\lfloor \frac{t}{T}\rfloor-1)\overline{T}}^{t} \exp(-2\mu(t-s-\varepsilon))ds\right]^{\frac{1}{2}}\left[\int_{(\lfloor \frac{t}{T}\rfloor-1)\overline{T}}^{t} \|h(s)\|_{L^\infty}^2 ds\right]^{\frac{1}{2}}\right\} \\
&\leq K\left[\sum_{i=1}^{\lfloor \frac{t}{T}\rfloor} \exp(-\mu(t-\varepsilon-i\overline{T}))\|h\|_{L^2((i-1)\overline{T},i\overline{T};L^\infty(\Omega))}\right. \\
&\quad \left. +\|h\|_{L^2(\lfloor \frac{t}{T}\rfloor,t;L^\infty(\Omega))}\right] \\
&\leq K\left[\sum_{i=1}^{+\infty} \exp(-\mu(t-\varepsilon-i\overline{T}))\max_{i=1,\dots,N}\|h\|_{L^2(((i-1)\overline{T},i\overline{T});L^\infty(\Omega))}\right. \\
&\quad \left. +\|h\|_{L^2(\lfloor \frac{t}{T}\rfloor,t;L^\infty(\Omega))}\right] \\
&\leq K\left[\|h\|_{L^2((i-1)\overline{T},i\overline{T};L^\infty(\Omega))}+\|h\|_{L^2(N\overline{T},T;L^\infty(\Omega))}\right]. \qquad (6.3.19)
\end{aligned}
$$

Then, by (6.3.18) and (6.3.19), for each $t\in[0,T]$

$$
\|\psi(t)\|_{L^\infty} \leq K\exp\left(-\mu(t-\delta)\right)\left[\|y_0\|_{L^\infty}+\|h\|_{L^2((i-1)\overline{T},i\overline{T};L^\infty(\Omega))}+\|h\|_{L^2(N\overline{T},T;L^\infty(\Omega))}\right]
$$

as desired. □

Remark 6.3.1. *Lemma 6.3.1 can be applied to a bounded solution y to* (6.1.2). *Indeed, set*

$$
c_y(t,x) := \begin{cases} \frac{f(y(t,x))}{y(t,x)} & y(t,x)\neq 0 \\ f'(0) & y(t,x)=0. \end{cases}
$$

Since f is increasing and $f(0)=0$, we have $c_y\geq 0$. Hence, we are in position to apply Lemma 6.3.1, with potential c_y.

Proof of Lemma 6.2.1

The proof of Lemma 6.2.1 follows the following scheme:

- divide the interval $[0, T]$ into subintervals of T-independent length;
- estimate the magnitude of the optima in each subinterval.

We consider those subintervals, where a given estimate is *not* satisfied. In such subintervals, we show that in fact the optima satisfy another estimate. To prove that, inspired by [72, Remark 5.1, section 5], we construct a quasi-optimal turnpike control. By comparing its performance, with the optimal one, we obtain the desired bound.

In order to carry out the proof of Lemma 6.2.1, we need some preliminary lemmas. We start by stating a known result about the controllability of a dissipative semilinear heat equation.

Lemma 6.3.3. *Let $y_0 \in L^\infty(\Omega)$ be an initial datum. Let $\hat{y} \in L^\infty((0,+\infty)\times\Omega)$ be a target trajectory, solution to* (6.1.2)*, with control $\hat{u} \in L^\infty((0,T)\times\omega)$. Let $R > 0$. Suppose $\|y_0\|_{L^\infty(\Omega)} \le R$ and $\|\hat{y}\|_{L^\infty((0,+\infty)\times\Omega)} \le R$. Then, there exists $T_R = T_R(\Omega, f, \omega, R)$, such that for any $T \ge T_R$ there exists $u \in L^\infty((0,T)\times\omega)$ such that the solution y to the controlled equation* (6.1.2)*, with initial datum y_0 and control u, verifies the final condition*

$$y(T,x) = \hat{y}(T,x) \quad \textit{in } \Omega \tag{6.3.20}$$

and

$$\|u\|_{L^\infty((0,T)\times\omega)} \le K\left[\|y_0 - \hat{y}(0)\|_{L^\infty(\Omega)} + \|\hat{u}\|_{L^\infty((0,T)\times\omega)}\right], \tag{6.3.21}$$

where the constant K depends only on Ω, f, ω and R.

The proof of the above lemma is classical (see, e.g. [53, 7]).

In order to prove Lemma 6.2.1, we introduce an optimal control problem, with specified terminal states. Let $t_1 < t_2$. Let $\hat{y}$ be a target trajectory, bounded solution to (6.1.2) in (t_1, t_2), i.e.

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + f(\hat{y}) = \hat{u}\chi_\omega & \text{in } (t_1,t_2)\times\Omega \\ \hat{y} = 0 & \text{on } (t_1,t_2)\times\partial\Omega. \end{cases} \tag{6.3.22}$$

Assume

$$\|\hat{y}\|_{L^\infty((t_1,t_2)\times\Omega)} \le R, \qquad \text{for some } R > 0.$$

For any control $u \in L^2((t_1,t_2)\times\omega)$, the corresponding state y is the solution to:

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (t_1,t_2)\times\Omega \\ y = 0 & \text{on } (t_1,t_2)\times\partial\Omega \\ y(t_1,x) = \hat{y}(t_1,x) & \text{in } \Omega. \end{cases} \tag{6.3.23}$$

We introduce the set of admissible controls

$$\mathscr{U}_{\text{ad}} := \left\{u \in L^2((t_1,t_2)\times\omega) \mid y(t_2) = \hat{y}(t_2)\right\}.$$

By definition, $\hat{u} \in \mathscr{U}_{\text{ad}}$. Hence, $\mathscr{U}_{\text{ad}} \neq \varnothing$. We consider the optimal control problem

$$\min_{u \in \mathscr{U}_{\text{ad}}} J_{t_1,t_2}(u) = \frac{1}{2}\int_{t_1}^{t_2}\int_{\omega} |u|^2 dx dt + \frac{\beta}{2}\int_{t_1}^{t_2}\int_{\omega_0} |y-z|^2 dx dt, \tag{6.3.24}$$

with running target $\|z\|_{L^\infty(\omega_0)} \leq R$. By the direct methods in the calculus of variations, the functional J_{t_1,t_2} admits a global minimizer in the set of admissible controls $\mathscr{U}_{\text{ad}}$.

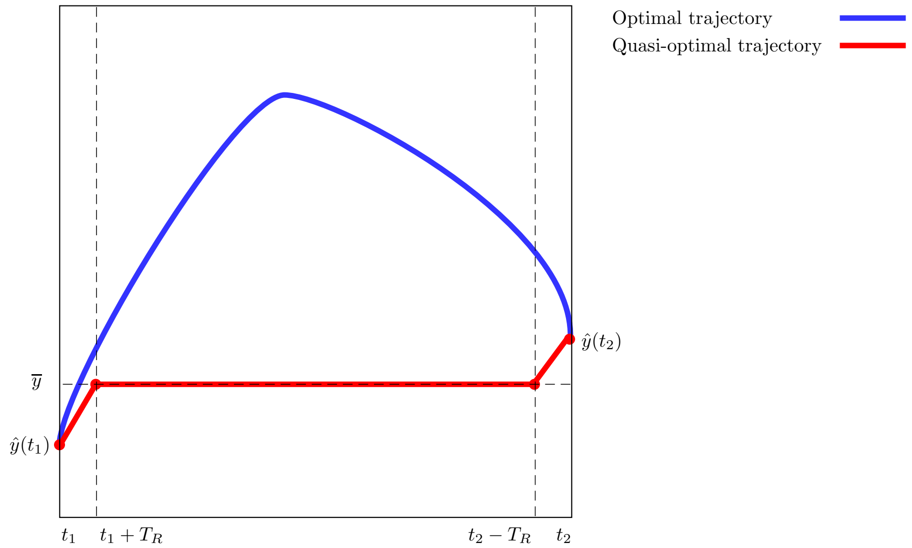


Figure 6.2: comparison between the optimal trajectory for (6.3.23)-(6.3.24) and a quasi-optimal turnpike trajectory.

The remark below is inspired by [72, Remark 5.1, section 5].

Remark 6.3.2. *Take into account the optimal control problem* (6.3.23)-(6.3.24), *with* $t_2 - t_1 \geq 2T_R$. *Let* $\overline{u}$ *be a steady optimum for* (6.1.4) *and let* $\overline{y}$ *be the solution to* (6.1.5), *with control* $\overline{u}$. *We are going to construct a turnpike control linking* $\hat{y}(t_1)$ *and* $\hat{y}(t_2)$, *whose corresponding trajectory* y *(see figure 6.2):*

1. *moves from* $\hat{y}(t_1)$ *to the turnpike* $\overline{y}$ *in time* $t_1 + T_R$;
2. *remains on the turnpike* $\overline{y}$, *for time* $t \in [t_1 + T_R, t_2 - T_R]$;
3. *leaves the turnpike and matches the final condition* $y(t_2) = \hat{y}(t_2)$, *at time* t_2.

By Lemma 6.3.3,

- *there exists* $u_1 \in L^\infty((t_1, t_1 + T_R) \times \omega)$, *steering* (6.1.2) *from* $\hat{y}(t_1)$ *to* $\overline{y}$ *in time* T_R;
- *there exists* $u_2 \in L^\infty((t_2 - T_R, t_2) \times \omega)$, *steering* (6.1.2) *from* $\overline{y}$ *to* $\hat{y}(t_2)$ *in time* T_R.

Set

$$u := \begin{cases} u_1 & \text{in } (t_1, t_1 + T_R) \\ \overline{u} & \text{in } (t_1 + T_R, t_2 - T_R) \\ u_2 & \text{in } (t_2 - T_R, t_2). \end{cases} \tag{6.3.25}$$

The above is the required turnpike control. By (6.3.21)*, we can bound the norm of the control, uniformly in* t_1 *and* t_2*,*

$$\begin{aligned}\|u\|_{L^\infty((t_1,t_2)\times\omega)} &\leq K\left[\|\hat{y}(t_1)-\overline{y}\|_{L^\infty(\Omega)}+\|\hat{y}(t_2-T_R)-\overline{y}\|_{L^\infty(\Omega)}\right.\\ &\quad\left.+\|\overline{u}\|_{L^\infty(\omega)}+\|\hat{u}\|_{L^\infty((t_2-T_R,t_2)\times\omega)}\right].\end{aligned} \tag{6.3.26}$$

Thanks to the above remark, we are in position to bound the minimal value of the functional (6.3.24).

Lemma 6.3.4. *Consider the optimal control problem* (6.3.23)-(6.3.24)*, with* $t_2-t_1\geq 2T_R$*. Then,*

$$\begin{aligned}\min_{\mathscr{U}_{ad}} J_{t_1,t_2} &\leq K\left[\|\hat{y}(t_1)\|^2_{L^\infty(\Omega)}+(t_2-t_1+1)\|z\|^2_{L^\infty(\omega_0)}\right.\\ &\quad\left.+\|\hat{u}\|^2_{L^\infty((t_2-T_R,t_2)\times\omega)}+\|\hat{y}(t_2-T_R)\|^2_{L^\infty(\Omega)}\right],\end{aligned} \tag{6.3.27}$$

the constant K *being independent of the time horizon* $t_2-t_1\geq 2T_R$*.*

Proof of Lemma 6.3.4. First of all, we observe that, by using the arguments in [116, subsection 3.2], we can estimate the uniform norm of the steady optima by the uniform norm of the target

$$\|\overline{u}\|_{L^\infty(\omega)}+\|\overline{y}\|_{L^\infty(\Omega)}\leq K\|z\|_{L^\infty(\omega_0)}.$$

Consider the control u introduced in (6.3.25) and let y be the solution to (6.3.23), with initial datum y_0 and control u. Then, we have

$$\begin{aligned}\min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2} &\leq J_{t_1,t_2}(u)\\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_\omega |u|^2dxdt+\frac{\beta}{2}\int_{t_1}^{t_2}\int_{\omega_0}|y-z|^2dxdt\\ &\leq \frac{1}{2}\int_{t_1}^{t_2}\int_\omega |u|^2dxdt+\beta\int_{t_1}^{t_2}\int_{\omega_0}|y|^2dxdt+\beta\int_{t_1}^{t_2}\int_{\omega_0}|z|^2dxdt\\ &\leq \frac{1}{2}\int_{t_1}^{t_2}\int_\omega |u|^2dxdt+\beta\int_{t_1}^{t_2}\int_{\omega_0}|y|^2dxdt+K(t_2-t_1)\|z\|^2_{L^\infty(\omega_0)}\\ &\leq K\left[\|u_1\|^2_{L^\infty((t_1,t_1+T_R)\times\omega)}+\|u_2\|^2_{L^\infty((t_2-T_R,t_2)\times\omega)}+(t_2-t_1)\|\overline{u}\|^2_{L^\infty(\omega)}\right.\\ &\quad+\|y\|^2_{L^\infty((t_1,t_1+T_R)\times\Omega)}+(t_2-t_1)\|\overline{y}\|^2_{L^\infty(\Omega)}+\|y\|^2_{L^\infty((t_2-T_R,t_2)\times\Omega)}\\ &\quad\left.+(t_2-t_1)\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\leq K\left[\|\hat{y}(t_1)\|^2_{L^\infty(\Omega)}+(t_2-t_1+1)\|z\|^2_{L^\infty(\omega_0)}\right.\\ &\quad\left.+\|\hat{u}\|^2_{L^\infty((t_2-T_R,t_2)\times\omega)}+\|\hat{y}(t_2-T_R)\|^2_{L^\infty(\Omega)}\right],\end{aligned}$$

as required. □

In the following Lemma we estimate the value of a function at some point, with the value of its integral.

Lemma 6.3.5. *Let $h \in L^1(c,d) \cap C^0(c,d)$, with $-\infty < c < d < +\infty$. Assume $h \geq 0$ a.e. in (c,d). Then,*

1. *there exists $t_c \in \left(c, c + \frac{d-c}{3}\right)$, such that*

$$h(t_c) \leq \frac{3}{d-c} \int_c^d h dt;$$

2. *there exists $t_d \in \left(d - \frac{d-c}{3}, d\right)$, such that*

$$h(t_d) \leq \frac{3}{d-c} \int_c^d h dt.$$

Proof of Lemma 6.3.5. By contradiction, for any $t \in \left(c, c + \frac{d-c}{3}\right)$, $h(t) > \frac{3}{d-c} \int_c^d h ds$. Then, we have

$$\int_c^d h dt \geq \int_c^{c+\frac{d-c}{3}} h dt > \int_c^{c+\frac{d-c}{3}} \left[\frac{3}{d-c} \int_c^d h ds\right] dt = \int_c^d h ds,$$

so obtaining a contradiction. The proof of (2.) is similar. □

We are now in position to prove Lemma 6.2.1.

Proof of Lemma 6.2.1. Step 1 **Estimates on subintervals**
Set $N_T := \lfloor \frac{T}{3T_R} \rfloor$. Arbitrarily fix $\theta > 0$, a degree of freedom, to be made precise later. Consider the indexes $i \in \{1, \dots, N_T\}$, such that

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \tag{6.3.28}$$

Set

$$\mathscr{I}_T := \left\{ i \in \{1, \dots, N_T\} \;\middle|\; \text{the estimate (6.3.28) is not verified} \right\}. \tag{6.3.29}$$

On the one hand, for any $i \in \{1, \dots, N_T\} \setminus \mathscr{I}_T$, by definition of $\mathscr{I}_T$

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].$$

On the other hand, for every $i \in \mathscr{I}_T$, we seek to prove the existence of a constant $K_\theta = K_\theta(\Omega, f, R, \theta)$, possibly larger than θ, such that

$$\int_{(i-1)3T_R}^{i3T_R} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq K_\theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \tag{6.3.30}$$

We start by considering the union of time intervals, where (6.3.28) is not verified

$$\mathscr{W}_T := \bigcup_{i \in \mathscr{I}_T} [(i-1)3T_R, i3T_R].$$

The above set is made of a finite union of disjoint closed intervals, namely there exists a natural M and $\{(a_j, b_j)\}_{j=1,\dots,M}$, such that

$$b_j < a_{j+1}, \quad j = 1, \dots, M-1$$

and

$$\mathscr{W}_T = \bigcup_{i \in \mathscr{I}_T} [(i-1)3T_R, i3T_R] = \bigcup_{j=1,\dots,M} [a_j, b_j].$$

For any $j = 1, \dots, M$, set

$$C_j := \{i \in \mathscr{I}_T \mid [(i-1)3T_R, i3T_R] \subseteq [a_j, b_j]\}. \tag{6.3.31}$$

We are going to prove (6.3.30), studying the optima in a neighbourhood of $[a_j, b_j]$, for $j = 1, \dots, M$. Four different cases may occur:

- **Case 1.** $a_1 = 0$ and $b_1 < 3T_R N_T$, namely the left end of the interval $[a_1, b_1]$ coincides with $t = 0$, while the right end is far from $t = T$;
- **Case 2.** $a_j > 0$ and $b_j < 3T_R N_T$, i.e. the left end of the interval $[a_j, b_j]$ is far from $t = 0$ and the right end is far from $t = T$;
- **Case 3.** $a_j = 0$ and $b_j = 3T_R N_T$, namely the left end of the interval $[a_j, b_j]$ coincides with $t = 0$ and the right end is close to $t = T$. Indeed, since $N_T = \lfloor \frac{T}{3T_R} \rfloor$, we have $T - b_j = T - 3T_R N_T < 3T_R$;
- **Case 4.** $a_j > 0$ and $b_j = 3T_R N_T$, i.e. the left end of the interval $[a_j, b_j]$ is far from $t = 0$, while the right end is close to $t = T$.

Case 1. $a_1 = 0$ and $b_1 < 3T_R N_T$.

Since $b_1 < 3T_R N_T$, we have $[b_1, b_1 + 3T_R N_T] \subseteq [0, T] \setminus \mathscr{W}_T$. Hence, by (6.3.29),

$$\int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)} \right] dt \le \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)} \right].$$

Set $c := b_1$, $d := b_1 + 3T_R$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma 6.3.5, there exist t_c and t_d,

$$b_1 < t_c < b_1 + T_R \quad \text{and} \quad b_1 + 2T_R < t_d < b_1 + 3T_R, \tag{6.3.32}$$

such that

$$\begin{aligned} \|q^T(t_c)\|^2_{L^\infty(\Omega)} + \|y^T(t_c)\|^2_{L^\infty(\Omega)} &\le \frac{1}{T_R} \int_{b_1}^{b_1+3T_R} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)} \right] dt \\ &\le \frac{\theta}{T_R} \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)} \right] \end{aligned}$$

and

$$\begin{aligned}\|q^T(t_d)\|^2_{L^\infty(\Omega)} + \|y^T(t_d)\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{b_1}^{b_1+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].\end{aligned}$$

Parabolic regularity in the optimality system (6.1.3) in the interval $[t_c, t_d]$ gives

$$\begin{aligned}\|y^T\|^2_{L^\infty((t_c,t_d)\times\Omega)} + \|q^T\|^2_{L^\infty((t_c,t_d)\times\Omega)} &\leq K\Big\{\|q^T(t_d)\|^2_{L^\infty(\Omega)} + \|y^T(t_c)\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\\ &\quad + \int_{b_1}^{b_1+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\Big\}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \qquad (6.3.33)\end{aligned}$$

where the constant K_θ is independent of the time horizon T, but depends on θ. At this point, we want to apply Lemma 6.3.4. To this purpose, we set up a control problem like (6.3.23)-(6.3.24) with specified final state

$$\begin{aligned}\hat{y} &:= y^T\\ t_1 &:= 0\\ t_2 &:= t_d.\end{aligned}$$

By (6.3.27) and (6.3.33),

$$\begin{aligned}\min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2} &\leq K\Big[\|y_0\|^2_{L^\infty(\Omega)} + (t_d+1)\|z\|^2_{L^\infty(\omega_0)}\\ &\quad + \|u^T\|^2_{L^\infty((t_d-T_R,t_d)\times\omega)} + \|y^T(t_d-T_R)\|^2_{L^\infty(\Omega)}\Big]\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma t_d\|z\|^2_{L^\infty(\omega_0)}, \qquad (6.3.34)\end{aligned}$$

where $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$. In our case the target trajectory for (6.3.23)-(6.3.24) is the state y^T associated to an optimal control u^T for (6.1.2)-(6.1.1). Then, by definition of (6.3.23)-(6.3.24),

$$J_{t_1,t_2}(u^T) \leq J_{t_1,t_2}(u), \quad \forall\, u \in \mathscr{U}_{\text{ad}}.$$

Hence, by (6.3.34),

$$\begin{aligned}J_{t_1,t_2}(u^T) &\leq \min_{\mathscr{U}_{\text{ad}}} J_{t_1,t_2}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma t_d\|z\|^2_{L^\infty(\omega_0)}. \qquad (6.3.35)\end{aligned}$$

By definition of $\mathscr{I}_T$ (6.3.29) and C_1 (6.3.31), we have

$$\begin{aligned}\int_0^{b_1}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\geq \sum_{i\in C_1}\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]\\ &= \frac{\theta b_1}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]\\ &> \frac{\theta(t_d-3T_R)}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right],\end{aligned}$$

where in the last inequality we have used (6.3.32), which yields $b_1 > t_d - 3T_R$. By the above inequality, Lemma 6.3.1, (6.3.33) and (6.3.35),

$$\begin{aligned}
&\frac{\theta(t_d-3T_R)}{6T_R}\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right] \\
&+\frac{1}{2}\int_0^{b_1}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \int_0^{b_1}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \\
&\qquad\leq K\left[J_{t_1,t_2}(u^T)+\|y_0\|^2_{L^\infty(\Omega)}+\|q^T(t_d)\|^2_{L^\infty(\Omega)}\right] \\
&\qquad\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]+\gamma t_d\|z\|^2_{L^\infty},
\end{aligned}$$

whence

$$\begin{aligned}
\int_0^{b_1}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right] \\
&\quad+2\left(\gamma t_d-\frac{\theta(t_d-3T_R)}{6T_R}\right)\|z\|^2_{L^\infty} \\
&\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right] \\
&\quad+2t_d\left(\gamma-\frac{\theta}{6T_R}\right)\|z\|^2_{L^\infty}.
\end{aligned}$$

If θ is large enough, we have $\gamma - \frac{\theta}{6T_R} < 0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_0^{b_1}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right].$$

Case 2. $a_j > 0$ and $b_j < 3T_R N_T$.

