Reliability Engineering and System Safety 213 (2021) 107694

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

A study on multi-level redundancy allocation in coherent systems formed by modules

Nuria Torrado [a,*], Antonio Arriaza [b], Jorge Navarro [c]

[a] Departamento de Análisis Económico: Economía Cuantitativa, Universidad Autónoma de Madrid, Spain
[b] Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Spain
[c] Departamento de Estadística e Investigación Operativa, Universidad de Murcia, Spain

ARTICLE INFO

Keywords:
Reliability
Hierarchical system structure
System/modular/component redundancy
Stochastic comparisons
Distribution-free comparisons
Optimal component allocation
Copulas

ABSTRACT

The present work studies the effect of redundancies on the reliability of coherent systems formed by modules. Different redundancies at components' level versus redundancies at modules' level are investigated, including active and standby redundancies. For that, a new model is presented. This model takes into account the dependence among the components, as well as, the dependence among the modules of the system. In both cases, the dependence structure is modeled by copula functions. Several results are provided to compare systems consisting of heterogeneous components. The comparisons are distribution-free with respect to the components. In particular, we consider the cases when the components in the modules are independent and connected (or not) in series, and when the components are dependent within the modules. In both cases, it is assumed that the modules can be dependent. Furthermore, the case in which the components in each module are identically distributed (dependent or independent) is also considered. We illustrate the theoretical results with several examples.

1. Introduction

Mainly, there exist two methods to improve the reliability of a coherent system. Firstly, designing and making higher quality and more reliable components and, secondly, including redundancies by adding spares, with similar or different reliability functions, to the original components. Frequently, the latter option is an efficient way to enhance the reliability of a system. However, redundancy allocation is not a trivial problem and it depends on the structure of the system, the dependence among the components, the reliability functions of the components and spares, economic restrictions, etc.

In order to deal with redundancy allocation problems, different approaches have been proposed in the literature. On the one hand, some authors study which components of the system should be assigned to be redundant. There are several ways to implement these redundant components. Some of the most used in the engineering field, among others, are the active (or hot) redundancy, which consists in adding to the original component one or more spares forming a parallel system (see [1–5] among others), and the standby (or cold) redundancy where a component is replaced, when it fails, by a spare which starts to work at the replacement moment. There exist many options of replacement for failed components. For example, in the case of perfect repairs, a new and identical unit is used as spare (see, for example, [6–8]). In the case of minimal repairs, the first model, proposed by Barlow and Hunter [9], states that a failed component is replaced by a spare whose reliability is the same as that of the original component just before the failure. Since then, many generalizations have been proposed in the literature (see for example Block et al. [10], Shaked and Shanthikumar [11], Aven [12], Aven and Jensen [13] and Finkelstein [14]). In some occasions the action of replacement is unsuccessful and the spare unit possess a worse reliability than the original component, in this case the replacement is known as imperfect repair (see Shaked and Shanthikumar [11], Zequeira and Berenguer [15] and Hollander et al. [16]). Some authors focus on obtaining optimal maintenance policies, which deal with cost functions associated to repairs or preventive maintenances of the system's components, see Hashemi et al. [17], Wang et al. [18] and Xu et al. [19]. Recent works which study the redundancy allocation problem are for instance Kim [20], Peiravi et al. [21], Li et al. [22], Wang et al. [23], Hsieh [24], Torrado [25], and Navarro and Fernández-Martínez [26], among others.

On the other hand, some authors study the convenience of carrying out a redundancy allocation at different levels of a system. Frequently, redundancies at components' level require more resources. Thus, it is interesting to find more efficient alternatives where allocate these redundancies. It is common in engineering areas, to find coherent systems

* Corresponding author.
E-mail address: nuria.torrado@uam.es (N. Torrado).

https://doi.org/10.1016/j.ress.2021.107694
Received 27 January 2021; Received in revised form 1 April 2021; Accepted 11 April 2021
Available online 20 April 2021
0951-8320/

which can be described by multiple layers. In this kind of systems, there exists a hierarchical structure where the whole system is located at the top and the components are set at the bottom. In the middle we find different subsystems or modules formed by some components of the system. A module is considered as a semi-coherent system (we provide the formal definition below) and two different modules do not share any common component. In this framework, redundancies can be allocated to any level (system, modules or components). Multilevel redundant designs have been widely used in different engineering areas such as communication systems, mechanical systems, computing systems, electrical systems and control systems among others (see, for example, [27–29]). As a particular case of multilevel redundancies, Barlow and Proschan [30] presented a result which states that active redundancies at components' level produce more reliable systems, in terms of the usual stochastic order, than active redundancies at system level in the case of independent components. This result is known in the literature as BP-principle and it has been extended in several ways. A detailed review about the successive generalizations of the BP-principle can be found in Yan and Wang [31].

In this work, we propose a new model to study redundancy mechanisms at multiple levels. We incorporate the possible dependency among the components or modules by using the Sklar's copula representation, which allows us to express the reliability function associated to a coherent system as a function of the corresponding reliability functions of the components or modules. This new approach allows us to prove that the BP-principle might hold for coherent systems with independent components, with heterogeneous or identically distributed components and with matching or non-matching spares. Indeed, comparing the reliability of the systems, we prove that active redundancies at components' level are better than active redundancies at modules' level for possibly dependent modules. The last comparison is also studied for any redundancy and some sufficient conditions are given to get distribution-free comparisons. The case of minimal repairs is also considered. Aven and Jensen [13] proposed a generalized model of minimal repair at systems level, which takes into account different levels of the system information. In a first level, all components are observed and we know in each moment which components are still working. When that information is available the minimal repair is called *physical minimal repair*. In a second level, we only know the age of the system at the moment of failing. In this case, the minimal repair is known as *black box minimal repair*. This latter option will be the one used in this article when we consider minimal repairs at module or system level. Finally, we study under which conditions the reliability of two systems, with the same number of components and modules, can be stochastically compared providing an optimal component allocation under some assumptions.

The present article is organized as follows. In Section 2 we provide some basic definitions and notations. Section 3 introduces the formulation of the proposed model and the expressions obtained for the reliability functions of the systems with redundancies at components' and modules' levels, respectively. These expressions are used in Sections 4 and 5 to compare the resulting systems under different assumptions. In Section 4 we deal with systems having heterogeneous components and we provide some results for independent components connected (or not) in series and dependent components within the modules. In both cases, we assume that the modules can be dependent. In Section 5 we consider the case of systems with identically distributed components within modules. The conclusions of the paper are presented in Section 6.

2. Definitions and preliminary results

In this section, we recall some well-known definitions. Throughout, we use increasing and decreasing to denote monotone nondecreasing and monotone nonincreasing, respectively. We denote by $\mathbb{R}^n$ the n-dimensional real vectorial space and $\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$. The notations used in this manuscript are presented in Table 1.

Definition 2.1. Given two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we say that the vector $\boldsymbol{x}$ majorizes the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{\text{m}}{\geq} \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}, \quad \text{for } j = 1, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$$

or if

$$\sum_{i=j}^{n} x_{i:n} \geq \sum_{i=j}^{n} y_{i:n}, \quad \text{for } j = 2, \dots, n \quad \text{and} \quad \sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}.$$

The vector $\boldsymbol{x}$ weakly supermajorizes the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{\text{w}}{\geq} \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}, \quad \text{for } j = 1, \dots, n,$$

where $x_{1:n}, \dots, x_{n:n}$ denote the components of the vector $(x_1, \dots, x_n)$ rearranged in increasing order. Thus, $x_{1:n}$ and $x_{n:n}$ represents the minimum and maximum of $(x_1, \dots, x_n)$, respectively. The same holds for $y_{i:n}$.

Bon and Păltănea [32] introduced the p-larger order, which is considered a preorder on $\mathbb{R}^n_+$. Here, we recall its definition.

Definition 2.2. Given two non-negative vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$, we say that $\boldsymbol{x}$ is p-larger than the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{\text{p}}{\geq} \boldsymbol{y}$, if

$$\prod_{i=1}^{j} x_{i:n} \leq \prod_{i=1}^{j} y_{i:n}, \quad \text{for } j = 1, \dots, n.$$

It is known that $\boldsymbol{x} \overset{\text{m}}{\geq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\text{w}}{\geq} \boldsymbol{y}$ and $\boldsymbol{x} \overset{\text{m}}{\geq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\text{p}}{\geq} \boldsymbol{y}$. The converses are, however, not always true.

Next, we introduce the notion of Schur-concave/convex functions related to the majorization order and also a result which can be found in [33].

Definition 2.3. A function $\psi : \mathcal{A} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-concave (Schur-convex) on $\mathcal{A}$ if, and only if, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$ such that $\boldsymbol{x} \overset{\text{m}}{\geq} \boldsymbol{y}$, one has $\psi(\boldsymbol{x}) \leq (\geq) \psi(\boldsymbol{y})$.

The following result is well-known.

Lemma 2.4. *A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ satisfies*

$$\boldsymbol{x} \overset{w}{\geq} \boldsymbol{y} \text{ on } \mathcal{A} \Rightarrow \phi(\boldsymbol{x}) \geq (\leq) \phi(\boldsymbol{y})$$

if, and only if, ϕ is decreasing (increasing) and Schur-convex (Schur-concave) on $\mathcal{A}$.

The following result is Lemma 2.1 in [34].

Lemma 2.5. *The function $\phi : \mathbb{R}^n_+ \to \mathbb{R}$ satisfies*

$$\boldsymbol{x} \overset{p}{\geq} \boldsymbol{y} \Rightarrow \phi(\boldsymbol{x}) \geq (\leq) \phi(\boldsymbol{y})$$

if, and only if, $\phi(e^{a_1}, \dots, e^{a_n})$ is Schur-convex (Schur-concave) in $(a_1, \dots, a_n)$ and decreasing (increasing) in a_i, for $i = 1, \dots, n$.

We will also need the following basic concepts of Reliability Theory. A (binary) **system** is a Boolean (structure) function $\psi : \{0,1\}^n \to \{0,1\}$. Here $x_i = 0$ means that the ith component does not work and $x_i = 1$ that it works. Then, the system state $\psi(x_1, \dots, x_n) \in \{0,1\}$ is completely determined by the structure function ψ and the component states $x_1, \dots, x_n \in \{0,1\}$. A system ψ is **semi-coherent** if it is increasing, $\psi(0, \dots, 0) = 0$ and $\psi(1, \dots, 1) = 1$. A system ψ is **coherent** if it is semi-coherent and all the components are relevant. We say that the ith component is relevant if ψ is strictly increasing in at least a point in the ith variable. In general, a semi-coherent system is not a coherent system. For example, the system $\psi(x_1, x_2) = x_2$ is semi-coherent but not coherent. The basic properties of systems can be seen in the classic book [30].

Table 1
Definitions of the used notations.

n	Total number of components in the system.
k	Total number of modules in the system.
M_j	The jth module, for $j = 1, \ldots, k$.
n_j	Number of components in the jth module.
$\mathbf{s}$	$\mathbf{s} = (n_1, \ldots, n_k)$ components' allocation vector in the k modules.
T	System lifetime without any redundancy.
X_i	Lifetime of component i.
X_{M_j}	Lifetime of module j.
$T_{\mathbf{s}}$	Lifetime of a system with redundancy at modules' level and components' allocation vector $\mathbf{s}$.
$\bar{F}_i$	Reliability function of component i.
$\bar{F}_{M_j}$	Reliability function of module j without any redundancy.
$\bar{F}_T$	Reliability function of T.
$\bar{G}_j$	Reliability function of module j with redundancy at modules' level.
$\bar{H}_j$	Reliability function of module j with redundancy at components' level.
R_1	Reliability function of a system when the redundancy is at components' level.
R_2	Reliability function of a system when the redundancy is at modules' level.
$R_2^{(\mathbf{s})}$	Reliability function of $T_{\mathbf{s}}$.
$\bar{Q}^*$	Distortion function defining the structure among the modules.
$\bar{Q}_{M_j}$	Distortion function defining the structure among the components within the module j.
$\bar{q}$	Redundancy distortion function.

Let T be the lifetime of a system with n components and let $X_1, \ldots, X_n$ be the lifetimes of the corresponding components. Let $\bar{F}_T(t) = \Pr(T > t)$ be the system reliability (or survival) function and let $\bar{F}_i(t) = \Pr(X_i > t)$ for $i = 1, \ldots, n$ be the components' reliability functions. If the system is semi-coherent, then it is well known, see, e.g., [35], that

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t)) \tag{2.1}$$

for all $t > 0$, where $\bar{Q} : [0,1]^n \to [0,1]$ is a generalized distortion function, that is, it is continuous, increasing and satisfies $\bar{Q}(0, \ldots, 0) = 0$ and $\bar{Q}(1, \ldots, 1) = 1$. Note that the respective distribution functions satisfy

$$F_T(t) = Q(F_1(t), \ldots, F_n(t)),$$

where

$$Q(u_1, \ldots, u_n) = 1 - \bar{Q}(1 - u_1, \ldots, 1 - u_n)$$

is another generalized distortion function. These functions depend on both the structure of the system and the dependency among the components. This possible dependency can be represented by the copula C in the representation of the joint distribution function of the components' lifetimes

$$\Pr(X_1 \le x_1, \ldots, X_n \le x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

or by the survival copula $\hat{C}$ in the representation of their joint reliability function

$$\Pr(X_1 > x_1, \ldots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)).$$

For example, for the series system $X_{1:n} = \min(X_1, \ldots, X_n)$, we have

$$\bar{F}_{1:n}(t) = \Pr(\min(X_1, \ldots, X_n) > t) = \hat{C}(\bar{F}_1(t), \ldots, \bar{F}_n(t)),$$

that is, $\bar{Q}_{1:n} = \hat{C}$, and for the parallel system $X_{n:n} = \max(X_1, \ldots, X_n)$, we have

$$\bar{F}_{n:n}(t) = 1 - \Pr(\max(X_1, \ldots, X_n) \le t) = 1 - C(1 - \bar{F}_1(t), \ldots, 1 - \bar{F}_n(t)),$$

that is, $\bar{Q}_{n:n}(u_1, \ldots, u_n) = 1 - C(1 - u_1, \ldots, 1 - u_n)$.

In particular, if the components are identically distributed (i.d.), that is, $\bar{F}_i = \bar{F}$ for $i = 1, \ldots, n$, then $\bar{F}_T(t) = \bar{q}_T(\bar{F}(t))$ with $\bar{q}_T(u) = \bar{Q}(u, \ldots, u)$ and $F_T(t) = q_T(F(t))$ with $q_T(u) = 1 - \bar{q}_T(1 - u)$ for $u \in [0,1]$.

