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Γ -convergence of polyconvex functionals involving s -fractional gradients to their local counterparts

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Abstract

In this paper we study localization properties of the Riesz s -fractional gradient $D^s u$ of a vectorial function u as $s \nearrow 1$. The natural space to work with s -fractional gradients is the Bessel space $H^{s,p}$ for $0 < s < 1$ and $1 < p < \infty$. This space converges, in a precise sense, to the Sobolev space $W^{1,p}$ when $s \nearrow 1$. We prove that the s -fractional gradient $D^s u$ of a function u in $W^{1,p}$ converges strongly to the classical gradient Du . We also show a weak compactness result in $W^{1,p}$ for sequences of functions u_s with bounded L^p norm of $D^s u_s$ as $s \nearrow 1$. Moreover, the weak convergence of $D^s u_s$ in L^p implies the weak continuity of its minors, which allows us to prove a semicontinuity result of polyconvex functionals involving s -fractional gradients defined in $H^{s,p}$ to their local counterparts defined in $W^{1,p}$. The full Γ -convergence of the functionals is achieved only for the case $p > n$.

Keywords: Riesz fractional gradient, localization of nonlocal gradient, Γ -convergence, Bessel spaces, polyconvex functionals

1 Introduction

Fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, with $0 < s < 1$ and $1 \leq p < \infty$, are nowadays of central importance in the analysis of partial differential equations, both of local or nonlocal (or fractional) nature [12]. These spaces provide, through the Gagliardo seminorm of a function u defined as

$$[u]_{W^{s,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}},$$

a measure of the fractional differentiability of order s of the function u . Indeed, in [6] (see also [20, Prop. 15.7]) it has been shown that for $p > 1$,

$$\lim_{s \nearrow 1} (1 - s)^{\frac{1}{p}} [u]_{W^{s,p}(\mathbb{R}^n)} = K(n, p) \|Du\|_{L^p(\mathbb{R}^n)}$$

for some constant $K(n, p) > 0$. This provides an interesting characterization of Sobolev spaces, which naturally leads to the question of the existence of a fractional differential object converging to the classical gradient as s goes to 1. The Riesz s -fractional gradient seems to be the right notion for such a differential object, and has been addressed by several authors in different situations in the recent years [4, 10, 23–25, 27]. The s -fractional gradient, $s \in (0, 1)$, of an L^1_{loc} function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as

$$D^s u(x) = c_{n,s} \text{pv}_x \iint_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy, \quad (1)$$

whenever it makes sense, where pv_x stands for the principal value centred at x , and $c_{n,s}$ is a suitable normalizing constant. As mentioned above, the first reason for which this object deserves attention is the fact that $D^s u$ converges to the classical gradient Du as $s \nearrow 1$. Indeed, $D^s u = D(I_{1-s} * u)$ for any $u \in C_c^\infty(\mathbb{R}^n)$ [10], where $I_{1-s} = \frac{c_{n,s}}{n+s-1} |\cdot|^{-n+1-s}$ is the classical Riesz potential [26]; therefore, applying Fourier transform,

$$\widehat{D^s u}(\xi) = 2\pi i \xi \widehat{I_{1-s}}(\xi) \hat{u}(\xi) = 2\pi i \xi |2\pi \xi|^{s-1} \hat{u}(\xi),$$

which converges to $\widehat{Du}(\xi)$ as $s \nearrow 1$.

Another remarkable reason why the fractional gradient seems to be the right differential object is given in [27]: it is shown that for $s \in (0, 1)$, definition (1) determines up to a multiplicative constant the unique object fulfilling some minimal consistency requirements from the physical and mathematical point of view, such as invariance under rotations and translations, s -homogeneity under dilations and some weak continuity properties. Furthermore, an s -fractional divergence may be defined, satisfying a similar characterization [27], as well as the integration by parts formula; consequently, the s -fractional gradient and the s -fractional divergence are dual operators [4, 10, 13, 18, 27].

The first reference dealing with this sort of differential objects seems to be [16]. More recently, nonlocal gradients defined as integrals in bounded domains have been investigated in [18] (see also [13]), showing some vector calculus facts, such as the integration by parts formula, and localization results in various topologies when the *horizon* of interaction among particles vanishes.

When dealing with variational problems or fractional PDE involving the s -fractional gradient, there appears naturally the space $H^{s,p}(\mathbb{R}^n)$, for $0 < s < 1$ and $1 \leq p < \infty$, defined as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)} + \|D^s u\|_{L^p(\mathbb{R}^n)}.$$

To be consistent with this definition of $H^{s,p}$ as a completion, given $u \in H^{s,p}(\mathbb{R}^n)$ we define $D^s u$ as the limit of $D^s u_j$ in L^p when $\{u_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ is any sequence converging to u in $H^{s,p}$. We leave for a subsequent work the study of whether this definition for $D^s u$ coincides with Riesz' fractional gradient (1).

In [24, 25] the space $H^{s,p}$ is extensively studied, showing Sobolev-type and compact embedding theorems; they also provide existence results for scalar convex variational problems involving the s -fractional gradient, and for the derived fractional PDE obtained as the first order equilibrium condition. See also [23] and the references therein for the case $p = 1$. In [10], the space of functions whose s -fractional total variation is finite is considered and, subsequently, an s -fractional Caccioppoli perimeter is addressed. In [4], the authors of this paper have addressed the study of polyconvex functionals depending on the s -fractional gradient, showing a fractional Piola identity, the weak continuity of the determinant of fractional gradients, a semicontinuity result for polyconvex functionals depending on $D^s u$, and finally an existence theory in that situation.

In this paper we continue with the study of polyconvex functionals depending on the s -fractional gradient by further exploring it through the study of its limit when $s \nearrow 1$. The main results of the

paper are described as follows. We prove the strong convergence in L^p of $D^s u$ to Du for functions $u \in W^{1,p}$, generalizing, and making the topology precise, the convergence mentioned above for smooth functions. Notice that this convergence is performed in the fractional parameter s rather than in the *horizon*, as done in [18]. This result is of interest in itself, as it provides a precise differential object converging to the distributional gradient. We also show a weak compactness result in $W^{1,p}$, establishing that if $\{u_s\}$ is a sequence such that $\{D^s u_s\}$ is bounded in L^p , then there exists a $u \in W^{1,p}$ such that u_s converges strongly and $D^s u_s$ converges weakly in L^p as $s \nearrow 1$ to u and Du , respectively. We also show the weak convergence of the minors of $D^s u_s$ to those of Du , whenever $D^s u_s$ converges weakly in L^p to Du ; as a consequence, we establish a new semicontinuity result for polyconvex functionals. Finally, we show that the family of vector variational problems based on minimization of

$$\mathcal{I}_s(u) = \iint_{\mathbb{R}^n} W(x, u(x), D^s u(x)) dx, \quad u \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$$

Γ -converges (see [7]) to the functional

$$\mathcal{I}(u) = \iint_{\mathbb{R}^n} W(x, u(x), Du(x)) dx, \quad u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$$

as $s \nearrow 1$, under the essential assumption of polyconvexity of $W(x, u, \cdot)$ [11]; we also need the extra assumption $p > n$ for the Γ -convergence. Other references dealing with Γ -convergence of variational functionals in the nonlocal setting are [19] (in the context of $W^{s,p}$), [5] (in nonlinear peridynamics) and [17] (in linear and geometrically nonlinear peridynamics).

After having finished this work we became aware that the article [9] (concurrently and independently) treats a closely related problem. Essentially, they address somewhat similar Γ -convergence questions than us but mainly in the special case $p = 1$ and without dealing with polyconvexity. More precisely, they prove some Γ -convergence properties in the space $BV^s(\mathbb{R}^n)$ of functions with bounded s -fractional variation, and study the fractional operators involved. Interestingly, they provide a different proof, not based on Fourier transform like our Theorem 3.2, of the convergence of $D^s u$ to Du in L^p for $u \in W^{1,p}$.

The outline of the paper is as follows. Section 2 is a preliminary section where we collect several results on the fractional differential operators D^s and div^s , as well as on the Bessel space $H^{s,p}$. We also provide a refinement on the Poincaré-Sobolev inequality in $H^{s,p}$ given in [24], with a constant independent of s (Theorem 2.9). In Section 3 we prove the localization of the fractional gradient; namely, that $D^s u$ converges to Du in L^p for a fixed $u \in W^{1,p}$ (Theorem 3.2). In Section 4 we prove the compactness result: for any sequence $\{u_s\}$ with a fixed complementary-value data and bounded $\{D^s u_s\}$ in L^p , there exist $u \in W^{1,p}$ and a subsequence strongly convergent to u in L^p , with the s -fractional gradients converging weakly to Du in L^p (Theorem 4.2). Section 5 is devoted to the weak convergence of the minors of $D^s u_s$ to the minors of Du when $D^s u_s$ converges weakly to Du in L^p (Theorem 5.4). Finally, in Section 6 we prove a novel semicontinuity result for polyconvex functionals (Theorem 6.1) and the Γ -convergence result of \mathcal{I}_s to \mathcal{I} under the assumption of polyconvexity (Theorem 6.2) and convexity (Theorem 6.3).

2 Preliminaries on Bessel spaces

In this section we introduce the fractional differential operators D^s and div^s , and the Bessel space $H^{s,p}$, together with some embedding theorems and a Poincaré-Sobolev inequality in this fractional context.

2.1 Fractional differential operators

We state the definition of the s -fractional gradient and divergence. Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $f \in L^1(B(x, r)^c)$ for every $r > 0$, we define the principal value centered at x of $\int_{\mathbb{R}^n} f$, denoted by

$$\text{pv}_x \int_{\mathbb{R}^n} f \quad \text{or} \quad \text{pv}_x \int \left(f, \right.$$

as

$$\lim_{r \rightarrow 0} \int_{B(x, r)^c} f,$$

whenever this limit exists. We have denoted by $B(x, r)$ the open ball centered at x of radius r , and by $B(x, r)^c$ its complement. As most integrals in this work are over \mathbb{R}^n , we will use the symbol \int as a substitute for $\int_{\mathbb{R}^n}$.

In order to avoid the principal value in (1), we first establish the following definition for C_c^∞ functions and then we extend it by density. The following definitions of s -fractional gradient and divergence are adapted from [4, 18, 24, 25]. Recall that Γ denotes Euler's gamma function.

Definition 2.1. Let $0 < s < 1$ and set

$$c_{n,s} := (n + s - 1) \frac{\Gamma\left(\frac{n+s-1}{2}\right)}{\pi^{\frac{n}{2}} 2^{1-s} \Gamma\left(\frac{1-s}{2}\right)}.$$

a) Let $u \in C_c^\infty(\mathbb{R}^n)$. We define $D^s u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$D^s u(x) := c_{n,s} \int \left(\frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy. \right. \quad (2)$$

b) Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We define $\text{div}^s \phi : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$\text{div}^s \phi(x) := -c_{n,s} \text{pv}_x \int \left(\frac{\phi(x) + \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy. \right.$$

The integral (2) is easily seen to be absolutely convergent for all $x \in \mathbb{R}^n$. Moreover, by odd symmetry,

$$-c_{n,s} \text{pv}_x \int \frac{\phi(x) + \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy = c_{n,s} \int \left(\frac{\phi(x) - \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy, \right. \quad (3)$$

and this last integral is absolutely convergent. Furthermore, $D^s u \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ and $\text{div}^s \phi \in L^q(\mathbb{R}^n, \mathbb{R})$ for all $q \in [1, \infty]$ for smooth and compactly supported u and ϕ ; see [4, Lemma 3.1], if necessary.

Definition 2.1 a) naturally extends to $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ by replacing (2) with

$$D^s u(x) := c_{n,s} \int \left(\frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy. \right. \quad (4)$$

Here, \cdot stands for the usual tensor product of vectors.

A crucial fact is the following fractional fundamental theorem of Calculus [10, Th. 3.11] (see also [24, Th. 1.12] or [21, Prop. 15.8]).

