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Primes represented by quadratic polynomials via exceptional characters

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Abstract. We estimate the number of primes represented by a general quadratic polynomial with discriminant Δ , assuming that the corresponding real character is exceptional.

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1. Introduction

Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with integer coefficients such that $(a, b, c) = 1$, $a+b$ or c odd and discriminant $\Delta \neq \square$. Conjecture F in the classic work [4] claims that there are infinitely many prime numbers of the form $f(n)$ when $a > 0$. Note that it is elementary that the imposed conditions on f are necessary to represent infinitely many primes. If $a < 0$, under the same conditions, we still expect to capture many primes if $\{n \in \mathbb{Z} : f(n) > 0\}$ is large and Conjecture H in [4] is an instance of it.

In [3] and in [2] special cases of these conjectures are addressed assuming the existence of exceptional characters. For instance, in the second paper it is proved that positive exceptional fundamental discriminants D can be written as $D = m^2 + p$ and that if D is “exceptional enough” we have an asymptotic formula for the number of primes of the form $D - m^2$. In our setting it corresponds to $\Delta = 4D > 0$ and $a = -1 < 0$. In [3] there are considered two families of polynomials with $\Delta < 0$ and $a > 0$. To interpret correctly these claims, it is important to keep in mind that the exceptional nature of a discriminant depends on our scale and in some sense an exceptional

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discriminant, zero or character is like a sequence. The existence of a real zero in $[1 - c/\log q, 1]$ only ruins the generic de la Vallée Poussin zero free region if c can be taken arbitrarily small when q grows. In [3] and [2] a bona fide asymptotic formula is only achieved if Δ is allowed to grow.

Our goal in this paper is to adapt the techniques of [2] to get a result valid for every f as above when Δ is exceptional. By the reasons explained before we prefer to present the result as a main term plus an error term instead of as an asymptotic formula resembling the original statement of the conjectures.

For each $N \geq 1$ we denote

$$\pi_f(N) = \#\{0 \leq f(n) \leq N : f(n) \text{ prime}, n \in \mathbb{Z}\}$$

with f as before and for each integer d , $\rho(d) = \#\{n : f(n) \equiv 0 \pmod{d}\}$. Let $\chi_\Delta(n)$ be the Kronecker character modulo Δ , $\chi_\Delta(n) = (\frac{\Delta}{n})$. We will also consider the L function associated to the character

$$L(s, \chi) = \sum_n \frac{\chi_\Delta(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi_\Delta(p)}{p^s}\right)^{-1}. \quad (1.1)$$

We will denote $\mathcal{A} = \{f(n) \in [0, N] : n \in \mathbb{Z}\}$ and $\mathcal{A}_d = \{k \in \mathcal{A} : d \mid k\}$ for $N, d \in \mathbb{Z}^+$ and A and A_d stand for their cardinality. Also, we denote $V(x) = \prod_{p < x} (1 - p^{-1}\rho(p))$. The exceptionality of the character χ_Δ will be measured by

$$\beta = -\log(L(1, \chi_\Delta) \log |\Delta|).$$

In the development of the proof it is convenient to introduce also

$$L = -\log(L(1, \chi_\Delta) \log A) \quad \text{and} \quad B = \frac{3 \log |\Delta|}{\log A}. \quad (1.2)$$

To state our main result we introduce the function

$$g(\Delta) = \begin{cases} \Delta e^{-\beta/2} & \text{if } \Delta > 0 \\ |\Delta| (4|a| - e^{-\beta/2})^{-1} & \text{if } \Delta < 0. \end{cases}$$

Theorem 1.1. *Let $1 \leq |a| \leq e^{\beta/5}$ and $g(\Delta) \leq N \leq |a||\Delta|^{\beta/2}$. Then*

$$\pi_f(N) = AV(A)(1 + O(e^{-\sqrt{\beta}/6}))$$

with an absolute O -constant.

Remark 1.2. An asymptotic formula is obtained only under $\beta \rightarrow +\infty$ or equivalently, under the usual definition of exceptionality $L(1, \chi_\Delta) \log |\Delta| \rightarrow 0$.