3. Module reliability modeling

We assume that the coherent system with n components that we want to study can be decomposed in k modules $M_1, \ldots, M_k$ of $n_1, \ldots, n_k$ components with $n_1 + \cdots + n_k = n$ (i.e., the modules do not contain common components). Without loss of generality, we can assume that the first n_1 components belong to module M_1, the components $n_1 + 1$ to $n_1 + n_2$ belong to module M_2, and so on. Each module has a semi-coherent structure and so the reliability function of the first module M_1 can be written as $\bar{F}_{M_1}(t) = \bar{Q}_{M_1}(\bar{F}_1(t), \ldots, \bar{F}_{n_1}(t))$, meanwhile, the reliability function of the module M_j for $j = 2, \ldots, k$, can be written as

$$\bar{F}_{M_j}(t) = \bar{Q}_{M_j}(\bar{F}_{n_1+\cdots+n_{j-1}+1}(t), \ldots, \bar{F}_{n_1+\cdots+n_j}(t)),$$

where $\bar{F}_1, \ldots, \bar{F}_n$ are the reliability functions of the components and $\bar{Q}_{M_1}, \ldots, \bar{Q}_{M_k}$ are generalized distortion functions $\bar{Q}_{M_j} : [0,1]^{n_j} \to [0,1]$. Note that, for a given module, the components in the other modules are irrelevant components and so we can extend these functions to $[0,1]^n$.

We also assume that the state of the system is determined by the states of the modules through a coherent structure. Hence the reliability function of the system lifetime T can be written as

$$\bar{F}_T(t) = \bar{Q}(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

for all $t > 0$, where $\bar{Q} = \bar{Q}^*(\bar{Q}_{M_1}, \ldots, \bar{Q}_{M_k})$ and $\bar{Q} : [0,1]^n \to [0,1]$ and $\bar{Q}^* : [0,1]^k \to [0,1]$ are two generalized distortion functions. Note that $\bar{Q}^*$ defines the modular structure, i.e, it contains all the information about the way in which the modules are connected to each other and the dependence among them. The dependence among the components in each module and the dependence among the modules will be modeled by copula functions. Several examples will be provided later.

Thus we can consider two options: the redundancy at the components' level or the redundancy at the modules' level. In both cases we assume that the redundancy is represented by a univariate distortion $\bar{q} : [0,1] \to [0,1]$ satisfying $\bar{q}(u) \ge u$ for all $u \in [0,1]$. This approach was introduced recently in [26] and allows us to represent different redundancy options in a unified way. For example, if a redundancy is applied to the first component, then this component is replaced with a "system" with reliability $\bar{q}(\bar{F}_1)$. If this component is reinforced by adding an independent component with a parallel structure having the same reliability, then

$$\begin{aligned}\Pr(\max(X_1, Y_1) > t) &= \Pr(X_1 > t) + \Pr(Y_1 > t) - \Pr(X_1 > t)\Pr(Y_1 > t) \\ &= \bar{q}_{2:2}\left(\bar{F}(t)\right),\end{aligned}$$

where Y_1 is the lifetime of the spare added to the first component, $\bar{F}$ is the common reliability function of X_1 and Y_1 (matching spares), and $\bar{q}_{2:2}(u) = 2u - u^2$ for $u \in [0,1]$. Later on we will consider other options, for example, when X_1 and Y_1 are dependent or when they are not identically distributed (not matching spares). The same procedure is applied to the other components and modules.

Hence, in the case of a redundancy $\bar{q}$ at the components' level, if n spares are added to the n components, then the reliability function of the improved system is

$$R_1(t) = \bar{Q}_1(\bar{F}_1(t), \dots, \bar{F}_n(t)),$$

where $\bar{Q}_1(u_1, \dots, u_n) = \bar{Q}(\bar{q}(u_1), \dots, \bar{q}(u_n))$ is another generalized distortion function. Clearly, $R_1 \geq \bar{F}_T$ since $\bar{Q}_1 \geq \bar{Q}$.

On the other hand, if we consider redundancies at the modules' level, the reliability of the jth module is replaced with $\bar{q}(\bar{F}_{M_j}(t))$ (see, e.g., Fig. 2 in [36]) and so the reliability of the resulting system is

$$R_2(t) = \bar{Q}^*(\bar{q}(\bar{F}_{M_1}(t)), \dots, \bar{q}(\bar{F}_{M_k}(t))) = \bar{Q}_2(\bar{F}_1(t), \dots, \bar{F}_n(t)), \tag{3.1}$$

where $\bar{Q}_2(u_1, \dots, u_n) = \bar{Q}^*(\bar{q}(\bar{Q}_{M_1}(u_1, \dots, u_n)), \dots, \bar{q}(\bar{Q}_{M_k}(u_1, \dots, u_n)))$. Again we get $R_2 \geq \bar{F}_T$ since $\bar{Q}_2 \geq \bar{Q}$.

The purpose is to compare these two redundancy options by comparing $\bar{Q}_1$ and $\bar{Q}_2$. Note that the redundancy at the system level is included in the second option when we just consider a module with all the components (i.e. $k = 1$). In this case we get

$$R_2(t) = \bar{q}(\bar{F}_T(t)) = \bar{Q}_2(\bar{F}_1(t), \dots, \bar{F}_n(t)),$$

where $\bar{Q}_2(u_1, \dots, u_n) = \bar{q}(\bar{Q}(u_1, \dots, u_n))$. Also note that if the system has different module decompositions, then they can also be compared by using the corresponding $\bar{Q}_2$ distortion functions obtained in each decomposition.

4. Systems with heterogeneous components

In this section, we compare the reliability functions of systems formed by possibly dependent modules consisting of heterogeneous components with redundancies at components' or modules' levels. Specifically, we consider two scenarios. Firstly, we investigate the case in which the components within the modules are independent. Secondly, we study the case in which the components within the modules are dependent. It is worth mentioning that, in both cases, the modular structure can be any type.

4.1. Independent components connected in series and dependent modules

Let us start studying comparisons between systems with modules having heterogeneous independent components connected in series. The modules can be dependent. In this case, the reliability function of the first module is

$$\bar{F}_{M_1}(t) = \prod_{i=1}^{n_1} \bar{F}_i(t),$$

and that of the jth module is

$$\bar{F}_{M_j}(t) = \prod_{i=1}^{n_j} \bar{F}_{n_1+\cdots+n_{j-1}+i}(t)$$

for $j = 2, \dots, k$. Therefore, if we apply the redundancy $\bar{q}$ at the module level, the resulting module reliability functions are given by

$$\bar{G}_1(t) = \bar{q}(\bar{F}_{M_1}(t)) = \bar{q}\left(\prod_{i=1}^{n_1} \bar{F}_i(t)\right)$$

and

$$\bar{G}_j(t) = \bar{q}(\bar{F}_{M_j}(t)) = \bar{q}\left(\prod_{i=1}^{n_j} \bar{F}_{n_1+\cdots+n_{j-1}+i}(t)\right)$$

for $j = 2, \dots, k$. Hence, the reliability function of the system with redundancy at the modules' level is

$$R_2(t) = \bar{Q}^*(\bar{G}_1(t), \dots, \bar{G}_k(t)). \tag{4.1}$$

In the following proposition we study comparisons between modular redundancy and redundancy at components' level.

Proposition 4.1. *If the components in each module are independent and are connected in series and the distortion* $\bar{q}$ *satisfies*

$$\bar{q}(u)\bar{q}(v) \geq (\leq) \bar{q}(uv) \tag{4.2}$$

for all $u, v \in [0,1]$*, then* $R_1 \geq (\leq) R_2$ *for any modular structure* $\bar{Q}^*$*.*

Proof. The reliability function of the system with redundancy at the modules' level is given in (4.1). On the other hand, we first note that, in this case, $\bar{Q}$ can be written as

$$\bar{Q}(u_1, \dots, u_n) = \bar{Q}^*\left(\prod_{i=1}^{n_1} u_i, \prod_{i=1}^{n_2} u_{n_1+i}, \dots, \prod_{i=1}^{n_k} u_{n_1+\cdots+n_{k-1}+i}\right).$$

Hence, the reliability function of the system with redundancy at the components' level is

$$R_1(t) = \bar{Q}^*(\bar{H}_1(t), \dots, \bar{H}_k(t)),$$

where

$$\bar{H}_1(t) := \prod_{i=1}^{n_1} \bar{q}\left(\bar{F}_i(t)\right) \text{ and } \bar{H}_j(t) := \prod_{i=1}^{n_j} \bar{q}\left(\bar{F}_{n_1+\cdots+n_{j-1}+i}(t)\right)$$

for $j = 2, \dots, k$. Note that if

$$\bar{q}(u_1) \cdots \bar{q}(u_i) \geq \bar{q}(u_1 \cdots u_i) \tag{4.3}$$

for all $u_1, \dots, u_i \in [0,1]$ and $i = 2, 3, \dots, n$, then $\bar{H}_j \geq \bar{G}_j$ and so $R_1 \geq R_2$. From (4.2), we know that $\bar{q}(u_1u_2) \leq \bar{q}(u_1)\bar{q}(u_2)$. Now, by induction, let us assume that (4.3) holds for $i - 1$. Then

$$\bar{q}(u_1 \dots u_i) \leq \bar{q}(u_1 \dots u_{i-1})\bar{q}(u_i) \leq \bar{q}(u_1) \dots \bar{q}(u_i)$$

and therefore (4.3) holds. The proof for the reverse inequality in (4.2) is analogous. ■

Remark 4.2. It is worth mentioning that if (4.2) holds, we can use Proposition 4.1 to any modular structure $\bar{Q}_*$. For instance, if we consider three modules forming a system with structure $T = \min(X_{M_1}, \max(X_{M_2}, X_{M_3}))$, where X_{M_i} is the lifetime of the ith module for $i = 1, 2, 3$, then the modular structure is

$$\bar{Q}^*(u_1, u_2, u_3) = \hat{C}(u_1, u_2, 1) + \hat{C}(u_1, 1, u_3) - \hat{C}(u_1, u_2, u_3),$$

with $u_1, u_2, u_3 \in [0,1]$ and $\hat{C}$ is the survival copula which determines the dependence between the three modules. In particular, if the modules are independent, then

$$\bar{Q}^*(u_1, u_2, u_3) = u_1u_2 + u_1u_3 - u_1u_2u_3,$$

since $\hat{C}$ is the product copula.

Condition (4.2) can be interpreted as follows. Consider a series system with two independent components (with arbitrary reliability functions $\bar{F}_1$ and $\bar{F}_2$) and redundancy function $\bar{q}$. Then, (4.2) means that in this system the redundancy at the components' level is better (in the usual stochastic order) than the redundancy at the system level, that is,

$$\bar{q}\left(\bar{F}_1(t)\right)\bar{q}\left(\bar{F}_2(t)\right) \geq \bar{q}\left(\bar{F}_1(t)\bar{F}_2(t)\right) \text{ for all } t \geq 0.$$

Proposition 4.1 shows that this condition can be extended to systems with any modular structure and components connected in series in each module. Property (4.2) holds for any hot standby independent redundancy (systems) and for perfect repairs (convolutions), see [30], page 187 (see also the following remark). As a consequence, the reverse property in (4.2) (with the reverse meaning) is not so common.

Remark 4.3. The condition (4.2) is equivalent to require that the distortion $\bar{q}$ preserves the new better than used (NBU) aging notion, see [37]. We recall that the lifetime X of a device is NBU if $\bar{F}(t_1+t_2) \leq \bar{F}(t_1)\bar{F}(t_2)$ for all $t_1, t_2 \geq 0$, where $\bar{F}$ is the reliability function of X. Thus, if $\bar{q}$ satisfies condition (4.2) and X is NBU, then

$$\bar{q}(\bar{F}(t_1+t_2)) \leq \bar{q}(\bar{F}(t_1)\bar{F}(t_2)) \leq \bar{q}(\bar{F}(t_1))\bar{q}(\bar{F}(t_2))$$

for all $t_1, t_2 \geq 0$. This means that the reliability function $\bar{q}(\bar{F}(t))$ is NBU and therefore $\bar{q}$ preserves this notion. On the other side, if $\bar{q}$ preserves the NBU notion, then

$$\bar{q}(\bar{F}(t_1+t_2)) \leq \bar{q}(\bar{F}(t_1))\bar{q}(\bar{F}(t_2))$$

for every NBU reliability function $\bar{F}$. In particular, the reliability function $\bar{F}(t) = e^{-t}$ for $t \geq 0$, associated to a standard exponential distribution is NBU. Then, $\bar{q}(e^{-t_1-t_2}) \leq \bar{q}(e^{-t_1})\bar{q}(e^{-t_2})$ for all $t_1, t_2 \geq 0$. Taking $u = e^{-t_1}$ and $v = e^{-t_2}$, we obtain condition (4.2) for all $u, v \in [0,1]$. Specifically, it can be proved that the NBU is preserved when the increasing failure rate (IFR) class is preserved. From [37], the IFR class is preserved if $u\bar{q}'(u)/\bar{q}(u)$ is decreasing. This last condition is sometimes easier to check than (4.2).

Remark 4.4. It is well known that the generalized distortion function $\bar{Q}$ of any coherent system with m independent components satisfies

$$\bar{Q}(u_1v_1,\dots,u_mv_m) \leq \bar{Q}(u_1,\dots,u_m)\bar{Q}(v_1,\dots,v_m)$$

for all $u_i, v_i \in [0,1]$ and $i = 1,\dots,m$, that is, these systems preserve the NBU property, see [30], pages 183 and 188 (Exercise 10) or (18) in [38]. Hence, if we add $m-1$ independent and identically distributed (i.i.d.) spares to a component with any coherent structure, then $\bar{q}(u) = \bar{Q}(u,\dots,u)$ and (4.2) holds. The same happen if the components are independent and have proportional hazard rates. In this case

$$\bar{q}(u) = \bar{Q}(u, u^{\alpha_2},\dots,u^{\alpha_m})$$

and so (4.2) holds for any $\bar{Q}$ (any structure) and any $\alpha_2,\dots,\alpha_m > 0$. So we can say that (4.2) is a weak condition. However, (4.2) is not always true (see Example 4.5 for redundancies with dependent i.d. spares or Remark 4.6 for redundancies not based on coherent system structures).

In the following example we consider dependent spares that satisfy condition (4.2) for some values of the dependence parameter.