Since $a_j > 0$ and $b_j < 3T_R N_T$, we have

$$\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right] \tag{6.3.36}$$

and

$$\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]. \tag{6.3.37}$$

In Case 2, we apply Lemma 6.3.5:

- in the interval $[a_j - 3T_R, a_j]$;
- in the interval $[b_j, b_j + 3T_R]$.

We start by applying Lemma 6.3.5 in $[a_j - 3T_R, a_j]$. To this end, set $c := a_j - 3T_R$, $d := a_j$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma 6.3.5, there exist $t_{a,c}$ and $t_{a,d}$,

$$a_j - 3T_R < t_{a,c} < a_j - 2T_R \quad \text{and} \quad a_j - T_R < t_{a,d} < a_j, \tag{6.3.38}$$

such that

$$\begin{aligned}\|q^T(t_{a,c})\|^2_{L^\infty(\Omega)} + \|y^T(t_{a,c})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]\end{aligned}$$

and

$$\begin{aligned}\|q^T(t_{a,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{a,d})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].\end{aligned}$$

By parabolic regularity in the optimality system (6.1.3) in the interval $[t_{a,c}, t_{a,d}]$, we have

$$\begin{aligned}\|y^T\|^2_{L^\infty((t_{a,c},t_{a,d})\times\Omega)} + \|q^T\|^2_{L^\infty((t_{a,c},t_{a,d})\times\Omega)} &\leq K\Big\{\|q^T(t_{a,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{a,c})\|^2_{L^\infty(\Omega)}\\ &\quad+\|z\|^2_{L^\infty(\omega_0)}\\ &\quad+\int_{a_j-3T_R}^{a_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\Big\}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \qquad (6.3.39)\end{aligned}$$

where the constant K_θ is independent of the time horizon T, but it depends on θ.

We apply Lemma 6.3.5 in $[b_j, b_j + 3T_R]$. To this extent, set $c := b_j$, $d := b_j + 3T_R$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma 6.3.5, there exist $t_{b,c}$ and $t_{b,d}$,

$$0 < t_{b,c} < b_j + T_R \quad\text{and}\quad b_j + 2T_R < t_{b,d} < b_j + 3T_R, \qquad (6.3.40)$$

such that

$$\begin{aligned}\|q^T(t_{b,c})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,c})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]\end{aligned}$$

and

$$\begin{aligned}\|q^T(t_{b,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,d})\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{b_j}^{b_j+3T_R}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right].\end{aligned}$$

By parabolic regularity in the optimality system (6.1.3) in the interval $[t_{b,c}, t_{b,d}]$, we have

$$\begin{aligned}\|y^T\|^2_{L^\infty((t_{b,c},t_{b,d})\times\Omega)} + \|q^T\|^2_{L^\infty((t_{b,c},t_{b,d})\times\Omega)} &\leq K\Big\{\|q^T(t_{b,d})\|^2_{L^\infty(\Omega)} + \|y^T(t_{b,c})\|^2_{L^\infty(\Omega)}\\ &\quad+\|z\|^2_{L^\infty(\omega_0)} + \int_{b_j}^{b_j+3T_R}\|q^T(t)\|^2_{L^\infty(\Omega)}dt\\ &\quad+\int_{b_j}^{b_j+3T_R}\|y^T(t)\|^2_{L^\infty(\Omega)}dt\Big\}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \qquad (6.3.41)\end{aligned}$$

where the constant K_θ is independent of the time horizon T, but it depends on θ.

At this point, we want to apply Lemma 6.3.4. To this purpose, we set up a control problem like (6.3.23)-(6.3.24) with specified final state

$$\begin{aligned}\hat{y} &:= y^T \\ t_1 &:= t_{a,c} \\ t_2 &:= t_{b,d}.\end{aligned}$$

By (6.3.27), (6.3.39) and (6.3.41),

$$\begin{aligned}\min_{\mathscr{U}_{\mathrm{ad}}} J_{t_1,t_2} &\leq K\left[\|y^T(t_{a,c})\|^2_{L^\infty(\Omega)} + (t_{b,d} - t_{a,c} + 1)\|z\|^2_{L^\infty(\omega_0)}\right. , \\ &\quad \left. + \|u^T\|^2_{L^\infty((t_{b,d}-T_R,t_{b,d})\times\omega)} + \|y^T(t_{b,d} - T_R)\|^2_{L^\infty(\Omega)}\right] \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma(t_{b,d} - t_{a,c})\|z\|^2_{L^\infty(\omega_0)}, \end{aligned} \tag{6.3.42}$$

where $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$. In our case the target trajectory for (6.3.23)-(6.3.24) is the state y^T associated to an optimal control u^T for (6.1.2)-(6.1.1). Then, by definition of (6.3.23)-(6.3.24),

$$J_{t_1,t_2}(u^T) \leq J_{t_1,t_2}(u), \quad \forall\, u \in \mathscr{U}_{\mathrm{ad}}.$$

Hence, by (6.3.42),

$$\begin{aligned}J_{t_1,t_2}(u^T) &\leq \min_{\mathscr{U}_{\mathrm{ad}}} J_{t_1,t_2} \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] + \gamma(t_{b,d} - t_{a,c})\|z\|^2_{L^\infty(\omega_0)}.\end{aligned}$$

By definition of $\mathscr{I}_T$ (6.3.29) and C_1 (6.3.31), we have

$$\begin{aligned}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\geq \sum_{i\in C_j}\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &= \frac{\theta(b_j - a_j)}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &> \frac{\theta(t_{b,d} - t_{a,c} - 6T_R)}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right],\end{aligned} \tag{6.3.43}$$

where in the last inequality we have used (6.3.38) and (6.3.40) to get $b_j - a_j > t_{b,d} - t_{a,c} - 6T_R$. By the above inequality, Lemma 6.3.1 and (6.3.43),

$$\begin{aligned}&\frac{\theta(t_{b,d} - t_{a,c} - 6T_R)}{6T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &+\frac{1}{2}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \\ &\leq K\left[J_{t_1,t_2}(u^T) + \|y_0\|^2_{L^\infty(\Omega)}\right] \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &\quad +\gamma(t_{b,d} - t_{a,c})\|z\|^2_{L^\infty(\omega_0)},\end{aligned}$$

whence

$$\begin{aligned}\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\quad +2\left(\gamma(t_{b,d}-t_{a,c})-\theta\frac{(t_{b,d}-t_{a,c}-6T_R)}{6T_R}\right)\|z\|^2_{L^\infty}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\quad +2(t_{b,d}-t_{a,c})\left(\gamma-\frac{\theta}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)}.\end{aligned}$$

If θ is large enough, we have $\gamma-\frac{\theta}{6T_R}<0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right].$$

Case 3. $a_j=0$ and $b_j=3T_RN_T$.

Let y be the solution to (6.1.2), with initial datum y_0 and control u. By definition of minimizer, we have

$$J_T(u^T)\leq J_T(0)\leq K\left[\|y_0\|^2_{L^\infty(\Omega)}+T\|z\|^2_{L^\infty(\omega_0)}\right]$$

whence, by Lemma 6.3.1 applied to the state and the adjoint equation in the optimality system (6.1.3),

$$\begin{aligned}\int_0^T\left[\|y^T(t)\|^2_{L^\infty(\Omega)}+\|q^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\leq K\left[J_T(u^T)+\|y_0\|_{L^\infty(\Omega)}\right]\\ &\leq K\left[\|y_0\|^2_{L^\infty(\Omega)}+T\|z\|^2_{L^\infty(\omega_0)}\right]\end{aligned}\tag{6.3.44}$$

Now, by definition of $\mathscr{I}_T$ (6.3.29) and by (6.3.44),

$$\begin{aligned}&\frac{\theta N_T}{2}\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &+\frac{1}{2}\int_0^T\left[\|y^T(t)\|^2_{L^\infty(\Omega)}+\|q^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \int_0^T\left[\|y^T(t)\|^2_{L^\infty(\Omega)}+\|q^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\qquad\leq K\left[\|y_0\|^2_{L^\infty(\Omega)}+T\|z\|^2_{L^\infty(\omega_0)}\right],\end{aligned}$$

Moreover, by definition of N_T, we have

$$\frac{T}{3T_R}\leq N_T+1.$$

Then,

$$\int_0^T\left[\|y^T(t)\|^2_{L^\infty(\Omega)}+\|q^T(t)\|^2_{L^\infty(\Omega)}\right]dt\leq K\|y_0\|^2_{L^\infty(\Omega)}+T\left(K-\frac{\theta}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)}.$$

If θ is large enough, $K - \frac{\theta}{6T_R} < 0$. Then, for θ large enough,we have the estimate

$$\int_0^T \left[\|y^T(t)\|^2_{L^\infty(\Omega)} + \|q^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq K\|y_0\|_{L^\infty(\Omega)}.$$

Case 4. $a_j > 0$ and $b_j = 3T_R N_T$.

Since $a_j > 0$, we have

$$\int_{a_j-3T_R}^{a_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \leq \theta \left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \tag{6.3.45}$$

We apply Lemma 6.3.5 in $[a_j - 3T_R, a_j]$. To this end, set $c := a_j - 3T_R$, $d := a_j$ and $h(t) := \|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}$. By Lemma 6.3.5, there exist t_c,

$$a_j - 3T_R < t_c < a_j - 2T_R \tag{6.3.46}$$

such that

$$\begin{aligned} \|q^T(t_c)\|^2_{L^\infty(\Omega)} + \|y^T(t_c)\|^2_{L^\infty(\Omega)} &\leq \frac{1}{T_R}\int_{a_j-3T_R}^{a_j} \left[\|q^T(t)\|^2_{L^\infty(\Omega)} + \|y^T(t)\|^2_{L^\infty(\Omega)}\right] dt \\ &\leq \frac{\theta}{T_R}\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right]. \end{aligned} \tag{6.3.47}$$

We introduce the control

$$u^0 := \begin{cases} u^T & \text{in } (0, t_c) \\ 0 & \text{in } (t_c, T) \end{cases}$$

Let y be the solution to (6.1.2), with initial datum y_0 and control u and y^0 be the solution to (6.1.2), with initial datum y_0 and control u^0. By definition of minimizer, we have

$$\begin{aligned} J_T(u^T) &\leq J_T(u^0) \\ &\leq \frac{1}{2}\int_0^T\int_\omega |u^0|^2 dxdt + \frac{\beta}{2}\int_0^T\int_{\omega_0} |y^0 - z|^2 dxdt \\ &= \frac{1}{2}\int_0^{t_c}\int_\omega \left|u^T\right|^2 dxdt + \frac{\beta}{2}\int_0^{t_c}\int_{\omega_0} \left|y^T - z\right|^2 dxdt \\ &\quad +\frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} |y^0 - z|^2 dxdt, \end{aligned}$$

whence,

$$\begin{aligned} \frac{1}{2}\int_{t_c}^T\int_\omega \left|u^T\right|^2 dxdt + \frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} \left|y^T - z\right|^2 dxdt &\leq \frac{\beta}{2}\int_{t_c}^T\int_{\omega_0} |y^0 - z|^2 dxdt \\ &\leq K\left[\|y(t_c)\|^2_{L^\infty(\Omega)} + (T - t_c)\|z\|^2_{L^\infty(\omega_0)}\right] \\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)} + \|z\|^2_{L^\infty(\omega_0)}\right] \\ &\quad +\gamma(T - t_c)\|z\|^2_{L^\infty(\omega_0)}, \end{aligned}$$

where we have used (6.3.47) and $K_\theta = K_\theta(\Omega, f, R, \theta)$ and $\gamma = \gamma(\Omega, f, R)$.

Now, on the one hand, by Lemma 6.3.1 applied to the state and the adjoint equation in (6.1.3), we have

$$\begin{aligned}\int_{t_c}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\quad+\gamma(T-t_c)\|z\|^2_{L^\infty(\omega_0)}. \qquad (6.3.48)\end{aligned}$$

On the other hand, by (6.3.46), $-a_j > -t_c - 3T_R$ and, since $b_j = 3T_R N_T$, $b_j \geq T - 3T_R$. Hence, $b_j - a_j > T - t_c - 6T_R$. Then, by (6.3.29),

$$\begin{aligned}\int_{a_j}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\geq \int_{a_j}^{b_j}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\geq \sum_{i\in C_j}\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &= \frac{\theta(b_j-a_j)}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &> \frac{\theta(T-t_c-6T_R)}{3T_R}\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right].\end{aligned}$$

By the above inequality and Lemma 6.3.1 and (6.3.48),

$$\begin{aligned}&\frac{\theta(T-t_c-6T_R)}{6T_R}\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &+\frac{1}{2}\int_{a_j}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq \int_{a_j}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt\\ &\qquad\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\qquad\quad+\gamma(T-t_c)\|z\|^2_{L^\infty(\omega_0)},\end{aligned}$$

whence

$$\begin{aligned}\int_{a_j}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\quad+2\left(\gamma(T-t_c)-\theta\frac{(T-t_c-6T_R)}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)}\\ &\leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right]\\ &\quad+2(T-t_c)\left(\gamma-\frac{\theta}{6T_R}\right)\|z\|^2_{L^\infty(\omega_0)}.\end{aligned}$$

If θ is large enough, we have $\gamma - \frac{\theta}{6T_R} < 0$. Hence, choosing θ large enough, we obtain the estimate

$$\int_{a_j}^{T}\left[\|q^T(t)\|^2_{L^\infty(\Omega)}+\|y^T(t)\|^2_{L^\infty(\Omega)}\right]dt \leq K_\theta\left[\|y_0\|^2_{L^\infty(\Omega)}+\|z\|^2_{L^\infty(\omega_0)}\right].$$

Step 2 **Conclusion**

The proof is concluded, with an application of Lemma 6.3.2 to the state and the adjoint equation in (6.1.3). □

Chapter 7

Non-uniqueness of minimizers for semilinear optimal control problems

7.1 Introduction

In this chapter, we produce a counterexample to the uniqueness of the optimal control in semilinear control. Both the case of internal control and boundary control are considered.

To simplify the notation, we have dropped the s subscript to denote steady controls/states.

In the context of boundary control, we consider the control problem

$$\min_{u\in L^2(\partial B(0,R))} J_s(u) = \frac{1}{2}\int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |y-z|^2 dx, \tag{7.1.1}$$

where:

$$\begin{cases} -\Delta y + f(y) = 0 & \text{in } B(0,R) \\ y = u & \text{on } \partial B(0,R). \end{cases} \tag{7.1.2}$$

$B(0,R)$ is a ball of $\mathbb{R}^n$ centered at the origin of radius R, with $n = 1,2,3$. The nonlinearity $f \in C^2(\mathbb{R})$ is strictly increasing and $f(0) = 0$. The target $z \in L^2(B(0,R))$. $\beta > 0$ is a penalization parameter. As β increases, the distance between the optimal state and the target decreases.

Theorem 7.1.1. *Consider the control problem* (7.1.2)-(7.1.1)*. Assume, in addition*

$$f''(y) \neq 0 \qquad \forall\, y \neq 0. \tag{7.1.3}$$

There exists a target $z \in L^\infty(B(0,R))$ such that the functional J_s defined in (7.1.1) *admits (at least) two global minimizers.*

The proof of Theorem 7.1.1 can be found in section 7.3.1. The main steps for that proof are:

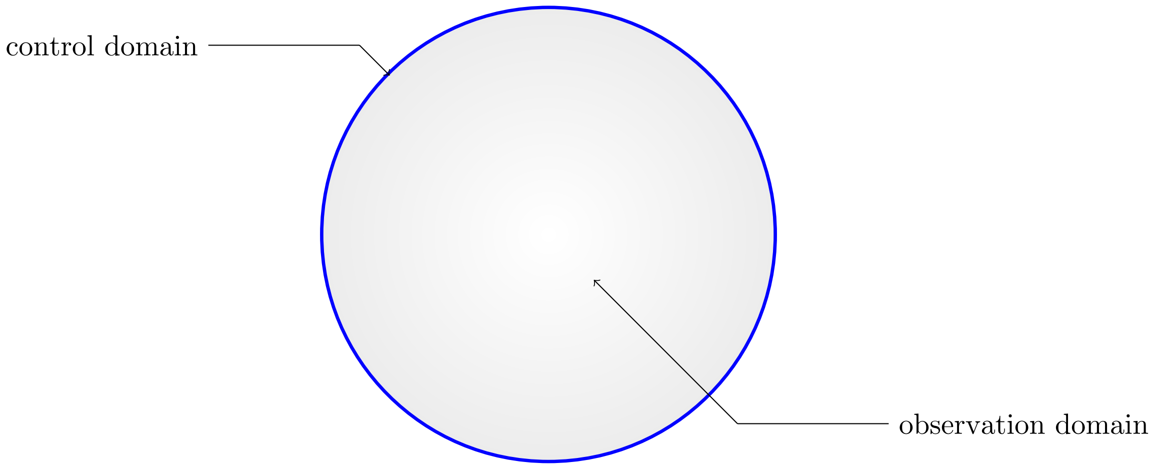
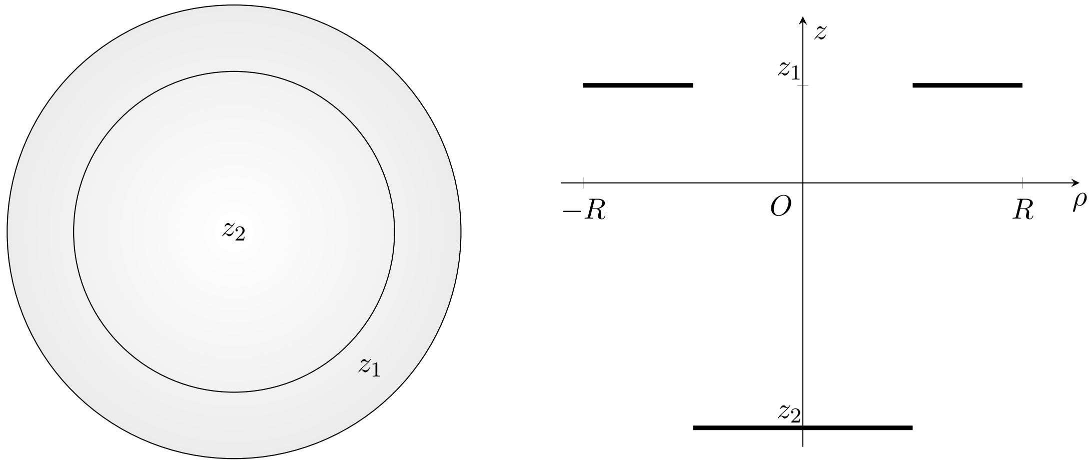


Figure 7.1: control and observation domains. The control domain is the blue boundary of the ball.



(a) level curves of the constructed radial target

(b) constructed radial target versus the radius ρ

Figure 7.2: target yielding nonuniqueness in boundary control

Step 1 **Reduction to constant controls**: by choosing radial targets and by using the rotational invariance of $B(0,R)$, we reduce to the the case the control set is made of constant controls;

Step 2 **Existence of two local minimizers**: we look for a target such that there exists two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the steady functional J_s;

Step 3 **Existence of two global minimizers**: by the former step and a bisection argument, we prove the existence of a target such that J_s admits two *global* minimizers.

The special target yielding nonuniqueness is a step function changing sign in the observation domain, as in figure 7.2.

The above techniques can be applied, with some modifications, to the internal control

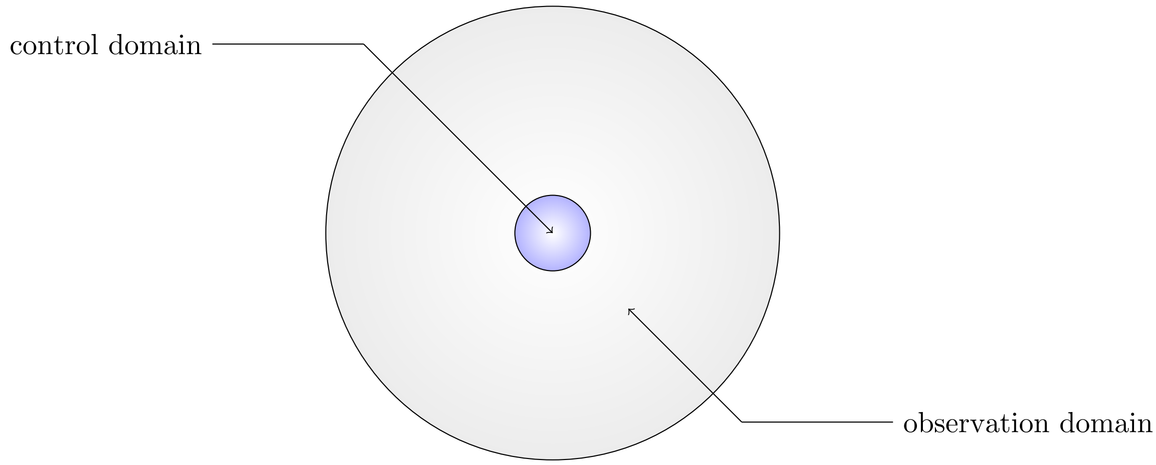


Figure 7.3: control and observation domains

problem

$$\min_{u \in L^2(B(0,r))} J_s(u) = \frac{1}{2}\int_{B(0,r)} |u|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |y-z|^2 dx, \tag{7.1.4}$$

where:

$$\begin{cases} -\Delta y + f(y) = u\chi_{B(0,r)} & \text{in } B(0,R) \\ y = 0 & \text{on } \partial B(0,R). \end{cases} \tag{7.1.5}$$

$B(0,R)$ is a ball of $\mathbb{R}^n$ centered at the origin of radius R, $n = 1,2,3$. The nonlinearity $f \in C^2(\mathbb{R})$ is strictly increasing and $f(0) = 0$. The control acts in $B(0,r)$, with $r \in (0,R)$. We observe in $B(0,R) \setminus B(0,r)$ (see figure 7.3). The target $z \in L^2(B(0,R) \setminus B(0,r))$. $\beta > 0$ is a penalization parameter. As β increases, the distance between the optimal state and the target decreases.

Theorem 7.1.2. *Consider the control problem* (7.1.5)-(7.1.4). *Assume, in addition,*

$$f''(y) \neq 0 \qquad \forall\ y \neq 0. \tag{7.1.6}$$

There exists a target $z \in L^\infty(B(0,R) \setminus B(0,r))$ such that the functional J_s defined in (7.1.4) *admits (at least) two global minimizers.*

The proof can be found in section 7.3.2.

A by-product of our nonuniqueness results is the lack of uniqueness of solutions $(\overline{y}, \overline{q})$ to the optimality system

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_{B(0,r)} & \text{in } B(0,R) \\ \overline{y} = 0 & \text{on } \partial B(0,R) \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{B(0,R)\setminus B(0,r)} & \text{in } B(0,R) \\ \overline{q} = 0 & \text{on } \partial B(0,R). \end{cases} \tag{7.1.7}$$

In the case of internal control, we can deduce the following corollary.