Theorem 2.1. Let $0 < s < 1$. For every $u \in C_c^\infty(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$ we have that

$$u(x) = c_{n,-s} \int \left(D^s u(y) \cdot \frac{x - y}{|x - y|^{n-s+1}} dy. \right.$$

The operators D^s and div^s enjoy the following duality property, which is a nonlocal integration by parts whose proof can be found in [4, Theorem 3.6] (see also [10, 18, 27]).

Lemma 2.2. *Let $0 < s < 1$. Then, for all $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int D^s u(x) \cdot \phi(x) dx = - \int \left(u(x) \operatorname{div}^s \phi(x) \right) dx.$$

We now extend Definition 2.1 to a broader class of functions.

Definition 2.2. *Let $0 < s < 1$ and $1 \leq p < \infty$.*

- a) *Let $u \in L^p(\mathbb{R}^n)$ be such that there exists a sequence of $\{u_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ converging to u in $L^p(\mathbb{R}^n)$ and for which $\{D^s u_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. We define $D^s u$ as the limit in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ of $D^s u_j$ as $j \rightarrow \infty$.*
- b) *Let $\phi \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ be such that there exists a sequence of $\{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ converging to ϕ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and for which $\{\operatorname{div}^s \phi_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. We define $\operatorname{div}^s \phi$ as the limit in $L^p(\mathbb{R}^n)$ of $\operatorname{div}^s \phi_j$ as $j \rightarrow \infty$.*

Of course, for a $u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, the definition of $D^s u$ is analogous taking into account (4).

The following result shows that the above definitions of $D^s u$ and $\operatorname{div}^s \phi$ are independent of the sequences $\{u_j\}_{j \in \mathbb{N}}$ and $\{\phi_j\}_{j \in \mathbb{N}}$, respectively, and of the exponent p .

Lemma 2.3. *Let $0 < s < 1$ and $1 \leq p, q < \infty$.*

- a) *Let $u \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ be such that there exist sequences $\{u_j\}_{j \in \mathbb{N}}$ and $\{v_j\}_{j \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n)$ such that $u_j \rightarrow u$ in $L^p(\mathbb{R}^n)$ and $v_j \rightarrow u$ in $L^q(\mathbb{R}^n)$, and for which $\{D^s u_j\}_{j \in \mathbb{N}}$ converges to some U in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\{D^s v_j\}_{j \in \mathbb{N}}$ converges to some V in $L^q(\mathbb{R}^n, \mathbb{R}^n)$. Then $U = V$.*
- b) *Let $\phi \in L^p(\mathbb{R}^n, \mathbb{R}^n) \cap L^q(\mathbb{R}^n, \mathbb{R}^n)$ be such that there exist sequences $\{\phi_j\}_{j \in \mathbb{N}}$ and $\{\theta_j\}_{j \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\phi_j \rightarrow \phi$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\theta_j \rightarrow \phi$ in $L^q(\mathbb{R}^n, \mathbb{R}^n)$, and for which $\{\operatorname{div}^s \phi_j\}_{j \in \mathbb{N}}$ converges to some Φ in $L^p(\mathbb{R}^n)$ and $\{\operatorname{div}^s \theta_j\}_{j \in \mathbb{N}}$ converges to some Θ in $L^q(\mathbb{R}^n, \mathbb{R}^n)$. Then $\Phi = \Theta$.*

Proof. We prove a), the proof of b) being analogous.

Let $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then, by Lemma 2.2,

$$\int U \cdot \phi = \lim_{j \rightarrow \infty} \int \left(D^s u_j \cdot \phi \right) = - \lim_{j \rightarrow \infty} \int u_j \operatorname{div}^s \phi = - \int \left(u \operatorname{div}^s \phi \right)$$

and, analogously,

$$\int V \cdot \phi = - \int \left(u \operatorname{div}^s \phi \right).$$

Thus,

$$\int U \cdot \phi = \int \left(V \cdot \phi \right)$$

for all $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$, whence $U = V$. □

We end this subsection by showing two uniform bounds on the constant $c_{n,s}$ with respect to s . We denote by ω_n the volume of the unit ball in \mathbb{R}^n .

Lemma 2.4. Let $n \in \mathbb{N}$. Consider the function $c_{n,\cdot} : [-1, 1] \rightarrow [0, \infty)$, defined as

$$c_{n,s} = \begin{cases} \frac{\Gamma(\frac{n+s+1}{2})}{\pi^{\frac{n}{2}} 2^{-s} \Gamma(\frac{1-s}{2})} & \text{if } -1 \leq s < 1, \\ 0 & \text{if } s = 1. \end{cases}$$

Then

$$\sup_{s \in [-1, 1]} c_{n,s} < \infty, \quad \sup_{s \in [-1, 1)} \frac{c_{n,s}}{1-s} < \infty \quad \text{and} \quad \lim_{s \nearrow 1} \frac{c_{n,s}}{1-s} = \frac{1}{\omega_n}.$$

Proof. The function $c_{n,\cdot}$ is clearly continuous in $[-1, 1)$. As $\Gamma(z) \rightarrow +\infty$ as $z \rightarrow 0^+$, we obtain that $c_{n,\cdot}$ is also continuous in $[-1, 1]$. Now, using the property $z\Gamma(z) = \Gamma(z+1)$ for $z > 0$, we find that

$$\frac{c_{n,s}}{1-s} = \frac{\Gamma(\frac{n+s+1}{2})}{\pi^{\frac{n}{2}} 2^{1-s} \Gamma(\frac{3-s}{2})},$$

and, hence, the function $s \mapsto \frac{c_{n,s}}{1-s}$ is continuous in $[-1, 1)$ and admits a continuous extension to $[-1, 1]$. In fact,

$$\lim_{s \nearrow 1} \frac{c_{n,s}}{1-s} = \frac{\Gamma(1 + \frac{n}{2})}{\pi^{\frac{n}{2}}} = \frac{1}{\omega_n}.$$

The conclusion follows. \square

Note that the function $c_{n,\cdot}$ of Lemma 2.4 is an extension of that of Definition 2.1.

2.2 Bessel spaces

Let $0 < s < 1$ and $1 \leq p < \infty$. For the sake of simplicity, we denote the norm in both $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n, \mathbb{R}^n)$ by $\|\cdot\|_p$. Given $u \in C_c^\infty(\mathbb{R}^n)$, we define $\|\cdot\|_{H^{s,p}(\mathbb{R}^n)}$ as

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = \|u\|_p + \|D^s u\|_p,$$

which is easily seen to be a norm. We define the space $H^{s,p}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^n)}$, and extend accordingly the definition of $\|\cdot\|_{H^{s,p}(\mathbb{R}^n)}$ to $H^{s,p}(\mathbb{R}^n)$. An analogous definition can be done for the space $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$.

This is the definition given in [24] (see also [10]). We leave for a future work the issue of whether this definition coincides with the set of $u \in L^p(\mathbb{R}^n)$ such that $D^s u$, as defined in (1), exists a.e. and is in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, which is the definition given in [25], without showing that both definitions are actually equivalent. What was established in [24, Th. 1.7] is the remarkable fact of the identification of $H^{s,p}$ with the classical Bessel potential spaces (see [1, 22, 26]).

For a $\phi \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ there is a natural relation between $D^s \phi$ and $\text{div}^s \phi$.

Lemma 2.5. Let $0 < s < 1$ and $1 \leq p < \infty$. Let $\phi \in H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$. Then $\text{div}^s \phi$ is well defined and $\text{tr } D^s \phi = \text{div}^s \phi$ a.e.

Proof. Let $\{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a sequence converging to ϕ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $\{D^s \phi_j\}_{j \in \mathbb{N}}$ converges to $D^s \phi$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$. By linearity, $\text{tr } D^s \phi_j \rightarrow \text{tr } D^s \phi$ in $L^p(\mathbb{R}^n)$ as $j \rightarrow \infty$. In view of Definition 2.2 b) and Lemma 2.3 b), it suffices to show that $\text{tr } D^s \phi_j = \text{div}^s \phi_j$ for all $j \in \mathbb{N}$. Having in mind that the integrals of (4) and of the right hand side of (3) are absolutely convergent, we obtain that

$$\begin{aligned} \text{tr } D^s \phi_j(x) &= c_{n,s} \text{tr} \left(\int \left(\frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \quad \frac{x-y}{|x-y|} \right) dy \right) = c_{n,s} \int \text{tr} \left(\frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \quad \frac{x-y}{|x-y|} \right) dy \\ &= c_{n,s} \int \left(\frac{\phi_j(x) - \phi_j(y)}{|x-y|^{n+s}} \cdot \frac{x-y}{|x-y|} \right) dy = \text{div}^s \phi_j(x), \end{aligned}$$

which concludes the proof. \square

The integration by parts formula of Lemma 2.2 can be extended to $H^{s,p}$ as follows. We denote by p' the conjugate exponent of p .

Lemma 2.6. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for all $u \in H^{s,p}(\mathbb{R}^n)$ and $\phi \in H^{s,p'}(\mathbb{R}^n, \mathbb{R}^n)$ we have*

$$\int D^s u(x) \cdot \phi(x) dx = - \iint (u(x) \operatorname{div}^s \phi(x) dx.$$

Proof. Let $\{u_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence converging to u in $H^{s,p}(\mathbb{R}^n)$, and let $\{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a sequence converging to ϕ in $H^{s,p'}(\mathbb{R}^n, \mathbb{R}^n)$. Then the following convergences hold as $j \rightarrow \infty$:

$$\begin{aligned} u_j &\rightarrow u \text{ in } L^p(\mathbb{R}^n), \quad D^s u_j \rightarrow D^s u \text{ in } L^p(\mathbb{R}^n, \mathbb{R}^n), \quad \phi_j \rightarrow \phi \text{ in } L^{p'}(\mathbb{R}^n, \mathbb{R}^n), \\ D^s \phi_j &\rightarrow D^s \phi \text{ in } L^{p'}(\mathbb{R}^n, \mathbb{R}^{n \times n}), \quad \operatorname{tr} D^s \phi_j \rightarrow \operatorname{tr} D^s \phi \text{ in } L^{p'}(\mathbb{R}^n), \quad \operatorname{div}^s \phi_j \rightarrow \operatorname{div}^s \phi \text{ in } L^{p'}(\mathbb{R}^n), \end{aligned}$$

the last convergence due to Lemma 2.5. As a consequence,

$$D^s u_j \cdot \phi_j \rightarrow D^s u \cdot \phi \quad \text{and} \quad u_j \operatorname{div}^s \phi_j \rightarrow u \operatorname{div}^s \phi \quad \text{in } L^1(\mathbb{R}^n). \quad (5)$$

By Lemma 2.2, for each $j \in \mathbb{N}$,

$$\int D^s u_j(x) \cdot \phi_j(x) dx = - \iint (u_j(x) \operatorname{div}^s \phi_j(x) dx.$$

This equality and the convergences (5) readily imply the conclusion. \square

2.3 Embeddings of Bessel spaces

It is known that, for $0 < s < 1$, the space $W^{1,p}(\mathbb{R}^n)$ embeds continuously into $H^{s,p}(\mathbb{R}^n)$ [1, Ch. 7]. In the next result we prove again this result but writing the dependence of the embedding constant with respect to s .

