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# Construction of copulas with hairpin support

Fernando Chamizo, Juan Fernández-Sánchez and Manuel Úbeda-Flores

**Abstract.** Consider a nondecreasing homeomorphisms  $f$  defined on  $[0, 1]$  such that  $f(x) < x$  for all  $x \in ]0, 1[$ . In this paper we provide necessary and sufficient conditions for such  $f$  to be part of a  $\mathcal{C}$ -hairpin that concentrates the mass of a bivariate copula. In addition, we study when copulas of this kind come from modular functions. Finally, under certain conditions, we give a multidimensional method that generalizes the bivariate case and allows to construct extreme points in the set of multidimensional copulas.

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**Keywords.** 1-Lipschitz, copula, hairpin.

## 1. Introduction

Copulas have proven to be remarkably useful in statistical modeling and in the study of dependence and association of random variables. In the investigations about extreme points in the class of copulas, which we denote by  $\mathcal{C}$ , the concept of copulas with hairpin support has been introduced (see [17, 18]): A couple of increasing homeomorphisms  $f, g: [0, 1] \rightarrow [0, 1]$  such that  $f(x) < x < g(x)$  for all  $x \in ]0, 1[$ —denoted  $(f, g)$  in the sequel—and whose graphs,  $G_f$  and  $G_g$ , can concentrate the mass of a copula is called a  $\mathcal{C}$ -*hairpin*. From the point of view of measure theory each copula corresponds to a unique doubly stochastic measure on the Borel  $\sigma$ -field  $\mathcal{B}([0, 1]^2)$ , (and copulas with hairpin support are noticeable because their associated measures are extremal doubly stochastic measures.

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On the other hand, relations between copulas and measure-preserving transformations and self-affine functions whose graphs are the support of a copula were established in [4], and the construction of copulas from modular functions in [5], where the copulas corresponding to  $\mathcal{C}$ -hairpins appear as special cases and give rise to a problem that generalizes the original approach in [17, 18].

In [2], the author demonstrates the difficulty of finding examples of couples  $(f, g)$  to be a  $\mathcal{C}$ -hairpin. The author herself proposes a method, but points out that the method is not applicable to all functions and gives the function  $f(x) = 1 - \sqrt{1 - x^2}$  as an example. Further, for  $g(x) = \sqrt{x}$  she cannot do it directly and looks for an alternative through the use of  $f(x) = x^2$ .

In this paper, after some preliminaries concerning (multivariate) copulas and measure theory, we find necessary and sufficient conditions for which, given a homeomorphism  $f: [0, 1] \rightarrow [0, 1]$  such that  $f(x) < x$  for all  $x \in ]0, 1[$ , there exists a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin and we solve the open problem stated in [5, p.125] (we quote it here with minor changes in the notation):

*Open Problem 1.* Given two nondecreasing surjective functions  $f, g: [0, 1] \rightarrow [0, 1]$  satisfying  $f(x) \leq x \leq g(x)$  for all  $x \in [0, 1]$ , when does there exist a nondecreasing, 1-Lipschitz continuous and modular function  $H: [0, 1]^2 \rightarrow \mathbb{R}$  satisfying  $H(0, 0) \geq 0$  and  $H(1, 1) \geq 1$ , such that the region where the corresponding copula  $H^*$  does not coincide with  $M$  (given by  $M(x, y) = \min(x, y)$  for all  $(x, y) \in [0, 1]^2$ ) is exactly the region between the graphs of  $f$  and  $g$ ? For example, does there exist a modular function  $H$  such that the copula  $C = \min(M, H)$  does not coincide with  $M$  on  $D = \{(x, y) : x \in [0, 1] \text{ and } x^2 < y < \sqrt{x}\}$  and coincides with  $M$  on  $[0, 1]^2 \setminus D$ ?

Finally, we obtain a higher dimensional generalization of copulas concentrating their mass on two curves. All of these copulas are extremal points of the convex set of all  $n$ -copulas.

## 2. Preliminaries

Let  $n \geq 2$  be a natural number. An  $n$ -dimensional copula ( $n$ -copula, for short) is a function  $C: [0, 1]^n \rightarrow [0, 1]$  satisfying:

- (C1) For every  $\mathbf{u} = (u_1, \dots, u_n)$  in  $[0, 1]^n$ ,  $C(\mathbf{u}) = 0$  if at least one coordinate of  $\mathbf{u}$  is 0, and  $C(\mathbf{u}) = u_k$  if all coordinates of  $\mathbf{u}$  are 1 except possibly  $u_k$ ;
- (C2)  $C$  is  $n$ -increasing, i.e. for all  $\mathbf{a}$  and  $\mathbf{b}$  in  $[0, 1]^n$  such that  $a_k \leq b_k$  for all  $k = 1, \dots, n$ ,  $V_C([\mathbf{a}, \mathbf{b}]) = \sum \text{sgn}(\mathbf{c})C(\mathbf{c}) \geq 0$ , for the  $n$ -box  $[\mathbf{a}, \mathbf{b}] = \times_{i=1}^n [a_i, b_i]$ , the sum is over the vertices  $\mathbf{c}$  of  $[\mathbf{a}, \mathbf{b}]$  i.e., each  $c_k$  is equal to either  $a_k$  or  $b_k$ , and  $\text{sgn}(\mathbf{c}) = 1$  if  $c_k = a_k$  for an even number of values of  $k$ , and  $-1$  if  $c_k = a_k$  for an odd number of values of  $k$ .

The function  $V_C$  in (C2) is called the  $C$ -volume of the  $n$ -box  $[\mathbf{a}, \mathbf{b}]$ . Note that, in particular, every  $n$ -copula  $C$  satisfies the 1-Lipschitz condition  $|C(\mathbf{u}) - C(\mathbf{v})| \leq \sum_{i=1}^n |u_i - v_i|$  for all  $\mathbf{u}, \mathbf{v}$  in  $[0, 1]^n$ .

The importance of copulas in statistics stems in part from Sklar's theorem [19]: If  $H$  is a multivariate distribution function with univariate margins  $F_i$ ,  $i = 1, 2, \dots, n$ , then there exists an  $n$ -copula  $C$  such that  $H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n))$  for all  $\mathbf{x}$  in  $[-\infty, \infty]^n$ ; moreover, if  $F_i$  are continuous for all  $i = 1, 2, \dots, n$ , then the copula is unique; otherwise, the copula is uniquely determined on  $\times_{i=1}^n \text{Range } F_i$  [3]. Thus  $n$ -copulas allow to recover in some way a multivariate distribution from its univariate margins. Relationships between copulas and measure-preserving transformations on the Borel sets of the unit interval are studied in [11]. For a complete survey on copulas, see [6, 16].

For any  $n$ -copula  $C$  we have the following inequality, known as the *Fréchet-Hoeffding bounds inequality*:

$$W^n(\mathbf{u}) := \max_{0 \leq \sum_{i=1}^n (\mu_i - n + 1)} \left( \leq C(\mathbf{u}) \leq \min(u_1, \dots, u_n) =: M^n(\mathbf{u}), \right.$$

where the superscripts denote dimension. The function  $M^n$  (in the bivariate case we will write  $M$ , for short) is an  $n$ -copula for all  $n$ , however  $W^n$  is not an  $n$ -copula for  $n \geq 3$  (note that  $V_{W^n}([1/2, 1]) = 1 - n/2 < 0$ ).