2. Guidelines.

Along the proof we follow the sieve techniques of [2]. As usual for $z \geq 2$ we write $P(z) = \prod_{p < z} p$ and $S(\mathcal{A}, z) = \#\{a \in \mathcal{A} : (a, P(z)) = 1\}$. We start with the trivial identity

$$\pi_f(N) = S(\mathcal{A}, \sqrt{N}) + O(1)$$

and use the well known Buchstab identity

$$S(\mathcal{A}, z) - S(\mathcal{A}, \sqrt{N}) = \sum_{z \leq p \leq \sqrt{N}} S(\mathcal{A}_p, p). \quad (2.1)$$

The main term of the theorem will come from $S(\mathcal{A}, z)$, while the sum on the right will be part of the error term. Then, the whole idea of the proof of Theorem 1.1 is to bound the right hand side of (2.1) using the exceptionality of the character χ_Δ . This comes by noting that for d squarefree $\rho(d) \leq \lambda(d)$ where $\lambda(n)$ is given by the convolution $\lambda = 1 * \chi_\Delta$, and in particular $S(\mathcal{A}_p, p) = 0$ if $\chi_\Delta(p) = -1$. If the character is exceptional, this will happen often, giving many zero terms in the right hand side of (2.1).

In order to estimate the sum, clearly we will need to have some control over A_p . Observe that

$$A_d = \frac{\rho(d)}{d} A + r_d, \quad \text{with } |r_d| \leq \rho(d).$$

Thanks to $\rho(d) \leq \lambda(d)$ finding good bounds for the sum in the right hand side of (2.1) will come from finding good estimates for the sum defined as

$$\delta(x) = \sum_{x \leq p \leq A} \frac{\lambda(p)}{p}. \quad (2.2)$$

3. Proof of Theorem 1.1.

Along the proof we will assume β large enough, since otherwise Theorem 1.1 is the classical upper bound from linear sieve theory. The size of A grows with Δ (see Lemma 3.3 below) then we can assume that A is bigger than a large constant. We will frequently use the inequality $\beta \leq \varepsilon \log |\Delta|$, which follows from Siegel's theorem.

It is important to remark that we will apply this inequality with specific values of ε , hence the ineffectiveness of the constant in Siegel's theorem does not affect the absolute nature of the O -constant in Theorem 1.1. More precisely, we could use an explicit fixed ε in all our applications of Siegel's theorem. In particular, this is the case of the previous inequality and β is larger than a concrete constant (which affects to the value of the O -constant). Of course, in our theoretical setting it would be cumbersome to provide a reasonable numerical value for the O -constant.

Let us start by finding the asymptotics of $S(\mathcal{A}, z)$. For that we use the fundamental lemma of sieve theory (see e.g. [1, Cor. 6.10]), with level of distribution $A^{2/3}$, $z^s = A^{2/3}$ and any $1 < s < \frac{2 \log A}{9 \log \log A}$ to get

$$S(\mathcal{A}, z) = AV(z)(1 + O(e^{-s})) + O(A^{2/3} \log A). \quad (3.1)$$

Observe that

$$V(z) = V(A)e^{O(\delta(z))} = V(A)(1 + O(\delta(z))),$$

provided that $\delta(z) \ll 1$, and hence we can replace $V(z)$ by $V(A)$ with an error term bounded by $\delta(z)$, which will be absorbed by the error term in (3.1).

The rest of the paper will be dedicated to bound the right hand side of (2.1), which we will split into three different sums, depending on the range of summation for the primes.

$$\begin{aligned} \sum_{z \leq p \leq \sqrt{N}} S(\mathcal{A}_p, p) &= \sum_{z \leq p \leq A/z^2} + \sum_{A/z^2 < p \leq A} + \sum_{A < p \leq \sqrt{N}} \\ &= S_1 + S_2 + S_3. \end{aligned} \quad (3.2)$$