Proposition 2.7. *Let $1 \leq p < \infty$. Then, there exists a constant $C = C(n, p) > 0$ such that for all $u \in W^{1,p}(\mathbb{R}^n)$ and $0 < s < 1$,*

$$\|D^s u\|_p \leq \frac{C}{s} \|u\|_{W^{1,p}(\mathbb{R}^n)},$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$,

$$|D^s u(x)| \leq c_{n,s} (A(x) + B(x)) \quad (6)$$

with

$$A(x) := \int_{B(x,1)} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dy, \quad B(x) := \iint_{\mathbb{R}^{(x,1)^c}} \frac{|u(x) - u(y)|}{|x - y|^{n+s}} dy,$$

so

$$\|D^s u\|_p \leq c_{n,s} \left(\|A\|_p + \|B\|_p \right) \left(\right.$$

Note that

$$A(x) = \int_{B(0,1)} \frac{|u(x+h) - u(x)|}{|h|^{n+s}} dh, \quad B(x) = \iint_{\mathbb{R}^{(0,1)^c}} \frac{|u(x+h) - u(x)|}{|h|^{n+s}} dh.$$

Applying Minkowski's integral inequality (see, e.g., [26, App. A.1]) we obtain

$$\|A\|_p \leq \int_{B(0,1)} \left(\int \left(\frac{|u(x+h) - u(x)|^p}{|h|^{(n+s)p}} dx \right)^{\frac{1}{p}} dh.$$

Now, for all $h \in B(0,1) \setminus \{0\}$,

$$\left(\int \left(\frac{|u(x+h) - u(x)|^p}{|h|^{(n+s)p}} dx \right)^{\frac{1}{p}} = \frac{1}{|h|^{n+s}} \left(\int |u(x+h) - u(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{1}{|h|^{n+s-1}} \|Du\|_p,$$

thanks to a classic inequality (see, e.g., [8, Prop. 9.3] and notice that it is still valid for $p = 1$). Therefore,

$$\|A\|_p \leq \|Du\|_p \int_{B(0,1)} \frac{1}{|h|^{n+s-1}} = \frac{\sigma_{n-1}}{1-s} \|Du\|_p, \quad (7)$$

where σ_{n-1} is the area of the unit sphere of \mathbb{R}^n .

As for B , we first notice that for all $x \in \mathbb{R}^n$, by Hölder's inequality

$$\begin{aligned} B(x) &\leq \int_{B(0,1)^c} \frac{|u(x+h)|}{|h|^{n+s}} dh + \int_{B(0,1)^c} \frac{|u(x)|}{|h|^{n+s}} dh \\ &\leq \left(\int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh \right)^{\frac{1}{p}} \left(\int_{B(0,1)^c} \frac{1}{|h|^{n+s}} dh \right)^{\frac{1}{p'}} + |u(x)| \int_{B(0,1)^c} \frac{1}{|h|^{n+s}} dh \\ &= \left(\int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh \right)^{\frac{1}{p}} \left(\frac{\sigma_{n-1}}{s} \right)^{\frac{1}{p'}} + |u(x)| \frac{\sigma_{n-1}}{s}, \end{aligned}$$

so, by Fubini's theorem,

$$\|B\|_p \leq \left(\frac{\sigma_{n-1}}{s} \right)^{\frac{1}{p'}} \left(\int \int_{B(0,1)^c} \frac{|u(x+h)|^p}{|h|^{n+s}} dh dx \right)^{\frac{1}{p}} + \frac{\sigma_{n-1}}{s} \|u\|_p = 2 \frac{\sigma_{n-1}}{s} \|u\|_p. \quad (8)$$

Putting together (6), (7) and (8), we obtain

$$\|D^s u\|_p \leq c_{n,s} \sigma_{n-1} \left(\frac{1}{1-s} \|Du\|_p + \frac{2}{s} \|u\|_p \right)$$

and, thanks to Lemma 2.4, the proof is finished. \square

Given an open set $\Omega \subset \mathbb{R}^n$ we define the subspace $H_0^{s,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $H^{s,p}(\mathbb{R}^n)$; of course, with this definition we are extending each function of $C_c^\infty(\Omega)$ by zero in Ω^c . In addition, given $g \in H^{s,p}(\mathbb{R}^n)$ we define the affine subspace $H_g^{s,p}(\mathbb{R}^n)$ as $g + H_0^{s,p}(\Omega)$. The affine subspace $W_g^{1,p}(\Omega)$ is defined in a similar way for $g \in W^{1,p}(\mathbb{R}^n)$. We note that in [25] the space $H_g^{s,p}(\Omega)$ was defined as the set of $u \in H^{s,p}(\mathbb{R}^n)$ such that $u = g$ in Ω^c . We leave for a future work the issue of the equality of both definitions, but we note that, for a $u \in H_g^{s,p}(\Omega)$ according to our definition, we trivially have that $u = g$ in Ω^c . In particular, the following compact embedding of $H_g^{s,p}(\Omega)$ into $L^q(\mathbb{R}^n)$ remains true (see [25, Th. 2.2]; the formulation is adapted from [4, Th. 2.3]). In what follows we set $p_s^* = \frac{pn}{n-sp}$, and \rightharpoonup denotes weak convergence.

Theorem 2.8. *Set $0 < s < 1$ and $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $g \in H^{s,p}(\mathbb{R}^n)$. Then for any sequence $\{u_j\}_{j \in \mathbb{N}} \subset H_g^{s,p}(\Omega)$ such that*

$$u_j \rightharpoonup u \quad \text{in } H^{s,p}(\mathbb{R}^n),$$

for some $u \in H^{s,p}(\mathbb{R}^n)$, one has $u \in H_g^{s,p}(\Omega)$ and

a) $u_j - g \rightarrow u - g$ in $L^q(\mathbb{R}^n)$ for every q satisfying

$$\begin{cases} q \in [1, p_s^*) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n, \end{cases}$$

b) $u_j \rightarrow u$ in $L^q(\mathbb{R}^n)$ for every q satisfying

$$\begin{cases} q \in [p, p_s^*) & \text{if } sp < n, \\ q \in [p, \infty) & \text{if } sp = n, \\ q \in [p, \infty] & \text{if } sp > n. \end{cases}$$

2.4 Poincaré-Sobolev inequality for $H_0^{s,p}(\Omega)$

In this section we prove the Poincaré-Sobolev inequality in $H^{s,p}$. This result is known (see [24, Th. 1.8]), but for the analysis of this work it is crucial to trace the dependence of the Poincaré-Sobolev constant on s , in the case of bounded domains. The proof we provide uses some ideas of [15, Lemma 7.12].

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then there exists $C = C(n, s, p)$ such that for all $0 < s < 1$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq \frac{C}{s} \|D^s u\|_p.$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\Omega)$. Let $R \in \mathbb{R}$, to be specified later, such that

$$R \geq 1, \quad \Omega \subset B(0, R). \quad (9)$$

Define $\Omega_1 := B(0, 2R)$.

Fix $x \in \Omega$. By Theorem 2.1 and Lemma 2.4,

$$|u(x)| \leq C(n) \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x - y|^{n-s}} dy + \int_{\Omega_1^c} \frac{|D^s u(y)|}{|x - y|^{n-s}} dy \right]. \quad (10)$$

Now $\Omega_1 \subset B(x, 3R)$, so

$$\int_{\Omega_1} \frac{1}{|x - y|^{n-s}} dy \leq \int_{B(x, 3R)} \frac{1}{|x - y|^{n-s}} dy = \frac{\sigma_{n-1}}{s} (3R)^s \leq C(n) \frac{1}{s} R. \quad (11)$$

Similarly, $\Omega_1^c \subset B(y, 3R)$ for every $y \in \Omega_1$, so

$$\int_{\Omega_1^c} \frac{1}{|x - y|^{n-s}} dx \leq C(n) \frac{1}{s} R. \quad (12)$$

By (11) and Hölder's inequality,

$$\int_{\Omega_1} \frac{|D^s u(y)|}{|x - y|^{n-s}} dy \leq \left[C(n) \frac{1}{s} R \right]^{\frac{1}{p'}} \left(\int_{\Omega_1} \frac{|D^s u(y)|^p}{|x - y|^{n-s}} dy \right)^{\frac{1}{p}}.$$

Therefore, using (12), we find

$$\begin{aligned} \left[\int \left(\int_{\mathbb{R}^n} \frac{|D^s u(y)|}{|x-y|^{n-s}} dy \right)^p dx \right]^{\frac{1}{p}} &\leq \left[C(n) \frac{1}{s} R \right]^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |D^s u(y)|^p \int \left(\frac{1}{|x-y|^{n-s}} dx dy \right)^{\frac{1}{p}} \right. \\ &\leq C(n) \frac{1}{s} R \|D^s u\|_p. \end{aligned} \quad (13)$$

Now, for any $y \in \mathbb{R}^n$, by Lemma 2.4,

$$|D^s u(y)| \leq C(n) \int \frac{|u(y) - u(z)|}{|y-z|^{n+s}} dz = C(n) \int \left(\frac{|u(z)|}{|y-z|^{n+s}} dz \right). \quad (14)$$

When $z \in \mathbb{R}^n$ we have

$$|y| \leq |y-z| + |z| \leq |y-z| + R \leq |y-z| + \frac{1}{2}|y|,$$

so $\frac{1}{2}|y| \leq |y-z|$ and, hence,

$$\frac{1}{|y-z|^{n+s}} \leq \left(\frac{2}{|y|} \right)^{n+s} \leq C(n) \frac{1}{|y|^{n+s}}. \quad (15)$$

Similarly, for each $x \in \mathbb{R}^n$ we have

$$\frac{1}{|x-y|^{n-s}} \leq C(n) \frac{1}{|y|^{n-s}}. \quad (16)$$

Using (15) we find that

$$\int \left(\frac{|u(z)|}{|y-z|^{n+s}} dz \right) \leq C(n) \frac{1}{|y|^{n+s}} \|u\|_{L^1(\Omega)} \leq C(n) | \mathbb{R}^n |^{\frac{1}{p'}} \frac{1}{|y|^{n+s}} \|u\|_{L^p(\Omega)},$$

whence we infer from (14) that

$$|D^s u(y)| \leq C(n) | \mathbb{R}^n |^{\frac{1}{p'}} \frac{1}{|y|^{n+s}} \|u\|_{L^p(\Omega)}. \quad (17)$$

Thus, using (16) as well,

$$\int_{\mathbb{R}^n} \frac{|D^s u(y)|}{|x-y|^{n-s}} dy \leq C(n) | \mathbb{R}^n |^{\frac{1}{p'}} \|u\|_{L^p(\Omega)} \int \left(\frac{1}{|y|^{n+s}} \frac{1}{|y|^{n-s}} dy \right) = C(n) | \mathbb{R}^n |^{\frac{1}{p'}} R^{-n} \|u\|_{L^p(\Omega)}.$$

This last inequality, combined with (10) and (13), implies by the triangular inequality that

$$\begin{aligned} \|u\|_{L^p(\Omega)} &\leq C(n) \frac{1}{s} R \|D^s u\|_p + C_1(n) | \mathbb{R}^n |^{\frac{2}{p'}} R^{-n} \|u\|_{L^p(\Omega)} \\ &\leq C(n) \frac{1}{s} R \|D^s u\|_p + C_1(n) \max\{1, | \mathbb{R}^n |^2\} R^{-n} \|u\|_{L^p(\Omega)}. \end{aligned}$$

Finally, we choose R such that, in addition to (9), satisfies $C_1(n) \max\{1, | \mathbb{R}^n |^2\} R^{-n} \leq \frac{1}{2}$, so that R depends on n and s_0 . We obtain that

$$\frac{1}{2} \|u\|_{L^p(\Omega)} \leq C(n) \frac{1}{s} R \|D^s u\|_p$$

and concludes the proof. \square

We will use the following immediate consequence of Theorem 2.9.

Corollary 2.10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $0 < s_0 < 1$. Then there exists $C = C(n, s_0)$ such that for all $s_0 < s < 1$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leq C \|D^s u\|_p.$$

3 Localization of fractional gradients

In this section we prove the convergence of the s -fractional gradient of a $W^{1,p}$ function to its local gradient as $s \nearrow 1$. This result is to be expected, and easy to obtain for smooth functions using the Fourier transform (see Lemma 3.1). In this section we provide a complete proof for functions in $W^{1,p}(\mathbb{R}^n)$. This result, which is of interest in its own right, is a first step to prove the Γ -convergence of the functional \mathcal{I}_s to \mathcal{I} (see the Introduction). It should be compared with [6, Cor. 2], where the convergence of the Gagliardo seminorm to the L^p norm of the fractional gradient is shown (see also [21, Prop. 15.7]).