Example 4.5. Let us assume that the spares are added in parallel and that the original component and spares are i.d. (matching spares) and dependent. Then the redundancy mechanism is defined by the following distortion

$$\bar{q}(u) = 1 - C(1-u, 1-u), \quad u \in [0,1], \tag{4.4}$$

where C is the distributional copula which defines the dependence structure. Let us assume that C is an Archimedean copula with generator $\psi(t) = (\theta t + 1)^{-1/\theta}$ for $\theta > 0$, which leads to a Clayton copula (see [39], page 117, expression (4.2.1)). Then

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad u, v \in [0,1]. \tag{4.5}$$

Note that the previous expression is defined as zero for the vectors $(0,v)$, $(u,0)$ and $(0,0)$, i.e., $C(0,v) = C(u,0) = C(0,0) := 0$. If $\theta = 1$, then the above copula can be rewritten as

$$C(u,v) = \frac{uv}{u+v-uv}, \quad u, v \in [0,1].$$

In this case, from (4.4), the redundancy distortion is

$$\bar{q}(u) = 1 - \frac{1-u}{1+u} = \frac{2u}{1+u}, \quad u \in [0,1]$$

and

$$\bar{q}(u)\bar{q}(v) - \bar{q}(uv) = \frac{2uv\,(2\,(1+uv) - (1+u)\,(1+v))}{(1+u)\,(1+v)\,(1+uv)} = \frac{2uv(1-u)(1-v)}{(1+u)\,(1+v)\,(1+uv)} \geq 0,$$

i.e, for $\theta = 1$, (4.2) holds. Then, from Proposition 4.1, we know that $R_1 \geq R_2$ holds for any $\bar{Q}^*$ and any $\bar{F}_1,\dots,\bar{F}_n$. On the other hand, taking $\theta = 4$ we have that (4.2) does not hold, see Fig. 1 (right). So, in this case, we cannot use Proposition 4.1 to compare R_1 and R_2.

Remark 4.6. As we have seen, there exist redundancies that do not satisfy neither condition (4.2) nor its reverse. Another example (not associated with a coherent structure) is the distortion $\bar{q}(u) = u(1-\log u)e^{u-1}$ for $u \in (0,1]$ and $\bar{q}(0) := 0$. It can be considered as a redundancy because $\bar{q}(u) \geq u$ for all $u \in [0,1]$. However, $\bar{q}(0.02)\,\bar{q}(0.01) - \bar{q}(0.02 \cdot 0.01) > 0$ and $\bar{q}(0.5)\,\bar{q}(0.2) - \bar{q}(0.5 \cdot 0.2) < 0$. Therefore neither condition (4.2) nor its reverse holds.

One method to enhance the reliability of a system is improving the quality of some components by reducing their failure rates by a factor α with $0 < \alpha < 1$. In that case, we can define the reliability function of the ith spare as $\bar{F}_i^{\alpha}$ (not matching spares) for all $i = 1, 2, \dots, n$. If we assume active redundancy and an independent spare in parallel, then the distortion function is

$$\bar{q}_{\alpha}(u) = 1 - (1-u)(1-u^{\alpha}) = u + u^{\alpha} - u^{\alpha+1}. \tag{4.6}$$

It is easy to check that this distortion satisfies condition (4.2) for any $\alpha > 0$ (as stated in Remark 4.4).

Observe that if $\alpha = 1$, then the redundancy method defined in (4.6) is the active redundancy $\bar{q}_{2:2}(u) = 2u - u^2$, i.e., only one spare is added in a parallel structure, independent and identically distributed as the original unit (matching spare). If $\alpha > 1$, then the spare is worse than the original component in the sense that the failure rate of the spare is greater than that of the original component (which is a reasonable assumption in practice). In addition, if $\alpha \in \mathbb{N}$, then the redundancy defined in (4.6) is equivalent to add in parallel α i.i.d. spares forming a series system. If $0 < \alpha < 1$, then the spare is better than the original component and the result also holds. Next, let us show an example on how to apply Proposition 4.1 to the redundancy mechanism defined in (4.6).

Example 4.7. We consider systems with two modules connected in parallel and we assume that each module has two independent components connected in series. The components in the first module have exponential distributions with hazard rates equal to 1 and in the second module, they have exponential distributions with hazard rates equal to 2. Next, we assume the redundancy defined in (4.6), that is, the spare is independent and it is added in parallel, for $\alpha = 0.5, 1$ and 2 (see Fig. 2 for theirs block diagrams). We consider two different cases, when the modules are independent, i.e., $\bar{Q}^*(u,v) = u + v - uv$, and when the modules are dependent with a distributional copula C, i.e.,

$$\bar{Q}^*(u,v) = 1 - C(1-u, 1-v), \quad \text{for } u, v \in [0,1].$$

We assume that C is a Clayton copula as defined in (4.5) for $\theta = 5$. In Fig. 3, we plot the reliability functions for the systems with redundancy at component level (R_1) and at module level (R_2) for independent modules (left) and dependent modules (right). As (4.2) holds in both cases, we always have $R_1 \geq R_2$. Also note that R_1 and R_2 decrease when α increases (as expected since the spares get worse).

Another redundancy, widely used in reliability theory, is the minimal repair. It is considered as a particular case of cold redundancy. A unit, with lifetime X and survival function $\bar{F}$, is replaced in case of failure by a used unit with the same reliability as X and the same age as the unit had when it failed. This is equivalent to assume that the unit is minimally repaired to be just as it was before its failure. Let X^* be the lifetime of a component with a minimal repair, then its reliability function is given by

$$\bar{F}_{X^*}(t) = \bar{q}_{mr}(\bar{F}(t)),$$

where $\bar{F}$ is the reliability function of X and $\bar{q}_{mr}(u) = u(1-\log u)$ is a distortion function, see, for example, formula (3.1) in [40] or [41]. The distortion function $\bar{q}_{mr}$ satisfies condition (4.2). Observe that this condition is equivalent to

$$u\,v[1 - \log(u\,v)] \leq u[1-\log(u)]\,v[1-\log(v)] \quad \text{for all } u, v \in [0,1]. \tag{4.7}$$

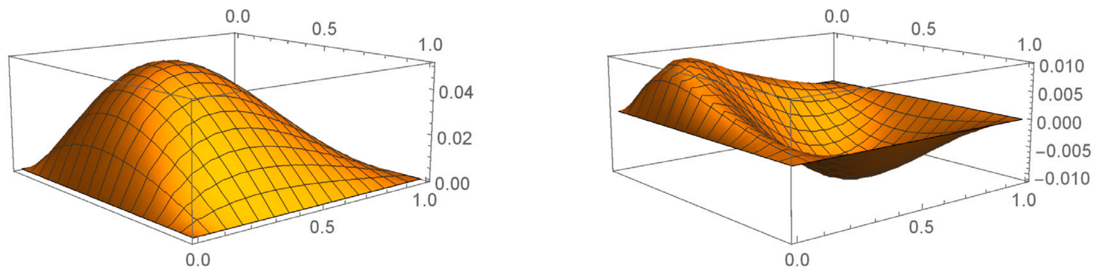


Fig. 1. Plot of $\bar{q}(u)\bar{q}(v) - \bar{q}(uv)$ for the systems in Example 4.5 when the components and spares are dependent with $\theta = 1$ (left) and $\theta = 4$ (right).

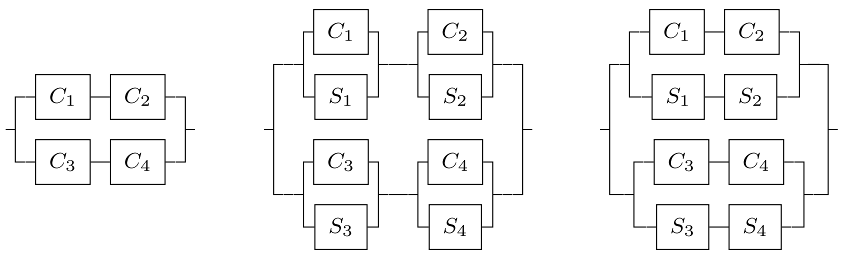


Fig. 2. Block diagrams of the parallel–series systems considered in Example 4.7 without any redundancy mechanism (left), when the redundancy is allocated at components' level (center) and at modules' level (right).

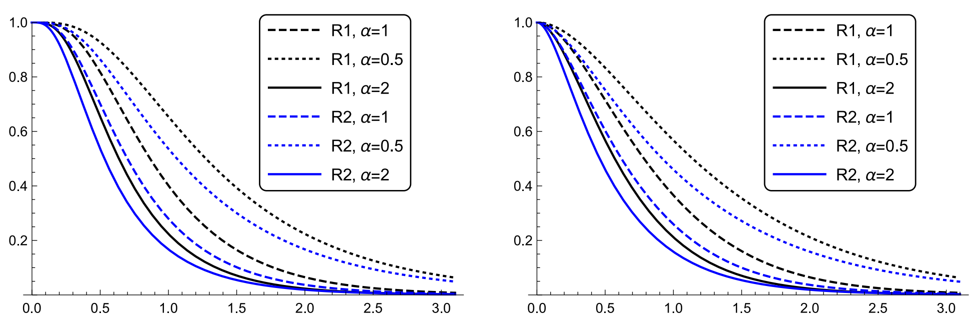


Fig. 3. Reliability functions for the parallel–series system in Example 4.7 when the modules are independent (left) or dependent (right) with a Clayton copula when $\theta = 5$.

If $u = 0$ or $v = 0$, then condition (4.2) is satisfied. Suppose now that $u, v \in (0, 1]$. Then, condition (4.7) can be rewritten as

$$\begin{aligned} 1 - \log(u\,v) &\leq [1 - \log(u)]\,[1 - \log(v)], \\ 1 - \log(u) - \log(v) &\leq [1 - \log(u)]\,[1 - \log(v)], \\ 1 - \log(u) - \log(v) &\leq 1 - \log(u) - \log(v) + \log(u)\log(v), \\ 0 &\leq \log(u)\log(v), \end{aligned}$$

where the last inequality holds for all $u, v \in (0, 1]$. Note that from Remark 4.3, this means that the distortion $\bar{q}_{mr}$ preserves the NBU notion.

Now, we need to establish what we understand by a minimal repair at module (system) level. Based on the definition of *black box minimal repair*, given by Aven and Jensen [13], we consider that a module (system) is minimally repaired when it is replaced by another used module (system) with the same distribution and the same age as the module (system) had when it failed. Let M_j^* be the resulting module after applying a minimal repair to the module M_j. Then, the reliability function associated to the lifetime of M_j^* can be expressed as

$$\bar{F}_{M_j^*}(t) = \bar{F}_{M_j}(t)\big(1 - \log \bar{F}_{M_j}(t)\big) = \bar{q}_{mr}(\bar{F}_{M_j}(t)). \quad (4.8)$$

In the particular case of a module with m components connected in series, we obtain that

$$\bar{F}_{M_j^*}(t) = \bar{q}_{mr}(\bar{F}_{1:m}(t)),$$

with $\bar{F}_{1:m}(t) = \hat{C}(\bar{F}_1(t), \ldots, \bar{F}_m(t))$, where $\bar{F}_1, \ldots, \bar{F}_m$ are the reliability functions of the components in that module and the survival copula $\hat{C}$ models the dependence among them. If we also assume independence among the components of the jth module, we obtain that

$$\bar{F}_{M_j^*}(t) = \bar{q}_{mr}\Big(\prod_{i=1}^{m} \bar{F}_i(t)\Big).$$

Let us see an example.

Example 4.8. Let X_{M_1}, X_{M_2} and X_{M_3} be the lifetimes of three modules M_1, M_2, and M_3 with 2, 2 and 3 independent components, respectively, connected in series. Let us assume that these modules define a system with lifetime $T = \max(\min(X_{M_1}, X_{M_2}), X_{M_3})$, then

$$\bar{Q}^*(u_1, u_2, u_3) = \hat{C}(1, 1, u_3) + \hat{C}(u_1, u_2, 1) - \hat{C}(u_1, u_2, u_3),$$

for $(u_1, u_2, u_3) \in [0, 1]^3$, where $\hat{C}$ is a survival copula, which models the dependence among M_1, M_2 and M_3, given by the following Gumbel–Hougaard copula:

$$\hat{C}(u_1, u_2, u_3) = \exp\left(-\left((-\log u_1)^{\theta} + (-\log u_2)^{\theta} + (-\log u_3)^{\theta}\right)^{1/\theta}\right) \quad (4.9)$$

with $\theta \geq 1$ and $u_1, u_2, u_3 \in [0, 1]$. This copula belongs to the well known family of Archimedean copulas, see [39], p. 118. The components 1 and 2, in the first module, have exponential distributions with hazard rates 1 and 2, respectively, i.e., $\bar{F}_i(t) = \exp(-i\,t)$ for $i = 1, 2$. The components 3 and 4 in the second module have Weibull distributions with shape parameter 1 and 2, respectively, and scale parameter 1 for both components, that is, $\bar{F}_i(t) = \exp(-t^{(i-2)})$ for $i = 3, 4$. Finally, the components 5, 6 and 7 in the last module have log-normal distributions with log-deviation 1, 2 and 3, respectively, all of them with log-mean 0, in this case, $\bar{F}_i(t) = 1 - \Phi(\log t/(i-4))$ for $i = 5,6,7$ with Φ the distribution function associated to a standard normal distribution. Next, we apply minimal repairs at modules' level and at components' level. Fig. 4 shows the plots of the reliability functions for the system without redundancy (blue), for the system with redundancy at module level (red) and for the system with redundancy at components' level (black), when the modules are independent (left) and when they are dependent (right) with copula parameter $\theta = 5$. From (4.7) and Proposition 4.1, we know that $R_1 \geq R_2$ holds for all $\bar{F}_1, \dots, \bar{F}_7$ (as can be seen in these particular plots).

Let $T_{\mathbf{s}}$ be the lifetime of a coherent system with redundancy at modules' level and let $R_2^{(\mathbf{s})}$ be its reliability function, where the vector $\mathbf{s} = (n_1, \dots, n_k)$ represents the number of components in each module. Next, we investigate comparisons for two types of modular redundancies between systems with dependent modules connected in series and whose dependency structure is defined by the family of Archimedean copulas. Then,

$$\bar{Q}^*(u_1, \dots, u_k) = \hat{C}_\psi(u_1, \dots, u_k) = \psi(\phi(u_1) + \cdots + \phi(u_k)),$$

for $u_1, \dots, u_k \in [0,1]$, where $\hat{C}_\psi$ is an Archimedean copula with generator ψ and $\phi = \psi^{-1}$. In addition, we consider that the components in each module are independent, connected in series and that theirs lifetime distributions are ordered. Thus we obtain the following result.

Proposition 4.9. *Let $T_{\mathbf{s}}$ and $T_{\mathbf{r}}$ be the lifetimes of two systems under modular redundancy with the same modular structure given by $\bar{Q}^* = \hat{C}_\psi$ an Archimedean copula with generator ψ, and with independent components connected in series in each module, where $\mathbf{s} = (n_1, \dots, n_k)$ and $\mathbf{r} = (m_1, \dots, m_k)$ are their respective component allocation vectors. Also assume that $\bar{F}_1 \geq \cdots \geq \bar{F}_n$, $n_1 \leq \cdots \leq n_k$ and $m_1 \leq \cdots \leq m_k$. If ψ is log-convex and*

$$\eta(u) = \frac{u\,\bar{q}'(u)}{\bar{q}(u)} \text{ is decreasing in } u \in (0,1], \tag{4.10}$$

then $\mathbf{s} \overset{m}{\geq} \mathbf{r}$ implies $R_2^{(\mathbf{s})} \leq R_2^{(\mathbf{r})}$.

