



# Network navigation with non-Lèvy superdiffusive random walks

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## ABSTRACT

We introduce a formalism based on a continuous time approximation to study the characteristics of random walks with jumps to random locations of the networks (Pagerank random walks). We find that the diffusion of the occupancy probability has a dynamics that exponentially “forgets” the initial conditions and settles to a steady state that depends only on the characteristics of the network. In the special case in which the walk begins from a single node, we show that the largest eigenvalue of the transition value ( $\lambda_1 = 1$ ) does not contribute to the dynamic and that the probability is constant in the direction of the corresponding eigenvector. We study the process of visiting new nodes, which we find to have a dynamic similar to that of the occupancy probability. Finally, we determine the average transit time between nodes  $\langle T \rangle$ .

We show that, for suitable values of the walk parameters, Pagerank walks behave similarly to Lèvy walks in terms of unique nodes visited and average transit time, while requiring much less information about the graph.

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## 1. Introduction

Transport and diffusion processes on graphs have been studied for their intrinsic interest as well as their applications to the study of epidemics [1,2], technical systems [3], animal [4,5] and human [6,7] behavior, and to information spread [8,9]. Random walks on graphs have been used for clustering [10,11] as well as sampling [12].

In a standard discrete-time random walk, the walker moves from a node to one of its neighbors picked at random, and the occupancy probability spreads from the initial point following a diffusion-like equation [13,14]. Other kinds of walk have also been studied, which lead to a different diffusion behavior. One well-known example is that of Lèvy walks (walks in which the walker makes jumps at a distance  $d$  from the current location with probability  $p(d) \sim d^{-\alpha}$  [15,16]), which exhibit a superdiffusive behavior similar to that of a common animal behavior known as *Area Restricted Search* [14,17]. Lèvy walks have been studied on graphs, and in this case the distance  $d$  that determines the probability  $p(d) \sim d^{-\alpha}$  is the length of the shortest path between nodes [18].

Lèvy walks guarantee a superdiffusive behavior [18,19] at the cost of requiring considerable global information about the graph. For each node where the walker is located, it is necessary to know the minimal path distance to all other nodes in the graph so that a successor can be picked with the prescribed probability.

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In this paper, we study the behavior of a random walk that does not require non-local information about the graph. If at time  $t$  the walker is on a node  $u$ , then at time  $t + 1$  it will move to one of the neighbors of  $u$  with probability  $1 - \beta$ , while it will jump to a random node in the graph with probability  $\beta$ . This kind of walk is known as *Pagerank* [10,20]. We obtain closed form equations for the occupancy probability, the number of unique nodes visited, and the average transit time between two nodes.

## 2. Basic equations

Let  $G = (V, E)$  be an undirected graph with  $n$  nodes and  $m$  edges ( $|E| = 2m$ ) with adjacency matrix  $\mathbf{A} = \mathbf{A}'$ , let  $d_i = \sum_j a_{ij}$  be the degree of node  $i$ , and  $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$ . A standard random walk on this graph [13] is governed by the matrix  $\mathbf{W} = \mathbf{D}^{-1}\mathbf{A}$  through the master equation

$$|p\rangle(t+1) = \mathbf{W}'|p\rangle(t), \quad (1)$$

where  $|p\rangle(t) = [p_1(t), \dots, p_n(t)]'$  is the vector of occupancy probabilities of the nodes. Some background on standard random walks is given in [Appendix A](#); here we only mention some properties of  $\mathbf{W}$  that we shall need in the following:

(i) For all  $i$ ,  $\sum_{j=1}^n w_{ij} = 1$ ;

(ii) The eigenvalues of  $\mathbf{W}$  are

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n > -1 \quad (2)$$

(bipartite graphs, and only them, have  $\lambda_n = -1$  [21], but we do not consider that case here);

(iii)  $\langle 1_n | = \langle 1_n | \mathbf{W}'$ , with  $\langle 1_n | = [1, \dots, 1]$ ;

(iv)  $\mathbf{W}' = \mathbf{W}'|\pi\rangle$ , with  $|\pi\rangle = [\pi_1, \dots, \pi_n]'$ ,  $\pi_k = d_k/2m$ .

In a random walk with random jumps (Pagerank), at each step we toss a coin with probability  $\beta$  of giving heads. If the result is tail, we pick a random neighbor and move to it. If the result is head, we jump to a random node in the graph. That is, the probability  $r_{ij}$  of jumping from a node  $i$  to a node  $j$  is

$$r_{ij} = (1 - \beta) \frac{a_{ij}}{d_i} + \beta \frac{1}{n} = (1 - \beta) w_{ij} + \frac{\beta}{n}. \quad (3)$$

The master equation for this walk follows the general schema of the standard walk, and is given by

$$\begin{aligned} p_i(t+1) &= \sum_j p_j(t) \omega_{ji} \\ &= \sum_j (1 - \beta) w_{ji} p_j(t) + \frac{\beta}{n} \sum_j p_j(t) \\ &= \sum_j (1 - \beta) w_{ji} p_j(t) + \frac{\beta}{n}, \end{aligned} \quad (4)$$

that is,

$$|p\rangle(t+1) = (1 - \beta) \mathbf{W}'|p\rangle(t) + \frac{\beta}{n} |1_n\rangle. \quad (5)$$

## 3. PageRank dynamics

The stationary point of (5) (viz., the occupancy probability at steady state) is the solution of

$$|p^*\rangle = (1 - \beta) \mathbf{W}'|p^*\rangle + \frac{\beta}{n} |1_n\rangle. \quad (6)$$

Since  $1 \geq \lambda_i > -1$  and  $\beta > 0$ , the matrix  $\mathbf{I} - (1 - \beta) \mathbf{W}$  has strictly positive eigenvalues and is therefore non singular. Eq. (6) can then be solved:

$$|p^*\rangle = [\mathbf{I} - (1 - \beta) \mathbf{W}']^{-1} \frac{\beta}{n} |1_n\rangle. \quad (7)$$

