# Weighted composition operators and weighted conformal invariance

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# Resumen


En esta tesis tratamos ciertos problemas relacionados con los operadores de composición ponderados. Estudiamos cómo actúan estos operadores en espacios de funciones analíticas en $\mathbb{D}$ o en un dominio acotado $\Omega \subset \mathbb{C}$.

En primer lugar nos centramos en una familia amplia de espacios de Hilbert de funciones analíticas en el disco unidad, los cuales satisfacen solamente un número reducido de axiomas y cuyo núcleo reproductor tiene la forma usual. A estos espacios se les llama espacios de Hardy con peso. En estos espacios caracterizamos los operadores de composición ponderados que son co-isométricos (equivalentemente, unitarios). El resultado principal nos revela una dicotomía al identificar una familia especifica de espacios de Hardy con peso como los únicos espacios en los cuales existen operadores no triviales de este tipo.

La segunda parte de la tesis está dedicada a explorar una clase de espacios de funciones analíticas los cuales comparten una cierta propiedad de invariancia conforme ponderada. Para ser más preciso, en esta parte presentamos una aproximación general a los espacios que son invariantes bajo los operadores $W_\varphi^\alpha$, definidos por $W_\varphi^\alpha f = (\varphi')^\alpha (f \circ \varphi)$ con $\alpha > 0$ y $\varphi \in Aut(\mathbb{D})$. Podemos observar que muchos de los espacios de Banach de funciones analíticas clásicos como los espacios de crecimiento de Korenblum, los espacios de Hardy, los espacios de Bergman con peso y ciertos espacios de Besov son invariantes bajo estos operadores. Entre otras cosas, en esta parte identificamos el espacio más grande, el más pequeño y el "único" espacio de Hilbert que satisface esta propiedad de invariancia ponderada para un $\alpha > 0$ dado.

En la última parte consideramos espacios de Banach abstractos de funciones analíticas en un dominio acotado general los cuales sólo satisfacen unos pocos axiomas. A continuación, describimos todos los operados de composición ponderados invertibles (equivalentemente, sobreyectivos) que actúan sobre estos espacios.

# Abstract

This thesis treats a number of problems related to weighted composition operators. We study how these operators act on the spaces of analytic functions in $\mathbb{D}$ or in a bounded domain $\Omega \subset \mathbb{C}$.

We first focus on a large family of Hilbert spaces of analytic functions in the unit disc which satisfy only a minimum number of axioms and whose reproducing kernels have the usual natural form. These spaces are called weighted Hardy spaces. In these spaces, we characterize the weighted composition operators which are *co-isometric* (equivalently, unitary). The main result reveals a dichotomy identifying a specific family of weighted Hardy spaces as the only ones that support non-trivial operators of this kind.

The second part of the thesis is devoted to exploring a class of spaces of analytic functions which share certain weighted invariant property. More precisely, in this part we present a general approach to the spaces which are invariant under the operators $W_\varphi^\alpha$, defined by $W_\varphi^\alpha f = (\varphi')^\alpha (f \circ \varphi)$ with $\alpha > 0$ and $\varphi \in Aut(\mathbb{D})$. We observe that many common examples of Banach spaces of analytic functions like Korenblum growth classes, Hardy spaces, standard weighted Bergman and certain Besov spaces are invariant under these operators. Among other things, we identify the largest and the smallest as well as the "unique" Hilbert space satisfying this weighted invariant property for a given $\alpha > 0$.

In the last part, we consider abstract Banach spaces of analytic functions on general bounded domains that satisfy only a minimum number of axioms. Then, we describe all invertible (equivalently, surjective) weighted composition operators acting on such spaces.

*Dedicado a mis padres,*
*a mi hermana y a Elvira*

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# Contents

# Introducción

En esta tesis trataremos ciertos problemas relacionados con los operadores de composición ponderados. Consideraremos el disco unidad $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ y estudiaremos cómo actúan estos operadores en espacios de funciones analíticas en el disco unidad.

Dada $f$, una función analítica en $\mathbb{D}$, se define el *operador de composición ponderado* con símbolos $F$ y $\phi$ como

$$W_{F,\phi}f = F(f \circ \phi),$$

donde $F$ y $\phi$ son funciones analíticas en el disco unidad y $\phi(\mathbb{D}) \subset \mathbb{D}$. Un caso particular de estos operadores que aparecerá de manera recurrente en muchos resultados será cuando $\phi$ es un automorfismo del disco, es decir, cuando $\phi = \varphi$ donde $\varphi$ es una aplicación fraccional lineal de la forma

$$\varphi(z) = \lambda \frac{z+a}{1+\bar{a}z},\ z \in \mathbb{D}, \quad a \in \mathbb{D},\ |\lambda| = 1,$$

que manda el disco unidad en sí mismo. Denotaremos el grupo de todos los automorfismos del disco por $Aut(\mathbb{D})$. Los operadores de composición ponderados son una generalización natural de los *operadores de multiplicación* y los *operadores de composición*, los cuales han sido estudiados ámpliamente en el contexto de los espacios de funciones analíticas en el disco unidad.

En general, dada una familia de operadores lineales en un espacio de Banach de funciones analíticas en el disco unidad siempre ha interesado conocer cuándo los operadores de esta familia cumplen ciertas propiedades como, por ejemplo, qué operadores son acotados, sobreyectivos, isométricos... En muchos espacios de Banach de funciones analíticas en el disco se conocen todos las isometrías o, al menos, todas las isometrías sobreyectivas (operadores unitarios). Por ejemplo, en analogía con el teorema clásico de Banach-Lamperti para los espacios $L^p$, Forelli en [46] probó que todos los operadores lineales isométricos que actúan sobre los espacios de Hardy, $H^p$, (excluyendo el caso $p = 2$) en sí mismos son precisamente operadores de composición ponderados cuyos símbolos satisfacen ciertas propiedades.

En los espacios de Hilbert las isometrías son mucho más numerosas e incluso existen muchos operadores unitarios y, por tanto, una pregunta natural es intentar describir todos los operadores unitarios que son de cierto tipo, por ejemplo, operadores de composición ponderados.

En el Capítulo 3 nos centraremos en responder a esta cuestión en una amplia familia de espacios de Hilbert de funciones analíticas en el disco unidad que satisfacen los siguientes axiomas:

(**A1**) Las evaluaciones puntuales son acotadas (así estamos ante un espacio de Hilbert con núcleo reproductor);

(**A2**) El núcleo reproductor $K_w(z)$ está normalizado por $K_w(0) = 1$ para todo $w \in \mathbb{D}$;

(**A3**) Los monomios $\{z^n : n = 0, 1, 2, \ldots\}$ están en nuestro espacio y forman un conjunto ortogonal y completo.

Estos espacios de Hilbert se denominan *espacios de Hardy con peso*. Más concretamente, en el Capítulo 3 caracterizaremos los operadores de composición ponderadores en los espacios de Hardy con peso los cuales son *co-isométricos* ($TT^* = I$ donde $T^*$ denota al adjunto del operador $T$ e $I$ denota el operador identidad). Ser co-isométrico *a priori* es más débil que la propiedad de ser unitario ($T^*T = TT^* = I$).

En la primera parte del Capítulo 3 se prueba que solamente con los axiomas definidos arriba podemos deducir múltiples propiedades para los espacios de Hardy con peso. Esto viene recogido en la Proposición 3.1 donde se prueba que si $\mathcal{H}$ es un espacio de Hilbert de funciones analíticas en el disco unidad que contiene todos los monomios y satisface los axiomas (**A1**) y (**A2**), entonces las siguientes condiciones son equivalentes:

(a) Se satisface el axioma $(\mathbf{A3})$, es decir, los monomios $\{z^n : n = 0, 1, 2, \ldots\}$ forman un conjunto ortogonal y completo en $\mathcal{H}$.

(b) El núcleo reproductor tiene la forma

$$K_w(z) = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n, \tag{1}$$

con $\gamma(n) = \|z^n\|^{-2}$.

(c) La norma de una función $f \in \mathcal{H}$ cuya serie de Taylor en $\mathbb{D}$ es $f(z) = \sum_{n=0}^{\infty} a_n z^n$ viene dada por

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \|z^n\|^2 \,.$$

(d) Las rotaciones $R_\lambda$ inducen operadores de composición isométricos $C_{R_\lambda}$ en $\mathcal{H}$.

(e) Las rotaciones $R_\lambda$ inducen operadores de composición unitarios $C_{R_\lambda}$ en $\mathcal{H}$.

(f) Las constantes de módulo uno ($|\mu| = 1$) y las rotaciones $R_\lambda$ inducen operadores de composición ponderados unitarios $W_{\mu, R_\lambda}$ en $\mathcal{H}$.

(g) $K_{\lambda w}(\lambda z) = K_w(z)$ para todo $z$, $w \in \mathbb{D}$ y todo $\lambda$ con $|\lambda| = 1$.

En la segunda parte del capítulo se probará el resultado principal, el Teorema 3.1. En este resultado juega un papel fundamental la familia de espacios de Hardy con peso $\mathcal{H}_\gamma$ con $\gamma > 0$ cuyo núcleo reproductor viene dado por la siguiente fórmula:

$$K_w^\gamma(z) = \frac{1}{(1 - \overline{w}z)^\gamma} = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n \,, \qquad z, w \in \mathbb{D} \,.$$

El Teorema 3.1 muestra que un operador de composición ponderado acotado en un espacio de Hardy con peso es co-isométrico si, y solo si, este es unitario, si, y solo si, se cumple uno de los siguientes casos:

- $\phi$ es un automorfismo del disco y $F$ está determinada por una fórmula explícita que depende del núcleo reproductor y de $\phi$, precisamente cuando $\mathcal{H}$ es uno de los espacios $\mathcal{H}_\gamma$,

- para todos los demás espacios considerados, el operador de composición ponderado debe ser trivial, es decir $\phi$ es una rotación y $F$ es una función constante de módulo uno.

Este resultado muestra el contraste que existe entre los espacios $\mathcal{H}_\gamma$ y el resto de espacios de Hardy con peso.

Debido a la generalidad considerada hay muchas herramientas estándares que no podremos aplicar en este caso. Además, la prueba se dividirá esencialmente en dos casos dependiendo si el núcleo reproductor es acotado (ver Teorema 3.2 ) o no acotado sobre la diagonal del bidisco (ver Teorema 3.3 y Teorema 3.4). Para este segundo caso será fundamental el uso de composiciones de operadores de composición ponderados con el fin de "mover puntos en el disco" (ver Lema 3.1).

En el Capítulo 4 estudiaremos cierta *invariancia conforme ponderada* en los espacios de Banach de funciones analíticas en el disco unidad. La invariancia conforme juega un papel crucial en la teoría de los espacios de Banach de las funciones analíticas en $\mathbb{D}$. En particular, esta invariancia es una poderosa herramienta para entender las funciones con oscilación media acotada en la frontera [16], o las funciones de Bloch [96]. En los años 90, estas ideas llevaron al desarrollo de la rica teoría de los llamados $Q_p$ espacios (ver el libro de Xiao [94]) y de sus generalizaciones naturales, los espacios $Q_K$ introducidos por Essén y Wulan (ver [42] y [43]).

Todos los espacios mencionados arriba pueden ser definidos de manera análoga, esto es, usando una seminorma conformemente invariante. Para ser más precisos, siguiendo las ideas en [6], dado un espacio de Banach $X$ de funciones analíticas en $\mathbb{D}$ el cual contiene las constantes y es invariante bajo los operadores de composición con símbolo $\varphi \in Aut(\mathbb{D})$, definimos

$$\mathcal{M}_0(X) = \{f \in X : \; \|f\|_0 = \sup_{\varphi \in Aut(\mathbb{D})} \|f \circ \varphi - f \circ \varphi(0)\|_X < \infty\}, \qquad (2)$$

y $\|f\|_{\mathcal{M}_0(X)} = |f(0)| + \|f\|_0$. Así tenemos que $BMOA = \mathcal{M}_0(H^2)$, el espacio de Bloch $\mathcal{B} = \mathcal{M}_0(A^2_0)$, $Q_p = \mathcal{M}_0(D^{2,\beta})$, $\beta \in (0,1)$, donde $H^2$ es el espacio de Hardy, $A^2$ el espacio de Bergman y $D^{2,p}$ denota el espacio de Dirichlet con peso estándar (ver el Capítulo 1 y la Sección 4.2 para la definición de todos estos espacios). Finalmente, $Q_K$ se construye de la misma manera empezando con el espacio de Dirichlet con peso $K$.

Hay bastantes resultados interesantes acerca de los espacios que son conformemente invariantes, también llamados espacios invariantes de Möbius. Por ejemplo, Rubel y Timoney [76] mostraron que el espacio de Bloch es el espacio más grande definido de esta manera y Arazy, Fisher y Peetre [11] probaron que el más pequeño es el espacio de

Besov $B^1$ el cual consiste en las funciones analíticas en $\mathbb{D}$ cuyas segundas derivadas son integrables con respecto a la medida del área. Arazy and Fisher [10] demostraron que, salvo equivalencia de normas, el espacio usual de Dirichlet es el único espacio de Hilbert conformemente invariante.

En este capítulo nos basaremos en estas ideas, pero usaremos la *invariancia conforme ponderada* en lugar de la invariancia conforme clásica, es decir, estudiaremos la invariancia bajo la composición ponderada

$$f \to (\varphi')^{\alpha}(f \circ \varphi),$$

para $\alpha > 0$. Debemos notar que la invariancia conforme ponderada es totalmente distinta a la invariancia conforme clásica. Esta propiedad se relaciona con el crecimiento de las funciones en lugar de con su oscilación. De hecho en la Proposición 4.1 c) se prueba que los espacios mencionados arriba, los cuales tienen invariancia conforme, no pueden satisfacer nuestra condición para la invariancia conforme ponderada.

Una motivación para el estudio de este tipo de invariancia es que muchos ejemplos comunes de espacios de Banach de funciones analíticas en $\mathbb{D}$, como los espacios de Korenblum, los espacios de Bergman y Besov con pesos estándares y los espacios de Hardy, satisfacen esta propiedad para un $\alpha > 0$ fijo, y resulta que este tipo de invariancia conforme es responsable de varias de sus propiedades comunes. Por ejemplo, los resultados recientes en [8] muestran que en los espacios que satisfacen la condición de la invariancia conforme ponderada para algún $\alpha \in (0,1)$ la matriz de Hilbert (actuando sobre la sucesión de coeficientes de Taylor) induce un operador lineal acotado cuyo espectro está totalmente determinado por $\alpha$.

Para ser más precisos, si denotamos por $Hol(\Omega)$ al espacio localmente convexo de las funciones analíticas en el conjunto abierto $\Omega \subset \mathbb{C}$, diremos que un espacio de Banach $X$ de funciones analíticas en $\mathbb{D}$ es *conformemente invariante de índice* $\alpha = \alpha(X)$ si cumple las siguientes propiedades:

1) $X$ está contenido continuamente en $Hol(\mathbb{D})$.

2) $X$ contiene a $Hol(\rho\mathbb{D})$, para todo $\rho > 1$.

3) Existen constantes $\alpha = \alpha(X)$ y $K = K(X) > 0$, tales que para cada $\varphi \in Aut(\mathbb{D})$, el operador lineal definido por $W_{\varphi}^{\alpha} f = (\varphi')^{\alpha}(f \circ \varphi)$, es acotado en $X$ y satisface que $\|W_{\varphi}^{\alpha}\| \leq K$.

En el Capítulo 4 mostraremos algunos ejemplos y métodos de construcción y estableceremos algunas propiedades básicas de estos espacios y de ciertos operadores que actúan sobre ellos.

La Sección 4.2 estará dedicada a enumerar una serie de espacios los cuales cumplen los tres axiomas anteriores.

En la Sección 4.3 empezaremos enfatizando la relación entre algunos objetos naturales de los espacios vistos antes, como los *multiplicadores puntuales* y los *productos débiles* (productos tensoriales proyectivos) con la propiedad de invariancia conforme ponderada. Otros objetos interesantes relacionados con estos espacios son dos grupos abelianos de operadores que emergen de 3), tales como el grupo de las composiciones con rotaciones $\{R_t : \ t \in [0, 2\pi)\}$ con $R_t f(z) = f(e^{it}z)$ y la representación en $\mathcal{B}(X)$ del grupo hiperbólico $\{W^{\alpha}_{\psi_a} : a \in (-1,1)\}$, donde $\psi_a(z) = \frac{z+a}{1+az}, \ a \in (-1,1)$, así como el semigrupo de las dilataciones definido por $D_r f(z) = f(rz)$, para $r \in [0,1]$. La acotación de $D_r, \ r \in [0,1]$ se sigue directamente de 2). En general, ninguno de estos grupos es fuertemente continuo en los espacios en cuestión, mientras que el semigrupo de las dilataciones no es necesariamente fuertemente continuo en $r = 1$. Sin embargo, en la Proposición 4.4 se probará que cuando $X$ es un espacio conformemente invariante de índice $\alpha > 0$, y los polinomios son densos en $X$, entonces el grupo $\{W^{\alpha}_{\varphi} : \ \varphi \in Aut(\mathbb{D})\}$ es fuertemente continuo con respecto a la topología relativa de $Hol(\mathbb{D})$ en $Aut(\mathbb{D})$.

En la segunda parte de esta sección, asumiendo solamente la acotación uniforme de $\{R_t : \ t \in [0, 2\pi)\}$ en $\mathcal{B}(X)$ en lugar de 3), probaremos un resultado curioso (Teorema 4.1) el cual nos dice que la densidad de los polinomios en $X$ es equivalente a cualquiera de las siguientes condiciones:

(i) $t \to R_t$ es fuertemente continuo en $[-\pi, \pi]$,

(ii) $r \to D_r$ en fuertemente continuo acercándonos por la izquierda a $r = 1$,

Una condición suficiente para la densidad de los polinomios en $X$ nos la da el Teorema 4.2: que las evaluaciones puntuales sean densas en el dual de $X$. Así, este resultado implica que los polinomios serán densos en $X$ para cualquier espacio $X$ reflexivo.

Además en la última parte de esta sección veremos que si $X$ es conformemente invariante de índice $\alpha > 0$ y los polinomios son densos en $X$ entonces podemos representar el dual de $X$ como un espacio conformemente invariante del mismo índice. Esto se logra usando el emparejamiento inducido por el espacio de Hilbert $H_\alpha$ determinado por el núcleo reproductor $k^{\alpha}(z, w) = (1 - \overline{w}z)^{-2\alpha}$ (Teorema 4.3).

En la primera parte de la Sección 4.4, dado un $\alpha > 0$, determinaremos el espacio de Banach conformemente invariante más grande y el más pequeño con este índice, lo cual extiende los resultados en [76] y [11] a este contexto. El espacio más grande será el espacio de Korenblum $\mathcal{A}^{-\alpha}$ mientras que el más pequeño será o bien un espacio de Bergman con peso o un espacio de Besov con peso (Teorema 4.4). Sin embargo, el principal resultado de esta sección, al cual dedicaremos la segunda parte de la misma, es la versión del teorema de Arazy y Fisher [10] adecuada a este contexto. En el Teorema 4.5 se probará que, salvo equivalencia de normas, el espacio de Hilbert $H_\alpha$ definido en el párrafo anterior es el único espacio de Hilbert conformemente invariante de índice $\alpha > 0$. Se puede ver que $H_\alpha$ es un espacio de Bergman con peso si $\alpha > \frac{1}{2}$, el espacio de Hardy $H^2$ si $\alpha = \frac{1}{2}$ y un espacio de Dirichlet (Besov) con peso si $\alpha < 1$. Podemos observar que este resultado está relacionado con el Teorema 3.1 el cual mostraba que bajo ciertas hipótesis los únicos espacios de Hilbert que tienen operadores de composición ponderados unitarios no triviales son los espacios $\mathcal{H}_\gamma$, los cuales se relacionan con nuestros espacios $H_\gamma$ con la relación $H_\gamma = \mathcal{H}_{2\gamma}$. Además estos operadores serán precisamente los $W^{\alpha}_{\varphi}$ que aparecen en 3). En el Teorema 3.1 trabajamos con operadores unitarios lo cual nos va a implicar una identidad para el núcleo reproductor del espacio lo que será una poderosa herramienta en nuestra prueba. Sin embargo en el Teorema 4.5, al no tener esta hipótesis, la manera de abordarlo será totalmente diferente y en cierto modo estará relacionada con la idea usada en [10] donde el paso clave es la amenabilidad del grupo hiperbólico. En nuestra prueba esta propiedad se usa sólo parcialmente ya que el argumento que usaremos estará esencialmente basado en estimaciones asintóticas de $\|W^{\alpha}_{\psi_a}\zeta^n\|$ cuando $a \to 1^-$.

En la Sección 4.5 mostraremos dos aplicaciones de los resultados previos. En la primera parte estudiaremos el análogo de (2) para nuestro contexto, es decir, dado un espacio de Banach $X$ que satisface 1) y 2) consideraremos el subespacio $\mathcal{M}_\alpha(X)$ formado por las funciones $f \in X$ con $W^{\alpha}_{\varphi} f \in X,\ \varphi \in Aut(\mathbb{D})$,

$$\|f\|_{\mathcal{M}_\alpha} = \sup_{\varphi \in Aut(\mathbb{D})} \|W^{\alpha}_{\varphi} f\|_X < \infty.$$

A partir de esto veremos que $\mathcal{M}_\alpha(X)$ tiene una estructura manejable en caso que nuestro espacio inicial $X$ sea conformemente invariante de índice $\beta > 0$. En este caso, $\mathcal{M}_\alpha(X)$ es o bien trivial, es decir $\mathcal{M}_\alpha(X) = \{0\}$, o coincide con $X$, o es un espacio de multiplicadores puntuales (Teorema 4.6). Por otra parte, si $X$ no es conformemente invariante para ningún índice $\beta > 0$, mostraremos mediante el Ejemplo 4.8 que $\mathcal{M}_\alpha(X)$ puede tener una estructura muy complicada, la cual difiere de los espacios vistos hasta el momento.

El segundo tema que trataremos en la Sección 4.5 será la interpolación compleja. Solo consideraremos el par de espacios dado por el espacio conformemente invariante de índice $\alpha > 0$ más grande y el más pequeño y usaremos la idea clásica de E.M. Stein [88] para mostrar que esta cadena de espacios está formada por espacios de Bergman con peso y para $\alpha < 1$, por espacios de Besov con peso (Teorema 4.6). Bastante sorprendente resulta que los espacios de Hardy no aparezcan en estas cadenas.

Por último, la Sección 4.6 estará dedicada a tres tipos de operadores los cuales actúan en espacios de Banach conformemente invariantes de índice $\alpha > 0$: diferenciación, tomar anti-derivadas y el operador de integración general de la forma $f \to T_g f$, donde

$$T_g f(z) = \int_0^z f(t) g'(t) dt,$$

con el símbolo $g \in Hol(\mathbb{D})$ fijado. Para los primeros dos casos consideraremos el rango de estos operadores con la norma inducida. La intuición basada en las llamadas identidades de Littlewood-Paley en los espacios de Bergman con peso, o $H^2$, sugiere que si $X$ es conformemente invariante de índice $\alpha > 0$, entonces

(i) El espacio de las derivadas $D(X) = \{f' :\ f \in X\}$ con la norma inducida es conformemente invariante de índice $\alpha + 1$.

(ii) Cuando $\alpha > 1$, el espacio de las anti-derivadas $A(X) = \{f :\ f' \in X\}$ con la norma inducida es conformemente invariante de índice $\alpha - 1$.

Sin embargo el Teorema 4.7 nos revela que, bajo la hipótesis de que los polinomios son densos en $X$, las afirmaciones anteriores dependen exclusivamente de las propiedades del operador que actúa sobre la sucesión de coeficientes de Taylor tomando las medias de Cesàro. Para ser más precisos su versión modificada

$$\mathcal{C}\left(\sum_{n\geq 0} f_n \zeta^n\right) = \sum_{n\geq 0} \zeta^{n+1}\left(\frac{1}{n+1}\sum_{k=0}^{n} f_k\right) \Leftrightarrow \mathcal{C}f(z) = \int_0^z \frac{f(t)}{1-t} dt.$$

El Teorema 4.7 nos mostrará que (i) es cierto si, y solo si, $\mathcal{C} \in \mathcal{B}(X)$. Además, en este caso se tendrá (ii) si, y solo si, $I_X - \mathcal{C}$ es invertible en $X$.

En la segunda parte de la Sección 4.6 veremos una aplicación del Teorema 4.7 a los operadores de integración. Existe mucha literatura que trata estos operadores (ver, por ejemplo, [1]). Incluso con la generalidad en la que estamos trabajando, la acotación de $T_g$ se puede caracterizar en términos de $g$ (ver Proposición 4.8). Siguiendo con la idea de Pommerenke [73] el cual usa la resolvente de estos operadores para deducir la conocida desigualdad de John-Nirenberg para las funciones en $BMOA$, nosotros mostraremos que una desigualdad similar también se satisface en el contexto general. Para ser más precisos, probaremos que si $T_g$ es acotado en un espacio de Banach $X$ conformemente invariante de índice $\alpha > 0$, entonces existe $\delta > 0$ tal que

$$\{\exp[\lambda(g \circ \varphi - g \circ \varphi(0))] : \ |\lambda| \leq \delta, \ \varphi \in Aut(\mathbb{D})\}$$

es un subconjunto acotado de $X$.

Finalmente, en el Capítulo 5 vamos a considerar espacios generales de Banach de funciones analíticas en un dominio acotado definidos de forma axiomática. En estos espacios vamos a estudiar ciertas propiedades de los operadores de composición ponderados, siendo una de ellas su invertibilidad (equivalentemente su sobreyectividad).

Para ser más precisos, en este capítulo vamos a considerar espacios de Banach $X \subset Hol(\Omega)$, donde $\Omega \subset \mathbb{C}$ es un dominio acotado, los cuales satisfacen los siguientes axiomas:

- **A1:** Todas las evaluaciones puntuales $l_z$ están acotadas en $X$.
- **A2:** $1 \in X$, donde $1(z) \equiv 1$.
- **A3:** Para cualquier $f \in X$, la función $\zeta f$ también está en $X$, donde $\zeta(z) = z$.
- **A4:** Para cada $f \in X$ y cada $u \in H^\infty(\Omega)$ que no se anula en $\Omega$, si $fu^n \in X$ para todo $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ entonces $fu^\alpha \in X$ para algún valor $\alpha > 0$ no entero.
- **A5:** Cada automorfismo de $\Omega$ induce un operador de composición acotado en $X$.

En la segunda parte de la Sección 5.2 veremos una serie de ejemplos de espacios clásicos de funciones analíticas, los cuales satisfacen nuestros axiomas. Entre ellos se encuentran los espacios de Hardy, Bergman con peso, Besov con peso, el espacio $BMOA$ y el de Bloch.

En la última parte de esta sección describiremos todos los operadores de composición ponderados invertibles que actúan en estos espacios generales de Banach (Teorema 5.1). Veremos que en un espacio de Banach $X \subset Hol(\Omega)$, que satisface los axiomas **A1** - **A4**, si un operador de composición ponderado acotado $W_{F,\phi}$ es invertible entonces su símbolo de composición $\phi$ es un automorfismo de $\Omega$, $F$ no se anula y $W_{F,\phi}^{-1}$ es otro operador de composición ponderado cuyos símbolos son

$$G = \frac{1}{F \circ \phi^{-1}}, \qquad \psi = \phi^{-1}.$$

Además, si el espacio $X$ también satisface **A5** obtenemos la siguiente caracterización. Un operador de composición ponderado acotado $W_{F,\phi}$ es invertible si y solo si $\phi$ es un automorfismo de $\Omega$, $F$ no se anula y $1/F \in Mult(X)$.

Por otro lado, en la Sección 5.3 solo consideraremos espacios de Banach de funciones analíticas en el disco unidad satisfaciendo los axiomas **A1** - **A4** y cuya norma viene dada por la fórmula $\|f\| = |f(0)| + \rho(f)$, donde $\rho(f)$ es una seminorma invariante por traslaciones. Por ejemplo, los espacios de Besov con peso, el espacio de Bloch y $BMOA$ cumplen estas propiedades. Para estos espacios probaremos que los únicos operadores de composición ponderados que son sobreyectivos e isométricos son los triviales, es decir, cuando el símbolo de multiplicación es una constante de módulo uno y el símbolo de composición es una rotación. Ver Teorema 5.2.

# Introduction

This thesis treats a number of problems related to weighted composition operators. We will consider the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and we will study how these operators act on the spaces of analytic functions in $\mathbb{D}$.

Let $f$ be an analytic function in $\mathbb{D}$. The *weighted composition operator* with symbols $F$ and $\phi$ is defined by

$$W_{F,\phi} f = F(f \circ \phi),$$

where $F$ and $\phi$ are analytic functions on $\mathbb{D}$ and $\phi(\mathbb{D}) \subset \mathbb{D}$. There is a particular case of these operators that will appear repeatedly in many results. It will be when $\phi$ is a disc automorphism, *i.e.* when $\phi = \varphi$ where $\varphi$ is a linear fractional map of the form

$$\varphi(z) = \lambda \frac{z + a}{1 + \overline{a} z}, \ z \in \mathbb{D}, \quad a \in \mathbb{D}, \ |\lambda| = 1,$$

mapping the unit disc onto itself. We will denote by $Aut(\mathbb{D})$ the group of automorphisms of the unit disc. The weighted composition operators are a natural generalization of the *pointwise multiplication operators* and the *composition operators*, which have been studied extensively in the context of spaces of analytic functions on the unit disc.

In general, given a family of linear operators acting on a Banach space of analytic functions in the unit disc it is interesting to know when the operators of this family satisfy certain properties. For instance, when the operators are bounded, surjective, isometries, and so on. In many Banach spaces of analytic functions on the unit disc all the isometries are known or, at least, all the surjective isometries (unitary operators). For example, in analogy with the classical Banach-Lamperti theorem for $L^p$ spaces, Forelli in [46] proved that all the isometric linear operators acting on the Hardy spaces $H^p$ (excluding the case $p = 2$) are weighted composition operators whose symbols satisfy certain properties.

The Hilbert spaces have plenty of isometric operators; they may even have many unitary operators. Therefore, a natural question is to try to characterize the unitary operators of a certain type, for instance weighted composition operators.

In the Chapter 3 we will focus on answering this question in a large family of Hilbert spaces of analytic functions on the unit disc which satisfy the following axioms:

**(A1)** The point evaluations are bounded (hence our space is a reproducing kernel Hilbert space);

**(A2)** The reproducing kernel $K_w(z)$ is normalized so that $K_w(0) = 1$ for all $w \in \mathbb{D}$;

(**A3**) The monomials $\{z^n : n = 0, 1, 2, \dots\}$ belong to $\mathcal{H}$ and form a complete orthogonal set.

These Hilbert spaces are called *weighted Hardy spaces*. To be precise, in Chapter 3 we will characterize the weighted composition operators in the weighted Hardy spaces which are *co-isometric* ($TT^* = I$ where $T^*$ denotes the adjoint operator of $T$ and $I$ denotes the identity operator). To be co-isometric is *a priori* weaker than to be unitary ($T^*T = TT^* = I$).

In the first part of Chapter 3, it will be proved that only with the axioms defined above we can deduce a number of properties that the weighted Hardy spaces satisfy. This can be found in Proposition 3.1 where it will be shown that if $\mathcal{H}$ is a Hilbert space of analytic functions on the unit disc that contains all monomials and satisfies the axioms (**A1**) and (**A2**), then the following statements are equivalent:

(a) Axiom $(\mathbf{A3})$ is fulfilled; that is, the monomials $\{z^n : n = 0, 1, 2, \dots\}$ form a complete orthogonal set in $\mathcal{H}$.

(b) The reproducing kernel has the form

$$K_w(z) = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n, \tag{3}$$

with $\gamma(n) = \|z^n\|^{-2}$.

(c) The norm of a function $f \in \mathcal{H}$ whose Taylor series in $\mathbb{D}$ is $f(z) = {}^{\infty}_{n=0}\, a_n z^n$ is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \|z^n\|^2 \,.$$

(d) The rotations $R_\lambda$ induce isometric composition operators $C_{R_\lambda}$ on $\mathcal{H}$.

(e) The rotations $R_\lambda$ induce unitary composition operators $C_{R_\lambda}$ on $\mathcal{H}$.

(f) The constant multipliers of modulus one ($|\mu| = 1$) and rotations $R_\lambda$ induce unitary weighted composition operators $W_{\mu, R_\lambda}$ on $\mathcal{H}$.

(g) $K_{\lambda w}(\lambda z) = K_w(z)$ for all $z$, $w \in \mathbb{D}$ and all $\lambda$ with $|\lambda| = 1$.

In the second part of this chapter, the main result will be proved, Theorem 3.1. In this result an important role is played by the family of weighted Hardy spaces $\mathcal{H}_\gamma$ with $\gamma > 0$ whose reproducing kernel is given by the formula

$$K_w^\gamma(z) = \frac{1}{(1 - \overline{w}z)^\gamma} = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n \,, \qquad z, w \in \mathbb{D} \,.$$

Theorem 3.1 shows that a bounded weighted composition operator in a weighted Hardy space is co-isometric if and only if it is unitary, if and only if one of the following cases occurs:

- $\phi$ is a disc automorphism and $F$ is determined by an explicit formula, depending on $\phi$ and the reproducing kernel, precisely when $\mathcal{H}$ is one of the spaces $\mathcal{H}_\gamma$,

- for all the remaining spaces considered, the weighted composition operator must be of trivial type: $\phi$ is a rotation and $F$ is a constant function of modulus one.

This shows a sharp contrast between the spaces $\mathcal{H}_\gamma$ and the remaining weighted Hardy spaces.

Because of the generality considered here several standard tools will be no longer available. Hence, the proof will become more involved. Moreover, the proof will be split essentially into two cases depending on whether the reproducing kernel is bounded (Theorem 3.2) or unbounded on the diagonal of the bidisc (Theorem 3.3 and Theorem 3.4). For this second case, we will use compositions of weighted composition operators in order to "move points around the disc" (Lemma 3.1).

In Chapter 4 we will study certain weighted conformal invariance in the Banach spaces of analytic functions on the unit disc. Conformal invariance plays a crucial role in the theory of Banach spaces of analytic functions on $\mathbb{D}$. In particular, it turns out to be a powerful tool in understanding analytic functions with bounded mean oscillation on the boundary [16], or Bloch functions [96]. In the 1990s, these ideas led to the rich theory of the so-called $Q_p$-spaces (see Xiao's book [94]) and their natural generalization, the $Q_K$-spaces introduced by Essén and Wulan (see [42], [43]).

All of the spaces mentioned here can be defined following a common pattern, that is, using a conformally invariant seminorm. More precisely, following the ideas in [6], let $X$ be a Banach space of analytic functions in $\mathbb{D}$ which contains the constants and is invariant under the operators of composition with symbol $\varphi \in Aut(\mathbb{D})$, and set

$$\mathcal{M}_0(X) = \{f \in X : \ \|f\|_0 = \sup_{\varphi \in Aut(\mathbb{D})} \|f \circ \varphi - f \circ \varphi(0)\|_X < \infty\}, \tag{4}$$

and $\|f\|_{\mathcal{M}_0(X)} = |f(0)| + \|f\|_0$. Then it turns out that $BMOA = \mathcal{M}_0(H^2)$, the Bloch space satisfies $\mathcal{B} = \mathcal{M}_0(A^2_0)$, $Q_p = \mathcal{M}_0(D^{2,\beta})$, $\beta \in (0,1)$, where $H^2$ is the Hardy space, $A^2$ the Bergman space and $D^{2,\beta}$ denotes the standard weighted Dirichlet space (see Chapter 1 and Section 4.2 for the definitions of these spaces). Finally, $Q_K$ is constructed in the same way starting with the weighted Dirichlet space with weight $K$.

There are a number of interesting results concerning such conformal invariant spaces also called *Möbius invariant spaces*. For example, Rubel and Timoney [76] showed that the Bloch space is the largest space defined this way and Arazy, Fisher and Peetre [11] proved that the smallest is the Besov space $B^1$ consisting of analytic functions in $\mathbb{D}$ whose second derivative is integrable with respect to area measure. Arazy and Fisher [10] proved that, up to equivalence of norms, the unweighted Dirichlet space is the only Hilbert space which occurs this way.

In this chapter, we will follow this line of investigation, but we will use the weighted conformal invariance property instead of the classical conformal invariance, that is, we will study the invariance under the weighted composition

$$f \to (\varphi')^\alpha (f \circ \varphi),$$

where $\alpha > 0$ is fixed. We should point out from the beginning that weighted conformal invariance is completely different from the classical conformal invariance. This type of condition is related to the growth rather than to the oscillation of functions. In fact, according to 4.1 c), the spaces which have conformal invariance cannot satisfy our conditions for weighted conformal invariance.

One motivation for the study of this invariance is the fact that most common examples of Banach spaces of analytic functions in $\mathbb{D}$ like Korenblum spaces, standard weighted Bergman and Besov spaces and Hardy spaces satisfy this property for a fixed $\alpha > 0$, and it turns out that this type of conformal invariance is responsible for a number of their common properties. For example, the recent results in [8] show that for spaces satisfying the weighted conformal invariance condition for some $\alpha \in (0,1)$ the usual Hilbert matrix (acting on the sequence of Taylor coefficients) induces a bounded linear operator whose spectrum is completely determined by $\alpha$.

To be more precise, if we denote $Hol(\Omega)$ the locally convex space of analytic functions in the open set $\Omega \subset \mathbb{C}$ we will say that a Banach space $X$ consisting of analytic functions in the unit disc $\mathbb{D}$ is *conformally invariant of index* $\alpha = \alpha(X)$ if $X$ satisfies the following properties:

1) $X$ is continuously contained in $Hol(\mathbb{D})$.

2) $X$ contains $Hol(\rho\mathbb{D})$, for all $\rho > 1$.

3) There exist constants $\alpha = \alpha(X),\ K = K(X) > 0$, such that for every $\varphi$ in $Aut(\mathbb{D})$, the linear map defined by $W_\varphi^\alpha f = (\varphi')^\alpha (f \circ \varphi)$, is bounded on $X$ and satisfies $\|W_\varphi^\alpha\| \leq K$.

In Chapter 4 we will show some examples and methods of construction and also establish some of the basic properties of such spaces and of some operators acting on them.

Section 4.2 will be devoted to giving a number of examples of natural spaces which fulfill the above axioms.

In Section 4.3 we will begin by emphasizing some natural objects related to such spaces, like the *pointwise multipliers* and their *weak products* (projective tensor products) and their relation with the weighted conformal invariant property. Other interesting objects related to these spaces are two Abelian groups of operators emerging from 3), namely the group of composition with rotations $\{R_t : \ t \in [0, 2\pi)\}$ with $R_t f(z) = f(e^{it}z)$, the representation on $\mathcal{B}(X)$ of the hyperbolic group $\{W_{\psi_a}^\alpha : a \in (-1,1)\}$, where $\psi_a(z) = \frac{z+a}{1+az},\ a \in (-1,1)$, together with the semigroup of dilations defined for $r \in [0,1]$ by $D_r f(z) = f(rz)$. The boundedness of $D_r,\ r \in [0,1]$ follows directly from 2). In general, none of these groups is strongly continuous on the spaces in question, while the semigroup is not necessarily strongly continuous at $r = 1$. However, Proposition 4.4 will show that when $X$ is a weighted conformal invariant space of index $\alpha > 0$, and the polynomials are dense in $X$, then the full group $\{W_\varphi^\alpha : \ \varphi \in Aut(\mathbb{D})\}$ becomes strongly continuous with respect to the relative topology of $Hol(\mathbb{D})$ on $Aut(\mathbb{D})$.

In the second part of this section, assuming only the uniform boundedness of $\{R_t : \ t \in [0, 2\pi)\}$ in $\mathcal{B}(X)$ instead of 3), we will arrive at the curious result (Theorem 4.1) that the density of the polynomials in $X$ is equivalent to any of the following statements:

(i) The strong continuity of $t \to R_t,\ t \in [0, 2\pi)$.

(ii) The strong continuity of $r \to D_r$ at $r = 1$ (from the left).

A sufficient condition for the density of polynomials in $X$ is given by Theorem 4.2: the point evaluations should be dense in the dual of $X$. At its turn, this result implies that polynomials are dense in $X$ whenever the space is reflexive.

Moreover, in the last part of this section, we will see that if $X$ is conformally invariant with index $\alpha > 0$ and polynomials are dense in $X$ then we can represent the dual of $X$ as a conformally invariant space of the same index. This is achieved using the pairing induced by the Hilbert space $H_\alpha$ determined by the reproducing kernel $k^\alpha(z, w) = (1 - \overline{w}z)^{-2\alpha}$ (Theorem 4.3).

In the first part of Section 4.4, we will determine the largest and smallest conformally invariant Banach space of a given index $\alpha > 0$, which extends the results in [76] and [11] to this context. The largest space is the Korenblum space $\mathcal{A}^{-\alpha}$ while the smallest is either a weighted Bergman or a weighted Besov space (Theorem 4.4). However, the main result of this section is the appropriate version of the Arazy-Fisher theorem [10] in this context. In Theorem 4.5 we will show that, up to equivalence of norms, the Hilbert space $H_\alpha$ defined above is the unique conformally invariant Hilbert space of index $\alpha > 0$. It can be seen that $H_\alpha$ is a weighted Bergman space when $\alpha > \frac{1}{2}$, $H_{\frac{1}{2}}$ is the Hardy space $H^2$, while for $\alpha < \frac{1}{2},\ H_\alpha$ is a weighted Dirichlet (Besov) space. We should note that this result is related to the Theorem 3.1, which claimed that under certain assumptions the unique Hilbert spaces that have no trivial unitary weighted composition operators are the spaces $\mathcal{H}_\gamma$, which are related with our spaces $H_\gamma$ with the relation $H_\gamma = \mathcal{H}_{2\gamma}$. Moreover, these operators will be the $W_\varphi^\alpha$ that appear in 3). In Theorem 3.1 we work with unitary operators. Hence, this implies an identity for the reproducing kernel of the space which will become a powerful tool in the proof of this theorem. However, in Theorem 4.5 we do not have this assumption so the approach is considerably more involved and is somewhat related to the idea in [10] where the key step is the amenability of the hyperbolic group. In our proof this property is only partly used, since our argument is essentially based on asymptotic estimates of $\|W_{\psi_a}^\alpha \zeta^n\|$ when $a \to 1^-$.

In Section 4.5 we will show two applications of the previous results. In the first application, we will focus on the analogue of (4) for our context, i.e. for a given Banach space $X$ satisfying 1) and 2) we will consider the subspace $\mathcal{M}_\alpha(X)$ consisting of the functions $f \in X$ with $W_\varphi^\alpha f \in X,\ \varphi \in Aut(\mathbb{D})$,

$$\|f\|_{\mathcal{M}_\alpha} = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|_X < \infty.$$

It turns out that $\mathcal{M}_\alpha(X)$ has a tractable structure in the case when the original space $X$ is itself conformally invariant of index $\beta > 0$. In this case, $\mathcal{M}_\alpha(X)$ is either trivial, *i.e.* $\mathcal{M}_\alpha(X) = \{0\}$, equal to $X$, or it is a space of pointwise multipliers (Proposition 4.6). On the other hand, if $X$ is not conformally invariant for any $\beta > 0$, we will show by the Example 4.8 that $\mathcal{M}_\alpha(X)$ may have a very complicated structure which differs from the examples presented until now.

The second topic in Section 4.5 will be the complex interpolation. We only consider the pair given by the largest and smallest conformally invariant spaces of a given index $\alpha > 0$ and use the classical idea of E.M. Stein [88] to show that this chain of spaces consists of weighted Bergman and for $\alpha < 1$, weighted Besov spaces (Theorem 4.6). Surprisingly enough, the Hardy spaces are excluded from the chains.

Finally, Section 4.6 will be devoted to three types of operators acting on conformally invariant Banach spaces of index $\alpha > 0$: differentiation, taking the anti-derivative and general integration operators of the form $f \to T_g f$, where

$$T_g f(z) = \int_0^z f(t)g'(t)dt,$$

with symbol $g \in Hol(\mathbb{D})$ is fixed. In the first two cases we will consider the ranges of these operators with the induced norm. The common intuition based on the so-called Littlewood-Paley identity (estimate) in weighted Bergman spaces, or $H^2$, suggests that if $X$ is conformally invariant of index $\alpha > 0$, then

(i) The space of derivatives $D(X) = \{f' : \ f \in X\}$ with the induced norm is conformally invariant of index $\alpha + 1$.

(ii) When $\alpha > 1$, the space of anti-derivatives $A(X) = \{f : \ f' \in X\}$ with the induced norm is conformally invariant of index $\alpha - 1$.

However, Theorem 4.7 will reveal that under the assumption the polynomials are dense in $X$, both assertions above depend entirely on the properties of the linear map which acts on the sequence of Taylor coefficients by taking the Cesàro means, or more precisely the modified version

$$\mathcal{C}\left(\sum_{n\geq 0} f_n \zeta^n\right) = \sum_{n\geq 0} \zeta^{n+1} \quad \frac{1}{n+1}\sum_{k=0}^{n} f_k \quad \Leftrightarrow \ \mathcal{C}f(z) = \int_0^z \frac{f(t)}{1-t}dt.$$

Theorem 4.7 will show that (i) holds true if and only if $\mathcal{C} \in \mathcal{B}(X)$. Moreover, in this case (ii) holds true if and only if $I_X - \mathcal{C}$ is invertible on $X$.

In the second part of Section 4.6, we will see an application of Theorem 4.7 regarding the integration operators defined above. There is a vast literature on the subject (see for example, [1]). Even in this generality, the boundedness of $T_g$ can be characterized in terms of $g$ (see Proposition 4.8). Following the idea of of Pommerenke [73] who used the resolvent of such operators to derive the well-known John-Nirenberg inequality for $BMOA$ functions. We will show that a similar inequality holds in the general context as well. More precisely, we prove that if $T_g$ is bounded on the conformally invariant Banach space $X$ then there exists $\delta > 0$ such that

$$\{\exp[\lambda(g \circ \varphi - g \circ \varphi(0))] : \ |\lambda| \leq \delta, \ \varphi \in Aut(\mathbb{D})\}$$

is a bounded subset of $X$.

Finally, in Chapter 5 we will consider general Banach spaces of analytic functions in a bounded domain defined axiomatically. In these spaces, we are going to study certain properties of the weighted composition operators, like their invertibility (equivalently, their surjectivity).

To be more precise, in this chapter we will consider Banach spaces $X \subset Hol(\Omega)$, where $\Omega \subset \mathbb{C}$ is a bounded domain, which satisfy the following axioms:

- **A1:** All point evaluation functionals $l_z$ are bounded on $X$.
- **A2:** $1 \in X$, where $1(z) \equiv 1$.
- **A3:** Whenever $f \in X$, the function $\zeta f$ is also in $X$, where $\zeta(z) = z$.
- **A4:** For every $f \in X$ and every $u \in H^\infty(\Omega)$ that do not vanish in $\Omega$, if $fu^n \in X$ for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ then $fu^\alpha \in X$ for some non-integer value $\alpha > 0$.
- **A5:** Each automorphism of $\Omega$ induces a bounded composition operator in $X$.

In the second part of Section 5.2, we will see some examples of classical spaces of analytic functions that satisfy our axioms. Between them, we can find the Hardy spaces, the weighted Bergman and weighted Besov spaces, the Bloch space, and the $BMOA$ space.

In the last part of this section, we will describe all invertible weighted composition operators acting on these general Banach spaces (Theorem 5.1). We will see that if $X \subset Hol(\Omega)$ is a Banach space that satisfies the Axioms **A1** - **A4**, if a bounded weighted composition operator $W_{F,\phi}$ is invertible, then its composition symbol $\phi$ is an automorphism of $\Omega$, $F$ does not vanish in $\Omega$ and $W_{F,\phi}^{-1}$ is a weighted composition operator with symbols

$$G = \frac{1}{F \circ \phi^{-1}}, \qquad \psi = \phi^{-1}.$$

Moreover, if the space $X$ also satisfies **A5** we will get the following characterization. A weighted composition operator $W_{F,\phi}$ is invertible if and only if $\phi$ is an automorphism of $\Omega$, $F$ does not vanish and $1/F \in Mult(X)$.

On the other hand, in Section 5.3 we will only consider Banach spaces of analytic functions in the unit disc, which satisfy the Axioms **A1** - **A4**, and whose norm is given by the formula $\|f\| = |f(0)| + \rho(f)$, where $\rho$ is a seminorm which is translation-invariant. For instance, the weighted Besov spaces, the Bloch space, and the $BMOA$ space satisfy these properties. For these spaces we will prove that the only surjective isometric weighted composition operators on $X$ are trivial, *i.e.* their multiplication symbol is a unimodular constant and their composition symbol is a rotation. See Theorem 5.2.

# 1 Function Spaces and Basic Notions

In this chapter we will introduce some spaces consisting of analytic functions in the unit disc, $Hol(\mathbb{D})$, that we will use throughout the thesis. There is a property shared by these spaces, the point evaluations are bounded *i.e.*, by a direct application of the uniform boundedness principle, the spaces are continuously included in $Hol(\mathbb{D})$. Thanks to this property we can use the closed graph theorem to get some interesting relation between these spaces.

## 1.1 F-spaces. Closed graph theorem

Usually, the closed graph theorem is used in Banach spaces, but for our purpose, we need the general version for $F$-spaces. The following definitions can be found in [78, Page 9] and the closed graph theorem in [78, Page 51].

**Definition 1.1.** A metric $d$ on a vector space $X$ will be called *translation invariant* if

$$d(x+z, y+z) = d(x, y)$$

for all $x, y, z \in X$.

**Definition 1.2.** Let $X$ be a topological vector space with topology $\tau$. $X$ is an $F$*-space* if its topology $\tau$ is induced by a complete invariant metric $d$.

In an $F$*-space*, continuity is understood in the usual sense of convergence induced by the metric.

**Theorem 1.1** (Closed graph theorem)**.** *Suppose $X$ and $Y$ are $F$-spaces, $\Lambda : X \to Y$ a linear map and $G = \{(x, \Lambda x) : x \in X\}$ is closed in $X \times Y$. Then $\Lambda$ is continuous.*

The following corollary will be useful for us later. In this corollary we will use the fact that if a metric space $X$ is continuously contained in $Hol(\mathbb{D})$, then the inclusion map is continuous from $X$ to $Hol(\mathbb{D})$. So, the convergence in $X$ implies the convergence in $Hol(\mathbb{D})$ (uniform convergence in compact sets of $\mathbb{D}$).