Corollary 7.1.1. *Under the assumptions of Theorem 7.1.2, there exists a target* $z \in L^\infty(B(0,R) \setminus B(0,r))$*, such that* (7.1.7) *admits (at least) two distinguished solutions* $(\overline{y}_1, \overline{q}_1)$ *and* $(\overline{y}_2, \overline{q}_2)$*.*

This follows from Theorem 7.1.2, together with the first order optimality condition for the functional (7.1.4) (see Proposition 3.1.7 in chapter 3).

Similarly, in the context of boundary control, the nonuniqueness for (7.1.1) leads to nonuniquness of solution to the optimality system

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = 0 & \text{in } B(0,R) \\ \overline{y} = \frac{\partial}{\partial n}\overline{q} & \text{on } \partial B(0,R) \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z) & \text{in } B(0,R) \\ \overline{q} = 0 & \text{on } \partial B(0,R). \end{cases} \tag{7.1.8}$$

Although the control problems we are treating are classical in the control literature (see e.g. [26, 8, 10, 139, 108, 25, 3, 66, 24, 89, 43, 125]), to the best of our knowledge, the issue of the uniqueness of the minimizer has not been addressed so far for large targets z. Indeed, the uniqueness of the optimal control has been proved under smallness conditions on the target [116, subsection 3.2] or on the adjoint state [2, Theorem 3.2]. In particular, in [2, Theorem 3.2] the uniqueness holds provided that the adjoint state is strictly smaller then a constant, explicitly determined (see [2, equation (3.6)]).

The issue of uniqueness of the minimizer for elliptic problems is of primary importance when studying the turnpike property for the corresponding time-evolution control problem (see, chapter 6). Indeed, the existence of multiple global minimizers for the steady problem generates multiple potential attractors for the time-evolution problem.

Before proving our main result on non-uniqueness of global minimizers, we observe that, for some targets, quadratic functionals of the optimal control governed by nonlinear state equations are not convex. This nonconvexity result applies to general control problems, with quadratic cost and nonlinear state equation.

Orientation

The rest of the chapter is organized as follows:

- section 7.2: Lack of convexity;
- section 7.3: Lack of uniqueness:
  - subsection 7.3.1: boundary control;
  - subsection 7.3.2: internal control;
- section 7.4: Numerical simulations;

- Appendix.

7.2 Lack of convexity

Before introducing our general Theorem, let us clarify what we mean by affine function.

Definition 7.2.1. *Let V_1 and V_2 be two real vector spaces. A function*

$$G : V_1 \longrightarrow V_2$$

is said to be affine if there exists a linear map $L : V_1 \to V_2$ and $d \in V_2$ such that

$$G(v) = L(v) + d, \qquad \forall\ v \in V_1.$$

We formulate now our result.

Theorem 7.2.1. *Let U and H be Hilbert spaces. Let*

$$G : U \longrightarrow H$$

be a function. Set:

$$J : U \longrightarrow H$$
$$J(u) := \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u) - z\|_H^2,$$

where $z \in H$.
Then, the following are equivalent:

1. *for any target $z \in H$, J is convex;*
2. *G is affine.*

In particular, in case G is not affine, there exists a target $z \in H$ such that the corresponding J is not convex.

Typically, in optimal control, H is the state space, U is the control space and G is the control-to-state map. Finally, $z \in H$ is the given target for the state. Note that the above theorem applies both to steady and time-evolution control problems.

In the proof of Theorem 7.2.1, we need the following lemma.

Lemma 7.2.1. *Let V_1 and V_2 be two vector spaces over $\mathbb{R}$. Take a function*

$$G : V_1 \longrightarrow V_2.$$

Then, G is affine if and only if, for any $\lambda \in [0,1]$ and $(v, w) \in V_1^2$

$$G((1-\lambda)v + \lambda w) = (1-\lambda)G(v) + \lambda G(w). \tag{7.2.1}$$

We prove Lemma 7.2.1 in the Appendix. We prove now Theorem 7.2.1.

Proof of Theorem 7.2.1. 2. $\implies$ 1. If G is affine, by direct computations and convexity of the square of Hilbert norms, J is convex for any $z \in H$.

1. $\implies$ 2. We assume now G is not affine and we want to show that there exist a target $z \in H$ such that J is not convex.

In what follows, we denote by $\langle \cdot, \cdot \rangle$ the scalar product of H.

Step 1 **Proof of the existence of $\tilde{\lambda} \in [0,1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:**

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle < (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle$$

First of all, we note that, up to change the sign of z^0, we can reduce to prove the existence of $\tilde{\lambda} \in [0,1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle \neq (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle. \tag{7.2.2}$$

Reasoning by contradiction, if (7.2.2) were not true, for any $z \in H$, for every $(u_1, u_2) \in U^2$ and for each $\lambda \in [0,1]$,

$$\langle z, G((1-\lambda)u_1 + \lambda u_2)\rangle = (1-\lambda)\langle z, G(u_1)\rangle + \lambda\langle z, G(u_2)\rangle.$$

By the arbitrariness of z, this leads to:

$$G((1-\lambda)u_1 + \lambda u_2) = (1-\lambda)G(u_1) + \lambda G(u_2),$$

for any $\lambda \in [0,1]$ and $(u_1, u_2) \in U^2$. Then, by Lemma 7.2.1, G is affine, which contradicts our hypothesis. This finishes this step.

Step 2 **Conclusion**

We remind that in the first step, we have proved the existence of $\tilde{\lambda} \in [0,1]$, $(\tilde{u}_1, \tilde{u}_2) \in U^2$ and $z^0 \in H$ such that:

$$\langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle < (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle.$$

Now, arbitrarily fix $k \in \mathbb{N}^*$. Set as target:

$$z^k := kz^0.$$

We develop J with target z^k, getting for any $u \in U$:

$$J(u) = \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u) - z^k\|_H^2$$

$$= \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2 + \frac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u)\rangle =$$

$$= P(u) + \frac{1}{2}\|z^k\|_H^2 - \langle z^k, G(u)\rangle,$$

where

$$P : U \longrightarrow \mathbb{R}$$
$$u \longmapsto \frac{1}{2}\|u\|_U^2 + \frac{1}{2}\|G(u)\|_H^2.$$

At this point, we introduce:

$$c_1 := (1-\tilde{\lambda})P(\tilde{u}_1) + \tilde{\lambda}P(\tilde{u}_2) - P((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)$$

and

$$c_2 := (1-\tilde{\lambda})\langle z^0, G(\tilde{u}_1)\rangle + \tilde{\lambda}\langle z^0, G(\tilde{u}_2)\rangle - \langle z^0, G((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2)\rangle.$$

Then, taking as target z^k,

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) - J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2) = c_1 - kc_2.$$

By the first step, $c_2 > 0$. Then, for k large enough, we have:

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) - J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2) = c_1 - kc_2 < 0,$$

which yields

$$(1-\tilde{\lambda})J(\tilde{u}_1) + \tilde{\lambda}J(\tilde{u}_2) < J((1-\tilde{\lambda})\tilde{u}_1 + \tilde{\lambda}\tilde{u}_2),$$

i.e. the desired lack of convexity of J. This concludes the proof. $\square$

Theorem 7.2.1 applies in semilinear control, both in the elliptic case and in the parabolic one. We show how to apply Theorem 7.2.1 for the control problem (7.1.5)-(7.1.4). Take

- control space $U = L^2(B(0,r))$;
- $H = L^2\left(B(0,R) \setminus B(0,r)\right)$ with scalar product $\langle v_1, v_2\rangle := \beta \int_{B(0,R)\setminus B(0,r)} v_1 v_2 dx$;
- $$G : L^2(B(0,r)) \longrightarrow L^2\left(B(0,R) \setminus B(0,r)\right)$$
  $$u \longrightarrow y_u\restriction_{B(0,R)\setminus B(0,r)},$$

  where y_u fulfills (7.1.5) with control u.

Then, by Theorem 7.2.1, we have two possibilities:

1. f is linear. Then, J_s is convex for any target $z \in L^2\left(B(0,R) \setminus B(0,r)\right)$.
2. f is not linear. Then, there exists a target $z \in L^2\left(B(0,R) \setminus B(0,r)\right)$ such that the corresponding J_s is not convex.

7.3 Lack of uniqueness

In this section, we prove our nouniqueness results. We start with boundary control (Theorem 7.1.2), to later deal with internal control (Theorem 7.1.1).

7.3.1 Boundary control

Hereafter, we will work with radial targets, defined below.

Definition 7.3.1. *A function $z : B(0,R) \longrightarrow \mathbb{R}$ is said to be radial if there exists $\phi : [0,R] \longrightarrow \mathbb{R}$, such that, for any $x \in B(0,R)$, we have $z(x) = \phi(\|x\|)$.*

We present our strategy to prove Theorem 7.1.1:

Step 1 **Reduction to constant controls**: by rotational invariance of $B(0,R)$ and radial targets, we reduce to the the case the control set is made of constant controls;

Step 2 **Existence of two local minimizers**: we look for a target $z^0 \in L^\infty(B(0,R))$ such that there exists two *local* minimizers ($u_1 < 0$ and $u_2 > 0$) for the steady functional J_s with target z^0;

Step 3 **Existence of two global minimizers**: by a bisection argument, we prove the existence of a target $\tilde{z} \in L^\infty(B(0,R))$ such the steady functional J_s with target $\tilde{z}$ admits (at least) two *global* minimizers.

Notation

First of all, we introduce the control-to-state map

$$G : L^2(\partial B(0,R)) \longrightarrow L^2(\Omega) \tag{7.3.1}$$

$$u \longmapsto y_u,$$

where y_u is the solution to (7.1.2) with control u. Then, set:

$$I : L^2(\partial B(0,R)) \times L^2(B(0,R)) \longrightarrow \mathbb{R} \tag{7.3.2}$$

$$I(u,z) := \frac{1}{2}\int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u)|^2 dx - \beta \int_{B(0,R)} G(u) z dx,$$

where G is the control-to-state map introduced in (7.3.1). One recognizes that, for any target $z \in L^\infty(B(0,R))$, $I(\cdot, z) + \frac{\beta}{2}\|z\|^2_{L^2(B(0,R))}$ coincides with the functional J_s defined in (7.1.1) with target z. Then, for any target $z \in L^\infty(B(0,R))$ minimizing $I(\cdot,z)$ is equivalent to minimize J_s with target z. Such translation is convenient, because $I(0,z) = 0$ for any target $z \in L^\infty(B(0,R))$.

We introduce:

$$h_1 : L^\infty(B(0,R)) \longrightarrow \mathbb{R}, \quad h_1(z) := \inf_{(-\infty,0]} [I(\cdot,z)] \tag{7.3.3}$$

and

$$h_2 : L^\infty(B(0,R)) \longrightarrow \mathbb{R}, \quad h_2(z) := \inf_{[0,+\infty)} [I(\cdot,z)]. \tag{7.3.4}$$

We formulate the first lemma.

Lemma 7.3.1. *Let $C \subseteq \mathbb{R}$ be a closed subset such that $0 \in C$. Then,*

1. *for any $z \in L^\infty(B(0,R))$, there exists $u_z \in C$ such that:*

$$I(u_z, z) = \inf_C [I(\cdot, z)].$$

   Furthermore, for any minimizer u_z

$$|u_z| \leq \sqrt{\frac{\beta}{R^{n-1} n\alpha(n)}} \|z\|_{L^2},$$

   where $n\alpha(n)$ is the surface area of $\partial B(0,1) \subset \mathbb{R}^n$, the unit sphere.

2. *the map*

$$h : L^\infty(B(0,R)) \longrightarrow \mathbb{R}, \quad h(z) := \inf_C [I(\cdot, z)]$$

   is continuous.

We prove Lemma 7.3.1 in the Appendix.

We now state the second lemma needed to prove Theorem 7.1.1.

Lemma 7.3.2. *Assume there exists $z^0 \in L^\infty(B(0,R))$ such that*

$$h_1(z^0) < 0 \qquad \textit{and} \qquad h_2(z^0) < 0,$$

where h_1 and h_2 are defined in (7.3.3) and (7.3.4) resp. Then, there exists $\tilde{z} \in L^\infty(B(0,R))$ such that

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The proof of Lemma 7.3.2 can be found in the Appendix. The following lemma is the key-point for the proof of two local minimizers for (7.1.1). At this point we employ the nonlinearity of the state equation (7.1.2).

Lemma 7.3.3. *Assume*

$$f''(y) \neq 0 \qquad \forall\, y \neq 0.$$

Fix $r_1 \in (0,R)$. For any $r_2 \in [r_1, R)$, let

$$M := \beta \begin{bmatrix} \int_{B(0,r_1)} G(-1)dx & \int_{B(0,R)\setminus B(0,r_2)} G(-1)dx \\ \int_{B(0,r_1)} G(2)dx & \int_{B(0,R)\setminus B(0,r_2)} G(2)dx \end{bmatrix}.$$

There exists $r_2 \in [r_1, R)$, such that rank$(M) = 2$.

Proof of Lemma 7.3.3. Let us assume, by contradiction, that for any $r_2 \in \mathbb{R}$, with $0 < r_1 \leq r_2 < R$ there exists $\lambda \in \mathbb{R}$ such that

$$\left[\int_{B(0,r_1)} G(2)dx; \int_{B(0,R)\setminus B(0,r_2)} G(2)dx\right] = \lambda \left[\int_{B(0,r_1)} G(-1)dx; \int_{B(0,R)\setminus B(0,r_2)} G(-1)dx\right]. \tag{7.3.5}$$

By strong maximum principle [57, Theorem 8.19 page 198],

$$\begin{aligned} &\int_{B(0,r_1)} G(2)dx > 0 \qquad &&\int_{B(0,R)\setminus B(0,r_2)} G(2)dx > 0 \\ &\int_{B(0,r_1)} G(-1)dx < 0 \qquad &&\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx < 0. \end{aligned} \tag{7.3.6}$$

From (7.3.5), we realize that

$$\lambda = \frac{\int_{B(0,r_1)} G(2)dx}{\int_{B(0,r_1)} G(-1)dx} \tag{7.3.7}$$

which leads to the independence of λ from r_2.

By (7.3.5), for any $r_2 \in [r_1, R)$,

$$\int_{B(0,R)\setminus B(0,r_2)} G(2)dx = \lambda \int_{B(0,R)\setminus B(0,r_2)} G(-1)dx, \tag{7.3.8}$$

whence

$$\int_{B(0,R)\setminus B(0,r_2)} \left[G(2) - \lambda G(-1)\right] dx, \quad \forall\ r_2 \in [r_1, R). \tag{7.3.9}$$

At this stage, we realize that, since the constant controls -1 and 2 are radial, the corresponding states $G(-1)$ and $G(2)$ are radial as well. Hence, the above equality together with measure theory yields

$$G(2) = \lambda G(-1), \quad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.10}$$

Then, by definition of the control operator, we have

$$-\Delta\left(G(-1)\right) + f(G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1) \tag{7.3.11}$$

and

$$-\Delta\left(G(2)\right) + f(G(2)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.12}$$

plugging $\lambda G(-1)$ in (7.3.12), we obtain

$$-\Delta\left(\lambda G(-1)\right) + f(\lambda G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1) \tag{7.3.13}$$

Similarly, multiplying (7.3.11) by λ, we get

$$-\Delta\left(\lambda G(-1)\right) + \lambda f(G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.14}$$

Subtracting (7.3.13) and (7.3.14), we obtain

$$f(\lambda G(-1)) = \lambda f(G(-1)) \quad \text{in } B(0,R)\setminus B(0,r_1). \tag{7.3.15}$$

Now, for any control $v \in \mathbb{R}$, $G(v) \in C^0\left(\overline{\Omega}\right)$, whence by strong maximum principle [57, Theorem 8.19 page 198], $G(-1) < 0$ in $B(0,R)\setminus B(0,r_1)$. Hence

$$-\Delta G(-1) = -f(G(-1)) > 0 \quad \text{in } B(0,R)\setminus B(0,r_1). \tag{7.3.16}$$

Therefore, $G(-1)$ is not constant in $B(0,R)\setminus B(0,r_1)$. Now, being $G(-1)$ non constant in $B(0,R)\setminus B(0,r_1)$, (7.3.15) leads to a contradiction with (7.1.3). Hence, for some $r_2 \in [r_1, R)$, $\mathrm{rank}(M) = 2$. □

We are now ready to prove Theorem 7.1.1.

Proof of Theorem 7.1.1. Step 1 **Reduction to constant controls.**
Suppose for some radial target z, the optimal control is not constant. Then, there exists an orthogonal matrix M, such that $u \circ M \neq u$. Now,

$$\begin{aligned} I\left(u \circ M, z)\right) &= \frac{1}{2}\int_{\partial B(0,R)} |u\circ M|^2 d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u\circ M)|^2 dx - \beta\int_{B(0,R)} G(u\circ M) z dx \\ &= \frac{1}{2}\int_{\partial B(0,R)} |u|^2 d\sigma(x) + \frac{\beta}{2}\int_{B(0,R)} |G(u)|^2 dx - \beta\int_{B(0,R)} G(u) z dx \qquad (7.3.17) \\ &= I(u,z), \end{aligned}$$

where in (7.3.17) we have employed the change of variable $\gamma(x) = Mx$. Then, u and $u\circ M$ are two distinguished global minimizers of $I(\cdot, z)$, as desired. It remains to prove the nonuniqueness in case, for any radial targets, all the optimal controls are constants.

Step 2 **Existence of a special target $z^0 \in L^\infty(B(0,R))$ such that $I(\cdot, z^0)$ admits (at least) two local minimizers.**

We start proving the existence of a special target $z^0 \in L^\infty(B(0,R))$ such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

For an arbitrary target $z^0 \in L^\infty(B(0,R))$, we have $I(-1,z^0) < 0$ and $I(2,z^0) < 0$ if and only if the following system of inequalities is fulfilled:

$$\begin{cases} \beta\int_{B(0,R)} G(-1) z^0 dx > \frac{R^{n-1} n\alpha(n)}{2} |-1|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(-1)|^2 dx \\ \beta\int_{B(0,R)} G(2) z^0 dx > \frac{R^{n-1} n\alpha(n)}{2} |2|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(2)|^2 dx, \end{cases} \tag{7.3.18}$$

where G is the control-to-state map introduced in (7.3.1) and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In the sequel, we work with *changing-sign* targets

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0,r_1) \\ 0 & \text{in } B(0,r_2)\setminus B(0,r_1) \\ z_2^0 & \text{in } B(0,R)\setminus B(0,r_2). \end{cases}$$

where $(z_1^0, z_2^0) \in \mathbb{R}^2$ and $0 < r_1 \le r_2 < R$. The radius $r_1 > 0$ is fixed, while r_2 and (z_1^0, z_2^0) are degrees of freedom we need in the remainder of the proof. With the above choice of the target, inequalities (7.3.18) are satisfied if the target $\left(z_1^0, z_2^0\right)$ satisfies the linear system below

$$\begin{cases} z_1^0\beta\displaystyle\int_{B(0,r_1)} G(-1)dx + z_2^0\beta\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx = \frac{R^{n-1}n\alpha(n)}{2}|-1|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(-1)|^2dx + 1 \\ z_1^0\beta\displaystyle\int_{B(0,r_1)} G(2)dx + z_2^0\beta\int_{B(0,R)\setminus B(0,r_2)} G(2)dx = \frac{R^{n-1}n\alpha(n)}{2}|2|^2 + \frac{\beta}{2}\int_{B(0,R)} |G(2)|^2dx + 1. \end{cases} \tag{7.3.19}$$

The 2×2 coefficients matrix for the above linear system reads as:

$$M = \beta \begin{bmatrix} \displaystyle\int_{B(0,r_1)} G(-1)dx & \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx \\ \displaystyle\int_{B(0,r_1)} G(2)dx & \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(2)dx \end{bmatrix}$$

By Lemma 7.3.3, there exists $r_2 \in [r_1, R)$, such that $\text{rank}(M) = 2$. Therefore, by Rouché-Capelli Theorem, there exists a solution to the linear system (7.3.19). Such solution (z_1^0, z_2^0) defines a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0, r_1) \\ 0 & \text{in } B(0, r_2) \setminus B(0, r_1) \\ z_2^0 & \text{in } B(0, R) \setminus B(0, r_2). \end{cases}$$

such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

We show now that $I\left(\cdot, z^0\right)$ admits (at least) two local minimizers. Indeed, by Lemma 7.3.1 (1.), there exist:

$$u_1 \le 0 \quad \text{such that} \quad I(u_1, z^0) = \inf_{(-\infty,0]} [I(\cdot, z^0)]$$

and

$$u_2 \ge 0 \quad \text{such that} \quad I(u_2, z^0) = \inf_{[0,+\infty)} [I(\cdot, z^0)].$$

Now,

$$I(u_1, z^0) = \inf_{(-\infty,0]} [I(\cdot, z^0)] \le I(-1, z^0) < 0 = I(0, z^0)$$

and

$$I(u_2, z^0) = \inf_{[0,+\infty)} [I(\cdot, z^0)] \le I(2, z^0) < 0 = I(0, z^0).$$

Then, the control u_1 minimizes $I(\cdot, z^0)$ in the half line $(-\infty, 0)$, while u_2 minimizes $I(\cdot, z^0)$ in the half line $(0, +\infty)$. We have found u_1 and u_2 two distinct local minimizers of $I(\cdot, z^0)$.

Step 3 **Conclusion**

We remind the definition of h_1 and h_2 given by (7.3.3) and (7.3.4) resp. In Step 2, we have determined $z^0 \in L^\infty(B(0, R))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from Lemma 7.3.2. □

Remark 7.3.1. *The thesis of Theorem 7.1.1 holds with nonlinearity*

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(y) := y|y|. \tag{7.3.20}$$

The proofs of Theorem 7.1.1 and related lemmas remains unchanged, except the proof of Lemma 7.3.3. Indeed in this case, since f is not C^2, we cannot use condition (7.1.3) *to conclude from* (7.3.15)*. However, for any $y \in \mathbb{R}$ and for any $\lambda \in \mathbb{R}$,*

$$f(\lambda y) = \lambda y|\lambda y| = \lambda|\lambda|y|y| = |\lambda|\left[\lambda f(y)\right]. \tag{7.3.21}$$

Then, $f(\lambda y) = \lambda f(y)$ if and only if $|\lambda|\left[\lambda f(y)\right] = \lambda f(y)$. Now,

$$\left\{(y,\lambda) \in \mathbb{R}^2 \mid |\lambda|\left[\lambda f(y)\right] = \lambda f(y)\right\} = \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \{\pm 1\} \times \mathbb{R}. \tag{7.3.22}$$

Therefore,

$$\left\{(y,\lambda) \in \mathbb{R}^2 \mid f(\lambda y) = \lambda f(y)\right\} = \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \cup \{\pm 1\} \times \mathbb{R}. \tag{7.3.23}$$

In (7.3.15)*, because of 7.3.7, $\lambda \notin \{0, \pm 1\}$ and $G(-1) \neq 0$. Hence,* (7.3.23) *together with* (7.3.15) *leads to a contradiction. The rest of the proof of Theorem 7.1.1 remains unchanged.*

7.3.2 Internal control

We introduce the concept of radial control.