Eq. (5) can also be rewritten as

$$\begin{aligned} |p\rangle(t+1) - |p\rangle(t) &= -[\mathbf{I} - (1 - \beta) \mathbf{W}']|p\rangle(t) + \frac{\beta}{n} |1_n\rangle \\ &= -\mathbf{Q}|p\rangle(t) + \frac{\beta}{n} |1_n\rangle, \end{aligned} \quad (8)$$

with

$$\mathbf{Q} \equiv [\mathbf{I} - (1 - \beta)\mathbf{W}'] \quad (9)$$

(here, as in the rest of the paper, “ $\equiv$ ” means “equal by definition”), and the steady state solution can be obtained as

$$|p^*\rangle = \mathbf{Q}^{-1} \frac{\beta}{n} |1_n\rangle. \quad (10)$$

The matrix  $\mathbf{Q} = [\mathbf{I} - (1 - \beta)\mathbf{W}']$  has the same eigenvectors as  $\mathbf{W}'$  and eigenvalues

$$\mu_i = 1 - (1 - \beta)\lambda_i, \quad (11)$$

with

$$\beta = \mu_1 < \mu_2 \leq \dots \leq \mu_n = 1 - (1 - \beta)\lambda_n < 2 - \beta. \quad (12)$$

Note that we place the eigenvalues in the order opposite to the usual:  $\mu_1$  is the smallest eigenvalue. We do this to maintain a simpler correspondence between  $\mu_i$  and  $\lambda_i$ .

If  $\beta = 0$  the matrix  $\mathbf{Q}$  is singular. We shall not consider this case, in which *Pagerank* reduces to the standard random walk. If  $\beta > 0$ , then  $\mathbf{Q}$  can be decomposed as

$$\begin{aligned} \mathbf{Q} &= \mathbf{T}\mathbf{L}\mathbf{T}' \\ \mathbf{L} &= \text{diag}(\mu_1, \dots, \mu_n), \end{aligned} \quad (13)$$

Here, the columns of the matrix  $\mathbf{T}$  consist of the right eigenvectors  $|\phi_i\rangle$  of  $\mathbf{W}'$  and the rows of  $\mathbf{T}^{-1}$  are the corresponding left eigenvectors  $\langle\psi_i|$  (see (66) and (67) in Appendix B).

If we iterate (8) $\Delta$  times, assuming that the right-hand side is kept constant, we have

$$|p\rangle(t + \Delta) - |p\rangle(t) = \Delta(-\mathbf{Q}|p\rangle(t) + \frac{\beta}{n}|1_n\rangle); \quad (14)$$

if we now consider  $t$  a continuous variable, divide by  $\Delta$  and take the limit for  $\Delta \rightarrow 0$  we obtain the continuous approximation of the walk, that is

$$\frac{d}{dt}|p\rangle = -\mathbf{Q}|p\rangle + \frac{\beta}{n}|1_n\rangle. \quad (15)$$

This is a diffusion equation under the spatial operator  $\mathbf{Q}$ . We are interested in studying the time evolution of the probability as it approaches the steady state.

The solution of (15) is

$$|p\rangle(t) = \frac{\beta}{n}\mathbf{Q}^{-1}|1_n\rangle + e^{-\mathbf{Q}t}|C\rangle, \quad (16)$$

where  $|C\rangle$  is an arbitrary vector that depends on the initial conditions. Using the initial probabilities  $|p^0\rangle$ , we have

$$\begin{aligned} |p\rangle(t) &= \frac{\beta}{n}(\mathbf{I} - e^{-\mathbf{Q}t})\mathbf{Q}^{-1}|1_n\rangle + e^{-\mathbf{Q}t}|p^0\rangle \\ &= \frac{\beta}{n}\mathbf{T}(\mathbf{I} - e^{-\mathbf{L}t})\mathbf{L}^{-1}\mathbf{T}^{-1}|1_n\rangle + \mathbf{T}e^{-\mathbf{L}t}\mathbf{T}^{-1}|p^0\rangle. \end{aligned} \quad (17)$$

In order to analyze more in detail the dynamics of the walk, we move to the eigenvector basis. Define  $|\zeta\rangle = \mathbf{T}^{-1}|p\rangle$ ,  $|\zeta^0\rangle = \mathbf{T}^{-1}|p^0\rangle$ , and

$$\mathbf{T}^{-1}|1_n\rangle = |b\rangle = [b_1, \dots, b_n]', \quad b_i = \sum_k \psi_{i,k}, \quad (18)$$

where  $\langle\psi_i|$  is the  $i$ th left eigenvector of  $\mathbf{W}'$  (see (68), Appendix B). With these we obtain

$$\frac{d}{dt}|\zeta\rangle = -\mathbf{L}|\zeta\rangle + \frac{\beta}{n}|b\rangle, \quad (19)$$

with solution

$$|\zeta\rangle(t) = \frac{\beta}{n}(\mathbf{I} - e^{-\mathbf{L}t})\mathbf{L}^{-1}|b\rangle + e^{-\mathbf{L}t}|\zeta^0\rangle. \quad (20)$$

The matrix that determines the dynamics is in this case diagonal, therefore each direction in the eigenvector space evolves independently. Consider the direction of the first eigenvector, corresponding to  $\mu_1 = u$ :

$$\begin{aligned}\zeta_1 &= \frac{\beta}{n} (1 - e^{-\beta t}) \mathbf{L}^{-1} |b\rangle|_1 + e^{-\beta t} \zeta_1^0 \\ &= \frac{\beta}{n} (1 - e^{-\beta t}) \frac{1}{\beta} \sum_k \psi_{1,k} + e^{-\beta t} \mathbf{T}^{-1} |p^0\rangle|_1 \\ &= 1 - e^{-\beta t} + e^{-\beta t} \mathbf{T}^{-1} |p^0\rangle|_1.\end{aligned}\quad (21)$$

If the walk begins at a specific node  $m$ , then  $p_k^0 = \delta_{k,m}$  that is

$$\mathbf{T}^{-1} |p^0\rangle|_1 = \sum_k \psi_{1,k} \delta_{k,m} = 1 \quad (22)$$

because of (76) in Appendix B, leading to  $\zeta_1 = 1$ . The first eigenvalue  $\mu_1$  does not contribute to the dynamics of the probability distribution, which is determined uniquely by  $\mu_2, \dots, \mu_n$ , the only eigenvalues that actually depend on the structure of the graph.

