Physics and Geometry of Gravity at High Energies

Memoria de la Tesis Doctoral realizada por

Ángel Jesús Murcia Gil

y presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para optar al Título de Doctor en Física Teórica

Tesis Doctoral dirigida por el **Prof. D. Tomás Ortín Miguel**[1]
y por el **Dr. D. Carlos Shahin Shahbazi Alonso**[2]

[1]Profesor de Investigación del Instituto de Física Teórica UAM-CSIC
[2]Investigador María Zambrano en la Universidad Nacional de Educación a Distancia

Departamento de Física Teórica
Universidad Autónoma de Madrid

Instituto de Física Teórica UAM-CSIC

Madrid, junio de 2022

Si volviera a nacer, si empezara de nuevo... Volvería a buscarte en mi nave del tiempo.
Eva Amaral y Juan Aguirre

Agradecimientos

Se agolpa en mi corazón una inefable sensación de alegría y melancolía al escribir estas palabras. Han sido cuatro años maravillosos, llenos de felices sorpresas, nuevas experiencias y, en especial, de aprendizaje. Poco queda de aquel timorato y bisoño chaval que llegó a Madrid sin saber qué depararía el futuro. Echando la vista atrás, no puedo hacer más que sonreír y sentirme privilegiado por todo lo que la vida ha tenido a bien regalarme: mi familia, mis amigos, mi vocación.

En primer lugar, me gustaría mentar a mis directores de tesis Tomás Ortín y Carlos Shahbazi. Querría darle las gracias a Tomás por la confianza depositada en mí al acogerme como estudiante de doctorado y por involucrarme en el grupo de investigación. Gracias a ello, he aprendido a colaborar científicamente con compañeros y amigos, a darme cuenta de que la física, y en definitiva la ciencia, no es más que una colaboración mundial entre toda la humanidad presente, pasada y futura para entender qué es y cómo funciona el mundo que nos rodea. Asimismo, quisiera agradecerle todos los buenos consejos que me ha facilitado a lo largo de todo el doctorado. En ocasiones, es difícil vislumbrar el camino a escoger y aprovecho estas letras para reconocer su desinteresada ayuda.

Por otro lado, me siento extremadamente afortunado por haber tenido también a Carlos como director de tesis. Además de aprender mucha física y geometría con él, ha estado siempre pendiente de mí, se ha preocupado e interesado por mí y ha procurado, con enorme afán y empeño, en que esta tesis saliera adelante. Ha sido un magnífico guía durante toda la travesía doctoral y confío poder seguir aprendiendo y trabajando con él en el futuro. Me ha transmitido esa pasión innata por el conocimiento, por el escudriñar y entender al detalle la materia bajo escrutinio. Hacer ciencia, al fin y al cabo, es replantearse lo que creemos saber, ponerlo en cuestión y analizarlo con rigor y método. Querría darle las gracias por revelarme esta gran verdad.

En estas líneas, quisiera también mencionar a Pablo Cano. Pablo, has sido un referente, un hermano mayor, un ejemplo y, de lo que más orgulloso estoy, un amigo. Más allá del brillante horizonte científico que te aguarda en un futuro muy próximo, no me considero merecedor de la extraordinaria suerte y la excepcional oportunidad que me ha supuesto poder trabajar contigo. Siempre recordaré con cariño nuestro primer artículo sobre las Gravedades Cuasitopológicas Generalizadas, de cómo incorporaste a un incipiente estudiante de doctorado que sentía que no terminaba de encajar. Espero haber correspondido a tus expectativas, que hayas disfrutado tanto como yo del resto de nuestros trabajos. Muchísimas gracias y deseo de corazón poder continuar nuestra investigación juntos.

Andererseits möchte ich Vicente Cortés für seine Hilfe und Liebenswürdigkeit danken. Ich freue mich sehr darüber, dass Sie mich in Ihre Arbeitsgruppe aufgenommen haben. Sie waren nicht nur der Betreuer meines wissenschaftlichen Aufenthalts in Hamburg, sondern haben sich auch immer um mich gekümmert. Außerdem habe ich viel von Ihnen gelernt: Mathematik, die Liebe zur Wissenschaft, die Leidenschaft zur Arbeit und fleißig zu sein. Sie waren immer geduldig mit mir and haben mir viele mathematische Begriff erklärt. Ich werde es für immer schätzen. En definitiva, quisiera agradecer a Vicente por su implicación conmigo, su confianza y su preocupación por mi estancia en Hamburgo y mi futuro científico. Es un científico magnífico, mejor persona y espero poder seguir aprendiendo y trabajando a su lado.

Para terminar la parte científica, también quisiera mencionar a todas aquellas personas que he conocido en el IFT o con relación directa con el doctorado. A mis hermanos mayores Pablo Bueno, Pedro F. Ramírez, Óscar Lasso, Alejandro Ruipérez, a los que les agradezco su implicación conmigo, su amistad y por haberme orientado en este intrincado y abstruso mundo académico. A José Juan Fernández-Melgarejo y a Miguel Montero, por su desinteresado afán en compartir sus conocimientos. A mis compañeros de despacho David Pereñíguez, Zach Elgood y Matteo Zatti por aguantarme como inquilino. A todos aquellos científicos con los que he tenido la gran suerte de trabajar: además de los ya mencionados, Javier Moreno, Andrei Moroianu, Alberto Rivadulla Sánchez y Xuao Zhang. En definitiva, a todos los amigos que el doctorado me ha permitido conocer: Francisco Arco, Juanra Balaguer, Romina Ballesteros, Thomas Biekötter, Ginevra Buratti, José Calderón, Alberto Castellano, Mattia Cèsaro, Jorge Dasilva, Matilda Delgado, Andrea Fontanella, Alejandro Gil, Eduardo Gonzalo, Álvaro Herráez, Gabriel Larios, Lucía Martín, Alessandro Minnino, Olalla Olea, Luca Romano, Arpan Saha, Raquel Santo, Martín Sasieta y Francisco Tello-Ortiz.

En lo que respecta al plano personal, me siento verdaderamente feliz de poder dedicar una parte de esta tesis a las personas que más quiero. Empiezo hablando de mis padres, Erundina y Ángel. Sois unas personas maravillosas y nunca podré agradeceros lo suficiente por vuestro cariño y preocupación por mí. Porque siempre habéis velado por mí. Porque siempre habéis estado ahí cuando lo necesitaba. Porque siempre me habéis ayudado con mis problemas del día a día. Sois un ejemplo de honestidad, bondad y sencillez, y sé que todo las buenas virtudes que yo pueda tener os las debo enteramente a vosotros. Mamá, papá, muchas gracias por todo. Os quiero mucho.

Asimismo, me considero realmente afortunado de haber contado con el apoyo del resto de mi familia. Quiero mencionar a mi tía María Jesús, a mis primas María y Lourdes y a mi tío José Luis. Quiero agradeceros de corazón vuestra ayuda, vuestros consejos y el tiempo en familia, en especial durante el largo confinamiento. Igualmente, le quiero dar las gracias a mi abuelita Paquita, que desde el cielo me cuida y me guía en mi vida.

Por otra parte, hasta no hace mucho, pensaba que no era compatible la calidad con la cantidad en la amistad. No obstante, estos años he tenido la suerte de descubrir cuán equivocado estaba. Amigos, muchas gracias por escucharme, por ayudarme, por los buenos (y divertidos, claro) ratos juntos. Sois los mejores y unas personas increíbles. ¡Qué bien me lo he pasado con vosotros!

Trataré de ser breve en mis agradecimientos (por orden alfabético). A Estrella Aínsa, por ser *mon étoile préférée* y mi mejor mentora de francés. A Tamara Benavides, por reservame siempre un hueco de su ajetreada agenda. A Ignacio Berdún, por ser una gran persona y de más confianza que IBM. A Javier Berdún, por enseñarme que los lunes son los nuevos viernes. A Andrea Bernal, por mostrarme lo bonita que es la Luna. A Javier Díez, mi abogado de preferencia para asuntos interestelares. A Elena Escario, por ser la mejor vecina que he tenido. A Marta Evangelista, por todas las cenas y fiestas juntos. A Elena de Francia, por su paciencia conmigo cada vez que decía *de France* (póngase acento parisino). A Claudia Giménez, mi geógrafa favorita. Ad Andrea Giovannoni, mio fratello italiano, per i suoi sempre saggi consigli e formare un duo impareggiabile ed inseparabile. A Teresa Güemes, por ser una amiga más dulce que el turrón. A Teresa Hernández, por ser una granadina tan estupenda y diplomática conmigo. A Pablo Idigoras, por mostrarme cómo es un vasco de los pies a la cabeza. A Nacho Márquez, por explicarme el espíritu del camero. A Elena Martín, por ser la mejor periodista deportiva que he conocido. A Mireya Moreno, por las noches en el Irish cantando a C Tangana (*un milagro como tú... ha tenido que caer del cielo*). A David Mota, el meu germà català, a qui estime molt i trobe molt a faltar (has de tornar a Madrid, tot i que hajam guanyat la 14!). A Mar Navarro, por ser una guía de Albacete tan buena. A Eduardo Naya, por encargarse de que todo esté siempre en orden en la segunda planta. To Penelope Piraino, for her *pennyfulness* and for always caring about me. A Joan Quirant, por los buenos momentos en Madrid y Guardamar. A Alberto Rivadulla, por poder contar con él en el *Iberian* y en Florencia. A Javier del Rosario, mi hermano canario, por aceptarme como amigo a pesar de ser *fleje godo* y acoger el maravilloso imán de Guardamar en su casa (¡Viva Guía!). A Paquito Sánchez-Porro, por ser el mejor guitarrista que jamás he conocido. A Silvia Traspuesto, por ser una cántabra tan simpática (aunque colchonera). A María Trujillo, por enseñarme lo maravilloso que es Bueu. Ich möchte auch Vanessa Vornbäumen dafür danken, so eine gute Freundin zu sein (du bist jederzeit nach Spanien eingeladen!). A Cristina Tabernero, por ser la mejor relaciones públicas de La Pocha. A Paz Vargas, por aquellas meriendas dominicales tan ricas (¡qué dicha!). A Rebeca Vicente, por enseñarme a bailar tan bien. A Celia Viejo, por esos desayunos con Tosta Rica. A Rafa Zamorano, por ser el físico más moderno y *hipster* que conozco.

También quisiera mencionar a otros muchos amigos con los que también he podido contar durante estos cuatro años: Rodrigo Adam, Pilar Alcaide, Rafa Aldeguer, David Aniorte, Teresa Babío, Víctor Bas, Sara Bustillo, Claudia Cerdán, José Antonio Domínguez, María Domínguez, Julia Fernández, Cristina Ferrer, Estrella Fuentes, Blanca García, Sofía de Gea, Teresa González, Josep Maria Gómez, Mario Gómez, Itziar Jara, Adrián Martínez, Ada Martínez, Fernando Merino, Pedro Merino, Ana Moreno, Alberto Moro, Teresa Olano, Miguel Pinilla, Xisca Pons, Pilar Román, Pedro Schwartz, Joan Just Senent, Jesús Senent, Borja Sierra, María Urra, Idoia Urra, Sergio Valero, Tamara Vega, Alessandra Vigotti...entre muchos otros que guardaré para siempre en mi corazón.

Finalmente, termino dando las gracias al Gobierno de España y a su ciudadanía en su conjunto, por la confianza depositada en mí a través de la beca FPU17/04964 durante los cuatro años de doctorado. Ich möchte auch dem DAAD und der Deutschen Regierung für die finanzielle Unterstützung während meines wissenschaftlichen Aufenthalts in Hamburg danken.

Me depara otro país, otra ciudad y otra vida. Solo espero que sea tan bonita como esta.

Madrid, Hamburgo y Guardamar, octubre 2018 - junio 2022

Declaration and List of Publications

The thesis is the result of four years of work. During this time, various themes and topics have been explored. The results presented here are based on the following articles (in chronological order):

[1] **All higher-curvature gravities as Generalized quasi-topological gravities**
P. Bueno, P. A. Cano, J. Moreno and Á. Murcia
JHEP **11** (2019) 062 and `arXiv:1906.00987` [hep-th].
Its contents appear in Chapter 1.

[2] **Contact metric three manifolds and Lorentzian geometry with torsion in six-dimensional supergravity**
Á. Murcia and C. S. Shahbazi
J. Geom. Phys. **158** (2020) 103868 and `arXiv:1912.08723` [math.DG].
Its contents appear in Chapter 7.

[3] **Resolution of Reissner-Nordström singularities by higher-derivative corrections**
P. A. Cano and Á. Murcia
Class. Quant. Grav. **38** (2021) 075014 and `arXiv:2006.15149` [hep-th].
Its contents appear in Chapter 2.

[4] **Electromagnetic Quasitopological Gravities**
P. A. Cano and Á. Murcia
JHEP **10** (2020) 125 and `arXiv:2007.04331` [hep-th].
Its contents appear in Chapter 2.

[5] **Parallel spinors on globally hyperbolic Lorentzian four-manifolds**
Á. Murcia and C. S. Shahbazi
Annals of Global Analysis and Geometry **61** (2022) 253 and `arXiv:2011.02423` [math.DG].
Its contents appear in Chapter 5.

[6] **Duality-invariant extensions of Einstein-Maxwell theory**
P. A. Cano and Á. Murcia
JHEP **08** (2021) 042 and `arXiv:2104.07674` [hep-th].
Its contents appear in Chapter 3.

[7] **Exact electromagnetic duality with nonminimal couplings**
P. A. Cano and Á. Murcia
JHEP **08** (2021) 042 and `arXiv:2105.09868` [hep-th].
Its contents appear in Chapter 3.

[8] **Parallel spinor flows on three-dimensional Cauchy hypersurfaces**
Á. Murcia and C. S. Shahbazi
`arXiv:2109.13906` [math.DG].
Its contents appear in Chapter 5.

[9] **Heisenberg-invariant self-dual Einstein manifolds**
V. Cortés and Á. Murcia
`arXiv:2111.11361` [math.DG].
Its contents appear in Chapter 6.

[10] **Higher-derivative holography with a chemical potential**
P. A. Cano, Á. Murcia, A. Rivadulla Sánchez and X. Zhang
Accepted for publication in JHEP, `arXiv:2202.10473` [hep-th].
Its contents appear in Chapter 4.

[11] **A universal feature of charged entanglement entropy**
P. Bueno, P. A. Cano, Á. Murcia and A. Rivadulla Sánchez
`arXiv:2203.04325` [hep-th].
Its contents appear in Chapter 4.

In addition, the following articles, unrelated to the contents of this thesis, were carried out by the candidate during the realization of this work:

[12] **Heterotic solitons on four-manifolds**
A. Moroianu, Á. Murcia and C. S. Shahbazi
`arXiv:2101.10309` [math.DG].

[13] **On small black holes, KK monopoles and solitonic 5-branes**
P. A. Cano, Á. Murcia, Pedro F. Ramírez and Alejandro Ruipérez
JHEP **05** (2021) 272 and `arXiv:2102.04476` [hep-th].

Finally, the following contribution to the proceedings of a conference was also written:

[14] **ε-contact structures and six-dimensional supergravity**
Á. Murcia
Accepted for publication in the Proceedings of the *X International Meeting on Lorentzian Geometry* (2021), `arXiv:2106.07602` [math.DG].

Abstract

In this thesis we study physical and geometric aspects of gravity at high energies. On the one hand, we carry out a detailed investigation of gravitational physics in this regime with the aid of higher-order gravities. These are extensions of General Relativity including terms in higher derivatives of the metric and other fields which, in addition to an effective field theory interpretation, possess as well an intrinsic interest by themselves. More concretely, we focus on higher-order gravities of the *(Generalized) Quasitopological* class, defined as those admitting static and spherically symmetric solutions characterized by a single function satisfying an equation of, at most, second order. The motivation for this is twofold: they are amenable to computations and, at the same time, are generic enough so as to capture effects and phenomena introduced by higher-order corrections, which one may use to learn properties of a putative theory of Quantum Gravity.

First, we restrict ourselves to theories of pure gravity and show that all higher-order gravities can be mapped, via (perturbative) field redefinitions, to a Generalized Quasitopological Gravity. Secondly, we consider the addition of a U(1) gauge vector field and identify infinite families of *Electromagnetic (Generalized) Quasitopological Gravities* (E(G)QGs). We establish several intriguing properties of these theories and explore their charged, static and spherically symmetric solutions. In particular, we prove that a subset of EQGs allows for completely regular electrically-charged black holes for arbitrary mass and non-vanishing charge. Next, we move to the analysis of higher-derivative extensions of Einstein-Maxwell theory which are duality-invariant. We classify all such theories up to eight derivatives and find that, up to the six-derivative level, they all can be mapped via field redefinitions to a higher-curvature gravity with a minimally coupled vector. Also, we are able to classify all exactly duality-invariant theories which are quadratic in the Maxwell field strength. Afterwards, we revisit EQGs and examine some of their holographic aspects. We manage to obtain fully analytic and non-perturbative results that motivate us to discover a new universal result valid for all $d(\geq 3)$-dimensional CFTs, which we rigorously prove.

On the other hand, we study geometric properties of gravity at high energies. We choose Supergravity and String Theory as the scenarios in which to test such properties and we inspect distinct topics on the subject, in an attempt to form a global picture of the type of geometric structures we may encounter in this context.

We start by exploring real parallel spinors on globally hyperbolic four-manifolds. We are able to reformulate the problem in terms of a system of differential equations for a family of functions and coframes on a Cauchy surface that we call the *parallel spinor flow*. Remarkably, we find that the parallel spinor and the Einstein flows coincide on common initial data, thus providing an initial data characterization of a real parallel spinor on a Ricci flat globally hyperbolic four-manifold. Then, we investigate self-dual Einstein four-manifolds admitting a principal and isometric action of the three-dimensional Heisenberg group with non-degenerate orbits and manage to classify all of them. Finally, we introduce ε-contact structures, which encompass the usual notions of (three-dimensional) contact Riemannian, contact Lorentzian and para-contact metric structures, but also allow for a lightlike Reeb vector field. We show explicitly how they can be used for the construction of solutions of six-dimensional Supergravity.

Resumen

En esta tesis se estudian aspectos físicos y geométricos de la gravedad a altas energías. Por un lado, se realiza una investigación detallada de la física gravitacional en este régimen con la ayuda de las gravedades de orden superior. Son extensiones de la Relatividad General que incluyen términos en derivadas superiores de la métrica y otros campos que, además de una interpretación como teorías efectivas, también poseen una relevancia intrínseca. En concreto, nos concentramos en gravedades de orden superior de la clase *Cuasitopológica (Generalizada)*. Se definen como aquellas que admiten soluciones estáticas y esféricamente simétricas caracterizadas por una sola función que cumple una ecuación de segundo orden o inferior en derivadas. La motivación para el estudio de dichas teorías es doble: permiten cálculos explícitos y son lo suficientemente genéricas como para capturar efectos de los términos de orden superior, quizá propios de la teoría subyacente de Gravedad Cuántica.

Primero, nos restringimos a teorías de gravedad pura y demostramos que todas las gravedades de orden superior se pueden escribir, mediante redefiniciones de campo perturbativas, como Gravedades Cuasitopológicas Generalizadas. En segundo lugar, añadimos un vector gauge U(1) e identificamos familias infinitas de Gravedades Electromagnéticas Cuasitopológicas (Generalizadas) (GEC(G)s). Establecemos diversas propiedades interesantes de estas teorías y exploramos las soluciones estáticas y esféricamente simétricas con carga. En particular, probamos que un subconjunto de las GECs admite agujeros negros con carga eléctrica completamente regulares para cualquier masa y carga no nula. A continuación, analizamos extensiones con derivadas superiores de la teoría de Einstein-Maxwell que son invariantes bajo dualidad. Clasificamos dichas teorías hasta octavo orden en derivadas y observamos que, hasta sexto orden, todas ellas se pueden reformular como una gravedad de orden superior con un vector acoplado mínimamente. Asimismo, somos capaces de clasificar todas las teorías exactamente invariantes bajo dualidad cuadráticas en el campo de Maxwell. Después, volvemos a las GECs y examinamos aspectos holográficos de las mismas. Logramos obtener resultados completamente analíticos y no perturbativos que nos ayudan a descubrir un nuevo resultado universal válido para todas las Teorías Conformes de Campos en dimensiones $d \geq 3$, que demostramos rigurosamente.

Por otro lado, estudiamos propiedades geométricas de la gravedad a altas energías. Elegimos Supergravedad y Teoría de Cuerdas como los escenarios en los que verificar estas propiedades y analizamos varias líneas de investigación, tratando de obtener una panorámica general del tipo de estructuras geométricas que podemos hallar en este contexto.

Nos centramos primero en espinores reales paralelos en cuatro-variedades globalmente hiperbólicas, reformulando el problema en términos de un sistema de ecuaciones diferenciales para una familia de funciones y bases ortonormales del espacio cotangente en una superficie de Cauchy que denominamos flujo de espinor paralelo. Observamos que el flujo de Einstein y el de espinor paralelo coinciden en datos iniciales comunes, lo cual nos proporciona una caracterización de los datos iniciales de un espinor real paralelo en una cuatro-variedad globalmente hiperbólica Ricci plana. Luego, reorientamos nuestras pesquisas hacia las cuatro-variedades autoduales Einstein que admiten una acción isométrica y principal del grupo de Heisenberg tres-dimensional con órbitas no-degeneradas, logrando clasificar todas ellas. Finalmente, introducimos las estructuras de ε-contacto, que además de englobar las nociones habituales de estructuras riemannianas y lorentzianas de contacto así como las estructuras métricas de para-contacto, incluyen la posibilidad de un vector de Reeb de tipo luz. Concluimos mostrando cómo se pueden emplear estas estructuras para la construcción de soluciones de Supergravedad en seis dimensiones.

Contents

I
Introduction

The current Standard Model (SM) of physics [15–18] is one of the greatest achievements of humanity. It has involved the most brilliant thinkers and scientists of History, from Ancient Greece to the today's world-wide collaborations. It explains the four interactions that we have identified so far: the electromagnetic force, the weak interaction, the strong force and gravity. Together with these four interactions, a suitable framework is needed to depict and understand the subsequent behaviour of Nature. Before the 20th century, such setup was provided by Classical Mechanics, which is extremely successful in describing macroscopic physics. Nevertheless, the observation of certain phenomena (photoelectric effect, black body radiation, absorption and emission spectra...) which could not be explained in the framework of Classical Mechanics triggered the discovery of Quantum Mechanics (QM).

The embedding of a special-relativistic Classical Field Theory into the quantum realm produces a Quantum Field Theory (QFT), which currently provides the most accurate and appropriate framework to describe Nature. It allows for a remarkably beautiful interpretation — matter is made up of particles (fermions) and interactions are a result of the exchange of particles (exchange bosons), both of them being continuously created and annihilated within an enormously dynamical and intricate vacuum.

In this context, it turns out that electromagnetism, the weak and the strong interactions fit into a QFT description. Nevertheless, gravity does not. In fact, the SM assumes that gravity is governed by Einstein's General Relativity (GR) and this theory cannot be properly *quantized* because it is not renormalizable [19–24], thus remaining in the realm of Classical Physics. Consequently, accepting that Nature respects the postulates of QM and QFT, this poses at the very least a quite substantial conceptual problem: why cannot GR be consistently quantized? What are we missing in our understanding of physics? These questions are relevant, since there are experiments and observations which showcase tensions with the SM.

On the one hand, many of these puzzles originate from the ΛCDM model for cosmology, which is the most robust description of the Universe at our disposal. The theoretical framework is that of GR together with the Cosmological Principle, which postulates the spatial sections of spacetime to be homogeneous and isotropic and large scales. However, it also possesses two key mysterious ingredients, whose existence is strongly supported by the investigation of the Cosmic Microwave Background (together with other observations): dark energy, considered to be responsible for the current accelerated expansion of the Universe, and dark matter, introduced to explain the velocity profile within galaxies. Unveiling their fundamental physical origin is one of the most longed-for objectives in theoretical physics, although diverse tantalizing ideas to describe their fundamental nature have already been put forward in the last years. Regarding dark matter, there have been several interesting proposals for its origin, such as weakly interacting massive particles [25, 26], ultra-light particles [27] or the fairly intriguing possibility of primordial black holes [28–32]. In the case of dark energy, apart from simply defining it as a (perfect) fluid with negative pressure[1], the covariant action or the Effective Field Theory approaches can be satisfactorily followed [32, 34, 35]. Nevertheless, it would also be quite elegant if a potential theory of Quantum Gravity provided a framework in which both dark energy and dark matter appeared naturally or which rendered their introduction unnecessary.

On the other hand, a plethora of theoretical challenges are posed as well by the most elegant and fascinating objects of the Universe, already predicted by GR: black holes. Leaving their precise definition for Section I.6, they are characterized by the existence of a region of spacetime from which even light cannot escape. The boundary of such a region is called the *event horizon*. Under physically reasonable conditions, Hawking and Penrose [36] discovered that the formation (and presence) of black holes is intimately connected to the existence of singularities which are hidden behind the black hole's event horizon. Again, such singularities are warning us that GR is breaking down and that a more sophisticated understanding of gravity, which would cure this unwanted behaviour, is needed. Also, the discovery of black hole thermodynamics [37, 38] and, more concretely, of black hole entropy [39–41], triggered very interesting questions. What does this entropy account for? What is the microscopic structure behind the event horizon of a black hole that generates such an entropy? In a similar fashion, it was shown [38] that black holes radiate, so that they have a definite temperature and could even evaporate and disappear. However, after a semiclassical study one would conclude that there is a loss of information through this process, what is known as the *information paradox* [42–44]. How can we reconcile black hole evaporation with *unitarity* (no information loss)?

The previous reasoning justifies the necessity of a theory of (Quantum) Gravity which answers some (if not all) of the questions posed above. Unfortunately, such a theory has not yet been found, although fairly promising candidates have been proposed. Among them, perhaps the most remarkable one is String Theory (see Section I.2 for a brief review), but there exist also other promising possibilities, such as Loop Quantum Gravity [45–48]. In any case, one could adopt an agnostic point of view about the UV completion to Quantum Gravity and try to examine and characterize generic effects to be expected from an improved description of gravity. This can be done through the use of *higher-order theories of gravity*, which we introduce in Section I.1.

In another vein, GR postulates that gravity is encoded in the spacetime curvature, and thus establishes a natural correspondence between geometry and gravity. Consequently, the study of geometry is crucial for its proper understanding. Given this fundamental connection, we expect quantum corrections not to break it, but rather to modify it by deforming or twisting the way gravity affects geometry and vice versa. Capturing the new geometric structures and properties appearing in potential candidates for quantum UV completions, such as String Theory, is essential to discover new features of gravity arising from quantum effects.

In this thesis we intend to contribute to the discovery of the quantum theory of gravity by exploring the physics and geometry of gravity at high-energies. More concretely, we will try to shed light on some of the theoretical puzzles explained above, in particular

[1] Interestingly enough, measurements are in agreement with dark energy coming effectively from a cosmological constant term [33], which corresponds to the equation of state $p = w\rho$ with $w = -1$, p and ρ being the pressure and the density of the fluid, respectively.

those concerning the mysterious and extraordinary physical properties of black holes. In this spirit, we have divided the thesis into two parts. In the First Part, we will concentrate on the study of higher-order theories of gravity, examining purely gravitational theories, non-minimal couplings to electromagnetism and analyzing holographic properties of these latter theories. In the Second Part we will study geometries that can be found in String Theory and Supergravity, such as those arising from the existence of parallel spinors, the intertwining of contact structures or from certain configurations in the context of scalar manifolds.

We provide in this introduction all the necessary preparatory material which we believe is needed for the understanding of the contents of this thesis. We begin by defining higher-order gravities in Section I.1. Next, a very brief review of String Theory is given, emphasizing certain physical and mathematical aspects of interest for the thesis. Later, some celebrated examples of higher-order gravities are provided. Afterwards, we say some words about duality rotations within theories of gravity and electromagnetism. Then, we present very succinctly the initial value problem in GR, which is followed by a concise introduction to black holes. This is continued by a short exposition about the role of higher-order gravities in holography. Finally, we study the (mathematical) definition of spinors and conclude with a summary of the main results of the thesis.

I.1 Introduction to higher-order theories of gravity

Einstein's General Relativity [49–52] conceives gravity as spacetime curvature. In particular, it postulates that the Universe can be modeled as a Lorentzian (four-dimensional) manifold (M, g) whose dynamics is encapsulated in a metric g playing the role of the gravitational field. The corresponding equations of motion are derived through the extremization of the Einstein-Hilbert (EH) action [53]

$$I_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}R\,, \tag{I.1}$$

where G is the Newton constant and R is the Ricci scalar associated to the Levi-Civita connection of g. The equations of motion state that (M, g) has to be Ricci flat, which in some local coordinates may be expressed as

$$R_{\mu\nu} = 0\,. \tag{I.2}$$

Despite their apparent simplicity, these equations pose a highly intricate and non-linear system of partial differential equations whose exact resolution is usually an inaccessible problem. Requiring the existence of symmetries allows one to obtain explicit solutions, many of them in the form of black holes (such as the Schwarzschild or Kerr solutions), which we will deal with afterwards.

GR arose as a result of the need of a more sophisticated theory which could solve some problems of Newton's theory of gravity, since the latter was not compatible with Special Relativity and had tensions with experimental observations — in particular, with that of the advance of the perihelion of Mercury[2]. Nevertheless, given the great accuracy

[2]This discrepancy was initially discovered by Le Verrier in the 19th century, who claimed it was due to the existence of a planet between Mercury and the Sun. Nonetheless, such an object was never observed.

and success of Newton's theory for the description of Solar System mechanics, it should be recovered when the gravitational field is sufficiently weak.

A quite interesting regime is given by gravitational fields which are strong enough to surpass the range of validity of the Newtonian theory, but weak enough so that a full-fledged use of GR is not required, sufficing to add to the Newton's Universal law of Gravitation some *corrections* (coming from GR). This formalism has already been developed in the literature and is called the Post-Newtonian expansion [54,55]. Within this regime of gravity, Newton's law for the motion of a test particle in the gravitational field of a massive body of mass M is modified [56] as follows[3]:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{GM}{r^2}\left\{-\hat{n} + \frac{1}{c^2}\left(\left\{4\frac{GM}{r} - v^2\right\}\hat{n} + 4\frac{\mathrm{d}r}{\mathrm{d}t}\vec{v}\right)\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{I.3}$$

where $\vec{v}$ is the velocity of the test particle, $\hat{n}$ the unit vector going from the massive body to the test particle, r the distance separating them and t the associated time coordinate. We observe that Newton's law gets corrected through velocity-dependent terms and higher powers of $\frac{1}{r}$, whose precise coefficients we may determine precisely since we know which is the putative theory that corrects Newtonian gravity — GR.

However, assume we did not know anything about GR and we were just aware of Newton's gravitational law and the necessity of finding an improved theory which solves the experimental discrepancy with the advance of the perihelion of Mercury. Then, as a first attempt to parametrize and capture effects of the correct theory replacing Newton's Gravity, we could think of just adding higher-order corrections to Newton's law as:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{GM}{r^2}\left\{-\hat{n} + \frac{1}{c^2}\left(\left\{\alpha_1\frac{GM}{r} + \alpha_2 v^2\right\}\hat{n} + \alpha_3\frac{\mathrm{d}r}{\mathrm{d}t}\vec{v}\right)\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{I.4}$$

where α_1, α_2 and α_3 would be *a priori* unknown constants to be fixed experimentally or by the parent theory of gravity from which such corrections can be derived (looking at (I.3), $\alpha_1 = \alpha_3 = 4$ and $\alpha_2 = -1$).

Now let us try to make some parallelism with the current situation between General Relativity and the potential theory of Quantum Gravity. First, exactly as with Newtonian gravity, GR has overcome a fairly large number of experimental tests, which include (apart from the advance of perihelion of Mercury [51]) light deflection [57, 58], Shapiro time delay [59, 60], the very recent detection of gravitational waves coming from black hole and neutron star binaries mergers [61–67] or the confirmation for the existence of black holes given by the first-ever images of the shadow of the supermassive black holes located at the center of the galaxy M87 [68,69] and the Milky Way [70,71], obtained by the Event Horizon Telescope collaboration. Secondly, in analogy with the situation between Newton's theory and Special Relativity, GR is not compatible with one of essential and fundamental pillars of physics —Quantum Mechanics, since the EH action (I.1) is not renormalizable. Thirdly, in the same way that the most well-founded suspicions with Newton's gravity arose from the advance of the perihelion of Mercury, the strongest issue with GR comes from the existence of singularities, where spacetime itself would break down. We summarize all these points in Table I.1.

[3]Here we have not set $c = 1$, since the Post-Newtonian expansion is in fact based on assuming a weak-field approximation and expanding around powers of $1/c^2 << 1$. However, unless otherwise stated, we will use units such that $c = 1$ all along the document.

	Newton's gravity	**General Relativity**
Successes	Earth gravity, Solar System Mechanics	Light deflection, black holes, gravitational waves
Incompatible with	Special Relativity	Quantum Mechanics
Major Problem	Advance perihelion of Mercury	Singularities
Replaced by	General Relativity	Quantum Gravity

Table I.1: Analogies between the status of Newton's Gravity in the beginning of the 20th century and the current situation with General Relativity.

Consequently, nowadays the situation with gravity is surprisingly similar to that existing just before the advent of GR: a theory which describes extremely well the physics we are observing in the Universe is known (GR), but, still, it is widely accepted that it has to be replaced by a more advanced theory (Quantum Gravity). However, the most challenging and exciting aspect of this problem is that such a theory of Quantum Gravity remains to be discovered[4]. And it is precisely at this moment, with the help of the gravitational wave detectors LIGO/VIRGO, the future space-based interferometer LISA [72] and the Event Horizon Telescope collaboration, when we are at the verge of testing GR with far more precision than ever. Therefore, it is mandatory to be ready for any possible deviation with respect to GR predictions that may be measured in the forthcoming years[5].

In case they are observed, such deviations will take place in a regime of gravity in which the gravitational field is strong enough to overpass the validity range of GR, but not sufficiently strong so as to require a complete Quantum Gravity description (see Figure I.1). Therefore, we could hope to study those phenomena by adding suitable *corrections*[6] to the EH action (I.1). But, since we do not know Quantum Gravity, which corrections should be added?

At this point, we may try to adopt a philosophy similar to that of (I.4): we can add to (I.1) corrections controlled by unknown couplings (coefficients) and then study which new phenomena and effects such terms introduce. This approach can be further justified from an Effective Field Theory (EFT) perspective [15, 73, 74], which requires the introduction of all possible terms compatible with the symmetries of the theory under consideration, each of them weighted by a characteristic length scale. They can be thought of as low-energy approximations to a more fundamental theory which may not even be described

[4]Assuming that this theory is indeed String Theory, at the very least it needs to be much better understood.

[5]With today's technology we are not yet able to probe ranges of energies in which Quantum Gravity dominates. Nonetheless, nowadays we are starting to test GR in a regime of moderately strong gravity and we may hope to detect deviations which trigger the discovery of an improved (effective) theory of gravity. This may take the form of a classical higher-order gravity which, going one step further, may be thought of as a low-energy effective theory arising from an underlying theory of Quantum Gravity and capturing the associated quantum effects.

[6]This might not provide us with the full theory of Quantum Gravity, but at the very least it can be extremely helpful to this aim. For instance, knowing the first effective terms with which to correct GR could be crucial to discard or validate potential candidates of Quantum Gravity.

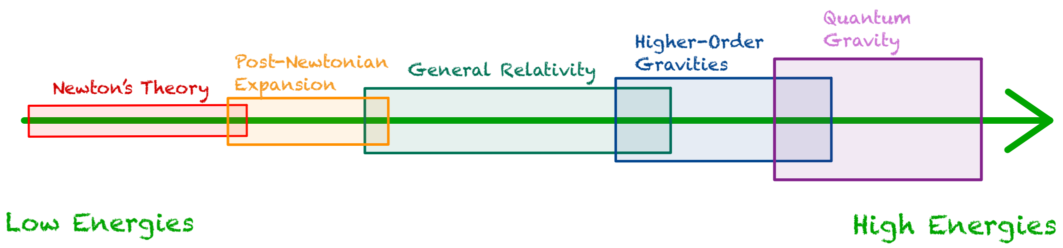


Figure I.1: We show in a diagrammatic fashion the most appropriate theory to apply in different energy regimes in terms of simplicity and accuracy. Of course, a full-fledged theory of Quantum Gravity should be able to describe all energies, but it is not necessary to use it for Solar System mechanics, since Newton's gravity already provides magnificent results in this framework.

by a field theory, thus dropping the worries about non-renormalizability or non-unitarity. This program has been successfully applied in the past, the paradigmatic example being that of the Fermi theory of β decays. Although it is non-renormalizable, such effective theory works extremely well for energies much lower than the mass of the $W^{\pm}$ bosons mediating the weak interaction, but it needs to be replaced by the proper electroweak theory incorporated in the SM when the energies involved are sufficiently high.

A similar scenario, we argue, might take place in the case of gravity. The EH action has been found to be not renormalizable, an issue which can be cured through the introduction of terms of higher order in derivatives[7] [75]. Nevertheless, the resulting theories turn out to be generically non-unitary. Hence a suitable position to adopt is that of interpreting the EH action as the leading term in an infinite expansion in powers of the spacetime curvature and its covariant derivatives, the first-order corrections being given by terms like R^2, $R_{\mu\nu}R^{\mu\nu}$ or $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor. Not knowing the full theory of Quantum Gravity, it is not possible to determine precisely which operators need to be included in the action (I.1), so it is reasonable to assume an EFT approach and add to the action all those terms compatible with the symmetries of the theory, with the hope of being able to capture and parametrize quantum effects. In the case of pure theories of gravity, these symmetries consist on the full diffeomorphism group, but in presence of other fields, such as a electromagnetic gauge field, gauge symmetry should also be respected.

The previous discussion leads naturally to the defintion of *higher-order gravities*.

Definition I.1. A *higher-order gravity* is a theory of gravity (possibly with non-minimally coupled matter[8]) in which the Einstein-Hilbert action (I.1) (maybe including minimally coupled matter terms) is corrected in a diffeomorphism-invariant fashion by terms of higher-order in the spacetime curvature (and, perhaps, in the matter fields as well):

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R + \mathcal{F}_0 + \sum_j \sum_{k,p=0}^{\infty}\sum_{n=1}^{\infty} \alpha_{n,k,p,j}\ell^{\sigma_{n,k,p}}(\nabla^k\mathcal{R}_n)_j^{\mu_1\dots\mu_s}\mathcal{F}^{p,j}_{\mu_1\dots\mu_s}\right], \quad (I.5)$$

[7]In particular, the introduction of quadratic curvature terms already yields a renormalizable action, while higher-curvature terms are super-renormalizable.

[8]Bosonic matter term is said to be minimally coupled to gravity if its only coupling to gravity is via the metric tensor. If it includes more generic couplings, such as contractions with curvature tensors, it is said to be non-minimally coupled.

where $(\nabla^k \mathcal{R}_n)_j{}^{\mu_1\cdots\mu_s}$ is constructed out of n curvature tensors and k covariant derivatives of it, labeling j all such possible inequivalent contractions[9], $\mathcal{F}_0$ (resp. $\mathcal{F}^{p,j}_{\mu_1\cdots\mu_s}$) stands for possible minimally coupled matter terms (resp. non-minimally coupled matter terms with p labeling every tensor that may be formed through matter fields and their covariant derivatives), ℓ is a certain length scale from which on the effects of Quantum Gravity need to be taken into account, the exponents $\sigma_{n,k,p}$ account for the dimension for the higher-order operators and the coefficients $\alpha_{n,k,p,j}$ are dimensionless couplings.

Higher-order gravities can be called equivalently *higher-derivative gravities* or *higher-curvature gravities*. We will regard them as synonyms. Similarly, we will indistinctly change the word "gravities" appearing before by "theories". In this thesis we will be mainly dealing with purely gravitational higher-order theories and higher-derivative gravities including a non-minimally coupled U(1) vector field (electromagnetism). When it is clear from the context, we will not write explicitly the adjective "purely gravitational" or the prepositional phrase "with a non-minimally coupled vector field" for the sake of simplicity. If some clarification is needed, it will be conveniently specified.

Example I.1. An instance of a (purely gravitational) higher-order gravity is given by:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R + \ell^2 R^2\right] . \tag{I.6}$$

This is Starobinsky's model [76], widely used to model inflation. On the other hand, an example of a higher-derivative gravity with a non-minimally coupled vector field is given by:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R - F^2 + \ell^2(2R_\mu{}^\alpha F^{\mu\nu}F_{\alpha\nu} - R^{\alpha\beta}{}_{\rho\sigma}F^{\rho\sigma}F_{\alpha\beta})\right] . \tag{I.7}$$

This theory will later be identified as an Electromagnetic Quasitopological Gravity in Chapter 2.

Adopting an EFT approach, one is entitled to study higher-order gravities and analyze the effects of higher-derivative corrections even from a classical perspective (*i.e.,* studying the associated action, the classical equations of motion and explicit solutions). However, their introduction causes the subsequent gravitational equations of motion to contain generically up to four-derivatives[10] of the metric, which brings about certain difficulties. On the one hand, having higher-order equations of motions is in general tantamount to the appearance of instabilities[11] by virtue of Ostrogradski's theorem [77], although we remind the reader that this issue is to be compensated with renormalizability. On the other hand, increasing the order of the equations of motion increments drastically the difficulty of their resolution, hampering the possibility of obtaining explicit solutions with which to explore and investigate the new physics introduced by such higher-derivative corrections, even in highly-symmetric configurations. Nevertheless, this problem has been circumvented in the last years with the advent of the so-called Generalized Quasitopological Gravities [78, 79],

[9] For the sake of concreteness, we exclude from our definition explicit couplings with a connection which cannot be rearranged in a curvature tensor, such as in the case of Heterotic String Theory that we briefly present in Example I.2.

[10] Except in the case of GR and the so-called Lanczos-Lovelock theories (see Section I.3), which have second-order equations of motion. In another vein, in presence of terms with covariant derivatives of the curvature, the order of the equations of motion is further increased.

[11] And of *ghosts*, which imply the loss of unitarity in the quantum setup.

characterized by admitting static and spherically symmetric solutions specified by a single function. From this defining feature, it can be seen that these theories satisfy a number of physically-intriguing properties which make them both amenable to computations and, at the same time, interesting from a phenomenological perspective. One of the main results of this thesis is the discovery that these theories turn out to form a (perturbative) spanning set of the space of all higher-curvature gravities once field redefinitions are considered (Chapter 1), so that they are quite more general than one could have *a priori* expected. Similarly, another interesting result is the proof of the existence of analogous higher-order theories with non-minimal couplings to the electromagnetic field, see Chapter 2.

I.1.1 Equations of motion and conserved charges

Given that our focus will be on the classical study of higher-order theories, it is of capital importance for us the derivation of the corresponding set of equations of motion. For the sake of simplicity and to illustrate the main ideas, we will restrict ourselves[12] to the consideration of four-dimensional higher-order gravities non-minimally coupled to a U(1) gauge field A_μ, invariant under diffeomorphisms and gauge transformations of A_μ and with algebraic dependence in the Riemann curvature and the field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. This type of theories are captured by the following action:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\mathcal{L}(R_{\alpha\beta\rho\sigma}, F_{\mu\nu})\,, \tag{I.8}$$

$$\begin{aligned}\mathcal{L}(R_{\alpha\beta\rho\sigma}, F_{\mu\nu}) = R - 2\Lambda - F^2 + \sum_{m=2}^{\infty}\alpha_m \ell^{2(m-1)}\mathcal{F}^{2m}\\ + \sum_j\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{n,m,j}\ell^{2(n+m-1)}(\mathcal{R}_{n,j}^{\mu_1\ldots\mu_s}\mathcal{F}^{2m,j}_{\mu_1\ldots\mu_s})\end{aligned} \tag{I.9}$$

where, following the notation of Definition I.1, the subindex j stands for the inequivalent ways in which the indices of a generic term with n Riemann tensors $\mathcal{R}_{n,j}^{\mu_1\ldots\mu_s}$ and $2m$ field strengths $\mathcal{F}^{2m,j}_{\mu_1\ldots\mu_s}$ can be contracted. Also, note that $\mathcal{F}_0$ would be given by $\mathcal{F}_0 = -2\Lambda - F^2 + \sum_{m=2}^{\infty}\alpha_m\ell^{2(m-1)}\mathcal{F}^{2m}$, where we have explicitly written the Maxwell term F^2 and a cosmological constant Λ. If we vary the action (I.8) and disregard boundary terms[13], we find the following set of equations of motion:

$$P_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} - \frac{1}{2}\mathcal{L}g_{\mu\nu} + 2\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho} + 2\star H_{(\mu}{}^{\alpha}F_{\nu)\alpha} = 0\,, \tag{I.10}$$

$$\mathrm{d}H = 0\,, \tag{I.11}$$

where we have defined

$$P^{\alpha\beta\rho\gamma} = \frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\rho\gamma}}\,, \quad H^{\alpha\beta} = -\star\frac{1}{2}\frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\,, \tag{I.12}$$

the Hodge dual operation $\star$ on two-forms being defined as

$$\star H_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}H^{\alpha\beta}\,, \tag{I.13}$$

[12]Along the thesis we will also deal with more intricate theories and in any number of dimensions.

[13]One must be careful at this point. Appropriate boundary terms have to be added in the action to make the variational problem set by (I.8) well posed, in a similar fashion to the Gibbons-Hawking-York term of GR [80, 81].

where $\varepsilon_{\mu\nu\alpha\beta}$ denotes the components of the canonical volume form with respect to the metric. We follow the convention that $\varepsilon_{0123} = \pm\sqrt{|g|}$ in some local coordinates, where the $\pm$ sign stands for the choice of orientation (we will use, unless otherwise indicated, the $+$ orientation if 0 is associated to a timelike coordinate).

Observe that the Einstein equation will generically contain four derivatives of the metric due to the term $\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho}$. Along with the equations of motion (I.10) and (I.11), we have to include the Bianchi identity for the field strength as well:

$$\mathrm{d}F = 0\,, \tag{I.14}$$

since $F = \mathrm{d}A$. On account of this, the set of equations of motion and the previous Bianchi identity can be conveniently rewritten as follows:

$$P_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} = \frac{1}{2}\mathcal{L}g_{\mu\nu} - 2\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho} - 2\star H_{(\mu}{}^{\alpha}F_{\nu)\alpha}\,, \tag{I.15}$$

$$\star H_{\mu\nu} = \frac{1}{2}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\,, \tag{I.16}$$

$$\mathrm{d}\begin{pmatrix} F \\ H \end{pmatrix} = 0\,. \tag{I.17}$$

First, we note that this rewriting of the set of equations of motion and Bianchi identity suggests that F and H can be promoted to fundamental variables, although the equations of motion impose that physical configurations cannot have arbitrary fields F and H, but rather they must be related through the so-called constitutive relation (I.16). Secondly, let us take a look at (I.17), which in this setup can be naturally identified as Bianchi identities for both F and H. If current three-forms $J_{\rm mag}$ and $J_{\rm elec}$ are placed on the right-hand-side of the Bianchi identities (I.17), then we would need to have $\mathrm{d}J_{\rm mag} = \mathrm{d}J_{\rm elec} = 0$, so the natural[14] definitions for electric Q and magnetic charge P are

$$Q = \frac{1}{4\pi}\int_{S^2_\infty} H\,, \qquad P = \frac{1}{4\pi}\int_{S^2_\infty} F\,. \tag{I.18}$$

The relevance of these electric and magnetic charges will be apparent in the study of black hole thermodynamics in the First Part of the thesis. We will say a few words about the circumstance of having F and H on an equal footing in the set of equations of motion and Bianchi identities in Section I.4, where we will introduce the notion of duality rotations.

Another fundamental physical magnitude which will play a key role is that of the spacetime mass. As a consequence of the Equivalence Principle, it is not possible to provide a local definition of gravitational energy, since the gravitational field can always be removed locally. Nevertheless, in some cases it is possible to define a notion of conserved mass M for the entire spacetime. In GR, the mass of asymptotically flat spacetimes is given by the celebrated Arnowitt-Deser-Misner (ADM) formula [82–85]:

$$M = \frac{1}{16\pi G}\int_{S^{D-2}_\infty} \mathrm{d}\Sigma^j(\partial_i h_{ij} - \partial_j h_{ii})\,, \tag{I.19}$$

[14] The fact that $J_{\rm elec}$ and $J_{\rm mag}$ are related to electric and magnetic charges respectively follows by comparison with the well-known situation in pure (Einstein-)Maxwell theory, in which one identifies the right-hand side of the Maxwell equation (resp. Bianchi identity) with the electric current (resp. magnetic current, associated to having non-trivial topology).

where the integral is carried out over a sphere located at infinity, D denotes the dimension of spacetime and $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the metric perturbation written in the Cartesian coordinates defined at infinity (i, j stand for spatial indices).

A more sophisticated formalism was later developed by Abbott and Deser [86], which allows to define a notion of mass which reduces to the ADM mass in case of asymptotically flat spacetimes. The Abbott-Deser formula is valid for asymptotically Anti-de Sitter (AdS) (and de Sitter[15] (dS)) spacetimes and furthermore admits a straightforward generalization to arbitrary higher-order gravities, since the modification is tantamount to the substitution of Newton's constant G by the so-called effective Newton's constant G_{eff}, which determines the coupling between gravity and matter [88, 89].

We will essentially be interested in studying asymptotically flat or asymptotically AdS static and spherically symmetric spacetimes, so it will suffice to provide an algorithm to compute the associated mass for these particular cases. Four-dimensional static and spherically symmetric (SSS) configurations are defined as those admitting four Killing vectors $k^{(A)}$, with $A = 0, 1, 2, 3$, satisfying the following algebra:

$$[k^{(0)}, k^{(a)}] = 0\,, \quad [k^{(a)}, k^{(b)}] = \varepsilon_{abc} k^{(c)}\,, \quad a, b, c = 1, 2, 3\,, \tag{I.20}$$

where ε_{abc} is the totally antisymmetric symbol defined as $\varepsilon_{123} = 1$ and where $k^{(0)}$ is timelike and the $k^{(a)}$ are spacelike. For such SSS spacetimes, it can be seen that the associated metric can be written in the following convenient form:

$$\mathrm{d}s^2_{\text{SSS}} = -N^2(r) f(r) \mathrm{d}t^2 + \frac{1}{f(r)} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2\,, \tag{I.21}$$

where $\mathrm{d}\Omega_2^2$ denotes the metric of the unit round two-sphere. This suggests the following natural generalization to D spacetime dimensions:

$$\mathrm{d}s^2_{\text{SSS}} = -N^2(r) f(r) \mathrm{d}t^2 + \frac{1}{f(r)} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{D-2}^2\,, \tag{I.22}$$

where $\mathrm{d}\Omega_{D-2}^2$ is the metric of the unit $(D-2)$-dimensional round sphere. Now, according to the Abbott-Deser prescription [90, 91], the spacetime mass can be derived by inspecting the coefficient of the term $1/r^{D-3}$ in the asymptotic expansion of $f(r)$:

$$f(r) = -\kappa r^2 + 1 - \frac{16\pi G_{\text{eff}} M}{(D-2) N_\infty \Omega_{D-2}} \frac{1}{r^{D-3}} + \ldots\,, \tag{I.23}$$

where κ is, up to a constant factor, the curvature of the asymptotic spacetime ($\kappa = 0$ in Minkowski, $\kappa < 0$ in AdS and $\kappa > 0$ in dS), Ω_{D-2} is the volume of the $(D-2)$-dimensional sphere and N_∞ is a constant factor that accounts for the normalization of the time coordinate at infinity.

I.2 String Theory

Having motivated the introduction of higher-order gravities as an effective approach to understand the implications of Quantum Gravity at low energies without really knowing its precise formulation, we now proceed to present an example of a consistent theory of

[15] There are some subtleties associated to the existence of a cosmological horizon, though [86, 87].

Quantum Gravity in which higher-derivative terms appear in the subsequent low-energy effective actions — String Theory (ST). Of course, its relevance in theoretical physics goes beyond the fact that its low-energy limits can be recast in the form of higher-order theories. In fact, it would be extremely unfair to just consider it as an instance of a higher-derivative gravity, specially if one considers its attractive and elegant properties (for example, anomaly cancellation[16] [92–94] or the regularization of UV/IR divergences [95, 96]) or the number of theoretical successes and new research areas it has catalyzed since its advent, such as the microscopic interpretation of black hole entropy [97, 98] or holography (see Section I.7). Nevertheless, it is instructive to show how higher-derivative theories arise in this setup, since this allows us, to some extent, to circumvent the enormous complexity of ST and have a way of parametrizing and determining its effects over energy ranges which we may be able to access in the near future.

In a nutshell, ST [95, 96, 99–102] postulates that the fundamental constituents of our Universe are strings which dwell in a D-dimensional target space, in principle undetermined. Wandering within such target space, strings sweep out a *worldsheet*, which is the two-dimensional analogue of point-particles' worldlines. Equipping the worldsheet W with an auxiliary two-dimensional metric γ_{ij}, the most general classical two-derivative action one can write for the string takes the form of a non-linear sigma model in which the dynamical variables are given by the *embedding coordinates* X^μ of the worldsheet into the ambient space [103]:

$$I_{\text{nlsm}} = \frac{1}{4\pi\alpha'}\int_W \mathrm{d}^2\xi\sqrt{|\gamma|}\left[\left(\gamma^{ij}g_{\mu\nu}(X) - \varepsilon^{ij}B_{\mu\nu}(X)\right)\partial_i X^\mu \partial_j X^\nu + \alpha'\phi(X)R(\gamma)\right] , \quad \text{(I.24)}$$

where $g_{\mu\nu}$ is the target space metric, $B_{\mu\nu}$ is a two-form called the Kalb-Ramond form, ϕ is a scalar field which receives the name of dilaton, $R(\gamma)$ is the Ricci scalar of the two-dimensional metric γ_{ij}, ε_{ij} is the totally antisymmetric two-dimensional symbol and α' is called the Regge slope and is related to the string tension T and to the string length l_s as $2\pi T = \alpha'^{-1} = l_s^{-2}$.

Several comments are in order. We identify the first term in the action (I.24) as the Polyakov action for a relativistic string. The second one is known as a Wess-Zumino term and is the integral of the pullback of the Kalb-Ramond form over the worldsheet. As a consequence, it is a purely topological term[17] (it is independent of the metric) and it is invariant, up to total derivatives, under gauge transformations $B \to B + d\rho$, ρ being a one-form. Thirdly, the last term is precisely the Euler characteristic of the worldsheet when the dilaton is constant.

Let us comment further about this last point. If we suppose that the (vacuum expectation value of the) dilaton barely varies across the worldsheet, it is a very reasonable approach to assume that over the worldsheet $\phi(X) \simeq \phi_0$, with $\phi_0 \in \mathbb{R}$. Then, in this context, one could think that the last term in (I.24) is dispensable, owing to the fact that it would be then a topological term. However, it does play a fundamental role in the quantization of the classical theory[18] given by (I.24), since the computation of string amplitudes requires

[16] This refers particularly to the miraculous cancellation of anomalies happening in Type IIB, Heterotic and Type I ST.

[17] By a topological term we mean one whose value (after integration) depends only on the topology of the underlying manifold. Abusing the nomenclature, one might also call a term topological if it does not depend on the metric.

[18] The Ricci scalar of a two-dimensional metric, apart from defining a topological invariant, is locally a total derivative, and total derivatives are generically relevant in path integrals.

to sum over all worldsheet topologies. If we denote $g_s \equiv e^{\phi_0}$, then each topology (with Euler characteristic χ) in such *genus expansion* would be weighted by a factor $g_s^{-\chi}$ so g_s becomes the string coupling constant and suggests a natural way to study perturbative quantum worldsheet theory.

On the other hand, we observe that the first two terms of the action (I.24) are invariant under Weyl rescalings of the two-dimensional metric γ_{ij}, while the last term is not. Nevertheless, even if we do not consider the Ricci scalar term in (I.24), we must demand conformal invariance to hold at the quantum level for consistency of the theory (in particular, to guarantee finiteness of string amplitudes), and this requires the introduction of such last term in (I.24) [103, 104]. In fact, considering the embedding coordinates as the variables of the theory, the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ can be understood as coupling functions, so we can rephrase the problem of conformal invariance at the quantum level in terms of the vanishing of the associated beta functionals. These beta functionals (anomaly Weyl coefficients) can be computed as a power series in α', the first terms being [103]:

$$\beta^g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\lambda\sigma} H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \phi + \mathcal{O}(\alpha')\,, \tag{I.25}$$

$$e^{-2\phi}\beta^B_{\mu\nu} = \nabla^\mu (e^{-2\phi} H_{\mu\rho\sigma}) + \mathcal{O}(\alpha')\,, \tag{I.26}$$

$$\beta^\phi_{\mu\nu} = \frac{D-26}{48\pi^2} + \frac{\alpha'}{16\pi^2}\left(4(\partial\phi)^2 - 4\nabla^2\phi - R + \frac{1}{12}H^2\right) + \mathcal{O}(\alpha'^2)\,, \tag{I.27}$$

where ∇, $R_{\mu\nu}$ and R denote the (Levi-Civita) covariant derivative, the Ricci tensor and Ricci scalar associated to the target space metric, respectively and $H = \mathrm{d}B$.

First of all, let us take a look on the beta functional β^ϕ associated to the dilaton. We observe the presence of a zeroth-order term given by $(48\pi^2)^{-1}(D-26)$. Conformal invariance then requires that $D = 26$, so it actually fixes the target space to be 26-dimensional. Such dimension ($D = 26$ for the bosonic string) is called the critical dimension. Secondly, working in this critical dimension, we can equivalently obtain Equations (I.25), (I.26) and (I.27) by extremizing the following action:

$$I_{\text{BS}} = \int \mathrm{d}^{26}x \sqrt{|g|} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]\,, \tag{I.28}$$

which coincides with the *effective action* of the bosonic string [103], defined from scattering amplitudes.

However, no fermions appear in the previous effective action nor in the theory itself, and their inclusion would be desirable since experiments have clearly shown that they exist in Nature. This problem can be solved by demanding the theory to be *supersymmetric*[19]. Very briefly, (linearly-realized) supersymmetry associates to every boson at each mass level a corresponding fermion and provides us with a canonical procedure to add fermions consistently into the theory. It turns out that there exist five different Superstring Theories, each of them constructed so as to ensure they are anomaly-free and do not contain tachyons in the physical spectrum. Interestingly enough, when working in the critical dimension (which is $D = 10$ for Superstring Theories), it can be seen that

[19] In particular, we demand spacetime supersymmetry. Demanding just worldsheet supersymmetry is insufficient, since there are examples of theories (Type 0 ST) which possess worldsheet supersymmetry and yet contain no spacetime fermions in their spectrum [102, 105].

the corresponding bosonic[20] effective actions of the Superstring Theories always contain a Neveu-Schwarz Neveu-Schwarz (NSNS) sector composed by a ten-dimensional metric $g_{\mu\nu}$, a two-form $B_{\mu\nu}$ appearing through its field strength $H = \mathrm{d}B$ and the dilaton ϕ, whose action at the two-derivative level is[21]

$$I_{\text{NSNS}} = \frac{g_s^2}{16\pi G^{(10)}} \int \mathrm{d}^{10}x \sqrt{|g|} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right] , \tag{I.29}$$

which is practically the same action as in (I.28) up to the change in spacetime dimensions. Also, we have included a factor before the integral in (I.29) to make connection with the Einstein-Hilbert action, g_s being the string coupling and $G^{(10)}$ the ten-dimensional Newton's constant which can be expressed in terms of g_s and the string length l_s as $G^{(10)} = 8\pi^6 g_s^2 l_s^8$.

At this point, it is important to clarify that the subsequent string effective actions are defined through the scattering amplitudes, analyzing which terms should be included in the action so as to obtain these amplitudes [107–109]. However, this procedure is typically very involved, the usual strategy being to infer some terms of the action via scattering amplitudes and find the rest of them (at a given order) through supersymmetrization[22]. In another vein, it has also been observed that such effective actions can be equivalently derived [110, 111] from the computation of the beta functionals [108, 109]. This is a highly non-trivial point and reflects the intricacy and beauty of ST.

In any case, whether we work with the bosonic string or with the any of the Superstring Theories, it is clear that they describe gravity, as can be naively observed from the fact that the effective action includes the Ricci scalar of the target space metric. Consequently, ST generalizes Einstein's GR and stands today as the most promising candidate for the theory of Quantum Gravity. Nevertheless, experimental evidence so far tells us that the Universe is four-dimensional, so we better find a way to convert ST into a four-dimensional theory. Restricting our analysis from now on to Superstring Theories, the usual procedure consists in performing a *spacetime compactification*, assuming that the ten-dimensional spacetime factorizes[23] as $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{Y}_6$, where $\mathcal{Y}_6$ is a compact six-dimensional space. Then, one decomposes the ten-dimensional fields in accordance to this compactification ansatz and ends up with a collection[24] of new scalars and vectors defined in the four-dimensional space $\mathcal{M}_4$ together with a gravitational sector which is always present and governed by (at leading order in α') $G_{\mu\nu} = 8\pi G^{(4)} T_{\mu\nu}$, where the four-dimensional Newton's constant $G^{(4)}$ is related to the ten-dimensional one as $G^{(10)} = V(\mathcal{Y}_6) G^{(4)}$, with $V(\mathcal{Y}_6)$ the volume of the compact space. In this way, we can obtain four-dimensional Einstein gravity coupled to different kinds of matter. However, given the essentially numberless

[20] It can be checked that setting to zero all fermionic fields is a consistent truncation. Furthermore, if one is interested in studying gravitational configurations, this is further supported by the fact that fermions are not observed macroscopically. In any case, it is important to start from a theory which already contains fermions — then, whether we truncate them or not is an approximation which could work extremely well in certain contexts, like gravitational ones.

[21] This discussion does not apply to Type I ST, whose NSNS sector is different [106].

[22] We thank Tomás Ortín and Ángel Uranga for clarifying these points.

[23] One could think as well of more complicated compactifications, such as those given by principal bundles or generic fibrations.

[24] Such procedure yields a bunch of massless modes as well as an infinite tower of massive ones, whose masses scale inversely proportional to the size of the compact dimensions. If such size is small enough, one can safely ignore these massive modes (which is equivalent to neglect the dynamics of the internal dimensions), since the typical energies of physical processes will be much lower than the mass of the least massive mode. This is known as dimensional reduction.

compactifications given by the possible choices of $\mathcal{Y}_6$, there are myriads of four-dimensional effective theories which actually come from ST compactifications. This is perhaps one of the biggest issues with ST and is called the Landscape Problem[25].

I.2.1 Higher-derivative corrections from String Theory

The discussion above regarding the effective actions was carried out working to leading order in the parameters α' and g_s, so, if one wants to incorporate further string corrections, one should consider the effects of higher-loop contributions. These corrections generically modify the beta functionals through the inclusion of higher-derivative terms, so that the full string effective action I_{ST} is a double expansion in the Regge slope α' and the string coupling g_s:

$$I_{\mathrm{ST}} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} g_s^{2k}(\alpha')^n I_{n,k}\,, \tag{I.30}$$

The higher-order terms in the previous double expansion are interpreted as the proper quantum corrections coming from ST and those with explicit powers of α' correspond to higher-derivative terms[26]. Therefore, if one compactifies down to four-dimensions, one does not longer obtain GR with a specific matter content, but a higher-order gravity.

Obtaining the precise higher-derivative terms appearing in the expansion (I.30) is an extraordinarily intricate task and only the first terms have been explicitly computed. As a matter of fact, g_s-corrections are much poorly understood than α'-corrections, and this is why the latter have been more extensively studied in the literature [13, 115–119]. Indeed, g_s-corrections turn out to generically appear at least at order α'^3, as it happens in Type II [120] and Heterotic theories [121, 122]. Let us show two explicit examples of stringy effective actions with α'- and g_s-corrections.

Example I.2. Heterotic String Theory. This case is very interesting, because up to order α'^2 the effective action takes a relatively simple expression. In fact, if we truncate the Yang-Mills fields for the sake of simplicity, such ten-dimensional action is given by [123, 124]

$$\begin{aligned} I_{\mathrm{Het}} = \frac{g_s^2}{16\pi G^{(10)}}\int \mathrm{d}^{10}x\sqrt{|g|}e^{-2\phi}\Big[R + 4\,(\partial\phi)^2 - \frac{1}{12}\hat{H}^2 \\ -\frac{\alpha'}{8}R_{(-)\mu\nu}{}^{a}{}_{b}R_{(-)}{}^{\mu\nu b}{}_{a}\Big] + \mathcal{O}(\alpha'^3)\,, \end{aligned} \tag{I.31}$$

where $R_{(-)\mu\nu}{}^{a}{}_{b}$ is the curvature of the torsionful spin connection $\Omega_{(-)}{}^{a}{}_{b} = \omega^{a}{}_{b} - \frac{1}{2}\hat{H}_{\mu}{}^{a}{}_{b}dx^{\mu}$, with $\omega^{a}{}_{b}$ the spin connection, and $\hat{H} = \mathrm{d}B + \frac{\alpha'}{4}\omega^{\mathrm{L}}_{(-)}$, where $\omega^{\mathrm{L}}_{(-)}$ is the Lorentz-Chern-Simons three-form of $\Omega_{(-)}{}^{a}{}_{b}$.

We explicitly observe in (I.31) the appearance of higher-order terms[27]. They become more and more involved as we go further in the expansion (I.30). In fact, quartic terms in

[25] A related problem is that investigated by the Swampland Program [112–114], which aims at discovering the criteria to decide whether a particular four-dimensional effective theory does descend from a UV-complete theory of Quantum Gravity (assuming it is given by ST) or not.

[26] Observe that g_s is dimensionless. Also, note that the leading term in the expansion (I.30) is given by two-derivative Supergravity actions.

[27] We observe the presence of terms directly coupled to a connection (not arising from expanding a Riemann curvature tensor), which was not contemplated explicitly in Definition I.1. Nevertheless, it is clear that there also appear terms quadratic in the Riemann curvature tensor, so it is completely reasonable to call (I.31) a higher-order gravity in the sense of Definition I.1.

the curvature already appear at order α'^3, which account for both tree-level and one-loop (g_s-)corrections as mentioned before.

When going down to four dimensions, one ends up with theories with different matter contents depending on the type of compactification under consideration, such as α'-corrected models with an axidilaton and/or gauge vector fields [125,126]. The appearance of gauge vector fields in the four-dimensional effective theory is of importance for us, since Chapters 2 and 3 will be devoted to the study of four-dimensional higher-order gravities in which a vector field is non-minimally coupled to gravity. Therefore, apart from their intrinsic relevance, this shows that such theories already appear in the context of ST, which illustrates the possibility of using higher-derivative theories to capture stringy corrections.

Example I.3. Type IIB String Theory. As it turns out, the first non-trivial corrections to the Type IIB Supergravity action arise at cubic order in α', containing as well g_s-corrections. In fact, it can be seen [127] that the corresponding effective action includes the following terms:

$$-\frac{\alpha'^3}{3\times 2^{10}}\int \mathrm{d}^{10}x\sqrt{|g|}\left[\left(e^{-2\phi}\zeta(3)+\frac{1}{2^6\pi^5}\right)\left(I_X-\frac{1}{8}I_Z\right)\right]\subset I_{\mathrm{IIB}}\,, \tag{I.32}$$

where I_X and I_Z are certain terms with quartic dependence on the curvature tensor whose precise expression can be read in [127]. A more manageable expression is obtained if we consider the theory on $(\mathcal{A}_5\times S^5, g_{\mathcal{A}_5}\oplus F g_{S^5})$, with $(\mathcal{A}_5, g_{\mathcal{A}_5})$ a five-dimensional Einstein manifold with negative curvature (asymptotically AdS$_5$), (S^5, g_{S^5}) the five-dimensional round sphere and $F\in C^\infty(\mathcal{A}_5)$. On this background, it is consistent to truncate all fields except the metric [128–130] and write the following effective action for the five-dimensional metric $g_{\mathcal{A}_5}$ [131,132]:

$$I_{\mathrm{IIB}_{\mathcal{A}_5\times S^5}}[g_{\mathcal{A}_5}]=\frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{|g_{\mathcal{A}_5}|}\left[R+\frac{12}{\ell^2}+\frac{\zeta(3)}{8}\alpha'^3W^4\right]\,, \tag{I.33}$$

where R denotes the Ricci scalar of $g_{\mathcal{A}_5}$, ℓ stands for the AdS radius and the radius of the sphere and W^4 is a particular combination of contractions of four Weyl tensors W_{abcd} of $g_{\mathcal{A}_5}$ given by

$$W^4=\left(W_{abcd}W^{ebcf}+\frac{1}{2}W_{adbc}W^{efbc}\right)W^{ag}{}_{he}\,W_{fg}{}^{hd}. \tag{I.34}$$

In Chapter 1 we will use the effective action (I.33) to illustrate the fact that any (purely gravitational) higher-curvature gravity can be mapped via field redefinitions to a Generalized Quasitopological Gravity, which is a specific and intriguing type of higher-order gravities that we introduce in Section I.3.

I.2.2 Geometric aspects of moduli spaces: the Supergravity c-map

When carrying out spacetime compactifications from the ten-dimensional ST or Supergravity, the decomposition of the ten-dimensional fields produce a collection of new fields in the (four-dimensional) spacetimes. Among these, one usually ends up with a bunch of scalars that define a non-linear sigma model on a target space, which is called *moduli space* or *scalar manifold*.

Interestingly enough, these moduli spaces or scalar manifolds turn out to possess extraordinarily beautiful geometric structures [106,133–135]. These spaces arise either from string compactifications or from direct Supergravity construction in the corresponding space-time dimension. We will be primarily interested in four-dimensional scalar manifolds with (para)hyperKähler or quaternionic (para)Kähler structures, which appear naturally in the ST and Supergravity context (see the previous Refs.). This justifies to introduce them here, for which we need to carry out some preliminary definitions. We shall follow [136–138].

Definition I.2. An almost (para)complex structure on a smooth manifold M is an endomorphism field $J \in \Gamma(\mathrm{End}(TM))$ such that:

1. $J \neq \mathrm{Id}_{TM}$ satisfies $J^2 = -\varepsilon \mathrm{Id}_{TM}$, with $\varepsilon = \pm 1$.
2. It has exactly two eigenspaces with the same dimension (if $\varepsilon = 1$, complexification of TM is required).

The pair (M, J) is called (almost) (para)complex manifold, which is forced to be even-dimensional. An almost (para)complex structure J is said to be integrable if the corresponding eigendistributions are both integrable. In such case J is called a (para)complex structure and (M, J) a (para)complex manifold. In all these cases, the prefix "para" is added if $\varepsilon = -1$, and dropped if $\varepsilon = 1$.

We will be interested in combining (para)complex structures with a (pseudo-)Riemannian metric in the form of (para)Kähler structures.

Definition I.3. A (pseudo-)Riemannian metric g on a (para)complex manifold (M, J) is (para)Hermitian if J is skew-symmetric with respect to g. For (M, J) (para)Hermitian, if the non-degenerate two-form ω defined as $\omega = g(\cdot, J\cdot)$ is closed, then (M, J, g) is said to be a (para)Kähler manifold[28]. Similarly, the symplectic form ω is called the (para)Kähler form of (M, J, g).

From the definition, we observe that:

$$g(J\cdot, J\cdot) = -g(\cdot, J^2\cdot) = \varepsilon g\,. \tag{I.35}$$

In another vein, it is important to remark that the (para)Kähler form ω in (para)Kähler manifolds (M, J, g) is parallel with respect to the Levi-Civita connection.

An interesting particular subclass of (para)Kähler manifolds is given by (para)hyperKähler manifolds, which we define next. It will suffice for our purposes to provide the definition for four-dimensional spaces.

Definition I.4. A four-dimensional (para)hyperKähler manifold (M, g, J_1, J_2, J_3) is a four-dimensional (pseudo-)Riemannian manifold (M, g) such that:

- (M, J_1, g) is Kähler and (M, J_l, g) with $l = 2, 3$ are both Kähler (hyperKähler case) or paraKähler (parahyperKähler case).
- The (para)complex structures (J_1, J_2, J_3) anticommute with each other and satisfy:

$$J_1 J_2 = J_3\,. \tag{I.36}$$

[28] Usually one finds in the literature that a pseudo-Riemannian manifold is called pseudo-Kähler if it is endowed with a complex structure whose musical two-form is closed. However, for the sake of simplicity we shall equally call it Kähler manifold.

In particular, (M, g) is either Riemannian or of neutral signature. In the latter case we include the prefix "para", while we drop it for the Riemannian case.

From its definition, one may infer that (para)hyperKähler manifolds are Ricci flat and possess a(n) (anti)self-dual Weyl tensor [139, 140].

Both (para)Kähler and (para)hyperKähler manifolds turn out to possess special holonomy, appearing explicitly in the subsequent classification of holonomy groups [141, 142]. Another class of (pseudo)-Riemannian manifolds with special holonomy we will be interested in will be that of quaternionic (para)Kähler four-manifolds, which we define next (again, restricting ourselves to four dimensions).

Definition I.5. Let (M, g) be a four-dimensional Riemannian (resp. neutral-signature) orientable four-manifold. It is said to be quaternionic Kähler (resp. quaternionic paraKähler) if and only if its Ricci tensor Ric^g satisfies $\mathrm{Ric}^g = \lambda g$ with $\lambda \in \mathbb{R}/\{0\}$ and its Weyl tensor if self-dual for one of the two possible orientations. We shall refer to them jointly as quaternionic (para)Kähler four-manifolds.

On the one hand, note that the previous definition is specific of dimension $D = 4$. If $D = 4n$ for $n > 1$, then the corresponding definition of quaternionic (para)Kähler manifold must be replaced by the existence of a parallel[29] subbundle $Q \subset \mathrm{End}(TM)$ such that $\forall p \in M$ there exist $I, J, K \in Q_p$ such that $Q_p = \mathrm{span}\{I, J, K\}$, $IJ = K$ and $I^2 = J^2 = K^2 = -\mathrm{Id}$ (resp. $I^2 = -J^2 = -K^2 = -\mathrm{Id}$). This definition is unsatisfactory for $D = 4$, since it can be seen that it would be fulfilled by every orientable (pseudo-)Riemannian four-manifold. An argument showing why Definition I.5 is the most appropriate in four dimensions one can be found in [143, 144]. On the other hand, while (para)hyperKähler manifolds are trivially quaternionic (para)Kähler manifolds, the converse is not true. In fact, quaternionic (para)Kähler manifolds need not to be even (para)Kähler [144].

Let us now depict how these structures arise in the context of Supergravity and ST for the purposes of this thesis. In the context of Type II compactifications, it was observed that the Lagrangians for the low-energy effective theories of IIA and IIB theories are related[30] through the so-called *Supergravity c-map* [145–147], nowadays understood from the mathematical setting[31] as a map associating a (para)Kähler manifold of restricted type —more concretely, a projective special (para)Kähler manifold— and dimension $2n$ with a quaternionic (para)Kähler manifold of dimension $4(n + 1)$ [148, 149]. This map is connected with the *rigid* c-map, which appears in the realm of supersymmetric field theories without gravity and associates particular classes of (para)Kähler manifolds — called affine special (para)Kähler manifolds— of dimension $(2n + 1)$ to (para)hyperKähler manifolds of dimension $4(n + 1)$. These two maps are intimately related, since the Supergravity c-map should be obtained from the rigid one by gauging the superconformal symmetry [149–151]. This inspired the search and discovery of the (para)HK/QK correspondence [148, 152–156], which maps certain (para)hyperKähler manifolds to a uniparametric family of quaternionic (para)Kähler ones.