**Corollary 1.1.** *If $X$ and $Y$ are $F$-spaces consisting of analytic functions in the unit disc $\mathbb{D}$ such that both are continuously contained in $Hol(\mathbb{D})$ and $Y \subset X$ then the inclusion is continuous.*

*Proof.* This is a direct consequence of the closed graph theorem. Let

$$\begin{aligned} I:\ & Y \to X \\ & f \to I(f) = f \end{aligned}$$

be the inclusion map. We suppose there exists a sequence $\{f_n\}$ in $Y$ such that $f_n \to_Y f$ and $I(f_n) \to_X g$. If we can prove $f = g$ we are done. Since $Y$ is continuously contained in $Hol(\mathbb{D})$, the convergence in $Y$ implies uniform convergence in compact sets of $\mathbb{D}$. In the same way in $X$. Then we have $f$ and $g$ are equals in each compact set of $\mathbb{D}$ so $g = f$. □

## 1.2 Automorphisms and basic conformal invariance

Another property of a space $X$ of analytic function in the unit disc that will play a important role later will be the conformal invariance. We consider $Aut(\mathbb{D})$, the group of conformal automorphisms of the unit disc in the complex plane *i.e.*, linear fractional maps of the form

$$\varphi(z) = \lambda \frac{z+a}{1+\bar{a}z},\ z \in \mathbb{D}, \quad a \in \mathbb{D},\ |\lambda| = 1,$$

mapping the unit disc onto itself. Therefore the space $X$ equipped with a seminorm $\rho$ is called *conformally invariant* if for each $f \in X$ we have $f \circ \varphi \in X$ for all $\varphi \in Aut(\mathbb{D})$ and there exists $C > 0$ such that

$$\rho(f \circ \varphi) \leq C\rho(f) \quad \text{for all } f \in X \text{ and } \varphi \in Aut(\mathbb{D}).$$

## 1.3 Poisson kernel

An important tool when we are working with analytic functions is the Poisson Kernel. The *Poissson Kernel* at $r \in [0, 1)$, is given by

$$P_r(e^{it}) = \frac{1-r^2}{|e^{it}-r|^2} = \frac{1-r^2}{1-2r\cos(t)+r^2}.$$

This kernel satisfies, among other properties, that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it})\,dt = 1$ for any $r \in [0, 1)$. The Poisson kernel is the key to solve the original *Dirichlet problem* posed for the Laplace equation. Given a continuous function $u$ on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ the *Poisson integral*, defined by

$$P[u](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)})u(e^{it})\,dt,$$

is the unique harmonic function on $\mathbb{D}$ continuous on $\overline{\mathbb{D}}$ such that $P[u]|_{\partial\mathbb{D}} = u$. For our purpose, this result tells us that given an analytic function, $f$, in the unit disc then, for $z = re^{i\theta} \in D(0,R) = \{z \in \mathbb{C} : |z| < R < 1\}$, it satisfies that

$$f(rRe^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)})f(Re^{it})\,dt.$$

## 1.4 Hardy spaces

The study of these spaces started with a question posed by H. Bohr and E. Landau to G. H. Hardy about the growth of the integral means. Concretely, Hardy in [52] proved that if $f$ is an analytic function on the unit disc then its integral mean

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

or

$$M_\infty(r,f) = \max_{0 \le t < 2\pi} |f(re^{it})|$$

is an increasing function of $r$ and $\log M_p(r,f)$ is a convex function of $\log r$ (for the proof see [40, Chapter 1]).

**Definition 1.3.** An analytic function $f$ on the unit disc is in the *Hardy space* $H^p$ if

$$\sup_{0<r<1} M_p(r,f) < \infty, \quad 0 < p < \infty$$

and $f$ is in $H^\infty$ if $\sup_{0<r<1} M_\infty(r,f) < \infty$.

The Hardy spaces are among the most important spaces of analytic functions and one in which many authors have worked in complex analysis, see for example [40] and [60] for basic monographs on theses spaces. By a direct aplication of Hölder inequality we obtain the following chain of inclusions $H^\infty \subsetneq H^p \subsetneq H^q$ for $p > q > 0$. The family of functions $f_a(z) = (1-z)^{-a}$, with $a > 0$, shows that the inclusions are strict since it is well known that $f_a$ is in $H^p$ if and only if $p < \frac{1}{a}$. Moreover, for $1 \le p \le \infty$ the space $H^p$ becomes a Banach space with the norm

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r) = \lim_{r\to 1^-} M_p(f,r) \quad \text{when } 1 \le p < \infty,$$

and

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

We also have that when $0 < p < \infty$ the polynomials are dense in $H^p$ and for all $z \in \mathbb{D}$

$$|f(z)| \le \frac{\|f\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}}},$$

(see [40, Exercise 8.4]) *i.e.* $H^p$ is continuously contained in $Hol(\mathbb{D})$. The particular case when $p = 2$, $H^2$ becomes a Hilbert space with the inner product

$$\langle f, g\rangle_{H^2} = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt$$

and if $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ it can be proved that

$$\langle f, g\rangle_{H^2} = \sum_{n=0}^{\infty} a_n\overline{b_n}$$

so $\|f\|_{H^2} = \sum_{n\geq 0} |a_n|^2$. One of the most important properties of Hardy spaces is the behaviour of their functions at the boundary. The existence of radial limits almost everywhere was proved in greater generality by Fatou.

**Theorem 1.2.** *If $f \in H^p$ with $p > 0$, then $\tilde{f}(e^{it}) := \lim_{r\to 1^-} f(re^{it})$ exists almost everywhere and $\tilde{f} \in L^p(\mathbb{T})$. Moreover,*

$$\|f\|_{H^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |\tilde{f}(e^{it})|^p\,dt\right)^{\frac{1}{p}} \quad \textit{when}\ 1 \leq p < \infty,$$

*and $\|f\|_{H^p} = \|\tilde{f}\|_{L^p}$.*

See [40, Chapters 1, 2] for the proof of these facts. Another property of Hardy spaces is the canonical factorization, obtained first by F. Riesz and then in a more refined form by Smirnov. This factorization shows that every function in a Hardy space can be written as a product of a *Blaschke product*, a *singular inner* function and an *outer* function (see [40, Theorem 2.8]).

The following relationship between the mean growth of an analytic function and that of its derivative (see [40, Theorem 5.5]) will be very useful.

**Theorem 1.3.** *Let $0 < p \leq \infty$, $\beta > 0$ and $f \in Hol(\mathbb{D})$. Then*

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\beta}}\right)$$

*if and only if*

$$M_p(r, f') = O\left(\frac{1}{(1-r)^{\beta+1}}\right).$$

## 1.5 Bergman spaces

Another family of important spaces of analytic functions in the unit disc are the Bergman spaces. The theory has its origin in the study of the reproducing kernel by S. Bergman but was only developed later, in the second half of the 20th century by a number of authors. Here the main references that we use are [41] and [54].

**Definition 1.4.** Given a function, $f$, analytic in $\mathbb{D}$ and $0 < p < \infty$, it is said to belong to the *Bergman space* $A^p$ if

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z) = \frac{1}{\pi} \, dx \, dy = \frac{1}{\pi} \, dt \, r \, dr$.

Thus, we can define the Bergman space as $A^p = Hol(\mathbb{D}) \cap L^p(\mathbb{D}, dA)$. When $1 \leq p < \infty$, $A^p$ becomes a Banach space with the norm $\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z)$. The following identity relates the Bergman norm with integral means used to define Hardy spaces:

$$\|f\|_{A^p}^p = 2 \int_0^1 r M_p^p(r, f) \, dr.$$

So, with this identity it is clear that $H^p \subset A^p$. In fact, Hardy and Littlewood proved that $H^p \subset A^{2p}$ (an elementary proof of this fact can be found in [93]). An important difference between Hardy and Bergman spaces is the behaviour of their functions on the boundary since we can find functions in all Bergaman spaces with a wild behaviour in the boundary. If we consider a non-negative integrable function $w$, a natural generalization of these spaces are the spaces called *weighted Bergman spaces* $A_w^p$, where $w$ is called a *weight* and

$$\|f\|_{A_w^p}^p = \int_{\mathbb{D}} |f(z)|^p w(z) \, dA(z), \quad 1 \leq p < \infty.$$

If $w(z) = (\beta + 1)(1 - |z|^2)^\beta$ with $\beta > -1$ we have the *standard weighted Bergman spaces* $A_\beta^p$ with

$$\|f\|_{A_\beta^p}^p = (\beta + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta \, dA(z), \quad 1 \leq p < \infty.$$

The particular case when $p = 2$, $A_\beta^2$ becomes a Hilbert space with the inner product

$$\langle f, g \rangle_{A_\beta^2} = (\beta + 1) \int_{\mathbb{D}} f(z) \overline{g(z)} (1 - |z|^2)^\beta \, dA(z).$$

Two basic properties of these spaces are the density of polynomials and the boundedness of the point evaluation.

**Theorem 1.4.** *The polynomials form a dense subset of the Bergman space $A_\beta^p$ where $0 < p < \infty$ and $\beta > -1$.*

**Theorem 1.5.** *Given $f$ in $A_\beta^p$, where $0 < p < \infty$ and $\beta > -1$, and $z \in \mathbb{D}$, then*

$$|f(z)| \leq \frac{\|f\|_{A_\beta^p}}{(1 - |z|^2)^{\frac{2+\beta}{p}}}.$$

On the one hand, a proof of the density of polynomials in the unweighted case can be found in [41, Chapter 2, Theorem 2] and the general result can be found in [54, Proposition 1.3], but the proof given there requires some corrections. On the other hand, the boundedness of point evaluations can be found in [92] where it is proven that this estimate is sharp in the context of a unit ball $\mathbb{C}^n$.

To work in the theory of the Bergman spaces it is useful to know some estimates for some integral operators. A well-known result is the following whose proof can be found in [54, Theorem 1.7] or [79, Proposition 1.4.10].

**Theorem 1.6.** *Given $\alpha > -1$ and $\beta \in \mathbb{R}$, if we define*

$$I_{\alpha,\beta}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{2+\alpha+\beta}}\, dA(w), \quad z \in \mathbb{D}$$

*and*

$$J_\beta(z) = \int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+\beta}}, \quad z \in \mathbb{D},$$

*then we have*

$$I_{\alpha,\beta}(z) \sim J_\beta(z) \sim \begin{cases} 1 & \text{if} \beta < 0, \\ \log\left(\frac{1}{1-|z|^2}\right) & \text{if} \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & \text{if} \beta > 0, \end{cases}$$

*when $|z| \to 1^-$.*

A relationship between functions and their derivatives in the context of the Bergman spaces is given by the Littlewood-Paley formula, which is related to Theorem 1.3 in the Hardy case.

**Theorem 1.7.** *Let $0 < p < \infty$ and let $\beta > -1$. Then,*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta\, dA(z) \sim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{\beta+p}\, dA(z)$$

*for all $f \in Hol(\mathbb{D})$.*

For the proof of this theorem see [44, Theorem 6]. We should note that there are some other results about this type of inequalities for more general weights, for instance [70] or [86].

An important tool in the Bergman spaces is the atomic decomposition, which was first proved by Coifman and Rochberg [33] for the $A^p_\beta$ spaces. For more information see also [15], [64] and a recent work by Constantin [36], in which a general atomic decomposition theorem is proved for weighted vector-valued Bergman spaces. We only will need the following case, see [33, Theorem 2]. Here we should note that their definition for the weighted Bergman spaces are a little different, see [33, Pages 14-15], so with the suitable change of notation we obtain the following result.

**Proposition 1.1.** *Given $f \in A^p_\beta$ with $0 < p < \infty$, $\beta > -1$, then there exist sequences $\{c_k\}_{k\geq 1} \in l^p$ and $\{a_k\}_{k\geq 1} \subset \mathbb{D}$ such that*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-\ |a_k|^2)^{\frac{2b+2+\beta}{p}}}{(1-\overline{a_k}z)^{\frac{2b+4+2\beta}{p}}}$$

*with*

$$b > \left(1 + \frac{\beta}{2}\right) \max\{-1, p-2\}.$$

*Furthermore,*

$$\|\{c_k\}_{k\geq 1}\|_{l^p}^p = \sum_{k=1}^{\infty} |c_k|^p \leq C\|f\|_{A_\beta^p}^p.$$

The converse is also true but we will only need this part.

### 1.5.1 Bergman projection

An important tool in the study of the standard weighted Bergman spaces $A_\beta^p$, with $\beta > -1$, is the operator called *the Bergman projection*. This operator, denoted by $P_\beta$, is defined originally only in the Hilbert space case by

$$P_\beta f(z) = (\beta+1) \int_{\mathbb{D}} \frac{f(w)(1-|w|^2)^\beta}{(1-z\overline{w})^2}\, dA(w)$$

for all $f \in L^2(\mathbb{D}, dA_\beta)$, where $dA_\beta(z) = (1-|z|^2)^\beta dA(z)$. To simplify the notation, we will denote this Lebesgue space by $L^2((1-|\zeta|^2)^\beta)$. The Bergman projection is a bounded linear operator from $L^2((1-|\zeta|^2)^\beta)$ onto $A_\beta^2$. The surjectivity of the projection is closely related with the reproducing kernels in the weighted Bergman spaces $A_\beta^2$ that we will see in Section 1.10; see also [54, Proposition 1.4].

Moreover, thanks to this integral formula $P_\beta$ is a well defined linear operator in the space $L^1((1-|\zeta|^2)^\beta)$. Thus, we can apply $P_\beta$ to any function in $L^p((1-|\zeta|^2)^\beta)$ with $1 \leq p < \infty$ and it can be proved that $P_\beta$ is a bounded linear operator from $L^p((1-|\zeta|^2)^\beta)$ onto $A_\beta^p$ for $1 < p < \infty$; see [41, Chapter 2, Theorem 5] for the unweighted case. A result that generalizes this claim can be found in [54, Proposition 1.10] and it is the following.

**Theorem 1.8.** *Suppose that $-1 < \beta, \gamma < \infty$ and $1 \leq p < \infty$. Then $P_\beta$ is a bounded projection from $L^p((1-|\zeta|^2)^\gamma)$ onto $A_\gamma^p$ if and only if $\gamma + 1 < (\beta+1)p$.*

### 1.5.2 Mixed norm spaces

A natural generalization of the standard weighted Bergman spaces are the *mixed norm spaces* $H(p,q,\beta)$ with $p,q,\beta > 0$ that consist of all functions $f$ in $Hol(\mathbb{D})$ for which

$$\|f\|_{p,q,\beta} = \left(\beta q \int_0^1 (1-r^2)^{\beta q-1} M_p^q(r,f) 2r\, dr\right)^{1/q} < \infty\,,$$

for $q < \infty$, and

$$\|f\|_{p,\infty,\beta} = \sup_{0\leq r<1} (1-r^2)^\beta M_p(r,f) < \infty\,.$$

Note that $H(p,p,\frac{\beta+1}{p}) = A_\beta^p$ and we can identify the Hardy space $H^p$ with the limit case $H(p,\infty,0)$. Moreover, if $p \geq 1$ and $q \geq 1$ the expressions above define "true" norms.

These spaces were explicitly defined by Flett in [44] and [45], but later they have been studied by many authors; see, for example, [12], [21], [48], [27]. In the mixed norm spaces we also have the boundedness of the point evaluations; see [58, Proposition 7.1.1].

**Proposition 1.2.** *If* $f \in H(p,q,\beta)$, $0 < p,q \leq \infty$, $0 < \beta < \infty$, *then*

$$|f(z)| \leq C \frac{\|f\|_{p,q,\beta}}{(1-|z|)^{\frac{\beta+1}{p}}},$$

*where* $C = C(p,q,\beta)$.

Moreover, these spaces are also related to other spaces that we will see in this chapter, like Korenblum spaces in Section 1.9.

# 1.6 Weighted Besov spaces

The *standard weighted Besov spaces* $B^{p,\beta}$, $p \geq 1, \beta > -1$ consist of analytic functions in $\mathbb{D}$ whose derivative belongs to $A^p_\beta$ and are normed by

$$\|f\|^p_{B^{p,\beta}} = |f(0)|^p + \|f'\|^p_{A^p_\beta}.$$

We should note that we will also use the equivalent norm defined by $\|f\|_1 = |f(0)| + \|f'\|_{A^p_\beta}$. In the literature, these spaces are also called weighted Dirichlet spaces, but we only use this terminology in the Hilbert case. This means that when $p = 2$ we will call $B^{2,\beta}$ a *weighted Dirichlet space* and will denote it by $D^{2,\beta}$. We have to differentiate this space with the *analytic Besov space* $B^p$, $1 < p < \infty$, the space of analytic functions in the unit disc such that

$$\|f\|^p_{B^p} = |f(0)|^p + (p-1) \int_{\mathbb{D}} (1-|z|^2)^{p-2} |f'(z)|^p \, dA(z),$$

and $B^1$ the space of analytic functions in the unit disc whose second derivatives are in $A^1$. Using the properties of Bergman spaces, we can deduce that the polynomials are dense in the weighted Besov spaces and the point evaluations are bounded.

As in the Bergman case, the atomic decomposition for the weighted Besov spaces is an important tool and it was proved by Peloso [71, Theorem 3.8]. Here we have to note that he uses a more general definition for the analytic Besov space, $B^s_p$, in the context of the unit ball $\mathbb{C}^n$ (see [71, Definition 1.1]) and the relation with our spaces in the unit disc is the following $B^s_p = B^{p,p(1-s)-1}$. We reformulate his result in our notation.

**Proposition 1.3.** *Given* $f \in B^{p,\beta}$ *with* $0 < p < \infty$ $\beta > -1$, *then there exist sequences* $\{c_k\}_{k \geq 1} \in l^p$ *and* $\{a_k\}_{k\geq 1} \subset \mathbb{D}$ *such that*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^b}{(1-\overline{a_k}z)^{b+\frac{\beta+2}{p}-1}}$$

*with*

$$b > \begin{cases} \max\left\{\left(\frac{\beta+2}{p} - 1\right)(p-1),\, 0\right\} & \textit{if } \ \beta > p-1, \\ \max\left\{\left(\frac{p-1}{p}\right),\, 0\right\} & \textit{if } \ \beta \leq p-1. \end{cases}$$

*Furthermore,*

$$\|\{c_k\}_{k\geq 1}\|_{l^p}^p = \sum_{k=1}^{\infty} |c_k|^p \leq C\|f\|_{B^{p,\beta}}^p.$$

The converse is also true but we will only use this part.

## 1.7 Basic duality

Before moving on to the next space of analytic functions we need to set some definitions and notations related to basic duality in Banach spaces.

Given $X$ and $Y$ two normed spaces, the vector space of all bounded operators of $X$ to $Y$ will be denote by $\mathcal{B}(X, Y)$. Moreover, in this space we can define a norm in the following natural way:

$$\|T\|_{\mathcal{B}(X,Y)} = \sup_{\|x\|_X \leq 1} \{\|T(x)\|_Y\}.$$

In addition, if $Y$ is a Banach space then $\mathcal{B}(X, Y)$ is also a Banach space with the norm defined above. See, for instance, [78, 4.1 Theorem]. We denote $\mathcal{B}(X)$ when $X = Y$.

The *dual space* of a Banach space $X$ will be the space consist of continuous linear functionals on $X$ (*i.e.* the continuous lineal mapping of $X$ into its scalar field). This space will be denoted by $X'$ and if it is equipped with the norm:

$$\|l\|_{X'} = \sup_{\|x\|_X \leq 1} \{|l(x)|\}$$

for each $l \in X'$, it becomes a Banach space. See [78, 4.3 Theorem].

## 1.8 Bloch spaces

The *Bloch space* $\mathcal{B}$ contains the analytic functions in the unit disc such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

This space has its origin in the classical Bloch theorem and the early discussion of the Bloch constant by Landau but was first seriously studied in the early 1970s by Pommerenke and his coauthors. Basic references for the theory of the Bloch function are [9], [41] and [96]. We will use the notation $\rho_{\mathcal{B}}(f) = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)|$, which is a seminorm. A well-known property of the Bloch space is its invariance under Möbius transformation *i.e.* the Bloch space is conformally invariant, in fact the seminorm $\rho_{\mathcal{B}}$ satisfies $\rho_{\mathcal{B}}(f \circ \varphi_a) = \rho_{\mathcal{B}}(f)$ where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. In the Bloch space we also have the boundedness of the point evaluations.

**Proposition 1.4.** *Given $f \in \mathcal{B}$ and $z \in \mathbb{D}$ we have*

$$|f(z) - f(0)| \leq \frac{1}{2}\rho_{\mathcal{B}}(f) \log \frac{1+|z|}{1-|z|}.$$

The proof can be found in [41, Chapter 2, Proposition 1]. The closure of the polynomials in the Bloch norm is the *little Bloch space*, $\mathcal{B}_0$. An analytic function $f$ belongs to the little Bloch space if $\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0$, see [41, Chapter 2, Proposition 4].

We can identify the dual of the Bergman space $A^1$ with the Bloch space (see [41, Chapter 2, Theorem 8])

**Theorem 1.9.** *The dual space of $A^1$ can be identified with the Bloch space, $\mathcal{B}$. Every bounded linear functional $l \in (A^1)'$ has a unique representation*

$$l(f) = l_g(f) = \lim_{r\to 1^-} \int_{r\mathbb{D}} f\overline{g}\, dA, \qquad f \in A^1,$$

*where $g \in \mathcal{B}$. Furthermore, the norm $\|l\|_{X'}$ is equivalent to the norm $\|g\|_{\mathcal{B}} = |g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(z)|$.*

## 1.9 Korenblum spaces

The *Korenblum spaces* (also called *growth spaces*) $\mathcal{A}^{-\gamma}$, with $\gamma > 0$, consist of analytic functions in the unit disc such that

$$\|f\|_{\mathcal{A}^{-\gamma}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\gamma}|f(z)| < \infty.$$

In the 1970s, Korenblum studied the zero sets of the union of all such spaces, which coincides with the union of all Bergman spaces of the disc. A basic reference for these spaces is [54, Section 4.3]. It is clear, by the definition of the Korenblum spaces, that each $\mathcal{A}^{-\gamma}$ contains all the bounded analytic functions and that the point evaluations are bounded.

**Remark 1.1.** Given $f \in \mathcal{A}^{-\gamma}$, $\gamma > 0$ and $z \in \mathbb{D}$ we have

$$|f(z)| \leq \frac{\|f\|_{\mathcal{A}^{-\gamma}}}{(1-|z|^2)^{\gamma}}.$$

Analogous to the little Bloch space $\mathcal{B}_0$, we will denote $\mathcal{A}_0^{-\gamma}$, with $\gamma > 0$, the closure of the polynomials in the Korenblum space $A^{-\gamma}$. Moreover, an analytic function $f$ belongs to $A_0^{-\gamma}$ if $\lim_{|z|\to 1^-}(1-|z|^2)^{\gamma}|f(z)| = 0$.

## 1.10 Weighted Hardy spaces

By the basic theorem of Riesz, every Hilbert space $\mathcal{H}$ of analytic functions in $\mathbb{D}$ on which all point evaluations are bounded has reproducing kernels. Given $w \in \mathbb{D}$, the *reproducing kernel* $K_w : \mathbb{D} \to \mathbb{C}$ is a function such that $f(w) = \langle f, K_w \rangle$ for all $f \in \mathcal{H}$. Two important historical references on reproducing kernels are [14] and [19]. Reproducing kernels are often viewed as functions of two complex variables (defined in the bidisc $\mathbb{D} \times \mathbb{D}$) by writing $K_w(z) = K(z, w)$. Clearly, $K_w(z) = \overline{K_z(w)}$, for all $z$, $w \in \mathbb{D}$.

Obviously, $K_z(z) = \langle K_z, K_z \rangle = \|K_z\|^2 \geq 0$ for all $z \in \mathbb{D}$. Actually, the restriction $K_z(w)$ to the diagonal $\{(z, w) : \; z = w\}$ of the bidisc $\mathbb{D} \times \mathbb{D}$ is often a radial function (meaning that it depends on $|z|$ only). Such spaces are of special interest. In what follows, we will consider a large family of Hilbert spaces $\mathcal{H}$ of analytic functions in the disc from which only the following axioms will be required:

(**A1**) The point evaluations are bounded (hence $\mathcal{H}$ is a reproducing kernel Hilbert space);

(**A2**) The reproducing kernel $K_w(z)$ is normalized so that $K_w(0) = 1$ for all $w \in \mathbb{D}$;

(**A3**) The monomials $\{z^n : \; n = 0, 1, 2, \ldots\}$ belong to $\mathcal{H}$ and form a complete orthogonal set (in the usual sense of maximal orthogonal sets in spaces with inner product).

The spaces $\mathcal{H}$ satisfying these axioms are often called *weighted Hardy spaces*. These spaces and some important operators on them were studied in detail by Shields [81] and later by many followers.

As Proposition 3.1 formulated in the Chapter 3 will show, if the above conditions are fulfilled then $\mathcal{H}$ will also satisfy several other conditions. Among them is the following representation of the reproducing kernel for $\mathcal{H}$:

$$K_w(z) = \sum_{n=0}^{\infty} \gamma(n) (\overline{w} z)^n \tag{1.1}$$

for certain numbers $\gamma(n) > 0$, where $\gamma(0) = 1$. Actually, computing the inner product of the monomial $z^n$ with the kernel easily yields that $\gamma(n) = \|z^n\|^{-2}$. Note also that the restriction $K_z(z) = \sum_{n=0}^{\infty} \gamma(n)|z|^{2n}$ is a positive and increasing function of $|z|$. By our normalization $(\mathbf{A2})$, we also have $\gamma(0) = K_w(0) = 1$; in view of $\gamma(0) = \|\mathbf{1}\|^{-2}$, where $\mathbf{1}$ denotes the constant function one, it also follows that $\|\mathbf{1}\| = 1$.

As can be seen, the representation of the kernel readily allows for the computation of [38, Section 2.1] that gives the norm of a function in $\mathcal{H}$:

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{\gamma(n)} |a_n|^2 , \quad \text{whenever } f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Regarding the notation used, it is worth mentioning that in Chapter 3 we are centered on the role played by the kernels and therefore use mainly the numbers $\gamma(n)$ whereas in many other texts the emphasis is on the norm formula in terms of the Taylor coefficients, so these spaces are denoted there by $H^2(\beta)$, where the obvious relationship between the numbers $\beta(n)$ and $\gamma(n)$ is as follows:

$$\gamma(n) = \frac{1}{\beta(n)^2}\,, \qquad n \geq 0\,.$$

We refer the reader to the standard reference [38].

### An important family of weighted Hardy spaces

An important family of weighted Hardy spaces $\mathcal{H}$ are the spaces $\mathcal{H}_\gamma$ whose reproducing kernel is given by the formula

$$K_w^\gamma(z) = \frac{1}{(1-\overline{w}z)^\gamma} = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n\,, \qquad z, w \in \mathbb{D}\,, \tag{1.2}$$

where the sequence $(\gamma(n))_{n=1}^{\infty}$ is defined as

$$\begin{aligned} &\gamma(0) = 1\,, \quad \gamma(1) = \gamma > 0 \\ &\gamma(n) = \binom{\gamma+n-1}{n} = \frac{\Gamma(n+\gamma)}{\Gamma(\gamma)n!} = \frac{1}{(n+\gamma)B(\gamma, n+1)}\,, \quad n \geq 1\,. \end{aligned} \tag{1.3}$$

This scale of spaces is formed precisely by the following well-known spaces:

- The standard Hardy space $H^2$, choosing $\gamma = 1$, which yields $\gamma(n) = 1$ for all $n \geq 0$, with the standard Szegő (Riesz) kernel $K_w^1(z) = (1-\overline{w}z)^{-1}$. Thus, for any $f \in H^2$

$$if(w) = \langle f, K_w^1\rangle_{H^2} = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{it})}{1-wre^{-it}}\,dt.$$

- The (larger) standard weighted Bergman spaces $A^2_{\gamma-2}$, when $\gamma > 1$, where the usual Bergman space $A^2$ corresponds to the values $\gamma(n) = n+1$ for $n \geq 0$, and the kernel is the standard Bergman kernel $K_w^2(z) = (1-\overline{w}z)^{-2}$. Thus, for any $f \in A^2_{\gamma-2}$

$$f(w) = \langle f, K_w^\gamma\rangle_{A^2_{\gamma-2}} = (\gamma-1)\int_{\mathbb{D}} \frac{f(z)}{(1-w\overline{z})^\gamma}(1-|z|^2)^{\gamma-2}\,dA(z).$$

- The (smaller) weighted Dirichlet spaces $D^{2,\gamma}$, when $\gamma < 1$.

Note that the standard unweighted Dirichlet spaces $D^{2,0}$ of the functions with square-integrable derivative (with respect to the area measure) does not belong to this scale.

To set notation, in Chapter 3 we will use the notation shown above, $\mathcal{H}_\gamma$ is the weighted Hardy space with reproducing kernel $K_w^\gamma(z) = \frac{1}{(1-\overline{w}z)^\gamma}$, this is the standard notation in the literature. However, in Chapter 4 we will be focus in certain weighted invariance property for spaces of analytic functions and for this purpose is suitable to work with the weighted Hardy spaces associate to the kernel $k_w^\gamma(z) = K_w^{2\gamma}(z) = \frac{1}{(1-\overline{w}z)^{2\gamma}}$, that we will denote $H_\gamma$ with the obvious relation $H_\gamma = \mathcal{H}_{2\gamma}$.

## 1.11 Carleson measures

The Carleson measures were defined by Lennart Carleson with the aim of characterizing the interpolating sequences of $H^\infty$, and to prove the Corona Theorem, see [29] and [40, Chapter 9]. Given a finite positive Borel measure $\mu$ on $\mathbb{D}$, we say that $\mu$ is a Carleson measure if there exists a constant $C_\mu$ such that

$$\mu(S_h(t)) \leq C_\mu h$$

for any Carleson box (or Carleson square)

$$S_h(t) = \{re^{is} :\ 0 < 1 - r \leq h,\ |t - s| \leq h\}.$$

In the next proposition we note the conformally invariant character of these measures, see [49, Lemma 3.3, page 239].

**Proposition 1.5.** *A positive measure $\mu$ on $\mathbb{D}$ is a Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2}\, d\mu(z) = M < \infty.$$

This result can be generalized to admit measures on $\overline{\mathbb{D}}$, see [22, Lemma 2.4]. We show this result with a corresponding changes of notation.

**Proposition 1.6.** *Let $\mu$ be a finite, positive Borel measure on $\overline{\mathbb{D}}$ and we denote by $\mu_{\mathbb{D}}$ and $\mu_{\mathbb{T}}$ its restrictions to the Borel subsets of $\mathbb{D}$ and $\mathbb{T}$, respectively. Then, if $I_h(t) = \{e^{is} :\ |t-s| \leq h\}$ and $0 < \gamma < \beta$ we have that*

$$\max\{\mu_{\mathbb{D}}(S_h(t)), \mu_{\mathbb{T}}(I_h(t))\} \leq Ch^\gamma,$$

*with $h \in (0,1)$ and $t \in [0, 2\pi)$, if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\overline{\mathbb{D}}} \frac{(1-|a|^2)^{\beta-\gamma}}{|1-\overline{a}z|^\beta}\, d\mu(z) = M < \infty.$$

# 2 Further Important Preliminaries

## 2.1 Transpose and adjoint operators

Let $X$ and $Y$ be two normed spaces and let $T$ be an operator in $\mathcal{B}(X, Y)$, then the *transpose operator* (also called *conjugate operator*) $T' : Y' \to X'$ of $T$ is the operator that sends any continuous linear functional $l \in Y'$ to the linear functional $lT$ in $X'$. Thus

$$T'l(x) = l(Tx),$$

for every $x \in X$ and $l \in Y'$.

Focusing on the Hilbert spaces, let $H_1$ and $H_2$ be two Hilbert spaces and let $A$ be a bounded linear operator from $H_1$ to $H_2$ then its *adjoint* is the bounded linear operator $A^* : H_2 \to H_1$ such that

$$\langle Ax, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1},$$

for any $x \in H_1$ and $y \in H_2$.

In this case of Hilbert spaces, we can observe that the notion of *transpose operator* is closely related to the notion of *adjoint operator* through the Riesz representation theorem and the antilinear identification between the Hilbert space and its conjugate.

## 2.2 Isometries and unitary operators

Given a Banach space $X$, a *linear isometry* of $X$ is a linear operator $T$ such that

$$\|Tx\| = \|x\|$$

for all $x \in X$. On a Hilbert space this is equivalent to $T^*T = I$, where $I$ is the identity operator and $T^*$ is the adjoint operator of $T$. Moreover if the Hilbert space isometry is also onto, it is called a *unitary operator* and is characterized by the property

$$T^*T = TT^* = I.$$

In Chapter 3 we will be interested in *co-isometric* operators that are the linear operators in the Hilbert space such that $TT^* = I$, *i.e.* such that $T^*$ is an isometry.

## 2.3 Invertible operators and resolvent set

Let $X$ be a Banach space and let $T$ be an operator in $\mathcal{B}(X)$. Then $T$ is *invertible* if there exists an operator $S$ in $\mathcal{B}(X)$ such that

$$ST = I = TS,$$

where $I$ is the identity map on $X$. In this case we write $S = T^{-1}$; see [78, 4.17 Definitions].

Let $T$ be an injective operator in $\mathcal{B}(X)$. By the closed graph theorem (Theorem 1.1) or the open mapping theorem [78, Theorem 2.11], $T$ is invertible if and only if it is surjective (onto).

On the other hand, the *spectrum* $\sigma(T)$ of an operator $T \in \mathcal{B}(X)$ is the set of all complex numbers $\lambda$ such that

$$T_\lambda = I - \lambda T$$

is not invertible. We should note that in many references $\sigma(T)$ consists of all $\mu \in \mathbb{C}$ such that $T - \mu I$ is invertible, hence we will need to reformulate their results; see, for example, [78], [95, Section VIII]. All complex numbers $\lambda$ not in $\sigma(T)$ form a set $\rho(T)$ called *the resolvent set* of $T$. These definitions can be extended to a more general context of Banach algebras. The following result will be useful for us in Chapter 4 and it is a direct consequence of [77, Corollary 18.3].

**Proposition 2.1.** *Let $T$ be an operator in $\mathcal{B}(X)$. If $|\lambda| < \frac{1}{\|T\|_{\mathcal{B}(X)}}$, then $\lambda \in \rho(T)$.*

## 2.4 Interpolation

The theory of interpolation started in the $L^p$ spaces, with the Riesz-Thorin theorem and the Marcinkiewicz theorem. The Riesz-Thorin theorem is also called the Riesz convexity theorem and it was proved by Riesz [74]. Later Thorin [91] gave another proof using convexity properties of analytic functions. The objective of this theory is, given two spaces, to find intermediate spaces that satisfy certain properties. There are some methods to achieve this goal, for example, the complex method and the real method. Thorin's proof of the Riesz-Thorin theorem is the starting point for the complex interpolation method, which that one we will use. Many influential authors have worked on this topic, for instance, Alberto Calderón [28], Elias Stein and Guido Weiss, [88], [89]. We will use, one of the basic references for the interpolation theory, the book by Jöran Bergh and Jörgen Löfström [18].

### 2.4.1 Interpolation spaces

**Definition 2.1.** Given two Banach spaces $A_0$ and $A_1$, we say that $A_0$ and $A_1$ are *compatible* (or the couple $(A_0, A_1)$ is *compatible*) if there exists $Z$, a Hausdorff topological vector space, such that $A_0$ and $A_1$ are both continuously contained in $Z$.

Thus, we can define $A_0 + A_1$ to consist of all elements $z = a_0 + a_1$ for some $a_0 \in A_0$ and $a_1 \in A_1$, and $A_0 \cap A_1$, which are subspaces of $Z$. Moreover, we can equip these spaces with the norms

$$\|a\|_{A_0 \cap A_1} = \max\left(\|a\|_{A_0}, \|a\|_{A_1}\right), \quad \|z\|_{A_0+A_1} = \inf\left(\|a_0\|_{A_0} + \|a_1\|_{A_1}\right)$$

where the infimum is taken over all representation of $z$ in the form $z = a_0 + a_1$. With these norms we have that $A_0 \cap A_1$ and $A_0 + A_1$ are complete; see [18, 2.3.1 Lemma] for the proof. The following definition can be found in [18, 2.4.1 Definition].

**Definition 2.2.** Given a compatible couple $(A_0, A_1)$ we say that $A$ is an *intermediate space* between $A_0$ and $A_1$, if

$$A_0 \cap A_1 \subset A \subset A_0 + A_1$$

with continuous inclusions. Moreover, the space $A$ is called an *interpolation space* between $A_0$ and $A_1$ (or with respect to $(A_0, A_1)$) if it satisfies the two following conditions:

- $A$ is an *intermediate space* between $A_0$ and $A_1$.
- For any linear operator $T$ from $A_0 + A_1$ to $A_0 + A_1$, which is bounded from $A_0$ to $A_0$ and from $A_1$ to $A_1$ respectively, we must have that $T : A \to A$ is bounded.

### 2.4.2 Complex method

We consider the following strips

$$\overline{S} = \{z \,:\, 0 \le \operatorname{Re} z \le 1\} \quad \text{and} \quad S = \{z \,:\, 0 < \operatorname{Re} z < 1\}.$$

So, for a given compatible couple $(A_0, A_1)$ we define the space $\mathcal{F}((A_0, A_1))$ that consists of all function $f$ with values in $A_0 + A_1$ such that $f$ is analytic on $S$, continuous and bounded on $\overline{S}$ and the functions $t \mapsto f(j + it)$, $j = 0, 1$, are continuous from the real line into $A_j$ and $\lim_{|t|\to\infty} \|f(j + it)\|_{A_j} = 0$. Thus, if we consider

$$\|f\|_{\mathcal{F}((A_0,A_1))} = \max\left(\sup_{t\in\mathbb{R}} \|f(ti)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1 + ti)\|_{A_1}\right),$$

we have that $\mathcal{F}((A_0, A_1))$ is a Banach space, see [18, 4.1.1 Lemma].

**Definition 2.3.** Given a compatible couple $(A_0, A_1)$, the complex interpolation space, $[A_0, A_1]_\theta$, for $\theta \in (0, 1)$, consists of all functions $a \in A_0 + A_1$ such that $a = f(\theta)$ for some $f \in \mathcal{F}((A_0, A_1))$. The norm in this space is defined by

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}((A_0,A_1))} \,:\, f(\theta) = a, f \in \mathcal{F}\},$$

where the infimum is taken over all $f \in \mathcal{F}((A_0, A_1))$ such that $f(\theta) = a$.

In the next theorem, [18, 4.1.2. Theorem], we observe that the complex interpolation space with the norm defined above is a Banach space and it is an interpolation space in the sense of Definition 2.2.

**Theorem 2.1.** *Given a compatible couple $(A_0, A_1)$ and $\theta \in (0,1)$, the complex interpolation space $[A_0, A_1]_\theta$ is a Banach space and it is an intermediate space with respect $(A_0, A_1)$. Moreover, if $(A_0, A_1)$ and $(B_0, B_1)$ are two compatible couples and $T$ is an operator from $A_0 + A_1$ to $B_0 + B_1$, which is bounded from $A_0$ to $B_0$ and $A_1$ to $B_1$ respectively, then $T : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$ is bounded with*

$$\|T\|_{B([A_0,A_1]_\theta,[B_0,B_1]_\theta)} \leq \|T\|^{1-\theta}_{B(A_0,B_0)} \|T\|^{\theta}_{B(A_1,B_1)}.$$

For our purpose we will need some basic properties of these complex interpolation spaces. Some of these properties and others can be seen in [18, 4.2.1. Theorem].

**Proposition 2.2.** *Given two compatible couples $(A_0, A_1)$ and $(B_0, B_1)$, they satisfy the following properties.*

$a)$ *If $A_0 \subset B_0$ and $A_1 \subset B_1$ with continuous inclusions, then $[A_0, A_1]_\theta \subset [B_0, B_1]_\theta$ for all $\theta \in (0,1)$.*

$b)$ *If $A = A_0 = A_1$ then $[A, A]_\theta = A$, for all $\theta \in (0,1)$.*

$c)$ *If $A_1 \subset A_0$ with continuous inclusion, then $[A_0, A_1]_\theta \subset A_0$, for all $\theta \in (0,1)$.*

*Proof.* $a)$ If we can prove that $\mathcal{F}((A_0, A_1)) \subset \mathcal{F}((B_0, B_1))$, then for $a \in [A_0, A_1]_\theta$ there exists $f \in \mathcal{F}((B_0, B_1))$ with $f(\theta) = a$, so $a \in [B_0, B_1]_\theta$. But if $f \in \mathcal{F}((A_0, A_1))$, then $f$ has values in $A_0 + A_1$ and

$$\|f\|_{\mathcal{F}((A_0,A_1))} = \max \left( \sup_{t\in\mathbb{R}} \|f(ti)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+ti)\|_{A_1} \right),$$

is finite. Thus, using that $A_0 + A_1 \subset B_0 + B_1$ and the continuity of the inclusions we conclude the desired result.

$b)$ This is obvious since $[A, A]_\theta$ has to be an intermediate space between $A$ and $A$.

$c)$ If we consider in $a)$ $B_0 = A_0$ and $B_1 = A_0$, applying $b)$ we obtain the result.

□

One of the most important applications of this theory is its use in $L^p$ spaces. Let $(U, \Sigma, \mu)$ be a $\sigma$-finite measure space. We denote by $L^p(d\mu)$ (or simply $L^p$) the Lebesgue space of all $\mu$-measurable functions $f$ on $U$, such that

$$\|f\|^p_{L^p} = \int_U |f|^p \, d\mu < \infty,$$

with $1 \leq p < \infty$. For the limit case, $L^\infty$ consist of all $\mu$-measurable functions $f$ on $U$, such that

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{z\in U} |f(z)|.$$

Thanks to the Theorem 2.1 we can get the Riesz-Thorin interpolation theorem as a Corollary of this next result, see [18, 5.1.1 Theorem].

**Theorem 2.2.** *Given $\theta \in (0,1)$ and $1 \leq p_0, p_1 \leq \infty$. Then, we have*

$$[L^{p_0}, L^{p_1}]_\theta = L^p$$

*with equal norms and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

More generally, we consider two different measures, $\mu_0$ and $\mu_1$ and further we suppose that both are absolutely continuous with respect $\mu$. Thus, we have

$$d\mu_0(x) = w_0(x)d\mu(x) \quad \text{ and } \quad d\mu_1(x) = w_1(x)d\mu(x),$$

and we denote by $L^p(w_0)$ and $L^p(w_1)$ the corresponding Lebesgue spaces. Moreover, we denote by $L^\infty(w_0)$ the space of all measurable functions $f$ such that $fw_0$ is essentially bounded. With these hypothesis we have the following result, see [18, 5.5.3. Theorem].

**Theorem 2.3.** *Given $\theta \in (0,1)$ and $1 \leq p_0, p_1 < \infty$. Then we have with equal norms*

$$[L^{p_0}(w_0), L^{p_1}(w_1)]_\theta = L^p(w) \quad 0 < \theta < 1,$$

*where*

$$w = w_0^{\frac{p(1-\theta)}{p_0}} w_1^{\frac{p\theta}{p_1}} \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

This theorem extends the classical interpolation theorem of Stein [88, Theorem 2], but it does not include the case $p_0 = \infty$ that it is what we will need. We have not been able to find any explicit reference for this fact, so we include a proof for this case, which is analogous to the proof of Theorem 2.3.

**Proposition 2.3.** *Given $\theta \in (0,1)$, $1 \leq p_1 < \infty$ and $p_0 = \infty$. Then, we have with equal norms*

$$[L^\infty(w_0), L^{p_1}(w_1)]_\theta = L^p(w) \quad 0 < \theta < 1,$$

*where*

$$w = w_0^{\frac{p_1(1-\theta)}{\theta}} w_1 \quad \textit{and} \quad p = \frac{p_1}{\theta}.$$

*Proof.* For a given $f \in \mathcal{F}((L^\infty(w_0), L^{p_1}(w_1)))$ we put

$$\tilde{f}(z,x) = w_0(x)^{(1-z)} w_1(x)^{z/p_1} f(z,x).$$

The mapping $f \to \tilde{f}$ is an isometric isomorphism between $\mathcal{F}((L^\infty(w_0), L^{p_1}(w_1)))$ and $\mathcal{F}((L^\infty, L^{p_1}))$. For example, to see the isometric property, since

$$\|\tilde{f}(it)\|_{L^\infty} = \|f(it)\|_{L^\infty(w_0)}, \quad \|\tilde{f}(1+it)\|_{L^\infty} = \|f(1+it)\|_{L^\infty(w_1)}$$

we have

$$\|\tilde{f}\|_{\mathcal{F}((L^\infty, L^{p_1}))} = \max \left( \sup \|\tilde{f}(it)\|_{L^\infty}, \sup \|\tilde{f}(1+it)\|_{L^{p_1}} \right) = \|f\|_{\mathcal{F}((L^\infty(w_0), L^{p_1}(w_1)))}.$$

Given $a \in [L^\infty(w_0), L^{p_1}(w_1)]_\theta$ we want to see that

$$\|a\|_\theta = \|a\|_{L^p(w)}.$$

On the one hand, if $a \in [L^\infty(w_0), L^{p_1}(w_1)]_\theta$ then there exists $f \in \mathcal{F}((L^\infty(w_0), L^{p_1}(w_1)))$ such that $f(\theta) = a$ and $\|f\|_{\mathcal{F}((L^\infty(w_0),L^{p_1}(w_1)))} < \|a\|_\theta + \varepsilon$. But thanks to the Theorem 2.2 ($[L^\infty, L^{p_1}]_\theta = L^p$) we have

$$\|a\|_{L^p(w)} = \|a\, w_0^{1-\theta}\, w_1^{\frac{\theta}{p_1}}\|_{L^p} \leq \|\tilde{f}\|_{\mathcal{F}((L^\infty,L^{p_1}))} = \|f\|_{\mathcal{F}((L^\infty(w_0),L^{p_1}(w_1)))} < \|a\|_\theta + \varepsilon.$$

On the other hand, let $b = w_0^{1-\theta} w_1^{\frac{\theta}{p_1}} a$, then $b \in [L^\infty, L^{p_1}]_\theta$. Therefore, there exists $\tilde{g} \in \mathcal{F}((L^\infty, L^{p_1}))$ such that $\|\tilde{g}\|_{\mathcal{F}((L^\infty,L^{p_1}))} < \|b\|_{[L^\infty,L^{p_1}]_\theta} + \varepsilon = \|a\|_{L^p(w)} + \varepsilon$. Hence

$$\|a\|_\theta \leq \|g\|_{\mathcal{F}((L^\infty(w_0),L^{p_1}(w_1)))} = \|\tilde{g}\|_{\mathcal{F}((L^\infty,L^{p_1}))} < \|a\|_{L^p(w)} + \varepsilon,$$

and we conclude the proof. □

## 2.5 Composition Operators and Weighted composition operators

Let $\phi$ be an analytic map of $\mathbb{D}$ into itself and let $X$ be a Banach space of analytic functions on $\mathbb{D}$. Then the *composition operator* $C_\phi$ is defined by

$$C_\phi f(z) = f(\phi(z))$$

for $z \in \mathbb{D}$ and $f \in X$. The boundedness of these operators in the classical spaces of analytic function in the unit disc has been studied by many authors, for example for the Hardy spaces, see [38, Corollary 3.7].

**Proposition 2.4.** *If $\phi$ is an analytic map of the disc into itself, then for $f \in H^p$ with $p \geq 1$, we have*

$${\frac{1}{1-|\phi(0)|^2}}^{\frac{1}{p}} \|f\|_{H^p} \leq \|C_\phi f\|_{H^p} \leq {\frac{1+|\phi(0)|}{1-|\phi(0)|}}^{\frac{1}{p}} \|f\|_{H^p}.$$

In this thesis we focus on a generalization of the composition operators. For a function $F$ analytic in $\mathbb{D}$ and an analytic map $\phi$ of $\mathbb{D}$ into itself, the *weighted composition operator* (or simply WCO) $W_{F,\phi}$ with *symbols* $F$ and $\phi$ is defined formally by the formula

$$W_{F,\phi} f = F(f \circ \phi) = M_F C_\phi f$$

as the composition followed by multiplication ($M_F f = Ff$). Such operators have been studied a great deal for various reasons. It is well-known that, in analogy with the classical theorem of Banach [17, Chapter XI] and Lamperti [61], the surjective linear isometries of all (non-Hilbert) Hardy and weighted Bergman spaces are operators of this type [46], [59]. Moreover, these operators have connections with some important problems, for example we should note their connections with Brennan's conjecture (see [82]), with the Hilbert matrix (see [39]) and with the Cesàro operator (see [30]).

## 2.6 Groups and Semigroups

Let $X$ be a Banach space and $\{T_t\}_{t\in\mathbb{R}}$ a family of operators contained in $\mathcal{B}(X)$ then $\{T_t\}_{t\in\mathbb{R}}$ forms a *one parameter group* if it satisfies the following conditions

1. $T_0 = I$
2. $T_{s+t} = T_s T_t$ for all $t, s \in \mathbb{R}$.

Note that by the second property these groups are always Abelian and with this definition we obtain $T_{-t}T_t = T_0 = I$ so the operator $T_t$ is invertible for all $t \in \mathbb{R}$. Moreover, we will say that the group $\{T_t\}_{t\in\mathbb{R}}$ is a *$C_0$-group* or it is *strongly continuous at* $0$ (usually in the literature it is said only *strongly continuous*) if for every $x \in X$ we have that

$$\lim_{t\to 0} \|T_t(x) - x\|_X = 0.$$

In general we will said that the group $\{T_t\}_{t\in\mathbb{R}}$ is *strongly continuous at a point* $t_1$ if $\lim_{t\to t_1} \|T_t(x) - T_{t_1}(x)\|_X = 0$. If $\lim_{t\to 0} \|T_t - I\|_{\mathcal{B}(X)} = 0$ then the group is *uniformly continuous*.

The definition for the one parameter semigroups is analogous but we only consider $t \geq 0$. Let $X$ be a Banach space and $\{T_t\}_{t\geq 0}$ a family of operators contained in $\mathcal{B}(X)$. $\{T_t\}_{t\geq 0}$ forms a *one parameter semigroup* if it satisfies the following conditions

1. $T_0 = I$
2. $T_{s+t} = T_s T_t$ for all $t, s \geq 0$.

Analogously, $\{T_t\}_{t\geq 0}$ is *strongly continuous from the right at* $0$ or it is a *$C_0$-semigroup* if for every $x \in X$ $\lim_{t\to 0^+} \|T_t(x) - x\|_X = 0$ and it is *uniformly continuous* if $\lim_{t\to 0^+} \|T_t - I\|_{\mathcal{B}(X)} = 0$.

The *infinitesimal generator* of a $C_0$-semigroup (analogously for a $C_0$-group) is an operator $A$ , in general unbounded, defined by

$$A(x) := \lim_{t\to 0^+} \frac{T_t(x) - x}{t} = \frac{d}{dt} T_t x|_{t=0},$$

whose domain is

$$D(A) = \left\{ x \in X \,:\, \lim_{t\to 0^+} \frac{T_t(x) - x}{t} \;\; \text{exists} \right\}.$$

It can be proved that $D(A)$ is dense in $X$; see [95, Chapter IX, Theorem 1]. Moreover, the infinitesimal operator is bounded if and only if $\{T_t\}_{t\geq 0}$ is *uniformly continuous*; see for example [25, Proposition 3.1.1]. The infinitesimal generator is also a closed operator; see [95, Chapter IX, Corollary 3].