Definition 7.3.2. *A control $u : B(0,r) \longrightarrow \mathbb{R}$ is said to be radial if there exists $\psi : [0,r] \longrightarrow \mathbb{R}$, such that, for any $x \in B(0,r)$, we have $u(x) = \psi(\|x\|)$.*

We present our strategy to prove Theorem 7.1.2:

Step 1 **Reduction to radial controls**: by rotational invariance of $B(0,R)$ and radial targets, we reduce to the the case the control set is made of radial controls;

Step 2 **Existence of two local minimizers**: we look for a target $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ such that there exists two *local* minimizers for the steady functional J_s with target z^0;

Step 3 **Existence of two global minimizers**: by a bisection argument, we prove the existence of a target $\tilde{z} \in L^\infty(B(0,R) \setminus B(0,r))$ such the steady functional J_s with target $\tilde{z}$ admits (at least) two *global* minimizers.

Notation

First of all, we define the control-to-state map

$$\begin{aligned} G : L^2(B(0,r)) &\longrightarrow L^2(B(0,R)) \\ u &\longmapsto y_u, \end{aligned} \tag{7.3.24}$$

where y_u is the solution to (7.1.5) with control u. Then, set:

$$I : L^2(B(0,r)) \times L^\infty(B(0,R) \setminus B(0,r)) \longrightarrow \mathbb{R} \tag{7.3.25}$$
$$I(u,z) := \frac{1}{2}\int_{B(0,r)} |u|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u)|^2 dx - \beta \int_{B(0,R)\setminus B(0,r)} G(u) z dx,$$

where G is the control-to-state map introduced in (7.3.24). One recognizes that, for any target $z \in L^\infty(B(0,R) \setminus B(0,r))$, $I(\cdot, z) + \frac{\beta}{2}\|z\|^2_{L^2(B(0,R)\setminus B(0,r))}$ coincides with the functional J_s defined in (7.1.4) with target z. Then, for any target $z \in L^\infty(B(0,R) \setminus B(0,r))$ minimizing $I(\cdot, z)$ is equivalent to minimize J_s with target z. Such translation is convenient, because $I(0,z) = 0$ for any target $z \in L^\infty(B(0,R) \setminus B(0,r))$.

We define

$$\mathscr{U}_r := \left\{ u \in L^2\left(B(0,r)\right) \ \mid u \text{ is radial} \right\}. \tag{7.3.26}$$

We have

$$\mathscr{U}_r = \mathscr{U}_r^- \cup \mathscr{U}_r^-, \tag{7.3.27}$$

with

$$\begin{aligned} \mathscr{U}_r^- &:= \left\{ u \in \mathscr{U}_r \ \middle| \ G(u)\restriction_{\partial B(0,r)} \leq 0 \right\} \\ \mathscr{U}_r^+ &:= \left\{ u \in \mathscr{U}_r \ \middle| \ G(u)\restriction_{\partial B(0,r)} \geq 0 \right\}. \end{aligned} \tag{7.3.28}$$

We introduce:

$$h_1 : L^\infty(B(0,R) \setminus B(0,r)) \longrightarrow \mathbb{R}, \quad h_1(z) := \inf_{\mathscr{U}_r^-} [I(\cdot, z)] \tag{7.3.29}$$

and

$$h_2 : L^\infty(B(0,R) \setminus B(0,r)) \longrightarrow \mathbb{R}, \quad h_2(z) := \inf_{\mathscr{U}_r^+} [I(\cdot, z)]. \tag{7.3.30}$$

We formulate the first Lemma.

Lemma 7.3.4. *Let $C = \mathscr{U}_r^-$ or $C = \mathscr{U}_r^+$. Then,*

1. *for any $z \in L^\infty(B(0,R) \setminus B(0,r))$, there exists $u_z \in C$ such that:*

$$I(u_z, z) = \inf_C [I(\cdot, z)].$$

   Furthermore, for any minimizer u_z

$$\|u_z\|_{L^2(B(0,r))} \leq \sqrt{\beta}\|z\|_{L^2}.$$

2. *the map*

$$h : L^\infty(B(0,R) \setminus B(0,r)) \longrightarrow \mathbb{R}$$
$$z \longmapsto \inf_C [I(\cdot, z)]$$

   is continuous.

We prove Lemma 7.3.4 in the Appendix.

We now state the second lemma needed to prove Theorem 7.1.2.

Lemma 7.3.5. *Assume there exists* $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ *such that*

$$h_1(z^0) < 0 \qquad \text{and} \qquad h_2(z^0) < 0,$$

where h_1 *and* h_2 *are defined in* (7.3.29) *and* (7.3.30) *resp. Then, there exists* $\tilde{z} \in L^\infty(B(0,R) \setminus B(0,r))$ *such that*

$$h_1(\tilde{z}) = h_2(\tilde{z}) < 0.$$

The proof of this lemma can be found in the Appendix. The next lemma is the foundation of the proof of two local minimizers for (7.1.4). The nonlinearity of the state equation (7.1.5) will play a key role in the proof.

Lemma 7.3.6. *Suppose*

$$f''(y) \neq 0 \qquad \forall\, y \neq 0.$$

Arbitrarily fix r_1*, such that* $r < r_1 < R$*. For any* $r_2 \in [r_1, R)$*, let*

$$M = \beta \begin{bmatrix} \displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(-1)dx & \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx \\ \displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(2)dx & \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(2)dx \end{bmatrix}.$$

There exists $r_2 \in [r_1, R)$*, such that rank*$(M) = 2$.

Proof of Lemma 7.3.6. Let us assume, by contradiction, that for any $r_2 \in \mathbb{R}$, with $0 < r_1 \leq r_2 < R$ there exists $\lambda \in \mathbb{R}$ such that

$$\begin{bmatrix} \displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(2)dx \\ \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(2)dx \end{bmatrix} = \lambda \begin{bmatrix} \displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(-1)dx \\ \displaystyle\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx \end{bmatrix}. \tag{7.3.31}$$

By strong maximum principle [57, Theorem 8.19 page 198],

$$\begin{aligned} &\int_{B(0,r_1)\setminus B(0,r)} G(2)dx > 0 \qquad &&\int_{B(0,R)\setminus B(0,r_2)} G(2)dx > 0 \\ &\int_{B(0,r_1)\setminus B(0,r)} G(-1)dx < 0 \qquad &&\int_{B(0,R)\setminus B(0,r_2)} G(-1)dx < 0. \end{aligned} \tag{7.3.32}$$

From (7.3.31), we realize that

$$\lambda = \frac{\displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(2)dx}{\displaystyle\int_{B(0,r_1)\setminus B(0,r)} G(-1)dx}, \tag{7.3.33}$$

which yields the independence of λ from r_2.

By (7.3.31), for any $r_2 \in [r_1, R)$,

$$\int_{B(0,R)\setminus B(0,r_2)} G(2)dx = \lambda \int_{B(0,R)\setminus B(0,r_2)} G(-1)dx, \tag{7.3.34}$$

whence

$$\int_{B(0,R)\setminus B(0,r_2)} [G(2) - \lambda G(-1)]\, dx = 0, \quad \forall\; r_2 \in [r_1, R). \tag{7.3.35}$$

Now, the constant controls -1 and 2 are radial, whence the states $G(-1)$ and $G(2)$ are radial as well. Then, the above inequality together with measure theory yields

$$G(2) = \lambda G(-1), \quad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.36}$$

Then, by definition of the control operator, we have

$$-\Delta\left(G(-1)\right) + f(G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1) \tag{7.3.37}$$

and

$$-\Delta\left(G(2)\right) + f(G(2)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.38}$$

plugging $\lambda G(-1)$ in (7.3.38), we obtain

$$-\Delta\left(\lambda G(-1)\right) + f(\lambda G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1) \tag{7.3.39}$$

Similarly, multiplying (7.3.37) by λ, we get

$$-\Delta\left(\lambda G(-1)\right) + \lambda f(G(-1)) = 0 \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.40}$$

Subtracting (7.3.39) and (7.3.40), we obtain

$$f(\lambda G(-1)) = \lambda f(G(-1)) \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.41}$$

Now, the dimension of the domain $B(0,r)$ is $n = 1, 2, 3$. Then, we have the continuous embedding $H^2(B(0,r)) \hookrightarrow C^0\left(\overline{B(0,r)}\right)$. By strong maximum principle [57, Theorem 8.19 page 198], $G(-1) < 0$ in $B(0,R)$. Hence

$$-\Delta G(-1) = -f(G(-1)) > 0 \qquad \text{in } B(0,R) \setminus B(0,r_1). \tag{7.3.42}$$

Therefore, $G(-1)$ is not constant in $B(0,R) \setminus B(0,r_1)$. Now, being $G(-1)$ non constant in $B(0,R) \setminus B(0,r_1)$, (7.3.41) leads to a contradiction with (7.1.6). □

We are now ready to prove Theorem 7.1.2.

Proof of Theorem 7.1.2. Step 1 **Reduction to radial controls.**
Suppose for some radial target z, the optimal control u is not radial, that is there exists an orthogonal matrix M, such that $u \circ M \neq u$. Now,

$$\begin{aligned} I\left(u \circ M, z\right)) &= \frac{1}{2}\int_{B(0,r)} |u \circ M|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u \circ M)|^2 dx - \beta \int_{B(0,R)\setminus B(0,r)} G(u \circ M) z dx \\ &= \frac{1}{2}\int_{B(0,r)} |u|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u)|^2 dx - \beta \int_{B(0,R)\setminus B(0,r)} G(u) z dx \qquad (7.3.43) \\ &= I(u,z), \end{aligned}$$

where in the last equality (7.3.43) we have employed the change of variable $\gamma(x) = Mx$. Then, u and $u \circ M$ are two distinguished global minimizers to $I\left(\cdot, z\right)$, as desired. It remains to prove the nonuniqueness in case, for any radial target, all the optimal controls are radial. Hereafter, for a radial target z, we will consider the restriction of the functional $I(\cdot, z)$ to $\mathscr{U}_r$.
Step 2 **Existence of a special target $z^0 \in L^\infty(B(0,R)\setminus B(0,r))$ such that $I(\cdot, z^0)$ admits (at least) two local minimizers.**
We start proving the existence of a special target $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

For an arbitrary target $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$, we have $I(-1, z^0) < 0$ and $I(2, z^0) < 0$ if and only if the following system of inequalities is fulfilled:

$$\begin{cases} \beta \int_{B(0,R)\setminus B(0,r)} G(-1) z^0 dx > \frac{r^n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(-1)|^2 dx \\ \beta \int_{B(0,R)\setminus B(0,r)} G(2) z^0 dx > \frac{r^n \alpha(n)}{2} |2|^2 + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(2)|^2 dx, \end{cases} \qquad (7.3.44)$$

where G is the control-to-state map introduced in (7.3.24) and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. In the sequel, we work with *changing-sign* targets

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0,r_1) \setminus B(0,r) \\ 0 & \text{in } B(0,r_2) \setminus B(0,r_1) \\ z_2^0 & \text{in } B(0,R) \setminus B(0,r_2). \end{cases}$$

where $(z_1^0, z_2^0) \in \mathbb{R}^2$ and $0 < r < r_1 \leq r_2 < R$. The pair of radii (r_1, r) is fixed, while r_2 and (z_1^0, z_2^0) are degrees of freedom we need in the remainder of the proof. With the above choice of the target, inequalities (7.3.44) are satisfied if the target $\left(z_1^0, z_2^0\right)$ satisfies the linear system below

$$\begin{cases} z_1^0 \beta \int_{B(0,r_1)} G(-1) dx + z_2^0 \beta \int_{B(0,R)\setminus B(0,r_2)} G(-1) dx = \frac{r^n \alpha(n)}{2} |-1|^2 + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(-1)|^2 dx + 1 \\ z_1^0 \beta \int_{B(0,r_1)} G(2) dx + z_2^0 \beta \int_{B(0,R)\setminus B(0,r_2)} G(2) dx = \frac{r^n \alpha(n)}{2} |2|^2 + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(2)|^2 dx + 1. \end{cases} \qquad (7.3.45)$$

The 2×2 coefficients matrix for the above linear system reads as:

$$M = \beta \begin{bmatrix} \int_{B(0,r_1)} G(-1)dx & \int_{B(0,R)\setminus B(0,r_2)} G(-1)dx \\ \int_{B(0,r_1)} G(2)dx & \int_{B(0,R)\setminus B(0,r_2)} G(2)dx \end{bmatrix}$$

By Lemma 7.3.6 there exists $r_2 \in [r_1, R)$, such that $\operatorname{rank}(M) = 2$. Therefore, by Rouché-Capelli Theorem, there exists a solution to the linear system (7.3.45). Such solution (z_1^0, z_2^0) defines a special target

$$z^0 := \begin{cases} z_1^0 & \text{in } B(0,r_1) \setminus B(0,r) \\ 0 & \text{in } B(0,r_2) \setminus B(0,r_1) \\ z_2^0 & \text{in } B(0,R) \setminus B(0,r_2). \end{cases}$$

such that $I(-1, z^0) < 0$ and $I(2, z^0) < 0$.

We show now that $I\left(\cdot, z^0\right)$ admits (at least) two local minimizers. Indeed, the set $\mathscr{U}_r$ (introduced in 7.3.26) splits

$$\mathscr{U}_r = \mathscr{U}_r^- \cup \mathscr{U}_r^+, \tag{7.3.46}$$

with

$$\begin{aligned} \mathscr{U}_r^- &= \left\{u \in \mathscr{U}_r \;\middle|\; G(u)\restriction_{\partial B(0,r)} \leq 0\right\} \\ \mathscr{U}_r^+ &= \left\{u \in \mathscr{U}_r \;\middle|\; G(u)\restriction_{\partial B(0,r)} \geq 0\right\}. \end{aligned} \tag{7.3.47}$$

By Lemma 7.3.4 (1.), there exist:

$$u_1 \in \mathscr{U}_r^- \quad \text{such that} \quad I(u_1, z^0) = \inf_{\mathscr{U}_r^-}[I(\cdot, z^0)]$$

and

$$u_2 \in \mathscr{U}_r^+ \quad \text{such that} \quad I(u_2, z^0) = \inf_{\mathscr{U}_r^+}[I(\cdot, z^0)].$$

Now, for any control $u \in \left\{u \in \mathscr{U}_r \;\middle|\; G(u)\restriction_{\partial B(0,r)} = 0\right\}$, we have

$$I(u_1, z^0) = \inf_{\mathscr{U}_r^-}[I(\cdot, z^0)] \leq I(-1, z^0) < 0 \leq I(u, z^0)$$

and

$$I(u_2, z^0) = \inf_{\mathscr{U}_r^+}[I(\cdot, z^0)] \leq I(2, z^0) < 0 \leq I(u, z^0).$$

Then, necessarily u_1 is a local minimizer for $I(\cdot, z^0)$ in the open set $\left\{u \in \mathscr{U}_r \;\middle|\; G(u)\restriction_{\partial B(0,r)} < 0\right\}$ and u_2 is a local minimizer for $I(\cdot, z^0)$ in the open set $\left\{u \in \mathscr{U}_r \;\middle|\; G(u)\restriction_{\partial B(0,r)} > 0\right\}$. Hence, we have found u_1 and u_2 two distinct local minimizers.

Step 3 **Conclusion**

We remind the definition of h_1 and h_2 given by (7.3.29) and (7.3.30) resp. In Step 2, we have determined $z^0 \in L^\infty(B(0,R) \setminus B(0,r))$ such that $h_1(z^0) < 0$ and $h_2(z^0) < 0$. To finish our proof it suffices to find $\tilde{z} \in \mathbb{R}^n$ such that $h_1(\tilde{z}) = h_2(\tilde{z}) < 0$. This follows from Lemma 7.3.5. □

The conclusions of Remark 7.3.1 holds as well in internal control.

7.4 Numerical simulations

We have performed a numerical simulation in the context of boundary control

- space dimension $n = 1$ and radius $R = 1$;
- nonlinearity $f(y) = y^3$;
- weighting parameter $\beta = 1$;
- step target

$$z := \begin{cases} 410000 & \text{in } \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right) \\ -10300000 & \text{in } \left(\frac{1}{4}, \frac{3}{4}\right). \end{cases}$$

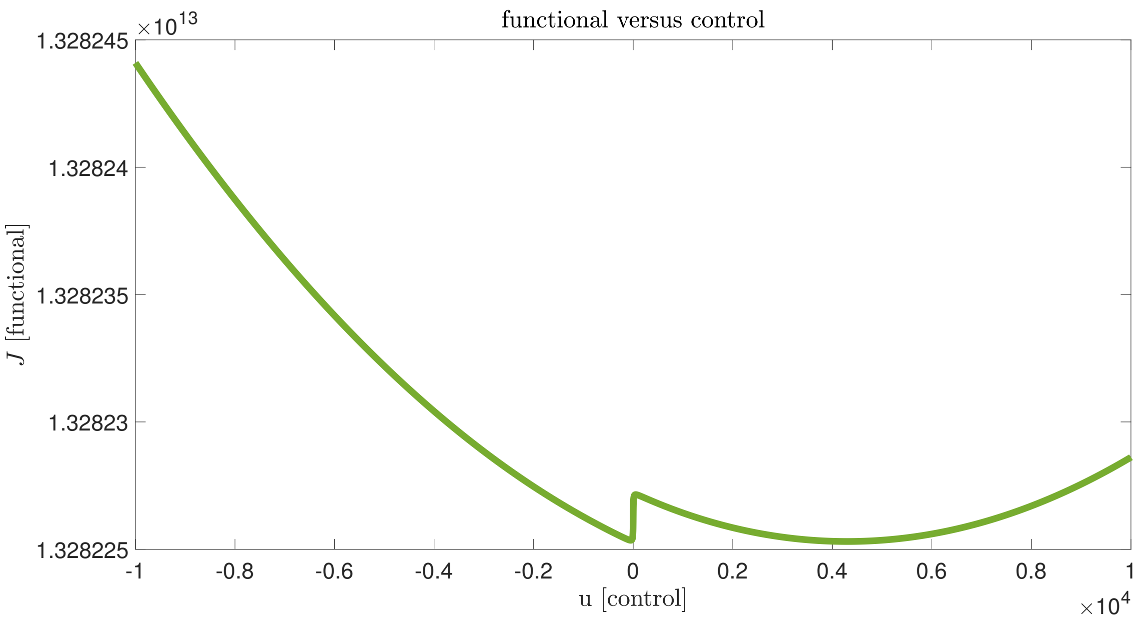


Figure 7.4: steady functional versus control (non-uniqueness of the *global* minimizer). This plot is obtained by drawing in MATLAB the graph of J_s defined in (8.3.2), with $R = 1$ and nonlinearity $f(y) = y^3$. The target $z = 410000\chi_{(0,\frac{1}{4})\cup(\frac{3}{4},1)} - 10300000\chi_{(\frac{1}{4},\frac{3}{4})}$.

As we have seen in the proof of Theorem 7.1.1, we can reduce to the case of constant controls on the boundary. In our case, the space dimension is $n = 1$. Then,we can then reduce to the case the same control acts on both endpoints $x = 0$ and $x = 1$. Hence, we plot in figure 7.4 the restriction $J_s\restriction_{\mathbb{R}} \mathbb{R} \longrightarrow \mathbb{R}$, where J_s as defined in (7.1.1)

There exist two distinguished global minimizers:

- a negative one $u_1 \cong -50$;
- a positive one $u_2 \cong 4298$.

The corresponding optimal states are depicted in figures 7.5 and 7.6.

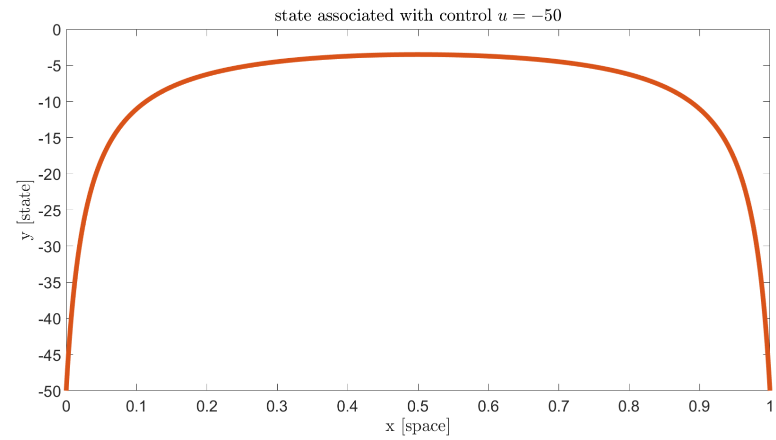


Figure 7.5: state associated with control $u = -50$.

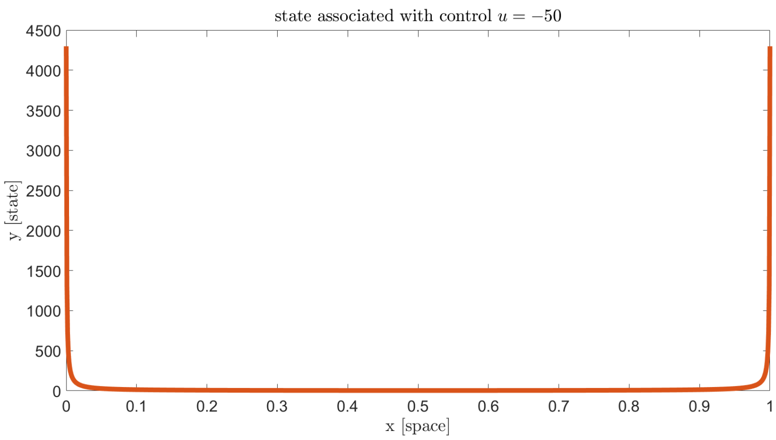


Figure 7.6: state associated with control $u = 4298$.

The idea behind this example is that two optimal strategies are available:

- take a large positive control u_2 to better approximate the target in $(0, \frac{1}{2})$;
- take a negative control u_1 to keep the state closer to the target in $(\frac{1}{2}, 1)$.