Proof. Let us denote $\beta_j = \bar{F}_{M_j}(t)$ for any fix $t > 0$ and $j = 1, \dots, k$, then $\beta_1 \geq \cdots \geq \beta_k$ by the assumptions $\bar{F}_1 \geq \cdots \geq \bar{F}_n$ and $n_1 \leq \cdots \leq n_k$. Analogously, we define $\gamma_j = \bar{F}_{\tilde{M}_j}(t)$ for $j = 1, \dots, k$, where

$$\bar{F}_{\tilde{M}_j}(t) = \prod_{i=1}^{m_j} \bar{F}_{m_1+\cdots+m_{j-1}+i}(t),$$

and therefore $\gamma_1 \geq \cdots \geq \gamma_k$. Then, the reliability functions of $T_{\mathbf{s}}$ and $T_{\mathbf{r}}$ at time t, defined in (4.1), can be rewritten as $R_2^{(\mathbf{s})}(t) = \bar{Q}^*\left(\bar{q}(\beta_1), \dots, \bar{q}(\beta_k)\right)$ and $R_2^{(\mathbf{r})}(t) = \bar{Q}^*\left(\bar{q}(\gamma_1), \dots, \bar{q}(\gamma_k)\right)$, respectively.

Now, observe that $\beta_k \leq \gamma_k$ when $n_k \geq m_k$ which holds by the assumption $\left(n_1, \dots, n_k\right) \overset{m}{\geq} \left(m_1, \dots, m_k\right)$. Analogously, it is easy to check that

$$\prod_{i=j}^{k} \beta_i \leq \prod_{i=j}^{k} \gamma_i \quad \text{when} \quad \sum_{i=j}^{k} n_i \geq \sum_{i=j}^{k} m_i,$$

for $j = 2, \dots, k-1$, and $\prod_{i=1}^{k} \beta_i = \prod_{i=1}^{n} \bar{F}_i(t) = \prod_{i=1}^{k} \gamma_i$ since $\sum_{i=1}^{k} n_i = \sum_{i=1}^{k} m_i$. Therefore, $\left(n_1, \dots, n_k\right) \overset{m}{\geq} \left(m_1, \dots, m_k\right)$ implies $\left(\beta_1, \dots, \beta_k\right) \overset{p}{\geq} \left(\gamma_1, \dots, \gamma_k\right)$. Thus, $R_2^{(\mathbf{s})} \leq R_2^{(\mathbf{r})}$ holds if

$$\bar{Q}^*\left(\bar{q}(\beta_1), \dots, \bar{q}(\beta_k)\right) \leq \bar{Q}^*\left(\bar{q}(\gamma_1), \dots, \bar{q}(\gamma_k)\right)$$

whenever $\left(\beta_1, \dots, \beta_k\right) \overset{p}{\geq} \left(\gamma_1, \dots, \gamma_k\right)$. To prove this, from Lemma 2.5, we need to show that the function

$$g(b_1, \dots, b_k) := \bar{Q}^*\left(\bar{q}(e^{b_1}), \dots, \bar{q}(e^{b_k})\right)$$

is increasing in b_i for $i = 1, \dots, k$ and Schur-concave in $\boldsymbol{b} = (b_1, \dots, b_k)$ where $b_i \in (-\infty, 0]$ for $i = 1, \dots, k$. Firstly, it is evident that g is increasing in b_i since it is the composition of three positive-valued and increasing functions. Secondly, to prove that

$$g(b_1, \dots, b_k) = \psi\left(\phi(\bar{q}(e^{b_1})) + \cdots + \phi(\bar{q}(e^{b_k}))\right)$$

is Schur-concave in $\boldsymbol{b}$, we obtain its first partial derivative with respect to b_i such as

$$\begin{aligned}
\frac{\partial g(\boldsymbol{b})}{\partial b_i} &= \psi'\left(\phi(\bar{q}(e^{b_1})) + \cdots + \phi(\bar{q}(e^{b_k}))\right)\phi'(\bar{q}(e^{b_i}))\bar{q}'(e^{b_i})e^{b_i} \\
&= \psi'\left(\phi(\bar{q}(e^{b_1})) + \cdots + \phi(\bar{q}(e^{b_k}))\right)\frac{\bar{q}'(e^{b_i})e^{b_i}}{\psi'(\phi(\bar{q}(e^{b_i})))} \\
&= \psi'\left(\phi(\bar{q}(e^{b_1})) + \cdots + \phi(\bar{q}(e^{b_k}))\right)\frac{\psi(\phi(\bar{q}(e^{b_i})))}{\psi'(\phi(\bar{q}(e^{b_i})))} \cdot \frac{\bar{q}'(e^{b_i})e^{b_i}}{\bar{q}(e^{b_i})} \\
&= \psi'\left(\phi(\bar{q}(e^{b_1})) + \cdots + \phi(\bar{q}(e^{b_k}))\right)s(b_i),
\end{aligned}$$

where $s(x) = s_1(x)s_2(x)$ with

$$s_1(x) = \frac{\psi(\phi(\bar{q}(e^x)))}{\psi'(\phi(\bar{q}(e^x)))} \quad \text{and} \quad s_2(x) = \frac{e^x\,\bar{q}'(e^x)}{\bar{q}(e^x)}.$$

Observe that s_1 is a negative and increasing function since $\log\psi$ is convex. Moreover, s_2 is a positive and decreasing function since (4.10) holds. Therefore, s is an increasing function and so $s(b_1) \geq s(b_2) \geq \cdots \geq s(b_k)$ for $b_1 \geq \cdots \geq b_k$. Consequently

$$\frac{\partial g(\boldsymbol{b})}{\partial b_1} \leq \frac{\partial g(\boldsymbol{b})}{\partial b_2} \leq \cdots \leq \frac{\partial g(\boldsymbol{b})}{\partial b_k},$$

since $\psi' \leq 0$. Then, from Theorem 3.A.3 in [33], we have that $g(\boldsymbol{b})$ is Schur-concave in $\boldsymbol{b}$ for $b_1 \geq \cdots \geq b_k$. As these inequalities hold for $\log(\beta_i)$ and $\log(\gamma_i)$, this completes the proof. ■

Note that if the modules are independent and connected in series then $\psi(x) = e^{-x}$, and therefore we can apply Proposition 4.9 to systems with independent modules. Next, we show how to apply Proposition 4.9 to systems with dependent modules assembled by an Archimedean copula.

Example 4.10. We consider three systems with two modules each one connected in series, that is, $\bar{Q}^*(u,v) = \hat{C}(u,u)$, where $\hat{C}$ is the survival copula that represents the possible dependence between these modules. The first module has n_1 independent components connected in series which have exponential distributions with hazard rates equal to 1. The second one has n_2 components of the same type (independent and connected in series) exponentially distributed with hazard rates equal to 2. Let us assume $n_1 = 1,2,3$ and $n_2 = 5,4,3$ for the three systems, respectively. It is easy to check that $(1,5) \overset{m}{\geq} (2,4) \overset{m}{\geq} (3,3)$. On the other hand, let $\bar{q}_{2:2}(u) = 2u - u^2$ be the distortion of the redundancy mechanism which satisfies (4.10). We show in Fig. 5 the reliability functions of the three systems for independent modules (left) and dependent modules (right). For the case of dependent modules, we suppose that the dependence structure is defined by the Clayton copula in (4.5) for $\theta = 5$ which satisfies that ψ is log-convex. Therefore, all the conditions in Proposition 4.9 hold. As expected from that proposition, R_2 decreases when (n_1, n_2) increases in the majorization order. This property will hold for any ordered reliability functions $\bar{F}_1 \geq \cdots \geq \bar{F}_6$.

Remark 4.11. The meaning of (4.10) is similar to that of (4.2) but replacing the usual stochastic order with the hazard rate order. Thus, if we consider a series system with two independent components X_1 and X_2 with arbitrary reliability functions $\bar{F}_1$ and $\bar{F}_2$, then the lifetime of the system when we apply the redundancy, represented by $\bar{q}$, at the components' level is $T_1 = \min(Y_1, Y_2)$ and its corresponding reliability function is

$$R_1(t) = \bar{q}(\bar{F}_1(t))\bar{q}(\bar{F}_2(t)) \text{ for all } t \geq 0,$$

where Y_i represents the lifetime of the ith unit with redundancy for $i = 1,2$. Then, its hazard rate function is

$$h_{T_1}(t) = h_{Y_1}(t) + h_{Y_2}(t) = \eta(\bar{F}_1(t))h_1(t) + \eta(\bar{F}_2(t))h_2(t),$$

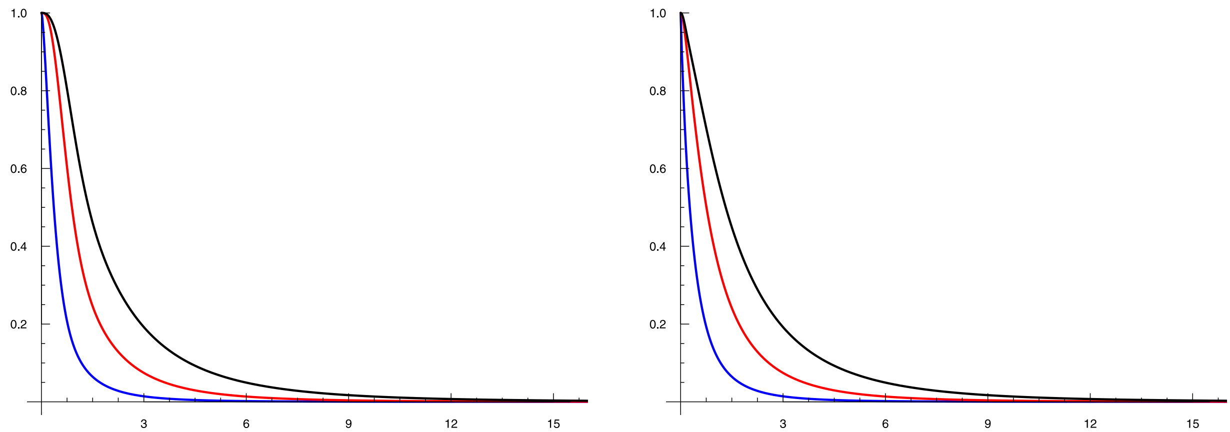


Fig. 4. Reliability functions for the systems in Example 4.8 without redundancy (blue), with redundancy at modules' level (red) and with redundancy at components' level (black), when the modules are independent (left) or dependent (right).

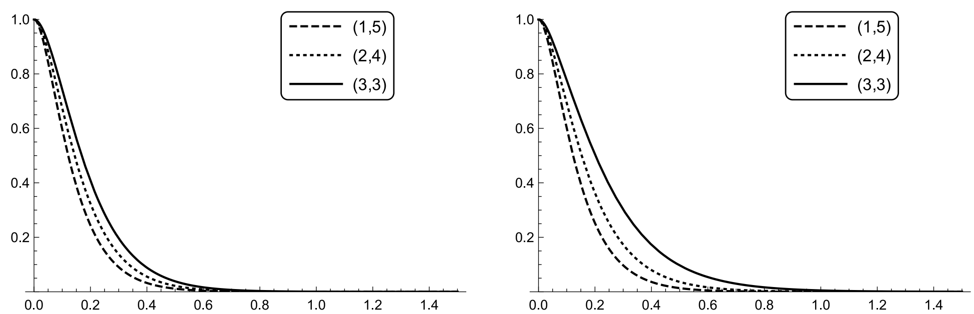


Fig. 5. Reliability functions R_2 for the series system in Example 4.10 when the modules are independent (left) or dependent (right) with a Clayton copula with $\theta = 5$.

where $\eta(u) = u\bar{q}'(u)/\bar{q}(u)$ for $u \in (0,1]$ and h_i is the hazard rate function of X_i for $i = 1,2$. Analogously, if we consider the lifetime T_2 of the system with redundancy at the system level, then its reliability is

$$R_2(t) = \bar{q}(\bar{F}_1(t)\bar{F}_2(t)) \text{ for all } t \geq 0,$$

and its hazard rate function is

$$h_{T_2}(t) = \eta(\bar{F}_1(t)\bar{F}_2(t))h_1(t) + \eta(\bar{F}_1(t)\bar{F}_2(t))h_2(t).$$

Hence, (4.10) implies that $h_{T_1} \leq h_{T_2}$, that is, the series system with redundancy at the components' level is better (in terms of the hazard rate order) than the one with redundancy at the system level.

From [37] we know that condition (4.10) is equivalent to the preservation of the increasing failure rate (IFR) class. Moreover, we also know from [42] that the IFR class is preserved in all k-out-of-n systems with i.i.d. components. Therefore, (4.10) holds for all these redundancy mechanisms which include parallel systems. Thus, if we consider active redundancy with $m-1$ spares, then $\bar{q}_{m:m}(u) = 1-(1-u)^m$ satisfies (4.10). Therefore, we can apply Proposition 4.9 to series systems with active redundancy. However, it is easy to check that the redundancy distortion $\bar{q}_\alpha$ in (4.6) does not satisfy (4.10) when $0 < \alpha < 1$ since

$$\left[\frac{u\,\bar{q}'(u)}{\bar{q}(u)}\right]' \overset{sign}{=} (\alpha-1)^2 - u(\alpha^2 + u^{\alpha-1})$$

and it takes positive and negative values in the interval $[0,1]$. On the other hand, Proposition 4.9 cannot be generalized to any copula as we show in the following example.