#### 4. Node visitation

Let  $p_k$  be the probability that a randomly chosen node of the graph had  $k$  neighbors. If, instead of choosing a random node we choose a random edge and follow it in a random direction, then the probability of arriving at a node will be proportional to the number of abutting edges, viz., to the number of its neighbors. That is, if we follow a random edge in a random direction, the probability of arriving at a node with  $k$  neighbors is  $q_k \sim kp_k$  [22] and, normalizing

$$q_k = \frac{kp_k}{\sum_h hp_h} = k \frac{p_k}{\bar{p}}. \quad (23)$$

The average number of neighbors of a node that we arrive at following a random edge is

$$\bar{q} = \sum_k k q_k = \frac{\sum_{k=1}^{n-1} k^2 p_k}{\sum_{k=1}^{n-1} k p_k}. \quad (24)$$

A case of special interest is that of *social networks*, graphs that describe the interactions among people in different social settings. Many of them have been shown to be *scale free* that is, their degree probability follows a power law  $p_k \sim k^{-\gamma}$  with  $\gamma \sim 2, \dots, 3$  [23]. The value  $\bar{q}$  is then

$$\bar{q} = \frac{\sum_{k=1}^{n-1} k^{-(\gamma-2)}}{\sum_{k=1}^{n-1} k^{-(\gamma-1)}}. \quad (25)$$

The value  $\bar{q}$  is relatively insensitive to  $n$ , at least for larger values of  $\gamma$ , and in most networks is of the order of 10.

Let  $m_t$  be the number of nodes visited at time  $t$ . If, at time  $t$ , we pick a random node of the graph, the probability that the node were already visited is  $m_t/n$ , and the probability that a node were a new one is  $1 - m_t/n$ .

Suppose now that at time  $t$  we have just arrived at a node  $v$ . There are two ways in which this can happen: either we were at a neighbor  $v'$  and from that we walked to one of the neighbors (this happened with probability  $1 - \beta$ ), ending up in  $v$ , or we decided to jump to a random node of the graph (this happened with probability  $\beta$ ) and with that jump we arrived at  $v$ . Once we are at  $v$ , we also have two alternatives as to what to do: we can either move to a neighbor (with probability  $1 - \beta$ ) or jump to a random node (with probability  $\beta$ ).

Let us consider separately the two ways in which we may have arrived at  $v$  and the possible ways in which we can leave  $v$ .

**Arrived from a neighbor (probability  $1 - \beta$ ):** the node  $v$  has on average  $\bar{q}$  neighbors and among these there is  $v'$ , the node we came from. At the node  $v$  we toss a coin:

**Move to a neighbor (probability  $1 - \beta$ ):** The average number of neighbors is  $\bar{q}$ ; with probability  $1/\bar{q}$  we shall go back to  $v'$ , and not visit any new node. With probability  $1 - 1/\bar{q}$  we shall move to a node other than the

neighbor we came from and in this case we have a probability  $1 - m_t/n$  of visiting a new neighbor. Considering the probability of choosing this option, we shall visit a new node with probability

$$(1 - \beta)^2 \frac{\bar{q} - 1}{\bar{q}} \left(1 - \frac{m_t}{n}\right). \quad (26)$$

**Jump to a random node (probability  $\beta$ ):** In this case, the fact that one of our neighbors has certainly be visited is irrelevant: we jump to a random node in the graph, so the probability of choosing this option and visiting a new node is

$$(1 - \beta)\beta \left(1 - \frac{m_t}{n}\right). \quad (27)$$

**Arrived from a jump (probability  $\beta$ ):** In this case, it does not matter how we leave the node, either with another jump or through a neighbor: we are in a random area of the graph, our neighbors are random nodes. The probability of visiting a new node using this option is

$$\beta \left(1 - \frac{m_t}{n}\right). \quad (28)$$

Putting it all together, the probability that with a step of the walk we visit a new neighbor is

$$\begin{aligned} & (1 - \beta) \left[ (1 - \beta)^2 \frac{\bar{q} - 1}{\bar{q}} \left(1 - \frac{m_t}{n}\right) + \beta \left(1 - \frac{m_t}{n}\right) \right] + \beta \left(1 - \frac{m_t}{n}\right) \\ &= \left[ (1 - \beta)^2 \frac{\bar{q} - 1}{\bar{q}} + (1 - \beta)\beta + \beta \right] \left(1 - \frac{m_t}{n}\right) \\ &\equiv \nu \left(1 - \frac{m_t}{n}\right), \end{aligned} \quad (29)$$

where  $\nu$  depends on  $\beta$  and can be written, after some manipulation, as

$$\nu \equiv -\frac{1}{\bar{q}}\beta^2 + \frac{2}{\bar{q}}\beta + \frac{\bar{q} - 1}{\bar{q}}. \quad (30)$$

From (29), we determine that the average increment of the number of nodes visited in the random walk is

$$m_{t+1} - m_t = \nu \left(1 - \frac{m_t}{n}\right). \quad (31)$$

Taking infinitesimal time steps as in the previous section, we transform this into a differential equation

$$\frac{d}{dt} m(t) = \nu \left(1 - \frac{m(t)}{n}\right) \quad (32)$$

that, assuming that we begin the walk from a single node ( $m(0) = 1$ ) gives

$$m(t) = n(1 - e^{-\nu t/n}) + e^{-\nu t/n}. \quad (33)$$

Note that this is a dynamic very similar to that of the probability spread (the two are, of course, related). The visit is faster if  $\nu$  is large and, deriving (30) with respect to  $\beta$ , we see that the maximum of  $\nu$  is for  $\beta = 1$ . That is, the completely random visit is the fastest. This is consistent with the results already obtained in [18] for Lévy random walks.

## 5. Transit time

We are now interested in determining the average time that it takes, if we start from a node  $i$ , to reach a node  $j$ . We begin by rewriting the master equation for continuous time. Let  $p_{ij}(t)$  be the probability of being in node  $j$  at time  $t$  given that we are in node  $i$  at time 0. The  $p_{ij}(t)$  are discrete probabilities on the nodes but probability densities in time. That is, if we start from node  $i$ , the probability of arriving at node  $j$  in the time interval  $[t, t + dt]$  is  $p_{ij}(t)dt$ . With this, we can write the master equation as

$$p_{ij}(t) = \delta_{ij}\delta(0) + \int_0^t p_{ij}(t - \tau)F_{ij}(\tau)d\tau, \quad (34)$$

where  $\delta_{ij}$  is the Kronecker delta,  $\delta(0)$  the Dirac's, and  $F_{ij}(t)$  is the probability density of first passage, that is,  $F_{ij}(t)dt$  is the probability that, starting from node  $i$ , the walk went for the first time through node  $j$  in the interval  $[t, t + dt]$ . Taking the Laplace transform of this relation we have

$$\tilde{p}_{ij}(s) = \delta_{ij} + \tilde{p}_{ij}(s)\tilde{F}_{ij}(s), \quad (35)$$

that is

$$\tilde{F}_{ij}(s) = \frac{\tilde{p}_{ij}(s) - \delta_{ij}}{\tilde{p}_{ij}(s)}. \quad (36)$$