We will be interested in the case when $X$ is a Banach space of analytic functions in the unit disc. In this context, Berkson and Porta [20] in 1978 started to study the semigroups of composition operators. Later, Siskakis in [83] and [84] studied these semigroups on the Bergman and the Dirichlet spaces. See also the survey [85] by Siskakis about this topic. For our purpose, we need to work with a generalized definition of groups. For us $\{T_t\}_{t\in G}$, a family of operators contained in $\mathcal{B}(X)$ with $G \subset \mathbb{R}$ an interval and $(G,\star)$ a locally compact Abelian group (with the usual topology), forms a *one parameter group* if it satisfies the following conditions

1. $T_e = I$, where $e$ is the identity element of $(G,\star)$.

2. $T_{s\star t} = T_s T_t$ for all $t, s \in G$.

$\{T_t\}_{t\in G}$ is *strongly continuous at* $e$ or it is a $C_0$*-group* if for every $x \in X$ $\lim_{t\to e} \|T_t(x) - x\|_X = 0$ and it is *uniformly continuous* if $\lim_{t\to e} \|T_t - I\|_{\mathcal{B}(X)} = 0$. The definitions are analogous for semigruoups.

**Example 2.1.** Let $\mathcal{D}^{2,0}$ be the usual Dirichlet space and we consider $G = (-1,1)$ and we define the operation $a \star b = \frac{a+b}{1+ab}$ for $a, b \in (-1,1)$. Then $(G,\star)$ is a locally compact Abelian group (with the usual topology). Now consider $\{T_a\}_{a\in G}$ where $T_a = C_{\psi_a}$ is the composition operator with symbol $\psi_a$ defined by $\psi_a(z) = \frac{z+a}{1+az}$. Thus, $T_a \in \mathcal{B}(\mathcal{D}^{2,0})$ and $\{T_a\}_{a\in G}$ is a group in the sense defined above.

## 2.7 Bochner integral

We base this part on [32, Appendix E] with the appropriate change of notation to adapt to the content of the thesis. The Bochner integral is defined analogously to the Lebesgue integral and it is essentially a vector-valued version of it. Let $(\Omega, \Sigma)$ be a measurable space and let $X$ be a real or complex Banach space. Now, we consider the $\sigma$-algebra of Borel subsets of $X$ denoted by $\mathfrak{B}(X)$. Then $f : \Omega \to X$ will be *Borel measurable* if it is measurable with respect to $\Sigma$ and $\mathfrak{B}(X)$ and it will be *strongly measurable* if it is Borel measurable and $f(\Omega)$ is separable.

**Remark 2.1.** In our case the polynomials will be dense in the Banach spaces $X$ of analytic functions on the unit disc, so a function $f : \Omega \to X$ will be Borel measurable if and only if it is strongly measurable.

In the next example we will see a particular case of the relation between the continuous function with strongly measurable functions. We will use this example in Section 4.3.2.

**Example 2.2.** If we consider $\Omega = [-\pi, \pi]$ with the Lebesgue $\sigma$-algebra and $X$ a separable Banach space, then a function $f : \Omega \to X$ continuous on $[-\pi, \pi]$ will be a strongly measurable functions since the Lebesgue $\sigma-$algebra contain the Borel sets.

In analogy with Lebesgue integral we define a simple function $f$ as a finite sum of the form

$$f(\omega) = \sum_{i=1}^{N} \chi_{A_i}(\omega) x_i$$

where $A_i$ are pairwise disjoint elements of $\Sigma$, $\chi_{A_i}$ is the characteristic function of $A_i$ and $x_i \in X$.

**Definition 2.4.** Let $(\Omega, \Sigma, \mu)$ be a measure space and let $X$ be a real or complex Banach space. Then a function $f : \Omega \to X$ is Bochner integrable if it is strongly measurable and

$$\int_\Omega \|f\|_X \, d\mu < \infty.$$

As in the Lebesgue case, the integral of a simple function $f(\omega) = \sum_{i=1}^N \chi_{A_i}(\omega) x_i$ will be defined as the sum $\sum_{i=1}^N \mu(A_i) x_i$ and for a general strongly measurable function $f : \Omega \to X$, the Bochner integral $\int_\Omega f \, d\mu$ is defined as following

$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega f_n \, d\mu$$

where $\{f_n\}_{n\geq 1} \subset X$ is a sequence of simple functions such that for all $\omega \in \Omega$ $\|f_n(\omega)\|_X \leq \|f(\omega)\|_X$ for all $n \geq 1$ and $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ (For the existence of this sequence see [32, E.2. (Proposition)]). Therefore, we have the following properties of the Bochner integral (see [32, E.4. and E.5. (Proposition)]).

**Proposition 2.5.** *Let $(\Omega, \Sigma, \mu)$ be a measure space, let $X$ be a real or complex Banach space and $f, g : \Omega \to X$ Bochner integrable functions. Then*

1. *$af + bg$ is an Bochner integrable function for any $a, b \in \mathbb{C}$.*
2. *$\| \int_\Omega f \, d\mu \|_X \leq \int_\Omega \|f\|_X \, d\mu$.*

Moreover, next proposition shows that we can interchange the integral and the continuous linear operators (see [32, E.11. (Proposition)]).

**Proposition 2.6.** *Let $(\Omega, \Sigma, \mu)$ be a measure space, let $X$ be a real or complex Banach space and let $f : \Omega \to X$ be Bochner integrable. Then*

$$\int_\Omega l(f) \, d\mu = l \left( \int_\Omega f \, d\mu \right)$$

*for all $l \in X'$.*

## 2.8 Pointwise multipliers

The *pointwise multipliers* can be defined in a more general context that which we will present below, but for our purpose we will consider spaces of analytic functions in the unit disc such that their point evaluations are bounded.

**Definition 2.5.** Given two Banach spaces $X, Y$ of analytic functions in the unit disc satisfying that the point evaluations are bounded on both spaces, the space $Mult(X, Y)$ consists of all analytic functions $u$ in $\mathbb{D}$ with $uX \subset Y$ with the norm

$$\|u\|_{Mult(X,Y)} = \sup_{\substack{f\in X \\ \|f\|_X \leq 1}} \|uf\|_Y.$$

We denote $Mult(X)$ when $X = Y$.

The functions in $Mult(X, Y)$ are usually called *pointwise multipliers from $X$ into $Y$*. By the closed graph theorem (using a standard argument with normal families) each $u \in Mult(X, Y)$ defines a bounded multiplication operator $M_u : X \to Y,\ M_u f = uf$, and $\|u\|_{Mult(X,Y)}$ equals the operator norm of $M_u$. In particular, it follows that $Mult(X, Y)$ is a Banach space.

## 2.9 Weak products of Banach spaces

**Definition 2.6.** Given two Banach spaces $X, Y$ satisfying that the point evaluations are bounded on both spaces, their *weak product* $X \odot Y$ consists of analytic functions $f$ in $\mathbb{D}$ which can be represented in the form

$$f = \sum_{n\geq 1} g_n h_n, \quad g_n \in X,\ h_n \in Y,\ \sum_{n\geq 1} \|g_n\|_X \|h_n\|_Y < \infty. \tag{2.1}$$

The norm of $f \in X \odot Y$ is defined by

$$\|f\|_{X\odot Y} = \inf \sum_{n\geq 1} \|g_n\|_X \|h_n\|_Y,$$

where the infimum is taken over all representations of $f$ in the form (2.1).

**Remark 2.2.** Given two Banach spaces $X, Y$ satisfying that the point evaluations are bounded on both spaces, $\|\cdot\|_{X\odot Y}$ is a norm.

*Proof.* It is clear that $\|f\|_{X\odot Y} \geq 0$ for all $f \in X \odot Y$.

1. $\|f\|_{X\odot Y} = 0$ if and only if $f \equiv 0$. If $f \equiv 0$, taking $g \equiv h \equiv 0$ we have $f = gh$, so $\|f\|_{X\odot Y} = 0$. Conversely, if $\|f\|_{X\odot Y} = 0$ given $\varepsilon > 0$ there exist $\{g_n^\varepsilon\}_{n\geq 1} \subset X, \{h_n^\varepsilon\}_{n\geq 1} \subset Y$ such that $f = {}_{n\geq 1} g_n^\varepsilon h_n^\varepsilon$ and ${}_{n\geq 1} \|g_n^\varepsilon\|_X \|h_n^\varepsilon\|_Y < \varepsilon$. Then, since the point evaluations are bounded on $X$ and $Y$, using the uniform boundedness principle for any compact $K \subset \mathbb{D}$ we have

   $$|f(z)| = |\sum_{n\geq 1} g_n^\varepsilon(z) h_n^\varepsilon(z)| \leq C_K \sum_{n\geq 1} \|g_n^\varepsilon\|_X \|h_n^\varepsilon\|_Y < C_K \varepsilon$$

   for all $z \in K$, so $f(z) = 0\, \forall z \in K$, therefore $f \equiv 0$.

2. Given $f \in X \odot Y$ and $\lambda \in \mathbb{C}$, any representation $f = {}_{n\geq 1} g_n h_n$ satisfies that $\lambda f = {}_{n\geq 1} (\lambda g_n) h_n$, so $\|\lambda f\|_{X\odot Y} = |\lambda| \|f\|_{X\odot Y}$.

3. Given $f^1, f^2 \in X \odot Y$ then $\|f^1 + f^2\| \leq \|f^1\| + \|f^2\|$. Given $\varepsilon > 0$ there exist $\{g_n^i\}_{n\geq 1} \subset X, \{h_n^i\}_{n\geq 1} \subset Y$ such that

   $$f^i = \sum_{n\geq 1} g_n^i h_n^i \quad \text{and} \quad \sum_{n\geq 1} \|g_n^i\|_X \|h_n^i\|_Y < \|f^i\|_{X\odot Y} + \frac{\varepsilon}{2}$$

   with $i = 1, 2$. Therefore, $f^1 + f^2 = {}_{k=1}^{\infty} G_k H_k$ and

   $$\|f^1 + f^2\|_{X\odot Y} \leq \sum_{k=1}^{\infty} \|G_k\|_X \|H_k\|_Y < \|f^1\|_{X\odot Y} + \|f^2\|_{X\odot Y} + \varepsilon.$$

Here we have used the bijection between $\mathbb{N}$ and $\mathbb{N}^2$ defined in the following way:

$$G_k = g^{i_k}_{\lceil \frac{k}{2} \rceil} \quad \text{and} \quad H_k = h^{i_k}_{\lceil \frac{k}{2} \rceil} \quad \text{with} \quad i_k = \begin{cases} 1, & \text{if } \frac{k}{2} \in \mathbb{N} \\ 2, & \text{if } \frac{k}{2} \notin \mathbb{N} \end{cases}. \tag{2.2}$$

□

**Remark 2.3.** Given two Banach spaces $X, Y$ satisfying that the point evaluations are bounded on both spaces, their weak product $X \odot Y$ is a Banach space.

*Proof.* Given a Cauchy sequence $\{f_n\}_{n\geq 1}$ on $X \odot Y$, we consider the subsequence $\{f_{n_k}\}_{k\geq 1}$ such that $\|f_{n_k} - f_{n_{k+1}}\| < 2^{-k}$ for all $k \geq 1$. Since $f_{n_k} - f_{n_{k+1}} \in X \odot Y$, given $\varepsilon > 0$ there exists $\{g^k_{n_i}\}_{i\geq 1}, \{h^k_{n_i}\}_{i\geq 1}$ such that

$$f_{n_k} - f_{n_{k+1}} = \sum_{i\geq 1} g^k_{n_i} h^k_{n_i} \quad \text{and} \quad \sum_{i\geq 1} \|g^k_{n_i}\|_X \|h^k_{n_i}\|_Y < 2^{-k} + \frac{\varepsilon}{2^k}.$$

Now, we define

$$f = f_{n_1} + \sum_{k\geq 1} (f_{n_{k+1}} - f_{n_k}) = f_{n_1} - \sum_{k\geq 1}\sum_{i\geq 1} g^k_{n_i} h^k_{n_i},$$

which is in $X \odot Y$. To see this assertion we only have to prove that $\sum_{k\geq 1}\sum_{i\geq 1} g^k_{n_i} h^k_{n_i}$ is in $X \odot Y$. Taking into account that the union of countable sets is countable, analogously to (2.2), $\sum_{k\geq 1}\sum_{i\geq 1} g^k_{n_i} h^k_{n_i} = \sum_{j\geq 1} G_j H_j$, with $G_j \in X, H_j \in Y$ and

$$\sum_{j\geq 1} \|G_j\|_X \|H_j\|_Y = \sum_{k\geq 1}\sum_{i\geq 1} \|g^k_{n_i}\|_X \|h^k_{n_i}\|_Y < \sum_{k\geq 1} 2^{-k} + \frac{\varepsilon}{2^k} = 1 + \varepsilon,$$

so $\sum_{k\geq 1}\sum_{i\geq 1} g^k_{n_i} h^k_{n_i} \in X \odot Y$. Therefore, for $j \geq 1$

$$\|f - f_n\|_{X\odot Y} \leq \|f - f_{n_j}\|_{X\odot Y} + \|f_{n_j} - f_n\|_{X\odot Y} < (1+\varepsilon)2^{-j+1} + \|f_{n_j} - f_n\|.$$

Here we have used that

$$\begin{aligned} f - f_{n_j} &= f_{n_1} - f_{n_j} + \sum_{k\geq 1} (f_{n_{k+1}} - f_{n_k}) = -\sum_{k=1}^{j-1} (f_{n_{k+1}} - f_{n_k}) + \sum_{k\geq 1} (f_{n_{k+1}} - f_{n_k}) \\ &= \sum_{k\geq j} (f_{n_{k+1}} - f_{n_k}) = \sum_{k\geq j}\sum_{i\geq 1} g^k_{n_i} h^k_{n_i}. \end{aligned}$$

Finally, since $\{f_n\}_{n\geq 1}$ is a Cauchy sequence, taking $j \to \infty$ we obtain that $f_n \to_{X\odot Y} f$, with $f \in X \odot Y$. □

The weak product can be identified with the projective tensor product $X \hat{\otimes} Y$. For more information about projective tensor product see, for example, [80]. In the Definition 2.6 we have supposed that the point evaluations are bounded on $X$ and $Y$. Then, by the uniform boundedness principle, this implies that $X \odot Y$ consist of analytic functions. If $f \in X \odot Y$, given $\varepsilon > 0$ there exists $\{g_n\}_{n\geq 1} \subset X, \{h_n\}_{n\geq 1} \subset Y$ such that

$$f = \sum_{n\geq 1} g_n h_n \quad \text{and} \quad \sum_{n\geq 1} \|g_n\|_X \|h_n\|_Y < \|f\|_{X\odot Y} + \varepsilon.$$

Thus for a compact $K \subset \mathbb{D}$, using that the point evaluations are bounded on $X$ and $Y$ and the uniform boundedness principle, we have

$$|f(z) - \sum_{n=1}^{N-1} g_n(z)h_n(z)| \leq \sum_{n\geq N} |g_n(z)||h_n(z)| \leq C_k \sum_{n\geq N} \|g_n\|_X \|h_n\|_Y \to 0$$

when $N \to \infty$ and for all $z \in K$. Therefore the sum $_{n\geq 1} g_n h_n$ is uniformly convergent to $f$ on the compact subsets of $\mathbb{D}$, so $f$ is analytic.

# 3 Co-isometric weighted composition operators on Hilbert spaces of analytic functions

## 3.1 Introduction

It is well known that Hilbert spaces have plenty of unitary operators (e.g., permuting the elements of an orthonormal basis gives such transformations) so it is of interest to know when an operator of some specific type, such as weighted composition operator (WCO), is unitary. See Section 2.2 and Section 2.5. Here we study the question of when a WCO has the (apparently weaker) property of being co-isometric: $TT^* = I$. This chapter is based on the paper [65] and it is devoted to showing that in this case the properties of being co-isometric and being unitary turn out to be equivalent and to obtaining a necessary and sufficient condition for a weighted composition operator to be co-isometric on a general weighted Hardy space, see Section 1.10.

### 3.1.1 Some recent results

Isometric multiplication operators on the Hardy spaces, weighted Bergman spaces, or weighted Besov spaces were characterized in [4]. Characterizations of composition operators on the Dirichlet space that are unitary (which is simple) and isometric (which is more involved) were given in [66]. Isometric WCOs on weighted Bergman spaces have been recently described by Zorboska [97]. Isometric WCOs on non-Hilbert weighted Bergman spaces were discussed in Matache's paper [69], expanding upon the classical work [46]. Li *et al.* [63] studied normal WCOs on weighted Dirichlet spaces.

Bourdon and Narayan [24] studied the normal WCOs and described the unitary WCOs on the Hardy space $H^2$. Le [62] considered the WCOs on a general class of weighted Hardy spaces, denoted by $\mathcal{H}_\gamma$ (see Section 1.10), whose reproducing kernel is of the form $(1-\overline{w}z)^{-\gamma}$ and which enjoy certain conformal properties. Besides describing when an adjoint of such an operator is again of the same type, he characterized the unitary operators among them (in the context of the unit ball), showing that this is equivalent to the property of being co-isometric. Zorboska [98] proved several related general results for Hilbert spaces of holomorphic functions in several variables defined by certain properties of their kernels. It should also be mentioned that Hartz [53] obtained several closely related results pertaining to the spaces with kernels of the type mentioned.

### 3.1.2 Some remarks

Since every unitary operator is invertible, results on invertibility of WCOs could in principle be relevant in this context. Most recently, in [13] two different theorems were proved, showing that in every functional Banach space in the disc that satisfies one of the two sets of five axioms listed there, a WCO is invertible if and only if its symbols $F$ and $\phi$ have certain properties that one would naturally expect. However, these theorems do not apply to every natural space of analytic functions. One example of a Hilbert space that satisfies these five axioms but is not included among the spaces considered in [62] is the Dirichlet space. Nonetheless, it can be seen that the invertibility results which give a lot of information on the symbols, even in this special case of the Dirichlet space, still require additional non-trivial work so as to deduce the complete information about the exact structure of $F$ and $\phi$. It is precisely this work that will be done here. We will make all the proof self-contained although the methods used by Hartz [53] could also give alternative proofs of some of our results.

## 3.2 A review of weighted Hardy spaces

In this section we focus on the consequences and restatements of the axioms that define a weighted Hardy space. Recall that a Hilbert space $\mathcal{H}$ of analytic functions in the unit disc is a weighted Hardy space if it satisfies the following axioms (see Section 1.10 for more details).

(**A1**) The point evaluations are bounded;

(**A2**) The reproducing kernel $K_w(z)$ is normalized so that $K_w(0) = 1$ for all $w \in \mathbb{D}$;

(**A3**) The monomials $\{z^n : n = 0, 1, 2, \ldots\}$ belong to $\mathcal{H}$ and form a complete orthogonal set (in the usual sense of maximal orthogonal sets in spaces with inner product).

To fix the notation, the rotations will be denoted by $R_\lambda$; for $|\lambda| = 1$, let $R_\lambda(z) = \lambda z$, for all $z \in \mathbb{D}$. The induced composition operator is denoted by $C_{R_\lambda}$: $C_{R_\lambda} f = f \circ R_\lambda$, for $f \in \mathcal{H}$.

The following simple result shows that one does not need to assume any further axioms that our spaces should satisfy in order to obtain the results that will be proved here. Moreover, it shows that under only a minimum set of assumptions we can produce examples of unitary WCOs on our spaces.

For our purpose it would actually suffice just to note that condition *(a)* below implies all the others (since this is all that is needed in the results that follow). This is either simple to prove or easy to find in other places; for example, *(a)* $\Rightarrow$ *(b)* is proved in [38, Theorem 2.10]. However, we have chosen to include a detailed proof of equivalences between all the conditions *(a)*–*(g)* below because this may have some independent interest. Since this result is not easy to find in one place and some of the implications are non-trivial, for the sake of completeness we also include a detailed proof, as self-contained as possible.

**Proposition 3.1.** *Let $\mathcal{H}$ be a Hilbert space of analytic functions in $\mathbb{D}$ that contains all monomials and satisfies our axioms $(\mathbf{A1})$ and $(\mathbf{A2})$: the point evaluations are bounded on $\mathcal{H}$ and the reproducing kernel is normalized so that $K_w(0) = 1$ for all $w \in \mathbb{D}$. Then the following statements are equivalent:*

*(a) Axiom $(\mathbf{A3})$ is fulfilled; that is, the monomials $\{z^n : n = 0, 1, 2, \ldots\}$ form a complete orthogonal set in $\mathcal{H}$.*

*(b) The reproducing kernel has the form*

$$K_w(z) = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n, \tag{3.1}$$

*with $\gamma(n) = \|z^n\|^{-2}$.*

*(c) The norm of a function $f \in \mathcal{H}$ whose Taylor series in $\mathbb{D}$ is $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is given by*

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \|z^n\|^2 .$$

*(d) The rotations $R_\lambda$ induce isometric composition operators $C_{R_\lambda}$ on $\mathcal{H}$.*

*(e) The rotations $R_\lambda$ induce unitary composition operators $C_{R_\lambda}$ on $\mathcal{H}$.*

*(f) The constant multipliers of modulus one ($|\mu| = 1$) and rotations $R_\lambda$ induce unitary weighted composition operators $W_{\mu, R_\lambda}$ on $\mathcal{H}$.*

*(g) $K_{\lambda w}(\lambda z) = K_w(z)$ for all $z$, $w \in \mathbb{D}$ and all $\lambda$ with $|\lambda| = 1$.*

*Proof.* It suffices to prove the following chain of implications:

*(b) ⇒ (a) ⇒ (c) ⇒ (d) ⇒ (e) ⇒ (g) ⇒ (b),*

in addition to the easy implications *(e) ⇒ (f) ⇒ (d)*.

*(b) ⇒ (a):* Assume that the kernel can be written as $K_w(z) = \sum_{n=0}^{\infty} \gamma(n)(\overline{w}z)^n$, with $\gamma(n) = \|z^n\|^{-2}$. We first check that the series converges in the norm of the space $\mathcal{H}$. To this end, write

$$p_{w,N}(z) = \sum_{n=0}^{N} \gamma(n)\overline{w}^n z^n , \quad z, w \in \mathbb{D} , \qquad N \geq 0 ,$$

for the partial sums of the kernel $K_w(z)$. Note that for $a \in \mathbb{D}$ we have $K_a(a) = \sum_{n=0}^{\infty} \gamma(n)|a|^{2n}$. Then clearly the infinite series $\sum_{n=0}^{\infty} |w|^n$ and $\sum_{n=0}^{\infty} \gamma(n)|w|^n$ both converge for all $w \in \mathbb{D}$. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that ${}_{n>N} |w|^n < \varepsilon/2$ and ${}_{n>N} \gamma(n)|w|^n < \varepsilon/2$. Then

$$\begin{aligned}
\|K_w - p_{w,N}\| &= \left\| \sum_{n>N} \gamma(n)\overline{w}^n z^n \right\| \leq \sum_{n>N} \gamma(n)|w|^n \|z^n\| \\
&= \sum_{n>N,\, \|z^n\|>1} \gamma(n)|w|^n \|z^n\| + \sum_{n>N,\, \|z^n\|\leq 1} \gamma(n)|w|^n \|z^n\| \\
&\leq \sum_{n>N,\, \|z^n\|>1} \frac{|w|^n}{\|z^n\|} + \sum_{n>N} \gamma(n)|w|^n \\
&\leq \sum_{n>N} |w|^n + \sum_{n>N} \gamma(n)|w|^n < \varepsilon\,.
\end{aligned}$$

Now, using the definition of the reproducing kernel $K_w$ (as a function of $z$), the following operation is justified for all $w \in \mathbb{D}$:

$$w^n = \langle z^n, K_w(z)\rangle = \langle z^n, \sum_{m=0}^{\infty} \gamma(m)\overline{w}^m z^m\rangle = \sum_{m=0}^{\infty} \overline{\gamma(m)}\langle z^n, z^m\rangle w^m\,, \qquad n \geq 0\,.$$

The uniqueness of the Taylor coefficients implies $\langle z^m, z^n\rangle = 0$ for all $m \neq n$. Hence the monomials form an orthogonal system.

It is only left to check that this orthogonal system is complete. To this end, it suffices to note that the linear span of the kernels is dense in $\mathcal{H}$: if a function is orthogonal to it, in particular it is orthogonal to each $K_w$ and hence vanishes at all $w \in \mathbb{D}$, so it must be the zero function. We have proved a little earlier that every kernel, and hence every function in the linear span of the kernels, can be approximated by polynomials in the norm of the space. Thus, the polynomials are dense in $\mathcal{H}$ and therefore form a complete orthogonal system.

*(a) ⇒ (c):* Let $f \in \mathcal{H}$, with the Taylor series $f(z) = {}_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. By assumption, the monomials form a complete orthogonal set in $\mathcal{H}$. As a consequence of our Axiom (**A1**), the development of $f$ in a series with respect to the orthonormal basis $\{z^n/\|z^n\| \,:\, n \geq 0\}$ must coincide with the Taylor series of $f$ at each point $z \in \mathbb{D}$. This justifies the following operations:

$$\|f\|^2 = \langle \sum_{m=0}^{\infty} a_m z^m, \sum_{n=0}^{\infty} a_n z^n\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m \overline{a_n}\langle z^m, z^n\rangle = \sum_{n=0}^{\infty} |a_n|^2 \|z^n\|^2\,.$$

*(c) ⇒ (d):* Let $\lambda$ be a complex number of modulus one. Since $C_{R_\lambda} f(z) = f(\lambda z)$, by the norm formula from the assumption *(c)* we have

$$\|C_{R_\lambda} f\|^2 = \sum_{n=0}^{\infty} |\lambda^n a_n|^2 \|z^n\|^2 = \|f\|^2\,.$$

*(d) ⇒ (e):* It suffices to notice that $C_{R_\lambda} C_{R_{\overline{\lambda}}} f = f$ for all $f \in \mathcal{H}$, hence $C_{R_\lambda}$ is onto. Being a surjective isometry, it is a unitary operator.

*(e) ⇒ (g):* Let $|\lambda| = 1$. Since $C_{R_\lambda}$ is unitary, we know that $C_{R_\lambda} C^*_{R_\lambda} = I$. Moreover,

$$\begin{aligned} C^*_{R_\lambda} K_w(z) &= \langle C^*_{R_\lambda} K_w, K_z \rangle = \langle K_w, C_{R_\lambda} K_z \rangle \\ &= \overline{\langle C_{R_\lambda} K_z, K_w \rangle} = \overline{C_{R_\lambda} K_z(w)} = \overline{K_z(\lambda w)} = K_{\lambda w}(z)\,. \end{aligned}$$

Hence

$$K_w(z) = C_{R_\lambda} C^*_{R_\lambda} K_w(z) = C_{R_\lambda} K_{\lambda w}(z) = K_{\lambda w}(\lambda z).$$

*(g) ⇒ (b):* It is a standard Hilbert space fact that the reproducing kernel $K(z,w) = K_w(z)$ is a positive definite function (in the usual sense that the corresponding quadratic form is positive semi-definite). We also have the normalization condition, Axiom (**A2**). Condition *(g)* tells us that

$$K_{\lambda w}(\lambda z) = K_w(z)\,, \quad z, w \in \mathbb{D}\,,\ |\lambda| = 1\,,$$

and since the rotations are easily verified to be the only $\mathbb{C}$-linear unitary maps of $\mathbb{C}$. Thus, we can apply [53, Lemma 2.2] to conclude that there exists a function $h$ analytic in $\mathbb{D}$ such that

$$K_w(z) = h(\overline{w}z)\,, \quad h(z) = \sum_{n=0}^{\infty} \gamma(n) z^n\,, \quad z, w \in \mathbb{D}\,, \quad \gamma(0) = 1\,,\ \gamma(n) \geq 0.$$

We still ought to show that $\gamma(n) = \|z^n\|^{-2}$ for all $n$, so further work is needed.

We follow the argument of [51, Proposition 4.1] and include the details for the sake of completeness. Let $I = \{n \in \mathbb{N} : \gamma(n) > 0\}$, so that $h(z) = \ {}_{n\in I} \gamma(n) z^n$. Consider the Hilbert space $H$ of analytic functions in the disc with orthonormal basis $\left\{e_n(z) = \sqrt{\gamma(n)} z^n : n \in I\right\}$. Our next objective is to show that the spaces $H$ and $\mathcal{H}$ coincide.

We first show that $H$ is a reproducing kernel Hilbert space by checking that the point evaluation functionals are bounded. Let $g \in H$, with $g = \ {}_{n\in I} b_n e_n$. Then, given $w \in \mathbb{D}$, the Cauchy-Schwarz inequality yields

$$\begin{aligned} |g(w)| &\leq \sum_{n\in I} |b_n|^2 {}^{1/2} \sum_{n\in I} |e_n(w)|^2 {}^{1/2} \\ &= \sum_{n\in I} |b_n|^2 {}^{1/2} \sum_{n\in I} \gamma(n) |w|^{2n} {}^{1/2} = \sqrt{h(|w|^2)} \|g\|_H\,. \end{aligned}$$

Next, we show that all functions in $H$ are analytic in $\mathbb{D}$. Since $h$ is analytic in $\mathbb{D}$ and has non-negative Taylor coefficients, $h(|z|)$ is an increasing function of $|z|$. Boundedness of point evaluations shows that convergence in $H$ implies uniform convergence on compact subsets of $\mathbb{D}$. Since the orthonormal basis of $H$ consists of polynomials, each function in $H$ is a uniform limit of polynomials on compact subsets of $\mathbb{D}$ and, hence, an analytic function in $\mathbb{D}$.

Finally, we check that the reproducing kernels of $H$ and $\mathcal{H}$ coincide. Denote by $E_w$ the reproducing kernel in $H$ for the point evaluation at $w \in \mathbb{D}$: $g(w) = \langle g, E_w \rangle_H$, for $g \in H$. If $E_w(z) = \quad {}_{n\in I} c_n e_n(z)$, for each $m \in I$ we have

$$e_m(w) = \langle e_m, E_w \rangle_H = \sum_{n\in I} \overline{c_n} \langle e_m, e_n \rangle_H = \overline{c_m}\,,$$

hence

$$E_w(z) = \sum_{n\in I} \overline{e_n(w)} e_n(z) = \sum_{n\in I} \gamma(n) \overline{w}^n z^n = K_w(z)$$

for all $z$, $w \in \mathbb{D}$.

Finally, both $\mathcal{H}$ and $H$ are Hilbert spaces with positive definite kernels that coincide. By the uniqueness part of the Moore-Aronszajn theorem [14, p. 344, (4)], we must have $H = \mathcal{H}$.

It is only left to determine the coefficients $\gamma(n)$ for all $n \geq 0$. For all $n \in I$ we easily compute $1 = \|e_n\|_H^2 = \|e_n\|_{\mathcal{H}}^2 = \langle \sqrt{\gamma(n)} z^n, \sqrt{\gamma(n)} z^n \rangle_{\mathcal{H}} = \gamma(n) \|z^n\|_{\mathcal{H}}^2$, hence $\gamma(n) = \|z^n\|_{\mathcal{H}}^{-2}$, as claimed.

Finally, it is only left to show that all the Taylor coefficients of $h$ must be non-zero: $I = \mathbb{N} \cup \{0\}$. Otherwise, there exists an index $N$ such that $\gamma(N) = 0$ and, since $z^N \in \mathcal{H} = H$, we have

$$z^N = \sum_{n\in I} b_n e_n(z) = \sum_{n\in I} b_n \sqrt{\gamma(n)} z^n\,,$$

with all the values $n \in I$ being different from $N$, which is clearly impossible.

*(e) ⇒ (f):* Trivially, multiplication by a constant of modulus one yields a surjective isometry (hence a unitary operator) and the product of two unitary operators is unitary.

*(f) ⇒ (d):* This assertion follows trivially by choosing the constant multiple to be one. □

## 3.3 Characterizations of co-isometric WCOs

### 3.3.1 Statement of the main theorem

We are now ready to formulate our main result of this chapter.

**Theorem 3.1.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms **(A1)**–**(A3)** and let $W_{F,\phi}$ be a bounded WCO on $\mathcal{H}$, where $F$ is a function analytic in $\mathbb{D}$ and $\phi$ an analytic map of $\mathbb{D}$ into itself. Then $W_{F,\phi}$ is unitary if and only if it is co-isometric (that is, if and only if $W_{F,\phi} W_{F,\phi}^* = I$).*

*Moreover, any of these two properties is further equivalent to the following:*

(a) *$\phi$ is a disc automorphism and $F = \mu(\phi')^{\gamma/2} = \nu\frac{K_a}{\|K_a\|}$, where $a = \phi^{-1}(0)$ and $\mu$ and $\nu$ are constants such that $|\mu| = |\nu| = 1$, in the case when $\mathcal{H}$ is one of the spaces $\mathcal{H}_\gamma$ considered in Section 1.10.*

(b) *$\phi$ is a rotation and $F$ is a constant function of modulus one, whenever $\mathcal{H}$ does not belong to the scale of spaces $\mathcal{H}_\gamma$.*

We shall refer to the operators given in the case (b) as to the trivial ones. It is interesting to notice that there are results in the literature with a similar flavor, although in a somewhat different context; *cf.*, for example, Proposition 4.3 and Corollary 9.10 of [53]. The recent paper [98] for a general class of spaces in the context of several variables (where certain conformal properties of the kernels are again assumed) contains some related ideas and similar results.

The rest of the sections is devoted to the proof of this result which we split up into a sequence of auxiliary statements in order to make it easier to follow. We begin by seeing that the assumption that $W_{F,\phi}$ is a co-isometry imposes certain important properties of the symbols.

### 3.3.2 Basic information on the symbols of $W_{F,\phi}$

As a preliminary information that will be needed later, we observe the following. Using the basic properties of the inner product and applying the operator to a reproducing kernel, we obtain

$$\begin{aligned}\langle f, W_{F,\phi}^* K_w\rangle &= \langle W_{F,\phi} f, K_w\rangle = \langle F(f\circ\phi), K_w\rangle = F(w) f(\phi(w)) \\ &= F(w)\langle f, K_{\phi(w)}\rangle = \langle f, \overline{F(w)} K_{\phi(w)}\rangle\,, \qquad f\in\mathcal{H}\,, \quad w\in\mathbb{D}\,.\end{aligned}$$

Hence we have the formula for the action of the adjoint of a WCO on reproducing kernels:

$$W_{F,\phi}^* K_w = \overline{F(w)} K_{\phi(w)}\,, \qquad w\in\mathbb{D}\,. \tag{3.2}$$

Using this, we can show that the assumption that a WCO is co-isometric already provides some rigid information on the symbols $F$ and $\phi$. As is usual, by a *univalent function* in $\mathbb{D}$ we mean a function analytic in the disc which is one-to-one there.

**Proposition 3.2.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms **(A1)**–**(A3)** and let $W_{F,\phi}$ be a bounded WCO on $\mathcal{H}$. If $W_{F,\phi}$ is co-isometric then $F$ is given by*

$$F(z) = \frac{1}{\overline{F(0)} K_{\phi(0)}(\phi(z))}\,, \qquad \text{for all } z\in\mathbb{D}\,, \tag{3.3}$$

*and $\phi$ is a univalent function.*

*Proof.* Since by assumption $W_{F,\phi} W_{F,\phi}^* = I$, using (3.2), we obtain

$$K_w = W_{F,\phi} W_{F,\phi}^* K_w = W_{F,\phi}\left(\overline{F(w)} K_{\phi(w)}\right) = \overline{F(w)} W_{F,\phi} K_{\phi(w)} = \overline{F(w)} F(K_{\phi(w)}\circ\phi)\,.$$

Thus,

$$F(z)\overline{F(w)}K_{\phi(w)}(\phi(z)) = K_w(z)\,, \qquad \text{for all } z, w \in \mathbb{D}\,. \tag{3.4}$$

Taking $w = 0$, the right-hand side is $\equiv 1$, so (3.3) follows immediately.

We now show that $\phi$ is univalent. To this end, let $z_1$, $z_2 \in \mathbb{D}$ be such that $\phi(z_1) = \phi(z_2)$. Then it follows that $F(z_1) = F(z_2)$: indeed, by (3.3) we have

$$F(z_1) = \frac{1}{\overline{F(0)}K_{\phi(0)}(\phi(z_1))} = \frac{1}{\overline{F(0)}K_{\phi(0)}(\phi(z_2))} = F(z_2)\,.$$

Next, using (3.4) three times but with different values of $z$ and $w$, we get

$$\begin{aligned} K_{z_1}(z_1) &= |F(z_1)|^2 K_{\phi(z_1)}(\phi(z_1))\,, \\ K_{z_2}(z_2) &= |F(z_2)|^2 K_{\phi(z_2)}(\phi(z_2))\,, \\ K_{z_2}(z_1) &= F(z_1)\overline{F(z_2)}K_{\phi(z_2)}(\phi(z_1))\,. \end{aligned}$$

By our choice of $z_1$ and $z_2$, the right-hand sides of the last three equations all coincide, hence

$$K_{z_1}(z_1) = K_{z_2}(z_2) = K_{z_2}(z_1)\,.$$

Recall that for any kernel given by (3.1) the function $K_z(z)$ is a strictly increasing function of $|z|$, hence from the first equality above it follows that $|z_1| = |z_2|$. If $z_1 = 0$, it follows that also $z_2 = 0$ and we are done. Thus, we may assume that $|z_1| > 0$.

Next, writing $z_2 = \lambda z_1$ with $|\lambda| = 1$ and using the remaining equality and equation (3.1), we obtain

$$\sum_{n=1}^{\infty} \gamma(n)|z_1|^{2n} = \sum_{n=1}^{\infty} \gamma(n)\overline{\lambda}^n|z_1|^{2n}\,,$$

which yields

$$\sum_{n=1}^{\infty} \gamma(n)\,\mathrm{Re}\,(1 - \overline{\lambda}^n)\,|z_1|^{2n} = 0\,.$$

Since each term in the sum on the left is non-negative and $\gamma(n) > 0$ and $|z_1| > 0$, it follows that

$$\mathrm{Re}\,(1 - \overline{\lambda}^n) = 0$$

for all $n \geq 1$, which implies $\lambda = 1$, hence $z_1 = z_2$. This proves that $\phi$ is univalent. $\square$

It should be noted that formula (3.4) has already appeared before in the literature; see Zorboska [98, Proposition 1].

### 3.3.3 Kernels bounded on the diagonal

Proposition 3.2 proved above will help us to handle the simpler case of the kernel bounded on the diagonal. We will refer to $\{(z, w) \in \mathbb{D} \times \mathbb{D} : z = w\}$ as the *diagonal* of the bidisc $\mathbb{D} \times \mathbb{D}$. It will be relevant to our proofs to distinguish between the kernels that are bounded on the diagonal and those that are not.

A simple example of the kernel of type (3.1) which is bounded on the diagonal is

$$K_w(z) = 1 + \overline{w}z + \sum_{n=2}^{\infty} \frac{\overline{w}z}{n(n-1)} = 1 + 2\overline{w}z - (1-\overline{w}z)\log\frac{1}{1-\overline{w}z}\,.$$

In relation to an argument mentioned in the proof below, it is convenient to recall that reproducing kernels may have zeros in the bidisc (on or off the diagonal); this question is relevant in the theory of one and several complex variables. In relation to the kernels considered here, we mention the recent reference [72].

**Theorem 3.2.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms **(A1)**–**(A3)** and whose reproducing kernel is bounded on the diagonal of the bidisc, and let $W_{F,\phi}$ be a bounded WCO on $\mathcal{H}$. Then the following statements are equivalent:*
*(a) $W_{F,\phi}$ is unitary.*
*(b) $W_{F,\phi}$ is co-isometric.*
*(c) $F$ is a constant function of modulus one and $\phi$ is a rotation.*

*Proof.* Trivially, (a) implies (b). It is clear from Proposition 3.1 that (c) implies (a). Thus, it only remains to show that (b) implies (c).

Suppose that $W_{F,\phi}$ is co-isometric. The assumption that the kernel is bounded on the diagonal of the bidisc is equivalent to saying that ${}_{n=0}^{\infty}\gamma(n) < +\infty$. The Weierstrass test and formula (3.1) readily imply that the kernel extends continuously to the closed bidisc $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Since in principle the kernel could have zeros, in view of (3.3) we need an additional argument in order to show that $F$ extends continuously to $\overline{\mathbb{D}}$. This can be seen as follows: for every fixed $z \in \mathbb{D}$, the function $K_z$ is continuous in $\overline{\mathbb{D}}$. Note also that

$$|F(z)| = |\langle F, K_z\rangle| \le \|F\|\|K_z\| = \|F\|\sqrt{K_z(z)} \le \|F\| \max_{\zeta\in\overline{\mathbb{D}}} \sqrt{K_\zeta(\zeta)}\,,$$

for all $z \in \mathbb{D}$, hence $F$ is bounded in the disc and therefore also in $\overline{\mathbb{D}}$. Equation (3.3) implies that $K_{\phi(0)}(\phi(z))$ is bounded away from zero in the disc and, since it is continuous in $\overline{\mathbb{D}}$, it is also bounded away from zero in the closed disc. This, together with (3.3), shows that $F$ is continuous in $\overline{\mathbb{D}}$.

In view of Proposition 3.2, setting $w = z$ in (3.4), we know that

$$|F(z)|^2 = \frac{K_z(z)}{K_{\phi(z)}(\phi(z))}\,, \qquad z \in \mathbb{D}\,. \tag{3.5}$$

Let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle. Since $\phi$ is analytic in $\mathbb{D}$ and bounded by one, the finite radial limits $\phi(\zeta) = \lim_{r\to 1^-}\phi(r\zeta)$ exist and also satisfy $|\phi(\zeta)| \le 1$ for almost every point $\zeta \in \mathbb{T}$ with respect to the normalized Lebesgue arc length measure on $\mathbb{T}$ [40] [60]. If $\zeta \in \mathbb{T}$ is a point where $\phi(\zeta)$ exists, since $K_z(z)$ is an increasing bounded function of $|z|$, we see that there exists $L < \infty$ such that

$$\lim_{r\to 1^-} K_{r\zeta}(r\zeta) = L.$$

Moreover, by monotonicity, it is clear that $\sup_{r<1} K_r(r) = L$. Hence, taking into account that $F$ is continuous in $\overline{\mathbb{D}}$ and (3.5), we deduce that there exists $L' < \infty$ such that

$$\lim_{r \to 1^-} K_{r\phi(\zeta)}(r\phi(\zeta)) = L'$$

and $L' \leq L = \sup_{r<1} K_r(r)$. Therefore, by (3.5), we have

$$\lim_{r \to 1^-} |F(r\zeta)|^2 = \frac{L}{L'} \geq 1,$$

so $|F(\zeta)| \geq 1$. Hence $|1/F| \leq 1$ almost everywhere on $\mathbb{T}$ and therefore $|1/F| \leq 1$ in $\mathbb{D}$ (by standard Hardy space arguments). On the other hand, from (3.5) we get

$$|F(0)|^2 = \frac{1}{K_{\phi(0)}(\phi(0))} \leq 1\,,$$

again because $K_z(z)$ is an increasing function of $|z|$. The maximum modulus principle applied to $1/F$ implies that $F$ is identically constant and has modulus one.

In view of (3.5) we have $K_z(z) = K_{\phi(z)}(\phi(z))$ for all $z \in \mathbb{D}$. Taking into account the form (3.1) of the kernel, this means that

$$1 + \sum_{n=1}^{\infty} \gamma(n)|z|^{2n} = 1 + \sum_{n=1}^{\infty} \gamma(n)|\phi(z)|^{2n}\,.$$

Since $1 + \sum_{n=1}^{\infty} \gamma(n) r^{2n}$ is a strictly increasing function of $r$, it follows that equality above is possible if and only if $|\phi(z)| = |z|$, for every $z \in \mathbb{D}$. But this implies that $\phi$ is a rotation. $\square$

### 3.3.4 Kernels unbounded on the diagonal

For the kernels of the general form (3.1) considered here, the assumption that $K_w(z)$ is unbounded on the diagonal obviously means that $\lim_{|z|\to 1^-} K_z(z) = +\infty$, which is easily seen to be equivalent to $\sum_{n=0}^{\infty} \gamma(n) = +\infty$.

It is relevant to note that for any of the spaces $\mathcal{H}_\gamma$ defined in Section 1.10 for which

$$\begin{aligned} &\gamma(0) = 1\,, \quad \gamma(1) = \gamma > 0 \\ &\gamma(n) = \binom{\gamma + n - 1}{n} = \frac{\Gamma(n+\gamma)}{\Gamma(\gamma) n!} = \frac{1}{(n+\gamma)B(\gamma, n+1)}\,, \quad n \geq 1\,, \end{aligned} \tag{3.6}$$

the reproducing kernel is always unbounded on the diagonal since $\gamma(1) > 0$. An example of a space $\mathcal{H}$ not in the family $\mathcal{H}_\gamma$ and whose kernel is unbounded on the diagonal is the classical Dirichlet space (renormed) with

$$\gamma(n) = \frac{1}{n+1}\,, \qquad K_w(z) = \frac{1}{\overline{w}z} \log \frac{1}{1 - \overline{w}z}\,.$$

Indeed, it can be checked that (3.6) is not fulfilled in this case.

We will see that, unlike in the previous case, for certain kernels of this type non-trivial unitary WCOs will exist.

**Theorem 3.3.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms (**A1**)–(**A3**) and whose reproducing kernel is unbounded on the diagonal. If $W_{F,\phi}$ is co-isometric on $\mathcal{H}$ then $\phi$ is a disc automorphism.*

*In the case when $\phi(0) = 0$ (in particular, whenever $\phi$ is a rotation), $F$ must be a constant of modulus one and the induced operator $W_{F,\phi}$ is unitary on $\mathcal{H}$.*

*Proof.* We recall that an inner function is a bounded function with radial limits of modulus one almost everywhere on the unit circle $\mathbb{T}$. From the basic factorization theory of Hardy spaces [40, Chapter 2 and Theorem 3.17], it follows that a univalent inner function must be a disc automorphism; note that this can also be concluded by post-composing with disc automorphisms and invoking Frostman's theorem see [47] or [40, Chapter 2, Exercise 8]. We already know from Proposition 3.2 that $\phi$ is univalent. Thus, it suffices to show that it is also an inner function.

We certainly know that $\phi$ is an analytic self-map of $\mathbb{D}$ so it must have radial limits almost everywhere and these limits have modulus at most one. Consider the set

$$E = \{\zeta \in \mathbb{T} : \ |\phi(\zeta)| < 1\}$$

and show that its arc length measure is $m(E) = 0$. Let us look again at formula (3.5). If $\zeta \in E$, since the kernel is unbounded on the diagonal, we have $\lim_{z\to\zeta} K_z(z) = +\infty$. On the other hand, $\phi(\zeta) \in \mathbb{D}$ by our definition of $E$, hence the value $K_{\phi(\zeta)}(\phi(\zeta))$ is defined and finite. It follows that $F(z) \to \infty$ as $z \to \zeta$. Now in view of (3.3), we obtain $K_{\phi(0)}(\phi(\zeta)) = 0$. Since $\zeta \in E$ was arbitrary, this holds for all $\zeta \in E$.

Now assume the contrary to our assumption: $m(E) > 0$. First note that, by our definition of $E$, the set $\phi(E)$ is contained in $\mathbb{D}$ and clearly

$$E \subset \bigcup_{a\in\phi(E)} \{\zeta \in \mathbb{T} : \ \phi(\zeta) = a\} . \tag{3.7}$$

Next, we claim that $m(\{\zeta \in \mathbb{T} : \ \phi(\zeta) = a\}) = 0$, for every $a \in \phi(E)$. We know that $\phi$ is univalent, hence it cannot be identically constant. And since $\phi \in H^\infty$, it is impossible for $\phi(\zeta) = a$ to hold on a set of positive measure on $\mathbb{T}$ by a theorem of Privalov (see [40, Theorem 2.2] or [60, Chapter III] for different versions of it). This proves the claim.

Now we can distinguish between two cases, depending on the cardinality of the set $\phi(E)$. If $\phi(E)$ is countable then $m(E) = 0$ by (3.7), as claimed. And if $\phi(E)$ is uncountable, we argue as follows. The set $\phi(E)$ is contained in $\mathbb{D}$ and has at least one accumulation point in $\mathbb{D}$. (Otherwise, each compact disc $D_n = \{z : \ |z| \le 1 - \frac{1}{n}\}$, $n \in \mathbb{N}$, would contain only finitely many points of $\phi(E)$ and since $\mathbb{D} = \cup_{n=1}^\infty D_n$, the set $\phi(E)$ would be countable.) But, as we have noticed above, the analytic function $K_{\phi(0)}$ vanishes on $\phi(E)$ and is therefore identically zero in $\mathbb{D}$ by the uniqueness principle, which is impossible.

Thus, we conclude that $m(E) = 0$, hence $\phi$ is an inner function. This completes the proof that $\phi$ is an automorphism.

As for the final part of the statement, if $\phi(0) = 0$ the function $F$ must be a constant of modulus one: indeed, equation (3.3) together with $K_0(z) \equiv 1$ shows that

$F \equiv 1/\overline{F(0)}$. Writing $F \equiv \lambda$, it is immediate that $|\lambda| = 1$. Now, it follows directly from Proposition 3.1 that the induced operator $W_{\lambda,\phi}$ is actually unitary. □

In what follows we will rely on the properties of the disc automorphisms. One basic type of automorphisms are the rotations $R_\lambda$, where $|\lambda| = 1$. Since our spaces are supposed to satisfy the axioms listed in Lemma 3.1, it follows that the induced composition operators $C_{R_\lambda}$ are unitary on $\mathcal{H}$. The other basic type of automorphisms are the maps $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$, $a \in \mathbb{D}$; such an automorphism is an involution and exchanges the point $a$ and the origin. As is well-known, every disc automorphism $\phi$ is of the form $\phi = R_\lambda \varphi_a$. To make the notation more compact, we will write $\varphi_{\lambda,a} = R_\lambda \varphi_a$.

The following lemma will be fundamental in proving the last theorem of this chapter. It will allow us to change from one co-isometric WCO to another operator of the same kind (acting on the same space) in a convenient way. We give an elementary proof below. However, it is convenient to remark that a stronger version of the statement could be derived from [53, Proposition 9.9].