We first recall the definition of Riesz potential, since it will be used in this section to relate the fractional gradient to the classical gradient. Given $0 < s < n$, the Riesz kernel $I_s : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ is

$$I_s(x) = \frac{1}{\gamma(s)} \frac{1}{|x|^{n-s}},$$

where the constant $\gamma(s)$ is given by

$$\gamma(s) = \frac{\pi^{\frac{n}{2}} 2^s \Gamma(\frac{s}{2})}{\Gamma(\frac{n-s}{2})}.$$

The Riesz potential of a locally integrable function f is given by

$$I_s * f(x) = \frac{1}{\gamma(s)} \iint \frac{f(y)}{|x-y|^{n-s}} dy.$$

Note the relationship between γ and $c_{n,s}$:

$$c_{n,s} = \frac{n+s-1}{\gamma(1-s)}. \quad (18)$$

It is interesting to regard the s -fractional gradient from a Fourier analysis perspective. As usual, the Fourier transform of an L^1 function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is defined as

$$\hat{f}(\xi) = \iint_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and then is extended by duality to the class of tempered distributions. We know that classical differentiation translates, when applying the Fourier transform, into multiplication of the Fourier transform of a function by a monomial. This also happens in a fractional sense in this situation. The following result was proved in [24, Th. 1.4] (but it was mistakenly written with a sign switch); we include here a proof for the reader's convenience.

Lemma 3.1. *Let $0 < s < 1$. Then, for all $u \in C_c^\infty(\mathbb{R}^n)$,*

$$\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi \xi|^{1-s}} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n.$$

Proof. By [24, Th. 1.2], $D^s u = I_{1-s} * Du$ for any $u \in C_c^\infty(\mathbb{R}^n)$. We compute the Fourier transform of $D^s u$ in the sense of distributions. We start by checking that $I_{1-s} \in \mathcal{S}'$, where \mathcal{S} is the Schwartz space. Given $\phi \in \mathcal{S}$,

$$\begin{aligned} \gamma(1-s) \langle I_{1-s}, \phi \rangle &= \int \frac{\phi(x)}{|x|^{n+s-1}} dx = \iint_{B(0,1)} \frac{\phi(x)}{|x|^{n+s-1}} dx + \iint_{B(0,1)^c} \frac{\phi(x)}{|x|^{n+s-1}} dx \\ &\leq \|\phi\|_\infty \frac{1}{|x|^{n+s-1}}_{L^1(B(0,1))} + \|\phi|x|\|_\infty \frac{1}{|x|^{n+s}}_{L^1(B(0,1)^c)} \\ &\leq [\|\phi\|_\infty + \|\phi|x|\|_\infty] \left[\frac{1}{|x|^{n+s-1}}_{L^1(B(0,1))} + \frac{1}{|x|^{n+s}}_{L^1(B(0,1)^c)} \right] \left(\right. \end{aligned}$$

which shows that the Riesz potential is a continuous linear map over the Schwartz space.

Now, $D^s u \in L^1(\mathbb{R}^n)$, since $u \in C_c^\infty(\mathbb{R}^n)$ (see, e.g., [4, Lemma 3.1]) and, so, $D^s u$ can also be regarded as a tempered distribution. Therefore, we can apply the Fourier transform to $D^s u = I_{1-s} * Du$ and, having in mind that the latter is a convolution of the Riesz potential with a Schwartz function, as well as that $\widehat{I_{1-s}}(\xi) = |2\pi\xi|^{-(1-s)}$ (see [26]), we have

$$\widehat{D^s u}(\xi) = \widehat{I_{1-s} * Du}(\xi) = \widehat{I_{1-s}}(\xi) \widehat{Du}(\xi) = |2\pi\xi|^{-(1-s)} \widehat{Du}(\xi) = \frac{2\pi i \xi}{|2\pi\xi|^{1-s}} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n,$$

as desired. \square

The main result of the section is the following. As mentioned in the introduction, similar results have been simultaneously proved in [9, Sect. 4.1] without the use of Fourier transform.

Theorem 3.2. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for each $u \in W^{1,p}(\mathbb{R}^n)$,*

$$D^s u \rightarrow Du \text{ in } L^p(\mathbb{R}^n) \text{ as } s \nearrow 1.$$

Proof. We first prove the result for smooth functions and then extend it by density to $W^{1,p}(\mathbb{R}^n)$.

Let $u \in C_c^\infty(\mathbb{R}^n)$. By Lemma 3.1,

$$\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi\xi|^{1-s}} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n,$$

so by the elementary inequality $t^s \leq 1 + t$ for all $t \geq 0$,

$$|\widehat{D^s u}(\xi)| = |2\pi\xi|^s |\hat{u}(\xi)| \leq (1 + |2\pi\xi|) |\hat{u}(\xi)|. \quad (19)$$

As \hat{u} is in the Schwartz space (because $u \in C_c^\infty(\mathbb{R}^n)$), both \hat{u} and $\xi \hat{u}(\xi)$ are in $L^1(\mathbb{R}^n)$. Therefore, $\widehat{D^s u} \in L^1(\mathbb{R}^n)$. On the other hand, by basic properties of the Fourier transform, $\widehat{Du}(\xi) = 2\pi i \xi \hat{u}(\xi)$, so clearly, $\widehat{D^s u} \rightarrow \widehat{Du}$ a.e. as $s \nearrow 1$. Thanks to the bound (19) and dominated convergence, $\widehat{D^s u} \rightarrow \widehat{Du}$ in $L^1(\mathbb{R}^n)$. As the inverse Fourier transform is continuous from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, we also have that

$$D^s u \rightarrow Du \text{ uniformly in } \mathbb{R}^n.$$

Now, using a standard interpolation inequality (or Hölder's), we get that

$$\begin{aligned} \|D^s u - Du\|_p &\leq \|D^s u - Du\|_1^{\frac{1}{p}} \|D^s u - Du\|_\infty^{\frac{1}{p'}} \\ &\leq (\|D^s u\|_1 + \|Du\|_1)^{\frac{1}{p}} \|D^s u - Du\|_\infty^{\frac{1}{p'}} \\ &\leq C \|u\|_{W^{1,1}(\mathbb{R}^n)}^{\frac{1}{p}} \|D^s u - Du\|_\infty^{\frac{1}{p'}}, \end{aligned}$$

where we have used Proposition 2.7, considering that, as $s \nearrow 1$, we can assume $s \geq \frac{1}{2}$, so the constant $C > 0$ does not depend on s . Thus, the convergence $D^s u \rightarrow Du$ in L^p follows and the result is true for C_c^∞ functions.

To conclude the proof, we extend this result through a density argument. Let us consider $u \in W^{1,p}(\mathbb{R}^n)$. Then, for every $\varepsilon > 0$ we can find $v \in C_c^\infty(\mathbb{R}^n)$ such that $\|v - u\|_{W^{1,p}(\mathbb{R}^n)} < \varepsilon$. Thus,

$$\begin{aligned} \|D^s u - Du\|_p &\leq \|D^s u - D^s v\|_p + \|D^s v - Dv\|_p + \|Dv - Du\|_p \\ &\leq (C + 1)\varepsilon + \|D^s v - Dv\|_p, \end{aligned}$$

where we have used again Proposition 2.7. Finally, when we take limits we obtain that

$$\limsup_{s \nearrow 1} \|D^s u - Du\|_p \leq (C + 1)\varepsilon,$$

for every $\varepsilon > 0$, which concludes the result. \square

Thanks to Lemma 2.5, the previous result also implies the convergence in L^p of the fractional divergence.

Corollary 3.3. *Let $0 < s < 1$ and $1 < p < \infty$. Then, for each $\phi \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$,*

$$\operatorname{div}^s \phi \rightarrow \operatorname{div} \phi \text{ in } L^p(\mathbb{R}^n) \text{ as } s \nearrow 1.$$

4 Compactness

In this section we establish that any sequence $\{u_s\}_{s \in (0,1)}$ with bounded $H_g^{s,p}(\Omega)$ norm is precompact in $L^q(\mathbb{R}^n)$ for a suitable $q \geq 1$.

Even though the continuous embedding of $H^{s,p}$ into $H^{\bar{s},p}$ for $0 < \bar{s} < s < 1$ is already known, we start by giving a new proof of this result, where we show that the embedding constant is independent of s . This proof follows the ideas of Theorem 2.9.

Proposition 4.1. *Let $0 < \bar{s} < s_0 < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then, there exists a constant $C = C(\Omega, n, s_0, \bar{s}) > 0$ such that for every $s \in [s_0, 1)$, $1 < p < \infty$ and $u \in H_0^{s,p}(\Omega)$ we have*

$$\|D^{\bar{s}}u\|_p \leq C\|D^s u\|_p. \quad (20)$$

Proof. By density, it is enough to prove the inequality for $u \in C_c^\infty(\Omega)$. We divide the proof into two steps.

Step 1. First, we prove that there exists $C = C(\Omega, n, s_0, \bar{s}) > 0$ such that

$$\|D^{\bar{s}}u\|_{L^p(\Omega)} \leq C\|D^s u\|_p. \quad (21)$$

Let $R \geq 1$ be such that $\Omega \subset B(0, R)$. Define $\Omega_1 := B(0, 2R)$ and fix $x \in \Omega$. Notice that, as a consequence of [24, Th. 1.2] and the semigroup property of the Riesz potential, we can write

$$D^{\bar{s}}u = I_{1-\bar{s}} * Du = (I_{1-s} * I_{s-\bar{s}}) * Du = I_{s-\bar{s}} * D^s u.$$

This equality, together with (18) and Lemma 2.4 yields

$$\begin{aligned} |D^{\bar{s}}u(x)| &\leq \frac{1}{\gamma(s-\bar{s})} \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy + \int_{\mathbb{R}^n \setminus \Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy \right] \\ &\leq C(n) \left[\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy + \int_{\mathbb{R}^n \setminus \Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy \right]. \end{aligned} \quad (22)$$

Now $\Omega_1 \subset B(x, 3R)$, so

$$\int_{\Omega_1} \frac{1}{|x-y|^{n-(s-\bar{s})}} dy \leq \int_{B(x, 3R)} \frac{1}{|x-y|^{n-(s-\bar{s})}} dy = \frac{\sigma_{n-1}}{s-\bar{s}} (3R)^{(s-\bar{s})} \leq C(n) \frac{1}{s-\bar{s}} R. \quad (23)$$

Similarly, $\Omega \subset B(y, 3R)$ for every $y \in \Omega_1$, so

$$\int_{\Omega} \frac{1}{|x-y|^{n-(s-\bar{s})}} dx \leq C(n) \frac{1}{s-\bar{s}} R. \quad (24)$$

By (23) and Hölder's inequality,

$$\int_{\Omega_1} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy \leq \left[C(n) \frac{1}{s-\bar{s}} R \right]^{\frac{1}{p'}} \left(\int_{\Omega_1} \frac{|D^s u(y)|^p}{|x-y|^{n-(s-\bar{s})}} dy \right)^{\frac{1}{p}}.$$

Therefore, using (24) and Fubini's theorem, we find, as in (13),

$$\left[\int \left(\int_{\mathbb{R}^n} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy \right)^p dx \right]^{\frac{1}{p}} \leq C(n) \frac{1}{s-\bar{s}} R \|D^s u\|_p. \quad (25)$$

Now, for any $y \in \mathbb{R}^n$, similarly to (16), for each $x \in \mathbb{R}^n$ we have

$$\frac{1}{|x-y|^{n-(s-\bar{s})}} \leq C(n) \frac{1}{|y|^{n-(s-\bar{s})}} \quad (26)$$

and, in fact, (17) also holds. Thus, using (26) and (17),

$$\int_{\mathbb{R}^n} \frac{|D^s u(y)|}{|x-y|^{n-(s-\bar{s})}} dy \leq C(n) | \cdot |^{\frac{1}{p'}} \|u\|_{L^p(\Omega)} \int_{\mathbb{R}^n} \frac{1}{|y|^{n+s}} \frac{1}{|y|^{n-(s-\bar{s})}} dy = C(n) | \cdot |^{\frac{1}{p'}} \frac{R^{-n-\bar{s}}}{n+\bar{s}} \|u\|_{L^p(\Omega)}.$$

This last inequality, combined with (22) and (25), implies by the triangular inequality that

$$\|D^{\bar{s}} u\|_{L^p(\Omega)} \leq C(n) \frac{1}{s-\bar{s}} R \|D^s u\|_p + | \cdot | C(n) \frac{R^{-n-\bar{s}}}{n+\bar{s}} \|u\|_{L^p(\Omega)}.$$

Finally, we apply Theorem 2.9 on the right hand side to obtain

$$\|D^{\bar{s}} u\|_{L^p(\Omega)} \leq C(n, | \cdot |) \left(\frac{1}{s-\bar{s}} + \frac{R^{-n-\bar{s}}}{n+\bar{s}} \frac{1}{s} \right) \left(\|D^s u\|_p \leq C(n, | \cdot |) \left(\left(\frac{1}{s_0-\bar{s}} + \frac{1}{s_0(n+\bar{s})} \right) \left(\|D^s u\|_p, \right. \right. \right.$$

which completes the proof of (21).