We recall some notions of measure theory. Any  $n$ -copula  $C$  induces on the class of Borel sets of  $[0, 1]^n$ , denoted by  $\mathcal{B}([0, 1]^n)$ , an  $n$ -stochastic measure (called *doubly stochastic measure* for the bivariate case)  $\mu_C$  satisfying  $\mu_C([\mathbf{a}, \mathbf{b}]) := V_C([\mathbf{a}, \mathbf{b}])$ . The *support* of an  $n$ -copula  $C$  is the complement of the union of all open subsets of  $[0, 1]^n$  with  $\mu_C$ -measure zero, and when we refer to “mass” on a set, we mean the value of  $\mu_C$  for that set. A set  $B$  concentrates the mass of  $C$  if  $\mu_C(B) = 1$ . The univariate *margins* of a measure  $\mu$  are defined by  $\mu_i(A) = \mu([0, 1]^{i-1} \times A \times [0, 1]^{n-i})$  for every  $i = 1, 2, \dots, n$  and for  $A \in \mathcal{B}([0, 1])$ .

Finally, we will need some additional notation. Given  $m: [0, 1] \rightarrow [0, 1]$ ,  $m_*$  denotes the function defined by  $m_*(x) = x - m(x)$  for all  $x \in [0, 1]$ .

### 3. Creation of $\mathcal{C}$ -hairpins

Let  $f, g: [0, 1] \rightarrow [0, 1]$  be two increasing homeomorphisms such that  $f(x) < x < g(x)$  for all  $x \in ]0, 1[$ , and let  $C$  be a 2-copula (copula, for short) with mass concentrated on  $G_f$  and  $G_g$ , i.e.  $(f, g)$  is a  $\mathcal{C}$ -hairpin. For  $x \in [0, 1]$ , we define the functions

$$A_C(x) = \mu_C(G_f \cap ([0, x] \times [0, 1])) \quad (3.1)$$

$$(A_C)_*(x) = x - A_C(x) = \mu_C(G_g \cap ([0, x] \times [0, 1])). \quad (3.2)$$

Observe that the function  $A_C$  (respectively,  $(A_C)_*$ ) is the first marginal distribution of the part of  $\mu_C$  that is distributed over  $G_f$  (respectively,  $G_g$ ).

Some noticeable properties concerning the function  $A_C$  are provided in the following result, in which 1-*Lipschitz function* means nonexpanding map i.e., its Lipschitz constant is less or equal than 1. The proof is straightforward and we omit it.

**Lemma 3.1.** *Let  $(f, g)$  be a  $\mathcal{C}$ -hairpin and let  $C$  be a copula with mass concentrated on  $G_f \cup G_g$ . Let  $A_C$  be the function given in (3.1). Then*

- (i)  $A_C(0) = 0$ ,
- (ii)  $A_C$  is monotone increasing, and
- (iii) the functions  $A_C$  and  $A_C \circ f^{-1}$  are 1-Lipschitz functions.

Observe that the function  $A_C \circ f^{-1}$  given in Lemma 3.1 is the second marginal of  $\mu_C$  that is distributed over  $G_f$ .

We firstly find necessary and sufficient conditions for a function  $f$  to be part of a  $\mathcal{C}$ -hairpin that concentrates the mass of a bivariate copula. With this purpose we introduce some additional notation.

Fibers of every monotone and continuous function  $A$  are either singletons or proper intervals. The union of those fibers that are proper intervals will be denoted  $C(A)$  and the set of values attained on these intervals, that is  $A(C(A))$ , will be denoted  $S(A)$ . In formulas, we have

$$C(A) = \bigcup \left\{ I : A|_I \text{ is constant} \quad \text{and} \quad S(A) = A(C(A)) \right\}$$

where  $I$  denotes a proper interval. The set  $A^{-1}(\{z\})$  is a closed interval for each  $z \in S(A)$  hence  $C(A)$  is a countable union of disjoint closed intervals. By the intermediate value theorem, for distinct  $x_1, x_2 \notin C(A)$  there exists  $x_3 \notin C(A)$  between them. Note that if  $A$  is a nondecreasing continuous function and  $f$  a homomorphism, as in the next result,  $S(A) = S(A \circ f^{-1})$ .

**Theorem 3.2.** *Given a homeomorphism  $f : [0, 1] \rightarrow [0, 1]$  such that  $f(x) < x$  for  $x \in ]0, 1[$ , it is possible to find  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin if and only if there exists a nondecreasing function  $A : [0, 1] \rightarrow [0, 1]$  with  $A(0) = 0$  verifying*

- (i)  $A$  and  $A \circ f^{-1}$  are 1-Lipschitz functions;
- (ii)  $S(A_*) = S((A \circ f^{-1})_*)$ ;
- (iii) For  $z \in S(A_*)$  if  $A_*^{-1}(\{z\}) = [a, b]$  and  $(A \circ f^{-1})_*^{-1}(\{z\}) = [c, d]$ , then  $a < c$  and  $b < d$ .

For the example  $f(x) = x^2$  in [2], the function  $A(x) = \frac{1}{2}x^2$  fulfills (i) while (ii) and (iii) become trivial. This choice gives  $g(x) = 2x - x^2$  in the proof below and corresponds to  $M = 2$  in [2].

*Proof.* We first prove that given  $A$  with the required properties it is possible to find a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin. In  $[0, 1]^2$  we define the measure that is induced by the function  $F(x, y) = M(A(x), A(f^{-1}(y)))$ . We now check the mass distribution of  $F$ . Let  $(x_0, y_0)$  be in  $[0, 1]^2$  satisfying  $A(x_0) > A(f^{-1}(y_0))$ , then there exists a neighborhood  $N$  of  $(x_0, y_0)$  such that  $A(x) > A(f^{-1}(y))$  for every  $(x, y)$  in  $N$ , so that  $F(x, y) = A(f^{-1}(y))$  for all  $(x, y)$  in  $N$ , which implies there exists a rectangle  $R \subseteq N$  such that  $V_F(R) = 0$ . We obtain a similar result assuming  $A(x_0) < A(f^{-1}(y_0))$ . Therefore,  $F$  has its mass concentrated on the set  $R = \{(x, y) \in [0, 1]^2 : A(x) = A(f^{-1}(y))\}$ . If  $A(x_0) = A(f^{-1}(y_0)) \in S(A) = S(A \circ f^{-1})$  then  $(x_0, y_0) \in N$

with  $N$  a closed rectangle in which  $F_N$  is constant, meaning that the mass is concentrated on

$$\{(x, f(x)) \in [0, 1]^2 : x \notin C(A) \subset G_f.$$

Note that  $A_*$  and  $(A \circ f^{-1})_*$  are nondecreasing by (i). Then a similar reasoning with  $G(x, y) = M(A_*(x), (A \circ f^{-1})_*(y))$  proves that the mass of  $G$  is concentrated on

$$\{(x, g(x)) \in [0, 1]^2 : x \notin C(A_*) \quad \text{where } g(x) = ((A \circ f^{-1})_*)^{-1} \circ A_*(x).$$

We now show that the definition of  $g$  can be extended to  $C(A_*)$  to obtain a homeomorphism with  $g(x) > x$  when  $x \in ]0, 1[$ .

If  $[a, b]$  is one of the disjoint closed intervals composing  $C(A_*)$ , then by (ii) we have  $A_*([a, b]) = (A \circ f^{-1})_*([c, d])$  for  $[c, d]$  one of the closed intervals composing  $C((A \circ f^{-1})_*)$ . Define  $g|_{[a, b]}(x) = \frac{d-c}{b-a}(x-a) + c$ . In this way we have a sound definition of  $g : [0, 1] \rightarrow [0, 1]$  with  $g(C(A_*)) = C((A \circ f^{-1})_*)$ . We have  $g(x) > x$  for  $x \in ]0, 1[ - C(A_*)$  because  $A_*(x) > (A \circ f^{-1})_*(x)$  there and by (iii) the same holds for  $x \notin C(A_*)$ . As  $A_*(x)$  and  $(A \circ f^{-1})_*$  are monotonic,  $g$  is strictly increasing.