**Lemma 3.1.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms **(A1)**–**(A3)** and let $W_{F,\phi}$ be a co-isometric WCO on $\mathcal{H}$ such that $\phi = \varphi_{\lambda,a}$, for some $\lambda$ with $|\lambda| = 1$, $a \in \mathbb{D}$. Then for all $b \in [0, |a|]$ there exists a point $c \in \mathbb{D}$ with $|c| = b$, a constant $\mu$ with $|\mu| = 1$, and an analytic function $G$ in the disc such that $W_{G,\varphi_{\mu,c}}$ is also a co-isometric WCO on $\mathcal{H}$.*

*Proof.* It is easy to check directly that the product of two co-isometric operators is again co-isometric. Thus, whenever $|\tau| = 1$ and $W_{F,\phi}$ is co-isometric, the operator $C_{R_\tau} W_{F,\lambda\varphi_a}$ is also co-isometric. In view of the simple identity for compositions of automorphisms:

$$\varphi_{\lambda,a}(\tau z) = \varphi_{\tau\lambda,\overline{\tau}a}(z)\,,$$

we obtain the following operator identity:

$$C_{R_\tau} W_{F,\varphi_{\lambda,a}} = W_{C_{R_\tau}F,\,\varphi_{\tau\lambda,\,\overline{\tau}a}}\,,$$

so the latter WCO is a co-isometric operator for every value of $\tau$ with $|\tau| = 1$. Thus, its square

$$W_{C_{R_\tau}F,\,\varphi_{\tau\lambda,\overline{\tau}a}} W_{C_{R_\tau}F,\,\varphi_{\tau\lambda,\overline{\tau}a}} = W_{G,\,\varphi_{\mu,c}}$$

is also co-isometric, where

$$G(z) = F(\tau z) \cdot F(\varphi_{\tau^2\lambda,\overline{\tau}a}(z))\,, \qquad \varphi_{\mu,c} = \varphi_{\tau\lambda,\,\overline{\tau}a} \circ \varphi_{\tau\lambda,\,\overline{\tau}a}\,. \tag{3.8}$$

(Note that the last map must equal some $\varphi_{\mu,c}$ for some $c \in \mathbb{D}$ and some $\mu$ with $|\mu| = 1$ since it is again a disc automorphism.)

Next, let $b$ be an arbitrary number such that $0 < b < |a|$. Let $F$ and $\phi$ be fixed as above. We have the freedom to choose $\tau$ arbitrarily in (3.8) and will now show that there exists $c$ as above (with $\tau$ chosen appropriately) so that $|c| = b$. In fact, such value of $c$ from the conditions above can easily be computed explicitly; indeed, we must have

$$(\varphi_{\tau\lambda,\overline{\tau}a} \circ \varphi_{\tau\lambda,\overline{\tau}a})(c) = 0\,,$$

hence $\varphi_{\tau\lambda,\overline{\tau}a}(c) = \overline{\tau}a$, meaning that

$$\tau\lambda\frac{\overline{\tau}a - c}{1 - \overline{a}\tau c} = \overline{\tau}a\,,$$

which yields

$$c = \overline{\tau}a\frac{\lambda\tau - 1}{\lambda\tau - |a|^2}\,.$$

Now note that

$$|c| = |a|\frac{|\lambda\tau - 1|}{|\lambda\tau - |a|^2|}$$

is a continuous function of the complex parameter $\tau$. When $\tau = \overline{\lambda}$, this function takes on value $0$ while choosing $\tau = -\overline{\lambda}$ yields

$$\frac{2|a|}{1 + |a|^2} > |a|$$

as the value of the function (recall that $a = 0$ since $\phi$ is not a rotation). Since the complex parameter $\tau = e^{it}$ can in turn be viewed as a continuous function of the real variable $t \in [0, 2\pi]$, the elementary Bolzano's intermediate value theorem from Calculus implies that there exists a value $\tau$ with $|\tau| = 1$ such that $|c| = b$, as claimed. (Of course, the last argument could have been made more explicit.) $\square$

We already know from T. Le's work [62, Theorem 3.1] that if $\phi$ is a disc automorphism and $F = \mu(\phi')^{\gamma/2}$, where $\gamma = \gamma(1)$ and $\mu$ is a constant such that $|\mu| = 1$, then the induced operator $W_{F,\phi}$ is unitary (hence, also co-isometric) on the space $\mathcal{H}_\gamma$. The next key statement identifies such spaces as the only ones on which $W_{F,\phi}$ can be co-isometric if the automorphism $\phi$ is not a rotation. It should be remarked that, using the following conformal property of the kernel:

$$K_{\phi(w)}(\phi(z)) = \frac{K_a(a)K_w(z)}{K_a(z)K_w(a)}\,,$$

for all disc automorphisms $\phi$ and $a = \phi^{-1}(0)$, the theorem below can also be deduced from a characterization of the spaces $\mathcal{H}_\gamma$ given by [53, Proposition 4.3].

**Theorem 3.4.** *Let $\mathcal{H}$ be a weighted Hardy space that satisfies axioms **(A1)**–**(A3)** and whose reproducing kernel is unbounded on the diagonal, and let $W_{F,\phi}$ be a co-isometric WCO on $\mathcal{H}$. If the automorphism $\phi$ is not a rotation, then there exists a positive $\gamma$ (namely, $\gamma = \gamma(1)$ in the formula for the kernel) such that $\mathcal{H} = \mathcal{H}_\gamma$, a space whose coefficients $\gamma(n)$ satisfy* (3.6) *and whose kernel, thus, is of the form* (1.2)*. Moreover, $F = \mu(\phi')^{\gamma/2} = \nu\frac{K_a}{\|K_a\|}$, where $a = \phi^{-1}(0)$ and $\mu$ and $\nu$ are constants such that $|\mu| = |\nu| = 1$.*

*Proof.* Let $\phi = \varphi_{\lambda,a}$, for some $a \in \mathbb{D}$, $a = 0$, and $\lambda$ with $|\lambda| = 1$. Put $w = a$ in (3.4) and recall that $\phi(a) = 0$ to obtain

$$F(z) = \frac{K_a(z)}{\overline{F(a)}}\,.$$

In what follows, we will always write simply $K'_a(z)$ instead of $\dfrac{\partial K_a}{\partial z}(z)$. After differentiation with respect to $z$, we obtain

$$F'(z) = \frac{K'_a(z)}{\overline{F(a)}}\,.$$

Also, note that $F(0) = 1/\overline{F(a)}$.

On the other hand, recalling that $\phi(0) = \lambda a$ and differentiating (3.3) with respect to $z$, we get

$$F'(z) = \frac{-\phi'(z)K'_{\lambda a}(\phi(z))}{\overline{F(0)}K^2_{\lambda a}(\phi(z))} = \frac{\lambda(1-|a|^2)K'_{\lambda a}(\phi(z))}{\overline{F(0)}(1-\overline{a}z)^2K^2_{\lambda a}(\phi(z))}\,.$$

Equating the right-hand sides of the last two equations, taking also into account the fact that $F(0) = 1/\overline{F(a)}$, yields

$$\frac{|F(a)|^2(1-|a|^2)}{(1-\overline{a}z)^2}K'_{\lambda a}(\phi(z)) = \overline{\lambda}K^2_{\lambda a}(\phi(z))K'_a(z)\,.$$

Setting $z = a$, we obtain

$$\frac{|F(a)|^2}{1-|a|^2}K'_{\lambda a}(0) = \overline{\lambda}K'_a(a)\,. \tag{3.9}$$

Differentiation of the formula for the kernel (3.1) with respect to $z$ yields

$$K'_w(z) = \gamma(1)\overline{w} + \sum_{n=2}^{\infty} n\gamma(n)\overline{w}^n z^{n-1}\,,$$

hence

$$K'_a(a) = \gamma(1)\overline{a} + \sum_{n=2}^{\infty} n\gamma(n)\overline{a}|a|^{2(n-1)}\,, \qquad K'_{\lambda a}(0) = \gamma(1)\overline{\lambda}\overline{a}\,.$$

Bearing in mind that

$$|F(a)|^2 = K_a(a) = 1 + \sum_{n=1}^{\infty}\gamma(n)|a|^{2n}$$

and using (3.9), it follows that

$$\begin{aligned} 1 + \sum_{n=1}^{\infty}\gamma(n)|a|^{2n} &= (1-|a|^2)\quad 1 + \sum_{n=2}^{\infty}\frac{n\gamma(n)}{\gamma(1)}|a|^{2(n-1)} \\ &= 1 + \sum_{n=1}^{\infty}\frac{(n+1)\gamma(n+1)}{\gamma(1)}|a|^{2n} - \sum_{n=1}^{\infty}\frac{n\gamma(n)}{\gamma(1)}|a|^{2n} \end{aligned}$$

(after regrouping the terms). From here we obtain that

$$\sum_{n=1}^{\infty} \gamma(n)(n+\gamma(1))|a|^{2n} = \sum_{n=1}^{\infty} (n+1)\gamma(n+1)|a|^{2n} . \tag{3.10}$$

Note that this holds only for one point $a$, for one given co-isometric operator $W_{F,\phi_{\lambda,a}}$. However, an application of Lemma 3.1 allows us to produce other WCOs $W_{G,\varphi_{\mu,c}}$ that are also co-isometric and with different values $c$ instead of $a$ so as to include all possible values of $|c|$ with $0 \le |c| \le |a|$, with conclusions analogous to (3.10). Note also an important point that, whenever $c = 0$, the symbol $\varphi_{\mu,c}$ is not a rotation. By this construction, we obtain

$$\sum_{n=1}^{\infty} \gamma(n)(n+\gamma(1))x^{2n} = \sum_{n=1}^{\infty} (n+1)\gamma(n+1)x^{2n}$$

for all $x \in [0,|a|]$, the case $x = 0$ being an obvious equality. The power series in the identity above are both even functions of $x$ that converge and coincide in $[-|a|,|a|]$. The uniqueness of the coefficients of a real power series in such an interval implies that

$$(n+1)\gamma(n+1) = \gamma(n)(n+\gamma(1)) , \qquad \text{for all } n \ge 1 .$$

This recurrence equation is easily solved: since

$$\gamma(n+1) = \gamma(n)\frac{n+\gamma(1)}{n+1} , \qquad \text{for all } n \ge 1 ,$$

and the formula trivially also extends to the case $n = 0$ (by the fact that $\gamma(0) = 1$), we have

$$\gamma(n) = \frac{n-1+\gamma(1)}{n}\gamma(n-1) , \qquad \text{for all } n \ge 1 .$$

Using the standard property $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$, from here we obtain by induction the desired formula (3.6), with $\gamma = \gamma(1)$.

Next, we derive the formula for $F$ in terms of $\phi'$. From (3.3), for the space is $\mathcal{H} = \mathcal{H}_\gamma$ (so, $K_a = K_a^\gamma$) and $\phi = \varphi_{\lambda,a}$, we have

$$F(z) = \frac{1}{\overline{F(0)}K_{\phi(0)}(\phi(z))} = \frac{1}{\overline{F(0)}} \;\; 1-\overline{\phi(0)}\phi(z) \;\; {}^{\gamma} = \frac{1}{\overline{F(0)}}\frac{(1-|a|^2)^\gamma}{(1-\overline{a}z)^\gamma} .$$

For $z = 0$ this shows that $|F(0)| = (1-|a|^2)^{\gamma/2}$, hence for appropriately chosen $\mu$ and $\nu$ with $|\mu| = |\nu| = 1$ we obtain

$$F(z) = \nu\frac{(1-|a|^2)^{\gamma/2}}{(1-\overline{a}z)^\gamma} = \nu\frac{K_a(z)}{\|K_a\|} = \mu(\phi'(z))^{\gamma/2} ,$$

which completes the proof. □

Finally, putting together Theorem 3.2, Theorem 3.3, and Theorem 3.4 and the comments on the results of Le preceding Theorem 3.4, we obtain the complete conclusions of Theorem 3.1.

# 4 Weighted conformal invariance of Banach spaces

## 4.1 Introduction

This chapter is devoted to exploring a class of spaces of analytic functions which share certain weighted invariant property and it is based on the paper [5]. First of all, we will define this property.

**Definition 4.1.** Let $X$ be a Banach space consisting of analytic functions in the unit disc $\mathbb{D}$ with the following properties:

1) $X$ is continuously contained in $Hol(\mathbb{D})$.

2) $X$ contains $Hol(\rho\mathbb{D})$, for all $\rho > 1$.

3) There exist constants $\alpha = \alpha(X),\ K = K(X) > 0$, such that for every $\varphi \in Aut(\mathbb{D})$, the linear map defined by $W_\varphi^\alpha f = (\varphi')^\alpha (f \circ \varphi)$, is bounded on $X$ and satisfies $\|W_\varphi^\alpha\| \leq K$.

Then we will say that $X$ is *conformally invariant of index* $\alpha$.

Before seeing a list of examples of conformally invariant Banach spaces, we make some basic properties which follow directly from the axioms and will be used frequently in what follows.

**Proposition 4.1.** *a) From 2) we have by the closed graph theorem that for every $\rho > 1$, the inclusion map from $Hol(\rho\mathbb{D})$ into $X$ is continuous.*

*b) All bounded operators as in 3) above are invertible and $(W_\varphi^\alpha)^{-1} = W_{\varphi^{-1}}^\alpha$.*

*c) The number $\alpha(X)$ in 3) is unique. The same argument that we will see bellow in the proof shows that the spaces $\mathcal{M}_0(X)$, defined by* (4) *in the Introduction, are not conformally invariant of index $\alpha > 0$ unless they are trivial i.e. unless $\mathcal{M}_0(X) = \{0\}$.*

*Proof.* a) Given $\rho > 1$, from 2) we know that $Hol(\rho\mathbb{D}) \subset X$. Thanks to Corollary 1.1 we obtain that $Hol(\rho\mathbb{D})$ is continuously contained in $X$. Here we are using that the topological vector space $Hol(\rho\mathbb{D})$ is an $F$-space considering the invariant metric defined in [37, Page 143]. For the completeness see [37, 2.3 Corollary].

b) By hypothesis $W_{\varphi^{-1}}^{\alpha}$ is bounded. Let $f \in X$. Then

$$W_{\varphi^{-1}}^{\alpha} W_{\varphi}^{\alpha} f = W_{\varphi^{-1}}^{\alpha} \left((\varphi')^{\alpha} f \circ \varphi\right) = [(\varphi^{-1})']^{\alpha} (\varphi' \circ (\varphi^{-1})')^{\alpha} f \circ \varphi \circ \varphi^{-1} = f$$

$$W_{\varphi}^{\alpha} W_{\varphi^{-1}}^{\alpha} f = W_{\varphi}^{\alpha} \left([(\varphi^{-1})']^{\alpha} f \circ \varphi^{-1}\right) = (\varphi')^{\alpha} ((\varphi^{-1})' \circ \varphi')^{\alpha} f \circ \varphi^{-1} \circ \varphi = f$$

Therefore, $W_{\varphi}^{\alpha}$ is an invertible operator with inverse $W_{\varphi^{-1}}^{\alpha}$.

c) If $W_{\varphi}^{\alpha}, W_{\varphi}^{\beta}$ are uniformly bounded and, say $\alpha < \beta$, then by 2) we know that $1 \in X$, so $W_{\varphi}^{\alpha} W_{\varphi^{-1}}^{\beta} 1 = (\varphi')^{\alpha - \beta}$ is uniformly bounded on $X$ which leads to a contradiction since the values at the origin of these functions are unbounded when $\varphi \in Aut(\mathbb{D})$. If $\varphi(z) = \lambda \frac{z+a}{1+\bar{a}z}$, with $a \in \mathbb{D}$ and $|\lambda| = 1$, then

$$(\varphi')^{\alpha-\beta}(z) = \left(\lambda \frac{1-|a|^2}{(1+\bar{a}z)^2}\right)^{\alpha-\beta} \Rightarrow (\varphi')^{\alpha-\beta}(0) = \left(\frac{1}{\lambda(1-|a|^2)}\right)^{\beta-\alpha}.$$

$\square$

We now turn to the examples.

## 4.2 Examples

The purpose of this section is to list a number of examples of conformally invariant Banach spaces in the unit disc.

**Example 4.1.** In many cases the operators defined in 3) are isometries on the spaces in question and this property follows by a change of variable. Such examples are the usual

- Hardy spaces $H^p,\ p \geq 1$, considered in Section 1.4, with $\alpha(H^p) = \frac{1}{p}$. If $f \in H^p$

$$\|W_{\varphi}^{\frac{1}{p}} f\|_{H^p}^p = \frac{1}{2\pi} \int_{\mathbb{T}} |f \circ \varphi|^p \, |\varphi'| \, dm = \frac{1}{2\pi} \int_{\mathbb{T}} |f|^p \, dm = \|f\|_{H^p}^p.$$

- Korenblum growth classes, $\mathcal{A}^{-\gamma}$, $\gamma > 0$, with $\alpha(\mathcal{A}^{-\gamma}) = \gamma$. First of all, as we had seen in Section 1.9, an analytic function $f$ is in the space $\mathcal{A}^{-\gamma}$ if and only if

$$\|f\|_{\mathcal{A}^{-\gamma}} = \sup_{z \in \mathbb{D}} (1-|z|^2)^{\gamma} |f(z)| < \infty.$$

If $f \in \mathcal{A}^{-\gamma}$, using the basic identity for disc automorphisms: $|\varphi'(z)|(1-|z|^2) = 1 - |\varphi(z)|^2$, we have

$$\begin{aligned} \|W_{\varphi}^{\gamma} f\|_{\mathcal{A}^{-\gamma}} &= \sup_{|z|<1} |f(\varphi(z))| |\varphi'(z)|^{\gamma} (1-|z|^2)^{\gamma} \\ &= \sup_{|z|<1} |f(\varphi(z))| (1-|\varphi(z)|^2)^{\gamma} \\ &= \sup_{|z|<1} |f(z)| (1-|z|^2)^{\gamma} \\ &= \|f\|_{\mathcal{A}^{-\gamma}}. \end{aligned}$$

The same holds for their "little oh" version $\mathcal{A}_0^{-\gamma}$.

- Standard weighted Bergman spaces $A^p_\beta$, $p \geq 1$, $\beta > -1$, considered in Section 1.5, with $\alpha(A^p_\beta) = \frac{2+\beta}{p}$. If $f \in A^p_\beta$, using again the identity $|\varphi'(z)|(1-|z|^2) = 1 - |\varphi(z)|^2$, we obtain

$$\begin{aligned} \|W_\varphi^{\frac{2+\beta}{p}} f\|^p_{A^p_\beta} &= (\beta+1) \int_{\mathbb{D}} |f \circ \varphi(z)|^p |\varphi'(z)|^{2+\beta} (1-|z|^2)^\beta dA(z) \\ &= (\beta+1) \int_{\mathbb{D}} |f \circ \varphi(z)|^p |\varphi'(z)|^2 (1-|\varphi(z)|^2)^\beta dA(z) \\ &= (\beta+1) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta dA(z) \\ &= \|f\|^p_{A^p_\beta}. \end{aligned}$$

Indeed, any Banach space $X$ satisfying 1)-3) can be endowed with the equivalent norm

$$\|f\|_\alpha = \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_\varphi f\|,$$

which makes these operators isometric. On the one hand, considering $\varphi = I$ we can see

$$\|f\|_\alpha = \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_\varphi f\| \geq \|f\|.$$

On the other hand, by 3)

$$\|f\|_\alpha = \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_\varphi f\| \leq \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_\varphi\| \|f\| \leq K\|f\|.$$

Moreover, if $\phi \in Aut(\mathbb{D})$ we have

$$\begin{aligned} \|W^\alpha_\phi f\|_\alpha &= \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_\phi W^\alpha_\varphi f\| = \sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha (\phi' \circ \varphi)^\alpha f \circ \phi \circ \varphi\| \\ &= \sup_{\varphi \in Aut(\mathbb{D})} \|W^\alpha_{\phi\circ\varphi} f\| = \sup_{\nu \in Aut(\mathbb{D})} \|W^\alpha_\nu f\| = \|f\|_\alpha. \end{aligned}$$

**Example 4.2.** The Banach space

$$\mathcal{A}^{\log} = \{f \in Hol(\mathbb{D}) : \|f\|_{\mathcal{A}^{\log}} = \sup_{|z|<1} \frac{1}{\log \frac{2}{1-|z|^2}} |f(z)| < \infty\},$$

satisfies 1) by a direct application of uniform boundedness principle, and 2), but fails to satisfy 3) for any $\alpha > 0$. Indeed, if we consider $\psi_a(z) = \frac{z-a}{1-\bar{a}z}$, then

$$\begin{aligned} \sup_{a \in \mathbb{D}} \|W^\alpha_{\psi_a^{-1}} 1\|_{\mathcal{A}^{\log}} &= \sup_{a \in \mathbb{D}} \|((\psi_a^{-1})')^\alpha\|_{\mathcal{A}^{\log}} = \sup_{a \in \mathbb{D}} \sup_{|z|<1} \frac{1}{\log \frac{2}{1-|z|^2}} |(\psi_a^{-1})'(z)|^\alpha \\ &= \sup_{a \in \mathbb{D}} \sup_{|w|<1} \frac{1}{\log \frac{2}{1-|\psi_a(w)|^2}} |\psi_a'(w)|^{-\alpha} \geq \sup_{a \in \mathbb{D}} \frac{1}{\log \frac{2}{1-|a|^2} (1-|a|^2)^\alpha} \\ &= \infty. \end{aligned}$$

**Example 4.3.** Recall that the standard weighted Besov spaces $B^{p,\beta},\ p \geq 1, \beta > -1$, considered in Section 1.6, consist of all analytic functions in $\mathbb{D}$ whose derivative belongs to $A^p_\beta$, in this chapter we will consider the norm

$$\|f\|_{B^{p,\beta}} = |f(0)| + \|f'\|_{A^p_\beta}.$$

These spaces satisfy 1)-3) if $p < \beta+2$ and in this case $\alpha(B^{p,\beta}) = \frac{\beta+2}{p} - 1$. The assertion will follow from a more general result, Theorem 4.7 below. The condition $p < \beta + 2$, is essential here. For example, $B^{2,0} = D^{2,0}$, the classical Dirichlet space, does not satisfy 3) for any $\alpha > 0$.

To prove that $D^{2,0}$ does not satisfy 3) for any $\alpha > 0$ we use a similar argument to the one used in Proposition 4.1 c). As we know, $D^{2,0}$ is conformally invariant. Thus, we have that the composition operator $C_\varphi$ is uniformly bounded for $\varphi \in Aut(\mathbb{D})$ and if we suppose that $W^\alpha_\varphi$ is also uniformly bounded, then since $1 \in D^{2,0}$, we have $C_\varphi W^\alpha_{\varphi^{-1}} 1 = (\varphi')^{-\alpha}$ is uniformly bounded on $D^{2,0}$ which leads to a contradiction.

**Example 4.4.** Let $\beta > -1, 0 < \gamma \leq 1,\ \beta - \gamma + 2 > 0,\ p \geq 1$, and consider the space $\mathcal{Q}_{p,\beta,\gamma}$, consisting of all analytic functions $f$ in $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} = \sup_{\substack{h\in(0,1)\\ t\in[0,2\pi)}} h^{-\gamma} \int_{S_h(t)} |f(z)|^p (1-|z|^2)^\beta dA(z) < \infty,$$

where $S_h$ is the usual Carleson box $S_h(t) = \{re^{is} :\ 0 < 1-r \leq h,\ |t-s| \leq h\}$. The "little oh" version $\mathcal{Q}^0_{p,\beta,\gamma}$ consists of all analytic functions $f$ in $\mathbb{D}$ such that

$$\lim_{h\to 0} h^{-\gamma} \sup_{t\in[0,2\pi)} \int_{S_h(t)} |f(z)|^p (1-|z|^2)^\beta dA(z) = 0,$$

and is a closed subspace of $\mathcal{Q}_{p,\beta,\gamma}$. Then $\mathcal{Q}_{p,\beta,\gamma},\ \mathcal{Q}^0_{p,\beta,\gamma}$ satisfy 1)-3) with $\alpha = \frac{\beta-\gamma+2}{p}$. Indeed, applying Proposition 1.6 we have

$$\|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} \sim \sup_{a\in\mathbb{D}} (1-|a|^2)^\gamma \int_{\mathbb{D}} \frac{|f(z)|^p}{|1-\overline{a}z|^{2\gamma}} (1-|z|^2)^\beta dA(z).$$

Moreover, for every $\varphi \in Aut(\mathbb{D})$ we have

$$\frac{1}{|1-\overline{a}z|^{2\gamma}} = \frac{|\varphi'(a)|^\gamma |\varphi'(z)|^\gamma}{|1-\overline{\varphi(a)}\varphi(z)|^{2\gamma}}.$$

Thus, with $\alpha$ as above we can use the fact that $W_\varphi^{\alpha+\frac{\gamma}{p}}$ is an isometry on $A^p_\beta$ to obtain

$$\begin{aligned}
\|W^\alpha_\varphi f\|^p_{\mathcal{Q}_{p,\beta,\gamma}} &\sim \sup_{a\in\mathbb{D}} (1-|a|^2)^\gamma |\varphi'(a)|^\gamma \int_{\mathbb{D}} \frac{|W_\varphi^{\alpha+\frac{\gamma}{p}} f(z)|^p}{|1-\overline{\varphi(a)}\varphi(z)|^{2\gamma}} (1-|z|^2)^\beta dA(z) \\
&= \sup_{a\in\mathbb{D}} (1-|\varphi(a)|^2)^\gamma \int_{\mathbb{D}} \frac{|f(z)|^p}{|1-\overline{\varphi(a)}z|^{2\gamma}} (1-|z|^2)^\beta dA(z) \\
&\sim \|f\|^p_{\mathcal{Q}_{p,\beta,\gamma}}.
\end{aligned}$$

The condition $0 < \gamma \leq 1$ only ensures that $\mathcal{Q}_{p,\beta,\gamma}$ is not a growth class. For $0 < \beta \leq 1$, the spaces $\mathcal{Q}_{2,\beta,\beta}$ consist of derivatives of functions in the standard $Q_\beta$-spaces. See [94] for more information about these spaces. In particular, $\mathcal{Q}_{2,1,1}$ consists of derivatives of $BMOA$-functions. The norms are equivalent to the original ones modulo constants.

## 4.3 Basic properties

### 4.3.1 Standard objects emerging from the definition

**Multipliers and weak products.**

In this section we will relate the notions of multipliers and weak product with the weighted conformal invariance property. See Section 2.8 and Section 2.9. Let us first note some basic remarks.

If $X, Y$ are Banach spaces with the properties 1) and 2) then $Mult(X, Y)$ also satisfies 1). Given $u \in Mult(X, Y)$, since by 2) $1 \in X$, $u \in Y$, so for a compact subset $K$ of $\mathbb{D}$ we have

$$|u(z)| \leq C_K \|u\|_Y = C_K \|1\|_X \left\| u \cdot \frac{1}{\|1\|_X} \right\|_Y \lesssim \|u\|_{Mult(X,Y)},$$

for all $z \in K$. However, $Mult(X, Y)$ may not satisfy 2), for example $1 \in Mult(X, Y)$ only if $X \subset Y$.

As far as weak product concerns, we can observe that if $X$ and $Y$ satisfy the property 1) then $X \odot Y$ also satisfies 1). To prove this claim we consider $f \in X \odot Y$, then for any $\varepsilon > 0$ there exist $\{g_n^\varepsilon\}_{n\geq 1} \subset X, \{h_n^\varepsilon\}_{n\geq 1} \subset Y$ such that $f = \sum_{n\geq 1} g_n^\varepsilon h_n^\varepsilon$ and $\sum_{n\geq 1} \|g_n^\varepsilon\|_X \|h_n^\varepsilon\|_Y < \|f\|_{X\odot Y} + \varepsilon$. Thus, using that $X$ and $Y$ satisfy 1) we obtain that

$$|f(z)| \leq \sum_{n\geq 1} |g_n^\varepsilon(z)||h_n^\varepsilon(z)| \leq C_K \sum_{n\geq 1} \|g_n^\varepsilon\|_X \|h_n^\varepsilon\|_Y < C_K(\|f\|_{X\odot Y} + \varepsilon)$$

for all $z$ in a compact set $K \subset \mathbb{D}$. Besides that, if $X$ and $Y$ satisfy 2) then, since $X, Y \subset X \odot Y$, it also satisfies 2).

The following result shows some relations between these spaces with our weighted conformal invariance property.

**Proposition 4.2.** *Let $X, Y$ be conformally invariant Banach spaces of indices $\alpha$, respectively $\beta$.*
*(i) If $\alpha > \beta$, then $Mult(X, Y) = \{0\}$.*

*(ii) If $\alpha < \beta$, then the space $Mult(X, Y)$ satisfies 3) with index $\beta - \alpha$. In addition, if $Mult(X, Y)$ satisfies 2), then it is conformally invariant of index $\beta - \alpha$.*

*(iii) $X \odot Y$ is conformally invariant of index $\alpha + \beta$.*

*Proof.* (i) If $u \in Mult(X, Y)$ we have

$$\begin{aligned}\sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\beta u W_{\varphi^{-1}}^\alpha 1\|_Y &\leq \sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \|u W_{\varphi^{-1}}^\alpha 1\|_Y \\ &\leq \sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \|u\|_{Mult(X,Y)} \|W_{\varphi^{-1}}^\alpha 1\|_X < \infty,\end{aligned}$$

hence by 1),

$$\sup_{\varphi\in Aut(\mathbb{D})} |W_\varphi^\beta u W_{\varphi^{-1}}^\alpha 1(0)| < \infty.$$

In particular, for the choice $\varphi(z) = \frac{z+a}{1+\bar{a}z}$, $a \in \mathbb{D}$, we get

$$\sup_{a\in\mathbb{D}} |u(a)|(1-|a|^2)^{\beta-\alpha} < \infty.$$

Then the maximum principle implies that $u = 0$.

(ii) A similar computation gives, for $f \in X$ and $u \in Mult(X,Y)$,

$$W_\varphi^\beta uf = (\varphi')^\beta u\circ\varphi\, f\circ\varphi = \quad (\varphi')^{\beta-\alpha} u\circ\varphi \quad ((\varphi')^\alpha f\circ\varphi) = \quad W_\varphi^{\beta-\alpha}u \quad W_\varphi^\alpha f \quad .$$

Since $W_\varphi^\alpha$ is invertible on $X$, for any $g \in X$ we consider $f = W_\varphi^\alpha g$ and we obtain

$$\begin{aligned}
\|W_\varphi^{\beta-\alpha}u\|_{Mult(X,Y)} &= \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \| \quad W_\varphi^{\beta-\alpha}u \quad f\|_Y = \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \| \quad W_\varphi^{\beta-\alpha}u \quad W_\varphi^\alpha g \quad \|_Y \\
&= \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|W_\varphi^\beta ug\|_Y \le \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|uW_{\varphi^{-1}}^\alpha f\|_Y \\
&= \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \quad u\frac{W_{\varphi^{-1}}^\alpha f}{\|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)}} \quad {}_Y .
\end{aligned}$$

Now, since $\|W_{\varphi^{-1}}^\alpha f\|_X \le \|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)}\|f\|_X$ we have

$$\begin{aligned}
\|W_\varphi^{\beta-\alpha}u\|_{Mult(X,Y)} &\le \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|uf\|_Y \\
&= \|W_\varphi^\beta\|_{\mathcal{B}(Y)} \|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)} \|u\|_{Mult(X,Y)} .
\end{aligned}$$

Using that $X, Y$ are conformally invariant of indices $\alpha$, respectively $\beta$, and taking supremum over the automorphisms we obtain the result.

(iii) If $f \in X \odot Y$, given $\varepsilon > 0$ there exists $\{g_n\}_{n\ge1} \subset X, \{h_n\}_{n\ge1} \subset Y$ such that

$$f = \sum_{n\ge1} g_n h_n \quad \text{and} \quad \sum_{n\ge1} \|g_n\|_X \|h_n\|_Y < \|f\|_{X\odot Y} + \varepsilon.$$

Therefore

$$\|W_\varphi^{\alpha+\beta} f\|_{X\odot Y} \le \sum_{n\ge1} \|W_\varphi^\alpha g_n\|_X \|W_\varphi^\beta h_n\|_Y < K\left(\|f\|_{X\odot Y} + \varepsilon\right).$$

□

**Remark 4.1.** 1) If $X$ is conformally invariant of index $\alpha > 0$, $Mult(X)$ is invariant under composition with conformal automorphisms. Indeed, if the multiplication operator $M_u$ is bounded on $X$, then $W_\varphi^\alpha M_u W_{\varphi^{-1}}^\alpha = M_{u\circ\varphi}$. Given $f \in X$

$$W_\varphi^\alpha M_u W_{\varphi^{-1}}^\alpha f = (\varphi')^\alpha u\circ\varphi \quad \varphi^{-1}{}' \circ\varphi {}^{\alpha} \quad f\circ\varphi^{-1}\circ\varphi = (u\circ\varphi)\, f.$$

Therefore if $u \in Mult(X)$ then $M_u$ is bounded, so, analogously to Proposition 4.2 (ii) we have

$$\begin{aligned} \|C_\varphi u\|_{Mult(X)} &= \sup_{\substack{f\in X \\ \|f\|_X \le 1}} \|M_{u\circ\varphi} f\|_X = \sup_{\substack{f\in X \\ \|f\|_X \le 1}} \|W_\varphi^\alpha M_u W_{\varphi^{-1}}^\alpha f\|_X \\ &\le \|W_\varphi^\alpha\|_{\mathcal{B}(X)} \|W_{\varphi^{-1}}^\alpha\|_{\mathcal{B}(X)} \sup_{\substack{f\in X \\ \|f\|_X \le 1}} \|uf\|_X\,. \end{aligned}$$

Using that $X$ is conformally invariant of index $\alpha$, and taking supremum over the automorphisms we obtain the result.

2) The spaces $Mult(B^{p,\beta}, A^p_\beta)$, $1 \le p < \beta + 2$, are of particular interest and they are not completely understood in full generality, but for example in [2] one can find a relation between some of these spaces with the Cauchy-predual of $Q_p$-spaces defined above. They consist of analytic functions $f$ with the property that $|f|^p(1-|z|^2)^\beta dA$ is a Carleson measure for $B^{p,\beta}$. Since $B^{p,\beta}$ and $A^p_\beta$ are conformally invariant Banach spaces of indices $\frac{2+\beta}{p}$, respectively $\frac{2+\beta}{p} - 1$, we have that $Mult(B^{p,\beta}, A^p_\beta)$ satisfies 1) and by Proposition 4.2 (ii) also 3) with $\alpha = 1$. Moreover if $g \in Hol(\rho\mathbb{D})$ for any $\rho > 1$ we can see that for any $f$ in $B^{p,\beta}$

$$\|fg\|^p_{A^p_\beta} = (\beta+1)\int_{\mathbb{D}} |g(z)|^p |f(z)|^p (1-|z|^2)^\beta dA(z) \le C_g \|f\|^p_{A^p_\beta}.$$

where $C_g = \sup_{z\in\mathbb{D}} |g(z)|$. Therefore $Mult(B^{p,\beta}, A^p_\beta)$ also satisfies 2), thus it is conformally invariant of index $1$.

**One-parameter Abelian operator groups.**

The group $Aut(\mathbb{D})$ contains several Abelian one-parameter subgroups. The generic examples are the group of rotations $\{\varphi_t : \ \varphi_t(z) = e^{it}z, \ t \in [0, 2\pi)\}$, and the hyperbolic group $\{\psi_a : \ \psi_a(z) = \frac{z+a}{1+az}, a \in (-1,1)\}$. Of course when $X$ is conformally invariant of index $\alpha > 0$, the corresponding operators $\{W^\alpha_{\varphi_t} : \ t \in [0, 2\pi)\}$, $\{W^\alpha_{\psi_a} : \ a \in (-1,1)\}$ form one-parameter Abelian groups of operators; see Section 2.6. However, in general, these groups fail to be strongly continuous. Another important object related to approximations is the semigroup of dilations $\{D_r : \ r \in [0,1]\}$ defined by

$$D_r f(z) = f(rz). \tag{4.1}$$

Sometimes we shall write $D_r f = f_r$.

**Proposition 4.3.** *Let $X$ be a conformally invariant Banach space of index $\alpha$, then the dilation operator $D_r$, $r > 0$, is bounded on $X$ and the semigroup is strongly continuous on $[0, 1)$.*

*Proof.* Since $D_r f \in Hol(\mathbb{D})$, by 2) we know that $D_r f \in X$. Now, if $\{f_n\}_{n\ge1} \subset X$ is such that $f_n \to_X f$ and $D_r f_n \to_X g$, we know by 1) $f_n$ converges to $f$ in compact sets of $\mathbb{D}$, then $D_r f_n$ converges to $D_r f$ in compact sets of $\frac{1}{r}\mathbb{D}$. Thus, $D_r f_n$ converges to

$D_r f$, so $g = D_r f$ and by the the closed graph theorem we obtain that $D_r$ is bounded on $X$.

To prove the strong continuity we have to see that for any $f \in X$ and $r_1 \in [0,1)$, we get $D_r f \to_X D_{r_1} f$, when $r \to r_1$. But, since $D_r f$ tends to $D_{r_1} f$ uniformly on compact subsets of $\rho\mathbb{D}$ for some $\rho > 1$, by 2) we have the convergence on $X$. □

The question of main interest is whether it is strongly continuous from the left at $r = 1$. Again, this property fails to hold in full generality.

An example for the above assertions is the space $X = \mathcal{A}^{-1}$. If $f(z) = (z-i)^{-1}$, it follows easily that the functions $t \to W^\alpha_{\varphi_t} f,\ a \to W^\alpha_{\psi_a} f, r \to D_r f$ are not norm-continuous in $\mathcal{A}^{-1}$ on $[0,2\pi)$, $(-1,1)$, or $[0,1]$. Let us see, for example, why the function $t \to W^\alpha_{\varphi_t} f$ is not norm-continuous in $\mathcal{A}^{-1}$ on $[0,2\pi)$ or, in other words, why the group $\{W^\alpha_{\varphi_t} :\ t \in [0,2\pi)\}$ is not strongly continuous in $\mathcal{A}^{-1}$.

Consider $f(\xi) = (\xi - i)^{-1}$. Noting that $f \in \mathcal{A}^{-1}$, if for some $t_1 \in [0,2\pi)$ we can prove that

$$\lim_{t\to t_1} \|W^\alpha_{\varphi_t} f - W^\alpha_{\varphi_{t_1}} f\|_{\mathcal{A}^{-1}} = 0,$$

we will be done. Considering $t_1 = 0$ we have

$$\begin{aligned}
\|W^\alpha_{\varphi_t} f - W^\alpha_{\varphi_0} f\|_{\mathcal{A}^{-1}} &= \sup_{z\in\mathbb{D}} (1-|z|^2) \left| \frac{e^{it}}{e^{it}z - i} - \frac{1}{(z-i)} \right| |e^{it\alpha}| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2) \left| \frac{1-e^{it}}{(e^{it}z - i)\,(z-i)} \right| \\
&\geq \sup_{0<|iz|=r<1} (1-r^2) \frac{|1-e^{it}|}{(1-r)|1-e^{it}r|} \\
&\geq \limsup_{r\to 1^-} (1+r) \frac{|1-e^{it}|}{|1-e^{it}r|} \\
&= 2.
\end{aligned}$$

Taking limits when $t \to 0$ we get the result.

If $X$ is conformally invariant of index $\alpha$ and under the additional assumption that polynomials are dense in the space, the whole group $\{W^\alpha_\varphi :\ \varphi \in Aut(\mathbb{D})\}$, as well as the above semigroup, become strongly continuous. For the semigroup $\{D_r :\ r \in [0,1]\}$ the assertion will be proved with some weaker conditions in Theorem 4.1, while for the group it is proved below after a well known lemma.

**Lemma 4.1.** *Given $\{\varphi_n\}_{n\geq 1}$ a sequence in $Aut(\mathbb{D})$ which converges uniformly on compact subsets of $\mathbb{D}$ to $\varphi \in Aut(\mathbb{D})$, then $\{\varphi_n\}_{n\geq 1}$ converges to $\varphi$ uniformly in $\rho\mathbb{D}$, for some $\rho > 1$.*

*Proof.* Since $\varphi_n$ converges to $\varphi$ uniformly on compact subsets of $\mathbb{D}$, $\{\varphi'_n\}_{n\geq 1}$ also converges to $\varphi'$ on compact subsets of $\mathbb{D}$. So $\varphi_n(0) \to \varphi(0)$ and $\varphi'_n(0) \to \varphi'(0)$ and if

$$\varphi_n(z) = \lambda_n \frac{z + a_n}{1 + \overline{a_n} z} \quad \text{and} \quad \varphi(z) = \lambda \frac{z+a}{1+\overline{a}z}$$

we obtain that $\lambda_n \to \lambda$ and $a_n \to a$. Now we know that $\varphi \in Hol\left(\frac{1}{|a|}\mathbb{D}\right)$. Thus, fixing $\rho \in \left(1, \frac{1}{|a|}\right)$ we can suppose that $\{\varphi_n\}_{n\geq 1} \subset Hol(\rho\mathbb{D})$. Therefore if $\rho_1 \in (1,\rho)$ we have

$$\begin{aligned}
\sup_{z\in\overline{\rho_1\mathbb{D}}} |\varphi_n(z)-\varphi(z)| &= \sup_{z\in\overline{\rho_1\mathbb{D}}} \left|\lambda_n \frac{z+a_n}{1+\overline{a_n}z} - \lambda\frac{z+a}{1+\overline{a}z}\right| \\
&= \sup_{z\in\overline{\rho_1\mathbb{D}}} \left|\frac{\lambda_n(z+a_n)(1+\overline{a}z)-\lambda(1+\overline{a_n}z)(z+a)}{(1+\overline{a_n}z)(1+\overline{a}z)}\right| \\
&\leq \frac{1}{\left(1-\frac{\rho_1}{\rho}\right)^2} \sup_{z\in\overline{\rho_1\mathbb{D}}} \left|\lambda_n a_n - \lambda a + \left(\lambda_n + \lambda a_n\overline{a} - \lambda - \lambda\overline{a_n}a\right)z + \left(\lambda_n\overline{a} - \lambda\overline{a_n}\right)z^2\right| \\
&\leq C\left(|\lambda_n a_n - \lambda a| + |\lambda_n + \lambda a_n\overline{a} - \lambda - \lambda\overline{a_n}a| + |\lambda_n\overline{a} - \lambda\overline{a_n}|\right)
\end{aligned}$$

where $C$ only depends on $\rho$ and $\rho_1$. Then taking limits when $n \to \infty$ the result follows. □

**Proposition 4.4.** *Assume that $X$ is conformally invariant of index $\alpha > 0$, and that polynomials are dense in $X$. If $(\varphi_n)$ is a sequence in $Aut(\mathbb{D})$ which converges uniformly on compact subsets of $\mathbb{D}$ to $\varphi \in Aut(\mathbb{D})$, then $W^\alpha_{\varphi_n}$ converges strongly to $W^\alpha_\varphi$.*

*Proof.* By Lemma 4.1, it follows that $(\varphi_n)$ converges to $\varphi$ uniformly in $\rho\mathbb{D}$, for some $\rho > 1$. Then for every polynomial $p$, $W^\alpha_{\varphi_n}p \to W^\alpha_\varphi p$ uniformly in $\rho\mathbb{D}$, hence $W^\alpha_{\varphi_n}p \to W^\alpha_\varphi p$ in $X$ by 2). Since polynomials are dense in $X$, given $\varepsilon > 0$ we can choose a polynomial $p$ such that

$$\|f-p\|_X < \frac{\varepsilon}{2\sup_{\varphi\in Aut(\mathbb{D})}\|W^\alpha_\varphi\|_{\mathcal{B}(X)}}.$$

Thus, using that the operator norms $\|W^\alpha_{\varphi_n}\|_{\mathcal{B}(X)}$ are bounded above, we obtain

$$\begin{aligned}
\|W^\alpha_{\varphi_n}f - W^\alpha_\varphi f\|_X &\leq \|W^\alpha_{\varphi_n}f - W^\alpha_{\varphi_n}p\|_X + \|W^\alpha_{\varphi_n}p - W^\alpha_\varphi p\|_X + \|W^\alpha_\varphi f - W^\alpha_\varphi p\|_X \\
&\leq \|W^\alpha_{\varphi_n}\|_{\mathcal{B}(X)}\|f-p\|_X + \|W^\alpha_{\varphi_n}p - W^\alpha_\varphi p\|_X + \|W^\alpha_\varphi\|_{\mathcal{B}(X)}\|f-p\|_X \\
&< \varepsilon + \|W^\alpha_{\varphi_n}p - W^\alpha_\varphi p\|_X.
\end{aligned}$$

Taking limits when $n \to \infty$ we get the result. □

If polynomials are dense in $X$, the one-parameter Abelian groups considered above have densely defined, closed infinitesimal generators. They are given by

$$A_\alpha f = \frac{d}{dt}W^\alpha_{\varphi_t}f|_{t=0}, \quad A_\alpha f(z) = izf'(z) + i\alpha f(z), \tag{4.2}$$

$$\mathcal{D}_\alpha f = \frac{d}{da}W^\alpha_{\psi_a}f|_{a=0}, \quad \mathcal{D}_\alpha f(z) = (1-z^2)f'(z) - 2\alpha z f(z). \tag{4.3}$$

The infinitesimal generator of $\{D_r : r \in [0,1]\}$ is $-iA_0$. All of these unbounded operators are considered on their maximal domain of definition. $\mathcal{D}_{\frac{1}{2}}$ plays a crucial role in the description of the spectrum of the Hilbert matrix on conformally invariant spaces of index $\alpha \in (0,1)$ obtained in [8].

### 4.3.2 Polynomial approximation

This is a central question regarding Banach spaces of analytic functions in $\mathbb{D}$, and in many cases it is addressed with help of the dilation semigroup. In the framework considered here, this is intimately related to the rotation group given by $R_t = e^{-i\alpha t} W^{\alpha}_{\varphi_t}$, $\varphi_t(z) = e^{it}z, t \in [0, 2\pi]$. The results below (partially related to the work of A.E. Taylor [90]) hold for any Banach space $X$ which satisfies 1), 2) and the weaker condition

3') $R_t \in \mathcal{B}(X)$, and $\|R_t\|_{\mathcal{B}(X)}$ is uniformly bounded in $t \in [-\pi, \pi]$.

Recall from Proposition 4.3 that the semigroup of dilations $\{D_r : r \in [0, 1]\}$ defined by (4.1) is contained in $\mathcal{B}(X)$ and is strongly continuous on $[0, 1)$.

**Theorem 4.1.** *Let $X$ satisfy 1), 2) and 3'). The following are equivalent:*

*(i) $t \to R_t$ is strongly continuous in $[-\pi, \pi]$,*

*(ii) $r \to D_r$ is strongly continuous from the left at $r = 1$,*

*(iii) Polynomials are dense in $X$.*

*Proof.* *(i) ⇒ (ii):* For $r \in (0, 1)$, let $P_r(e^{it}) = \frac{1-r^2}{|e^{it}-r|^2}$ be the Poisson kernel at $r \in (0, 1)$; see Section 1.3. Then for $f \in X$, $t \to P_r(e^{it})R_t f$ is a continuous $X$-valued function on $[-\pi, \pi]$. To see this we fix $t_1 \in [-\pi, \pi]$ and we have to prove that

$$\left\| P_r(e^{it})R_t f - P_r(e^{it_1})R_{t_1} f \right\|_X \to 0,$$

when $t \to t_1$. First,

$$\begin{aligned}
&\left\| P_r(e^{it})R_t f - P_r(e^{it_1})R_{t_1} f \right\|_X = (1-r^2) \left\| R_t f \frac{1}{|e^{it}-r|^2} - R_{t_1} f \frac{1}{|e^{it_1}-r|^2} \right\|_X \\
&= (1-r^2) \left\| R_t f \frac{1}{|e^{it}-r|^2} - R_{t_1} f \frac{1}{|e^{it}-r|^2} + R_{t_1} f \left( \frac{1}{|e^{it}-r|^2} - \frac{1}{|e^{it_1}-r|^2} \right) \right\|_X \\
&\leq (1-r^2) \frac{1}{|e^{it}-r|^2} \|R_t f - R_{t_1} f\|_X + (1-r^2) \left\| R_{t_1} f \left( \frac{1}{|e^{it}-r|^2} - \frac{1}{|e^{it_1}-r|^2} \right) \right\|_X
\end{aligned}$$

On the one hand by 3'),

$$\left| \frac{1}{|e^{it}-r|^2} - \frac{1}{|e^{it_1}-r|^2} \right| \|R_{t_1} f\|_X \leq K \left| \frac{1}{|e^{it}-r|^2} - \frac{1}{|e^{it_1}-r|^2} \right| \|f\|_X \to 0,$$

when $t \to t_1$. We obtain the conclusion to the other part in the inequality above using that $t \to R_t$ is strongly continuous in $[-\pi, \pi]$. Therefore, the function $t \to P_r(e^{it})R_t f$ is Bochner integrable. See Example 2.2 and Definition 2.4. Indeed, its Bochner integral satisfies for all $z \in \mathbb{D}$,

$$\left( \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it})R_t f \, dt \right)(z) = f(rz) = D_r f(z).$$

(See Proposition 2.6). Noting that if $z = \rho e^{i\theta} \in \mathbb{D}$ using the Poisson integral we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it}) R_t f(\rho e^{i\theta})\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it}) f(\rho e^{i(\theta+t)})\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{i(\theta-t)}) f(\rho e^{it})\,dt \\ &= f(r\rho e^{i\theta}).\end{aligned}$$

Thus,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it}) R_t f dt = D_r f.$$

Then from the standard estimates for such integrals (see Proposition 2.5) and the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it})\,dt = 1$, we obtain for every $\delta > 0$,

$$\begin{aligned}\|D_r f - f\|_X &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(e^{it})\|R_t f - f\|_X\,dt \\ &= \frac{1}{2\pi}\int_{|t|<\delta} P_r(e^{it})\|R_t f - f\|_X\,dt + \frac{1}{2\pi}\int_{\delta\le|t|<\pi} P_r(e^{it})\|R_t f - f\|_X\,dt \\ &\le \sup_{|t|<\delta}\|R_t f - f\|_X + \frac{1+\sup_{t\in[0,2\pi]}\|R_t\|_{\mathcal{B}(X)}\ \ \|f\|_X}{2\pi}\int_{|t|>\delta} P_r(e^{it})\,dt.\end{aligned}$$

Here we have used that $\|R_t f - f\|_X \le (1+\|R_t\|_{\mathcal{B}(X)})\|f\|_X$. Given $\varepsilon > 0$ we choose, by (i), $\delta > 0$ such that $\sup_{|t|<\delta}\|R_t f - f\|_X < \varepsilon$ and let $r \to 1^-$ in the above inequality to obtain

$$\limsup_{r\to 1^-}\|D_r f - f\|_X \le \varepsilon,$$

*i.e.* $D_r f \to f$ in $X$.

*(ii) ⇒ (iii)*: From 2) it follows immediately that for fixed $r \in (0,1)$, $f_r = D_r f$ can be approximated by polynomials in $X$, which gives *(iii)*.