Particularly appealing is the study of the Supergravity c-map after the consideration of string one-loop corrections, which require the introduction of α'^3-terms into the effec-

[29] That is, preserved under the Levi-Civita connection.

[30] Today this is interpreted as an instance of *T-duality* between IIA and IIB string compactifications.

[31] From a physical perspective, it is a map between the manifold of vector multiplet scalars of four-dimensional IIA theory to the manifold of hypermultiplet scalars of four-dimensional IIB theory [137].

tive action, as mentioned before. Interestingly enough, it can be shown that the subsequent *one-loop deformed* c-map provides a one-parameter family of deformed quaternionic (para)Kähler manifolds [148,156], so that the property of being quaternionic (para)Kähler is respected by quantum corrections.

In the four-dimensional setting, the uncorrected metric is called *universal hypermultiplet metric*. Its one-loop deformation [120,157] has been scrutinized in the last years [158–160], specially in the Riemannian setting. Indeed, it has been found [159] that the isometry group of the deformed quaternionic Kähler manifold is $\mathrm{O}(2) \ltimes \mathrm{H}$, with H the three-dimensional Heisenberg group. This motivated to complete the classification of all Riemannian Einstein metrics of non-positive scalar curvature which are invariant under the action of $\mathrm{SO}(2) \ltimes \mathrm{H}$ in $\mathbb{R}^4$ [160].

Nevertheless, the previous work concerned just positive-definite metrics and was restricted to the study of manifolds with symmetry group $\mathrm{SO}(2) \ltimes \mathrm{H}$. Thus, it would be intriguing to examine both Riemannian and neutral-signature four-manifolds and reduce the isometry group to be, for example, just H. Such analysis is carried out in Chapter 6, where we investigate Heisenberg-invariant self-dual Einstein four-manifolds. When the Ricci tensor is not identically zero, we discover that the only possibilities[32] are the one-loop deformed universal hypermultiplet metrics (and neutral-signature versions of it) [149, 156] together with positively-curved (resp. negatively) counterparts in the Riemannian (resp. neutral-signature) context, which seem not to have been previously considered in the literature. Similarly, (para)hyperKähler manifolds with Heisenberg symmetry are studied, being able to present a classification result in Section 6.3.

I.2.3 Contact structures and Supergravity

All the mathematical definitions we presented before, including (para)complex manifolds, (para)Hermitian manifolds, (para)Kähler manifolds, (para)hyperKähler manifolds or quaternionic (para)Kähler manifolds, only apply in even dimensions (and some of them only for $D = 4n$). Therefore, it is natural to ask: is it possible to define, up to some extent, odd-dimensional analogues? If so, do they have any potential application to Supergravity and ST?

We begin by addressing the first question with the introduction of the notions of contact Riemannian structure, contact Lorentzian structure and para-contact metric structure [161–164]. They conform the appropriate particularizations of the concept of contact structure, defined as a $(2n+1)$-dimensional smooth manifold M endowed with a one-form α such that $\alpha\wedge(\mathrm{d}\alpha)^n \neq 0$ everywhere, in the presence of Riemannian and pseudo-Riemannian metrics.

Definition I.6. Let (M, g) be an oriented smooth (pseudo-Riemannian) manifold of dimension $(2n+1)$ with $n \geq 1$ and let $\alpha \in \Omega^1(M)$. Assume that:

$$\phi^2 = \sigma_g(-\varepsilon \mathrm{Id} + \xi \otimes \alpha)\,, \quad g(\mathrm{Id} \otimes \phi) = \mathrm{d}\alpha\,, \quad \xi = \alpha^\sharp\,, \quad \eta(\xi) = g(\xi, \xi) = \varepsilon\,, \tag{I.37}$$

where $\sigma_g = \mathrm{sign}(\det(g))$ and $\varepsilon = \pm 1$. Then:

- If g is Riemannian, then necessarily $\varepsilon = \sigma_g = 1$ and the triple (M, g, α) defines a contact Riemannian structure on M.

[32] Additionally, we also discover a family of Lorentzian and neutral-signature conformally flat manifolds in which the Heisenberg center is lightlike. See Subsection 6.2.3 for more details.

- If g is Lorentzian and $\varepsilon = \sigma_g = -1$, the triple (M, g, α) defines a contact Lorentzian structure on M.
- If g is pseudo-Riemannian of signature $(n+1, n)$ and $\varepsilon = -\sigma_g = 1$, the triple (M, g, α) defines a para-contact metric structure on M.

Following conventions, ξ is usually called the Reeb vector field, ϕ the characteristic endomorphism and α the contact form. Given a contact Riemannian, contact Lorentzian or para-contact metric structure on M, it is usually included as part of their definition the formula:

$$g \circ \phi \otimes \phi = \sigma_g(\varepsilon g - \alpha \otimes \alpha)\,, \tag{I.38}$$

However, we note that it can be easily derived from the skew-symmetry of the characteristic endomorphism ϕ and the first equation in (I.37). Next we display explicit instances of contact Riemannian, contact Lorentzian and para-contact metric structures.

Example I.4. Consider $M = \mathbb{R}^{2n+1}$ with Cartesian coordinates (z, x^i, y^i) with $i = 1, \dots, n$. Define $\alpha = \mathrm{d}z - \sum_{i=1}^n y^i \mathrm{d}x^i$. Define $g = \varepsilon_1 \alpha \otimes \alpha + \sum_{i=1}^n \left((\mathrm{d}x^i)^2 + \varepsilon_2 (\mathrm{d}y^i)^2\right)$ with $\varepsilon_1, \varepsilon_2 = \pm 1$. The endomorphism $\phi : T\mathbb{R}^{2n+1} \to T\mathbb{R}^{2n+1}$ defined as $g(X, \phi(Y)) = \mathrm{d}\alpha(X, Y)$ is given by:

$$\phi = -\varepsilon_2 \sum_{i=1}^n \frac{\partial}{\partial y^i} \otimes \mathrm{d}x^i + \sum_{i=1}^n \left(y^i \frac{\partial}{\partial z} + \frac{\partial}{\partial x^i} \right) \otimes \mathrm{d}y^i\,. \tag{I.39}$$

Then $\phi^2 = -\varepsilon_2 \mathrm{Id} + \varepsilon_2 \varepsilon_1 \xi \otimes \alpha$, with $\xi = \alpha^\sharp$. Consequently, if $\varepsilon_1 = \varepsilon_2 = 1$ the triple $(\mathbb{R}^{2n+1}, g, \alpha)$ defines a contact Riemannian structure on $\mathbb{R}^{2n+1}$, if $\varepsilon_1 = -\varepsilon_2 = -1$ we have that $(\mathbb{R}^{2n+1}, g, \alpha)$ conforms a contact Lorentzian structure on $\mathbb{R}^{2n+1}$ and if $\varepsilon_1 = -\varepsilon_2 = 1$, $(\mathbb{R}^{2n+1}, g, \alpha)$ defines a para-contact metric structure on $\mathbb{R}^{2n+1}$.

Let $\mathcal{N}_\phi \in \mathrm{End}(TM)$ be the Nijenhuis torsion tensor associated to an endomorphism ϕ, which is defined as:

$$\mathcal{N}_\phi(X, Y) = \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)]\,, \qquad X, Y \in \mathfrak{X}(M)\,. \tag{I.40}$$

A very important subclass of contact Riemannnian, contact Lorentzian and para-contact metric structures is obtained by imposing them to be *K-contact* [165] or *Sasakian* [166–168].

Definition I.7. Let (M, g, α) be a contact Riemannian, contact Lorentzian or para-contact metric structure on M. It is said to be K-contact if the subsequent Reeb vector field is Killing. Similarly, (M, g, α) is Sasakian if $\mathcal{N}_\phi + \xi \otimes \mathrm{d}\alpha = 0$.

Two comments are in order. First of all, while the K-contact and Sasaki conditions are equivalent in three-dimensions for the contact structures we have considered so far, this is no longer true in higher dimensions, for which the Sasaki condition is stronger than K-contactness (in fact, if (M, g, α) is Sasakian it is automatically K-contact). Second, Sasakianity arises from demanding the integrability of the almost (para)complex structure J defined on the $2(n+1)$-dimensional manifold $M \times \mathbb{R}$ and given by:

$$J : T(M \times \mathbb{R}) \to T(M \times \mathbb{R})\,, \quad (v, c\,\partial_q) \mapsto (\phi(v) - \varepsilon \sigma_g c \xi, \alpha(v) \partial_q)\,, \tag{I.41}$$

where q is the canonical coordinate on $\mathbb{R}$ and $c \in \mathbb{R}$. Observe that $J^2 = -\varepsilon \sigma_g \mathrm{Id}_{T(M\times\mathbb{R})}$. On the other hand, this condition can be restated equivalently as the metric cone $(\hat{M}, \hat{g})$

over (M, g) being (para)Kähler [168, 169]. From this perspective, it is quite natural to think about Sasakian condition as the odd-dimensional analogue to the (para)Kähler one. Likewise, non-Sasakian contact Riemannian, contact Lorentzian or para-contact metric structures would be in correspondence with almost (para)Kähler structures[33].

Example I.5. In Example I.4, the contact Riemannian, contact Lorentzian and para-contact metric structures there presented can be easily seen to be Sasakian. In fact, on account of (I.39), routine computations show directly that $\mathcal{N}_\phi + \mathrm{d}\alpha \otimes \xi = 0$, proving their Sasakianity. *A fortiori*, these contact structures are K-contact as well.

There are many examples of non-Sasakian and non K-contact contact structures, as it will become clear in Chapter 7.

So far, we have introduced special classes of contact structures characterized by their properties with respect to a (pseudo-)Riemannian metric, but nothing has been said about the subsequent curvature. In particular, many intriguing properties, specially in the K-contact and Sasakian case, can be derived [161, 162, 170, 171], but for us it will suffice to introduce the notion of η-Einstein contact structure, which generalizes the concept of Einstein metrics.

Definition I.8. Let (M, g, α) be either a contact Riemannian, contact Lorentzian or para-contact structure. It is said to be η-Einstein if the following condition holds:

$$\mathrm{Ric}^g = a g + b \alpha \otimes \alpha \,, \tag{I.42}$$

where $a, b \in C^\infty(M)$.

Example I.6. The contact Riemannian, contact Lorentzian and para-contact metric structures of Example I.4 can be seen to belong, by direct computation, to the η-Einstein class, since the associated Ricci tensor of (M, g, α) reads:

$$\mathrm{Ric}^g = -\frac{\sigma_g}{2} g + \frac{n+1}{2} \varepsilon \sigma_g \alpha \otimes \alpha \,. \tag{I.43}$$

Of course, there are many contact Riemannian, contact Lorentzian and para-contact metric structures which are not η-Einstein. We will illustrate this point in Chapter 7.

Applications of contact structures to the realm of physics are extremely abundant: in Hamiltonian mechanics [172–174], thermodynamics [175–179] or in Supergravity and ST [180–186], among others. In this thesis, we will focus on widening this range of possibilities within the setup of minimal six-dimensional Supergravity. More concretely, we will be interested in $\mathcal{N} = (1, 0)$ minimal six-dimensional Supergravity coupled to a tensor multiplet with constant dilaton [187–191]. The bosonic degrees of freedom are described by an oriented smooth (Lorentzian) manifold g, a two-form $B \in \Omega^2(M)$ and the dilaton. The set of equations of motion and Bianchi identity for (g, B) after imposing the dilaton to be constant can be seen to be[34]:

$$\mathrm{Ric}^g - \frac{1}{4} H \circ H = 0 \,, \quad \mathrm{d}H = 0 \,, \quad \mathrm{d} \star H = 0 \,, \quad |H|^2_g = 0 \,, \tag{I.44}$$

[33] We have not defined them before, but they are almost (para)Hermitian manifolds for which the two-form $\omega = g(\cdot, J\cdot)$ is closed.

[34] The set of equations of motion and Bianchi identity for (g, B) and generically non-constant dilaton can be obtained equivalently from trivial dimensional reduction of the ten-dimensional NS-NS action (I.29) down to four-dimensions.

where $H = \mathrm{d}B$ and where, in some local coordinates, $(H \circ H)_{\mu\nu} = H_\mu{}^{\alpha\beta} H_{\nu\alpha\beta}$.

When it comes to the resolution of the previous set of equations, one customarily concentrates on the subset of supersymmetric solutions, which has been extensively studied in the literature [192–204]. Apart from their phenomenological interest, supersymmetric solutions are generically more simple to obtain since typically it turns out to that configurations solving the conditions for spacetime supersymmetry (the so-called Killing spinor equations) and the corresponding Bianchi identities satisfy automatically all the set of equations of motion and Bianchi identities [205].

In this context, in Chapter 7 we devote ourselves to the resolution of the system of equations (I.44) without resorting to any supersymmetric condition, which will allow us to derive generically non-supersymmetric solutions. Of course, rather than studying the problem of classifying solutions to minimal Supergravity coupled to a tensor multiplet in its full generality, which seems to be currently out of reach, we will assume that the six-dimensional Lorentzian manifold (M, g) splits in the form of a direct product $(N \times X, \chi \oplus h)$, where (N, χ) and (X, h) are three-dimensional oriented Lorentzian and Riemannian manifolds, respectively. Then, we will find that the product of so-called $\varepsilon\eta$-Einstein contact structures, which we will also introduce, yields a bi-parametric family of solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton. In particular, such $\varepsilon\eta$-Einstein contact structures encompass particular η-Einstein Riemannian, Lorentzian and para-contact metric structures, but also allow for the possibility of a lightlike Reeb vector field. This latter case seems not to have been previously studied in the literature and will motivate the definition of *null contact structures*, which we will examine in detail.

I.3 Examples of higher-order gravities

Now we proceed to introduce some of the most popular examples of higher-order gravities in the literature. In particular, we will provide a brief introduction to $f(R)$ theories, Lanczcos-Lovelock gravities and Generalized Quasitopological Gravities.

I.3.1 $f(R)$ theories

$f(R)$ theories are one of the most celebrated examples and more widely-studied higher-order gravities. They were first rigorously studied by Buchdahl back in 1970 [206] and are given by the following type of actions [207, 208]:

$$I_f = \frac{1}{16\pi G} \int \mathrm{d}^D x \sqrt{|g|} \left[f(R) + \mathcal{L}_{\text{matter}} \right] , \tag{I.45}$$

where $f \in C^\infty(\mathbb{R})$ is a certain smooth function of the Ricci scalar and $\mathcal{L}_{\text{matter}}$ is the Lagrangian associated to matter minimally-coupled to gravity. The associated equations of motion are given by [207, 208]:

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right] f'(R) = \frac{1}{2} T_{\mu\nu} \,, \tag{I.46}$$

where the stress-energy tensor is defined as usual[35],

$$T_{\mu\nu} = -2\frac{\delta \mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}\,. \tag{I.47}$$

We observe that the Einstein equation (I.46) is clearly of fourth order in derivatives of $g_{\mu\nu}$, so one could expect *a priori* the appearance of Ostrogradski instabilities. Nevertheless, it can be shown [209, 210] that $f(R)$ theories, as long as $f''(R) \neq 0$, are equivalent to the following Brans-Dicke theory with a scalar potential:

$$I_{\text{BD}} = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{|g|}\left[\varphi R - V(\varphi) + \mathcal{L}_{\text{matter}}\right]\,, \tag{I.48}$$

where the scalar potential V is given by:

$$V(\varphi) = \chi(\varphi)\varphi - f(\chi(\varphi))\,, \tag{I.49}$$

χ being obtained by inverting $\varphi = f'(\chi)$. Since the action (I.48) has second-order equations of motion, we conclude that $f(R)$ theories avoid the Ostrogradski instability [211]. On account of this property, these models have been very popular in cosmology, specially for the exploration of inflationary and late-time cosmic acceleration scenarios [76, 212–214].
Nevertheless, being equivalent to Brans-Dicke theories, $f(R)$ models do not actually introduce new purely gravitational phenomena. In fact, the Schwarzschild and Kerr black holes, which are solutions of the vacuum GR Einstein's equation, solve as well[36] the equations of motion (I.46). Consequently, these models do not actually tell us how higher-derivative terms correct GR solutions. Furthermore, it can be seen that the Schwarzschild-(A)dS solution (which is an exact solution of $f(R)$ theories in absence of matter [215]) cannot correspond to the exterior solution in presence of a spherical distribution of mass [216–220], so it would be convenient to find other higher-derivative theories which do fulfill this condition.

Finally, before closing this example, it is convenient to elucidate a subtle point. In GR, apart from the usual variation of the metric to obtain Einstein's equations, there is an equivalent way of deriving it which receives the name of Palatini formalism [221] (see [222] for a modern treatment of the topic), in which one considers the connection and the metric as independent variables and varies them independently. The corresponding equation of motion for the connection fixes it to be the Levi-Civita connection and, after substitution on the equation for the metric, one recovers Einstein's equation. However, for $f(R)$ gravities (and, in fact, for all higher-order gravities with the exception of Lanczos-Lovelock gravities to be studied next) this is no longer true and the so-called metric formalism does not yield the same equations of motion as the Palatini formalism after solving the equation for the connection [207], so that the two possibilities give rise to different equations of motion and different theories. In our case, all along the thesis we will always work in the metric formalism and assume the connection is given by the Levi-Civita connection.

[35] Note, however, that in Chapter 3, we will write the Einstein equation for Einstein-Maxwell theory in the form $G_{\mu\nu} = 2T_{\mu\nu}$, where $T_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2$ is the Maxwell stress-energy tensor.

[36] Assuming that $f(R)$ admits a polynomial expansion around $R = 0$ and that there is no cosmological constant (*i.e.*, $f(0) = 0$), this will always be the case.

I.3.2 Lanczos-Lovelock gravities

Another prominent example of higher-order gravities is provided by Lanczos-Lovelock theories [223–226]. They are defined as follows:

$$\begin{aligned} I_{\rm LL} &= \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{|g|}\left[R+\sum_{k=2}^{[D/2]}\alpha_k \ell^{2k-2}\mathcal{L}_{\rm LL}^{(k)}+\mathcal{L}_{\rm matter}\right], \\ \mathcal{L}_{\rm LL}^{(k)} &= \frac{1}{2^k}\delta^{\nu_1\ldots\nu_{2k}}_{\mu_1\ldots\mu_{2k}}R^{\mu_1\mu_2}_{\nu_1\nu_2}\ldots R^{\mu_{2k-1}\mu_{2k}}_{\nu_{2k-1}\nu_{2k}}\,, \end{aligned} \tag{I.50}$$

where $\delta^{\nu_1\ldots\nu_r}_{\mu_1\ldots\mu_r}=(r)!\delta^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\ldots\delta^{\nu_r}_{\mu_r]}$, ℓ is a length scale, α_k are dimensionless couplings and $[x]$ denotes the biggest integer not bigger than $x>0$. These theories are relevant since some of the densities $\mathcal{L}_{\rm LL}^{(k)}$ appear in effective string actions, like the Gauss-Bonnet density $\mathcal{L}_{\rm LL}^{(2)}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2$ [110,227,228]. In another vein, note that the number of densities included for each dimension is different, since $\mathcal{L}_{\rm LL}^{(k)}$ vanishes identically whenever $k>D/2$. Furthermore, when $D=2k$, the associated density $\mathcal{L}_{\rm LL}^{(D)}$ is topological, since it coincides with the Euler characteristic of a compact, oriented (pseudo-)Riemannian manifold by virtue of the Chern-Gauss-Bonnet theorem [229]. Consequently, by direct inspection, we observe that Lanczos-Lovelock gravities in four-dimensions just reduce to Einstein gravity.

As it turns out, Lanczos-Lovelock theories provide the most general diffeomorphism-invariant Lagrangians with second-order gravitational equations of motion (Lovelock's theorem, [225, 226]) and propagate the same degrees of freedom as GR[37]. The corresponding Einstein's equations are given by [230]:

$$G_{\mu\nu}-\sum_{k=2}^{[D/2]}\frac{\alpha_k\ell^{2k-2}}{2^{k+1}}g_{\kappa\mu}\delta^{\kappa\sigma_1\ldots\sigma_{2k}}_{\nu\lambda_1\ldots\lambda_{2k}}R^{\lambda_1\lambda_2}_{\sigma_1\sigma_2}\ldots R^{\lambda_{2k-1}\lambda_{2k}}_{\sigma_{2k-1}\sigma_{2k}}=\frac{1}{2}T_{\mu\nu}\,, \tag{I.51}$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$. Differently from $f(R)$ gravities, the equations of motion of Lanczos-Lovelock gravities (I.51) in vacuum are no longer solved by Ricci flat (or more generally, Einstein) metrics, and thus they do modify the intrinsic dynamics of Gravity (see [227,231–234] for explicit black hole solutions differing from GR's ones). Also, Lanczos-Lovelock gravities are the unique theories of gravity for which the metric and Palatini approaches are strictly equivalent [235–237].

I.3.3 Generalized Quasitopological Gravities

By Lovelock's theorem, the only possible theories of gravity with second-order equations of motion are just the Lanczos-Lovelock gravities. As mentioned before, in four-dimensions they reduce to Einstein's GR, so the only four-dimensional diffeomorphism-invariant metric theory of gravity with second-order equations of motion is GR. However, although it is not possible to find additional theories with this property, we may wonder if there exist theories whose gravitational equations of motion are second-order *under certain circumstances.*

A first step in this direction was given by Oliva and Ray in 2010, who identified a cubic theory of gravity in five dimensions with second-order traced field equations [238]. Shortly

[37]Furthermore, for sufficiently weak gravitational fields one recovers precisely the Newton's law, up to a renormalized gravitational constant (when non-flat maximally-symmetric backgrounds are considered).

after, Myers and Robinson further characterized that theory and generalized it for arbitrary dimension $D \geq 5$, naming it *Quasitopological Gravity* [239]. It is given by the addition to the Einstein-Hilbert action (I.1) of the following density [239] (weighted by a dimensionful coupling):

$$\begin{aligned}\mathcal{Z}_D = &R_{\mu\nu}{}^{\rho\sigma}R^{\nu\lambda}{}_{\sigma\kappa}R_{\lambda}{}^{\mu\kappa}{}_{\rho} + \frac{1}{(2D-3)(D-4)}\left(\frac{3(3D-8)}{8}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R\right.\\ &- 3(D-2)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\kappa}R^{\sigma\kappa} + 3DR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}\\ &\left.+ 6(D-2)R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} - \frac{3(3D-4)}{2}R_{\mu\nu}R^{\mu\nu}R + \frac{3D}{8}R^3\right).\end{aligned} \tag{I.52}$$

These theories possess black hole solutions with spherical, planar or hyperbolic horizons characterized by a single function determined by an algebraic equation of motion and whose linearized spectrum only contains a transverse and traceless graviton on maximally symmetric backgrounds, as in Einstein gravity [238–240]. These features have been shown to hold as well for the so-called quartic [241] and quintic Quasitopological Gravities [242], which motivates the definition of a Quasitopological theory as one admitting static and spherically symmetric (or planar or hyperbolic) solutions completely specified by a single function with algebraic field equation. Further intriguing properties of these theories which mimic or generalize aspects of Einstein gravity have been studied in the last years [216, 241, 243–247].

Nevertheless, Quasitopological Gravities turn out to be trivial in four dimensions and they just collapse into Einstein gravity, just like the Lanczos-Lovelock models. Therefore, we should relax some of the previous properties to try to find special theories with physically-interesting features but still with certain amenability to computations. An important step in this direction was given by Bueno and Cano in 2016 [78], who discovered a four-dimensional higher-curvature gravity with the same linear spectrum on maximally symmetric backgrounds as Einstein gravity. Such theory received the name of *Einsteinian Cubic Gravity* (ECG) and is obtained by adding to the Einstein-Hilbert Lagrangian the following density $\mathcal{P}$ with a dimensionful coupling:

$$\mathcal{P} = 12R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}{}^{\gamma}{}_{\sigma}{}^{\delta}R_{\gamma}{}^{\mu}{}_{\delta}{}^{\nu} + R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\gamma\delta}R_{\gamma\delta}{}^{\mu\nu} - 12R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + 8R_{\mu\nu}R^{\nu\rho}R_{\rho}{}^{\mu}\,. \tag{I.53}$$

Later on, it was observed that this theory admits static black holes with spherical, planar or hyperbolic horizons and fully-specified by a single metric function with second-order equation of motion [216, 248]. This fact triggered the identification of Generalized Quasitopological Gravities [79] (GQGs), defined precisely as higher-curvature theories admitting solutions with spherical, planar or hyperbolic symmetry characterized by a single function satisfying an ordinary differential equation of second or lower order. Einsteinian Cubic Gravity (I.53) is the paradigmatic example, but instances of such theories have been constructed to all orders and dimensions [249–251]. Naturally, GQGs encompass those of the Quasitopological type, and thus they include as well Lanczos-Lovelock gravities and Einstein gravity, see Figure I.2.

GQGs have been extensively studied in the literature from many points of view, which include cosmology, black holes, holography and phenomenology [89, 249, 252–284]. However, the strict definition of GQGs does not involve matter, so an important extension would be to allow for the inclusion of matter compatible with the single-function condition

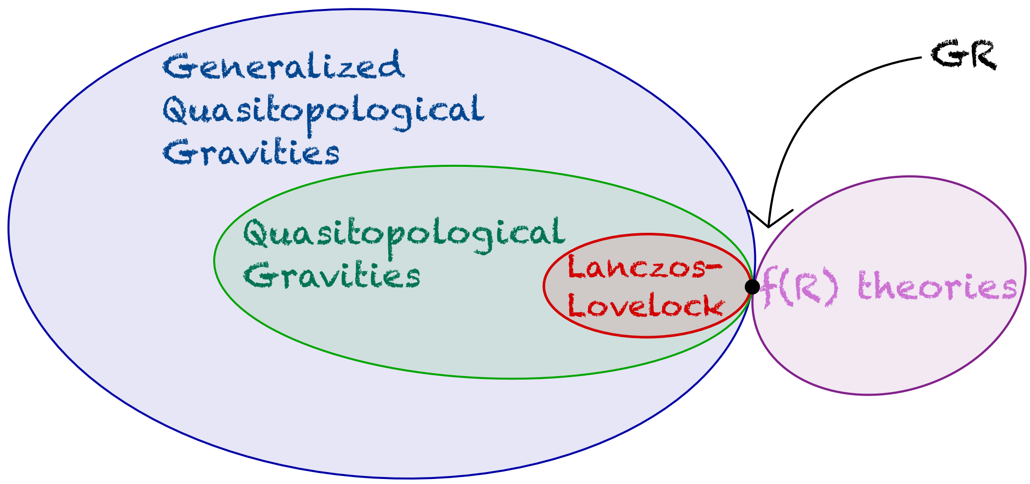


Figure I.2: We show in a pictorial way how the different (purely gravitational) higher-order gravities we have worked so far are embedded (or not) into the different classes. Observe that all of them intersect at a point, which stands for GR.

on backgrounds with static and spherical, planar or hyperbolic symmetry [285]. A simple but relevant instance of this is given by a minimally coupled Maxwell field, explored in [248, 254, 268, 271] in the context of ECG and higher-order GQGs. It is possible also to couple these theories to non-linear electrodynamics, as in [286]. However, these examples just involve minimally-coupled gauge fields, which is a highly restricted way of coupling a vector to gravity. In general, higher-derivative effective actions might contain all sorts of couplings between the fields present in them.

Thus, one interesting question is whether it is possible to identify a family of theories analogous to Generalized Quasitopological Gravities with non-minimal couplings between the curvature and a gauge field. We will answer this question in the positive in Chapter 2, where we introduce and study the so-called *Electromagnetic (Generalized) Quasitopological Gravities*[38] (E(G)QGs), defined as those admitting electrically- or magnetically-charged static solutions with spherical, planar or hyperbolic symmetry. Notwithstanding, in contradistinction to the pure-gravity case, we will be able to find theories of the Quasitopological type (that is, with algebraic field equation for the function which fully specifies the solution when we consider backgrounds with spherical, planar or hyperbolic symmetry) already in four dimensions, making it worth keeping oneself in such spacetime dimension all along Chapter 2. The study of E(G)QGs in higher dimensions is postponed to Chapter 4.

[38]Not to be confused with the recently constructed "Quasitopological Electromagnetism" [287,288], which provides a non-linear extension of Maxwell's electromagnetism with intriguing properties and explicit black hole solutions when minimally coupled to gravity.

I.4 Duality rotations in higher-order gravities with electromagnetism

Now let us go back momentarily to Eqs. (I.15), (I.16) and (I.17), in the context of higher-order theories of gravity and electromagnetism. As commented in Subsection I.1.1, it seems natural to consider the fields F and H on an equal footing[39], assuming them to be two *a priori* independent variables (at the level of the equations of motion) which in turn have to satisfy the constitutive relation (I.16). From this point of view, it is licit to wonder if the set of equations of motion and Bianchi identities (I.15), (I.16) and (I.17) possess any symmetry under transformations of F and H, such as invariance under continuous rotations of F and H.

Let us motivate this problem by taking a look at the case of pure Einstein-Maxwell theory (with $\Lambda = 0$), for which the set of equations of motion and Bianchi identities reduce to:

$$G_{\mu\nu} = -2 \star H_{\langle\mu}{}^{\alpha} F_{\nu\rangle\alpha} \,, \tag{I.54}$$

$$\star H_{\mu\nu} = -F \,, \tag{I.55}$$

$$\mathrm{d} \begin{pmatrix} F \\ H \end{pmatrix} = 0 \,, \tag{I.56}$$

where $\langle\mu\nu\rangle$ represents the symmetric and traceless part of a tensor, $X_{\langle\mu\nu\rangle} = X_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} X^{\alpha}{}_{\alpha}$. We directly observe that under the SO(2) rotations

$$\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} F' \\ H' \end{pmatrix} , \tag{I.57}$$

Eqs. (I.54), (I.55) and (I.56) remain formally invariant. It is natural to call these transformations *duality rotations*. Noting this invariance in the case of the Einstein-Maxwell theory, one may wonder whether it is possible to find analogous duality rotations which leave invariant the set of equations of motion and Bianchi identities of more generic theories.

On the one hand, it has been found that duality rotations can indeed be generalized to be symmetries of $\mathrm{U}(1)^N$ gauge theories coupled to scalar fields [289], which are omnipresent in Supergravity and ST [106, 290–299]. On the other hand, restricting ourselves to theories with a single vector field coupled to gravity, we may wonder at this point whether higher-derivative corrections to Einstein-Maxwell theory will preserve such invariance. Indeed, the idea that the laws of Nature are invariant under certain transformations has turned out to be one of the most fundamental and essential principles of physics, so it is a pertinent question. Nevertheless, higher-order corrections generically break this invariance, so that assuming invariance under duality rotations constrains the possible terms to be included in the effective action. This is particularly appealing from the EFT perspective, since it allows to reduce the number of terms that can appear in the action. Such idea has been particularly successful in ST, whose spectrum and amplitudes are believed to be invariant under T-duality[40]. This motivates the claim that T-duality is a symmetry of the effective action [300] at all orders in the α'-expansion[41], which restricts significantly the higher

[39] This is only possible when there are no currents which couple directly to the vector potential and if we are only interested in the gravitational effects of the electromagnetic field.

[40] This transformation was discovered in string compactifications, interchanging winding modes with KK modes and compactification radii with their inverses.

[41] This fact has been confirmed explicitly, at least, for the lowest-order terms [301–306].

derivative corrections that may take place in the stringy effective actions[42] [312–314]. Therefore, it is clear that the study of higher-order theories which are invariant under duality rotations can be extremely useful.

The possibility of finding duality-invariant higher-derivative theories has been explored in the literature only within the restricted subclass of theories called non-linear electrodynamics (NLE), which are made of polynomials with arbitrary powers of the gauge field strength (so the Maxwell equation is no longer linear in $F_{\mu\nu}$). They are usually constructed in flat space, although the inclusion of minimal couplings to gravity is typically straightforward. The most representative examples of NLE we may mention today are Born-Infeld electrodynamics, whose discovery dates back to almost a century ago, and ModMax electrodynamics, found just a couple of years ago.

Example I.7. Born-Infeld electrodynamics. This theory was identified in 1934 by Born and Infeld [315]. The motivation for its discovery was to find a theory that guaranteed the finiteness of the electric field self-energy of charged particles. This theory was first described in flat space and the Lagrangian is given by:

$$\mathcal{L}_{\rm BI} = \frac{1}{b^2}\left(1-\sqrt{\det\left(\eta_{\mu\nu}-2bF_{\mu\nu}\right)}\right) . \tag{I.58}$$

where b is a constant with units of length squared. In the $b \to 0$ limit, we recover Maxwell theory, while in the $b \to \infty$ limit one would obtain the so-called Bialynicki-Birula electrodynamics [316, 317]. Born-Infeld theory (I.58) can be shown [318, 319] to be exactly invariant under duality rotations and it has acquired special relevance in ST as well, since the Born-Infeld action turns out to appear naturally in the context of ST [320, 321].

Example I.8. Mod-Max electrodynamics. Discovered by Bandos, Lechner, Sorokin and Townsend in 2020 [322], it has attracted a lot of attention [323–327]. It is characterized for being the most general duality- and conformally-invariant theory. Its Lagrangian is given by [322, 323, 328]:

$$\mathcal{L}_{\rm ModMax} = -\cosh\gamma F^2 + \sinh\gamma\sqrt{(F^2)^2+(F^{\mu\nu}\star F_{\mu\nu})^2}\,, \tag{I.59}$$

where $\gamma \geq 0$ is a dimensionless parameter. First, we note that in the $\gamma \to 0$ limit one recovers Maxwell electrodynamics, while $\gamma \to \infty$ yields the aforementioned Bialynicki-Birula electrodynamics. Second, the theory possesses conformal symmetry, as can be observed by noticing that there is no length scale appearing explicitly in the Lagrangian.

With the notable exception of NLE theories including minimal couplings to gravity, such as in the case of Einstein-Born-Infeld theory

$$I_{\rm BI} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R + b^{-2}\left(1-\frac{1}{\sqrt{|g|}}\sqrt{|\det(g_{\mu\nu}-2bF_{\mu\nu})|}\right)\right] \tag{I.60}$$

or Einstein-ModMax theory [324], the study of general effective extensions of the Einstein-Maxwell theory with non-minimal couplings between the spacetime curvature and the gauge field strength (like $R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$) seems to be largely missing in the literature. Therefore, since non-minimal couplings generically appear in stringy effective actions [304], we have found relevant to study in Chapter 3 duality-invariant higher-order theories of electromagnetism which are non-minimally coupled to gravity.

[42] Sometimes it is even possible to constrain all α'-corrections in special contexts like cosmological ones [307–311].

I.5 Initial value problem in General Relativity

The ultimate goal of physics is to describe the behaviour of Nature. In particular, this implies discovering the laws that allow us to predict the behaviour of a physical system once we know its state at a certain *initial time*. This is precisely the way Classical Mechanics or Quantum Mechanics work. However, if one takes a look at GR, it is not clear whether we have such initial value formulation. In fact, one may wonder: which set of initial conditions do we need in order to state a well-defined Cauchy problem? Where would these initial conditions be defined? Does it make sense even to consider this problem, given the general coordinate-invariance of GR and the subsequent relative definition of time? These are highly non-trivial questions, since Einstein's field equations intertwine space and time in an intrinsically non-linear fashion.

These questions were answered by Choquet-Bruhat and Geroch [329,330], who showed that the initial value (or Cauchy) problem of GR is *well posed*. We will momentarily explain in greater detail what we mean by well-posedness, but let us remark at this point that the initial value formulation of GR, which can be identified with a Hamiltonianian approach [85, 331–334], has become an essential tool both in Mathematical Relativity [335–337] and Numerical Relativity [338–340], thus justifying the need of understanding the Cauchy problem of GR in the present era of Gravitation.

To outline the main features of the GR Cauchy problem, we have found convenient to follow [335, 341–343]. We begin by introducing the notion of Cauchy hypersurface and globally hyperbolic manifold, which play a fundamental role in the initial value formulation of GR.

Definition I.9. Let (M, g) be an oriented and connected four-dimensional[43] Lorentzian manifold. A *Cauchy hypersurface* (or Cauchy surface, for short) Σ is an embedded submanifold $i : \Sigma \hookrightarrow M$ which every non-spacelike curve intersects exactly once. In particular, it has to be a spacelike hypersurface. If (M, g) possesses a Cauchy hypersurface Σ, then (M, g) is said to be *globally hyperbolic*.

Globally hyperbolic manifolds are the natural arena on which one could pose the Cauchy problem of GR. Indeed, given a Cauchy surface Σ on (M, g), one could predict the state of the universe at any time from the knowledge of some appropriate *initial data* on Σ, since every point of M could then be reached by causal (timelike and lightlike) curves departing from Σ.

For the sake of simplicity, let us restrict our study to the Einstein field equation in vacuum. We focus now on the problem of determining what initial data we have to provide on a Cauchy surface Σ so as to obtain its time evolution, which will yield in turn the corresponding Ricci flat globally hyperbolic development. For that, let us establish, first, that every globally hyperbolic manifold (M, g) splits as [344–346]

$$(M, g) = (\mathcal{I} \times \Sigma, -\beta_t^2 \mathrm{d}t^2 + h_t)\,, \tag{I.61}$$

where $\mathcal{I} \subset \mathbb{R}$, $t \in \mathcal{I}$ is identified with the time coordinate, $\{h_t\}_{t\in\mathcal{I}}$ is a family of Riemannian metrics on Σ and $\{\beta_t\}_{t\in\mathcal{I}}$ is a family of positive functions on Σ. Assume that $t = 0$ is the initial time on which we want to pose the subsequent Cauchy problem. By direct inspection of (I.61), it is very natural to expect the induced metric $h = h_t|_{t=0}$ on Σ to

[43] We will consider four spacetime dimensions in this section for the sake of simplicity.

be part of the initial data. Also, it would be also reasonable to expect the information of how *curved* (Σ, h) is within (M, g) as part of these initial data. A suitable measure of this *extrinsic curvature* is given by the so-called *shape operator* or second fundamental form $\Theta \in \mathrm{End}(T\Sigma)$, which is given by:

$$\Theta = -\frac{1}{2\beta_t}\partial_t h_t\Big|_{t=0} . \tag{I.62}$$

Interestingly enough, (Σ, h, Θ) conform all the necessary initial data to determine uniquely a Ricci flat globally hyperbolic manifold (M, g) such that for every $t \in \mathcal{I}$ the corresponding induced metric on $\{t\} \times \Sigma$ is precisely h_t, as we will see in Theorem I.1. Nevertheless, it is convenient to point out first that not every set of initial data (Σ, h, Θ) is allowed. In fact, *admissible* initial values must satisfy the following constraint equations:

$$\mathrm{R}_h = |\Theta|_h^2 - \mathrm{Tr}_h(\Theta)^2 , \quad \mathrm{d}\mathrm{Tr}_h(\Theta) = \mathrm{div}_h(\Theta) , \tag{I.63}$$

where R_h denotes the Ricci scalar of (Σ, h). The first equation in (I.63) is called the *Hamiltonian constraint* while the second one receives the name of *momentum constraint*. These constraints are obtained by decomposing the Einstein field equation $\mathrm{Ric}^g = 0$ according to the $3+1$-splitting canonically induced by (I.61). A triple (Σ, h, Θ) satisfying (I.63) will be called admissible or allowed indistinctly. If (Σ, h, Θ) denotes some admissible initial data, the equations determining the temporal evolution $\{\beta_t, h_t\}_{t\in\mathcal{I}}$ are obtained from the remaining components of $\mathrm{Ric}^g = 0$.

Note that the constraints (I.63) could be obtained at every time t, *i.e.,* for every Cauchy surface $\Sigma_t := \{t\} \times \Sigma$. Consequently, we have to make sure that the subsequent time evolution preserves the constraints (I.63). In other words, if we start from some initial data satisfying (I.63) and we time-evolve them according to the Einstein field equation for $t \in \mathcal{I}$, we have to make sure that the corresponding constraint equations obtained on every Σ_t hold as well. Together with ensuring the existence and uniqueness of solutions, this is what we mean by the well-posedness of the GR initial value problem. These are extremely subtle and non-trivial points, which may fail to be true.

Remarkably enough, Choquet-Bruhat and Geroch [329, 330] were able to show that the Cauchy problem of GR is well posed, so that the solution is unique, exists and the restriction at any time of some evolved admissible initial data does also satisfy the subsequent constraint equations.

Theorem I.1 (Choquet-Bruhat, Geroch). *Let (Σ, h) be a smooth Riemannian three-manifold and let $\Theta \in \mathrm{End}(T\Sigma)$ be a certain smooth tensor on Σ. Suppose that the triple (Σ, h, Θ) satisfies the constraint equations (I.63). Then there exists a unique smooth spacetime (M, g), called the maximal Cauchy development of (Σ, h, Θ), which fulfills the following properties:*

1. *(M, g) is Ricci flat, i.e., it solves the Einstein field equations*[44] $\mathrm{Ric}^g = 0$.
2. *(M, g) is globally hyperbolic with Cauchy surface Σ.*
3. *The induced metric on Σ and the shape operator of Σ are given by (h, Θ), respectively.*

[44] It is possible to generalize Theorem I.1 in the presence of a cosmological constant and/or matter fields whose (matter) Lagrangian contains no more than first derivatives of the fields [342].

4. *Every other spacetime satisfying (1)-(3) can be mapped isometrically into a subset of* (M, g).

Finally, the solution (M, g) *depends continuously on the initial data* (Σ, h, Θ).

However, observe that it might be possible to extend the maximal Cauchy development (M, g) of certain allowed initial data (Σ, h, Θ) into a proper subset of a different spacetime, although Σ would no longer be a Cauchy surface for this enlarged manifold and, thus, it would not be globally hyperbolic. In another vein, note that there are quite important instances of spacetimes which are not globally hyperbolic, such as Anti-de Sitter space, the Taub-NUT space or the Reissner-Nordström solution. These spaces do not have a Cauchy surface, but it is possible to formulate the initial value problem and obtain a subsequent globally hyperbolic development which would just be a proper subset of the total spacetime.

In this thesis, the formulation of GR as an initial value problem will make its appearance in Chapter 5, where we will study the Cauchy problem of a real parallel spinor (see Section I.8) on a globally hyperbolic manifold. Although not studying the Cauchy problem of Ricci flat metrics in itself, we will be able to prove a very powerful result: given some initial data fulfilling the constraint equations (I.63) and the constraint equations associated to the existence of a real parallel spinor, then the evolution posed by the parallel spinor (which is first order in time) matches exactly with the second-order evolution prescribed by Einstein's field equations. This is a quite intriguing result, since this suggests the tantalizing possibility of using first-order differential equations to construct special solutions of GR.

I.6 Black holes

Shortly after the discovery of GR in 1915 by Einstein, Karl Schwarzschild presented the first non-trivial exact solution to Einstein's field equations [347]. Today known as the Schwarzschild solution, it is static, possesses spherical symmetry and represents the gravitational field outside any matter distribution with these symmetries [348, 349]. However, two issues were observed. On the one hand, a divergence appears at the point $r = 0$, where r is a variable for which the spheres selected by the reigning symmetry have area equal to $4\pi r^2$, which was at first somewhat omitted and just interpreted as the relativistic analogue of the $1/r^2$ divergence in the Newton's law of Universal Gravitation. On the other hand, another problem was observed to occur at $r = r_G = 2GM$, M being the mass of the object we are describing[45]. Indeed, any light ray emitted from $r < r_G$ would not be seen by observers in the region $r > r_G$, since the gravitational field would be so strong than even light could not escape from the region $r < r_G$. Although originally ostracized, these *dark* objects were not properly understood until almost 50 years after their first appearance, during the so-called Golden Age of GR, when the term *black hole* was coined. For more historical details, we refer the reader to [350–352].

They are one of the most extraordinary, impressive and elegant predictions of GR, and today we have compelling evidence of their existence thanks to the gravitational wave detectors LIGO/Virgo [61–67] and the Event Horizon Telescope collaboration [68–71]. With the additional aid of the future space-based interferometer LISA [72], in the forthcoming years it is expected that experimental data will be accurate enough to permit precision tests on black holes. Consequently, nowadays is the best moment to make progress in

[45] As comparison with the Newtonian limit or the ADM formula (reviewed in (I.19)) may reveal.

the theoretical understanding of black holes, specially in the study of how their physical properties get modified after the inclusion of higher-derivative corrections [353]. To this aim, we commit ourselves in this section to introduce some of the most intriguing and attractive properties of black holes in GR, which we will examine afterwards in the thesis in the context of higher-order theories.

I.6.1 Definition of a black hole. Event horizon

Before going through the definition of a black hole, let us introduce some preliminary concepts. We will follow the analyses of [341, 354–356]. First, we begin by defining the notion of asymptotically simple space. A connected, oriented and time-oriented Lorentzian four-manifold $(\mathcal{M}, g)$ is said to be an asymptotically simple spacetime if there exists another Lorentzian four-manifold $(\bar{\mathcal{M}}, \bar{g})$ and an embedding $\theta : \mathcal{M} \to \bar{\mathcal{M}}$ which embeds $\mathcal{M}$ as a manifold with smooth boundary $\partial\mathcal{M}$ into $\bar{\mathcal{M}}$ such that:

1. There is a smooth function $\Omega \in C^\infty(\bar{\mathcal{M}})$ such that on $\theta(\mathcal{M})$, $\Omega > 0$ and $\theta^* \bar{g} = \Omega^2 g$.
2. $\Omega|_{\partial\mathcal{M}} = 0$ and $\mathrm{d}\Omega|_{\partial\mathcal{M}} \neq 0$.
3. All null geodesics in $\mathcal{M}$ have two endpoints in $\partial\mathcal{M}$.

If $(\bar{\mathcal{M}}, \bar{g})$ fulfills at least the first two conditions above, then $(\bar{\mathcal{M}}, \bar{g})$ is said to be a *conformal compactification* of $(\mathcal{M}, g)$, from where the so-called Carter-Penrose diagrams arise, of extreme utility in understanding the causal structure of the spacetime. On other hand, if we add to the previous three points the condition:

4. $R_{\mu\nu} = 0$ on a open neighbourhood of $\partial\mathcal{M}$ in $\bar{\mathcal{M}}$,

then $(\mathcal{M}, g)$ is said to be asymptotically empty and simple. Examples of these spaces include Minkowski space or asymptotically simple spaces harbouring compact objects like planets or stars that have not gone through gravitational collapse. Nevertheless, there are popular examples of spacetimes, such as the Schwarzschild, Reissner-Nordström or Kerr ones, which do not satisfy the definition of being asymptotically simple, since there exist null geodesics with no endpoints on $\partial\mathcal{M}$.

To overcome this setback, we relax the previous definition and say that a spacetime $(\mathcal{M}, g)$ is *weakly asymptotically simple* is there exists an asymptotically simple space $(\mathcal{M}', g')$ and some open subset U' of $\mathcal{M}'$ with $\partial\mathcal{M}' \subset U'$ such that $U' \cap \mathcal{M}'$ is isometric to an open set U of $\mathcal{M}$. In simple terms, a weakly asymptotically spacetime has the conformal infinity of an asymptotically simple one but it may possess additional infinities too. Finally, a weakly asymptotically simple spacetime if *asymptotically flat* if it satisfies condition 4 above.

Now we are in a position to present the definition of a *black hole*.

Definition I.10. Let $(\mathcal{M}, g)$ be an asymptotically flat spacetime[46]. It is said to be or to contain a black hole if the following region $\mathcal{B} \subset \mathcal{M}$ is non-empty:

$$\mathcal{B} \equiv \mathcal{M} \setminus J^-(\mathscr{I}^+)\,, \tag{I.64}$$

[46] The generalization to asymptotically AdS spacetimes is by no means direct or trivial, since the rigorous definition of being asymptotically AdS is already quite involved [357]. However, for the purposes of this introduction, we believe it is enough to have stated the precise definition of an asymptotically flat black hole.

where $J^-(\mathscr{I}^+)$ denotes the chronological past of the future null infinity $\mathscr{I}^+ \subset \partial\mathcal{M}$, which is defined as the set of points $p \in \mathcal{M}$ which can be reached from a past-directed null curve starting from $\mathscr{I}^+$. The boundary of $\mathcal{B}$ will be denoted by $\mathcal{H}$ and is called the *event horizon*.

Observe that the concept of black hole is a global one — one actually needs the knowledge of the future history in order to determine it. Thus, the notion of event horizon lacks a local significance, as it is reflected by the fact that an observer freely falling into a black hole would feel nothing special while crossing it.

Nevertheless, from its definition it seems highly intricate to find event horizons and to study spacetimes with black holes. The situation dramatically improves when we consider stationary asymptotically flat spacetimes, defined in terms of the existence of a Killing vector field ξ^α which becomes timelike asymptotically. Note that this is a more general setup that the one we will be mainly dealing with in the thesis, which corresponds to static spacetimes. These are defined as stationary spacetimes for which the Killing vector ξ^α is hypersurface orthogonal, *i.e.* $\xi \wedge d\xi = 0$, where ξ is the metric-dual one-form of the Killing vector. According to the so-called rigidity theorems [356, 358, 359], originally proven by Hawking and Carter, the event horizon of a stationary black hole, under some generic conditions, is a Killing horizon (defined as a null hypersurface on which a Killing vector k^μ becomes null, which in turn is said to generate the horizon). In particular, Carter [358] showed by purely geometrical arguments that the event horizon of stationary and axisymmetric black holes is indeed a Killing horizon (the associated Killing vector being asymptotically timelike), while Hawking [356] proved that for any stationary (vacuum or electrovacuum) black hole in GR the associated event horizon must be a Killing horizon. Therefore, given a stationary spacetime which we suspect contains a black hole, we may look for its event horizon by studying the potential Killing horizons and then analyzing if such Killing horizons are event horizons, which greatly simplifies the problem of identifying event horizons.

In case a Killing horizon $\mathcal{H}$ exists, we have that the associated Killing vector satisfies

$$k^\nu \nabla_\nu k^\mu|_{\mathcal{H}} = \kappa k^\mu \,, \tag{I.65}$$

where κ is *a priori* a non-constant function on $\mathcal{H}$ called surface gravity and accounts for how much the integral curves of k^μ fail to be affinely parametrized. In the case of static spacetimes, the Killing vector generating the horizon coincides with the one associated to static symmetry. For axisymmetric stationary spacetimes with a rotating horizon, it turns out that $k^\mu = t^\mu + \Omega_H \omega^\mu$, where t^μ and ω^μ are the Killing vectors associated to time translations and rotations around the axis respectively and Ω_H is the angular velocity of the event horizon.

Although κ could be an arbitrary function on $\mathcal{H}$, it can be proved under quite general conditions that it is constant on the horizon. In fact, this can be shown for stationary and axisymmetric black holes without relying on Einstein's field equations [358,360], or for any black hole in GR using Einstein's field equations and the dominant energy condition [37, 361]. This (generic) constancy of κ on the horizon is called the *zeroth law of black hole mechanics*, a name which we will justify very shortly. If $\kappa \neq 0$, one can show that k^μ generates a bifurcate Killing horizon [360,362] while if $\kappa = 0$, the horizon is degenerate and the black hole is said to be *extremal*.

Before presenting explicit examples of black holes, we delve into their thermodynamic properties and present a brief review of the singularity theorems.

I.6.2 Black hole thermodynamics

The discovery and development of black hole thermodynamics was not a linear nor ordered process, since some results were hypothesized and proposed before they were really proven and better understood. In fact, some aspects of black hole thermodynamics remain yet to be understood and lie precisely at the core of the problem of finding the theory of Quantum Gravity.

Let us start from the work of Bardeen, Carter and Hawking of 1973 [37], in which they established the *four laws of black hole mechanics*[47], named this way because of their resemblance with the four laws of thermodynamics. Firstly, the zeroth law of black hole mechanics was stated, which claims that the surface gravity of a black hole is constant (generically non-zero) over the event horizon, as already mentioned. This is in analogy with the zeroth law of thermodynamics, by which thermodynamic systems in equilibrium have uniform temperature. Nevertheless, from a purely classical perspective, this is just a mere coincidence, since the temperature of a classical black hole has to be strictly zero (classically, they would just absorb particles and they would not radiate).

It was Hawking who rigorously proved that the surface gravity can be properly interpreted as the black hole temperature [38], showing through semiclassical computations that black holes do radiate as black bodies with a temperature given by

$$T_{\mathrm{BH}} = \frac{\hbar\kappa}{2\pi}\,. \tag{I.66}$$

Regarding the first law, it claims that under perturbations of a black hole solution, the following equality holds:

$$\delta M = \frac{\kappa}{8\pi G}\delta A_H + \Omega_H \delta J + \ldots\,, \tag{I.67}$$

where M stands for the black hole mass, A_H is the area of the event horizon, Ω_H is the rotational velocity of the black hole, J is its angular momentum and the ellipsis stands for work terms associated to other variables on which the black hole may depend. For instance, in the case of electrically- and/or magnetically-charged black holes (Reissner-Nordström solution), we would have to include[48] $\phi_h \delta Q + \psi_h \delta P$, where Q is the electric charge, ϕ_h the electrostatic potential at the horizon[49], P the magnetic charge and ψ_h the electrostatic potential of the dual vector field strength (it could be also called magnetostatic potential or just magnetic potential).

If the zeroth law states that the surface gravity is proportional to the temperature, by comparison of (I.67) with the first law of thermodynamics we conclude that black holes should have an entropy S_{BH} given by:

$$S_{\mathrm{BH}} = \frac{A_H}{4\hbar G}\,, \tag{I.68}$$

where the proportionality constant is fixed by using (I.66). This is the black hole entropy or the Bekenstein-Hawking entropy [38, 39, 41].

[47] The third law was just conjectured in [37].

[48] We will find this result explicitly in Chapters 2, 3 and 4. Nevertheless, it was argued in [363] (and references therein) that such first law could be better expressed as $\phi_h \delta Q - \psi_h \delta P$.

[49] In Chapters 2 and 3 we will follow conventions by which the first law will look precisely this way. However, in Chapter 4, ϕ_h and ψ_h are replaced by their asymptotic values because of different conventions for the associated electrostatic potentials.

The second law of thermodynamics states that the entropy of an isolated thermodynamic system never decreases. Hence, to have a complete analogy, the black hole entropy, or equivalently the area of a black hole, should never decrease. Remarkably enough, this is precisely the content of the Hawking area theorem [356], which implies that $\delta A_H \geq 0$ with time[50]. Nevertheless, this theorem is violated on account of the Hawking radiation predicted by semiclassical computations (the same forcing black holes to have the temperature (I.66)), since this implies a loss of black hole's mass and, subsequently, a decrease in the event horizon area. A resolution to this problem is provided by defining the generalized entropy [41, 364] $S_{\rm gen} = S_{\rm BH} + S_{\rm out}$, where $S_{\rm out}$ stands for the entropy of the matter surrounding the black hole, and establish that for all physical processes $\delta S_{\rm gen} \geq 0$, since the entropy of the radiated particles would compensate the loss associated to the decrease of the area.

Finally, the third law of black hole mechanics (proved by Israel [365] under reasonable energy conditions) claims that the surface gravity (*i.e.,* the black hole temperature) cannot be lowered down to zero in a finite time, in resemblance with the usual third law of thermodynamics which states that the absolute zero of temperature cannot be reached in a finite number of steps.

These laws of black hole dynamics were discovered in the context of GR, so a relevant and pertinent question in the context of this thesis is whether these laws are expected to hold for higher-derivative gravities. Firstly, Hawking's computation of the black hole temperature only relies on the notion of event horizon, without making any reference to the gravitational field equations. Consequently, the identification of the surface gravity with the temperature is fully general and must be valid as well in the realm of higher-order theories. Moreover, since the constancy of κ on the horizon can be rigorously proven at least for every stationary and axysimmetric black hole [360], this provides strong evidence for the zeroth law to hold as well for any higher-derivative theory.

Secondly, let us now turn our attention to the validity of the first law for generic higher-order theories. In these theories, the black hole solutions will be modified with respect to the GR ones. This implies that the expressions for the temperature or the area of the black hole in the corrected theory will change with respect to the GR solution, so we cannot expect the first law (I.67) to hold as it stands for higher-derivative theories. Therefore, if we want to have any chance of discovering a general first law of black hole thermodynamics in this context, we should consider the possibility that the entropy is no longer proportional to the area. In fact, one could even wonder why we should expect such first law to hold for higher-order theories.

This skepticism goes away thanks to Wald [366], who proved the existence of a first law of black hole mechanics in any theory of gravity preserving diffeomorphism invariance. In fact, following this formalism, the black hole entropy in generic theories of gravity is no longer given by the event horizon area but rather by the Wald's entropy formula[51] [366, 370, 371]:

$$S_{\rm Wald} = -2\pi \int_{\mathcal{H}} \mathrm{d}^{D-2}x\sqrt{h}\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{I.69}$$

where the integral is carried out over the bifurcation surface of the horizon, h is the corre-

[50] This observation together with the apparent violation of the second law of thermodynamics in processes in which black holes are present led Bekenstein to claim that the event horizon area should play the role of the entropy in a usual thermodynamic system [39, 40].

[51] Notwithstanding, some subtleties arise when defining the entropy for theories which include matter with internal gauge freedom [367–369].

sponding induced metric, $\epsilon_{\mu\nu}$ is the volume form transverse to the horizon (the binormal) normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ and where $\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}$ stands for the Euler-Lagrange variation of $\mathcal{L}$ with respect to the Riemann curvature tensor, considered as an independent variable:

$$\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \nabla_\alpha R_{\mu\nu\rho\sigma}} \right) + \dots . \tag{I.70}$$

Using this definition for black hole entropy, Wald then showed that the following first law of black hole thermodynamics:

$$\delta M = \frac{\kappa}{8\pi G} \delta S_{\text{Wald}} + \Omega_H \delta J + \dots \tag{I.71}$$

holds.

Regarding the extension of the second law to generic higher-order theories, things do not turn out to be that simple. In fact, the mere consideration of four-dimensional Lovelock terms into the Einstein-Hilbert action may already violate this law [372]. As a potential resolution to such problem, it has been suggested that one should replace Wald's entropy by a more general quantity which coincides with Dong's formula [373] for the entanglement entropy defined in the holographic context (which we define in Section I.7), so that the sum of this new entropy and the entropy of the surrounding matter should never decrease [374, 375]. Finally, the third law of black hole thermodynamics has been examined in the context of higher-order theories, finding instances which violate it [376], although it has been shown that the corresponding solutions are unstable and thus they can be, to some extent, ruled out [377, 378].

I.6.3 Singularity theorems

One of the greatest achievements of the GR's scientific community in the past century was the proof of the so-called *singularity theorems*. They constitute a historical landmark[52] and a key contribution to the understanding of gravity and GR. Many theoretical physicists were involved in their development, but perhaps one could mention Penrose, Hawking and Geroch as the pioneers of this line of research.

These theorems establish generic conditions under which spacetime singularities appear. Although it is a highly non-trivial task to find the most appropriate definition of singularities, it is accepted that they are linked with the existence of inextendible geodesics with finite affine length, which is called *geodesic incompleteness* [379, 380]. Indeed, the presence of incomplete geodesics is a clear proxy of singularities, since they signal the possible appearance of particles from nothing or/and the sudden vanishing of freely-falling test particles. However, even if one eliminates the problem of considering singular spacetimes obtained by excision of a point (to this aim, one may work with inextendible spacetimes), there are some issues with this definition. For example, there exist spacetimes which are spacelike and null geodesically complete, but not timelike complete, and vice versa. Similarly, there exist geodesically complete spacetimes possessing timelike curves of bounded acceleration which have finite proper length [380]. Nevertheless, despite these subtleties, it is conventionally assumed that a spacetime is singular if it contains inextendible timelike

[52] Indeed, half of the Nobel Prize in Physics in 2020 was awarded to Penrose's work on singularity theorems.

or null geodesics, this property being the one that is proved in the so-called singularity theorems.

From the first singularity theorem proven by Penrose [379], many other singularity theorems and generalizations of it have been found [36, 341, 381–383]. They commonly provide conditions under which a given spacetime possesses singularities. More concretely, they generically state [352] that if a spacetime of sufficient differentiability satisfies a constraint on the curvature, a causality condition and an appropriate initial and/or boundary condition, then there exist inextendible incomplete causal geodesics. The paradigmatic singularity theorem is the following one [36].

Theorem I.2 (Hawking and Penrose, 1970). *A spacetime is not timelike and null geodesically complete if:*

1. $R_{\mu\nu}k^\mu k^\nu \geq 0$ *for every non-spacelike vector* k^μ.
2. *Every non-spacelike geodesic contains a point at which* $k_{[\mu}R_{\nu]\rho\sigma[\lambda}k_{\eta]}k^\rho k^\sigma \neq 0$, *where* k^μ *is the tangent vector to the geodesic.*
3. *There are no closed timelike curves.*
4. *One of the following properties holds:*
   (a) *There exists a compact achronal set (that is, a set in which no pair of two points are connected by a timelike curve) without edge*[53].
   (b) *There is a trapped surface, understood as a two-dimensional closed surface whose two families of orthogonal null geodesics are converging.*
   (c) *There exists a point such that the expansion of future-directed (or past-directed) null geodesics emanating from this point becomes negative for every of these geodesics.*

We believe conditions (1), (2), (3) and (4)c for past-directed null geodesics of Theorem I.2 to hold for our Universe[54], so that according to this Theorem, the Universe must have a(n) (initial) singularity [342].

On the other hand, these conditions turn out to hold as well for the usual GR black hole spacetimes, justifying, thus, the presence of singularities. Although they are hidden behind the event horizon as required by the Cosmic Censorship Conjecture [384], it is expected that a UV-complete theory of Quantum Gravity resolves such singularities and yields *regular black holes*. In such a case, one would hope to capture this phenomenon through the associated effective field theory or, equivalently, the corresponding higher-order theory.

Indeed, there has been a keen interest in understanding the properties of these hypothetical regular black holes [385–393], although most of the literature so far has only focused on modelizing these geometries. When it comes to describing a dynamical foundation for regular black holes, things are much more involved. As explained before, one would ideally

[53]See [341] for a precise definition of this feature.

[54]Indeed, if we assume the Universe is described by a Friedmann-Lemaître-Robertson-Walker (FLRW) model (not deviating very much from the case of flat time slices), it is known that for these models the expansion of past-directed null geodesics emanating from the event associated to our present time becomes negative more recently than the matter-radiation decoupling time.

wish to find an effective high-energy modification of GR whose black hole solutions were naturally singularity-free. There have been some interesting attempts toward this goal in the literature [394–410], but all of them have certain limitations, such as an unreasonable amount of fine-tuning [394–396, 406] or the introduction of ad hoc matter [398, 402, 405]. More promising approaches have also been followed [399, 400, 404, 407, 408, 410], although obtaining exact solutions is usually challenging in those cases. Despite these problems, in Chapter 2, we will present the very first instances of theories of gravity with a non-minimally coupled U(1) gauge vector field which, to the best of our knowledge, possess electrically-charged black hole solutions with fully regular gravitational and electromagnetic fields for any value of the mass and non-vanishing charge.

Next we proceed to study two of the paradigmatic black hole solutions in GR: the Schwarzschild and the Reissner-Nordström solutions.

I.6.4 The Schwarzschild black hole

As previously stated, this solution was discovered by Schwarzschild [347] and it is the only spherically symmetric vacuum solution of GR [348, 349]. In the so-called Schwarzschild coordinates, the metric is given by:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,, \tag{I.72}$$

where $r > 2GM > 0$ and where M is the total mass of the spacetime, as can be checked with (I.19). Since the previous expression exactly fits into the form of (I.21), we explicitly observe that this metric is static and spherically symmetric. We impose the condition $r > 2GM$ to guarantee asymptotic flatness and to avoid the metric singularity present at $r = 2GM$. Indeed, it can be readily seen that this apparent singularity is not physical and it is due to a bad choice of coordinates. To this aim, let us introduce the retarded and advanced Eddington-Finkelstein coordinates [354, 379]:

$$u = t - r_*\,, \qquad v = t + r_*\,, \tag{I.73}$$

where r_* is the so-called tortoise coordinate given by:

$$r_* = \int \frac{\mathrm{d}r}{1 - 2GM/r} = r + 2GM \log\left|\frac{r}{2GM} - 1\right|\,. \tag{I.74}$$

Observe that the coordinates (u, v) are constant for outgoing and ingoing null radial geodesics respectively. Using coordinates (v, r, θ, ϕ), the metric takes the form g' with the following coordinate expression:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,. \tag{I.75}$$

The metric (I.75) is defined for every $r > 0$. Therefore, we can *extend* the initial Lorentzian manifold $(\mathcal{M}, g)$ in which the Schwarzschild metric was originally defined to a larger one $(\mathcal{M}', g')$ admitting arbitrary positive values for r. In this larger manifold $\mathcal{M}'$ we clearly see that $r = 2GM$ defines a null hypersurface on which the Killing vector $k = \partial_v$ becomes null, defining a Killing horizon. This alerts us about the possibility of this Killing horizon being actually an event horizon, and careful examination reveals that this is indeed the case, since future-directed timelike and null curves only cross this Killing horizon from the outside ($r > 2GM$) to the inside ($r < 2GM$). The surface gravity reads

$$\kappa = \frac{1}{4GM} \,. \tag{I.76}$$

On the other hand, had we considered the use of the coordinates (u, r, θ, ϕ), then we would have derived the following form g'' for the metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,. \tag{I.77}$$

Analogously, this defines another possible extension $(\mathcal{M}'', g'')$ of the original Schwarzschild manifold $(\mathcal{M}, g)$, since the metric (I.77) remains perfectly well defined for $r > 0$. However, there is a crucial difference with respect to the previous case: now the null hypersurface $r = 2GM$ just allows past-directed causal curves cross from the outside ($r > 2GM$) to the inside ($r < 2GM$), defining thus a "white hole".

As a matter of fact, it is possible to define a still larger extension $(\mathcal{M}^*, g^*)$ into which both $(\mathcal{M}', g')$ and $(\mathcal{M}'', g'')$ can be isometrically embedded, coinciding over the region $r > 2GM$ with the starting $(\mathcal{M}, g)$. This construction is due to Kruskal and Szekeres [411, 412], who introduced the following coordinates:

$$U = -\exp\left[-\frac{u}{4GM}\right]\,, \qquad V = \exp\left[\frac{v}{4GM}\right]\,. \tag{I.78}$$

In terms of this coordinates, the metric takes the form g^* given by:

$$\mathrm{d}s^2 = -32\frac{(GM)^3}{r}e^{-r/2GM}\mathrm{d}U\mathrm{d}V + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \tag{I.79}$$

where r is now a function of the new coordinates U and V, implicitly defined as

$$-UV = \left(\frac{r}{2GM} - 1\right)e^{r/2GM}\,. \tag{I.80}$$

According to (I.78), $-U$ and V should be positive, but if we allow them to take negative values as well, then we are actually extending $(\mathcal{M}, g)$ into a larger Lorentzian manifold $(\mathcal{M}^*, g^*)$ in such a way that $(\mathcal{M}, g)$, $(\mathcal{M}', g')$ and $(\mathcal{M}'', g'')$ are each appropriately isometrically embedded. As a matter or fact, the Kruskal-Szekeres extension $(\mathcal{M}^*, g^*)$ is the unique analytic and locally inextendible extension of the Schwarzschild solution [341].

In order to study the causal properties of the maximally-extended Schwarzschild solution, it is convenient to find a suitable conformal compactification of $(\mathcal{M}^*, g^*)$. For that, let us perform the change of coordinates:

$$\bar{U} = \arctan U\,, \qquad \bar{V} = \arctan V\,. \tag{I.81}$$

In these new coordinates the metric reads

$$\mathrm{d}s^2 = -32\frac{(GM)^3}{r}e^{-r/2GM}\sec^2\bar{U}\sec^2\bar{V}\mathrm{d}\bar{U}\mathrm{d}\bar{V} + r^2(\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2)\,, \tag{I.82}$$

Now, multiplying the previous expression by the conformal factor $\Omega^{-2} = \cos^2\bar{U}\cos^2\bar{V}$, we obtain precisely the conformal compactification $(\bar{\mathcal{M}}, \bar{g})$ of $(\mathcal{M}^*, g^*)$, which we can conveniently depict in the form of a Carter-Penrose diagram as in Figure I.3. There are four different regions. Region I ($U < 0, V > 0$) corresponds to the black hole exterior originally described by the Schwarzschild coordinates (I.72). Region II ($U > 0, V > 0$) represents the black hole interior and it was covered by the advanced Eddington-Finkelstein coordinates (I.75). Region IV ($U < 0, V < 0$) describes the "white hole" interior, being charted by the retarded Eddington-Finkelstein coordinates (I.77). Region III ($U > 0, V < 0$) corresponds to another asymptotically flat Universe isometric to the exterior Schwarzschild solution (I.72) and has been discovered thanks to the Kruskal-Szkeres extension. The event horizon bifurcates into two null hypersurfaces given by $U = 0$ and $V = 0$ which intersect at the so-called *bifurcation sphere* at $U = V = 0$. The wiggling lines at the top and bottom of the diagram represent the geometric locus of points with $r = 0$, which correspond respectively to a past and a future singularity, since there are timelike and lightlike geodesics which emerge or disappear at finite affine distance, respectively. This singularity could have been expected from the fact that the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$ diverges as $r \to 0$.

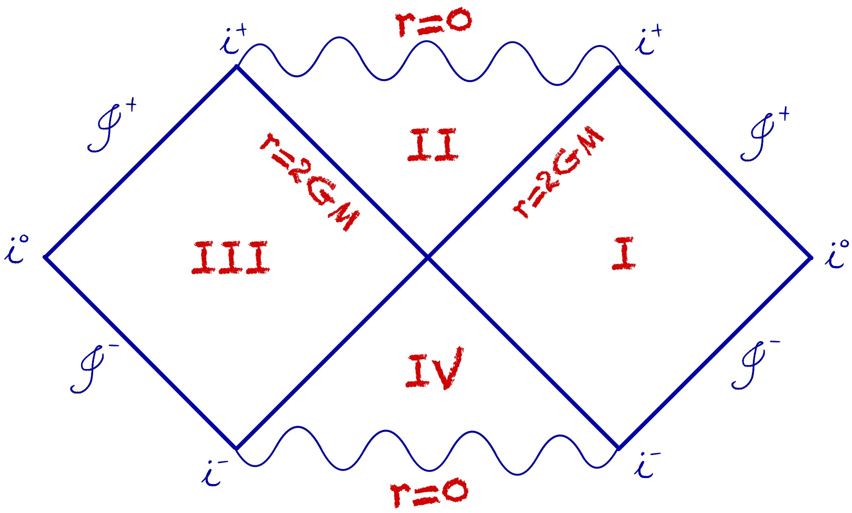


Figure I.3: Carter-Penrose diagram of the maximally-extended Schwarzschild spacetime. $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) denotes the future (past) null infinity, i^+ (i^-) stands for the future (past) timelike infinity while i^0 is the spacelike infinity. The orientation of the diagram is such that time increases as we move upwards and light rays correspond to straight lines with slope of 45°.

I.6.5 The Reissner-Nordström black hole

Before we introduced the Schwarzschild solution and commented about some of its main properties. However, it is a vacuum configuration and it would be interesting to obtain an analogous solution in presence of matter. A very important case is given by the addition of a minimally coupled Abelian gauge field A_μ with field strength $F_{\mu\nu}$, which defines the Einstein-Maxwell action:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R - F^2\right] . \tag{I.83}$$

The set of equations of motion and Bianchi identity is given by:

$$
\begin{aligned}
G_{\mu\nu} &= 2T_{\mu\nu}\,, \text{ where } \quad T_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2\,, \\
\mathrm{d}F &= 0\,, \qquad \mathrm{d}\star F = 0\,.
\end{aligned}
\tag{I.84}
$$

Within the realm of static and spherically symmetric solutions, Reissner, Weyl and Nordström were able to find independently [413–415] a solution to the previous system of PDEs, which is today known as the Reissner-Nordström (RN) solution. Adding both electric and magnetic charges, the complete solution is given by:

$$
\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \qquad f(r) = 1 - \frac{2GM}{r} + \frac{Q^2+P^2}{r^2}\,, \tag{I.85}
$$

$$
F = \frac{Q}{r^2}\mathrm{d}t\wedge\mathrm{d}r + P\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{I.86}
$$

The function $f(r)$ may at most have two zeros located at $r_\pm = GM \pm \sqrt{G^2M^2 - (Q^2+P^2)}$. In analogy with the Schwarzschild black hole, the zeros of $f(r)$ indicate the presence of horizons. However, since the argument of the square root present in $r_\pm$ might not be positive, this motivates us to distinguish the following three different possibilities:
Subextremal RN solution. This corresponds to $GM > \sqrt{Q^2+P^2}$, so that $f(r)$ haves two zeros. In this case, the metric is singular in $r = r_\pm$ and $r = 0$. As in the Schwarzschild case, we can show that the points $r = r_\pm$ do not correspond to physical singularities. To see this, we define:

$$
\begin{aligned}
r_* = \int \frac{\mathrm{d}r}{1 - 2GM/r + (Q^2+P^2)/r^2} &= r + \frac{r_+^2}{r_+ - r_-}\log\left|\frac{r}{r_+} - 1\right| \\
&\quad - \frac{r_-^2}{r_+ - r_-}\log\left|\frac{r}{r_-} - 1\right|\,.
\end{aligned}
\tag{I.87}
$$

Then, performing the change of coordinates:

$$
\begin{aligned}
U_1 &= \arctan\left(-\exp\left(\frac{-r_+ + r_-}{2r_+^2}(t - r_*)\right)\right)\,, \\
V_1 &= \arctan\left(\exp\left(\frac{r_+ - r_-}{2r_+^2}(t + r_*)\right)\right)
\end{aligned}
\tag{I.88}
$$

one arrives to

$$
\mathrm{d}s^2 = 16f(r)\frac{r_+^4}{(r_+ - r_-)^2}\mathrm{cosec}(2U_1)\mathrm{cosec}(2V_1)\mathrm{d}U_1\mathrm{d}V_1 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \tag{I.89}
$$

r being a function of U_1 and V_1 implicitly defined as

$$
\tan U_1 \tan V_1 = -\exp\left(\frac{r_+ - r_-}{r_+^2}r\right)\left(\frac{r}{r_+} - 1\right)\left(\frac{r}{r_-} - 1\right)^{-r_-^2/r_+^2}\,. \tag{I.90}
$$

The coordinates U_1 and V_1 allow us to extend the original RN spacetime by admitting negative values of $-U_1$ and V_1. However, they fail at $r = r_-$, so a different set of coordinates must be chosen. A suitable choice of coordinates to cover an open set containing r_- could be given by:

$$\begin{aligned} U_2 &= \arctan\left(-\exp\left(\frac{-r_- + r_+}{2r_-^2}(t - r_*)\right)\right) , \\ V_2 &= \arctan\left(\exp\left(\frac{r_- - r_+}{2r_-^2}(t + r_*)\right)\right) . \end{aligned} \tag{I.91}$$

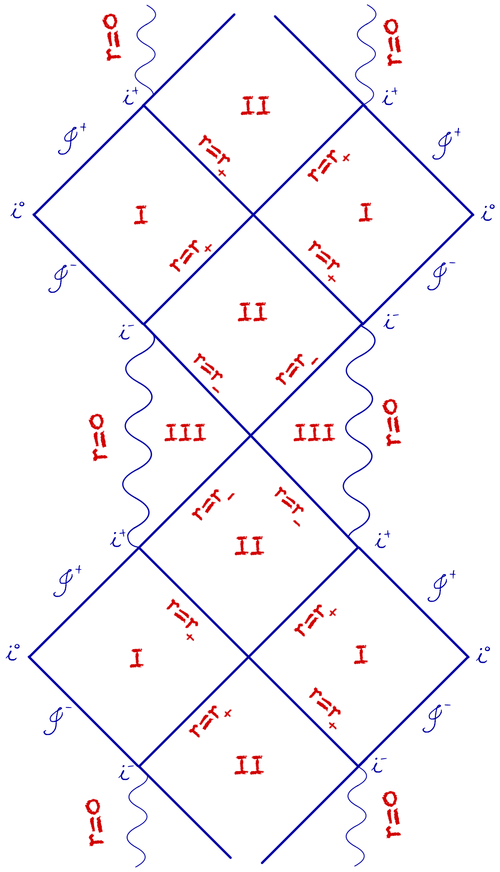
Now, in the coordinates (U_2, V_2), it can be checked that the RN metric is analytic everywhere except at $r = r_+$.