The only remaining issue is to prove that  $g$  has no jump discontinuities. If one occurs at  $x = x_0$  then for  $x_n^- \uparrow x_0$  and  $x_n^+ \downarrow x_0$  we would have

$$g(x_n^-) < l < L < g(x_n^+). \quad (3.3)$$

Clearly  $x_n^\pm$  cannot lie eventually in the same closed interval of  $C(A_*)$  because  $g$  is an affine function there. Then by the intermediate value property we can assume  $l, L \notin C((A \circ f^{-1})_*)$ . If  $x_0 \notin C(A_*)$  we can take  $x_n^\pm \notin C(A_*)$  and applying  $(A \circ f^{-1})_*$  to (3.3) we have  $A_*(x_n^-) < l' < L' < A_*(x_n^+)$  that contradicts the continuity of  $A_*$ . If  $x_0 \in C(A_*)$ , say  $x_0 \in [a, b]$  with  $[a, b]$  a closed interval of  $C(A_*)$  as before, The only possibilities are  $x_0 = a, b$  (because  $x_n^\pm$  do not lie eventually in  $[a, b]$ ). Both cases are similar. For instance, if  $x_0 = a$  take  $x_n^+ = a$  and  $x_n^- \notin C(A_*)$ . Applying  $(A \circ f^{-1})_*$  to (3.3) as before we have  $A_*(x_n^-) < l' < L' < A_*(a)$  leading to the same contradiction.

Finally, observe that the function  $F + G$  is a copula whose mass is concentrated on  $G_f \cup G_g$ .

Conversely, let  $(f, g)$  be a  $\mathcal{C}$ -hairpin and let  $C$  be a copula whose mass  $(\mu_C)$  is concentrated on  $G_f \cup G_g$ . We define the function  $A : [0, 1] \rightarrow [0, 1]$  given by  $A(x) = \mu_C(G_f \cap [0, x] \times [0, 1])$ . For  $y < x$  we have  $A(x) - A(y) \leq \mu_C([y, x] \times [0, 1]) = x - y$ , and since

$$A(f^{-1}(x)) - A(f^{-1}(y)) \leq \mu_C([0, 1] \times [y, x]) = x - y,$$

we have that  $A$  and  $A \circ f^{-1}$  are 1-Lipschitz functions, whence we obtain condition (i). Condition (ii) follows from the fact that  $A^{-1}(z) = [a_z, b_z]$  is equivalent to  $(A \circ f^{-1})^{-1}(z) = [f(a_z), f(b_z)]$ . Finally, conditions (ii) and (iii) follow from the equality  $A_*(x) = \mu_C(G_g \cap [0, 1] \times [0, x])$  and  $(A \circ f^{-1})_*(x) = A_* \circ g^{-1}(x)$ , and this completes the proof.  $\square$

It is possible to find examples of  $\mathcal{C}$ -hairpins  $(f, g)$  corresponding to a copula with support differing from  $G_f \cup G_g$  (see, e.g. [7, Example 5]). So the following question arises: Given a homeomorphism  $f$  defined on  $[0, 1]$  and such that  $f(x) < x$  for all  $x \in ]0, 1[$ , is it possible to find a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin and the support of the copula is exactly  $G_f \cup G_g$ ? If  $f$  is seen as a distribution function with mass concentrated on  $[0, 1]$ , the answer depends on the absolutely continuous component of  $f$ , which we denote by  $f_{ac}$ . Recall that a real function  $f$  is said to be *singular* if it is monotone nondecreasing with null derivative in a set of full measure. The following result (see, e.g. [10]) will be useful for our purposes.

**Lemma 3.3.** *Given a non-decreasing and singular function  $f$  defined on  $[c, d]$ , there exists  $D \subset [c, d]$  such that its measure is  $d - c$  and the measure of  $f(D)$  is zero.*

*Remark 3.4.* Observe that, as a consequence of Lemma 3.3, if  $f$  is a homeomorphism then we have  $f([c, d] \setminus D) = [f(c), f(d)] \setminus f(D)$ , i.e. the set  $[c, d] \setminus D$  has measure zero and via  $f$  we obtain a set of measure  $f(d) - f(c)$ .

We are now in position to answer our question.

**Theorem 3.5.** *Given an increasing homeomorphism  $f$  defined on  $[0, 1]$ , there exists a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin with support  $G_f \cup G_g$  if and only if  $f_{ac}$  is strictly increasing.*

*Proof.* Suppose  $f_{ac}$  is strictly increasing. We define the functions  $\alpha(x) = \min\{f'(x), 1\}/2$  and  $A(x) = \int_0^x \alpha(t) dt$ . Observe that  $\alpha$  is defined almost everywhere, and that  $A$  is 1-Lipschitz and  $A(0) = 0$ . To check that  $A \circ f^{-1}$  is also 1-Lipschitz, note that, for  $x < y$ , we have

$$\begin{aligned} A(f^{-1}(y)) - A(f^{-1}(x)) &= \int_{f^{-1}(x)}^{f^{-1}(y)} \alpha(t) dt \leq \frac{f_{ac}(f^{-1}(y)) - f_{ac}(f^{-1}(x))}{2} \\ &\leq f(f^{-1}(y)) - f(f^{-1}(x)) = y - x. \end{aligned}$$

Moreover, the functions  $A$  and  $f$  are strictly increasing, and hence conditions (i), (ii) and (iii) of Theorem 3.2 hold, so that there exists a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin. Since the mass is concentrated on  $G_f \cup G_g$ , for any  $\varepsilon > 0$ , consider the open ball with center  $(x, f(x))$  and radius  $\varepsilon$ , i.e.  $B((x, f(x)), \varepsilon)$ . Then  $B((x, f(x)), \varepsilon)$  contains a piece of the graph  $G_f$  included between  $(x, f(x))$  and  $(y, f(y))$ , with  $y > x$ , so that  $\mu_C(B((x, f(x)), \varepsilon)) \geq A(y) - A(x) > 0$  (and similarly  $\mu_C(B((x, g(x)), \varepsilon)) > 0$ ).

Conversely, assume  $f_{ac}$  is constant on the interval  $[c, d]$ . This means that  $f_{ac}$  applies a set  $D$  of measure  $d - c$  (respectively, zero) to another set of measure zero (respectively,  $f(d) - f(c)$ ): recall Lemma 3.3 and Remark 3.4. If a function  $g$  existed such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin, with copula  $C$ , and  $\mu_C(\{(x, f(x)) : x \in D\}) > 0$ , then we obtain a contradiction, because this involves  $0 = \lambda(f(D)) = \mu_C([0, 1] \times f(D)) > 0$ . If  $\mu_C(\{(x, f(x)) : x \in D\}) =$

0, the contradiction comes from the fact that we have  $0 = \lambda([c, d[ \setminus D) = \mu_C([c, d[ \setminus D) \times [0, 1]) > 0$ , and this completes the proof.  $\square$

We now apply Theorems 3.2 and 3.5 to provide an example of a construction of a  $\mathcal{C}$ -hairpin.

*Example 1.* Consider the function  $f(x) = 1 - \sqrt{1 - x^2}$  defined on  $[0, 1]$  (note  $f(x) < x$  for all  $x \in ]0, 1[$ ), and let  $A(x)$  be the function defined by  $A(x) = \frac{x^2}{2}$  for all  $x \in [0, 1]$ . We have  $A(f^{-1}(x)) = \frac{2x-x^2}{2}$  for all  $x \in ]0, 1[$ . Note that  $A_*$  and  $(A \circ f^{-1})_*$  are 1-Lipschitz and strictly increasing functions, so that, by Theorem 3.2, there exists a function  $g$  such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin. In this case, the copula  $C$  corresponds to that given in [16, Example 3.5], and where  $g = f^{-1}$ . Namely, it is the diagonal copula  $C^{\text{FN}}(x, y) = \min(x, y, \frac{1}{2}(x^2 + y^2))$ . For details about  $C^{\text{FN}}$  copulas we refer the reader to [6, 16].