Step 2. Now we prove (20). Let us call $c_C = | \cdot | + B(0, 1)$. Then,

$$\|D^{\bar{s}} u\|_p \leq \|D^{\bar{s}} u\|_{L^p(\Omega_C)} + \|D^{\bar{s}} u\|_{L^p(\Omega_C^c)}.$$

By (21) there exists $C > 0$ (depending on c_C, n, s_0, \bar{s} , so, ultimately, on $| \cdot |, n, s_0, \bar{s}$) such that

$$\|D^{\bar{s}} u\|_p \leq C \|D^s u\|_p + \|D^{\bar{s}} u\|_{L^p(\Omega_C^c)}. \quad (27)$$

Now, for $x \in \mathbb{R}^n$,

$$D^{\bar{s}} u(x) = -c_{n,\bar{s}} \int \frac{u(y)}{|x-y|^{n+\bar{s}}} \frac{x-y}{|x-y|} dy,$$

so, by Lemma 2.4,

$$|D^{\bar{s}} u(x)| \leq C(n) \int \left(\frac{|u(y)|}{|x-y|^{n+\bar{s}}} dy, \right.$$

and, hence, by Minkowski's integral inequality,

$$\|D^{\bar{s}} u\|_{L^p(\Omega_C^c)} \leq C(n) \int_{\mathbb{R}^n} \left(\int \left(\frac{|u(y)|}{|x-y|^{n+\bar{s}}} dy \right)^p dx \right)^{\frac{1}{p}} \leq C(n) \int |u(y)| \int_{\mathbb{R}^n} \frac{1}{|x-y|^{(n+\bar{s})p}} dx \Big)^{\frac{1}{p}} dy. \quad (28)$$

Now, for every $y \in \mathbb{R}^n$ we have $\mathbb{R}^n - y \subset B(0, 1)^c$, and hence

$$\int_{\mathbb{R}^n} \frac{1}{|x-y|^{(n+\bar{s})p}} dx = \int_{\mathbb{R}^n - y} \frac{1}{|z|^{(n+\bar{s})p}} dz \leq \int_{B(0,1)^c} \frac{1}{|z|^{(n+\bar{s})p}} dz = \frac{\sigma_{n-1}}{(n+\bar{s})p-n} \leq \frac{\sigma_{n-1}}{\bar{s}}.$$

Thus, continuing from (28) we find that

$$\|D^{\bar{s}} u\|_{L^p(\Omega_C^c)} \leq C(n) \max\{1, \frac{\sigma_{n-1}}{\bar{s}}\} \int \left(|u(y)| dy \leq C(n) \max\{1, \frac{\sigma_{n-1}}{\bar{s}}\} \max\{1, | \cdot | \} \|u\|_{L^p(\Omega)}. \quad (29)$$

Inequalities (27), (29) and Theorem 2.9 finish the proof. \square

Now we present the main result of this section. The proof of the following compactness result is partly inspired by that of [17, Lemma 3.6]. This result should be compared with [19, Th. 1.2], in which a $W^{s,p}$ version is done. In what follows, given $p \in [1, n)$ we denote by p^* its Sobolev conjugate exponent, i.e., $p^* = \frac{pn}{n-p}$. Recall also the notation p_s^* from Theorem 2.8.

Theorem 4.2. *Let $1 < p < \infty$ and $g \in W^{1,p}(\mathbb{R}^n)$. For each $s \in (0, 1)$, let $u_s \in H_g^{s,p}(\Omega)$ be such that the family $\{D^s u_s\}_{s \in (0,1)}$ is bounded in $L^p(\mathbb{R}^n)$. Then, there exist $u \in W^{1,p}(\mathbb{R}^n)$ and an increasing sequence $\{s_j\}_{j \in \mathbb{N}} \subset (0, 1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ such that for every q satisfying*

$$\begin{cases} q \in [p, p^*) & \text{if } p < n, \\ q \in [p, \infty) & \text{if } p = n, \\ q \in [p, \infty] & \text{if } p > n, \end{cases}$$

there exists $j_q \in \mathbb{N}$ for which $\{u_{s_j}\}_{j \geq j_q} \subset L^q(\mathbb{R}^n)$ and the convergences

$$u_{s_j} \rightarrow u \text{ in } L^q(\mathbb{R}^n) \quad \text{and} \quad D^{s_j} u_{s_j} \rightharpoonup Du \text{ in } L^p(\mathbb{R}^n)$$

hold as $j \rightarrow \infty$.

Proof. Thanks to Theorem 3.2 and the Sobolev embedding, we can assume, without loss of generality, that $g = 0$.

Fix $0 < \bar{s} < s_0 < 1$. By hypothesis and Proposition 4.1, $\{D^{\bar{s}} u_s\}_{s \in [s_0, 1)}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, and consequently, by Corollary 2.10, $\{u_s\}_{s \in [s_0, 1)}$ is bounded in $H_0^{\bar{s},p}(\Omega)$. Since $H_0^{\bar{s},p}(\Omega)$ is reflexive, there exist $u \in H_0^{\bar{s},p}(\Omega)$ and an increasing sequence $\{s_j\}_{j \geq 1} \subset [s_0, 1)$, with $\lim_{j \rightarrow \infty} s_j = 1$, such that

$$u_{s_j} \rightharpoonup u \quad \text{in } H_0^{\bar{s},p}(\Omega).$$

Now, if $p \leq n$, given $q \in [p, p^*)$ (defining $p^* = \infty$ if $p = n$), there exists $j_q \in \mathbb{N}$ such that for all $j \geq j_0$ we have $q < p_{s_j}^*$. Arguing as above we obtain that $u_{s_j} \rightharpoonup u$ in $H_0^{s_{j_0},p}(\Omega)$, so applying Theorem 2.8, we have that $\{u_{s_j}\}_{j \geq j_q} \subset L^q(\mathbb{R}^n)$ and

$$u_{s_j} \rightarrow u \quad \text{in } L^q(\mathbb{R}^n).$$

If $p > n$, there exists $j_0 \in \mathbb{N}$ such that $s_{j_0} p > n$, and arguing as above using again Theorem 2.8, we have that $\{u_{s_j}\}_{j \geq j_0} \subset L^q(\mathbb{R}^n)$, for any $q \in [p, +\infty]$, and

$$u_{s_j} \rightarrow u \quad \text{in } L^q(\mathbb{R}^n).$$

Next, as $\{D^{s_j} u_{s_j}\}_{j \geq j_0}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, there exists $V \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $D^{s_j} u_{s_j} \rightharpoonup V$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \rightarrow \infty$, in principle up to a subsequence, but we will see that in fact it holds true for the whole sequence. Given $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, using the fractional integration by parts, Lemma 2.6, we get

$$\int \left(D^{s_j} u_{s_j}(x) \cdot \varphi(x) \, dx = - \int u_{s_j}(x) \operatorname{div}^{s_j} \varphi(x) \, dx, \right.$$

and passing to the limit as $j \rightarrow \infty$, having in mind that both u_{s_j} and $\operatorname{div}^{s_j} \varphi$ are strongly convergent (Corollary 3.3), we obtain

$$\int \left(V(x) \cdot \varphi(x) \, dx = - \int u(x) \operatorname{div} \varphi(x) \, dx, \right.$$

and hence $Du = V$ and $u \in W^{1,p}(\mathbb{R}^n)$. Since this V is unique, this shows that $D^{s_j} u_{s_j} \rightharpoonup V$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \rightarrow \infty$ without the need of taking a subsequence. This finishes the proof. \square

5 Weak continuity of the minors for varying s

In this section we prove the analogue in this context of the weak continuity of minors, namely, that if we have a sequence $\{u_s\}_{s \in (0,1)}$ such that $u_s \in H^{s,p}$ for each s and $D^s u_s \rightharpoonup Du$ in L^p as $s \nearrow 1$ for some $u \in W^{1,p}$ then the minors of $D^s u_s$ converge weakly in some L^q to the minors of Du . For this, we follow the general guidelines of [4], where the analogue convergence for a fixed s is proved. In essence, this section consists of an adaptation of many results of [4] with bounds that do not depend on s .

We start with an analogue of [4, Lemma 3.1].

Lemma 5.1. *Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and let $\bar{\alpha} \in (0, 1)$. Then*

$$\sup_{s \in (\bar{\alpha}, 1)} \sup_{x \in \mathbb{R}^n} c_{n,s} \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy \right) < \infty \quad \text{and} \quad \sup_{s \in (\bar{\alpha}, 1)} \sup_{r \in [1, \infty]} \|D^s \varphi\|_r < \infty.$$

Proof. Fix $\alpha \in (0, \bar{\alpha})$. Let L and C be, respectively, the Lipschitz and α -Hölder constants of φ . Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \int \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy &\leq \iint_{B(x,1)} \frac{L}{|x - y|^{n+s-1}} dy + \iint_{B(x,1)^c} \frac{C}{|x - y|^{n+s-\alpha}} dy \\ &= \int_{B(0,1)} \frac{L}{|z|^{n+s-1}} dz + \iint_{B(0,1)^c} \frac{C}{|z|^{n+s-\alpha}} dz = \frac{\sigma_{n-1}L}{1-s} + \frac{\sigma_{n-1}C}{s-\alpha}. \end{aligned} \quad (30)$$

By Lemma 2.4, we find that

$$\sup_{s \in (\bar{\alpha}, 1)} \sup_{x \in \mathbb{R}^n} c_{n,s} \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy \right) < \infty,$$

and, as a consequence,

$$\sup_{s \in (\bar{\alpha}, 1)} \|D^s \varphi\|_\infty < \infty.$$

Denote by F the support of φ . Then

$$\int c_{n,s} \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy \right) dx \leq |c_{n,s}| (A + B), \quad (31)$$

where

$$A := \int \iint_F \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy dx, \quad B := \int \iint_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy dx.$$

Now, we observe that, applying Fubini's Theorem and (30),

$$A = \int_F \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dx dy \right) \leq \left(\frac{\sigma_{n-1}L}{1-s} + \frac{\sigma_{n-1}C}{s-\alpha} \right) (F|), \quad (32)$$

where $|F|$ denotes the measure of the set F .

We notice that $|\varphi(x) - \varphi(y)| = 0$ for every $(x, y) \in F^c \times F^c$. Therefore, applying again (30) we get

$$B = \int_F \iint_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy dx \leq \int_F \int \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dy dx \leq \left(\frac{\sigma_{n-1}L}{1-s} + \frac{\sigma_{n-1}C}{s-\alpha} \right) (F|). \quad (33)$$

Putting together (31), (32) and (33) we have that

$$\|D^s \varphi\|_1 \leq 2c_{n,s} \left(\frac{\sigma_{n-1}L}{1-s} + \frac{\sigma_{n-1}C}{s-\alpha} \right) (F|).$$

Applying now Lemma 2.4 we infer that

$$\sup_{s \in (\bar{\alpha}, 1)} \|D^s \varphi\|_1 < \infty.$$

Finally, through a standard interpolation argument, we get that

$$\sup_{s \in (\bar{\alpha}, 1)} \sup_{r \in [1, \infty]} \|D^s \varphi\|_r < \infty.$$

□

We now recall a nonlocal operator related to the fractional gradient introduced in [4]. Moreover, we adapt [4, Lemma 3.2] to show bounds of this operator independent of s .