Example 4.12. We consider two systems with three dependent modules connected in series assembled by a Farlie–Gumbel–Morgenstern (FGM) copula, then

$$\bar{Q}^*(u_1,u_2,u_3) = \hat{C}(u_1,u_2,u_3) = u_1u_2u_3(1+\theta(1-u_1)(1-u_2)(1-u_3)),$$

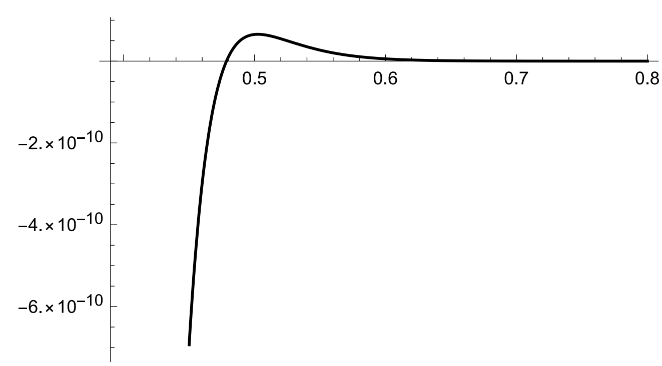


Fig. 6. Function $R_2^{(\mathbf{s})} - R_2^{(\mathbf{r})}$ for the series system in Example 4.12 with different redundancy allocation vectors when the components are heterogeneous, independent and connected in series.

for $(u_1,u_2,u_3) \in [0,1]^3$ and $\theta \in [-1,1]$. Each module has heterogeneous, independent components connected in series. Let us assume that $\mathbf{s} = (1,3,3)$ and $\mathbf{r} = (2,2,3)$ are the allocation vectors of components by modules for both systems, respectively. It is easy to check that $\mathbf{s} \overset{m}{\geq} \mathbf{r}$. We suppose that the component lifetimes have exponential distributions with hazard rate $\lambda \in \{0.1, 2, 6, 7, 8, 9, 10\}$, then $\bar{F}_1 \geq \cdots \geq \bar{F}_7$. Finally, we consider that the redundancy mechanism is $\bar{q}_{2:2}(u) = 2u - u^2$ and we take $\theta = -0.9$ in the FGM copula. In Fig. 6 we plot the function $R_2^{(\mathbf{s})} - R_2^{(\mathbf{r})}$ and it is evident that it takes positive and negative values. Therefore, these systems are not ordered.

If, for example, we consider a system with seven components and we are interesting in splitting the components in three modules, according to Proposition 4.9 (and under the assumptions made in that proposition), it is better to put two components in the first two modules and three components in the last one, since $(1,1,5) \overset{m}{\geq} (1,2,4) \overset{m}{\geq} (1,3,3) \overset{m}{\geq} (2,2,3)$. In the following result, we provide the optimal component allocation vector for systems with redundancy at module level satisfying these assumptions.

Corollary 4.13. *Under the assumption of Proposition 4.9, the best system with k modules and redundancy at modules' level is that formed by modules with components distributed according to the vector*

$$\mathbf{s}^\star = (\overbrace{s, s, \dots, s}^{k-r}, \overbrace{s+1, s+1, \dots, s+1}^{r}),$$

where s and $r \in \mathbb{Z}_+$ are the unique integers such that $n = sk + r$ and $0 \le r < k$.

Proof. We need to prove that

$$\mathbf{s} \overset{m}{\geq} \mathbf{s}^\star \tag{4.11}$$

for all vectors $\mathbf{s} = (n_1, \dots, n_k)$ with $n_1 \le \dots \le n_k$ and $\sum_{i=1}^k n_i = n$.

Given a vector $\mathbf{s}$, then there exists a number $\upsilon \in \mathbb{N}$ and a finite sequence of vectors $\{\mathbf{s}_i\}_{i\in\{1,\dots,\upsilon\}}$ with $\mathbf{s}_i = (n_1^{(i)}, \dots, n_k^{(i)})$, $n_1^{(i)} \le \dots \le n_k^{(i)}$ and $\sum_{j=1}^k n_j^{(i)} = n$ for all $i \in \{1, \dots, \upsilon\}$, such that $\mathbf{s}_1 = \mathbf{s}$, $\mathbf{s}_\upsilon = \mathbf{s}^\star$, and for any pair of consecutive vectors $\mathbf{s}_i$ and $\mathbf{s}_{i+1}$ there exist j_0 and $j_1 \in \mathbb{N}$ (which depend on i) with $1 \le j_0 < j_1 \le k$ such that

$$\begin{aligned} n_{j_0}^{(i+1)} &= n_{j_0}^{(i)} + 1, \\ n_{j_1}^{(i+1)} &= n_{j_1}^{(i)} - 1, \\ n_j^{(i+1)} &= n_j^{(i)}, \text{ for all } j \in \{1, 2, \dots, k\} \backslash \{j_0, j_1\}. \end{aligned}$$

Note that $\mathbf{s}_{i+1}$ is obtained from $\mathbf{s}_i$ by moving one component from the module M_{j_1} to the module M_{j_0} with $j_0 < j_1$. Furthermore, it is cleat that

$$\begin{aligned} \sum_{j=1}^m n_j^{(i)} &= \sum_{j=1}^m n_j^{(i+1)} \text{ for all } m = 1, 2, \dots, j_0 - 1; \\ \sum_{j=1}^m n_j^{(i)} &< \sum_{j=1}^m n_j^{(i+1)} \text{ for all } m = j_0, j_0 + 1, \dots, j_1 - 1; \\ \sum_{j=1}^m n_j^{(i)} &= \sum_{j=1}^m n_j^{(i+1)} \text{ for all } m = j_1, j_1 + 1, \dots, k. \end{aligned}$$

Therefore, the condition $\mathbf{s}_i \overset{m}{\geq} \mathbf{s}_{i+1}$ holds for all $i = 1, 2, \dots, \upsilon - 1$ and we have that

$$\mathbf{s} = \mathbf{s}_1 \overset{m}{\geq} \mathbf{s}_2 \overset{m}{\geq} \dots \overset{m}{\geq} \mathbf{s}_{\upsilon-1} \overset{m}{\geq} \mathbf{s}_\upsilon = \mathbf{s}^\star.$$

Finally, from (4.11) and Proposition 4.9, we obtain that $R_2^{(\mathbf{s})} \le R_2^{(\mathbf{s}^\star)}$, where $R_2^{(\mathbf{s}^\star)}$ represents the reliability function of the system with redundancy at module level and components distributed in the modules according to the vector $\mathbf{s}^\star$. ∎

4.2. Independent components and dependent modules

In this subsection, we present a result for the active redundancy with $m-1$ independent spares, $\bar{q}_{m:m}(u) = 1 - (1-u)^m$, applied to systems with heterogeneous and independent components not necessarily connected in series in each module. This result generalizes Theorem 1 in [31] for active redundancies, even more, it proves that the BP-principle, mentioned in the introduction section, holds for a more general case.

Proposition 4.14. *If we consider an active redundancy with $m-1$ independent spares, $\bar{q}_{m:m}(u) = 1 - (1-u)^m$, and the components in each module are independent, then $R_1 \ge R_2$.*

Proof. We provide here the proof for active redundancy with $m = 2$, that is, $\bar{q}(u) = \bar{q}_{2:2}(u) = 2u - u^2$ for $u \in [0,1]$ (as mentioned above, this is equivalent to add an independent spare in parallel to each component/module). A similar reasoning can be followed for proving the general case ($m \ge 3$). The reliability function of the system with redundancy at the components' level is

$$\begin{aligned} R_1(t) = \bar{Q}^* \Big(&\bar{Q}_{M_1}\left(\bar{q}(\bar{F}_1(t)), \dots, \bar{q}(\bar{F}_{n_1}(t))\right), \dots, \\ &\bar{Q}_{M_k}\left(\bar{q}(\bar{F}_{n_1+\dots+n_{k-1}+1}(t)), \dots, \bar{q}(\bar{F}_n(t))\right)\Big). \end{aligned}$$

On the other hand, the reliability function of the system with redundancy at the modules' level is

$$\begin{aligned} R_2(t) = \bar{Q}^* \Big(&\bar{q}\left(\bar{Q}_{M_1}(\bar{F}_1(t), \dots, \bar{F}_{n_1}(t))\right), \dots, \\ &\bar{q}\left(\bar{Q}_{M_k}(\bar{F}_{n_1+\dots+n_{k-1}+1}(t), \dots, \bar{F}_n(t))\right)\Big). \end{aligned}$$

To prove $R_1(t) \ge R_2(t)$, we only need to show that

$$\bar{q}\left(\bar{Q}_{M_1}(\bar{F}_1(t), \dots, \bar{F}_{n_1}(t))\right) \le \bar{Q}_{M_1}\left(\bar{q}(\bar{F}_1(t)), \dots, \bar{q}(\bar{F}_{n_1}(t))\right). \tag{4.12}$$

A similar reasoning can be done for the rest of modules. Let us assume that the module M_1 has got r_1 minimal path sets $\{P_1, P_2, \dots, P_{r_1}\}$, and P_i has got m_i components for $i = 1, \dots, r_1$. Note that these minimal path sets can share some components. If we apply the active redundancy at the module M_1, the resulting system (with lifetime $T_{M_1}^{(2)}$) has got $2r_1$ minimal path sets $\{P_1', P_2', \dots, P_{2r_1}'\}$. It is not difficult to see that $P_i' = P_i$ for all $i = 1, 2, \dots, r_1$ and $P_{r_1+i}' = L_i$ for all $i = 1, 2, \dots, r_1$, where L_i coincides with the minimal path set P_i but using the spares instead of the original components.

On the other hand, if we apply the active redundancy to M_1 at the components' level, then the resulting system (with lifetime $T_{M_1}^{(1)}$) has got $s_1 = \sum_{i=1}^{r_1} 2^{m_i}$ minimal path sets $\{P_1'', P_2'', \dots, P_{s_1}''\}$. It is straightforward to show that the minimal path sets $\{P_1, P_2, \dots, P_{r_1}\}$ and $\{L_1, L_2, \dots, L_{r_1}\}$ are included in $\{P_1'', P_2'', \dots, P_{s_1}''\}$. Therefore, we have proved that

$$T_{M_1}^{(1)} = \max(T_{M_1}^{(2)}, W),$$

with $W = \max_{P \subseteq \mathcal{P}}\left(\min_{X_i \in P}(X_i)\right)$, where $\mathcal{P} = \{P_1'', \dots, P_{s_1}''\} \backslash \{P_1, \dots, P_{r_1}, L_1, \dots, L_{r_1}\}$. Note that the components in each minimal path set are independent. Then, the corresponding reliability functions of $T_{M_1}^{(1)}$ and $T_{M_1}^{(2)}$ are ordered and the inequality (4.12) holds. ∎

This result means that the active redundancy at the components' level is always better than that at the module level for any $\bar{F}_1, \dots, \bar{F}_n$, any $\bar{Q}_{M_1}, \dots, \bar{Q}_{M_k}$ and $\bar{Q}^*$ (i.e. any structure and any dependence among modules) whenever components within modules and spares are independent. The following example illustrates the theoretical result of Proposition 4.14.

Example 4.15. Let us consider that the spares are independent and they are added in a parallel configuration, then $\bar{q}_{2:2}(u) = 2u - u^2$ for $u \in [0,1]$. We consider three dependent modules forming a 2-out-of-3 system with lifetime T. The three modules have the same structure and they are formed by three independent but not identically distributed components. The modules' lifetimes are given by $X_{M_j} = \min(X_j, \max(Y_j, Z_j))$ for $j = 1, 2, 3$ (see Fig. 7). Then, the structure among the modules and the structure among the components within the modules are

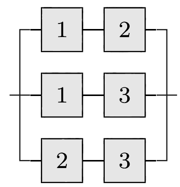
$$\bar{Q}^*(u_1, u_2, u_3) = \hat{C}(u_1, u_2, 1) + \hat{C}(u_1, 1, u_3) + \hat{C}(1, u_2, u_3) - 2\hat{C}(u_1, u_2, u_3)$$

and

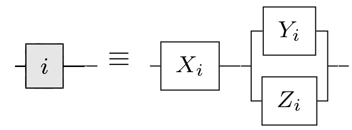
$$\bar{Q}_M(u_1, u_2, u_3) = u_1u_2 + u_1u_3 - u_1u_2u_3,$$

respectively, where $u_1, u_2, u_3 \in [0,1]$ and $\hat{C}$ is a survival copula which models the dependence among modules.

We assume that X_i, Y_i, Z_i have exponential distributions with hazard rate i, Weibull distributions with scale parameter 1 and shape parameter $i/10$ and Weibull distributions with scale parameter 2 and shape

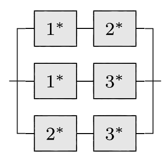


(a) Modules 1, 2 and 3 forming a 2-out-of-3 system

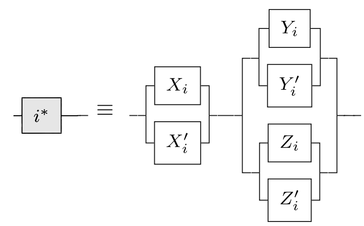


(b) Modular structure for $i = 1, 2, 3$

Fig. 7. Block diagram of three modules forming a 2-out-of-3 system without redundancy (a). Structure in each module (b).



(a) Modules 1,2 and 3 with redundancy at components' level forming a 2-out-of-3 system.



(b) Modular structure with redundancy at components' level for $i = 1, 2, 3$

Fig. 8. Block diagram of three modules with redundancy at component level forming a 2-out-of-3 system (a). Modular structure with redundancy at component level (b).

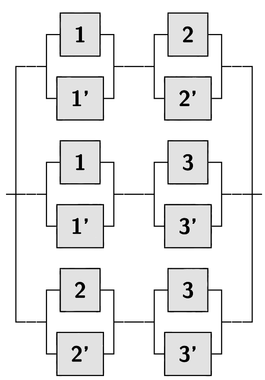


Fig. 9. Block diagram of three modules forming a 2-out-of-3 system with redundancy at module level.

parameter $i/2$, for $i = 1, 2, 3$, respectively. Then, the reliability function of the system T is

$$\bar{F}_T(t) = \bar{Q}^*\left(\bar{Q}_M\left(e^{-t}, e^{-t^{0.1}}, e^{-2t^{0.5}}\right), \bar{Q}_M\left(e^{-2t}, e^{-t^{0.2}}, e^{-2t}\right), \bar{Q}_M\left(e^{-3t}, e^{-t^{0.3}}, e^{-2t^{1.5}}\right)\right), \ t \geq 0.$$

Hence the reliability function of the system with redundancy at the components' level (see Fig. 8) is

$$R_1(t) = \bar{Q}^*\left(\bar{H}_1(t), \bar{H}_2(t), \bar{H}_3(t)\right)$$

where

$$\bar{H}_j(t) = \bar{Q}_M\left(\bar{q}(e^{-jt}), \bar{q}(e^{-t^{j/10}}), \bar{q}(e^{-2t^{j/2}})\right),$$

for $j = 1, 2, 3$.

Finally, the reliability function of the system with modular redundancy (see Fig. 9) is

$$R_2(t) = \bar{Q}^*\left(\bar{G}_1(t), \bar{G}_2(t), \bar{G}_3(t)\right),$$

where $\bar{G}_j(t) = \bar{q}\left(\bar{Q}_M(e^{-jt}, e^{-t^{j/10}}, e^{-2t^{j/2}})\right)$ for $j = 1, 2, 3$.