Furthermore, since  $f'(x) > 0$  for all  $x \in ]0, 1[$ , applying the method described in the proof of Theorem 3.5, we have

$$\alpha(x) = \begin{cases} \begin{pmatrix} x \\ 2\sqrt{1-x^2} \end{pmatrix}, & \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2}, \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, & \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1, \end{cases}$$

and

$$A(x) = \begin{cases} \begin{pmatrix} 1 - \sqrt{1-x^2} \\ \frac{x^2}{2} \end{pmatrix}, & \text{if } 0 \leq x \leq \frac{\sqrt{2}}{2}, \\ \begin{pmatrix} x + 1 - \sqrt{2} \\ \frac{x^2}{2} \end{pmatrix}, & \text{if } \frac{\sqrt{2}}{2} \leq x \leq 1, \end{cases}$$

so we can obtain a function  $g$ , given by

$$g(x) = \begin{cases} \begin{pmatrix} \sqrt{1-x^2} + 2x - 1, \\ h(x), \end{pmatrix} & \text{if } 0 \leq x \leq P, \\ \begin{pmatrix} h(x), \\ \frac{1}{5}\sqrt{-x^2 + 4x + 1} + \frac{1}{5}(2x + 1), \end{pmatrix} & \text{if } P \leq x \leq \frac{\sqrt{2}}{2}, \\ \begin{pmatrix} \frac{1}{5}\sqrt{-x^2 + 4x + 1} + \frac{1}{5}(2x + 1), \\ \frac{1}{5}(2x + 1), \end{pmatrix} & \text{if } \frac{\sqrt{2}}{2} < x \leq 1, \end{cases}$$

such that  $(f, g)$  is a  $\mathcal{C}$ -hairpin, where

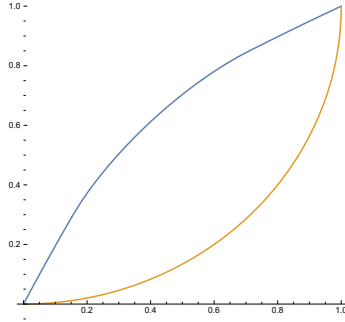
$$P = \frac{1}{5} (4 - \sqrt{2}) - \frac{1}{5} \sqrt{\frac{1}{2} (4\sqrt{2} + 1)} \left( \right.$$

and

$$h(x) = \frac{1}{5} \left( \sqrt{(4 + 2\sqrt{2} - 4x)\sqrt{1-x^2} + x(-3x + 4\sqrt{2} + 8)} \left( -4\sqrt{2} - 2 \right. \right. \\ \left. \left. + 4x - 2\sqrt{2} + 1 + 2\sqrt{1-x^2} \right) \right).$$

The set  $G_f \cup G_g$  is shown in Figure 1.



FIGURE 1.  $G_f \cup G_g$  for the copula  $C$  in Example 1.

A slightly more general problem is posed in [5], where modular functions are involved. Before tackling this problem, we recall some preliminary notions.

Let  $R = [a_1, a_2] \times [b_1, b_2]$  be a rectangle in  $\mathbb{R}^2$ . A function  $H: R \rightarrow \mathbb{R}$  is called *modular* if

$$H(x_1, y_1) + H(x_2, y_2) = H(\max(x_1, x_2), \max(y_1, y_2)) + H(\min(x_1, x_2), \min(y_1, y_2))$$

for any  $(x_1, y_1), (x_2, y_2) \in R$ . Also,  $H$  is modular if there exist two functions  $r: [a_1, a_2] \rightarrow \mathbb{R}$  and  $s: [b_1, b_2] \rightarrow \mathbb{R}$  such that  $H(x, y) = r(x) + s(y)$  for any  $(x, y) \in R$  (see [20]).

In [5] it is illustrated that a simple operation, such as truncating by means of a Fréchet-Hoeffding bound, can transform a suitable modular function  $H$  into a copula  $H^* = \min(M, H)$ . The region where the copula  $H^*$  coincides with  $M$  is separated from the region where it does not by means of two curves. The support of the copula  $H^*$  is contained in the union of these curves. This leads to the Open Problem 1 in Introduction. We want to stress the strong connection between the study of copulas with hairpin support and this problem, namely if  $f$  and  $g$  are two homeomorphisms such that  $f(x) < x < g(x)$  for all  $x \in ]0, 1[$  and the problem has an affirmative answer, then  $(f, g)$  is a  $\mathcal{C}$ -hairpin. On the other hand, there are pairs of functions  $(f, g)$  forming a  $\mathcal{C}$ -hairpin for what Open Problem 1 has a negative answer: For instance, [7, Example 5] does not satisfy that the set of points at which  $H$  does not match  $M$  is exactly between the graphs of  $f$  and  $g$ : For example,  $H^*(1/2, 1/2) = 1/2 = M(1/2, 1/2)$ .

With the ideas developed in this work, it is possible to provide an answer to Open Problem 1 to the case in which  $g(x) = \sqrt{x}$  and  $f(x) = x^2$ . Let us see that these functions do not give a  $\mathcal{C}$ -hairpin. Note  $g(x) = \sqrt{x} > x$  for all  $x \in ]0, 1[$ . If we had a copula with mass concentrated on  $G_{\sqrt{x}} \cup G_{x^2}$  the corresponding function  $(A_C)_*: [0, 1] \rightarrow [0, 1]$  given by (3.1) would satisfy (we will write  $A_*$  instead of  $(A_C)_*$ )

$$x - A_*(x) = x^2 - A_*(x^4). \quad (3.4)$$

Note that if  $B(x)$  denotes the mass concentrated on  $G_{\sqrt{x}}$  between the points  $(0, 0)$  and  $(x, \sqrt{x})$ , its margins are  $A_*(x)$  and  $A_*(y^2)$ , so that in  $G_{x^2}$  we have a mass with margins  $x - A_*(x)$  and  $y - A_*(y^2)$ , whence, for  $y = x^2$ , we obtain the equality in (3.4), which is equivalent to  $A_*(x) = x - x^2 + A_*(x^4)$ . Iterating this last formula, we obtain

$$A_*(x) = \sum_{n=0}^{\infty} (-1)^n x^{2^n}.$$

Observe that  $A_*$  is differentiable on  $]0, 1[$ , with

$$A'_*(x) = \sum_{n=0}^{\infty} (-2)^n x^{2^n-1}.$$

The function  $A_*$  must be increasing however a numerical calculation shows  $A'_*(0.985) \simeq -0.498$  and this leads to a contradiction. We want to stress that in [12] the authors obtain this result by stating, without proof, that the function  $x - A_*(x)$  is not monotone.

In [17, Proposition 3] the authors claim that it is not possible to find a copula whose mass is concentrated on the  $\mathcal{C}$ -hairpin given by the functions  $\phi_a(x) = x^a$  and  $\phi_{1/a}(x) = x^{1/a}$ , with  $a > 1$ . Unfortunately, their proof is not completely correct. To be precise, it is not correctly proven that the function

$$A_C(x) = \sum_{k=1}^{\infty} (-1)^{k+1} x^{a^k}, \quad 0 \leq x < 1,$$

where  $a > 1$ , does not have a limit as  $x$  approaches 1 from the left. We now provide a correct proof, in which we will write  $A$  instead of  $A_C$ .