Lemma 5.2. *Let $1 \leq q < \infty$ and $0 < s < 1$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$, $k \in \mathbb{N}$ and $r \in [1, q]$. Then, the operator $K_\varphi^s : L^q(\mathbb{R}^n, \mathbb{R}^{k \times n}) \rightarrow L^r(\mathbb{R}^n, \mathbb{R}^k)$ defined as*

$$K_\varphi^s(U)(x) = c_{n,s} \int \left(\frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} U(y) \frac{x - y}{|x - y|} dy, \quad a.e. \ x \in \mathbb{R}^n,$$

is linear and bounded. Moreover, given $0 < \bar{\alpha} < 1$, there exists a constant $C = C(n, q, \bar{\alpha}, \varphi)$ such that for every $s \in (\bar{\alpha}, 1)$, every $r \in [1, q]$ and $U \in L^q(\mathbb{R}^n, \mathbb{R}^{k \times n})$,

$$\|K_\varphi^s(U)\|_r \leq C \|U\|_q.$$

Proof. The operator K_φ^s is clearly linear. Let $U \in L^q(\mathbb{R}^n, \mathbb{R}^{k \times n})$. For all $x \in \mathbb{R}^n$ we have

$$K_\varphi^s(U)(x) \leq |c_{n,s}| \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| dy,$$

so

$$K_\varphi^s(U)(x)^q \leq 2^{q-1} |c_{n,s}|^q (g(x) + h(x)), \quad (34)$$

with

$$g(x) := \left(\int_{B(x,1)} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| dy \right)^q, \quad h(x) := \left(\int_{\mathbb{R}^{(x,1)^c}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)| dy \right)^q.$$

Fix $\alpha \in (0, \bar{\alpha})$. Let L and L_α be the Lipschitz and α -Hölder semi-norm, respectively, of φ . Then, applying Hölder's inequality, we get

$$\begin{aligned} g(x) &\leq L^q \left(\int_{\mathbb{R}^{(x,1)}} \frac{|U(y)|}{|x - y|^{n+s-1}} dy \right)^q = L^q \left(\int_{\mathbb{R}^{(0,1)}} \frac{|U(x - z)|}{|z|^{n+s-1}} dz \right)^q \\ &\leq L^q \int_{\mathbb{R}^{(0,1)}} \frac{|U(x - z)|^q}{|z|^{n+s-1}} dz \left(\int_{B(0,1)} \frac{1}{|z|^{n+s-1}} dz \right)^{q-1} \\ &= L^q \left(\frac{\sigma_{n-1}}{1-s} \right)^{q-1} \int_{\mathbb{R}^{(0,1)}} \frac{|U(x - z)|^q}{|z|^{n+s-1}} dz, \end{aligned}$$

where σ_{n-1} is the area of the unit sphere of \mathbb{R}^n . Integrating,

$$\int \left(g(x) dx \leq L^q \left(\frac{\sigma_{n-1}}{1-s} \right)^{q-1} \int_{B(0,1)} \frac{1}{|z|^{n+s-1}} \int \left(|U(x-z)|^q dx dz \right) = L^q \left(\frac{\sigma_{n-1}}{1-s} \right)^q \|U\|_q^q. \quad (35)$$

As for the term h , applying Hölder's inequality,

$$\begin{aligned} h(x) &\leq L_\alpha^q \left(\int_{\mathbb{R}(x,1)^c} \frac{|U(y)|}{|x-y|^{n+s-\alpha}} dy \right)^q \\ &\leq L_\alpha^q \int_{\mathbb{R}(0,1)^c} \frac{|U(x-z)|^q}{|z|^{n+s-\alpha}} dz \left(\int_{B(0,1)^c} \frac{1}{|z|^{n+s-\alpha}} dz \right)^{q-1} \\ &= L_\alpha^q \left(\frac{\sigma_{n-1}}{s-\alpha} \right)^{q-1} \int_{\mathbb{R}(0,1)^c} \frac{|U(x-z)|^q}{|z|^{n+s-\alpha}} dz. \end{aligned}$$

Integrating,

$$\int \left(h(x) dx \leq L_\alpha^q \left(\frac{\sigma_{n-1}}{s-\alpha} \right)^{q-1} \int_{B(0,1)^c} \frac{1}{|z|^{n+s-\alpha}} \int \left(|U(x-z)|^q dx dz \right) = L_\alpha^q \left(\frac{\sigma_{n-1}}{s-\alpha} \right)^q \|U\|_q^q. \quad (36)$$

Putting together (34), (35) and (36) we obtain

$$K_\varphi^s(U) \quad {}^q \leq 2^{q-1} |c_{n,s}|^q \left(L^q \left(\frac{\sigma_{n-1}}{1-s} \right)^q + L_\alpha^q \left(\frac{\sigma_{n-1}}{s-\alpha} \right)^q \right) \left(\|U\|_q^q, \right.$$

so applying Lemma 2.4 we find that

$$K_\varphi^s(U) \quad {}_q \leq C \|U\|_q \quad (37)$$

for some constant C independent of $s \in (\bar{\alpha}, 1)$ and U .

Next, we are going to check the boundedness of $K_\varphi^s : L^q(\mathbb{R}^n, \mathbb{R}^{k \times n}) \rightarrow L^1(\mathbb{R}^n, \mathbb{R}^k)$. Denote by F the support of φ . Then

$$\int \left(K_\varphi^s(U)(x) dx \leq |c_{n,s}| (A + B), \quad (38)$$

where

$$A := \int \int_F \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)| dy dx, \quad B := \int \int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)| dy dx.$$

Now, we observe that, applying Fubini's Theorem, Hölder's inequality and Lemma 5.1 there exists $C_0 > 0$ independent of $s \in (\bar{\alpha}, 1)$ such that

$$A \leq \int_F |U(y)| \int \left(\frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} dx dy \leq C_0 |F|^{\frac{1}{q'}} \left(\int_F |U(y)|^q dy \right)^{\frac{1}{q}} \leq C_0 |F|^{\frac{1}{q'}} \|U\|_q. \quad (39)$$

Since $|\varphi(x) - \varphi(y)| = 0$ for every $(x, y) \in F^c \times F^c$, in view of Hölder's inequality and Lemma 5.1 we get

$$\begin{aligned} B &= \int_F \int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)| dy dx \\ &\leq \int_F \left(\int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} dy \right)^{\frac{1}{q'}} \left(\int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)|^q dy \right)^{\frac{1}{q}} dx \\ &\leq C_0^{\frac{1}{q'}} \int_F \left(\int_{F^c} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^{n+s}} |U(y)|^q dy \right)^{\frac{1}{q}} dx. \end{aligned}$$

Using again Hölder's inequality, Lemma 5.1 and Fubini's Theorem, we obtain

$$\begin{aligned}
B &\leq C_0^{\frac{1}{q'}} \left(\int_F \int_{\mathbb{R}^c} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} |U(y)|^q dy dx \right)^{\frac{1}{q}} |F|^{\frac{1}{q'}} \\
&= (C_0 |F|)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^c} |U(y)|^q \int_F \frac{|\varphi(x) - \varphi(y)|}{|x - y|^{n+s}} dx dy \right)^{\frac{1}{q}} \\
&\leq (C_0 |F|)^{\frac{1}{q'}} C_0^{\frac{1}{q}} \left(\iint_{\mathbb{R}^c} |U(y)|^q dy \right)^{\frac{1}{q}} \leq C \|U\|_q,
\end{aligned} \tag{40}$$

where $C > 0$ is a constant independent of s and U . Inequalities (38), (39) and (40) lead us to

$$K_\varphi^s(U) \leq C \|U\|_q, \tag{41}$$

for some constant C independent of $s \in (\bar{\alpha}, 1)$ and U . The conclusion of the theorem is obtained through an interpolation of inequalities (37) and (41). \square

The following result is the key to adapt the continuity of minors of [4] to our case. It establishes the relationship between the operators K_φ^s and $D\varphi$ when $s \nearrow 1$.

Lemma 5.3. *Let $p > 1$ and $0 < s < 1$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $w \in L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$. Consider a family $\{w_s\}_{s \in (0,1)}$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$ such that $w_s \rightharpoonup w$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$. Then, for all $r \in (1, p]$,*

$$K_\varphi^s(w_s) \rightharpoonup w D\varphi \quad \text{in } L^r(\mathbb{R}^n, \mathbb{R}^n) \text{ as } s \nearrow 1.$$

Proof. Assume first that $w_s \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^{n \times n})$ for all $s \in (0, 1)$. Fix two indexes $1 \leq i, j \leq n$, let u_s be the (i, j) -th entry of w_s and let u be the (i, j) -th entry of w . Let $\theta \in C_c^\infty(\mathbb{R}^n)$. We apply the product formula of [4, Lemma 3.4] and then Lemma 2.6 to obtain

$$\int \theta K_\varphi^s(u_s I) = \int \theta D^s(\varphi u_s) - \iint (\theta \varphi D^s u_s = - \int \varphi u_s \operatorname{div}^s \theta + \int (u_s \operatorname{div}^s(\theta \varphi).$$

Now we have from Corollary 3.3 that $\operatorname{div}^s \theta \rightarrow \operatorname{div} \theta$ and $\operatorname{div}^s(\theta \varphi) \rightarrow \operatorname{div}^s(\theta \varphi)$ in $L^q(\mathbb{R}^n)$ for every $q \in (1, \infty)$ as $s \nearrow 1$. As $u_s \rightharpoonup u$ in $L^p(\mathbb{R}^n)$ we obtain

$$\iint (\theta K_\varphi^s(u_s I) \rightarrow - \int \varphi u \operatorname{div} \theta + \int u \operatorname{div}(\theta \varphi) = \iint (\theta u D\varphi.$$

This shows that $K_\varphi^s(u_s I) \rightharpoonup u D\varphi$ in the sense of distributions. Now, by Lemma 5.2, for every $r \in (1, p]$

$$K_\varphi^s(u_s I) \leq_r C \|u_s\|_p \leq C_1$$

for some $C, C_1 > 0$ independent of s , which implies that $K_\varphi^s(u_s I) \rightharpoonup u D\varphi$ in $L^r(\mathbb{R}^n)$.

Now, we remove the assumption $w_s \in W^{1,p}(\mathbb{R}^n)$. Fix $r \in (1, p]$. For each $s \in (0, 1)$, let $v_s \in W^{1,p}(\mathbb{R}^n)$ be such that $\|u_s - v_s\|_r \leq 1 - s$. By Lemma 5.2,

$$\|K_\varphi^s(u_s I) - K_\varphi^s(v_s I)\|_r = \|K_\varphi^s(u_s I - v_s I)\|_r \leq C \|u_s - v_s\|_p \rightarrow 0,$$

which implies that $K_\varphi^s(u_s I) \rightharpoonup u D\varphi$ in $L^r(\mathbb{R}^n)$. In other words, for each $j \in \{1, \dots, n\}$, the family of functions

$$x \mapsto c_{n,s} \iint \left(\frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} u_s(y) \frac{x_j - y_j}{|x - y|} dy \right)$$

converges weakly to $u(D\varphi)_j$ as $s \nearrow 1$. Therefore, for each $i \in \{1, \dots, n\}$, the family of functions

$$x \mapsto (K_\varphi^s(w_s))_{ij}(x) = \sum_{j=1}^n \left(\int \frac{\varphi(x) - \varphi(y)}{|x - y|^{n+s}} (w_s)_{ij}(y) \frac{x_j - y_j}{|x - y|} dy \right)$$

converges weakly to $\sum_{j=1}^n w_{ij}(D\varphi)_j = (w D\varphi)_i$. This concludes the proof. \square

We now show a convenient notation for submatrices, which is taken from [4, Def. 4.1].

Definition 5.1. Let $k \in \mathbb{N}$ be with $1 \leq k \leq n$. Consider indices $1 \leq i_1 < \dots < i_k \leq n$ and $1 \leq j_1 < \dots < j_k \leq n$.