We study two cases, when modules are independent, i.e.,

$$\hat{C}(u_1, u_2, u_3) = u_1 u_2 u_3$$

for all $u_1, u_2, u_3 \in [0, 1]$, and when modules are dependent assembled by the following Clayton copula

$$\hat{C}(u_1, u_2, u_3) = \left(u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} - 2\right)^{-1/\theta} \quad u_1, u_2, u_3 \in [0, 1] \tag{4.13}$$

and $\theta > 0$. Note that $\hat{C}(u_1, u_2, 0) = \hat{C}(u_1, 0, u_3) = \hat{C}(0, u_2, u_3) = \hat{C}(0, 0, 0) := 0$. Fig. 10 shows the respective reliability functions $\bar{F}_T$ (blue), R_1 (black) and R_2 (red) for the case of independent modules (left) or when they are dependent (right), with a Clayton survival copula, defined as in (4.13), with $\theta = 16$. As it can be seen, $R_1 \geq R_2$ which is according to Proposition 4.14.

4.3. Dependent components within the modules and dependent modules

Presently, let us consider that the components within each module are dependent, then the reliability function of the jth module is

$$\bar{F}_{M_j}(t) = \bar{Q}_{M_j}\left(\bar{F}_{n_1+\cdots+n_{j-1}+1}(t), \ldots, \bar{F}_{n_1+\cdots+n_j}(t)\right),$$

where $\bar{Q}_{M_j}$ defines the structure within the jth module, i.e., it indicates the way in which the components are connected to each other and the dependence among them. If we apply the redundancy $\bar{q}$ to the jth module, the reliability of the resulting module is defined as in (4.1) where

$$\bar{G}_j(t) = \bar{q}(\bar{F}_{M_j}(t)) = \bar{q}\left(\bar{Q}_{M_j}\left(\bar{F}_{n_1+\cdots+n_{j-1}+1}(t), \ldots, \bar{F}_{n_1+\cdots+n_j}(t)\right)\right). \tag{4.14}$$

In the following result we obtain sufficient conditions in order to compare both redundancy methods (components versus modules) in the case of dependent components within each module.

Proposition 4.16. *If the components in each module are dependent and the distortion $\bar{q}$ satisfies*

$$\bar{Q}_{M_j}\left(\bar{q}(v_1), \ldots, \bar{q}(v_{n_j})\right) \geq (\leq) \bar{q}\left(\bar{Q}_{M_j}(v_1, \ldots, v_{n_j})\right) \tag{4.15}$$

for all $v_1, \ldots, v_{n_j} \in [0, 1]$ and $j = 1, \ldots, k$, then $R_1 \geq (\leq) R_2$ for any modular structure $\bar{Q}^*$.

Proof. Observe that the reliability function of the system with redundancy at the components' level is

$$R_1(t) = \bar{Q}^*(\bar{H}_1(t), \ldots, \bar{H}_k(t))$$

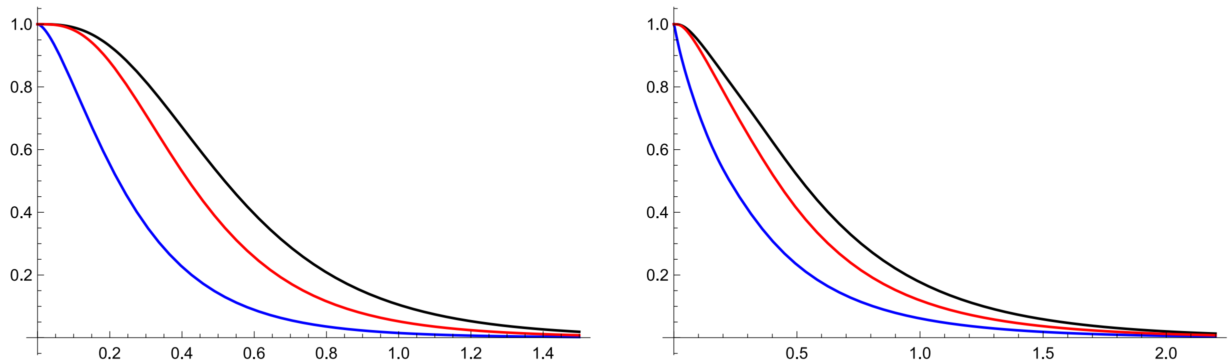


Fig. 10. Reliability functions $\bar{F}_T$ (blue), R_1 (black) and R_2 (red) for the system considered in Example 4.15, when the modules are independent (left) or dependent (right) with a Clayton copula.

where

$$\bar{H}_j(t) = \bar{Q}_{M_j}\left(\bar{q}(\bar{F}_{n_1+\cdots+n_{j-1}+1}(t)), \dots, \bar{q}(\bar{F}_{n_1+\cdots+n_j}(t))\right),$$

for $j = 1, \dots, k$. On the other hand, the reliability function of the system with modular redundancy is given in (4.1) and $\bar{G}_j$ is defined in (4.14). Then, from (4.15), we have that $\bar{H}_j \geq (\leq) \bar{G}_j$ for $j = 1, \dots, k$ and hence $R_1 \geq (\leq) R_2$. ■

The next example shows how to apply Proposition 4.16.

Example 4.17. Let us consider that the components are connected in series in each module, then $\bar{Q}_{M_j} = \hat{C}_j$, where $\hat{C}_j$ is the survival copula which models the dependence between the components in the jth module, for $j = 1, \dots, k$. Specifically, we assume that all the modules have two dependent components and that $\hat{C}_j$ is the following FGM copula

$$\hat{C}_j(u, v) = uv(1 + \theta(1-u)(1-v)), \quad u, v \in [0,1],$$

where $\theta \in [-1,1]$, for $j = 1, \dots, k$. On the other hand, we suppose that the redundancy method is $\bar{q}_\alpha$ as defined in (4.6). Then we plot $\hat{C}(\bar{q}_\alpha(u), \bar{q}_\alpha(v)) - \bar{q}_\alpha(\hat{C}(u,v), \hat{C}(u,v))$ for $\alpha \in \{0.3, 0.6, 1, 2\}$ and $\theta = 0.5$ in Fig. 11 and it can be seen that condition (4.15) holds. Then, from Proposition 4.16, we know that $R_1 \geq R_2$ for any $\bar{Q}^*$ and any $\bar{F}_1, \dots, \bar{F}_n$.

In the following example, we show that, in some cases, R_1 and R_2 are not ordered and therefore, the systems cannot be compared.

Example 4.18. We consider two modules with two dependent components connected in series in each one. Suppose that the components in the first module have exponential distributions with hazard rates 1 and 3, and those in the second module have exponential distributions with hazard rates 2 and 4. Thus,

$$\bar{G}_1(t) = \bar{q}\left(\hat{C}(e^{-t}, e^{-3t})\right), \quad \bar{G}_2(t) = \bar{q}\left(\hat{C}(e^{-2t}, e^{-4t})\right)$$

and

$$\bar{H}_1(t) = \hat{C}\left(\bar{q}(e^{-t}), \bar{q}(e^{-3t})\right), \quad \bar{H}_2(t) = \hat{C}\left(\bar{q}(e^{-2t}), \bar{q}(e^{-4t})\right),$$

where $\hat{C}$ defines the dependence structure between the components in each module. We consider the redundancy $\bar{q}_\alpha$, defined as in (4.6), with $\alpha = 0.3$ and $\hat{C}$, defined as in (4.5), with $\theta = 1$. We assume that the modules are connected in series under two different cases, when modules are independent and when they are dependent. For the case of independent modules, we get $R_2(0.1) = 0.9067 \leq 0.9291 = R_1(0.1)$ and $R_2(1) = 0.1286 \geq 0.1051 = R_1(1)$, so the reliability functions cross each other. For the case of dependent modules connected by a Clayton copula, as defined in (4.5), for $\theta = 5$, we have $R_2(0.1) = 0.9146 \leq 0.9339 = R_1(0.1)$ and $R_2(1) = 0.2929 \geq 0.2558 = R_1(1)$, so the reliability functions are not ordered.

5. Identically distributed components within modules

In this section, we consider that modules and components can be dependent and that the components of the jth module are identically distributed (i.d.), i.e,

$$F_{n_1+\cdots+n_{j-1}+1} = \cdots = F_{n_1+\cdots+n_j} = F_j^*,$$

for $j = 1, \dots, k$. In this case, the reliability function of the jth module can be written as

$$\bar{F}_{M_j}(t) = \bar{q}_{M_j}(\bar{F}_j^*(t)),$$

where $\bar{q}_{M_j}(u) = \bar{Q}_{M_j}(u, \dots, u)$ for $u \in [0,1]$, is a univariate distortion function determined by the modular structure and the dependence between its components. For instance, if we assume that the jth module has n_j dependent components connected in series where its dependence structure is defined by a FGM copula, then

$$\bar{q}_{M_j}(u) = u^{n_j}(1 + \theta(1-u)^{n_j}), \quad u \in [0,1], \tag{5.1}$$

with $\theta \in [-1,1]$. Another example is to consider that the jth module is a 2-out-of-3 system with independent components, then

$$\bar{q}_{M_j}(u) = 3u^2 - 2u^3, \quad u \in [0,1],$$

(see Table 1 in [43]). Other structures can be found in Table 2 in [44].

In the following result we compare systems with redundancy at component and modular levels.

Proposition 5.1. *If the components in each module are i.d. and*

$$\bar{q}_{M_j}(\bar{q}(u)) \geq (\leq)\, \bar{q}(\bar{q}_{M_j}(u)) \tag{5.2}$$

for all $u \in [0,1]$ and $j = 1, \dots, k$, then $R_1 \geq (\leq) R_2$ for any $\bar{F}_1^*, \dots, \bar{F}_k^*$ and for any modular and dependence structure $\bar{Q}^*$.

Proof. From (4.1), the reliability function of the system with redundancy at module level is

$$R_2(t) = \bar{Q}^*\left(\bar{q}\left(\bar{q}_{M_1}(\bar{F}_1^*(t))\right), \dots, \bar{q}\left(\bar{q}_{M_k}(\bar{F}_k^*(t))\right)\right),$$

meanwhile, the reliability function of the system with redundancy at component level is

$$R_1(t) = \bar{Q}^*\left(\bar{q}_{M_1}\left(\bar{q}(\bar{F}_1^*(t))\right), \dots, \bar{q}_{M_k}\left(\bar{q}(\bar{F}_k^*(t))\right)\right).$$

Then, from (5.2), we get $\bar{q}_{M_j}(\bar{q}(\bar{F}_j^*(t))) \geq (\leq)\, \bar{q}(\bar{q}_{M_j}(\bar{F}_j^*(t)))$ and therefore $R_1 \geq (\leq) R_2$. ■

Of course, if $\bar{q} = \bar{q}_{M_j}$ for all j, then $R_1 = R_2$. In the following example, we show how Proposition 5.1 can be applied to compare the reliability functions between coherent systems with modules where theirs components are d.i.d. and are connected in series.

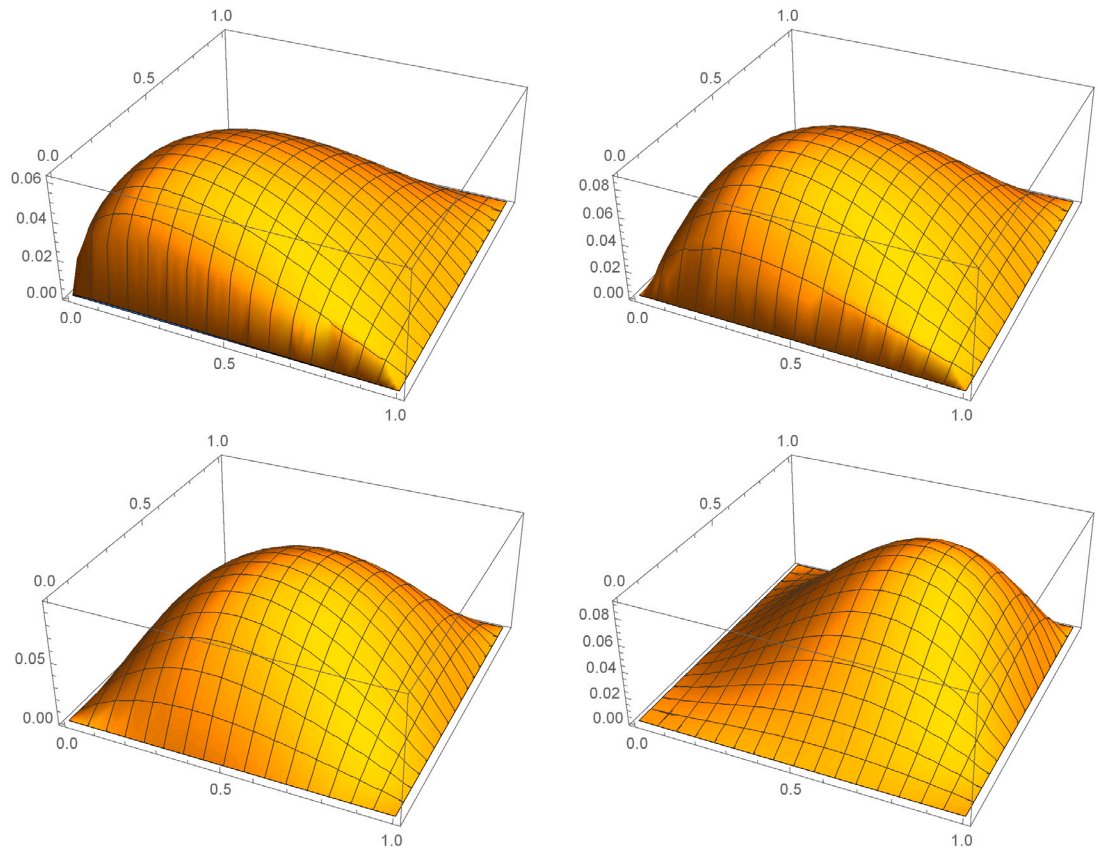


Fig. 11. Plots of $\hat{C}(\bar{q}_\alpha(u), \bar{q}_\alpha(v)) - \bar{q}_\alpha(\hat{C}(u,v), \hat{C}(u,v))$ in Example 4.17 for $\alpha \in \{0.3, 0.6, 1, 2\}$ (top left, top right, bottom left, bottom right).

Example 5.2. Let us consider that the first module has two i.d. components connected in series with a FGM survival copula as defined in (5.1) for $\theta \in [-1, 1]$, then

$$\bar{q}_{M_1}(u) = \hat{C}(u,u) = u^2(1 + \theta(1-u)^2), \quad u \in [0,1].$$

The other modules have the same structure. On the other hand, we assume that the redundancy method is minimal repair, i.e., $\bar{q}(u) = \bar{q}_{mr}(u) = u(1 - \log u)$ for $u \in [0,1]$. Note that

$$\bar{q}_{mr}(\bar{q}_{M_j}(u)) = u^2(1 + \theta(1-u)^2)(1 - 2\log u - \log(1 + \theta(1-u)^2))$$

and

$$\bar{q}_{M_j}(\bar{q}_{mr}(u)) = u^2(1 - \log u)^2(1 + \theta(1 - u + u\log u)^2).$$

In Fig. 12 (left), we plot $\bar{q}_{M_j}(\bar{q}_{mr}(u)) - \bar{q}_{mr}(\bar{q}_{M_j}(u))$ for $u \in [0,1]$ and $\theta \in \{-1, -0.5, 0, 0.5, 1\}$. It can be seen that (5.2) holds and, therefore, we can apply Proposition 5.1 obtaining $R_1 \geq R_2$ for any $\bar{F}_1^*, \ldots, \bar{F}_k^*$ and any $\bar{Q}^*$.