**Theorem 3.6.** *For  $a > 1$  the pair  $(\phi_a, \phi_{1/a})$  is not a  $\mathcal{C}$ -hairpin.*

*Proof.* For  $a > 1$ , consider the function

$$A(x) = - \sum_{n=1}^{\infty} (-1)^n x^{a^n} = - \sum_{n=1}^{\infty} (x^{a^{2n}} - x^{a^{2n-1}}) \quad (3.5)$$

for all  $x \in [0, 1[$ . We will prove the theorem in three steps.

*Step 1.* Let  $A$  be the function given by (3.5). If  $A(1^-)$  exists then  $A(1^-) = 1/2$ .

Consider the functional equation  $A(x) + A(x^a) = x^a$ . Then the result follows by taking  $x \rightarrow 1^-$ .

*Step 2.* Let  $A$  be the function given by (3.5). If  $A(1^-) = 1/2$  then the function defined for every  $x \in \mathbb{R}$  by

$$h(x) = - \sum_{k \in \mathbb{Z}} \left( e^{-a^{2k+2x}} - e^{-a^{2k+2x-1}} \right) \quad (3.6)$$

is identically  $1/2$ .

Let  $h_K$  be the partial sum of  $h$ :

$$h_K(x) = - \sum_{k=-2K}^{2K} (-1)^k e^{-a^{k+2x}}.$$

Then we have  $h_K(x) + h_K(x + 1/2) = e^{-a^{-2K+2x-1}} - e^{-a^{2K+2x+1}}$ , and taking  $K \rightarrow +\infty$ , we obtain  $h(x) + h(x + 1/2) = 1$ . Suppose that the function  $h$  is not identically  $1/2$ . Then there exists  $y_0 \in \mathbb{R}$  such that  $h(y_0) \neq 1/2$ . If  $h(y_0) > 1/2$ , we take  $x_0 = y_0$ ; and if  $h(y_0) < 1/2$ , then  $h(1/2 + y_0) = 1 - h(y_0) > 1 - 1/2$ , and we take  $x_0 = 1/2 + y_0$ . In any case,  $h(x_0) > 1/2$ . Since  $-e^{-a^{2k+2x}} + e^{-a^{2k+2x-1}} > 0$  for all  $x \in \mathbb{R}$ , there exists  $K_0 \in \mathbb{Z}$  such that

$$h_{K_0}(x_0) = - \sum_{k=-2K_0-1}^{2K_0} (-1)^k e^{-a^{k+2x_0}} > \frac{1}{2}.$$

Let  $x_N = 1 - a^{2(x_0-N)}$  with  $N > K_0$ . For  $K_1 \in \mathbb{Z}$  such that  $N > K_1 > K_0$  we have

$$\begin{aligned} A(x_N) &> - \sum_{n=N-K_1}^{N+K_1} \left( \left( 1 - a^{2(x_0-N)} \right)^{a^{2n-1}} + \left( 1 - a^{2(x_0-N)} \right)^{a^{2n}} \right) \\ &= - \sum_{k=-K_1}^{K_1} \left( \left( 1 - a^{2(x_0-N)} \right)^{a^{2N+2k-1}} + \left( 1 - a^{2(x_0-N)} \right)^{a^{2N+2k}} \right), \end{aligned}$$

and letting  $N \rightarrow \infty$  we obtain

$$A(1^-) = \lim_{N \rightarrow +\infty} A(x_N) \geq h_{K_0}(x_0) > \frac{1}{2},$$

which is a contradiction; so that the function  $h(x)$  is identically  $1/2$  for every  $x \in \mathbb{R}$ .

*Step 3. The function  $h$  given by (3.6) is not constant.*

Define

$$I := \int_0^1 h(x) e^{2\pi i x} dx = 0.$$

Note that

$$\begin{aligned} I &= - \sum_{k \in \mathbb{Z}} \int_k^{k+1} \left( e^{-a^{2x}} - e^{-a^{2x-1}} \right) e^{2\pi i x} dx \\ &= - \int_{-\infty}^{\infty} \left( e^{-a^{2x}} - e^{-a^{2x-1}} \right) e^{2\pi i x} dx. \end{aligned}$$

With the change  $a^{2x} = u$  we have  $2\pi i e^{2\pi i x} dx = s_a u^{s_a-1} du$ , where  $s_a = \pi i / \ln a$ , whence

$$\begin{aligned} I &= \frac{-s_a}{2\pi i} \int_0^\infty \left( e^{-u} - e^{-u/a} \right) u^{s_a-1} du = \frac{-s_a}{2\pi i} (1 - a^{s_a}) \Gamma(s_a) = \frac{-s_a}{\pi i} \Gamma(s_a) \\ &= \frac{-\Gamma(s_a + 1)}{\pi i} \neq 0, \end{aligned}$$

since  $\Gamma$  does not vanish in its domain (see [13] for details). Then  $h$  cannot be a constant.

This completes the proof.  $\square$

As a byproduct of the last part of the proof, the usual bounds for  $\Gamma$  prove that for  $a$  close to 1,  $|h(x) - 1/2|$  exceeds  $e^{-C/(a-1)}$  for some  $x$  (because  $|I|$  does). Numerical calculations suggest that this tiny number with a different  $C$  is also an upper bound for  $\sup |h(x) - 1/2|$ . Then it is very hard to detect numerically that  $h$  is not constant for  $a$  small.

#### 4. A generalization of Open Problem 1

Our aim now is to solve a problem slightly more general than Open Problem 1. Essentially we relax the condition of having graphs of functions, allowing vertical lines.

Consider the sets of the form

$$\beta^\blacktriangle = \{(x, y) \in [0, 1]^2 : y \leq f_\beta(x)\}$$

and

$$\alpha^\blacktriangle = \{(x, y) \in [0, 1]^2 : x \leq g_\alpha(y)\}$$

where  $f_\beta$  and  $g_\alpha$  are two nondecreasing right-continuous functions such that  $f_\beta(0) = g_\alpha(0) = 0$ ,  $f_\beta(1) = g_\alpha(1) = 1$  and  $\max\{f_\beta(t), g_\alpha(t)\} < t$  for all  $t \in ]0, 1[$ . Assuming right-continuity is just a convenient way of assuring the compactness of  $\beta^\blacktriangle$  and  $\alpha^\blacktriangle$  and we could state our results for arbitrary non-decreasing functions fixing the end points and replacing the sets  $\beta^\blacktriangle$  and  $\alpha^\blacktriangle$  by their closures.

To motivate the notation and understand the geometric situation, keep in mind that the graphs of  $f$  and  $g$  in Open Problem 1 are replaced in  $]0, 1[^2$  by the sets  $\beta$  and  $\alpha$  where  $\beta$  is the graph of  $f_\beta$  union all segment joining  $(t, f_\beta(t^-))$  and  $(t, f_\beta(t))$  at jump singularities, and  $\alpha$  is defined in the same way with  $(g_\beta(t), t)$  i.e., reversing the role of the variables (it can be thought as a graph seen from the  $y$ -axis). In other words,  $\beta$  and  $\alpha$  are the boundaries of  $\beta^\blacktriangle$  and  $\alpha^\blacktriangle$  restricted to  $]0, 1[^2$ .

A final comment is that in principle the analogy with Open Problem 1 suggests to impose  $\max\{f_\beta(t), g_\alpha(t)\} \leq t$  instead of  $\max\{f_\beta(t), g_\alpha(t)\} < t$ . We note that if there exists a diagonal point  $(t, t) \in \alpha \cup \beta$  the copula we want to construct takes the value  $M(t, t) = t$  at that point and [16, Theorem 3.2.1] assures that it is an ordinal sum, so it is possible to study each component of the ordinal sum separately.