This quantity is related to the expected transit time. The expected time to go from  $i$  to  $j$  is the average of the first passage, that is

$$\langle T_{ij} \rangle = \int_0^\infty t F_{ij}(t) dt = -\tilde{F}'_{ij}(0). \quad (37)$$

In order to compute the derivative, we write

$$\begin{aligned} \tilde{p}_{ij}(s) &= \int_0^\infty e^{-st} p_{ij}(t) dt \\ &= p_j^* \int_0^\infty e^{-st} + \int_0^\infty e^{-st} [p_{ij}(t) - p_j^*] dt \\ &= p_j^* \int_0^\infty e^{-st} + \int_0^\infty e^{-st} \hat{p}_{ij}(t) dt, \end{aligned} \quad (38)$$

where we have defined  $\hat{p}_{ij}(t) \equiv p_{ij}(t) - p_j^*$ . Computing the first integral and expressing the exponential in the second as a series, we have

$$\begin{aligned} \tilde{p}_{ij}(s) &= \frac{p_j^*}{s} + \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n \int_0^\infty t^n \hat{p}_{ij}(t) dt \\ &= \frac{p_j^*}{s} + \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n R_{ij}^n \\ &= \frac{p_j^*}{s} B_{ij}(s), \end{aligned} \quad (39)$$

where we have defined the moments

$$R_{ij}^n \equiv \int_0^\infty t^n \hat{p}_{ij}(t) dt = \int_0^\infty t^n (p_{ij}(t) - p_j^*) dt \quad (40)$$

and the shortcut

$$B_{ij}(s) \equiv \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n R_{ij}^n. \quad (41)$$

Note that  $B_{ij}(0) = R_{ij}^{(0)}$ . We can now rewrite (36) as

$$\begin{aligned} \tilde{F}_{ij}(s) &= \frac{1}{\frac{p_j^*}{s} + B_{ij}(s)} \left[ \frac{p_j^*}{s} + B_{ij}(s) - \delta_{ij} \right] \\ &= \frac{p_j^* + s(B_{ij}(s) - \delta_{ij})}{p_j^* + sB_{ij}(s)}. \end{aligned} \quad (42)$$

Computing the derivative in  $s = 0$  we obtain

$$\langle T_{ij} \rangle = \frac{1}{p_j^*} [R_{ij}^{(0)} - R_{ij}^{(0)} + \delta_{ij}]; \quad (43)$$

taking the average over all pairs of nodes, we have

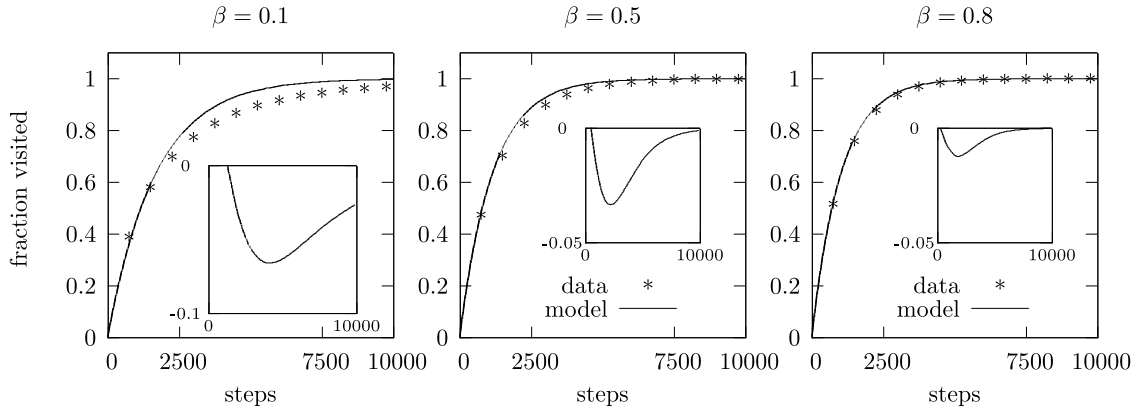
$$\langle T \rangle = \sum_{i \neq j} \langle T_{ij} \rangle p_j^* = \sum_i R_{ii}^{(0)}. \quad (44)$$

In order to compute the values  $R_{ii}^{(0)}$  we need to express  $p_{ij}(t)$ . From (16) we have

$$p_{ij}(t) = \frac{\beta}{N} \langle e_j | (\mathbf{I} - e^{-\mathbf{Q}t}) \mathbf{Q}^{-1} | 1_n \rangle + \langle e_j | e^{-\mathbf{Q}t} | e_i \rangle, \quad (45)$$

that is,

$$\begin{aligned} \hat{p}_{ij}(t) &= \langle e_j | e^{-\mathbf{Q}t} | e_i \rangle - \frac{\beta}{N} \langle e_j | e^{-\mathbf{Q}t} \mathbf{Q}^{-1} | 1_n \rangle \\ &= \langle e_j | \mathbf{T} e^{-\mathbf{L}t} \mathbf{T}^{-1} | e_i \rangle - \frac{\beta}{N} \langle e_j | \mathbf{T} e^{-\mathbf{L}t} \mathbf{L}^{-1} \mathbf{T}^{-1} | 1_n \rangle. \end{aligned} \quad (46)$$



**Fig. 1.** The number of new nodes visited in the power-law graphs during the simulation (indicated as “\*”) and the predicted value (continuous curve) for  $\beta = 0.1, 0.5, 0.8$ . The insets show the error (simulated minus predicted). The values are averages over 50 walks on each of the 50 graphs generated for the test. The variance of the data was very small, and is not shown to avoid cluttering the figures.