Let us consider a freely-falling observer in the RN spacetime initially located at $r > r_+$. After some finite proper time, she will cross the hypersurface $r = r_+$. Inside this region, r becomes a timelike coordinate, and thus the motion must proceed with r decreasing (this implies that the hypersurface $r = r_+$ is an event horizon, so the RN spacetime contains a black hole). Later, she will cross the hypersurface $r = r_-$ as well and she will be able to reverse course and continue with r increasing, which is allowed since now r is again a spacelike coordinate. Consequently, she will emerge into a new asymptotically flat region, and this procedure can be repeated *ad infinitum*. Therefore, the maximal analytic extension of RN black hole is obtained by considering infinite tuples of coordinates (U_1, V_1, θ, ϕ) and (U_2, V_2, θ, ϕ), constructing, thus, a corresponding atlas.

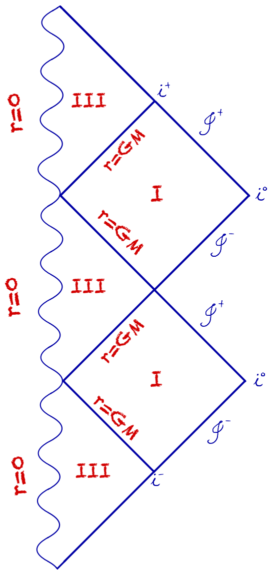
The situation can be better understood by taking a look at the associated Carter-Penrose diagram depicted in Figure I.4a (note that the conformal factor can be easily guessed from (I.89)). In the Schwarzschild black hole, the use of Kruskal-Szekeres coordinates teaches us the existence of two asymptotically flat regions isometric to the original coordinates (I.85). Here we find an infinite number of such regions, due to the reasoning above. Every such region is connected to the others by intermediate regions II and III, of which there are likewise infinitely many. The hypersurface $r = 0$, located in the interior of Region III, is an irremovable physical singularity, but its nature is quite different to that of the Schwarzschild case. In fact, it is a timelike singularity (note that it is a vertical line in the Carter-Penrose diagram), meaning that timelike and null curves can avoid it[55]. As a consequence, there exist past-directed timelike curves in Region III which do not cross the surface $r = r_-$, causing a loss of predictability for observers in other Regions (located in the past). This implies that $r = r_-$ is actually a Cauchy horizon, since the solution beyond $r = r_-$ cannot be determined from any spacelike hypersurface at a previous time.

Extremal RN solution. It occurs when $GM = \sqrt{Q^2 + P^2}$. It is a very interesting case, since for this precise value of the mass in terms of the electromagnetic charges, both outer and inner horizons r_+ and r_- merge into a single degenerate horizon located at $r = GM$. Extremal black holes play a fundamental role not only within the study of the classical GR Reissner-Nordström solution, but also in the context of ST and supersymmetric solutions, since their very special properties generically render them supersymmetric and, having vanishing temperature (since they satisfy that $\kappa = 0$), they are expected to have a simpler quantum-mechanical description [106]. Furthermore, they provide an extraordinary arena to test ST and the subsequent higher-derivative effective theories, owing to the fact that the computation of quantities such as the extremal charge-to-mass ratio (which is equal to 1 in the RN solution, in some appropriate units) allows one to make contact with the Weak

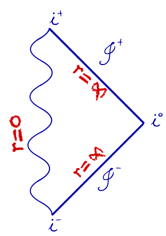
[55] It turns out that it is a repulsive singularity, since every timelike geodesic evades it (although non-geodesic timelike or radial null geodesics can hit it).



(a) Subextremal RN black hole.



(b) Extremal RN black hole.



(c) Overextremal RN black hole.

Figure I.4: Carter-Penrose diagram of the maximally-extended subextremal RN black hole, extremal RN black hole and overextremal RN black hole. We use the notation of Fig. I.3.

Gravity Conjecture [416] (WGC) and decide which effective theories do actually have the possibility of arising from a UV complete theory of Quantum Gravity, in the spirit of the Swampland Program [112–114].

Focusing on the extremal RN solution, let us indicate how one may arrive to the maximally-extended solution. In analogy with the previous cases, the hypersurface $r = GM$ does not correspond to an actual singularity and signals the presence of an event horizon. To see this, let us define:

$$r_* = \int \frac{\mathrm{d}r}{1 - 2GM/r + G^2M^2/r^2} = r + GM \log\left(\frac{r}{GM} - 1\right)^2 - \frac{G^2M^2}{r - GM}\,. \tag{I.92}$$

Now, following equivalent steps to those carried out before with the subextremal solution[56], we end up with the Carter-Penrose diagram presented in Figure I.4b. It is qualitatively different to that of the subextremal case, since there is no region II whatsoever, and it consists on an infinite array of consecutively-connected regions I and III. As in the non-extremal case, the hypersurface $r = 0$ is again a timelike singularity which can be avoided

[56]Nevertheless, it was shown by Carter [417] that one can find a single coordinate patch to cover the whole extremal RN spacetime.

by timelike geodesics.
Overextremal RN solution. This happens when $GM < \sqrt{Q^2 + P^2}$. For these values of the mass, no horizon exists and we are left with a naked singularity at $r = 0$. Such type of scenarios are excluded according to the Cosmic Censorship Conjecture [384], by which singularities visible from infinity cannot arise from the gravitational collapse of any object with a physically acceptable energy-momentum tensor. The corresponding Carter-Penrose diagram is given by I.4c.

I.7 Higher-order gravities and holography

Perhaps, the most exciting and intriguing discovery in high-energy theoretical physics in the last 25 years corresponds to the foundation and development of *holography*. Its crucial importance nowadays stems from the pioneering work of Maldacena [418], who conjectured the celebrated AdS/CFT correspondence[57]. It states the physical equivalence between Type IIB ST on $\mathrm{AdS}_5 \times S^5$ with radius of curvature L and N units of the flux of the self-dual five-form[58] on S^5 and four-dimensional $\mathcal{N} = 4$ Super-Yang-Mills theory with gauge group $\mathrm{SU}(N)$. More concretely, the correspondence claims the partition functions of both theories to be exactly identical,

$$\mathcal{Z}_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{ST}} \,. \tag{I.93}$$

Note that the previous identity has no meaning unless we find a way of identifying the fields between each of the sides of the correspondence or, in other words, unless we possess a *holographic dictionary* relating quantities of both theories. The correspondence is conjectured to be true for all values of the parameters, but there is a very interesting regime in which the ST part can be described with great accuracy by a classical Supergravity theory, obtained by taking $N \to \infty$, $g_s \to 0$ and assuming the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ to be sufficiently large. In such a case, the partition function of the bulk theory may be obtained through the saddle-point approximation:

$$\mathcal{Z}_{\mathrm{ST}} \sim e^{-I_{\mathrm{grav}}} \,, \tag{I.94}$$

where I_{grav} denotes the Supergravity effective action evaluated on a certain configuration and where Euclidean signature has been employed. The massive amount of evidence gathered so far, specially in the large N limit, provides enormously strong support for the validity of the conjecture.

This AdS/CFT correspondence [418–420] embodies the first realization of the holographic principle [421–423]. More concretely, under the name of gauge/gravity duality or holography we understand the web of firmly established correspondences between d-dimensional CFTs and $d+1 = D$-dimensional theories of gravity, not necessarily within the ST setup. It is interpreted that the CFT is defined on the boundary of an asymptotically AdS space, and then the physics in the bulk (the spacetime interior) is determined by the physics of the boundary CFT and vice versa. This is a quite amazing picture, since the D-dimensional spacetime is emerging as a hologram of the d-dimensional physics.

[57] CFT stands for Conformal Field Theory.
[58] Type IIB ST/Supergravity contains in its spectrum a self-dual (with respect to the ten-dimensional spacetime) five-form [106].

Interestingly enough, holography can be employed in a two-fold way. On the one hand, one may capture aspects of Quantum Gravity by studying the boundary or dual quantum field theory, whose treatment is more manageable than that of a UV-complete theory of Quantum Gravity, which we have not even yet discovered[59]. On the other hand, this duality may be used in the opposite direction, and we could try to learn aspects of CFTs by exploring theories of gravity. These points might seem in contradiction, but they can be reconciled on noting that the gravitational theory becomes classical when working in certain limits as mentioned before, so that bulk computations within these regimes have the possibility of being much more simple than those in the CFT dual. In this thesis, we will actually exploit this second direction and we will devote ourselves to the study of features and properties of CFTs through the examination of gravitational dual configurations in higher-derivative theories. As we will briefly explain afterwards, the validity and interest of this strategy may be justified from the fact that higher-order gravities allow us to explore different *universality classes* and discover universal relationships which hold for every CFT.

I.7.1 CFT correlators from the bulk

Ever since the advent of the AdS/CFT correspondence, studies of the holographic principle in the wider context of higher-derivative gravities have been developed (see [243, 265, 266, 281, 424–430], among others), which have allowed the construction of generic prescriptions to derive properties of CFTs from the associated (higher-derivative) dual bulk theories. Since CFTs are determined by correlators involving the stress-energy tensor T_{ab} and, in case of couplings to a vector field (as our case will be), currents J_a, these are the quantities we would like to access from the gravitational side. For example, according to the holographic dictionary, if we consider the following perturbation of Euclidean AdS space in $d+1$ dimensions expressed in Poincaré coordinates:

$$\mathrm{d}s^2 = \frac{L}{r^2}\mathrm{d}r^2 + \frac{r^2}{L^2}\mathrm{d}x^a\mathrm{d}x^b(\delta_{ab} + h_{ab}(x))\,, \tag{I.95}$$

then the correlator of two stress-energy tensors, also called 2-point function, in the dual CFT can be computed through [431, 432]:

$$\left\langle T_{ab}(x)T_{cd}(x')\right\rangle = 4\,\left.\frac{\delta^2\mathcal{W}_{\mathrm{grav}}}{\delta h^{ab}(x)h^{cd}(x')}\right|_{h=0}\,, \tag{I.96}$$

where $\mathcal{W}_{\mathrm{grav}}$ denotes the generating functional associated to the Euclidean gravitational action I_{grav}. Conformal symmetry constrains the form of the 2-point function to be[60]

$$\left\langle T_{ab}(x)T_{ce}(x')\right\rangle = \frac{C_T}{|x-x'|^{2d}}\mathcal{I}_{ab,ce}(x-x')\,, \tag{I.97}$$

where $\mathcal{I}_{ab,cd}$ is a tensorial structure given by:

$$\mathcal{I}_{ab,cd}(y) = \frac{1}{2}\left(I_{ac}(y)I_{bd}(y) + I_{ad}(y)I_{bc}(y)\right) - \frac{1}{d}\delta_{ab}\delta_{cd}\,, \qquad I_{ab}(y) = \delta_{ab} - 2\frac{y_a y_b}{y^2}\,. \tag{I.98}$$

[59] In fact, it has been proposed to define a quantum theory of gravity by the application of the holographic principle.

[60] We just show the terms in the correlators which become the leading singularities as the spacetime points (where the CFT is defined) tend to be coincident.

Consequently, the only theory-dependent information is encoded in the quantity C_T, which receives the name of *central charge*. For instance, for Einstein gravity with a negative cosmological constant:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[\frac{d(d-1)}{L^2}+R\right], \tag{I.99}$$

where L is a length scale representing the AdS radius, the central charge is

$$C_T^{\mathrm{E}} = \frac{\Gamma(d+2)L^{d-1}}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}G}. \tag{I.100}$$

When one considers modifications of GR, such as adding a Gauss-Bonnet term $\mathcal{X}_4$ in dimensions $d \geq 4$:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[\frac{d(d-1)}{L^2}+R+\frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4\right], \tag{I.101}$$

then it turns out that the central charge differs from the pure Einstein value and reads [426]:

$$C_T^{\mathrm{GB}} = \frac{\Gamma(d+2)\tilde{L}^{d-1}}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}G}(1-2\lambda f_\infty), \qquad f_\infty = \frac{1-\sqrt{1-4\lambda}}{2\lambda}. \tag{I.102}$$

where $\tilde{L} = L/\sqrt{f_\infty}$ is the new AdS radius. Consequently, we see explicitly that higher-order gravities have the ability to modify the value of the CFT central charge.

An analogous situation takes place for correlators $\langle J_a J_b\rangle$ of CFTs dual to higher-order theories of gravities and electromagnetism, where J_a stands for the current that couples to the CFT vector field. Conformal symmetry completely fixes the structure of this correlator up to a constant C_J:

$$\langle J_a(x)J_b(x')\rangle = \frac{C_J}{|x-x'|^{2(d-1)}}I_{ab}(x-x'), \tag{I.103}$$

where C_J is called the central charge of the current C_J. In the case of Einstein-Maxwell theory with a negative cosmological constant:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[\frac{d(d-1)}{L^2}+R-F^2\right], \tag{I.104}$$

the associated C_J is given by:

$$C_J^{\mathrm{EM}} = \frac{\Gamma(d)}{\Gamma(d/2-1)}\frac{\ell_*^2 L^{d-3}}{4\pi^{d/2+1}G}, \tag{I.105}$$

where ℓ_* is a length scale which relates the bulk gauge field A_μ with the CFT vector field $\tilde{A}_\mu = \ell_*^{-1}A_\mu$ that couples to the current. In Chapter 4 we will illustrate how higher-derivative theories are capable of modifying this value.

Let us now turn into the analysis of 3-point functions. It can be shown that the correlator $\langle TTT\rangle$ is uniquely specified by conformal symmetry up to three constants[61] [433,434], one of which can be chosen to be the central charge C_T, while the correlator $\langle TJJ\rangle$ is fixed up

[61] At least, in parity-preserving theories.

to two constants [433, 434], being possible to pick one of them as C_J. To determine the remaining constants for both correlators, a convenient way to proceed is by examination of the energy fluxes in the AdS boundary after the insertion of operators associated to local perturbations [435]. Following this reference, the operator for the energy flux in the direction $\vec{n}$ is given by:

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^{d-2} \int_{-\infty}^{\infty} \mathrm{d}x^0 T^0{}_i(x^0, r\vec{n}) n^i \,, \tag{I.106}$$

with $r^2 = \delta_{ij} x^i x^j$ and where i denotes a spatial index on the boundary. Assume the operator representing the local perturbation to be given by:

$$\mathcal{O}_E = \int \mathrm{d}^d x\, \mathcal{O}(x)\, e^{-iEx^0} \psi(x/\sigma) \,, \tag{I.107}$$

where $\psi(x/\sigma)$ is a distribution function localizing the perturbation at $x^a = 0$ as $\sigma \to 0$ (for the sake of simplicity, we could think of a Gaussian). Then, the expectation value for the energy flux after such insertion is provided by:

$$\langle \mathcal{E}(\vec{n}) \rangle_{\mathcal{O}_E} = \frac{\langle 0|\mathcal{O}_E^\dagger\, \mathcal{E}(\vec{n})\, \mathcal{O}_E|0\rangle}{\langle 0|\mathcal{O}_E^\dagger\, \mathcal{O}_E|0\rangle} \,. \tag{I.108}$$

If one first considers $\mathcal{O} \sim \varepsilon_{ij} T^{ij}$, then from the subsequent correlator $\langle \mathcal{E}(\vec{n}) \rangle_T$ it is possible to extract the remaining coefficients of the 3-point function $\langle TTT \rangle$ [426]. In the case of Einstein gravity, these constants turn out to be exactly zero and $\langle TTT \rangle$ is completely fixed by C_T, so to get a generic 3-point correlator we need to explore CFTs belonging to different *universality classes*. This can be achieved via higher-derivative theories [243, 426, 428, 436–438] such as Quasitopological Gravities, which already provide dual CFTs with the most general $\langle TTT \rangle$ structure[62].

Analogously, if we consider $\mathcal{O} \sim \varepsilon_i J^i$, we will show in Chapter 4 that it is possible to obtain the remaining coefficient characterizing the correlator $\langle TJJ \rangle$ through the expression of $\langle \mathcal{E}(\vec{n}) \rangle_J$. Again, in the case of Einstein gravity, this correlator is fully specified by C_J, so to find CFTs with a generic $\langle TJJ \rangle$ we must find different (higher-order) bulk theories which allow us to probe different universality classes. In Chapter 4 we will present explicit examples of higher-order theories of gravity and electromagnetism for which the corresponding dual CFTs do have the most general correlator $\langle TJJ \rangle$.

I.7.2 Holographic entanglement entropy

Another important quantity we may compute through the use of the holographic principle is entanglement entropy (EE). For a bipartition of the Hilbert space into two complementary subspaces A and B, we define the EE of A with respect to B as [439, 440]

$$S_{\mathrm{EE}}(A) = -\mathrm{Tr}\left[\rho_A \log \rho_A\right] \,, \tag{I.109}$$

where ρ_A is the reduced density matrix of A obtained by summing over the degrees of freedom of B. By definition, it can be checked that $S_{\mathrm{EE}}(A) = S_{\mathrm{EE}}(B)$. It is possible to generalize the EE with the notion of Rényi entropies (RE) [441, 442], which are defined as:

$$S_n(A) = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n \,. \tag{I.110}$$

[62] Nevertheless, it is necessary to resort to GQGs such as ECG for the case of CFTs in $d = 3$ [257], for which the correlator $\langle TTT \rangle$ is just specified just by two constants, one of which can be chosen to be C_T.

In particular, observe that $\lim_{n\to 1} S_n(A) = S_{\text{EE}}(A)$. The algebraic properties of RE and, thus, of EE, make them specially suited to provide a quantitative measure of the amount of entanglement between complementary subsets [441–444].

Working in the context of CFTs, we will be interested in the case in which A and B are two spatial regions (at a given time) with common boundary given by an entangling surface $\mathcal{S}$. Nevertheless, the direct computation of the corresponding RE or EE is typically quite involved, even in the weakly coupled regime, so it turns out to be much more convenient to resort to the holographic principle and calculate these entropies from the gravitational counterpart. Furthermore, this can be very useful, since the holographic computation may provide further insights into the properties of RE and EE.

Let us illustrate how the holographic principle may help us compute RE. For that, choose the CFT to be defined in flat space and consider $\mathcal{S}$ to be a sphere of radius R (so that A can be taken to be its interior). In such a case, it can be shown, through the so-called Casini-Huerta-Myers map [445], that the RE are in correspondence with the thermal entropy S_{thermal} of the same theory defined on a cylinder with hyperbolic slices of curvature scale R [446]:

$$S_n(A) = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S_{\text{thermal}}(T)\mathrm{d}T\,, \tag{I.111}$$

where $T_0 = (2\pi R)^{-1}$ denotes the temperature of the thermal bath in the hyperbolic cylinder to which the vacuum of the CFT in the original spacetime is mapped. Therefore, if we now make use of the holographic principle, then $S_{\text{thermal}}(T)$ equals the Wald entropy of a black hole with hyperbolic horizon, whose computation is usually more manageable. Consequently, by considering families of higher-order gravities in the bulk, we will obtain a collection of black hole solutions and RE which will allow us to probe and analyze different classes of CFTs.

In presence of matter fields charged under global symmetries, the appropriate generalization to RE is given by the so-called charged RE [447]. In Chapter 4 we will derive the charged RE of certain higher-order theories of gravity and electromagnetism and observe that the conjugate chemical potential always increases the amount of entanglement and that the usual properties of RE are preserved if sensible physical constraints are met. Furthermore, inspired by the examination of these theories, we will show that for a general $d(\geq 3)$-dimensional CFT, the leading correction to the uncharged entanglement entropy across a spherical entangling surface is quadratic in the chemical potential, positive definite, and universally controlled (up to fixed d-dependent constants) by the coefficients C_J and a_2, being the latter a coefficient characterizing the energy flux $\langle\mathcal{E}(\vec{n})\rangle_J$ associated to $\mathcal{O}\sim\varepsilon_i J^i$.

I.8 Spinors

We devote this section to the physical and mathematical introduction of spinors, which are indispensable to describe Nature. In fact, electrons, which are present in every atom we are made of, happen to be properly described by spinors. We will begin with a historical introduction.

Shortly after the discovery of the (non-relativistic) Schrödinger equation in 1926 [448], the quest of its proper special-relativistic generalization started off. The first candidate was the Klein-Gordon equation [449, 450]:

$$(\Box - m^2)\phi = 0\,, \tag{I.112}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ for $\eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and where ϕ is a complex scalar. This equation is clearly Lorentz-invariant, although an essential problem with this equation was soon spotted — the subsequent would-be probability density is not positive definite and, thus, it is not well defined[63] [451, 452].

Dirac attributed this behaviour to the fact that (I.112) was second-order in (time) derivatives, so he committed himself to the search of some relativistic wave equation which was first-order in derivatives. The genial idea happened to be to look for the *square root* of the Klein-Gordon equation. More concretely, Dirac attempted to look for a first-order differential operator $i\Gamma^\mu\partial_\mu$ such that:

$$i^2(\Gamma^\mu\partial_\mu)(\Gamma^\nu\partial_\nu) = -\Gamma^\mu\Gamma^\nu\partial_\mu\partial_\nu = \Box\,. \tag{I.113}$$

For the latter to hold, we must demand:

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = -2\eta^{\mu\nu}\,, \tag{I.114}$$

On observing this expression, Dirac realized that these Γ^μ needed to be matrices. In particular, he found that it was necessary at least for the matrices to be 4×4. A particular (purely imaginary) representation for these matrices is given by $\Gamma^\mu = -i\gamma^\mu$, where

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0\end{pmatrix}\,, \quad \gamma^1 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix}\\ \gamma^2 &= \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0\end{pmatrix}\,, \quad \gamma^3 = -\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1\end{pmatrix}\,,\end{aligned} \tag{I.115}$$

where σ^i for $i = 1,2,3$ stands for the Pauli matrices given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\,. \tag{I.116}$$

This way, the square root of the equation $\Box\psi = m^2\psi$ turns out to be

$$(i\Gamma^\mu\partial_\mu - m)\psi = 0\,. \tag{I.117}$$

Dirac discovered this equation in 1928 [453], today known as the Dirac equation. Its finding was one of the greatest successes of the 20th century physics, since it stands as the dynamic equation to be satisfied[64] by every fermion in the Standard Model. Indeed, such equation admits a well-defined probability density (positive-definite) and, furthermore, motivated Dirac to conjecture the existence of positrons [454], which was awarded with the Nobel Prize in Physics in 1933.

Nevertheless, we also observe a key feature — the field ψ must have four (complex) components, no longer being a scalar. Furthermore, after a very careful and patient examination

[63] Nowadays, it is understood that the solution to this problem is to reinterpret the Klein-Gordon equation as one for an operator, so that the mistakenly-considered probability density should be thought of as a charge density. In fact, the Klein-Gordon equation has proved to be very useful for the description of scalar particles like pions.

[64] Up to a slight generalization of it to account for gauge invariance.

of the Lorentz invariance of (I.117), one can realize that the field ψ acquires a minus sign under rotations by 2π, what is absolutely counter-intuitive! This extremely exotic behaviour has amazed physicists and mathematicians ever since and nowadays such ψ receives the name of a *spinor*. We proceed in the following to provide all the necessary mathematical material to understand the present concept of a spinor[65] and the elements appearing in (I.117). Our exposition will be based on [455–457].

I.8.1 Clifford algebras

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ or $\mathbb{C}$ and let V be a finite-dimensional $\mathbb{K}$-vector space. We remind that a unital associative $\mathbb{K}$-algebra is a finite-dimensional $\mathbb{K}$-vector space A together with a bilinear and associative product $\cdot : A \times A \to A$ and an element $1 \in A$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.

Definition I.11. Let (V, Q) be a finite-dimensional $\mathbb{K}$-vector space equipped with a symmetric bilinear form $Q : V \times V \to \mathbb{K}$. A Clifford algebra for (V, Q) is a pair $(\mathrm{Cl}(V, Q), \iota)$ such that:

1. $\mathrm{Cl}(V, Q)$ is a unital associative $\mathbb{K}$-algebra.
2. $\iota : V \to \mathrm{Cl}(V, Q)$ is a linear map such that $\iota(v)^2 = Q(v, v) \cdot 1$ for every $v \in V$.
3. It satisfies the universal property: if A' is another unital associative $\mathbb{K}$-algebra with a linear map $\delta : V \to A'$ satisfying $\delta(v)^2 = Q(v, v) \cdot 1$, then there exists a unique algebra homomorphism $\phi : \mathrm{Cl}(V, Q) \to A'$ such that the diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\iota"] \arrow[dr, "\delta"'] & \mathrm{Cl}(V,Q) \arrow[d, "\phi"] \\ & A' \end{tikzcd}$$

From now on, let us denote by $(\mathrm{Cl}(V, Q), \iota)$ a Clifford algebra for (V, Q). The condition[66] $\iota^2(v) = Q(v, v)$ turns out to be equivalent to $\iota(v)\iota(w) + \iota(w)\iota(v) = 2Q(v, w)$ for every $v, w \in V$, as can be shown by expanding $\iota^2(v + w) = Q(v + w, v + w)$. We observe this condition is highly reminiscent of Eq. (I.114), but for the moment let us leave this fact as an aside comment and continue.

It can be checked that Clifford algebras for (V, Q) exist and are unique for each (V, Q).

Proposition I.1. *For every* (V, Q) *there exists a unique Clifford algebra* $(\mathrm{Cl}(V, Q), \iota)$.

Proof. Let $T(V)$ stand for the tensor algebra of V:

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n} = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots , \tag{I.118}$$

[65] Note, however, that our spinors are not anticommuting objects, as required by Quantum Mechanics. In some sense, we could say that we are working with classical spinors.

[66] Whenever no possible confusion may arise, we may write $\lambda \cdot 1 \in \mathrm{Cl}(V, Q)$ for $\lambda \in \mathbb{K}$ simply as λ.

Consider the two-sided ideal $I(Q)$ in $T(V)$ generated by the set of elements $\{v \otimes v - Q(v, v) | v \in V\}$. Thus:

$$I(Q) = \left\{ \sum_{i=1}^{n} \kappa_i \otimes (v_i \otimes v_i - Q(v_i, v_i)) \otimes \sigma_i \,|\, \kappa_i, \sigma_i \in T(V)\,, v_i \in V\,, n \in \mathbb{N} \right\} . \qquad \text{(I.119)}$$

Define the quotient space $\mathrm{Cl}(V, Q) = T(V)/I(Q)$ through the equivalence relation $a \sim b+x$ if $x \in I(Q)$ for any $a, b \in T(V)$. This is naturally equipped with a well-defined associative product $\cdot$ given by $[a] \cdot [b] = [a \otimes b]$, since:

$$(a+x)\otimes(b+y) = a\otimes b+a\otimes y+x\otimes b+x\otimes y \sim a\otimes b\,, \qquad a, b \in T(V)\,,\ x, y \in I(Q)\,. \quad \text{(I.120)}$$

Consider the inclusion of V into $T(V)$ $i : V \to T(V)$ and the canonical projection $\pi : T(V) \to \mathrm{Cl}(V, Q)$ given by $\pi(a) = [a]$ for every $a \in T(V)$. Define:

$$\iota = \pi \circ i : V \to \mathrm{Cl}(V, Q)\,. \qquad \text{(I.121)}$$

This map ι is clearly linear and satisfies $\iota(v)^2 = [v]\cdot[v] = [v\otimes v] = Q(v, v)$ for every $v \in V$. Furthermore, $\iota(V)$ generates multiplicatively $\mathrm{Cl}(V, Q)$ since $T(V)$ is generated by tensor products of V.

Consequently, we just need to check that $\mathrm{Cl}(V, Q)$ satisfies the universal property from the definition of Clifford algebras to prove existence. For that, assume there is another associative unital $\mathbb{K}$-algebra with a $\mathbb{K}$-linear map $\delta : V \to A'$ with $\delta(v)^2 = Q(v, v)$ for all $v \in V$. We can trivially extend δ to be an algebra homomorphism $\Delta : T(V) \to A'$ by imposing that $\delta(c) = c$ for $c \in \mathbb{K}$ and $\delta(v\otimes v) = \delta(v)\cdot_{A'}\delta(v)$, where $\cdot_{A'}$ denotes the product operation in A'. By construction, we note that $\Delta(w) = 0$ for every $w \in I(Q)$, so the map Δ descends to a homomorphism $\phi : \mathrm{Cl}(V, Q) \to A'$ with $\phi \circ \iota = \delta$. Consequently, the homomorphism ϕ is uniquely specified, on account of the fact that $\iota(V)$ generates $\mathrm{Cl}(V, Q)$ and ϕ is fixed on $\iota(V)$. This proves existence of a Clifford algebra $\mathrm{Cl}(V, Q)$ for every (V, Q). Uniqueness of $\mathrm{Cl}(V, Q)$ is then guaranteed by the universal property we have just proven and we conclude. □

It can be shown that any Clifford algebra $\mathrm{Cl}(V, Q)$ is isomorphic as a vector space[67] to the exterior algebra Λ^*V of V, so the dimension of $\mathrm{Cl}(V, Q)$ is $\dim_{\mathbb{K}}\mathrm{Cl}(V, Q) = 2^n$, where $n = \dim_{\mathbb{K}} V$. In particular, if $\{e_1, \ldots , e_n\}$ denotes an orthonormal basis for (V, Q), then the set of elements

$$\iota(e_{i_1}) \cdot \iota(e_{i_2}) \cdots \iota(e_{i_k})\,, \qquad 1 \leq i_1 < i_2 < \cdots i_k \leq n \qquad \text{(I.122)}$$

with $k \leq n$ span $\mathrm{Cl}(V, Q)$ as a vector space, if we take the convention that $k = 0$ implies taking the unit element. This shows that the image $\iota(V) \subset \mathrm{Cl}(V, Q)$ is isomorphic to V (as a vector space).

Let $\rho : \mathrm{Cl}(V, Q) \to \mathrm{End}_{\mathbb{K}}(W)$ be a representation of $\mathrm{Cl}(V, Q)$ in the n-dimensional $\mathbb{K}$-vector space W — *i.e.,* an algebra homomorphism ρ between $\mathrm{Cl}(V, Q)$ and the algebra of matrices on W. We define the *mathematical gamma matrices*:

$$\gamma_a = \rho \circ \iota(e_a)\,, \qquad a = 1, \ldots , n\,, \qquad \text{(I.123)}$$

[67] More precisely, if $Q = 0$ there exists an algebra isomorphism between $\mathrm{Cl}(V, 0)$ and the exterior algebra $(\Lambda^*V, \wedge)$ equipped with the wedge product. Therefore, we may understand Clifford algebras as exterior algebras with a *twisted* wedge product.

where $\{e_a\}_{a=1,\dots,n}$ denotes and orthonormal basis for (V, Q). If we define $\gamma^a = \eta^{ab}\gamma_b$ and assume Q is of Lorentzian signature, then we observe:

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}I_n\,, \tag{I.124}$$

where I_n is the n-dimensional identity matrix. We note that this expression is, up to a minus sign, equivalent to (I.114). In order to precisely obtain (I.114), we define the *physical* gamma matrices:

$$\Gamma^a = -i\gamma^a\,, \tag{I.125}$$

so that now $\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = -2\eta^{ab}I_n$ coincides with (I.114).

Example I.9. Let us consider $(V, Q) = (\mathbb{R}^2, \eta)$, where η is the (two-dimensional) Minkowski metric. Let (e_0, e_1) be an orthonormal basis with $\eta(e_0, e_0) = -1$. The associated Clifford algebra $\mathrm{Cl}(\mathbb{R}^2, \eta)$ is spanned by $\{1, \iota(e_0), \iota(e_1), \iota(e_0)\iota(e_1)\}$ and satisfies:

$$\iota(e_0)^2 = -1\,, \quad \iota(e_0)\iota(e_1) = -\iota(e_1)\iota(e_0)\,, \quad \iota(e_1)^2 = 1\,. \tag{I.126}$$

An irreducible representation $\rho : \mathrm{Cl}(\mathbb{R}^2, \eta) \to \mathrm{End}_{\mathbb{R}^2}(\mathbb{R}^2)$ is provided in terms of the Pauli matrices (I.116) by $\rho(\iota(e_0)) = \gamma_1 = i\sigma_2$ and $\rho(\iota(e_1)) = \sigma_1$. In fact, $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^2) = \mathrm{Span}_{\mathbb{R}}\{I_2, \rho(\iota(e_0)), \rho(\iota(e_1)), \rho(\iota(e_0)\iota(e_1)) = \sigma_3\}$.

Example I.10. Let us consider $(V, Q) = (\mathbb{R}^4, \eta)$, with η the (four-dimensional) Minkowski metric. Let (e_0, e_1, e_2, e_3) be an orthonormal basis with $\eta(e_0, e_0) = -1$. The associated Clifford algebra $\mathrm{Cl}(\mathbb{R}^4, \eta)$ is spanned by

$$\mathcal{B} = \{1, \iota(e_i), \iota(e_i)\iota(e_j), \iota(e_i)\iota(e_j)\iota(e_k), \iota(e_0)\iota(e_1)\iota(e_2)\iota(e_3)\}\,, \quad 1 \leq i < j < k \leq 4 \tag{I.127}$$

and satisfies $\iota(e_i)\iota(e_j) + \iota(e_j)\iota(e_i) = 2\eta_{ij}$. An irreducible representation $\rho : \mathrm{Cl}(\mathbb{R}^4, \eta) \to \mathrm{End}_{\mathbb{R}^4}(\mathbb{R}^4)$ is given by the matrices $\gamma_a = \eta_{ab}\gamma^b$, being γ^b as in (I.115). Moreover, we have that $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^4) = \mathrm{Span}_{\mathbb{R}}\{\rho(\mathcal{B})\}$.

I.8.2 Spinor bundles and spinors

The analysis so far was restricted to a single quadratic vector space, and we would actually like to have a way to generalize these definitions when we substitute the vector space by the tangent bundle of an orientable (pseudo-)Riemannian manifold (M, g). To this aim, we introduce the following definitions [458, 459].

Definition I.12. A Clifford bundle $\mathrm{Cl}(M, g)$ over (M, g) is a smooth bundle over M whose fibers are Clifford algebras $\mathrm{Cl}(T_pM, g_p)$ for all $p \in M$. In particular, the fiberwise multiplication in $\mathrm{Cl}(M, g)$ gives an algebra structure to the space of sections of $\mathrm{Cl}(M, g)$.

Equipped with the notion of Clifford bundle, we would like to define a notion of bundle of representations of $\mathrm{Cl}(M, g)$.

Definition I.13. Let $\mathrm{Cl}(M, g)$ be a Clifford bundle over (M, g). A spinor bundle is a pair (S, ρ), where S is a $\mathbb{K}$-vector bundle over (M, g) and $\rho : \mathrm{Cl}(M, g) \to \mathrm{End}_{\mathbb{K}}(\mathrm{S})$ is a smooth morphism of vector bundles such that all fiber maps $\rho_p : \mathrm{Cl}(T_pM, g_p) \to \mathrm{End}_{\mathbb{K}}(\mathrm{S}_p)$ define isomorphic representations for all $p \in M$. In case they are irreducible, (S, ρ) is said to be an irreducible spinor bundle.

After these preliminary definitions, we are finally ready to present the definition of a *spinor*.

Definition I.14. A(n) (irreducible) spinor $\psi \in \Gamma(\mathrm{S})$ is a section of a(n) (irreducible) spinor bundle (S, ρ) over (M, g). Such a spinor is additionally said to be real if $\rho_p : \mathrm{Cl}(T_pM, g_p) \to \mathrm{End}_{\mathbb{R}}(\mathrm{S}_p)$ defines a real representation for all $p \in M$.

Observe that the existence of a spinor bundle on an orientable (pseudo-)Riemannian manifold is not always guaranteed. In signature $p - q = 0, 2 \bmod 8$, where p (resp. q) denotes the number of spacelike (resp. timelike) directions, it can be shown [459] that such obstruction is equivalent to the existence of a spin structure[68], understood as a lift of the frame bundle with respect to the double-covering of $\mathrm{SO}(p, q)$. In this case, (S, ρ) can be interpreted as the vector bundle naturally associated to the frame bundle.

Once we have specified the notion of a spinor, our next objective will be to understand, from a mathematical perspective, the remaining elements taking place in the Dirac equation (I.117). The first step in this direction is provided by the following definition.

Definition I.15. Let (S, ρ) be a spinor bundle over (M, g). We define Clifford multiplication or Clifford product between a vector field $X \in \mathfrak{X}(M)$ and a spinor $\psi \in \Gamma(\mathrm{S})$ as:

$$\begin{aligned} \mathfrak{X}(M) \times \Gamma(\mathrm{S}) &\to \quad \Gamma(\mathrm{S}) \\ (X, \psi) \quad &\mapsto X \cdot \psi = \rho(\iota(X))\psi\,. \end{aligned} \tag{I.128}$$

If $\{e_a\}_{a=1,\dots,n}$ denotes a local orthonormal frame on (M, g), then locally we have

$$e_a \cdot \psi = \gamma_a \cdot \psi\,, \qquad \gamma_a = \rho(\iota(e_a)) \tag{I.129}$$

In another vein, since a spinor bundle (S, ρ) is a vector bundle, it is natural to consider the possibility of defining covariant derivatives on it. Of course, there exist infinitely many different covariant derivatives one could define on (S, ρ), but there exists a very canonical choice of connection associated to the Levi-Civita one defined on the tangent bundle. Such connection is called the spin connection.

Definition I.16. The spin connection $\nabla : \Gamma(\mathrm{S}) \to \Gamma(T^*M \otimes \mathrm{S})$ is defined as the equivariant lift of the Levi-Civita one into the spinor bundle (S, ρ).

Locally, the covariant derivative $\nabla_X \psi$ of ψ along X takes the following form:

$$\nabla_X \psi = \mathrm{d}\psi(X) + \frac{1}{4}\omega_{ab}(X)\gamma^{ab}\psi\,, \tag{I.130}$$

where ω_{ab}, in some local orthonormal frame $\{e_a\}_{a=1,\dots,n}$, is given by $\nabla e_a = \omega_{ab}\eta^{bc} \otimes e_c$, $\gamma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and $\mathrm{d}\psi(X) = X^\alpha \partial_\alpha \psi$. In particular, a spinor ψ is *parallel* if $\nabla_X \psi = 0$ for every $X \in \mathfrak{X}(M)$.

The spin connection satisfies the following Leibniz rule with respect to Clifford product:

$$\nabla_X(Y \cdot \psi) = \nabla_X Y \cdot \psi + Y \cdot \nabla_X \psi\,, \tag{I.131}$$

which makes manifest the naturalness of the spin connection and its compatibility with the Clifford product. In fact, it is precisely the combination of these two concepts which leads us to the definition of *Dirac operator*.

[68] A spin structure exists if and only if the so-called second Karoubi Stiefel-Whitney class [455, 460, 461] vanishes.

Definition I.17. The Dirac operator $D : \Gamma(\mathrm{S}) \to \Gamma(\mathrm{S})$ is defined as the composition $D = c \circ \nabla$, where $c : \Gamma(T^*M \otimes \mathrm{S}) \to \Gamma(\mathrm{S})$ is given in terms of a local coordinate frame $\{e_a\}_{a=1,\dots,n}$ by:

$$c(\beta \otimes \psi) = \beta_a \gamma^a \psi\,, \quad \beta \in \Omega^1(M)\,. \tag{I.132}$$

The specific choice of orthonormal frame does not affect c, and thus both c and D are well defined.

In some local orthonormal frame, the Dirac operator on a spinor ψ takes the form:

$$D\psi = \gamma^a \nabla_{e_a} \psi\,. \tag{I.133}$$

Using the physical gamma matrices Γ^a:

$$D\psi = i\Gamma^a \nabla_{e_a} \psi\,. \tag{I.134}$$

Observe that if we consider (M, g) to be the Minkowski space, then the previous operator matches precisely that of (I.117), so have managed our goal of contextualizing within the mathematical and differential-geometric framework the Dirac equation.

Example I.11. Now that we have a deeper understanding of (I.117), let us provide the most canonical examples of solutions to (I.117) in four-dimensional Minkowski spacetime $(\mathbb{R}^4, \eta)$ which are usually presented in the literature [15, 17, 18]. Introducing the *slash* notation $\Gamma^\mu a_\mu = \not{a}$ for any covector a_μ, we want to solve the equation:

$$(i\not{\partial} - m)\psi = 0\,. \tag{I.135}$$

Let us assume the following ansatz for ψ:

$$\psi = u(\vec{p})e^{ip_\mu x^\mu} + v(\vec{p})e^{-ip_\mu x^\mu}\,, \tag{I.136}$$

where $p_\mu = (-p^0, \vec{p})$ is the four-momentum satisfying that $p^0 = \sqrt{m^2 + (\vec{p})^2}$ and u, v are momentum-dependent spinors. If we substitute (I.136) into (I.135), we find the following two independent equations:

$$(\not{p} + m)u(\vec{p}) = 0\,, \quad (\not{p} - m)v(\vec{p}) = 0\,. \tag{I.137}$$

At this point, it is convenient to distinguish between the massive and massless cases. If $m \neq 0$, we can go to the rest frame in which $\vec{p} = 0$ with no loss of generality. Then, $\not{p} = -m\Gamma^0$ and, on account of (I.115), the solutions to (I.137) are trivially given by:

$$u(\vec{0}) = (a_1, a_2, -ia_2, ia_1)^T\,, \qquad v(\vec{0}) = (b_1, b_2, ib_2, -ib_1)^T\,, \tag{I.138}$$

where $a_i, b_i \in \mathbb{C}$. By performing arbitrary boosts, we obtain the solution for arbitrary $\vec{p}$. If $m = 0$, we can choose a reference frame such that $p^z = \pm p^0$ and $p^1 = p^2 = 0$. In such a case, denoting $u(p^z = \pm p^0) = u(\pm p^0)$ and $v(p^z = \pm p^0) = v(\pm p^0)$, (I.137) reduces to:

$$(-\Gamma^0 \pm \Gamma^3)u(\pm p^0) = 0\,, \quad (-\Gamma^0 \pm \Gamma^3)v(\pm p^0) = 0\,. \tag{I.139}$$

Evidently, both $u(\pm p^0)$ and $v(\pm p^0)$ satisfy the same equations and the solutions now correspond to:

$$\begin{aligned} u(p^0) &= (a_1, a_2, -a_1, a_2)^T\,, \quad v(p^0) = (b_1, b_2, -b_1, b_2)^T\,, \\ u(-p^0) &= (a_1, a_2, a_1, -a_2)^T\,, \quad v(-p^0) = (b_1, b_2, b_1, -b_2)^T\,, \end{aligned} \tag{I.140}$$

with $a_i, b_i \in \mathbb{C}$. As before, by appropriate boosts one gets the most general solution for any $\vec{p}$.

In the previous example, we have obtained explicit solutions in the case (M, g) is given by four-dimensional Minkowski space. Nevertheless, one may consider the Dirac equation in generic curved backgrounds, where $\not{\partial}$ has to be replaced by the Dirac operator (I.134). Even if we restrict ourselves to macroscopic configurations, spinorial equations such as the parallel condition $\nabla\psi = 0$ turn out to be ubiquitous in the Supergravity and ST context, since they naturally arise as the conditions to ensure the supersymmetry[69] of the underlying solution [106, 133]. However, such spinorial equations are terribly involved, the usual procedure being to try to solve these equations by finding a set of equivalent conditions which do not involve spinors but tensors, which are much more manageable [204, 462–472].

In this thesis, we are going to apply this idea to the case of globally hyperbolic four-manifolds endowed with a real parallel spinor. Indeed, by the seminal work of Cortés, Lazaroiu and Shahbazi [473], a parallel real spinor can be equivalently described by a pair of one-forms called *parabolic pair* satisfying a prescribed system of first-order partial differential equations. We will observe in Chapter 5 that such description will prove to be very useful, since it will allow us to study properties of globally hyperbolic four-manifolds harboring real parallel spinors without even making reference to the spinor itself.

I.9 Summary of main results

This thesis in essentially based on References [1–11]. Its main goal consists in contributing to the understanding of the physics and geometry of gravity at high energies. To this aim, it has been convenient to split the manuscript into two parts. Firstly, we shall devote ourselves to the study of higher-order theories from a bottom-up approach, focusing mostly on their physical properties. Secondly, we will adopt a more mathematical attitude and try to understand the geometric structures underlying in diverse setups which arise in the context of high-energy physics.

In the spirit of this Robert Louis Stevenson's[70] duality, we intent to follow the formalism, notation and conventions best suited for each part. For instance, in the study of higher-order gravities it will suffice to mainly work in a certain local coordinate system, so that expressions will be very often accompanied by *indices*. However, when changing into a more mathematical position, we will be generically more interested in coordinate-invariant expressions, dealing with the corresponding tensors and differential forms rather than with the associated components in a given (coordinate) basis. Also, we have tried to balance uniformization of notation along the thesis with following the usual conventions of theoretical physics and differential geometry in the corresponding chapters, in an attempt of presenting computations and results in the most transparent way. We shall describe some conventions employed along the document at the very end of this introduction, while other specific notation taking place in the thesis will be introduced in the (hopefully) most appropriate place of the manuscript.

We now proceed to provide a summary of the main results presented in each chapter of the thesis. Chapters 1, 2, 3 and 4 conform the First Part of the thesis, *Higher-order gravities*,

[69]One is usually interested in supersymmetric solutions because of their intriguing properties, such as amenability to computations, stability or very special behaviour under quantum corrections.

[70]We leave up to the reader to decide which part is to be associated to Dr. Jekyll and which to Mr. Hyde.

while the Second Part, *Geometric aspects of Supergravity and String Theory*, is made up of Chapters 5, 6 and 7.

Chapter 1 (based on [1])

We begin by carrying out a thorough study of Generalized Quasitopological Gravities (GQGs), which we already introduced back in Section I.3. We then consider the possibility of performing (perturbative) field redefinitions of the metric, which at the end of the day are tantamount to a change of variables and, therefore, should leave physical observables invariant. In particular, we justify that black hole thermodynamics should remain unaffected under field redefinitions [371]. Then, we exhibit how field redefinitions generically modify higher-derivative Lagrangians. More concretely, we show that invariants including Ricci curvatures —or, more generally, those becoming a total derivative when evaluated on Ricci flat backgrounds— can always be removed from the action.

Later, we present the most general quadratic and cubic higher-order gravities one may write after identifying two theories as equivalent if they are related by field redefinitions. Interestingly enough, it is found that such general quadratic and cubic theories belong to the GQG type, motivating the (highly non-trivial) idea that all higher-curvature gravities may be mapped, by field redefinitions, to a GQG. This is rigorously proved in a perturbative fashion shortly after for any higher-derivative gravity with no explicit covariant derivatives of the curvature, on taking into account that there exists a non-trivial GQG at each order in curvature [250]. Next we move to the more general case of higher-curvature densities with covariant derivatives of the Riemann tensor, showing explicitly that all quartic gravities as well as densities constructed from an arbitrary number of Riemann tensors and two covariant derivatives can be mapped to GQGs. In all cases, we observe that the resulting GQGs are equivalent to other GQGs without covariant derivatives as long as static and spherically symmetric solutions are concerned. This motivates us two state the following two conjectures:

1. Any higher-derivative gravity Lagrangian can be mapped, order by order, to a sum of GQGs by implementing redefinitions of the metric.

2. Any higher-derivative gravity Lagrangian can be mapped, order by order, to a sum of GQGs which, when evaluated on a SSS metric, are equivalent to GQGs with no explicit covariant derivatives of the curvature.

Finally, we illustrate some of the generic statements presented in the chapter for the gravity sector of Type IIB ST truncated at order α'^3 on $\mathrm{AdS}_5 \times S^5$, showcasing the explicit (perturbative) field redefinition to map the associated effective action to a GQG.

Chapter 2 (based on [3, 4])

In this chapter we discover a set of (four-dimensional) higher-derivative extensions of Einstein-Maxwell theory with very intriguing properties which render them as the natural generalizations of GQGs in the presence of a non-minimally coupled vector field. We name these theories *Electromagnetic (Generalized) Quasitopological Gravities* (E(G)QGs). They satisfy the following features:

1. They allow for electrically- or magnetically-charged static and spherically symmetric (SSS) solutions characterized by a single metric function $f(r) = -g_{tt} = 1/g_{rr}$.

2. The equation for the function f can be integrated once yielding at most a second-order equation where the mass appears as an integration constant.

3. The only gravitational mode propagated on maximally symmetric backgrounds is a massless graviton.

4. (Conjecture) The thermodynamic properties of the subsequent charged SSS black holes may be obtained analytically.

We establish the definition of E(G)QGs and the properties presented above after reviewing some basic aspects of general $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ theories such as their equations of motion, dualization, conserved charges or the first law of black hole mechanics. We observe that E(G)QGs can be naturally divided into two subclasses: those for which the equation for f is algebraic (EQGs) and those for which it is differential (EGQGs). Regarding the case of algebraic equation for f, we discover two infinite families of Lagrangians belonging to this class and we exactly solve the equations for magnetically charged SSS solutions. We show that in many cases the solutions are non-singular, corresponding to regular black holes or smooth horizonless geometries. Afterwards, we examine the thermodynamic properties of black holes in these theories, showing that the first law of black hole mechanics holds exactly. In addition, we analyze the properties of extremal black holes. Next we introduce infinite family of proper EGQGs, *i.e.,* those for which the equation for f is not algebraic. We study how this equation could be solved and we manage to determine analytically the thermodynamic properties of the subsequent black hole solutions. The extremal limit is discussed as well.

Finally, through the dualization of EQGs with completely regular magnetically-charged SSS solutions, we are able to identify a non-minimal higher-derivative extension of Einstein-Maxwell theory (belonging to the EQG type by definition) in which electrically-charged black holes and point charges have globally regular gravitational and electromagnetic fields. We present an exact SSS solution of this theory which reduces to the Reissner-Nordström one at weak coupling, but in which the singularity at $r = 0$ is regularized for arbitrary mass and (non-vanishing) charge. We then discuss the properties of these solutions and comment on the physical significance of these results.

Chapter 3 (based on [6, 7])

We study higher-order extensions of Einstein-Maxwell theory which are invariant under electromagnetic duality rotations, allowing for non-minimal couplings between gravity and the gauge field. For that, we start by determining the necessary and sufficient conditions on the most general 4-,6- and 8- derivative Lagrangians to preserve electromagnetic duality. Afterwards, using this result, we obtain explicitly the most general duality-invariant theory up to eight derivatives. Next we examine the effect of metric redefinitions on duality-invariant theories, showing that, to the six-derivative level, all the higher-order terms involving Maxwell field strengths can be removed via (perturbative) field redefinitions, proving along the way that the number of (gravitational) higher-order operators in the six-derivative action can be further reduced to five, of which one is topological. Therefore, the most general six-derivative duality-invariant action can be mapped via field redefinitions to Maxwell theory minimally coupled to a purely gravitational higher-derivative theory, which motivates us to conjecture the same to hold at higher orders. Later we study the charged SSS black hole solutions of this special six-derivative theory and compute their thermodynamic properties, paying special attention to the corrections to the extremal charge-to-mass ratio and discussing several additional bounds on the couplings by using the recently proposed mild form of the WGC [474, 475].

Finally, we focus on the family of higher-order gravities whose action is quadratic in the (non-minimally coupled) Maxwell field strength. Any such theory involves an infinite tower of higher-derivative terms whose computation and summation usually poses an incredibly challenging problem. Despite that, we manage to derive a closed form for the action of all the theories with a quadratic dependence on the vector field strength, finding a very peculiar expression for such action, which is reminiscent of that of Born-Infeld Lagrangians. Then we study the SSS black hole solutions of the simplest of these models with non-minimal couplings, observing that the corresponding equations of motion are invariant under rotations of the electric and magnetic charges. We work out the perturbative corrections to the Reissner-Nordström solution in this theory, determining the near-horizon geometry as well as the entropy in the case of extremal black holes. Remarkably, the entropy just possesses a constant correction despite the action containing an infinite number of terms. In addition, we discover that there is a lower bound for the charge and the mass of extremal black holes. When the sign of the coupling is such that the WGC is satisfied, the area and the entropy of extremal black holes vanish at the minimal charge.

Chapter 4 (based on [10, 11])

We present an extensive study of holographic aspects of any-dimensional higher-order generalizations of Einstein-Maxwell theories in a fully analytic and non-perturbative fashion. We achieve this by introducing the D-dimensional version of EQGs, most naturally written in terms of a $(D-3)$-form B and characterized by admitting SSS magnetically-charged solutions under B whose metric depends on a single function (the B-field can be properly dualized into a vector field, in terms of which the solutions become electrically charged). We are able to find EQGs at arbitrary order in the field strength and in the curvature, but for the sake of simplicity and concreteness we opt to restrict ourselves to a four-derivative EQG including four different operators, in which we focus for the rest of the chapter. Then we examine asymptotically AdS black holes with spherical, planar and hyperbolic horizons of the four-derivative EQG and establish various basic entries of the holographic dictionary associated to these theories. Next we carry out a detailed computation of the central charge C_J associated to $\langle JJ \rangle$ as well as of the parameter a_2 controlling the angular distribution of energy radiated after a local insertion of J [435], by means of which we obtain explicitly the coefficients of the 3-point function $\langle TJJ \rangle$. Afterwards we aim to constrain the couplings of the bulk higher-derivative theory by imposing several physical conditions. We examine unitarity and positivity-of-energy bounds on the boundary and we show that the latter are exactly equivalent to forbidding superluminal propagation of electromagnetic waves in the bulk. We further study the constraints coming from the mild form of the WGC, recently explored in the case of AdS [476].

Next we study the charged Rényi entropies S_n [447] for our holographic EQGs, which are functions of the chemical potential μ conjugate to the charge contained in the entangling region and reduce to the usual notion when $\mu \to 0$. We prove that, as long as the unitarity constraints are met, a small chemical potential always increases the amount of entanglement. Furthermore, we show that, if the WGC bounds are also satisfied, then the Rényi entropies satisfy a series of standard inequalities as a function of the index n, but we observe that these can be violated if the WGC does not hold. We then concentrate on the very special case given by $n = 1$, which we naturally identify with a notion of charged entanglement entropy. We are able to show in complete generality that for any $d(\geq 3)$-dimensional CFT, the leading correction to the (uncharged) entanglement entropy across a spherical entangling surface is quadratic in the chemical potential, positive definite, and universally controlled (up to fixed d-dependent constants) by the coefficients C_J and a_2. This proof follows from already known identities involving the magnetic response of twist operators [447] and fundamental thermodynamic relations.

Chapter 5 (based on [5, 8])

In this chapter we inaugurate the Second Part of the thesis and examine the evolution problem posed by a real parallel and irreducible spinor field defined on a globally hyperbolic Lorentzian four-manifold (M, g), which is well posed by the results of Leistner and Lischewski [477, 478]. The starting point is the theory of parabolic pairs [473] which allows to study first-order spinorial equations on pseudo-Riemannian manifolds in terms of differential systems for an algebraically constrained pair of one-forms, which is especially convenient for the study of global geometric and topological aspects of such equations. By means of this formalism, we are able to reformulate the evolution problem of a parallel spinor as a system of flow equations for a family of functions $\{\beta_t\}_{t\in\mathcal{I}}$ and a family of coframes $\{\mathfrak{e}^t\}_{t\in\mathcal{I}}$ on an appropriately chosen Cauchy hypersurface $\Sigma \subset M$, which defines the notion of parallel spinor flow. Such initial value problem imposes constraint equations on Σ, which conform the parallel Cauchy differential system and whose variables are the so-called parallel Cauchy pairs $(\mathfrak{e}, \Theta)$ given by a coframe $\mathfrak{e}$ and a symmetric two-tensor Θ on Σ. Comparing the flow dictated by the vacuum Einstein field equations together with the parallel spinor flow on common admissible initial data, we are able to state an initial data characterization of parallel spinors on Ricci flat Lorentzian four-manifolds, discovering the following result.

Theorem I.3. *A globally hyperbolic Lorentzian four-manifold (M, g) admitting a parallel spinor is Ricci flat if and only if there exists an adapted Cauchy hypersurface whose Hamiltonian constraint vanishes.*

Afterwards we investigate in more detail the parallel Cauchy differential system, characterizing all parallel Cauchy pairs on simply connected Cauchy surfaces and classifying all compact three-manifolds admitting parallel Cauchy pairs. Later we find all left-invariant parallel Cauchy pairs on simply connected Lie groups, specifying when they are allowed initial data for the Ricci flat equations. Next we classify all left-invariant parallel spinor flows on simply connected three-dimensional Lie groups, elaborating on some of their properties. We obtain furthermore the necessary and sufficient conditions for such flows to be immortal. Finally, we study a particularly simple subclass of parallel spinor flows, which we characterize geometrically and solve explicitly in some concrete cases.

Chapter 6 (based on [9])

We classify all self-dual Einstein four-manifolds invariant under a principal action of the (three-dimensional) Heisenberg group H with non-degenerate orbits. This is motivated by the fact that the one-loop deformed universal hypermultiplet metrics [120, 157] have as isometry group $\mathrm{O}(2) \ltimes \mathrm{H}$ [159], so that it is interesting to elucidate whether it suffices to recover these deformed metrics to only demand symmetry under the Heisenberg group together with a different condition, which we choose to be self-duality. We allow for Riemannian and neutral-signature metrics, permitting additionally in the latter case the possibility for the Heisenberg center to be of any causal character (timelike, lightlike or spacelike). Up to an overall sign in the metric, the manifolds under consideration can be decomposed as $(\mathcal{I} \times \mathrm{H}, \varepsilon \mathrm{d}t^2 + \chi_t)$, being $\mathcal{I} \subset \mathbb{R}$ an open interval parametrized by time t, $\varepsilon = \pm 1$ and $\{\chi_t\}_{t\in\mathcal{I}}$ a family of Riemannian or Lorentzian metrics on H, respectively.

After introducing the notation to be used in the chapter, we begin by showing that the causal character of the Heisenberg center in Einstein manifolds with Heisenberg symmetry is preserved along time, which suggests to divide the study of the neutral-signature case in terms of the causal character of the Heisenberg center. Then we determine all quaternionic (para)Kähler four-manifolds admitting an isometric principal action of the Heisenberg group with non-degenerate orbits. In addition to the lightlike case for neutral-signature metrics, which we also explore, we identify the negatively-curved (respectively positively-curved) in the Riemannian (resp. neutral-signature) case with the quaternionic (para)Kähler geometries arising from the one-loop deformed spatial Supergravity c-map (resp. temporal and Euclidean Supergravity c-maps) [136] metrics, finding as well counterparts with positive (resp. negative) curvature. Later we also classify all (para)hyperKähler four-manifolds with an isometric principal action of the Heisenberg group with non-degenerate orbits. Both for (para)hyperKähler and quaternionic (para)Kähler geometries, the metric typically possesses the following schematic form:

$$g = \varepsilon \mathrm{d}t^2 + \sum_{j,k=1}^{3} a_{jk}(t)\, \eta_{jk} \mathfrak{e}^j_{t_0} \odot \mathfrak{e}^k_{t_0}\,, \tag{I.141}$$

where $(\mathfrak{e}^i_{t_0})$ denotes a certain left-invariant coframe of the Heisenberg group, η_{jk} stands for the three-dimensional Euclidean or Minkowski metric (in an orthonormal or Witt basis), $t \in \mathcal{I}$ and $a_{jk} \in C^\infty(\mathcal{I})$ for $i, j = 1, 2, 3$ is a symmetric matrix of positive-definite functions. Finally, we also study when the corresponding (Riemannian or neutral-signature) metrics are (geodesically) complete.

Chapter 7 (based on [2])

In the last chapter of the thesis we explore the possibility of using three-dimensional contact structures for the construction of Lorentzian six-manifolds with Ricci flat metric-compatible connections with isotropic, totally antisymmetric, closed and co-closed torsion, which in turn provide solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton. For that, we begin by introducing the notion of ε-contact structure, which encompasses as particular cases the usual three-dimensional contact Riemannian, contact Lorentzian and para-contact metric structures, but which also allows for a null Reeb vector field. Next we devote ourselves to the study of ε-contact structures with lightlike Reeb vector field, which seem not to have been previously explored in the literature. We call them null contact structures. We introduce the associated notions of Sasakianity and K-contactness and classify all null contact structures on simply connected three-dimensional Lie groups which are left-invariant.

Afterwards we introduce the concept of $\varepsilon\eta$-Einstein contact structures, which include particular cases of the usual η-Einstein Riemannian and Lorentzian (para-)contact metric three-manifolds for non-null Reeb vector fields. We also classify all left-invariant $\varepsilon\eta$-Einstein contact structures on simply connected three-dimensional Lie groups. Later we prove the following theorem, which establishes the link between $\varepsilon\eta$-Einstein contact structures and solutions of Supergravity in six dimensions:

Theorem I.4. *Let $(N, \chi, \alpha_N, \varepsilon_N)$ and (X, h, α_X) be $\varepsilon\eta$-Einstein contact structures, where (N, χ) and (X, h) are Lorentzian and Riemannian three-manifolds, respectively. Then, for appropriate choices of parameters in the $\varepsilon\eta$-Einstein condition (specified in Theorem 7.5), the oriented Cartesian product manifold*

$$M = N \times X\,, \tag{I.142}$$

carries a family of solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton given by:

$$g = \chi \oplus h\,, \qquad H_{\lambda,l} = \lambda\,\nu_\chi + \frac{l}{3}\,(*_\chi \alpha_N) \wedge \alpha_X + \frac{l}{3}\,\alpha_N \wedge (*_h \alpha_X) + \lambda\,\nu_h \tag{I.143}$$

and parametrized by $(\lambda, l) \in \mathbb{R}^2$. Equivalently, $(M = N \times X, g = \chi \oplus h)$ admits a bi-parametric family of metric-compatible Ricci flat connections $\nabla^{H_{\lambda,l}}$ with totally skew-symmetric, isotropic, closed and co-closed torsion $H_{\lambda,l}$.

Finally, we illustrate the type of solutions of six-dimensional Supergravity which are obtained through the previous theorem by using the left-invariant $\varepsilon\eta$-Einstein contact structures previously mentioned.

Note on conventions

Unless otherwise stated, we use natural units $c = \hbar = 1$ all along the thesis, leaving Newton's gravitational constant G explicit. As explained before, D will stand for the number of spacetime dimensions, while $d = D - 1$ will be the dimension of the boundary CFT theory in the holographic context (Chapter 4). Similarly, we will be using the mostly-plus signature for Lorentzian metrics, $(-, +, +, \ldots, +)$. We will be following two definitions of the Riemann tensor that differ by a minus sign. In the First Part of the thesis, we follow Wald's convention [342] for the Riemann tensor:

$$\nabla_\mu \nabla_\nu \omega_\rho - \nabla_\nu \nabla_\mu \omega_\rho = R_{\mu\nu\rho}{}^{\sigma} \omega_\sigma\,. \tag{I.144}$$

On the other hand, in the Second Part of the thesis, the Riemann curvature tensor R^g is given by the conventions of Kobayashi and Nomizu [479]:

$$\mathrm{R}^g(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w\,, \qquad u, v, w \in \mathfrak{X}(M)\,. \tag{I.145}$$

Nonetheless, the subsequent Ricci curvatures are defined so as they coincide. In particular, $R_{\mu\nu} = R_{\mu\alpha\nu}{}^{\alpha}$ and $\mathrm{Ric}^g(X, Y) = \mathrm{Tr}(\mathrm{R}^g(\cdot, X)Y)$. The scalar curvature is denoted by either

$R = R_\mu{}^\mu$ or $\mathrm{Scal}^g = \mathrm{Tr}_g(\mathrm{Ric}^g)$. Unless otherwise stated, we will assume the connection is given by the Levi-Civita one. Also, while the Lie derivative of a tensor T along a vector ξ will be expressed as $L_\xi T$ in the First Part of the thesis, we will denote such derivative as $\mathcal{L}_\xi T$ in the Second Part.

On the other hand, given a certain (gravitational) action functional:

$$I[g_{\mu\nu}, \phi^i] = \int \mathrm{d}^D x \sqrt{|g|} \mathcal{L}(g^{\mu\nu}, R_{\mu\nu\rho\sigma}, \phi^i, \nabla\phi^i)\,, \tag{I.146}$$

depending on generic (possibly tensorial) fields ϕ^i, the corresponding equations of motion are given by the corresponding Euler-Lagrange equations, which we denote as:

$$\frac{1}{\sqrt{|g|}} \frac{\delta I}{\delta g^{\mu\nu}} = 0\,, \quad \frac{1}{\sqrt{|g|}} \frac{\delta I}{\delta \phi^i} = 0\,. \tag{I.147}$$

Analogously, in case we want to emphasize the Lagrangian (density) from which the action functional is constructed, we may equivalently write the corresponding equations of motion as $\left(\sqrt{|g|}\right)^{-1} \dfrac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\mu\nu}} = 0$ (or even just as $\dfrac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = 0$) and $\dfrac{\delta \mathcal{L}}{\delta \phi^i} = 0$, denoting the derivative operation with δ the functional or Euler-Lagrange derivative.

First Part

Higher-order gravities

1

All higher-order gravities as Generalized Quasitopological Gravities

Even in the purely gravitational case, there exist infinitely many instances of higher-order gravities one may think of. Moreover, the corresponding equations of motion are typically of fourth-order in derivatives, which obscures the study of their physical properties. These two features have generically led to the belief that the problem of understanding generic features of higher-derivative gravities is practically unmanageable. However, as an attempt to try to circumvent this problem, we may wonder: is it possible to find a very special class of higher-order gravities which enjoy for amenability to computations and, furthermore, span —in some sense— the whole set of higher-curvature gravities?

In this first chapter of the thesis[1], we will devote ourselves to answer this question in the positive. In particular, we will show that such special class of theories is provided by the so-called *Generalized Quasitopological Gravities* (GQGs) [78, 79, 216, 248, 249, 252, 253], which were already described in Section I.3.3. They are characterized by admitting non-hairy generalizations of the (static and spherically symmetric) Schwarzschild black hole determined by a single metric function $f(r) = -g_{tt} = 1/g_{rr}$ whose associated equation of motion is at most second-order. These properties, already known in the literature, guarantee that GQGs satisfy the first requirement above of being manageable enough so as to make explicit computations, at least in situations with a sufficient amount of symmetry.

Interestingly enough, we will be able to show that GQGs also fulfill the second condition previously proposed — they conform as well a generating set of the space of all higher-derivative gravities. In particular, we will prove that if a higher-order gravity contains no explicit covariant derivatives of the curvature or, in case it does possess such covariant derivatives, it is either an eight-derivative theory at most or just contains terms with up to two covariant derivatives of the curvature, then it can be mapped via (perturbative) field redefinitions to a GQG. Since field redefinitions, at the end of the day, are no more than a change of variables, physical observables should remain invariant and in this way we may interpret that GQGs provide a very convenient spanning set for the study of generic properties of any (purely gravitational) higher-derivative theory.

Having said this, the chapter is organized as follows. First, we present the definition of GQGs and their most relevant properties. Next, we study the generic effects of field re-

[1] As aforementioned, the content of this chapter is mainly based on [1]. One of the authors of the previous work presented his PhD thesis [89] including selected parts of [1] before the submission of the present thesis, while other author of [1] will defend his PhD thesis [480] with contents from [1] after the submission of the present PhD thesis but before its public defense.

definitions in higher-curvature gravities. Then we show explicitly how to map all quadratic and cubic gravities to GQGs via field redefinitions and that any higher-order gravity with no covariant derivatives of the curvature is equivalent to a GQG once field redefinitions are considered. In the case of terms of covariant derivatives, we prove that all theories with at most eight-derivative terms or containing densities with at most two covariant derivatives of the curvature can be mapped as well to GQGs. Afterwards we illustrate some of these points by mapping the higher-order theory arising from the gravity sector of Type IIB Supergravity in AdS_5 up to order $O(\alpha'^3)$ to a GQG and we conclude with a discussion of our findings.

1.1 Generalized Quasitopological Gravities (GQGs)

The action of any GQG [79] can be written schematically as

$$I = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{|g|}\left[-2\Lambda + R + \sum_{n=2}\sum_{i_n}\ell^{2(n-1)}\mu_{i_n}^{(n)}\mathcal{R}_{i_n}^{(n)}\right], \tag{1.1}$$

where ℓ is some length scale, $\mu_{i_n}^{(n)}$ are dimensionless couplings, and $\mathcal{R}_{i_n}^{(n)}$ are particular linear combinations of densities constructed in each case from contractions of n Riemann tensors and the metric. The subindex i_n refers to the number of independent GQG invariants at each order n.

The technical requirement which makes a generic $\mathcal{L}(g^{ab}, R_{abcd}, \nabla_a R_{bcde}, \dots)$ theory belong to the GQG class is the following. Consider a general static and spherically symmetric ansatz (SSS),

$$\mathrm{d}s^2_{\mathrm{SSS}} = -N(r)^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{(D-2)}, \tag{1.2}$$

and let $L_{N,f}$ be the effective Lagrangian which results from evaluating $\sqrt{|g|}\mathcal{L}$ in (1.2), namely

$$L_{N,f}(r, f(r), N(r), f'(r), N'(r), \dots) = N(r) r^{D-2}\mathcal{L}|_{\mathrm{SSS}}, \tag{1.3}$$

(up to an irrelevant angular contribution). Also, let $L_f = L_{1,f}$, *i.e.*, the expression resulting from imposing $N = 1$ in $L_{N,f}$.

Definition 1.1. We say that $\mathcal{L}(g^{ab}, R_{abcd}, \nabla_a R_{bcde}, \dots)$ belongs to the GQG family if the Euler-Lagrange equation of L_f vanishes identically, *i.e.*, if

$$\frac{\partial L_f}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial L_f}{\partial f'} + \frac{\mathrm{d}^2}{\mathrm{d}r^2}\frac{\partial L_f}{\partial f''} - \dots = 0, \quad \forall\, f(r). \tag{1.4}$$

The consequences of imposing (1.4) have been explored quite extensively by now, and they can be summarized as follows.

1. When linearized around any maximally symmetric background, the equations of motion of GQGs become second-order *i.e.,* they only propagate the usual massless and traceless graviton characteristic of Einstein gravity on such backgrounds [78, 79, 216, 248, 249, 252, 253].[2]

[2] Note that higher-curvature gravities satisfying property "1." —and not necessarily the rest of properties appearing in the list, nor condition (1.4)— have been studied in several other papers, *e.g.,* [88, 481–490].

2. They have a continuous and well-defined Einstein gravity limit, which corresponds to setting $\mu^{(n)}_{i_n} \to 0$ for all n and i_n.

3. They admit generalizations of the (asymptotically flat, de Sitter or Anti-de Sitter) Schwarzschild black hole —*i.e.,* solutions which reduce to it in the Einstein gravity limit— characterized by a single function $f(r)$ [79,216,248,249,252,253]. For them, $N(r) = 1$ (or some other constant) in (1.2) and $g_{tt}g_{rr} = -1$.

4. The metric function $f(r)$ is determined from a differential equation of order ≤ 2 —which can be obtained from the Euler-Lagrange equation of $N(r)$ associated to the effective Lagrangian $L_{N,f}$ defined in (1.3)[3]— when the action does not include covariant derivatives of the Riemann tensor.[4] Schematically, $\mathcal{E}[r, f(r), f'(r), f''(r); \mu^{(n)}_{i_n}] = 0$. In that case, there are typically three situations:

   - The corresponding density does not contribute at all to the equation and then we call it *trivial.*
   - The density contributes to the equation with an algebraic dependence on $f(r)$ — namely, with terms involving powers of $f(r)$. This is the case of Quasitopological [238,239,241,242,249] and Lovelock [225,226] terms. This kind of contributions only exist for $D \geq 5$.
   - The density contributes to the equation with terms containing up to two derivatives of $f(r)$. This is the case of Einsteinian Cubic Gravity in $D = 4$ [78,248,252].

It has been proven that non-trivial GQGs exist in any number of dimensions, including $D = 4$, and for arbitrarily high orders of curvature [250,251]. These references also showed that the equation that determines $f(r)$ can only be modified in a single way at each order in curvature in $D = 4$ and in two ways in $D \geq 5$. Namely, given a curvature order n, in $D = 4$ there is a linear combination of parameters $\mu^{(n)} = \sum_{i_n} c_{i_n}\mu^{(n)}_{i_n}$ such that the contribution to the equation of $f(r)$ will only depend on $\mu^{(n)}$: as long as the equation of $f(r)$ is concerned, we can turn on and off as many densities as we want, provided at least one of them (corresponding to a nontrivial density) is nonzero at each order in curvature [253]. The explicit form of the equation reads [250,251,253]

$$\begin{aligned}(1-f) - \frac{2GM}{r} - \frac{\Lambda r^2}{3} - \sum_n \mu^{(n)}\ell^{2(n-1)}\frac{f'^{(n-3)}}{r^{n-2}}\Bigg[\frac{f'^3}{n} + \frac{(n-3)f+2}{(n-1)r}f'^2 \\ -\frac{2}{r^2}f(f-1)f' - \frac{1}{r}ff''\left(f'r - 2(f-1)\right)\Bigg] = 0\,,\end{aligned} \tag{1.5}$$

where M stands for the ADM mass of the solution [82,491,492].

In $D \geq 5$ we can split the couplings in two sums of couplings. The first group of densities, belonging to the Quasitopological subset, will modify the equation of $f(r)$ algebraically, whereas the second group will introduce derivatives of $f(r)$. The

[3]Namely, the equation reads $\frac{\delta L_{N,f}}{\delta N}\big|_{N=1} = 0$, and it can be proven that it takes the form of a total derivative for any theory satisfying (1.4).

[4]When it does, the differential equation would be of order $\leq 2m+2$, where m is the number of covariant derivatives of the term with the greatest number of them.

equation of $f(r)$ will only depend on a particular combination of couplings of each one of these groups [79, 249–251]. Schematically we have

$$\mathcal{E}_{\rm E}[r,f(r)]+\sum_n\left[\mu^{(n)}_{\rm QG}\,\mathcal{E}^{\rm QG}_n[r,f(r)^n]+\mu^{(n)}_{\rm GQG}\,\mathcal{E}^{\rm GQG}_n[r,f(r),f'(r),f''(r)]\right]=0\,, \quad (1.6)$$

where $\mathcal{E}_{\rm E}[r,f(r)]$ is the Einstein gravity contribution

$$\mathcal{E}_{\rm E}[r,f(r)]=(1-f)-\frac{16\pi GM}{(D-2)\Omega_{(D-2)}r^{D-3}}-\frac{2\Lambda r^2}{(D-1)(D-2)}\,, \quad (1.7)$$

where $\Omega_{(D-2)}=2\pi^{(D-1)/2}/\Gamma[(D-2)/2]$ and $\mu^{(n)}_{\rm QG}$ and $\mu^{(n)}_{\rm GQG}$ stand for the sums of all couplings corresponding to densities contributing algebraically and with derivatives of $f(r)$ to the equation respectively. For planar and hyperbolic horizons, exactly the same story holds with small modifications in the corresponding equations for the metric function.