Note that $|u_1| < |u_2|$. Indeed, the control domain is $(0, \frac{1}{2})$. Then, the effect of the control is stronger in $(0, \frac{1}{2})$ than in $(\frac{1}{2}, 1)$. For this reason, it is worth to take a large positive control to better approximate the target in $(0, \frac{1}{2})$. On the other hand, it is less convenient to take a very negative control to approximate the target in $(\frac{1}{2}, 1)$.

Numerical simulations are performed in MATLAB. We explain now the numerical methods employed.

Firstly choose an interval of controls $[-M, M]$, where to study the functional J_s. Then, our goal is to plot $J_s\restriction_{[-M,M]} \colon [-M, M] \longrightarrow \mathbb{R}$.

For the interval $[-M, M]$, we choose an equi-spaced grid $v_i = -M + (i-1)\frac{2M}{N_c-1}$, with $i = 1, \ldots, N_c$ and $N_c \in \mathbb{N} \setminus \{0\}$.

Now, for each control v_i, we need to find numerically the corresponding state y_i, solution to the following PDE with cubic nonlinearity

$$\begin{cases} -(y_i)_{xx} + (y_i)^3 = 0 \quad & x \in (0,1) \\ y_i(0) = y_i(1) = v_i. \end{cases} \tag{7.4.1}$$

Following [15, subsubsection 4.3.2], we solve (7.4.1) by a fixed-point type algorithm with relaxation. Namely, in any iteration k, we determine the solution $y_{i,k}$ to the linear PDE

$$\begin{cases} -(y_{i,k})_{xx} + (\theta_{i,k-1})^2 y_{i,k} = 0 \quad & x \in (0,1) \\ y_{i,k}(0) = y_{i,k}(1) = v_i \end{cases} \tag{7.4.2}$$

and we set $\theta_k \coloneqq \frac{1}{2}\theta_{i,k-1} + \frac{1}{2}y_k$. The initial guess $\theta_{i,0}$ is taken to be y_{i-1}, i.e. the solution to (7.4.1), with control v_{i-1}.

To compute the solution to the linear PDE (7.4.2), we choose a finite difference scheme with uniform space grid $x_j = \frac{j-1}{\Delta x}$, where $j = 1, \ldots, N_x$, $N_x \in \mathbb{N} \setminus \{0\}$ and $\Delta x \coloneqq \frac{1}{N_x - 1}$. Then, $y_{i,k} = \left(y_{i,k,j}\right)_j$ is a N_x-dimensional discrete vector solution to

$$\begin{cases} \frac{-y_{i,k,j-1} + 2y_{i,k,j} - y_{i,k,j+1}}{(\Delta x)^2} + (\theta_{i,k-1,j})^2 y_{i,k,j} = 0 \quad & j = 2, \ldots, N_x - 1 \\ y_{i,k,1} = y_{i,k,N_x} = v_i. \end{cases}$$

Once we have determined the state y_i, we evaluate the functional J_s at the control v_i. The integral appearing in (7.1.1) can be computed by quadrature methods. We are now in position to plot the functional $J_s\restriction_{[-M,M]} \colon [-M, M] \longrightarrow \mathbb{R}$.

Note that, as long as we know, the actual convergence of the fixed-point method described has not been proved. However, for any control v_i, we are able to check that the state computed solves the finite difference version of the nonlinear problem (7.4.1) up to a small error.

An extensive literature is available on the numerical approximation of solutions to (7.4.1) (see, for instance, [58] for a survey). Let us mention two alternative numerical methods. The first one is a finite difference-Newton method presented in [88, subsection 2.16.1]. The

idea is to discretize directly (7.4.1). This leads to a nonlinear equation in finite dimension. This equation can be solved by a Newton method.
Another option is to find the solution to (7.4.1), by minimizing the convex functional

$$K(y) = \frac{1}{2}\int_0^1 |y_x|^2 dx + \frac{1}{4}\int_0^1 y^4 dx \tag{7.4.3}$$

over the affine space

$$\mathcal{A} := \left\{ y \in H^1(0,1) \mid y(0) = y(1) = v \right\}. \tag{7.4.4}$$

Appendix

Proof of Lemma 7.2.1

As announced, we prove Lemma 7.2.1.

Proof of Lemma 7.2.1. If G is affine, (7.2.1) holds.

Now, let us suppose (7.2.1) is verified.

Step 1 **Definition of the intercept and $\hat{G}$**

We set $d := G(0)$ and

$$\hat{G} : V_1 \longrightarrow V_2$$
$$v \longmapsto G(v) - b = G(v) - G(0).$$

It remains to prove that $\hat{G}$ is linear.

Step 2 **Proof of: $\hat{G}(\alpha v) = \alpha \hat{G}(v)$, for any v in V and $\alpha \in \mathbb{R}$**

First of all, for any $v \in V_1$,

$$\frac{1}{2}\hat{G}(v) + \frac{1}{2}\hat{G}(-v) = \hat{G}\left(\frac{1}{2}v - \frac{1}{2}v\right) = \hat{G}(0) = 0.$$

Then, for every $v \in V_1$,

$$\hat{G}(-v) = -\hat{G}(v). \tag{7.4.5}$$

Secondly, for each $\alpha \in [0,1]$,

$$\hat{G}(\alpha v) = \hat{G}(\alpha v + (1-\alpha)0) = \alpha \hat{G}(v) + (1-\alpha)\hat{G}(0) = \alpha \hat{G}(v). \tag{7.4.6}$$

Finally, for any $\alpha > 1$ and $v \in V_1$:

$$\hat{G}(v) = \hat{G}\left(\frac{\alpha v}{\alpha}\right) = \frac{1}{\alpha}\hat{G}(\alpha v).$$

Then, for any $\alpha > 1$,

$$\hat{G}(\alpha v) = \alpha \hat{G}(v), \qquad \forall\, v \in V_1. \tag{7.4.7}$$

Combining (7.4.5), (7.4.6) and (7.4.7), one has

$$\hat{G}(\alpha v) = \alpha \hat{G}(v), \qquad \forall\, v \in V_1 \quad \text{and} \quad \forall\, \alpha \in \mathbb{R}.$$

Step 3 $\hat{G}(v+w) = \hat{G}(v) + \hat{G}(w), \qquad \forall\ (v,w) \in V_1^2$
For any $(v,w) \in V_1^2$,

$$\hat{G}(v+w) = \hat{G}\left(\frac{1}{2}(2v) + \frac{1}{2}(2w)\right)$$
$$= \frac{1}{2}\hat{G}(2v) + \frac{1}{2}\hat{G}(2w) = \hat{G}(v) + \hat{G}(w),$$

where the second equality follows from (7.2.1) and the third equality comes from the homogeneity proved in the first step.

Combining Step 2 and Step 3, we conclude.

□

Proof of Lemma 7.3.1

We prove Lemma 7.3.1.

Proof of Lemma 7.3.1. Step 1 **Proof of 1.**
Arbitrarily fix $z \in L^\infty(B(0,R))$. The existence of a minimizer u_z is a consequence of the direct methods in the Calculus of Variations. Moreover, by (7.3.2), definition of minimizer and $G(0) = 0$:

$$\frac{1}{2}R^{n-1}n\alpha(n)|u_z|^2 \le I(u_z,z) + \frac{\beta}{2}\int_{B(0,R)}|z|^2dx \le I(0,z) + \frac{\beta}{2}\int_{B(0,R)}|z|^2dx = \frac{\beta}{2}\int_{B(0,R)}|z|^2dx,$$

which yields $\frac{1}{2}|u_z|^2 \le \frac{\beta}{2R^{n-1}n\alpha(n)}\int_{B(0,R)}|z|^2dx$, as required.
Step 2 **Proof of 2.**
Arbitrarily fix $M \in \mathbb{R}^+$. For any pair of targets $(z_1,z_2) \in L^\infty(B(0,R))^2$ such that:

$$\|z_1\|_{L^2} \le M \qquad \text{and} \qquad \|z_2\|_{L^2} \le M.$$

For each control $u \in C$ such that $|u| \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}M$, we have:

$$I(u,z_2) - I(u_{z_1},z_1) = I(u,z_2) - I(u,z_1) + I(u,z_1) - I(u_{z_1},z_1)$$
$$\ge -|I(u,z_2) - I(u,z_1)| + 0 = -\beta\left|\int_{B(0,R)}G(u)(z_1-z_2)dx\right|$$
$$\ge -K\|z_2 - z_1\|_{L^\infty},$$

where the last inequality is justified by $|u| \le \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}M$ and the continuity of the control-to-state map G.

Then, one has that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that:

$$I(u,z_2) - I(u_{z_1},z_1) > -\varepsilon,$$

whenever $\|z_2 - z_1\|_{L^\infty} < \delta_\varepsilon$.

Now, by the first step, any minimizer u_{z_2} for $I(\cdot,z_2)$ verifies $|u_{z_2}| \leq \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}\|z_2\|_{L^2} \leq \sqrt{\frac{\beta}{R^{n-1}n\alpha(n)}}M$. Then, we have proved that:

$$\inf_C[I(\cdot,z_2)] - \inf_C[I(\cdot,z_1)] = I(u_{z_2},z_2) - I(u_{z_1},z_1) > -\varepsilon.$$

Exchanging the role of z_1 and z_2, one can get:

$$\inf_C[I(\cdot,z_1)] - \inf_C[I(\cdot,z_2)] > -\varepsilon.$$

This yields the continuity of h. □

Proof of Lemma 7.3.2

We prove Lemma 7.3.2.

Proof of Lemma 7.3.2. If $h_1(z^0) = h_2(z^0)$, we take $\tilde{z} := z^0$, thus concluding. Let us now suppose $h_1(z^0) \neq h_2(z^0)$.

We start by considering the case $h_1(z^0) < h_2(z^0)$.

Step 1 **Proof of the existence of $\mu_0 \geq 0$ such that:**

- $\forall \mu \in [0,\mu_0]$, $h_2(z^0+\mu) < 0$;
- $h_1\left(z^0+\mu_0\right) = 0$.

First of all, we observe that for any $\mu \geq 0$, $h_2(z^0+\mu) < 0$. Indeed, since $h_2(z^0) < 0$, there exists $u_2 > 0$ such that $I(u_2,z^0) < 0$. Then,

$$h_2(z^0+\mu) \leq I(u_2,z^0+\mu) = \frac{R^{n-1}n\alpha(n)}{2}|u_2|^2 + \frac{\beta}{2}\int_{B(0,R)}|G(u_2)|^2dx - \beta\int_{B(0,R)}(z^0+\mu)G(u_2)dx$$

$$= I(u_2,z^0) - \mu\beta\int_{B(0,R)}G(u_2)dx \leq I(u_2,z^0) < 0,$$

where we have used that $G(u_2) \geq 0$ a.e. in $B(0,r)$.

We prove now that $h_1\left(z^0+\mu_0\right) = 0$, for $\mu_0 = \|z^0\|_{L^\infty}$. Indeed, for any $v \leq 0$:

$$I(v,z^0+\mu_0) = \frac{R^{n-1}n\alpha(n)}{2}|v|^2 + \frac{\beta}{2}\int_{B(0,R)}|G(v)|^2dx - \beta\int_{B(0,R)}(z^0+\mu_0)G(v)dx \geq 0,$$

since $z^0+\mu_0 \geq 0$ and $G(v) \leq 0$ a.e. in $B(0,r)$. This finishes the first step.

Step 2 **Conclusion**

Set:

$$g : [0,\mu_0] \longrightarrow \mathbb{R}$$

$$\mu \longmapsto h_2(z^0+\mu) - h_1(z^0+\mu).$$

Since $h_1(z^0) < h_2(z^0)$, $g(0) > 0$ and by Step 1 $g(\mu_0) < 0$. Then, by continuity, there exists $\mu_1 \in (0,\mu_0)$ such that $g(\mu_1) = 0$. Hence,

$$\tilde{z} := z^0 + \mu_1$$

is the desired target. Indeed, by definition of g and μ_1, $h_1(\tilde{z}) = h_2(\tilde{z})$. Furthermore, since $\mu_1 \in (0, \mu_0)$, by Step 1, $h_2(\tilde{z}) < 0$. This concludes the proof for the case $h_1(z^0) < h_2(z^0)$. The proof for the remaining case $h_1(z^0) > h_2(z^0)$ is similar. □

Proof of Lemma 7.3.4

We prove Lemma 7.3.4.

Proof of Lemma 7.3.4. Step 1 **Proof of 1.**
Arbitrarily fix $z \in L^\infty(B(0,R) \setminus B(0,r))$. The existence of a minimizer u_z is a consequence of the direct methods in the Calculus of Variations. Moreover, by (7.3.25), definition of minimizer and $G(0) = 0$:

$$\frac{1}{2}\int_{B(0,r)} |u_z|^2 dx \leq I(u_z, z) + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |z|^2 dx \leq I(0,z) + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |z|^2 dx = \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |z|^2$$

which yields $\frac{1}{2}\int_{B(0,r)} |u_z|^2 dx \leq \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |z|^2 dx$, as required.
Step 2 **Proof of 2.**
Arbitrarily fix $M \in \mathbb{R}^+$. For any pair of targets $(z_1, z_2) \in L^\infty(B(0,R) \setminus B(0,r))^2$ such that:

$$\|z_1\|_{L^2} \leq M \qquad \text{and} \qquad \|z_2\|_{L^2} \leq M.$$

For each control $u \in C$ such that $\|u_z\|_{L^2(B(0,r))} \leq \sqrt{\beta} M$, we have:

$$\begin{aligned} I(u, z_2) - I(u_{z_1}, z_1) &= I(u, z_2) - I(u, z_1) + I(u, z_1) - I(u_{z_1}, z_1) \\ &\geq -|I(u, z_2) - I(u, z_1)| + 0 = -\beta \left| \int_{B(0,R)\setminus B(0,r)} G(u)(z_1 - z_2) dx \right| \\ &\geq -K\|z_2 - z_1\|_{L^\infty}, \end{aligned}$$

where the last inequality is justified by $\|u_z\|_{L^2(B(0,r))} \leq \sqrt{\beta} M$, the continuous embedding $H^2(B(0,r)) \hookrightarrow C^0\left(\overline{B(0,r)}\right)$ and the continuity of the control-to-state map $G : L^2(B(0,r)) \longrightarrow H^2(B(0,r))$.

Then, one has that for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that:

$$I(u, z_2) - I(u_{z_1}, z_1) > -\varepsilon,$$

whenever $\|z_2 - z_1\|_{L^\infty} < \delta_\varepsilon$.

Now, by the first step, any minimizer u_{z_2} for $I(\cdot, z_2)$ verifies
$|u_{z_2}| \leq \sqrt{\beta}\|z_2\|_{L^2} \leq \sqrt{\beta} M$. Then, we have proved that:

$$\inf_C [I(\cdot, z_2)] - \inf_C [I(\cdot, z_1)] = I(u_{z_2}, z_2) - I(u_{z_1}, z_1) > -\varepsilon.$$

Exchanging the role of z_1 and z_2, one can get:

$$\inf_C [I(\cdot, z_1)] - \inf_C [I(\cdot, z_2)] > -\varepsilon.$$

This yields the continuity of h. □

Proof of Lemma 7.3.5

We prove Lemma 7.3.5.

Proof of Lemma 7.3.5. If $h_1(z^0) = h_2(z^0)$, we take $\tilde{z} := z^0$, thus concluding. Let us now suppose $h_1(z^0) \neq h_2(z^0)$.

We start by considering the case $h_1(z^0) < h_2(z^0)$.

Step 1 **Proof of the existence of $\mu_0 \geq 0$ such that:**

- $\forall \mu \in [0, \mu_0]$, $h_2(z^0 + \mu) < 0$;
- $h_1\left(z^0 + \mu_0\right) = 0$.

First of all, we observe that for any $\mu \geq 0$, $h_2(z^0 + \mu) < 0$. Indeed, since $h_2(z^0) < 0$, there exists $u_2 \in \mathscr{U}_r^+ \setminus \{0\}$ such that $I(u_2, z^0) < 0$. Then,

$$h_2(z^0+\mu) \leq I(u_2, z^0+\mu) = \frac{1}{2}\int_{B(0,r)} |u_2|^2 dx + \frac{\beta}{2}\int_{B(0,R)} |G(u_2)|^2 dx - \beta \int_{B(0,R)\setminus B(0,r)} (z^0+\mu) G(u_2) dx$$

$$= I(u_2, z^0) - \mu\beta \int_{B(0,R)\setminus B(0,r)} G(u_2) dx \leq I(u_2, z^0) < 0,$$

where we have used that $G(u_2) \geq 0$ a.e. in $B(0,R) \setminus B(0,r)$.

We prove now that $h_1\left(z^0 + \mu_0\right) = 0$, for $\mu_0 = \|z^0\|_{L^\infty}$. Indeed, for any $u \in \mathscr{U}_r^-$:

$$I(u, z^0+\mu_0) = \frac{1}{2}\int_{B(0,r)} |u|^2 dx + \frac{\beta}{2}\int_{B(0,R)\setminus B(0,r)} |G(u)|^2 dx - \beta \int_{B(0,R)\setminus B(0,r)} (z^0+\mu_0) G(u) dx \geq 0,$$

since $z^0 + \mu_0 \geq 0$ and $G(u) \leq 0$ a.e. in $B(0,R) \setminus B(0,r)$. This finishes the first step.

Step 2 **Conclusion**

Set:

$$g : [0, \mu_0] \longrightarrow \mathbb{R}$$
$$\mu \longmapsto h_2(z^0 + \mu) - h_1(z^0 + \mu).$$

Since $h_1(z^0) < h_2(z^0)$, $g(0) > 0$ and by Step 1 $g(\mu_0) < 0$. Then, by continuity, there exists $\mu_1 \in (0, \mu_0)$ such that $g(\mu_1) = 0$. Hence,

$$\tilde{z} := z^0 + \mu_1$$

is the desired target. Indeed, by definition of g and μ_1, $h_1(\tilde{z}) = h_2(\tilde{z})$. Furthermore, since $\mu_1 \in (0, \mu_0)$, by Step 1, $h_2(\tilde{z}) < 0$. This concludes the proof for the case $h_1(z^0) < h_2(z^0)$. The proof for the remaining case $h_1(z^0) > h_2(z^0)$ is similar. □

Chapter 8

Rotors imbalance suppression by optimal control

This chapter corresponds to [59] and it is part of the outcome of a secondment in the company "Marposs S.p.A.".

8.1 Introduction

Imbalance vibration affects several rotor dynamic systems. Indeed, often times, rotors' mass distribution is not homogeneous, due to wear, damage and other reasons. The purpose of this paper is to present a control theoretical approach to rotors imbalance suppression. A balancing device, made of moving masses, is given. We look for the optimal movement of a system of balancing masses to minimize the vibrations.

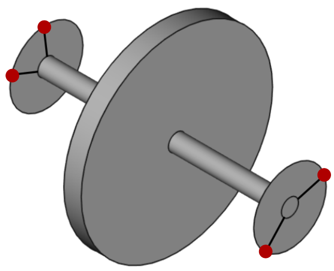


Figure 8.1: the rotor and the balancing device are represented. In the special case represented, the balancing heads are located at the endpoints of the spindle. The four balancing masses (two for each balancing head) are drawn in red.

The topic is very classical in engineering literature. Indeed, balancing devices are ubiquitous in rotor dynamic systems. For instance, grinding machines often get deteriorated during their operational life-cycle. This leads to dangerous imbalance vibrations, which affects their performances while shaping objects (see, for instance, [68, 75, 149, 30]). Imbalance is a significant concern for wind turbines as well. In this case, the imbalance may affect the efficiency of power production and the life-cycle of the turbine. If the vibrations become too large, the turbine may collapse. For this reason, vibration detection and correction systems have been developed (see the U.S. patent [80]). Balancing devices have been developed to stabilize CD-ROM drives and washing machines (see [32, 119, 28, 29, 82]). Another classical topic in engineering is car's wheels balance. Indeed, easily the wheels can go out of alignment from encountering potholes and/or striking raised objects. Misalignment may cause irregular wear of the tyres. Suspensions components may be damaged as well. For this reason, refined machines have been designed for wheel balancing (see, e.g., [46, chapter 44]). The classical engineering literature on imbalance suppression is concerned with imbalance detection and/or imbalance correction.

In the present chapter, we address the imbalance correction problem. The imbalance is an input. We consider an imbalanced rotor rotating about a fixed axis at constant angular velocity. We work in the general case of dynamical imbalance, where the imbalanced rotor exert both a force and a torque on the rotation axle. In this context, we suppose that two balancing heads are mounted along two planes orthogonal to the rotation axis. It is assumed that the balancing heads are integral with the rotor, i.e. they rotate together with the rotor. Each balancing head is made of two masses, free to rotate about the rotation axis. Their angular movements are measured with respect to a rotor-fixed reference frame.
An initial configuration of the balancing masses is given. Our goal is to determine four angular trajectories steering the masses from their initial configuration to a steady configuration, where the balancing masses compensate the imbalance. Note that, differently from the classical wheel balancing machines, our balancing device rotates together with the rotor and the rotor is moving while the balancing procedure is accomplished. This motivates us to formulate the problem as a dynamic optimization problem so that transient responses are also taken into account.

A control problem is formulated. We exhibit an open-loop control strategy to move the balancing heads from their initial configuration to a steady configuration, where they compensate the imbalance of the rotor. First of all, viewing the problem in the framework of the Calculus of Variations, the existence of the optimum is proved and the related Euler-Lagrange optimality conditions have been derived. By Łojasiewicz inequality, the stabilization of the optimal trajectories towards steady optima is proved in any condition. In case the imbalance is below a given threshold, we provide an exponential estimate of the stabilization. The estimate is obtained, by seeing the problem as an optimal control problem, thus writing the Optimality Condition as a first order Pontryagin system. In this context, we prove the hyperbolicity of the Pontryagin system around steady optima, to apply the stable manifold theorem (see [109, Corollary page 115] and [129]). Our conclusions fit in the general framework of Control Theory and, in particular, of stabilization, turnpike and controllability (see e.g.

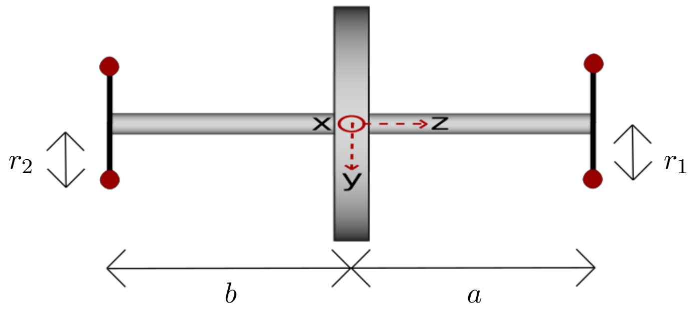


Figure 8.2: front view of the system made of rotor and balancing device.