*(iii) ⇒ (i)*:. Again by 2) we conclude that $t \to R_t f$ is strongly continuous in $[-\pi,\pi]$, whenever $f$ is a polynomial, hence by *(iii)* this holds true for any $f \in X$. Fix $f \in X$ and $t_1 \in [-\pi,\pi]$. Given $\varepsilon > 0$ we can choose a polynomial $p$ in $X$ and, for this polynomial, $\delta > 0$ such that

$$\|f-p\|_X < \frac{\varepsilon}{4\sup_{t\in[-\pi,\pi]}\|R_t\|_{\mathcal{B}(X)}} \quad \text{and} \quad \|R_t p - R_{t_1} p\| < \frac{\varepsilon}{2}$$

for all $|t - t_1| < \delta$. Thus

$$\begin{aligned}\|R_t f - R_{t_1} f\|_X &\le \|R_t f - R_t p\|_X + \|R_t p - R_{t_1} p\|_X + \|R_{t_1} f - R_{t_1} p\|_X \\ &\le 2\sup_{t\in[-\pi,\pi]}\|R_t\|_{\mathcal{B}(X)}\|f-p\|_X + \|R_t p - R_{t_1} p\|_X \\ &\le \varepsilon\end{aligned}$$

for all $|t - t_1| < \delta$, so we deduce *(i)*. □

There is an important sufficient condition for the density of polynomials in such spaces. The result is interesting in its own right and also in view of its applications. Throughout in what follows we shall denote by $X'$ the dual of the Banach space $X$ and by $T'$ the transpose of $T \in \mathcal{B}(X)$, $T'l(f) = l(Tf)$, $f \in X, l \in X'$; see Section 1.7 and Section 2.1.We can observe that if $T \in \mathcal{B}(X)$ then

$$\begin{aligned}\|T'\|_{\mathcal{B}(X')} &= \sup_{\substack{l\in X'\\ \|l\|'_X\le 1}} \|T'l\|_{X'} = \sup_{\substack{l\in X'\\ \|l\|'_X\le 1}} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} |T'l(f)| = \sup_{\substack{l\in X'\\ \|l\|'_X\le 1}} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} |l(Tf)| \\ &\le \sup_{\substack{l\in X'\\ \|l\|'_X\le 1}} \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|l\|_{X'}\|Tf\|_X \le \sup_{\substack{f\in X\\ \|f\|_X\le 1}} \|Tf\|_X = \|T\|_{\mathcal{B}(X)}.\end{aligned} \tag{4.4}$$

**Theorem 4.2.** *Let $X$ satisfy 1), 2) and 3'). If the linear span of point evaluations $l_w(f) = f(w)$, $f \in X, w \in \mathbb{D}$, is dense in $X'$, then polynomials are dense in $X$.*

*Proof.* Note first that for fixed $w \in \mathbb{D}$, we have $R'_t l_w = l_{e^{it}w}$, and that $t \to R'_t l_w$ is continuous on $[-\pi,\pi]$, this means that fixing $t_1 \in [-\pi,\pi]$

$$\|R'_t l_w - R'_{t_1} l_w\|_{X'} \to 0$$

when $t \to t_1$. Indeed, using 1) we have, $|f(e^{it}w) - f(e^{is}w)| \le |t-s| c_w \|f\|_X$, with $c_w > 0$ independent of $f$. To see this, we suppose $t \ge s$ and we obtain

$$|f(e^{it}w) - f(e^{is}w)| = \left| \int_s^t f'(e^{iu}w) i e^{iu} w \, du \right| \le \int_s^t |f'(e^{iu}w)||w| \, du.$$

Thus, fixing $u \in [s,t]$ and let $\gamma$ be the circle defined by the equation $\{z(\theta) = we^{iu} + re^{i\theta}, \quad \theta \in [0,2\pi)\}$, with $r = \frac{1-|w|}{2}$, which ensures that the circle is contained on $\mathbb{D}$. So, using the Cauchy's formula we obtain

$$\begin{aligned}|f'(we^{iu})| &= \left| \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-we^{iu})^2} \, dz \right| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(we^{iu} + re^{i\theta}) i r e^{i\theta}}{r^2 e^{2i\theta}} \, d\theta \right| \\ &\le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(we^{iu} + re^{i\theta})| r}{r^2} \, d\theta \le c_w \max_{z\in K} |f(z)|\end{aligned}$$

where $K$ is a compact of $\mathbb{D}$, which contains $\overline{r\mathbb{D}}$. Therefore, by 1) and the uniform boundedness principle we obtain $|f'(we^{iu})| \le c_w \|f\|_X$. Thus

$$|f(e^{it}w) - f(e^{is}w)| \le \int_s^t |f'(e^{iu}w)||w| \, du \le (t-s) c_w \|f\|_X.$$

In other words, we have proved that

$$\begin{aligned}\|R'_t l_w - R'_s l_w\|_{X'} &= \sup_{\substack{f\in X\\ \|f\|_X\le 1}} |R'_t l_w f - R'_s l_w f| \\ &= \sup_{\substack{f\in X\\ \|f\|_X\le 1}} |f(e^{it}w) - f(e^{is}w)| \le c_w |t-s|,\end{aligned} \tag{4.5}$$

which proves the claim. Now 3') together with the density of $span\{l_w: \ w \in \mathbb{D}\}$ in $X'$ implies that $t \to R'_t$ is strongly continuous on $[-\pi, \pi]$. Given $l \in X'$ we want to prove that fixing $t_1 \in [-\pi, \pi]$,

$$\|R'_t l - R'_{t_1} l\|_{X'} \to 0$$

when $t \to t_1$. Fixing $\varepsilon > 0$, by density there exists $l^\varepsilon = \sum_{i=1}^N a_i l_{w_i} \in X'$, with $a_i \in \mathbb{C}$ and $w_i \in \mathbb{D}$, which satisfies

$$\|l - l^\varepsilon\|_{X'} < \frac{\varepsilon}{4 \sup_{t\in[-\pi,\pi]} \|R_t\|_{\mathcal{B}(X)}}.$$

Thus, using (4.4) and (4.5)

$$\begin{aligned}
\|R'_t l - R'_{t_1} l\|_{X'} &\leq \|R'_t l - R'_t l^\varepsilon\|_{X'} + \|R'_t l^\varepsilon - R'_{t_1} l^\varepsilon\|_{X'} + \|R'_{t_1} l - R'_{t_1} l^\varepsilon\|_{X'} \\
&\leq \|R'_t\|_{B(X')} \|l - l^\varepsilon\|_{X'} + |t - t_1| \sum_{i=1}^N |a_i| c_{w_i} + \|R'_{t_1}\|_{B(X')} \|l - l^\varepsilon\|_{X'} \\
&< \varepsilon,
\end{aligned}$$

when $|t - t_1| < \frac{\varepsilon}{2 \sum_{i=1}^N |a_i| c_{w_i}}$. Thus, $t \to R'_t$ is strongly continuous on $[-\pi, \pi]$, so the Bochner integral

$$T_r l = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R'_t l \, dt,$$

defines a bounded linear operator on $X'$. This can be seen using (4.4) and 3')

$$\|T_r l\|_{X'} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) \|R'_t l\|_{X'} \, dt \leq \left(\sup_{t\in[-\pi,\pi]} \|R_t\|_{\mathcal{B}(X)}\right) \|l\|_{X'} \leq K \|l\|_{X'}.$$

This operator also satisfies that for any $f \in X$ and $w = \rho e^{i\theta}$,

$$\begin{aligned}
T_r l_w(f) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) R'_t l_w f \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) f(e^{it} w) \, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{i(t-\theta)}) f(\rho e^{it}) \, dt = f(r\rho e^{i\theta}) = f_r(w) = D'_r l_w(f).
\end{aligned}$$

Therefore, since by definition

$$\|T_r l_w - D'_r l_w\|_{X'} = \sup_{\substack{f\in X \\ \|f\|_X \leq 1}} |T_r l_w(f) - D'_r l_w(f)|,$$

we have $T_r l_w = D'_r l_w$. By density, we get $T_r l = D'_r l$ for all $l \in X'$. Thus, $\{D'_r: \ r \in \ [0, 1]\}$ is bounded in $\mathcal{B}(X')$. As in Theorem 4.1 using that for $\delta > 0$

$$\begin{aligned}
\|D'_r l - l\|_{X'} &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(e^{it}) \|R'_t l - l\|_{X'} \, dt \\
&\leq \sup_{|t|<\delta} \|R'_t l - l\|_{X'} + \frac{\left(1 + \sup_{t\in[0,2\pi]} \|R_t\|_{\mathcal{B}(X)}\right) \|l\|_{X'}}{2\pi} \int_{|t|>\delta} P_r(e^{it}) \, dt,
\end{aligned}$$

we deduce that $D_r'l \to_{X'} l,\ r \to 1^-$, for any $l \in X'$. This proves that $D_r f \to_X f, r \to 1^-$, weakly in $X$, since for any $l \in X'$ and $f \in X$

$$|l(D_r f) - l(f)| = |D_r' l(f) - l(f)| \leq \|D_r' l - l\|_{X'} \|f\|_X.$$

Now, by 2) we conclude that polynomials are weakly dense in $X$. Finally, thanks to the Hahn-Banach theorem the result follows. To see this, we denote $Y$ the closure of polynomials in $X$. Therefore, we know that $Y$ is weakly dense in $X$ and if there exists $f \in X$ such that $f \notin Y$, using the Hahn-Banach theorem, see [78, Theorem 3.7], there exists $l \in X'$ such that $|l(g)| \leq 1$ for all $g \in Y$ and $l(f) > 1$. Thus if $W = l^{-1}\left(\overline{\mathbb{D}}\right)$, $W$ is a weakly closed subset of $X$ such that $Y \subset W$ and $f \notin W$, which is impossible. □

In general, the density of polynomials in $X$ does not imply that the linear span of point evaluations is dense in $X'$. The following example shows this.

**Example 4.5.** The Bergman space $A^1$ satisfies that the polynomials are dense in $A^1$ but the linear span of point evaluations is not dense in $(A^1)'$.

First, by Theorem 1.4 we know the polynomials are dense in $A^1$. Furthermore, using Theorem 1.9 we can identify $(A^1)'$ with the Bloch space $\mathcal{B}$ which is a non-separable space. If we suppose the linear span of point evaluations is dense in $(A^1)'$ and if we can see that any point evaluation $l_w$ can be approximated by point evaluations $l_{a+ib}$ with $a, b \in \mathbb{Q}$ we will obtain a contradiction with the non-separability of Bloch space. This happens since if we use the identification used in Theorem 1.9 with these $l_{a+ib}$ we could find a countable dense subset of $\mathcal{B}$.

Let $l_w$ be a point evaluation with $w \in \mathbb{D}$ and $q = a + ib$ with $a, b \in \mathbb{Q}$. Then using the formula

$$f(w) = \int_{\mathbb{D}} f(z) k_w(z)\, dA(z), \quad \text{with} \quad k_w(z) = \frac{1}{(1 - \overline{z}w)^2}$$

for all $f \in A^1$, we can deduce

$$\begin{aligned}
\|l_w - l_q\|_{X'} &= \sup_{\substack{f \in A^1 \\ \|f\|_{A^1} \leq 1}} |l_w(f) - l_q(f)| = \sup_{\substack{f \in A^1 \\ \|f\|_{A^1} \leq 1}} \left| \int_{\mathbb{D}} f(z)(k_w(z) - k_q(z))\, dA(z) \right| \\
&\leq \sup_{\substack{f \in A^1 \\ \|f\|_{A^1} \leq 1}} \|k_w(z) - k_q(z)\|_\infty \|f\|_{A^1} \leq \|k_w(z) - k_q(z)\|_\infty \\
&= \sup_{z \in \mathbb{D}} \left| \frac{1}{(1 - \overline{z}w)^2} - \frac{1}{(1 - \overline{z}q)^2} \right| = \sup_{z \in \mathbb{D}} \left| \frac{2\overline{z}(w - q) + \overline{z}^2(q^2 - w^2)}{(1 - \overline{z}w)^2(1 - \overline{z}q)^2} \right| \\
&\leq \frac{2|w - q| + |q^2 - w^2|}{(1 - |w|)^2(1 - |q|^2)}.
\end{aligned}$$

So, since we can find a sequence $\{q_n\}_{n \geq 1}$ such that $q_n = a_n + ib_n$ with $a_n, b_n \in \mathbb{Q}$ for all $n \geq 1$ and $q_n \to w$, we deduce that any $l_w$ can be approximated by point evaluations $l_{a+ib}$ with $a, b \in \mathbb{Q}$.

A direct application of Theorem 4.2 is as follows.

**Corollary 4.1.** *Assume that $X$ satisfies 1), 2) and 3'). If $X$ is reflexive then polynomials are dense in $X$.*

*Proof.* If $\Lambda \in X''$ annihilates all point evaluations, from reflexivity we have $\Lambda(l) = l(f)$, for some $f \in X$, and since $\Lambda(l_w) = f(w) = 0,\ w \in \mathbb{D}$, it follows that $f = 0$, hence $\Lambda = 0$. Thus, applying the Hahn-Banach theorem we deduce that the linear span of point evaluations is dense in $X'$ and Theorem 4.2 gives the desired result. ☐

### 4.3.3 Duality

It is a well-known fact that if $X$ is a Banach space of analytic functions in $\mathbb{D}$ containing the polynomials as a dense subset, then its dual $X'$ can be represented as a Banach space of analytic functions as well. We are interested in a representation which preserves conformal invariance of index $\alpha$ which can be achieved with a suitable pairing. Given $\alpha > 0$, let $H_\alpha$ denote the Hilbert space with reproducing kernel

$$k^\alpha(z,w) = k^\alpha_w(z) = (1-\overline{w}z)^{-2\alpha}, \quad z,w \in \mathbb{D}.$$

It should be noted that here we are using a slightly different notation from Section 1.10. The reader should note that $H_\alpha = \mathcal{H}_{2\alpha}$. When $\alpha > \frac{1}{2}$ we have $H_\alpha = A^2_{2\alpha-2}$, $H_{\frac{1}{2}} = H^2$, and when $\alpha < \frac{1}{2}$, we have $H_\alpha = D^{2,2\alpha}$. Let $\langle \cdot, \cdot \rangle_\alpha$ be the scalar product induced by the kernel $k^\alpha$. The reason for choosing this kernel (and the corresponding pairing) is the identity

$$(1-\overline{w}z)^{-2\alpha} = \overline{\varphi'^{\alpha}(w)}\, \varphi'^{\alpha}(z)(1-\overline{\varphi(w)}\varphi(z))^{-2\alpha}, \quad z,w \in \mathbb{D},\ \varphi \in Aut(\mathbb{D}), \tag{4.6}$$

which says that $W^\alpha_\varphi$ is unitary on $H_\alpha$ (see [62, Proposition 3.1]). Note that

$$k^\alpha(z,w) = 1 + \sum_{n\ge 1} \binom{2\alpha+n-1}{n} \overline{w}^n z^n = 1 + \sum_{n \ge 1} \frac{2\alpha \cdots (2\alpha+n-1)}{n!} \overline{w}^n z^n, \tag{4.7}$$

and that for fixed $w \in \mathbb{D}$, the series on the right converges in any Banach space $X$ which satisfies 1) and 2). Indeed, 2) implies that $Hol(\rho\mathbb{D})$ is continuously contained in $X,\ \rho > 1$. In particular, if $\zeta(z) = z$, $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} = 1$. To see this claim we observe the following, by 1), if we consider the compact $K = \overline{t\mathbb{D}}$ with $t < 1$ we obtain

$$t = \left(\sup_{z \in K} |z^n|\right)^{\frac{1}{n}} \le C_t^{\frac{1}{n}} \|\zeta^n\|^{\frac{1}{n}}$$

where $C_t$ only depends on $t$. Then,

$$t \le \limsup_{n\to\infty} C_t^{\frac{1}{n}} \|\zeta^n\|^{\frac{1}{n}} = \limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}}$$

for all $t < 1$, thus $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} \ge 1$. On the other part, by 2) using Proposition 4.1 a), if we consider $K = \overline{t\mathbb{D}}$ with $t > 1$ we have

$$\|\zeta^n\|^{\frac{1}{n}} \le C_t^{\frac{1}{n}} \left(\sup_{z\in K} |z^n|\right)^{\frac{1}{n}} = C_t^{\frac{1}{n}} t$$

where $C_t$ only depends on $t$ and analogously we deduce $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} \leq 1$. Consequently, using the root test we can deduce that the series on the right in (4.7) converges absolutely in $X$ for a fixed $w \in \mathbb{D}$.

Therefore, we obtain that if $l \in X'$, the function

$$Ul(w) = l\left(k^\alpha(\cdot, \overline{w})\right) = l\left((1 - w\zeta)^{-2\alpha}\right), \quad w \in \mathbb{D},$$

is analytic in $\mathbb{D}$. In fact, from (4.7) we have

$$Ul(w) = l(1) + \sum_{n\geq 1} \frac{2\alpha \cdots (2\alpha + n - 1)}{n!} w^n l(\zeta^n), \tag{4.8}$$

and from $\limsup_{n\to\infty} \|\zeta^n\|^{\frac{1}{n}} = 1$ we see that the series converges uniformly on each compact subset of $\mathbb{D}$. Given a compact $K$ in $\mathbb{D}$ and $w \in K$ we have

$$\limsup_{n\to\infty} \left(\frac{2\alpha \cdots (2\alpha + n - 1)}{n!}\right)^{\frac{1}{n}} |w||l(\zeta^n)|^{\frac{1}{n}} \leq \sup_{z\in K} |z| \limsup_{n\to\infty} C^{\frac{1}{n}} \|l\|_{X'}^{\frac{1}{n}} \|\zeta^n\|^{\frac{1}{n}} < 1$$

and applying the root test we obtain the desired result.

This gives a linear map $U : X' \to Hol(\mathbb{D})$. We shall denote by $X'_\alpha$ its range: $X'_\alpha = UX'$.

**Theorem 4.3.** *Let $X$ be conformally invariant of index $\alpha > 0$ and assume that polynomials are dense in $X$. Then with respect to the norm $\|Ul\|_{X'_\alpha} = \|l\|_{X'}$, $X'_\alpha$ becomes a Banach space of analytic functions which is conformally invariant of index $\alpha$. Moreover, every $l \in X'$ can be represented in the form*

$$l(f) = \lim_{r\to 1^-} \langle f_r, g_r\rangle_\alpha, \quad f \in X,$$

*with $g = Ul \in X'_\alpha$.*

*Proof.* If polynomials are dense in $X$ then $U$ is injective, since by (4.8) $Ul = 0$, implies that $l(\zeta^n) = 0,\ n \geq 0$, i.e. $l = 0$. Then $\|Ul\| = \|l\|_{X'}$ defines a norm on $X'_\alpha$ which becomes isometrically isomorphic to $X'$, in particular, it is a Banach space. The fact that $X'_\alpha$ satisfies 1) follows also directly from (4.8). To verify 2), let $\rho > 1$, let $g \in Hol(\rho^2\mathbb{D})$, and set $g^*(z) = \overline{g}(\overline{z})$. Then the dilation $g^*_\rho \in H_\alpha$, since it verifies 2), and by 1), $f \to f_{\frac{1}{\rho}}$ defines a bounded linear map from $X$ into $H_\alpha$. Thus

$$l(f) = \langle f_{\frac{1}{\rho}}, g^*_\rho\rangle_\alpha, \quad f \in X$$

defines an element $l \in X'$ since for any $f \in X$ we have

$$|l(f)| = |\langle f_{\frac{1}{\rho}}, g^*_\rho\rangle_\alpha| \leq \|g^*_\rho\|_\alpha \|f_{\frac{1}{\rho}}\|_\alpha \leq C\|g^*_\rho\|_\alpha \|f\|_X.$$

Furthermore, a direct calculation gives $Ul(w) = g(w)$, if $g(z) = \sum_{n\geq 0} a_n z^n$ we have for all $n \geq 1$

$$l(\zeta^n) = \langle \zeta^n_{\frac{1}{\rho}}, g^*_\rho\rangle_\alpha = a_n \frac{n!}{2\alpha \cdots (2\alpha + n - 1)} \quad \text{and} \quad l(1) = a_0.$$

Thus, we get

$$Ul(w) = l(1) + \sum_{n\geq 1} \frac{2\alpha\cdots(2\alpha+n-1)}{n!} w^n l(\zeta^n) = a_0 + \sum_{n\geq 1} a_n w^n = g(w).$$

To see 3), we use the identity (4.6) in the form

$$k^{\alpha}(z,\overline{w}) = \overline{\varphi'^{\,\alpha}(\overline{w})}\, \varphi'^{\,\alpha}(z)(1-\overline{\varphi(\overline{w})}\varphi(z))^{-2\alpha}, \quad z,w\in\mathbb{D},\ \varphi\in Aut(\mathbb{D}).$$

If $z=\varphi^{-1}(\lambda)$, from $\varphi'(z)(\varphi^{-1})'(\lambda)=1$ and the above equality, for $\lambda, w\in\mathbb{D}$ we get

$$\begin{aligned}
W^{\alpha}_{\varphi^{-1}} k^{\alpha}(\cdot,\overline{w})(\lambda) &= \left(\varphi^{-1}\right)'^{\alpha}(\lambda) k^{\alpha}(\varphi^{-1}(\lambda),\overline{w})\\
&= \left(\varphi^{-1}\right)'^{\alpha}(\lambda)\, (\varphi^*)'^{\,\alpha}(w)\, (\varphi')^{\alpha}(z)(1-\varphi^*(w)\lambda)^{-2\alpha}\\
&= (\varphi^*)'^{\,\alpha}(w)(1-\varphi^*(w)\lambda)^{-2\alpha}\\
&= (\varphi^*)'^{\,\alpha}(w) k^{\alpha}(\cdot,\varphi(\overline{w}))
\end{aligned}$$

where, as before, $\varphi^*(w)=\overline{\varphi(\overline{w})}$. This leads to

$$W^{\alpha}_{\varphi^*} Ul = U(W^{\alpha}_{\varphi^{-1}})'l,$$

since

$$\begin{aligned}
W^{\alpha}_{\varphi^*} Ul(w) &= (\varphi^*)'^{\,\alpha}(w) Ul(\varphi^*(w)) = (\varphi^*)'^{\,\alpha}(w) l(k^{\alpha}(\cdot,\varphi(\overline{w})))\\
U(W^{\alpha}_{\varphi^{-1}})'l(w) &= (W^{\alpha}_{\varphi^{-1}})'l(k^{\alpha}(\cdot,\overline{w})) = l\left(W^{\alpha}_{\varphi^{-1}} k^{\alpha}(\cdot,\overline{w})\right) = (\varphi^*)'^{\,\alpha}(w) l\left(k^{\alpha}(\cdot,\varphi(\overline{w}))\right).
\end{aligned}$$

Now, observing that the map $f\to f^*$ defines a bijection in the group of automorphisms, 3) follows from the fact that $X$ satisfies 3) since for any $\varphi^*\in Aut(\mathbb{D})$ and $l\in X'$ we have

$$\|W^{\alpha}_{\varphi^*} Ul\|_{X'_{\alpha}} = \|U(W^{\alpha}_{\varphi^{-1}})'l\|_{X'_{\alpha}} = \|(W^{\alpha}_{\varphi^{-1}})'l\|_{X'} \leq \|W^{\alpha}_{\varphi^{-1}}\|_{\mathcal{B}(X)}\|l\|_{X'} \leq C\|Ul\|_{X'_{\alpha}}.$$

Finally, (4.8), together with another direct computations, gives

$$\begin{aligned}
\langle f_r, (Ul)_r\rangle_{\alpha} &= \left\langle \sum_{n\geq 0} f_n r^n \zeta^n, l(1) + \sum_{n\geq 1} \frac{2\alpha\cdots(2\alpha+n-1)}{n!} r^n w^n l(\zeta^n)\right\rangle_{\alpha}\\
&= \sum_{n\geq 0} r^{2n} f_n l(\zeta^n)\\
&= l(f_{r^2}),
\end{aligned}$$

for $f=\sum_{n\geq 0} f_n\zeta^n$, $r\in(0,1)$. Thus, since polynomials are dense in $X$, by Theorem 4.1 we have $r\to D_r$ is strongly continuous from the left at $r=1$. Thus

$$\lim_{r\to 1^-}\langle f_r, (Ul)_r\rangle_{\alpha} = l(f),$$

since

$$\lim_{r\to 1^-}|l(f)-l(f_{r^2})| \leq \|l\|_{X'}\lim_{r\to 1^-}\|f-f_{r^2}\|_X = 0.$$

Therefore, the result follows. □

## 4.4 The largest and the smallest space. The Hilbert space case

In this section we show that there is a largest and a smallest Banach space of analytic functions in $\mathbb{D}$, conformally invariant of a given index $\alpha > 0$, and that amongst such spaces there exists a unique Hilbert space.

### 4.4.1 Largest and smallest space

If $X$ is conformally invariant of index $\alpha > 0$, it contains the constants, hence by 3)

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\| = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha 1\| < K\|1\|. \tag{4.9}$$

Thus, a good candidate for the smallest space with this property is

$$X_\alpha^{min} = \left\{ f \in Hol(\mathbb{D}) : \ f = \sum_j a_j (\varphi_j')^\alpha, \ \varphi_j \in Aut(\mathbb{D}), \ a_j \in \mathbb{C}, \ \sum_j |a_j| < \infty \right\}, \tag{4.10}$$

with the norm

$$\|f\|_{X_\alpha^{min}} = \inf \left\{ \sum_j |a_j| : \ f = \sum_j a_j (\varphi_j')^\alpha \right\}. \tag{4.11}$$

**Remark 4.2.** $\|\cdot\|_{X_\alpha^{min}}$ is a norm.

*Proof.* It is clear that $\|f\|_{X_\alpha^{min}} \geq 0$ for all $f \in X_\alpha^{min}$.

1. If $\|f\|_{X_\alpha^{min}} = 0$ then, given $\varepsilon > 0$ there exists $\{c_j^\varepsilon\}_{j\geq 1} \in l^1$ and $\{\varphi_j^\varepsilon\}_{j\geq 1} \subset Aut(\mathbb{D})$, such that $f = \ _{j\geq 1} c_j^\varepsilon \varphi_j^\varepsilon$ and $\|\{c_j^\varepsilon\}_{j\geq 1}\|_{l^1} < \varepsilon$. Then, if $K$ is a compact of $\mathbb{D}$ we have
$$\sup_{z\in K} |f(z)| \leq \sup_{z \in K} \sum_{j\geq 1} |c_j^\varepsilon| | \ \varphi_j^{\varepsilon\,\prime}(z)| \leq C_K \|\{c_j^\varepsilon\}_{j\geq 1}\|_{l^1} < C_K \varepsilon,$$
therefore, $f(z) = 0$ for all $z \in K$, so $f \equiv 0$. The other direction is clear.

2. If $\lambda \in \mathbb{C}$, then $\ _{j\geq 1} c_j \ \ \varphi_j' \ ^\alpha$ is a representation of $f$ if and only if $\ _{j\geq 1} \lambda c_j \ \ \varphi_j' \ ^\alpha$ is a representation of $\lambda f$, so $\|\lambda f\|_{X_\alpha^{min}} = |\lambda| \|f\|_{X_\alpha^{min}}$.

3. Given $f^1, f^2 \in X_\alpha^{min}$ then given $\varepsilon > 0$ there exists $\{c_j^i\}_{j\geq 1} \in l^1$, $\{\varphi_j^i\}_{j\geq 1} \subset Aut(\mathbb{D})$ such that
$$f^i = \sum_{j\geq 1} c_j^i \ \ \varphi_j^{i\,\prime} \ ^\alpha \quad \text{and} \quad \|\{c_j^i\}_{j\geq 1}\|_{l^1} < \|f^i\|_{X_\alpha^{min}} + \frac{\varepsilon}{2}$$
with $i = 1, 2$. Therefore, $f^1 + f^2 = \ _{k=1}^\infty c_k (\varphi_k')^\alpha$ and
$$\|f^1 + f^2\|_{X_\alpha^{min}} \leq \|\{c_k\}_{k\geq 1}\|_{l^1} \leq \|f^1\|_{X_\alpha^{min}} + \|f^2\|_{X_\alpha^{min}} + \varepsilon.$$

Here we have used the bijection between $\mathbb{N}$ and $\mathbb{N}^2$ in the following way

$$c_k = c^{i_k}_{\frac{k}{2}} \quad \text{and} \quad \varphi_k = \varphi^{i_k}_{\frac{k}{2}} \quad \text{with} \quad i_k = \begin{cases} 1, & \text{if } \frac{k}{2} \in \mathbb{N} \\ 2, & \text{if } \frac{k}{2} \notin \mathbb{N} \end{cases}. \tag{4.12}$$

□

The next step is to prove $X_\alpha^{min}$ with the norm defined above is a Banach space, for this purpose we need the following well-known lemma, see for example [18, Lemma 2.2.1] or [75, Chapter 6, 5. Proposition].

**Lemma 4.2.** *Let $X$ be a normed vector space. Then $X$ is complete if and only if for any sequence $\{f_n\}_{n\geq1} \subset X$ such that*

$$\sum_{n=1}^{\infty} \|f_n\|_X < \infty$$

*there exists $f \in X$ such that*

$$f - \sum_{n=1}^{N} f_n \quad {}_X \to 0 \quad \textit{when} \quad N \to \infty.$$

**Remark 4.3.** $X_\alpha^{min}$ with the norm defined in (4.11) is a Banach space.

*Proof.* To prove this claim we will use Lemma 4.2, so let $\{f_n\}_{n\geq1}$ be a sequence in $X$ such that ${}_{n\geq1} \|f_n\|_{X_\alpha^{min}} < \infty$. For each $n$ there exists $\{c_j^n\}_{j\geq1} \in l^1$ and $\{\varphi_j^n\}_{j\geq1} \subset Aut(\mathbb{D})$ such that

$$f_n = \sum_{j=1}^{\infty} c_j^n \quad \varphi_j^{n\,\prime} \quad {}^{\alpha} \quad \text{and} \quad \|\{c_j^n\}_{j\geq1}\|_{l^1} < \|f_n\|_{X_\alpha^{min}} + 2^{-n}.$$

But, since the union of countable sets is countable there exists $\{\hat{c}_k^n\}_{k\geq1} \in l^1$ and $\{\varphi_k\}_{k\geq1} \subset Aut(\mathbb{D})$ with

$$f_n = \sum_{k=1}^{\infty} \hat{c}_k^n \left(\varphi_k'\right)^\alpha \quad \text{and} \quad \|\{\hat{c}_k^n\}_{k\geq1}\|_{l^1} = \|\{c_j^n\}_{j\geq1}\|_{l^1} < \|f_n\|_{X_\alpha^{min}} + 2^{-n}.$$

Thus,

$$\sum_{n=1}^{\infty} \|\{\hat{c}_k^n\}_{k\geq1}\|_{l^1} < \sum_{n=1}^{\infty} \quad \|f_n\|_{X_\alpha^{min}} + 2^{-n} \quad < \infty.$$

Therefore ${}_{n\geq1}\{\hat{c}_k^n\}_{k\geq1}$ is an absolutely convergent series in $l^1$, so applying Lemma 4.2 we have there exists $\{\hat{c}_k\}_{k\geq1}$ in $l^1$ such that ${}_{n\geq1}\{\hat{c}_k^n\}_{k\geq1}$ converges to $\{\hat{c}_k\}_{k\geq1}$ in $l^1$. Now, we consider

$$f = \sum_{k=1}^{\infty} \hat{c}_k \left(\varphi_k'\right)^\alpha$$

which is in $X_\alpha^{min}$ since $\{\hat{c}_k\}_{k\geq1} \in l^1$ and

$$f - \sum_{n=1}^{N} f_n \quad {}_{X_\alpha^{min}} \leq \sum_{k=1}^{\infty} \quad \hat{c}_k - \sum_{n=1}^{N} \hat{c}_k^n \quad = \quad \{\hat{c}_k\}_{k\geq1} - \sum_{n=1}^{N} \{\hat{c}_k^n\}_{k\geq1} \quad {}_{l^1} \to 0,$$

when $N \to \infty$, which completes the proof. □

**Remark 4.4.** The space $X_\alpha^{min}$ satisfies 1) and 3). Given $K$ compact of $\mathbb{D}$, $f \in X_\alpha^{min}$ and a representation $f = \quad_j a_j(\varphi_j')^\alpha$ we have

$$\sup_{z\in K} |f(z)| \leq \sup_{z\in K} \sum_j |a_j||\varphi_j'(z)| \leq \sum_j |a_j| \sup_{z\in K} \frac{1}{(1-|z|)^2} \leq C_K \sum_j |a_j|$$

and since this is true for all representation of $f$, we get $\sup_{z\in K} |f(z)| \leq C_K \|f\|_{X_\alpha^{min}}$, thus $X_\alpha^{min}$ satisfies 1). Now, given a representation $f = \quad_j a_j(\varphi_j')^\alpha$

$$W_\varphi^\alpha f = \sum_j a_j(\varphi_j' \circ \varphi)^\alpha (\varphi')^\alpha = \sum_j a_j \left[(\varphi_j \circ \varphi)'\right]^\alpha$$

so for any representation of $f$ we can find a representation of $W_\varphi^\alpha f$ with the same coefficients $a_j$, so the norms of $f$ and $W_\varphi^\alpha$ are equal. Therefore, $X_\alpha^{min}$ satisfies 3).

Furthermore by (4.9), $X_\alpha^{min}$ is continuously contained in any conformally invariant space of index $\alpha$. Let $X$ be a Banach space conformally invariant of index $\alpha$ and $f \in X_\alpha^{min}$. We can find a representation $f = \quad_j a_j(\varphi_j')^\alpha$ such that $\quad_j |a_j| < \|f\|_{X_\alpha^{min}} + \varepsilon$. Thus, by (4.9)

$$\|f\|_X = \|\sum_j a_j(\varphi_j')^\alpha\|_X \leq \sum_j |a_j|\|\varphi_j'\|_X \leq C \sum_j |a_j| \leq C \quad \|f\|_{X_\alpha^{min}} + \varepsilon \quad .$$

So, we have $X_\alpha^{min}$ is continuously contained in $X$.

It turns out that $X_\alpha^{min}$ can be identified either with a weighted Bergman space, or a weighted Besov space.

**Lemma 4.3.** *If $\alpha > 1$, then $X_\alpha^{min} = A^1_{\alpha-2}$, and if $\alpha \leq 1$, $X_\alpha^{min} = B^{1,\alpha-1}$. In all cases the norms are equivalent.*

*Proof.* A standard estimate shows that $\{(\varphi')^\alpha : \ \varphi \in Aut(\mathbb{D})\}$ is bounded in $A^1_{\alpha-2}$, when $\alpha > 1$, and in $B^{1,\alpha-1}$, when $\alpha \leq 1$. Given $\varphi(z) = \lambda\varphi_a = \lambda\frac{z-a}{1-\bar{a}z}$ we have for $\alpha > 1$

$$\begin{aligned}
\|(\varphi')^\alpha\|_{A^1_{\alpha-2}} &= (\alpha-1) \quad_{\mathbb{D}} |\varphi'(z)|^\alpha (1-|z|^2)^{\alpha-2}\, dA(z) \\
&= (\alpha-1) \quad_{\mathbb{D}} (1-|\varphi(z)|^2)^{\alpha-2} |\varphi'(z)|^2\, dA(z) \\
&= (\alpha-1) \quad_{\mathbb{D}} (1-|z|^2)^{\alpha-2}\, dA(z) = \|1\|_{A^1_{\alpha-2}}
\end{aligned}$$

and for $\alpha \leq 1$

$$\begin{aligned}
\|(\varphi')^\alpha\|_{B^{1,\alpha-1}} &= |\varphi'(0)|^\alpha + \alpha \quad_{\mathbb{D}} \alpha|\varphi'(z)|^{\alpha-1}|\varphi''(z)|(1-|z|^2)^{\alpha-1}\, dA(z) \\
&= (1-|a|^2)^\alpha + 2|a|\alpha^2 \quad_{\mathbb{D}} \frac{(1-|a|^2)^\alpha}{|1-\bar{a}z|^{2\alpha+1}}(1-|z|^2)^{\alpha-1}\, dA(z) \\
&\leq C.
\end{aligned}$$

In last inequality we have applied Theorem 1.6. Thus $X_\alpha^{min}$ is continuously contained in the space indicated in the statement. The reverse (continuous) inclusion follows directly from the atomic decomposition theorems. In the first case, using Proposition 1.1, if $f \in A^1_{\alpha-2}$ with $\alpha > 1$ then there exist a sequences $\{c_k\}_{k\geq 1} \in l^1$ and $\{a_k\}_{k\geq 1} \subset \mathbb{D}$ such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{\left(1 - \ |a_k|^2\right)^{2b+\alpha}}{(1 - \overline{a_k}z)^{2b+2\alpha}}$$

with $b > -\frac{\alpha}{2}$ and $\|\{c_k\}_{k\geq 1}\|_{l^1} \leq C\|f\|_{A^1_{\alpha-2}}$. Now, choosing $b = 0$ we obtain

$$f(z) = \sum_{k=1}^{\infty} c_k \ \ \varphi'_{a_k}{}^{\alpha}$$

with $\varphi_{a_k}(z) = \frac{z-a_k}{1-\overline{a_k}z}$ and $\|\{c_k\}_{k\geq 1}\|_{l^1} \leq C\|f\|_{A^1_{\alpha-2}}$. In the second case, using Proposition 1.3, if $f \in B^{1,\alpha-1}$ with $\alpha \leq 1$, there exist sequences $\{c_k\}_{k\geq 1} \in l^1$ and $\{a_k\}_{k\geq 1} \subset \mathbb{D}$ such that

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{\left(1 - \ |a_k|^2\right)^{b}}{(1 - \overline{a_k}z)^{b+\alpha}}$$

with $b > 0$ and $\|\{c_k\}_{k\geq 1}\|_{l^1} \leq C\|f\|_{B^{1,\alpha-1}}$. Now, choosing $b = \alpha$ we obtain

$$f(z) = \sum_{k=1}^{\infty} c_k \ \ \varphi'_{a_k}{}^{\alpha}$$

with $\varphi_{a_k}(z) = \frac{z-a_k}{1-\overline{a_k}z}$ and $\|\{c_k\}_{k\geq 1}\|_{l^1} \leq C\|f\|_{B^{1,\alpha-1}}$. Thus in each case, the spaces indicated in the statement are continuously contained in $X_\alpha^{min}$. $\square$

With the lemma in hand we can prove the main result of this section.

**Theorem 4.4.** *If $X$ is conformally invariant of index $\alpha > 0$, then $X$ is continuously contained in $X_\alpha^{max} = \mathcal{A}^{-\alpha}$, and $X_\alpha^{min}$ is continuously contained in $X$.*

*Proof.* We have already seen at Lemma 4.3 that $X_\alpha^{min}$ is continuously contained in $X$. For the remaining part, let $\varphi_a(z) = \frac{a+z}{1+\bar{a}z}$, $a, z \in \mathbb{D}$, and use 1) and 3) to conclude that there exists $K_1 > 0$, such that for all $f \in X$,

$$\|f\|_{\mathcal{A}^{-\alpha}} = \sup_{a\in\mathbb{D}}(1 - |a|^2)^\alpha |f(a)| = \sup_{a\in\mathbb{D}} |W^\alpha_{\varphi_a} f(0)| \leq K \sup_{a\in\mathbb{D}} \|W^\alpha_{\varphi_a} f\|_X \leq K_1\|f\|_X,$$

which completes the proof. $\square$

### 4.4.2 The Hilbert space case.

We shall prove that the only Hilbert space which is conformally invariant of index $\alpha > 0$ is the space $H_\alpha$ introduced in Section 4.3.3, i.e. $H_\alpha = A^2_{2\alpha-2}$ when $\alpha > \frac{1}{2}$, $H_{\frac{1}{2}} = H^2$, and $H_\alpha = D^{2,2\alpha}$ when $\alpha < \frac{1}{2}$.

We begin with a useful observation derived from the results in Section 4.3.2.

**Lemma 4.4.** *If $X$ is a conformally invariant Hilbert space of index $\alpha > 0$, then there exists a scalar product on $X$ which induces an equivalent norm and has the property that monomials form an orthogonal basis in $X$.*

*Proof.* $X$ is reflexive, hence polynomials are dense in $X$, by Corollary 4.1. Consequently, by Theorem 4.1 (i), the group $\{R_t :\ t \in [0, 2\pi]\}$ is strongly continuous. Set

$$\|f\|_1^2 = \int_0^{2\pi} \|R_t f\|_X^2 \, dt.$$

First, $\|\cdot\|_1$ is equivalent to the original norm since if $f \in X$ and $\varphi_t(z) = e^{it}z$, using 3) we obtain

$$\|f\|_1^2 = \int_0^{2\pi} \|R_t f\|_X^2 \, dt = \int_0^{2\pi} \|e^{-it\alpha} W_{\varphi_t}^\alpha f\|_X^2 \, dt \le \int_0^{2\pi} \|W_{\varphi_t}^\alpha\|_{\mathcal{B}(X)}^2 \|f\|_X^2 \, dt \le C\|f\|_X^2,$$

$$\|f\|_X^2 = \frac{1}{2\pi}\int_0^{2\pi} \|f\|_X^2 \, dt = \frac{1}{2\pi}\int_0^{2\pi} \|W_{\varphi_{-t}}^\alpha W_{\varphi_t}^\alpha f\|_X^2 \, dt \le C_1 \int_0^{2\pi} \|W_{\varphi_t}^\alpha f\|_X^2 \, dt = C_1 \|f\|_1^2.$$

The induced scalar product is

$$\langle f, g\rangle_1 = \int_0^{2\pi} \langle R_t f, R_t g\rangle \, dt,$$

hence for $n = m$,

$$\langle \zeta^n, \zeta^m\rangle_1 = \int_0^{2\pi} e^{i(n-m)t} \langle \zeta^n, \zeta^m\rangle \, dt = 0,$$

which completes the proof. □

Using the equivalent norm given by Lemma 4.4, it follows that $X$ consists of all analytic functions $f = \sum_{n\ge 0} f_n \zeta^n$ in $\mathbb{D}$ with

$$\|f\|^2 = \sum_{n\ge 0} |f_n|^2 v_n < \infty, \tag{4.13}$$

where $v_n = \|\zeta^n\|^2 > 0$. From Section 1.10 we have that $H_\alpha$ consists of all analytic functions $f = \sum_{n\ge 0} f_n \zeta^n$ in $\mathbb{D}$ with

$$\|f\|_{H_\alpha}^2 = \sum_{n\ge 0} |f_n|^2 v_{n,\alpha} < \infty,$$

where

$$\begin{aligned} v_{0,\alpha} &= 1, \\ v_{n,\alpha} &= \frac{1}{\binom{2\alpha+n-1}{n}} = \frac{\Gamma(2\alpha)\Gamma(n+1)}{\Gamma(2\alpha+n)} = \frac{n!}{2\alpha(2\alpha+1)\cdots(2\alpha+n-1)}, \quad n \ge 1. \end{aligned} \tag{4.14}$$

Here is a simple observation regarding the weights $v_n,\ n \ge 0$.

**Lemma 4.5.** *There exists $c > 0$ such that*

$$\sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2} \leq c(n+1)^{2\alpha},$$

*for all $n \geq 0$.*

*Proof.* With the rotationally invariant norm considered above, we can deduce the estimate

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\| = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha 1\| \leq \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha\| v_0^{\frac{1}{2}} = c_1.$$

Now, since $\frac{1}{(1-az)^{2\alpha}} = \sum_{k=0}^{\infty} \genfrac{}{}{0pt}{}{2\alpha+k-1}{k} a^k z^k$ we have

$$\frac{1}{(1-az)^{2\alpha}}^{\;2} = \sum_{k=0}^{\infty} \genfrac{}{}{0pt}{}{2\alpha+k-1}{k}^{2} a^{2k} v_k = \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k},$$

and therefore,

$$\|(\varphi')^\alpha\|^2 = \quad \frac{(1-a^2)^\alpha}{(1-az)^{2\alpha}}, \frac{(1-a^2)^\alpha}{(1-az)^{2\alpha}} \quad = (1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k}.$$

Thus, the above estimate reduces to

$$\sup_{a \in (0,1)} (1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k} \leq c_1^2.$$

Then, using the fact that for all $k \leq n$ we have the inequality $\;1 - \frac{1}{n+1}\;^{k} > \frac{1}{e}$, we can deduce that for $a^2 = 1 - \frac{1}{n+1}$

$$(1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k} \geq \frac{1}{(n+1)^{2\alpha}} \sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2} \;\; 1 - \frac{1}{n+1}\;^{k} \geq \frac{1}{e} \frac{1}{(n+1)^{2\alpha}} \sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2}.$$

Thus,

$$\frac{1}{(n+1)^{2\alpha}} \sum_{k=0}^{n} \frac{v_k}{v_{k,\alpha}^2} \leq e \sup_{a \in (0,1)} (1-a^2)^{2\alpha} \sum_{k=0}^{\infty} \frac{v_k}{v_{k,\alpha}^2} a^{2k} \leq e c_1^2,$$

which completes the proof. $\square$

For $a \in (0,1)$, let $\psi_a(z) = \frac{z+a}{1+az},\; z \in \mathbb{D}$. We shall use some identities and estimates for the scalar products

$$C_{n,k,\alpha}(a) = \langle W_{\psi_a}^\alpha \zeta^n, \zeta^k \rangle_{H_\alpha} = v_{k,\alpha} \frac{[\psi_a^n (\psi_a')^\alpha]^{(k)}(0)}{k!}. \tag{4.15}$$

**Lemma 4.6.** *(i) $C_{n,k,\alpha}(a) \in \mathbb{R},\; k, n \geq 0, a \in (0,1)$, and for fixed $n \geq 0,\; a \in (0,1)$,*

$$|C_{n,k,\alpha}(a)|, |\frac{d}{da} C_{n,k,\alpha}(a)| = o(b^k),\; k \to \infty$$

*for any $b \in (a,1)$, while for fixed $k, n \geq 0$, $\lim_{a \to 1^-} C_{n,k,\alpha}(a) = 0$.*

*(ii) If* $1 \le n \le k,\ a \in (0,1)$*, we have*

$$C_{n,k+1,\alpha}(a) = -aC_{n,k,\alpha}(a) + \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha} C_{n-1,k,\alpha+\frac{1}{2}}(a).$$

*(iii) For* $k,n \ge 1,\ a \in (0,1)$*,*

$$C_{n,k,\alpha}(a) - \frac{k+1+2\alpha}{k+1} C_{n,k+2,\alpha}(a) = -\frac{1-a^2}{k+1}\frac{d}{da} C_{n,k+1,\alpha}(a).$$

*(iv) Consequently, for* $1 \le n \le k,\ a \in (0,1)$

$$\begin{aligned} C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a) \le{}& \frac{n^2(1-a^2)}{4\alpha^2 a^2} C^2_{n-1,k,\alpha+\frac{1}{2}}(a) + \frac{1}{a(k+1)}\frac{d}{da}[(1-a^2)C^2_{n,k+1,\alpha}(a)] \\ &+ \frac{2}{k+1} C^2_{n,k+1,\alpha}(a) - \frac{4\alpha}{a(k+1)} C_{n,k+1,\alpha}(a) C_{n,k+2,\alpha}(a). \end{aligned}$$

*Proof.* (i) For $k,n \ge 0$ and $a \in (0,1)$ by definition (4.15) it follows that $C_{n,k,\alpha}(a) \in \mathbb{R}$. Now, for fixed $n \ge 0,\ a \in (0,1)$, we will see that $|C_{n,k,\alpha}(a)| = o(b^k),\ k \to \infty$ for any $b \in (a,1)$. On the one hand, using the identity $\lim_{k\to\infty} \frac{\Gamma(k+c)}{\Gamma(k)k^c} = 1$ for all $c \in \mathbb{R}$, we have $v_{k,\alpha} \le Ck^{1-2\alpha}$. On the other hand, $\frac{[\psi_a^n(\psi_a')^\alpha]^{(k)}(0)}{k!}$ is the $k$-th coefficient of an analytic function in $\frac{1}{a}\mathbb{D}$. Therefore, by the Cauchy integral formula, we deduce that

$$v_{k,\alpha}\frac{[\psi_a^n(\psi_a')^\alpha]^{(k)}(0)}{k!} = o(b^k),\ k \to \infty$$

for any $b \in (a,1)$. The proof is analogous for $|\frac{d}{da}C_{n,k,\alpha}(a)|$ since $\frac{d}{da}W^\alpha_{\psi_a}\zeta^n$, for a fixed $n \ge 0,\ a \in (0,1)$, is also an analytic function in $\frac{1}{a}\mathbb{D}$.