Of course Proposition 5.1 can be used for systems with modules whose structure is different from components connected in series. Let us see an example.

Example 5.3. Now, we consider that the first module has lifetime $X_{M_1} = \min(X_1, \max(X_2, X_3))$, where X_1, X_2, X_3 are i.i.d. and that the other $k-1$ modules have the same structure. Then $\bar{q}_{M_j}(u) = 2u^2 - u^3$ for $u \in [0,1]$. Next, we assume that the original component and its spare are dependent and that the redundancy distortion is defined as in (4.4) with a Clayton copula with $\theta = 1$, i.e., $\bar{q}(u) = 2u/(1+u)$. Hence,

$$\bar{q}(\bar{q}_{M_j}(u)) = \frac{2\left(2u^2 - u^3\right)}{1 + \left(2u^2 - u^3\right)} = \frac{2u^2(2-u)}{1 + 2u^2 - u^3}$$

and

$$\bar{q}_{M_j}(\bar{q}(u)) = 2\left(\frac{2u}{1+u}\right)^2 - \left(\frac{2u}{1+u}\right)^3 = \frac{8u^2}{(1+u)^3}.$$

Then,

$$\bar{q}_{M_j}(\bar{q}(u)) - \bar{q}(\bar{q}_{M_j}(u)) = \frac{2u^2(1-u)^2}{(1+u)^3\left(1 + 2u^2 - u^3\right)}\left(2 + u^2 - u\right) \geq 0,$$

and therefore (5.2) holds (see Fig. 12 (right)). Then, from Proposition 5.1, we obtain $R_1 \geq R_2$ for any $\bar{F}_1^*, \ldots, \bar{F}_k^*$ and any $\bar{Q}^*$.

Remark 5.4. Proposition 5.1 can be easily generalized for two different distortions $\bar{q}_i$ for $i = 1, 2$, where the first one is applied to the components and the second one to the modules. Hence, condition (5.2) can be rewritten as

$$\bar{q}_{M_j}(\bar{q}_1(u)) \geq (\leq)\, \bar{q}_2(\bar{q}_{M_j}(u)), \tag{5.3}$$

for all $u \in [0,1]$ and $j = 1, \ldots, k$. If (5.3) holds then $R_1 \geq (\leq) R_2$. Let us see an example. We assume, as in Example 5.3, that the lifetime of each module is $X_{M_j} = \min(X_1, \max(X_2, X_3))$, where X_1, X_2, X_3 are i.i.d. Now, we suppose that $\bar{q}_1(u) = 2u - u^2$ and $\bar{q}_2(u) = u + u^\alpha - u^{\alpha+1}$ for $u \in [0,1]$ and $0 < \alpha < 1$. Observe that if $\alpha = 1$ then $\bar{q}_1 = \bar{q}_2$ and, for $0 < \alpha < 1$, we have $\bar{q}_2 \geq \bar{q}_1$ since $\bar{q}_2(u) = 1 - (1-u)(1-u^\alpha)$ is decreasing in α. However, depending on the parameter α, the modular redundancy could be better than the redundancy at component level, as it can be seen in Fig. 13. In particular, from Fig. 13, we get that $R_1 \leq R_2$ for $\alpha = 0.3, 0.4, 0.5, 0.6$ and $R_1 \geq R_2$ for $\alpha = 1$. For $\alpha = 0.7, 0.8, 0.9$ (red, green and orange lines), the reliability functions are not ordered.

Table 2 contains comparisons between R_1 and R_2 when active redundancy $\bar{q}_1(u) = 2u - u^2$ is applied at components' level (R_1), and

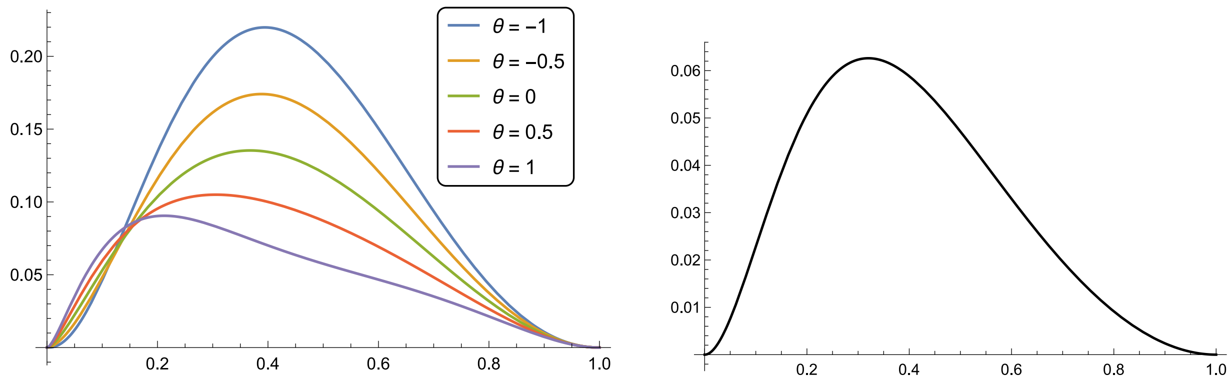


Fig. 12. Plots of $\bar{q}_{M_j}(\bar{q}_{mr}(u)) - \bar{q}_{mr}(\bar{q}_{M_j}(u))$ for $u \in [0,1]$ and $\theta \in \{-1, -0.5, 0, 0.5, 1\}$ in Example 5.2 (left) and plot of $\bar{q}_{M_j}(\bar{q}(u)) - \bar{q}(\bar{q}_{M_j}(u))$ for $u \in [0,1]$ in Example 5.3 (right).

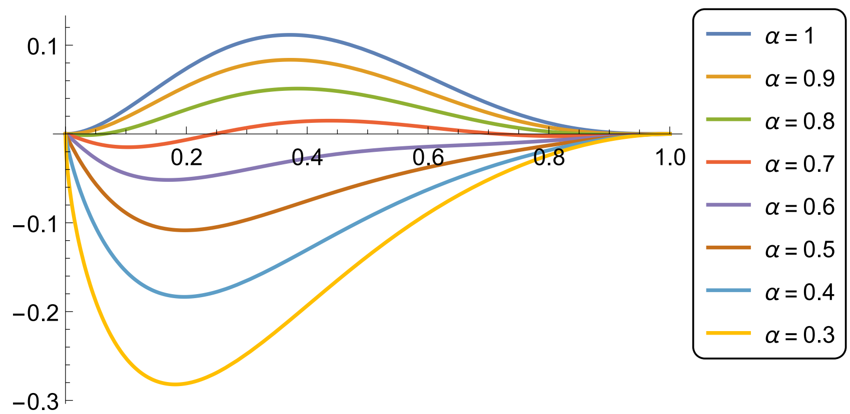


Fig. 13. Plots of $\bar{q}_{M_j}(\bar{q}_1(u)) - \bar{q}_2(\bar{q}_{M_j}(u))$ for $u \in [0,1]$ and $\alpha \in \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ in Remark 5.4.

Table 2
Comparisons for coherent systems with three components within modules with the same module structure under component and module level redundancies. The value 1 indicates that $R_1 \geq R_2$, the value 2 means that $R_1 \leq R_2$ and the value 0 indicates that R_1 and R_2 are not ordered.

	Module	$\bar{q}_{M_j}(u)$	α values for $\bar{q}_2$							
			0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1	Series	u^3	2	0	0	0	0	0	0	1
2	$\min(X_1, \max(X_2, X_3))$	$2u^2 - u^3$	2	2	2	2	0	0	0	1
3	2-out-of-3	$3u^2 - 2u^3$	2	2	2	0	0	0	0	1
4	$\max(X_1, \min(X_2, X_3))$	$u + u^2 - u^3$	2	2	2	0	0	0	0	1
5	Parallel	$3u - 3u^2 + u^3$	2	2	2	2	2	2	2	1&2

when redundancy $\bar{q}_2$ defined by (4.6) is applied at modules' level (R_2) for coherent systems with three i.i.d. components within the k modules. Therefore, we need to study if (5.3) holds. The value 1 indicates that $R_1 \geq R_2$ holds, the value 2 means that $R_1 \leq R_2$ holds and the value 0 indicates that R_1 and R_2 are not ordered. Note that, for parallel modules and $\alpha = 1$, $\bar{q}_{M_j}(\bar{q}_1(u)) = \bar{q}_2(\bar{q}_{M_j}(u))$ for $u \in [0,1]$.

The last column ($\alpha = 1$) in Table 2 is according to Proposition 4.14. As it is clear from Table 2, one redundancy type is not superior to the other for all values of α and all types of module distortions. In particular, if $\alpha = 0.3$, then the module level redundancy becomes better for any structure at the modules and any lifetime distribution at the components.

Table 3 displays comparisons between R_1 and R_2 when the redundancy method is the same but the module structure is different. Thus, we need to study if (5.2) holds. We consider three different redundancy methods: active redundancy (one i.i.d. spare is added in parallel), minimal repair and a spare added in parallel assembled to the original one by a Clayton copula as defined in (4.5) for $\theta = 5$. We consider the same module structures as those in Table 2.

The first main diagonals of the left and center tables in Table 3 are according to Proposition 4.14. Observe that there is no difference between the results obtained for the active redundancy and minimal repair (except in the case 5-5 where we get $R_1 = R_2$ in the left table). The redundancy at component level is more effective when the components within the modules are connected in parallel in the system with components' redundancy (last rows in Table 3) and also when the components form a series module in the system with modular redundancy (first column in Table 3). However, the redundancy at module level is better when the components have a parallel configuration in the system with modular redundancy (last columns in Table 3).

The following result is similar to Proposition 4.9 but for modules with i.i.d. components connected in parallel.

Proposition 5.5. *Let $T_{\mathbf{s}}$ and $T_{\mathbf{r}}$ be the lifetimes of two systems under modular redundancy with the same modular structure, given by $\bar{Q}^*$, and possibly dependent modules with i.i.d. components connected in parallel and common reliability function $\bar{F}$ for all components. Let $\mathbf{s} = (n_1, \ldots, n_k)$ and $\mathbf{r} = (m_1, \ldots, m_k)$ be the allocation vectors of components by modules and, $R_2^{(\mathbf{s})}$ and $R_2^{(\mathbf{r})}$ the reliability functions for both systems $T_{\mathbf{s}}$ and $T_{\mathbf{r}}$, respectively. Assume that $n_1 \leq \cdots \leq n_k$, $m_1 \leq \cdots \leq m_k$, that the distortion $\bar{Q}^*$ is Schur-concave and that $\bar{q}$ is concave. If $\mathbf{s} \overset{m}{\geq} \mathbf{r}$ then $R_2^{(\mathbf{s})} \leq R_2^{(\mathbf{r})}$.*

Proof. Let us consider a fixed value $t \geq 0$, and denote $\beta_j = \bar{F}_{M_j}(t) = 1 - (1 - \bar{F}(t))^{n_j}$ for all $j = 1, \ldots, k$, then $\beta_1 \leq \cdots \leq \beta_k$, because $n_1 \leq \cdots \leq n_k$. Analogously, we define $\gamma_j = \bar{F}_{\tilde{M}_j}(t) = 1 - (1 - \bar{F}(t))^{m_j}$ for all $j = 1, \ldots, k$, and therefore, $\gamma_1 \leq \cdots \leq \gamma_k$. Thus, the reliability functions of $T_{\mathbf{s}}$ and $T_{\mathbf{r}}$ at time t can be rewritten as $R_2^{(\mathbf{s})}(t) = \bar{Q}^*\left(\bar{q}(\beta_1), \ldots, \bar{q}(\beta_k)\right)$ and $R_2^{(\mathbf{r})}(t) = \bar{Q}^*\left(\bar{q}(\gamma_1), \ldots, \bar{q}(\gamma_k)\right)$, respectively. Firstly, we observe

Table 3
Comparisons for coherent systems with three components within modules with different module structure under component and module level redundancies. the value 1 indicates that $R_1 \geq R_2$ holds, the value 2 means that $R_1 \leq R_2$ holds and the value 0 indicates that R_1 and R_2 are not ordered. the values 1–5 represents the structures given in Table 2.

	$\bar{q}_{2:2}(u) = 2u - u^2$					$\bar{q}_{mr}(u) = u(1 - \log u)$					$\bar{q}(u) = 1 - \left(2(1-u)^5 - 1\right)^{-1/5}$				
	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
1	1	2	2	2	2	1	2	2	2	2	0	2	2	2	2
2	1	1	0	2	2	1	1	0	2	2	1	0	0	2	2
3	1	1	1	0	2	1	1	1	0	2	1	1	1	2	2
4	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2
5	1	1	1	1	1&2	1	1	1	1	1	1	1	1	1	1

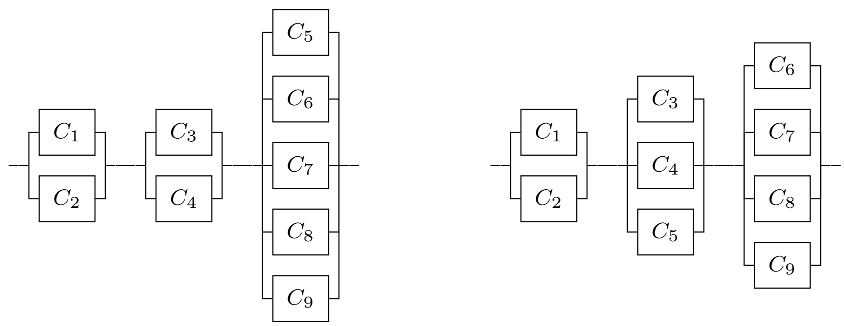


Fig. 14. Block diagrams of the two series–parallel systems without any redundancy mechanism considered in Example 5.6 with components allocation vector $\mathbf{s} = (2, 2, 5)$ (left) and $\mathbf{r} = (2, 3, 4)$ (right).