[54, 134, 153, 115, 138, 151]).

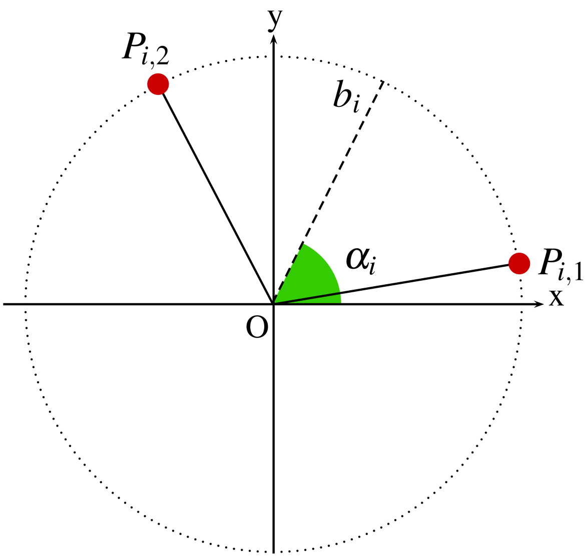
The remainder of the manuscript is organized as follows. In section 8.2, we conceive a physical model of the rotor together with the balancing device. In section 8.3, we formulate a control problem to determine stabilizing trajectories for the balancing masses. We summarize our achievements in Proposition 8.3.1. The steady problem is analyzed in subsection 8.3.2, where the steady optima are determined. In subsection 8.3.3, we prove some general results. In Proposition 8.3.3, the existence of the global minimizer is proved. In Proposition 8.3.4, the Optimality Conditions are deduced in the form of Euler-Lagrange equations or equivalently as a state-adjoint state Pontryagin system. In Proposition 8.3.5 the asymptotic behaviour of the optima is analyzed in the spirit of stabilization and turnpike theory (see [115, 138, 129]). The Łojasiewicz inequality is employed to show that, in any condition, the optima stabilize towards a steady configuration. In case the imbalance does not violate a computed threshold, the stabilization is exponentially fast. This is shown as a consequence of the hyperbolicity of the Pontryagin system around steady optima and the stable manifold theorem. Numerical simulations are performed in subsection 8.3.5. The exponential stabilization of the optima emerges, thus validating the theoretical results. The notation is introduced at the end of the Appendix.

8.2 The model

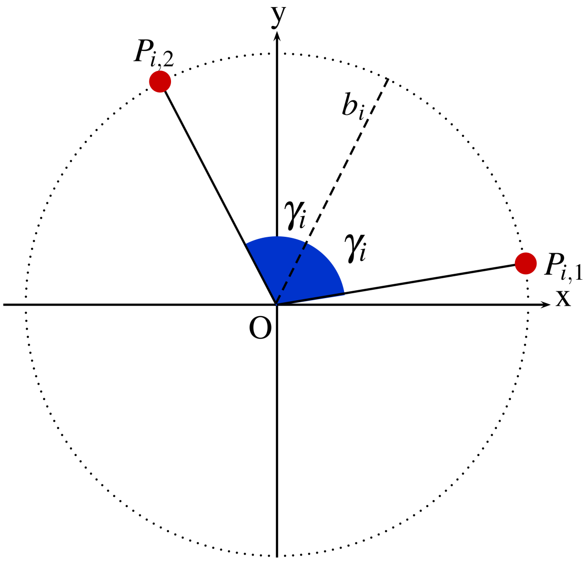
Assume the rotor is a rigid body $\Omega \subset \mathbb{R}^3$ rotating about an axis at a constant angular velocity ω. Often times the rotor mass distribution is not homogeneous, producing imbalance in the rotation. This leads to dangerous vibrations. Our goal is to manage a system of balancing masses in order to minimize the imbalance.

Consider $(O; (x, y, z))$ Ω-fixed reference frame. By definition, the axes (x, y) rotate about axis z at a constant angular velocity ω.

The balancing device (see figures 8.1 and 8.2) is made up two heads lying in two planes



(a) *intermediate* angle



(b) *gap* angle

Figure 8.3: one balancing head is considered. The balancing masses $(m_i, P_{i,1})$ and $(m_i, P_{i,2})$ are drawn in red. The bisector of the angle generated by $\overrightarrow{OP}_{i,1}$ and $\overrightarrow{OP}_{i,2}$ is the dashed line. The *intermediate* angle α_i and the *gap* angle γ_i give the position of the balancing masses in each balancing head.

orthogonal to the rotation axis z. Each head is made of a pair of balancing masses, which are free to rotate on a plane orthogonal to the rotation axis z. Namely, we have

- two planes $\pi_1 := \{z = -a\}$ and $\pi_2 := \{z = b\}$, with a, $b \geq 0$;

- two mass-points $(m_1, P_{1,1})$ and $(m_1, P_{1,2})$ lying on π_1 at distance r_1 from the axis z, i.e., in the reference frame $(O; (x, y, z))$

$$\begin{cases} P_{1,1;x} = & r_1 \cos(\alpha_1 - \gamma_1) \\ P_{1,1;y} = & r_1 \sin(\alpha_1 - \gamma_1) \\ P_{1,1;z} = & -a, \end{cases}$$
$$\text{and} \tag{8.2.1}$$
$$\begin{cases} P_{1,2;x} = & r_1 \cos(\alpha_1 + \gamma_1) \\ P_{1,2;y} = & r_1 \sin(\alpha_1 + \gamma_1) \\ P_{1,2;z} = & -a; \end{cases}$$

- two mass-points $(m_2, P_{2,1})$ and $(m_2, P_{2,2})$ lying on π_2 at distance r_2 from the axis z, namely, in the reference frame $(O;(x,y,z))$

$$\begin{cases} P_{2,1;x} = & r_2\cos(\alpha_2-\gamma_2)\\ P_{2,1;y} = & r_2\sin(\alpha_2-\gamma_2)\\ P_{2,1;z} = & b,\end{cases}$$
and
$$\begin{cases} P_{2,2;x} = & r_2\cos(\alpha_2+\gamma_2)\\ P_{2,2;y} = & r_2\sin(\alpha_2+\gamma_2)\\ P_{2,2;z} = & b.\end{cases} \tag{8.2.2}$$

For any $i=1,2$, let b_i be the bisector of the angle generated by $\overrightarrow{OP_{i,1}}$ and $\overrightarrow{OP_{i,2}}$ (see figure 8.3). For any $i=1,2$, the *intermediate* angle α_i is the angle between the x-axis and the bisector b_i, while the *gap* angle γ_i is the angle between $\overrightarrow{OP_{i,1}}$ and the bisector b_i. Note that the angles α_i and γ_i are defined with respect to the Ω-fixed reference frame $(O;(x,y,z))$. Indeed, the balancing device described above is integral with the body Ω. Furthermore, we observe that on the one hand, in view of avoiding the generation of torque in each single head, the two balancing masses composing a single head are placed on a single plane. On the other hand, the available balancing heads are placed on two separate planes and torque may be generated by the composed action of the heads.

Following a classical approach, the imbalance may be described as the force F and the momentum N exerted by the imbalanced body Ω on the rotation axis. The force is applied at the origin O. The momentum is computed with respect to the pole O. Both the force and the momentum are supposed to be orthogonal to the rotation axis z. As we mentioned, F and N are given data.

In $(O;(x,y,z))$, set $P_1 := (0,0,-a)$, $P_2 := (0,0,b)$, $F := (F_x,F_y,0)$ and $N := (N_x,N_y,0)$. By imposing the equilibrium condition on forces and momenta, the force F and the momentum N can be decomposed into a force F_1 exerted at P_1 contained in plane π_1 and a force F_2 exerted at P_2 contained in π_2

$$F_1 = \frac{1}{a+b}\begin{bmatrix} bF_x - N_y \\ bF_y + N_x \\ 0 \end{bmatrix} \quad \text{and} \quad F_2 = \frac{1}{a+b}\begin{bmatrix} aF_x + N_y \\ aF_y - N_x \\ 0 \end{bmatrix}. \tag{8.2.3}$$

In each plane, we are able to generate a force to balance the system, by moving the balancing masses described in (8.2.1) and (8.2.2).

In particular, by trigonometric formulas

- in plane π_1, we compensate force F_1 by the centrifugal force:

$$B_1 = 2m_1r_1\omega^2\cos(\gamma_1)\left(\cos(\alpha_1),\sin(\alpha_1)\right); \tag{8.2.4}$$

- in plane π_2, we compensate force F_2 by the centrifugal force:

$$B_2 = 2m_2 r_2 \omega^2 \cos(\gamma_2) \left(\cos(\alpha_2), \sin(\alpha_2)\right). \tag{8.2.5}$$

The overall imbalance of the system is then given by the resulting force in π_1

$$F_{ris,1} = B_1 + F_1$$

and the resulting force in π_2

$$F_{ris,2} = B_2 + F_2.$$

Note that, if the balancing masses are moved incorrectly, we may increase the imbalance on the system.

We introduce the imbalance indicator

$$G := \|B_1 + F_1\|^2 + \|B_2 + F_2\|^2. \tag{8.2.6}$$

The above quantity measures the imbalance on the overall system made of rotor and balancing heads.

By (8.2.4) and (8.2.5), we observe that

$$G(\alpha_1, \gamma_1, \alpha_2, \gamma_2) = G_1(\alpha_1, \gamma_1) + G_2(\alpha_2, \gamma_2), \tag{8.2.7}$$

where

$$\begin{aligned} G_1(\alpha_1, \gamma_1) \quad &:= \quad \Big[\left|2m_1 r_1 \omega^2 \cos(\gamma_1)\cos(\alpha_1) + F_{1,x}\right|^2 \\ &\qquad + \left|2m_1 r_1 \omega^2 \cos(\gamma_1)\sin(\alpha_1) + F_{1,y}\right|^2\Big] \end{aligned}$$

and

$$\begin{aligned} G_2(\alpha_2, \gamma_2) \quad &:= \quad \Big[\left|2m_2 r_2 \omega^2 \cos(\gamma_2)\cos(\alpha_2) + F_{2,x}\right|^2 \\ &\qquad + \left|2m_2 r_2 \omega^2 \cos(\gamma_2)\cos(\alpha_2) + F_{2,y}\right|^2\Big]. \end{aligned}$$

8.3 The control problem

An initial configuration Φ_0 for the balancing masses is given.
Our goal is to find a control strategy such that:

- the balancing masses move from Φ_0 to a final configuration $\overline{\Phi}$, where they compensate the imbalance;
- the imbalance should not increase and velocities of the masses are kept small during the correction process.

We suppose that we do not have a real-time feedback concerning the imbalance of the system. For this reason, we design an open-loop control.

Accordingly, we introduce a control problem to steer our system to a stable configuration, which minimizes the imbalance. In the context of the model described in section 8.2, we choose as *state* $\Phi(t) := (\alpha_1(t), \gamma_1(t); \alpha_2(t), \gamma_2(t))$, where $\alpha_i(t)$ and $\gamma_i(t)$ are the angles regulating the position of the four balancing masses, as illustrated in (8.2.1) and (8.2.2).
The *control* $\psi(t) := (\psi_1(t), \psi_2(t); \psi_3(t), \psi_4(t))$ is the time derivative of the state, i.e. its components are the time derivatives of the angles $\Phi_i(t)$. Namely, the state equation is

$$\begin{cases} \dfrac{d}{dt}\Phi = \psi & t \in (0, +\infty) \\ \Phi(0) = \Phi_0. \end{cases}$$

Note that we are in the particular case of the Calculus of Variations. The time interval is infinite and special attention has to be paid for the limiting behavior of the solution.

The Lagrangian $L : \mathbb{T}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ reads as

$$L(\Phi, \psi) := \frac{1}{2}\left[\|\psi\|^2 + \beta \hat{G}(\Phi)\right],$$

where $\beta > 0$ is a parameter to be fixed and $\hat{G} = G - \inf G$, G being the imbalance indicator introduced in (8.2.6). Note that for any $\overline{\Phi} \in \mathscr{S} := \text{argmin}(G)$, $\hat{G}(\overline{\Phi}) = G(\overline{\Phi}) - \inf G = \inf G - \inf G = 0$, namely $\mathscr{S}$ coincides with the zero set of $\hat{G}$. We have introduced $\hat{G}$ to guarantee the integrability of the Lagrangian along admissible trajectories over the half-line $(0, +\infty)$.
In the above Lagrangian, there is a trade-off between the cost of controlling the system to a stable regime and the velocity of the balancing masses, with respect to the rotor. If β is large, the primary concern for the optimal strategy is to minimize the cost of controlling, while if β is small our priority is to minimize the velocities.

Let $\Phi_0 \in \mathbb{T}^4$ be an initial configuration. We introduce the space of admissible trajectories

$$\mathscr{A} := \left\{\Phi \in H^1_{loc}([0, +\infty); \mathbb{T}^4) \;\middle|\; \Phi(0) = \Phi_0, \quad \text{and} \quad L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)\right\}, \tag{8.3.1}$$

where the Sobolev space $H^1_{loc}((0, +\infty); \mathbb{T}^4)$ is defined in (8.7.1). Note that the requirement $L(\Phi, \dot{\Phi}) \in L^1(0, +\infty)$ is equivalent to

$$\dot{\Phi} \in L^2(0, +\infty) \quad \text{and} \quad G(\Phi) - \inf G \in L^1(0, +\infty).$$

Our goal is to minimize the functional $J : \mathscr{A} \longrightarrow \mathbb{R}$

$$J(\Phi) := \frac{1}{2}\int_0^{\infty} \left[\|\dot{\Phi}\|^2 + \beta \hat{G}(\Phi)\right] dt. \tag{8.3.2}$$

8.3.1 Statement of the main result

We state now our main result.

Proposition 8.3.1. *Consider the functional* (8.3.2). *For* $i = 1, 2$, *set*

$$c^i := \frac{1}{2m_i r_i \omega^2} (F_{i,x}, F_{i,y}) \tag{8.3.3}$$

Then,

1. *there exists* $\Phi \in \mathscr{A}$ *minimizer of* J;

2. $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ *is* C^∞ *smooth and, for* $i = 1, 2$, *the following Euler-Lagrange equations are satisfied, for* $t > 0$

$$\begin{cases} -\ddot{\alpha}_i = \beta \cos(\gamma_i) \left[-c_1^i \sin(\alpha_i) + c_2^i \cos(\alpha_i)\right] \\ -\ddot{\gamma}_i = -\beta \sin(\gamma_i) \left[c_1^i \cos(\alpha_i) + c_2^i \sin(\alpha_i) - \cos(\gamma_i)\right] \\ \alpha_i(0) = \alpha_{0,i}, \ \gamma_i(0) = \gamma_{0,i}, \ \dot{\Phi}(T) \underset{T\to+\infty}{\longrightarrow} 0. \end{cases} \tag{8.3.4}$$

(3) for any optimal trajectory Φ *for* (8.3.2), *there exists* $\overline{\Phi} \in \mathscr{S}$ *such that*

$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi}, \tag{8.3.5}$$

$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{8.3.6}$$

and

$$\left|\hat{G}(\Phi(t))\right| \underset{t\to+\infty}{\longrightarrow} 0. \tag{8.3.7}$$

If, in addition

$$m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \text{and} \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}, \tag{8.3.8}$$

we have the exponential estimate for any $t \geq 0$

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |G(\Phi(t))| \leq C \exp(-\mu t), \tag{8.3.9}$$

with C, $\mu > 0$ *independent of* t.

In the following subsection, we analyze the corresponding steady problem. In subsection 8.3.3, we develop general tools to prove the above result. In subsection 8.3.4, we prove Proposition 8.3.1. Finally, in subsection 8.3.5, we perform some numerical simulations validating the theory.

8.3.2 The steady problem

First of all, we address the *steady* problem:

Find a 4-tuple of angles $(\overline{\alpha}_1, \overline{\gamma}_1; \overline{\alpha}_2, \overline{\gamma}_2)$ *such that the imbalance indicator* G *is minimized.*

A solution to the above steady problem is called *steady optimum.* We recall that the set of steady optima is denoted by $\mathscr{S} = \operatorname{argmin}(G)$.

Remark 8.3.1. *We observe that by using* (8.2.7)*,*

$$\mathscr{S} = \mathit{argmin}(G_1) \times \mathit{argmin}(G_2),$$

namely we can reduce our 4-dimensional problem to a 2-dimensional problem.

Therefore, we have reduced to find minimizers of a function of the form:

$$g(\alpha, \gamma) := |\cos(\gamma)\cos(\alpha) - c_1|^2 + |\cos(\gamma)\sin(\alpha) - c_2|^2 .$$

This task is accomplished in lemma below.

Lemma 8.3.1. *Let* $c = (c_1, c_2) \in \mathbb{R}^2$*. Set*

$$g(\alpha, \gamma) := |\cos(\gamma)\cos(\alpha) - c_1|^2 + |\cos(\gamma)\sin(\alpha) - c_2|^2 . \tag{8.3.10}$$

Let $\mathit{argmin}\,(g)$ *be the set of minimizers of* g*. Then,*

1. *if* $c = 0$*, then*
$$\mathit{argmin}(g) = \left\{ \left(\theta, \frac{\pi}{2}\right) \;\middle|\; \theta \in \mathbb{T} \right\};$$
2. *if* $c \neq 0$*, set* $d := \min\{1, \|c\|\}$*. Then,*
$$\mathit{argmin}(g) = (\arg(c_1 + ic_2), \arccos(d)) \cup \tag{8.3.11}$$
$$\cup\, (\arg(c_1 + ic_2) + \pi, \arccos(-d)),$$
where $\arg(c_1 + ic_2)$ *denotes the argument of the complex number* $c_1 + ic_2$*.*
Moreover, if $c \neq 0$*, there exists a unique* (α, γ) *minimizer of* g*, with* $0 \le \alpha < 2\pi$ *and* $0 \le \gamma \le \frac{\pi}{2}$*;*
3. $\inf g = 0$ *if and only if* $\|c\| \le 1$*;*
4. $\inf g = \begin{cases} 0, & \text{if } \|c\| \le 1 \\ |\|c\| - 1|^2, & \text{if } \|c\| > 1. \end{cases}$

Lemma 8.3.1 can be proved by trigonometric calculus.

Now, let $\overline{\Phi} \in \mathscr{S}$ be a minimizer of the imbalance indicator G. We highlight that two circumstances may occur:

- $\inf G = 0$, namely, the overall system made of rotor and balancing masses can be fully balanced, by placing the four balancing masses as

$$\begin{aligned} P_{1,1} &= r_1 \left(\cos\left(\overline{\alpha}_1 - \overline{\gamma}_1\right), \sin\left(\overline{\alpha}_1 - \overline{\gamma}_1\right)\right) \\ P_{1,2} &= r_1 \left(\cos\left(\overline{\alpha}_1 + \overline{\gamma}_1\right), \sin\left(\overline{\alpha}_1 + \overline{\gamma}_1\right)\right) \end{aligned} \tag{8.3.12}$$

  and

$$\begin{aligned} P_{2,1} &= r_2 \left(\cos\left(\overline{\alpha}_2 - \overline{\gamma}_2\right), \sin\left(\overline{\alpha}_2 - \overline{\gamma}_2\right)\right) \\ P_{2,2} &= r_2 \left(\cos\left(\overline{\alpha}_2 + \overline{\gamma}_2\right), \sin\left(\overline{\alpha}_2 + \overline{\gamma}_2\right)\right). \end{aligned} \tag{8.3.13}$$

- $\inf G > 0$, i.e. the imbalance of the rotor is too large to be compensated by the available balancing masses. Despite that, $(\overline{\alpha}_1, \overline{\gamma}_1; \overline{\alpha}_2, \overline{\gamma}_2)$ is a minimizer of G. Hence, by locating the balancing masses in configuration (8.3.12)-(8.3.13), we do our best to balance the system, being aware full balance cannot be achieved.

In the proposition below, we illustrate when the circumstance $\inf G = 0$ occurs.

Proposition 8.3.2. *The imbalance indicator G admits zeros* ($\inf G = 0$) *if and only if*

$$m_1 r_1 \geq \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \textit{and} \quad m_2 r_2 \geq \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}.$$

Proof of Proposition 8.3.2. We have $G(\alpha_1, \gamma_1, \alpha_2, \gamma_2) = 0$ if and only if

$$\begin{cases} 2m_1 r_1 \omega^2 \cos(\gamma_1)\cos(\alpha_1) & = F_{1,x} \\ 2m_1 r_1 \omega^2 \cos(\gamma_1)\sin(\alpha_1) & = F_{1,y} \end{cases}$$

$$\begin{cases} 2m_2 r_2 \omega^2 \cos(\gamma_2)\cos(\alpha_2) & = F_{2,x} \\ 2m_2 r_2 \omega^2 \cos(\gamma_2)\sin(\alpha_2) & = F_{2,y}. \end{cases}$$

Note that the first two equations are decoupled with respect to the second ones. By Lemma 8.3.1 (3), the above system admits a solution if and only if

$$\begin{cases} m_1 r_1 & \geq \dfrac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \\ m_2 r_2 & \geq \dfrac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2}, \end{cases}$$

as required. □

As we have seen at the beginning of section 8.3, an initial configuration $\Phi_0 = (\alpha_{0,1}, \gamma_{0,1}; \alpha_{0,2}, \gamma_{0,2})$ of the balancing masses is given. A key issue is to determine a trajectory $\Phi(t) = (\alpha_{0,1}(t), \gamma_{0,1}(t); \alpha_{0,2}(t), \gamma_{0,2}(t))$ joining the initial configuration Φ_0 with a steady optimum $\overline{\Phi} \in \mathscr{S}$ minimizing the imbalance in the meanwhile. For this reason, the *dynamical* control problem has to be addressed. Our main result Proposition 8.3.1 asserts the *steady* problem and the *dynamical* one are interlinked.

8.3.3 General results

The purpose of this section is to provide some general tools to prove Proposition 8.3.1. We introduce a generalized version of our functional (8.3.2).

Consider the Lagrangian $L : \mathbb{T}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$L(\Phi, \psi) := \frac{1}{2}\|\psi\|^2 + Q(\Phi),$$

where $Q : \mathbb{T}^n \longrightarrow \mathbb{R}^+$ is real analytic.