We start with the sum S_1 . The trivial bound $S(\mathcal{A}_p, p) \leq A_p$ is not good enough for the small primes in the sum S_1 , and we need a better bound gotten, as in [2], using an upper bound sieve of dimension 2 and level of distribution A/pz (see e.g. [1, Cor. 6.10]). This gives us

$$S(\mathcal{A}_p, p) \ll \frac{\rho(p)}{p} AV(z) + \sum_{d < A/pz} \rho(d),$$

where the sum runs over squarefree integers d . The last term is trivially bounded by

$$\sum_{d < A/pz} \rho(d) \leq \sum_{d < A/pz} (1 * \chi_\Delta)(d) \ll \sum_{d < A/pz} \tau(d) \ll \frac{A}{pz} \log A.$$

Noting that $z \geq (\log A)^3$, which follows by our assumption in s , and that $V(z) \geq V(A) \gg (\log A)^{-2}$, since $\rho(p) \leq 2$ for any prime p , we end up with $S(\mathcal{A}_p, p) \ll p^{-1} \lambda(p) AV(z)$ and hence

$$S_1 \ll AV(z) \delta(z). \quad (3.3)$$

To bound $\delta(z)$ we use Lemma 3.4 of [2], which we include for reader's convenience.

Lemma 3.1. *Let $2 \leq u \leq y \leq x$. Then,*

$$\sum_{\substack{y \leq n \leq x \\ (n, P(u))=1}} \frac{\lambda(n)}{n} \ll W(u) L(1, \chi_\Delta) \log \left(\frac{x}{y} \right) + |\Delta|^{1/8+\varepsilon} y^{-1/3} u^{1/3} \log u,$$

where $W(u) = \prod_{p < u} (1 - p^{-1}) (1 - p^{-1} \chi_\Delta(p))$.

Remark 3.2. Observe that, since $W(u) \leq C \prod_{p < u} (1 - p^{-1} \lambda(p)) \leq CV(u)$, for some absolute constant C , we can write either $W(u)$ or $V(u)$ indistinctly.

Further we will use the formula, also proved in [2, p.1106],

$$\delta(z)^k \ll k k! W(z) L(1, \chi_\Delta) \log A + k! |\Delta|^{1/8+\varepsilon} z^{(1-k)/3} \quad (3.4)$$

valid for any integer $k \geq 1$. It is worth to note that in order to establish the previous formula it is needed a bound of the type $\log z \ll \Delta^\epsilon$, which in our case follows assuming $L > 0$ in (1.2). Indeed

$$\log z < \log A < \frac{1}{L(1, \chi_\Delta)} \ll \Delta^\epsilon.$$

Expanding a previous comment, we are not going to let $\epsilon \rightarrow 0^+$ but choosing a specific value of ϵ , and it saves the effectiveness of the \ll constant. Dropping the contribution of $W(z)$ in (3.4) we get the more convenient form

$$\delta(z) \ll k \left(kL(1, \chi_\Delta) \log A + |\Delta|^{1/8+\epsilon} z^{(1-k)/3} \right)^{1/k}.$$

Our goal is to prove

$$\delta(z) \ll e^{-s} \tag{3.5}$$

with a proper selection of s .

Taking any positive integer $k \geq (\frac{3}{16} + 3\epsilon)Bs + 1$, with B as in (1.2), and noting that $z = |\Delta|^{2/Bs}$ we obtain

$$|\Delta|^{1/8+\epsilon} z^{(1-k)/3} \leq |\Delta|^{-\epsilon}.$$

On the other hand

$$kL(1, \chi_\Delta) \log A \geq |\Delta|^{-\epsilon}$$

follows by Siegel's theorem (recall that A is bigger than a large constant), and then

$$\delta(z) \ll k(L(1, \chi_\Delta) \log A)^{1/k}.$$

Observe that, assuming again $L > 0$, we have that the previous bound is increasing in k , and so we can relax the condition of k being an integer. In particular, we can take $k = Bs$, which is possible assuming Bs greater than a constant greater than $16/13$ (we get later $Bs > 16/13 + 1/4$). Then, to prove (3.5) we need to select some

$$s \leq -\log k + \frac{L}{k},$$

which gives, replacing the value of k ,

$$Bs^2 + Bs \log(Bs) \leq L. \tag{3.6}$$

The error term in Theorem 1.1 is in terms of β instead of L . The comparison between both quantities comes from a proper control in A . We have the following lemma.