5. Both when the equation is algebraic and when it is differential of order 2, given a fixed set of $\mu^{(n)}_{i_n}$, the equation admits a single black-hole solution representing a smooth deformation of Schwarzschild's one, which is completely characterized by its ADM energy. For spherically symmetric configurations, the corresponding metric describes the exterior field of matter distributions [216].

6. The thermodynamic properties of black holes can be computed analytically by solving a system of algebraic equations without free parameters. At least in $D=4$, black holes typically become stable below certain mass, which substantially modifies their evaporation process [253].

7. A certain subset of GQGs admit additional solutions of the Taub-NUT/Bolt class in even dimensions [258]. Similarly to black holes, these are also characterized by a single metric function and their thermodynamic properties can be computed analytically.

8. A (generally) different subset of four-dimensional GQGs also gives rise to second-order equations for the scale factor when evaluated on a FLRW ansatz, giving rise to a well-posed cosmological evolution [260, 262, 263]. Remarkably, an inflationary period smoothly connected with late-time standard ΛCDM evolution is naturally generated by the higher-curvature terms.

In addition to this more or less structural properties, GQGs have been considered in various contexts, and many interesting additional properties and applications explored —see *e.g.,* [254–257, 259, 261, 264–267, 493, 494].

At cubic order in curvature, the most general (nontrivial) GQG can be written as

$$I=\int\frac{\mathrm{d}^Dx\sqrt{|g|}}{16\pi G}\left[-2\Lambda+R+\ell^2\mu^{(2)}_1\mathcal{X}_4+\ell^4\left(\mu^{(3)}_1\mathcal{X}_6+\mu^{(3)}_2\mathcal{Z}_D+\mu^{(3)}_3\mathcal{S}_D\right)\right]\,, \quad (1.8)$$

where we used the notation of (1.1) to denote the couplings. Here, $\mathcal{X}_4$ and $\mathcal{X}_6$ stand for the dimensionally-extended Euler quadratic and cubic densities, also known as Gauss-Bonnet and cubic Lovelock terms, respectively,

$$\mathcal{X}_4=+R^2-4R_{ab}R^{ab}+R_{abcd}R^{abcd}\,, \quad (1.9)$$

$$\begin{aligned}\mathcal{X}_6 = & -8R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b}+4R_{ab}^{\ \ cd}R_{cd}^{\ \ ef}R_{ef}^{\ \ ab}-24R_{abcd}R^{abc}{}_{e}R^{de}\\ & +3R_{abcd}R^{abcd}R+24R_{abcd}R^{ac}R^{bd}+16R_a^bR_b^cR_c^a-12R_{ab}R^{ab}R+R^3\,.\end{aligned} \tag{1.10}$$

$\mathcal{X}_4$ is topological in $D=4$ and trivial for $D\leq 3$, while $\mathcal{X}_6$ is topological in $D=6$ and trivial for $D\leq 5$. On the other hand, $\mathcal{Z}_D$ is the so-called Quasitopological Gravity density [238, 239]

$$\begin{aligned}\mathcal{Z}_D = & +R_{a\ c}^{\ b\ d}R_{b\ d}^{\ e\ f}R_{e\ f}^{\ a\ c}+\frac{1}{(2D-3)(D-4)}\left[\frac{3(3D-8)}{8}R_{abcd}R^{abcd}R\right.\\ & -\frac{3(3D-4)}{2}R_a{}^cR_c{}^aR-3(D-2)R_{acbd}R^{acb}{}_{e}R^{de}+3DR_{acbd}R^{ab}R^{cd}\\ & \left.+6(D-2)R_a{}^cR_c{}^bR_b{}^a+\frac{3D}{8}R^3\right]\,.\end{aligned} \tag{1.11}$$

Note that when only the above three terms are included in addition to the usual Einstein-Hibert action, the equation satisfied by the metric function $f(r)$ is algebraic —which partially explains why they were identified before the last term, $\mathcal{S}_D$. We also stress that for $D\geq 6$, $\mathcal{Z}_D$ affects the equation of $f(r)$ in the same way as $\mathcal{X}_6$ does. For $D=5$, $\mathcal{X}_6$ is trivial, and the effect of $\mathcal{Z}_5$ is nontrivial —from this perspective, we could have just omitted $\mathcal{X}_6$ from (1.8). These observations are in agreement with our comments in "4." above regarding the fact that at each order and for each D there is a single way of modifying the equation of $f(r)$ algebraically (and another single way involving derivatives of $f(r)$ —see below).

When $\mathcal{S}_D$ is included, the equation becomes differential of order 2. The explicitly form of this density can be chosen to be [79]

$$\begin{aligned}\mathcal{S}_D = & +14R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b}+2R_{abcd}R^{abc}{}_{e}R^{de}-\frac{(38-29D+4D^2)}{4(D-2)(2D-1)}R_{abcd}R^{abcd}R\\ & -\frac{2(-30+9D+4D^2)}{(D-2)(2D-1)}R_{abcd}R^{ac}R^{bd}-\frac{4(66-35D+2D^2))}{3(D-2)(2D-1)}R_a^bR_b^cR_c^a\\ & +\frac{(34-21D+4D^2)}{(D-2)(2D-1)}R_{ab}R^{ab}R-\frac{(30-13D+4D^2)}{12(D-2)(2D-1)}R^3\,.\end{aligned} \tag{1.12}$$

The explicit form of the equation of $f(r)$ corresponding to (1.8) can be found *e.g.,* in [79]. In $D=4$, $\mathcal{S}_4$ is usually rewritten in terms of the so-called Einsteinian Cubic Gravity density,[5] defined as [78]

$$\mathcal{P}=12R_{a\ b}^{\ c\ d}R_{c\ d}^{\ e\ f}R_{e\ f}^{\ a\ b}+R_{ab}^{cd}R_{cd}^{ef}R_{ef}^{ab}-12R_{abcd}R^{ac}R^{bd}+8R_a^bR_b^cR_c^a\,, \tag{1.13}$$

which was in fact the first GQG identified beyond the Lovelock and Quasitopological ones [248, 252]. Both densities are connected through [79]

$$\mathcal{S}_4-\frac{1}{4}\mathcal{X}_6+4\mathcal{C}=\mathcal{P}\,, \tag{1.14}$$

[5]The original construction of Einsteinian Cubic Gravity in [78] was based on the fact that it satisfies properties "1." and "2." for general dimensions, and it does so in a way such that the relative coefficients appearing in its definition in (1.13) are the same for general D —just like for Lovelock theories. It was later realized that the four-dimensional version of the theory satisfies the rest of properties listed.

where $\mathcal{C}$ is an example of a trivial GQG, in the sense that it has no effect on the equation of $f(r)$, as its contribution to it vanishes identically. It is given by

$$\mathcal{C} = \frac{1}{2} R_a^b R_b^a R - 2R^{ac} R^{bd} R_{abcd} - \frac{1}{4} R R_{abcd} R^{abcd} + R^{de} R_{abcd} R^{abc}{}_e \,. \tag{1.15}$$

Although we will not be particularly interested in trivial GQGs, we emphasize that those terms are only trivial for SSS metrics, but they can —and they do [260, 262, 263]— play an important role when other kinds of metrics are considered.

As we mention later on, the structure of GQGs above described seems to extend to general dimensions and arbitrary orders in curvature. So far, examples of GQGs including covariant derivatives of the Riemann tensor have not appeared in the literature, but we are confident that they do exist as well —see Sections 1.5 and 1.7 for discussions on the role played by invariants containing covariant derivatives of the Riemann tensor in our setup.

1.2 Field redefinitions in higher-curvature gravities

In this section we explore some of the effects resulting from redefining the metric tensor on higher-curvature gravities. In subsection 1.2.1, we make some technical comments regarding metric redefinitions involving derivatives of the metric itself and explain how on-shell actions evaluated on solutions related by metric redefinitions agree with each other. Then, in subsection 1.2.2, we explain how higher-curvature densities involving Ricci curvatures —or, more generally, densities which become a total derivative when evaluated on Ricci flat metrics— can be removed from the gravitational effective action by convenient metric redefinitions.

1.2.1 On-shell action invariance

Let us consider the most general metric-covariant theory of gravity[6]

$$I[g_{ab}] = \int \mathrm{d}^D x \sqrt{|g|} \mathcal{L} \left(g^{ab}, R_{abcd}, \nabla_e R_{abcd}, \nabla_e \nabla_f R_{abcd}, \ldots \right) \,. \tag{1.16}$$

We are interested in determining how (1.16) transforms under a redefinition of the metric tensor g_{ab} of the form

$$g_{ab} = \tilde{g}_{ab} + \tilde{Q}_{ab} \,, \tag{1.17}$$

where $\tilde{Q}_{ab}$ is a symmetric tensor constructed from the new metric $\tilde{g}_{ab}$. Ideally, we would like the field redefinition to be algebraic, so that the relation between g_{ab} and $\tilde{g}_{ab}$ is functional. However, the most general tensor we can build using the metric without introducing higher derivatives is proportional to the metric itself. Hence, $\tilde{Q}_{ab}$ generically involves curvature tensors, and (1.17) is a differential relation. The action $\tilde{I}$ for the new metric $\tilde{g}_{ab}$ is simply obtained by substituting (1.17) in the original action, namely

$$\tilde{I}[\tilde{g}_{ab}] = I[\tilde{g}_{ab} + \tilde{Q}_{ab}] \,. \tag{1.18}$$

Observe that since (1.17) involves derivatives of the metric, extremizing the action with respect to $\tilde{g}_{ab}$ is, in general, inequivalent from extremizing it with respect to g_{ab}. Whenever

[6]We also assume that parity is preserved so that we do not have to include terms containing the Levi-Civita symbol. Nevertheless, all results in this section also apply when those terms are included.

$g_{ab}^{\rm sol}$ is a solution of the original theory, the relation (1.17) always produces a solution $\tilde{g}_{ab}^{\rm sol}$ of the transformed theory when we invert it. However, the converse is not true: there exist solutions of the equations of motion obtained from the variation with respect to $\tilde{g}_{ab}$ which do not produce a solution of the original theory when we apply the map (1.17). The reason behind this is the presence of extra derivatives in the field redefinition. This increases the number of derivatives in the equations of motion derived from $\tilde{I}$, which introduces spurious solutions that need be discarded. This issue is further discussed in Appendix 1.A. Provided it is taken into account, both theories, I and $\tilde{I}$, are equivalent.

Note that when we keep only the meaningful solutions —*i.e.,* those which are related by (1.17)— the corresponding on-shell actions match,

$$\tilde{I}\left[\tilde{g}_{ab}^{\rm sol}\right] = I\left[g_{ab}^{\rm sol}\right] . \tag{1.19}$$

Since, *e.g.,* black hole thermodynamics can be determined —in the Euclidean path-integral approach [81]— by evaluating the on-shell action, this simple observation proves that black hole thermodynamics can be equivalently computed in both frames. The same conclusion can be reached [371] using Wald's formula [366] —see [495–499] for additional discussions regarding this issue.[7]

Of particular interest for us will be situations in which both $g_{ab}^{\rm sol}$ and $\tilde{g}_{ab}^{\rm sol}$ represent static and spherically symmetric black holes. As argued in [371], field redefinitions of the form (1.17) preserve both the asymptotic and horizon structures of $g_{ab}^{\rm sol}$, so they map black holes into black holes. Particularizing even more, from the following subsection on, we will consider higher-derivative theories controlled by small parameters and perturbative field redefinitions weighted by them. Ultimately, one of the reasons for considering redefinitions mapping generic higher-derivative theories to GQGs is the fact that the equations of motion of the latter on static and spherically symmetric configurations become particularly simple and universal. In this particular setup, (1.19) will relate the on-shell action corresponding to a certain generalization of the Schwarzschild-(A)dS black hole (continuously connected to it) for a given higher-derivative theory at leading order in the corresponding coupling to the on-shell action of the black hole solution corresponding to the transformed GQG. We will provide an explicit example of this match between on-shell actions in Section 1.6.

1.2.2 Ricci curvatures and reducible densities

Let us now determine how the redefinition (1.17) changes the action (1.16). For that, we assume the redefinition to be perturbative, *i.e.,* we treat $\tilde{Q}_{ab}$ as a perturbation and we work at linear order. This is enough for our purposes, since, following the EFT approach, we will also expand the action in a perturbative series of higher-derivative terms. Observe that in this case the relation (1.17) can be inverted as

$$\tilde{g}_{ab} = g_{ab} - Q_{ab} + \mathcal{O}(Q^2) , \tag{1.20}$$

where Q_{ab} has the same expression as $\tilde{Q}_{ab}$ but replacing $\tilde{g}_{ab} \to g_{ab}$. Let us introduce the equations of motion of the original theory as

$$\mathcal{E}_{ab} = \frac{1}{\sqrt{|g|}}\frac{\delta I}{\delta g^{ab}} . \tag{1.21}$$

[7] In order to prove this statement rigorously, it is necessary to assume some mild conditions on $\tilde{Q}_{ab}$, namely, its fall-off at infinity should be fast enough. All redefinitions we will consider are well-behaved in this sense.

Then, at linear order in $\tilde{Q}_{ab}$, the transformed action $\tilde{I}$ reads

$$\tilde{I} = \int \mathrm{d}^D x \sqrt{|\tilde{g}|} \left[\tilde{\mathcal{L}} - \tilde{\mathcal{E}}_{ab}\tilde{Q}^{ab} + \mathcal{O}(Q^2)\right] . \tag{1.22}$$

where the tildes denote evaluation on $\tilde{g}_{ab}$. Thus, the redefinition introduces a term in the action proportional to the equations of motion of the original theory. Let us be more explicit about the form of the Lagrangian by expanding it as a sum over all possible higher-derivative terms

$$I = \frac{1}{16\pi G}\int \mathrm{d}^D x \sqrt{|g|} \left[R + \sum_{n=2}^{\infty} \ell^{2(n-1)} \mathcal{L}^{(n)}\right] , \tag{1.23}$$

where ℓ is a length scale and $\mathcal{L}^{(n)}$ represents the most general Lagrangian involving $2n$ derivatives of the metric. The explicit form of the invariants at orders $n=2$ and $n=3$ can be found below in (1.29) and (1.30) respectively. The number of terms grows very rapidly, and the $n=4$ Lagrangian already contains 92 terms [500].[8]

Let $\tilde{Q}^{(k)}_{ab}$ be a symmetric tensor containing $2k$ derivatives of the metric. Then, we perform the following field redefinition

$$g_{ab} = \tilde{g}_{ab} + \ell^{2k}\tilde{Q}^{(k)}_{ab} . \tag{1.24}$$

Then, the transformed action (1.22) reads

$$\tilde{I} = \int \frac{\mathrm{d}^D x\sqrt{|\tilde{g}|}}{16\pi G} \left[\tilde{R} + \sum_{n=2}^{k} \ell^{2(n-1)}\tilde{\mathcal{L}}^{(n)} + \ell^{2k}\left(\tilde{\mathcal{L}}^{(k+1)} - \tilde{R}^{ab}\hat{Q}^{(k)}_{ab}\right) + \sum_{n=k+2}^{\infty} \ell^{2(n-1)}\tilde{\mathcal{L}}'^{(n)}\right] , \tag{1.25}$$

where all quantities are evaluated on $\tilde{g}_{ab}$, and[9]

$$\hat{Q}^{(k)}_{ab} = \tilde{Q}^{(k)}_{ab} - \frac{1}{2}\tilde{g}_{ab}\tilde{Q}^{(k)} , \quad \tilde{Q}^{(k)} = \tilde{g}^{ab}\tilde{Q}^{(k)}_{ab} . \tag{1.26}$$

Hence, all terms containing up to $2k$ derivatives of the metric remain unaffected, while those with $2(k+1)$ derivatives receive a correction of the form $-\tilde{R}^{ab}\hat{Q}^{(k)}_{ab}$. The higher-order terms also get corrections which depend in a more complicated way on $\tilde{Q}^{(k)}_{ab}$. If the starting action already contained all possible terms, the net effect of these corrections is just to change the couplings in the Lagrangian. We denote these modified terms as $\tilde{\mathcal{L}}'^{(n)}$.

From this, it is clear that performing this type of field redefinitions order by order, starting at $k=1$, we can remove all terms in the action which involve contractions of the Ricci tensor —except, of course, the Einstein-Hilbert term. At each order, it suffices to choose $\tilde{Q}^{(k)}_{ab}$ in (1.24) such that $\hat{Q}^{(k)}_{ab}$ equals the tensorial structure which appears contracted

[8]Ref. [500] provides the number of *linearly independent* invariants, but many of them differ by total derivative terms, which are irrelevant for the action. The number of relevant terms is, in general, much smaller —yet quite large. For instance, besides the 3 quadratic densities and the 10 cubic densities which we include in (1.29) and (1.30), [500] adds $\nabla_a\nabla^a R$ to the former list, and 7 more terms of the form: $\nabla_a\nabla^a\nabla_b\nabla^b R$, $R\nabla_a\nabla^a R$, $\nabla^a\nabla^b R R_{ab}$, $R^{ab}\nabla_c\nabla^c R_{ab}$, $\nabla^a\nabla^b R^{cd}R_{cadb}$, $\nabla^a R^{bc}\nabla_c R_{ba}$, $\nabla^a R^{bcde}\nabla_a R_{bcde}$ to the latter. All these terms are either total derivatives or can be written in terms of the others plus total derivatives, so they can be discarded —see *e.g.,* [239, 240].

[9]We have $\mathcal{E}^{ab}\tilde{Q}^{(k)}_{ab} = (R^{ab} - \frac{1}{2}g^{ab}R)\tilde{Q}^{(k)}_{ab} = R^{ab}\hat{Q}^{(k)}_{ab}$.

with R^{ab} in the corresponding density. In other words, any term containing Ricci curvatures is meaningless from the EFT point of view, and we are free to add or remove terms of that type. From a different perspective, it has been argued —*e.g.,* in [501]— that if some higher-curvature correction controlled by ℓ^{2k} involves operators which vanish on the equations of motion produced by the lower-order action, the relevant physics is not affected at $\mathcal{O}(\ell^{2k})$, and we can just ignore it. For the gravitational effective action, this is equivalent to the possibility of removing all terms involving Ricci curvatures.

Observe that in (1.23) we (intentionally) did not include a cosmological constant. When we add it, the effect of the redefinition (1.24) is

$$\begin{aligned}\tilde{I}=\int\frac{\mathrm{d}^Dx\sqrt{|\tilde{g}|}}{16\pi G}&\left[-2\Lambda+\tilde{R}+\sum_{n=2}^{k-1}\ell^{2(n-1)}\tilde{\mathcal{L}}^{(n)}+\ell^{2(k-1)}\left(\tilde{\mathcal{L}}^{(k)}+\frac{2(\Lambda\ell^2)}{(D-2)}\hat{Q}^{(k)}\right)\right.\\&\left.+\ell^{2k}\left(\tilde{\mathcal{L}}^{(k+1)}-\tilde{R}^{ab}\hat{Q}^{(k)}_{ab}\right)+\sum_{n=k+2}^{\infty}\ell^{2n}\tilde{\mathcal{L}}'^{(n)}\right].\end{aligned}\tag{1.27}$$

Namely, not only the terms involving $2(k+1)$ derivatives of the metric get modified, those involving $2k$ derivatives also receive a correction. This is a complication with respect to the case without cosmological constant. If we remove terms involving Ricci curvatures at a given order, the field redefinition of the following order will introduce a correction of the form $\frac{2(\Lambda\ell^2)}{(D-2)}\hat{Q}^{(k)}$ which will generically include again terms involving Ricci curvatures. Hence, the process cannot be carried out order-by-order because all steps are coupled. If one wants to remove all the terms with Ricci curvature up to order $2k$, it is necessary to consider the most general field redefinition up to that order, *i.e.,* including all the terms $\tilde{Q}^{(m)}_{ab}$ of order $m\leq k$ at the same time. Nevertheless, we stress that this is just a technical complication: finding the precise field redefinition that removes the corresponding Ricci curvature terms is more involved, but it can certainly be done.

Motivated by the above analysis, let us close this section with a definition which will be useful in the remainder of the chapter.

Definition 1.2. A curvature invariant is said to be *reducible* if it is a total derivative when evaluated on any Ricci flat metric. The rest of them are said to be *irreducible.*

Note that this trivially contains the case in which the invariant vanishes on Ricci flat metrics. Intuitively, the irreducible terms correspond to those formed purely from contractions of the Riemann tensor, without explicit factors of Ricci curvature. As we have explained, all reducible terms can be removed or introduced by using field redefinitions, whereas the irreducible ones cannot. Therefore, the most general higher-derivative gravitational effective action is obtained by including all possible irreducible terms. Then, we are free to add as many reducible terms as we wish: these would simply correspond to different frame choices.

1.3 All quadratic and cubic gravities as GQGs

In the absence of cosmological constant, the gravitational effective action can be written as a series of operators with an increasing number of derivatives of the metric:

$$I=\frac{1}{16\pi G}\int\mathrm{d}^Dx\sqrt{|g|}R+\sum_{n=2}^{\infty}\frac{\ell^{2n-2}}{16\pi G}I^{(2n)}\,.\tag{1.28}$$

Again, ℓ is some length scale, and $I^{(2n)}$ is the most general action involving curvature invariants of order n. Ignoring total derivatives, the four- and six-derivative actions read

$$I^{(4)} = \int \mathrm{d}^D x \sqrt{|g|} \left[\alpha_1 R^2 + \alpha_2 R_{ab} R^{ab} + \alpha_3 R_{abcd} R^{abcd} \right] , \tag{1.29}$$

$$\begin{aligned} I^{(6)} = \int \mathrm{d}^D x \sqrt{|g|} \Big[& \beta_1 R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} + \beta_2 R_{ab}^{\ \ cd} R_{cd}^{\ \ ef} R_{ef}^{\ \ ab} + \beta_3 R_{abcd} R^{abc}{}_{e} R^{de} \\ & + \beta_4 R_{abcd} R^{abcd} R + \beta_5 R_{abcd} R^{ac} R^{bd} + \beta_6 R_a^{\ b} R_b^{\ c} R_c^{\ a} + \beta_7 R_{ab} R^{ab} R \\ & + \beta_8 R^3 + \beta_9 \nabla_d R_{ab} \nabla^d R^{ab} + \beta_{10} \nabla_a R \nabla^a R \Big] . \end{aligned} \tag{1.30}$$

In the case of the four-derivative action, the Riemann-squared term can be traded by the Gauss-Bonnet density (1.9), so that the most general action reads[10]

$$I^{(4)} = \int \mathrm{d}^D x \sqrt{|g|} \left[\alpha_1 R^2 + \alpha_2 R_{ab} R^{ab} + \alpha_3 \mathcal{X}_4 \right] . \tag{1.31}$$

Similarly to the quadratic case, we can trade two of the cubic invariants involving contractions of the Riemann tensor alone by the cubic Lovelock density $\mathcal{X}_6$, defined in (1.10), and one of the cubic Generalized Quasitopological densities, $\mathcal{S}_D$, defined in (1.12). Therefore, $I^{(6)}$ can be alternatively written as

$$\begin{aligned} I^{(6)} = \int \mathrm{d}^D x \sqrt{|g|} \Big[& \beta_1 \mathcal{X}_6 + \beta_2 \mathcal{S}_D + \beta_3 R_{abcd} R^{abc}{}_{e} R^{de} + \beta_4 R_{abcd} R^{abcd} R + \beta_5 R_{abcd} R^{ac} R^{bd} \\ & + \beta_6 R_a^{\ b} R_b^{\ c} R_c^{\ a} + \beta_7 R_{ab} R^{ab} R + \beta_8 R^3 + \beta_9 \nabla_d R_{ab} \nabla^d R^{ab} + \beta_{10} \nabla_a R \nabla^a R \Big] . \end{aligned} \tag{1.32}$$

Note that in $D \geq 5$, we can alternatively replace either $\mathcal{S}_D$ or $\mathcal{X}_6$ by the cubic Quasitopological term $\mathcal{Z}_D$ defined in (1.11). Also, in $D = 4$ we can replace $\mathcal{S}_4$ by the Einsteinian Cubic Gravity density (1.13) using (1.14). Regardless of these choices, we observe that in addition to the first two terms, belonging to the GQG family, we are left with a series of reducible terms which, as we have argued in the previous section, can be removed by convenient field redefinitions of the metric.

The explicit redefinition which removes all terms but $\mathcal{X}_4$, $\mathcal{X}_6$ and $\mathcal{S}_D$ goes as follows. First, in order to remove the R^2 and $R_{ab}R^{ab}$ terms, we perform

$$g_{ab} = \tilde{g}_{ab} + \alpha_2 \ell^2 \tilde{R}_{ab} - \frac{\ell^2 \tilde{R}}{D-2} \tilde{g}_{ab} (2\alpha_1 + \alpha_2) . \tag{1.33}$$

Then:

$$I^{(4)} \to \tilde{I}^{(4)} = \int \mathrm{d}^D x \sqrt{|\tilde{g}|} \alpha_3 \tilde{\mathcal{X}}_4 . \tag{1.34}$$

Now, this redefinition also affects the higher-order terms, but since we are starting from the most general theory, the only effect is to change the coefficients of these terms. In particular, for the six-derivative ones: $\beta_i \to \tilde{\beta}_i$. Then, the following redefinition of the metric

$$\tilde{g}_{ab} = \tilde{\tilde{g}}_{ab} + \ell^4 \Big[\tilde{\beta}_3 \tilde{\tilde{R}}_{aecd} \tilde{\tilde{R}}_b^{\ ecd} + \tilde{\beta}_5 \tilde{\tilde{R}}^{ef} \tilde{\tilde{R}}_{aebf} + \tilde{\beta}_6 \tilde{\tilde{R}}_a^{\ e} \tilde{\tilde{R}}_{be} + \tilde{\beta}_7 \tilde{\tilde{R}} \tilde{\tilde{R}}_{ab} - \tilde{\beta}_9 \tilde{\tilde{\nabla}}^2 \tilde{\tilde{R}}_{ab} \tag{1.35}$$

[10]The coefficients α_i are not the same as in the previous action, but we prefer not to introduce additional unnecessary notation whenever possible.

$$-\frac{\tilde{\tilde{g}}_{ab}}{D-2}\left(\tilde{\tilde{R}}_{efcd}\tilde{\tilde{R}}^{efcd}(\tilde{\beta}_3+2\tilde{\beta}_4)+\tilde{\tilde{R}}_{ef}\tilde{\tilde{R}}^{ef}(\tilde{\beta}_5+\tilde{\beta}_6)+\tilde{\tilde{R}}^2(\tilde{\beta}_7+2\tilde{\beta}_8)-\tilde{\tilde{\nabla}}^2\tilde{\tilde{R}}(\tilde{\beta}_9-2\tilde{\beta}_{10})\right)\Bigg]\,,$$

leaves the four-derivative terms unaffected, while cancelling all six-derivative terms that contain Ricci curvatures,

$$\tilde{I}^{(6)}\to\tilde{\tilde{I}}^{(6)}=\int \mathrm{d}^Dx\sqrt{|\tilde{\tilde{g}}|}\left[\tilde{\beta}_1\tilde{\tilde{\mathcal{X}}}_6+\tilde{\beta}_2\tilde{\tilde{\mathcal{S}}}_D\right]\,. \tag{1.36}$$

Hence, the most general action can be written, after all, as

$$\tilde{\tilde{I}}=\frac{1}{16\pi G}\int \mathrm{d}^Dx\sqrt{|\tilde{\tilde{g}}|}\left[\tilde{\tilde{R}}+\ell^2\alpha_3\tilde{\tilde{\mathcal{X}}}_4+\ell^4\left(\tilde{\beta}_1\tilde{\tilde{\mathcal{X}}}_6+\tilde{\beta}_2\tilde{\tilde{\mathcal{S}}}_D\right)+\mathcal{O}(\ell^6)\right]\,, \tag{1.37}$$

which only contains GQG terms, as anticipated —compare with (1.8). In $D=4$, the cubic Lovelock density vanishes identically and the Gauss-Bonnet term is topological, which leaves us with

$$I=\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R+\beta\ell^4\mathcal{P}+\mathcal{O}(\ell^6)\right]\,, \tag{1.38}$$

where we traded $\mathcal{S}_4$ by the ECG density $\mathcal{P}$ using (1.14) and renamed the gravitational coupling. Hence, Einsteinian Cubic Gravity [78] is (up to field redefinitions) the most general four-dimensional gravitational effective action we can write including up to six derivatives of the metric.[11]

1.4 All $\mathcal{L}(g^{ab}, R_{abcd})$ gravities as GQGs

Let us now move on to a more general case, namely, general higher-curvature gravities constructed from arbitrary contractions of the metric and the Riemann tensor. In addition to the notion of reducible densities introduced in Section 1.2, it is convenient to define here another concept:

Definition 1.3. We say that a curvature invariant $\mathcal{L}$ is *completable to a Generalized Quasitopological density* (or just *completable* for short), if there exists a GQG density $\mathcal{Q}$ such that $\mathcal{L}-\mathcal{Q}$ is reducible.

In other words, $\mathcal{L}$ is completable if by adding reducible terms to it, we are able to obtain a GQG term. Note that reducible terms are trivially completable to 0. Then, the question whether any higher-derivative gravity can be expressed as a sum of GQG terms is equivalent to the following question: *Are all irreducible densities completable to a GQG?* We have just found that the answer is positive at least up to six-derivative terms. The reason is that there exist more independent GQG densities than irreducible terms, which allowed us to "complete" all of them. In the case of the four-derivative terms, the only irreducible density is the Riemann-squared term, and this can be completed to the Gauss-Bonnet density. For the six-derivative terms, we saw that all terms containing derivatives of the Riemann tensor are reducible, and that the only irreducible terms are the two Riemann-cube contributions respectively controlled by β_1 and β_2 in (1.30). In general

[11] This is consistent with the result in [501], where $\mathcal{P}$ appears traded by the density $\sim R_{ab}^{cd}R_{ef}^{ab}R_{cd}^{ef}$. That is also the kind of term which appears in the two-loop effective action of perturbative quantum gravity [23, 24].

dimensions D there are 3 GQGs involving different combinations of these cubic terms, so they can always be completed.

Observe that the problem of completing irreducible invariants depends on the number of spacetime dimensions. In lower dimensions, many of the densities are not linearly independent, so the number of irreducible densities is significantly smaller, and this simplifies the problem of completing them to GQGs. As a consequence, on general grounds we expect that if all irreducible invariants are completable for high enough D, they will also be completable for smaller D. For instance, going back to the six-derivative example, we find that the two cubic densities are independent when $D \geq 6$. In $D = 4, 5$ only one of them is linearly independent, and in $D < 4$ there is only Ricci curvature so all theories are reducible to Einstein gravity. On the other hand, the number of independent GQGs in $D = 4$ is four, whereas in $D > 4$ there are only three of them. Therefore, in lower dimensions there are less irreducible terms and more ways to complete them to a GQG theory. The lower the dimension, the easier the task.

As we will see in a moment, the problem of completing all invariants constructed from an arbitrary contraction of metrics and n Riemann tensors —a number which grows very rapidly with n— can be drastically simplified. In order to formulate this result, we will need the following somewhat surprising result:

Lemma 1.1 (Deser, Ryzhov, 2005 [502]). *When evaluated on a general static and spherically symmetric ansatz (1.2), all possible contractions of n Weyl tensors*[12] *are proportional to each other. More precisely, let $(W^n)_i$ be one of the possible independent ways of contracting n Weyl tensors. Then for all i*

$$(W^n)_i|_{\text{SSS}} = F(r)^n c_i \, , \tag{1.40}$$

where c_i is some constant which depends on the particular contraction, and $F(r)$ is an i-independent function of r given in terms of the functions appearing in the SSS ansatz (1.2). In other words, the ratio $[(W^n)_1/(W^n)_2]|_{\text{SSS}}$ for any pair of contractions of n Weyl tensors is a constant which does not depend on the radial coordinate r.

Proof. When evaluated on (1.2) the Weyl tensor (with two covariant and two contravariant indices) takes the form

$$W^{ab}{}_{cd}\Big|_{\text{SSS}} = -2\chi(r)\frac{(D-3)}{(D-1)} w^{ab}{}_{cd} \, , \tag{1.41}$$

where

$$\chi(r) = \frac{(-2 + 2f - 2rf' + r^2 f'')}{2r^2} + \frac{N'}{2rN}(-2f + 3rf') + \frac{fN''}{N} \tag{1.42}$$

is a function which contains the full dependence on the radial coordinate. On the other hand, $w^{ab}{}_{cd}$ is a r-independent tensorial structure which can be written as [502]

$$w^{ab}{}_{cd} = 2\tau^{[a}_{[c}\rho^{b]}_{d]} - \frac{2}{(D-2)}\left(\tau^{[a}_{[c}\sigma^{b]}_{d]} + \rho^{[a}_{[c}\sigma^{b]}_{d]}\right) + \frac{2}{(D-2)(D-3)}\sigma^{[a}_{[c}\sigma^{b]}_{d]} \, , \tag{1.43}$$

[12]Recall that the Weyl tensor is defined as

$$W_{abcd} = R_{abcd} - \frac{2}{(D-2)}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(D-2)(D-1)} R g_{a[c}g_{d]b} \, . \tag{1.39}$$

where τ, ρ and σ are orthogonal projectors defined as[13]

$$\tau_a^b = \delta_a^0 \delta_0^b\,, \quad \rho_a^b = \delta_a^1 \delta_1^b\,, \quad \sigma_a^b = \sum_{m=2}^{D} \delta_a^m \delta_m^b\,. \tag{1.44}$$

The precise form of the projectors is not particularly relevant for our purposes. The important point is that any possible invariant $(W^n)_i$ constructed from the contraction of n Weyl tensors will be given by

$$(W^n)_i|_{\text{SSS}} = \left(-2\chi(r)\frac{(D-3)}{(D-1)}\right)^n (w^n)_i\,, \tag{1.45}$$

where $(w^n)_i$ stands for the constant resulting from the contraction induced on the w tensors, which we can identify with c_i in (1.40). Therefore, $(W^n)_i|_{\text{SSS}}$ takes the form (1.40) with $F(r)$ given by the function between brackets. □

Now, we are ready to formulate the following result:

Proposition 1.1. *Every irreducible curvature invariant constructed out of any number of Riemann curvature tensors is completable.*

Proof. Let us consider first the case of irreducible curvature invariants with any number of Riemann tensors and no explicit covariant derivatives of them. Let the order of these invariants be $2n$ in derivatives of the metric, *i.e.,* n in curvature. Since they are irreducible and they do not contain derivatives of the curvature, they are formed from contractions of a product of n Riemann tensors. We can write schematically $\mathcal{L}_i = (\text{Riem}^n)_i$, where the subscript i denotes again a specific way of contracting the indices. We can consider an alternative basis by replacing the Riemann tensor by the Weyl tensor in the expressions of these densities. Both ways of expressing these invariants are equivalent since they differ by terms containing Ricci curvatures, which are reducible. We denote the densities resulting from replacing $R_{abcd} \to W_{abcd}$ everywhere in the $\mathcal{L}_i$ by $\tilde{\mathcal{L}}_i = (W^n)_i$. Let $\tilde{\mathcal{L}}_{i_0}$ denote a non-trivial GQG constructed out of the contraction of n Riemann tensors with no covariant derivatives, which is known to always exist [250]. As we explained in Section 1.1, the condition that determines if a given density belongs to the GQG class exclusively depends on the evaluation of the density on the general static and spherically symmetric (SSS) metric ansatz (1.2), *i.e.,* on the way the corresponding density depends on the radial coordinate r. But from Lemma 1.1 we know that all order-n invariants constructed from contractions of the Weyl tensor are proportional to each other when evaluated on (1.2), in the sense that the dependence on the radial coordinate is identical for all i, and given by a fixed function —which we called $F(r)^n$ in (1.40). Then, since by assumption $\tilde{\mathcal{L}}_{i_0}\big|_{\text{SSS}} \neq 0$, all invariants $\tilde{\mathcal{L}}_i$ are proportional to $\tilde{\mathcal{L}}_{i_0}$ when evaluated on SSS metrics. As a consequence, the fact that $\tilde{\mathcal{L}}_{i_0}$ is completable implies that all the rest of densities of order n are. □

The result can be reformulated as follows:

Corollary 1.1. *Any higher-derivative gravity Lagrangian built out of curvature invariants without covariant derivatives can be mapped, order by order, to a sum of GQG terms by implementing redefinitions of the metric of the form* (1.17).

[13] Namely, they satisfy $\tau\tau = \tau$, $\rho\rho = \rho$, $\sigma\sigma = \sigma$, $\tau\rho = \tau\sigma = \rho\sigma = 0$. Also, their traces read $\text{Tr}\tau = \text{Tr}\rho = 1$, $\text{Tr}\sigma = D-2$.

Let us note that the result above shows the existence of a field redefinition that takes the Lagrangian $\mathcal{L}(g^{ab}, R_{abcd})$ to a sum of GQGs, but it does not guarantee uniqueness. Indeed, if at a given order one has several types of non-trivial GQGs —namely, Quasitopological and proper GQG— it is possible to map the Lagrangian to a sum of terms whose equations for SSS metrics match the ones of the chosen theory (again, Quasitopological or proper GQG). More generally, the Lagrangian can be mapped to a combination of those terms. Note that this implies that Quasitopological Gravities and GQGs are related by field redefinitions.[14]

Before closing this section, let us mention that our conclusions also hold if one includes parity-breaking terms in the effective action, *i.e.,* those that involve the Levi-Civita symbol $\epsilon_{a_1 \ldots a_D}$. In fact, all such terms vanish for spherically symmetric configurations, hence all of them trivially belong to the GQG family.

1.5 Terms involving covariant derivatives of the Riemann tensor

In the previous section we proved that all $\mathcal{L}(g_{ab}, R_{abcd})$ gravities can be either removed from the action or written as GQGs using field redefinitions. Let us now see what happens with higher-curvature terms involving covariant derivatives of the Riemann tensor. The role of these terms is less clear. In particular, they have not been used to construct GQGs so far —although, for what we know, this type of theories should exist as well. On the other hand, as we saw in Section 1.3, up to six-order in derivatives all these terms are actually reducible. This is no longer the case at quartic order in curvature.

In order to gain some insight about the general behavior of this kind of terms, let us consider what happens at that order. There exist 26 independent quartic invariants which do not involve covariant derivatives of the Riemann tensor, namely —see *e.g.,* [88,249,500],

$$
\begin{aligned}
I^{(8)} = \int \mathrm{d}^D x \sqrt{|g|} \Big[& \gamma_1 R^{abcd} R_{a\ c}^{\ e\ f} R_{e\ b}^{\ g\ h} R_{fgdh} + \gamma_2 R^{abcd} R_{a\ c}^{\ e\ f} R_{e\ f}^{\ g\ h} R_{bgch} \\
& + \gamma_3 R^{abcd} R_{ab}^{\ \ ef} R_{c\ e}^{\ g\ h} R_{dgfh} + \gamma_4 R^{abcd} R_{ab}^{\ \ ef} R_{ce}^{\ \ gh} R_{dfgh} + \gamma_5 R^{abcd} R_{ab}^{\ \ ef} R_{ef}^{\ \ gh} R_{cdgh} \\
& + \gamma_6 R^{abcd} R_{abc}^{\ \ \ e} R_{fghd} R^{fgh}{}_{e} + \gamma_7 (R_{abcd} R^{abcd})^2 + \gamma_8 R^{ab} R^{cdef} R_{c\ ea}^{\ g} R_{dgfb} \\
& + \gamma_9 R^{ab} R^{cdef} R_{cd\ a}^{\ \ g} R_{efgb} + \gamma_{10} R^{ab} R_{a\ b}^{\ c\ d} R_{efgc} R^{efg}{}_{d} + \gamma_{11} R R_{a\ b}^{\ c\ d} R_{c\ d}^{\ e\ f} R_{e\ f}^{\ a\ b} \\
& + \gamma_{12} R R_{ab}^{cd} R_{cd}^{ef} R_{ef}^{ab} + \gamma_{13} R^{ab} R^{cd} R^{e\ f}_{\ a\ c} R_{ebfd} + \gamma_{14} R^{ab} R^{cd} R^{e\ f}_{\ a\ b} R_{ecfd} \\
& + \gamma_{15} R^{ab} R^{cd} R^{ef}{}_{ac} R_{efbd} + \gamma_{16} R^{ab} R_b^c R^{def}{}_{a} R_{defc} + \gamma_{17} R_{ef} R^{ef} R_{abcd} R^{abcd} \\
& + \gamma_{18} R R_{abcd} R^{abc}{}_{e} R^{de} + \gamma_{19} R^2 R_{abcd} R^{abcd} + \gamma_{20} R^{ab} R_{acbd} R^{ec} R_e^d \\
& + \gamma_{21} R R_{abcd} R^{ac} R^{bd} + \gamma_{22} R_a^b R_b^c R_c^d R_d^a + \gamma_{23} \left(R_{ab} R^{ab} \right)^2 + \gamma_{24} R R_a^b R_b^c R_c^a
\end{aligned} \tag{1.46}
$$

[14] Imagine, for instance, that we start with a Quasitopological density $\mathcal{Z}$ and a GQG density $\mathcal{S}$ of certain order. Replacing all Riemann tensors by Weyl tensors gives rise to new densities $\tilde{\mathcal{Z}} = \mathcal{Z} + \mathrm{RC}_{\mathcal{Z}}$ and $\tilde{\mathcal{S}} = \mathcal{S} + \mathrm{RC}_{\mathcal{S}}$ where $\mathrm{RC}_{\mathcal{S},\mathcal{Z}}$ are certain reducible densities involving Ricci curvatures. Now, from Lemma 1.1 we know that $\tilde{\mathcal{Z}}|_{\mathrm{SSS}} = c\tilde{\mathcal{S}}|_{\mathrm{SSS}}$, for some constant c. Then, it follows that $\mathcal{Z} = c(\mathcal{S} + \mathrm{RC}_{\mathcal{S}}) - \mathrm{RC}_{\mathcal{Z}} + \mathcal{T}$, where $\mathcal{T}$ is a trivial GQG density, *i.e.,* one such that $\mathcal{T}|_{\mathrm{SSS}} = 0$. Naturally, $\mathcal{S}' \equiv c\mathcal{S} + \mathcal{T}$ is another GQG density. It follows that Quasitopological densities can be mapped to GQG densities of the same order —and viceversa— via field redefinitions, the mapping generically involving trivial GQG densities (which play no role as far as the equations of SSS metrics are concerned).

$$+\gamma_{25} R^2 R_{ab} R^{ab} + \gamma_{26} R^4 \Big] \, .$$

Of these, at most the first 7 are irreducible —this happens for $D > 7$. Now, in [249] several non-trivial and irreducible GQG theories were constructed using those invariants. Consequently, by virtue of Corollary 1.1, it is clear that the 26 invariants can always be written as a sum of GQGs using field redefinitions. Hence, just like in the quadratic and cubic cases, all quartic gravities of the form $\mathcal{L}(g_{ab}, R_{abcd})$ can be written as GQGs.

What about terms with covariant derivatives of the Riemann tensor? Looking at [500], we find five apparently irreducible terms of that kind, namely

$$\mathcal{L}_1 = R^{abcd} \nabla_b R^{efg}{}_a \nabla_d R_{efgc} \, , \tag{1.47}$$
$$\mathcal{L}_2 = R^{abcd} \nabla_c R^{efg}{}_a \nabla_d R_{efgb} \, , \tag{1.48}$$
$$\mathcal{L}_3 = R^{abcd} \nabla^g R^{e}{}_a{}^f{}_c \nabla_g R_{ebfd} \, , \tag{1.49}$$
$$\mathcal{L}_4 = R^{abcd} R_a{}^{efg} \nabla_d \nabla_g R_{becf} \, , \tag{1.50}$$
$$\mathcal{L}_5 = \nabla_e \nabla_f R_{abcd} \nabla^e \nabla^f R^{abcd} \, . \tag{1.51}$$

However, a careful analysis —using commutation of covariant derivatives, the symmetries of the Riemann tensor and the Bianchi identities[15]— reveals that all of them can be decomposed as a sum of total derivative terms plus quartic curvature terms (without covariant derivatives) plus terms with Ricci curvature (hence reducible). This is, they can be expressed (for each i) as

$$\mathcal{L}_i = \nabla_a J^a_{(i)} + \mathcal{Q}_{(i)} + R_{ab} F^{ab}_{(i)} \, , \tag{1.52}$$

for certain tensors $J^a_{(i)}$ and $F^{ab}_{(i)}$ and some quartic density $\mathcal{Q}_{(i)}$. In order to illustrate this, let us show how $\mathcal{L}_1$ is reduced to an expression of the form (1.52). First, we have

$$\begin{aligned}\mathcal{L}_1 &= R^{abcd} \nabla_b R^{efg}{}_a \nabla_d R_{efgc} = \frac{1}{4} R^{abcd} \nabla^g R^{ef}{}_{ab} \nabla_g R_{efcd} \\ &= \frac{1}{4} \nabla_g \left(R^{abcd} \nabla^g R^{ef}{}_{ab} R_{efcd} \right) - \frac{1}{4} R^{abcd} \nabla^2 R^{ef}{}_{ab} R_{efcd} - \frac{1}{4} \nabla_g R^{abcd} \nabla^g R^{ef}{}_{ab} R_{efcd} \, ,\end{aligned} \tag{1.53}$$

where in the first equality we applied the differential Bianchi identity twice, and in the second we integrated by parts. Now we note that the last term in the second line is actually $-\mathcal{L}_1$, so we get

$$\mathcal{L}_1 = \frac{1}{8} \nabla_g \left(R^{abcd} \nabla^g R^{ef}{}_{ab} R_{efcd} \right) - \frac{1}{8} R^{abcd} \nabla^g \nabla_g R^{ef}{}_{ab} R_{efcd} \, . \tag{1.54}$$

Then we are done, because the Laplacian of the Riemann tensor decomposes, using a schematic notation, as $\nabla^2 \mathrm{Riem} = \nabla\nabla \mathrm{Ricci} + \mathrm{Riem}^2$,[16] so we can indeed express $\mathcal{L}_1$ as in (1.52). Proceeding similarly with the rest of terms we arrive at the same conclusion.

[15] Recall that these read: $R_{abcd} + R_{acdb} + R_{adbc} = 0$ and $\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$ respectively.

[16] Explicitly, one has [503]

$$\begin{aligned}\nabla^e \nabla_e R_{abcd} = &+ 2\nabla_{[a|} \nabla_c R_{|b]d} + 2\nabla_{[b|} \nabla_d R_{|a]c} - 4 \left[R^p{}_a{}^q{}_b R_{p[c|q|d]} + R^p{}_a{}^q{}_{[c|} R_{pbq|d]} \right] \\ &+ g^{pq} \left[R_{qbcd} R_{pa} + R_{aqcd} R_{pb} \right] \, .\end{aligned} \tag{1.55}$$

Since total derivatives are irrelevant for the action, and since we can remove all terms containing Ricci curvatures by means of field redefinitions, the terms with covariant derivatives of the Riemann tensor only change the coefficients of the quartic terms, which are already present in the action. Hence, from the point of view of effective field theory, these densities are meaningless and can be removed. In addition, we conclude that all eight-derivative terms can be recast as a sum of GQGs by implementing field redefinitions.

Let us now turn to a more general case. Any higher-derivative gravity can be written as the span of all monomials formed from contractions of ∇_a, W_{abcd} and R_{ab}. Such a set can be written schematically as $\mathcal{A} = \cup_{q,n,r\in\mathbb{N}}\mathcal{A}_{q,n,r}$ where $\mathcal{A}_{q,n,r} = \{\nabla^q \times W^n \times \mathrm{Ric}^r\}$. Out of these subsets, the only ones susceptible of containing irreducible terms are $\mathcal{A}_{q,n,0}$, so the ultimate goal would be to prove that all elements in

$$\mathcal{I}_q = \bigcup_{n\in\mathbb{N}} \mathcal{A}_{q,n,0} \tag{1.56}$$

are completable to a GQG. First, let us note that these sets can be split according to the partitions of the number of covariant derivatives, q,

$$\mathcal{I}_q = \bigcup_{k=1}^{p(q)} \mathcal{I}_q^{P_k(q)}\,, \tag{1.57}$$

where $p(q)$ is the the number of partitions of q and $P_k(q)$ denotes the k-th partition of q (we assume partitions to be ordered in some way). For instance, the first few cases are: $\mathcal{I}_0$, which is the set of monomials formed from general contractions of Weyl tensors; $\mathcal{I}_2$, which is the set of monomials formed from Weyl tensors and two covariant derivatives —this can be in turn split as the union of $\mathcal{I}_2^{\{1,1\}}$ and $\mathcal{I}_2^{\{2\}}$: in the former set the two covariant derivatives act on two different Weyl tensors, while in the second the two derivatives act on the same Weyl; $\mathcal{I}_4$, which contains terms with four covariant derivatives and an arbitrary number of Weyl tensors —this can be decomposed as $\mathcal{I}_4 = \mathcal{I}_2^{\{1,1,1,1\}} \cup \mathcal{I}_2^{\{2,1,1\}} \cup \mathcal{I}_2^{\{2,2\}} \cup \mathcal{I}_2^{\{3,1\}} \cup \mathcal{I}_2^{\{4\}}$. Observe that not all subsets are independent. For example, we see that any term belonging to $\mathcal{I}_2^{\{2\}}$ can be written as a sum of terms in $\mathcal{I}_2^{\{1,1\}}$ upon integration by parts. For the same reason, for $q = 4$ it is enough to keep the subsets $\mathcal{I}_2^{\{1,1,1,1\}}$, $\mathcal{I}_2^{\{2,1,1\}}$ and $\mathcal{I}_2^{\{2,2\}}$.

We know that all terms in $\mathcal{I}_0$ can be completed to GQGs, and the purpose of the remainder of this section is to show explicitly that all terms in $\mathcal{I}_2$ satisfy the same property. We expect the trend to go on for all sets $\mathcal{I}_q$ but a general proof seems quite challenging —not so much a case-by-case partial proof for the following $\mathcal{I}_{q\geq 4}$.

As we have said, the only subset of $\mathcal{I}_2$ which needs to be considered is $\mathcal{I}_2^{\{1,1\}}$. Any term belonging to this subset can be written schematically as

$$\mathcal{I}_2^{\{1,1\}} \ni \mathcal{R}_2^{\{1,1\}} = W^n \nabla W \nabla W\,, \tag{1.58}$$

for some value of n. We saw in (1.41) that, when evaluated on a SSS metric the Weyl tensor has a very simple structure so that any scalar formed from it is proportional to the same quantity. In Appendix 1.B we show that any term of the form (1.58) can be written in turn as

$$\mathcal{R}_2^{\{1,1\}}\Big|_{\mathrm{SSS}} = f(r)\chi^n \left(c_1(\chi')^2 + c_2\frac{\chi\chi'}{r} + c_3\frac{\chi^2}{r^2}\right)\,, \tag{1.59}$$

where $\chi' = \mathrm{d}\chi(r)/\mathrm{d}r$ and $c_{1,2,3}$ are constants. Thus, there are at most three linearly independent terms in $\mathcal{I}_2^{\{1,1\}}$ when one considers SSS metrics. Hence, if we are able to find three independent terms in $\mathcal{I}_2^{\{1,1\}}$ which are completable to a GQG, that will imply that all densities in $\mathcal{I}_2^{\{1,1\}}$ are completable. Three possible terms of that type are

$$\begin{aligned}
\mathcal{W}_1^{\{1,1\}} =& \sum_{k=0}^{n} \nabla_b W_{a_1a_2}{}^{a_3a_4} (W^{n-k})_{a_3a_4}{}^{a_5a_6} \nabla^b W_{a_5a_6}{}^{a_7a_8} (W^n)_{a_7a_8}{}^{a_1a_2}\,, && (1.60)\\
\mathcal{W}_2^{\{1,1\}} =& \nabla_b W^{bcd}{}_{a_1} \nabla_c W_{da_2}{}^{a_3a_4} W_{a_3a_4}{}^{a_5a_6} \dots W_{a_{2n+1}a_{2n+2}}{}^{a_1a_2}\,, && (1.61)\\
\mathcal{W}_3^{\{1,1\}} =& \nabla_b W^{bcde} \nabla_f W^f{}_{cde} W_{a_1a_2}{}^{a_3a_4} \dots W_{a_{2n-1}a_{2n}}{}^{a_1a_2}\,, && (1.62)
\end{aligned}$$

where $(W^n)_{bc}{}^{df}$ denotes a n-Weyl product of the form $W_{bc}{}^{a_1a_2} W_{a_1a_2}{}^{a_3a_4} \dots W_{a_{2n}a_{2n+1}}{}^{df}$. We can check that when evaluated on a SSS metric the previous terms are linearly independent. For instance, in $D = 4$ we obtain the expressions

$$\begin{aligned}
\mathcal{W}_1^{\{1,1\}} =& \frac{3^{-n-2} 4 \left((-1)^n + 2^{n+1}\right) f(r) (-\chi)^n \left((n+1) r^2 (\chi')^2 + 6\chi^2\right)}{r^2}\,, && (1.63)\\
\mathcal{W}_2^{\{1,1\}} =& 3^{-n-1} \left(2^n - (-1)^n\right) f(r) (-\chi)^n \left(\frac{\chi'\chi}{r} + 3\frac{\chi^2}{r^2}\right)\,, && (1.64)\\
\mathcal{W}_3^{\{1,1\}} =& f(r) \left(\chi' + 3\frac{\chi}{r}\right)^2 (-\chi)^n \frac{(2 - (-1)^{n-1} 2^{2-n})}{3}\,, && (1.65)
\end{aligned}$$

which are linearly independent for any integer value of n. Hence, any term of the form (1.59) can be expressed a sum of these three combinations (the same conclusion holds for arbitrary D). Therefore all invariants in $\mathcal{I}_2^{\{1,1\}}$ can be expressed as a linear combination of these terms when evaluated on SSS metrics. This can be alternatively written as

$$\mathcal{R}_2^{\{1,1\}} = C_1 \mathcal{W}_1^{\{1,1\}} + C_2 \mathcal{W}_2^{\{1,1\}} + C_3 \mathcal{W}_3^{\{1,1\}} + \dots\,, \qquad (1.66)$$

where the ellipsis denote terms that vanish on SSS metrics —which are trivially completable to a GQG. Now, it is easy to check that, by means of field redefinitions, the densities $\mathcal{W}_{1,2,3}^{\{1,1\}}$ are completable. Actually, both $\mathcal{W}_2^{\{1,1\}}$ and $\mathcal{W}_3^{\{1,1\}}$ are reducible because they are proportional to the divergence of Weyl tensor, which depends only on Ricci curvatures

$$\nabla_c W^c{}_{abd} = \frac{2(D-3)}{D-2} \left[\nabla_{[b} R_{d]a} - \frac{1}{2(D-1)} g_{a[d} \nabla_{b]} R\right]. \qquad (1.67)$$

On the other hand, $\mathcal{W}_1^{\{1,1\}}$ can be written as

$$\begin{aligned}
\mathcal{W}_1^{\{1,1\}} =& \nabla_b \left(\nabla^b W_{a_1a_2}{}^{a_3a_4} W_{a_3a_4}{}^{a_5a_6} W_{a_5a_6}{}^{a_7a_8} \dots W_{a_{2n+3}a_{2n+4}}{}^{a_1a_2}\right)\\
&- \nabla^2 W_{a_1a_2}{}^{a_3a_4} W_{a_3a_4}{}^{a_5a_6} W_{a_5a_6}{}^{a_7a_8} \dots W_{a_{2n+3}a_{2n+4}}{}^{a_1a_2}\,.
\end{aligned} \qquad (1.68)$$

Since the Laplacian of the Weyl tensor can be expressed as $\nabla^2\text{Weyl} = \nabla\nabla\text{Ricci} + \text{Riem}^2$, we conclude that, by means of field redefinitions, $\mathcal{W}_1^{\{1,1\}}$ can be reduced to a sum of terms without covariant derivatives. We know that those terms are completable, so the densities $\mathcal{W}_{1,2,3}^{\{1,1\}}$ and any other $\mathcal{R}_2^{\{1,1\}}$ are also completable. The result is actually stronger than that:

since the densities $\mathcal{W}_{1,2,3}^{\{1,1\}}$ can be completed to a GQG without covariant derivatives of the Riemann tensor, this implies that any other $\mathcal{R}_2^{\{1,1\}}$ can be completed to a GQG which, when evaluated on a SSS metric, is equivalent to a GQG without covariant derivatives.

Collecting all the results of this section, together with Proposition 1.1, we are ready to formulate the main result of this chapter:

Theorem 1.1. *Assume that an irreducible curvature invariant satisfies one of the following conditions:*

1. *It is constructed out of any number of Riemann curvature tensors with no covariant derivatives.*
2. *It is an eight-derivative term.*
3. *It contains at most two covariant derivatives.*

Then it is completable to a GQG which does not involve covariant derivatives when evaluated on SSS metrics. Equivalently, consider a certain higher-derivative gravity Lagrangian built out of curvature invariants fulfilling each of them at least one of the conditions above. Then, by implementing redefinitions of the metric of the form (1.17)*, it can be mapped, order by order, to a sum of GQG terms which evaluated on a SSS background are equivalent to GQGs without covariant derivatives.*

In sum, we have shown that, at least for densities including eight (or less) derivatives of the metric as well as for densities constructed from an arbitrary number of Riemann tensors and two covariant derivatives, all densities can be mapped to GQGs. In all cases, those GQGs become equivalent to GQGs which do not involve covariant derivatives when evaluated on SSS metrics. We postpone a discussion on the role of this kind of terms in ever more general situations to Section 1.7. Before doing so, we wish to illustrate, for a particularly charismatic higher-derivative gravity action, how the mapping to a GQG is done and how this preserves the thermodynamic properties of black hole solutions.

1.6 Type IIB effective action at $\mathcal{O}(\alpha'^3)$ as a GQG

In this section we show how the gravitational sector of the Type IIB ST effective action on $\mathrm{AdS}_5 \times S^5$ truncated at (sub)leading order in α' can be mapped to a generic quartic GQG. Then we show, in spite of the very different appearance of the equations of motion evaluated on a SSS ansatz in both frames —and therefore of the corresponding black hole metrics—, that their thermodynamic properties exactly match, as expected.

The usual ten-dimensional Type IIB Supergravity action receives stringy corrections weighted by powers of α'. The first correction appears at α'^3 order [108, 504], so schematically we have

$$I_{\mathrm{IIB}} = I_{\mathrm{IIB}}^{(0)} + \alpha'^3 I_{\mathrm{IIB}}^{(1)} + \dots, \tag{1.69}$$

where $I_{\mathrm{IIB}}^{(0)}$ is the usual two-derivative Supergravity action [505], and the dots stand for subleading corrections in α'. When the theory is considered in $\mathcal{A}_5 \times S^5$ where $\mathcal{A}_5$ is a negatively curved Einstein manifold, it is consistent to truncate all fields except the metric and it is possible to write an effective action for the five-dimensional metric [128–130]. This is given by [131, 132]

$$I_{\mathrm{IIB}_{\mathcal{A}_5\times S^5}}[g_{ab}] = \frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{|g|}\left[R + \frac{12}{\ell^2} + \frac{\zeta(3)}{8}\alpha'^3 W^4\right], \tag{1.70}$$

where W^4 is a particular combination of contractions of four Weyl tensors given by

$$W^4 = \left(W_{abcd}W^{ebcf} + \frac{1}{2}W_{adbc}W^{efbc}\right)W^{ag}{}_{he}W_{fg}{}^{hd}. \tag{1.71}$$

As we mentioned in Section 1.5, at quartic order in curvature, there are 26 invariants involving contractions of the Riemann tensor of the metric —see (1.46). The last 19 densities involve explicit Ricci tensors, so they are reducible and we can use them to complete the Type IIB effective action in (1.70) to GQGs by means of field redefinitions. The structure of quartic GQGs was completely characterized in [249]. As usual in $D \geq 5$, there exist three kinds of terms: those which belong to the Quasitopological class (including the one previously constructed in [241] and the quartic Lovelock density $\mathcal{X}_8$) —namely, their contribution to the equation which determines the metric function $f(r)$ when the SSS ansatz (1.2) is considered is algebraic —, those which contribute with up to two derivatives to the equation of $f(r)$ and those which do not contribute to the equation of $f(r)$ at all. As explained in Section 1.1, in spite of the degeneracy of GQG densities, there are only two functional modifications of the equation of $f(r)$ at each curvature order, so when the full set of $n = 4$ GQG invariants is introduced, the different couplings only appear summed to each other in two groups in front of each kind of contribution to the equation as in (1.6).

Let us now consider the following metric redefinition

$$g_{ab} \to g_{ab} - \frac{\zeta(3)}{8}\alpha'^3\left(\hat{C}_{ab} - \frac{1}{3}\hat{C}g_{ab}\right), \tag{1.72}$$

where $\hat{C}_{ab}$ is some cubic-curvature rank-2 symmetric tensor and $\hat{C} = \hat{C}_{ab}g^{ab}$. The original action (1.70) is transformed to

$$\tilde{I} = \frac{1}{16\pi G}\int \mathrm{d}^5x\sqrt{|g|}\left[\frac{12}{\ell^2} + R + \frac{\zeta(3)\alpha'^3}{2\ell^2}\hat{C} + \frac{\zeta(3)}{8}\alpha'^3\left(W^4 + R^{ab}\hat{C}_{ab}\right)\right], \tag{1.73}$$

up to subleading terms in α'. Observe that the presence of the cosmological constant gives rise to the appearance of a cubic contribution. The most general $\hat{C}_{ab}$ we can write is[17]

$$\begin{aligned}\hat{C}_{ab} = {}& a_8R^{cdef}R_{c}{}^{g}{}_{ea}R_{dgfb} + a_9R^{cdef}R_{cd}{}^{g}{}_{a}R_{efgb} + a_{10}R_{a}{}^{c}{}_{b}{}^{d}R_{efgc}R^{efg}{}_{d} \\ &+ a_{11}g_{ab}R_{g}{}^{c}{}_{h}{}^{d}R_{c}{}^{e}{}_{d}{}^{f}R_{e}{}^{g}{}_{f}{}^{h} + a_{12}g_{ab}R_{gh}{}^{cd}R_{cd}{}^{ef}R_{ef}{}^{gh} + a_{13}R^{cd}R^{e}{}_{a}{}^{f}{}_{c}R_{ebfd} \\ &+ a_{14}R^{cd}R^{e}{}_{a}{}^{f}{}_{b}R_{ecfd} + a_{15}R^{cd}R^{ef}{}_{ac}R_{efbd} + a_{16}R^{c}_{b}R^{def}{}_{a}R_{defc} + a_{17}R_{ab}R_{cdef}R^{cdef} \\ &+ a_{18}g_{ab}R_{ghcd}R^{ghc}{}_{e}R^{de} + b_{18}RR_{ghca}R^{ghc}{}_{b} + a_{19}R_{ab}RR_{ghcd}R^{ghcd} + a_{20}R_{acbd}R^{ec}R^{d}_{e} \\ &+ b_{20}R^{gh}R_{gahd}R^{d}_{b} + a_{21}g_{ab}R_{ghcd}R^{gc}R^{hd} + b_{21}RR_{gacb}R^{gc} + a_{22}R_{a}{}^{c}R_{bc}\end{aligned} \tag{1.74}$$

[17] We denote the different coefficients by a_i and b_i. The a_i correspond to terms which, when contracted with R^{ab}, produce a scalar numbered as in (1.46); the b_i are used in cases in which there is a second term which produces the same scalar.

$$+ a_{23} R_{ab} R_{ef} R^{ef} + a_{24} g_{ab} R_c{}^d R_d{}^e R_e{}^c + b_{24} R R_a{}^c R_{bc} + a_{25} g_{ab} R R_{ef} R^{ef} + b_{25} R^2 R_{ab} + a_{26} g_{ab} R^3 \,.$$

Then we have

$$\begin{aligned} R^{ab}\hat{C}_{ab} = {} & a_8 R^{ab} R^{cdef} R_c{}^g{}_{ea} R_{dgfb} + a_9 R^{ab} R^{cdef} R_{cd}{}^g{}_a R_{efgb} + a_{10} R^{ab} R_a{}^c{}_b{}^d R_{efgc} R^{efg}{}_d \\ & + a_{11} R R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + a_{12} R R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + a_{13} R^{ab} R^{cd} R^e{}_a{}^f{}_c R_{ebfd} \\ & + a_{14} R^{ab} R^{cd} R^e{}_a{}^f{}_b R_{ecfd} + a_{15} R^{ab} R^{cd} R^{ef}{}_{ac} R_{efbd} + a_{16} R^{ab} R^c_b R^{def}{}_a R_{defc} \\ & + a_{17} R_{ef} R^{ef} R_{abcd} R^{abcd} + (a_{18} + b_{18}) R R_{abcd} R^{abc}{}_e R^{de} + a_{19} R^2 R_{abcd} R^{abcd} \\ & + (a_{20} + b_{20}) R^{ab} R_{acbd} R^{ec} R^d_e + (a_{21} + b_{21}) R R_{abcd} R^{ac} R^{bd} + a_{22} R^b_a R^c_b R^d_c R^a_d \\ & + a_{23} \left(R_{ab} R^{ab}\right)^2 + (a_{24} + b_{24}) R R^b_a R^c_b R^a_c + (a_{25} + b_{25}) R^2 R_{ab} R^{ab} + a_{26} R^4 \,, \end{aligned} \tag{1.75}$$

as well as

$$\begin{aligned} \hat{C} = {} & (a_8 + 5a_{11}) R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + (a_9 + 5a_{12}) R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} \\ & + (a_{10} + a_{13} + a_{15} + a_{16} + 5a_{18}) R_{abcd} R^{abc}{}_e R^{de} + (a_{17} + b_{18} + 5a_{19}) R_{abcd} R^{abcd} R \\ & + (a_{14} + b_{20} + 5a_{21}) R_{abcd} R^{ac} R^{bd} + (a_{20} + a_{22} + 5a_{24}) R_a{}^b R_b{}^c R_c{}^a \\ & + (b_{21} + a_{23} + b_{24} + 5a_{25}) R_{ab} R^{ab} R + (b_{25} + 5a_{26}) R^3 \,. \end{aligned} \tag{1.76}$$

Imposing the terms $\hat{C}$ and $W^4 + R^{ab}\hat{C}_{ab}$ to be of the GQG type independently, we find the following constraints

$$a_{10} = -\frac{43}{32} - \frac{13\sigma}{32} - \frac{a_8}{10} - \frac{6a_9}{5} \,, \tag{1.77}$$

$$a_{12} = \frac{1}{16} + \frac{\sigma}{16} - \frac{a_8}{10} - \frac{a_9}{5} - \frac{a_{11}}{2} \,, \tag{1.78}$$

$$a_{17} = \frac{3451}{2880} + \frac{1241\sigma}{2880} + \frac{3a_8}{100} + \frac{3a_9}{50} - \frac{7a_{13}}{40} - \frac{25a_{14}}{72} - \frac{19a_{15}}{180} - \frac{11a_{16}}{45} \,, \tag{1.79}$$

$$a_{18} = -\frac{113}{640} - \frac{233\sigma}{640} + \frac{31a_8}{50} + \frac{6a_9}{25} + 3a_{11} - \frac{a_{13}}{5} - \frac{a_{15}}{5} - \frac{a_{16}}{5} \,, \tag{1.80}$$

$$b_{18} = -\frac{43}{640} - \frac{13\sigma}{640} + \frac{7a_8}{100} - \frac{9a_9}{25} - \frac{a_{13}}{5} - \frac{a_{15}}{5} - \frac{a_{16}}{5} \,, \tag{1.81}$$

$$a_{19} = -\frac{449}{2880} - \frac{17\sigma}{1440} - \frac{19a_8}{200} + \frac{3a_9}{50} - \frac{3a_{11}}{8} + \frac{3a_{13}}{40} + \frac{5a_{14}}{72} + \frac{11a_{15}}{180} + \frac{4a_{16}}{45} \,, \tag{1.82}$$

$$b_{20} = -\frac{439}{144} - \frac{83\sigma}{48} + \frac{3a_8}{5} - \frac{24a_9}{5} - 2a_{13} + \frac{4a_{14}}{3} - \frac{8a_{15}}{3} - \frac{4a_{16}}{3} - a_{20} \,, \tag{1.83}$$

$$a_{21} = \frac{1553}{1440} + \frac{391\sigma}{480} - \frac{18a_8}{25} + \frac{24a_9}{25} - 3a_{11} + \frac{2a_{13}}{5} - \frac{7a_{14}}{15} + \frac{8a_{15}}{15} + \frac{4a_{16}}{15} + \frac{a_{20}}{5} \,, \tag{1.84}$$

$$b_{21} = \frac{253}{288} + \frac{41\sigma}{96} - \frac{a_8}{5} + \frac{6a_9}{5} + \frac{3a_{13}}{10} - \frac{11a_{14}}{30} + \frac{a_{15}}{3} + \frac{4a_{16}}{15} - \frac{a_{20}}{5} \,, \tag{1.85}$$

$$a_{23} = -\frac{539}{810} - \frac{11\sigma}{90} - \frac{4a_8}{25} + \frac{56a_9}{75} + \frac{a_{13}}{3} + \frac{a_{14}}{90} + \frac{17a_{15}}{45} + \frac{13a_{16}}{45} - \frac{7a_{22}}{30} \,, \tag{1.86}$$

$$a_{24} = \frac{27}{64} + \frac{27\sigma}{64} - \frac{2a_8}{5} - 2a_{11} - \frac{a_{20}}{5} - \frac{a_{22}}{5} \,, \tag{1.87}$$

$$b_{24} = \frac{439}{960} + \frac{83\sigma}{320} - \frac{7a_8}{50} + \frac{18a_9}{25} + \frac{a_{13}}{5} - \frac{a_{14}}{5} + \frac{2a_{15}}{5} + \frac{a_{20}}{5} - \frac{3a_{22}}{5}\,, \tag{1.88}$$

$$a_{25} = -\frac{599}{1296} - \frac{127\sigma}{288} + \frac{2a_8}{5} - \frac{8a_9}{15} + \frac{3a_{11}}{2} - \frac{a_{13}}{6} + \frac{a_{14}}{9} - \frac{2a_{15}}{9} - \frac{a_{16}}{9} + \frac{a_{22}}{6}\,, \tag{1.89}$$

$$b_{25} = -\frac{317}{5184} - \frac{11\sigma}{192} + \frac{a_8}{20} - \frac{7a_9}{30} - \frac{a_{13}}{24} + \frac{a_{14}}{24} - \frac{a_{15}}{12} + \frac{a_{22}}{6}\,, \tag{1.90}$$

$$a_{26} = \frac{1127}{25920} + \frac{41\sigma}{960} - \frac{7a_8}{200} + \frac{7a_9}{150} - \frac{a_{11}}{8} + \frac{a_{13}}{120} - \frac{a_{14}}{120} + \frac{a_{15}}{60} - \frac{a_{22}}{30}\,. \tag{1.91}$$

We have 10 free parameters, which can be chosen to be a_8, a_9, a_{10}, a_{11}, a_{13}, a_{14}, a_{15}, a_{16}, a_{20}, a_{22}. However, we rewrote one of them, a_{10}, in terms of another constant that we called σ —this is convenient when studying black hole solutions as we show below.

1.6.1 Black hole solutions in the original frame

Let us first study the black hole solutions of the Type IIB action in the original frame (1.70). We extend the spherical symmetry of (1.2) to planar and hyperbolic geometries as well, so that we search for solutions of the form

$$\mathrm{d}s^2 = -N(r)^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + \frac{r^2}{\ell^2}\mathrm{d}\Sigma_k^2\,, \quad \mathrm{d}\Sigma_k^2 = \begin{cases} \ell^2\mathrm{d}\Omega_3^2\,, & \text{for} \quad k=1\,, \\ \mathrm{d}\vec{x}_3^2\,, & \text{for} \quad k=0\,, \\ \ell^2\mathrm{d}\Xi_3^2\,, & \text{for} \quad k=-1\,. \end{cases} \tag{1.92}$$

The simplest way of computing the equations of motion is to use the reduced action method —see *e.g.*, [216,248,506,507]. After plugging the metric ansatz (1.92) into (1.70) and taking the corresponding functional derivatives with respect to $N(r)$ and $f(r)$, we proceed to solve the subsequent equations of motion perturbatively in α'. Keeping only the leading $(\alpha')^3$ correction, we find the following expressions for $N(r)$ and $f(r)$:[18]

$$\begin{aligned} f(r) &= k + \frac{r^2}{\ell^2}\left[1 - \frac{\omega^4}{r^4} + \gamma\left(\frac{360\omega^{12}}{r^{12}} + \frac{320\omega^{12}k\ell^2}{r^{14}} - \frac{285\omega^{16}}{r^{16}}\right)\right]\,, \\ N(r) &= N_k\left(1 - \gamma\frac{120\omega^{12}}{r^{12}}\right)\,, \end{aligned} \tag{1.93}$$

where we introduced $\gamma = \zeta(3)\alpha'^3/(8\ell^6)$ and where N_k and ω^4 are integration constants. In particular, ω^4 defined in this way is proportional to the total energy of the solutions. In the $k=-1$ case, the expressions in (1.93) can be seen to agree with those appearing in [130], but one should take into account that the integration constants have been chosen differently. Now, the temperature T of any black hole solution of the type considered is given by

$$T = \frac{N(r_h)f'(r_h)}{4\pi}\,, \tag{1.94}$$

where $r_h \equiv \max\{r_i | f(r_i) = 0\}$ is the value of the radial coordinate at which the event horizon is located. As a function of r_h, the temperature and the parameter ω^4 read

$$T = \frac{N_k}{4\pi}\left[\frac{2k}{r_h} + \frac{4r_h}{\ell^2} + \gamma\left(\frac{60r_h}{\ell^2} - \frac{20k^4\ell^6}{r_h^7} + \frac{120k^2\ell^2}{r_h^3} + \frac{160k}{r_h}\right)\right]\,, \tag{1.95}$$

[18]Note that for general values of γ, the equations of motion of (1.70) evaluated on the ansatz (1.92) become two fourth-order coupled differential equations for $N(r)$ and $f(r)$. This is in contrast with the GQG frame, in which the corresponding equations of motion reduce to a single second-order equation for a single metric function —see (1.101) below— for general γ.

$$\omega^4 = r_h^4 + k\ell^2 r_h^2 + 5\gamma \left(r_h^2 + k\ell^2\right)^3 \left(\frac{15}{r_h^2} + \frac{7k\ell^2}{r_h^4}\right) , \tag{1.96}$$

where again we are working perturbatively in γ. Let us now compute the on-shell action of these solutions in order to determine their free energy, from which we can obtain the rest of relevant thermodynamic quantities. In order to do this, we need to include an appropriate generalized Gibbons-Hawking-York term [81] as well as counterterms for the action (1.70). To the best of our knowledge, specific boundary terms have not been constructed for this theory. However, we can use the effective boundary terms introduced in [257]. In that reference, it was argued that for theories with second-order linearized equations of motion around maximally symmetric backgrounds, one can write an effective boundary term that works for asymptotically AdS solutions. The prescription is that the same GHY term and counterterms that appear for Einstein gravity must be multiplied by an overall constant, which in the holographic context is identified with the universal contribution to the entanglement entropy across a spherical region, a^* —see *e.g.,* [445, 508]. In the case of the theory (1.70), the condition of second-order linearized equations is satisfied —in fact, the Weyl^4 term does not contribute to the linearized equations at all— and the charge a^* coincides with the Einstein gravity one. Therefore, we can use directly the same boundary terms and counterterms as for Einstein gravity [509–511], and the Euclidean action reads

$$I^{\rm E}_{\rm IIB_{\mathcal{A}_5\times S^5}} = -\int_{\mathcal{M}} \frac{d^5x\sqrt{|g|}}{16\pi G}\left[R + \frac{12}{\ell^2} + \gamma\ell^6 W^4\right] - \int_{\partial\mathcal{M}} \frac{d^4x\sqrt{h}}{8\pi G}\left[K - \frac{3}{\ell} - \frac{\ell}{4}\mathcal{R}\right] . \tag{1.97}$$

The computation is more or less straightforward, and we get the result

$$I^{\rm E}_{\rm IIB_{\mathcal{A}_5\times S^5}} = \frac{\beta N_k V_k}{16\pi G\ell^5}\left[\frac{3k^2\ell^4}{4} + k\ell^2 r_h^2 - r_h^4 + \frac{5\gamma}{r_h^4}\left(k\ell^2 - 15r_h^2\right)\left(k\ell^2 + r_h^2\right)^3\right] , \tag{1.98}$$

where V_k is the dimensionful volume of the transverse space (for instance, $V_1 = 2\pi^2\ell^3$) and β is the inverse temperature, corresponding to the Euclidean time periodicity. When we express this result in terms of the black hole temperature we get

$$\begin{aligned} I^{\rm E}_{\rm IIB_{\mathcal{A}_5\times S^5}} =& \frac{V_k}{32G}\Bigg[3kx - x^3 \mp \left(x^2 - 2k\right)^{3/2} \\ & - \frac{15}{2}\gamma\left(\frac{k^2}{x} - 28kx + 34x^3 \pm (6k - 30x^2)\sqrt{x^2 - 2k}\right)\Bigg] , \end{aligned} \tag{1.99}$$

where we have introduced the notation $x = \pi\ell T/N_k$.