Let $\Phi_0 \in \mathbb{T}^n$ be an initial condition. Set the space of admissible trajectories

$$\mathscr{A} := \left\{ \Phi \in H^1_{loc}([0, +\infty); \mathbb{T}^n) \;\middle|\; \Phi(0) = \Phi_0 \quad \text{and} \quad L(\Phi, \dot{\Phi}) \in L^1(0, +\infty) \right\}. \tag{8.3.14}$$

The zero set of Q is denoted by $\mathscr{Z}$.

Our goal is to minimize the functional $K : \mathscr{A} \longrightarrow \mathbb{R}$

$$K(\Phi) := \int_0^\infty \frac{1}{2}\|\dot{\Phi}\|^2 + Q(\Phi)\; dt. \tag{8.3.15}$$

Remark 8.3.2. *If $\mathscr{Z} \neq \varnothing$, then the space of admissible trajectories $\mathscr{A}$ is nonempty.*

Proof. Take $\overline{\Phi} \in \mathscr{Z}$. Consider the trajectory

$$\Phi(t) := \begin{cases} (1-t)\Phi_0 + t\overline{\Phi} & t \in [0, 1) \\ \overline{\Phi} & t \in [1, +\infty). \end{cases} \tag{8.3.16}$$

Now, $\Phi \in \mathscr{A}$, thus showing that $\mathscr{A} \neq \varnothing$. □

In Proposition 8.3.3, we are concerned with the existence of minimizer of (8.3.15). The proof can be found in the Appendix.

Proposition 8.3.3. *There exists $\Phi \in \mathscr{A}$ global minimizer of* (8.3.15)*.*

We now derive to optimality conditions for (8.3.15). Let $\Phi \in \mathscr{A}$ be an admissible trajectory. We consider directions $v \in C_c^\infty((0, +\infty); \mathbb{R}^n)$. We can compute the directional derivative of K at Φ along the direction v, obtaining

$$\langle dK(\Phi), v \rangle = \int_0^\infty \dot{\Phi}\dot{v} + \nabla Q(\Phi) v dt. \tag{8.3.17}$$

From the above computation of the directional derivative and Fermat's theorem, we derive the first order Optimality Conditions.

Proposition 8.3.4. *Take Φ minimizer of* (8.3.15)*. Then, we have:*

1. $\Phi \in C^\infty([0, +\infty); \mathbb{T}^n)$;

2. *the Euler-Lagrange equations are satisfied*

$$\begin{cases} -\ddot{\Phi} = \nabla Q\left(\Phi\right) & t \in (0, +\infty) \\ \Phi(0) = \Phi_0, \;\; \dot{\Phi}(T) \underset{T\to+\infty}{\longrightarrow} 0; \end{cases} \tag{8.3.18}$$

3. *the energy is conserved, i.e.*

$$E(t) := \|\dot{\Phi}(t)\|^2 - Q\left(\Phi(t)\right) \equiv 0. \tag{8.3.19}$$

Note that (8.3.18) can be seen as a system of two coupled elliptic PDEs, with a Dirichlet condition at time $t = 0$ and a Neumann condition at $t = +\infty$. We prove the above proposition in the Appendix.

Equivalently, we can formulate the first order optimality conditions as a state-adjoint state first order system.

$$\begin{cases} \dot{\Phi} = -q & t \in (0, +\infty) \\ -\dot{q} = \nabla Q(\Phi) & t \in (0, +\infty) \\ \Phi(0) = \Phi_0, \;\; q(T) \underset{T\to+\infty}{\longrightarrow} 0. \end{cases} \tag{8.3.20}$$

Now, in the spirit of stabilization-turnpike theory (see [115, 138, 129]), we show that the time-evolution optima converges as $t \to \infty$ to steady optima.

Proposition 8.3.5. *Assume $\mathscr{X} \subset \mathbb{T}^n$ is finite and Q real analytic. Consider $\Phi \in \mathscr{A}$ global minimizer of* (8.3.15). *Then,*

1. *there exists $\overline{\Phi} \in argmin\,(Q)$ such that*

$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi},$$

$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0$$

and

$$\left|Q\left(\Phi(t)\right)\right| \underset{t\to+\infty}{\longrightarrow} 0.$$

2. *if, in addition*

$$\nabla^2 Q(\overline{\Phi}) \quad \textit{is (strictly) positive definite,} \tag{8.3.21}$$

we have the exponential estimate, for any $t \geq 0$

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + \left|Q\left(\Phi(t)\right)\right| \leq C \exp\left(-\mu t\right), \tag{8.3.22}$$

with C, $\mu > 0$ independent of t.

Proof of Proposition 8.3.5 (1). We start proving (1).
Let Φ be a minimizer of (8.3.15). By (8.3.18), we immediately have $\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0$, whence by (8.3.19)

$$Q(\Phi(t)) \underset{t\to+\infty}{\longrightarrow} 0. \tag{8.3.23}$$

By Lojasiewicz inequality (see, e.g. [99, Théorème 2 page 62]), there exists d, $N > 0$ such that, for each $\Phi \in \mathbb{T}^n$,

$$|Q(\Phi)| \geq d\,\mathrm{dist}(\Phi, \mathscr{Z})^N.$$

where $\mathscr{Z}$ denotes the zero set of Q and $\mathrm{dist}(\Phi, \mathscr{Z}) := \inf_{\theta\in\mathscr{Z}} \|\Phi - \theta\|$. Now, we take $\Phi \in \mathscr{A}$ a minimizer for (8.3.15) and we plug it in the above Lojasiewicz inequality, getting

$$\mathrm{dist}(\Phi(t), \mathscr{Z}) \leq d^{-\frac{1}{N}} |Q(\Phi(t))|^{\frac{1}{N}} \underset{t\to+\infty}{\longrightarrow} 0, \tag{8.3.24}$$

by (8.3.23). Since $\mathscr{Z} \subset \mathbb{T}^n$ is finite and Φ is continuous, there exists a unique $\overline{\Phi} \in \mathscr{Z}$, such that

$$\mathrm{dist}(\Phi(t), \mathscr{Z}) = \|\Phi(t) - \overline{\Phi}\|, \quad \forall\, t \geq \bar{t},$$

with $\bar{t}$ large enough. By the above equality and (8.3.24), we have

$$\|\Phi(t) - \overline{\Phi}\| \leq d^{-\frac{1}{N}} |Q(\Phi(t))|^{\frac{1}{N}} \underset{t\to+\infty}{\longrightarrow} 0,$$

as required.

The proof of (2) is a consequence of Lemma 8.3.2 stated and proved below. □

The next lemma is inspired by [138] and [129].

Lemma 8.3.2. *Assume the condition* (8.3.21) *holds. Let* $\Phi_0 \in \mathbb{T}^n$ *and* $\Phi \in C^\infty(\mathbb{R}^+; \mathbb{T}^n)$ *solution to*

$$\begin{cases} -\ddot{\Phi} + \nabla Q(\Phi) = 0 & t \in (0, +\infty) \\ \Phi(0) = \Phi_0, \ \dot{\Phi}(T) \underset{T\to+\infty}{\longrightarrow} 0. \end{cases} \tag{8.3.25}$$

Assume there exists $\overline{\Phi} \in \mathscr{Z}$ *such that*

$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi} \tag{8.3.26}$$

and

$$\dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{8.3.27}$$

Then,

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |g(\Phi(t))| \leq C \exp(-\mu t), \quad \forall\, t \geq 0,$$

with $\mu > 0$.

Proof. Take any Φ solution to (8.3.25). Then, the function

$$\mathbf{x} := \begin{bmatrix} \Phi - \overline{\Phi} \\ \dot{\Phi} \end{bmatrix} \tag{8.3.28}$$

solves the first order problem

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) & t \in (0, +\infty) \\ \mathbf{x}(T) \underset{T \to +\infty}{\longrightarrow} 0. \end{cases} \tag{8.3.29}$$

where

$$\mathbf{f}(\mathbf{x}) := \begin{bmatrix} x_{n+1} \\ \vdots \\ x_{2n} \\ \nabla Q\left((x_1, \ldots, x_n) + \overline{\Phi}\right) \end{bmatrix}. \tag{8.3.30}$$

We observe that $\mathbf{f}(0) = 0$, since $\overline{\Phi}$ is a zero of Q. Moreover, the Jacobian of $\mathbf{f}$ at $\mathbf{x} = 0$ is a block matrix

$$D\mathbf{f}(0) = \begin{pmatrix} 0 & I_n \\ \nabla^2 Q\left(\overline{\Phi}\right) & 0, \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. By assumption (8.3.21), $\nabla^2 Q\left(\overline{\Phi}\right)$ is positive definite. Then, there exists C symmetric positive definite, such that $C^2 = \nabla^2 Q\left(\overline{\Phi}\right)$. Following [130, subsection III.B], we introduce the matrix

$$\Lambda := \frac{1}{2} \begin{pmatrix} 2I_n & -C^{-1} \\ 2C & I_n \end{pmatrix}. \tag{8.3.31}$$

Since $\nabla^2 Q\left(\overline{\Phi}\right)$ is (strictly) positive definite, Λ is invertible[8.1] and

$$\Lambda^{-1} D\mathbf{f}(0)\, \Lambda = \begin{pmatrix} C & 0 \\ 0 & -C. \end{pmatrix}$$

Hence, the spectrum of the jacobian $D\mathbf{f}(0)$ does not intersect the imaginary axis, whence $\mathbf{0}$ is an hyperbolic equilibrium point for (8.3.29), as required.

Step 3 **Conclusion by applying the stable manifold theorem**

As we have seen in step 2, $\mathbf{0}$ is an hyperbolic equilibrium point for (8.3.29). Then, by the stable manifold theorem (see e.g. [109, section 2.7] or [129]), the stable and unstable manifolds for (8.3.29) exist in a neighborhood of 0. Besides, thanks to (8.3.26) and (8.3.27), $\mathbf{x} = \left(\Phi - \overline{\Phi}, \dot{\Phi}\right)$ belongs to the stable manifold of the above problem.

8.1

$$\Lambda^{-1} = \begin{pmatrix} \frac{1}{2} I_2 & \frac{1}{2} C^{-1} \\ -C & I_2. \end{pmatrix}$$

Hence, by Stable Manifold theory (see, e.g. [109, Corollary page 115] or [129]), we have for some $\mu > 0$

$$\|\mathbf{x}(t)\| \leq C\exp(-\mu t), \quad \forall\, t \geq 0, \tag{8.3.32}$$

which yields

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| \leq C\exp\left(-\mu t\right), \quad \forall\, t \geq 0.$$

To conclude the proof, we observe that Q is globally Lipschitz and $Q\left(\overline{\Phi}\right) = 0$. Then,

$$\begin{aligned} |Q\left(\Phi(t)\right)| &= \left|Q\left(\Phi(t)\right) - Q\left(\overline{\Phi}\right)\right| \leq L\|\Phi(t) - \overline{\Phi}\| \\ &\leq C\exp\left(-\mu t\right). \end{aligned}$$

where in the last inequality we have employed (8.3.32). □

8.3.4 Proof of Proposition 8.3.1

We prove Proposition 8.3.1 employing the general results of subsection 8.3.3.

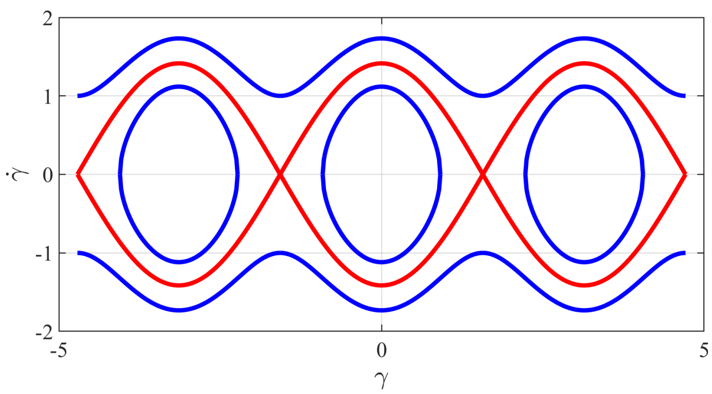


Figure 8.4: phase portrait for the Euler-Lagrange equations in the balanced case. The red curve is the separatrix.

Proof of Proposition 8.3.1. The existence of minimizers for (8.3.2) follows from Proposition 8.3.3, with $K = J$.

Step 1 **Reduction to two angles**

By (8.2.7), the imbalance indicator splits as $G(\alpha_1, \gamma_1, \alpha_2, \gamma_2) = G_1(\alpha_1, \gamma_1) + G_2(\alpha_2, \gamma_2)$, whence

$$\hat{G}(\alpha_1, \gamma_1, \alpha_2, \gamma_2) = \hat{G}_1(\alpha_1, \gamma_1) + \hat{G}_2(\alpha_2, \gamma_2),$$

with $\hat{G}_1(\alpha_1, \gamma_1) := G_1(\alpha_1, \gamma_1) - \inf G_1$ and $\hat{G}_2(\alpha_2, \gamma_2) := G_2(\alpha_2, \gamma_2) - \inf G_2$. Then, the functional

$$J(\Phi) = J_1(\alpha_1, \gamma_1) + J_2(\alpha_2, \gamma_2), \tag{8.3.33}$$

where

$$J_1(\alpha_1,\gamma_1) := \frac{1}{2}\int_0^{\infty} \left[|\dot{\alpha}_1|^2 + |\dot{\gamma}_1|^2 + \beta\hat{G}_1(\alpha_1,\gamma_1)\right] dt \tag{8.3.34}$$

and

$$J_2(\alpha_2,\gamma_2) := \frac{1}{2}\int_0^{\infty} \left[|\dot{\alpha}_2|^2 + |\dot{\gamma}_2|^2 + \beta\hat{G}_2(\alpha_2,\gamma_2)\right] dt. \tag{8.3.35}$$

This enables us to work on J_1 and J_2 separately. From the physical viewpoint, the functional J_1 is related to the first balancing head, while J_2 is related to the second balancing head. Both J_1 and J_2 fit in a general class of functionals (8.3.15), defining

$$\begin{aligned} Q_i(\alpha_i,\gamma_i) \quad := \quad & \frac{\beta}{2}\left[\left|\cos(\gamma_i)\cos(\alpha_i) - c_1^i\right|^2\right. \\ & \left.+\left|\cos(\gamma_i)\sin(\alpha_i) - c_2^i\right|^2\right], \end{aligned} \tag{8.3.36}$$

possibly remaining β after the absorption of the coefficient $\frac{1}{2m_ir_i\omega^2}$ and

$$c^i = \frac{1}{2m_ir_i\omega^2}\left(F_{i,x},F_{i,y}\right). \tag{8.3.37}$$

Step 2 **Proof of (2)**

For any $\Phi = (\alpha_1,\gamma_1;\alpha_2,\gamma_2)$ minimizer of (8.3.2), (α_1,γ_1) minimizes J_1 and (α_2,γ_2) minimizes J_2. We apply Proposition 8.3.4 to J_1 and J_2, computing the gradient of Q_i defined in (8.3.36)

$$\begin{aligned} \frac{\partial Q_i}{\partial \alpha_i}(\alpha_i,\gamma_i) \quad &= \quad \beta\cos(\gamma_i)\left[c_1^i\sin(\alpha_i) - c_2^i\cos(\alpha_i)\right] \\ \frac{\partial Q_i}{\partial \gamma_i}(\alpha_i,\gamma_i) \quad &= \quad \beta\sin(\gamma_i)\left[c_1^i\cos(\alpha_i) + c_2^i\sin(\alpha_i)\right. \\ & \qquad \left.-\cos(\gamma_i)\right]. \end{aligned} \tag{8.3.38}$$

Step 3

Proof of (3) and (4)

By Step 1, we reduce to prove the assertion for minimizers of J_1 and J_2. Let (α_i,γ_i) be a minimizer of J_i, for some $i = 1,2$.

Case 1. argmin $(Q_i) \subset \mathbb{T}^2$ is finite.

If argmin $(Q_i) \subset \mathbb{T}^2$ is finite, we directly apply Proposition 8.3.5 (1) to $K := J_i$, getting the required convergences. If, in addition, (8.3.8) is verified, we want to prove that the Hessian of Q_i at the steady optimum is positive definite. To this end, we compute $\nabla^2 Q_i(\alpha_i,\gamma_i)$

$$\begin{aligned} \frac{\partial^2 Q_i}{\partial \alpha^2}(\alpha_i,\gamma_i) \quad &= \quad \beta\cos(\gamma_i)\left[c_1^i\cos(\alpha_i) + c_2^i\sin(\alpha_i)\right] \\ \frac{\partial^2 Q_i}{\partial \gamma_i^2}(\alpha_i,\gamma_i) \quad &= \quad \beta\cos(\gamma_i)\left[c_1^i\cos(\alpha_i) + c_2^i\sin(\alpha_i)\right. \\ & \qquad \left.-\cos(\gamma_i)\right] + \beta\sin(\gamma_i)^2 \\ \frac{\partial^2 Q_i}{\partial \gamma_i\partial \alpha_i}(\alpha_i,\gamma_i) \quad &= \quad \beta\sin(\gamma_i)\left[-c_1^i\sin(\alpha_i) + c_2^i\cos(\alpha_i)\right] \end{aligned} \tag{8.3.39}$$

Now, let $\overline{\Phi} \in \operatorname{argmin}(Q_i)$. Since $\overline{\Phi} \in \operatorname{argmin}(Q_i)$ and (8.3.8) holds, by Lemma 8.3.1,

$$c^i = \cos(\overline{\gamma}_i)\left(\cos(\overline{\alpha}_i), \sin(\overline{\alpha}_i)\right).$$

and $\sin(\overline{\gamma}_i) \neq 0$. Hence, by (8.3.38), $c_1 \cos(\overline{\alpha}_i) + c_2 \sin(\overline{\alpha}_i) - \cos(\overline{\gamma}_i) = 0$. We plug these results into (8.3.39), obtaining

$$\begin{aligned}
\frac{\partial^2 Q_i}{\partial \alpha_i^2}(\overline{\alpha}_i, \overline{\gamma}_i) &= \beta \cos(\overline{\gamma}_i)^2 \left[\cos(\overline{\alpha}_i)^2 + \sin(\overline{\alpha}_i)^2\right] \\
&= \beta \cos(\overline{\gamma}_i)^2 \\
\frac{\partial^2 Q_i}{\partial \gamma_i^2}(\overline{\alpha}_i, \overline{\gamma}_i) &= \beta \sin(\overline{\gamma}_i)^2 \\
\frac{\partial^2 Q_i}{\partial \gamma_i \partial \alpha_i}(\overline{\alpha_i}, \overline{\gamma}_i) &= \beta \|c\| \sin(\overline{\gamma}_i)\left[-\cos(\overline{\alpha}_i)\sin(\overline{\alpha}_i)\right. \\
&\quad \left. + \sin(\overline{\alpha}_i)\cos(\overline{\alpha}_i)\right] = 0,
\end{aligned} \tag{8.3.40}$$

namely the Hessian of Q_i computed at $(\overline{\alpha}_i, \overline{\gamma}_i)$ is diagonal. Using once more (8.3.8) and by Lemma 8.3.1, we have both $\cos(\overline{\gamma}) \neq 0$ and $\sin(\overline{\gamma}) \neq 0$. Then, the Hessian of Q_i computed at $(\overline{\alpha}_i, \overline{\gamma}_i)$ is (strictly) positive definite. We apply Proposition 8.3.5 (2) to conclude.

Case 2. $\operatorname{argmin}(Q_i) \subset \mathbb{T}^2$ is a continuum.

From the physical viewpoint, this occurs when in the plane π_i there is no imbalance, namely $F_i = 0$. Now, by Lemma 8.3.1, $\operatorname{argmin}(Q_i) \subset \mathbb{T}^2$ is a continuum if and only if $c^i = 0$, namely

$$Q_i(\alpha_i, \gamma_i) = \frac{\beta}{2}\left[|\cos(\gamma_i)\cos(\alpha_i)|^2 + |\cos(\gamma_i)\sin(\alpha_i)|^2\right].$$

and the Euler-Lagrange equations satisfied by (α_i, γ_i) read as

$$\begin{cases}
-\ddot{\alpha}_i = 0 & t \in (0, +\infty) \\
-\ddot{\gamma}_i = \dfrac{\beta}{2}\sin(2\gamma) & t \in (0, +\infty) \\
\alpha(0) = \alpha_0, \ \ \dot{\alpha}(T) \underset{T \to +\infty}{\longrightarrow} 0 & \\
\gamma(0) = \gamma_0, \ \ \dot{\gamma}(T) \underset{T \to +\infty}{\longrightarrow} 0. &
\end{cases} \tag{8.3.41}$$

This entails that

$$\alpha(t) \equiv \alpha_0. \tag{8.3.42}$$

Furthermore, for any integer k, $\cos((2k+1)\pi) < 0$. Therefore, we are in position to conclude applying Proposition 8.3.5 to the functional

$$K(\gamma_i) := \frac{1}{2}\int_0^\infty \left[\|\dot{\gamma}_i\|^2 + \beta\,|\cos(\gamma)|^2\right] dt.$$

In case $\operatorname{argmin}(Q_i) \subset \mathbb{T}^2$ is a continuum, the above proof can be seen from the point of view of phase analysis. Indeed, the Euler-Lagrange equations reduce to the pendulum-like equation

$$\begin{cases}
-\ddot{\gamma} = \dfrac{\beta}{2}\sin(2\gamma) & t \in (0, +\infty) \\
\gamma(0) = \gamma_0, \ \ \dot{\gamma}(T) \underset{T \to +\infty}{\longrightarrow} 0. &
\end{cases} \tag{8.3.43}$$

We have the end condition $\dot{\gamma}(T) \underset{T\to+\infty}{\longrightarrow} 0$. Then, any solution γ of (8.3.43) lies on the separatrix (the red curve in figure 8.4), so that it must stabilize towards some steady state. □

8.3.5 Numerical simulations

In order to perform some numerical simulations, we firstly discretize our functional (8.3.15) and then we run `AMPL-IPOpt` to minimize the resulting discretized functional.