Finally, we want to see that for fixed $k,n \ge 0$, $\lim_{a\to 1^-} C_{n,k,\alpha}(a) = 0$. First, we will observe that $W^\alpha_{\psi_a}\zeta^n$ converges weakly to $0$ in $H_\alpha$ when $a \to 1^-$. Since $H_\alpha$ is a Hilbert space satisfying 1), the desired weak convergence is equivalent to proving that $W^\alpha_{\psi_a}\zeta^n$ is bounded in $H_\alpha$ and $W^\alpha_{\psi_a}\zeta^n$ converges pointwise to 0. For this equivalence see for example [38, Corollary 1.3]. Fixing $z \in \mathbb{D}$ we obtain

$$W^\alpha_{\psi_a}\zeta^n(z) = (\psi_a'(z))^\alpha\,\psi_a^n(z) = \frac{(1-a^2)^\alpha\,(z+a)^n}{(1+az)^{n+2\alpha}} \to 0$$

when $a \to 1^-$. Furthermore,

$$\|W^\alpha_{\psi_a}\zeta^n\|_{H_\alpha} = \|\zeta^n\|_{H_\alpha}.$$

Thus, $W_{\psi_a}\zeta^n$ converges weakly to $0$ in $H_\alpha$ when $a \to 1^-$ and therefore

$$\lim_{a\to 1^-} C_{n,k,\alpha}(a) = \lim_{a\to 1^-}\langle W^\alpha_{\psi_a}\zeta^n, \zeta^k\rangle_{H_\alpha} = 0.$$

(ii) Since $\psi_a(z) = a + \frac{(1-a^2)z}{1+az}$, it follows for any $f \in H_\alpha$,

$$\begin{aligned}\langle (\psi_a')^\alpha \psi_a^n, f\rangle_{H_\alpha} &= \left\langle \frac{(1-a^2)^\alpha}{(1+a\zeta)^{2\alpha}} \left(a + \frac{(1-a^2)\zeta}{1+a\zeta}\right)^n, f \right\rangle_{H_\alpha} \\ &= \sum_{j=0}^{n} \binom{n}{j} a^{n-j}(1-a^2)^{j+\alpha} \left\langle \frac{\zeta^j}{(1+a\zeta)^{j+2\alpha}}, f \right\rangle_{H_\alpha}.\end{aligned}$$

Furthermore,

$$\left\langle \frac{1}{(1+a\zeta)^{2\alpha}}, f \right\rangle_{H_\alpha} = \overline{\left\langle f, \frac{1}{(1-(-a)\zeta)^{2\alpha}} \right\rangle_{H_\alpha}} = \overline{f(-a)},$$

and if $j \geq 1$,

$$\begin{aligned}\left\langle \frac{\zeta^j}{(1+a\zeta)^{j+2\alpha}}, f \right\rangle_{H_\alpha} &= \frac{(-1)^j}{2\alpha \cdots (2\alpha+j-1)} \frac{d^j}{da^j} \left\langle \frac{1}{(1+a\zeta)^{2\alpha}}, f \right\rangle_{H_\alpha} \\ &= \frac{(-1)^j}{2\alpha \cdots (2\alpha+j-1)} \frac{d^j}{da^j} \overline{f(-a)} \\ &= \frac{1}{2\alpha \cdots (2\alpha+j-1)} \overline{f^{(j)}(-a)}.\end{aligned}$$

Thus for $f = \zeta^k$, since $k \geq n$, we obtain

$$C_{n,k,\alpha}(a) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{k-j} a^{n+k-2j} (1-a^2)^{j+\alpha} \prod_{0 \leq l < j} \frac{k-l}{2\alpha+l},$$

where, as is usual, we set the product over the empty set to be 1, i.e. the first term in the above sum is $(-1)^k a^{n+k}(1-a^2)^\alpha$. This implies

$$\begin{aligned}C_{n,k+1,\alpha}(a) &= \sum_{j=0}^{n} \binom{n}{j} (-1)^{k+1-j} a^{n+k+1-2j} (1-a^2)^{j+\alpha} \prod_{0 \leq l < j} \frac{k+1-l}{2\alpha+l} \\ &= -aC_{n,k,\alpha}(a) \\ &+ \sum_{j=1}^{n} \binom{n}{j} (-1)^{k+1-j} a^{n+k+1-2j} (1-a^2)^{j+\alpha} \left( \prod_{0 \leq l < j} \frac{k+1-l}{2\alpha+l} - \prod_{0 \leq l < j} \frac{k-l}{2\alpha+l} \right).\end{aligned}$$

Now, with the above convention, it can be verified that for $j \geq 1$

$$\begin{aligned}\prod_{0 \leq l < j} \frac{k+1-l}{2\alpha+l} - \prod_{0 \leq l < j} \frac{k-l}{2\alpha+l} &= \left( \prod_{0 \leq l < j} \frac{1}{2\alpha+l} \prod_{0 \leq l < j-1} (k-l) \right) ((k+1)-(k-j+1)) \\ &= \frac{j}{2\alpha} \prod_{0 \leq l < j-1} \frac{k-l}{2\alpha+1+l}.\end{aligned}$$

This leads to

$$
\begin{aligned}
&C_{n,k+1,\alpha}(a) = -aC_{n,k,\alpha}(a)\\
&+ \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha} \sum_{j=1}^{n} \binom{n-1}{j-1} (-1)^{k-j+1} a^{n-1+k-2(j-1)} (1-a^2)^{j-1+\alpha+\frac{1}{2}} \prod_{0\le l<j-1} \frac{k-l}{2\alpha+1+l}\\
&= -aC_{n,k,\alpha}(a)\\
&+ \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha} \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^{k-m} a^{n-1+k-2m} (1-a^2)^{m+\alpha+\frac{1}{2}} \prod_{0\le l<m} \frac{k-l}{2\alpha+1+l}\\
&= -aC_{n,k,\alpha}(a) + \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha} C_{n-1,k,\alpha+\frac{1}{2}}(a),
\end{aligned}
$$

which proves the identity in the statement.

(iii) Recall that $\{W^\alpha_{\psi_a} \;:\; a \in (-1,1),\; \psi_a = \frac{z+a}{1+az}\}$ is a unitary group on $H_\alpha$ with infinitesimal generator $\mathcal{D}_\alpha$ given by (4.3). Then $i\mathcal{D}_\alpha$ is selfadjoint on $H_\alpha$, i.e. $\mathcal{D}_\alpha^* = -\mathcal{D}_\alpha$ on this space, since for any $f, g$ in the domain of $\mathcal{D}_\alpha$ if we consider the equality

$$\langle f, g\rangle_{H_\alpha} = \left\langle W^\alpha_{\psi_a} f, W^\alpha_{\psi_a} g\right\rangle_{H_\alpha}$$

and take derivatives with respect $a$ at $0$, we obtain

$$0 = \left\langle \frac{d}{da} W^\alpha_{\psi_a} f \Big|_{a=0}, W^\alpha_{\psi_0} g \right\rangle + \left\langle W^\alpha_{\psi_0} f, \frac{d}{da} W^\alpha_{\psi_a} g \Big|_{a=0} \right\rangle = \langle \mathcal{D}_\alpha f, g\rangle + \langle f, \mathcal{D}_\alpha g\rangle .$$

Moreover, by a direct calculation, it follows that

$$\mathcal{D}_\alpha W^\alpha_{\psi_a} f = W^\alpha_{\psi_a} \mathcal{D}_\alpha f = (1-a^2)\frac{d}{da} W^\alpha_{\psi_a} f, \tag{4.16}$$

whenever $f$ is in the domain of $\mathcal{D}_\alpha$, since

$$
\begin{aligned}
\mathcal{D}_\alpha W^\alpha_{\psi_a} f &= \mathcal{D}_\alpha \left[ (\psi_a')^\alpha (f\circ\psi_a)\right] = (1-\zeta^2)\left[ (\psi_a')^\alpha (f\circ\psi_a)\right]' - 2\alpha\zeta\, (\psi_a')^\alpha (f\circ\psi_a)\\
&= (1-\zeta^2)\left[\alpha\, (\psi_a')^{\alpha-1}\, \psi_a''(f\circ\psi_a) + (\psi_a')^{\alpha+1} (f'\circ\psi_a)\right] - 2\alpha\zeta\, (\psi_a')^\alpha (f\circ\psi_a)\\
&= (1-\zeta^2)\,(\psi_a')^{\alpha+1} (f'\circ\psi_a) - 2\alpha\, (\psi_a')^\alpha (f\circ\psi_a) \left[\frac{(1-\zeta^2)a}{1+a\zeta} + \zeta\right]\\
&= (1-\zeta^2)\,(\psi_a')^{\alpha+1} (f'\circ\psi_a) - 2\alpha\, (\psi_a')^\alpha\, \psi_a (f\circ\psi_a),
\end{aligned}
$$

$$
\begin{aligned}
W^\alpha_{\psi_a}\mathcal{D}_\alpha f &= W^\alpha_{\psi_a}\left[(1-\zeta^2)f' - 2\alpha\zeta f\right] = (\psi_a')^\alpha (1-\psi_a^2)(f'\circ\psi_a) - 2\alpha(\psi_a')^\alpha \psi_a (f\circ\psi_a)\\
&= (1-\zeta^2)\,(\psi_a')^{\alpha+1} (f'\circ\psi_a) - 2\alpha\, (\psi_a')^\alpha\, \psi_a (f\circ\psi_a),
\end{aligned}
$$

$$
\begin{aligned}
(1-a^2)\frac{d}{da} W^\alpha_{\psi_a} f &= (1-a^2)\frac{d}{da}\left[(\psi_a')^\alpha (f\circ\psi_a)\right]\\
&= (1-a^2)\left[\alpha(\psi_a')^{\alpha-1} \left(\frac{d}{da}\psi_a'\right) (f\circ\psi_a) + (\psi_a')^\alpha (f'\circ\psi_a)\frac{d}{da}\psi_a\right]\\
&= -2\alpha(\psi_a')^{\alpha-1}\frac{(1-a^2)(\zeta+a)}{(1+a\zeta)^3}(f\circ\psi_a) + (1-\zeta^2)\frac{1-a^2}{(1+a\zeta)^2}\,(\psi_a')^\alpha (f'\circ\psi_a)\\
&= (1-\zeta^2)\,(\psi_a')^{\alpha+1} (f'\circ\psi_a) - 2\alpha\, (\psi_a')^\alpha\, \psi_a (f\circ\psi_a).
\end{aligned}
$$

In particular, we have (4.16) when $f$ is a polynomial. Since

$$\mathcal{D}_\alpha \zeta^{k+1} = (k+1)\zeta^k - (k+1+2\alpha)\zeta^{k+2}, \quad k \geq 1,$$

we obtain

$$\begin{aligned}
C_{n,k,\alpha}(a) - \frac{k+1+2\alpha}{k+1} C_{n,k+2,\alpha}(a) &= \langle W^\alpha_{\psi_a}\zeta^n, \zeta^k\rangle_{H_\alpha} - \frac{k+1+2\alpha}{k+1}\langle W^\alpha_{\psi_a}\zeta^n, \zeta^{k+2}\rangle_{H_\alpha} \\
&= \frac{1}{k+1}\langle W^\alpha_{\psi_a}\zeta^n, \mathcal{D}_\alpha\zeta^{k+1}\rangle_{H_\alpha} \\
&= -\frac{1}{k+1}\langle \mathcal{D}_\alpha W^\alpha_{\psi_a}\zeta^n, \zeta^{k+1}\rangle_{H_\alpha} \\
&= -\frac{1-a^2}{k+1}\langle \frac{d}{da} W^\alpha_{\psi_a}\zeta^n, \zeta^{k+1}\rangle_{H_\alpha} \\
&= -\frac{1-a^2}{k+1}\frac{d}{da} C_{n,k+1,\alpha}(a).
\end{aligned}$$

(iv) By the elementary computation $A^2 - B^2 = 2A(A-B) - (A-B)^2$ for $A, B \in \mathbb{R}$ we can deduce

$$\begin{aligned}
C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a) &= 2C_{n,k,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) \\
&\quad - (C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a))^2.
\end{aligned}$$

Now, adding and subtracting $\frac{2}{a}C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a))$ we obtain

$$\begin{aligned}
C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a) &= 2(C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a))(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) \\
&\quad - \frac{2}{a}C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) - (C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a))^2.
\end{aligned}$$

Finally, using the elementary inequality $2AB - C - B^2 = -(A-B)^2 + A^2 - C \leq A^2 - C$ for $A, B, C \in \mathbb{R}$ we get

$$\begin{aligned}
C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a) &\leq (C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a))^2 \\
&\quad - \frac{2}{a}C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)).
\end{aligned} \tag{4.17}$$

By (ii) we have $C_{n,k,\alpha}(a) + \frac{1}{a}C_{n,k+1,\alpha}(a) = \frac{n(1-a^2)^{\frac{1}{2}}}{2\alpha a}C_{n-1,k,\alpha+\frac{1}{2}}(a)$, and by (iii),

$$\begin{aligned}
&C_{n,k+1,\alpha}(a)(C_{n,k,\alpha}(a) - C_{n,k+2,\alpha}(a)) \\
&= C_{n,k+1,\alpha}(a) \quad C_{n,k,\alpha}(a) - \frac{k+1+2\alpha}{k+1}C_{n,k+2,\alpha}(a) \quad + \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a) \\
&= -\frac{1-a^2}{k+1}C_{n,k+1,\alpha}(a)\frac{d}{da}C_{n,k+1,\alpha}(a) + \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a) \\
&= -\frac{1}{2(k+1)}\frac{d}{da}[(1-a^2)C^2_{n,k+1,\alpha}(a)] - \frac{a}{k+1}C^2_{n,k+1,\alpha}(a) \\
&+ \frac{2\alpha}{k+1}C_{n,k+1,\alpha}(a)C_{n,k+2,\alpha}(a).
\end{aligned}$$

By replacing these expressions in (4.17), we obtain the desired inequality.

□

**Lemma 4.7.** *Let $X$ be a Hilbert space with scalar product*

$$\langle f, g\rangle = \sum_{n\geq 0} f_n\, \overline{g_n}\, v_n,$$

*with $v_n > 0$ for all $n \geq 0$ and $v_0 = 1$. Then $X'_\alpha = \tilde{X}$ with equality of norms, where*

$$\tilde{X} = \left\{ f = \sum_{n\geq 0} f_n \zeta^n \ : \ \|f\|^2_{\tilde{X}} = \sum_{n\geq 0} |f_n|^2 \frac{v^2_{n,\alpha}}{v_n} < \infty \right\}.$$

*Proof.* Let $g \in X'_\alpha$. Then there exists $l \in X'$ such that $Ul = g$ with $\|g\|_{X'_\alpha} = \|l\|_{X'}$. See Section 4.3.3. By the Riesz representation theorem there exists $h_l = \sum_{n\geq 0} h_n\zeta^n \in X$ such that

$$l(f) = \langle h_l, f\rangle \quad \text{for all } f \in X \quad \text{and} \quad \|h_l\|_X = \|l\|_{X'}.$$

Since $\langle h_l, \zeta^n\rangle = h_n v_n$ we have

$$g(w) = Ul(w) = l(1) + \sum_{n\geq 1} \frac{1}{v_{n,\alpha}} w^n l(\zeta^n) = \sum_{n\geq 0} \frac{v_n}{v_{n,\alpha}} h_n w^n,$$

so

$$\|g\|^2_{\tilde{X}} = \sum_{n\geq 0} |h_n|^2 v_n = \|h_l\|^2_X = \|l\|^2_{X'} = \|g\|^2_{X'_\alpha}.$$

Reciprocally, let $g = \sum_{n\geq 0} g_n\zeta^n \in \tilde{X}$. Then $h_l = \sum_{n\geq 0} g_n \frac{v_{n,\alpha}}{v_n}\zeta^n \in X$. Next, let $l \in X'$ be such that

$$l(f) = \langle h_l, f\rangle \quad \text{for all } f \in X.$$

Then, we have $Ul = g$. Hence, $g \in X'_\alpha$. □

We can now turn to our result about conformally invariant Hilbert spaces.

**Theorem 4.5.** *Let $X$ be a Hilbert space which is conformally invariant of index $\alpha > 0$. Then $X = H_\alpha$ and the corresponding norms are equivalent.*

*Proof.* Without loss of generality, we can assume that $X$ is equipped with the norm given by Lemma 4.4. As above, let $v_n = \|\zeta^n\|^2$, and let $v_{n,\alpha}$ be given by (4.14). It will be sufficient to show that there exists $\eta > 0$, depending only on $X$, such that

$$v_n \leq \eta v_{n,\alpha}, \quad n \geq 0. \tag{4.18}$$

Indeed, Lemma 4.7 shows that $X'_\alpha$ consists of all analytic functions $f = \sum_{n\geq 0} f_n\zeta^n$ in $\mathbb{D}$ with

$$\|f\|^2_{X'_\alpha} = \sum_{n\geq 0} |f_n|^2 \frac{v^2_{n,\alpha}}{v_n} < \infty.$$

If (4.18) holds for any space $X$ as in the statement, thanks to Theorem 4.3 we know that it also holds for $X'_\alpha$, which implies $\frac{v_{n,\alpha}^2}{v_n} \le \eta' v_{n,\alpha}$ for all $n$, that is,

$$\frac{1}{\eta'} v_{n,\alpha} \le v_n \le \eta v_{n,\alpha}, \quad n \ge 0,$$

and the result follows.

To verify the claim (4.18), we consider the disc automorphisms $\psi_a(z) = \frac{z+a}{1+az}$, $z \in \mathbb{D}$, $a \in (0,1)$, and use the conformal invariance of $X$ to conclude that for $a \in (0,1)$,

$$\begin{aligned} v_n &= \|\zeta^n\|^2 = \|W^\alpha_{\psi_a^{-1}} W^\alpha_{\psi_a} \zeta^n\|^2 \le c_1 \|W^\alpha_{\psi_a} \zeta^n\|^2 \\ &= c_1 \sum_{k=0}^\infty \left| \frac{[\psi_a^n (\psi_a')^\alpha]^{(k)}(0)}{k!} \right|^2 v_k = c_1 \sum_{k=0}^\infty C^2_{n,k,\alpha}(a) \frac{v_k}{v^2_{k,\alpha}}, \end{aligned} \tag{4.19}$$

where $C_{n,k,\alpha}(a)$ are given by (4.15). For $n \ge 2$, $a \in (0,1)$, let

$$S_n(a) = \sum_{k=n}^\infty (C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)) \sum_{j=n}^k \frac{v_j}{v^2_{j,\alpha}}.$$

For fixed $a \in (0,1)$, by Lemma 4.5 and Lemma 4.6 (i), we can interchange the order of summation since by Lemma 4.6 (i) there exits $N_0 \in \mathbb{N}$ such that $|C_{n,k,\alpha}(a)| < b^k$ for $b \in (a,1)$ and for all $k \ge N_0$. Thus,

$$\begin{aligned} \sum_{k=n}^\infty |C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)| \sum_{j=n}^k \frac{v_j}{v^2_{j,\alpha}} &\le \sum_{k=n}^\infty |C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)| \sum_{j=0}^k \frac{v_j}{v^2_{j,\alpha}} \\ &\le c \sum_{k=n}^\infty |C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)| (k+1)^{2\alpha} \\ &\le c \sum_{k=n}^\infty \left( |C_{n,k,\alpha}(a)|^2 + |C_{n,k+2,\alpha}(a)|^2 \right) (k+1)^{2\alpha} \\ &\le c \sum_{k=n}^{N_0} \left( |C_{n,k,\alpha}(a)|^2 + |C_{n,k+2,\alpha}(a)|^2 \right) (k+1)^{2\alpha} + 2c \sum_{k=N_0+1}^\infty b^k (k+1)^{2\alpha} \\ &< \infty. \end{aligned}$$

With this interchange, we obtain

$$\begin{aligned} S_n(a) &= \sum_{j=n}^\infty \frac{v_j}{v^2_{j,\alpha}} \sum_{k \ge j} (C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)) \\ &= \sum_{j=n}^\infty \frac{v_j}{v^2_{j,\alpha}} (C^2_{n,j,\alpha}(a) + C^2_{n,j+1,\alpha}(a)) \\ &= \sum_{j=0}^\infty \frac{v_j}{v^2_{j,\alpha}} (C^2_{n,j,\alpha}(a) + C^2_{n,j+1,\alpha}(a)) - \sum_{j=0}^{n-1} \frac{v_j}{v^2_{j,\alpha}} (C^2_{n,j,\alpha}(a) + C^2_{n,j+1,\alpha}(a)). \end{aligned}$$

Thus, by another application of Lemma 4.6 (i), we conclude that

$$v_n = \|\zeta^n\|^2 \le c_1 \sum_{k=0}^{\infty} C^2_{n,k,\alpha}(a) \frac{v_k}{v^2_{k,\alpha}} \le c_1 S_n(a) + o(1), \quad a \to 1^-. \tag{4.20}$$

In order to estimate $S_n(a)$ when $a \to 1^-$, for $k \ge n$, let

$$\begin{aligned} B_{n,k,\alpha}(a) = {} & \frac{n^2(1-a^2)}{4\alpha^2 a^2} C^2_{n-1,k,\alpha+\frac{1}{2}}(a) + \frac{1}{a(k+1)} \frac{d}{da}[(1-a^2)C^2_{n,k+1,\alpha}(a)] \\ & + \frac{2}{k+1} C^2_{n,k+1,\alpha}(a) - \frac{4\alpha}{a(k+1)} C_{n,k+1,\alpha}(a) C_{n,k+2,\alpha}(a), \end{aligned}$$

and use the inequality in Lemma 4.6 (iv) to obtain

$$S_n(a) = \sum_{k=n}^{\infty} (C^2_{n,k,\alpha}(a) - C^2_{n,k+2,\alpha}(a)) \sum_{j=n}^{k} \frac{v_j}{v^2_{j,\alpha}} \le \sum_{k=n}^{\infty} B_{n,k,\alpha}(a) \sum_{j=n}^{k} \frac{v_j}{v^2_{j,\alpha}}. \tag{4.21}$$

We want to use the estimate in Lemma 4.5, but at this stage it cannot be applied since the numbers $B_{n,k,\alpha}(a)$ might be negative. However, it turns out that their (weighted) averages can be controlled. If $a \in (0,1)$, then

$$\begin{aligned} \frac{1}{a-a^2} \int_{a^2}^{a} s B_{n,k,\alpha}(s) ds = {} & \frac{n^2}{4\alpha^2(a-a^2)} \int_{a^2}^{a} \frac{(1-s^2)}{s} C^2_{n-1,k,\alpha+\frac{1}{2}}(s) ds \\ & + \frac{1}{(k+1)(a-a^2)} \int_{a^2}^{a} \frac{d}{ds}[(1-s^2)C^2_{n,k+1,\alpha}(s)] ds \\ & + \frac{2}{(k+1)(a-a^2)} \int_{a^2}^{a} s C^2_{n,k+1,\alpha}(s) ds \\ & - \frac{4\alpha}{(k+1)(a-a^2)} \int_{a^2}^{a} C_{n,k+1,\alpha}(s) C_{n,k+2,\alpha}(s) ds. \end{aligned}$$

Note that

$$\int_{a^2}^{a} \frac{d}{ds}[(1-s^2)C^2_{n,k+1,\alpha}(s)] ds \le (1-a^2) C^2_{n,k+1,\alpha}(a),$$

and

$$- \int_{a^2}^{a} C_{n,k+1,\alpha}(s) C_{n,k+2,\alpha}(s) ds \le \int_{a^2}^{a} |C_{n,k+1,\alpha}(s) C_{n,k+2,\alpha}(s)| ds.$$

From (4.21) we infer that

$$\frac{1}{a-a^2} \int_{a^2}^{a} s S_n(s) ds \le \sum_{k=n}^{\infty} \sum_{j=n}^{k} \frac{v_j}{v^2_{j,\alpha}} \frac{1}{a-a^2} \int_{a^2}^{a} s B_{n,k,\alpha}(s) ds,$$

where the interchange of the sum and the integral is also justified by Lemma 4.6 (i) and Lemma 4.5. Now use the above estimates in order to replace $\frac{1}{a-a^2} \int_{a^2}^{a} s B_{n,k,\alpha}(s) ds$ by a sum of four nonnegative terms, and then apply Lemma 4.5 to arrive at

$$\frac{1}{a-a^2} \int_{a^2}^{a} s S_n(s) ds \le c \frac{1}{a-a^2} \int_{a^2}^{a} \left( S_n^{I}(s) + S_n^{II}(s) + S_n^{III}(s) \right) ds + c S_n^{IV}(a), \tag{4.22}$$

where $c$ is the constant in Lemma 4.5, and

$$
\begin{aligned}
S_n^I(s) &= \frac{n^2(1-s^2)}{4\alpha^2 s}\sum_{k=0}^{\infty}(k+1)^{2\alpha}C^2_{n-1,k,\alpha+\frac{1}{2}}(s),\\
S_n^{II}(s) &= 2s\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(s),\\
S_n^{III}(s) &= 4\alpha\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}|C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)|,\\
S_n^{IV}(a) &= \frac{1-a^2}{(a-a^2)}\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(a).
\end{aligned}
$$

Using the property $\lim_{k\to\infty}\frac{\Gamma(k+b)}{\Gamma(k)k^b}=1$ for all $b\in\mathbb{R}$, we can observe that there exists a constant $c_2$ such that, for all $k\geq 0$

$$
v_{k,\alpha}^{-1}=\frac{\Gamma(k+2\alpha)}{\Gamma(2\alpha)\Gamma(k+1)}=\frac{1}{\Gamma(2\alpha)}\frac{\Gamma(k+1+2\alpha-1)}{\Gamma(k+1)}\geq c_2(k+1)^{2\alpha-1}.
$$

So, we can deduce the estimate $(k+1)^{2\alpha}\leq c_3\min\{v^{-1}_{k,\alpha+\frac{1}{2}},(k+1)v^{-1}_{k+1,\alpha},(k+1)v^{-1}_{k+2,\alpha}\}$, valid for some fixed constant $c_3>0$ and all integers $k\geq 0$. Therefore, we conclude that

$$
\begin{aligned}
S_n^I(s) &\leq c_3\frac{n^2(1-s^2)}{4\alpha^2 s}\sum_{k=0}^{\infty}v^{-1}_{k,\alpha+\frac{1}{2}}C^2_{n-1,k,\alpha+\frac{1}{2}}(s)=c_3\frac{n^2(1-s^2)}{4\alpha^2 s}\|W_{\psi_a}^{\alpha+\frac{1}{2}}\zeta^{n-1}\|^2_{H_{\alpha+\frac{1}{2}}}\\
&= c_3\frac{n^2(1-s^2)}{4\alpha^2 s}\|\zeta^{n-1}\|^2_{H_{\alpha+\frac{1}{2}}}=c_3\frac{n^2(1-s^2)}{4\alpha^2 s}v_{n-1,\alpha+\frac{1}{2}},
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
S_n^{II}(s) &= 2s\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(s)\leq 2sc_3\sum_{k=0}^{\infty}v^{-1}_{k+1,\alpha}C^2_{n,k+1,\alpha}(s)\\
&\leq 2c_3 s\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha}=2c_3 s\|\zeta^n\|^2_{H_\alpha}=2c_3 s v_{n,\alpha},
\end{aligned}
$$

$$
\begin{aligned}
S_n^{III}(s) &= 4\alpha\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}|C_{n,k+1,\alpha}(s)C_{n,k+2,\alpha}(s)|\\
&\leq 4\alpha\quad\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(s)\quad{}^{\frac{1}{2}}\quad\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+2,\alpha}(s)\quad{}^{\frac{1}{2}}\\
&\leq 4\alpha c_3\quad\sum_{k=0}^{\infty}v^{-1}_{k+1,\alpha}C^2_{n,k+1,\alpha}(s)\quad{}^{\frac{1}{2}}\quad\sum_{k=0}^{\infty}v^{-1}_{k+2,\alpha}C^2_{n,k+2,\alpha}(s)\quad{}^{\frac{1}{2}}\\
&\leq 4\alpha c_3\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha}=4\alpha c_3 v_{n,\alpha},
\end{aligned}
$$

where we have used the Cauchy-Schwartz inequality, and

$$
\begin{aligned}
S_n^{IV}(a) &= \frac{1-a^2}{(a-a^2)}\sum_{k=0}^{\infty}(k+1)^{2\alpha-1}C^2_{n,k+1,\alpha}(a)\leq c_3\frac{1+a}{a}\sum_{k=0}^{\infty}v^{-1}_{k+1,\alpha}C^2_{n,k+1,\alpha}(a)\\
&\leq c_3\frac{1+a}{a}\|W^{\alpha}_{\psi_a}\zeta^n\|^2_{H_\alpha}=c_3\frac{1+a}{a}v_{n,\alpha}.
\end{aligned}
$$

In particular, we can see that for all $n \geq 2$ and all $a \in (\frac{1}{2}, 1)$

$$\begin{aligned}
\int_{a^2}^{a} S_n^I(s)\, ds &= c_3 \frac{n^2 v_{n-1,\alpha+\frac{1}{2}}}{4\alpha^2} \frac{1}{a-a^2} \int_{a^2}^{a} \frac{1-s^2}{s}\, ds \\
&\leq c_3 \frac{n^2 v_{n-1,\alpha+\frac{1}{2}}}{4\alpha^2} \frac{2}{a^2(a-a^2)} \int_{a^2}^{a} 1 - s\, ds \\
&= c_3 \frac{n^2 v_{n-1,\alpha+\frac{1}{2}}}{4\alpha^2} \frac{(1-a)(a+2)}{a^2} \to 0
\end{aligned}$$

when $a \to 1^-$, and there exists $c_4 > 0$ such that

$$\frac{1}{a-a^2} \int_{a^2}^{a} S_n^{II}(s)\, ds,\ \frac{1}{a-a^2} \int_{a^2}^{a} S_n^{III}(s)\, ds,\ S_n^{IV}(a) \leq c_4 v_{n,\alpha}.$$

Thus, from (4.20) and the estimate

$$\frac{1}{a-a^2} \int_{a^2}^{a} s\, ds = \frac{a(1+a)}{2} > \frac{3}{8}$$

for $a \in (\frac{1}{2}, 1)$, we deduce

$$v_n = \|\zeta^n\|^2 \leq \frac{8}{3} c_1 \frac{1}{a-a^2} \int_{a^2}^{a} s S_n(s)\, ds + o(1) \leq c_5 v_{n,\alpha} + o(1), \quad a \to 1^-,$$

and the claim (4.18) follows by letting $a \to 1^-$. This completes the proof of the theorem. □

# 4.5 Applications

## 4.5.1 Conformally invariant subspaces

Following the idea in [6] we attempt to construct a conformally invariant space starting with an arbitrary Banach space $X$ of analytic functions in $\mathbb{D}$ which satisfies 1) and 2). We simply set for $\alpha > 0$,

$$\mathcal{M}_\alpha(X) = \{f \in X : (\varphi')^\alpha\, (f \circ \varphi) \in X,\ \varphi \in Aut(\mathbb{D}),\ \|f\|_{\mathcal{M}_\alpha} = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|_X < \infty\}.$$

Clearly, $\mathcal{M}_\alpha(X) \subset X$, with equality if and only if $X$ is conformally invariant of index $\alpha$. Moreover, $\mathcal{M}_\alpha(X)$ is a Banach space satisfying 1) and 3), but it is not clear whether it satisfies the condition 2) .

**Proposition 4.5.** *Let $X$ be a Banach space satisfying 1) and 2). Then $\mathcal{M}_\alpha(X)$ is conformally invariant of index $\alpha$ if and only if $\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\|_X < \infty$. Moreover, if $\mathcal{A}^{-\alpha} \subset X$ then $\mathcal{M}_\alpha(X) = \mathcal{A}^{-\alpha}$.*

*Proof.* This is a direct application of Theorem 4.4. First, we will prove that the condition $\sup_{\varphi\in Aut(\mathbb{D})}\|(\varphi')^\alpha\|_X < \infty$ is equivalent to $X_\alpha^{min}\subset\mathcal{M}_\alpha(X)$. On the one hand, if $f\in X_\alpha^{min}$ and we assume $\sup_{\varphi\in Aut(\mathbb{D})}\|(\varphi')^\alpha\|_X = C<\infty$, then given $\varepsilon>0$ there exist $\{c_j\}_{j\geq1}\in l^1$, $\{\varphi_j\}_{j\geq1}\subset Aut(\mathbb{D})$ such that

$$f=\sum_{j\geq1}c_j\left(\varphi_j'\right)^\alpha \quad\text{and}\quad \|\{c_j\}_{j\geq1}\|_{l^1}<\|f\|_{X_\alpha^{min}}+\varepsilon.$$

Therefore,

$$\begin{aligned}\|f\|_{\mathcal{M}_\alpha(X)} &= \sup_{\phi\in Aut(\mathbb{D})}\|W_\phi^\alpha f\|_X\leq\sum_{j\geq1}|c_j|\sup_{\phi\in Aut(\mathbb{D})}\left\|W_\phi^\alpha\left(\varphi_j'\right)^\alpha\right\|_X\\ &=\sum_{j\geq1}|c_j|\sup_{\phi\in Aut(\mathbb{D})}\left\|\left((\varphi_j\circ\phi)'\right)^\alpha\right\|_X=\sum_{j\geq1}|c_j|\sup_{\mu\in Aut(\mathbb{D})}\left\|(\mu')^\alpha\right\|_X\\ &\leq C\left(\|f\|_{X_\alpha^{min}}+\varepsilon\right).\end{aligned}$$

On the other hand, if $X_\alpha^{min}\subset\mathcal{M}_\alpha(X)$, since both satisfy 1) we conclude, by Corollary 1.1, that the inclusion is continuous. Thus,

$$\sup_{\varphi\in Aut(\mathbb{D})}\|(\varphi')^\alpha\|_X=\|1\|_{\mathcal{M}_\alpha(X)}\leq C\|1\|_{X_\alpha^{min}}<\infty.$$

So, if $X_\alpha^{min}\subset\mathcal{M}_\alpha(X)$, then $\mathcal{M}_\alpha(X)$ satisfies 2) because $X_\alpha^{min}$ does. Conversely, if $\mathcal{M}_\alpha(X)$ satisfies 2), it is conformally invariant of index $\alpha$, hence it must contain $X_\alpha^{min}$. If $\mathcal{A}^{-\alpha}\subset X$, an application of the closed graph theorem (see Corollary 1.1) gives that the inclusion is continuous, hence $\mathcal{A}^{-\alpha}\subset\mathcal{M}_\alpha(X)$. Thus, $\mathcal{M}_\alpha(X)$ is conformally invariant of index $\alpha$, and by Theorem 4.4 we have $\mathcal{A}^{-\alpha}=\mathcal{M}_\alpha(X)$. □

Some interesting examples occur in this way.

**Example 4.6.** If $p\geq1$, and $\alpha<\frac{1}{p}$, a direct computation based on a change of variable reveals that $\mathcal{M}_\alpha(H^p)$ consists of $f\in H^p$ with

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\alpha p}\int_0^{2\pi}\frac{|f(e^{it})|^p}{|e^{it}-a|^{2-2\alpha p}}dt<\infty.$$

If $f\in\mathcal{M}_\alpha(H^p)$ then

$$\begin{aligned}2\pi\|f\|^p_{\mathcal{M}_\alpha(H^p)} &= \left[\sup_{\varphi\in Aut(\mathbb{D})}\left[\int_0^{2\pi}|\varphi'(e^{it})|^{\alpha p}|f(\varphi(e^{it}))|^p\,dt\right]^{\frac{1}{p}}\right]^p\\ &=\sup_{\varphi\in Aut(\mathbb{D})}\int_0^{2\pi}|\varphi'(e^{it})|^{\alpha p-1}|f(\varphi(e^{it}))|^p|\varphi'(e^{it})|\,dt\\ &=\sup_{\varphi\in Aut(\mathbb{D})}\int_0^{2\pi}\left|\left(\varphi^{-1}\right)'(e^{iw})\right|^{1-\alpha p}|f(e^{iw})|^p\,dw\\ &=\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\alpha p}\int_0^{2\pi}\frac{|f(e^{iw})|^p}{|1-\overline{a}e^{iw}|^{2-2\alpha p}}\,dw\\ &=\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\alpha p}\int_0^{2\pi}\frac{|f(e^{iw})|^p}{|e^{iw}-a|^{2-2\alpha p}}\,dw.\end{aligned}$$

Moreover, using Proposition 1.6 (in this case the measure $\mu$ is only supported on $\mathbb{T}$), we deduce that the above condition is equivalent to

$$\sup_{\substack{h\in(0,1)\\ t\in[0,2\pi)}} h^{\alpha p-1} \int_{I_h(t)} |f(z)|^p \, dz < \infty.$$

**Example 4.7.** In a similar way we can see for $p \geq 1$, $\beta > -1$, and $\frac{\beta+1}{p} \leq \alpha < \frac{\beta+2}{p}$, that $\mathcal{M}_\alpha(A^p_\beta) = \mathcal{Q}_{p,\beta,\beta+2-\alpha p}$. If $f \in \mathcal{M}_\alpha(A^p_\beta)$ then

$$\begin{aligned}
\frac{1}{\beta+1}\|f\|^p_{\mathcal{M}_\alpha(A^p_\beta)} &= \sup_{\varphi\in Aut(\mathbb{D})} \int_{\mathbb{D}} |\varphi'(z)|^{\alpha p-2}|f(\varphi(z))|^p(1-|z|^2)^\beta|\varphi'(z)|^2 \, dz \\
&= \sup_{\varphi\in Aut(\mathbb{D})} \int_{\mathbb{D}} \left|\left(\varphi^{-1}\right)'(w)\right|^{2-\alpha p}|f(w)|^p(1-|\varphi^{-1}(w)|^2)^\beta \, dw \\
&= \sup_{\varphi\in Aut(\mathbb{D})} \int_{\mathbb{D}} \left|\left(\varphi^{-1}\right)'(w)\right|^{2-\alpha p+\beta}|f(w)|^p(1-|w|^2)^\beta \, dw \\
&= \sup_{a\in\mathbb{D}}(1-|a|^2)^{2-\alpha p+\beta} \int_{\mathbb{D}} \frac{|f(w)|^p(1-|w|^2)^\beta}{|1-\overline{a}w|^{4-2\alpha p+2\beta}} \, dw.
\end{aligned}$$

Now, by Proposition 1.6, and from Example 4.4 we conclude that

$$\|f\|^p_{\mathcal{M}_\alpha(A^p_\beta)} \sim \sup_{\substack{h\in(0,1)\\ t\in[0,2\pi)}} h^{-(\beta+2-\alpha p)} \int_{S_h(t)} |f(z)|^p(1-|z|^2)^\beta dA(z) = \|f\|^p_{\mathcal{Q}_{p,\beta,\beta+2-\alpha p}}.$$

Let us also note that for $\gamma > 0$, we have $\mathcal{M}_\alpha(\mathcal{A}^{-\gamma}) = \{0\}$, when $\alpha > \gamma$, $\mathcal{M}_\gamma(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma}$, and $\mathcal{M}_\alpha(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\alpha}$, when $0 < \alpha < \gamma$. All these examples are actually consequences of a general fact which is proved below and Proposition 4.5.

**Proposition 4.6.** *Let $X$ be conformally invariant of index $\gamma > 0$. Then $\mathcal{M}_\alpha(X) = \{0\}$, when $\alpha > \gamma$, $\mathcal{M}_\gamma(X) = X$, and when $0 < \alpha < \gamma$, $\mathcal{M}_\alpha(X) = Mult(X^{min}_{\gamma-\alpha}, X)$.*

*Proof.* Let $\alpha > \gamma$, and let $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$, $a, z \in \mathbb{D}$. If $f \in M_\alpha(X)$ and $z \in K$, with $K$ a compact subset of $\mathbb{D}$, by 1) and using that $\varphi_a^{-1} = \varphi_a$ we have

$$\begin{aligned}
\sup_{a\in\mathbb{D}}(1-|a|^2)^{\gamma-\alpha}|f(z)| &\lesssim \sup_{a\in\mathbb{D}} |(\varphi_a'(z))^{\gamma-\alpha}f(z)| = \sup_{a\in\mathbb{D}} |W^\gamma_{\varphi_a}W^\alpha_{\varphi_a}f(z)| \\
&\lesssim \sup_{a\in\mathbb{D}} \|W^\gamma_{\varphi_a}W^\alpha_{\varphi_a}f\|_X \lesssim \sup_{a\in\mathbb{D}} \|W^\alpha_{\varphi_a}f\|_X \leq \|f\|_{\mathcal{M}_\alpha(X)}.
\end{aligned}$$

Hence, $f \equiv 0$. If $0 < \alpha < \gamma$ and $\varphi \in Aut(\mathbb{D})$, for $f$ analytic in $\mathbb{D}$ we have

$$W^\gamma_\varphi(f([\varphi^{-1}]')^{\gamma-\alpha}) = W^\alpha_\varphi f$$

or equivalently

$$W^\gamma_{\varphi^{-1}}(f(\varphi')^{\gamma-\alpha}) = W^\alpha_{\varphi^{-1}}f.$$

On the one hand, if $f \in Mult(X^{min}_{\gamma-\alpha}, X)$, then the left hand side is bounded on $X$, uniformly in $\varphi \in Aut(\mathbb{D})$ since using that $X$ and $X^{\min}_{\gamma-\alpha}$ are conformally invariant of index

$\gamma$ respectively $\gamma - \alpha$, we have

$$\begin{aligned}\sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\gamma(f([\varphi^{-1}]')^{\gamma-\alpha})\|_X \quad & \sup_{\varphi \in Aut(\mathbb{D})} \|f([\varphi^{-1}]')^{\gamma-\alpha}\|_X \\ & \leq \|f\|_{Mult(X_{\gamma-\alpha}^{min}, X)} \sup_{\varphi \in Aut(\mathbb{D})} \|W_{\varphi^{-1}}^{\gamma-\alpha} 1\|_{X_{\gamma-\alpha}^{min}} \\ & \quad \|f\|_{Mult(X_{\gamma-\alpha}^{min}, X)}.\end{aligned}$$

Thus,

$$\|f\|_{\mathcal{M}_\alpha(X)} = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\alpha f\|_X = \sup_{\varphi \in Aut(\mathbb{D})} \|W_\varphi^\gamma(f([\varphi^{-1}]')^{\gamma-\alpha})\|_X \quad \|f\|_{Mult(X_{\gamma-\alpha}^{min}, X)}.$$

Conversely, if $f \in \mathcal{M}_\alpha(X)$, then the right hand side is bounded in $X$, uniformly in $\varphi \in Aut(\mathbb{D})$, and by 3), since

$$\|f(\varphi')^{\gamma-\alpha}\|_X = \|W_\varphi^\gamma W_{\varphi^{-1}}^\alpha f\|_X \quad \|W_{\varphi^{-1}}^\alpha f\|_X \leq \|f\|_{\mathcal{M}_\alpha(X)},$$

and the same holds for $f(\varphi')^{\gamma-\alpha}$. Therefore, let $g$ be a function in the space $X_{\gamma-\alpha}^{min}$ with $g = \ \ _j a_j(\varphi_j')^{\gamma-\alpha}$ and $\ \ _j |a_j| < \infty$. Then, we have

$$\|fg\|_X \leq \sum_j |a_j| \|f(\varphi_j')^{\gamma-\alpha}\|_X \quad \|f\|_{\mathcal{M}_\alpha(X)} \sum_j |a_j| < \infty,$$

which gives $f \in Mult(X_{\gamma-\alpha}^{min}, X)$, by (4.10). □

The situation is more complicated in the case when $X$ is not conformally invariant. We close the paragraph with an example of this type.

**Example 4.8.** Let

$$A_{\log^{-2}}^1 = \left\{ f \in Hol(\mathbb{D}) : \|f\|_{A_{\log^{-2}}^1} = \ \ _{\mathbb{D}} |f(z)| \frac{1}{\log^2 \frac{2}{1-|z|^2}} dA(z) < \infty \right\}.$$

Then:

a) $A_{\log^{-2}}^1$ is not conformally invariant of any index $\alpha > 0$,

b) $\mathcal{M}_\alpha(A_{\log^{-2}}^1) = \{0\}$, when $\alpha > 2$, and $\mathcal{M}_\alpha(A_{\log^{-2}}^1) = \mathcal{A}^{-\alpha}$ when $0 < \alpha \leq 1$,

c) If $1 < \alpha \leq 2$ we have for every $\varepsilon > 0$,

$$\begin{aligned} Mult(X_{2-\alpha}^{min}, A_0^1) \subset \mathcal{M}_\alpha(A_{\log^{-2}}^1) \subset Mult(X_{2-\alpha+\varepsilon}^{min}, A_\varepsilon^1) \quad & 1 < \alpha < 2, \\ A_0^1 \subset \mathcal{M}_\alpha(A_{\log^{-2}}^1) \subset Mult(X_\varepsilon^{min}, A_\varepsilon^1) \quad & \alpha = 2. \end{aligned}$$

We were not able to relate the formal definition of $\mathcal{M}_\alpha(A_{\log^{-2}}^1)$ to standard objects in this area, for example those considered in Section 4.3.1. It remains a challenging question to do so. For this reason, we have appealed to the fact that there exists $C > 0$, and for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that for all $g \in Hol(\mathbb{D})$, we have

$$\|g\|_{A_\varepsilon^1} \leq C_\varepsilon \|g\|_{A_{\log^{-2}}^1} \quad \text{and} \quad \|g\|_{A_{\log^{-2}}^1} \leq C\|g\|_{A_0^1}. \tag{4.23}$$

*Proof.* b) Note that for $0 < \alpha \leq 1$, we have $\mathcal{A}^{-\alpha} \subset A^1_{\log^{-2}}$, if $f \in \mathcal{A}^{-\alpha}$ then

$$\begin{aligned}
\|f\|_{A^1_{\log^{-2}}} &= \int_{\mathbb{D}} |f(z)| \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z) \leq \|f\|_{\mathcal{A}^{-\alpha}} \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^\alpha \log^2 \frac{2}{1-|z|^2}}\, dA(z) \\
&= 2\|f\|_{\mathcal{A}^{-\alpha}} \int_0^1 \frac{r}{(1-r^2)^\alpha \log^2 \frac{2}{1-r^2}}\, dr \leq C\|f\|_{\mathcal{A}^{-\alpha}}.
\end{aligned}$$

Therefore, Proposition 4.5 gives the second part of b). Now, if $\alpha > 2$ and we choose $\varepsilon < \alpha - 2$, by (4.23), we know that $A^1_{\log^{-2}} \subset A^1_\varepsilon$ and $A^1_\varepsilon$ is conformally invariant of index $2+\varepsilon$, (see Example 4.1). Thus, $\mathcal{M}_\alpha(A^1_{\log^{-2}}) \subset \mathcal{M}_\alpha(A^1_\varepsilon)$ and by Proposition 4.6, $\mathcal{M}_\alpha(A^1_\varepsilon) = \{0\}$.

c) By (4.23), we have $A^1_0 \subset A^1_{\log^{-2}} \subset A^1_\varepsilon$. Therefore, $\mathcal{M}_\alpha(A^1_0) \subset \mathcal{M}_\alpha(A^1_{\log^{-2}}) \subset \mathcal{M}_\alpha(A^1_\varepsilon)$ and if $1 < \alpha < 2$, by Proposition 4.6, $Mult(X^{min}_{2-\alpha}, A^1_0) = \mathcal{M}_\alpha(A^1_0)$ and $Mult(X^{min}_{2-\alpha}, A^1_\varepsilon) = \mathcal{M}_\alpha(A^1_\varepsilon)$. Note that in the case $\alpha = 2$, since $A^1_0$ is conformally invariant of index $2$, we have $\mathcal{M}_2(A^1_0) = A^1_0$.

a) By b) we know $A^1_{\log^{-2}}$ is not confomally invariant of order $\alpha$ with $\alpha > 2$ or $0 < \alpha \leq 1$. If for any $\alpha$ such that $1 < \alpha \leq 2$, $A^1_{\log^{-2}}$ was conformally invariant of index $\alpha$ we would have, by Theorem 4.4, $A^1_{\log^{-2}} \subseteq \mathcal{A}^{-\alpha}$ and this is a contradiction. For example, if we consider $f(z) = \frac{1}{(1-z)^2} \log\log \frac{2e}{1-z}$, which is in $A^1_{\log^{-2}}$, but not in $\mathcal{A}^{-\alpha}$ for $1 < \alpha \leq 2$. On the one hand,

$$\begin{aligned}
\|f\|_{\mathcal{A}^{-\alpha}} &= \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha \left| \frac{\log\log \frac{2e}{1-z}}{(1-z)^2} \right| \geq \sup_{r \in (0,1)} (1-r^2)^\alpha \frac{\log\log \frac{2e}{1-r}}{(1-r)^2} \\
&\geq \sup_{r \in (0,1)} \frac{\log\log \frac{2e}{1-r}}{(1-r)^{2-\alpha}} = \infty,
\end{aligned}$$

when $1 < \alpha \leq 2$. On the other hand,

$$\begin{aligned}
\|f\|_{A^1_{\log^{-2}}} &= \int_{\mathbb{D}} \frac{|\log\log \frac{2e}{1-z}|}{|1-z|^2} \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z) \leq \int_{\mathbb{D}} \frac{\log|\log \frac{2e}{1-z}| + \pi}{|1-z|^2} \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z) \\
&\leq \int_{\mathbb{D}} \frac{\log\left(\log \frac{2e}{|1-z|} + \pi\right) + \pi}{|1-z|^2} \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z) \\
&\leq \int_{\mathbb{D}} \frac{\log\left(\log \frac{4e}{1-|z|^2} + \pi\right)}{|1-z|^2} \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z) + \pi \int_{\mathbb{D}} \frac{1}{|1-z|^2} \frac{1}{\log^2 \frac{2}{1-|z|^2}}\, dA(z).
\end{aligned}$$

We only have to prove that the first integral is finite since the second integral can be controlled by the first. Thus, using Theorem 1.6 we obtain

$$\begin{aligned}
\int_{\mathbb{D}} \frac{\log\left(\log\frac{4e}{1-|z|^2}+\pi\right)}{|1-z|^2}\frac{1}{\log^2\frac{2}{1-|z|^2}}\,dA(z) &= \frac{1}{\pi}\int_0^1 \frac{r\log\left(\log\frac{4e}{1-r^2}+\pi\right)}{\log^2\frac{2}{1-r^2}}\int_0^{2\pi}\frac{dt\,dr}{|1-re^{it}|^2}\\
&\sim \frac{1}{\pi}\int_0^1 \frac{r\log\left(\log\frac{4e}{1-r^2}+\pi\right)}{(1-r^2)\log^2\frac{2}{1-r^2}}\,dr\\
&= \frac{1}{2\pi}\int_2^\infty \frac{\log\left(\log(2es)+\pi\right)}{s\log^2 s}\,ds\\
&< \infty,
\end{aligned}$$

where we have used the change of variable $s = \frac{2}{(1-r^2)}$.

□

### 4.5.2 Interpolation

Interpolating between conformally invariant Banach spaces of analytic functions is certainly meaningful and might lead to interesting examples. In view of 1), any pair of weighted conformal invariant spaces of index $\alpha$ is *compatible*, see Definition 2.1. In general, describing the intermediate spaces is a difficult task. We shall consider the extreme case, that is, we are going to apply the complex interpolation method to the couple $(X_\alpha^{max}, X_\alpha^{min})$, $\alpha > 0$. Our result is essentially based on the following lemma which is actually a well known result. Given a positive measurable function $v$ on $\mathbb{D}$ recall that we denote by $L^p(v) = L^p(vdA)$, $1 \le p < \infty$, and $L^\infty(v) = v^{-1}L^\infty(dA)$.

**Lemma 4.8.** *Let* $\gamma > -1$, $\delta > 0$, $p \in [1,\infty)$. *For* $\beta > \max\{\gamma,\delta\}$, $f \in L^1((1-|\zeta|^2)^\beta)$, *and* $z \in \mathbb{D}$, *define*

$$P_\beta f(z) = (\beta+1)\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\beta+2}}(1-|w|^2)^\beta\,dA(w), \quad Q_\beta f(z) = P_\beta((1-|\zeta|^2)f)(z).$$

*Then*

*a)* $P_\beta$ *is a bounded projection from* $L^\infty((1-|\zeta|^2)^\delta)$ *onto* $\mathcal{A}^{-\delta}$, *and from* $L^p\left((1-|\zeta|^2)^\gamma\right)$ *onto* $A^p_\gamma$.

*b)* $Q_\beta$ *extends to a continuous bijection from* $\mathcal{A}^{-\delta-1}$ *onto* $\mathcal{A}^{-\delta}$, *from* $A^p_\gamma$ *onto* $A^p_{\gamma-p}$, *when* $\gamma > p-1$, *and from* $A^p_\gamma$ *onto* $B^{p,\gamma}$, *when* $\gamma \le p-1$.