1.6.2 Black hole solutions in the GQG frame

Let us now compare this result with the one corresponding to the transformed GQG frame (1.73). This theory possesses black hole solutions characterized by a single function, namely, of the form

$$ds^2 = -N_k^2 f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{\ell^2}d\Sigma_k^2 , \quad d\Sigma_k^2 = \begin{cases} \ell^2 d\Omega_3^2 , & \text{for} \quad k = 1 , \\ d\vec{x}_3^2 , & \text{for} \quad k = 0 , \\ \ell^2 d\Xi_3^2 , & \text{for} \quad k = -1 , \end{cases} \tag{1.100}$$

where N_k is now a constant. It is convenient to write $f = k+g(r)r^2/\ell^2$. Then, the equation which determines the metric function reads

$$
\begin{aligned}
g-\left(1-\frac{\omega^4}{r^4}\right)=&-\frac{5\zeta(3)\alpha'^3}{2048\ell^6}g'\Big[-8\left(2r^3(1+\sigma)+3gr^3(1+2\sigma)+2k\ell^2r(3+5\sigma)\right)g'^2\\
&+3r^4(1+2\sigma)g'^3-48(-1+g)r\left(k\ell^2+gr^2\right)(1+\sigma)g''\\
&-12g'\left(10g^2r^2(1+\sigma)+k\ell^2\left(-14(1+\sigma)+r^2(1+2\sigma)g''\right)\right.\\
&\left.+g\left(2\left(7k\ell^2-5r^2\right)(1+\sigma)+r^4(1+2\sigma)g''\right)\right)\Big]\,,
\end{aligned}
\tag{1.101}
$$

where again ω^4 is an integration constant related to the ADM energy of the solution. Observe that, while there are a lot of independent free couplings, they all affect the equation of $g(r)$ in a very universal way controlled by the combination of coefficients given by σ. Solving perturbatively the above equation one is left with

$$
\begin{aligned}
f(r)=k+\frac{r^2}{\ell^2}\Bigg[&1-\frac{\omega^4}{r^4}\\
&+\gamma\left(\frac{5(5-13\sigma)\omega^{12}}{2r^{12}}+\frac{5k\ell^2(3-11\sigma)\omega^{12}}{2r^{14}}+\frac{15(-1+3\sigma)\omega^{16}}{2r^{16}}\right)\Bigg]\,.
\end{aligned}
\tag{1.102}
$$

Besides, it is not difficult to solve exactly the equation above using numerical methods. However, the most interesting aspect about GQGs is that the thermodynamic properties of black holes can be determined exactly — namely, nonperturbatively in γ — and analytically. First, expanding $f(r)$ near the horizon, according to

$$
f(r)=\frac{4\pi T}{N_k}(r-r_h)+\mathcal{O}\left((r-r_h)^2\right)\,,
\tag{1.103}
$$

and plugging this in (1.101), we get two equations that relate ω^4, T and r_h:

$$
\begin{aligned}
\omega^4=&k\ell^2r_h^2+r_h^4-\frac{5\ell^3\gamma}{16r_h^4}\left(k\ell+2xr_h\right)^2\left[k^2\ell^3(3+2\sigma)\right.\\
&\left.+4k\ell r_h\left(-\ell x(3+2\sigma)+(1+\sigma)r_h\right)+4xr_h^2\left(3\ell x(1+2\sigma)-4(1+\sigma)r_h\right)\right]\,,
\end{aligned}
\tag{1.104}
$$

$$
\begin{aligned}
0=&2r_h\left(kL^2+2r_h\left(-\ell x+r_h\right)\right)+\frac{5\ell^3\gamma}{4r_h^5}\left(k\ell+2xr_h\right)^2\left(k^2\ell^3(3+2\sigma)\right.\\
&\left.+2k\ell r_h\left(-\ell x\sigma+r_h+\sigma r_h\right)-2x(1+\sigma)r_h^3\right)\,,
\end{aligned}
\tag{1.105}
$$

where we defined again $x=\pi\ell T/N_k$. These equations are analogous to the ones in (1.95), but now they are exact for the theory (1.73). However, we only expect the thermodynamic relations to match in both frames at first order in γ. At that order, one finds

$$
T=\frac{N_k}{4\pi}\left[\frac{2k}{r_h}+\frac{4r_h}{\ell^2}+\gamma\frac{5\left(r_h^2+k\ell^2\right)^3\left(k\ell^2(\sigma+3)-2(\sigma+1)r_h^2\right)}{4\pi\ell^2r_h^7}\right]\,,
\tag{1.106}
$$

$$
\omega^4=r_h^4+k\ell^2r_h^2-\frac{5\gamma}{r_h^4}\left(kL^2+r_h^2\right)^3\left(kL^2\sigma+r_h^2(2\sigma-1)\right)\,.
\tag{1.107}
$$

These are different from the ones in (1.95). However, note that the relations $T(r_h)$ or $\omega(r_h)$ are not really physically meaningful. $T(\omega)$ is though, since ω^4 is defined in both frames as (proportional to) the total energy. One can check that, at leading order in γ this relation has the same form in both frames. Finally, we compute the Euclidean action, for which the same boundary terms as before are valid, namely

$$\begin{aligned}\tilde{I}^{\mathrm{E}} =& -\frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^5x\sqrt{|g|}\left[\frac{12}{\ell^2}+R+\frac{\zeta(3)\alpha'^3}{2\ell^2}\hat{C}+\frac{\zeta(3)}{8}\alpha'^3\left(W^4+R^{ab}\hat{C}_{ab}\right)\right] \\ & -\frac{1}{8\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^4x\sqrt{h}\left[K-\frac{3}{\ell}-\frac{\ell}{4}\mathcal{R}\right].\end{aligned} \tag{1.108}$$

Due to the properties of the GQG theory, the action can be computed exactly: the Lagrangian is a total derivative and the integration only requires knowing the solution near the horizon (1.103) and asymptotically — see [257] for a similar explicit computation. Since in both limits we know the exact form of $f(r)$, we obtain the following exact result

$$\begin{aligned}\tilde{I}^{\mathrm{E}} =& \frac{V_k}{16G\ell^4 x}\left[\frac{3}{4}k^2\ell^4+3k\ell^2r_h^2+r_h^3\left(3r_h-4\ell x\right)\right. \\ & \left.-\frac{15\ell^3\gamma}{16r_h^4}\left(k\ell+2xr_h\right)^3\left(k\ell^2(3+2\sigma)-2\ell x(1+2\sigma)r_h+4(1+\sigma)r_h^2\right)\right].\end{aligned} \tag{1.109}$$

The last step is to use relation (1.105) to express $\tilde{I}^{\mathrm{E}}$ as a function of the temperature. We see that in general the action depends on σ. However, when we expand it at leading order in γ the dependence on σ disappears and we get exactly the same result as in the original frame given in (1.99).

1.7 Discussion

We close the chapter with some additional comments and conjectures. Firstly, based on the evidence presented here we state the following:

Conjecture 1.1. *Any higher-derivative gravity Lagrangian can be mapped, order by order, to a sum of GQG terms by implementing redefinitions of the metric of the form* (1.17).

We know there are many theories satisfying the GQG condition (1.4), and the amount of terms we can modify in the action with field redefinitions is also very large. All in all, there is so much freedom that field redefinitions seem to be able to bring the most general action (1.23) into a sum of GQG terms, order by order in the curvature.

Our main result is Theorem 1.1, which essentially tells us that all densities of the form $\mathcal{L}(g_{ab}, R_{abcd})$ are completable to a GQG, as well as those either belonging to the eight-derivative level or possessing at most two covariant derivatives. Regarding these last cases, we have seen that, interestingly, densities containing explicit covariant derivatives of the Riemann tensor do not seem to play any role. In fact, we have checked that, up to eighth order, all terms involving derivatives of the Riemann tensor are irrelevant —they can always be mapped to other terms which already appear in the action. More generally, we have been able to prove that any term with two covariant derivatives can be completed to a GQG which is equivalent to a GQG of the form $\mathcal{L}(g_{ab}, R_{abcd})$ when evaluated on a SSS metric. Note that the last claim is slightly different from stating that the original term can be completed to a GQG of the form $\mathcal{L}(g_{ab}, R_{abcd})$. It means that the GQG to which the original density is completed may, in principle, contain covariant derivatives of the curvature, but it is guaranteed that those terms vanish for a SSS metric. We argued that the previous conclusion may, very likely, extend to densities with an arbitrary number of covariant derivatives, which suggests a stronger conjecture:

Conjecture 1.2. *Any higher-derivative gravity Lagrangian can be mapped, order by order, to a sum of GQG terms which, when evaluated on a SSS metric, are equivalent to GQGs of the* $\mathcal{L}(g_{ab}, R_{abcd})$ *type.*

If true, the second statement in this conjecture implies that we can study the spherically symmetric black holes of the most general higher-derivative gravity effective action by analyzing only the solutions of the GQGs of the form $\mathcal{L}(g_{ab}, R_{abcd})$ —like in the example of Section 1.6. While, in general, the profile of the solutions will be different in every frame, recall that black hole thermodynamics is invariant under the change of frame. That kind of analysis was already performed in $D = 4$ for a general GQG involving arbitrarily high curvature terms [253]. It revealed a high degree of universality for the thermodynamic behavior of the Schwarzschild black hole generalizations, including asymptotically flat stable small black holes and infinite evaporation times. Our findings here suggest that those results may actually extend to arbitrary higher-derivative theories.

The conclusion is that theories of the GQG class are not just toy models with interesting properties. According to our results, they capture, at the very least, a very large part of all possible effective theories of gravity, and very likely —if Conjecture 1.2 is true— they capture all of them. From this point of view, we could think of GQGs as the most general EFT expressed in a frame in which the study of spherically symmetric black holes is particularly simple and universal.

As mentioned in Section 1.1, a certain subset of four-dimensional GQGs possess second-order equations for the scale factor when evaluated on a FLRW ansatz, which gives rise to a well-posed cosmological evolution [260, 262, 263]. The possibility that in fact all $D = 4$ higher-derivative effective actions can be mapped to GQGs belonging to this particular subset does not sound unreasonable and deserves further exploration. More generally, assuming Conjecture 1.2 and/or Conjecture 1.1 hold, one could try to impose further constraints on the GQG family of theories targeted by the field redefinitions and then provide refinements of those conjectures.

In another vein, it would be highly interesting to obtain a complete characterization of GQGs at each level in derivatives. In the case of theories with no covariant derivatives, it has been recently proven [251] that at each order n there exist $n-1$ inequivalent GQGs, one of them being of the Quasitological type and the remaining $n-2$, proper GQGs. Out of the $n-2$ inequivalent GQGs, it was possible to obtain the precise covariant expression of one of them [250] for every n, so it would be intriguing to find the explicit covariant form of every inequivalent GQG at arbitrary order n. Finally, it could also be intriguing to investigate the possibility of GQGs with covariant derivatives[19].

[19]In a very recent work [512], it has been argued that higher-order gravities with explicit covariant derivatives cannot have an Einstein-like linearized spectrum (that is, they would always contain ghosts). Nevertheless, by private communication with Pablo Bueno, we have learned that GQGs with covariant derivatives do exist, being quite abundant. This clearly shows that this is a very active topic of research and that further investigation is needed.

Appendix 1.A Redefining the metric

Implementing a differential change of variables directly in the action can be problematic if one is not careful enough. In order to see this, let us consider the equations of motion of $\tilde{g}_{ab}$ —defined so that $g_{ab} = \tilde{g}_{ab} + K_{ab}$— by computing the variation of the new action $\tilde{I}[\tilde{g}_{ab}] = I[g_{ab}]$:[20]

$$\frac{\delta \tilde{I}}{\delta \tilde{g}_{ab}} = \frac{\delta I}{\delta g_{ab}} + \frac{\delta I}{\delta g_{ef}}\frac{\delta K_{ef}}{\delta \tilde{g}_{ab}}\bigg|_{g_{ab}=\tilde{g}_{ab}+K_{ab}} . \tag{1.111}$$

Now, it is clear that we can always solve these equations if

$$\frac{\delta I}{\delta g_{ab}}\bigg|_{g_{ab}=\tilde{g}_{ab}+K_{ab}} = 0 . \tag{1.112}$$

In other words, implementing the change of variables directly in the equations of the original theory produces an equation that solves the equations of $\tilde{I}$. However, the equations of $\tilde{I}$ contain more solutions. These additional solutions are spurious and appear as a consequence of increasing the number of derivatives in the action, so they should not be considered. A possible way to formalize this intuitive argument consists in introducing auxiliary fields so that the redefinition of the metric becomes algebraic. Let us consider the following action

$$\begin{aligned} I_\chi = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{|g|}\Big[&-2\Lambda + R + f\left(g^{ab}, \chi_{abcd}, \chi_{e_1,abcd}, \chi_{e_1e_2,abcd}, \ldots\right) \\ &+ \frac{\partial f}{\partial \chi_{abcd}}\left(R_{abcd} - \chi_{abcd}\right) + \frac{\partial f}{\partial \chi_{e_1,abcd}}\left(\nabla_{e_1} R_{abcd} - \chi_{e_1,abcd}\right) \\ &+ \frac{\partial f}{\partial \chi_{e_1e_2,abcd}}\left(\nabla_{e_1}\nabla_{e_2} R_{abcd} - \chi_{e_1e_2,abcd}\right) + \ldots \Big] , \end{aligned} \tag{1.113}$$

where we have introduced some auxiliary fields χ_{abcd}, $\chi_{e_1,abcd}$, $\ldots \chi_{e_1\ldots e_n,abcd}$. Let us convince ourselves that this action is equivalent to (1.16). When we take the variation with respect to $\chi_{e_1\ldots e_i,abcd}$, we get

$$\sum_{j=0} \frac{\partial^2 f}{\partial \chi_{e_1\ldots e_i,abcd}\partial \chi_{a_1\ldots a_j,ghmn}}\left(\nabla_{a_1}\ldots\nabla_{a_j} R_{ghmn} - \chi_{a_1\ldots a_j,ghmn}\right) = 0 . \tag{1.114}$$

In this way, we get a system of algebraic equations for the variables $\chi_{e_1\ldots e_i,abcd}$ that always has the following solution

$$\chi_{abcd} = R_{abcd} , \tag{1.115}$$

$$\chi_{e_1,abcd} = \nabla_{e_1} R_{abcd} , \tag{1.116}$$

$$\chi_{e_1e_2,abcd} = \nabla_{e_1}\nabla_{e_2} R_{abcd} , \tag{1.117}$$

$$\ldots \tag{1.118}$$

[20]Note that in the second term we used the chain law for the functional derivative, which is in general given by

$$\frac{\delta I}{\delta \phi}\frac{\delta \phi}{\delta \psi} = \frac{\delta I}{\delta \phi}\frac{\partial \phi}{\partial \psi} - \partial_a\left(\frac{\delta I}{\delta \phi}\frac{\partial \phi}{\partial_a \psi}\right) + \ldots \tag{1.110}$$

This is the unique solution if the matrix of the system is invertible, and this is the expected case if f is general. When we plug this solution back in the action we recover (1.16) (with explicit Einstein-Hilbert and cosmological constant terms), so that both formulations are equivalent.

Now let us perform the following redefinition of the metric in I_χ:

$$g_{ab} = \tilde{g}_{ab} + \alpha K_{ab}\,, \quad \text{where } K_{ab} = K_{ab}\left(\tilde{g}^{ef}, \chi_{efcd}, \chi_{a_1,efcd}, \ldots\right)\,, \tag{1.119}$$

this is, K_{ab} is a symmetric tensor formed from contractions of the χ variables and the metric, but it contains no derivatives of any field. In this way, the change of variables is algebraic and can be directly implemented in the action. We therefore get

$$\tilde{I}_\chi\left[\tilde{g}_{ab}, \chi\right] = I_\chi\left[\tilde{g}_{ab} + \alpha K_{ab}, \chi\right]\,, \tag{1.120}$$

where, for simplicity, we are collectively denoting all auxiliary variables by χ. Now, both actions are equivalent and so are the field equations:

$$\frac{\delta \tilde{I}_\chi}{\delta \tilde{g}_{ab}} = \left.\frac{\delta I_\chi}{\delta g_{ab}}\right|_{g_{ab}=\tilde{g}_{ab}+\alpha K_{ab}}\,, \tag{1.121}$$

$$\frac{\delta \tilde{I}_\chi}{\delta \chi} = \left.\frac{\delta I_\chi}{\delta \chi} + \alpha \frac{\delta I_\chi}{\delta g_{ab}}\frac{\delta K_{ab}}{\delta \chi}\right|_{g_{ab}=\tilde{g}_{ab}+\alpha K_{ab}}\,. \tag{1.122}$$

Substituting the first equation into the second one, we see that the equations for the auxiliary variables become $\delta I_\chi/\delta\chi = 0$, which of course have the same solution as before (1.115). When we take that into account, K_{ab} becomes a tensor constructed from the curvature of the original metric g_{ab}, so that we get

$$g_{ab} = \tilde{g}_{ab} + \alpha K_{ab}\left(\tilde{g}^{ef}, R_{efcd}, \nabla_{\alpha_1} R_{efcd}, \ldots\right)\,. \tag{1.123}$$

Then, according to Eq. (1.121), the equation for the metric $\tilde{g}_{ab}$ is simply obtained from the equation of g_{ab} by substituting the change of variables:

$$\left.\frac{\delta I_\chi}{\delta g_{ab}}\right|_{g_{ab}=\tilde{g}_{ab}+\alpha K_{ab}} = 0\,. \tag{1.124}$$

However, note that this is not the same as substituting (1.115) in the action and taking the variation. This would yield instead

$$\begin{aligned}\frac{\delta \tilde{I}_\chi\left[\tilde{g}_{ab}, \chi(\tilde{g}_{ab})\right]}{\delta \tilde{g}_{ab}} &= \frac{\delta \tilde{I}_\chi}{\delta \tilde{g}_{ab}} + \frac{\delta \tilde{I}_\chi}{\delta \chi}\frac{\delta \chi}{\delta \tilde{g}_{ab}} \\ &= \left.\frac{\delta I_\chi}{\delta g_{ab}}\right|_{g_{ab}=\tilde{g}_{ab}+\alpha K_{ab}} + \left.\alpha\frac{\delta I_\chi}{\delta g_{ef}}\frac{\delta K_{ef}}{\delta \chi}\frac{\delta \chi}{\delta \tilde{g}_{ab}}\right|_{g_{ab}=\tilde{g}_{ab}+\alpha K_{ab}}\,.\end{aligned} \tag{1.125}$$

This equation is formally different to (1.121) due to the second term, and it is equivalent to (1.111). The second term appears because the auxiliary variables $\chi(\tilde{g}_{\mu\nu})$ do not solve the equation $\delta\tilde{I}_\chi/\delta\chi = 0$, but $\delta I_\chi/\delta\chi = 0$. However, we must solve $\delta\tilde{I}_\chi/\delta\chi = 0$ in order to get a solution of $\tilde{I}_\chi\left[\tilde{g}_{ab}, \chi\right]$, and according to (1.122) this would only happen if $(\delta I_\chi/\delta g_{ab})(\delta K_{ab}/\delta\chi) = 0$, so that the only consistent solutions of (1.125) are those which satisfy (1.124). This explains why the only solutions of (1.111) we should consider are the ones satisfying (1.112).

Appendix 1.B $W^n\nabla W\nabla W$ terms on SSS backgrounds

In this appendix we show that (1.59) holds. In order to do that, it is convenient to carry out the following change or radial coordinate in the SSS ansatz (1.2):

$$\mathrm{d}\tilde{r}^2=\frac{\mathrm{d}r^2}{r^2 f(r)}\,. \tag{1.126}$$

In these coordinates, the SSS metric reads

$$\mathrm{d}s^2=r(\tilde{r})^2\Big[-\tilde{N}(\tilde{r})^2\tilde{f}(\tilde{r})\mathrm{d}t^2+\mathrm{d}\tilde{r}^2+\mathrm{d}\Omega^2_{(D-2)}\Big]\,, \tag{1.127}$$

where we denoted $\tilde{N}(\tilde{r})=N(r(\tilde{r}))$ and $\tilde{f}(\tilde{r})=f(r(\tilde{r}))$.

We use a tilde to denote tensor components in the new coordinates. Direct computation shows that the components of the Weyl tensor in these new coordinates have formally the same expression as in the original ones, namely,

$$\tilde{W}^{ab}{}_{cd}=-2\tilde{\chi}(\tilde{r})\frac{(D-3)}{(D-1)}\tilde{w}^{ab}{}_{cd}\,, \tag{1.128}$$

where the tensorial structure $\tilde{w}^{ab}{}_{cd}$ is given by

$$\tilde{w}^{ab}{}_{cd}=2\tilde{\tau}^{[a}_{[c}\tilde{\rho}^{b]}_{d]}-\frac{2}{(D-2)}\left(\tilde{\tau}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}+\tilde{\rho}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}\right)+\frac{2}{(D-2)(D-3)}\tilde{\sigma}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}\,, \tag{1.129}$$

and where $\tilde{\rho}^b_a$ denotes the projection onto our new radial coordinate $\tilde{r}$. If we define $\tilde{H}^b_a=\tilde{\tau}^b_a+\tilde{\rho}^b_a$, we may express $\tilde{w}^{ab}{}_{cd}$ as

$$\tilde{w}^{ab}{}_{cd}=\tilde{H}^{[a}_{[c}\tilde{H}^{b]}_{d]}-\frac{2}{(D-2)}\tilde{H}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}+\frac{2}{(D-2)(D-3)}\tilde{\sigma}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}\,. \tag{1.130}$$

Consequently, the covariant derivative of the Weyl tensor turns out to be

$$\nabla_e\tilde{W}^{ab}{}_{cd}\Big|_{\mathrm{SSS}}=-2\frac{(D-3)}{(D-1)}\left[\frac{\mathrm{d}\tilde{\chi}}{\mathrm{d}\tilde{r}}\delta^1_e\tilde{w}^{ab}{}_{cd}+\tilde{\chi}(\tilde{r})\,\nabla_e\tilde{w}^{ab}{}_{cd}\Big|_{\mathrm{SSS}}\right]\,, \tag{1.131}$$

where we are denoting the components of the covariant derivative of any tensor T in our new coordinates as $\nabla_e\tilde{T}^{cd\ldots}_{ab\ldots}$. Hence we just need to work out $\nabla_e\tilde{w}^{ab}{}_{cd}|_{\mathrm{SSS}}$. Using (1.130), we find

$$\begin{aligned}\nabla_e\tilde{w}^{ab}{}_{cd}=&\,2\nabla_e\tilde{H}^{[a}_{[c}\tilde{H}^{b]}_{d]}-\frac{2}{(D-2)}\nabla_e\tilde{H}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}\\&-\frac{2}{(D-2)}\nabla_e\tilde{\sigma}^{[a}_{[c}\tilde{H}^{b]}_{d]}+\frac{4}{(D-2)(D-3)}\nabla_e\tilde{\sigma}^{[a}_{[c}\tilde{\sigma}^{b]}_{d]}\,.\end{aligned} \tag{1.132}$$

Since $\nabla_e\tilde{H}^b_a+\nabla_e\tilde{\sigma}^b_a=0$, we just need to compute $\nabla_e\tilde{H}^b_a$. A straightforward calculation produces

$$\nabla_e\tilde{H}^b_a=\frac{1}{(r(\tilde{r}))^3}\frac{\mathrm{d}r}{\mathrm{d}\tilde{r}}\tilde{g}_{eg}\tilde{\sigma}^f_a\delta^b_1+\frac{1}{r(\tilde{r})}\frac{\mathrm{d}r}{\mathrm{d}\tilde{r}}(D-2)\tilde{\sigma}^b_e\delta^1_a\,. \tag{1.133}$$

Using this, the covariant derivative of the Weyl tensor gives

$$\nabla_e\tilde{W}^{ab}{}_{cd}\Big|_{\mathrm{SSS}}=-2\frac{(D-3)}{(D-1)}\frac{\mathrm{d}\tilde{\chi}}{\mathrm{d}\tilde{r}}\delta^1_e\tilde{w}^{ab}{}_{cd}-2\frac{(D-3)}{(D-1)}\tilde{\chi}(\tilde{r})\frac{\mathrm{d}r}{\mathrm{d}\tilde{r}}\bigg[\frac{2}{(r(\tilde{r}))^3}\tilde{g}_{ef}\tilde{\sigma}^f_{[c|}\delta^{[a}_1\tilde{H}^{b]}_{|d]} \tag{1.134}$$

$$
\begin{aligned}
&+\frac{2(D-2)}{r(\tilde r)}\tilde\sigma_e^{[a|}\delta^1_{[c}\tilde H_{d]}^{|b]}-\frac{2}{(D-2)(r(\tilde r))^3}\tilde g_{ef}\tilde\sigma^f_{[c|}\delta^{[a}_1\tilde\sigma^{b]}_{|d]}\\
&-\frac{2}{r(\tilde r)}\tilde\sigma_e^{[a|}\delta^1_{[c}\tilde\sigma_{d]}^{|b]}+\frac{2}{(D-2)(r(\tilde r))^3}\tilde g_{ef}\tilde\sigma^f_{[c|}\delta^{[a}_1\tilde H^{b]}_{|d]}+\frac{2}{r(\tilde r)}\tilde\sigma_e^{[a|}\delta^1_{[c}\tilde H_{d]}^{|b]}\\
&-\frac{4}{(D-2)(D-3)(r(\tilde r))^3}\tilde g_{ef}\tilde\sigma^f_{[c|}\delta^{[a}_1\tilde\sigma^{b]}_{|d]}-\frac{4}{(D-3)r(\tilde r)}\tilde\sigma_e^{[a|}\delta^1_{[c}\tilde\sigma_{d]}^{|b]}\Bigg].
\end{aligned}
$$

Equipped with (1.134), we may infer the general form of any invariant $\mathcal{R}_2^{\{1,1\}}\Big|_{\rm SSS}$ as defined in (1.58). Since the $\mathcal{R}_2^{\{1,1\}}\Big|_{\rm SSS}$ are scalars, we can obtain them expressed in the original coordinates by performing all calculations in the new ones and then substituting any dependence on $\tilde r$ by the initial radial coordinate r.

We notice the following facts: a) any $\mathcal{R}_2^{\{1,1\}}\Big|_{\rm SSS}$ will have three types of terms: those carrying a factor $\tilde\chi^n\,(\mathrm{d}\tilde\chi/\mathrm{d}\tilde r)^2$, those involving a factor $\tilde\chi^{n+1}(\mathrm{d}\tilde\chi/\mathrm{d}\tilde r)(\mathrm{d}r/\mathrm{d}\tilde r)$ and a third type of terms with the common factor $\tilde\chi^{n+2}(\mathrm{d}r/\mathrm{d}\tilde r)^2$; b) since $\tilde r$ is dimensionless, we infer that the first type of terms is not weighted by any power of r, the second type is accompanied by r^{-1} and the third type, by r^{-2}. An additional overall factor of r^{-2} is required by dimensional analysis. Using these observations, it follows that

$$
\mathcal{R}_2^{\{1,1\}}\Big|_{\rm SSS}=\frac{\tilde\chi^n(\tilde r)}{r(\tilde r)^2}\left[c_1\left(\frac{\mathrm{d}\tilde\chi}{\mathrm{d}\tilde r}\right)^2+c_2\frac{\mathrm{d}\tilde\chi}{\mathrm{d}\tilde r}\frac{\mathrm{d}r}{\mathrm{d}\tilde r}\frac{\tilde\chi(\tilde r)}{r(\tilde r)}+c_3\left(\frac{\tilde\chi(\tilde r)}{r(\tilde r)}\right)^2\left(\frac{\mathrm{d}r}{\mathrm{d}\tilde r}\right)^2\right], \tag{1.135}
$$

for some constants c_1, c_2, c_3 which will depend on the specific term. Taking into account that $\mathrm{d}\tilde r/\mathrm{d}r=1/(r\sqrt{f(r)})$ we finally find

$$
\mathcal{R}_2^{\{1,1\}}\Big|_{\rm SSS}=\chi^n f(r)\left(c_1(\chi')^2+c_2\frac{\chi\chi'}{r}+c_3\frac{\chi^2}{r^2}\right), \tag{1.136}
$$

where $\chi'=\mathrm{d}\chi/\mathrm{d}r$.

2

Electromagnetic Quasitopological Gravities

In the previous chapter we focused on purely gravitational higher-order gravities and found that GQGs are a very special subset of higher-curvature theories which are amenable to computations and, additionally, span the set of all higher-derivative theories of pure gravity. Nevertheless, by definition GQGs are higher-order theories that do not involve matter, so it would be interesting to find an extension which admits the possibility of considering couplings to other fields. This is an intriguing exercise if the matter fields also respect the property of producing SSS solutions satisfying $g_{tt}g_{rr} = -1$ [285]. A nice and important instance of this is the addition of a minimally coupled Maxwell field, which has proven to be a successful strategy [248, 254, 268, 271, 286]. Notwithstanding, limiting ourselves to the consideration of minimal couplings to gravity is highly restrictive — in fact, if one already includes purely gravitational higher-derivative terms, why should not one add higher-order terms in which matter is non-minimally coupled to gravity?

We will show in this chapter that such generalization is indeed possible. The subsequent higher-order theories of gravity and electromagnetism receive the name of *Electromagnetic (Generalized) Quasitopological Gravities* (E(G)QGs), and are characterized by admitting electrically- or magnetically-charged, static and spherically symmetric solutions completely specified by a single function $f(r) = -g_{tt} = 1/g_{rr}$ whose associated equation of motion is (at most) second-order. As with the purely gravitational counterparts, it is natural to divide E(G)QGs into two subclasses: those for which the equation of motion for $f(r)$ is algebraic (corresponding to the EQG type) and those possessing a second-order equation for $f(r)$ (these would be the proper EGQGs). We will work in four dimensions all along the chapter and we will present explicit non-trivial examples of both EQGs and proper EGQGs, studying the properties of the subsequent charged black hole solutions. Interestingly enough, there is a crucial difference with respect to the pure gravity case— theories of the Quasitopological type, with algebraic equation of motion, already exist in four dimensions.

More concretely, the most natural way to construct E(G)QGs is through the consideration of theories canonically admitting for magnetically-charged, static and spherically symmetric solutions (as we will justify), while those with electric solutions are obtained by *dualization* of the vector field. When studying the subsequent (magnetically-)charged black hole solutions in theories of the EQG subclass, we will observe that, quite generally, the singularity at the core of the black hole is removed by the higher-derivative corrections, yielding a solution which describes a globally regular geometry. Consequently, this suggests the intriguing possibility of constructing, through dualization, EQGs which allow for electrically-charged black hole solutions with completely regular electromagnetic and gravitational fields for arbitrary values of the mass and non-vanishing charge. Remarkably, we will explicitly show that this feature is realizable, by writing down the first (to the best of our knowledge) such theory. This is an explicit confirmation of the *proof of principle* that higher-derivative corrections possess the ability to resolve singularities.

The chapter is divided as follows. First, general aspects of higher-order theories of gravity and electromagnetism are examined. Secondly, we focus on the study of static and spherically solutions and present the definition of E(G)QGs. Afterwards, we study charged black hole solutions in explicit instances of EQGs and proper EGQGs. Then we show an explicit example of an EQ which regularizes the singularities existing in the Reissner-Nordström black hole and we conclude with a discussion of the main results.

2.1 Aspects of $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ theories

In this Section we will present and review some generalities of non-minimal extensions of Einstein-Maxwell theory (EM). For that, we shall consider the most general gauge- and diffeomorphism-invariant theory for the metric $g_{\mu\nu}$ and a U(1) gauge field A_μ, whose Lagrangian must necessarily be constructed from contractions of the Riemann tensor $R_{\mu\nu\rho\sigma}$ and the gauge field strength[1] $F = \mathrm{d}A$. Consequently, the corresponding action $I[g, A]$ will adopt the form:

$$I[g, A] = \frac{1}{16\pi G}\int_M \mathrm{d}^n x\sqrt{|g|}\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})\,, \tag{2.1}$$

where n denotes the spacetime dimension which we will fix to $n = 4$ all along the chapter with the exception of Subsection 2.1.1, where we will keep it arbitrary. We will begin by deriving the the equations of motion of the theory (2.1), afterwards we will introduce the Electromagnetic Duality Map (EDM) and finally we will review some notions of black hole thermodynamics.

2.1.1 Equations of motion

In the case of pure theories of gravity, a general formula for the equation of motion associated to the gravitational field (the Einstein equation) is already known [230, 513]. Therefore, our first objective will be to derive an analogous formula for the Einstein equation of theories $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ which include arbitrary non-minimal couplings between gravity and electromagnetism. To this aim, we consider the variation of the action (2.1) with respect to the inverse metric:

$$\delta I[g, A](\delta g^{\mu\nu}, 0) = \frac{1}{16\pi G}\int_M \mathrm{d}^n x\sqrt{|g|}\left\{-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\rho\gamma}}\delta R_{\alpha\beta\rho\gamma}\right\}, \tag{2.2}$$

where we remind that the term $\delta R_{\alpha\beta\rho\gamma}$ is not independent from $\delta g^{\mu\nu}$. Let us define

$$P^{\alpha\beta\rho\gamma} = \frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\rho\gamma}}\,, \quad \mathcal{M}^{\alpha\beta} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\,, \tag{2.3}$$

Up to total derivatives, one observes that

$$P^{\alpha\beta\rho\gamma}\delta R_{\alpha\beta\rho\gamma} = -2\nabla^\sigma\nabla^\beta P_{\mu\sigma\beta\nu}\delta g^{\mu\nu} - P_\beta{}^{\sigma\mu\nu}R_{\rho\sigma\mu\nu}\delta g^{\rho\beta}\,. \tag{2.4}$$

[1]For the sake of simplicity we will be assuming no covariant derivatives of the Riemann tensor nor of the gauge field strength.

On the other hand, it is possible to work out the Lie derivative $L_\xi \mathcal{L}$ with respect to an arbitrary vector field $\xi \in \mathfrak{X}(M)$ in the following two different ways:

$$L_\xi \mathcal{L} = \xi^\mu \nabla_\mu \mathcal{L} = \xi^\mu P^{\nu\rho\sigma\beta} \nabla_\mu R_{\nu\rho\sigma\beta} - 2\xi^\mu \mathcal{M}^{\alpha\beta} \nabla_\mu F_{\alpha\beta} \,, \tag{2.5}$$

$$L_\xi \mathcal{L} = P^{\nu\rho\sigma\beta} L_\xi R_{\nu\rho\sigma\beta} + \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} L_\xi g_{\alpha\beta} - 2\mathcal{M}^{\alpha\beta} L_\xi F_{\alpha\beta} \,. \tag{2.6}$$

Taking into account that

$$P^{\nu\rho\sigma\beta} L_\xi R_{\nu\rho\sigma\beta} = \xi^\mu P^{\nu\rho\sigma\beta} \nabla_\mu R_{\nu\rho\sigma\beta} + 4(\nabla_\mu \xi_\nu) P^{\mu\rho\sigma\beta} R^\nu{}_{\rho\sigma\beta} \,, \tag{2.7}$$

$$L_\xi g_{\alpha\beta} = 2\nabla_{(\alpha}\xi_{\beta)} \,, \tag{2.8}$$

$$\mathcal{M}^{\alpha\beta} L_\xi F_{\alpha\beta} = \xi^\mu \nabla_\mu F_{\alpha\beta} \mathcal{M}^{\alpha\beta} + 2\nabla_\alpha \xi^\mu F_{\mu\beta} \mathcal{M}^{\alpha\beta} \,, \tag{2.9}$$

we find that[2]

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = 2P_\mu{}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} - 2\mathcal{M}_\mu{}^\alpha F_{\nu\alpha} \,. \tag{2.10}$$

Therefore, Eq. (2.2) may be rewritten as

$$\begin{aligned} \delta I[g,A](\delta g^{\mu\nu},0) = \frac{1}{16\pi G} \int_M \mathrm{d}^n x \sqrt{|g|} \Big\{ & -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \mathcal{L} + 2P_\mu{}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} \delta g^{\mu\nu} - 2\mathcal{M}_\mu{}^\alpha F_{\nu\alpha} \delta g^{\mu\nu} \\ & -2\nabla^\sigma \nabla^\beta P_{\mu\sigma\beta\nu} \delta g^{\mu\nu} - P_\mu{}^{\sigma\alpha\beta} R_{\nu\sigma\alpha\beta} \delta g^{\mu\nu} \Big\} \,, \end{aligned} \tag{2.11}$$

Hence the Einstein equation of any theory given by the action (2.1), which represents the most general theory of gravity and electromagnetism and includes all possible terms constructed out of curvature tensors, metrics and field strengths, is given by[3]

$$\mathcal{E}_{\mu\nu} = P_{(\mu}{}^{\rho\sigma\gamma} R_{\nu)\rho\sigma\gamma} - \frac{1}{2} g_{\mu\nu} \mathcal{L} + 2\nabla^\sigma \nabla^\rho P_{(\mu|\sigma|\nu)\rho} - 2\mathcal{M}_{(\mu}{}^\alpha F_{\nu)\alpha} = 0 \,. \tag{2.12}$$

In another vein, the derivation of the generalized Maxwell equation is direct and it yields

$$\mathcal{E}^\nu = \nabla_\mu \mathcal{M}^{\mu\nu} = 0 \,. \tag{2.13}$$

2.1.2 Dualization

As before, let $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ be a generic theory of gravity and electromagnetism. Observe that the Maxwell equation (2.13) and the Bianchi identity of F can be written in the following compact fashion:

$$\mathrm{d} \begin{pmatrix} F \\ H \end{pmatrix} = 0 \,, \tag{2.14}$$

where we have implicitly defined

$$H_{\mu\nu} = -\frac{1}{2} \star \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} \,, \tag{2.15}$$

[2] Note that

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = -g_{\mu\alpha} \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} g_{\nu\beta} \,.$$

[3] In the case of pure gravity we do not have to include an explicit symmetrization in the $\mu\nu$ indices because the terms $P_\mu{}^{\rho\sigma\gamma} R_{\nu\rho\sigma\gamma}$ and $\nabla^\sigma \nabla^\rho P_{\mu\sigma\nu\rho}$ are automatically symmetric [230]. However, in the case at hands we have not been able to show that the different structures appearing in $\mathcal{E}_{\mu\nu}$ are necessarily symmetric — although we highly suspect it — and thus we have symmetrized explicitly.

denoting $\star$ the Hodge dual map. The two-form H receives the name of dual field strength and (2.15) is the constitutive relation. We note that (2.14) suggests to consider F and H on an equal footing. In particular, we may think about the possibility of finding a new theory $\mathcal{L}_{\text{dual}}(R_{\mu\nu\rho\sigma}, H_{\alpha\beta})$ which depends solely the dual field strength H (and gravity, of course), being independent of F. This can be done through the process of *dualization* or *duality transformation*, which consists in a Legendre transformation of $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ that takes a theory $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ into its *dual* given by[4]

$$\mathcal{L}_{\text{dual}}(R_{\mu\nu\rho\sigma}, H_{\alpha\beta}) = \mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta}(\star H_{\lambda\kappa}, R_{\varepsilon\gamma\pi\zeta})) - 2F^{\mu\nu}(\star H_{\lambda\kappa}, R_{\varepsilon\gamma\pi\zeta})) \star H_{\mu\nu}\,, \tag{2.16}$$

where the expression of $F_{\alpha\beta} = F_{\alpha\beta}(\star H_{\lambda\kappa}, R_{\varepsilon\gamma\pi\zeta}))$ is obtained by inverting the constitutive relation (2.15), which on general grounds can be a rather challenging (if not impossible) problem. Dualization satisfies the following nice property.

Proposition 2.1. *Let $(g_{\mu\nu}, F_{\alpha\beta})$ be a solution of the set of equations of motion and Bianchi identity of $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$. Then*

$$(g_{\mu\nu}, H_{\alpha\beta})\,, \quad H_{\alpha\beta} = -\frac{1}{2} \star \frac{\partial \mathcal{L}}{\partial F^{\alpha\beta}}$$

is a solution of the set of equations of motion and Bianchi identity of $\mathcal{L}_{\text{dual}}(R_{\mu\nu\rho\sigma}, H_{\alpha\beta})$.

Proof. Assume that $(g_{\mu\nu}, F_{\alpha\beta})$ is a solution of the set of equations of motion and Bianchi identity of $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$. This automatically implies that $\mathrm{d}H = 0$, since this corresponds to the Maxwell equation for F, as checked by direct inspection of (2.15). Regarding the Maxwell equation for H, the variation of the action I_{dual} with respect to B, $H = \mathrm{d}B$, reveals[5] that:

$$\begin{aligned}\delta I_{\text{dual}}(0, \delta B_\mu) = \frac{1}{8\pi G}\int_M \mathrm{d}^4x\sqrt{|g|}\bigg\{ &\frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}}\frac{\partial F_{\alpha\beta}}{\partial H_{\mu\nu}}\partial_\mu \delta B_\nu - 2\frac{\partial F_{\alpha\beta}}{\partial H_{\mu\nu}} \star H^{\alpha\beta}\partial_\mu \delta B_\nu \\ &- 2\star F^{\mu\nu}\partial_\mu \delta B_\nu\bigg\} = \frac{1}{4\pi G}\int_M \mathrm{d}^4x\sqrt{|g|}\partial_\mu \star F^{\mu\nu}\delta B_\nu\,,\end{aligned} \tag{2.17}$$

where we discarded total derivatives and where we are taking into account that F depends on H through (inverting) Eq. (2.15). Consequently, we learn that the Maxwell equation for H is nothing else than the Bianchi identity for F. Finally, regarding the Einstein equation, since $\sqrt{|g|}F^{\mu\nu} \star H_{\mu\nu}$ is a topological term, we infer that the Einstein equation of $\mathcal{L}_{\text{dual}}(R_{\mu\nu\rho\sigma}, H_{\alpha\beta})$ would be that of $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ after exchanging $F_{\alpha\beta} \to F_{\alpha\beta}(H)$. This proves that $(g_{\mu\nu}, H_{\alpha\beta})$ is a solution of the set of equations of motion and Bianchi identity of $\mathcal{L}_{\text{dual}}(R_{\mu\nu\rho\sigma}, H_{\alpha\beta})$ and we conclude. $\square$

Furthermore, it can be checked that dualization is an involutive map, in the sense that the dual of the dual theory is the theory itself.

[4] Here we do not consider the issue of boundary terms in the dualization process. They will be explicitly included in Chapter 4.

[5] At least locally, $\mathrm{d}H = 0$ implies that $H = \mathrm{d}B$ for a certain 1-form.

2.1.3 Mass, charges and thermodynamics

For the purposes of the chapter, it will be very convenient to briefly review the issue of conserved charges and black hole thermodynamics. In the case of electric and magnetic charges, one obtains them directly from the equations of motion. In particular, the generalized Maxwell equation can be written as

$$\mathrm{d}\star\mathcal{M}=0\,,\quad \text{where}\quad \mathcal{M}=-\frac{1}{4}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}dx^{\mu}\wedge dx^{\nu}\,. \tag{2.18}$$

If we place a current three-form J on the right-hand-side, the equation implies that the current is conserved, $dJ=0$. Therefore, the natural definition of electric charge is

$$Q=\frac{1}{4\pi G}\int_{S^2_\infty}\star\mathcal{M}\,, \tag{2.19}$$

where the integration is taken at spatial infinity. In asymptotically flat spacetimes, with a Lagrangian $\mathcal{L}=R-F^2$+higher-order, we have $\mathcal{M}\to F$ asymptotically, so in practice we can exchange $\mathcal{M}$ by F as long as the integral is performed at infinity. In the asymptotically AdS case, there is a theory-dependent constant c_q such that $\mathcal{M}\to c_q F$ at the boundary of AdS, as we will see in Chapter 4. On the other hand, the magnetic charge is defined in the standard way:

$$P=\frac{1}{4\pi}\int_{S^2_\infty}F\,. \tag{2.20}$$

Gravitational conserved charges in higher-order gravities were examined in Refs. [90,91,514], but here we are just interested in the total mass. In the case of asymptotically flat spacetimes, the mass can be formally computed using the same prescriptions as for GR, for instance via the ADM [85,492] or the Abbott-Deser [86] formulae. Thus, the mass can be computed by studying the asymptotic behaviour of the metric in the usual way, *i.e.*, identifying the term $2GM/r\in g_{tt}$. In the AdS case, a global theory-dependent factor should be added to those formulas, corresponding to the replacement of the Newton's constant by the so-called effective Newton's constant G_{eff} [88]. However, for $\mathcal{L}(R_{\mu\nu\rho\sigma},F_{\alpha\beta})$ theories we do not expect these results to be affected, since the higher-order operators formed from $F_{\alpha\beta}$ decay too fast at infinity to contribute to the mass.

Regarding black hole thermodynamics, it is known that higher-curvature gravities minimally coupled to a Maxwell Lagrangian F^2 satisfy the following first law of black hole mechanics [37,515]

$$\mathrm{d}M=T\mathrm{d}S+\Phi_h\mathrm{d}Q+\Psi_h\mathrm{d}P\,. \tag{2.21}$$

Here M, Q and P are the mass and charges computed as specified above, T is the Hawking temperature of the black hole [38] and S is Wald's entropy [366,370], given by[6]

$$S=-2\pi\int_\Sigma \mathrm{d}^2x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{2.22}$$

where the integral is carried out on the bifurcation surface of the event horizon and $\epsilon_{\mu\nu}$ is the binormal to this surface. In another vein, Φ_h is the electrostatic potential at the

[6]There are some subtleties when defining the Noether charge for theories involving fields with internal gauge freedom — see *e.g.* Refs. [367,368] for recent discussions on this topic. Nevertheless, we will observe afterwards that defining the entropy according to the Wald prescription will yield an entropy which satisfies the first law of black hole thermodynamics as defined in (2.21). This is a highly non-trivial consistency check.

horizon and Ψ_h is the electrostatic potential of the dual vector field whose field strength is given by (2.15). These can be computed according to

$$\xi^\nu F_{\mu\nu} = \partial_\mu \Phi\,, \quad \xi^\nu H_{\mu\nu} = \partial_\mu \Psi\,, \tag{2.23}$$

where ξ^ν is the Killing vector that generates the horizon, with the condition that Φ and Ψ vanish asymptotically[7].

It was shown in Ref. [516] that the form of the first law remains unmodified when instead of the Maxwell Lagrangian one considers non-linear electrodynamics minimally coupled to Einstein gravity. Nevertheless, such analysis does not include the case of non-minimally coupled terms, and one may wonder if the form of the first law could change. This is, just like the entropy is no longer proportional to the area in presence of higher-curvature terms, $S \neq A/4$, the question is whether the quantities Φ or Q in (2.21) could have to be replaced by others in order for the first law to hold. Another non-trivial question is whether the Noether-charge and the Euclidean path integral approaches yield equivalent results for black hole thermodynamics [517, 518]. We will provide strong arguments for a positive answer to both questions in the following sections.

2.2 Static and spherically symmetric solutions

Once we have reviewed some aspects of generic higher-order theories of gravity and electromagnetism, now we commit ourselves to the problem of finding static, spherically symmetric (SSS) solutions of such theories. Obviously, the equations of motion are far too general to be solved without specifying a Lagrangian, so instead our aim is to understand the structure of those equations and to describe a class of theories for which the problem can be simplified.

If we have a SSS configuration for a metric g and a field strength F, then the following conditions must hold:

$$L_{k^{(A)}} g_{\mu\nu} = L_{k^{(A)}} F_{\mu\nu} = 0\,, \tag{2.24}$$

where $k^{(A)}$ with $A = 0, 1, 2, 3$ are the Killing vectors associated to these spacetime symmetries. Working with canonical spherical coordinates (t, r, θ, ϕ), they can be chosen to be:

$$\begin{aligned} k^{(0)} = \partial_t\,, \quad k^{(1)} = \partial_\phi\,, \quad k^{(2)} &= -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\,\partial_\phi\,, \\ k^{(3)} &= \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi\,. \end{aligned} \tag{2.25}$$

The most general ansatz for the subsequent SSS metric takes the form of Eq. (I.21):

$$\mathrm{d}s^2_{N,f} = -N^2(r) f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,, \tag{2.26}$$

which depends on two functions N and f. Imposing $L_{k_{(i)}} F_{\mu\nu} = 0$ together with the Bianchi condition for[8] F, the most general form for F consistent with static and spherical symmetry

[7] This is the convention we follow here and in Chapter 3. However, in Chapter 4 we will find convenient to define the electrostatic potentials in such a way that they vanish at the black hole horizon (whenever it exists).

[8] Without imposing $\mathrm{d}F = 0$, we would have $F = \Phi(r)\mathrm{d}t \wedge \mathrm{d}r + \chi(r)\cos\theta\mathrm{d}\theta \wedge \mathrm{d}\phi$. Then $\mathrm{d}F = 0$ implies $\chi(r) = P$, for constant P.

is:

$$F_{\text{SSS}} = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r + P\cos\theta \mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{2.27}$$

Note that F_{SSS} is the linear combination of an electric ansatz A^{e} and a magnetic ansatz A^{m} for the vector field:

$$A^e = \Phi(r)\mathrm{d}t \quad \Rightarrow \quad F^e = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r\,, \tag{2.28}$$

$$A^m = -P\cos\theta \mathrm{d}\phi \quad \Rightarrow \quad F^m = P\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{2.29}$$

Interestingly enough, while the electric ansatz is undetermined up to a choice of electric potential $\Phi(r)$ (to be fixed by the Maxwell equation), a magnetic field strength is completely fixed by static and spherical symmetry and the Bianchi identity, being the only degree of freedom the choice of magnetic charge[9] P.

We will assume in the following either an electric or a magnetic ansatz for the vector field, whose field strengths F^e and F^m are given above. One could also consider dyonic vectors, but this increases the difficulty of the problem because of the non-linearity of the associated Maxwell equations, so we will restrict ourselves to purely electric or purely magnetic configurations.

2.2.1 The reduced Lagrangian

If one is interested in the study of SSS solutions, one could try to obtain them by first deriving the equations of motion in full generality and, afterwards, setting the SSS ansatz (2.26) in these equations. However, as seen by direct inspection of (2.12) and (2.13), this is a quite intricate procedure and it would be convenient to have at disposal a more amenable method for SSS solutions.

In fact, it is possible to derive the equations of motion associated to these configurations by just varying the Lagrangian evaluated on a SSS ansatz, which we call reduced Lagrangian. On electric and magnetic SSS configurations, this is defined as[10]

$$L_{N,f,\Phi} = \sqrt{|g|}\mathcal{L}\Big|_{\mathrm{d}s^2_{N,f},F_{\text{SSS}}}\,. \tag{2.30}$$

Proposition 2.2. *For any theory $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$, the Einstein and Maxwell equations evaluated on a SSS ansatz are equivalent to*

$$\mathcal{E}_N = \frac{\delta L_{N,f,\Phi}}{\delta N} = 0\,, \quad \mathcal{E}_f = \frac{\delta L_{N,f,\Phi}}{\delta f} = 0\,, \quad \mathcal{E}_\Phi = \frac{\delta L_{N,f,\Phi}}{\delta \Phi} = 0\,. \tag{2.31}$$

Proof. Let us show first the equivalence between the Maxwell equation and $\mathcal{E}_\Phi = 0$. The SSS condition implies that $H = -\frac{1}{2}\star\frac{\partial\mathcal{L}}{\partial F}$ must have the structure

$$H = H_{tr}(r)\mathrm{d}t \wedge \mathrm{d}r + H_{\theta\phi}(r)\mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{2.32}$$

[9]This is due to the fact of choosing a SSS field strength, what is natural if one already imposes this condition on the metric. In [4], this condition was not imposed, but it was found afterwards that the resolution of the subsequent Maxwell equation for the magnetic field strength requires precisely F to have the form (2.29).

[10]It is important to distinguish such reduced Lagrangian $L_{N,f,\Phi}$ from the Lie derivative of a certain tensor T along a vector ξ, given by $L_\xi T$. It should be clear from the context the actual meaning implied.

Consequently, the only component of the Maxwell equation $\mathcal{E}^\nu = (\star \mathrm{d}H)^\nu = 0$ which is not automatically solved is $\mathcal{E}^0$ (note that $\mathrm{d}H = H'_{\theta\phi}(r)\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$, so $H'_{\theta\phi}(r) = 0$ would be the equation of motion). On the other hand, if $I_{N,f,\Phi}$ denotes the evaluation of the action on the SSS ansatz given by (2.26) and (2.27):

$$\mathcal{E}_\Phi = \frac{\delta I_{N,f,\Phi}}{\delta \Phi} = \frac{\delta I_{N,f,\Phi}}{\delta A_\mu}\frac{\partial A_\mu}{\partial \Phi} = \frac{\delta I_{N,f,\Phi}}{\delta A_0} = \mathcal{E}^0 \,. \tag{2.33}$$

Therefore, varying the action evaluated on the SSS ansatz with respect to $\Phi(r)$ yields the zeroth component of the Maxwell equation, which is the only non-trivial one on a SSS ansatz. This proves the equivalence between the Maxwell equation and $\mathcal{E}_\Phi = 0$. Regarding the Einstein equation, let $\mathcal{E}_{\mu\nu}$ denote the gravitational equations:

$$\mathcal{E}_{\mu\nu} = \frac{\delta I}{\delta g^{\mu\nu}} \,. \tag{2.34}$$

Using the chain rule, we find that:

$$\begin{aligned} \mathcal{E}_N &= \frac{\delta I_{N,f,\Phi}}{\delta N} = \mathcal{E}_{\mu\nu}|_{\rm SSS}\frac{\partial g^{\mu\nu}}{\partial N} = \frac{2}{N^3 f}\mathcal{E}_{tt} \,, \\ \mathcal{E}_f &= \frac{\delta I_{N,f,\Phi}}{\delta f} = \mathcal{E}_{\mu\nu}|_{\rm SSS}\frac{\partial g^{\mu\nu}}{\partial f} = \frac{1}{N^2 f^2}\mathcal{E}_{tt} + \mathcal{E}_{rr} \,. \end{aligned} \tag{2.35}$$

We see that $\mathcal{E}_N = 0$ and $\mathcal{E}_f = 0$ imply $\mathcal{E}_{tt} = \mathcal{E}_{rr} = 0$. Next, let us check that all the off-diagonal components of the gravitational equations are trivial. The easiest way to show this is by taking into account that

$$L_{k^{(A)}}\mathcal{E}_{\mu\nu} = 0 \,, \tag{2.36}$$

as demanded by static and spherical symmetry, where $k^{(A)}$ are the Killing vectors (2.25). On the one hand, both $L_{k^{(0)}}\mathcal{E}_{\mu\nu} = L_{k^{(1)}}\mathcal{E}_{\mu\nu} = 0$ directly imply that all components of $\mathcal{E}_{\mu\nu}$ are independent of time t and azimuthal coordinate ϕ. Computing now $\mathcal{E}_{k,\phi} = \cos\phi L_{k^{(2)}}\mathcal{E}_{\mu\nu} + \sin\phi L_{k^{(3)}}\mathcal{E}_{\mu\nu}$, we find that

$$\mathcal{E}_{k,\phi} = \begin{pmatrix} 0 & 0 & \csc^2\theta\mathcal{E}_{t\phi} & -\mathcal{E}_{t\theta} \\ 0 & 0 & \csc^2\theta\mathcal{E}_{r\phi} & -\mathcal{E}_{r\theta} \\ \csc^2\theta\mathcal{E}_{t\phi} & \csc^2\theta\mathcal{E}_{r\phi} & 2\csc^2\theta\mathcal{E}_{\theta\phi} & -\mathcal{E}_{\theta\theta} + \csc^2\theta\mathcal{E}_{\phi\phi} \\ -\mathcal{E}_{t\theta} & -\mathcal{E}_{r\theta} & -\mathcal{E}_{\theta\theta} + \csc^2\theta\mathcal{E}_{\phi\phi} & -2\mathcal{E}_{\theta\phi} \end{pmatrix} . \tag{2.37}$$

Since the latter must be identically zero, we learn that $\mathcal{E}_{\theta\theta} = \csc^2\theta\mathcal{E}_{\phi\phi}$ and that all off-diagonal components of $\mathcal{E}_{\mu\nu}$, except for $\mathcal{E}_{tr}$, vanish. In order to show that $\mathcal{E}_{tr}$ also vanishes, we can perform a direct computation using (2.12).

The term proportional to the metric in (2.12) is trivially diagonal, as well as that corresponding to $\mathcal{M}_{(\mu}{}^{\alpha}F_{\nu)\alpha}$, after taking into account that $\mathcal{M}_{\mu\nu}$ has necessarily the same components as $F_{\mu\nu}$. Indeed, both antisymmetric tensors take the same schematic form after the consideration of the electric/magnetic SSS ansatz:

$$F_{\mu\nu} = q(r)\tau^t_{[\mu}\rho^r_{\nu]} + p(r)\sigma^\theta_{[\mu}\sigma^\phi_{\nu]} \,, \quad \mathcal{M}_{\mu\nu} = \tilde{q}(r)\tau^t_{[\mu}\rho^r_{\nu]} + \tilde{p}(r)\sigma^\theta_{[\mu}\sigma^\phi_{\nu]} \,, \tag{2.38}$$

where $q(r), \tilde{q}(r), p(r)$ and $\tilde{p}(r)$ are some radial functions and where we defined the projectors

$$\tau_\mu^\nu = \delta^t_\mu\delta^\nu_t \,, \quad \rho_\mu^\nu = \delta^r_\mu\delta^\nu_r \,, \quad \sigma_\mu^\nu = \sum_{i=1}^{2}\delta^i_\mu\delta^\nu_i \,, \tag{2.39}$$

where the index i runs over the angular coordinates.

On the other hand, the Riemann tensor, and thence the $P_{\mu\nu\alpha\beta}$ tensor, only have the following type of components[11] when evaluated on (2.26) [502]:

$$
\begin{aligned}
R_{\mu\nu}{}^{\alpha\beta} &= A(r)\tau^{[\alpha}_{[\mu}\rho^{\beta]}_{\nu]} + B(r)\tau^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]} + C(r)\rho^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]} + D(r)\sigma^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]}\,,\\
P_{\mu\nu}{}^{\alpha\beta} &= \tilde{A}(r)\tau^{[\alpha}_{[\mu}\rho^{\beta]}_{\nu]} + \tilde{B}(r)\tau^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]} + \tilde{C}(r)\rho^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]} + \tilde{D}(r)\sigma^{[\alpha}_{[\mu}\sigma^{\beta]}_{\nu]}\,,
\end{aligned}
\tag{2.40}
$$

where we used the projectors defined above and where $A(r), \tilde{A}(r), B(r)\ldots$ are certain radial functions. Taking into account (2.40), we check by direct computation that both $P_{(\mu}{}^{\alpha\beta\gamma}R_{\nu)\alpha\beta\gamma}$ and $\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho}$ have vanishing off-diagonal components.

Finally we must check that the $\theta\theta$ and $\phi\phi$ components of the Einstein equations are satisfied once the tt and rr are. For that, consider the Bianchi identity associated to diffeomorphism invariance of any action of the form (2.2). Evidently this Bianchi identity is different from the usual $\nabla^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0$ (which holds too) since there are higher-order terms in the curvature in addition to non-trivial couplings to electromagnetism. However we can equally apply Noether's second theorem to (2.2) in order to obtain the following off-shell identity:

$$
\nabla_\mu\left(\frac{\delta I}{\delta g_{\mu\nu}}\right) + \mathcal{B} = 0\,,
\tag{2.41}
$$

where $\mathcal{B}$ is a certain quantity that vanishes when $\frac{\delta I}{\delta A_\nu} = 0$. Therefore, if the vector field is a solution of the generalized Maxwell equation (2.13) — for instance, the magnetic vector given by (2.29) — we obtain that

$$
\nabla_\mu\left(\frac{\delta I}{\delta g_{\mu\nu}}\right)\bigg|_{F=F_{\rm sol}} = 0\,.
\tag{2.42}
$$

Thus, we can assume the Bianchi identity $\nabla_\mu\mathcal{E}^{\mu\nu} = 0$ once the Maxwell equation is solved. Evaluating this identity on the SSS metric (2.26) and taking into account that $\mathcal{E}_{\mu\nu}$ has no off-diagonal components, we may find after some algebra that the $\nu = r$ component of the divergence of the Einstein equation (2.12) takes the form:

$$
\frac{\mathrm{d}\mathcal{E}^{rr}}{\mathrm{d}r} + \left(\frac{2}{r} - \frac{1}{2}f^{-1}f' + N^{-1}N'\right)\mathcal{E}^{rr} + \left(f^2NN' + \frac{1}{2}N^2ff'\right)\mathcal{E}^{tt} - \frac{f}{r}g_{ij}\mathcal{E}^{ij} = 0\,,
\tag{2.43}
$$

where i, j are the angular components. Since due to spherical symmetric $\mathcal{E}^{\theta\theta}$ and $\mathcal{E}^{\phi\phi}$ are proportional to each other, then this identity implies that whenever $\mathcal{E}^{rr} = \mathcal{E}^{tt} = 0$, then $\mathcal{E}^{\theta\theta} = \mathcal{E}^{\phi\phi} = 0$, and the equations of motion are solved.

□

Proposition 2.2 is quite powerful, since it allows us to obtain the equations of motion through a much simplified procedure, by which we just have to vary the action evaluated on the SSS ansatz. Furthermore, we note that, by direct inspection of the proof of the equivalence between the Maxwell equation and $\mathcal{E}_\Phi = 0$, a pure magnetic ansatz F^m *always* solves the Maxwell equation. Indeed, since any term in the Lagrangian must contain

[11] One may see, upon use of the conditions $L_{k_{(i)}}R_{\mu\nu\rho\sigma} = L_{k_{(i)}}P_{\mu\nu\rho\sigma} = 0$ for $i = 1, \ldots, 4$, that any of the two expressions given at (2.40) represents the most general tensor with the same symmetries as the Riemann and consistent with the static and spherical symmetry.

an even number of field strengths (assuming that the Lagrangian admits a polynomial expansion in terms of $F_{\alpha\beta}$ and $R_{\mu\nu\rho\sigma}$) by virtue of diffeomorphism and gauge invariance, then $\mathcal{E}_\Phi = 0$ is always satisfied if we choose $\Phi = 0$, and hence a pure magnetic ansatz is an automatic solution of the Maxwell equation.

Notwithstanding, this is no longer the case when we set an electric ansatz for the vector field. This is easily illustrated if we suppose as before that the Lagrangian is a polynomial in $F_{\alpha\beta}$ and $R_{\mu\nu\rho\sigma}$, *i.e*, it is composed of terms of the form $F^{2m}R^n$, with the indices contracted appropriately. In fact, by studying the structure of the curvature tensor on a SSS metric [502], it is not hard to show that a monomial built out from $2m$ field strengths[12] and n curvatures has the following structure when evaluated on such electric configuration:

$$(F^e)^{2m}R^n \sim \left(\frac{\Phi'}{N}\right)^{2m}\sum_{i=0}^{i_{\max}}\sum_{j=0}^{j_{\max}}\frac{(N')^i(N'')^j}{N^{i+j}}\mathcal{F}_{ij}(f,f',f'',r)\,. \tag{2.44}$$

where the sum is always finite and the functions $\mathcal{F}_{ij}$ are polynomial in f, f' and f''. Now, schematically the reduced Lagrangian is $L_{N,f,\Phi} = Nr^2\sin\theta\sum_{n,m}F^{2m}R^n$. Therefore, the Maxwell equation, obtained from variation with respect to Φ, reads

$$\mathcal{E}_\Phi = -\frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial L_{N,f,\Phi}}{\partial\Phi'} = 0 \Rightarrow \frac{\partial L_{N,f,\Phi}}{\partial\Phi'} = Q\,, \tag{2.45}$$

where Q is an integration constant. Since the left-hand-side of the last equation is a (polynomial) function of Φ', we can in principle invert it so that we get[13]

$$\Phi' = \Phi'_{\rm sol}(f,f',f'',N,N',N'',r,Q). \tag{2.46}$$

In this way we have eliminated one of the variables in the system of equations. Now we have to plug the value of Φ' in the equations for N and f and we get two differential equations for these functions,

$$\mathcal{E}_N\Big|_{\Phi'=\Phi'_{\rm sol}} = 0\,,\quad \mathcal{E}_f\Big|_{\Phi'=\Phi'_{\rm sol}} = 0\,. \tag{2.47}$$

Since $\Phi'_{\rm sol}$ is generically a highly nonlinear (not even polynomial) function of the variables f and N and their derivatives, solving these equations is in general an inaccessible problem.

2.2.2 The condition $g_{tt}g_{rr} = -1$: Generalized Quasitopological theories

The lesson we extract from the previous discussion is that magnetic solutions are much simpler to study than electric ones. However, even if we restrict ourselves to these magnetic solutions, the equations for N and f are typically too complicated to obtain relevant information in general, so further simplification is desirable.

In the case of pure gravity, an intriguing class of theories has been identified in recent years. These theories are known as *Generalized Quasitopological Gravities*[14] (GQGs) [79, 216] and they are characterized by possessing SSS solutions of the form

$$\mathrm{d}s_f^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_{(2)}^2\,, \tag{2.48}$$

[12] Whenever no possible confusion may arise, we use the same letter m to denote the power of field strengths F present in a given monomial and the label for the magnetic ansatz.

[13] Of course, there may be more than one solution.

[14] The works [79, 216] could be considered as the ones establishing the general properties of the new family of theories, but these were motivated by earlier works on Einsteinian Cubic Gravity [78, 248, 252].

i.e., with $N = 1$, and where in addition the equation of motion for f can be partially integrated. Other remarkable properties of these theories are that the thermodynamic properties of black holes can be studied in a completely analytic fashion and that they only propagate a massless graviton on constant curvature backgrounds.

Interestingly, GQGs can be nicely combined with minimally coupled vector fields, since they respect the property of having solutions with $g_{tt}g_{rr} = -1$. In this context, it is tantalizing to wonder about the possibility of generalizing these theories in the case of non-minimally coupled vector fields. The defining property of GQGs, from which all the rest are derived[15] [79, 216], is that their reduced Lagrangian becomes a total derivative when evaluated on the single-function ansatz (2.48), so we may try to extend this definition when a non-minimally coupled vector is present.

However, the main problem is that the reduced Lagrangian $L_{N,f,A}$ for arbitrary electromagnetic vector field A_μ will not in general enjoy the same structure as for pure gravities, since it will strongly depend on A_μ. Thus, for instance, it does not seem possible to impose $L_{1,f,A}$ to be a total derivative without specifying the form of A_μ. A more reasonable property to demand would be that the Euler-Lagrange equation of f vanishes identically when it is evaluated on $N = 1$ and on a gauge field that solves the Maxwell equation:

$$\left.\frac{\delta L_{1,f,A}}{\delta f}\right|_{A=A_{\rm sol}} = 0\,. \tag{2.49}$$

Still, in the electric case we have seen that $A_{\rm sol}$ generically has a non-polynomial dependence on f and its derivatives, so it seems complicated to find by brute force theories satisfying this property.

Fortunately enough, the situation is different for magnetic configurations. In fact, we have seen that in the magnetic case we can work with a reduced Lagrangian that depends only on N and f, since the Maxwell equation is automatically solved. In fact, the reduced Lagrangian $L_{N,f}$ for non-minimally coupled theories $F^{2p}R^n$ with a magnetic vector field has the same structure as for pure gravity theories, since it can be seen that

$$(F^m)^{2p}R^n \sim \sum_{i=0}^{i_{\rm max}}\sum_{j=0}^{j_{\rm max}} \frac{(N')^i(N'')^j}{N^{i+j}}\mathcal{F}_{ij}(f, f', f'', r)\,, \tag{2.50}$$

where $\mathcal{F}_{ij}$ are polynomials in f, f' and f''. Therefore one can extend straightforwardly the definition of Generalized Quasitopological Gravities to these non-minimally coupled terms. In particular, we are able to state the following result.

Theorem 2.1. *Let us consider a theory with a Lagrangian $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ of the form*

$$\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta}) = R - F^2 + \text{higher-derivative terms}, \tag{2.51}$$

where the higher-derivative terms are formed from monomials of the Riemann tensor and the field strength,[16] schematically R^nF^{2m}. Let us consider a SSS configuration given by the metric (2.48) and by a magnetic vector field with field strength (2.29) and let us define the reduced Lagrangian of the system as

$$L_f = r^2\mathcal{L}\Big|_{\mathrm{d}s^2_f, F^m}. \tag{2.52}$$

[15] See also Chapter 3 of [89] for a refinement on some of the results in [216].
[16] With this we mean that we do not allow terms such as *e.g.* F^2/R.

If the Euler-Lagrange equation for the reduced Lagrangian L_f vanishes identically, i.e.,

$$\frac{\partial L_f}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial L_f}{\partial f'} + \frac{\mathrm{d}}{\mathrm{d}r^2}\frac{\partial L_f}{\partial f''} = 0\,, \tag{2.53}$$

then the following properties hold:

1. *the theory allows for magnetically-charged SSS solutions of the form* (2.48), (2.29),
2. *the equation for the function f can be integrated once yielding at most a second-order equation where the mass appears as an integration constant,*
3. *the only gravitational mode propagated on maximally symmetric backgrounds is the spin-2 massless graviton, and*
4. (Conjecture) *the thermodynamic properties of magnetically-charged static black holes can be obtained analytically.*

The points 1 and 2 follow from the results in [216] by noticing that the reduced Lagrangian has the same structure as in the case of pure gravity. Point 3 is also a consequence of the results there — see also [89] —, but it is somewhat trivial, since terms $R^n F^{2m}$ with $m > 0$ do not contribute to the linearized equations of the metric[17], while the pure gravity terms satisfying (2.53) are known to produce Einstein-like linearized equations. Furthermore, let us note that the degrees of freedom corresponding to the vector field are the same as in Maxwell theory, since the Maxwell equation for any theory of the form (2.51) is of second-order in A_μ. Regarding point 4, it technically stands as a conjecture since no formal proof has been offered so far, though we strongly believe this conjecture to be true due to the large evidence collected in the GQG case [79, 248, 249, 252, 253]. Moreover, we will show in the next sections that it also holds for all the non-minimally coupled theories we construct.

These results involve magnetically-charged black hole solutions. Nonetheless, by using the dualization procedure explained in Section 2.1.2, one can dualize any theory satisfying (2.53) and obtain a new theory with electric solutions of the form (2.48). Consequently all the items in the Theorem 2.1 hold as well for the dual theories after replacing "magnetically-charged" by "electrically-charged". This motivates the following definition of *Electromagnetic Generalized Quasitopological Gravities* (EGQGs):

Definition 2.1. A theory $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ belongs to the family of *Electromagnetic Generalized Quasitopological Gravities (EGQG)* if and only if its Lagrangian or the Lagrangian of its dual theory satisfies the condition (2.53).

One could name these theories as "Magnetic" and "Electric" GQGs respectively, but we shall refer to them collectively by Electromagnetic GQGs for two reasons. Firstly, because it makes sense to use the adjective electromagnetic to express the fact that these theories are (non-minimally) coupled to an electromagnetic field. Secondly, because theories of one and another class are simply related by dualization, and therefore they are equivalent. However, even though the condition (2.53) can be fulfilled by simple polynomial Lagrangians (see next sections), the dual (electric) theory is generically much more involved and will typically contain an infinite number of terms.

[17] Note that, in the vacuum, $F = 0$.

In complete analogy to their pure gravity counterparts[18], we may distinguish two different classes of EGQGs: the general case in which the equations of motion of these theories for charged SSS configurations take the form of a second-order equation for the function f and the special subset of theories for which the order of this equation can be reduced twice again, being left with an algebraic equation. In this case we say that the theories belong to the Quasitopological class (without the "generalized" adjective). For pure gravity theories, Quasitopological Gravities only exist $D \geq 5$, but as we show below, infinitely many Electromagnetic Quasitopological Gravities (EQGs) exist in $D = 4$.

2.3 Electromagnetic Quasitopological Gravities

As stated above, the family of theories admitting single-function SSS solutions can be further divided into two classes: those for which the equation for f is algebraic ("Quasitopological") and those for which f satisfies a 2nd order equation ("Generalized Quasitopological"). In order to discover theories of the E(G)QG class, one first writes down a generic Lagrangian (including for example all the densities of the form $F^{2m}R^n$ up to a given order), then evaluates the Lagrangian on the configuration given by (2.48), (2.29) and finally demands the condition (2.53) to be satisfied, which yields constraints on the couplings of the higher-derivative terms.

In this section we shall focus on Electromagnetic Quasitopological Gravities, so we will be interested in the subset of theories with an algebraic equation for f. At low orders in the derivative expansion, one can easily find all the theories of this type. However, the process becomes more and more intricate at higher orders, since the number of independent densities one can include grows very fast. Thus, a completely general analysis does not seem a priori accessible. Nevertheless, from the analysis of the lower-order densities we can probably extract a general conclusion on the structure of EQGs. Indeed, in the Appendix 2.A we observe the following two facts. First, that there are only two Lagrangians of the form F^2R belonging to the Electromagnetic Quasitopological class[19]. Second, that at order F^2R^2, despite the larger number of linearly independent invariants one can construct, there are only two different ways in which these densities modify the equation of f. Thus, if we are only interested in studying SSS solutions, it suffices to keep two representative EQG densities at a given order. By repeating a similar analysis at higher orders, we observe that the situation appears to be general: there are many independent EQGs but their equations on SSS metric are degenerate, owing to the fact that there are only two different contributions at every order. So, instead of studying the whole set of EQGs, we will provide a set of representative theories which — we conjecture — span and capture all the possible modifications to SSS solutions. This will be enough for our goal, which is to study black holes in these theories.