For the purpose of the numerical simulations, it is convenient to rewrite (8.3.15) as

$$\widetilde{K}(\psi,\Phi) := \int_0^\infty \frac{1}{2}\|\dot{\Phi}\|^2 + Q(\Phi)\,dt, \tag{8.3.44}$$

subject to the state equation

$$\begin{cases} \dfrac{d}{dt}\Phi = \psi & t \in (0,+\infty) \\ \Phi(0) = \Phi_0. \end{cases}$$

8.3.5.1 Discretization

Choose T sufficiently large and $Nt \in \mathbb{N} \setminus \{0,1\}$. Set
$\Delta t := \frac{T}{Nt-1}$. The discretized state is $(\Phi_i)_{i=0,\dots,Nt-1}$, whereas the discretized control (velocity) is
$(\psi_i)_{i=0,\dots,Nt-2}$. The discretized functional reads as

$$\widetilde{K_d}(\psi,\Phi) := \Delta t \sum_{i=0}^{Nt-1} \left[\frac{1}{2}\|\psi_i\|^2 + Q(\Phi_i)\right], \tag{8.3.45}$$

subject to the discretized state equation

$$\frac{\Phi_i - \Phi_{i-1}}{\Delta t} = \psi_{i-1}, \quad i = 1,\dots,Nt-1. \tag{8.3.46}$$

8.3.5.2 Algorithm execution

By (8.3.46) and (8.3.45), the discretized minimization problem is

$$\text{minimize } \widetilde{K_d}, \quad \text{subject to (8.3.46)}. \tag{8.3.47}$$

We address the above minimization problem by employing the interior-point optimization routine `IPOpt` (see [143] and [144]) coupled with `AMPL` [55], which serves as modelling language and performs the automatic differentiation. The interested reader is referred to [135, Chapter 9] and [130] for a survey on existing numerical methods to solve an optimal control problem.

In figures 8.5, 8.6, 8.7 and 8.8, we plot the computed optimal trajectory for (8.3.2), with initial datum $\Phi_0 = (\alpha_{0,1}, \gamma_{0,1}; \alpha_{0,2}, \gamma_{0,2}) := (2.6, 0.6, 2.5, 1.5)$. We choose F, N and m_i, such that the condition (8.3.8) is fulfilled. The exponential stabilization proved in Proposition 8.3.1 emerges. In figure 8.9, we depict the imbalance indicator versus time, along the computed trajectories. As expected, it decays to zero exponentially.

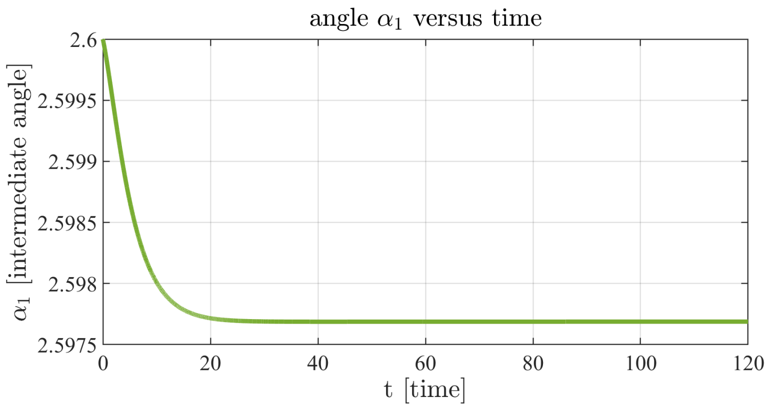


Figure 8.5: intermediate angle α_1 versus time

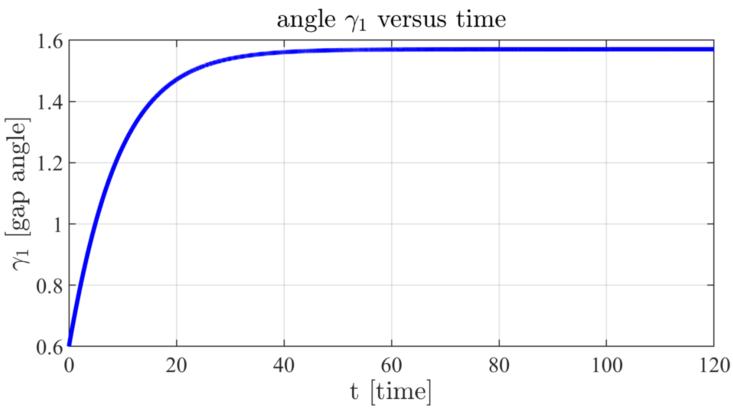


Figure 8.6: gap angle γ_1 versus time

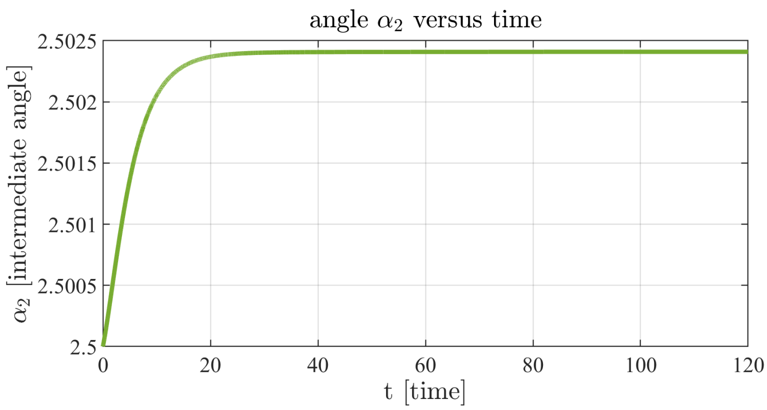


Figure 8.7: intermediate angle α_2 versus time

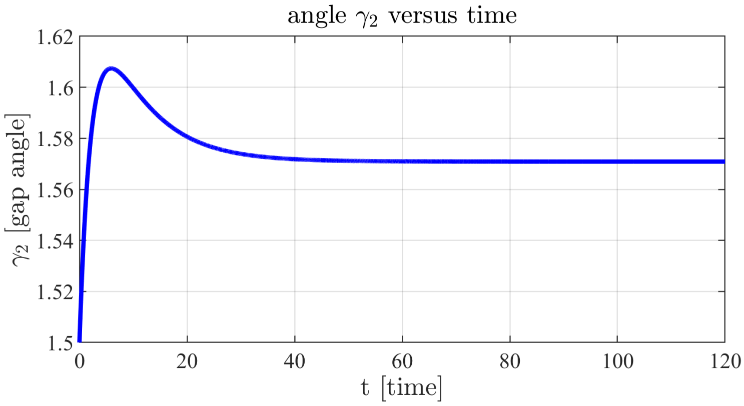


Figure 8.8: gap angle γ_2 versus time

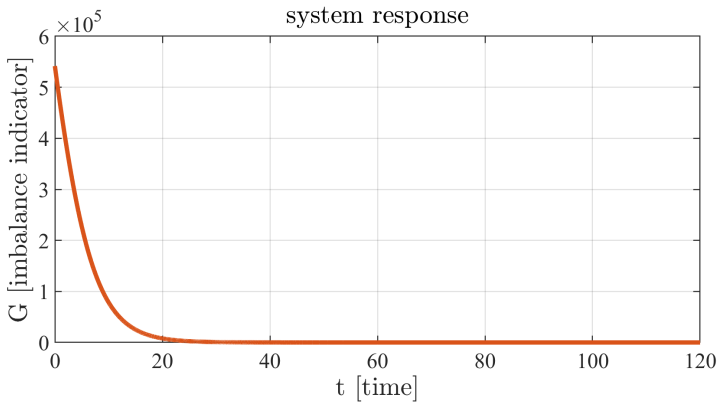


Figure 8.9: the imbalance indicator G along the computed trajectory versus time.

8.3.6 Conclusions and perspectives

In this paper, we addressed a problem of rotors imbalance suppression. We conceived a physical model. We formulated a problem of the Calculus of Variations in an infinite time horizon. We introduced a general class of variational problems, which contains ours as a particular case. In this general framework, we proved well-posedness in infinite-time and we derived Optimality Condition both in the form of Euler-Lagrange equations and in the form of Pontryagin system. The Lojasiewicz inequality was employed to prove convergence of the time optima towards the steady optima. In case the imbalance is below a given threshold, we used the Stable Manifold theory to obtain an exponential estimate of the speed of convergence.

The optimal controller we designed is open-loop. In case feedback information is available, a closed loop should be determined. To this end, Hamilton-Jacobi theory may be employed (see e.g. [130] and [11]). The Hamilton-Jacobi equation for our functional (8.3.15) reads as

$$\|\nabla V(\theta)\|^2 = 2Q(V(\theta)) \qquad \theta \in \mathbb{T}^n, \tag{8.3.48}$$

where

$$V(\theta) = \inf_{\mathscr{A}} J = \inf_{\mathscr{A}} \int_0^\infty \frac{1}{2}\|\dot{\Phi}\|^2 + Q(\Phi)\, dt,$$

with

$$\mathscr{A} := \Big\{\Phi \in H^1_{loc}([0,+\infty);\mathbb{T}^n) \;\Big|\; \Phi(0) = \theta \text{ and } L(\Phi,\dot{\Phi}) \in L^1(0,+\infty)\Big\}.$$

Appendix

The appendix is devoted to the proof of Lemma 8.3.1, Proposition 8.3.3 and Proposition 8.3.4.

8.4 Proof of Lemma 8.3.1

We start by proving Lemma 8.3.1, which is the building block of section 8.3.2.

Proof of Lemma 8.3.1. Step 1 **Gradient of g**
We have

$$\begin{aligned} \frac{\partial g}{\partial \alpha}(\alpha,\gamma) &= 2\cos(\gamma)\left[c_1\sin(\alpha) - c_2\cos(\alpha)\right] \\ \frac{\partial g}{\partial \gamma}(\alpha,\gamma) &= 2\sin(\gamma)\left[c_1\cos(\alpha) + c_2\sin(\alpha) - \cos(\gamma)\right]. \end{aligned} \tag{8.4.1}$$

Step 2 **Proof of (1) and (2)**
The case $c = 0$ follows from (8.3.10). If $c \neq 0$ take any

$$\tilde{\alpha} \in \arg(c_1 + ic_2).$$

We have

$$c = \|c\| \left(\cos(\tilde{\alpha}), \sin(\tilde{\alpha})\right). \tag{8.4.2}$$

Hence, by (8.4.1),

$$\begin{aligned} \frac{\partial g}{\partial \alpha}(\alpha, \gamma) &= 2\|c\| \cos(\gamma) \left[\cos(\tilde{\alpha}) \sin(\alpha) - \sin(\tilde{\alpha}) \cos(\alpha)\right] \\ &= 2\|c\| \cos(\gamma) \sin(\alpha - \tilde{\alpha}). \end{aligned} \tag{8.4.3}$$

Let $(\overline{\alpha}, \overline{\gamma})$ be a minimizer of g. By Fermat theorem and (8.4.3),

$$\|c\| \cos(\gamma) \sin(\overline{\alpha} - \tilde{\alpha}) = 0.$$

Now, since $c \neq 0$, we have either

$$\tilde{\alpha} = \overline{\alpha}, \tag{8.4.4}$$

or

$$\tilde{\alpha} = \overline{\alpha} + \pi. \tag{8.4.5}$$

In the first case, we plug (8.4.2) and (8.4.4) in (8.4.1), getting

$$\begin{aligned} \frac{\partial g}{\partial \gamma}(\overline{\alpha}, \overline{\gamma}) &= 2\sin(\overline{\gamma}) \left[\|c\| \cos(\overline{\alpha})^2 + \|c\| \sin(\overline{\alpha})^2 - \cos(\overline{\gamma})\right] \\ &= 2\sin(\overline{\gamma}) \left[\|c\| - \cos(\overline{\gamma})\right] \end{aligned} \tag{8.4.6}$$

Since $c \neq 0$, we have $\overline{\gamma} \neq \pi$, whence, by the above computations and Fermat condition $\frac{\partial g}{\partial \gamma}(\overline{\alpha}, \overline{\gamma}) = 0$,

$$\overline{\gamma} \in \arccos(d),$$

as desired. If $\tilde{\alpha} = \overline{\alpha} + \pi$, we plug (8.4.2) and (8.4.5) in (8.4.1), getting

$$\begin{aligned} \frac{\partial g}{\partial \gamma}(\overline{\alpha}, \overline{\gamma}) &= 2\sin(\overline{\gamma}) \left[-\|c\| \cos(\overline{\alpha})^2 - \|c\| \sin(\overline{\alpha})^2 - \cos(\overline{\gamma})\right] \\ &= -2\sin(\overline{\gamma}) \left[\|c\| + \cos(\overline{\gamma})\right] \end{aligned} \tag{8.4.7}$$

Since $c \neq 0$, we have $\overline{\gamma} \neq \pi$, whence, by the above computations and Fermat condition $\frac{\partial g}{\partial \gamma}(\overline{\alpha}, \overline{\gamma}) = 0$,

$$\overline{\gamma} \in \arccos(-d),$$

as required.

Furthermore, if we impose the conditions

$$0 \leq \alpha < 2\pi \quad \text{and} \quad 0 \leq \gamma \leq \frac{\pi}{2},$$

the desired uniqueness is guaranteed.
Step 3 **Proof of (3)**
On the one hand, if $\|c\| \leq 1$, we have $d = \|c\|$. Then, by (8.3.11), we have

$$\inf g = g\left(\arg(c_1 + ic_2) - \arccos\left(\|c\|\right), \arg(c_1 + ic_2) + \arccos\left(\|c\|\right)\right) = 0.$$

On the other hand, if $\|c\| > 1$, suppose, by contradiction, that $\inf g = 0$. Then, for some α and γ in $\mathbb{T}$, $g(\alpha, \gamma) = 0$, which yields

$$\cos(\gamma)\left(\cos(\alpha), \sin(\alpha)\right) = c.$$

Hence, we have

$$1 \geq \cos(\gamma)^2 = \|\cos(\gamma)\left(\cos(\alpha), \sin(\alpha)\right)\|^2 = \|c\|^2 > 1,$$

so obtaining a contradiction.
Step 4 **Proof of (4)**
The case $\|c\| \leq 1$ is a consequence of Step 3. If $\|c\| > 1$,

$$\inf g = g\left(\arg\left(c_1 + ic_2\right), 0\right) = \left|\|c\| - 1\right|^2.$$

This finishes the proof. □

8.5 Proof of Proposition 8.3.3

Now, we prove the well posedness of the time-evolution problem, by employing the direct methods in the Calculus of Variations.

Proof of Proposition 8.3.3. Step 1 **Boundedness of the minimizing sequence.**

Let $\{\Phi^m\}_{m\in\mathbb{N}} \subset \mathscr{A}$ be a minimizing sequence for (8.3.15). We wish to prove that $\left\{\dot{\Phi}^m\right\}_{m\in\mathbb{N}} \subset L^2((0, +\infty); \mathbb{R}^n)$ is bounded.

By definition of minimizing sequence, if m is large enough,

$$\frac{1}{2}\|\dot{\Phi}^m\|_{L^2}^2 \leq K(\Phi^m) \leq \inf_{\mathscr{A}} K + 1.$$

Then, $\|\dot{\Phi}^m\|_{L^2} \leq M$ for any natural m, as desired.
Step 2 **Weak convergence of the minimizing sequence in $\mathscr{A}$.**
Now, for any $t \geq 0$,

$$\Phi^m(t) = \Phi_0 + \int_0^t \dot{\Phi}^m(s)ds.$$

Then, by Cauchy-Schwarz inequality, for any $T > 0$, $\|\Phi^m\|_{L^2((0,T);\mathbb{T}^n)} \leq M\left(\sqrt{T} + 1\right)$. Hence, by Banach-Alaoglu theorem, there exists $\Phi \in H^1_{loc}((0, +\infty); \mathbb{T}^n)$ with $\dot{\Phi} \in L^2((0, +\infty); \mathbb{R}^n)$ such that, up to subsequences,

$$\Phi^m \underset{m\to\infty}{\longrightarrow} \Phi$$

weakly in $H^1((0,T);\mathbb{T}^n)$ for any $T>0$ and

$$\dot{\Phi}^m \underset{m\to\infty}{\longrightarrow} \dot{\Phi},$$

weakly in $L^2((0,+\infty);\mathbb{R}^n)$. Furthermore, the above convergence occurs point-wise. Indeed, for $t\geq 0$ and $T\geq t$, the linear operator

$$\delta_t : H^1((0,T);\mathbb{T}^n) \longrightarrow \mathbb{R}^n$$

$$\Phi \longmapsto \Phi(t)$$

is continuous. Hence, by the definition of weak convergence,
$\Phi^m(t)=\delta_t(\Phi^m)\longrightarrow \delta_t(\Phi)=\Phi(t)$. Since, for any natural m, $\Phi^m(0)=\Phi_0$, we have $\Phi(0)=\Phi_0$, whence $\Phi\in\mathscr{A}$, as required.

Step 3 **Conclusion**

By the lower semicontinuity of the norm with respect to the weak convergence

$$\int_0^\infty \|\dot{\Phi}\|^2 dt \leq \liminf_{m\to+\infty}\int_0^\infty \|\dot{\Phi}^m\|^2 dt. \tag{8.5.1}$$

At this stage, we want to prove the inequality

$$\int_0^\infty Q(\Phi)dt \leq \liminf_{m\to+\infty}\int_0^\infty Q(\Phi^m)dt. \tag{8.5.2}$$

Now, as we have shown in Step 2, Φ^m converges to Φ point-wise, whence

$$Q(\Phi^m(t)) \longrightarrow Q(\Phi(t))$$

for any $t\geq 0$. Furthermore, by Weierstrass theorem $Q:\mathbb{T}^n\longrightarrow\mathbb{R}^+$ is bounded. Then, for every $T>0$, by the Dominated Convergence theorem,

$$Q\left(\Phi^m\right) \longrightarrow Q\left(\Phi\right)$$

in the $L^1((0,T);\mathbb{R})$ norm, whence

$$\begin{aligned}\int_0^T Q\left(\Phi\right)dt &= \lim_{m\to+\infty}\int_0^T Q\left(\Phi^m\right)dt\\ &\leq \liminf_{m\to+\infty}\int_0^\infty Q\left(\Phi^m\right)dt.\end{aligned}$$

Hence, by arbitrariness of $T>0$,

$$\int_0^\infty Q\left(\Phi\right)dt \leq \liminf_{m\to+\infty}\int_0^\infty Q\left(\Phi^m\right)dt, \tag{8.5.3}$$

i.e. (8.5.2).

In conclusion, by (8.5.1) and (8.5.3), we have

$$\begin{aligned}K(\Phi) &= \frac{1}{2}\int_0^\infty \|\dot{\Phi}\|^2 + Q\left(\Phi\right)dt\\ &\leq \liminf_{m\to+\infty}\frac{1}{2}\int_0^\infty \|\dot{\Phi}^m\|^2 + Q\left(\Phi^m\right)dt\\ &= \inf_{\mathscr{A}} K,\end{aligned}$$

whence $\Phi\in\mathscr{A}$ is the required minimizer. This finishes the proof. □

8.6 Proof of Proposition 8.3.4

After proving the existence of minimizers for 8.3.15, we derive the Optimality Conditions.

Proof of Proposition of 8.3.4. Step 1 **Regularity of Φ by the fundamental lemma of the calculus of variations**

Take Φ a minimizer of (8.3.15). By (8.3.17) and Fermat's theorem, for any direction $v \in C_c^\infty((0,+\infty);\mathbb{R}^n)$, we have

$$\int_0^\infty \dot{\Phi}\dot{v} + \nabla Q(\Phi) \cdot v dt = \langle dK(\Phi), v\rangle = 0. \tag{8.6.1}$$

Then, by the fundamental lemma in the calculus of variations (see [73]), $\Phi \in C^2([0,T];\mathbb{T}^n)$.

Step 2 **Proof of (2)**

Since $\Phi \in C^2$, we are allowed to integrate by parts in (8.6.1), getting

$$\begin{aligned} 0 &= \int_0^\infty \dot{\Phi}\dot{v} + \nabla Q(\Phi)\, v dt \\ &= \lim_{T\to+\infty} \dot{\Phi}(T)v(T) + \int_0^\infty \left[-\ddot{\Phi} + \nabla Q(\Phi)\right] v\, dt, \end{aligned}$$

which, thanks to the arbitrariness of v, leads to (8.3.18). Furthermore, by bootstrapping in (8.3.18), we have the C^∞ regularity of the minimizer Φ.

Step 3 **Proof of (3)**

For any $s \geq 0$, we multiply (8.3.18) by $\dot{\Phi}$, getting

$$-\ddot{\Phi}(s)\cdot\dot{\Phi}(s) + \nabla Q(\Phi(s))\cdot\dot{\Phi}(s) = 0,$$

which yields, for each $t \geq 0$

$$\frac{d}{ds}\left[\|\dot{\Phi}(s)\|^2\right] = \frac{d}{ds} Q(\Phi(s)).$$

For any $t \geq 0$, we integrate over $[t,+\infty)$, obtaining

$$\lim_{T\to+\infty} \|\dot{\Phi}(T)\|^2 - \|\dot{\Phi}(t)\|^2 = \left[\lim_{T\to+\infty} Q(\Phi(T)) - Q(\Phi(t))\right]. \tag{8.6.2}$$

Now, $Q(\Phi) \in L^2((0,+\infty);\mathbb{R})$. Therefore, there exists a sequence $\{T_q\} \subset (0,+\infty)$ such that $T_q \underset{q\to\infty}{\longrightarrow} +\infty$ and

$$Q(\Phi(T_q)) \underset{q\to+\infty}{\longrightarrow} 0.$$

Taking $T = T_q$ in (8.6.2), we get

$$\left[\lim_{q\to+\infty} \|\dot{\Phi}(T_q)\|^2 - \|\dot{\Phi}(t)\|^2\right] = \lim_{q\to+\infty} Q(\Phi(T_q))^2 - \|Q(\Phi(t))\|^2,$$

whence

$$Q(\Phi(t)) = \|\dot{\Phi}(t)\|,$$

whence

$$E(t) = \|\dot{\Phi}(t)\| - Q\left(\Phi(t)\right) \equiv 0,$$

as required.

□

8.7 Notation

The circumference is denoted by

$$\mathbb{T} := \mathbb{R}/\sim,$$

where $\varphi_1 \sim \varphi_2$ if and only if there exists an integer k such that $\varphi_2 = \varphi_1 + 2k\pi$.

We introduce the following function spaces:

$$L^2_{loc}((0,+\infty);\mathbb{R}^n) := \bigcap_{T>0} L^2((0,T);\mathbb{R}^n).$$

$$H^1((0,T);\mathbb{T}^n) := \Big\{\Phi \in L^2((0,T);\mathbb{T}^n) \mid \\ \Phi \text{ is weakly differentiable and } \dot{\Phi} \in L^2((0,T);\mathbb{T}^n)\Big\}.$$

$$H^1_{loc}([0,+\infty);\mathbb{T}^n) := \left\{\Phi \in H^1((0,T);\mathbb{T}^n), \quad \forall T > 0\right\}; \tag{8.7.1}$$

$$C^\infty_c((0,+\infty);\mathbb{R}^n) := \{\Phi : [0,+\infty) \longrightarrow \mathbb{R}^n \mid \\ \Phi \text{ is infinitely many times differentiable} \\ \text{and } \mathrm{supp}(\Phi) \subset\subset (0,+\infty)\}.$$

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