Using the definitions (66), (67), and (18), we can expand this equation as

$$\hat{p}_{ij}(t) = \sum_k e^{-\mu_k t} \phi_{k,j} \psi_{k,i} - \frac{\beta}{N} \frac{e^{-\mu_k t}}{\mu_k} \phi_{k,j} b_k, \quad (47)$$

that is

$$\begin{aligned} R_{ij}^{(0)} &= \int_0^\infty \hat{p}_{ij}(t) dt = \sum_k \frac{1}{\mu_k} \left[ \phi_{k,j} \psi_{k,i} - \frac{\beta}{N \mu_k} \phi_{k,j} b_k \right] \\ &= \sum_k \frac{1}{\mu_k} \psi_{k,i} \psi_{k,i} - \frac{\beta}{N \mu_k^2} \sum_k \phi_{k,i} b_k; \end{aligned} \quad (48)$$

therefore, from (44) we have

$$\begin{aligned} \langle T \rangle &= \sum_i \sum_k \frac{1}{\mu_k} \phi_{k,i} \psi_{k,i} - \sum_i \sum_k \frac{\beta}{N \mu_k^2} \phi_{k,i} b_k \\ &= \sum_k \frac{1}{\mu_k} \sum_i \phi_{k,i} \psi_{k,i} - \sum_k \frac{\beta}{N \mu_k^2} b_k \sum_i \phi_{k,i}. \end{aligned} \quad (49)$$

The first summation over  $i$  is equal to 1 because of (70), while of the second only the first term remains, the others being zero because of Lemma B.1 (Appendix B). Considering that  $\mu_1 = \beta$  and applying (77), we have

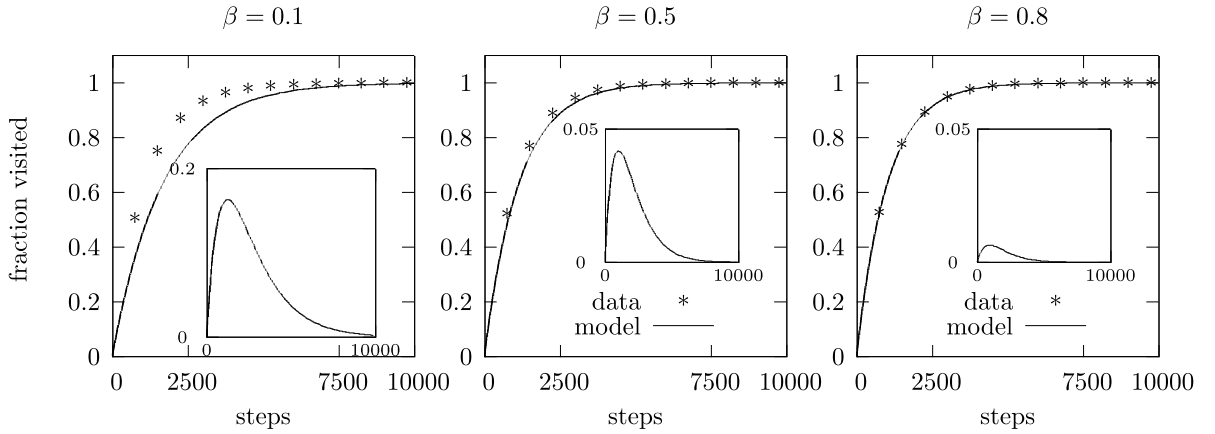
$$\begin{aligned} \langle T \rangle &= \sum_k \frac{1}{\mu_k} - \frac{u}{N \mu_1^2} \sum_i \phi_{1,i} \sum_i \psi_{k,1} \\ &= \sum_k \frac{1}{\mu_k} - \frac{1}{\mu_1} \\ &= \sum_{k=2}^n \frac{1}{\mu_k} \\ &= \sum_{k=2}^n \frac{1}{1 - (1 - \beta) \lambda_k}. \end{aligned} \quad (50)$$

If  $\beta = 0$ , the result is the same as that of [18] when  $\alpha \rightarrow \infty$ . Note however that in [18] the dependence on the parameter  $\alpha$  is hidden in the dependence of  $\lambda_k$  while, in our case,  $\lambda_k$  is a constant depending only on the structure of the graph, and the dependence on  $\beta$  is explicit.

The result does not depend on  $\mu_1$ , but only on the eigenvalues  $\mu_2, \dots, \mu_n$  which describe the structure of the graph.

## 6. Simulations

In this section, we present simulations to verify our findings. We use two types of random graphs: Erdős–Rényi [24], and power-law graphs created using the algorithm in [25]. All graphs have 1,000 nodes and, for the test, 50 graphs of



**Fig. 2.** The number of new nodes visited in the Erdős-Rényi graphs during the simulation (indicated as “\*”) and the predicted value (continuous curve) for  $\beta = 0.1, 0.5, 0.8$ . The insets show the error (simulated minus predicted). The values are averages over 50 walks on each of the 50 graphs generated for the test. The variance of the data was very small, and is not shown to avoid cluttering the figures.

each type were generated with the same parameters. All the tests that we show in the following are averaged over the 50 graphs of the respective type, so as to avoid dependencies on the particularities and the idiosyncrasies of a single graph. The power-law graphs have a node degree distribution  $p_k = k^{-\gamma_L}$ , with an average  $\bar{\gamma}_L = 1.27$  ( $\sigma^2 = 0.005$ ), an average node degree (number of neighbor per node)  $\bar{d}_L = 5.95$  ( $\sigma^2 = 0.45$ ) and an average clustering coefficient  $\bar{\chi}_L = 0.96$  ( $\sigma^2 = 0.03$ ). The averages and variances are computed over the set of 50 graphs. The Erdős-Rényi graphs were created with a parameter of  $d_E = 6$  neighbors per node (this value is a parameter of the algorithm, so all the Erdős-Rényi graphs average  $d_E$  neighbor per node). These graphs do not exhibit a power-law distribution ( $\bar{\gamma}_E = 10^{-4}$ ), and the average clustering coefficient is  $\bar{\chi}_E = 0.08$  ( $\sigma^2 = 10^{-6}$ ).

The first test is on the number of nodes visited. On each graph, we execute 50 random walks starting at a randomly selected node and terminating when all the nodes have been visited; the results are averaged over all walks and graphs. Figs. 1 and 2 show the results for the two classes of graphs and various values of the parameter  $\beta$ . The data shown are averages over 50 walks on each of the 50 graphs. The variance is very small and is not shown, the continuous curve is the theoretical prediction.