It turns out that it is not difficult to provide a set of two representative Lagrangians of the EQG type at every order. Let us introduce the following notation:

$$(R^n)^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\alpha_1\beta_1} R^{\alpha_1\beta_1}{}_{\alpha_2\beta_2} \dots R^{\alpha_{n-1}\beta_{n-1}}{}_{\rho\sigma} \,, \tag{2.54}$$

[18] There is an important qualitative difference between EGQGs and purely gravitational GQGs though. While Lovelock gravities belong naturally to the GQG family, EGQGs do not include Lovelock-like theories in which a gauge field is non-minimally coupled to gravity, like the ones defined at Refs. [518–520].

[19] In this very particular case it even happens that the proper EGQG family coincides with the Quasitopological one. This is of course not a general feature.

with the convention that $\left(R^0\right)^{\mu\nu}{}_{\rho\sigma} = \delta^{\mu\nu}{}_{\rho\sigma} \equiv \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}$. Define the following Lagrangians of order $R^n F^{2m}$:

$$\mathcal{L}^{(a)}_{n,m} = \left(2nR_\mu{}^\alpha\delta_\nu{}^\beta - (3n-3+4m)R^{\alpha\beta}{}_{\mu\nu}\right)\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma}F^{\rho\sigma}F_{\alpha\beta}\left(F^2\right)^{m-1}, \tag{2.55}$$

$$\begin{aligned}\mathcal{L}^{(b)}_{n,m} = {} & \left(F^2\right)^{m-1}F_{\mu\nu}F^{\rho\sigma}\left(\frac{n}{2}R\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma} + \frac{1}{4}(n+4-4m)(3n-3+4m)\left(R^n\right)^{\mu\nu}{}_{\rho\sigma}\right) \\ & - n\left(F^2\right)^{m-1}F_{\alpha\nu}F^{\rho\sigma}R_\mu{}^\alpha\left((1+2n)\left(R^{n-1}\right)^{\mu\nu}{}_{\rho\sigma} - (n-1)R^\beta{}_\rho\left(R^{n-2}\right)^{\mu\nu}{}_{\beta\sigma}\right). \end{aligned}\tag{2.56}$$

Note that for $n=0$ we have $(m-1)\mathcal{L}^{(a)}_{0,m} = \mathcal{L}^{(b)}_{0,m} = (m-1)(3-4m)(F^2)^m$, so that these Lagrangians are well defined for all integers $n \geq 0$ and $m \geq 1$.

Proposition 2.3. *The theories given by (2.55) and (2.56) are Electromagnetic Quasitopological Gravities.*

Proof. In order to show that the Lagrangians (2.55) and (2.56) belong to the EQG family, we have to check that they become a total derivative when evaluated on the single-function ansatz (2.48) with a vector field strength given by (2.29). For that, we evaluate the Lagrangians on the general SSS metric ansatz given by (2.26) and we get:

$$\mathcal{L}^{(a)}_{n,m}\Big|_{\mathrm{d}s^2_{N,f},F^m} = \frac{2^{m+n}\psi^{n-1}P^{2m}}{r^{4m}}\left[nH - (3n-3+4m)\psi\right], \tag{2.57}$$

$$\begin{aligned}\mathcal{L}^{(b)}_{n,m}\Big|_{\mathrm{d}s^2_{N,f},F^m} = {} & \frac{2^{m+n-2}\psi^{n-2}P^{2m}}{r^{4m}}\Big[n\psi(F+G+2\hat{H}) + (n+4-4m)(3n-3+4m)\psi^2 \\ & - 2n(1+2n)\psi\hat{H} + n(n-1)\hat{H}^2\Big], \end{aligned}\tag{2.58}$$

where, following the notation of [502], we have introduced

$$\psi = \frac{1-f}{r^2}, \quad \hat{H} = -\frac{fN'}{rN} + \frac{1-f-rf'}{r^2}, \tag{2.59}$$

$$F = \frac{-4fN' - 2rfN'' - 3rN'f' - N(2f'+rf'')}{2rN}, \tag{2.60}$$

$$G = \frac{-2rfN'' - 3rN'f' - N(2f'+rf'')}{2rN}. \tag{2.61}$$

which represent several components of the Riemann and Ricci tensors. Evaluating (2.57) and (2.58) on $N=1$, we find:

$$r^2\mathcal{L}^{(a)}_{n,m}\Big|_{\mathrm{d}s^2_f,F^m} = \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{I}^{(a)}_{n,m}, \tag{2.62}$$

$$r^2\mathcal{L}^{(b)}_{n,m}\Big|_{\mathrm{d}s^2_f,F^m} = \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{I}^{(b)}_{n,m}, \tag{2.63}$$

where

$$\mathcal{I}^{(a)}_{n,m} = 2^{m+n}P^{2m}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{3-4m}\psi^n\right], \tag{2.64}$$

$$\mathcal{I}^{(b)}_{n,m} = 2^{m+n-2} P^{2m} \frac{\mathrm{d}}{\mathrm{d}r}\left[(-4+2n+4m) r^{3-4m}\psi^{n} + n r^{4-4m}\psi'\psi^{n-1}\right] . \tag{2.65}$$

Since these Lagrangians are total derivatives, the corresponding Euler-Lagrange equations for the single-function SSS ansatz vanish identically, showing that they are truly EGQG theories. Finally, Proposition 2.4 (right after this proof) ensures that (2.55) and (2.56) belong to the Quasitopological type after noticing that the equation of motion for the metric function $f(r)$ is algebraic, so we conclude. □

Proposition 2.4. *Let us consider the theory built from the Einstein-Maxwell action and the most general linear combination of* (2.55) *and* (2.56)*:*

$$\begin{aligned} I &= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\mathcal{L}^{\mathrm{EQG}} , \\ \mathcal{L}^{\mathrm{EQG}} &= R + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \ell^{2(n+m-1)}\left(\lambda_{n,m}\mathcal{L}^{(a)}_{n,m} + \gamma_{n,m}\mathcal{L}^{(b)}_{n,m}\right) , \quad \lambda_{n,m},\gamma_{n,m}\in\mathbb{R} , \quad \lambda_{0,1}=1 . \end{aligned} \tag{2.66}$$

When evaluated on (2.26)*, the equation of motion for f is algebraic and takes the form:*

$$1 - f - \frac{2M}{r} + \sum_{n=0}^{\infty}(1-f)^{n-1}\left[\alpha_n(r) + \beta_n(r) f\right] = 0 , \tag{2.67}$$

where M is the mass[20] *of the solution and*

$$\alpha_n(r) = \sum_{m=1}^{\infty}\frac{2^{m+n-1}P^{2m}\ell^{2(n+m-1)}}{r^{4m+2n-2}}\left[\lambda_{n,m} + (m-1)\gamma_{n,m}\right] , \tag{2.68}$$

$$\begin{aligned} \beta_n(r) = \sum_{m=1}^{\infty}\frac{2^{m+n-2}P^{2m}\ell^{2(n+m-1)}}{r^{4m+2n-2}}\Big[&2(n-1)\lambda_{n,m} \\ &+ (n^2-4n+2+m(-2+4n))\gamma_{n,m}\Big] . \end{aligned} \tag{2.69}$$

Proof. We already know that the Maxwell equation is always solved by magnetic vectors with field strength (2.29) and Proposition 2.3 guarantees that the Einstein's equations allow for solutions with $N=1$. Thus, we only have to determine the equation of the function f in the SSS metric (2.26), which can be obtained by evaluating the action on the SSS ansatz (2.26) (with the help of (2.58)), varying with respect to N, and then evaluating at $N=1$. The resulting equation takes the form of a total derivative, $\mathrm{d}\hat{\mathcal{E}}(f,r)/\mathrm{d}r = 0$, and upon integration we obtain the algebraic equation (2.67) and we conclude. □

Note that we set $\lambda_{0,1}=1$ to ensure that the usual Maxwell term $-F^2$ in included in the action (2.66). Also, as we remarked earlier, the equation of motion for $f(r)$ (2.67) seems to be the most general equation one can get for the set of all Electromagnetic Quasitopological Gravities. This is, we suspect that any other EQG will only have the effect of changing the value of the coefficients $\alpha_{n,m}$ and $\beta_{n,m}$ above. We explicitly show in

[20] Properly speaking, the mass of the solution would be given by M/G. However, for the sake of simplicity, in the present chapter and in Chapter 3 we will refer to M as the mass. In Chapter 4 we will substitute $M \to GM$ and write G explicitly.

Appendix 2.A that any Quasitopological theory built with two curvature tensors and two gauge field strengths indeed satisfies this property.

Along with the metric and the (magnetic) field strength there is an additional physical magnitude of interest: the electric potential associated to the dual field strength. Indeed, if $(g_{\mu\nu}, F_{\mu\nu})$ is a magnetic solution of a given theory, then $(g_{\mu\nu}, H_{\mu\nu})$, where $H_{\mu\nu}$ is the dual field strength as defined in Equation (2.15), is a solution of the associated dual theory. In this latter theory, the potential of $H_{\mu\nu}$ will be electric. This electric potential will make its appearance in the subsequent first law of black hole thermodynamics (see Subsection 2.3.2), so it will be useful to compute it. For that, using the notation employed in Eq. (2.3), first we notice that

$$-2\mathcal{M}_{\mu\nu} = Y_{\mu\nu\alpha\beta}F^{\alpha\beta}(F^2)^{m-1} + (m-1)Y_{\lambda\eta\alpha\beta}F^{\alpha\beta}F^{\lambda\eta}F_{\mu\nu}(F^2)^{m-2}\,, \tag{2.70}$$

where we have implicitly defined

$$\begin{aligned}
Y_{\lambda\eta\alpha\beta} = &\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} 2\lambda_{n,m}\left(nR_{\mu[\alpha}(R^{n-1})^{\mu}{}_{\beta]\lambda\eta} + nR_{\mu[\lambda}(R^{n-1})^{\mu}{}_{\eta]\alpha\beta} - (3n-3+4m)(R^n)_{\lambda\eta\alpha\beta}\right)\\
&+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\gamma_{n,m}\left(nR(R^{n-1})_{\lambda\eta\alpha\beta} + \frac{1}{2}(n+4-4m)(3n-3+4m)(R^n)_{\lambda\eta\alpha\beta}\right.\\
&- n(1+2n)R_{\mu[\alpha}(R^{n-1})^{\mu}{}_{\beta]\lambda\eta} - n(1+2n)R_{\mu[\lambda}(R^{n-1})^{\mu}{}_{\eta]\alpha\beta}\\
&\left.+2n(n-1)R^{\mu}{}_{[\lambda}(R^{n-2})_{\eta]\mu\sigma[\alpha}R^{\sigma}{}_{\beta]}\right)\,.
\end{aligned} \tag{2.71}$$

Imposing the field strength to be magnetic (2.29) and evaluating on the general SSS ansatz (2.26), we find that the dual field strength H, given by Equation (2.15), takes the form

$$H = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\ell^{2(n+m-1)}2^{n+m-3}mP^{2m-1}\mathcal{H}_{n,m}\mathrm{d}t\wedge\mathrm{d}r\,, \tag{2.72}$$

where we defined

$$\begin{aligned}
\mathcal{H}_{n,m} = -\frac{\psi^{n-2}}{r^{4m-2}}&\left(2nr(n\gamma_{n,m}+2\lambda_{n,m})\psi\psi' + (n-1)nr^2\gamma_{n,m}(\psi')^2\right.\\
&\left.+\psi(-2(4m-3)((n+2m-2)\gamma_{n,m}+2\lambda_{n,m})\psi + nr^2\gamma_{n,m}\psi''\right)\right)\,.
\end{aligned} \tag{2.73}$$

Amusingly, the latter can be explicitly rephrased as $H = -\Psi'(r)\mathrm{d}t\wedge\mathrm{d}r$, Ψ being the electric potential. It takes the form

$$\Psi = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{mP^{2m-1}\ell^{2(n+m-1)}}{2^{3-n-m}r^{4m-3}}\psi^n\left[4\lambda_{n,m} + \left((-4+2n+4m)+nr\frac{\psi'}{\psi}\right)\gamma_{n,m}\right]\,. \tag{2.74}$$

Once we have under control all the physically relevant magnitudes, we are going to present next some explicit examples of SSS solutions of particular Electromagnetic Quasitopological Gravities, since it will help us illustrate some of the most important features of this new type of theories. In fact, with the exception of the $n = m = 1$ case, for which the existence of Lagrangians with simple magnetic spherically symmetric solutions has been previously noticed in the literature [401,521,522], general theories with magnetic SSS solutions with general n, m have not been constructed to the best of our knowledge.

2.3.1 Explicit non-singular solutions in quadratic-curvature theories

In general, an explicit solution of Eq. (2.67) in a theory involving an arbitrary number n of Riemann curvature tensors is not available, since one would need to obtain the roots of an nth-degree polynomial. Therefore, for the sake of simplicity, let us study the solutions of theories that are just of second order in the curvature ($n \leq 2$) but including an arbitrary number of gauge field strengths ($m \geq 1$). In this case, the equation of motion for the metric function $f(r)$ turns out to be:

$$1 - f - \frac{2M}{r} + \alpha_0(r) + [\alpha_1(r) + \beta_1(r) f] + (1 - f)\,[\alpha_2(r) + \beta_2(r) f] = 0\,. \tag{2.75}$$

where we have taken into account that $\alpha_0(r) = -\beta_0(r)$. Since this is a quadratic polynomial in f one may solve the equation directly to obtain two possible solutions

$$\begin{aligned} f_\pm(r) = {} & \frac{-\alpha_2(r) + \beta_1(r) + \beta_2(r) - 1}{2\beta_2(r)} \\ & \pm \frac{\sqrt{(-\alpha_2(r) + \beta_2(r) + \beta_1(r) - 1)^2 + 4\beta_2(r)\left[\alpha_2(r) + \alpha_1(r) + \alpha_0(r) - \frac{2M}{r} + 1\right]}}{2\beta_2(r)}\,. \end{aligned} \tag{2.76}$$

Now, one can check that the solution f_+ is asymptotically flat and that it satisfies $f_+(r) = 1 - 2M/r + \ldots$ when $r \to \infty$. In addition, this solution reduces to Reissner-Nordström in the limit in which the corrections vanish $\ell \to 0$,

$$\lim_{\ell \to 0} f_+(r) = 1 - \frac{2M}{r} + \frac{P^2}{r^2}\,, \tag{2.77}$$

where one has to take into account that $\alpha_0 \to P^2/r^2$ while α_1, α_2, β_1 and β_2 vanish in that limit. On the other hand, f_- has an exotic asymptotic behaviour, $f_-(r) \sim -1/\beta_2(r)$, and it does not have an Einstein gravity limit, so we will consider only f_+ as the physically relevant solution.

A quite remarkable property of these solutions is that, in many cases, the curvature singularity at $r = 0$ is regularized by the higher-order corrections. To analyze the behaviour of the solution near $r = 0$, let us assume that we only include terms containing up to $2m_c$ field strengths (so that $1 \leq m \leq m_c$). Then, the functions α_i and β_i read

$$\alpha_n(r) = \sum_{m=1}^{m_c} \frac{\alpha_{n,m}}{r^{4m+2n-2}}\,, \quad \beta_n(r) = \sum_{m=1}^{m_c} \frac{\beta_{n,m}}{r^{4m+2n-2}}\,, \tag{2.78}$$

and we can see that, except for certain fine-tuned values for the couplings, $\alpha_2(r)$ and $\beta_2(r)$ are the dominant terms in the limit $r \to 0$. Hence, from (2.76) we get

$$f_+(r) = \frac{-\alpha_{2,m_c} + \beta_{2,m_c} + |\alpha_{2,m_c} + \beta_{2,m_c}|}{2\beta_{2,m_c}} + \mathcal{O}(r^2) \quad \text{when} \quad r \to 0\,. \tag{2.79}$$

Thus, whenever $\alpha_{2,m_c} + \beta_{2,m_c} > 0$ and $\beta_{2,mc} \neq 0$ we have $f_+(r) \sim 1 + \mathcal{O}(r^2)$ near $r = 0$, implying that the geometry is regular there. Notice that this is rather remarkable, since we do not need to fine-tune the couplings, just require that they satisfy a bound. In order for the solution to be globally regular we also have to make sure that there are no other singularities, *e.g.*, the term inside the square root in (2.76) should not become negative, but again this is easily achievable (for instance, if all the couplings are positive). Let us note that the regularization is also possible if we only include linear curvature terms ($n = 1$) — in fact this has been previously observed in the literature in the case of F^2R theories [401], although in that case the couplings must be related in a specific way. By analyzing the solutions of Eq. (2.67) near $r = 0$, one can see that the regularization at $r = 0$ can also be achieved in the general case in which we include terms $F^{2m}R^n$ of arbitrary order. The form of the equation (2.67) forces $f(r) \sim 1 + \mathcal{O}(r^2)$ near $r = 0$ in most of the cases in a quite natural way. Therefore we conclude that, without the need of much tuning on the coupling constants, the magnetically-charged SSS solutions of the EQG theories (2.66) are singularity-free.[21]

Notice that we have made not reference yet to black hole solutions, because, as in Einstein-Maxwell theory, not all the charged solutions are black holes. If the charge is too large compared to the mass, the solution does not have a horizon and in GR this means that we have a naked singularity. However, in our theories the gravitational field does not diverge, so horizonless regular solutions exist. In Fig. 2.1a we show the profile of $f(r)$ for a black hole solution while in Fig. 2.1b we show the one corresponding to a gravitating point charge. In both cases, the gravitational field is regular everywhere.

Since regular black holes are not possible in GR due to the singularity theorems by Hawking and Penrose [36, 341], as explained in Subsection I.6.3, it follows that some of the hypotheses of these theorems are broken by our higher-derivative theories. In particular, these theorems rely on different energy conditions, and these may not be satisfied. One of them is the null energy condition (NEC), which is satisfied if for any future-pointing null vector field k^μ it holds that $T_{\mu\nu}k^\mu k^\nu \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor, defined —as if we were working in GR — by the equation $G_{\mu\nu} = \frac{1}{2}T_{\mu\nu}$. When evaluated on the metric (2.48), it is not difficult to show that, for any null vector k^μ we have

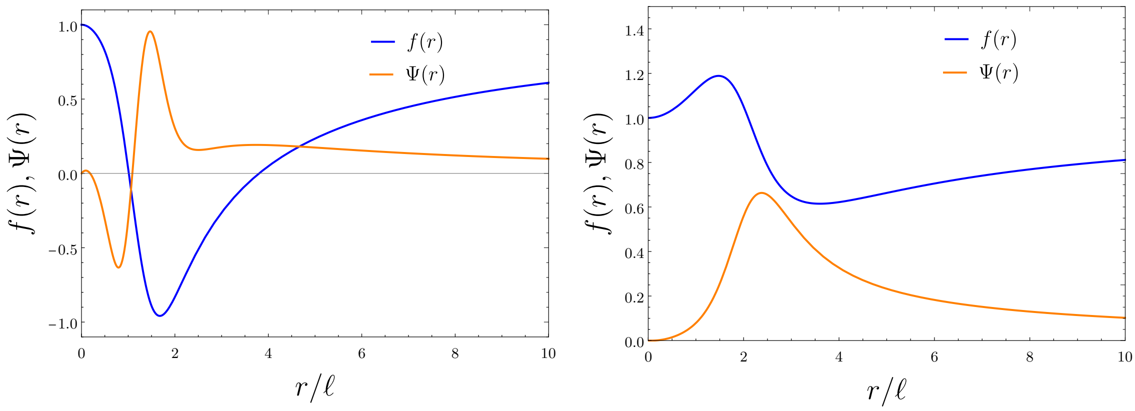
$$T_{\mu\nu}k^\mu k^\nu = a^2 \left(f''(r) + \frac{2(1 - f(r))}{r^2} \right) , \tag{2.80}$$

where a^2 is a non-negative quantity related to the normalization and direction of the null vector. We have checked that this quantity does in fact become negative for our regular black hole solutions for some intervals of r, and hence the NEC is violated. Let us mention anyway that there is no objective way of distinguishing what goes into the right-hand-side or left-hand-side of Einstein's equations in the theory (2.66), so that the definition of the stress-energy tensor is somewhat arbitrary.

Another physical quantity of interest is the dual electric potential Ψ, which was calculated back in Eq. (2.74). In the particular Electromagnetic Quasitopological theories we are considering, it turns out that the electric potential takes the form

$$\begin{aligned}\Psi(r) = &\sum_{m=1}^{\infty} \frac{\alpha_{0,m} m}{P\, r^{4m-3}} + \sum_{m=1}^{\infty} \frac{m}{P\, r^{4m-1}}(1-f) \left[(\alpha_{1,m} + \beta_{1,m}) - \frac{\beta_{1,m}}{2(2m-1)} \frac{2 - 2f + rf'}{1-f} \right] \\ &+ \sum_{m=1}^{\infty} \frac{m}{P\, r^{4m+1}}(1-f)^2 \left[\alpha_{2,m} + \frac{\beta_{2,m} - \alpha_{2,m}}{2m} - \frac{\beta_{2,m} - \alpha_{2,m}}{4m} \frac{2 - 2f + rf'}{1-f} \right] .\end{aligned} \tag{2.81}$$

[21] See, *e.g.*, Refs. [288, 394–396, 398, 400, 402, 405, 406] for other examples of non-singular black holes in different setups.



(a) A black hole solution. We have set $2P = M = 2\ell$, $2\gamma_{1,1} = 2 = -2\gamma_{1,2} = \lambda_{1,1}$ and $\lambda_{1,2} = \frac{25}{16}$.

(b) A non-black hole solution. We have set $P = M = \ell$, $3\gamma_{1,1} = 6 = -6\gamma_{1,2} = \lambda_{1,1}$ and $\lambda_{1,2} = \frac{53}{4}$.

Figure 2.1: The metric function $f(r)$ and the potential Ψ (in appropriate units) for two given particular sets of couplings, magnetic charge P and mass M. Note that Fig. 2.1a represents a black hole solution with an inner and outer horizon while Fig. 2.1b is an instance of a horizonless solution. In both cases, couplings have been chosen so that Ψ is regular at $r = 0$, and we see that it vanishes in this limit.

While the geometry is generally regular, this is not the always case for the electric potential. If we want to have a regular potential at $r = 0$, not any set of couplings is allowed. Indeed, since around $r = 0$ the metric function $f(r)$ can be approximated by $f(r) \underset{r\sim 0}{\sim} 1 + Ar^2$, we have

$$\lim_{r\to 0} \Psi(r) = \sum_{m=1}^{\infty} \frac{m}{P\, r^{4m-3}} \left[\alpha_{0,m} - A(\alpha_{1,m} + \beta_{1,m}) - A^2 \left(\alpha_{2,m} + \frac{\beta_{2,m} - \alpha_{2,m}}{2m}\right)\right]. \tag{2.82}$$

Therefore regularity requires that $\alpha_{0,m} - A(\alpha_{1,m} + \beta_{1,m}) - A^2(\alpha_{2,m} + \frac{1}{2m}(\beta_{2,m} - \alpha_{2,m})) = 0$ for all m. This will happen for a certain subset of the whole moduli space of couplings, but it is a realizable feature, as it is shown in Fig. 2.1a and Fig. 2.1b. Thus, in these cases black holes and horizonless solutions have regular gravitational and electromagnetic fields everywhere. We will explore in more detail this phenomenon in Section 2.5 for a particular quadratic theory.

2.3.2 Black hole thermodynamics

After describing some generic aspects of SSS solutions of Electromagnetic Quasitopological Gravities (2.66), in this subsection we focus on black holes and their thermodynamic description. One of our goals is to check that the first law of black hole mechanics holds in these theories and to identify the relevant thermodynamic potentials.

Let us begin with the solution given by (2.48) and assume the metric function $f(r)$ has some zero for $r \in \mathbb{R}^+$. The black hole horizon would be consequently located at $r_h = \max\{r \in \mathbb{R}^+ | f(r) = 0\}$. Using the equation of motion for $f(r)$ (2.67), after evaluation on $r = r_h$ we find that

$$1 - \frac{2M}{r_h} + \sum_{n=0}^{\infty} \alpha_n(r_h) = 0\,. \tag{2.83}$$

From here we can solve for the mass M of the black hole and get

$$2M = r_h + r_h \sum_{n=0}^{\infty} \alpha_n(r_h)\,. \tag{2.84}$$

We can also obtain the temperature, for which we first work out the derivative of (2.67) at $r = r_h$,

$$-f'(r_h) + \frac{2M}{r_h^2} + \sum_{n=0}^{\infty}\left[-f'(r_h)(n-1)\alpha_n(r_h) + \alpha_n'(r_h) + \beta_n(r_h)f'(r_h)\right] = 0\,. \tag{2.85}$$

Substituting the expression for the mass found at (2.84) and taking into account that the temperature T of the black hole is given by $4\pi T = f'(r_h)$, we are left with

$$T = \frac{1}{4\pi r_h}\frac{1+\sum_{n=0}^{\infty}\left[\alpha_n(r_h) + r_h\alpha_n'(r_h)\right]}{1-\sum_{n=0}^{\infty}\left[\beta_n(r_h) - (n-1)\alpha_n(r_h)\right]}\,. \tag{2.86}$$

Our next objective is the computation of the black hole entropy S, which is given by Iyer-Wald's formula [366, 370]

$$S = -2\pi\int_{\Sigma} \mathrm{d}^2x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{2.87}$$

where $\epsilon_{\mu\nu}$ denotes the binormal to the horizon Σ. After contraction with the binormals, we realize that we just have to care about the component $\dfrac{\partial\mathcal{L}}{\partial R_{trtr}}$, which turns out to be

$$\left.\frac{\partial\mathcal{L}}{\partial R_{trtr}}\right|_{r=r_h} = -\frac{1}{16\pi}\left[\frac{1}{2} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} 2^{n+m-3} n\,\ell^{2(n+m-1)}\gamma_{n,m}\frac{P^{2m}}{r_h^{4m+2n-2}}\right]. \tag{2.88}$$

Plugging this result into Eq. (2.87) and integrating over the angular variables, we find the black hole entropy:

$$S = \pi r_h^2\left[1 + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} 2^{n+m-2} n\,\ell^{2(n+m-1)}\gamma_{n,m}\frac{P^{2m}}{r_h^{4m+2n-2}}\right]. \tag{2.89}$$

We see that the entropy is no longer just the black hole area divided by 4 and we generically have corrections. In particular, we check that the type (a) theory described by (2.55) does not introduce any corrections to the entropy, being just the type (b) theory (and more concretely, the term involving a Ricci scalar) (2.56) the one which causes deviations from the Bekenstein-Hawking result [40].

In order to check the first law of black hole mechanics we need to bear in mind that we have a magnetically-charged solution. Consequently, if we consider the dual theory, this magnetic solution will become an electric one after dualization. However, we know that electric solutions satisfy a first law which includes the associated electric potential (at least, in the Reissner-Nordström solution). Therefore, taking into account that black hole thermodynamics remain unchanged under duality transformations (since solutions in one frame are mapped into solutions of the other), we conclude that the dual of any electric solution, which will be magnetic, will satisfy a first law including the aforementioned electric potential. Hence we can consider this argument backwards to justify that it is the dual electric potential the one entering in the first law of thermodynamics of these magnetic solutions.

From (2.74) we get the following value of the dual electrostatic potential evaluated at the horizon:

$$\Psi_h = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \ell^{2(n+m-1)} m \frac{2^{n+m-1} P^{2m-1}}{r_h^{4m+2n-3}} \left[\lambda_{n,m} + (m-1)\gamma_{n,m} - n\pi r_h T \gamma_{n,m}\right], \tag{2.90}$$

where at the same time we may use the expression for T in (2.86). At this point we have the quantities M, T, S and Ψ_h expressed explicitly as functions of r_h and P, and it is not hard to check, by using (2.84), (2.86), (2.89) and (2.90) that the following relations hold:

$$T = \frac{\partial_{r_h} M}{\partial_{r_h} S}, \quad \Psi_h = \partial_P M - \partial_P S \frac{\partial_{r_h} M}{\partial_{r_h} S}, \tag{2.91}$$

where ∂_{r_h} and ∂_P denote partial differentiation with respect to r_h and P, respectively. When expressed this way, one may directly check that the following first law

$$\mathrm{d}M = T\mathrm{d}S + \Psi_h \mathrm{d}P, \tag{2.92}$$

is satisfied. Hence we have shown that there exists a first law of thermodynamics for the Electromagnetic Quasitopological Gravities given by the action (2.66) which holds exactly. Despite the presence of non-minimally coupled terms, this result shows that the first law is formally unchanged, with the effect of the charge appearing through the standard term $\Psi_h dP$, where Ψ_h is the electrostatic potential at the horizon. We will observe this behaviour for EQGs in generic dimensions in Chapter 4, where Ψ_h will be identified with the chemical potential[22].

In order to complete our study of the thermodynamic properties of these black holes, let us compute the free energy. This can be defined from the rest of thermodynamic potentials as

$$F = M - TS. \tag{2.93}$$

As a consistency check, we will obtain F from the on-shell evaluation of the Euclidean action according to $F = TI^{\mathrm{E}}$ and observe that it matches exactly with (2.93).

To this aim, we will have to add to the action an appropriate boundary term and suitable counterterms. Finding these boundary terms for higher-curvature theories of gravity is a highly non-trivial issue, *e.g.* [523–527], but nevertheless one can see that, whatever these terms are, they do not contribute to the on-shell evaluation of the action in the case at hands. On general grounds, we expect that the boundary terms will be proportional to the first derivative of the Lagrangian with respect to the curvature [257], and since we are considering asymptotically flat situations, all such terms decay too fast at infinity to make a finite contribution.[23] Thus, we may use as a boundary term the standard Gibbons-Hawking-York term [80, 81] minus its background contribution. Therefore we propose the

[22] Up to a minus sign. See Chapter 4.

[23] The situation is different for asymptotically AdS solutions, but in that case one may introduce an effective boundary term which is proportional to the Gibbons-Hawking-York term [257]. This procedure is known to work at least for theories of the GQG class and has been tested in several occasions [258, 259, 270].

following Euclidean action

$$I^{\mathrm{E}} = -\frac{1}{16\pi G}\int_M \mathrm{d}^4x\,\sqrt{|g|}\mathcal{L}^{\mathrm{EQG}} - \frac{1}{8\pi G}\int_{\partial M}\mathrm{d}^3x\left[\sqrt{h}K - \sqrt{h^{\mathrm{flat}}}K^{\mathrm{flat}}\right], \tag{2.94}$$

where we have already Wick-rotated the time coordinate. In order to evaluate this action on our black hole solutions, note that the on-shell Lagrangian takes the form of an explicit total derivative when evaluated on the single-function metric (2.48). This follows from (2.62), (2.63) and from a similar property satisfied by the Ricci scalar. Thus we have

$$\begin{aligned}\mathcal{L}^{\mathrm{EQG}}\Big|_{\mathrm{d}s_f^2, F^m} &= \frac{1}{r^2}\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}r}\,,\\ \mathcal{I} &= 2r(1-f) - r^2 f' + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\ell^{2(n+m-1)}\left(\lambda_{n,m}\mathcal{I}^{(a)}_{n,m} + \gamma_{n,m}\mathcal{I}^{(b)}_{n,m}\right).\end{aligned} \tag{2.95}$$

and the Euclidean action takes the form

$$I^{\mathrm{E}} = -\frac{\beta}{4G}\mathcal{I}(r)\Big|_{r_h}^{\infty} - \frac{1}{8\pi G}\int_{\partial M}\mathrm{d}^3x\left[\sqrt{h}K - \sqrt{h^{\mathrm{flat}}}K^{\mathrm{flat}}\right], \tag{2.96}$$

where $\beta = 1/T$. Then, one can check that the evaluation of $\mathcal{I}(r)$ at $r \to \infty$ is exactly cancelled by the boundary terms, so we are left with the evaluation at the horizon, $I^{\mathrm{E}} = \beta\mathcal{I}(r_h)/4$. This yields the following value:

$$\begin{aligned}I^{\mathrm{E}}/\beta = \frac{r_h}{2} &+ \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\ell^{2(n+m-1)}2^{m+n-2}P^{2m}}{r_h^{4m+2n-3}}\Big(\lambda_{n,m} + (m-1)\gamma_{n,m}\Big)\\ &- T\left(\pi r_h^2 + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{\ell^{2(m+n-1)}2^{m+n-2}P^{2m}}{r_h^{4m+2n-4}}\pi n\gamma_{n,m}\right).\end{aligned} \tag{2.97}$$

By comparison with (2.93) we check that $F = I^{\mathrm{E}}/\beta$, and consequently, we show that the Noether-charge and the Euclidean path integral approaches to black hole thermodynamics give equivalent results.

Finally, let us work out the specific heat C_P at constant magnetic charge. A direct application of the inverse function theorem shows that

$$C_P = \left(\frac{\partial T}{\partial r_h}\right)^{-1}\frac{\partial M}{\partial r_h} = \frac{T(\partial_{r_h}S)^2}{\partial^2_{r_h}M - T\,\partial^2_{r_h}S}\,. \tag{2.98}$$

We check that C_P generally vanishes in the extremal limit $T \to 0$, which we study in more detail next.

2.3.3 Extremal black holes

We conclude the study of black holes in Electromagnetic Quasitopological theories by pinpointing some characteristic aspects of their extremal limit. For simplicity, we restrict our analysis to the particular class of theories which are quadratic in the vector field strength ($m = 1$). Defining the dimensionless parameter $\rho = r_h/\ell$, then we can express the black hole mass M for this class of theories as

$$\frac{2M}{\ell} = \rho - U(\rho)\frac{P^2}{\ell^2}\,, \tag{2.99}$$

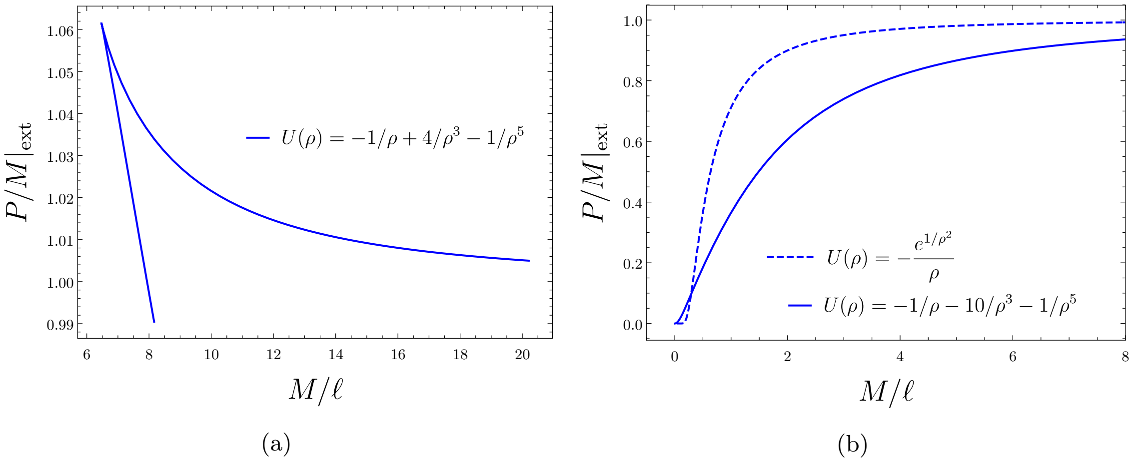


Figure 2.2: The extremal charge-to-mass ratio of particular EQGs as a function of the mass (in units of ℓ). On the one hand, in Fig. 2.2a we notice the existence of two branches (just the upper one would be connected to the Reissner-Nordström black hole), and the charge-to-mass ratio is monotonically decreasing with the mass for both branches. However, there are no extremal black holes below the mass at which both branches merge. On the other hand, in Fig. 2.2b we consider two different functions $U(\rho)$ and we realize that in both cases the extremal charge-to-mass ratio is monotonically increasing.

where we have introduced the function

$$U(\rho) = -\sum_{n=0}^{\infty} \frac{2^n}{\rho^{2n+1}} \lambda_{n,1} \,. \tag{2.100}$$

From (2.86), we see that the extremality condition $T = 0$ consequently takes the form

$$0 = 1 - \frac{P^2}{\ell^2} U'(\rho) \,. \tag{2.101}$$

From here one can solve for P^2 and obtain the following extremal charge-to-mass ratio:

$$\left.\frac{P}{M}\right|_{\rm ext} = \frac{2\sqrt{U'(\rho)}}{(\rho U'(\rho) - U(\rho))} \,. \tag{2.102}$$

Thus, we have an explicit a non-perturbative expression for the extremal charge-to-mass ratio in terms of the radius. Note that if only a finite number of terms is included in the action, the function U is a polynomial in $1/\rho$. However, if an infinite number of them is added, U can actually be any function of the form $U(\rho) = u(\rho^{-2})/\rho$, where $u(x)$ is an arbitrary analytic function (to recover Einstein-Maxwell theory at low energies it must satisfy $u(x) \to 1$ when $x \to 0$, though). In Fig. 2.2 we show $P/M|_{\rm ext}$ as a function of the mass for several choices of this function.

The effect of higher-derivative corrections on extremal black holes has a particular interest in the context of the Weak Gravity Conjecture (WGC) [416]. In fact, a mild form of the WGC states that the extremal charge-to-mass ratio in a consistent theory of Quantum Gravity must not decrease as the mass decreases. Thus, $P/M\big|_{\rm ext}$ must be a growing (or

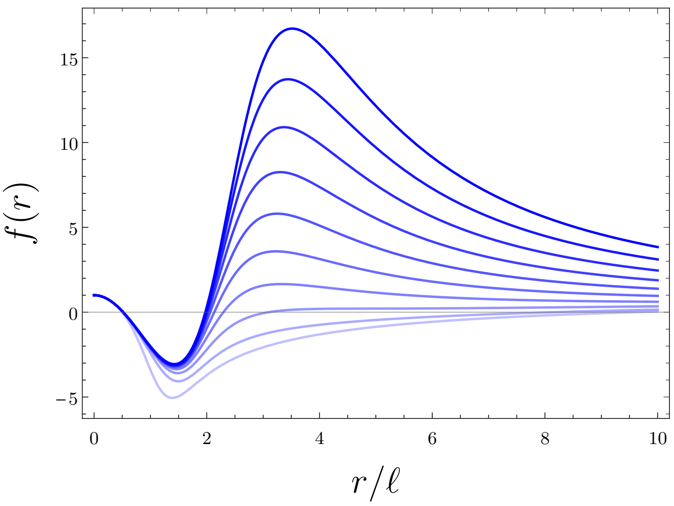


Figure 2.3: Profile of the metric function $f(r)$ for black holes corresponding to the same theory as in Fig. 2.2a. We show the solutions for a mass $M = 5\ell$ (which is below the minimal extremal mass) and, from less to more opacity, $P/\ell = 2, 4, \ldots, 20$. As we see, extremality is not reached.

constant) function when when we move from larger to smaller masses[24]. This condition ensures that the decay of an extremal black hole into a set of smaller black holes is possible, at least from the point of view of energy and charge conservation. Perturbative higher-derivative corrections to the extremal charge-to-mass ratio have been recently explored in a number of papers, *e.g.* [125, 126, 474, 475, 528–533]. Nevertheless, our study is fully non-perturbative, so we can analyze what happens when the corrections become important.

According to the WGC, just the theory depicted at Fig. 2.2a would be acceptable, since it satisfies (for the two branches) that the charge-to-mass ratio decreases when the mass grows. However, note that extremal black hole solutions cease to exist below a minimal mass (when the two branches merge). Although this might seem to be a peculiar feature of the particular model considered, such behaviour turns out to appear quite generally. Additional examples of this situation are shown in Section 2.4.2 for a different family of theories. One should wonder what happens with the evaporation process of black holes at this point. For that, let us consider an initially large ($M >> \ell$) non-extremal black hole. Due to Hawking radiation, it loses mass until it approaches extremality. At that moment, it also needs to lose charge in order to continue evaporating, and this is achieved if the WGC holds by emitting a particle with charge-to-mass ratio $p/m \geq 1$. Note that since our black holes satisfy $P/M\big|_{\text{ext}} > 1$ and this quantity becomes larger for smaller black holes, evaporation is not obstructed. In addition, a process by which an extremal black hole decays into a set of smaller, non-extremal black holes would be in principle allowed in terms of energy and charge conservation. Through this process, the black hole evaporates down to arbitrarily small masses, following approximately the line of extremal black holes in Fig. 2.2a (we may assume the black hole remains near-extremal during the

[24]In following chapters we will implement the WGC by demanding that $M_{\text{ext}}(Q_1 + Q_2) \geq M_{\text{ext}}(Q_1) + M_{\text{ext}}(Q_2)$. We consider this last condition to be more accurate, since it connects very clearly with the physical content of the WGC. However, for asymptotically flat black holes, dimensional analysis reveals that $M_{\text{ext}} \sim Q_{\text{ext}} + c\ell^2/Q_{\text{ext}}$ for certain coefficient c, so requiring $P/M\big|_{\text{ext}}$ to decrease for larges masses and superadditivity of M_{ext} are equivalent, at least perturbatively. Since we intent to remain close to the argumentation of [4], this suffices for our purposes.

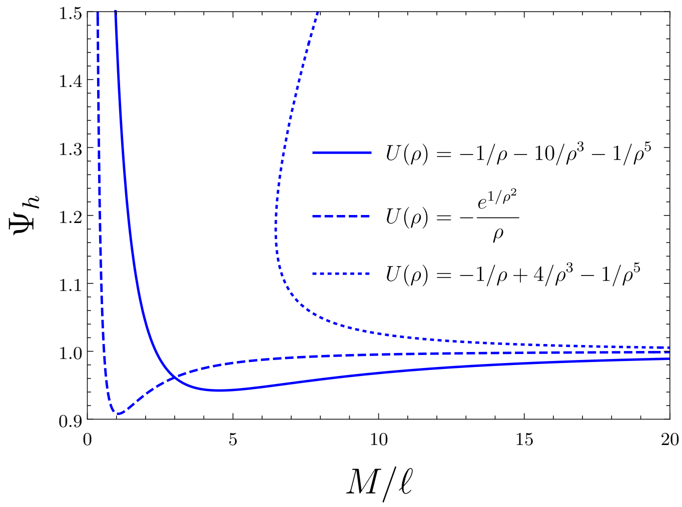


Figure 2.4: The electrostatic potential at the event horizon of extremal black holes as a function of the black hole mass. We plot three different EQGs, specified at the legend of the graph.

evaporation process). Then, it will reach the minimal mass in order for extremal black holes to exist, and this can have several meanings. One possibility is that below that line there are no black holes at all, *e.g.*, all the solutions are naked singularities or horizonless smooth configurations. Black holes could then transition to one of these objects, but this is highly speculative. A more interesting possibility is that below that mass any black hole is non-extremal, and this is precisely the case with the model depicted in Fig. 2.2a, corresponding to $\lambda_{1,1} = -2$, $\lambda_{2,1} = 1/4$. The black hole solutions of that theory with $M = 5\ell$ (below the minimal extremal mass) are shown in Fig. 2.3. We see that, no matter how large the charge is, the black hole is non-extremal. Thus, in this case, the black hole can always lose mass by means of Hawking radiation without imposing any conditions on kind of particles emitted. The fact that there is no obstruction to achieve the black hole's evaporation is in fully agreement with the spirit of the WGC.

Finally, let us comment on another interesting property that we can study in the extremal limit, namely, the value of the electrostatic potential Ψ at the event horizon. Indeed, in the limit $T \to 0$ the quantity Ψ_h takes the following simple expression:

$$\Psi_h = -\frac{U(\rho)}{\sqrt{U'(\rho)}}\,. \tag{2.103}$$

Let us remark that $\Psi_h = 1$ for extremal Reissner-Nordström black holes, but this is no longer the case for our black holes. In Fig. 2.4 we represent the electrostatic potential for the same three theories we considered in Fig. 2.2. We observe that in general such electric potential does not necessarily monotonically increase or decrease with the mass. We discover a rather counter-intuitive fact: rapid decreases in the charge-to-mass ratio plots seem to correspond to increases of the electric potential. Indeed, one would expect that as the charge diminishes, the potential to decrease as well, but we are finding precisely the opposite behaviour. This phenomenon not only happens for the theories forbidden by the WGC, but it also takes place for theories which, a priori, would be allowed. This is explicitly seen for one branch (the one disconnected from Reissner-Nordström) of the theory with $U(\rho) = -1/\rho + 4/\rho^3 - 1/\rho^5$.

2.4 Electromagnetic Generalized Quasitopological Gravities

In the previous section we have studied two families of EGQGs that belong to the Quasitopological subset, since they yield an algebraic equation for the metric function f when evaluated on the (magnetic) single-function ansatz (2.48). However, these are not the only type of EGQGs that exist. Analogously to the case for pure gravity, there are some theories for which f does not satisfy an algebraic equation, but a 2nd order differential equation — these are the proper *Generalized* Quasitopological theories. One may wonder why study these theories since we have already described two infinite classes of theories with simpler black hole solutions. There is a good reason, though. It turns out that, despite the fact that it is not usually possible to provide explicit black hole solutions for proper EGQGs, we can obtain the thermodynamic properties of black holes exactly. As we show below, the thermodynamic relations can have a quite different form with respect to the Quasitopological case, so that these new theories yield qualitatively different modifications of the Reissner-Nordström solution.

Based on our previous experience with (purely gravitational) GQGs, we expect that there are many of these theories at each order. Consequently, we will not attempt to provide a complete classification of this family of theories. Instead, our goal is to show that these theories indeed exist and to study some of their properties. A general characterization of EGQG theories may be addressed elsewhere.

Define the following family of Lagrangians:

$$\mathcal{L}_{n,m}^{\mathrm{EGQG}} = \left(R^{n-1}\right)^{\mu\nu}\left[nRg^{\alpha\beta} - (4n+4m-3)R^{\alpha\beta}\right]F_{\mu\alpha}F_{\nu\beta}\left(F^2\right)^{m-1}\,, \tag{2.104}$$

where $(R^n)^{\mu\nu}$ is the n-power of the Ricci tensor,

$$(R^n)^{\mu}{}_{\nu} = R^{\mu}{}_{\alpha_1}R^{\alpha_1}{}_{\alpha_2}\dots R^{\alpha_{n-1}}{}_{\nu}\,, \tag{2.105}$$

with the convention that $\left(R^0\right)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$.

Proposition 2.5. *Every Lagrangian of the family (2.104) defines an Electromagnetic Generalized Quasitopological Gravity for every $n, m \in \mathbb{N}$.*

Proof. If we evaluate (2.104) on the magnetic ansatz SSS given by (2.26), that we rewrite here for the sake of convenience:

$$\begin{aligned} \mathrm{d}s^2_{N,f} =& -N^2(r)f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,, \\ F =& P\mathrm{d}\theta\sin\theta\wedge\mathrm{d}\phi\,, \end{aligned} \tag{2.106}$$

we find that:

$$\mathcal{L}_{n,m}^{\mathrm{EGQG}}\Big|_{\mathrm{d}s^2_{N,f}} = \left(\frac{2P^2}{r^4}\right)^m \hat{H}^{n-1}\left[nR - (4n+4m-3)\hat{H}\right]\,, \tag{2.107}$$

where

$$R = -f'' - \frac{4f'}{r} - \frac{2f}{r^2} + \frac{2}{r^2} - \frac{N'}{N}\left(3f' + \frac{4f}{r}\right) - \frac{2fN''}{N}\,, \tag{2.108}$$

$$\hat{H} = \frac{1-f-rf'}{r^2} - \frac{N'f}{Nr}\,. \tag{2.109}$$

Further evaluation on $N=1$ shows that the Lagrangian becomes a total derivative,

$$\mathcal{L}_{n,m}^{\rm EGQG}\Big|_{{\rm d}s^2_{1,f}}=\frac{1}{r^2}\frac{\rm d}{{\rm d}r}\mathcal{I}_{n,m}\,,\quad \mathcal{I}_{n,m}=\left(\frac{2P^2}{r^4}\right)^m r^3\left(\frac{1-f-rf'}{r^2}\right)^n\,. \tag{2.110}$$

and therefore it belongs to the EGQG class. Proposition 2.6 (right after this proof) guarantees that the equation of motion for f is of second-order, defining thus a proper EGQG, and we conclude. □

Proposition 2.6. *Let us consider an extension of Einstein-Maxwell theory with the terms of the EGQG class (2.104) above:*

$$I=\frac{1}{16\pi}\int{\rm d}^4x\sqrt{|g|}\left[R+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\ell^{2(n+m-1)}\mu_{n,m}\mathcal{L}_{n,m}^{\rm EGQG}\right]\,,\quad \mu_{n,m}\in\mathbb{R}\,, \tag{2.111}$$

where ℓ is an overall length scale while $\mu_{n,m}$ are dimensionless couplings such that $\mu_{0,1}=1$. The equation of motion for f is second-order[25] *and takes the form:*

$$\mathcal{G}_f=2M\,, \tag{2.112}$$

where M is the mass of the solution and

$$\begin{aligned}\mathcal{G}_f=r(1-f)+\sum_{n,m}\mu_{n,m}\frac{\ell^{2(n+m-1)}}{2r}\left(\frac{2P^2}{r^4}\right)^m\left(\frac{1-f-rf'}{r^2}\right)^{n-2}\Big[-(n-1)r^2f'^2\\ +(n-2)rf'+f\left((n-1)nr^2f''+r(2-4mn)f'+4mn+2n^2-3n-2\right)\\ +f^2\left((3-4m)n-2n^2+1\right)+1\Big]\,.\end{aligned} \tag{2.113}$$

Proof. Proposition 2.5 ensures that the theory (2.111) admits magnetic SSS solutions characterized by a single function f, whose equation of motion we may obtain by direct variation of the reduced action evaluated on the SSS ansatz (2.26). Such equation is given by the evaluation on $N=1$ of the Euler-Lagrange equation for N, and this must be a total derivative by construction. We encounter that such equation is given by $\frac{\rm d}{{\rm d}r}\mathcal{G}_f$, which yields (2.112) after direct integration and we conclude. □

Note that the usual Maxwell term $-F^2$ is included in the sum, since we have $\mathcal{L}_{0,m}^{\rm EGQG}=-(4m-3)(F^2)^m$ and we have set $\mu_{0,1}=1$. By definition, this theory has magnetically-charged solutions of the form (2.48), and hence we only have to deal with the equation of motion for f, which is given by (2.112). Since in general this equation is a differential equation of second order, there are two more integration constants which need to be fixed by the boundary conditions. This is analogous to the recently studied case of purely gravitational GQG theories [79, 248, 249, 253], so let us just comment briefly on it.

First, we impose that the solution is asymptotically flat (we do not have a cosmological constant), which implies that $f(r)\to 1$ when $r\to\infty$. In the asymptotic region, we may expand the general solution in the following form:

$$f(r)=f_p(r)+f_h(r)\,, \tag{2.114}$$

[25] Whenever not all $\mu_{n,m}$ vanish.

where f_p is a particular solution while f_h represents a deviation with respect to that solution (and will satisfy a homogeneous equation). We can obtain a particular solution by assuming a $1/r$ expansion, which yields the following result

$$f_p(r) = 1 - \frac{2M}{r} + \frac{P^2}{r^2} + 3\mu_{1,1}\frac{\ell^2 P^2}{r^4} + \mathcal{O}(r^{-5})\,. \tag{2.115}$$

On the other hand, the boundary conditions imply that $f_h \to 0$ asymptotically, and hence we can assume that it is arbitrarily small. Thus, plugging (2.114) into (2.112) and expanding linearly in f_h we get

$$af_h'' + bf_h' + cf_h = 0\,, \tag{2.116}$$

where the asymptotic expansion of the coefficient reads

$$a = \frac{2\mu_{2,1}\ell^4 P^2}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\,, \quad b = -\frac{6\mu_{2,1}\ell^4 P^2}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)\,, \quad c = -r + \mathcal{O}\left(\frac{1}{r^3}\right)\,. \tag{2.117}$$

The equation above can be solved in terms of Bessel functions, but for our purposes it suffices to observe that the asymptotic solution behaves as

$$f_h(r) \sim A\exp\left[\frac{r^3}{3P\ell^2\sqrt{2\mu_{2,1}}}\right] + B\exp\left[-\frac{r^3}{3P\ell^2\sqrt{2\mu_{2,1}}}\right]\,, \tag{2.118}$$

where A and B are integration constants. Thus, when $\mu_{2,1} > 0$ one of the modes is exponentially growing and the other one is exponentially decaying. By setting the appropriate constant to 0 we achieve an asymptotically flat solution with a free integration constant. When $\mu_{2,1} < 0$, the solutions become highly oscillating at infinity and the only way to obtain a regular solution is to set $A = B = 0$, thus there are no further boundary conditions that one can fix. This is problematic because we cannot impose regularity at the horizon (see below), and therefore there are no regular black hole solutions in this case. Thus, we only consider $\mu_{2,1} > 0$. If this coefficient is 0, the constraint will appear in the next coefficient in the expansion.

On the other hand, we impose the existence of a regular horizon, *i.e.*, a point r_h at which $f(r_h) = 0$ and around which f is analytic. In particular, we assume that f has a Taylor expansion of the form

$$f(r) = 4\pi T(r - r_h) + \sum_{n=2}^{\infty} b_n (r - r_h)^n\,, \tag{2.119}$$

where we are making explicit that $f'(r_h) = 4\pi T$, where T is Hawking's temperature. When we insert this expansion into the equation (2.112), we get a system of equations that relate the coefficients b_n. Nonetheless, the first two equations are special, since they only involve r_h and T. These read

$$M = \frac{r_h}{2}\left[1 + \frac{1}{2}\sum_{n,m}\mu_{n,m}\left(\frac{\ell^2(1-4\pi T r_h)}{r_h^2}\right)^{n-1}\left(\frac{2\ell^2 P^2}{r_h^4}\right)^m (1 + (n-1)4\pi T r_h)\right]\,, \tag{2.120}$$

$$\begin{aligned} 0 = {}& 1 - 4\pi T r_h + \frac{1}{2}\sum_{n,m}\mu_{n,m}\left(\frac{\ell^2(1-4\pi T r_h)}{r_h^2}\right)^{n-1}\left(\frac{2\ell^2 P^2}{r_h^4}\right)^m \Big(3 - 4m - 2n \\ & + (n + 4m - 3)4\pi T r_h\Big)\,. \end{aligned} \tag{2.121}$$

These two equations allow one to get (implicitly) the temperature T and the radius r_h once M and P are given. The rest of the equations provide relations for the coefficients b_n. A simple inspection reveals the only free parameter in the expansion is b_2, and the rest of the b_n are fixed in terms of it. Finally, b_2 is fixed by demanding that the solution be asymptotically flat. The full solution $f(r)$ can be obtained by a numeric integration of (2.112) using (2.119) as initial condition, and implementing a shooting algorithm to search for the value of b_2 that yields the correct asymptotic behaviour. Such numeric resolution will be carried out elsewhere, but comparing with previous works on neutral black holes in GQG theories, we expect that the solution exists providing the condition on the couplings discussed above is satisfied. Fortunately, a great deal of information about these black holes can be obtained without resorting to the numeric solution.

2.4.1 Black hole thermodynamics

Even though the profile of solutions has to be determined numerically, one remarkable property of the theories in Eq. (2.111) — which is shared by all the theories of the GQG class — is that the thermodynamic properties of black holes can be found analytically. First, observe that (2.120), (2.121) above give us the relation between M, P and T. Although such relation cannot be written explicitly due to the complicated form of the equations, it is nevertheless possible to solve the system of equations parametrically. To this aim, let us first introduce two dimensionless parameters p and x defined as

$$p = \frac{2\ell^2 P^2}{r_h^4}\,, \qquad x = \frac{\ell^2(1-4\pi T r_h)}{r_h^2}\,. \tag{2.122}$$

Then, we can define the 2-variable function

$$\mathcal{W}(x,p) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \mu_{n,m} x^n p^m\,. \tag{2.123}$$

In terms of these quantities we can rewrite (2.120) and (2.121) as

$$M = \frac{r_h}{2}\left[1 + \left(1 - \frac{x r_h^2}{\ell^2}\right)\partial_x \mathcal{W} + \frac{r_h^2}{\ell^2}\mathcal{W}\right]\,, \tag{2.124}$$

$$0 = r_h^2\left(x\partial_x\mathcal{W} + 4p\partial_p\mathcal{W} - 3\mathcal{W} - x\right) + \ell^2\partial_x\mathcal{W}\,. \tag{2.125}$$

Whenever $\partial_x\mathcal{W} \neq 0$, we can obtain explicitly $r_h(x,p)$ from the second equation. On the other hand, if $\mathcal{W}$ does not depend on x, the same equation determines the relation $x(p)$, while r_h remains free. Note that this only happens in the trivial case in which the higher-order Lagrangians do not depend on the curvature, so it is not relevant for our purposes. Then, inserting $r_h(x,p)$ in (2.124) we derive the explicit relation $M(x,p)$, and analogously, we get $T(x,p)$ and $P(x,p)$ from (2.122), namely,

$$P = \frac{r_h^2}{\ell}\sqrt{\frac{p}{2}}\,, \qquad T = \frac{1}{4\pi r_h}\left(1 - \frac{x r_h^2}{\ell^2}\right)\,. \tag{2.126}$$

Thus, we have been able to write all these thermodynamic quantities, as well as the radius, in terms of two independent parameters x and p. This is a convenient way to study the thermodynamic phase space of these theories. Let us now work out the rest of thermodynamic properties of these black holes.

The entropy is computed by Wald's formula, as introduced previously in Section 2.3

$$S = -2\pi \int \mathrm{d}^2x\sqrt{h}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,. \tag{2.127}$$

When computing the derivative with respect to the curvature of the Lagrangians (2.104), each time we differentiate one of the Ricci tensors $R_{\alpha\beta}$ we end up generating a contraction between the binormal $\epsilon_{\alpha\beta}$ and a field strength. Note that such contractions are always 0 for magnetic configurations, since ϵ and F are orthogonal in that case. Thus, only the derivative on the Ricci scalar appearing in (2.104) will contribute and yield a non-vanishing contribution. The result reads

$$S = \pi r_h^2 \left[1 + \sum_{n,m} \mu_{n,m} n \left(R^{n-1}\right)^{\mu\nu} F_{\mu\alpha}F_{\nu}{}^{\alpha} \left(F^2\right)^{m-1}\right]\Bigg|_{r=r_h}\,, \tag{2.128}$$

where we have already performed the integration on the horizon. Evaluating this expression at $r = r_h$ and using the parameters (2.122) and the function (2.123), we get the simple result

$$S = \pi r_h^2 \left[1 + 2\partial_x \mathcal{W}\right]\,. \tag{2.129}$$

Again, using (2.125) we obtain the explicit relation $S(x, p)$.

Let us now compute the electrostatic potential at the horizon. As we saw in Section 2.1.2, the dual field strength is given by (2.15). The derivative of our Lagrangians (2.104) with respect to the field strength is

$$\frac{\partial \mathcal{L}_{n,m}}{\partial F^{\mu\nu}} = 2(m-1)F_{\mu\nu}(F^2)^{m-2}F_{\rho\alpha}F_{\sigma\beta}Z^{\rho\sigma\alpha\beta} + 2(F^2)^{m-1}F^{\alpha\beta}Z_{[\mu|\alpha|\nu]\beta}\,, \tag{2.130}$$

where

$$Z_{\rho\sigma\alpha\beta} = \left(R^{n-1}\right)_{\rho\sigma}\left(nRg_{\alpha\beta} - (4n+4m-3)R_{\alpha\beta}\right)\,. \tag{2.131}$$

Evaluating this expression for a magnetic vector field (2.29) and for the metric (2.48) we obtain the value of the dual field strength,

$$H = \mathrm{d}t \wedge \mathrm{d}r \sum_{n,m} \mu_{n,m} m \frac{P}{r^2}\left(\frac{2\ell^2 P^2}{r^4}\right)^{m-1} \ell^{2n}\hat{H}^{n-1}\left(-nR + (4n+4m-3)\hat{H}\right)\,, \tag{2.132}$$

where R and $\hat{H}$ are given by (2.108) and (2.109). Remarkably enough, this expression takes the form of an explicit total derivative, namely $H = -\Psi'(r)\mathrm{d}t \wedge \mathrm{d}r$, where the electrostatic potential reads

$$\Psi(r) = \sum_{n,m} \mu_{n,m} m \frac{P}{r}\left(\frac{2\ell^2 P^2}{r^4}\right)^{m-1}\left(\frac{\ell^2(1-f-rf')}{r^2}\right)^n\,. \tag{2.133}$$

Finally, evaluating at the horizon and using (2.122) and (2.123), we may express the result as

$$\Psi_h = \frac{r_h}{\ell}\sqrt{2p}\,\partial_p \mathcal{W}\,. \tag{2.134}$$

Additionally, we can derive the free energy from the on-shell Euclidean action. The computation can be done using the same prescription for the boundary terms as in Eq. (2.94). The bulk action can be evaluated right away thanks to our on-shell Lagrangians being total derivatives (2.110). Then, we find that the evaluation at infinity gets cancelled with the contribution from the boundary terms and we are left with the evaluation of the quantities $\mathcal{I}_{n,m}$ in (2.110) at the horizon. This yields the following result for the free energy, $F = I^{\mathrm{E}}/\beta$:

$$F = \frac{r_h}{4}\left(1 + x\frac{r_h^2}{\ell^2} + 2\mathcal{W}\frac{r_h^2}{\ell^2}\right) . \tag{2.135}$$

In sum, the equations (2.124), (2.126), (2.129), (2.134) and (2.135), together with the relation (2.125), give us explicit expressions for all the thermodynamic quantities M, P, T, S, Ψ_h, F and the radius r_h in terms of two independent parameters x and p. The theory-dependence of all these formulas is encoded in the function $\mathcal{W}$, defined in (2.123). Let us now check that these quantities satisfy consistent thermodynamic relations. In particular, they should satisfy the 1st law of black hole mechanics,

$$\mathrm{d}M = T\mathrm{d}S + \Psi_h \mathrm{d}P . \tag{2.136}$$

This relation is in fact verified. The easiest way to prove this consists in assuming first that r_h is an independent variable in the expressions of M, S and P (Eqs. (2.124), (2.129) and (2.126), respectively). Then, the variations of those quantities with respect to *just* x and p automatically satisfy the first law above. Afterwards, we may take the variation only with respect to r_h (assuming that now x and p are independent variables) and we check that it also satisfies the first law once we take into account the constraint (2.125). Hence when the dependence of r_h on x and p is taken into account, the first law holds too for arbitrary variations of the free parameters.

On the other hand one can also check that $F = M - TS$, which is a non-trivial consistency test of our results, indicating that the Wald's entropy (Noether charge) and the Euclidean action approaches are equivalent.

2.4.2 Extremal and near-extremal black holes

Let us study how the corrections present in (2.111) affect extremal black holes. In terms of the variable x, the extremality condition $T = 0$ implies that x and r_h are related according to

$$x = \frac{\ell^2}{r_h^2} . \tag{2.137}$$

Due to this, (2.125) becomes a complicated equation relating p and x at extremality:

$$2x\partial_x \mathcal{W} + 4p\partial_p \mathcal{W} - 3\mathcal{W} - x = 0 . \tag{2.138}$$

To simplify the discussion, let us consider the subset of theories that are only quadratic in the Maxwell field strength (but which have an arbitrary number of higher-curvature terms). In such case, the function $\mathcal{W}$ takes the form

$$\mathcal{W} = \frac{p}{2}U(x) , \quad \text{where} \quad U(x) = 1 + \sum_{n=1}^{\infty} \mu_{n,1} x^n . \tag{2.139}$$

For this function, it is possible to solve (2.138) explicitly to obtain $p(x)$ at extremality:

$$p = \frac{2x}{2xU' + U} \, . \tag{2.140}$$

Then, from (2.124) and (2.126) we obtain the mass and the charge[26]

$$M_{\rm ext} = \frac{\ell}{\sqrt{x}} \left[\frac{xU' + U}{2xU' + U} \right] , \qquad P_{\rm ext} = \frac{\ell}{\sqrt{x}} \frac{1}{\sqrt{2xU' + U}} \, , \tag{2.141}$$

and the extremal charge-to-mass ratio

$$\left. \frac{P}{M} \right|_{\rm ext} = \frac{\sqrt{2xU' + U}}{xU' + U} \, . \tag{2.142}$$

The entropy in turn reads

$$S_{\rm ext} = \frac{\pi \ell^2}{x} \left[\frac{4xU' + U}{2xU' + U} \right] \, . \tag{2.143}$$

Then, as we did in Subsection 2.3.3 we can check some particular cases to study if it is possible to satisfy the mild form of the Weak Gravity Conjecture at a non-perturbative level.

Since $x = \ell^2/r_h^2$, we must demand that $P/M|_{\rm ext}$ is monotonically growing with x, although in that case we also have to make sure that M is a decreasing function of x. As an example, let us consider the case in which there is a single higher-derivative term in the action so that $U = 1 + \mu_n x^n$. Then we have

$$M_{\rm ext} = \frac{\ell}{\sqrt{x}} \left[\frac{1 + (n+1)\mu_n x^n}{1 + (2n+1)\mu_n x^n} \right] , \quad \left. \frac{P}{M} \right|_{\rm ext} = \frac{\sqrt{1 + (2n+1)\mu_n x^n}}{1 + (n+1)\mu_n x^n} \, , \tag{2.144}$$

For $\mu_n > 0$, the extremal charge-to-mass ratio and the mass are actually monotonically decreasing with x, so this case should be discarded according to the WGC. On the other hand, if we take $\mu_n < 0$ we observe that for small x (large M) the charge-to-mass ratio is in fact growing with x. However, it soon reaches a maximum value and then decreases again. Moreover, M has a minimum value, so there are no extremal black holes below certain mass — see Fig. 2.5. One can also consider other choices of higher-derivative terms that produce different forms of the function $U(x)$, and a few examples are shown in Fig. 2.5. We find the same qualitative behaviour in all of these cases, namely, P/M has a maximum value which happens for the minimum mass. Thus, it seems quite difficult for $P/M|_{\rm ext}$ to be a growing function all the way down to $M = 0$, at least within this family of theories. Nevertheless, this can be interesting from the point of view of the WGC, since, as explained in Subsection 2.3.3, it may imply that below the minimal mass all the solutions are non-extremal black holes, and hence there is no obstacle to prevent the evaporation of these black holes.

Another intriguing fact about these examples is that the corrections to the extremal entropy are negative. For instance, in the case of $U = 1 + \mu_n x^n$ we have

$$S_{\rm ext} = \frac{\pi \ell^2}{x} \left[\frac{1 + (4n+1)\mu_n x^n}{1 + (2n+1)\mu_n x^n} \right] < \pi P^2 \quad \text{if} \quad \mu_n < 0. \tag{2.145}$$

[26]We stumble upon the following fact: the results for the extremal mass and charge given by Eq. (2.141) coincide exactly with those for EQGs — see Eqs. (2.99) and (2.101) — after performing the replacement $U(x) \to -\rho U(\rho)$, where $U(x)$ and $U(\rho)$ are given by Eqs. (2.139) and (2.100) respectively and where $x = \frac{1}{\rho^2}$.

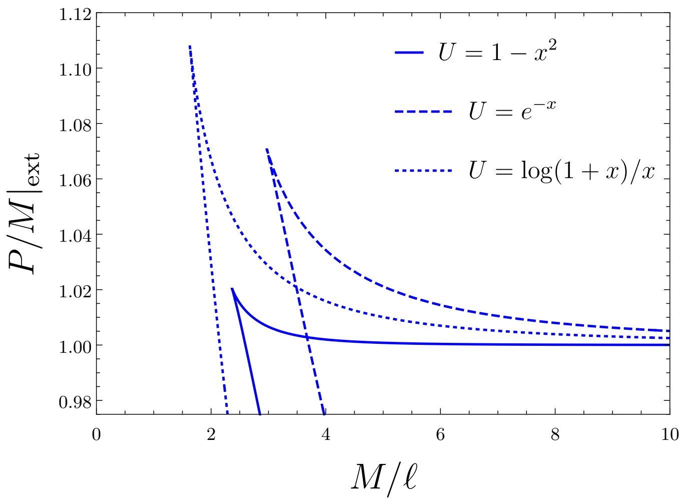


Figure 2.5: Extremal charge-to-mass ratio for some higher-derivative theories. The couplings of the higher-derivative terms are chosen so that P/M increases when M decreases, but we see it is not possible to continue this trend all the way down to $M = 0$. There is a minimum mass below which extremal black holes do not exist. In these examples we observe that each curve has two branches, but only the upper one is smoothly connected with the Reissner-Nordström solution when the higher-derivative couplings are set to 0.

This seems in contradiction with claims and results in the literature [528,531] which relate positive corrections to the extremal charge-to-mass ratio with positive corrections to the entropy. However, the contradiction is not such, since, as noted in Ref. [125], the comparison must be done with the corrections to the near extremal entropy, while the corrections to the extremal entropy are independent. This is an example of that situation.

Finally, let us also comment on near-extremal black holes. A characteristic feature of extremal black holes in Einstein gravity is that the specific heat at constant charge goes to zero, while its first derivative is positive. This means that near-extremal black holes satisfy $M - M_{\rm ext} = cT^2$ with $c > 0$, and therefore the black hole mass grows as we increase the temperature. Interestingly enough, this is not always the case for our black holes with higher-derivative corrections. The specific heat, defined as $C_P = \left(\frac{\partial M}{\partial T}\right)_P$, vanishes at extremality, but its first derivative reads instead

$$\left(\frac{\partial^2 M}{\partial T^2}\right)_P \bigg|_{\rm ext} = \frac{4\ell^3\pi^2\left(-6x^2UU'' - 10x^3U'U'' + U^2\right)}{x^{3/2}\left(2xU' + U\right)\left(x\left(2xU'' + 5U'\right) + U\right)}\,. \tag{2.146}$$

This quantity can have either sign, depending on the model and on the value of x. If it is positive, then near-extremal black holes behave as in Einstein-Maxwell theory and they are stable, in the sense that when we increase the temperature (hence we depart from extremality), the mass also increases. The case in which this quantity is negative is quite intriguing. It implies that in order to get away from extremality, the black hole must lose mass. Therefore, extremal black holes are thermodynamically unstable and they do not represent the minimal mass state for a given charge. Instead, the minimal mass state will take place at a different point in which $C_P = 0$, and this is the solution to which the black hole tends when it evaporates. An example of this situation is represented in Fig. 2.6, where we show T vs M at fixed charge for a particular set of higher-derivative

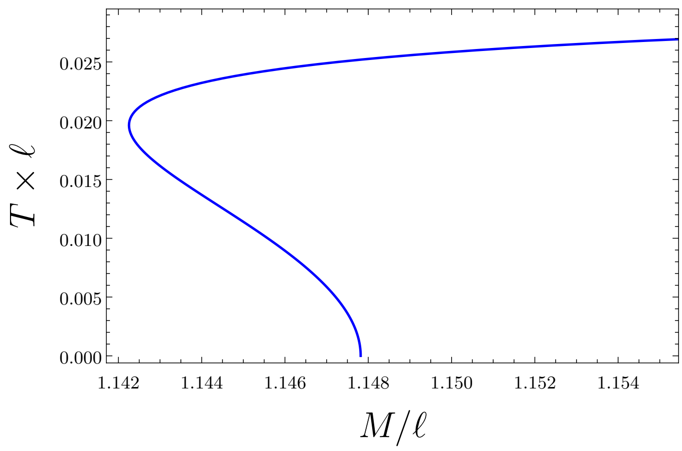


Figure 2.6: Temperature vs mass diagram at fixed charge for near-extremal black holes with $\left(\partial^2 M/\partial T^2\right)_P\big|_{\rm ext} < 0$. The extremal black hole is not the state with the minimal mass. We consider the model $U(x) = 1 + x^2$, but nevertheless the profile of the curve will be similar for any other case in which $\left(\partial^2 M/\partial T^2\right)_P\big|_{\rm ext} < 0$.

terms. Another consequence of this effect is that, in a certain region of the parameter space, there exists more than one black hole solution with the same mass and charge. This non-uniqueness of solutions can be thought of as a discrete violation of the no-hair conjecture and is analogous to the situation with charged black holes in Einsteinian Cubic Gravity that was recently reported in Ref. [271].

2.5 Resolution of Reissner-Nordström singularities: electric black holes

In previous sections we introduced the notion of Electromagnetic (Generalized) Quasitopological Gravities and presented infinite instances of such theories. Afterwards, both in the Quasitopological (algebraic equation for $f(r)$) and in the Generalized Quasitopological (second-order equation for $f(r)$) cases, we focused on magnetic SSS configurations. Nevertheless, as indicated in Definition 2.1, a given theory $\mathcal{L}(R_{\mu\nu,\rho,\sigma}, F_{\alpha\beta})$ is an E(G)QG if the theory itself or its electromagnetic dual (obtained through dualization (2.16)) becomes a total derivative when evaluated on the magnetic SSS ansatz given by (2.26) and (2.29) with $N = 1$, a condition reflected in Eq. (2.53). Consequently, if we consider an E(G)QG with magnetic solutions, the corresponding dual theory will allow by construction for electric SSS solutions characterized by a single metric function f, which furthermore will be the same as in the magnetic frame.

We will devote ourselves in this section to the study of Electromagnetic Quasitopological Gravities with electric solutions. Of course, the most direct way to construct such theories will be through the dualization of EQG theories admitting magnetic solutions. In general, this requires to express the original field strength in terms of the dual one, which is in general a inaccessible process. This is why we restrict ourselves to the dualization of EQG theories of quadratic order in F — a process which we study in Subsection 2.5.1. More concretely, we will be interested in studying such EQGs allowing for magnetic SSS solutions with completely regular gravitational field and electric potential (defined as the potential associated to the dual field strength), since then the corresponding dual theory will naturally admit regular electric solutions. Following this procedure, we are able to find which is, to the best of our knowledge, the first explicit example of a theory that fully regularizes both gravitational and electromagnetic fields for solutions of arbitrary mass and non-vanishing charge.

It is important to remark the relevance of this discovery. In fact, one of the most demanded features of a theory of Quantum Gravity is its ability to resolve the singularities that arise in General Relativity. Assuming that Nature should have no singularities, we interpret these singularities as a signal of the failure of this theory. Thus, ideally one would wish to find an effective high-energy modification of General Relativity whose black hole solutions were singularity-free. Therefore, finding a higher-order theory with completely regular gravitational and electromagnetic fields would give us a proof of principle of the fact that singularities can be indeed regularized by higher-derivative corrections. We present the first instance of such a theory with fully regular electric solutions in Subsection 2.5.2, computing explicit, exact and regular electric SSS solutions and studying some physical aspects of them.