We are now in position to pose:

*Open Problem 2.* When does there exist an increasing, 1-Lipschitz continuous and modular function  $H: [0, 1]^2 \rightarrow \mathbb{R}$  such that the region where the corresponding copula  $H^*$  does not coincide with  $M$  is exactly  $[0, 1]^2 \setminus (\alpha^\blacktriangle \cup \beta^\blacktriangle)$ ?

The following result provides an answer to Open Problem 2.

**Theorem 4.1.** *There exists an increasing, 1-Lipschitz continuous modular function  $H: [0, 1]^2 \rightarrow \mathbb{R}$  such that the region where the corresponding copula  $H^*$  does not coincide with  $M$  is exactly  $[0, 1]^2 \setminus (\alpha^\blacktriangle \cup \beta^\blacktriangle)$  if and only if the following two conditions hold:*

i. *The functions*

$$\begin{aligned} a(x) &= f_\beta(x) - (g_\alpha \circ f_\beta)(x) + (f_\beta \circ g_\alpha \circ f_\beta)(x) \\ &\quad - (g_\alpha \circ f_\beta \circ g_\alpha \circ f_\beta)(x) + \cdots \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} b(y) &= g_\alpha(y) - (f_\beta \circ g_\alpha)(y) + (g_\alpha \circ f_\beta \circ g_\alpha)(y) \\ &\quad - (f_\beta \circ g_\alpha \circ f_\beta \circ g_\alpha)(y) + \cdots \end{aligned} \quad (4.2)$$

defined for  $x \in [0, 1]$ , are nondecreasing, 1-Lipschitz and satisfy

$$\lim_{t \rightarrow 1^-} (a(t) + b(t)) = 1.$$

ii. *The function  $a$  (respectively,  $b$ ) is constant in an interval if, and only if,  $f_\beta$  (respectively,  $g_\alpha$ ) is constant.*

*In such a case, we have  $H(x, y) = a(x) + b(y)$ .*

*Proof.* Assume that such a function  $H$  exists. As  $\{(x, f_\beta(x)) : x \in [0, 1]$  and  $\{(g_\beta(y), y) : y \in [0, 1]$  are part of the boundary of  $\alpha^\blacktriangle \cup \beta^\blacktriangle$ , we must have  $H = M$  there. We know  $H(x, y) = a(x) + b(y)$  for  $a, b$  nondecreasing and 1-Lipschitz. Then  $a + b \circ f_\beta = f_\beta$  and  $a \circ g_\alpha + b = g_\alpha$ . Composing the second identity with  $f_\beta$  and subtracting the first one, we get  $a - a \circ F = f_\beta - F$  with  $F = g_\alpha \circ f_\beta$ . The sequence  $x_{n+1} = F(x_n)$  with  $x_0 \in [0, 1]$  is nonincreasing (because  $F(x) < x$  for  $x \in ]0, 1[$ ) and  $x_n \rightarrow 0$  except for  $x_0 = 1$  because 0 is the only fixed point of  $F$  in  $[0, 1[$ . We have

$$a(x_0) - a(x_{N+1}) = \sum_{k=0}^N (a(x_k) - a(x_{k+1})) = \sum_{k=0}^N (f_\beta(x_k) - F(x_k)).$$

When  $N \rightarrow \infty$  we get (4.1) with  $x_0 = x \in [0, 1[$ . Note that  $f_\beta(x_k) \rightarrow 0$  and  $F(x_k) \rightarrow 0$  and the convergence is assured without pairing the terms. A symmetric argument exchanging the variables proves (4.2).

The copula  $H^*$  verifies  $H^*(x, x) = a(x) + b(x)$  and  $H^*(1, y) = y$  then the continuity of  $H^*$  assures  $a(x) + b(x) \rightarrow 1$  as  $x \rightarrow 1^-$ .

By (4.1),  $a$  is constant whenever  $f_\beta$  is constant. For the converse, note  $f_\beta(x) = \inf\{y \in [0, 1] : H(x, y) < M(x, y)\}$  that equals  $\inf\{y \in [0, 1] : a(x) + b(y) < y\}$  and it is clear that this is constant when  $a$  is constant.

Conversely, define  $H(x, y) = a(x) + b(y)$ , then  $H^*$  is a copula [5, Theorem 1]. Clearly  $H^*(0, 0) = M(0, 0)$  and  $H^*(1, 1) = M(1, 1)$  because

$$\lim_{x \rightarrow 1^-} (a(x) + b(x)) = 1.$$

We want to prove  $f_\beta(x) = \sup\{y \in [0, x] : H^*(x, y) = M(x, y)\}$  and  $f_\alpha(y) = \sup\{x \in [0, y] : H^*(x, y) = M(x, y)\}$  or, equivalently,  $f_\beta(x) = \sup\{y \in [0, 1] : a(x) + b(y) = y\}$  and analogously for  $g_\alpha$ . We focus on this equality for  $f_\beta$  because a similar argument works for  $g_\alpha$  exchanging  $x$  and  $y$ .

If for certain  $x = x_0$  the latter supremum is not  $f_\beta(x_0)$  then there exists  $y_* > f_\beta(x_0)$  such that  $a(x_0) + b(y_*) = y_*$ . On the other hand,  $a(x_0) + b(f_\beta(x_0)) = f_\beta(x_0)$  and the 1-Lipschitz continuity of  $b$  assures  $a(x_0) + b(y) = y$

for  $y \in [f_\beta(x_0), y_*)$ . If  $y = f_\beta(x)$ , comparing this to  $a(x) + b(y) = y$  we conclude

$$a(x) = a(x_0) \text{ for every } x \in \mathcal{F} \text{ where } \mathcal{F} = \{x \geq x_0 : f_\beta(x) \in [f_\beta(x_0), y_*)\}. \quad (4.3)$$

Let  $x_1 = \sup\{x : a(x) = a(x_0)\}$ , necessarily  $x_1 \neq x_0$  because the right-continuity of  $f_\beta$  implies that  $\mathcal{F}$  contains infinitely many points. Since  $a$  is nondecreasing and continuous, it is constant in  $[x_0, x_1]$ . Therefore  $f_\beta$  is also constant there (note that  $x_1 \neq 1$ ). Again by the right-continuity, there exists  $x_2 > x_1$  with  $x_2 \in \mathcal{F}$  and (4.3) contradicts the definition of  $x_1$ .  $\square$

**Corollary 4.2.** *There exists a modular function solving Open Problem 1 if and only if  $f_\beta = f$  and  $g_\alpha(y) = \sup\{x : g(x) \leq y\}$  satisfy the conditions i and ii in Theorem 4.1.*

*Proof.* Clearly  $g_\alpha$  is nondecreasing. As  $g$  is monotonic and continuous, the inverse image of a closed interval is a closed interval and we have  $g^{-1}([0, y]) = [0, g_\alpha(y)]$ . Hence the right continuity of  $g_\alpha$  is a consequence of the following equalities for  $y_n \rightarrow y_0^+$

$$g^{-1}([0, y_0]) = g^{-1}\left(\bigcap [0, y_n]\right) = \bigcap [0, g_\alpha(y_n)] = [0, \lim_n g_\alpha(y_n)].$$

Note that  $x \in g^{-1}([0, y])$  is equivalent to  $x \leq g_\alpha(y)$ . Then the region between the graphs of  $f$  and  $g$  is exactly  $[0, 1]^2 \setminus (\alpha^\blacktriangle \cup \beta^\blacktriangle)$ .  $\square$

*Remark 4.3.* Note that if in Theorem 4.1 condition i holds but condition ii is not satisfied, then we have that it is possible to construct a copula whose mass is concentrated on  $\alpha \cup \beta$ , but it is not possible to find a copula  $H^*$  such that coincides with  $M$  in  $\alpha^\blacktriangle \cup \beta^\blacktriangle$ . Moreover, if condition i is not satisfied, then it is not possible to construct a copula with mass concentrated on  $\alpha \cup \beta$ .