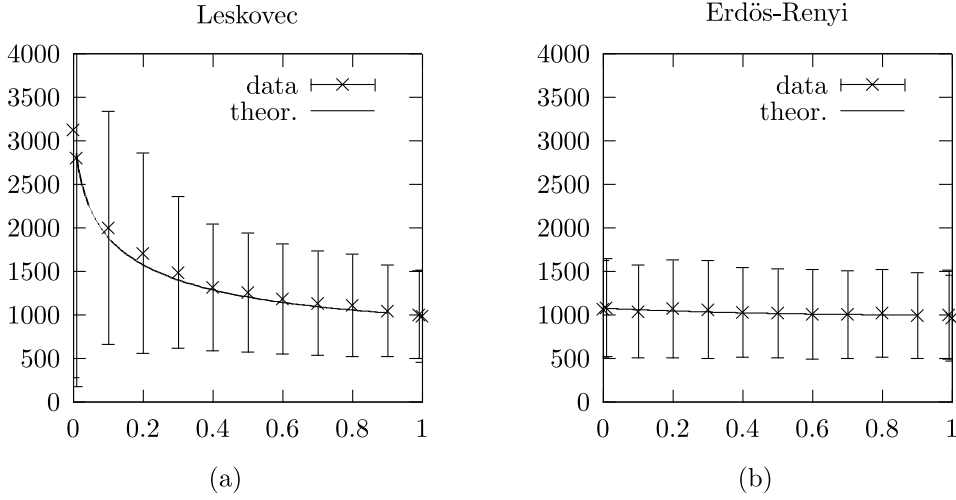
Two phenomena are noteworthy. First, in the case of power-law graphs, the theoretical curve slightly overestimates the actual number of nodes visited, while in the Erdős-Rényi graphs it underestimates it. Second, in both cases the error is larger for small  $\beta$ s than for large  $\beta$ s, that is, for walks with less jumps to a random node of the graph.

The underestimation is due to an underestimation of the consequences of consecutive short jumps. While we consider, in (26), the probability of going back and visiting the node we just came from, we do not consider that of revisiting a node that we already visited in the recent past. In the case of walks with low  $\beta$  in highly clustered graphs, this is often the case: nodes visited in the recent past are re-visited. As a consequence, the underestimation is especially evident in graphs with high clustering (the power-law, with a clustering factor one order of magnitude higher than the Erdős-Rényi) and low  $\beta$  (higher probability of local jumps). This problem is virtually absent in the Erdős-Rényi graph, in which another phenomenon dominates, leading to the overestimation: in the continuous time approximation, between time  $t$  and  $t + 1$  the theoretical function keeps “visiting” nodes, while the simulation does not.

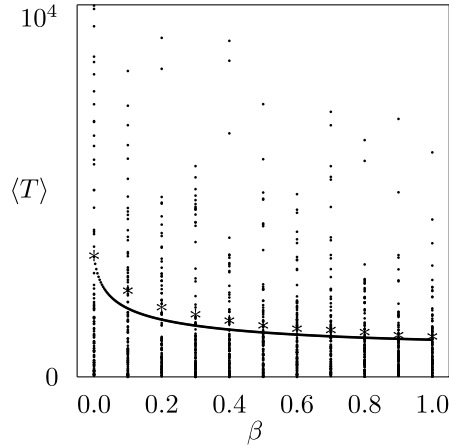
Figs. 3a and 3b show the average transit time for the power-law and the Erdős-Rényi graphs, respectively. The tests were carried out in such a way as to compute ensemble statistics. For each run, a graph and two random nodes were selected, the walk started from the first node and the time necessary to reach the second was recorded. The reason for this arrangement is that in the case of superdiffusive random walks on an infinite line, a break of ergodicity has been noted [26–28]. It is not clear whether the same applies to finite, but long, walks on finite graphs but, in order to avoid possible errors, the decision was made to use ensemble statistics.

The most noteworthy feature of Fig. 3 is the high variance for low  $\beta$  in the power-law graph. This variance is inherent to the walk: executing a series of walks between two fixed nodes in the same graph gives, with very high probability, a high variance (exceptions are mostly very special cases such as pairs of neighboring nodes, especially if part of a dense cluster). This high variance appears to be due to the presence of “outlier walks” that take a long time to go from one node to another. The scatter diagram of Fig. 4 clarifies the problem. Most walks go from the origin to the destination in a fairly short time, but a relatively small number of walks take a very long time, causing the high variance. The fact that the variance is higher in power-law graphs, in which the clustering coefficient is one order of magnitude higher than in the Erdős-Rényi graph suggests that these long-time walks might “get stuck” in one or more cluster. The variance is lower for high  $\beta$  since the long range random jumps allow the walk to get “unstuck” and leave the cluster. For the same reason, the value of  $\langle T \rangle$  for the Erdős-Rényi graph is virtually independent on  $\beta$ : given the flat distribution of the node degree ( $\gamma_E \sim 0$ ), local jumps have the same statistical characteristics as random jumps.





**Fig. 3.** The average time to transit between nodes,  $\langle T \rangle$ , as a function of  $\beta$ : simulation data vs. model predictions. In (a), the average for all the 50 power-law graphs, in (b), the average for all the 50 Erdős-Rényi graphs. See the text for the details of the calculation.



**Fig. 4.** The average transit time  $\langle T \rangle$  is shown in the form of a scatter plot for various values of  $\beta$ . The averages (indicated as “”) have been computed on 50 walks for each of the 50 graphs but, for the sake of clarity, the scatter plot contains only 200 points corresponding to 200 walks randomly chosen among those executed.

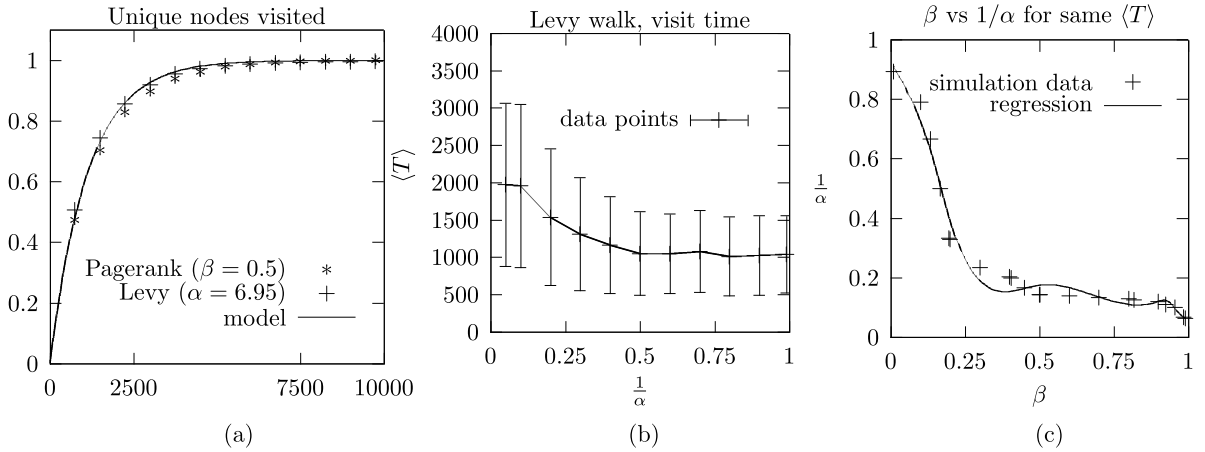
In order to obtain the model prediction, the eigenvalues of  $\mathbf{W}$  were computed using the iterative method in [29], modified with the application of Gram–Schmidt [30] to impose orthogonality between the eigenvector currently determined and all those found so far. If  $\lambda$  and  $v$  are the eigenvalue and the eigenvector that we are computing, the approximation error was determined as