2.5.1 Dualization of theories of quadratic order in F

As explained above, we will be interested in finding the electromagnetic dual of EQGs with quadratic dependence on the gauge field strength F, since this will allow us to construct theories which canonically admit electrically-charged solutions. To this aim, let us consider a theory of gravity coupled to electromagnetism given by the following action:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R - \mathcal{Q}_{\mu\nu\rho\sigma}(g, R_{\alpha\beta\lambda\gamma})F^{\mu\nu}F^{\rho\sigma}\right] , \tag{2.147}$$

where $\mathcal{Q}_{\mu\nu\rho\sigma}$ depends exclusively on the metric $g_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$. Note the following symmetry properties of $\mathcal{Q}_{\mu\nu\rho\sigma}$:

$$\mathcal{Q}_{\mu\nu\rho\sigma} = -\mathcal{Q}_{\nu\mu\rho\sigma} = -\mathcal{Q}_{\mu\nu\sigma\rho} = \mathcal{Q}_{\rho\sigma\mu\nu} . \tag{2.148}$$

From (2.147), one can easily compute that

$$\mathcal{M}_{\mu\nu} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} = \mathcal{Q}_{\mu\nu\rho\sigma}F^{\rho\sigma} . \tag{2.149}$$

Taking into account Eq. (2.15), we have:

$$H_{\mu\nu} = (\star\mathcal{M})_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathcal{Q}^{\alpha\beta\rho\sigma}F_{\rho\sigma} . \tag{2.150}$$

Therefore the action I_{dual} dual to (2.147) turns out to be

$$I_{\mathrm{dual}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R + \mathcal{M}_{\mu\nu}F^{\mu\nu}\right] , \tag{2.151}$$

which is the result one gets from direct application of (2.16). Define now an inverse tensor $\mathcal{Q}^{-1}_{\mu\nu\rho\sigma}$ with the same symmetries as $\mathcal{Q}_{\mu\nu\rho\sigma}$ and satisfying

$$\mathcal{Q}^{-1}_{\mu\nu\rho\sigma}\mathcal{Q}^{\rho\sigma\alpha\beta} = \delta_{\mu\nu}{}^{\alpha\beta} . \tag{2.152}$$

Using this inverse tensor of $\mathcal{Q}$, it is clear that

$$F_{\mu\nu} = \mathcal{Q}^{-1}_{\mu\nu\rho\sigma}\mathcal{M}^{\rho\sigma}\,. \tag{2.153}$$

Therefore, on taking into account Eq. (2.150), we infer that the dual theory may be expressed in the following compact form:

$$I_{\rm dual} = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{|g|}\left[R - \chi_{\mu\nu\rho\sigma}H^{\mu\nu}H^{\rho\sigma}\right]\,, \tag{2.154}$$

where we have defined the tensor $\chi_{\mu\nu\rho\sigma}$ as

$$\chi_{\mu\nu\rho\sigma} = -\frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}(\mathcal{Q}^{-1})^{\alpha\beta\lambda\eta}\varepsilon_{\rho\sigma\lambda\eta} = 6\delta_{\mu\nu[\rho\sigma}\mathcal{Q}^{-1}{}_{\alpha\beta]}{}^{\alpha\beta}\,. \tag{2.155}$$

Consequently, the problem of dualizing a theory with quadratic dependence on $F_{\mu\nu}$ is equivalent to finding the inverse of the tensor $\mathcal{Q}_{\mu\nu\rho\sigma}$. We will exploit this fact in the following subsection.

2.5.2 A higher-order theory with fully regular electric solutions

Now we proceed to construct the first-ever theory of gravity and electromagnetism with completely regular electrically-charged black hole solutions. As described above, such a theory can be found by first studying magnetic SSS solutions with regular geometry and electric potential (associated to the dual field strength), and then applying the dualization procedure to obtain a theory with regular electric solutions.

2.5.2.1 Regular magnetic SSS configurations

Let us start by finding an instance of a higher-order theory of electromagnetism non-minimally-coupled to gravity with regular SSS magnetic solutions, in the sense explained above. One should bear in mind that the ultimate goal of such process is to find a theory with regular electric configurations through dualization, so it is convenient to work with theories for which the computation of their electromagnetic dual is, to some extent, a manageable task, such as theories with quadratic dependence on the gauge field strength. In fact, we presented in Subsection 2.5.1 a procedure to dualize any theory quadratic in $F_{\mu\nu}$, so we can use the results exposed there.

Among this subset of theories, we must find one (at a minimum) with magnetic SSS solutions whose geometry and electric potential are fully regular. We demonstrated, via explicit examples in Subsection 2.3.1, that this is indeed a realizable feature, at least in the context of theories with quadratic dependence on the Riemann curvature tensor. Let us show this is possible as well for theories quadratic in $F_{\mu\nu}$.

Again, our objective is to provide a proof of principle of the fact that there exist theories with regular electric solutions, so it suffices to find one concrete instance of such a theory. In particular, let us consider the quadratic (on $F_{\mu\nu}$) theory:

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R + \mathcal{L}^{(a)}_{0,1} + \alpha\left(\frac{3}{2}\mathcal{L}^{(a)}_{1,1} - \mathcal{L}^{(b)}_{1,1}\right) + \alpha^2\left(\frac{1}{2}\mathcal{L}^{(a)}_{2,1} - \frac{1}{4}\mathcal{L}^{(b)}_{2,1}\right)\right\}, \tag{2.156}$$

where $\mathcal{L}^{(a)}_{n,m}$ and $\mathcal{L}^{(b)}_{n,m}$ are defined as in Eqs. (2.55) and (2.56) and where $\alpha > 0$ is a constant with units of length squared . These theories were shown to be EQGs, admitting magnetic

SSS solutions characterized by a single metric-function $f(r)$. Setting the magnetic SSS ansatz (2.26) with $N=1$, the equation of motion for f can be found by setting in Eq. (2.67) $\lambda_{0,1}=\frac{2}{3}\lambda_{1,1}=-\gamma_{1,1}=2\lambda_{2,1}=4\gamma_{2,1}=1$ and the remaining $\lambda_{n,m},\gamma_{n,m}$ to zero:

$$1-f-\frac{2M}{r}+\frac{P^2}{r^2}+\frac{\alpha P^2(3-f)}{r^4}+\frac{2\alpha^2P^2(1-f)}{r^6}=0\,, \tag{2.157}$$

from where we can solve for f and obtain:

$$f=\frac{r^4(r^2-2Mr+Q^2)+\alpha Q^2(3r^2+\alpha)}{r^6+\alpha Q^2(r^2+2\alpha)}\,. \tag{2.158}$$

Similarly, the electric potential Ψ can be found by particularizing (2.74) for our theory (2.156). One finds:

$$\Psi=\frac{P}{r}\left(1+\frac{\alpha(1-f)}{r^2}\right)\left(1+\frac{\alpha(4-4f+rf')}{2r^2}\right)\,. \tag{2.159}$$

It is convenient to pinpoint three aspects. First of all, note that the denominator of f in Eq. (2.158) is never-vanishing. Second, observe that $f(r)\underset{r\sim 0}{=}1+r^2/\alpha^2+\mathcal{O}(r^3)$. Third, the electrostatic potential and its first derivative vanish at the origin $r=0$. Noticing that f asymptotes to the (magnetic) Reissner-Nordström solution, we conclude that we have encountered a completely regular magnetic SSS configuration. Dualization of such theory will produce a theory which canonically admits regular electric solutions.

2.5.2.2 Regular electric solutions

Expanding (2.156) , we can rewrite it in the following way:

$$I=\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R-\mathcal{Q}^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}\,, \tag{2.160}$$

where

$$\begin{aligned}\mathcal{Q}^{\mu\nu}{}_{\rho\sigma}=\delta^{\mu\nu}{}_{\rho\sigma}+\alpha\left(6R^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]}+7R^{\mu\nu}{}_{\rho\sigma}+\frac{1}{2}R\delta^{\mu\nu}{}_{\rho\sigma}\right)+\alpha^2\left(\frac{9}{4}R_{\alpha}{}^{[\mu}R^{\nu]\alpha}{}_{\rho\sigma}\right.\\ \left.+\frac{9}{4}R^{\alpha}{}_{[\rho}R^{\mu\nu}{}_{\sigma]\alpha}+\frac{1}{4}RR^{\mu\nu}{}_{\rho\sigma}+\frac{35}{8}R^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\sigma}+\frac{1}{2}R_{\lambda}{}^{[\mu}\delta^{\nu]\lambda}{}_{\beta[\rho}R_{\sigma]}{}^{\beta}\right)\,.\end{aligned} \tag{2.161}$$

We observe that this is formally equivalent to Eq. (2.147), so the dual theory to (2.160) is given by:

$$I=\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R-\chi^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right\}\,, \tag{2.162}$$

where, as in Eq. (2.155),

$$\chi_{\mu\nu\rho\sigma}=-\frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}(\mathcal{Q}^{-1})^{\alpha\beta\lambda\eta}\varepsilon_{\rho\sigma\lambda\eta}=6\delta_{\mu\nu[\rho\sigma}\mathcal{Q}^{-1}{}_{\alpha\beta]}{}^{\alpha\beta}\,, \tag{2.163}$$

being $\mathcal{Q}^{-1}$ the inverse tensor of $\mathcal{Q}$, in the sense of (2.152). It is important to note that the theory (2.162) contains a free coupling constant α and when it is set to zero one recovers Einstein-Maxwell theory. In fact, at low energies we have

$$I=\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R-F^2+\alpha\left(7F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma}-22F_{\mu\alpha}F_{\nu}{}^{\alpha}R^{\mu\nu}+\frac{9}{2}F^2R\right)+\mathcal{O}(\alpha^2)\right\}, \tag{2.164}$$

so that the action reduces to the Einstein-Maxwell one when the curvature and the field strength are small enough (or when $\alpha \to 0$). Thus, we can think of this theory as a toy model for a UV-completion of GR containing an infinite tower of higher-derivative terms. On the other hand, as we will see below, for any value of the coupling we are able to find exact solutions of arbitrary mass and charge (we do not need to tune M and Q to particular values), making this a very useful theory for practical purposes.

In order to find SSS electric solutions of (2.162), we may resort to Proposition 2.1, according to which the metric and the dual field strength of a magnetic configuration produce an electric solution of the dual theory. Consequently, from the results of Subsubsection 2.5.2.1, the electric SSS configuration

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2_{(2)}\,, \quad A = \Phi(r)\mathrm{d}t\,, \tag{2.165}$$

with

$$\begin{aligned} f(r) &= \frac{r^4\left(r^2 - 2Mr + Q^2\right) + \alpha Q^2(3r^2 + 2\alpha)}{r^6 + \alpha Q^2(r^2 + 2\alpha)}\,, \\ \Phi(r) &= \frac{Q}{r}\left(1 + \frac{\alpha(1-f)}{r^2}\right)\left(1 + \frac{\alpha(4 - 4f + rf')}{2r^2}\right) \end{aligned} \tag{2.166}$$

is an *exact* solution to the theory given by (2.162), with M and Q being two integration constants that will be (re)identified later with the mass and the electric charge, respectively. We would like to remark at this point that one could have arrived to the solution (2.166) without need of resorting to the theory (2.160). As a matter of fact, one can derive the Einstein and Maxwell equations of (2.166) in an exact fashion by varying directly the action. Defining the auxiliary tensor $\hat{F}^{\mu\nu} = \chi^{\mu\nu\rho\sigma}F_{\rho\sigma}$, these equations take the form:

$$\begin{aligned} 2\mathcal{E}^E_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 12\hat{F}_{\mu}{}^{\alpha}\hat{F}_{[\nu\alpha}\,\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma} + 3\,g_{\mu\nu}\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\,\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma} \\ &\quad + 6\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\frac{\partial\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma}}{\partial R^{\mu\lambda\tau\gamma}}R_{\nu}{}^{\lambda\tau\gamma} + 12\nabla^{\lambda}\nabla^{\gamma}\left(\hat{F}^{\alpha\beta}\hat{F}_{[\alpha\beta}\frac{\partial\mathcal{Q}_{\rho\sigma]}{}^{\rho\sigma}}{\partial R^{\mu\lambda\nu\gamma}}\right) + (\mu\leftrightarrow\nu)\,, \end{aligned} \tag{2.167}$$

$$\mathcal{E}^M_{\nu} = \nabla_{\mu}\hat{F}^{\mu}{}_{\nu}\,, \tag{2.168}$$

where the proper field strength $F_{\mu\nu}$ can be recovered from $\hat{F}$ using [27]

$$F_{\mu\nu} = 6\hat{F}_{[\rho\sigma}\,\mathcal{Q}_{\mu\nu]}{}^{\rho\sigma}\,. \tag{2.169}$$

Thus, we realize that the equations of motion can be expressed solely in terms of $\hat{F}$ and $\mathcal{Q}$, hence circumventing the highly intricate task of computing the inverse tensor $\mathcal{Q}^{-1}$. Indeed, we are able to show explicitly in Appendix 2.B the process of finding the SSS solution (2.165) by direct resolution of the Einstein and Maxwell equations (2.167) and (2.168).

Let us now explore the physical properties of the electric solution (2.166), whose profile is shown in Fig. 2.7 for specific values of M and Q. We discussed very briefly the regularity properties of the magnetic SSS solutions of the theory (2.156), so let us be more explicit in the context of electric solutions. First, notice that the fields asymptotically

[27] This follows from the fact that $(\chi^{-1})^{\mu\nu}{}_{\rho\sigma} = 6\delta^{[\mu\nu}{}_{\rho\sigma}\mathcal{Q}^{\alpha\beta]}{}_{\alpha\beta}$.

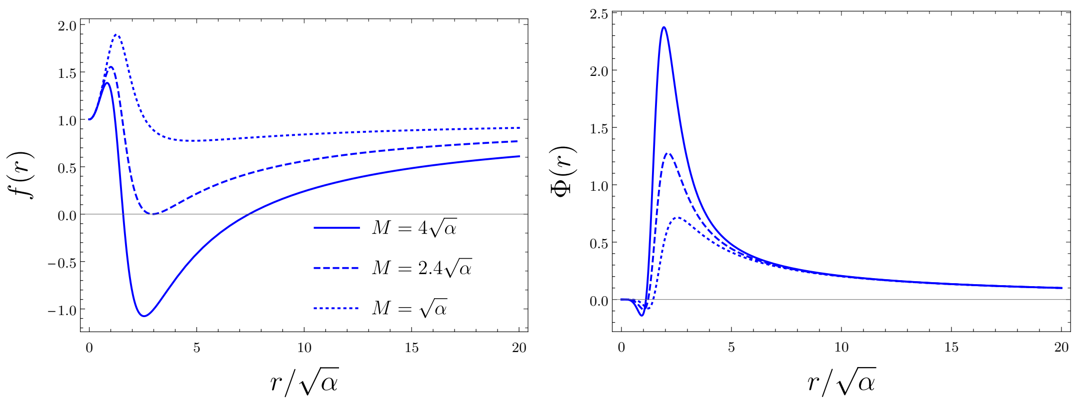


Figure 2.7: Profile of the metric function $f(r)$ and the electrostatic potential $\Phi(r)$ appearing in (2.166) as functions of the radial coordinate. In these plots we use $Q = 2\sqrt{\alpha}$ and various values of the mass. Solid line: black hole with outer and inner horizons. Dashed line: extremal black hole, which in this case takes place for $M_{\text{ext}} \simeq 2.4\sqrt{\alpha}$. Dotted line: horizonless solution. In all cases the point $r = 0$ is a smooth cap of the geometry and both $f(r)$ and $\Phi(r)$ are finite everywhere.

behave as in the Reissner-Nordström solution, $f(r) \sim 1-2M/r+Q^2/r^2$, $\Phi(r) \sim Q/r$, from where one can identify M with the mass and Q with the electric charge. Also, the full RN solution is recovered when we set $\alpha = 0$. Thus, this solution is a continuous deformation of the RN one and the deviations with respect to it are small as long as the curvature and the field strength take sufficiently small values. On the other hand, the corrections have a drastic effect near the would-be singularities, where these quantities would diverge. Indeed, the most remarkable property of this solution, as can be easily seen from (2.166), is that the geometry is smooth everywhere. More precisely, as discussed above with the magnetic solution (2.158), we observe that $f(r)$ has no divergences, and we find that around $r = 0$ it behaves as

$$f(r) = 1 + \frac{r^2}{\alpha} + \mathcal{O}(r^3)\,, \tag{2.170}$$

which implies that the point $r = 0$ is a smooth cap of the geometry. In particular, the region near $r = 0$ is a locally AdS space of radius $\sqrt{\alpha}$. Interestingly enough, the electromagnetic field is also finite everywhere and one can see that near the origin the electrostatic potential is given by

$$\Phi(r) \sim -\frac{M^2 r^5}{2Q^3\alpha^2} + \mathcal{O}(r^7)\,. \tag{2.171}$$

The field strength $F = -\Phi' \mathrm{d}t \wedge \mathrm{d}r$ is also finite and vanishes at $r = 0$. Thus, whenever $Q \neq 0$ these solutions are free of singularities. In the absence of charge, the theory (2.162) effectively reduces to GR, so the solution (2.166) becomes the Schwarzschild black hole. This clearly signals that the non-minimal coupling to electromagnetism is crucial for singularity-resolution. In fact, regularization of neutral black holes remains elusive, and the evidence so far seems to indicate that black hole solutions in purely gravitational higher-order theories are still singular, although their curvature divergence usually gets softened — see *e.g.* [227, 231, 233, 239, 248].

Depending on the relative values of the mass and the charge, these solutions have a different nature. As we can see in Eq. (2.166), only the term proportional to the mass comes with a negative sign, so if M is large enough compared to the charge, $f(r)$ will vanish at certain point. In that case the solution contains a horizon and hence it is a black

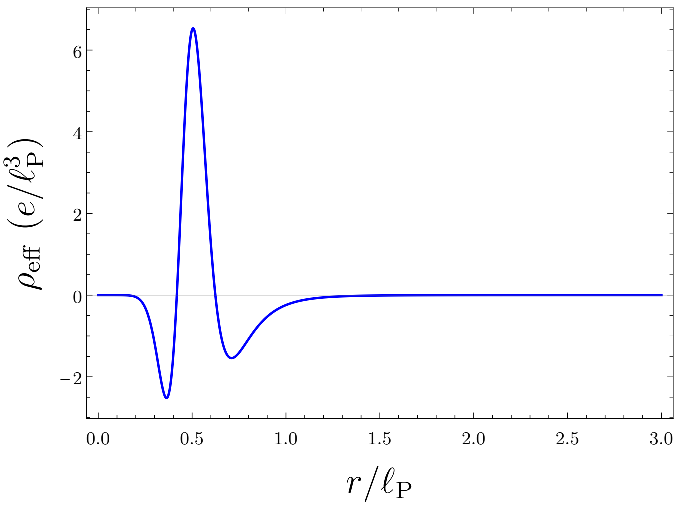


Figure 2.8: Effective charge density of an electron (defined as in Eq. (2.173)) predicted by the theory (2.162). We assume that the corrections appear at Planck scale, $\alpha=\ell_{\rm P}^2$.

hole — see the solid lines in Fig. 2.7. In addition, this black hole has always a second, inner horizon (except in the extremal limit), so the causal structure is very similar to that of the RN black hole, with the difference that the timelike singularity at $r=0$ is removed. The Penrose diagram of this regular black hole is identical to that of other models already discussed in the literature [388, 390], so we refer to those works for further details. For a specific value of the mass, $M=M_{\rm ext}(Q)$, both horizons merge into a degenerate horizon and we have a extremal black hole — depicted by the dashed lines in Fig. 2.7. Let us note that the extremality condition is modified with respect to the case of the Reissner-Nordström black hole, so that $M_{\rm ext}\neq Q$. Finally, if the mass is below the extremal value, then the effect of the charge dominates and the solution does not possess a horizon. This situation is represented by the dotted lines in Fig. 2.7. This horizonless smooth solution is particularly interesting and, in principle, one could interpret it as a soliton or a fuzzball.

One intriguing question is that about the origin of the charge and the mass in these solutions. Apparently, there are no matter sources involved, so one might conclude that the mass and charge arise due to the non-linear interactions between gravity and electromagnetism. However, a closer look reveals that this is not entirely correct. While the geometry is smooth ($\mathcal{C}^\infty$) everywhere, one can check that the potential Φ (and hence the vector A) is only $\mathcal{C}^4$ at[28] $r=0$. This means that some of the equations of motion may not be satisfied at $r=0$, which typically indicates the presence of point-like sources. In fact, in the case of the Maxwell equation (2.168) it is easy to see that we have a Dirac delta on the right-hand-side,

$$\nabla_\mu \hat{F}^{\mu\nu}=4\pi Q\delta^{(3)}(r)\delta^\nu_t\,, \tag{2.172}$$

and hence, these solutions do have a point-like source of electric charge. This means that the horizonless solutions we have found should be really interpreted in terms of fields of charged point particles rather than as solitons. Thus, higher-derivatives seem to have the effect of "smearing" the charge, so that, even though it is concentrated at a single point, the

[28]While the potential is a smooth function when expressed in terms of the radial coordinate r, after expressing $r=\sqrt{x^2+y^2+z^2}$ for some local Cartesian coordinates $(x,y,z)\in\mathbb{R}^3$, we observe that the potential Φ is just of class $\mathcal{C}^4$ when understood as a function from $\mathbb{R}^3$ to $\mathbb{R}$.

fields are finite. As an interesting example, we may consider the case of an electron. Let us assume that the corrections appear at Planck scale and that $\alpha = \ell_{\rm P}^2$. In Planck units the charge of the electron is $Q = -eG^{1/2}(4\pi\epsilon_0)^{-1/2} \approx -1/\sqrt{137}$, where e is the elementary charge. We may also approximate $m_e \approx 0$, since it is much smaller than Planck's mass. Then, even though we treat the electron as a point particle, we may define an effective charge density in the usual way, $\mathrm{d}\star F = 4\pi\rho_{\rm eff} V_3$, where V_3 is the volume form of constant-t spatial slices. This leads to the identification

$$4\pi\rho_{\rm eff} = -\frac{\sqrt{f}}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) . \tag{2.173}$$

The profile of this effective charge density is shown in Fig. 2.8, where we can check that it is finite everywhere and concentrated around the region $r < \ell_{\rm P}$. Thus, higher-derivative corrections delocalize the charge, yielding the electron some apparent structure. In this sense, by direct inspection of Fig. 2.8, we may associate the outermost minimum of $\rho_{\rm eff}$ with the electron charge-radius, since it expresses the distance from which the charge density starts to asymptote to zero. We observe that such charge-radius is of the order of the Planck length, but one should be aware that this is related to the fact that we have chosen $\alpha = \ell_{\rm P}^2$; in general the effective charge-radius will be a given function of α, so it can be always tuned to an appropriate value. A similar discussion would apply in the case of the effective mass density.

On the other hand, the gravitational and electromagnetic potentials $f(r)$ and $\Phi(r)$ have qualitatively similar profiles to those shown represented by dotted lines in Fig. 2.7. All of this provides us with a remarkable physical picture. In the first place, the electron sources the electromagnetic field, which at the same time creates a gravitational field. Then, due to the non-minimal couplings between them, a non-linear backreaction is produced which at the end renders both fields finite.

Consequently, we have shown that the theory (2.162) is able to resolve the singularities of charged black holes and point-like charged particles. The regular black holes we have obtained have similar properties to some of the models analyzed in the literature [388, 390], with the timelike singularity of RN black holes replaced by a smooth "AdS core". On the other hand, point charges acquire an effective structure of finite size due to the short-distance modifications of gravity and electromagnetism implied by the theory (2.162). Thus, we have proven that the regularization of singularities *is* possible within the framework of Einstein-Maxwell theory with higher derivatives. In this sense, it suffices to show that it can be achieved by some theories to prove that effective actions can capture this property of a UV-complete theory. In fact, this makes the action (2.162) a very interesting model for a UV-completion of Einstein-Maxwell theory.

2.6 Discussion

In this chapter we have introduced a new class of non-minimally coupled higher-derivative extensions of Einstein-Maxwell theory. These theories are characterized by possessing magnetic or electric SSS solutions characterized by a single metric function f (see (2.48)) whose equation of motion is (at least partially) integrable. In addition, within this set of theories, the thermodynamic properties of black holes can be computed exactly. Such theories are analogous to GQGs and thus we refer to them as EGQGs. As in the case of pure gravity, we have seen EGQGs come in two main classes: those for which the SSS equations of motion can be reduced to an algebraic equation for f belong to the Quasitopological class, while if the equation is of second order we say that the theory is properly of the Generalized Quasitopological class. We have constructed an infinite number of densities of both types, although we suspect that there are many others, especially in Generalized Quasitopological case. Determining the most general structure of these Lagrangians would be an interesting problem.

In the case of Quasitopological theories, we have shown some explicit examples of black hole and non-black hole solutions — see Section 2.3. We observed that, in a quite remarkable and general way, these solutions possess globally regular geometries, *i.e.*, the timelike singularity at $r = 0$ characteristic of charged black holes or point charges is smoothed away by the higher-derivative corrections. In slightly more restrictive cases, we showed that the electrostatic potential of the dual theory also remains finite everywhere, thus making these solutions particularly appealing. In particular, in the horizonless case, one may regard these objects as solitons or even as four-dimensional fuzzballs.

For both Quasitopological and Generalized Quasitopological theories, we have performed a detailed study of black hole thermodynamics — see Sections 2.3.2 and 2.4.1. We have been able to provide explicit expressions for all the relevant thermodynamic potentials and we have shown that the first law of black hole mechanics,

$$\mathrm{d}M = T\mathrm{d}S + \Psi_h \mathrm{d}P\,, \tag{2.174}$$

holds exactly. Here S is Wald's entropy and Ψ_h is the electrostatic potential of the dual theory evaluated on the event horizon. Thus, the first law is formally unchanged with respect to the case of a minimally coupled gauge field. This is not a general proof of the first law, but rather a check of it for a large class of theories. It would be interesting to actually attempt a proof in the case of general $\mathcal{L}(R_{\mu\nu\rho\sigma}, F_{\alpha\beta})$ theories, as done in Ref. [516] in the case of non-linear electromagnetism coupled to Einstein gravity[29]. In addition, we have checked that the Euclidean methods provide the same answer for black hole thermodynamics than the Noether-charge approach. In particular, we have seen that the on-shell Euclidean action yields indeed the free energy, $TI^{\mathrm{E}} = F = M - TS$. Due to the large space of theories that we consider, we have not made a general analysis of the features of the new thermodynamic relations, so a more detailed study is left for future work.

Motivated by the WGC, we studied the properties of extremal and near-extremal black holes in these theories. A mild form of the WGC states that, in a consistent quantum theory of gravity, the charge-to-mass ratio of extremal black holes should grow monotonically as the mass decreases. This would allow for the decay of extremal black holes in terms of energy and charge conservation. Previous literature had studied perturbative corrections to the extremality bound in a variety of theories, ranging from general EFTs to stringy effective actions [125, 126, 474, 475, 528–533]. Although our theories do not belong (a priori) to those categories, they have the advantage of allowing us to perform exact, non-perturbative computations. Thus, they may be used to learn about the corrections to extremality at large coupling. As we observed in Sections 2.3.3 and 2.4.2, it is always easy (*e.g.*, by choosing the signs of the couplings appropriately) to impose $P/M\big|_{\mathrm{ext}}$ to satisfy the WGC when the mass is large (*i.e.*, in the perturbative regime). However, when the

[29]By private communication with T. Ortín and D. Pereñíguez, we have been informed that it is actually possible to show that the first law of black hole thermodynamics holds for any higher-order gravity with a non-minimally coupled vector field. They hope to publish their findings soon.

curve $P/M\big|_{\rm ext}$ vs $M_{\rm ext}$ is continued to lower masses, one often finds that it stops at a minimal mass, meaning that there are no extremal black holes below that mass. There can be different reasons for this behaviour, but we have shown with an example (see Fig. 2.3) that a possibility is that below that mass all solutions are non-extremal black holes, regardless the value of the charge. This is actually appealing from the point of view of the WGC, since it implies that, below the minimal mass, charged black holes find no obstruction to evaporate.

Higher-derivative corrections can introduce new effects into the game, and in particular we observed another situation that has implications for black hole evaporation. In some instances — as shown in Fig. 2.6 — it may occur that the extremal black hole is not the one with a minimum mass for a given charge. In those cases, (near-)extremal black holes are unstable, and tend to decay to this minimum mass black hole, which has a non-vanishing temperature. Thus, in that situation one does not have to worry about the charge-to-mass ratio of the extremal black hole, but about the one of the minimum mass black hole. An analogous example has been recently reported in Ref. [271] in the context of Einsteinian cubic gravity with a (minimally-coupled) Maxwell field.

Via dualization of EQGs possessing magnetically-charged solutions with regular gravitational field and smooth electrostatic potential (associated to the dual vector field), in Section 2.5 we have been able to identify the very first instance, to the best of our knowledge, of a higher-order theory which fully regularizes the gravitational and electromagnetic fields for arbitrary values of the mass and (non-vanishing) charge. These solutions are interpreted as generalizations of the Reissner-Nordström solution and provide a proof of principle of the long-lasted dream of curing singularities through the introduction of higher-derivative corrections. Probably, many other higher-derivative theories (not necessarily EQGs) are also singularity-free, but one cannot check this easily due to the complicated form of the equations of motion in the general case. In any case, given the infinite amount of EQGs with magnetic regular solutions found in Section 2.3, this suggests that the regularization of singularities by non-minimal higher-derivative terms could be a more general phenomenon than expected. Nevertheless, the regularization of the Schwarzschild black hole remains elusive, being highly interesting to find a purely gravitational higher-order gravity with regular black holes in vacuum.

The new theories offer various possibilities since they allow us to perform many explicit computations that are inaccessible in general higher-derivative theories. Thus, let us close the chapter by commenting on future directions. As we have already mentioned, it would be interesting to complete the characterization of EGQG Lagrangians to find the most general action of this type. On the other hand, here we have focused on asymptotically flat solutions, so one could extend this work by including a non-vanishing cosmological constant. The asymptotically anti-de Sitter case is particularly relevant due to its connection to holography. In fact, it is known that higher-derivative gravities with a negative cosmological constant are very useful holographic toy models that can be used to learn non-trivial information about CFTs — see Refs. [259, 270] for recent results involving GQGs. Since EGQGs contain higher-derivatives not only of the metric but also of a vector field, these may be used to probe additional aspects of a CFT. We will explore these features in Chapter 4, where we will also define higher-dimensional examples of the (four-dimensional) EQGs we have here identified.

One could also characterize subsets of these theories satisfying additional properties. For instance, we expect some EGQGs to allow for single-function Taub-NUT solutions whose thermodynamic properties can be studied exactly, as the ones in [258]. In particular, some theories of the Quasitopological subclass might allow for explicit Taub-NUT solutions. In addition, non-minimally coupled electrodynamics may be of interest in cosmology — see *e.g.* [534] — so we may wonder if some of these theories could be useful in that context, as the ones in [260, 262, 263].

Finally, it would be interesting to explore the effect of field redefinitions in E(G)QGs. In particular, since GQGs turn out to span the space of all purely gravitational higher-curvature theories once field redefinitions are taken into account, it is natural to investigate whether this behaviour replicates somehow for E(G)QGs.

Appendix 2.A All E(G)QGs of the form RF^2 and R^2F^2

In this appendix we are going to build all Electromagnetic (Generalized) Quasitopological Gravities constructed by linear combinations of terms up to quadratic order both in the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ and in the gauge field strength $F_{\mu\nu}$. For that, we first classify all E(G)QGs of the form RF^2, made up of scalar terms with one Riemann and two field strength and, afterwards, we proceed analogously for E(G)QG theories R^2F^2, whose constituent terms contain exactly two Riemann tensors and two field strengths.

RF^2 theories

Our first task is to find a set of diffeomorphism-invariant terms which span all possible scalars of the form RF^2. This can be done straightforwardly and we find the following basis of invariants containing one Riemann tensor and two field strengths:

$$\mathcal{I}_1 = RF^2\,, \quad \mathcal{I}_2 = R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\,, \quad \mathcal{I}_3 = R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,. \tag{2.175}$$

Now we build the Lagrangian density

$$\mathcal{L} = R + \ell^2 \sum_{i=1}^{3} a_i \mathcal{I}_i\,, \quad a_i \in \mathbb{R} \tag{2.176}$$

and wonder when the corresponding theory belongs to the Electromagnetic (Generalized) Quasitopological type. For that, we just need to check when the Definition 2.1 is fulfilled. Setting $L_f = r^2\,\mathcal{L}|_{\mathrm{d}s_f^2, F^m}$, where $\mathrm{d}s_f^2$ and F^m are given by (2.48) and (2.29) respectively, we have that

$$\frac{\partial L_f}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial L_f}{\partial f'} + \frac{\mathrm{d}}{\mathrm{d}r^2}\frac{\partial L_f}{\partial f''} = -\frac{4P^2\ell^2(10a_1 + 2a_2 + a_3)}{r^4}\,. \tag{2.177}$$

For the theory to be of the (Generalized) Quasitopological type, we must ensure that the latter expression vanishes. This is accomplished by

$$a_3 = -10a_1 - 2a_2\,. \tag{2.178}$$

Now the equation of motion for the metric function $f(r)$ is obtained by evaluating the Lagrangian (2.176) on the general SSS ansatz (2.26) with a magnetic vector (2.29), varying the subsequent action with respect to N and, finally, imposing the condition $N = 1$. Through this procedure, one finds the following equation of motion for $f(r)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[2r(1-f) + \frac{2P^2\ell^2(6a_1 + a_2 + 2a_1 f(r))}{r^3}\right] = 0\,. \tag{2.179}$$

This equation can be directly integrated to yield

$$1 - f - \frac{2M}{r} + \frac{P^2\ell^2}{r^4}(6a_1 + a_2 + 2a_1 f(r)) = 0\,, \tag{2.180}$$

where M is an integration constant appropriately chosen to be identified with the mass, as done in the main text.

There are two important conclusions to extract from Eq. (2.180). Firstly, we recognize precisely the same structure as in Eq. (2.67), if we limit ourselves to the Einstein-Hilbert term and the terms with $n=1, m=2$. Secondly, we check that the set of EGQGs and EQGs coincide for theories of the form RF^2, since Eq. (2.180) is algebraic. This property does not hold generally of course and is very particular of RF^2 theories. As a matter of fact, the special properties of these Lagrangians had been previously noticed in the literature [401, 521, 522].

R^2F^2 theories

Again, first of all we shall concentrate on finding a set of invariants spanning all possibles scalars built out with precisely two Riemanns and two field strengths. After some work, it is possible to choose such set to be[30]:

$$
\begin{aligned}
&\mathcal{I}_1 = R^2F^2\,, \quad \mathcal{I}_2 = RR_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\,, \quad \mathcal{I}_3 = R_{\mu\nu}R^{\mu\nu}F^2\,, \quad \mathcal{I}_4 = R_{\mu\nu}R_{\alpha}{}^{\nu}F^{\mu\beta}F^{\alpha}{}_{\beta}\,,\\
&\mathcal{I}_5 = R_{\mu\nu}R_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta}\,, \quad \mathcal{I}_6 = RR_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,, \quad \mathcal{I}_7 = R_{\mu\nu}R^{\mu\alpha\nu\beta}F_{\alpha\rho}F_{\beta}{}^{\rho}\,,\\
&\mathcal{I}_8 = R_{\mu\nu}R^{\mu}{}_{\alpha\beta\sigma}F^{\beta\sigma}F^{\alpha\nu}\,, \quad \mathcal{I}_9 = R_{\mu\nu}R^{\mu}{}_{\alpha\beta\sigma}F^{\alpha\beta}F^{\sigma\nu}\,, \quad \mathcal{I}_{10} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}F^2\\
&\mathcal{I}_{11} = R_{\mu\nu\rho\alpha}R^{\mu\nu\rho\beta}F^{\alpha\lambda}F_{\beta\lambda}\,, \quad \mathcal{I}_{12} = R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}F^{\rho\sigma}F_{\alpha\beta}\,, \quad \mathcal{I}_{13} = R_{\mu\nu\rho\sigma}R^{\mu\nu\alpha\beta}F^{\rho}{}_{\alpha}F^{\sigma}{}_{\beta}\,,\\
&\mathcal{I}_{14} = R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}F^{\nu\sigma}F_{\alpha\beta}\,, \quad \mathcal{I}_{15} = R_{\mu\nu\rho\sigma}R^{\mu\alpha\rho\beta}F^{\nu}{}_{\alpha}F^{\sigma}{}_{\beta}\,.
\end{aligned}
\tag{2.181}
$$

Proceeding in the same way as with RF^2 theories, now we consider the Lagrangian density

$$
\mathcal{L} = R + \ell^4\sum_{i=1}^{15} b_i\mathcal{I}_i\,, \quad b_i\in\mathbb{R}
\tag{2.182}
$$

and investigate when the corresponding theory belongs to the E(G)QG type. Defining as before $L_f = r^2\,\mathcal{L}|_{\mathrm{d}s^2_f, F^m}$, we have that

$$
\begin{aligned}
&\frac{\partial L_f}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial L_f}{\partial f'} + \frac{\mathrm{d}}{\mathrm{d}r^2}\frac{\partial L_f}{\partial f''}\\
&\qquad = \frac{P^2\ell^4}{r^6}\left(\mathcal{A}_1 - \mathcal{A}_1 f + \mathcal{A}_2 r f' + \mathcal{A}_3 r^2 f'' - 4\mathcal{A}_4 r^3 f^{(3)} + \mathcal{A}_4 r^4 f^{(4)}\right)\,,
\end{aligned}
\tag{2.183}
$$

where we have defined

$$
\begin{aligned}
\mathcal{A}_1 &= -2(168b_1 + 8b_{10} + 4b_{11} + 8b_{12} + 4b_{13} + 2b_{14} + 54b_2 + 24b_3 + 12b_4\\
&\quad + 12b_5 + 88b_6 + 7(b_7 - 2b_8 + b_9))\,,\\
\mathcal{A}_2 &= 4(96b_1 + 2(8b_{10} + 2b_{11} + b_{15} + 9b_2 + 2(7b_3 + b_4 + b_5 - 2b_6)) + 7b_7)\,,\\
\mathcal{A}_3 &= -2(36b_1 - 4b_{10} + 2b_{11} + b_{15} + 9b_2 + 2(4b_3 + b_4 + b_5 - 2b_6)) - 7b_7\,,\\
\mathcal{A}_4 &= 4(b_1 + b_{10}) + 2b_3\,,
\end{aligned}
\tag{2.184}
$$

The theory is a (Generalized) Quasitopological one if all $\mathcal{A}_i$ vanish simultaneously. Such a system of linear equations is solved by:

$$
b_3 = -2b_1 - 2b_{10}\,, \quad 7b_7 = -40b_1 + 40b_{10} - 4b_{11} - 2b_{15} - 18b_2 - 4b_4 - 4b_5 + 8b_6\,,
$$

[30] Note that we are not claiming that all terms in this set are linearly independent.

$$7b_9 = -80b_1 - 8b_{12} - 4b_{13} - 2b_{14} + 2b_{15} - 36b_2 - 8b_4 - 8b_5 - 96b_6 + 14b_8 \,. \tag{2.185}$$

Imposing these constraints, one obtains the generic expression for any EGQG constructed out of terms with two Riemanns and two field strengths. However, we still need to figure out which of these EGQGs are actually Quasitopological.

For that, we must learn when the equation for $f(r)$ is algebraic. As aforementioned, this equation is derived after evaluating the Lagrangian (2.182) on our magnetic SSS ansatz, varying the subsequent action respect to N and afterwards setting $N = 1$. One obtains:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[2r(1-f) + \frac{P^2\ell^4}{7r^5}\left(\mathcal{B}_1 + \mathcal{B}_2 f + \mathcal{B}_3 f^2 + 6\mathcal{B}_4 r f f' + \mathcal{B}_4 r^2 f'^2 - 2\mathcal{B}_4 r^2 f''\right)\right] = 0\,, \tag{2.186}$$

where

$$\begin{aligned}
\mathcal{B}_1 &= 2(24b_1 - 8b_{10} - 2b_{11} - 4b_{12} - 2b_{13} - b_{14} + 8b_2 + b_4 + b_5 + 12b_6)\,,\\
\mathcal{B}_2 &= 56(4b_1 + b_2 + 2b_6)\,,\\
\mathcal{B}_3 &= -2(136b_1 - 8b_{10} - 2b_{11} - 4b_{12} - 2b_{13} - b_{14} + 36b_2 + b_4 + b_5 + 68b_6\,,\\
\mathcal{B}_4 &= -8b_1 + 8b_{10} + 2b_{11} + b_{15} + 2b_2 + 2b_4 + 2b_5 - 4b_6\,.
\end{aligned} \tag{2.187}$$

EQGs are characterized by having an algebraic equation of motion for the metric function $f(r)$. Interestingly enough, this is achieved if we just impose the vanishing of $\mathcal{B}_4$. Therefore, setting $\mathcal{B}_4 = 0$, we get the most general form for the equation of motion of $f(r)$ in any EQG built out of linear combinations of scalars with at most two Riemann curvature tensors and two gauge field strengths. This equation reads as follows:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[2r(1-f) + \frac{2P^2\ell^4}{7r^5}(1-f)(\mathcal{C}_1 + \mathcal{C}_2 f)\right] = 0\,, \tag{2.188}$$

where

$$\begin{aligned}
\mathcal{C}_1 &= 16b_{10} + 4b_{11} - 4b_{12} - 2b_{13} - b_{14} + 3b_{15} + 14b_2 + 7b_4 + 7b_5\,,\\
\mathcal{C}_2 &= 128b_{10} + 32b_{11} - 4b_{12} - 2b_{13} - b_{14} + 17b_{15} + 70b_2 + 35b_4 + 35b_5\,.
\end{aligned} \tag{2.189}$$

Upon direct integration of the previous expression, choosing appropriately the constant of integration M, we end up with

$$1 - f - \frac{2M}{r} + \frac{P^2\ell^4}{7r^6}(1-f)(\mathcal{C}_1 + \mathcal{C}_2 f) = 0\,, \tag{2.190}$$

and we recognize the same structure as in Eq. (2.67) after restricting ourselves to those terms with $n = 2, m = 1$. Hence we have proven that the equation for $f(r)$ of the most general EQG constructed from terms with at most two Riemann tensors and two gauge field strengths is indeed represented by Eq. (2.67), after an appropriate choice of couplings.

Appendix 2.B Solving the equations of motion of the theory (2.162)

First, one can see that, due to the form of the ansatz in Eqs (2.26) and (2.28), the only non-vanishing component of $\hat{F}_{\mu\nu}$ is $\hat{F}_{tr} = -\hat{F}_{rt}$. Then, Eq. (2.13) implies that this tensor must have the form

$$\hat{F} = \frac{QN(r)}{r^2}\mathrm{d}t \wedge \mathrm{d}r\,, \tag{2.191}$$

where Q is an integration constant that represents the electric charge. The next step is to substitute (2.191) on the Einstein equation (2.12). Before that, we note that the static condition and spherical symmetry imply that all its off-diagonal components vanish identically. Furthermore, using the Bianchi identity one may deduce that the angular components are satisfied once the tt and rr components hold. Therefore these are the only non-trivial equations we obtain from (2.12). Imposing the static and spherically symmetric ansatz (2.26) together with the expression of $\hat{F}$ presented in Eq. (2.191), we find:

$$\begin{aligned}\mathcal{E}^E_{tt} =& -\frac{fN^2}{r^4}\left(r^3 f' + (f-1)r^2 + Q^2\right) - \frac{\alpha f N^2 Q^2}{r^6}\left(rf' - 3f + 9\right)\\ &+\frac{\alpha^2 f Q^2}{2r^8}\left(-f^2N'^2r^2 + 4N^2\left(-rf' + 5f - 5\right)\right.\\ &\left.+2fNr\left(2N'rf' - 4fN' + fN''r\right)\right) = 0\,,\end{aligned} \tag{2.192}$$

$$\mathcal{E}^E_{tt} + f^2N^2\mathcal{E}^E_{rr} = 2NN'f^2\left(\frac{1}{r} + \frac{Q^2\alpha}{r^5} - \frac{Q^2\alpha^2(-2N + rfN')}{Nr^7}\right) = 0\,. \tag{2.193}$$

Despite the intricateness of Eq. (2.192), we see by direct inspection that the combination of $\mathcal{E}^E_{tt}$ and $\mathcal{E}^E_{rr}$ in Eq. (2.193) imposes $N =$ constant (another possible solution could be obtained by setting the quantity between brackets to zero, but this yields an unphysical solution which is not asymptotically flat). In particular, we may always set $N = 1$ after an appropriate rescaling of the time coordinate. Imposing $N = 1$ simplifies Eq. (2.192), which takes the form

$$\begin{aligned}\mathcal{E}^E_{tt}\big|_{N=1} =& -\frac{f}{r^4}\left(r^3 f' + (f-1)r^2 + Q^2\right) - \frac{\alpha Q^2 f}{r^6}\left(rf' - 3f + 9\right)\\ &-\frac{2\alpha^2 Q^2 f}{r^8}\left(5 - 5f + rf'\right) = 0\,.\end{aligned} \tag{2.194}$$

Interestingly enough, the combination $\dfrac{r^2\mathcal{E}^E_{tt}\big|_{N=1}}{f}$ can be easily integrated. One finds that

$$\frac{1}{2}\int \frac{r^2\mathcal{E}^E_{tt}\big|_{N=1}}{f}\mathrm{d}r = r(1-f) + \frac{Q^2}{r} + \frac{3\alpha Q^2}{r^3} - \frac{\alpha Q^2 f}{r^3} + \frac{(1-f)}{r^5}2\alpha^2Q^2 = 2M\,, \tag{2.195}$$

where M is an integration constant that we identify with the mass of the solution. Thus, we have a linear equation for the metric function f whose solution reads

$$f(r) = \frac{r^4\left(r^2 - 2Mr + Q^2\right) + \alpha Q^2(3r^2 + 2\alpha)}{r^6 + \alpha Q^2(r^2 + 2\alpha)}\,. \tag{2.196}$$

This is the expression for f given at Eq. (2.166). The following task is to derive the original gauge field strength F. Using Eq. (2.169) we have that

$$F_{\mu\nu} = \mathcal{Q}_{\mu\nu}{}^{\rho\sigma}\hat{F}_{\rho\sigma} + 4\mathcal{Q}^{\alpha\beta}{}_{\alpha[\mu}\hat{F}_{\nu]\beta} + \mathcal{Q}_{\alpha\beta}{}^{\alpha\beta}\hat{F}_{\mu\nu}\,. \tag{2.197}$$

On the one hand, $\hat{F}$ was already obtained back at Eq. (2.191). On the other hand, we have just derived the expression for the metric functions f and N, so the tensor $\mathcal{Q}_{\mu\nu}{}^{\rho\sigma}$ is also determined as well. Assuming the electric ansatz (2.28), which implies that

$$F = -\Phi'(r)\mathrm{d}t \wedge \mathrm{d}r\,, \tag{2.198}$$

we find, after equating this last expression with Eq. (2.197), a first-order ODE for $\Phi(r)$ in terms of the electric charge Q and the metric functions f and N. Such equation reads

$$\begin{aligned}\Phi'(r) = &-\frac{QN}{r^2} + \frac{\alpha Q}{2r^4}\Big[r\left(3rf'N' + f\left(2rN'' - 8N'\right)\right) + N\left(r^2 f'' - 8rf' + 18f - 18\right)\Big]\\ &-\frac{\alpha^2 Q}{2r^5}\Big[-3rf'N' + f\left(\left(5rf' + 12\right)N' - 2rN''\right) + 2f^2\left(rN'' - 6N'\right)\Big]\\ &-\frac{\alpha^2 Q}{2r^6}\left[N\left(r^2 f'^2 - r^2 f'' + 12rf' + f\left(r^2 f'' - 12rf' - 40\right) + 20f^2 + 20\right) + r^2 f^2\frac{N'^2}{N}\right],\end{aligned} \tag{2.199}$$

where we remark that we have not replaced yet the expressions obtained for N and f. After setting $N = 1$, however, we find that a great simplification takes place and (2.199) boils down to

$$\begin{aligned}\Phi'(r) = &-\frac{Q}{r^2} + \frac{\alpha Q}{2r^4}\left(r^2 f'' - 8rf' + 18f - 18\right)\\ &-\frac{\alpha^2 Q}{2r^6}\Big[\left(20 - r^2 f'' + r^2 f'^2 + 12rf'\right) + f\left(r^2 f'' + 20f - 12rf' - 40\right)\Big].\end{aligned} \tag{2.200}$$

Imposing a vanishing electric potential at infinity, this equation can be integrated to yield

$$\Phi(r) = \frac{Q}{r}\left(1 + \frac{\alpha(1-f)}{r^2}\right)\left(1 + \frac{\alpha(4 - 4f + rf')}{2r^2}\right), \tag{2.201}$$

which is precisely what appears in Eq. (2.166).

3

Duality-invariant extensions of Einstein-Maxwell theory

In the previous chapter we studied an interesting class of higher-order theories of gravity with a non-minimally coupled gauge vector field which satisfies two fairly remarkable properties: amenability to computations and some reasonable physical requirements. This special set of theories, named E(G)QGs, are defined precisely by admitting electrically- or magnetically-charged, static and spherically symmetric solutions characterized by a single metric function $f(r)$ with (at most) second-order equation of motion. However, while theories with such magnetic configurations can be typically obtained by adding higher-derivative monomials R^nF^{2m} to the Einstein-Hilbert action, this is no longer the case for electric solution. Despite that, one can circumvent this difficulty by dualization of theories with magnetic solutions.

Now we will be interested in investigating higher-derivative extensions of Einstein-Maxwell theory which are invariant under duality rotations. Indeed, the set of equations of motion and Bianchi identity of Einstein-Maxwell theory is invariant under the continuous group of SO(2) duality rotations of the field strength into its dual and vice versa, as explained in Section I.4. Nevertheless, such invariance under duality rotations is generically broken after the addition of higher-derivative terms into the action, so demanding a theory with higher-order terms to be invariant under continuous duality rotations constrains drastically the type of terms that may appear in the action. Since the idea that the laws of Nature must be invariant under certain transformations is one of the most fundamental principles of modern physics, this motivates the study of those particular theories whose set of equations of motion and Bianchi identity enjoy invariance under duality rotations.

In the case of purely electromagnetic theories, for which gravity is absent, the existence of duality-invariant higher-derivative extensions of the Maxwell Lagrangian is known [315–317, 322]. However, the characterization of generic effective extensions of Einstein-Maxwell theory, in which one considers the addition of non-minimal couplings between curvature and the gauge field strength, seems not to have been previously addressed in the literature, so in this chapter we will commit ourselves to a thorough characterization of such theories.

In particular, the presence of non-minimal couplings makes it virtually impossible to obtain exactly invariant Lagrangians, so we will begin to approach this problem by assuming a derivative expansion of the action. We will obtain the conditions on the 4-, 6- and 8-derivative Lagrangians that ensure that the theory is a truncation of a duality-preserving one. In addition, we will see that, due to the coupling to gravity, metric field redefinitions acquire a very interesting role in the case of duality-invariant theories. In fact, we will show that, to six-derivatives, one can get rid of all the higher-derivative terms involving field strengths in any duality-preserving theory by performing such redefinitions, and we conjecture the same to be true at all orders. We will furthermore study the charged, static and spherically symmetric black hole solutions of this six-derivative theory.

Despite the apparent impossibility of constructing a higher-order theory with exact electromagnetic duality invariance, we will observe that the task becomes dramatically more manageable if we restrict ourselves to the subset of theories which have at most quadratic dependence on the vector field strength. Remarkably enough, we will be able to obtain a closed form of the action for all such theories, noting that the Maxwell field couples to gravity through a curvature-dependent susceptibility tensor that takes a peculiar form, reminiscent of that of Born-Infeld Lagrangians. We will then particularize to the most simple of these models and study the corresponding static and spherically symmetric black hole solutions.

This chapter is organized as follows. First we obtain the conditions for (up to) eight-derivative theories to be (perturbatively) duality-invariant and then we construct the most general eight-derivative theory coming from a truncation of an exactly duality-invariant one. Afterwards we examine the effect of metric redefinitions and, up to the six-derivative level, we prove that any duality-invariant action can be mapped to a higher-derivative gravity minimally coupled to the Maxwell Lagrangian. Later we study the black hole solutions of this special six-derivative theory. Next we focus on theories whose action is quadratic in the Maxwell field strength and obtain a closed form for all such theories (with non-minimal couplings) which are exactly duality-invariant, studying the subsequent physical properties of black holes in the most tractable of these theories. Finally we conclude with a discussion of our findings and future directions.

3.1 Duality-invariant actions

As announced in the Preamble of the chapter, our goal is the study of higher-order theories whose set of equations of motion and Bianchi identity is invariant under continuous duality rotations. To this aim, in this section we determine the necessary and sufficient conditions for a higher-derivative theory to be invariant under duality rotations.

Let us start by writing a general higher-derivative extension of Einstein-Maxwell theory,

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R - F^2 + \mathcal{L}\left(g^{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \ldots; F_{\mu\nu}, \nabla_\alpha F_{\mu\nu}, \ldots\right)\right] , \quad (3.1)$$

where as usual $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength of the gauge field A_μ and $R_{\mu\nu\rho\sigma}$ is the curvature tensor of the metric $g_{\mu\nu}$. On the other hand, $\mathcal{L}$ represents a general invariant formed out of these quantities and their derivatives, and we will assume it allows for a polynomial expansion in terms with an increasing number of derivatives. The equations of motion coming from the variation of this action read

$$G_{\mu\nu} = 2T_{\mu\nu} - \frac{1}{\sqrt{|g|}}\frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\mu\nu}} , \quad (3.2)$$

$$0 = \nabla_\nu\left(F^{\mu\nu} - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta F_{\mu\nu}}\right) , \quad (3.3)$$

where

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F^2 \,, \tag{3.4}$$

is the Maxwell stress-energy tensor, and

$$\frac{\delta\mathcal{L}}{\delta F_{\mu\nu}} = \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} - \nabla_\alpha \frac{\partial\mathcal{L}}{\partial\nabla_\alpha F_{\mu\nu}} + \nabla_\beta\nabla_\alpha \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\nabla_\beta F_{\mu\nu}} - \dots \,. \tag{3.5}$$

For our purposes, it is convenient to rewrite this system of equations by introducing the dual field strength H as follows

$$G_{\mu\nu} = 2\hat{T}_{\mu\nu} + \mathcal{E}_{\mu\nu} \,, \tag{3.6}$$

$$\star H_{\mu\nu} = -F_{\mu\nu} + \frac{1}{2}\frac{\delta\mathcal{L}}{\delta F^{\mu\nu}} \,, \tag{3.7}$$

$$\mathrm{d}\begin{pmatrix} F \\ H \end{pmatrix} = 0 \,, \tag{3.8}$$

where now

$$\hat{T}_{\mu\nu} = -F_{\langle\mu|\alpha} \star H_{|\nu\rangle}{}^{\alpha} \,, \tag{3.9}$$

$$\mathcal{E}_{\mu\nu} = -\frac{1}{\sqrt{|g|}}\frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\mu\nu}} + F_{\langle\mu|\alpha}\frac{\delta\mathcal{L}}{\delta F^{|\nu\rangle}{}_{\alpha}} \,, \tag{3.10}$$

representing $\langle\mu\nu\rangle$ the symmetric and traceless part of a tensor, this is

$$X_{\langle\mu\nu\rangle} = X_{(\mu\nu)} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}X_{\alpha\beta} \,. \tag{3.11}$$

In addition, the Hodge dual is defined as

$$\star\, H_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}H^{\alpha\beta} \,, \tag{3.12}$$

where the Levi-Civita symbol $\epsilon_{\mu\nu\alpha\beta}$ is such that $\epsilon_{0123} = \sqrt{|g|}$.

Eq. (3.6) is the Einstein equation, Eq.(3.7) is the so-called constitutive relation that defines the dual field strength H in terms of the original one F and Eq. (3.8) includes the Bianchi identities of F and H. In this formulation, F and H are taken as independent fundamental variables, and the equations of motion impose that they are closed 2-forms and related by the constitutive relation (3.7).

Let us now analyze if the equations (3.6), (3.7) and (3.8) have any symmetry. It is clear that the form of the two Bianchi identities (3.8) is preserved if we consider any $\mathrm{GL}(2,\mathbb{R})$ transformation of F and H. In the case without corrections, it can be easily checked, as justified in Section I.4, that the the constitutive relation (3.7) and the Einstein equation are invariant under the SO(2) transformations

$$\begin{pmatrix} F \\ H \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} F' \\ H' \end{pmatrix} \,. \tag{3.13}$$

Once the corrections are taken into consideration, however, the equations are generally not invariant under this rotation, so, our goal is to determine which Lagrangians do give rise to duality-invariant equations of motion.

Let us first start by stating that, if the Lagrangian $\mathcal{L}$ depends non-degenerately on the derivatives of $F_{\mu\nu}$, then the relation $H(F)$ given by (3.7) is differential. This means that the inverse relation $F(H)$ involves integration. Now imagine, for instance, a rotation with an angle $\alpha = \pi/2$. In that case, the new fields F', H' satisfy the relation

$$\star F'_{\mu\nu} = H'_{\mu\nu} - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta F^{\mu\nu}}\bigg|_{F\to H'} . \tag{3.14}$$

Then, if the equations of motion are invariant under duality rotations, this relation should be equivalent to the one obtained by inverting (3.7). However, we see that this is not possible since, as we mentioned, $F(H)$ must involve integration while (3.14) is again differential. The conclusion is that the equations of motion cannot be duality-invariant if the Lagrangian contains derivatives of the field strength, so duality restricts the set of allowed Lagrangians to be of the form

$$\mathcal{L}\left(g^{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \ldots ; F_{\mu\nu}\right) . \tag{3.15}$$

It is important to clarify a subtle point, though. If one assumes a perturbative expansion of the Lagrangian, then one may, in fact, invert (3.7) in a perturbative fashion, obtaining a differential relation for $F(H)$. Thus, one can find theories with a differential relation (3.7) that are invariant under duality rotations in this perturbative sense [535]. Nonetheless, by the argument above, those theories cannot arise from the truncation of a complete theory that is *exactly* invariant. In other words, this means that the summation of the whole perturbative series would not give rise to a well-defined theory. Thus, we will restrict ourselves to theories that depend only algebraically on the field strength $F_{\mu\nu}$. Let us now study which further constraints duality invariance imposes on the Lagrangian.

3.1.1 Invariance of the constitutive relation

Let us focus first on the constitutive relation (3.7). After some algebraic manipulations one can show that the rotated fields F' and H' in (3.13) satisfy the relation

$$\star H'_{\mu\nu} = -F'_{\mu\nu} + \frac{1}{2}\hat{R}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to F'\cos\alpha + H'\sin\alpha} , \tag{3.16}$$

where we have tacitly defined the operator

$$\hat{R} = \cos\alpha + \star\sin\alpha \, . \tag{3.17}$$

Duality invariance requires the transformed fields F' and H' to be also a solution of the original equation (3.7). This will happen if

$$\hat{R}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to F'\cos\alpha + H'\sin\alpha} = \frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to F'} . \tag{3.18}$$

Of course, this equality can never hold off-shell since the left-hand-side depends on H', while the right-hand-side does not. However, we only require that both quantities be equal on-shell, which ensures that F' and H' indeed solve the original equation. Thus, we can conveniently write this consistency equation as

$$\hat{R}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to F'\cos\alpha + \star\left(F' - \frac{1}{2}\frac{\partial\mathcal{L}}{\partial F}\right)\sin\alpha} = \frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to F'} . \tag{3.19}$$

This is a highly nonlinear equation that constrains the form of $\mathcal{L}$. In order to make additional progress, at this point it is convenient to expand the Lagrangian in a derivative expansion, as

$$\mathcal{L} = \ell^2\mathcal{L}_{(4)} + \ell^4\mathcal{L}_{(6)} + \ell^6\mathcal{L}_{(8)} + \ldots\,, \tag{3.20}$$

where ℓ is a length scale and each Lagrangian $\mathcal{L}_{(2n)}$ contains $2n$ derivatives of the fields. Then, we will impose duality invariance order by order, assuming that the full Lagrangian defines an exactly invariant theory. We can solve (3.16) perturbatively in ℓ and we get

$$\begin{aligned}
\star H'_{\mu\nu} = &-F'_{\mu\nu} + \frac{\ell^2}{2}\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F'} \\
&+\ell^4\left[\frac{1}{2}\hat{R}\frac{\partial\mathcal{L}_{(6)}}{\partial F^{\mu\nu}} - \frac{1}{4}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}}\right]\bigg|_{F\to\hat{R}F'} \\
&+\ell^6\Bigg[\frac{1}{2}\hat{R}\frac{\partial\mathcal{L}_{(8)}}{\partial F^{\mu\nu}} - \frac{1}{4}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(6)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}} - \frac{1}{4}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(6)}}{\delta F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}} \\
&+\frac{1}{16}\hat{R}\frac{\partial}{\partial F^{\mu\nu}}\left\{\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\rho\sigma}}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\rho\sigma}\right\}\Bigg]\bigg|_{F\to\hat{R}F'} + \mathcal{O}(\ell^8)\,,
\end{aligned} \tag{3.21}$$

where $s \equiv \sin\alpha$. Note that the operator $\hat{R}$ in the left always acts on the indices $\mu\nu$. Now, from (3.18) we derive the following necessary conditions in order for the theory to be invariant under duality rotations (we remove the F' notation for clarity)

$$\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\mu\nu}} = \hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F}\,, \tag{3.22}$$

$$\frac{\partial\mathcal{L}_{(6)}}{\partial F^{\mu\nu}} = \left[\hat{R}\frac{\partial\mathcal{L}_{(6)}}{\partial F^{\mu\nu}} - \frac{1}{2}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}}\right]\bigg|_{F\to\hat{R}F}\,, \tag{3.23}$$

$$\begin{aligned}
\frac{\partial\mathcal{L}_{(8)}}{\partial F^{\mu\nu}} = &\Bigg[\hat{R}\frac{\partial\mathcal{L}_{(8)}}{\partial F^{\mu\nu}} - \frac{1}{2}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(6)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}} - \frac{1}{2}\hat{R}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(6)}}{\partial F}\right)^{\alpha\beta}\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\mu\nu}} \\
&+\frac{1}{8}\hat{R}\frac{\partial}{\partial F^{\mu\nu}}\left\{\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\rho\sigma}}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\left(s\star\hat{R}\frac{\partial\mathcal{L}_{(4)}}{\partial F}\right)^{\rho\sigma}\right\}\Bigg]\bigg|_{F\to\hat{R}F}\,.
\end{aligned} \tag{3.24}$$

Let us further investigate the implications of these relations for the Lagrangian. First, notice that each Lagrangian is built out of monomials that can be schematically denoted by $F^n\nabla^qR^p$. Clearly, duality rotations do not mix terms with different values of n, q and p, and hence, if the theory preserves duality, the relations above are satisfied by each of these families of monomials independently. Now, let us note that, for every such monomial we have the identity

$$F^{\mu\nu}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}} = n\mathcal{L}\,, \tag{3.25}$$

since they are homogeneous functions of degree n in F. Likewise, we have

$$F^{\mu\nu}\left(\hat{R}\frac{\partial\mathcal{L}}{\partial F^{\mu\nu}}\bigg|_{F\to\hat{R}F}\right) = n\mathcal{L}\Big|_{F\to\hat{R}F}\,. \tag{3.26}$$

Let us then apply these results to the four-derivative case $\mathcal{L}_{(4)}$. Since the equation (3.22) must hold for every type of monomial, we conclude that the four-derivative Lagrangian must satisfy the condition

$$\mathcal{L}_{(4)}(\hat{R}F) = \mathcal{L}_{(4)}(F)\,, \tag{3.27}$$

so $\mathcal{L}_{(4)}$ remains invariant under a rotation of F and $\star F$. Let us now consider the case of $\mathcal{L}_{(6)}$. First, it is convenient to rewrite Eq. (3.23) as follows:

$$\frac{\partial \mathcal{L}_{(6)}}{\partial F^{\mu\nu}} = \left[\hat{R}\frac{\partial \mathcal{L}_{(6)}}{\partial F^{\mu\nu}} - \frac{1}{4}\hat{R}\frac{\partial}{\partial F^{\mu\nu}}\left(\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\right)\right]\Bigg|_{F\to\hat{R}F}. \tag{3.28}$$

Then, proceeding with the same logic as before, we can split this expression into monomials of degree n in F, and contracting with $F^{\mu\nu}$ we find that

$$\mathcal{L}_{(6)}(\hat{R}F) - \mathcal{L}_{(6)}(F) = \frac{1}{4}\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\Bigg|_{F\to\hat{R}F}. \tag{3.29}$$

This tells us that $\mathcal{L}_{(6)}$ does not remains invariant under a rotation of F and $\star F$ since there is an inhomogeneous term associated to $\mathcal{L}_{(4)}$. Clearly, this can be traced back to the fact that duality is non-linearly realized in the Lagrangian formulation. The general solution to this equation can be expressed as

$$\mathcal{L}_{(6)} = \mathcal{L}^{\rm H}_{(6)} + \mathcal{L}^{\rm IH}_{(6)}\,, \tag{3.30}$$

where $\mathcal{L}^{\rm H}_{(6)}$ is the general solution of associated homogeneous equation, and therefore satisfies

$$\mathcal{L}^{\rm H}_{(6)}(\hat{R}F) = \mathcal{L}^{\rm H}_{(6)}(F)\,, \tag{3.31}$$

and $\mathcal{L}^{\rm IH}_{(6)}$ is a particular solution of the full inhomogeneous equation. Let us show that a particular solution is given by

$$\mathcal{L}^{\rm IH}_{(6)} = -\frac{1}{8}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial \mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\,. \tag{3.32}$$

In fact, using the property of $\mathcal{L}_{(4)}$ in (3.22), we have

$$\begin{aligned}\mathcal{L}^{\rm IH}_{(6)}(F) &= -\frac{1}{8}\left(\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\left(\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)_{\alpha\beta}\Bigg|_{F\to\hat{R}F}\\ &= -\frac{1}{8}\left(\hat{R}^2\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\Bigg|_{F\to\hat{R}F},\end{aligned} \tag{3.33}$$

and, on the other hand,

$$\mathcal{L}^{\rm IH}_{(6)}(\hat{R}F) = -\frac{1}{8}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial \mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\Bigg|_{F\to\hat{R}F} = -\frac{1}{8}\left(\hat{R}^{-1}\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\Bigg|_{F\to\hat{R}F}. \tag{3.34}$$

Thus, combining both expressions and using that $\hat{R}^{-1} = \cos\alpha - \star\sin\alpha$ one easily checks that (3.29) is satisfied. Let us finally turn to the case of the eight-derivative Lagrangian.

After making use of the decomposition of $\mathcal{L}_{(6)}$ in (3.30), we can write the equation (3.24) as:

$$\frac{\partial \mathcal{L}_{(8)}}{\partial F^{\mu\nu}} = \Bigg[\hat{R}\frac{\partial \mathcal{L}_{(8)}}{\partial F^{\mu\nu}} - \frac{1}{2}\hat{R}\frac{\partial}{\partial F^{\mu\nu}}\left(\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}^{\mathrm{H}}_{(6)}}{\partial F^{\alpha\beta}}\right) + \frac{1}{8}\hat{R}\frac{\partial}{\partial F^{\mu\nu}}\left\{\frac{\partial^2 \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\rho\sigma}}\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\left(c\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\rho\sigma}\right\}\Bigg]\Bigg|_{F\to\hat{R}F}. \quad (3.35)$$

Splitting this expression into monomials and contracting with $F^{\mu\nu}$ we arrive at the equation

$$\mathcal{L}_{(8)}(\hat{R}F) - \mathcal{L}_{(8)}(F) = \Bigg[\frac{1}{2}\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\frac{\partial \mathcal{L}^{\mathrm{H}}_{(6)}}{\partial F^{\alpha\beta}} - \frac{1}{8}\frac{\partial^2 \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\rho\sigma}}\left(s\star\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\alpha\beta}\left(c\hat{R}\frac{\partial \mathcal{L}_{(4)}}{\partial F}\right)^{\rho\sigma}\Bigg]\Bigg|_{F\to\hat{R}F}. \quad (3.36)$$

The general solution can again be written as

$$\mathcal{L}_{(8)} = \mathcal{L}^{\mathrm{H}}_{(8)} + \mathcal{L}^{\mathrm{IH}}_{(8)}\,, \quad (3.37)$$

where $\mathcal{L}^{\mathrm{H}}_{(8)}$ satisfies

$$\mathcal{L}^{\mathrm{H}}_{(8)}(\hat{R}F) = \mathcal{L}^{\mathrm{H}}_{(8)}(F) \quad (3.38)$$

and $\mathcal{L}^{\mathrm{IH}}_{(8)}$ is a particular solution of the complete inhomogeneous equation. In this case, one can check that a particular solution is given by

$$\mathcal{L}^{\mathrm{IH}}_{(8)} = -\frac{1}{4}\frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial \mathcal{L}^{\mathrm{H}}_{(6)}}{\partial F_{\alpha\beta}} + \frac{1}{32}\frac{\partial^2 \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}\partial F^{\rho\sigma}}\frac{\partial \mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\frac{\partial \mathcal{L}_{(4)}}{\partial F_{\rho\sigma}}\,. \quad (3.39)$$

In this way, the theory is determined up to the eight-derivative level once the set of Lagrangians $\mathcal{L}_{(4)}$, $\mathcal{L}^{\mathrm{H}}_{(6)}$, $\mathcal{L}^{\mathrm{H}}_{(8)}$, is specified, so our final task consist in characterizing these.

In order to characterize the Lagrangians that are invariant under a rotation of F and $\star F$, it is useful to introduce the vector of 2-forms

$$\mathcal{F}^A = \begin{pmatrix} F \\ \star F \end{pmatrix}\,, \quad (3.40)$$

where A is an SO(2) index. The only way of obtaining SO(2) invariant quantities is by considering the contraction $\mathcal{F}^A_{\mu\nu}\mathcal{F}^B_{\alpha\beta}\delta_{AB}$. Note that the contraction with the symplectic matrix σ_{AB} also yields an invariant, but however it is not independent since $\mathcal{F}^A_{\mu\nu}\mathcal{F}^B_{\alpha\beta}\sigma_{AB} = \mathcal{F}^A_{\mu\nu}\star\mathcal{F}^B_{\alpha\beta}\delta_{AB}$, and thus it has the same effect as applying the Hodge dual on the indices $\alpha\beta$. Let us then evaluate this contraction explicitly,

$$\begin{aligned}\mathcal{F}^A_{\mu\nu}\mathcal{F}^{B\,\alpha\beta}\delta_{AB} &= F_{\mu\nu}F^{\alpha\beta} + \star F_{\mu\nu}\star F^{\alpha\beta} \\ &= F_{\mu\nu}F^{\alpha\beta} - 6F^{\rho\sigma}F_{[\rho\sigma}\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu]} = 4T^{[\alpha}{}_{[\mu}\delta^{\beta]}{}_{\nu]}\,,\end{aligned} \quad (3.41)$$

where $T^{\alpha}{}_{\mu}$ is the Maxwell stress tensor as defined in (3.4). Consequently, this result indicates that any SO(2) invariant quantity must depend on $F_{\mu\nu}$ only through the stress tensor $T_{\mu\nu}$. Therefore, we conclude that the homogeneous part of the Lagrangians $\mathcal{L}^{\rm H}_{(n)}$ (including $\mathcal{L}_{(4)}=\mathcal{L}^{\rm H}_{(4)}$) must be of the form

$$\mathcal{L}^{\rm H}_{(n)}=\mathcal{L}^{\rm H}_{(n)}\left(g^{\mu\nu},T_{\mu\nu},R_{\mu\nu\rho\sigma},\nabla_{\alpha}R_{\mu\nu\rho\sigma},\ldots\right)\,. \tag{3.42}$$

This result together with the relations (3.32) and (3.39) fully characterizes the set of theories that have a duality-invariant constitutive relation (3.7) to the eight-derivative level.

3.1.2 Invariance of Einstein's equations

We have just obtained the conditions the Lagrangian must satisfy for the constitutive relation (3.7) to be invariant under a duality rotation. Now the question is whether these necessary conditions ensure the invariance of Einstein's equations as well, so that they are actually sufficient for the existence of duality invariance. For that, we see first in Eq. (3.6) that the energy-momentum tensor $\hat{T}_{\mu\nu}$ (see (3.9)) is exactly invariant under a rotation, so we just have to make sure that the quantity $\mathcal{E}_{\mu\nu}$ — defined in (3.10) — remains invariant. At this point it is convenient to study first those theories which are algebraic in the curvature tensor, since the proof for generic theories with covariant derivatives of the curvature is a direct generalization of that.

We derived back in Eq. (2.12) the Einstein equation for the most general theory of gravity and electromagnetism with algebraic dependence in the curvature and the gauge field strength. Consequently, we have that the tensor $\mathcal{E}_{\mu\nu}$ defined in Eq. (3.10) can be written as

$$\mathcal{E}_{\mu\nu}=-P_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma}-2\nabla^{\sigma}\nabla^{\rho}P_{(\mu|\sigma|\nu)\rho}+\frac{1}{2}g_{\mu\nu}\left(\mathcal{L}-\frac{1}{2}F_{\alpha\beta}\frac{\partial\mathcal{L}}{\partial F_{\alpha\beta}}\right)\,, \tag{3.43}$$

where

$$P_{\mu\nu\rho\sigma}=\frac{\partial\mathcal{L}}{\partial R^{\mu\nu\rho\sigma}}\,. \tag{3.44}$$

Let us expand this in powers of ℓ,

$$\mathcal{E}_{\mu\nu}=\ell^2\mathcal{E}^{(4)}_{\mu\nu}+\ell^4\mathcal{E}^{(6)}_{\mu\nu}+\ell^6\mathcal{E}^{(8)}_{\mu\nu}\ldots\,, \tag{3.45}$$

where every term $\mathcal{E}^{(n)}_{\mu\nu}$ is computed from the corresponding Lagrangian $\mathcal{L}_{(n)}$. Let us examine these terms. The Lagrangian $\mathcal{L}_{(4)}$ only depends on $F_{\mu\nu}$ through the Maxwell stress-energy tensor. Since every monomial $\mathcal{L}_i$ in the Lagrangian satisfies that $F_{\alpha\beta}\frac{\partial\mathcal{L}_i}{\partial F_{\alpha\beta}}\propto\mathcal{L}_i$, by looking at (3.43) we conclude that the tensor $\mathcal{E}^{(4)}_{\mu\nu}$ also depends on $F_{\mu\nu}$ through $T_{\mu\nu}$ only. We can express this fact by writing $\mathcal{E}^{(4)}_{\mu\nu}=\mathcal{E}^{(4)}_{\mu\nu}(T)$. Now, since under a duality transformation, $T_{\mu\nu}$ is invariant up to terms of order ℓ^2, we already conclude that the Einstein's equations are invariant up to terms of order ℓ^4.