*Proof.* a) That $P_\beta$ is a bounded projection from $L^p\left((1-|\zeta|^2)^\gamma\right)$ onto $A^p_\gamma$ follows by Theorem 1.8 ( [54, Theorem 1.10]). Now, we will see that $P_\beta$ is a bounded projection from $L^\infty((1-|\zeta|^2)^\delta)$ onto $\mathcal{A}^{-\delta}$. Let $f$ be a function in $L^\infty((1-|\zeta|^2)^\delta)$. Then using

Theorem 1.6 we obtain

$$
\begin{aligned}
\|P_\beta f\|_{\mathcal{A}^{-\delta}} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\delta |P_\beta f(z)| \\
&\le (\beta+1)\sup_{z\in\mathbb{D}}(1-|z|^2)^\delta \int_{\mathbb{D}} \frac{|f(w)|}{|1-\overline{w}z|^{\beta+2}}(1-|w|^2)^\beta \, dA(w) \\
&\le (\beta+1)\|f\|_{L^\infty((1-|\zeta|^2)^\delta)} \sup_{z\in\mathbb{D}}(1-|z|^2)^\delta \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta-\delta}}{|1-\overline{w}z|^{\beta+2}}\, dA(w) \\
&\sim \|f\|_{L^\infty((1-|\zeta|^2)^\delta)} \sup_{z\in\mathbb{D}}(1-|z|^2)^\delta(1-|z|^2)^{-\delta} \\
&= \|f\|_{L^\infty((1-|\zeta|^2)^\delta)}.
\end{aligned}
$$

Thus, we have seen $P_\beta$ is bounded. To see that it is also surjective we consider $f \in \mathcal{A}^{-\delta}$ with $f(z) = \sum_{n\ge 0} a_n z^n$. Hence, if we see that $P_\beta f = f$ we are done. Given $z \in K$ where $K$ is a compact subset of $\mathbb{D}$ we have

$$
\begin{aligned}
P_\beta f(z) &= (\beta+1)\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\beta+2}}(1-|w|^2)^\beta \, dA(w) \\
&= (\beta+1)\int_{\mathbb{D}} \sum_{n\ge0} f(w)\binom{\beta+n+1}{n}\overline{w}^n z^n (1-|w|^2)^\beta \, dA(w) \\
&= \frac{\beta+1}{\pi}\int_0^1 (1-r^2)^\beta r \int_0^{2\pi} \sum_{n\ge0} f(re^{it})\binom{\beta+n+1}{n} r^n e^{-itn} z^n \, dt\, dr \\
&= \frac{\beta+1}{\pi}\int_0^1 \sum_{n\ge0}(1-r^2)^\beta r^{n+1}\binom{\beta+n+1}{n} z^n \int_0^{2\pi} f(re^{it})e^{-itn}\, dt\, dr.
\end{aligned}
$$

The interchange of integration and summations is justified, because for fixed $r \in (0,1)$ and for all $z \in K$ we can use Fubini's theorem, since

$$
\int_0^{2\pi} \sum_{n\ge0} |f(re^{it})| \binom{\beta+n+1}{n} r^n |z|^n \, dt < \infty.
$$

Now, using the Taylor series of $f$ we have

$$
\begin{aligned}
P_\beta f(z) &= \frac{\beta+1}{\pi}\int_0^1 \sum_{n\ge0}(1-r^2)^\beta r^{n+1}\binom{\beta+n+1}{n} z^n \int_0^{2\pi} \sum_{m\ge0} a_m r^m e^{it(m-n)}\, dt\, dr \\
&= \frac{\beta+1}{\pi}\int_0^1 \sum_{n\ge0}(1-r^2)^\beta r^{n+1}\binom{\beta+n+1}{n} z^n \sum_{m\ge0} a_m r^m \int_0^{2\pi} e^{it(m-n)}\, dt\, dr,
\end{aligned}
$$

and noting that if $m = n$ then $\int_0^{2\pi} e^{it(m-n)}\, dt = 0$ and $\int_0^{2\pi} e^{it(m-n)}\, dt = 1$ when $m = n$, we obtain that

$$\begin{aligned}
P_\beta f(z) &= 2(\beta+1)\int_0^1 \sum_{n\geq 0}(1-r^2)^\beta r^{2n+1}\binom{\beta+n+1}{n} a_n z^n\,dr\\
&= 2(\beta+1)\sum_{n\geq 0}\binom{\beta+n+1}{n}\left(\int_0^1 (1-r^2)^\beta r^{2n+1}\,dr\right) a_n z^n\\
&= (\beta+1)\sum_{n\geq 0}\binom{\beta+n+1}{n} B(n+1,\beta+1)a_n z^n\\
&= \sum_{n\geq 0} a_n z^n\\
&= f(z),
\end{aligned}$$

where $B(x,y)$ is the Beta function and the interchanges of integration and summations are also justified by Fubini's theorem. Thus,

$$(\beta+1)\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\beta+2}}(1-|w|^2)^\beta\,dA(w) = f(z),$$

and the integral converges uniformly for $z$ in every compact subset of $\mathbb{D}$.

b) First, we will prove that

$$(Q_\beta f)' = (\beta+1)P_{\beta+1}\overline{\zeta}f. \tag{4.24}$$

We have

$$Q_\beta f(z) = P_\beta((1-|\zeta|^2)f)(z) = (\beta+1)\int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{\beta+2}}(1-|w|^2)^{\beta+1}\,dA(w).$$

Therefore,

$$(Q_\beta f)'(z) = (\beta+1)(\beta+2)\int_{\mathbb{D}} \frac{\overline{w}f(w)}{(1-\overline{w}z)^{\beta+3}}(1-|w|^2)^{\beta+1}\,dA(w) = (\beta+1)P_{\beta+1}\overline{\zeta}f(z).$$

• The next step is to prove that $Q_\beta$ is a bounded linear operator between the spaces in the statement. On the one hand, given $f\in\mathcal{A}^{-\delta-1}$, we have

$$\|Q_\beta f\|_{\mathcal{A}^{-\delta}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\delta|Q_\beta f(z)| = \sup_{r\in(0,1)}(1-r^2)^\delta M_\infty(r,Q_\beta f)$$

where

$$M_\infty(r,Q_\beta f) = \max_{t\in[0,2\pi)}|Q_\beta f(re^{it})|.$$

Now, noting that

$$M_\infty(r,Q_\beta f) \leq |Q_\beta f(0)| + \int_0^r M_\infty(s,(Q_\beta f)')\,ds,$$

with

$$|Q_\beta f(0)| = \left|(\beta+1)\int_{\mathbb{D}} f(w)(1-|w|^2)^{\beta+1}\,dA(w)\right| \lesssim |f(0)| \le \|f\|_{\mathcal{A}^{-\delta-1}},$$

and by (4.24)

$$\begin{aligned} M_\infty(s,(Q_\beta f)') &= \max_{t\in[0,2\pi)} |(Q_\beta f)'(se^{it})| = (\beta+1)\max_{t\in[0,2\pi)} |P_{\beta+1}\overline{\zeta} f(se^{it})| \\ &\lesssim \frac{\|P_{\beta+1}\overline{\zeta} f\|_{\mathcal{A}^{-\delta-1}}}{(1-s^2)^{\delta+1}} \lesssim \frac{\|\overline{\zeta} f\|_{L^\infty((1-|\zeta|^2)^{\delta+1})}}{(1-s^2)^{\delta+1}} \\ &\lesssim \frac{\|f\|_{L^\infty((1-|\zeta|^2)^{\delta+1})}}{(1-s^2)^{\delta+1}} = \frac{\|f\|_{\mathcal{A}^{-\delta-1}}}{(1-s^2)^{\delta+1}}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \|Q_\beta f\|_{\mathcal{A}^{-\delta}} &\le \sup_{r\in(0,1)} (1-r^2)^\delta \left(|Q_\beta f(0)| + \int_0^r M_\infty(s,Q_\beta f)\,ds\right) \\ &\lesssim \|f\|_{\mathcal{A}^{-\delta-1}} \sup_{r\in(0,1)} (1-r^2)^\delta \left(1+\int_0^r \frac{1}{(1-s^2)^{\delta+1}}\,ds\right) \\ &\lesssim \|f\|_{\mathcal{A}^{-\delta-1}}. \end{aligned}$$

On the other hand, given $f\in A^p_\gamma$, $\gamma\le p-1$ and using that

$$|Q_\beta f(0)| = \left|(\beta+1)\int_{\mathbb{D}} f(w)(1-|w|^2)^{\beta+1}\,dA(w)\right| \lesssim |f(0)| \lesssim \|f\|_{A^p_\gamma},$$

by (4.24), which says that

$$\begin{aligned} \|(Q_\beta f)'\|_{A^p_\gamma} &= (\beta+1)\|P_{\beta+1}\overline{\zeta} f\|_{A^p_\gamma} \lesssim \|\overline{\zeta} f\|_{L^p(1-|\zeta|^2)^\gamma)} \\ &\lesssim \|f\|_{L^p((1-|\zeta|^2)^\gamma)} = \frac{1}{(\gamma+1)^{\frac{1}{p}}}\|f\|_{A^p_\gamma}, \end{aligned}$$

we obtain that

$$\|Q_\beta f\|_{B^{p,\gamma}} = |Q_\beta f(0)| + \|(Q_\beta f)'\|_{A^p_\gamma} \lesssim \|f\|_{A^p_\gamma}.$$

If $f\in A^p_\gamma$ and $\gamma>p-1$, we will use the Littlewood-Paley formula, Theorem 1.7, to conclude that

$$\|Q_\beta f\|^p_{A^p_{\gamma-p}} \sim |Qf(0)|^p + \|(Qf)'\|^p_{A^p_\gamma} \lesssim \|f\|_{A^p_\gamma}.$$

• We have already proved that $Q_\beta$ is bounded between the spaces in the statement. Its injectivity is clear if we see that for $f$ belongs to the spaces in the statement, and $Q_\beta f = 0$, we have that all Taylor coefficients of $f$ are zero. Given $f$ as above, by (4.24),

$(Q_\beta f)' = (\beta+1)P_{\beta+1}\overline{\zeta} f$ but

$$
\begin{aligned}
P_{\beta+1}\overline{\zeta} f(z) &= (\beta+2)\int_{\mathbb{D}} \frac{\overline{w} f(w)}{(1-\overline{w}z)^{\beta+3}}(1-|w|^2)^{\beta+1}\, dA(w) \\
&= \frac{\beta+2}{\pi}\int_0^1 (1-r^2)^{\beta+1} r^2 \int_0^{2\pi} \sum_{m=0}^{\infty} a_m r^m e^{i(m-1)t} \sum_{n=0}^{\infty} \binom{\beta+n+2}{n} r^n e^{-int} z^n \, dt\, dr \\
&= \sum_{n=0}^{\infty} (\beta+2)\binom{\beta+n+2}{n} a_{n+1} z^n B(n+2, \beta+2) \\
&= \sum_{n=0}^{\infty} \frac{n+1}{n+\beta+3} a_{n+1} z^n,
\end{aligned} \tag{4.25}
$$

where the interchanges of integration and summations are justified by Fubini's theorem as in the above part. Thus, if $Q_\beta f = 0$ then $f$ is a constant and noting that

$$
Q_\beta f(0) = (\beta+1)\int_{\mathbb{D}} f(w)(1-|w|^2)^{\beta+1}\, dA(w) = C_\beta f(0),
$$

we conclude that $f \equiv 0$. So, $Q_\beta$ is injective.

• To see the surjectivity, note that (4.24) implies the following calculation. Whenever $u\zeta^{-1} \in L^\infty((1-|\zeta|^2)^{\delta+1})$, or $u\zeta^{-1} \in L^p\left((1-|\zeta|^2)^{\gamma}\right)$, we have

$$
(\beta+1)P_{\beta+1}u = (\beta+1)P_{\beta+1}\overline{\zeta} P_{\beta+1}\overline{\zeta}^{-1} u = (Q_\beta P_{\beta+1}\overline{\zeta}^{-1} u)', \tag{4.26}
$$

where the second equality is due to (4.24). To prove the first one, given $z \in K$ compact of $\mathbb{D}$ we have

$$
\begin{aligned}
&P_{\beta+1}\overline{\zeta} P_{\beta+1}\overline{\zeta}^{-1} u(z) = (\beta+2)\int_{\mathbb{D}} \frac{\overline{\mu} P_{\beta+1}(\overline{\zeta}^{-1} u)(\mu)(1-|\mu|^2)^{\beta+1}}{(1-\overline{\mu}z)^{\beta+3}}\, dA(\mu) \\
&= (\beta+2)^2 \int_{\mathbb{D}} \frac{\overline{\mu}\int_{\mathbb{D}} \frac{\overline{w}^{-1} u(w)}{(1-\overline{w}\mu)^{\beta+3}}(1-|w|^2)^{\beta+1}\, dA(w)}{(1-\overline{\mu}z)^{\beta+3}}(1-|\mu|^2)^{\beta+1}\, dA(\mu) \\
&= (\beta+2)\int_{\mathbb{D}} \overline{w}^{-1} u(w)(1-|w|^2)^{\beta+1}\left((\beta+2)\int_{\mathbb{D}} \frac{\frac{\mu}{(1-\mu\overline{z})^{\beta+3}}(1-|\mu|^2)^{\beta+1}}{(1-\overline{\mu}w)^{\beta+3}}\, dA(\mu)\right) dA(w) \\
&= (\beta+2)\int_{\mathbb{D}} \overline{w}^{-1} u(w)(1-|w|^2)^{\beta+1}\frac{\overline{w}}{(1-\overline{w}z)^{\beta+3}}\, dA(w) \\
&= P_{\beta+1}u(z).
\end{aligned}
$$

Here we have used that $P_{\beta+1}f = f$ for $f(w) = \frac{w}{(1-w\overline{z})}$ and the interchange of the integrals is justified by Fubini's theorem in the following way. If $u\zeta^{-1} \in L^\infty((1-|\zeta|^2)^{\delta+1})$, using Theorem 1.6 we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|\mu|^2)^{\beta+1}}{|1-\mu z|^{\beta+3}} \int_{\mathbb{D}} \frac{|w^{-1}u(w)|(1-|w|^2)^{\beta+1}}{|1-\overline{w}\mu|^{\beta+3}}\, dA(w)\, dA(\mu) \\
&\le C_K \|u\zeta^{-1}\|_{L^\infty((1-|\zeta|^2)^{\delta+1})} \int_{\mathbb{D}} (1-|\mu|^2)^{\beta+1} \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta-\delta}}{|1-\overline{w}\mu|^{\beta+3}}\, dA(w)\, dA(\mu) \\
&\sim C_K \|u\zeta^{-1}\|_{L^\infty((1-|\zeta|^2)^{\delta+1})} \int_{\mathbb{D}} \frac{(1-|\mu|^2)^{\beta+1}}{(1-|\mu|^2)^{\delta+1}}\, dA(\mu) \\
&< \infty.
\end{aligned}$$

If $u\zeta^{-1} \in L^p((1-|\zeta|^2)^\gamma)$, using Theorem 1.6 and Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{(1-|\mu|^2)^{\beta+1}}{|1-\mu z|^{\beta+3}} \int_{\mathbb{D}} \frac{|w^{-1}u(w)|(1-|w|^2)^{\beta+1}}{|1-\overline{w}\mu|^{\beta+3}}\, dA(w)\, dA(\mu) \\
&\le C_K \|u\zeta^{-1}\|_{L^p((1-|\zeta|^2)^\gamma)} \int_{\mathbb{D}} (1-|\mu|^2)^{\beta+1} \left[ \int_{\mathbb{D}} \frac{(1-|w|^2)^{(\beta+1-\frac{\gamma}{p})q}}{|1-\overline{w}\mu|^{(\beta+3)q}}\, dA(w) \right]^{\frac{1}{q}} dA(\mu) \\
&\sim C_K \|u\zeta^{-1}\|_{L^p((1-|\zeta|^2)^\gamma)} \int_{\mathbb{D}} \frac{(1-|\mu|^2)^{\beta+1}}{(1-|\mu|^2)^{\frac{2+\gamma}{p}}}\, dA(\mu) \\
&< \infty.
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and we have used that $\beta+1-\frac{2+\gamma}{p} > -1$ for $p \ge 1$ and $\beta > \gamma$. Therefore, if $f \in \mathcal{A}^{-\delta-1}$, or $f \in A^p_\gamma$ with $f(0)=0$, by (4.26) we have

$$(\beta+1)f = (Q_\beta P_{\beta+1}\overline{\zeta}^{-1} f)'.$$

Moreover, by (4.24) and (4.25), for $n \ge 1$ we can see that $Q_\beta \zeta^n = c_{n,\beta}\zeta^n$, with $c_{n,\beta} = 0$ this follows thanks to

$$(Q_\beta \zeta^n)' = (\beta+1)P_{\beta+1}\overline{\zeta}\zeta^n = c_{n,\beta}\zeta^{n-1},$$

and $Q_\beta\zeta^n(0)=0$. In particular, we have that the range of $Q_\beta$ contains all polynomials of degree one. Thus, the range of $Q_\beta$ contains all anti-derivatives of functions in $\mathcal{A}^{-\delta-1}$, respectively in $A^p_\gamma$. Note that the range of $Q_\beta$ also contains the constants since $Q_\beta 1$ is a constant different from zero. Therefore, if $f \in A^p_{\gamma-p}$ by Theorem 1.7 we have $f' \in A^p_\gamma$ and $f(z) = \int_0^z f'(w)\,dw + f(0)$, so $f$ is in the range of $Q_\beta$. In the same way, if $f \in \mathcal{A}^{-\delta}$ by Theorem 1.3 we have $f' \in \mathcal{A}^{-\delta-1}$, so $f$ is in the range of $Q_\beta$. Then the surjectivity of $Q_\beta$ follows.

□

Recall that for a compatible couple of Banach spaces $(X,Y)$ we denote by $[X,Y]_\theta$, $\theta \in (0,1)$, the corresponding complex interpolation space, see Section 2.4.

**Theorem 4.6.** *For* $0<\theta<1$ *if* $\alpha>\theta$*, then* $[X^{max}_\alpha, X^{min}_\alpha]_\theta = A^{\frac{1}{\theta}}_{\frac{\alpha}{\theta}-2}$*, and if* $\alpha \le \theta$*, then* $[X^{max}_\alpha, X^{min}_\alpha]_\theta = B^{\frac{1}{\theta},\frac{\alpha+1}{\theta}-2}$.

*Proof.* We want to find $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta,\ \theta\in(0,1)$. By Proposition 2.3, if $\theta\in(0,1)$

$$\begin{aligned}[L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta &= L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{1-\theta}{\theta}(\alpha+1)+\alpha-1})\\ &= L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2}).\end{aligned}$$

Since $\mathcal{A}^{-\alpha-1}\subset L^\infty((1-|\zeta|^2)^{\alpha+1}),\ A^1_{\alpha-1}\subset L^1\left((1-|\zeta|^2)^{\alpha-1}\right)$, it follows by Proposition 2.2 that for $\theta\in(0,1)$

$$[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta\subset[L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta = L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2}).$$

Moreover, using again Proposition 2.2, since $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta\subset\mathcal{A}^{-\alpha-1}$, $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta$ consists of analytic functions in $\mathbb{D}$, hence $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta\subset A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$. On the other hand, by Theorem 2.1 and Lemma 4.8, for $\beta>\max\{\alpha+1,\frac{\alpha+1}{\theta}-2\}$ we have that

$$P_\beta:[L^\infty((1-|\zeta|^2)^{\alpha+1}), L^1((1-|\zeta|^2)^{\alpha-1})]_\theta = L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2})\to[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta,$$

is bounded for all $\theta\in(0,1)$. Since $P_\beta: L^{\frac{1}{\theta}}((1-|\zeta|^2)^{\frac{\alpha+1}{\theta}-2})\to A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$ is onto, we obtain $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta\supset A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$. The fact that the norms on these two spaces are equivalent follows by the closed graph theorem, see Corollary 1.1.
Finally another application of Lemma 4.8 and Theorem 2.1 shows that $Q_\beta$ is an invertible linear operator from $[\mathcal{A}^{-\alpha-1}, A^1_{\alpha-1}]_\theta$ onto $[X^{max}_\alpha, X^{min}_\alpha]_\theta$ for all $\theta\in(0,1)$, hence the result follows from applying Lemma 4.8 $b)$ to the space $A^{\frac{1}{\theta}}_{\frac{\alpha+1}{\theta}-2}$. □

## 4.6 Derivatives, anti-derivatives and integration operators

### 4.6.1 Spaces of derivatives and anti-derivatives

Let $X$ be a conformally invariant space of index $\alpha>0$. We are interested in the spaces

$$D(X)=\{f':\ f\in X\},\quad A(X)=\{f\in Hol(\mathbb{D}):\ f'\in X\}.$$

They are endowed with the norms

$$\|g\|_{D(X)}=\|G\|_X,\ G(z)=\int_0^z g(w)\,dw,\quad \|g\|_{A(X)}=|g(0)|+\|g'\|_X.$$

The norm on $D(X)$ is actually equivalent to the standard one

$$\|g\|_{D(X),1}=\inf_{c\in\mathbb{C}}\|G+c\|_X,$$

where $g, G$ are related as above. The following result shows some properties about these spaces related to the conformal invariant property.

**Proposition 4.7.** *Let $X$ be a conformally invariant space of index $\alpha > 0$. Then the spaces $D(X)$, $A(X)$ satisfy 1), 2) and $D(A(X)) = A(D(X)) = X$. Moreover,*

*(i) $X^{min}_{\alpha+1} \subset D(X) \subset X^{max}_{\alpha+1} = \mathcal{A}^{-\alpha-1}$, and the inclusions are continuous.*

*(ii) For $\alpha > 1$, $X^{min}_{\alpha-1} \subset A(X) \subset X^{max}_{\alpha-1} = \mathcal{A}^{-\alpha+1}$, and the inclusions are continuous.*

*Proof.* First we will prove that $D(X)$, $A(X)$ satisfy 1), 2). We will only treat the case $D(X)$, for $A(X)$ is analogous. By definition, $D(X) \subset Hol(\mathbb{D})$. Now, it only remain to prove that the inclusion is continuous, but this is direct using the closed graph theorem, Theorem 1.1. If $\{f_n\}_{n\geq 1} \subset D(X)$ such that $f_n \to_{D(X)} f$ by definition we have $F_n \to_X F$ where $F_n(z) = \int_0^z f_n(w)\,dw$ and $F(z) = \int_0^z f(w)\,dw$. Thus, since $X$ satisfies 1), $F_n$ converges to $F$ uniformly on compact subsets of $\mathbb{D}$, so $f_n$ converges to $f$ uniformly on compact subsets of $\mathbb{D}$. To prove that $D(X)$ and $A(X)$ satisfy 2) it is enough to observe that $X$ satisfies 2). Then given $f \in Hol(\rho\mathbb{D})$ with $\rho > 1$, then $f'$ and $F$ are in $Hol(\rho\mathbb{D})$ where $F$ is as above. Therefore, applying Corollary 1.1 we obtain the result.

- $D(A(X)) = X$. Given $f \in D(A(X))$, then if $F(z) = \int_0^z f(w)\,dw$ we have
$$\|f\|_{D(A(X))} = \|F\|_{A(X)} = \|F'\|_X = \|f\|_X.$$
- $A(D(X)) = X$. Given $f \in D(A(X))$, using that $X$ satisfies 1)
$$\|f\|_{A(D(X))} = |f(0)| + \|f'\|_{D(X)} = |f(0)| + \|f - f(0)\|_X \sim \|f\|_X.$$

(i) First, to prove $X^{min}_{\alpha+1} \subset D(X)$ it suffices to prove $\sup_{\varphi\in Aut(\mathbb{D})} \| (\varphi')^{\alpha+1} \|_{D(X)} < \infty$; see Section 4.4.1. Given $\varphi(z) = \lambda\varphi_a(z) = \lambda\frac{z-a}{1-\bar{a}z}$, we consider

$$\begin{aligned} F(z) &= \int_0^z (\varphi')^{\alpha+1}(w)\,dw = \lambda^{\alpha+1}(1-|a|^2)^{\alpha+1}\int_0^z \left(\frac{1}{1-\bar{a}w}\right)^{2\alpha+2} dw \\ &= \frac{-\lambda^{\alpha+1}}{2\alpha+1}\left[\frac{\frac{(1-|a|^2)^{\alpha+1}}{(1-\bar{a}z)^{2\alpha+1}} - (1-|a|^2)^{\alpha+1}}{\bar{a}}\right] = \frac{-\lambda^{\alpha+1}}{2\alpha+1}\left[\frac{(\varphi_a')^{\alpha}\frac{1-|a|^2}{1-\bar{a}z} - (1-|a|^2)^{\alpha+1}}{\bar{a}}\right]. \end{aligned}$$

Thus, given $\delta \in (0,1)$ fixed, if $|a| \geq \delta$ using that $X$ satisfies $3)$ we obtain

$$\begin{aligned} \|(\varphi')^{\alpha+1}\|_{D(X)} = \|F\|_X &= C_\alpha \left\| \frac{(\varphi_a')^{\alpha}\frac{1-|a|^2}{1-\bar{a}\zeta} - (1-|a|^2)^{\alpha+1}}{\bar{a}} \right\|_X \\ &\leq \frac{C_\alpha}{\delta}\|W^{\alpha}_{\varphi_a}(1+\bar{a}\zeta)\|_X + \frac{C_\alpha}{\delta}\|1\|_X \\ &\leq \frac{C_\alpha K}{\delta}(\|1\|_X + \|\zeta\|_X) + \frac{C_\alpha}{\delta}\|1\|_X. \end{aligned}$$

Now, if $|a| < \delta$ the problem appears when $a$ tends to $0$, but if we consider the function

$$G_a(z) = \frac{\frac{1}{(1-\bar{a}z)^{2\alpha+1}} - 1}{\bar{a}}$$

which is in $Hol(\frac{1}{\delta}\mathbb{D})$, in view its Taylor series, $G_a$ tends to $(2\alpha+1)\zeta$ when $|a| \to 0$ in $Hol(\frac{1}{\delta}\mathbb{D})$ and by 2) we also have convergence in $X$. Therefore, there exists $M > 0$ such that if $\delta$ is small enough we have

$$\|(\varphi')^{\alpha+1}\|_{D(X)} = \frac{(1-|a|^2)^{\alpha+1}}{2\alpha+1}\|G_a\|_X < M,$$

for all $|a| < \delta$.

On the other hand, if $f \in D(X)$ and we define as above $F(z) = \int_0^z f(w)\,dw$, then $F \in X$, so by Theorem 4.4 $F \in \mathcal{A}^{-\alpha}$ and, by Theorem 1.3, we obtain that $f \in \mathcal{A}^{-\alpha-1}$.

(ii) Given $\alpha > 1$, as above, to prove that $X_{\alpha-1}^{min} \subset A(X)$ we only need to prove that the condition $\sup_{\varphi\in Aut(\mathbb{D})} \| (\varphi')^{\alpha-1} \|_{A(X)} < \infty$ is satisfied. Given $\varphi(z) = \lambda\varphi_a(z) = \lambda\frac{z-a}{1-\bar{a}z}$ we have

$$\begin{aligned}\| (\varphi')^{\alpha-1} \|_{A(X)} &= |\varphi'(0)|^{\alpha-1} + (\alpha-1)\|(\varphi')^{\alpha-2}\varphi''\|_X \\ &\leq 1 + C \quad W_{\varphi_a}^{\alpha}\frac{1}{1+\bar{a}\zeta} \quad {}_X .\end{aligned}$$

Now, using that $X$ satisfies 3), Lemma 4.3 and Theorem 1.6, we obtain

$$\begin{aligned} W_{\varphi_a}^{\alpha}\frac{1}{1+\bar{a}\zeta} \quad {}_X \qquad \frac{1}{1+\bar{a}\zeta} \quad {}_X \qquad \frac{1}{1+\bar{a}\zeta} \quad {}_{A^1_{\alpha-2}} \\ = \quad {}_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-2}}{|1+\bar{a}z|} dA(z) < \infty. \end{aligned}$$

Thus, $X_{\alpha-1}^{min} \subset A(X)$. On the other hand if $f \in A(X)$ then $f' \in X$, so by Theorem 4.4 $f' \in \mathcal{A}^{-\alpha}$ and, by Theorem 1.3, we obtain $f \in \mathcal{A}^{-\alpha+1}$ and we conclude the proof.

□

The purpose of this section is to investigate the conformal invariance of $D(X)$ and $A(X)$. For the standard examples we have (see Theorem 1.3 and Theorem 1.7)

$$D(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma-1}, \quad D(A^p_\beta) = A^p_{\beta+p}, \; p \geq 1, \beta > -1,$$

and

$$A(\mathcal{A}^{-\gamma}) = \mathcal{A}^{-\gamma+1}, \; \gamma > 1, \quad A(A^p_\beta) = A^p_{\beta-p}, \; p \geq 1, \beta > p-1,$$

or $A(A^p_\beta) = B^{p,\beta}$, $p \geq 1, \beta \leq p-1$. In other words (assuming for the moment the assertions in Example 4.3), if $X$ is any of the spaces listed above and $\alpha > 0$ is its index of conformal invariance, then $D(X)$ is conformally invariant of index $\alpha+1$ and when $\alpha > 1$, $A(X)$ is conformally invariant of index $\alpha-1$.

It turns out that the result continues to hold for many other conformally invariant spaces. Surprisingly enough, this property is closely related to the behaviour on the spaces in question of the modified Cesàro $\mathcal{C}$, operator, formally defined by

$$\mathcal{C}f(z) = \; {}_0^z \frac{f(w)}{1-w}dw, \quad f \in Hol(\mathbb{D}).$$

Our result is as follows.

**Theorem 4.7.** *Let $X$ be a conformally invariant space of index $\alpha > 0$, such that polynomials are dense in $X$.*

*(i) $D(X)$ is conformally invariant of index $\beta > 0$, if and only if $\beta = \alpha + 1$ and $\mathcal{C} \in \mathcal{B}(X)$.*

*(ii) Assume that $\mathcal{C} \in \mathcal{B}(X)$. Then $A(X)$ is conformally invariant of index $\beta > 0$, if and only if $\alpha > 1$, $\beta = \alpha - 1$, and $I_X - \mathcal{C}$ is invertible.*

Note that part (ii) implies the assertions in Example 4.3. Indeed, it is known that $\mathcal{C}$ is bounded on $A^p_\beta$, $p \geq 1, \beta > -1$ (use for example Theorem 1.7), and its resolvent set consists of points $\lambda \in \mathbb{C} \setminus \{0\}$, such that $(1-\zeta)^{-\frac{1}{\lambda}} \in A^p_\beta$ (see [3], Theorem 5.2). In particular, $I - \mathcal{C}$ is invertible on $A^p_\beta$ if and only if $\beta + 2 > p$. In this case, by part (ii) of the above theorem $B^{p,\beta}$ is conformally invariant with index $\frac{\beta+2}{p} - 1$ (see Theorem 1.6).

Our argument involves two families of linear operators formally defined for $a \in \mathbb{D}$ by

$$\mathcal{C}_a f(z) = \int_0^z \frac{f(w)}{1-aw} dw, \quad T_a f(z) = \frac{1}{1-az} \int_0^z f(w) dw, \quad z \in \mathbb{D}, f \in Hol(\mathbb{D}). \tag{4.27}$$

Their relation to the modified Cesàro operator $\mathcal{C}$ is explained in the next two lemmas.

**Lemma 4.9.** *Let $\sigma \in \{0, 1\}$ and let $X$ be conformally invariant of index $\alpha > \sigma$, such that polynomials are dense in $X$. For $f \in Hol(\mathbb{D})$ and $a \in \mathbb{D}$ let*

$$T^\sigma f(z) = (1-z)^{-\sigma} \int_0^z \frac{f(w)}{(1-w)^{1-\sigma}} dw, \quad T^\sigma_a f(z) = (1-az)^{-\sigma} \int_0^z \frac{f(w)}{(1-aw)^{1-\sigma}} dw.$$

*Then the following are equivalent:*

*i) $T^\sigma \in \mathcal{B}(X)$.*

*ii) $T^\sigma_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ and $\sup_{a \in \mathbb{D}} \|T^\sigma_a\|_{\mathcal{B}(X)} < \infty$.*

*iii) There exists $\delta \in (0,1)$ such that $T^\sigma_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ with $\delta \leq |a| < 1$ and $\sup_{\delta \leq |a| < 1} \|T^\sigma_a\|_{\mathcal{B}(X)} < \infty$.*

*Proof.* $\boxed{\text{i)} \Rightarrow \text{ii) :}}$ For every $f \in Hol(\mathbb{D})$ and $a \in \mathbb{D}$ we have

$$T^\sigma_a f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a(e^{it}) e^{-it} R_t T^\sigma R_{-t} f(z)\, dt, \tag{4.28}$$

where $P_a(e^{it}) = \frac{1-|a|^2}{|a-e^{it}|^2}$ is the Poisson kernel at $a$; see Section 1.3. To see this, first given $z \in \mathbb{D}$, after a change of variable we get

$$T^\sigma R_{-t} f(z) = (1-z)^{-\sigma} \int_0^z \frac{R_t f(w)}{(1-w)^{1-\sigma}} dw = (1-z)^{-\sigma} \int_0^{e^{-it}z} \frac{f(\mu) e^{it}}{(1-e^{it}\mu)^{1-\sigma}} d\mu,$$

so

$$e^{-it} R_t T^\sigma R_{-t} f(z) = (1-e^{it}z)^{-\sigma} \int_0^z \frac{f(\mu)}{(1-e^{it}\mu)^{1-\sigma}} d\mu.$$

Thus, we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})e^{-it}R_tT^\sigma R_{-t}f(z)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})(1-e^{it}z)^{-\sigma}\int_0^z \frac{f(\mu)}{(1-e^{it}\mu)^{1-\sigma}}\,d\mu\,dt\\ &= \int_0^z f(\mu)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})\frac{(1-e^{it}z)^{-\sigma}}{(1-e^{it}\mu)^{1-\sigma}}\,dt\right)d\mu\\ &= \int_0^z f(\mu)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})g(e^{it})\,dt\right)d\mu,\end{aligned}$$

where, the interchange of integrals is justified by Fubini's theorem, since for fixed $z \in \mathbb{D}$

$$\int_{-\pi}^{\pi}\int_0^z |f(\mu)|\frac{|1-e^{it}z|^{-\sigma}}{|1-e^{it}\mu|^{1-\sigma}}\,d\mu\,dt < \infty,$$

and fixed $z, \mu \in \mathbb{D}$ $g(b) = \frac{(1-bz)^{-\sigma}}{(1-b\mu)^{1-\sigma}}$ is an analytic function in $\mathbb{D}$. Therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_a(e^{it})e^{-it}R_tT^\sigma R_{-t}f(z)\,dt &= \int_0^z f(\mu)g(a)\,d\mu\\ &= (1-az)^{-\sigma}\int_0^z \frac{f(\mu)}{(1-a\mu)^{1-\sigma}}\,d\mu\\ &= T_a^\sigma f(z).\end{aligned}$$

If $T^\sigma \in \mathcal{B}(X)$, then, since the polynomials are dense by Theorem 4.1, we can deduce that $t \to P_a(e^{it})e^{-it}R_tT^\sigma R_{-t}$ is strongly continuous on $[-\pi,\pi]$. To see this, let $f \in X$ and fixing $t_1 \in [-\pi,\pi]$ we have to show that

$$\lim_{t\to t_1}\|P_a(e^{it})e^{-it}R_tT^\sigma R_{-t}f - P_a(e^{it_1})e^{-it_1}R_{t_1}T^\sigma R_{-t_1}f\|_X = 0.$$

First, if we consider $G_t = R_tT^\sigma R_{-t}$ and $G_t^1 = e^{-it}R_tT^\sigma R_{-t}$, which are in $\mathcal{B}(X)$ and

$$\|G_t^1\|_{\mathcal{B}(X)} = \|G_t\|_{\mathcal{B}(X)} \le \left(\sup_{t\in[-\pi,\pi]}\|R_t\|_{\mathcal{B}(X)}\right)^2\|T^\sigma\|_{\mathcal{B}(X)},$$

we have

$$\begin{aligned}\|P_a(e^{it})G_t^1f - P_a(e^{it_1})G_{t_1}^1f\|_X &= (1-|a|^2)\left\|G_t^1f\frac{1}{|e^{it}-a|^2} - G_{t_1}^1f\frac{1}{|e^{it_1}-a|^2}\right\|_X\\ \le (1-|a|^2)\frac{1}{|e^{it}-a|^2}\left\|G_t^1f - G_{t_1}^1f\right\|_X &+ (1-|a|^2)\left\|G_{t_1}^1f\left(\frac{1}{|e^{it}-a|^2} - \frac{1}{|e^{it_1}-a|^2}\right)\right\|_X.\end{aligned}$$

Since $G_{t_1}^1 \in \mathcal{B}(X)$, the second part tends to $0$ when $t \to t_1$ (for more details see the proof of Theorem 4.1), so we only have to prove that $\left\|G_t^1f - G_{t_1}^1f\right\|_X$ tends to $0$ when $t \to t_1$. But, by analogous calculations as above this is equivalent to prove

$\|G_t f - G_{t_1} f\|_X$ tends to $0$ when $t \to t_1$. Thus, if we consider $g = T^\sigma R_{-t_1} f$, which is in $X$ since $T^\sigma$ and $R_{-t_1}$ are in $\mathcal{B}(X)$, we have

$$\begin{aligned}\|G_t f - G_{t_1} f\|_X &= \|R_t T^\sigma R_{-t} f - R_{t_1} T^\sigma R_{-t_1} f\|_X \\ &= \|R_t T^\sigma (R_{-t} - R_{-t_1}) f - (R_{t_1} - R_t) T^\sigma R_{-t_1} f\|_X \\ &\leq \sup_{t\in[-\pi,\pi]} \|R_t\|_{\mathcal{B}(X)} \quad \|T^\sigma\|_{\mathcal{B}(X)} \|(R_{-t} - R_{-t_1}) f\|_X + \|(R_t - R_{t_1}) g\|_X.\end{aligned}$$

Moreover, by Theorem 4.1 $t \to R_t$ is strongly continuous in $[-\pi, \pi]$. Hence, we have that $\|(R_{-t} - R_{-t_1}) f\|_X$ and $\|(R_t - R_{t_1}) g\|_X$ tends to $0$ when $t \to t_1$. Therefore $t \to P_a(e^{it}) e^{-it} R_t T^\sigma R_{-t}$ is strongly continuous on $[-\pi, \pi]$. Hence for $f \in X$ the right hand side of (4.28) becomes a Bochner integral, see Section 2.7. Thus

$$T_a^\sigma f = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a(e^{it}) e^{-it} R_t T^\sigma R_{-t} f \, dt, \quad f \in X,\, a \in \mathbb{D},$$

and

$$\|T_a^\sigma f\|_X \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_a(e^{it}) \|R_t T^\sigma R_{-t} f\|_X \, dt \leq (\sup_{t\in[-\pi,\pi]} \|R_t\|_{\mathcal{B}(X)})^2 \|T^\sigma\|_{\mathcal{B}(X)} \|f\|_X,$$

for all $f \in X$ and $a \in \mathbb{D}$.

*ii) ⇒ iii) :* It is trivial.

*iii) ⇒ i) :* Assume that there exists $\delta \in (0,1)$ such that $\{T_a^\sigma : \ \delta \leq |a| < 1\}$ is bounded in $\mathcal{B}(X)$. The boundedness of $T^\sigma$ will follow directly from the closed graph theorem once we show that $T^\sigma f \in X$, whenever $f \in X$. To this end, we verify that $\mathcal{D}_{T^\sigma} = \{f \in X : \ T^\sigma f \in X\}$, is both closed and dense in $X$.
To prove that $\mathcal{D}_{T^\sigma}$ is closed, we use the equality

$$(T^\sigma f)_r(z) = (1 - rz)^{-\sigma} \int_0^1 \frac{rz f(srz)}{(1 - srz)^{1-\sigma}} \, ds = r T_r^\sigma f_r(z). \tag{4.29}$$

Let $\{f_n\}_{n\geq 1}$ be a sequence in $\mathcal{D}_{T^\sigma}$, $f \in X$ with $\|f_n - f\|_X \to 0$. Given $\varepsilon > 0$, choose $n_\varepsilon \geq 1$, such that $\|f_n - f_m\|_X < \varepsilon,\ n, m > n_\varepsilon$. For such $m, n$ use Theorem 4.1 to find $r = r(m,n) \in (\delta, 1)$, with

$$\|T^\sigma f_n - (T^\sigma f_n)_r\|_X < \varepsilon, \quad \|T^\sigma f_m - (T^\sigma f_m)_r\|_X < \varepsilon.$$

By another application of Theorem 4.1, there exists $c > 0$, independent of $m, n, r$ such that $\|(f_n)_r - (f_m)_r\|_X \leq c\|f_n - f_m\|_X < c\varepsilon$. Thus, by (4.29)

$$\begin{aligned}\|T^\sigma f_n - T^\sigma f_m\|_X &= \|T^\sigma f_n - (T^\sigma f_n)_r + (T^\sigma f_n)_r - (T^\sigma f_m)_r + (T^\sigma f_m)_r - T^\sigma f_m\|_X \\ &< 2\varepsilon + \|(T^\sigma f_n)_r - (T^\sigma f_m)_r\|_X = 2\varepsilon + r\|T_r^\sigma (f_n)_r - T_r^\sigma (f_m)_r\|_X \\ &\leq 2\varepsilon + c \sup_{\rho\in(\delta,1)} \|T_\rho^\sigma\|_{\mathcal{B}(X)} \|f_n - f_m\|_X < (2 + c \sup_{\rho\in(\delta,1)} \|T_\rho^\sigma\|_{\mathcal{B}(X)})\varepsilon,\end{aligned}$$

i.e. $\{T^\sigma f_n\}_{n\geq 1}$ is a Cauchy sequence in $X$. Since $T^\sigma f_n(z) \to T^\sigma f(z),\ z \in \mathbb{D}$, we obtain that $f \in \mathcal{D}_{T^\sigma}$, that is, $\mathcal{D}_{T^\sigma}$ is closed.
To verify that $\mathcal{D}_{T^\sigma}$ is dense in $X$, set

$$\mathcal{D}_0 = \{f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) :\ f(1) = 0\},\ \mathcal{D}_1 = \left\{f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) :\ \int_0^1 f(w)\,dw = 0\right\},$$

and observe that if $f \in \mathcal{D}_\sigma$, then $T^\sigma f \in \cup_{\rho>1} Hol(\rho\mathbb{D}) \subset X$, i.e. $f \in \mathcal{D}_{T^\sigma}$. We claim that $\mathcal{D}_\sigma$ is a dense subspace of $X$. Indeed, if $l \in \mathcal{D}_\sigma^\perp$, and $g \in \cup_{\rho>1} Hol(\rho\mathbb{D})$, then if $\sigma = 0$,

$$l(g) = l(g(1) + g - g(1)) = l(1)g(1),$$

and similarly, if $\sigma = 1$,

$$l(g) = l\left(\int_0^1 g(w)dw + g - \int_0^1 g(w)dw\right) = l(1)\int_0^1 g(w)dw.$$

If $l(1) = 0$ we can see in both cases that the restriction of $l$ to the bounded set $\{(\varphi')^\alpha :\ \varphi \in Aut(\mathbb{D})\} \subset X$ is unbounded: if $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ we have, for $\sigma = 0$

$$\sup_{a\in\mathbb{D}} |l(\varphi_a')| = |l(1)| \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\alpha}{|1-\bar{a}|^{2\alpha}} \geq |l(1)| \sup_{r\in(0,1)} \frac{(1-r^2)^\alpha}{|1-r|^{2\alpha}} = \infty,$$

and for $\sigma = 1$ (recall that $\alpha > \sigma$)

$$\begin{aligned}\sup_{a\in\mathbb{D}} |l(\varphi_a')| &= |l(1)| \sup_{a\in\mathbb{D}} \left|\int_0^1 \frac{(1-|a|^2)^\alpha}{(1-\bar{a}w)^{2\alpha}}\,dw\right| = |l(1)| \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\alpha}{|a|(2\alpha-1)} \left|\frac{1}{(1-\bar{a})^{2\alpha-1}} - 1\right| \\ &\geq |l(1)| \sup_{r\in(\frac{1}{2},1)} \frac{(1-r^2)^\alpha}{r(2\alpha-1)} \left(\frac{1}{(1-r)^{2\alpha-1}} - 1\right) = \infty,\end{aligned}$$

which is a contradiction ($|l\left((\varphi')^\alpha\right)| \leq \|l\|_{X'} \|(\varphi')^\alpha\|_X$). Hence $l(1) = 0$ which implies that $l = 0$ since it annihilates all polynomials which are dense.
Finally, to see that the graph of $T^\sigma$ is closed, assume $\|f_n - f\|_X \to 0,\ \|T^\sigma f_n - g\|_X \to 0$, with $f_n, f, g \in X$. Then, since $T^\sigma f_n(z) \to T^\sigma f(z),\ z \in \mathbb{D}$ we conclude $T^\sigma f = g$.

□

**Lemma 4.10.** *Let $X$ be conformally invariant of index $\alpha > 1$ and such that polynomials are dense in $X$, and assume that $\mathcal{C} \in \mathcal{B}(X)$. Then $I_X - \mathcal{C}$ is invertible if and only if $T_a \in \mathcal{B}(X)$ for all $a \in \mathbb{D}$ and $\sup_{a\in\mathbb{D}} \|T_a\|_{\mathcal{B}(X)} < \infty$.*

*Proof.* With the notation in Lemma 4.9 we have $T_a = T_a^1,\ a \in \mathbb{D}$. We will prove the identity

$$T^1\mathcal{C}f = T^1 f - \mathcal{C}f = \mathcal{C}T^1 f, \quad f \in Hol(\mathbb{D}).$$

Using integration by parts we obtain

$$\begin{aligned}
T^1\mathcal{C}f(z) &= \frac{1}{1-z}\int_0^z \mathcal{C}f(w)\,dw = \frac{1}{1-z}\int_0^z\int_0^w \frac{f(\mu)}{(1-\mu)}\,d\mu\,dw \\
&= \frac{z}{1-z}\int_0^z \frac{f(w)}{1-w}\,dw - \frac{1}{1-z}\int_0^z \frac{wf(w)}{1-w}\,dw \\
&= \frac{1}{1-z}\int_0^z f(w)\,dw - \int_0^z \frac{f(w)}{1-w}\,dw \\
&= T^1 f(z) - \mathcal{C}f(z),
\end{aligned}$$

and

$$\begin{aligned}
T^1 f(z) - \mathcal{C}f(z) &= \frac{1}{1-z}\int_0^z f(w)\,dw - \int_0^z \frac{f(w)}{1-w}\,dw \\
&= \frac{1}{1-z}\int_0^z f(w)\,dw - \frac{1}{1-z}\int_0^z f(w)\,dw + \int_0^z \frac{1}{(1-w)^2}\int_0^w f(\mu)\,d\mu\,dw \\
&= \int_0^z \frac{\frac{1}{1-w}\int_0^w f(\mu)\,d\mu}{1-w}\,dw \\
&= \mathcal{C}T^1 f(z).
\end{aligned}$$

From these identities we deduce

$$(I_{Hol(\mathbb{D})} + T^1)(I_{Hol(\mathbb{D})} - \mathcal{C}) = (I_{Hol(\mathbb{D})} - \mathcal{C})(I_{Hol(\mathbb{D})} + T^1) = I_{Hol(\mathbb{D})}. \tag{4.30}$$

If $I_X - \mathcal{C}$ is invertible, then by (4.30) it follows that $(I_X - \mathcal{C})^{-1} = I_X + T^1$. In particular, $T^1 \in \mathcal{B}(X)$, and by Lemma 4.9, $\{T_a : a \in \mathbb{D}\}$ is bounded in $\mathcal{B}(X)$. Conversely, if $\{T_a : a \in \mathbb{D}\}$ is bounded in $\mathcal{B}(X)$, by Lemma 4.9 we have that $T^1 \in \mathcal{B}(X)$, and by (4.30) we obtain $I_X + T^1 = (I_X - \mathcal{C})^{-1}$. □

*Proof of Theorem 4.7.*
(i) Assume that $D(X)$ is conformally invariant of index $\beta > 0$. Then

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{D(X)} < \infty.$$

By Proposition 4.7, $D(X)$ is continuously contained in $\mathcal{A}^{-\alpha-1}$, hence

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{\mathcal{A}^{-\alpha-1}} < \infty,$$

which implies that $\beta \leq \alpha + 1$. On the other hand,

$$\sup_{\varphi \in Aut(\mathbb{D})} \|((\varphi')^\alpha)'\|_{D(X)} = \sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha - (\varphi')^\alpha(0)\|_X \leq \sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\|_X < \infty,$$

and, by Theorem 4.4, $D(X)$ is continuously contained in $\mathcal{A}^{-\beta}$, hence

$$\sup_{\varphi \in Aut(\mathbb{D})} \|((\varphi')^\alpha)'\|_{\mathcal{A}^{-\beta}} < \infty,$$

which implies $\alpha+1 \leq \beta$. To see this, if $\beta < \alpha+1$, we can consider $\varphi(z) = \lambda\varphi_a(z) = \lambda\frac{\overline{a}-z}{1-az}$, hence

$$\begin{aligned}
\sup_{\varphi\in Aut(\mathbb{D})} \|((\varphi')^\alpha)'\|_{\mathcal{A}^{-\beta}} &= \sup_{\varphi\in Aut(\mathbb{D})} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |((\varphi')^\alpha)'(z)| \\
&\geq \sup_{-1<a<1} \sup_{0<r<1} (1-r^2)^\beta |((\varphi_a')^\alpha)'(r)| \\
&\geq \sup_{0<r<1} (1-r^2)^\beta |((\varphi_r')^\alpha)'(r)| \\
&= \sup_{0<r<1} \frac{2\alpha r}{(1-r^2)^{\alpha+1-\beta}} \\
&= \infty.
\end{aligned}$$

Therefore, with both inequalities at hand we obtain $\beta = \alpha+1$.