- a) We define $M = M_{i_1, \dots, i_k; j_1, \dots, j_k} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{k \times k}$ as the map such that $M(F)$ is the submatrix of $F \in \mathbb{R}^{n \times n}$ formed by the rows i_1, \dots, i_k and the columns j_1, \dots, j_k .
- b) We define $\bar{M} = \bar{M}_{i_1, \dots, i_k; j_1, \dots, j_k} : \mathbb{R}^{k \times k} \rightarrow \mathbb{R}^{n \times n}$ as the map such that $\bar{M}(F)$ is the matrix whose rows i_1, \dots, i_k and columns j_1, \dots, j_k coincide with those of F , whereas the rest of the entries are zero.
- c) We define $\tilde{N} = \tilde{N}_{i_1, \dots, i_k} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as the map such that $\tilde{N}(v)$ is the vector whose entries i_1, \dots, i_k coincide with the corresponding entries of v , whereas the rest of the entries are zero.

The following is the main result of this section, and shows the weak convergence of the minors of $D^s u_s$ to those of Du , whenever u_s converges weakly to u . Of course, by a minor we mean the determinant of a submatrix. Its proof is an adaptation of [4, Th. 5.2].

Theorem 5.4. Let $p \geq n - 1$ and $0 < s < 1$. Let $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $\{u_s\}_{s \in (0,1)}$ be a family such that $u_s \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ for each $s \in (0, 1)$, while $u_s \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $D^s u_s \rightharpoonup Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$. Then

- a) If $k \in \mathbb{N}$ with $1 \leq k \leq n - 2$ and μ is a minor of order k then $\mu(D^s u_s) \rightharpoonup \mu(Du)$ in $L^{\frac{p}{k}}(\mathbb{R}^n)$ as $s \nearrow 1$.
- b) If $\text{cof } D^s u_s \rightharpoonup \vartheta$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$ for some $q \in [1, \infty)$ and $\vartheta \in L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ then $\vartheta = \text{cof } Du$.
- c) Assume $\det D^s u_s \rightharpoonup \theta$ in $L^\ell(\mathbb{R}^n)$ as $s \nearrow 1$ for some $\ell \in [1, \infty)$ and some $\theta \in L^\ell(\mathbb{R}^n)$. If $p < n$ assume, in addition, that $\text{cof } D^s u_s \rightharpoonup \text{cof } Du$ in $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$ for some $q \in (\frac{p^*}{p^*-1}, \infty)$. Then $\theta = \det Du$.

Proof. We will prove a) by induction on k . For $k = 1$ there is nothing to prove. Assume it holds for some $k \leq n - 3$ and let us prove it for $k + 1$. Let μ be a minor of order $k + 1$. In the notation of Definition 5.1, $\mu(F) = \det M(F)$ for all $F \in \mathbb{R}^{n \times n}$, where $M = M_{i_1, \dots, i_{k+1}; j_1, \dots, j_{k+1}}$ for some $1 \leq i_1 < \dots < i_{k+1} \leq n$ and $1 \leq j_1 < \dots < j_{k+1} \leq n$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. By induction assumption, $\text{cof } M(D^s u_s) \rightharpoonup \text{cof } M(Du)$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{(k+1) \times (k+1)})$ as $s \nearrow 1$, so $\bar{M}(\text{cof } M(D^s u_s)) \rightharpoonup \bar{M}(\text{cof } M(Du))$ in $L^{\frac{p}{k}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$. By Lemma 5.3, $K_\varphi^s(\bar{M}(\text{cof } M(D^s u_s))) \rightharpoonup \bar{M}(\text{cof } M(Du)) D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{k}]$. By Theorem 4.2, $\tilde{N}(u_s) \rightarrow \tilde{N}(u)$ in $L^p(\mathbb{R}^n)$, so

$$\tilde{N}(u_s) \cdot K_\varphi^s(\bar{M}(\text{cof } M(D^s u_s))) \rightharpoonup \tilde{N}(u) \cdot (\bar{M}(\text{cof } M(Du)) D\varphi) \quad \left(\text{in } L^1(\mathbb{R}^n) \right) \quad (42)$$

since $\frac{k}{p} + \frac{1}{p} \leq 1$. Now, the nonlocal integration by parts for the determinant given in [4, Lemma 5.1] as well as the classical (local) one state that

$$-\frac{1}{k} \int \tilde{N}(u_s)(x) \cdot K_\varphi(\bar{M}(\text{cof } M(D^s u_s)))(x) dx = \int \left(\det M(D^s u_s)(x) \varphi(x) dx \right) \quad (43)$$

and

$$-\frac{1}{k} \int \left(\tilde{N}(u)(x) \cdot \left(\bar{M}(\operatorname{cof} M(Du))(x) D\varphi(x) \right) dx = \int \left(\det M(Du(x)) \varphi(x) dx, \quad (44)$$

respectively, so

$$\int \left(\det M(D^s u_s(x)) \varphi(x) dx \rightarrow \int \left(\det M(Du(x)) \varphi(x) dx. \quad (45)$$

This shows that $\det M(D^s u_s) \rightharpoonup \det M(Du)$ in the sense of distributions. As $\{\det M(D^s u_s)\}_{s \in (0,1)}$ is bounded in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$ and $p > k+1$, we have that $\det M(D^s u_s) \rightharpoonup \det M(Du)$ in $L^{\frac{p}{k+1}}(\mathbb{R}^n)$ as $s \nearrow 1$.

The proof of *b)* follows the lines of *a)*. Let μ be a minor of order $n-1$. As before, $\mu(F) = \det M(F)$ for all $F \in \mathbb{R}^{n \times n}$, where $M = M_{i_1, \dots, i_{n-1}; j_1, \dots, j_{n-1}}$ for some $1 \leq i_1 < \dots < i_{n-1} \leq n$ and $1 \leq j_1 < \dots < j_{n-1} \leq n$. Let $\phi \in C_c^\infty(\Omega)$. By part *a)*, $\operatorname{cof} M(D^s u_j) \rightharpoonup \operatorname{cof} M(D^s u)$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{(n-1) \times (n-1)})$, so $\bar{M}(\operatorname{cof} M(D^s u_s)) \rightharpoonup \bar{M}(\operatorname{cof} M(Du))$ in $L^{\frac{p}{n-2}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$. By Lemma 5.3, $K_\varphi^s(\bar{M}(\operatorname{cof} M(D^s u_s))) \rightharpoonup \bar{M}(\operatorname{cof} M(Du)) D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{n-2}]$. By Theorem 4.2, $\tilde{N}(u_s) \rightarrow \tilde{N}(u)$ in $L^p(\mathbb{R}^n)$, so convergence (42) is also valid since $\frac{n-2}{p} + \frac{1}{p} \leq 1$. Again thanks to (43)–(44), we conclude that convergence (45) holds. This shows that $\mu(D^s u_s) \rightharpoonup \mu(Du)$ in the sense of distributions. As this is true for every minor μ of order $n-1$, we obtain that $\operatorname{cof} D^s u_s \rightharpoonup \operatorname{cof} Du$ in the sense of distributions. Due to the assumption, $\vartheta = \operatorname{cof} Du$.

We finally show part *c)*. Let $\phi \in C_c^\infty(\Omega)$. Assume first $p < n$. By the assumption and Lemma 5.3, $K_\varphi^s(\operatorname{cof} D^s u_s) \rightharpoonup \operatorname{cof} Du D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, q]$. By Theorem 4.2, $u_s \rightarrow u$ in $L^t(\mathbb{R}^n)$ for every $t \in [p, p^*)$, so

$$u_s \cdot K_\varphi^s(\operatorname{cof} D^s u_s) \rightharpoonup u \cdot (\operatorname{cof} Du D\varphi) \quad \text{in } L^1(\mathbb{R}^n) \quad (46)$$

since $\frac{1}{q} + \frac{1}{p^*} < 1$.

Assume now $p \geq n$. Then $\{\operatorname{cof} D^s u_s\}_{s \in (0,1)}$ is bounded in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$ so, thanks to part *b)*, $\operatorname{cof} D^s u_s \rightharpoonup \operatorname{cof} Du$ in $L^{\frac{p}{n-1}}(\mathbb{R}^n, \mathbb{R}^{n \times n})$. By Lemma 5.3, $K_\varphi^s(\operatorname{cof} D^s u_s) \rightharpoonup \operatorname{cof} Du D\varphi$ in $L^r(\mathbb{R}^n, \mathbb{R}^n)$ for every $r \in (1, \frac{p}{n-1}]$. By Theorem 4.2, $u_s \rightarrow u$ in $L^t(\mathbb{R}^n)$ for every $t \in [1, \infty)$, so convergence (46) holds since $p > n-1$.

In either case, we have convergence (46), so by the analogue of (43)–(44) with $k = n$ we obtain

$$\int \det D^s u_s(x) \varphi(x) dx \rightarrow \int \left(\det Du(x) \varphi(x) dx.$$

This shows that $\det D^s u_s \rightharpoonup \det Du$ in the sense of distributions, so $\theta = \det Du$. \square

6 Γ -convergence

Γ -convergence is the main conceptual tool for studying the variational convergence of families of functionals defined on metric spaces [7]. In this section we show that the functional

$$\mathcal{I}_s(u) = \int \left(W(x, u(x), D^s u(x)) dx,$$

defined on $H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$, Γ -converges, as $s \nearrow 1$, to the functional

$$\mathcal{I}(u) = \int \left(W(x, u(x), Du(x)) dx,$$

defined on $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ under the assumption of W being polyconvex. We recall the concept of polyconvexity (see, e.g. [2, 11]). Let τ be the number of submatrices of an $n \times n$ matrix. We fix a

function $\vec{\mu} : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^\tau$ such that $\vec{\mu}(F)$ is the collection of all minors of an $F \in \mathbb{R}^{n \times n}$ in a given order. A function $W_0 : \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ is polyconvex if there exists a convex $\Phi : \mathbb{R}^\tau \rightarrow \mathbb{R} \cup \{\infty\}$ such that $W_0(F) = \Phi(\vec{\mu}(F))$ for all $F \in \mathbb{R}^{n \times n}$. Polyconvexity of $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ means polyconvexity in the last variable.

It is convenient to consider both \mathcal{I}_s and \mathcal{I} defined on the same functional space independent of s , so we consider both functionals defined on $L^p(\mathbb{R}^n, \mathbb{R}^n)$. The extension of \mathcal{I}_s to $L^p(\mathbb{R}^n, \mathbb{R}^n) \setminus H^{s,p}(\mathbb{R}^n, \mathbb{R}^n)$ and of \mathcal{I} to $L^p(\mathbb{R}^n, \mathbb{R}^n) \setminus W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ is done by infinity. Recalling the definition of Γ -convergence in this particular situation, we say that \mathcal{I}_s Γ -converges to \mathcal{I} as $s \nearrow 1$ in the strong topology of $L^p(\mathbb{R}^n, \mathbb{R}^n)$ if the following two conditions hold:

- *Liminf inequality:* For every family $\{u_s\}_{s \in (0,1)}$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_s \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $s \nearrow 1$, we have

$$\mathcal{I}(u) \leq \liminf_{s \nearrow 1} \mathcal{I}_s(u_s).$$

- *Limsup inequality:* For each $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, there exists a family $\{u_s\}_{s \in (0,1)} \subset L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_s \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $s \nearrow 1$ and

$$\limsup_{s \nearrow 1} \mathcal{I}_s(u_s) \leq \mathcal{I}(u).$$

Although not in the definition of Γ -convergence, it is customary to attach a compactness property to the conditions above, which, in this context, reads as follows:

- *Compactness:* For every $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and every family $\{u_s\}_{s \in (0,1)}$ with $u_s = g$ in \mathcal{C}^∞ for all $s \in (0,1)$ such that $\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) < \infty$, there exist an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ in $(0,1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ and a $u \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that $u_{s_j} \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \rightarrow \infty$.

The *limsup inequality* will be a consequence of Theorem 3.2, while the compactness property will follow of Theorem 4.2. The *liminf inequality*, on the other hand, is a novel semicontinuity result, which improves that of [4] done for a fixed s , and is singled out in the following proposition. As we will see, the growth conditions for proving the *liminf* and *limsup* inequalities are compatible only in the range $p > n$.