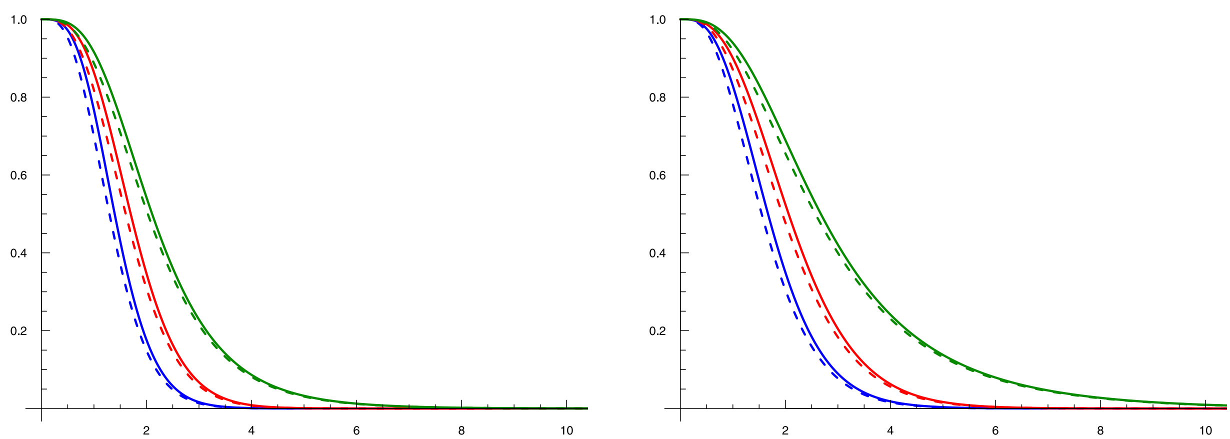


Fig. 15. Reliability functions $R_2^{(\mathbf{s})}$ (dashed line) and $R_2^{(\mathbf{r})}$ (solid line) for the redundancy mechanisms $\bar{q}_{2:2}$ (blue), $\bar{q}_{mr}$ (red) and $\bar{q}_\alpha$ for $\alpha = 0.3$ (green) when the modules are independent (left) or dependent (right) with a Gumbel–Hougaard copula.

that $1 - (1 - \bar{F}(t))^d$ is concave in d for each t. Then, from Theorem 5.A.1 in [33], we know that $\mathbf{s} \overset{m}{\geq} \mathbf{r}$ implies $(\beta_1, \ldots, \beta_k) \overset{w}{\geq} (\gamma_1, \ldots, \gamma_k)$. Let us denote $\varphi(\beta_1, \ldots, \beta_k) = \bar{Q}^*(\bar{q}(\beta_1), \ldots, \bar{q}(\beta_k))$ and $\varphi(\gamma_1, \ldots, \gamma_k) = \bar{Q}^*(\bar{q}(\gamma_1), \ldots, \bar{q}(\gamma_k))$. Therefore, we need to prove that

$$\varphi(\beta_1, \ldots, \beta_k) \leq \varphi(\gamma_1, \ldots, \gamma_k)$$

whenever $(\beta_1, \ldots, \beta_k) \overset{w}{\geq} (\gamma_1, \ldots, \gamma_k)$. To do this, from Lemma 2.4, we need to show that the function φ is increasing and Schur-concave in $(\beta_1, \ldots, \beta_k)$. It is clearly that φ is increasing in β_i since both functions, $\bar{Q}^*$ and $\bar{q}$, are positive-valued and increasing. Now, from Table 2 in [33] and from the assumptions $\bar{Q}^*$ Schur-concave and $\bar{q}$ concave, we have that φ is Schur-concave. ∎

Observe that if the modules are independent and connected in series then $\bar{Q}^*(u_1, \ldots, u_k) = \prod_{i=1}^k u_i$ and this distortion is Schur-concave. In the case of dependent modules connected in series $\bar{Q}^* = \hat{C}$ where $\hat{C}$ is a survival copula. From [39], pages 104 and 134, we know that the most common copulas, like for instance the family of Archimedean copulas, are Schur-concave. Then, we can apply Proposition 5.5 to this type of systems.

If we consider active redundancies with $m-1$ spares, i.e., we allocate $m-1$ i.i.d. spares in parallel, then $\bar{q}_{m:m}(u) = 1 - (1-u)^m$ and it is easy to check that this function is concave for all $m \geq 2$. Therefore, we can use Proposition 5.5 for this type of redundancy.

Another redundancy method is determined by the distortion $\bar{q}_\alpha(u) = u + u^\alpha - u^{\alpha+1}$ defined in (4.6). Its second derivative is

$$\bar{q}_\alpha''(u) = \alpha u^{\alpha-2}(\alpha - 1 - u - \alpha u) \leq 0,$$

if $0 < \alpha \leq 1$. So we can also apply Proposition 5.5 for this redundancy. Note that for $\alpha > 1$, the function $\bar{q}_\alpha$ is neither concave nor convex.

Finally, it is straightforward to prove that the distortion associated to minimal repairs, $\bar{q}_{mr}(u) = u(1 - \log u)$, is also concave, and therefore, it can be used in examples where Proposition 5.5 applies. Next, we provide an illustrative example on how to apply this proposition.

Example 5.6. Let us consider two systems with three modules, each of them connected in series. Each module has independent components connected in parallel which have exponential distributions with hazard rates equal to 1. Let us assume that $\mathbf{s} = (2, 2, 5)$ and $\mathbf{r} = (2, 3, 4)$ are the allocation vectors of components by modules for both systems. In

Fig. 14, we plot the block diagrams of the two series–parallel systems without any redundancy mechanism. It is easy to check that $(2,2,5) \overset{m}{\geq} (2,3,4)$. On the other hand, let $\bar{q}_{2:2}(u) = 2u - u^2$, $\bar{q}_{\alpha}(u) = u + u^{\alpha} - u^{\alpha+1}$ and $\bar{q}_{mr}(u) = u(1 - \log u)$ be the distortions of the redundancy mechanisms available. Fig. 15 shows the reliability functions $R_2^{(\mathbf{s})}$ and $R_2^{(\mathbf{r})}$ of both systems, under the three redundancies $\bar{q}_{2:2}$, $\bar{q}_{\alpha}$ with $\alpha = 0.3$ and $\bar{q}_{mr}$, for independent modules (left) and dependent modules (right). For the case of dependent modules, we suppose that the dependence structure is defined by a Gumbel–Hougaard copula as in (4.9) for $\theta = 2$. As expected from Proposition 5.5, $R_2^{(\mathbf{s})} \leq R_2^{(\mathbf{r})}$ holds for all the concave redundancies $\bar{q}$.

6. Conclusions

The main novelty of this paper is to bring a new model to study redundancy mechanisms in systems composed of modules. Both the modules and the components in the modules can be dependent. These possible dependencies are represented by copulas and then the different systems' reliabilities are represented by distortions. The different redundancy mechanisms are also represented by distortions. This approach includes the classical ones, an independent spare in parallel (hot redundancy) and minimal repair (cold or standby redundancy) but it can also be used to study other redundancy mechanisms (as e.g. dependent spares).

This approach allows us to obtain several general results under different assumptions. First we consider the cases of independent or dependent components within the modules with different distributions. Then we also study the cases in which the components in each module are identically distributed (dependent or independent). In this way we are able to determine the best redundancy options. In many cases, these results do not depend on the components distributions, and even on the modules structures.

This paper is just a first step. We have studied results for general families of copulas (Schur-concave or Archimedean) and for particular ones (Clayton, FGM, Gumbel–Hougaard, etc.). More results could be obtained for other families of copulas or for more specific systems/modules structures. Furthermore, we left as a future research project to address a cost-allocation problem under the proposed approach.

CRediT authorship contribution statement

Nuria Torrado: Conceptualization, Methodology, Writing and reviewing. **Antonio Arriaza:** Conceptualization, Methodology, Writing and reviewing. **Jorge Navarro:** Conceptualization, Methodology, Writing and reviewing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We would like to thank the Associated Editor and the anonymous reviewers for their helpful suggestions. NT is partially supported by Ministerio de Ciencia e Innovación of Spain under grant PID2019-108079GB-C22/AEI/10.13039/501100011033. AA was supported by Ministerio de Economía y Competitividad of Spain under grant MTM2017-89577-P. Finally, JN is partially supported by Ministerio de Ciencia e Innovación of Spain under grant PID2019-103971GB-I00/AEI/10.13039/501100011033.

References

[1] Boland PJ, El-Neweihi E, Proschan F. Active redundancy allocation in coherent systems. Probab Engrg Inform Sci 1988;2:343–53.
[2] Singh H, Misra N. On redundancy allocation in systems. J Appl Probab 1994;31:1004–14.
[3] Belzunce F, Martínez-Puertas H, Ruiz JM. On allocation of redundant components for systems with dependent components. European J Oper Res 2013;230:573–80.
[4] Zhao P, Zhang Y, Chen J. Optimal allocation policy of one redundancy in a n-component series system. European J Oper Res 2017;257:656–68.
[5] Hadipour H, Amiri M, Sharifi M. Redundancy allocation in series-parallel systems under warm standby and active components in repairable subsystems. Reliab Eng Syst Saf 2019;192:106048.
[6] Misra N, Misra AK, Dhariyal ID. Standby redundancy allocations in series and parallel systems. J Appl Probab 2011;48:43–55.
[7] You Y, Li, X. On allocating redundancies to k-out-of-n reliability systems. Appl Stoch Models Bus Ind 2014;30:361–71.
[8] Eryilmaz S. The effectiveness of adding cold standby redundancy to a coherent system at system and component levels. Reliab Eng Syst Saf 2017;165:331–5.
[9] Barlow R, Hunter L. Optimum preventive maintenance policies. Oper Res 1960;8:90–100.
[10] Block H, Borges W, Savits T. Age-dependent minimal repair. J Appl Probab 1985;22:370–85.
[11] Shaked M, Shanthikumar G. Multivariate imperfect repair. Oper Res 1986;34:437–48.
[12] Aven T. A counting process approach to replacement models. Optimization 1987;18:285–96.
[13] Aven T, Jensen U. A general minimal repair model. J Appl Probab 2000;37:187–97.
[14] Finkelstein MS. Minimal repair in heterogeneous populations. J Appl Probab 2004;41:281–6.
[15] Zequeira RI, Berenguer C. Periodic imperfect preventive maintenance with two categories of competing failure modes. Reliab Eng Syst Saf 2006;91:460–8.
[16] Hollander M, Samaniego FJ, Sethuraman J. Imperfect repair, encyclopedia of statistics in quality and reliability. Chichester: John Wiley & Sons; 2007.
[17] Hashemi M, Asadi M, Zarezadeh S. Optimal maintenance policies for coherent systems with multi-type components. Reliab Eng Syst Saf 2020;195:106674.
[18] Wang C, Wang X, Xing L, Guan Q, Yang C, Yu M. A fast and accurate reliability approximation method for heterogeneous cold standby sparing systems. Reliab Eng Syst Saf 2021;212:107596.
[19] Xu J, Liang Z, Li Y-F, Wang K. Generalized condition-based maintenance optimization for multi-component systems considering stochastic dependency and imperfect maintenance. Reliab Eng Syst Saf 2021;211:107592.
[20] Kim H. Optimal reliability design of a system with k-out-of-n subsystems considering redundancy strategies. Reliab Eng Syst Saf 2017;167:572–82.
[21] Peiravi A, Karbasian M, Ardakan MA, Coit DW. Reliability optimization of series-parallel systems with K-mixed redundancy strategy. Reliab Eng Syst Saf 2019;183:17–28.
[22] Li XY, Li YF, Huang HZ. Redundancy allocation problem of phased-mission system with non-exponential components and mixed redundancy strategy. Reliab Eng Syst Saf 2020;199:106903.
[23] Wang W, Lin M, Fu Y, Luo X, Chen H. Multi-objective optimization of reliability-redundancy allocation problem for multi-type production systems considering redundancy strategies. Reliab Eng Syst Saf 2020;193:106681.
[24] Hsieh TJ. Component mixing with a cold standby strategy for the redundancy allocation problem. Reliab Eng Syst Saf 2021;206:107290.
[25] Torrado N. On allocation policies in systems with dependence structure and random selection of components. J Comput Appl Math 2021;388:113274.
[26] Navarro J, Fernández-Martínez P. Redundancy in systems with heterogeneous dependent components. European J Oper Res 2021;290:766–78.
[27] Kuo W, Prasad V. An annotated overview of system-reliability optimization. IEEE Trans Reliab 2000;49(2):176–87.
[28] Wang W, Loman NJ, Vassiliou. Reliability importance of components in a complex system. In *Proceedings of the annual reliability and maintainability symposium*, LA, 2004, p. 6–11.
[29] Kuo W, Wang R. Recent advances in optimal reliability allocation. IEEE Trans Syst Man Cybern A Syst Hum 2007;37(2):143–56.
[30] Barlow RE, Proschan F. Statistical theory of reliability and life testing. International series in decision processes, New York: Holt, Rinehart and Winston, Inc.; 1975.
[31] Yan R, Wang J. Component level versus system level at active redundancies for coherent systems with dependent heterogeneous components. Comm Statist Theory Methods 2020. http://dx.doi.org/10.1080/03610926.2020.1767140.
[32] Bon JL, Păltănea E. Ordering properties of convolutions of exponential random variables. Lifetime Data Anal 1999;5(2):185–92.
[33] Marshall AW, Olkin I, Arnold BC. Inequalities: Theory of majorization and its applications. New York: Springer; 2011.
[34] Khaledi B-L, Kochar S. Dispersive ordering among linear combinations of uniform random variables. J Statist Plann Inference 2002;100:13–21.

[35] Navarro J, Spizzichino F. Aggregation and signature based comparisons of multi-state systems via decompositions of fuzzy measures. Fuzzy Sets and Systems 2020;396:115–37.

[36] Yun WY, Songa YM, Kim HG. Multiple multi-level redundancy allocation in series systems. Reliab Eng Syst Saf 2007;92:308–13.

[37] Navarro J, del Águila Y, Sordo MA, Suárez-Llorens A. Preservation of reliability classes under the formation of coherent systems. Appl Stoch Models Bus Ind 2014;30:444–54.

[38] Navarro J. Distribution-free comparisons of residual lifetimes of coherent systems based on copula properties. Statist Papers 2018;59:781–800.

[39] Nelsen RB. An introduction to copulas. Springer series in statistics, 2nd ed.. New York: Springer; 2006.

[40] Krakowski M. The relevation transform and a generalization of the gamma distribution function. Rev Franç Autom Inform Rech Opér Mai (V-2) 1973;10:7–120.

[41] Navarro J, Arriaza A, Suárez-Llorens A. Minimal repair of failed components in coherent systems. European J Oper Res 2019;279:951–64.

[42] Esary J, Proschan F. Relationship between system failure rate and component failure rates. Technometrics 1963;5:183–9.

[43] Navarro J, del Águila Y. Stochastic comparisons of distorted distributions, coherent systems and mixtures with ordered components. Metrika 2017;80:627–48.

[44] Navarro J, Torrado N, del Águila Y. Comparisons between largest order statistics from multiple-outlier models with dependence. Methodol Comput Appl Probab 2018;20:411–33.