Lemma 3.3. *Let $A \geq 1$ and assume the hypotheses in Theorem 1.1. Given $\epsilon > 0$, we have*

$$|\Delta|^{1/2-\epsilon} \leq A < \frac{4\sqrt{N}}{\sqrt{|a|}},$$

for $|\Delta|$ large enough (depending on ϵ).

Remark 3.4. For the application of this in the proof of the main result we are going to choose $1/2 - \varepsilon = 7/16$. Again, this saves the effectiveness of the lower bound for $|\Delta|$ which guarantees the inequality.

Proof. The inequalities $0 \leq f(x) \leq N$ define one or two intervals for x , depending on the real zeros of f and the sign of a and Δ , and it is straightforward to measure the length of those intervals to be

$$X = \begin{cases} \frac{\sqrt{\Delta+4aN}}{a} & \text{if } \Delta < 0, a > 0, N > \frac{|\Delta|}{4|a|}, \\ \frac{4N}{\sqrt{\Delta+4aN}+\sqrt{\Delta}} & \text{if } \Delta > 0, a > 0 \text{ or if } \Delta > 0, a < 0, N \leq \frac{|\Delta|}{4|a|}, \\ \frac{\sqrt{\Delta}}{|a|} & \text{if } \Delta > 0, a < 0, N > \frac{|\Delta|}{4|a|}. \end{cases}$$

The cases not listed above give empty intervals. From here the upper bound $X \leq 2\sqrt{N/|a|}$ is trivial. Then, noting

$$X - 2 < A < X + 2, \quad (3.7)$$

we deduce $A \leq 4\sqrt{N/|a|}$ for $|a| < N$, whenever $X \geq 2$, which follows from our assumptions $a \leq e^{\beta/5}$, $N \geq g(\Delta)$ by Siegel's theorem because β and Δ can be assumed sufficiently large.

We now prove $X > |\Delta|^{1/2-\varepsilon}$. In the last case in the definition of X , the result follows again from $|a| \leq e^{\beta/5}$. If $\Delta < 0$, then $X > a^{-1}e^{-\beta/4}\sqrt{N}$, since $N \geq g(\Delta)$. Further if $\Delta > 0$ and $N \leq \frac{\Delta}{4|a|}$, we have $X \gg N/\sqrt{\Delta}$, finally if $\Delta > 0$, $a > 0$ and $N > \frac{\Delta}{4|a|}$ we have the stronger bound $X \gg \sqrt{N/|a|}$. In any case $X > \Delta^{1/2-\varepsilon}$ is a consequence of $N/\Delta \gg |\Delta|^{-\varepsilon}$ and $a \ll |\Delta|^\varepsilon$. \square

Now, $A \leq \frac{4\sqrt{N}}{\sqrt{|a|}}$ and the upper bound for N give

$$\beta = L + \log \left(\frac{\log A}{\log |\Delta|} \right) < 2L.$$

We select $s = \frac{1}{2}\sqrt{\beta/B}$. By Lemma 3.3 B is bounded, namely with the choice of ε as in the Remark we have $B < 7$. Then s is arbitrarily large, in particular $s > 1$. It is important to check that this selection of s is compatible with the rest of our previous assumptions:

$$\left(\frac{16}{13} + \frac{1}{4} \right) \frac{1}{B} < s < \frac{2 \log A}{9 \log \log A}.$$

The first inequality is consequence of $A < 4\sqrt{N/|a|}$ and the upper bound in N . The second is equivalent to

$$1 < \frac{4\sqrt{B} \log A}{9\sqrt{\beta} \log \log A},$$

which follows for A large enough, by the definition of B and Siegel's Theorem.