$$\frac{\|\mathbf{W}v - \lambda v\|_\infty}{\|v\|_\infty} \quad (51)$$

The error of the first 100 eigenvalues was around 0.01, then increased until reaching around 0.2 in the last ones, especially for pairs of eigenvalues with very similar value. Note however that small eigenvalues have the least impact on the value of  $\langle T \rangle$  according to (50).

### 6.1. Comparison with Lévy walks

Lévy walks [18] are characterized by a transition probability  $p_{ij} \sim d_{ij}^{-\alpha}$ , where  $d_{ij}$  is the shortest path distance between nodes  $i$  and  $j$ , and  $\alpha > 0$  is the parameter that characterizes the walk. One of the contentions of the paper is that there is a correspondence  $\beta \leftrightarrow \alpha$  such that a Lévy walk with parameter  $\alpha$  behaves similarly, in terms of nodes visited and average transit time, to a PageRank walk with a suitable parameter  $\beta$ . We present some data to support this hypothesis. Fig. 5a shows the number of nodes visited for PageRank with  $\beta = 0.5$  and for a Lévy walk with  $\alpha = 6.95$  (the value that



**Fig. 5.** In (a), the number of unique node visits as a function of time for a PAGERank walk with  $\beta = 0.5$  and for a Lévy walk with the value of  $\alpha$  that gives a similar behavior ( $\alpha = 6.95$ ). In (b), the average time between nodes,  $\langle T \rangle$  for Lévy walks as a function of  $1/\alpha$ , this graph should be compared with Fig. 3. In (c), the relation between  $\beta$  and  $1/\alpha$  that gives the same value of  $\langle T \rangle$  for the two walks. The crosses are the points determined through simulation, the continuous line is a cubic spline regression with 6 nodes.

corresponds to  $\beta = 0.5$ ). Fig. 5b shows the value of  $\langle T \rangle$  as a function of  $1/\alpha$  (compare this with Fig. 3). Fig. 5c shows the relation between  $\beta$  and  $1/\alpha$  that causes the two types of walk to have the same value of  $\langle T \rangle$ .

These results show that PageRank and Lévy walks do behave similarly for certain measures (nodes visited and  $\langle T \rangle$ ) and suitable values of the parameters. Generalization of these results to other aspects of the walks should be done with extreme care. The arguments in [31] show that different types of random walk can coincide in crucial aspects (the PDF, in that case) and still be different in others. While [31] considers specific types of sub-diffusive walks and, therefore, its results are not directly applicable to our case, these considerations suggest extreme care in order to avoid unwarranted generalizations.

## 7. Conclusions

We have used a continuous approximation and spectral analysis to determine the diffusion characteristics of random walks with random jumps. We have derived closed-form time functions for quantities like the occupancy probability and the number of new nodes visited, as well as a closed for expression for the average time between nodes.

In this, we have found some parallel with Lévy Random walks, which also give raise to superdiffusion [17] and whose behavior exhibits similarities with that studied here in measures such as the number of unique nodes visited and the average transit time. This result is especially significant since the walks that we study here assume much less information about the graph: while Lévy walks assume that we know the shortest path distance between any pair of nodes, we have shown that similar characteristics can be obtained with just local information (viz., the neighbors of a node).

In addition, in our model we can express  $\langle T \rangle$  in closed form as a function of the walk parameter  $\beta$ , depending simply on the computation of the eigenvalues of  $\mathbf{W}$ . By contrast, the results of [18] require the computation of the eigenvalues of a matrix  $\mathbf{W}(\alpha)$  with  $w_{ij} \sim d_{ij}^{-\alpha}$ , which depend on the walk parameter and whose dependence cannot, in general, be expressed in closed form.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Appendix A. Random walks on graphs

Let  $G = (V, E)$  be an undirected graph with  $n$  nodes and  $m$  edges ( $V = \{1, \dots, n\}$ ,  $E \subseteq V \times V$ ,  $|E| = 2m$ ), with adjacency matrix  $\mathbf{A} = \mathbf{A}'$  ( $a_{ij} = 1$  if  $(i, j) \in E$ ,  $a_{ij} = 0$  otherwise), with  $a_{ii} = 0$  (we do not allow self-loops). Let  $d_i = \sum_j a_{ij}$  be the degree of node  $i$ , viz., the number of its neighbors, and  $\mathbf{D} = \text{diag}(d_1, \dots, d_n)$  the degree matrix of the graph.

In a random walk [13], the walker starts at time  $t = 0$  from a node  $v_0 \in V$  and, if at time  $t$  it is located in node  $v_t$ , at time  $t + 1$  it moves to one of its neighbors with probability  $1/d_{v_t}$ . Define the matrix  $\mathbf{W} = \mathbf{D}^{-1}\mathbf{A}$  with elements

$$w_{ij} = \frac{a_{ij}}{d_i} = \begin{cases} \frac{1}{d_i} & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases} \quad (52)$$

Then  $w_{ij}$  is the probability of moving from node  $i$  to node  $j$  in a single step. Let  $p_i(t) = \mathbb{P}[v_t = i]$  be the probability that the walker be in node  $i$  at time  $t$ . The evolution of this probability is given by the master equation

$$p_i(t+1) = \sum_{j:(i,j) \in E} \frac{1}{d_j} p_j(t) = \sum_{j=1}^n \frac{a_{ij}}{d_j} p_j(t) = \sum_{j=1}^n w_{ji} p_j(t) \quad (53)$$