We now provide an example.

*Example 2.* Given  $s, t \in ]0, 1[$ , define for  $x, y \in [0, 1[$

$$f_\beta(x) = \max(0, x - s), \quad g_\alpha(y) = \max(0, y - t) \quad \text{and} \quad \sigma = (t + s)^{-1}$$

and complete the definition of the functions with  $f_\beta(1) = g_\alpha(1) = 1$ . The corresponding  $\beta$  and  $\alpha$  sets are the graphs of  $f_\beta$  and  $g_\alpha$  completed with the segments joining  $(1, 1)$  to  $(1, 1 - s)$  and to  $(1 - t, 1)$ , respectively.

Composing  $g_\alpha \circ f_\beta$  with itself  $k$  times, we get  $\max(0, x - k\sigma^{-1})$  (for  $x \in [0, 1[$ . Hence in this range, according to (4.1),

$$\begin{aligned} a(x) &= \sum_{k=1}^{\infty} \left( f_\beta(\max(0, x - (k-1)\sigma^{-1})) - \max(0, x - k\sigma^{-1}) \right) \\ &= \sum_{k=1}^{\infty} \left( \max(0, x + t - k\sigma^{-1}) - \max(0, x - k\sigma^{-1}) \right). \end{aligned}$$

Each term in the sum is  $t$  if  $k \leq \sigma x$  and it is 0 for  $k > \sigma x + \sigma t$ . The same applies for  $b$  exchanging  $s$  and  $t$ . Therefore

$$a(x) = \begin{cases} \lfloor \sigma x \rfloor t & \text{if } \lfloor \sigma x \rfloor = \lfloor \sigma x + \sigma t \rfloor, \\ x - \lfloor \sigma x + \sigma t \rfloor s & \text{otherwise} \end{cases}$$

and

$$b(y) = \begin{cases} \lfloor \sigma y \rfloor s & \text{if } \lfloor \sigma y \rfloor = \lfloor \sigma y + \sigma s \rfloor, \\ y - \lfloor \sigma y + \sigma s \rfloor t & \text{otherwise,} \end{cases}$$

where  $\lfloor \cdot \rfloor$  is the usual floor function  $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$ . Note that  $\lfloor \sigma x + \sigma t \rfloor - \lfloor \sigma x \rfloor \in \{0, 1\}$  and the same holds for  $\lfloor \sigma y + \sigma s \rfloor - \lfloor \sigma y \rfloor$  because  $\sigma t, \sigma s \in ]0, 1[$ . These functions  $a$  and  $b$  are monotonic and continuous. For the continuity, for instance of  $a$ , note that  $\sigma x \rightarrow n \in \mathbb{Z}^+$  implies  $\lfloor \sigma x \rfloor \neq \lfloor \sigma x + \sigma t \rfloor$  and  $\sigma x + \sigma t \rightarrow n \in \mathbb{Z}^+$  implies  $\lfloor \sigma x \rfloor = \lfloor \sigma x + \sigma t \rfloor$ . Using that the floor function is piecewise constant, it is not difficult to prove that the graphs of  $a$  and  $b$  are polygonal lines composed by horizontal segments and segments of slope 1. In particular,  $a$  and  $b$  are 1-Lipschitz.

To check the last property in Theorem 4.1.i we observe that

$$a(1^-) + b(1^-) = (1 - s) + (1 - t) > 1 \quad \text{for } \sigma < 1$$

and the property does not hold in this case. On the other hand, for  $\sigma \geq 1$  it is easy to see that  $a(1^-) + b(1^-) < 1$  except when we choose the second possibility in the definition of  $a$  and  $b$ . Namely,

$$a(1^-) + b(1^-) = 2 - \lfloor \sigma^- \rfloor \sigma^{-1} - \sigma^{-1} \quad \text{for } \lfloor \sigma^- \rfloor \neq \lfloor \sigma^- + \sigma t \rfloor = \lfloor \sigma^- + \sigma s \rfloor, \quad \sigma \geq 1.$$

Here we use  $\lfloor x^- + y \rfloor$  as an abbreviation of  $\lim_{h \rightarrow 0^-} \lfloor x + h + y \rfloor$ . If  $\sigma = 1$  this is immediately satisfied and we get a shuffle of Min. In fact,  $2 - \lfloor \sigma^- \rfloor \sigma^{-1} - \sigma^{-1} = 1$  if and only if  $\sigma \in \mathbb{Z}^+$  and any of these integral values satisfies the conditions required above for  $\sigma$ . In conclusion, the conditions in Theorem 4.1.i hold if and only if  $\sigma \in \mathbb{Z}^+$ . Hence if  $\sigma \notin \mathbb{Z}^+$ , it is not possible to have a copula having mass concentrated on  $\alpha \cup \beta$ .

If  $\sigma \in \mathbb{Z}_{>1}$ , there exists a copula whose mass is concentrated on  $\alpha \cup \beta$ , but there does not exist a copula of type  $H^*$  such that it is equal to  $M$  in a set which contains  $\alpha^\blacktriangle \cup \beta^\blacktriangle$ . The reason is that there are initial intervals in which  $f_\beta$  and  $g_\alpha$  are equal to zero, and there are also intervals that do not contain the zero in which the functions  $a$  and  $b$  are constants, but this is absurd since  $f_\beta$  and  $g_\alpha$  are not constant beyond the initial intervals.

In general these copulas for  $\sigma \in \mathbb{Z}_{>1}$  are always ordinal sums of a copula  $A$  with itself with respect to the intervals  $\left\{ \left[ \frac{k}{\sigma}, (k+1)/\sigma \right] \right\}_{k=0}^{\sigma-1}$ . The result is trivial for  $\sigma = 1$ , let us prove it by induction assuming that it is true for  $\sigma = n - 1$  and deducing it for  $H^*$ ,  $s$ ,  $t$ ,  $a$  and  $b$  with  $\sigma = (s + t)^{-1} = n$ . By the formulas for  $a$  and  $b$

$$H^*(\sigma^{-1}, \sigma^{-1}) = a(\sigma^{-1}) + b(\sigma^{-1}) \neq \sigma^{-1} - s + \sigma^{-1} - t = \sigma^{-1}.$$

Then [16, Theorem 3.2.1] assures that  $H^*$  is an ordinal sum of two copulas  $A$  and  $B$  with respect to the intervals  $[0, \sigma^{-1}]$  and  $[\sigma^{-1}, 1]$ . Some calculations show that  $A$  is a shuffle of Min supported on the segment joining the points

$(s\sigma, 0)$  and  $(1, 1 - s\sigma, 0)$  and the segment joining the points  $(0, t\sigma)$  and  $(1 - t\sigma, 1)$ .

On the other hand  $B$  is the copula corresponding to  $f_\beta(x) = \max(0, x - s')$  and  $g_\alpha(y) = \max(0, y - t')$  with  $s' = \sigma s/(\sigma - 1)$  and  $t' = \sigma t/(\sigma - 1)$ . Since  $\sigma' = (s' + t')^{-1}$ , we can apply the induction hypothesis to conclude that  $B$  is an ordinal sum of a certain  $A'$  with itself with respect to the intervals  $\{[k/\sigma', (k+1)/\sigma']\}_{k=0}^{\sigma'-1}$ . Focusing on the first interval and using the formulas for  $a$  and  $b$  we obtain  $A' = A$ . Then the ordinal sum of  $A$  and  $B$  is as expected.