Proposition 6.1. *Let $p \geq n - 1$ satisfy $p > 1$ and $0 < s < 1$. Let Ω be a bounded open subset of \mathbb{R}^n and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

- W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n \times n}$ -measurable, where \mathcal{L}^n denotes the Lebesgue sigma-algebra in \mathbb{R}^n , whereas \mathcal{B}^n and $\mathcal{B}^{n \times n}$ denote the Borel sigma-algebras in \mathbb{R}^n and $\mathbb{R}^{n \times n}$, respectively.*
- $W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.*
- For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.*
- There exist a constant $c > 0$, an $a \in L^1(\mathbb{R}^n)$ and a Borel function $h : [0, \infty) \rightarrow [0, \infty)$ such that*

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \infty$$

and

$$\begin{cases} W(x, y, F) \geq a(x) + c|F|^p + c|\operatorname{cof} F|^q + h(|\det F|) & \text{for some } q > \frac{p^*}{p^*-1}, & \text{if } p < n, \\ W(x, y, F) \geq a(x) + c|F|^p + h(|\det F|), & & \text{if } p = n, \\ W(x, y, F) \geq a(x) + c|F|^p, & & \text{if } p > n, \end{cases}$$

for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n \times n}$.

For each $s \in (0, 1)$, let $u_s \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^n)$ satisfy $u_s \rightarrow u$ in $L^p(\Omega, \mathbb{R}^n)$ as $s \nearrow 1$. Then

$$\mathcal{I}(u) \leq \liminf_{s \nearrow 1} \mathcal{I}_s(u_s) \quad (47)$$

Proof. We can assume that

$$\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) < \infty, \quad (48)$$

hence by assumption *d*), there exists an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ in $(0, 1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ such that $\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) = \lim_{j \rightarrow \infty} \mathcal{I}_{s_j}(u_{s_j})$ and the sequence $\{D^{s_j}u_{s_j}\}_{j \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, so by Theorem 4.2, for a subsequence (not relabelled),

$$u_{s_j} \rightarrow u \quad \text{and} \quad D^{s_j}u_{s_j} \rightharpoonup Du \quad \text{in } L^p \text{ as } j \rightarrow \infty. \quad (49)$$

By Theorem 5.4, for any minor μ of order $k \leq n - 2$, we have that

$$\mu(D^{s_j}u_{s_j}) \rightharpoonup \mu(Du) \text{ in } L^{\frac{p}{k}}(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (50)$$

If $p < n$ then, by assumption *d*), $\{\text{cof } D^s u_s\}_{0 < s < 1}$ is bounded in $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$, whereas if $p \geq n$ we set $q := \frac{p}{n-1}$ and have that $\{\text{cof } D^s u_s\}_{0 < s < 1}$ is bounded in $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$. In either case we have that $q > 1$, so for a subsequence $\{\text{cof } D^{s_j}u_{s_j}\}_{j \in \mathbb{N}}$ converges weakly in $L^q(\mathbb{R}^n, \mathbb{R}^{n \times n})$ and, by Theorem 5.4,

$$\text{cof } D^{s_j}u_{s_j} \rightharpoonup \text{cof } Du \text{ in } L^q(\mathbb{R}^n, \mathbb{R}^{n \times n}) \text{ as } j \rightarrow \infty. \quad (51)$$

If $p \leq n$ then, by assumption *d*) and de la Vallée Poussin's criterion, $\{\det D^s u_s\}_{0 < s < 1}$ is equiintegrable, whereas if $p > n$ we have that $\{\det D^s u_s\}_{0 < s < 1}$ is bounded in $L^{\frac{p}{n}}(\mathbb{R}^n)$ and $\frac{p}{n} > 1$. In either case we have that, for a subsequence, $\{\det D^{s_j}u_{s_j}\}_{j \in \mathbb{N}}$ converges weakly in $L^\ell(\mathbb{R}^n)$ with

$$\begin{cases} \ell = 1 & \text{if } p \leq n, \\ \ell = \frac{p}{n} & \text{if } p > n, \end{cases}$$

and, hence, by Theorem 5.4,

$$\det D^{s_j}u_{s_j} \rightharpoonup \det Du \text{ in } L^\ell(\mathbb{R}^n) \text{ as } j \rightarrow \infty. \quad (52)$$

Convergences (49)–(52) imply, thanks to a standard lower semicontinuity result for polyconvex functionals (see, e.g., [3, Th. 5.4] or [14, Th. 7.5]), that for any $R > 0$,

$$\iint_{B(0,R)} W(x, u(x), Du(x)) \, dx \leq \liminf_{j \rightarrow \infty} \iint_{B(0,R)} W(x, u_{s_j}(x), D^{s_j}u_{s_j}(x)) \, dx. \quad (53)$$

Therefore,

$$\iint_{B(0,R)} [W(x, u(x), Du(x)) - a(x)] \, dx \leq \liminf_{j \rightarrow \infty} \iint_{B(0,R)} [W(x, u_{s_j}(x), D^{s_j}u_{s_j}(x)) - a(x)] \, dx.$$

By monotone convergence,

$$\iint [W(x, u(x), Du(x)) - a(x)] \, dx \leq \liminf_{j \rightarrow \infty} \int [W(x, u_{s_j}(x), D^{s_j}u_{s_j}(x)) - a(x)] \, dx,$$

so

$$\mathcal{I}(u) \leq \liminf_{j \rightarrow \infty} \mathcal{I}_{s_j}(u_{s_j}),$$

as desired. \square

We finally present the main result of this paper, which shows the Γ -convergence of polyconvex functionals defined on Bessel spaces, involving s -fractional gradients, to a classical local polyconvex functional defined on a Sobolev space. Unfortunately, we crucially need the extra assumption $p > n$ in order to prove the *limsup inequality*. This is because the coercivity conditions of W in Proposition 6.1 are compatible with the standard upper bound by $|F|^p$ (which makes the functional \mathcal{I} continuous in $W^{1,p}$; see [11]) only in the case $p > n$.

Theorem 6.2. *Let $p > n$ and $0 < s < 1$. Let Ω be a bounded open subset of \mathbb{R}^n and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

- a) *W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n \times n}$ -measurable.*
- b) *$W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.*
- c) *For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is polyconvex.*
- d) *Assume there exist $c > 0$ and $a \in L^1(\mathbb{R}^n)$ such that*

$$W(x, y, F) \geq a(x) + c|F|^p, \quad \text{a.e. } x \in \mathbb{R}^n, \text{ all } y \in \mathbb{R}^n, \text{ all } F \in \mathbb{R}^{n \times n},$$

and for every $R > 0$ there exist $a_R \in L^1(\mathbb{R}^n)$ and $c_R > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ with $|y| \leq R$ and all $F \in \mathbb{R}^{n \times n}$,

$$W(x, y, F) \leq a_R(x) + c_R|F|^p.$$

The following statements hold:

- i) *For each $s \in (0, 1)$, let $u_s \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ satisfy*

$$\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) < \infty. \quad (54)$$

Then there exist $u \in W_g^{1,p}(\Omega, \mathbb{R}^n)$ and an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ in $(0, 1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ such that $u_{s_j} \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \rightarrow \infty$.

- ii) *For each $s \in (0, 1)$, let $u_s \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ and $u \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ satisfy $u_s \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\mathcal{I}(u) \leq \liminf_{s \nearrow 1} \mathcal{I}_s(u_s).$$

- iii) *For each $u \in W_g^{1,p}(\Omega, \mathbb{R}^n)$ and $s \in (0, 1)$, there exists $u_s \in H_g^{s,p}(\Omega, \mathbb{R}^n)$ such that $u_s \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ and*

$$\limsup_{s \nearrow 1} \mathcal{I}_s(u_s) \leq \mathcal{I}(u). \quad (55)$$

Proof. For proving i), just notice that by assumption d), (54) implies that there is an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ in $(0, 1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ such that $\{D^{s_j} u_{s_j}\}_{j \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. Therefore, by Theorem 4.2, there exists $u \in W_g^{1,p}(\Omega, \mathbb{R}^n)$ such that, for a subsequence $u_{s_j} \rightarrow u$ in $L^p(\mathbb{R}^n, \mathbb{R}^n)$ as $j \rightarrow \infty$.

Part ii) is a particular case of Proposition 6.1.

Finally we show iii), so we let $u \in W_g^{1,p}(\Omega, \mathbb{R}^n)$. By Theorem 3.2, $D^s u \rightarrow Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$. Assumption c) implies in particular the continuity of $W(x, y, \cdot)$ for a.e. $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^n$ (see, e.g., [11]). The Sobolev embedding shows that u is bounded. By the growth conditions and dominated convergence,

$$\lim_{s \nearrow 1} \int W(x, u, D^s u) = \int W(x, u, Du), \quad (56)$$

which proves (55). □

Although the bulk of this article has been focused on the assumption of polyconvexity, with the stronger assumption of convexity we can achieve the analogue result of Theorem 6.2 for the full range of exponents $p \in (1, \infty)$. Since the proof is analogous (and in some steps, simpler) than that of Theorem 6.2, it will only be sketched.

Theorem 6.3. *Let $1 < p < \infty$ and $0 < s < 1$. Let Ω be a bounded open subset of \mathbb{R}^n and $g \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Let $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \cup \{\infty\}$ satisfy the following conditions:*

a) *W is $\mathcal{L}^n \times \mathcal{B}^n \times \mathcal{B}^{n \times n}$ -measurable.*

b) *$W(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \mathbb{R}^n$.*

c) *For a.e. $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^n$, the function $W(x, y, \cdot)$ is convex.*

d1) *If $p < n$ assume there exist $c \geq 1$ and $a \in L^1(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n \times n}$,*

$$-a(x) + \frac{1}{c} |F|^p \leq W(x, y, F) \leq a(x) + c \left(|y|^p + |y|^{p^*} + |F|^p \right).$$

d2) *If $p = n$ assume there exist $r \in [p, \infty)$, $c \geq 1$ and $a \in L^1(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ and all $F \in \mathbb{R}^{n \times n}$,*

$$-a(x) + \frac{1}{c} |F|^p \leq W(x, y, F) \leq a(x) + c \left(|y|^p + |y|^r + |F|^p \right).$$

d3) *If $p > n$ assume there exist $c > 0$ and $a \in L^1(\mathbb{R}^n)$ such that*

$$W(x, y, F) \geq a(x) + c |F|^p, \quad \text{a.e. } x \in \mathbb{R}^n, \text{ all } y \in \mathbb{R}^n, \text{ all } F \in \mathbb{R}^{n \times n},$$

and for every $R > 0$ there exist $a_R \in L^1(\mathbb{R}^n)$ and $c_R > 0$ such that for a.e. $x \in \mathbb{R}^n$, all $y \in \mathbb{R}^n$ with $|y| \leq R$ and all $F \in \mathbb{R}^{n \times n}$,

$$W(x, y, F) \leq a_R(x) + c_R |F|^p.$$

Then, statements i)–iii) of Theorem 6.2 hold.

Proof. The proof of i) is the same as that of Theorem 6.2.

For the proof of ii) we initially follow that of Proposition 6.1. We can assume inequality (48), so there exists an increasing sequence $\{s_j\}_{j \in \mathbb{N}}$ in $(0, 1)$ with $\lim_{j \rightarrow \infty} s_j = 1$ such that $\liminf_{s \nearrow 1} \mathcal{I}_s(u_s) = \lim_{j \rightarrow \infty} \mathcal{I}_{s_j}(u_{s_j})$ and the sequence $\{D^{s_j} u_{s_j}\}_{j \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n, \mathbb{R}^n)$. By Theorem 4.2, for a subsequence, convergences (49) hold. By a standard lower semicontinuity result for convex functionals (see, e.g., [14, Th. 7.5]), for any $R > 0$, inequality (53) holds, and we conclude (47) as in Proposition 6.1.

To show iii) we apply Theorem 3.2 and obtain $D^s u \rightarrow Du$ in $L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})$ as $s \nearrow 1$. Assumption c) implies in particular the continuity of $W(x, y, \cdot)$ for a.e. $x \in \mathbb{R}^n$ and all $y \in \mathbb{R}^n$. The Sobolev embedding in the three cases ($p < n$, $p = n$ and $p > n$) shows that the growth conditions allow us to apply dominated convergence and conclude inequality (56), as desired. \square

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