If we collect the probabilities in a vector  $|p\rangle(t) = [p_1(t), \dots, p_n(t)]'$  then the master equation can be written as

$$|p\rangle(t+1) = \mathbf{W}'|p\rangle(t) \quad (54)$$

That is, if  $|p^0\rangle$  is the initial probability distribution of the walker

$$|p\rangle(t) = (\mathbf{W}')^t |p^0\rangle \quad (55)$$

It is easy to check that the probability distribution  $|\pi\rangle$ , with

$$\pi_i = \frac{d_i}{\sum_k d_k} = \frac{d_1}{2m} \quad (56)$$

satisfies

$$|\pi\rangle = \mathbf{W}'|\pi\rangle \quad (57)$$

and is therefore a stationary distribution of the walk. It is also possible to show that this distribution is unique [13]. From (57) it follows that  $|\pi\rangle$  is an eigenvector of  $\mathbf{W}'$  corresponding to  $\lambda_1 = 1$  (or, equivalently, a right eigenvector of  $\mathbf{W}$ :  $\langle\pi| = \langle\pi|\mathbf{W}$ ) and the unicity of the stationary distribution implies that the eigenvalue  $\lambda_1 = 1$  has multiplicity 1. Since the eigenvector  $|\pi\rangle$  has all positive components, it follows from the Frobenius–Perron theorem [32] that

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq -1 \quad (58)$$

It is possible to show that  $\lambda_n = -1$  only for bipartite graphs [21], a case that we do not consider, so we always assume  $\lambda_n > -1$ . If we are at a node  $i$  at time  $t$ , at time  $t+1$  we will certainly be in one of its neighbors, consequently

$$\sum_j w_{ij} = \sum_j \frac{a_{ij}}{d_i} = \frac{1}{d_i} \sum_{(i,j) \in E} 1 = 1 \quad (59)$$

If  $|1_n\rangle = [1, \dots, 1]'$ , it is easy to see from (59) that  $\mathbf{W}'|1_n\rangle = |1_n\rangle$ , or

$$\langle 1_n| = \langle 1_n|\mathbf{W}' \quad (60)$$

That is, the constant vector  $|1_n\rangle$  is the right eigenvector of  $\mathbf{W}'$  corresponding to  $\lambda_1 = 1$ .

## Appendix B. Properties of the eigenvectors

The right eigenvectors other than the first correspond to eigenvalues with  $|\lambda| < 1$ . These have a property that will be relevant in the paper:

**Lemma B.1.** *Let  $|v\rangle$  be a right eigenvector of  $\mathbf{W}'$  corresponding to an eigenvalue  $\lambda \neq 1$ . Then*

$$\sum_i v_i = 0 \quad (61)$$

**Proof.** The eigenvector equation gives

$$\sum_j w_{ji} v_j = \lambda v_i \quad (62)$$

Adding up the components, we have

$$\lambda \sum_i v_i = \sum_i \sum_j w_{ji} v_j = \sum_j \underbrace{\sum_i w_{ji}}_1 v_j = \sum_j v_j \quad (63)$$

where the sum over  $j$  in the third expression is equal to 1 because of (59). The equation

$$\lambda \sum_i v_i = \sum_j v_j \quad (64)$$

with  $\lambda \neq 1$  has (61) as the only solution  $\square$

The matrix  $\mathbf{W}'$  can be diagonalized as

$$\mathbf{W}' = \mathbf{T} \mathbf{\Lambda} \mathbf{T}^{-1} \quad (65)$$

Let  $\mathbf{T}$  and  $\mathbf{T}^{-1}$  be defined as

$$\mathbf{T} = [|\phi_1\rangle, \dots, |\phi_n\rangle] \quad (66)$$

and

$$\mathbf{T}^{-1} = \begin{bmatrix} \langle\psi_1| \\ \text{---} \\ \vdots \\ \text{---} \\ \langle\psi_n| \end{bmatrix} \quad (67)$$

Then  $|\phi_i\rangle$  are right eigenvectors and  $\langle\psi_i|$  left eigenvectors of  $\mathbf{W}'$ , that is

$$\begin{aligned} \mathbf{W}'|\phi_i\rangle &= \lambda_i|\phi_i\rangle \\ \langle\psi_i|\mathbf{W}' &= \lambda_i\langle\psi_i| \end{aligned} \quad (68)$$

Let  $t_{ij}$  be the  $i, j$  element of  $\mathbf{T}$  and  $\tau_{ij}$  that of  $\mathbf{T}^{-1}$ , then  $t_{ij} = \langle e_i|\phi_j\rangle = \phi_{j,i}$  (the  $i$ th component of  $|\phi_j\rangle$ ), and  $\tau_{ij} = \langle\psi_i|e_j\rangle = \psi_{i,j}$  (the  $j$ th component of  $\langle\psi_i|$ ). We have

$$\delta_{ij} = (\mathbf{T}^{-1}\mathbf{T})_{ij} = \sum_k \tau_{ik} t_{kj} = \sum_k \psi_{i,k} \phi_{j,k} = \langle\psi_i|\phi_j\rangle \quad (69)$$

and

$$\delta_{ij} = (\mathbf{T}\mathbf{T}^{-1})_{ij} = \sum_k t_{ik} \tau_{kj} = \sum_k \phi_{i,k} \psi_{j,k} \quad (70)$$

The matrix  $\mathbf{W}'$  is that of the standard random walk, and its only equilibrium point is the probability vector  $|\pi\rangle$ . This vector satisfies the equation  $\mathbf{W}'|\pi\rangle = |\pi\rangle$ , that is,  $\pi$  is an eigenvector relative to  $\lambda = 1$  and therefore  $\phi_1 \sim \pi$ , or

$$\phi_{1,i} = b d_i \quad (71)$$

with  $b > 0$ . From (60) we have

$$\langle 1_n | \mathbf{W}' = \langle 1_n | \quad (72)$$

that is,

$$\langle\psi_1| = a \langle 1_n | \quad (73)$$

with  $a > 0$ . One of the constants  $a$  and  $b$  can be determined by condition (69) (the other one is arbitrary):

$$1 = \langle\psi_1|\phi\rangle = a \cdot b \cdot \sum_i d_i \quad (74)$$

We can choose  $b = 1/\sum d_i$ , leading to

$$\phi_{1,i} = \frac{d_i}{\sum_k d_k} = \pi_i \quad (75)$$

and  $a = 1$ , that is

$$\langle\psi_1| = \langle 1_n |. \quad (76)$$

Note that

$$\begin{aligned} \sum_k \phi_{1,k} &= 1 \\ \sum_k \psi_{1,k} &= n \end{aligned} \quad (77)$$

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