As a particular case, if  $s = t = 1/4$ , we have

$$a(h) = b(h) = \begin{cases} 0, & \text{if } 0 \leq h \leq \frac{1}{4}, \\ h - \frac{1}{4}, & \text{if } \frac{1}{4} < h \leq \frac{1}{2}, \\ \frac{1}{4}, & \text{if } \frac{1}{2} < h \leq \frac{3}{4}, \\ h - \frac{1}{2}, & \text{if } \frac{3}{4} < h \leq 1. \end{cases}$$

The copula  $C$ , whose mass is concentrated on  $\alpha \cup \beta$  is a shuffle of  $\text{Min}$ , the ordinal sum of  $C_1$  with itself with respect to the intervals  $[0, 1/2]$  and  $[1/2, 1]$  which is given by

$$C_1(x, y) = \min\{x, y, \max\{0, x - 1/2, y - 1/2, x + y - 1\}\}$$

for every  $(x, y) \in [0, 1]^2$  (see Figure 2).

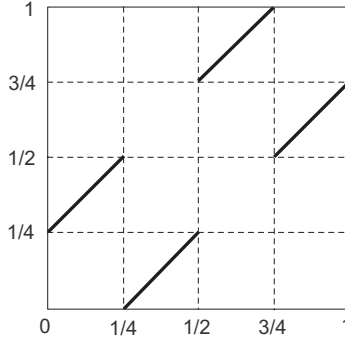


FIGURE 2. Support (in thick lines) of the copula  $C$  in Example 2.

In the case  $s = 1/3$  and  $t = 1/6$  we obtain a copula  $\tilde{C}$  with support shown in Figure 3.



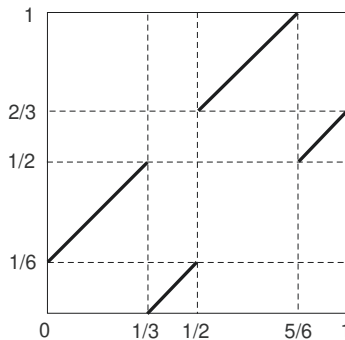


FIGURE 3. Support (in thick lines) of the copula  $\tilde{C}$  in Example 2.

## 5. The multidimensional case

In this section we extend some of the results in the preceding sections to the multivariate case.

Let  $\varphi_2, \dots, \varphi_n$  be  $n-1$  absolutely continuous homeomorphisms on  $[0, 1]$  with  $\varphi_j(x) < x$  for all  $x \in ]0, 1[$  and for every  $j = 2, \dots, n$ , and such that  $\lambda(\{x : \varphi'_j(x) = 0\}) \neq 0$ . We define the function

$$A(x) = \frac{1}{2} \int_0^x \min(\varphi'_2(t), \dots, \varphi'_n(t), 1) dt.$$

With this function  $A$ , we define the  $n$ -copulas

$$F(x_1, \dots, x_n) = M^n(A(x_1), A(\varphi_2^{-1}(x_2)), \dots, A(\varphi_n^{-1}(x_n))) \quad (5.1)$$

and

$$G(x_1, \dots, x_n) = M^n(A_*(x_1), (A \circ \varphi_2^{-1})_*(x_2), \dots, (A \circ \varphi_n^{-1})_*(x_n)) \quad (5.2)$$

We have the following result. Its proof is based on the observations described above and on the proof of Theorem 3.5, so we omit it.

**Theorem 5.1.** *Let  $n$  be a natural number such that  $n \geq 2$ , and let  $F$  and  $G$  be the functions given by (5.1) and (5.2), respectively. Then the function  $C = F + G$  is an  $n$ -copula whose support is the union of two curves given by strictly increasing functions in each component.*

*Remark 5.2.* In Theorem 5.1 we have required  $\varphi_j(x) < x$  for all  $x \in ]0, 1[$  and for every  $j = 2, \dots, n$ . We want to stress that this is important in the bidimensional case, but not to higher dimensions; it suffices to ask that there does not exist  $x$  such that  $\varphi_j(x) = x$  for all  $j = 2, \dots, n$ .

We recall that a *generalized hairpin* is the union of the graphs of  $y = L(x)$  and  $x = U(y)$ , where  $L$  and  $U$  are non-decreasing functions from  $[0, 1]$  into  $[0, 1]$  such that  $(L \circ U)(t) < t$  and  $(U \circ L)(t) < t$  whenever  $0 < t < 1$ .

(see [14]). In [8], the authors show that at most one 2-copula can have its mass concentrated on the union of the graphs of two increasing functions and hence that 2-copula must be extremal. In the following result, we provide an analogue result for higher dimensions.

**Corollary 5.3.** *The  $n$ -copula  $C$  given in Theorem 5.1 is an extreme point in the set of all  $n$ -copulas.*

*Proof.* Firstly, note that the  $n$ -copula  $C$  is determined by the bidimensional margins  $C(x_1, 1, \dots, 1, x_j, 1, \dots, 1)$ , which are 2-copulas. Since these 2-copulas are generalized hairpins, and these have their mass uniquely determined by their two graphs, the result follows.  $\square$

We conclude this section with an example for the trivariate case.

*Example 3.* Consider the functions  $\varphi_2(x) = x^2$  and  $\varphi_3(x) = x^3$  for all  $x \in [0, 1]$ . Applying Theorem 5.1, we obtain the 3-copula  $C$  whose support is the union of the curve  $\{(t, t^2, t^3) : t \in [0, 1]\}$  and another more complicated curve, both depicted in Figure 4.

## 6. Conclusions

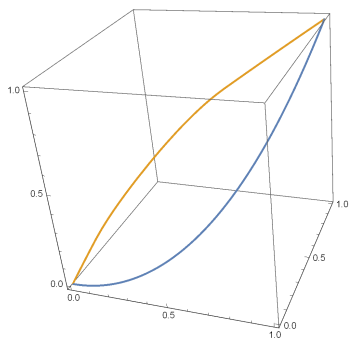
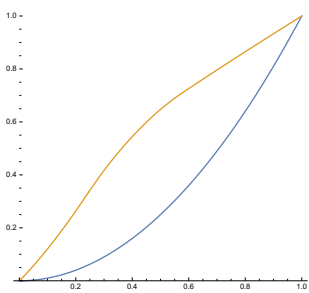
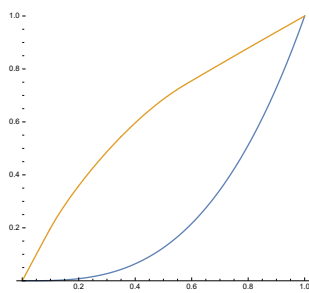
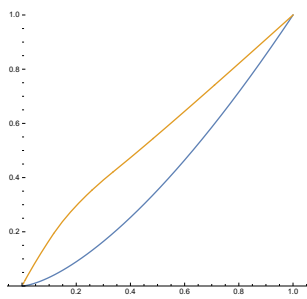
In this paper we have provided necessary and sufficient conditions for which the graph of an increasing homeomorphism  $f$  defined on  $[0, 1]$  such that  $f(x) < x$  for all  $x \in ]0, 1[$  can be part of a  $\mathcal{C}$ -hairpin that concentrates the mass of a bivariate copula. We generalize the classic result in [17] and [18] and we extend them to the general nondecreasing case without assuming bijectivity. This generalization allows us to solve the open problem posed in [5]. Furthermore, under certain conditions, we give a multidimensional method that generalizes the bivariate case and allows to construct extreme points in the set of multidimensional copulas.

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(a) Support of the 3-copula  $C$ (b) Support of the margin  
 $X, Y$ (c) Support of the margin  
 $X, Z$ (d) Support of the margin  
 $Y, Z$ FIGURE 4. Supports of the 3-copula  $C$  and its bivariate margins in Example 3.

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