For $\varphi = \lambda\varphi_a \in Aut(\mathbb{D})$, with $\varphi_a(z) = \frac{\overline{a}-z}{1-az}$, as above, and $f \in X$ we have

$$(W_\varphi^\alpha f)' = W_\varphi^{\alpha+1} f' + ((\varphi')^\alpha)' f\circ\varphi = W_\varphi^{\alpha+1} f' + \frac{((\varphi')^\alpha)'}{(\varphi')^\alpha} W_\varphi^\alpha f.$$

Now, using that $\frac{((\varphi')^\alpha)'(z)}{(\varphi')^\alpha(z)} = \frac{2a\alpha}{1-az}$ and $(\mathcal{C}_a W_\varphi^\alpha f)'(z) = \frac{W_\varphi^\alpha f(z)}{1-az}$ the above equality can be rewritten as

$$W_\varphi^{\alpha+1} f' = (W_\varphi^\alpha f)' - 2a\alpha(\mathcal{C}_a W_\varphi^\alpha f)'. \tag{4.31}$$

Replace $f \in X$ by $W_{\varphi^{-1}}^\alpha f$ (recall that $W_\varphi^\alpha$ is an invertible operator, see Proposition 4.1 b)) to obtain

$$W_\varphi^{\alpha+1}(W_{\varphi^{-1}}^\alpha f)' = f' - 2a\alpha(\mathcal{C}_a f)'.$$

Since $D(X)$ is conformally invariant of index $\alpha+1$,

$$\|W_\varphi^{\alpha+1}(W_{\varphi^{-1}}^\alpha f)'\|_{D(X)} \quad \|(W_{\varphi^{-1}}^\alpha f)'\|_{D(X)} = \|W_{\varphi^{-1}}^\alpha f - W_{\varphi^{-1}}^\alpha f(0)\|_X \quad \|f\|_X,$$

the left hand side stays bounded in $D(X)$ when $\varphi \in Aut(\mathbb{D})$ and $f \in X$ with $\|f\|_X \leq 1$. This implies that, setting $\delta \in (0,1)$, $\|(\mathcal{C}_a f)'\|_{D(X)}$ stays bounded when $\varphi, f$ are as above and $\delta \leq |a| < 1$. This can be seen as follows

$$\begin{aligned}
\|(C_a f)'\|_{D(X)} &= \frac{1}{2\alpha|a|} \|f' - W_\varphi^{\alpha+1}(W_{\varphi^{-1}}^\alpha f)'\|_{D(X)} \\
&\leq \frac{1}{2\alpha\delta} \quad \|f'\|_{D(X)} + \|W_\varphi^{\alpha+1}(W_{\varphi^{-1}}^\alpha f)'\|_{D(X)} \quad \|f\|_X.
\end{aligned}$$

Hence,

$$\|C_a f\|_X = \|(C_a f)'\|_{D(X)} \quad \|f\|_X$$

i.e. $\sup_{\delta\leq|a|<1} \|\mathcal{C}_a\|_{\mathcal{B}(X)} < \infty$. By Lemma 4.9 with $\sigma = 0$ we obtain $\mathcal{C} \in \mathcal{B}(X)$. Conversely, if $\mathcal{C}$ is bounded on $X$, use again Lemma 4.9 with $\sigma = 0$, to conclude that the second term on the right hand side of (4.31) stays bounded in $D(X)$ when $\varphi \in Aut(\mathbb{D})$ and $f \in X$, $\|f\| \leq 1$, which implies that $\|W_\varphi^{\alpha+1}\|_{\mathcal{B}(D(X))}$ stays bounded when $\varphi \in Aut(\mathbb{D})$.

(ii) Assume that $A(X)$ is conformally invariant of index $\beta > 0$ and that $\mathcal{C} \in \mathcal{B}(X)$. If $\alpha \leq 1$, by direct integration we will see that $A(X)$ is continuously contained in the growth class $\mathcal{A}^{\log}$ from Example 4.2. The proof of this fact is analogous to the proof of Theorem 1.3, but this theorem does not include these cases. Let $\alpha \leq 1$ and $f \in A(X)$, then $f' \in X$, hence by Theorem 4.4 $f' \in \mathcal{A}^{-\alpha}$, so for $z = re^{i\theta} \in \mathbb{D}$, using Proposition 4.7, we obtain

$$\begin{aligned} |f(z)| &= \left| \int_0^z f'(w)\, dw + f(0) \right| \leq \int_0^r |f'(\rho e^{i\theta})|\, d\rho + \|f\|_{A(X)} \\ &\leq \|f'\|_{\mathcal{A}^{-\alpha}} \int_0^r \frac{1}{(1-\rho)^\alpha}\, d\rho + \|f\|_{A(X)}. \end{aligned}$$

Thus, as $\alpha \leq 1$ we have $f \in \mathcal{A}^{\log}$. But then, we can verify that

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{A(X)} = \infty,$$

which is a contradiction, see Example 4.2. Thus, $\alpha > 1$. The proof of the equality $\beta = \alpha - 1$ is analogous to the corresponding argument in the proof of (i). Assume that $A(X)$ is conformally invariant of index $\beta > 0$. Then by Proposition 4.7

$$\sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{\mathcal{A}^{-\alpha-1}} \leq \sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\beta\|_{A(X)} < \infty,$$

which implies that $\beta \leq \alpha - 1$. On the other hand, if $\Phi_\varphi(z) = \int_0^z (\varphi')^\alpha(w)\, dw$, we have

$$\sup_{\varphi \in Aut(\mathbb{D})} \|\Phi_\varphi\|_{A(X)} = \sup_{\varphi \in Aut(\mathbb{D})} \|(\varphi')^\alpha\|_X < \infty,$$

and, by Theorem 4.4, $A(X)$ is continuously contained in $\mathcal{A}^{-\beta}$, hence

$$\sup_{\varphi \in Aut(\mathbb{D})} \|\Phi_\varphi\|_{\mathcal{A}^{-\beta}} < \infty,$$

which implies $\alpha - 1 \leq \beta$. To see this, if $\beta < \alpha - 1$, we can consider $\varphi(z) = \lambda \varphi_a(z) = \lambda \frac{\overline{a} - z}{1 - az}$, hence

$$\begin{aligned} \sup_{\varphi \in Aut(\mathbb{D})} \|\Phi_\varphi\|_{\mathcal{A}^{-\beta}} &= \sup_{\varphi \in Aut(\mathbb{D})} \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\Phi_\varphi(z)| \\ &\geq \sup_{-1 < a < 1} \sup_{0 < r < 1} (1 - r^2)^\beta |\Phi_{\varphi_a}(r)| \\ &\geq \sup_{\frac{1}{2} < r < 1} (1 - r^2)^\beta |\Phi_{\varphi_r}(r)| \\ &\geq \frac{1}{2\alpha - 1} \sup_{\frac{1}{2} < r < 1} (1 - r^2)^{\beta + \alpha} \left( \frac{1}{(1 - r^2)^{2\alpha - 1}} - 1 \right) \\ &= \infty. \end{aligned}$$

Therefore, from these inequalities we obtain $\beta = \alpha - 1$.

The remaining part of the proof is similar to the above as well. For a function $f \in A(X)$, $\varphi = \lambda\varphi_a \in Aut(\mathbb{D})$, with $\varphi_a(z) = \frac{\overline{a}-z}{1-az}$ and $\varphi^{-1} = \mu\varphi_b$, we write

$$(W_\varphi^{\alpha-1} f)' = W_\varphi^\alpha f' + ((\varphi')^{\alpha-1})'[f \circ \varphi - f(0)] + f(0)[(\varphi')^{\alpha-1}]'.$$

Using that $\varphi' \circ \varphi^{-1} = \frac{1}{(\varphi^{-1})'}$, $\varphi'' \circ \varphi^{-1} = \frac{-(\varphi^{-1})''}{[(\varphi^{-1})']^3}$ and $\frac{(\varphi^{-1})''(z)}{(\varphi^{-1})'(z)} = \frac{2b}{1-bz}$ we obtain

$$\begin{aligned} W_{\varphi^{-1}}^\alpha \ ((\varphi')^{\alpha-1})'(f\circ\varphi - f(0)) &= (\alpha-1)[(\varphi^{-1})']^\alpha(\varphi'\circ\varphi^{-1})^{\alpha-2}(\varphi''\circ\varphi^{-1})(f-f(0)) \\ &= -(\alpha-1)\frac{(\varphi^{-1})''}{(\varphi^{-1})'}(f-f(0)) \\ &= -2b(\alpha-1)\frac{f-f(0)}{1-b\zeta} \\ &= -2b(\alpha-1)T_b f'. \end{aligned}$$

Hence, using $W_\varphi^\alpha W_{\varphi^{-1}}^\alpha f = f$, for all $f \in A(X)$, the above equality becomes

$$(W_\varphi^{\alpha-1} f)' = W_\varphi^\alpha f' - 2b(\alpha-1)W_\varphi^\alpha T_b f' + f(0)((\varphi')^{\alpha-1})'. \tag{4.32}$$

Let $f(0) = 0$ and apply $W_{\varphi^{-1}}^\alpha$ on both sides to obtain

$$W_{\varphi^{-1}}^\alpha (W_\varphi^{\alpha-1} f)' = f' - 2b(\alpha-1)T_b f'.$$

By assumption $A(X)$ is conformally invariant of index $\alpha - 1$ so

$$\|W_{\varphi^{-1}}^\alpha (W_\varphi^{\alpha-1} f)'\|_X \quad \|(W_\varphi^{\alpha-1} f)'\|_X = \|W_\varphi^{\alpha-1} f\|_{A(X)} \quad \|f\|_{A(X)}.$$

Hence, the left hand side stays bounded in $X$ when $\varphi \in Aut(\mathbb{D})$, and $\|f\|_{A(X)} \leq 1$, $f(0) = 0$, and so does the first term on the right. Since the condition on $f$ is equivalent to $\|f'\|_X \leq 1$ it follows that setting $\delta \in (0,1)$, $T_b \in \mathcal{B}(X)$, $\delta \leq |b| < 1$. To see this claim, let $g \in X$ and $G(z) = \int_0^z g(w)\,dw$, then $G \in A(X)$ with $G(0) = 0$ and $\|G\|_{A(X)} = \|g\|_X$. Thus

$$\begin{aligned} \|T_b g\|_X = \|T_b G'\|_X &= \frac{1}{2b(\alpha-1)}\|G' - W_{\varphi^{-1}}^\alpha (W_\varphi^{\alpha-1} G)'\|_X \\ &\leq \frac{1}{2\delta(\alpha-1)} \quad \|G\|_{A(X)} + \|W_{\varphi^{-1}}^\alpha (W_\varphi^{\alpha-1} G')'\|_X \\ &\quad \|g\|_X. \end{aligned}$$

Therefore, $\sup_{\delta\leq|b|<1}\|T_b\|_{\mathcal{B}(X)} < \infty$. Thus by Lemma 4.9 with $\sigma = 1$ and Lemma 4.10, $I_X - \mathcal{C}$ is invertible on $X$. To see the converse, first we can use Proposition 4.7 (ii) to conclude that $|W_\varphi^{\alpha-1} f(0)|$ and $|f(0)|\|((\varphi')^{\alpha-1})'\|_X$ stay bounded when $\varphi \in Aut(\mathbb{D})$ and $\|f\|_{A(X)} \leq 1$ since

$$|W_\varphi^{\alpha-1} f(0)| = (1 - |\varphi(0)|^2)^{\alpha-1}|f(\varphi(0))| \leq \|f\|_{\mathcal{A}^{-\alpha+1}} \quad \|f\|_{A(X)},$$

and

$$|f(0)|\|((\varphi')^{\alpha-1})'\|_X \leq \|f\|_{A(X)}\|(\varphi')^{\alpha-1}\|_{A(X)} \quad \|f\|_{A(X)}\|(\varphi')^{\alpha-1}\|_{X^{min}_{\alpha-1}} \quad \|f\|_{A(X)}.$$

If $\mathcal{C} \in \mathcal{B}(X)$, and $I_X - \mathcal{C}$ is invertible, then by Lemma 4.10 we have $\sup_{a\in\mathbb{D}} \|T_a\|_{\mathcal{B}(X)} < \infty$ and we conclude that the right hand side of (4.32) stays bounded in $X$ when $\varphi \in Aut(\mathbb{D})$ and $\|f\|_{A(X)} \leq 1$, this can be seen as following

$$\begin{aligned}
\|(W_\varphi^{\alpha-1}f)'\|_X &= \|W_\varphi^\alpha f' - 2b(\alpha-1)W_\varphi^\alpha T_b f' + f(0)((\varphi')^{\alpha-1})'\|_X \\
&\leq \|W_\varphi^\alpha f'\|_X + 2(\alpha-1)\|W_\varphi^\alpha T_b f'\|_X + |f(0)|\|((\varphi')^{\alpha-1})'\|_X \\
&\quad \|f'\|_X + 2(\alpha-1)\|T_b f'\|_X + \|f\|_{A(X)} \\
&\quad \|f\|_{A(X)}.
\end{aligned}$$

Thus for $f \in A(X)$

$$\|W_\varphi^{\alpha-1}f\|_{A(X)} = |W_\varphi^{\alpha-1}f(0)| + \|(W_\varphi^{\alpha-1}f)'\|_X \quad \|f\|_{A(X)}.$$

Hence $W_\varphi^{\alpha-1} \in \mathcal{B}(A(X))$ and $\sup_{\varphi\in Aut(\mathbb{D})} \|W_\varphi^{\alpha-1}\|_{\mathcal{B}(A(X))} < \infty$, which completes the proof. □

### 4.6.2 Integration operators

We shall apply our results to investigate a class of integration operators containing the modified Cesàro operator from the previous paragraph. These operators are formally defined by

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \quad f \in Hol(\mathbb{D}),$$

where the *symbol* $g \in Hol(\mathbb{D})$ is fixed. There is a vast literature on the subject concerning boundedness and compactness of such operators, for instance in [7] it is proved that for a large class of weights, $T_g$ is bounded on $A^p_w$ if and only if $g$ is in the Bloch space. More recently, even their spectral properties have been studied (see for example, [1], [3] and the references therein). Here we shall only discuss boundedness in the general context of conformal invariant Banach spaces of analytic functions. We start with a simple result whose statement is self-explanatory. However, in many cases it turns out to be an important observation related to the characterization of the symbols $g$ which generate bounded operators $T_g$. We shall use the notations from the previous paragraph for arbitrary Banach spaces satisfying 1) and 2).

**Proposition 4.8.** *Let $X, Y$ be Banach spaces satisfying 1) and 2). Then $T_g : X \to Y$ is bounded if and only if $g' \in Mult(X, D(Y))$, and the norms $\|T_g\|_{\mathcal{B}(X,Y)}$, $\|g'\|_{Mult(X,D(Y))}$ are equal.*

*Proof.* By the closed graph theorem $T_g \in \mathcal{B}(X, Y)$ if and only if $T_g f \in Y$ whenever $f \in X$, or equivalently, $g'f \in D(Y)$ whenever $f \in X$, in fact

$$\|T_g\|_{\mathcal{B}(X,Y)} = \sup_{\|f\|_X\leq 1} \|T_g f\|_Y = \sup_{\|f\|_X\leq 1} \|g'f\|_{D(Y)} = \|g'\|_{Mult(X,D(Y))}.$$

□

**Remark 4.5.** If $X, D(Y)$ are conformally invariant of indices $\alpha > 0$, respectively $\beta > \alpha$ and $T_g \in \mathcal{B}(X,Y)$, then by Proposition 4.2 it follows that the integration operators generated by $g_\varphi = \int_0^z W_\varphi^{\beta-\alpha} g'(w)dw,\ \varphi \in Aut(\mathbb{D})$ are uniformly bounded in $\mathcal{B}(X,Y)$

$$\|T_{g_\varphi}\|_{\mathcal{B}(X,Y)} = \|g'_\varphi\|_{Mult(X,D(Y))} = \|W_\varphi^{\beta-\alpha} g'\|_{Mult(X,D(Y))}$$
$$\|g'\|_{Mult(X,D(Y))} = \|T_g\|_{\mathcal{B}(X,Y)}.$$

We shall be concerned with the case when $X = Y$. The following result provides a necessary condition for boundedness of such operators.

**Corollary 4.2.** *Let $X$ be conformally invariant of index $\alpha > 0$ such that polynomials are dense in $X$. Assume also that $\mathcal{C} \in \mathcal{B}(X)$. If $T_g \in \mathcal{B}(X)$, then there exist $c, \delta > 0$ such that for all $\lambda \in \mathbb{C},\ |\lambda| \le \delta$, and all $\varphi \in Aut(\mathbb{D})$, $\exp(\lambda(g\circ\varphi - g\circ\varphi(0))) \in X$, with*

$$\|\exp(\lambda(g\circ\varphi - g\circ\varphi(0)))\|_X \le c.$$

*Proof.* By Theorem 4.7, $D(X)$ is conformally invariant of index $\alpha+1$. Then Remark 4.5 applies and we obtain that the family $\{T_{g_\varphi}:\ \varphi \in Aut(\mathbb{D})\}$ is bounded in $\mathcal{B}(X)$, where

$$T_{g_\varphi} f(z) = \int_0^z f(w) W_\varphi^1 g'(w)dw = \int_0^z f(w)(g\circ\varphi)'(w)dw, \quad z \in \mathbb{D},\ f \in X.$$

Choose $\delta > 0$ such that

$$\delta \|T_{g_\varphi}\|_{\mathcal{B}(X)} < \frac{1}{2}, \quad \varphi \in Aut(\mathbb{D}).$$

Differentiating and solving an ordinary linear differential equation of first order we obtain that the unique solution $f_\lambda$ of

$$f - \lambda T_{g_\varphi} f = 1,$$

is given by $f_\lambda = \exp(\lambda(g\circ\varphi - g\circ\varphi(0)))$. Now for $|\lambda| < \delta$, $I_X - \lambda T_{g_\varphi}$ is invertible (see Proposition 2.1) with $\|(I_X - \lambda T_{g_\varphi})^{-1}\|_{\mathcal{B}(X)} < 2$, so that,

$$\|f_\lambda\|_X = \|(I_X - \lambda T_{g_\varphi})^{-1} 1\|_X < 2\|1\|_X,$$

and the result follows. □

The idea of exponentiating via the resolvents of $T_g$ is due to Pommerenke [73]. When $X = H^2$, one can use it to prove the John-Nirenberg inequality for $BMO$ (see [49]). In the general context considered here, necessary condition for boundedness of $T_g$ provided by Corollary 4.2 is probably not sufficient. It would be a problem of interest (but probably difficult) to find a condition that is necessary and sufficient for boundedness of $T_g$ in this context.

# 5 Invertible and isometric weighted composition operators

## 5.1 Introduction

In this chapter, we focus on abstract Banach spaces of analytic functions on general bounded domains that satisfy certain axioms. The chapter is based on the reference [68]. Our goal is to study the properties of weighted composition operators in a unified way by working with general Banach spaces of analytic functions that satisfy only a handful of axioms while still obtaining that same conclusions as in the known special situations, thus covering many cases in one stroke. Recall that a weighted composition operator (WCO) $W_{F,\phi}$ is defined formally by the formula

$$W_{F,\phi}f = F(f \circ \phi) = M_F C_\phi f.$$

See Section 2.5. We should note several papers that have guided by a similar philosophy, for example, [13], [23], [35], [57].

The first step towards understanding the spectrum of an operator consists in understanding its invertibility. Since a non-trivial weighted composition operator (one for which $\phi \not\equiv$ const and $F \not\equiv 0$) is injective, as was noted in Section 2.3, it is an invertible operator if and only if it is surjective. Two general theorems regarding invertibility were proved in [13], assuming different sets of five axioms that the space should satisfy. The main result of this chapter, Theorem 5.1, proves a more general result under slightly modified axioms that seem easier to verify than those in [13].

Isometries of various spaces defined in terms of derivatives were characterized in [56]. This had been done earlier in [31] for the quotient Bloch space, identified with $\{f \in \mathcal{B} : f(0) = 0\}$, and all isometries turn out to be WCO. However, in the true Bloch space $\mathcal{B}$ there are more isometries so it is still a question of interest to characterize all isometric WCO on $\mathcal{B}$. In the special case when $W_{F,\phi}$ is a composition operator ($F \equiv 1$) the isometries were characterized in two different ways in [34] and [67]. The case of isometric multipliers ($\phi(z) = z$) is simpler and was covered in [4]. Theorem 5.2 bellow characterizes the surjective isometries among the WCOs on a rather general class of functional Banach spaces with a translation-invariant seminorm. The results include various cases that, to the best of our knowledge, were not covered before in the literature.

## 5.2 Surjective WCOs on functional Banach spaces

### 5.2.1 A set of axioms

The axioms considered in this chapter will be slightly changed with respect to [13] in order to include some new spaces, notably the Bloch space, $BMOA$, and Korenblum-type spaces. For instance, in [13] one axiom considered in the conditions of Theorem 5 is the density of the polynomials that here will be not assumed. This change is fundamental since the class of admissible spaces will be much wider, yet at the same time the proofs will be notably shorter.

Here we will consider general Banach spaces of analytic functions on a bounded domain $\Omega \subset \mathbb{C}$. We shall write $Hol(\Omega)$ for the algebra of functions analytic in $\Omega$. Of course, the case of main interest is $\Omega = \mathbb{D}$, the unit disc.

We will consider arbitrary Banach spaces $X \subset Hol(\Omega)$ that satisfy the following axioms:

- **A1:** All point evaluation functionals $l_z$ are bounded on $X$.
- **A2:** $1 \in X$, where $1(z) \equiv 1$.
- **A3:** Whenever $f \in X$, the function $\zeta f$ is also in $X$, where $\zeta(z) = z$.
- **A4:** For every $u \in H^\infty(\Omega)$ that does not vanish in $\Omega$ and every $f \in X$, if $fu^n \in X$ for all $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then $fu^\alpha \in X$ for some positive non-integer value $\alpha$.
- **A5:** Each automorphism of $\Omega$ induces a bounded composition operator in $X$.

The first axiom is essentially equivalent to the requirement that the space be Banach as it shows that convergence in norm implies uniform convergence on compact subsets of $\Omega$. Also, this allows the use of the closed graph theorem to show that Axiom **A3** implies that the shift operator is bounded on $X$.

Recall that a function $u$ is said to be a *pointwise multiplier* of $X$ if $uf \in X$ for all $f \in X$; we write $u \in Mult(X)$ to denote this. See Section 2.8. Thus, **A3** can be rephrased by saying that the identity function $\zeta \in Mult(X)$. Together with Axiom **A2**, this shows that the space $X$ contains the polynomials.

Axiom **A4** may not look so natural at first sight but can be explained as follows. Axiom **A1** implies that each pointwise multiplier is a bounded function analytic in $\Omega$. However, the converse is false in many spaces. For example, for the Bloch space $\mathcal{B}$ of the disc, it is well known that $u \in Mult(\mathcal{B})$ if and only if $u \in H^\infty$ and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)| < \infty\,. \tag{5.1}$$

This was proved in several papers by different authors around 1990; see, for example, [26]. The situation is more complicated in the Dirichlet space where other conditions have to be added to the boundedness assumption on $u$; we refer the reader to [87]. The spaces of the disc for which $Mult(X) = H^\infty$ have been characterized in [13] in terms of a domination property.

In view of the above, assuming the condition $Mult(X) = H^\infty(\Omega)$ (that is: every bounded analytic function in $\Omega$ is a multiplier of $X$ into itself) would not provide a remedy as this would not cover the Bloch or the Dirichlet space. The next step would be to think of a weaker condition, assuming that every bounded non-vanishing function could be "compressed" in order to be made into a multiplier:

- For every $u \in H^\infty(\Omega)$ that does not vanish in the disc, $fu^\alpha \in X$ for some non-integer value $\alpha > 0$.

However, we can check that again this property is not satisfied in the Bloch space. Indeed, let $v \in H^\infty$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\log\frac{1}{1-|z|^2}|v'(z)| = \infty,$$

*i.e.* $v \notin Mult(\mathcal{B})$. Then, there exists $f \in \mathcal{B}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(vf)'(z)| = \infty.$$

Thus, if we consider $u = v + 2\|v\|_{H^\infty}$, we observe that the function $u \in H^\infty$ does not vanish in the unit disc. Indeed,

$$3\|v\|_{H^\infty} \geq |u(z)| \geq 2\|v\|_{H^\infty} - |v(z)| \geq \ \|v\|_{H^\infty}, \quad z \in \mathbb{D}.$$

Therefore, for all $\alpha > 0$

$$\begin{aligned}
|(fu^\alpha)'(z)| = &\ \alpha u^{\alpha-1}(z)u'(z)f(z) + u^\alpha(z)f'(z) \\
\geq &\ \frac{1}{3}\|v\|_{H^\infty}^{\alpha-1}\,|\alpha v'(z)f(z) + v(z)f'(z) + 2\|v\|_{H^\infty}f'(z)| \\
= &\ \frac{1}{3}\|v\|_{H^\infty}^{\alpha-1}\,|\alpha v'(z)f(z) + \alpha v(z)f'(z) + (1-\alpha)v(z)f'(z) + 2\|v\|_{H^\infty}f'(z)| \\
\geq &\ \frac{1}{3}\|v\|_{H^\infty}^{\alpha-1}\,(\alpha|(fv)'(z)| - |\alpha-1||v(z)||f'(z)| - 2\|v\|_{H^\infty}|f'(z)|) \\
\geq &\ \frac{1}{3}\alpha\|v\|_{H^\infty}^{\alpha-1}\,|(fv)'(z)| - \frac{1}{3}(|\alpha-1|+2)\|v\|_{H^\infty}^{\alpha}|f'(z)|.
\end{aligned}$$

Thus,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(fu^\alpha)'(z)| = \infty.$$

Therefore, we have that $fu^\alpha$ does not belong to $\mathcal{B}$ for any $\alpha > 0$.

Nevertheless, the condition above can be weakened further:

- For every $u \in H^\infty(\Omega)$ that does not vanish in the disc and for every $f \in X$, if $fu \in X$ then $fu^\alpha \in X$ for some non-integer value $\alpha > 0$.

It turns out that this property is satisfied in $\mathcal{B}$ but it is not obvious how to verify it for the minimal analytic Besov space $B^1$. However, one can weaken the above requirement a little more, thus formulating our Axiom **A4**. It turns out that it is fulfilled in most "reasonable" spaces of the disc.

### 5.2.2 Spaces that satisfy our axioms

We first review a partial list of spaces that satisfy all of the above axioms:

- $H^\infty$, the (Hardy) space of all *bounded analytic functions* in $\mathbb{D}$, equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, and the *disc algebra* $\mathcal{A} = H^\infty \cap C(\overline{\mathbb{D}})$, its subspace with the same norm.

- The standard *Hardy spaces* $H^p$, $1 \le p < \infty$; see Section 1.4.

- The *weighted Bergman spaces* $A^p_\beta$, $1 \le p < \infty$, $-1 < \beta < \infty$; see Section 1.5.

- The general mixed norm spaces $H(p, q, \beta)$, $0 < p, q \le \infty$, $0 < \beta < \infty$; see Section 1.5.2.

- The *Korenblum spaces (growth spaces)* $\mathcal{A}^{-\gamma}$, $\gamma > 0$; see Section 1.9.

- The *weighted Bloch spaces* $\mathcal{B}^\beta$, $\beta > 0$, defined as the set of all $f \in Hol(\mathbb{D})$ such that
$$\|f\|_{\mathcal{B}^\beta} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\beta |f'(z)| < \infty.$$
Of course, this scale includes the standard *Bloch space* (see Section 1.8), obtained for the value $\beta = 1$.

- The *logarithmic Bloch spaces* $\mathcal{B}_{\log^\gamma}$, $\gamma \in \mathbb{R}$, where
$$\|f\|_{\mathcal{B}_{\log^\gamma}} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2) \log^\gamma \quad \frac{2}{1 - |z|^2} \quad |f'(z)| < \infty.$$
(For $\gamma = 1$, the reader will recognize the familiar condition (5.1), which is part of the motivating for studying these spaces while the value $\gamma = 0$ gives the standard Bloch space.)

- The *weighted Besov spaces* $B^{p,\beta}$ with $1 \le p < \infty$ and $-1 < \beta < \infty$ with the norm
$$\|f\|_{B^{p,\alpha}} = |f(0)| + \|f'\|_{A^p_\beta}, \qquad f \in Hol(\mathbb{D});$$
see Section 1.6. This includes the conformally invariant *analytic (diagonal) Besov spaces* $B^{p,p-2}$, $1 < p < \infty$; a further special case $p = 2$ yields the *Dirichlet space*.

- The *minimal Besov space* $B^1$ which can be defined in terms of atomic decomposition (infinite sums of disc automorphisms with $\ell^1$ coefficients) but it is more conveniently seen as the space of all analytic functions in $\mathbb{D}$ such that $f'' \in A^1$, equipped with the norm
$$\|f\|_{B^1} = |f(0)| + |f'(0)| + {}_{\mathbb{D}} |f''(z)|\, dA(z)\,.$$

- The space $BMOA$ of analytic functions of *bounded mean oscillation*, defined by

$$\|f\|_{BMOA} = |f(0)| + \sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^2} < \infty,$$

 where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the standard disc automorphism which is an involution. It is convenient to use the following equivalent norm in $BMOA$

$$\|f\|_\star = |f(0)| + \sup_{a\in\mathbb{D}} \left( \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dA(z) \right)^{\frac{1}{2}};$$

 see [50, Theorem 6.2].

Now, we check, or give several indications to show, that the spaces from the above list fulfill the five axioms **A1**-**A5**.

For all of the spaces listed, Axiom **A1** is satisfied in view of the appropriate and well-known pointwise estimates (actually, they are required in order to show that the space in question is complete). For Hardy and Bergman spaces, as well as for the Bloch space, see Section 1.4, Theorem 1.5 and Proposition 1.4. For analytic Besov spaces, see [55]. Just to illustrate some arguments, for $f\in\mathcal{B}^\beta$ and $z=re^{it}\in\mathbb{D}$, the standard pointwise estimate:

$$\begin{aligned} |f(z)-f(0)| &\le \int_0^r \left|f'(\rho e^{it})\right| d\rho \le \|f\|_{\mathcal{B}^\beta} \int_0^r \frac{1}{(1-\rho^2)^\beta}\, d\rho \\ &\le \begin{cases} C\|f\|_{\mathcal{B}^\beta} & \text{if } \beta<1, \\ \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\|f\|_{\mathcal{B}^\beta} & \text{if } \beta=1, \\ \frac{C}{(1-r^2)^{\beta-1}}\|f\|_{\mathcal{B}^\beta} & \text{if } \beta>1, \end{cases} \end{aligned}$$

shows that Axiom **A1** is satisfied.

Since $BMOA$ and analytic Besov spaces are contained in $\mathcal{B}$ and the inclusion is continuous, the above estimate for $\beta=1$ can be used.

For the logarithmic Bloch space, one can actually produce a unified estimate, because an associated integral is convergent independently of $\gamma$:

$$|f(z)-f(0)| \le \|f\|_{\mathcal{B}_{\log^\gamma}} \int_0^r \frac{1}{(1-\rho^2)\log^\gamma\left(\frac{2}{1-\rho^2}\right)}\, d\rho \quad \frac{\|f\|_{\mathcal{B}_{\log^\gamma}}}{(1-r^2)^\varepsilon},$$

for $f\in\mathcal{B}_{\log^\gamma}$, $z=re^{it}\in\mathbb{D}$, and $\varepsilon>0$.

Axiom **A2** holds trivially in all spaces from our list.

Axiom **A3** is trivially verified in Hardy, weighted Bergman, and Korenblum spaces. In Bloch-type and Besov-type spaces, essentially, one has to use the fact that a function has better properties than its derivative (in terms of boundedness and integrability). In order to estimate $|(\zeta f)'(z)| = |f(z)+zf'(z)|$, it then suffices to add up the obvious estimates for the derivative and the function.

In $BMOA$, we can argue as follows:

$$
\begin{aligned}
\|\zeta f\|_\star^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |zf'(z)+f(z)|^2(1-|\varphi_a(z)|^2)\,dA(z)\\
&\quad \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)\,dA(z) + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2(1-|\varphi_a(z)|^2)\,dA(z)\\
&\quad \left(1+\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left(1+\frac{1}{2}\log\frac{1+|z|}{1-|z|}\right)^2 (1-|\varphi_a(z)|^2)\,dA(z)\right)\|f\|_\star^2\\
&\le \left(1+\int_{\mathbb{D}} \left(1+\frac{1}{2}\log\frac{1+|z|}{1-|z|}\right)^2 dA(z)\right)\|f\|_\star^2,
\end{aligned}
$$

where the integral in the last line is convergent.

In $B^1$, it is convenient to use the Littlewood-Paley formula to check that the axiom is satisfied. See Theorem 1.7.

The main issue is, of course, checking our Axiom **A4**. This is quite clear for the disc algebra, Hardy spaces, weighted Bergman spaces or Korenblum spaces.

Given $f \in \mathcal{B}^\beta$ and a non-vanishing function $u \in H^\infty$ such that $fu^n \in \mathcal{B}^\beta$ for all $n \in \mathbb{N}$, if $\alpha > 1$ then

$$\|fu^\alpha\|_{\mathcal{B}^\beta} = |f(0)u^\alpha(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)u^\alpha(z)+\alpha u^{\alpha-1}(z)u'(z)f(z)|.$$

Since $|f(0)u^\alpha(0)| \le \|f\|_{\mathcal{B}^\beta}\|u\|_{H^\infty}^\alpha$ and

$$
\begin{aligned}
|f'(z)u^\alpha(z)+&\alpha u^{\alpha-1}(z)u'(z)f(z)|\\
&= |(1-\alpha)f'(z)u^\alpha(z)+\alpha u^{\alpha-1}(z)\left(f'(z)u(z)+u'(z)f(z)\right)|\\
&\le (\alpha-1)|f'(z)u^\alpha(z)| + \alpha\|u\|_{H^\infty}^{\alpha-1}|(fu)'(z)|,
\end{aligned}
$$

we have

$$\|fu^\alpha\|_{\mathcal{B}^\beta} \le \|u\|_{H^\infty}^\alpha\|f\|_{\mathcal{B}^\beta} + \alpha\|u\|_{H^\infty}^{\alpha-1}\|fu\|_{\mathcal{B}^\beta} + (\alpha-1)\|u\|_{H^\infty}^\alpha\|f\|_{\mathcal{B}^\beta}.$$

A similar argument works for logarithmic Bloch spaces and $BMOA$.

It is the case of $B^1$ that requires most work. It also explains why we may need values $n > 1$ in Axiom **A4**. Given $f \in B^1$ and a non-vanishing function $u \in H^\infty$ such that $fu^n \in B^1$ for all $n \in \mathbb{N}$, if $\alpha > 2$ then

$$
\begin{aligned}
|(u^\alpha f)''(z)| &= |\alpha(\alpha-1)u^{\alpha-2}(z)(u'(z))^2f(z)+\alpha u^{\alpha-1}(z)u''(z)f(z)\\
&\quad +2\alpha u^{\alpha-1}(z)u'(z)f'(z)+u^\alpha(z)f''(z)|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|(u^\alpha f)''(z)| \le &\frac{\alpha(\alpha-1)}{2}\|u\|_{H^\infty}^{\alpha-2}\Big|(fu^2)''(z) - 2\left(1-\frac{1}{\alpha-1}\right)u(z)u''(z)f(z)\\
&-4\left(1-\frac{1}{\alpha-1}\right)u(z)u'(z)f'(z) - \left(1-\frac{2}{\alpha(\alpha-1)}\right)u^2(z)f''(z)\Big|.
\end{aligned}
$$

Hence,

$$|(u^\alpha f)''(z)| \quad \|u\|_{H^\infty}^{\alpha-2}|(fu^2)''(z)| + \|u\|_{H^\infty}^{\alpha-2}\,|g(z)|\,,$$

where

$$\begin{aligned} g(z) = -2 \quad 1 - \frac{1}{\alpha-1} \quad u(z)u''(z)f(z) - 4 \quad 1 - \frac{1}{\alpha-1} \quad u(z)u'(z)f'(z) \\ - \quad 1 - \frac{2}{\alpha(\alpha-1)} \quad u^2(z)f''(z). \end{aligned}$$

A direct computation yields

$$\begin{aligned} |g(z)| \quad & \|u\|_{H^\infty}|u''(z)f(z) + 2u'(z)f'(z) + u(z)f''(z)| + \|u\|_{H^\infty}^2|f''(z)| \\ &= \|u\|_{H^\infty}|(uf)''(z)| + \|u\|_{H^\infty}^2|f''(z)|. \end{aligned}$$

Therefore

$$_{\mathbb{D}}|(u^\alpha f)''(z)|\,dA(z) \quad \|u\|_{H^\infty}^{\alpha-2}\|fu^2\|_{B^1} + \|u\|_{H^\infty}^{\alpha-1}\|fu\|_{B^1} + \|u\|_{H^\infty}^{\alpha}\|f\|_{B^1}.$$

In Hardy and weighted Bergman spaces, Axiom **A5** relies on the Littlewood subordination principle. In the growth spaces and the Bloch space, this follows from the Schwarz-Pick lemma. In weighted Besov spaces one also has to use the fact that the derivative of a fixed disc automorphism is bounded from above and bounded away from zero. The Bloch space, $BMOA$, $B^1$, and the disc algebra are conformally invariant; see Section 1.2.

### 5.2.3 Invertible weighted composition operators

We now prove the main result of this chapter. It should be noted that the assumption in Axiom **A4** that $\alpha$ be a non-integer is relevant in the proof (in order to produce a non-analytic function and obtain a contradiction).

**Theorem 5.1.** *Let $X \subset Hol(\Omega)$ be any functional Banach space on a bounded planar domain $\Omega$ in which the Axioms **A1** - **A4** are satisfied, and suppose that a weighted composition operator $W_{F,\phi}$ is bounded in $X$.*

*(a) If $W_{F,\phi}$ is invertible in $X$ then its composition symbol $\phi$ is an automorphism of $\Omega$, the multiplication symbol $F$ does not vanish in $\Omega$, and the inverse operator $W_{F,\phi}^{-1}$ is another weighted composition operator $W_{G,\psi}$, whose symbols are:*

$$G = \frac{1}{F \circ \phi^{-1}}, \qquad \psi = \phi^{-1}. \tag{5.2}$$

*(b) Assuming that all Axioms **A1** - **A5** hold, we have the following characterization.*

*The weighted composition operator $W_{F,\phi}$ is invertible on $X$ if and only if its composition symbol $\phi$ is an automorphism of $\Omega$, the multiplication symbol $F$ does not vanish in $\Omega$, and $1/F \in Mult(X)$.*

*If this is the case, then $F$ is also a self-multiplier of $X$ and the inverse operator is $W_{G,\psi}$ whose symbols are given by (5.2).*

*Proof.* We first prove part *(a)*, using only Axioms **A1** - **A4**.

Since $1 \in X$ by Axiom **A2** and $W_{F,\phi}$ is onto by assumption, $F(f \circ \phi) = 1$ must hold for some $f \in X$, hence $F$ cannot vanish in $\Omega$. We now show that $\phi$ is an automorphism of $\Omega$.

Since $F = W_{F,\phi} 1 \in X$, we know that $\zeta F \in X$ by Axiom **A3**. Since the operator $W_{F,\phi}$ is onto, there exists a function $f \in X$ such that $\zeta F = F(f \circ \phi)$. It follows that $f \circ \phi \equiv \zeta$ and from here it is immediate that $\phi$ is univalent: if $\phi(a) = \phi(b)$ then $a = f(\phi(a)) = f(\phi(b)) = b$.

We next show that $\phi(\Omega) = \Omega$. Suppose that, on the contrary, $\phi$ omits some value $w \in \Omega$. Then the function $\phi(z) - w$ is bounded and does not vanish in $\Omega$. (This is where we use the assumption about boundedness of $\Omega$.) Note that by Axiom **A2** and Axiom **A3**, the function $\zeta^k \in X$, hence

$$F \in X, \qquad F(\phi - w)^n = \sum_{k=0}^{n} \binom{n}{k} w^{n-k} W_{F,\phi}(\zeta^k) \in X, \qquad n \in \mathbb{N}.$$

Hence, by Axiom **A4**, there exists a non-integer value $\alpha > 0$ such that $F(\phi - w)^\alpha \in X$.

The operator $W_{F,\phi}$ is onto, hence for some $f \in X$ we have $F(f \circ \phi) = F(\phi - w)^\alpha$. But $\phi$ is univalent, hence not constant. Therefore the equality $f(z) = (z - w)^\alpha$ holds on the non-empty open set $\phi(\Omega)$, hence in all of $\Omega$, by the uniqueness principle. However, the function on the right is not analytic in $\Omega$, hence $f \notin X$, which is absurd.

We have, thus, shown that $\phi$ is an automorphism of $\Omega$.

Given $g \in X$ , from the assumption that the operator is invertible and solving the equation $F(f \circ \phi) = g$ for $f$ yield by straightforward computation that formulas (5.2) hold.

*(b)* This part is similar to the proof given in [13] but we can still simplify the arguments given there.

We first prove the forward implication, starting from the assumption that $W_{F,\phi}$ is invertible. By the first part of the theorem, we know that $\phi$ is an automorphism of $\Omega$.

To see that $1/F \in Mult(X)$, let $g \in X$ be arbitrary. Since $W_{F,\phi}$ is onto, there exists a function $f \in X$ such that $F(f \circ \phi) = g$. But $f \circ \phi \in X$ by Axiom **A5**, hence $g/F$ is analytic in $\Omega$ and $g/F \in X$.

Now for the reverse implication. Assuming that all five axioms are satisfied and that $\phi$ is an automorphism of $\Omega$, $F$ does not vanish, and $1/F \in Mult(X)$, we argue as follows. Since $\phi$ is an automorphism of $\Omega$, so is its inverse function $\phi^{-1}$. Given an arbitrary function $f \in X$, we also have $f/F \in X$ and therefore also $(f \circ \phi^{-1})/(F \circ \phi^{-1}) \in X$. Thus, the operator $W_{G,\psi}$ where

$$G = \frac{1}{F \circ \phi^{-1}}, \qquad \psi = \phi^{-1}$$

maps $X$ into itself. By an application of the closed graph theorem, which is possible thanks to Axiom **A1** and the principle of uniform boundedness, $W_{G,\psi}$ is a bounded operator on $X$. Now one easily checks directly that

$$W_{G,\psi} W_{F,\phi} f = W_{F,\phi} W_{G,\psi} f = f$$

for all $f \in X$, hence the operator $W_{F,\phi}$ has bounded inverse $W_{G,\psi}$. $\square$

## 5.3 Surjective isometries among WCOs on spaces with translation-invariant seminorm

The result of this section focuses on the spaces of Bloch or Besov type. We show that in such spaces the only onto linear isometries among the weighted composition operators are the most obvious ones. This was not explicitly stated, even for the Bloch space and the usual composition operators, in [34] or [67], though in this particular case it can be deduced from the results obtained there after some discussion.

Specifically, we consider function spaces of the disc with a seminorm $\rho$ that is translation-invariant and also has the additional property that $\rho(f) = 0$ implies that $f$ is a constant function. This is precisely the case with the Bloch type spaces, weighted Besov spaces or the $BMOA$ space where the seminorm involves expressions like

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|, \qquad {}_{\mathbb{D}}\, |f'(z)|^p(1-|z|^2)^\beta \, dA(z)$$

or

$$\sup_{a\in\mathbb{D}} \|f\circ\varphi_a - f(a)\|_{H^2}.$$

It is convenient to observe that if the norm is given by the formula $\|f\| = |f(0)| + \rho(f)$, where $\rho(f)$ is a seminorm which is translation-invariant: $\rho(f + C) = \rho(f)$, for all $f \in X$ and all $C \in \mathbb{C}$, then $\rho(f) = 0$ implies that $f$ is a constant function. Indeed, the function $f - f(0)$ vanishes at the origin, hence in view of the norm formula and translation invariance we have

$$\|f - f(0)\| = \rho(f - f(0)) = \rho(f) = 0$$

which implies that $f = f(0)$, a constant function.

**Theorem 5.2.** *Let $X \subset Hol(\mathbb{D})$ be a functional Banach space in which the Axioms **A1** - **A4** are satisfied and in which the norm is given by the formula $\|f\| = |f(0)| + \rho(f)$, where $\rho(f)$ is a seminorm which is translation-invariant. If $W_{F,\phi}$ is a surjective isometric weighted composition operator on $X$, then $F$ is a unimodular constant and $\phi$ is a rotation.*

*Proof.* By assumption, our operator is onto, hence there exists a function $f_0 \in X$ such that $F(f_0 \circ \phi) \equiv 1$, so trivially $F$ does not vanish in $\mathbb{D}$.

Next, we show that $\phi(0) = 0$. To this end, consider the function $f_\lambda$ given by $f_\lambda(z) = 1 + \lambda z$, with $|\lambda| = 1$. By our assumptions on the space, $f_\lambda \in X$ and, by the translation invariance and homogeneity of the seminorm and the assumption that $W_{F,\phi}$ is an isometry, we have

$$\begin{aligned} 1 + \|\zeta\| = 1 + \rho(\zeta) = 1 + \rho(f_\lambda) = \|f_\lambda\| &= \|F + \lambda F\phi\| \\ &\le \|F\| + \|F\phi\| = \|1\| + \|\zeta\| = 1 + \|\zeta\|. \end{aligned}$$

Thus, equality must hold throughout, meaning that $\|F + \lambda F\phi\| = \|F\| + \|F\phi\|$. Hence

$$|(F + \lambda F\phi)(0)| + \rho\,(F + \lambda F\phi) = |F(0)| + \rho(F) + |(F\phi)(0)| + \rho(F\phi).$$

Since $\rho\left(F+\lambda F\phi\right)\leq\rho(F)+\rho(F\phi)$, it follows that

$$|(F+\lambda F\phi)(0)|\geq|F(0)|+|(F\phi)(0)|\,.$$

We know that $F(0)=0$, hence

$$|(1+\lambda\phi)(0)|\geq 1+|\phi(0)|\,,$$

for all $\lambda$ with $|\lambda|=1$, which is only possible when $\phi(0)=0$. On the other hand, by Theorem 5.1, $\phi$ must be a disc automorphism. Hence, it is a rotation.

Now, recalling that the function $f_0\in X$ chosen earlier has the property $F(f_0\circ\phi)\equiv 1$, we obtain $F(0)f_0(0)=1$. On the other hand, the same property implies that

$$|f_0(0)|\leq\|f_0\|=\|W_{F,\phi}(f_0)\|=\|1\|=1.$$

Also, $|F(0)|\leq\|F\|=\|W_{F,\phi}(1)\|=\|1\|=1$, hence we must have $|F(0)|=|f_0(0)|=1$. Therefore $\rho(F)=0$, hence $F$ is a constant of modulus one. This proves the statement. $\square$

# Conclusiones

Los resultados de esta tesis se pueden encontrar en los artículos de investigación:

- A. Aleman and A. Mas. Weighted conformal invariance of Banach spaces of analytic functions. *J. Funct. Anal.*, 280(9):108946, 2021
- M. J. Martín, A. Mas, and D. Vukotić. Co-Isometric Weighted Composition Operators on Hilbert Spaces of Analytic Functions. *Results Math.*, 75(3):128, 2020
- A. Mas and D. Vukotić. Invertible and isometric weighted composition operators. *Preprint*

El trabajo desarrollado en esta tesis puede servir como base para una futura investigación. Por ejemplo, una cuestión que surge de manera natural del Capítulo 3 sería intentar entender los operadores de composición ponderados que son isométricos sobre los espacios considerados en dicho capítulo. Esta cuestión es obviamente mas complicada de responder. Por ejemplo, no parece claro cómo se podría conseguir en este caso una fórmula como (3.3); ver Proposición 3.2 en la página 33. Trabajando con la hipótesis de una isometría usual en espacios de Hilbert: $W^*_{F,\phi}W_{F,\phi} = I$, no parece que se pueda obtener mucho más que la fórmula

$$W^*_{F,\phi}K_w = \overline{F(w)}K_{\phi(w)}$$

probada en el Capítulo 3. Pero esto no parece implicar ninguna fórmula general obvia parecida a (3.3) para $W^*_{F,\phi}f$, $f \in \mathcal{H}$.

Aún más difícil sería intentar describir los operadores de composición ponderados normales que actúan en un espacio de Hilbert general de funciones analíticas. En [24] han sido estudiados estos operadores pero no se han descrito en su totalidad, incluso en el espacio $H^2$, excepto en el caso cuando el punto fijo del símbolo de composición pertenece al disco. Debemos notar que en esta línea de trabajo existen resultados interesantes como [62] para los espacios $\mathcal{H}_\gamma$ o [98] para espacios más generales. Hasta lo que sabemos, incluso en una variable, la respuesta completa está lejos de ser conocida para la familia general de espacios considerada en este capítulo de la tesis.

Por otra parte, una pregunta que podría ser interesante sería si los resultados obtenidos en el Capítulo 4 pueden generalizarse a la bola unidad en varias variables. Otra cuestión que queda por responder en este capítulo sería entender los espacios que surgen al considerar el subespacio $\mathcal{M}_\alpha(X)$ cuando $X$ es un espacio de Banach de funciones analíticas no conformemente invariante para ningún índice $\beta > 0$. Por último, siguiendo las ideas desarrolladas en la Sección 4.6, otra futura linea de investigación podría ser estudiar cómo otros tipos de operadores actúan sobre los espacios de Banach conformemente invariantes de índice $\alpha > 0$.

Finalmente, siguiendo con los resultados vistos en el Capítulo 5, podríamos intentar describir todos los operadores de composición ponderados isométricos en espacios de Banach de funciones analíticas que satisfacen ciertos axiomas. Particularmente, podría ser interesante caracterizar estos operadores en el espacio de Bloch.

# Conclusions

The results of this thesis are contained in the following research papers:

- A. Aleman and A. Mas. Weighted conformal invariance of Banach spaces of analytic functions. *J. Funct. Anal.*, 280(9):108946, 2021
- M. J. Martín, A. Mas, and D. Vukotić. Co-Isometric Weighted Composition Operators on Hilbert Spaces of Analytic Functions. *Results Math.*, 75(3):128, 2020
- A. Mas and D. Vukotić. Invertible and isometric weighted composition operators. *Preprint*

The work developed in this thesis could provide a basis for further research. For instance, a natural question that may arise from Chapter 3 would be to understand the isometric weighted composition operators on the spaces considered in this chapter. This is obviously a more difficult question. For example, it does not seem clear how one could obtain a formula like (3.3) in this case; see Proposition 3.2 on page 33. Working with the typical assumption of the isometries: $W_{F,\phi}^* W_{F,\phi} = I$, one does not seem to get much more than the formula

$$W_{F,\phi}^* K_w = \overline{F(w)} K_{\phi(w)}$$

already proved here. But this does not seem to imply in any obvious way a general formula for $W_{F,\phi}^* f$, $f \in \mathcal{H}$, nor a formula like (3.3).

Still a harder question would be to describe the normal weighted composition operators on the general Hilbert spaces of analytic functions. Such operators have been studied but not fully described even on $H^2$ in [24], except in the case when the fixed point of the composition symbol belongs to the disc. Interesting general results have been obtained for the spaces $\mathcal{H}_\gamma$ in [62] and also for general spaces in [98]. To the best of our knowledge, even in one variable, a complete answer is far from being known for the general family of spaces considered in Chapter 3.

On the other hand, a question of interest would be if the results obtained in Chapter 4 could be generalized for the unit ball in several variables. Another question that remained regarding this chapter would be to understand the spaces that arise from considering the subspace $\mathcal{M}_\alpha(X)$ when $X$ is a Banach space of analytic functions not conformally invariant for any $\beta > 0$. Finally, another line of further investigation may be, following the ideas developed in Section 4.6, to study how some other types of operators acting on conformally invariant Banach spaces of index $\alpha > 0$.

Finally, following the results seen in Chapter 5, we could try to describe all the isometric weighted composition operators, acting on Banach spaces of analytic functions in the unit disc which satisfy certain axioms. It would be of particular interest to characterize these operators on the Bloch space.

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