Let us prove with this selection of s that

$$Bs^2 + Bs \log(Bs) \leq \frac{\beta}{2} < L. \quad (3.8)$$

As $B < 7$ and s is arbitrarily large, we can suppose $Bs^2 \geq Bs \log(Bs)$ and then (3.8) follows directly from our choice of s . This proves (3.8), and (3.5) with $s = \frac{1}{2}\sqrt{\beta/B}$ (and assures $L > 0$ as assumed), which gives by (3.2)

$$S_1 \ll AV(z)e^{-\sqrt{\beta/4B}} \ll AV(A)e^{-\sqrt{\beta}/6} \quad (3.9)$$

since $B < 7$.

It remains to bound $S(\mathcal{A}_p, p)$ for medium and large p . If $A/z^2 < p < A$ then

$$S(\mathcal{A}_p, p) \leq A_p \leq \frac{\rho(p)}{p}A + \lambda(p) \ll \frac{\lambda(p)}{p}A,$$

so

$$S_2 \ll A\delta(A/z^2). \quad (3.10)$$

We apply Lemma 3.1 $x = A$ and $y = A/z^2$ and $u = Az^{-2}|\Delta|^{-3/8-4\epsilon}$. Observe that $u = A^\gamma$ with $\gamma = 1 - \frac{4}{3s} - \frac{B}{8} - \frac{4}{3}B\epsilon$ and $\gamma > 0$ for ϵ sufficiently small and s sufficiently large, since $B < 7$. With this selection the first term in the sum in Lemma 3.1 dominates the second and we deduce

$$\delta(A/z^2) \ll V(u)L(1, \chi_\Delta) \log(z^2) \ll V(A)L(1, \chi_\Delta) \log(z^2).$$

For the last inequality we have used that u is a positive power of A . Now,

$$\log(z^2)L(1, \chi_\Delta) = \frac{4}{3s}e^{-L} < e^{-\beta/2} \leq e^{-\sqrt{\beta}/6},$$

since $2L > \beta$, and putting everything together we get the desired result

$$S_2 \ll AV(A)e^{-\sqrt{\beta}/6}. \quad (3.11)$$

Finally, it remains to bound S_3 corresponding to the primes $A < p < \sqrt{N}$. We have

$$S(\mathcal{A}_p, p) \leq A_p \leq \frac{\rho(p)}{p}A + \lambda(p) \ll \frac{\lambda(p)}{p}\sqrt{N},$$

and again using Lemma 3.1 with the same parameter u as before, $x = \sqrt{N}$ and $y = A$, we get

$$\sum_{A \leq p \leq \sqrt{N}} S(\mathcal{A}_p, p) \ll \sqrt{N}V(A)L(1, \chi_\Delta) \log \sqrt{N}$$

We separate the different cases giving $f(\mathbb{Z}) \cap \mathbb{Z}^+ \neq \emptyset$ (see the definition of X in the proof of Lemma 3.3).

If $\Delta < 0$, $a > 0$, by hypothesis $N \geq g(\Delta)$, which gives, $\sqrt{N} \ll ae^{\beta/4}A$ by (3.7) and, hence, since $f(x) = x \log x$ is increasing for $x > 2$, we get

$$\sqrt{N} \log \sqrt{N} \ll ae^{\beta/4}A \log(ae^{\beta/4}A) \ll ae^{\beta/4}A \log A$$

by our assumption in a and Lemma 3.3. Also $a < e^{\beta/5}$ implies

$$a \ll e^{L-\beta/4-\sqrt{\beta}/6},$$

since $\beta < 2L$, which gives

$$S_3 \ll AV(A)e^{-\sqrt{\beta}/6}, \quad (3.12)$$

in this case as desired.

If $\Delta > 0$, $a < 0$, we just need to consider the case $\Delta e^{-\beta/2} \leq N \leq \frac{\Delta}{4|a|}$, since $f(n) \leq \frac{\Delta}{4|a|}$. Then $A \gg N/\sqrt{\Delta} > e^{-\beta/4}\sqrt{N}$ and the proof of (3.12) follows in the same way as before.

Finally, if $\Delta > 0$, $a > 0$ we have $A \gg \sqrt{N/a} \gg e^{-\beta/10}\sqrt{N}$ and the same proof applies getting again (3.12).

Substituting (3.9), (3.11) and (3.12) in (3.2) and recalling (2.1) and (3.1), the proof of Theorem 1.1 is complete.

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