Let us see now what happens with the $\mathcal{O}(\ell^4)$ and $\mathcal{O}(\ell^6)$ terms. First, we remind that we can split $\mathcal{L}_{(6)}$ and $\mathcal{L}_{(8)}$ in a homogeneous plus an inhomogeneous part, $\mathcal{L}_{(6)}=\mathcal{L}^{\rm H}_{(6)}+\mathcal{L}^{\rm IH}_{(6)}$ and $\mathcal{L}_{(8)}=\mathcal{L}^{\rm H}_{(8)}+\mathcal{L}^{\rm IH}_{(8)}$, where

$$\mathcal{L}^{\rm IH}_{(6)}=-\frac{1}{8}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\,,\qquad \mathcal{L}^{\rm IH}_{(8)}=-\frac{1}{4}\frac{\partial\mathcal{L}^{\rm H}_{(6)}}{\partial F^{\alpha\beta}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}+\frac{1}{32}\frac{\partial^2\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}F^{\rho\sigma}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}}\,. \tag{3.46}$$

Correspondingly, each $\mathcal{E}^{(n)}_{\mu\nu}$ splits as

$$\mathcal{E}^{(6)}_{\mu\nu} = \mathcal{E}^{\mathrm{H}(6)}_{\mu\nu} + \mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu} \,, \qquad \mathcal{E}^{(8)}_{\mu\nu} = \mathcal{E}^{\mathrm{H}(8)}_{\mu\nu} + \mathcal{E}^{\mathrm{IH}(8)}_{\mu\nu} \,. \tag{3.47}$$

It is useful to rewrite the terms coming from the homogeneous parts, $\mathcal{E}^{\mathrm{H}\,(n)}_{\mu\nu}$, in terms of the exactly-invariant tensor $\hat{T}_{\mu\nu}$. We recall that it is related to the Maxwell stress-energy tensor $T_{\mu\nu}$ by

$$T_{\mu\nu} = \hat{T}_{\mu\nu} + \frac{1}{2} F_{\langle\mu|\alpha} \frac{\partial \mathcal{L}}{\partial F^{|\nu\rangle}{}_{\alpha}} \,. \tag{3.48}$$

Since $\mathcal{L}^{\mathrm{H}}_{(6)}$ and $\mathcal{L}^{\mathrm{H}}_{(8)}$ are built out only of the Maxwell stress-energy tensor $T_{\mu\nu}$, for the same reason as before it follows that $\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}$ and $\mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}$ depend on $F_{\mu\nu}$ only through $T_{\mu\nu}$. Expressing $T_{\mu\nu} = \hat{T}_{\mu\nu} + \delta T_{\mu\nu}$, by expanding around $\hat{T}$ we find:

$$\begin{aligned}
\mathcal{E}^{(4)}_{\mu\nu}(T) = {} & \mathcal{E}^{(4)}_{\mu\nu}(\hat{T}) + \frac{\ell^2}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(\hat{T}) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\sigma}} + \frac{\ell^4}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(\hat{T}) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(6)}}{\partial F^{|\beta\rangle}{}_{\sigma}} \\
& + \frac{\ell^4}{8} \frac{\delta^2 \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta} \delta T_{\rho\sigma}}(\hat{T}) \circ F_{\langle\alpha|\lambda} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\lambda}} \circ F_{\langle\rho|\gamma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\sigma\rangle}{}_{\gamma}} + \mathcal{O}(\ell^6) \,,
\end{aligned} \tag{3.49}$$

$$\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(T) = \mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(\hat{T}) + \frac{\ell^2}{2} \frac{\delta \mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}}{\delta T_{\alpha\beta}}(\hat{T}) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\sigma}} + \mathcal{O}(\ell^4) \,, \tag{3.50}$$

$$\mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(T) = \mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(\hat{T}) + \mathcal{O}(\ell^2) \,, \tag{3.51}$$

where we have defined:

$$\begin{aligned}
\frac{\delta}{\delta \mathcal{B}_{\mu_1 \ldots \mu_p}} \circ \mathcal{C}_{\mu_1 \ldots \mu_p} = {} & \frac{\partial}{\partial \mathcal{B}_{\mu_1 \ldots \mu_p}} \mathcal{C}_{\mu_1 \ldots \mu_p} + \frac{\partial}{\partial \nabla_\nu \mathcal{B}_{\mu_1 \ldots \mu_p}} \nabla_\nu \mathcal{C}_{\mu_1 \ldots \mu_p} \\
& + \frac{\partial}{\partial \nabla_{\nu_1} \nabla_{\nu_2} \mathcal{B}_{\mu_1 \ldots \mu_p}} \nabla_{\nu_1} \nabla_{\nu_2} \mathcal{C}_{\mu_1 \ldots \mu_p} + \ldots \,,
\end{aligned} \tag{3.52}$$

being $\mathcal{B}$ and $\mathcal{C}$ arbitrary tensors. Taking into account that $\hat{T}_{\mu\nu} = T_{\mu\nu} + \mathcal{O}(\ell^2)$, we can express (3.49), (3.50) and (3.51) in the form:

$$\begin{aligned}
\mathcal{E}^{(4)}_{\mu\nu}(T) = {} & \mathcal{E}^{(4)}_{\mu\nu}(\hat{T}) + \frac{\ell^2}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(\hat{T}) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\sigma}} + \frac{\ell^4}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(T) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(6)}}{\partial F^{|\beta\rangle}{}_{\sigma}} \\
& + \frac{\ell^4}{8} \frac{\delta^2 \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta} \delta T_{\rho\sigma}}(T) \circ F_{\langle\alpha|\lambda} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\lambda}} \circ F_{\langle\rho|\gamma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\sigma\rangle}{}_{\gamma}} + \mathcal{O}(\ell^6) \,,
\end{aligned} \tag{3.53}$$

$$\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(T) = \mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(\hat{T}) + \frac{\ell^2}{2} \frac{\delta \mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}}{\delta T_{\alpha\beta}}(T) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\sigma}} + \mathcal{O}(\ell^4) \,, \tag{3.54}$$

$$\mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(T) = \mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(\hat{T}) + \mathcal{O}(\ell^2) \,, \tag{3.55}$$

Let us further rewrite (3.53) as follows,

$$\mathcal{E}^{(4)}_{\mu\nu}(T) = \mathcal{E}^{(4)}_{\mu\nu}(\hat{T}) + \frac{\ell^2}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(T) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\sigma}} + \frac{\ell^4}{2} \frac{\delta \mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}}(T) \circ F_{\langle\alpha|\sigma} \frac{\partial \mathcal{L}_{(6)}}{\partial F^{|\beta\rangle}{}_{\sigma}}$$

$$-\frac{\ell^4}{8}\frac{\delta^2\mathcal{E}^{(4)}_{\mu\nu}}{\delta T_{\alpha\beta}\delta T_{\rho\sigma}}(T)\circ F_{\langle\alpha|\lambda}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{|\beta\rangle}{}_{\lambda}}\circ F_{\langle\rho|\gamma}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{|\sigma\rangle}{}_{\gamma}}+\mathcal{O}(\ell^6)\,, \tag{3.56}$$

where we have replaced $\hat{T}=T-\delta T$ in the $\mathcal{O}(\ell^2)$ term and expanded in δT once again. Now, taking into account the following identity:

$$\frac{\delta}{\delta T_{\alpha\beta}}\circ F_{\langle\alpha|\sigma}\mathcal{A}_{|\beta\rangle}{}^{\sigma}=\frac{1}{2}\frac{\delta}{\delta F_{\alpha\beta}}\circ\mathcal{A}_{\alpha\beta}\,, \tag{3.57}$$

which is valid for any antisymmetric tensor $\mathcal{A}_{\alpha\beta}$, we find that

$$\begin{aligned}\mathcal{E}_{\mu\nu}=&\,\ell^2\mathcal{E}^{(4)}_{\mu\nu}(\hat{T})+\ell^4\left(\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(\hat{T})+\mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu}(T)+\frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\right)+\\ &\ell^6\left(\mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(\hat{T})+\mathcal{E}^{\mathrm{IH}(8)}_{\mu\nu}(T)+\frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(6)}}{\partial F_{\alpha\beta}}-\frac{1}{16}\frac{\delta^2\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}\delta F_{\rho\sigma}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\rho\sigma}}\right.\\ &\left.+\frac{1}{4}\frac{\delta\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\right)+\mathcal{O}(\ell^8)\,,\end{aligned} \tag{3.58}$$

Finally, one can prove the following identities:

$$\mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu}(T)+\frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}=0\,, \tag{3.59}$$

$$\begin{aligned}&\mathcal{E}^{\mathrm{IH}(8)}_{\mu\nu}(T)+\frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(6)}}{\partial F_{\alpha\beta}}-\frac{1}{16}\frac{\delta^2\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}\delta F_{\rho\sigma}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\rho\sigma}}\\ &+\frac{1}{4}\frac{\delta\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}=0\,.\end{aligned} \tag{3.60}$$

In order to show this, it is useful to take into account the following property,

$$\frac{\delta\nabla_{\mu_1}\nabla_{\mu_2}\mathcal{Q}_{\nu_1\ldots\nu_p}}{\delta F_{\alpha\beta}}\circ\mathcal{A}_{\alpha\beta}=\nabla_{\mu_1}\nabla_{\mu_2}\left(\frac{\partial\mathcal{Q}_{\nu_1\ldots\nu_p}}{\partial F_{\alpha\beta}}\mathcal{A}_{\alpha\beta}\right)\,, \tag{3.61}$$

where $\mathcal{Q}_{\nu_1\ldots\nu_p}$ is an arbitrary tensor which depends algebraically on $F_{\mu\nu}$ and where $\mathcal{A}_{\alpha\beta}$ is any antisymmetric tensor. Let us illustrate how to use (3.61) to prove the first identity (3.59). On the one hand, since $\mathcal{L}_{(4)}$ depends algebraically on the curvature, we have, explicitly,

$$\begin{aligned}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}=&-\hat{P}^{(4)\rho\sigma\gamma}_{(\mu}R_{\nu)\rho\sigma\gamma}+\frac{1}{2}g_{\mu\nu}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial}{\partial F_{\alpha\beta}}\left(\mathcal{L}_{(4)}-\frac{1}{2}F_{\rho\sigma}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}}\right)\\ &-2\frac{\delta\nabla^\sigma\nabla^\rho P^{(4)}_{(\mu|\sigma|\nu)\rho}}{\delta F_{\alpha\beta}}\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\\ =&-\hat{P}^{(4)\rho\sigma\gamma}_{(\mu}R_{\nu)\rho\sigma\gamma}-2\nabla^\sigma\nabla^\rho\hat{P}^{(4)}_{(\mu|\sigma|\nu)\rho}\\ &+\frac{1}{2}g_{\mu\nu}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial}{\partial F_{\alpha\beta}}\left(\mathcal{L}_{(4)}-\frac{1}{2}F_{\rho\sigma}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}}\right)\,,\end{aligned} \tag{3.62}$$

where

$$P^{(4)\mu\nu\rho\sigma} = \frac{\partial\mathcal{L}_{(4)}}{\partial R_{\mu\nu\rho\sigma}}\,, \quad \hat{P}^{(4)\mu\nu\rho\sigma} = \frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial P^{(4)\mu\nu\rho\sigma}}{\partial F_{\alpha\beta}}\,. \tag{3.63}$$

After some direct computations, we recognize at the last line of (3.62) the term $-4\mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu}(T)$, so we prove (3.59). Showing that (3.60) holds is a more intricate task, but it can be done by using (3.61) and following a completely analogous procedure to that of the proof of (3.59). Consequently, we have shown that the Einstein's equations can be written as

$$\mathcal{E}_{\mu\nu} = \ell^2\mathcal{E}^{(4)}_{\mu\nu}(\hat{T}) + \ell^4\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}(\hat{T}) + \ell^6\mathcal{E}^{\mathrm{H}(8)}_{\mu\nu}(\hat{T}) + \mathcal{O}(\ell^8)\,, \tag{3.64}$$

so that they are manifestly duality-invariant to order ℓ^6. Thus, the conclusion is that the invariance of the constitutive relation implies the invariance of Einstein's equations, so the conditions found in the previous subsection are necessary and sufficient in order for a theory with algebraic dependence on the Riemann tensor to be duality-invariant.

These results can be generalized to the case of theories that contain covariant derivatives of the curvature. The proof of the invariance of Einstein's equations in that case can be obtained along similar lines, although it is slightly more technical and thus we include it in Appendix 3.

3.2 All duality-invariant theories up to eight derivatives

The goal of this section is to obtain explicitly the Lagrangian of the most general duality-invariant theory up to eight-derivative terms. To this aim, we will use Eqs. (3.32), (3.39) and (3.42). In addition, we will also assume that parity is preserved, so that we will discard parity-breaking operators.

We start by analyzing the fourth-derivative terms. Since all the dependence of $\mathcal{L}_{(4)}$ on $F_{\mu\nu}$ must be through the Maxwell stress-energy tensor $T_{\mu\nu}$, we observe that the most general duality-invariant four-derivative Lagrangian will take the form:

$$\mathcal{L}_{(4)} = \alpha_1 T_{\mu\nu}T^{\mu\nu} + \alpha_2 R^{\mu\nu}T_{\mu\nu} + \alpha_3\mathcal{X}_4 + \alpha_4 R^{\mu\nu}R_{\mu\nu} + \alpha_5 R^2, \tag{3.65}$$

where $\mathcal{X}_4$ denotes the topological Gauss-Bonnet density, given by $\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$. Now we move into the most general six-derivative Lagrangian $\mathcal{L}_{(6)}$. In the previous section we decomposed it into a homogeneous part $\mathcal{L}^{\mathrm{H}}_{(6)}$ which takes the functional form given at (3.42) and an inhomogeneous part $\mathcal{L}^{\mathrm{IH}}_{(6)}$ which is obtained from the four-derivative Lagrangian and whose particular form we rewrite here:

$$\mathcal{L}^{\mathrm{IH}}_{(6)} = -\frac{1}{8}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\,. \tag{3.66}$$

Now, taking into account that

$$\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}} = 4\alpha_1 T^{\mu[\rho}F_{\mu}{}^{\sigma]} + 2\alpha_2\hat{R}^{\mu[\rho}F_{\mu}{}^{\sigma]}\,, \tag{3.67}$$

where $\hat{R}_{\mu\nu} = R_{\langle\mu\nu\rangle}$, we obtain the following expression for $\mathcal{L}^{\mathrm{IH}}_{(6)}$:

$$\mathcal{L}^{\mathrm{IH}}_{(6)} = -2\alpha_1^2 T^{\mu[\rho}F_{\mu}{}^{\sigma]}T_{\nu\rho}F^{\nu}{}_{\sigma} - 2\alpha_1\alpha_2 T^{\mu[\rho}F_{\mu}{}^{\sigma]}\hat{R}_{\nu\rho}F^{\nu}{}_{\sigma} - \frac{1}{2}\alpha_2^2\hat{R}^{\mu[\rho}F_{\mu}{}^{\sigma]}\hat{R}_{\nu\rho}F^{\nu}{}_{\sigma}\,. \tag{3.68}$$

Once we have obtained the inhomogeneous part, now we write down the most general homogeneous Lagrangian $\mathcal{L}^{\rm H}_{(6)}$ that preserves duality and parity. This can be seen to have the following form:

$$\mathcal{L}^{\rm H}_{(6)} = \mathcal{R}^{\mu\nu\alpha\beta}_{(6)} T_{\mu\nu} T_{\alpha\beta} + \mathcal{R}^{\mu\nu}_{(6)} T_{\mu\nu} + \mathcal{R}_{(6)} \, , \tag{3.69}$$

where $\mathcal{R}^{\mu\nu\alpha\beta}_{(6)}$ and $\mathcal{R}^{\mu\nu}_{(6)}$ are tensors formed out of the curvature and $\mathcal{R}_{(6)}$ is the general six-derivative Lagrangian for the metric. After discarding trivial terms and total derivatives, the general form of these quantities reads [500]:

$$\mathcal{R}^{\mu\nu\alpha\beta}_{(6)} = \beta_1 R^{\mu\alpha\nu\beta} + \beta_2 R^{\mu\alpha} g^{\beta\nu} + \beta_3 R g^{\mu\alpha} g^{\beta\nu} \, , \tag{3.70}$$

$$\begin{aligned} \mathcal{R}^{\mu\nu}_{(6)} &= \beta_4 R_{\alpha\beta\gamma}{}^{\mu} R^{\alpha\beta\gamma\nu} + \beta_5 R^{\rho\mu\alpha\nu} R_{\rho\alpha} + \beta_6 R^{\mu\alpha} R_{\alpha}{}^{\nu} + \beta_7 R R^{\mu\nu} \\ &\quad + \beta_8 \nabla^{(\mu}\nabla^{\nu)} R + \beta_9 \nabla^2 R^{\mu\nu} \, , \end{aligned} \tag{3.71}$$

$$\begin{aligned} \mathcal{R}_{(6)} &= \beta_{10} R^{\rho\sigma\mu\nu} R_{\mu\nu}{}^{\lambda\eta} R_{\lambda\eta\rho\sigma} + \beta_{11} R R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \beta_{12} R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + \beta_{13} R^{\mu\nu} R_{\mu\alpha} R_{\nu}{}^{\alpha} \\ &\quad + \beta_{14} R R^{\mu\nu} R_{\mu\nu} + \beta_{15} R^3 + \beta_{16} \nabla^{\sigma} R \nabla_{\sigma} R + \beta_{17} \nabla_{\sigma} R_{\mu\nu} \nabla^{\sigma} R^{\mu\nu} \, . \end{aligned} \tag{3.72}$$

The eight-derivative Lagrangian $\mathcal{L}_{(8)}$ also decomposes in a homogeneous part $\mathcal{L}^{\rm H}_{(8)}$ and an inhomogeneous one $\mathcal{L}^{\rm IH}_{(8)}$. This last piece could be expressed in terms of the lower-derivative Lagrangians $\mathcal{L}_{(4)}$ and $\mathcal{L}^{\rm H}_{(6)}$ as

$$\mathcal{L}^{\rm IH}_{(8)} = -\frac{1}{4} \frac{\partial \mathcal{L}_{(4)}}{\partial F^{\alpha\beta}} \frac{\partial \mathcal{L}^{\rm H}_{(6)}}{\partial F_{\alpha\beta}} + \frac{1}{32} \frac{\partial^2 \mathcal{L}_{(4)}}{\partial F^{\alpha\beta} \partial F^{\rho\sigma}} \frac{\partial \mathcal{L}_{(4)}}{\partial F_{\alpha\beta}} \frac{\partial \mathcal{L}_{(4)}}{\partial F_{\rho\sigma}} \, . \tag{3.73}$$

Taking into account that

$$\frac{\partial \mathcal{L}^{\rm H}_{(6)}}{\partial F_{\rho\sigma}} = 4 T_{\alpha\beta} \hat{\mathcal{R}}^{\alpha\beta\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} + 2 \hat{\mathcal{R}}^{\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} \, , \tag{3.74}$$

$$\begin{aligned} \frac{\partial^2 \mathcal{L}_{(4)}}{\partial F_{\alpha\beta} F^{\rho\sigma}} &= 4\alpha_1 F^{\alpha}{}_{[\rho} F^{\beta}{}_{\sigma]} - 2\alpha_1 F_{\rho\sigma} F^{\alpha\beta} + 8\alpha_1 T_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} + \alpha_1 F^2 \delta_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} \\ &\quad + 2\alpha_2 \hat{R}_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} \, , \end{aligned} \tag{3.75}$$

where a hat over $\mathcal{R}^{\alpha\beta\nu\rho}_{(6)}$ means that we take its traceless part over each pair of indices, we conclude that (3.39) is given by:

$$\begin{aligned} \mathcal{L}^{\rm IH}_{(8)} &= -4\alpha_1 T_{\alpha\beta} \hat{\mathcal{R}}^{\alpha\beta\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} T_{\mu\rho} F^{\mu}{}_{\sigma} - 2\alpha_2 T_{\alpha\beta} \hat{\mathcal{R}}^{\alpha\beta\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} \hat{R}_{\mu\rho} F^{\mu}{}_{\sigma} - 2\alpha_1 \hat{\mathcal{R}}^{\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} T_{\mu\rho} F^{\mu}{}_{\sigma} \\ &\quad - \alpha_2 \hat{\mathcal{R}}^{\nu[\rho}_{(6)} F_{\nu}{}^{\sigma]} \hat{R}_{\mu\rho} F^{\mu}{}_{\sigma} + 2\alpha_1^3 T_{\mu\alpha} F^{\mu}{}_{\beta} T^{\nu\rho} F_{\nu}{}^{\sigma} F^{\alpha}{}_{[\rho} F^{\beta}{}_{\sigma]} - \alpha_1^3 T_{\mu\alpha} F^{\mu}{}_{\beta} T^{\nu\rho} F_{\nu}{}^{\sigma} F_{\rho\sigma} F^{\alpha\beta} \\ &\quad + 4\alpha_1^3 T_{\mu\alpha} F^{\mu}{}_{\beta} T^{\nu\rho} F_{\nu}{}^{\sigma} T_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} + 1/2\alpha_1^3 F^2 T^{\mu[\rho} F_{\mu}{}^{\sigma]} T_{\nu\rho} F^{\nu}{}_{\sigma} \\ &\quad + 2\alpha_1^2 \alpha_2 T^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} F^{\alpha}{}_{[\rho} F^{\beta}{}_{\sigma]} - \alpha_1^2 \alpha_2 T^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} F_{\rho\sigma} F^{\alpha\beta} \\ &\quad + 4\alpha_1^2 \alpha_2 T^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} T_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} + \alpha_1^2 \alpha_2 T_{\mu\alpha} F^{\mu}{}_{\beta} T^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}^{[\alpha}_{[\rho} \delta_{\sigma]}{}^{\beta]} \\ &\quad + 1/2\alpha_1^2 \alpha_2 F^2 T^{\mu[\rho} F_{\mu}{}^{\sigma]} \hat{R}_{\nu\rho} F^{\nu}{}_{\sigma} + 1/2\alpha_1 \alpha_2^2 \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} \hat{R}^{\nu\rho} F_{\nu}{}^{\sigma} F^{\alpha}{}_{[\rho} F^{\beta}{}_{\sigma]} \\ &\quad - 1/4\alpha_1 \alpha_2^2 \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} \hat{R}^{\nu\rho} F_{\nu}{}^{\sigma} F_{\rho\sigma} F^{\alpha\beta} + \alpha_1 \alpha_2^2 \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} \hat{R}^{\nu\rho} F_{\nu}{}^{\sigma} T_{[\rho}{}^{[\alpha} \delta_{\sigma]}{}^{\beta]} \\ &\quad + \alpha_1 \alpha_2^2 T^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} \hat{R}^{[\alpha}_{[\rho} \delta_{\sigma]}{}^{\beta]} + 1/8\alpha_1 \alpha_2^2 F^2 \hat{R}^{\mu[\rho} F_{\mu}{}^{\sigma]} \hat{R}_{\nu\rho} F^{\nu}{}_{\sigma} \end{aligned}$$

$$+\, 1/4\alpha_2^3 \hat{R}_{\mu\alpha} F^{\mu}{}_{\beta} \hat{R}^{\nu\rho} F_{\nu}{}^{\sigma} \hat{R}^{[\alpha}_{[\rho} \delta_{\sigma]}{}^{\beta]}\,. \tag{3.76}$$

Finally, the homogeneous part $\mathcal{L}^{\rm H}_{(8)}$ must take the form (3.42). Hence, it can be written as

$$\begin{aligned}\mathcal{L}^{\rm H}_{(8)} &= \gamma_1 (T_{\mu\nu}T^{\mu\nu})^2 + \gamma_2 T_{\mu}{}^{\nu} T_{\nu}{}^{\alpha} T_{\alpha}{}^{\beta} T_{\beta}{}^{\mu} + \mathcal{R}^{\mu\nu\rho\sigma\alpha\beta}_{(8)} T_{\mu\nu} T_{\rho\sigma} T_{\alpha\beta} \\ &\quad + \mathcal{R}^{\mu\nu\rho\sigma}_{(8)} T_{\mu\nu} T_{\rho\sigma} + \mathcal{R}^{\mu\nu}_{(8)} T_{\mu\nu} + \mathcal{R}_{(8)}\,.\end{aligned} \tag{3.77}$$

A list with all the possible tensors appearing in these expressions can be obtained from [500]. We get the following general expressions:

$$\mathcal{R}_{(8)\mu\nu\rho\sigma\alpha\beta} = \gamma_3 R_{\mu\rho\nu\alpha} g_{\sigma\beta} + \gamma_4 R_{\mu\nu} g_{\rho\alpha} g_{\sigma\beta} + \gamma_5 R_{\mu\rho} g_{\alpha\nu} g_{\sigma\beta}\,, \tag{3.78}$$

$$\begin{aligned}\mathcal{R}_{(8)\mu\nu\rho\sigma} &= \gamma_6 R_{\mu\nu} R_{\rho\sigma} + \gamma_7 R_{\mu\rho} R_{\nu\sigma} + \gamma_8 R R^{\mu\rho\nu\sigma} + \gamma_9 R^{\alpha\beta}{}_{\mu\rho} R_{\alpha\beta\nu\sigma} + \gamma_{10} R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} R_{\alpha\beta\rho\sigma} \\ &\quad + \gamma_{11} R^{\alpha}{}_{\mu}{}^{\beta}{}_{\rho} R_{\alpha\nu\beta\sigma} + \gamma_{12} \nabla_\mu \nabla_\rho R g_{\nu\sigma} + \gamma_{13} \nabla^2 R_{\mu\rho} g_{\nu\sigma} + \gamma_{14} R R_{\mu\rho} g_{\nu\sigma} \\ &\quad + \gamma_{15} R^{\alpha}{}_{\mu} R_{\alpha\rho} g_{\nu\sigma} + \gamma_{16} R^{\alpha\beta} R_{\alpha\mu\beta\rho} g_{\nu\sigma} + \gamma_{17} R^{\alpha\beta\gamma}{}_{\mu} R_{\alpha\beta\gamma\rho} g_{\nu\sigma} + \gamma_{18} \nabla^2 R g_{\mu\rho} g_{\nu\sigma} \\ &\quad + \gamma_{19} R^2 g_{\mu\rho} g_{\nu\sigma} + \gamma_{20} R^{\alpha\beta} R_{\alpha\beta} g_{\mu\rho} g_{\nu\sigma} + \gamma_{21} R^{\alpha\beta\lambda\eta} R_{\alpha\beta\lambda\eta} g_{\mu\rho} g_{\nu\sigma}\,,\end{aligned} \tag{3.79}$$

$$\begin{aligned}\mathcal{R}_{(8)\mu\nu} &= \gamma_{22} \nabla_\mu \nabla_\nu \nabla^2 R + \gamma_{23} \nabla^2 \nabla^2 R_{\mu\nu} + \gamma_{24} R \nabla_\mu \nabla_\nu R + \gamma_{25} R_{\mu\nu} \nabla^2 R \\ &\quad + \gamma_{26} R^{\alpha\beta} \nabla_\alpha \nabla_\beta R_{\mu\nu} + \gamma_{27} R \nabla^2 R_{\mu\nu} + \gamma_{28} R^{\alpha\beta} \nabla_\mu \nabla_\nu R_{\alpha\beta} + \gamma_{29} R^{\alpha\beta} \nabla_\nu \nabla_\beta R_{\mu\alpha} \\ &\quad + \gamma_{30} \nabla^\alpha \nabla_\mu R R_{\alpha\nu} + \gamma_{31} \nabla^\rho \nabla^\sigma R R_{\rho\mu\sigma\nu} + \gamma_{32} \nabla^2 R^{\rho\sigma} R_{\rho\mu\sigma\nu} \\ &\quad + \gamma_{33} \nabla^\rho \nabla_\mu R^{\alpha\beta} R_{\alpha\rho\beta\nu} + \gamma_{34} \nabla^\beta \nabla^\rho R^{\alpha}{}_{\mu} R_{\alpha\beta\rho\nu} + \gamma_{35} R^{\alpha\beta\rho\sigma} \nabla_\mu \nabla_\nu R_{\alpha\beta\rho\sigma} \\ &\quad + \gamma_{36} \nabla_\mu R^{\alpha\beta} \nabla_\nu R_{\alpha\beta} + \gamma_{37} \nabla_\alpha R_{\beta\mu} \nabla^\alpha R^{\beta}{}_{\nu} + \gamma_{38} \nabla^\beta R^{\alpha}{}_{\mu} \nabla_\alpha R_{\beta\nu} \\ &\quad + \gamma_{39} \nabla^\rho R_{\alpha\beta} \nabla_\nu R_{\alpha\rho\beta\mu} + \gamma_{40} \nabla_\mu R \nabla_\nu R + \gamma_{41} \nabla_\alpha R \nabla_\nu R_{\mu}{}^{\alpha} + \gamma_{42} \nabla_\alpha R \nabla^\alpha R_{\mu\nu} \\ &\quad + \gamma_{43} \nabla_\mu R^{\alpha\beta} \nabla_\beta R_{\alpha\nu} + \gamma_{44} \nabla^\sigma R^{\alpha\beta} \nabla_\sigma R_{\alpha\mu\beta\nu} + \gamma_{45} \nabla_\mu R^{\alpha\beta\sigma\lambda} \nabla_\nu R_{\alpha\beta\sigma\lambda} \\ &\quad + \gamma_{46} \nabla^\lambda R^{\alpha\beta\sigma}{}_{\mu} \nabla_\lambda R_{\alpha\beta\sigma\nu} + \gamma_{47} R^2 R_{\mu\nu} + \gamma_{48} R R^{\alpha}{}_{\mu} R_{\alpha\nu} + \gamma_{49} R_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} \\ &\quad + \gamma_{50} R^{\alpha\beta} R_{\alpha\mu} R_{\beta\nu} + \gamma_{51} R R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \gamma_{52} R_{\alpha}{}^{\sigma} R_{\beta\mu\sigma\nu} + \gamma_{53} R^{\alpha\beta} R^{\sigma}{}_{\mu} R_{\alpha\sigma\beta\nu} \\ &\quad + \gamma_{54} R R^{\alpha\beta\sigma}{}_{\mu} R_{\alpha\beta\sigma\nu} + \gamma_{55} R_{\mu\nu} R^{\alpha\beta\lambda\sigma} R_{\alpha\beta\lambda\sigma} + \gamma_{56} R^{\alpha}{}_{\mu} R^{\beta\rho\sigma}{}_{\alpha} R_{\beta\rho\sigma\nu} \\ &\quad + \gamma_{57} R^{\alpha\beta} R^{\rho\sigma}{}_{\alpha\mu} R_{\rho\sigma\beta\nu} + \gamma_{58} R^{\alpha\beta} R_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} R_{\rho\mu\sigma\nu} + \gamma_{59} R^{\alpha\beta} R^{\rho}{}_{\alpha}{}^{\sigma}{}_{\mu} R_{\rho\beta\sigma\nu} \\ &\quad + \gamma_{60} R^{\alpha\beta\rho\sigma} R_{\alpha\beta}{}^{\lambda}{}_{\mu} R_{\rho\sigma\lambda\nu} + \gamma_{61} R^{\alpha\beta\rho\sigma} R_{\alpha}{}^{\lambda}{}_{\rho\mu} R_{\beta\lambda\sigma\nu} + \gamma_{62} R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho}{}^{\lambda} R_{\sigma\mu\lambda\nu}\,.\end{aligned} \tag{3.80}$$

On the other hand, the list of the eight-derivative curvature invariants appearing in $\mathcal{R}_{(8)}$ can also be checked in [500].

Let us note that here and also in the case of $\mathcal{L}^{\rm H}_{(6)}$ we are including a set of densities that spans the set of all duality-invariant Lagrangians, but they may be not linearly independent. Even though we are removing redundant terms in the $\mathcal{R}^{\mu_1\mu_2\cdots}_{(n)}$ tensors following [500], the densities formed by contracting these tensors with $T_{\mu\nu}$ may still be linearly dependent or be related up to total derivatives. The determination of a linearly independent set of generating densities for the Lagrangians $\mathcal{L}^{\rm H}_{(6)}$ and $\mathcal{L}^{\rm H}_{(8)}$ may be carried out elsewhere.

3.3 Field redefinitions

In the previous section we obtained the most general duality-invariant action up to the eight derivative level. This can be understood as the truncation of an exactly duality-invariant theory that necessarily contains an infinite tower of higher-derivative terms. But at the same time, it can be interpreted as the effective field theory of some underlying UV-complete theory that respects electromagnetic duality. From this point of view, it is very natural to consider field redefinitions of the metric and vector fields, since these correspond simply to different choices of renormalization schemes of the hypothetical quantum theory, leaving the physics invariant. One can of course redefine the variables of any given theory, and if the original theory possess a symmetry, so must the transformed one. However, if the change of variables is not invariant under that symmetry, then the new action will not be manifestly symmetric. The goal of this section is to investigate this question in the case of the duality-invariant theories under consideration. We would like to find field redefinitions that map these theories into other of the same class, with the final aim of removing as many parameters as we can from the Lagrangian.

Let us start by writing down our duality-invariant action up to the six-derivative level:

$$I=\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left\{R-F^2+\ell^2\mathcal{L}_{(4)}+\ell^4\left(\mathcal{L}^{\mathrm{H}}_{(6)}-\frac{1}{8}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\alpha\beta}}\right)+\mathcal{O}(\ell^6)\right\}\,, \tag{3.81}$$

where we recall that $\mathcal{L}_{(4)}$ reads

$$\mathcal{L}_{(4)}=\alpha_1 T_{\mu\nu}T^{\mu\nu}+\alpha_2 T_{\mu\nu}R^{\mu\nu}+\alpha_3\mathcal{X}_4+\alpha_4 R_{\mu\nu}R^{\mu\nu}+\alpha_5 R^2\,, \tag{3.82}$$

and $\mathcal{L}^{\mathrm{H}}_{(6)}$ is given by (3.69). Let us then consider a redefinition of the metric of the form

$$g_{\mu\nu}\to g_{\mu\nu}+\ell^2 h_{\mu\nu}\,, \tag{3.83}$$

where $h_{\mu\nu}$ is some symmetric 2-derivative tensor. Performing such field redefinition, expanding in powers of ℓ^2, and neglecting total derivatives, the action I undergoes the transformation

$$\begin{aligned}I'=I+\frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\Bigg\{&-\ell^2 h^{\mu\nu}(G_{\mu\nu}-2T_{\mu\nu})+\ell^4\Bigg[\frac{1}{8}\left(2h_{\alpha\beta}h^{\alpha\beta}-h^2\right)(R-F^2)\\
&\left(h^{\mu\alpha}h^{\nu}{}_{\alpha}-\frac{1}{2}hh^{\mu\nu}\right)(G_{\mu\nu}-2T_{\mu\nu})-h^{\mu\nu}h^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}-\frac{1}{4}\nabla_\mu h_{\alpha\beta}\nabla^\mu h^{\alpha\beta}+\frac{1}{4}\nabla_\mu h\nabla^\mu h\\
&+\frac{1}{2}\nabla_\mu h^{\alpha\beta}\nabla_\beta h_\alpha{}^{\mu}-\frac{1}{2}\nabla_\mu h^{\mu\alpha}\nabla_\alpha h-h^{\mu\nu}\Bigg(R^{\lambda}{}_{\mu}\frac{\partial\mathcal{L}_{(4)}}{\partial R^{\nu\lambda}}+\frac{1}{2}g_{\mu\nu}\nabla^\alpha\nabla^\beta\frac{\partial\mathcal{L}_{(4)}}{\partial R^{\alpha\beta}}\\
&-\nabla^\alpha\nabla_\mu\frac{\partial\mathcal{L}_{(4)}}{\partial R^{\nu\alpha}}+\frac{1}{2}\nabla^2\frac{\partial\mathcal{L}_{(4)}}{\partial R^{\mu\nu}}-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{(4)}+F_{\mu\alpha}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\nu}{}_{\alpha}}\Bigg)\Bigg]\Bigg\}\,.\end{aligned} \tag{3.84}$$

Then, the idea is to choose a tensor $h_{\mu\nu}$ which simplifies the Lagrangian. Note that, in the four-derivative Lagrangian, the redefinition introduces terms proportional to $T_{\mu\nu}$, allowing one to remove all the terms depending on the Maxwell stress tensor. This is achieved by the redefinition

$$h_{\mu\nu}=-\frac{\alpha_1}{2}T_{\mu\nu}-\frac{\alpha_1+2\alpha_2}{4}\hat{R}_{\mu\nu}+\sigma R g_{\mu\nu}\,, \tag{3.85}$$

where $\hat{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ and σ is a free parameter. This has the following effect on $\mathcal{L}_{(4)}$:

$$\mathcal{L}'_{(4)} = \alpha_3\mathcal{X}_4 + \alpha'_4 R_{\mu\nu}R^{\mu\nu} + \alpha'_5 R^2\,, \tag{3.86}$$

where

$$\alpha'_4 = \alpha_4 + \frac{\alpha_1}{4} + \frac{\alpha_2}{2}\,, \qquad \alpha'_5 = \alpha_5 - \frac{\alpha_1}{16} - \frac{\alpha_2}{8} + \sigma\,. \tag{3.87}$$

Therefore, we have removed all the terms containing field strengths in the four-derivative Lagrangian, and hence, the new theory is obviously duality-invariant at that order. However, this redefinition also affects the 6- and higher-derivative Lagrangians, so we must investigate if duality is also preserved at those higher orders. Let us note that, in this new frame, the Maxwell stress tensor $T_{\mu\nu}$ is invariant to order ℓ^2 under a duality rotation due to the absence of field strengths in $\mathcal{L}_{(4)}$. Thus, it follows that the redefinition (3.83) is invariant to order ℓ^4, and hence once would expect the transformed theory to be also duality-invariant at that order. We recall that the presence of terms with $T_{\mu\nu}$ in $\mathcal{L}_{(4)}$ induces inhomogenous terms in $\mathcal{L}_{(6)}$ that are required by duality. Now, since the redefinition removes the terms with $T_{\mu\nu}$ in $\mathcal{L}_{(4)}$, our intuition is that it should also remove the inhomogeneous terms associated with these, since otherwise duality symmetry would be broken. In fact, this is almost exactly what happens.

After a somewhat lengthy computation, we arrive at the following expression for the transformed action up to $\mathcal{O}(\ell^4)$:

$$\begin{aligned} I' =& \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\Bigg\{R - F^2 + \ell^2\left(\alpha_3\mathcal{X}_4 + \alpha'_4 R_{\mu\nu}R^{\mu\nu} + \alpha'_5 R^2\right) + \\ & \ell^4\left(\left(\mathcal{L}^{\mathrm{H}}_{(6)}\right)' + \mathcal{O}\left(\nabla T\right) - \frac{\alpha_1^2}{64}\mathcal{E}_{\mu\nu}\mathcal{E}^{\langle\mu\nu\rangle}F^2 - \frac{\alpha_1^2}{16}\mathcal{E}^{\langle\mu\nu\rangle}\mathcal{E}^{\langle\alpha\beta\rangle}F_{\alpha\mu}F_{\beta\nu}\right) + \mathcal{O}(\ell^6)\Bigg\}\,, \end{aligned} \tag{3.88}$$

where $\mathcal{E}_{\mu\nu} = G_{\mu\nu} - 2T_{\mu\nu}$ are the zeroth order equations of motion. Here, $\left(\mathcal{L}^{\mathrm{H}}_{(6)}\right)'$ is the homogeneous duality-invariant Lagrangian (3.69) with renormalized couplings, while $\mathcal{O}\left(\nabla T\right)$ represents new terms that, besides depending on the curvature and the Maxwell stress tensor, also contain derivatives of the latter. These terms were not included in the original action because, as we argued, they cannot arise in the truncation of a theory that is *exactly* invariant under duality rotations. However, the field redefinitions we are considering are defined only perturbatively and they generically introduce this type of terms, which indeed preserve duality in a perturbative sense. In any case, as we show below, one can easily get rid of them. Thus, the only problematic terms are those that depend explicitly on the field strength. Apparently, these terms break duality invariance, but a closer look reveals that this is not so. Indeed, since they are proportional to the square of the zeroth-order equations of motion, this implies that, on-shell, their contribution is of order ℓ^6. Hence the equations of the new theory are actually invariant under duality rotations at order ℓ^4. Moreover, we can simply remove those terms from the action by performing an additional redefinition of the metric,

$$g_{\mu\nu} \to g_{\mu\nu} + \ell^4 h^{(4)}_{\mu\nu}\,. \tag{3.89}$$

This yields,

$$I'' = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\Bigg\{R - F^2 + \ell^2\left(\alpha_3\mathcal{X}_4 + \alpha'_4 R_{\mu\nu}R^{\mu\nu} + \alpha'_5 R^2\right) + \tag{3.90}$$

$$\ell^4 \left(\left(\mathcal{L}^{\rm H}_{(6)} \right)' + \mathcal{O}\left(\nabla T\right) - \frac{\alpha_1^2}{64}\mathcal{E}_{\mu\nu}\mathcal{E}^{\langle\mu\nu\rangle}F^2 - \frac{\alpha_1^2}{16}\mathcal{E}^{\langle\mu\nu\rangle}\mathcal{E}^{\langle\alpha\beta\rangle}F_{\alpha\mu}F_{\beta\nu} - \mathcal{E}^{\mu\nu}h^{(4)}_{\mu\nu} \right) + \mathcal{O}(\ell^6) \Bigg\} ,$$

so that, by choosing,

$$h^{(4)}_{\mu\nu} = -\frac{\alpha_1^2}{64}\mathcal{E}_{\langle\mu\nu\rangle}F^2 - \frac{\alpha_1^2}{16}\mathcal{E}^{\langle\alpha\beta\rangle}F_{\alpha\langle\mu|}F_{\beta|\nu\rangle} , \tag{3.91}$$

we cancel those terms. Note that, again, this redefinition is of order ℓ^6 on-shell, and hence it is trivially invariant at order ℓ^4. In fact, it makes sense that duality invariance is preserved only on-shell, since this is a symmetry of the equations of motion, not of the action.

We can now perform additional $\mathcal{O}(\ell^4)$ redefinitions in order to simplify the six-derivative Lagrangian. Since these introduce terms of the form $G^{\mu\nu}h^{(4)}_{\mu\nu} - 2T^{\mu\nu}h^{(4)}_{\mu\nu}$, this means that we can remove all the terms depending on the stress-energy tensor $T_{\mu\nu}$, or more precisely, we can simply perform the replacement $T_{\mu\nu} \to G_{\mu\nu}/2$. Note that this works, too, for the terms that contain derivatives of $T_{\mu\nu}$, since, by integration by parts, it is always possible to remove the derivatives from one stress-energy tensor, and then the replacement above can be applied. Since all pure-metric higher-derivative terms were already included in the original action, these redefinitions have the only effect of renormalizing their couplings while removing the dependence on $T_{\mu\nu}$ — and hence, $F_{\mu\nu}$ — in the six-derivative Lagrangian.

Thus, we have reached a quite remarkable result: to the six-derivative level, the most general duality-invariant extension of Einstein-Maxwell theory is equivalent to the most general higher-order gravity minimally coupled to Maxwell theory. Explicitly, the action reads[1]

$$\begin{aligned} I''' =& \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\Bigg\{ R - F^2 + \ell^2\left(\alpha_3\mathcal{X}_4 + \alpha_4' R_{\mu\nu}R^{\mu\nu} + \alpha_5' R^2\right) + \\ & \ell^4 \Big(\beta_1 R_{\mu\ \nu}^{\ \rho\ \sigma} R_{\rho\ \sigma}^{\ \alpha\ \beta} R_{\alpha\ \beta}^{\ \mu\ \nu} + \beta_2 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + \beta_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\alpha}R^{\sigma\alpha} \\ & + \beta_4 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + \beta_5 R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + \beta_6 R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} + \beta_7 R_{\mu\nu}R^{\mu\nu}R \\ & + \beta_8 R^3 + \beta_9 \nabla_\sigma R_{\mu\nu}\nabla^\sigma R^{\mu\nu} + \beta_{10}\nabla_\mu R \nabla^\mu R \Big) + \mathcal{O}(\ell^6) \Bigg\} , \end{aligned} \tag{3.92}$$

where we are including all the six-derivative Riemann invariants modulo total derivatives [500]. However, we can further decrease the number of terms in this action. To begin with, not all the cubic invariants are independent, since they satisfy two constraints [500], and this allows us to set, for instance $\beta_1 = \beta_3 = 0$. On the other hand, there are residual redefinitions of the metric that cancel some of these curvature terms without introducing field strengths. In the four-derivative Lagrangian, we recall that α_5' is given by (3.87), where σ is arbitrary. Thus, we are free to choose $\sigma = -\alpha_5 + \frac{\alpha_1}{16} + \frac{\alpha_2}{8}$, so that $\alpha_5' = 0$. In the case of the six-derivative terms, notice that a redefinition of the form $g_{\mu\nu} \to g_{\mu\nu}(1 + \ell^4 h^{(4)})$ adds the term $\ell^4 h^{(4)} R$ to the Lagrangian. Hence, we can cancel all the terms that contain at least one Ricci scalar, and thus we can set $\beta_4 = \beta_7 = \beta_8 = \beta_{10} = 0$. Finally, since we can transform all Ricci tensors into Maxwell stress-energy tensors via metric redefinitions (up to terms that involve Ricci scalars, and that hence can be removed), we have the map

$$R_{\mu}{}^{\nu}R_{\nu}{}^{\rho}R_{\rho}{}^{\mu} \to 8T_{\mu}{}^{\nu}T_{\nu}{}^{\rho}T_{\rho}{}^{\mu} = 0 . \tag{3.93}$$

[1]Note that the numbering in the β_i couplings is different from the one used in Section 3.2.

Therefore, we can also set $\beta_6 = 0$. In sum, the action is simplified to[2]

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\Bigg\{R - F^2 + \ell^2\left(\alpha_1\mathcal{X}_4 + \alpha_2 R_{\mu\nu}R^{\mu\nu}\right) + \ell^4\left(\beta_1 R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu} + \beta_2 R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + \beta_3\nabla_\sigma R_{\mu\nu}\nabla^\sigma R^{\mu\nu}\right) + \mathcal{O}(\ell^6)\Bigg\}, \quad (3.94)$$

and it only contains five independent operators, of which one is topological. Thus, duality invariance together with field redefinitions is a powerful tool that removes most of the higher-order terms one can include in the action.

One may wonder if this result extends to even higher-derivative terms. As we have discussed, at order n, the Lagrangian of a duality-invariant theory can be decomposed as $\mathcal{L}_{(2n)} = \mathcal{L}^{\mathrm{H}}_{(2n)} + \mathcal{L}^{\mathrm{IH}}_{(2n)}$, where $\mathcal{L}^{\mathrm{H}}_{(2n)}$ contains new independent terms that depend on $F_{\mu\nu}$ only through $T_{\mu\nu}$, and $\mathcal{L}^{\mathrm{IH}}_{(2n)}$ is determined by the lower-order Lagrangians. Now, if up to order $n-1$ we have been able to cancel all the higher-order terms containing $F_{\mu\nu}$ via field redefinitions, we expect that those redefinitions also cancel $\mathcal{L}^{\mathrm{IH}}_{(2n)}$ up to terms which are duality-invariant at that order, and that hence depend on $T_{\mu\nu}$. This is precisely what happened with $\mathcal{L}^{\mathrm{IH}}_{(6)}$ when we performed the redefinition that cancels the F-dependent terms in $\mathcal{L}_{(4)}$. If this is the case, then the corresponding transformed Lagrangian $\mathcal{L}'_{(2n)}$ will depend on $F_{\mu\nu}$ only through $T_{\mu\nu}$ and therefore there is an additional metric redefinition of $2n-2$ derivatives that maps that Lagrangian into a pure gravity one. By induction, one would conclude that this process can be carried out to all orders. As we have seen, things are not so simple, since in this process of field redefinitions duality is only preserved on-shell, which adds some additional complications to the argument. Still, the evidence gathered so far makes us confident to propose the following

Conjecture 3.1. *Any duality-invariant theory that allows for a derivative expansion around Einstein-Maxwell theory is perturbatively equivalent to Maxwell theory minimally coupled to a higher-derivative gravity at any order in the derivative expansion.*

As a further support of this conjecture, it would be interesting to carry out the explicit computation for the eight-derivative terms, although this is a highly challenging task that entails the computation of the third variation of the Einstein-Maxwell action as well as the second variation of the Lagrangian $\mathcal{L}_{(4)}$. Interestingly, if the conjecture holds, then it means that, when coupled to Einstein gravity, non-linear duality-invariant electromagnetic theories such as Born-Infeld theory are in fact perturbatively equivalent to Maxwell theory coupled to higher-derivative gravity. Another interesting question is whether one can find a fully non-perturbative equivalence.

Let us close this section with a few additional comments. In our approach to field redefinitions we have not included a cosmological constant since it introduces a new length scale besides ℓ. This makes the computations more involved since now a redefinition of order n affects linearly the Lagrangians of order n and $n+1$ and new scales $\Lambda\ell^{2n}$ are generated. However, upon the assumption that $\Lambda\ell^2 << 1$ we believe our qualitative results for the asymptotically flat case can be applied as well — see Chapter 1. Finally, we have been able to map any duality-invariant action to a theory that only contains metric

[2] With respect to (3.92), we are relabeling the couplings as $\alpha_3 \to \alpha_1$, $\alpha'_4 \to \alpha_2$, $\beta_2 \to \beta_1$, $\beta_5 \to \beta_2$ and $\beta_9 \to \beta_3$.

higher-derivative terms, so one may wonder if one can make an analogous transformation to a frame in which the theory takes the form of Einstein gravity coupled to non-linear electrodynamics. However, this is not the case since terms that depend explicitly on the Riemann curvature such as $R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$ or $R^{\mu\nu\alpha\beta}T_{\mu\alpha}T_{\nu\beta}$ cannot be mapped into terms containing only field strengths. Therefore, the action (3.94) represents possibly the simplest form for a general six-derivative duality-invariant theory.

3.4 Black holes

As a final application of our results, in this section we study the spherically symmetric black hole solutions of (3.94), which, as we have shown, is equivalent to the most general duality-invariant theory to the six-derivative level. A general static and spherically symmetric ansatz can be written as

$$\mathrm{d}s^2 = -N(r)^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{3.95}$$

$$A = A_t(r)\mathrm{d}t - P\cos\theta\mathrm{d}\phi\,, \tag{3.96}$$

where $N(r)$, $f(r)$ and $A_t(r)$ are functions and P is a constant. The field strength F then reads

$$F = -A_t'(r)\mathrm{d}t\wedge\mathrm{d}r + P\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \tag{3.97}$$

and the Maxwell equation, which does not receive any corrections, reads simply $\mathrm{d}\star F = 0$. The magnetic part of F automatically satisfies this equation, while A_t satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2A_t'}{N}\right) = 0\,, \quad \Rightarrow \quad A_t' = -\frac{NQ}{r^2}\,, \tag{3.98}$$

where Q is an integration constant. This fully characterizes the field strength in terms of the parameters Q and P, which are the electric and magnetic charges, defined as

$$Q = \frac{1}{4\pi}\int_\Sigma \star F\,, \qquad P = \frac{1}{4\pi}\int_\Sigma F\,, \tag{3.99}$$

where Σ is any surface that encloses $r = 0$. Finally, Einstein's equations can be easily solved if one assumes a perturbative expansion of the functions $N(r)$ and $f(r)$ as

$$N(r) = \sum_{n=0}^{\infty} N_n(r)\ell^{2n}\,, \qquad f(r) = \sum_{n=0}^{\infty} f_n(r)\ell^{2n}\,. \tag{3.100}$$

In this way, at each order the functions $\{f_n, N_n\}$ with $n > 0$ satisfy the inhomogeneous linearized Einstein's equations, whose resolution is straightforward. The integration constants in this process are fixed so that $N(r\to\infty) = 1$ and $f(r\to\infty) = 1 - 2M/r + \ldots$. Then, the result reads

$$\begin{aligned} N(r)^2 = 1 &+ \frac{2\ell^2\mathcal{Q}^2\alpha_2}{r^4} + \ell^4\left(\mathcal{Q}^2\alpha_2^2\left(\frac{128M}{r^7} - \frac{80\mathcal{Q}^2}{r^8} - \frac{48}{r^6}\right) + \mathcal{Q}^2\beta_2\left(\frac{32M}{r^7} - \frac{60\mathcal{Q}^2}{r^8}\right)\right.\\ &\left.+12\beta_1\left(\frac{416M\mathcal{Q}^2}{7r^7} - \frac{51\mathcal{Q}^4}{r^8} - \frac{18M^2}{r^6}\right) + \mathcal{Q}^2\beta_3\left(\frac{192M}{r^7} - \frac{150\mathcal{Q}^2}{r^8} - \frac{48}{r^6}\right)\right), \end{aligned} \tag{3.101}$$

$$
\begin{aligned}
f(r) = 1 &- \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2} + \ell^2 \mathcal{Q}^2 \alpha_2 \left(-\frac{12\mathcal{Q}^2}{5r^6} + \frac{6M}{r^5} - \frac{4}{r^4} \right) \\
&+ \ell^4 \Bigg[\alpha_2^2 \mathcal{Q}^2 \left(\frac{1408\mathcal{Q}^4}{15r^{10}} - \frac{351M\mathcal{Q}^2}{r^9} + \frac{320M^2 + \frac{1192\mathcal{Q}^2}{7}}{r^8} - \frac{304M}{r^7} + \frac{72}{r^6} \right) \\
&+ \beta_1 \left(\frac{1724\mathcal{Q}^6}{3r^{10}} - \frac{1884M\mathcal{Q}^4}{r^9} + \frac{\frac{11064M^2\mathcal{Q}^2}{7} + 672\mathcal{Q}^4}{r^8} - \frac{8\left(49M^3 + 96M\mathcal{Q}^2\right)}{r^7} \right. \\
&\left. + \frac{216M^2}{r^6} \right) + \beta_2 \mathcal{Q}^2 \left(\frac{521\mathcal{Q}^4}{9r^{10}} - \frac{158M\mathcal{Q}^2}{r^9} + \frac{72M^2 + 68\mathcal{Q}^2}{r^8} - \frac{40M}{r^7} \right) \\
&+ \beta_3 \mathcal{Q}^2 \left(\frac{1472\mathcal{Q}^4}{9r^{10}} - \frac{566M\mathcal{Q}^2}{r^9} + \frac{464M^2 + \frac{1752\mathcal{Q}^2}{7}}{r^8} - \frac{384M}{r^7} + \frac{72}{r^6} \right) \Bigg] ,
\end{aligned}
\tag{3.102}
$$

where

$$
\mathcal{Q} = \sqrt{Q^2 + P^2} \, . \tag{3.103}
$$

This solution obviously reduces to the (dyonic) Reissner-Nordström solution when all the couplings are set to zero. Note also that in the case of vanishing charge the only non-trivial correction is the one associated to β_1, since the rest of the interactions involve Ricci curvature. Let us now study some properties of this solution. When the charge-to-mass ratio is small enough, it represents a black hole, whose horizon r_+ corresponds to the largest root of the equation $f(r_+) = 0$. Since near extremality the zeroth-order solution has a double root, one has to be careful when studying the solutions to the corrected equation. Thus, let us first consider the case in which we are far from extremality, meaning that $0 < M^2 - \mathcal{Q}^2 >> \ell^2$. Note that, if $\mathcal{Q} >> \ell$, this condition still allows us to get relatively close to extremality, in the sense that we can have $M^2 - \mathcal{Q}^2 << \mathcal{Q}^2$, but not *too* close. In this regime the horizon radius receives corrections of order ℓ^2, and it is not difficult to solve the equation $f(r_+) = 0$ perturbatively in ℓ in order to get

$$
\begin{aligned}
r_+ = &M(1+\zeta) + \frac{\ell^2\left(1 + 3\zeta - 4\zeta^2\right)\alpha_2}{5M\zeta(1+\zeta)^2} - \frac{\ell^4(1-\zeta)^2\left(21 + 147\zeta + 773\zeta^2 + 1984\zeta^3\right)\alpha_2^2}{1050M^3\zeta^3(1+\zeta)^5} \\
&+ \frac{2\ell^4\left(4 - 39\zeta + 336\zeta^2 - 511\zeta^3\right)\beta_1}{21M^3\zeta(1+\zeta)^5} + \frac{\ell^4(1-\zeta)(-1+7\zeta)(-1+13\zeta)\beta_2}{18M^3\zeta(1+\zeta)^5} \\
&+ \frac{\ell^4(1-\zeta)(5 + \zeta(35 + 464\zeta))\beta_3}{63M^3\zeta(1+\zeta)^5} + \mathcal{O}(\ell^6) \, ,
\end{aligned}
\tag{3.104}
$$

where ζ is the "extremality parameter"

$$
\zeta = \sqrt{1 - \frac{\mathcal{Q}^2}{M^2}} \, , \tag{3.105}
$$

which ranges from 1 in the uncharged case to (near) 0 at extremality. Note that the corrections seem to diverge when $\zeta \to 0$, but this is only because the assumption that the corrections are of order ℓ^2 is no longer correct. Indeed, when $M^2 - \mathcal{Q}^2 \sim \ell^2$ one can see that the leading correction to r_+ is of order ℓ rather than ℓ^2. Thus, the expression above is reliable for $\zeta \gtrsim \ell/M$, which can in fact be very small.

Let us then study what happens exactly at extremality. This is achieved when r_+ is a double root of f, and hence we also have the condition $f'(r_+) = 0$. We find that the radius and mass at extremality read

$$r_+^{\rm ext} = \mathcal{Q} - \frac{3\ell^4(4\beta_1+\beta_2)}{2\mathcal{Q}^3} + \mathcal{O}(\ell^6)\,, \tag{3.106}$$

$$M^{\rm ext} = \mathcal{Q} - \frac{\alpha_2\ell^2}{5\mathcal{Q}} - \frac{\ell^4(3\alpha_2^2+48\beta_1+7\beta_2+10\beta_3)}{126\mathcal{Q}^3} + \mathcal{O}(\ell^6)\,. \tag{3.107}$$

3.4.1 Black hole thermodynamics

Once we have found perturbatively the most general static and spherically solution to the theory given by (3.94), our next objective is to study the thermodynamics of the corresponding black hole solutions. To this aim, we are interested in several physical magnitudes, namely: the black hole mass M, its temperature T, its entropy S, and the electric and "magnetic" potentials at the horizon, $\Phi(r_+)$ and $\Psi(r_+)$ respectively[3]. For convenience, we shall express all these quantities in terms of r_+ and $\mathcal{Q}$.

We begin by obtaining the black hole mass as a function $M = M(r_+, \mathcal{Q})$. This can be done by imposing the condition $f(r_+) = 0$, and one gets

$$\begin{aligned}M =& \frac{r_+}{2} + \frac{\mathcal{Q}^2}{2r_+} + \frac{\mathcal{Q}^2(3\mathcal{Q}^2-5r_+^2)\alpha_2\ell^2}{10r_+^5} + \frac{\alpha_2^2\mathcal{Q}^4\ell^4}{84r_+^9}(7\mathcal{Q}^2-9r_+^2)\\ &+\frac{\beta_1\ell^4}{42r_+^9}(-445\mathcal{Q}^6+909\mathcal{Q}^4r_+^2-585\mathcal{Q}^2r_+^4+105r_+^6) - \frac{\mathcal{Q}^2\beta_2\ell^4}{18r_+^9}(28\mathcal{Q}^4-45\mathcal{Q}^2r_+^2+18r_+^4)\\ &-\frac{\mathcal{Q}^2\beta_3\ell^4}{126r_+^9}(217\mathcal{Q}^4-459\mathcal{Q}^2r_+^2+252r_+^4)+\mathcal{O}(\ell^6)\,.\end{aligned} \tag{3.108}$$

The black hole temperature is given by the general formula

$$T = \frac{f'(r_+)N(r_+)}{4\pi}\,, \tag{3.109}$$

and when expressed in terms of r_+ and $\mathcal{Q}$ yields the following result,

$$\begin{aligned}T =& \frac{r_+^2-\mathcal{Q}^2}{4\pi r_+^3} + \frac{\mathcal{Q}^2(r_+^2-\mathcal{Q}^2)\alpha_2\ell^2}{4\pi r_+^7} - \frac{\mathcal{Q}^4\alpha_2^2\ell^4}{8\pi r_+^{11}}(r_+^2-\mathcal{Q}^2)\\ &+\frac{3\beta_1\ell^4}{28\pi r_+^{11}}(109\mathcal{Q}^6-147\mathcal{Q}^4r_+^2+73\mathcal{Q}^2r_+^2-7r_+^6)+\frac{\beta_2\mathcal{Q}^2\ell^4}{4\pi r_+^{11}}(7\mathcal{Q}^4-6\mathcal{Q}^2r_+^2+2r_+^4)\\ &+\frac{\beta_3\mathcal{Q}^2\ell^4}{4\pi r_+^{11}}(7\mathcal{Q}^4-11\mathcal{Q}^2r_+^2+4r_+^4)+\mathcal{O}(\ell^6)\,.\end{aligned} \tag{3.110}$$

Our next goal is the computation of the black hole entropy. According to the Iyer-Wald prescription [366, 370], the entropy S of a black hole arising as a solution of a theory with Lagrangian density $\mathcal{L}$ is given by

$$S = -2\pi\int_\Sigma \mathrm{vol}_\Sigma \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{3.111}$$

[3] It is also possible to compute the on-shell action, which can be seen to be invariant under rotations of the electric and magnetic charges when appropriate boundary terms are included [536].

where Σ is the bifurcation surface of the horizon and vol_Σ its (induced) volume form. Being the same formula as in (I.69), $\epsilon_{\mu\nu}$ denotes the binormal to the Σ, which is nothing else than the components of the volume form in the orthogonal space to the horizon, and

$$\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}}=\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}-\nabla_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha R_{\mu\nu\rho\sigma}}+\ldots \tag{3.112}$$

is the functional derivative of the Lagrangian with respect to the Riemann tensor. From the spherical symmetry of the black hole horizon, it follows that we just have to work out the component $\dfrac{\delta\mathcal{L}}{\delta R_{trtr}}$, which reads:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta R_{trtr}}\bigg|_\Sigma&=-\frac{1}{32\pi}-\frac{\alpha_1\ell^2}{8\pi r_+^2}+\frac{\alpha_2\ell^2\mathcal{Q}^2}{8\pi r_+^4}+\frac{\alpha_1\alpha_2\ell^4\mathcal{Q}^2}{4\pi r_+^6}+\frac{\alpha_2^2\ell^4\mathcal{Q}^2}{4\pi r_+^8}(-3\mathcal{Q}^2+2r_+^2)\\&+\frac{\beta_1\ell^4}{\pi r_+^8}\left(-\frac{1252\mathcal{Q}^4}{112}+\frac{519\mathcal{Q}^2r_+^2}{56}-\frac{33r_+^4}{16}\right)+\frac{\beta_2\ell^4\mathcal{Q}^2}{8\pi r_+^8}\left(-\frac{51}{4}\mathcal{Q}^2+5r_+^2\right)\\&+\frac{\beta_3\ell^4\mathcal{Q}^2}{16\pi r_+^8}(-31\mathcal{Q}^2+32r_+^2)\,.\end{aligned} \tag{3.113}$$

After these computations, direct application of the Iyer-Wald formula (3.111) yields

$$\begin{aligned}S=&\pi r_+^2+4\pi\alpha_1\ell^2-\frac{2\pi\mathcal{Q}^2\alpha_2\ell^2}{r_+^2}+\frac{12\pi\beta_1\ell^4}{r_+^6}(r_+^2-2\mathcal{Q}^2)^2+\frac{\pi\mathcal{Q}^2\beta_2\ell^4}{r_+^6}(7\mathcal{Q}^2-4r_+^2)\\&+\frac{8\pi\mathcal{Q}^2\beta_3\ell^4}{r_+^6}(\mathcal{Q}^2-r_+^2)+\mathcal{O}(\ell^6)\,.\end{aligned} \tag{3.114}$$

While the Gauss-Bonnet term, being topological, does not affect at all to the equations of motion of the theory, we observe that it does contribute to the corresponding black hole entropy by introducing a constant term which, in turn, does not have any influence on the first law of black hole thermodynamics.

Let us finally compute the electrostatic and magnetic potentials at the horizon, $\Phi(r_+)$ and $\Psi(r_+)$. These are defined as

$$F=\mathrm{d}(\Phi(r)\mathrm{d}t)-\star\mathrm{d}(\Psi(r)\mathrm{d}t)\,, \tag{3.115}$$

and comparing to (3.97), we have that

$$\Phi(r)=Q\chi(r)\,,\quad \Psi(r)=P\chi(r)\,,\quad \text{where}\quad \chi'(r)=\frac{N(r)}{r^2}\,. \tag{3.116}$$

Imposing both $\Phi(r)$ and $\Psi(r)$ to vanish at infinity, and using the perturbative expression for $N(r)$ found in (3.101), we encounter

$$\begin{aligned}\chi(r)=&\frac{1}{r}+\frac{\mathcal{Q}^2\alpha_2\ell^2}{5r^5}+\frac{\mathcal{Q}^2\alpha_2^2\ell^4}{14r^9}(16(7M-3r)r-63\mathcal{Q}^2)+\frac{2\mathcal{Q}^2\beta_2\ell^4}{3r^9}(3Mr-5\mathcal{Q}^2)\\&-\frac{2\beta_1\ell^4}{7r^9}(119\mathcal{Q}^4-156M\mathcal{Q}^2r+54M^2r^2)-\frac{\mathcal{Q}^2\beta_3\ell^4}{21r^9}(175\mathcal{Q}^2+36r(2r-7M)))+\mathcal{O}(\ell^6)\,.\end{aligned} \tag{3.117}$$

Evaluation at the black hole horizon yields:

$$\chi(r_+)=\frac{1}{r_+}+\frac{\mathcal{Q}^2\alpha_2\ell^2}{5r_+^5}-\frac{\mathcal{Q}^2\alpha_2^2\ell^4}{14r_+^9}(7\mathcal{Q}^2-8r_+^2)-\frac{\beta_1\ell^4}{7r_+^9}(109\mathcal{Q}^4-102\mathcal{Q}^2r_+^2+27r_+^4)$$

$$+\frac{\mathcal{Q}^2\beta_2\ell^4}{3r_+^9}(3r_+^2-7\mathcal{Q}^2)+\frac{\mathcal{Q}^2\beta_3\ell^4}{21r_+^9}(-49\mathcal{Q}^2+54r_+^2)+\mathcal{O}(\ell^6)\,. \tag{3.118}$$

Making use of (3.108), (3.114), (3.110) and (3.118) we observe that the following identities hold:

$$\frac{\partial M}{\partial r_+}=T\frac{\partial S}{\partial r_+}\,,\quad \frac{1}{\mathcal{Q}}\frac{\partial M}{\partial \mathcal{Q}}=\frac{T}{\mathcal{Q}}\frac{\partial S}{\partial \mathcal{Q}}+\chi(r_+)\,. \tag{3.119}$$

From this, it immediately follows that the first law of black hole thermodynamics,

$$\mathrm{d}M=T\mathrm{d}S+\Phi(r_+)\mathrm{d}Q+\Psi(r_+)\mathrm{d}P\,, \tag{3.120}$$

is identically satisfied.

3.4.2 Constraints from the Weak Gravity Conjecture

The string Swampland Program [112] aims at finding universal features of low-energy effective theories with a consistent UV completion, so that one can discard those that do not satisfy those properties. In this respect, one of the most successful proposals is that of the Weak Gravity Conjecture (WGC) [416], which has recently motivated the study of higher-derivative corrections to charged extremal black holes [4,125,126,474,475,528–530,532,533].

Corrections to the charge-to-mass ratio

Let us briefly review how the WGC can be used to constrain effective gravitational theories. A heuristic form of this conjecture states that all black holes, including extremal ones, should be able to decay. In order for (near-) extremal black holes to decay, it follows that there must exist a particle with a charge-to-mass ratio larger than the one of a extremal black hole — otherwise the black hole cannot be discharged due to the extremality bound in GR: $M\geq|\mathcal{Q}|$. However, in the presence of higher-derivative corrections, the extremal charge-to-mass ratio is not a constant but a function of the charge, and hence this statement depends on the size of the black hole. To see this, let us consider an extremal black hole with charge $\mathcal{Q}$ and mass $M_{\rm ext}(\mathcal{Q})$, and let us assume that it discharges by emitting a particle with charge q and mass m. The resulting black hole will have mass and charge $M'=M_{\rm ext}(\mathcal{Q})-m$, $\mathcal{Q}'=\mathcal{Q}-q$, but, on account of the extremality bound, the mass must satisfy $M'\geq M_{\rm ext}(\mathcal{Q}')$. Then, assuming that $q<<\mathcal{Q}$, one can see that this condition is satisfied only if

$$\frac{m}{q}\leq\frac{\mathrm{d}M_{\rm ext}}{\mathrm{d}\mathcal{Q}}\,. \tag{3.121}$$

This provides the bound on the particle's charge-to-mass ratio in order for the black hole to discharge. For large black holes $\frac{\mathrm{d}M_{\rm ext}}{\mathrm{d}\mathcal{Q}}=1$ and hence it is enough to have a particle with $q/m\geq 1$. However, if $\frac{\mathrm{d}M_{\rm ext}}{\mathrm{d}\mathcal{Q}}$ decreases as the black hole discharges, the bound (3.121) may be violated at certain point, and hence the evaporation is obstructed. A simple way to avoid this problem consists in demanding $\frac{\mathrm{d}M_{\rm ext}}{\mathrm{d}\mathcal{Q}}$ to grow as $\mathcal{Q}$ decreases, or in other words,

$$\frac{\mathrm{d}^2M_{\rm ext}}{\mathrm{d}\mathcal{Q}^2}\leq 0\,. \tag{3.122}$$

A more robust argument can be obtained from a slightly different form of the WGC which states that the decay of a black hole into a set of smaller black holes should be possible in terms of energy and charge conservation [474]. This is like saying that smaller black holes play the role of the particle hypothesized by the WGC. Let us suppose that the initial black hole is extremal (or arbitrarily close to extremality), so that it has mass $M_{\rm ext}(\mathcal{Q})$, while the final black holes are not necessarily extremal and have charges $\mathcal{Q}_1$, $\mathcal{Q}_2$ with $\mathcal{Q}_1+\mathcal{Q}_2=\mathcal{Q}$, and masses M_1, M_2. A necessary condition in order for this process to be admissible is that $M_{\rm ext}(\mathcal{Q}_1+\mathcal{Q}_2)\geq M_1+M_2$, which, upon use of the extremality bound $M_i\geq M_{\rm ext}(\mathcal{Q}_i)$, yields the following constraint

$$M_{\rm ext}(\mathcal{Q}_1+\mathcal{Q}_2)\geq M_{\rm ext}(\mathcal{Q}_1)+M_{\rm ext}(\mathcal{Q}_2)\,. \tag{3.123}$$

Thus, the higher-derivative corrections to $M_{\rm ext}$ must be such that this condition is satisfied.

When applied to our result (3.107), either condition (3.122) or (3.123) yields equivalent constraints. For instance, from (3.122) we get

$$-\frac{2\alpha_2\ell^2}{5\mathcal{Q}^3}-\frac{2\ell^4(3\alpha_2^2+48\beta_1+7\beta_2+10\beta_3)}{21\mathcal{Q}^5}+\mathcal{O}(\ell^6)\leq 0\,. \tag{3.124}$$

Obviously, in the regime where $\mathcal{Q}>>\ell$ the first term is dominant and this implies that

$$\alpha_2\geq 0\,. \tag{3.125}$$

Now, the second term only becomes relevant when $\mathcal{Q}\sim\ell$, but this point marks the limit of applicability of the perturbative expansion, so it is not clear what constraint one should impose on the additional couplings. Still, one may argue that, if the coefficient of $1/\mathcal{Q}^5$ is positive, then the bound may be eventually violated. In order to avoid this problem, it seems reasonable to impose as well the condition

$$3\alpha_2^2+48\beta_1+7\beta_2+10\beta_3\geq 0\,. \tag{3.126}$$

In this way, we guarantee that the mild form of the WGC is satisfied.

Positivity of the corrections to the entropy

In another vein, it has been argued [528] that negative corrections to the mass of extremal black holes are in correlation with positive corrections to the black hole entropy. Although such connection has been proven in [531], the relation is not complete. Strictly speaking, only the corrections to the near-extremal entropy are related to the corrections to the extremal mass, while the corrections to the extremal entropy are independent, as explained in [125]. Likewise, the corrections for neutral BHs are also independent. On the other hand, the relation proven in [531] only applies for the leading order corrections, but it will probably not hold for the subleading ones. This motivates us to study the range of values of the couplings of the theory defined in (3.94) for which the corrections to the entropy are non-negative. We will demand that, for any charge and mass, the corrections of each order are non-negative independently, which is the strongest condition one may impose.

We shall study the corrections in the non-extremal and extremal regimes. We begin by considering this latter case, since it is simpler. In fact, by replacing (3.106) in (3.114) we find the following remarkably compact expression for the extremal entropy $S_{\rm ext}$:

$$S_{\rm ext}=\pi\mathcal{Q}^2+2\pi(2\alpha_1-\alpha_2)\ell^2+\mathcal{O}(\ell^6)\,, \tag{3.127}$$

which contains no ℓ^4 corrections. These extremal corrections are non-negative whenever

$$\alpha_1 \geq \frac{\alpha_2}{2}\,. \tag{3.128}$$

Although the Gauss-Bonnet term is topological, we observe that demanding the corrections to extremal entropy to be non-negative imposes a bound on α_1 — in particular, it cannot vanish if $\alpha_2 > 0$. This condition did not appear when studying corrections to the extremal charge-to-mass ratio, since, as we anticipated, the corrections to the extremal black hole entropy are unrelated to those of the extremal charge-to-mass ratio [125].

Let us now study the entropy of non-extremal black holes. For that, we need to express first the black hole entropy in terms of the mass and charge, $S = S(M, \mathcal{Q})$. Note that back in (3.114) we wrote the black hole entropy as a function of r_+ and $\mathcal{Q}$, but we must bear in mind that the truly thermodynamic variables in which to express the entropy are the mass and charge. This is an important issue since the corrections at fixed r_+ are not the same as those at fixed M, the reason for this being that for a fixed mass M the horizon radius r_+ is altered after the inclusion of corrections, and vice-versa. In terms of the extremality parameter $\zeta = \sqrt{1-\frac{\mathcal{Q}^2}{M^2}}$, we find that the entropy $S_{\rm ne}$ in the non-extremal regime reads

$$\begin{aligned} S_{\rm ne} = {} & \pi M^2(1+\zeta)^2 + 4\pi\ell^2\alpha_1 + \frac{2\pi\ell^2\alpha_2(1-\zeta)^2}{5\zeta(1+\zeta)} + \frac{a\pi\ell^4(\zeta-1)^2(8\zeta+1)}{63M^2\zeta(1+\zeta)^4} \\ & + \frac{8\beta_1\pi\ell^4(3+4\zeta)\zeta}{7M^2(1+\zeta)^4} + \frac{\pi(1-\zeta)^2\ell^4\alpha_2^2}{525M^2\zeta^3(1+\zeta)^4}(-21-126\zeta-185\zeta^2+32\zeta^3) + \mathcal{O}(\ell^6)\,, \end{aligned} \tag{3.129}$$

where we have defined $a = 48\beta_1 + 7\beta_2 + 10\beta_3$. We recall that this expression is valid for $\zeta \gtrsim \ell/M$, which in practice can be very small. Regarding the $\mathcal{O}(\ell^2)$ corrections, we see that the term with α_2 is dominant for small ζ, and therefore positivity demands that

$$\alpha_2 \geq 0\,, \tag{3.130}$$

which is the same condition as the one from the corrections to the extremality bound. On the other hand, (3.128) ensures that the $\mathcal{O}(\ell^2)$ corrections are positive for any other value of ζ. If we now demand that the $\mathcal{O}(\ell^4)$ corrections be non-negative independently, we see that the following constraints are obtained,

$$\alpha_2 = 0\,, \quad 48\beta_1 + 7\beta_2 + 10\beta_3 \geq 0\,, \quad \beta_1 \geq 0\,. \tag{3.131}$$

The first two are obtained by examining the corrections for small ζ while the bound on β_1 is obtained in the opposite limit $\zeta = 1$. On the one hand, note that we recover (3.126) with $\alpha_2 = 0$. On the other hand, we observe that these constraints are quite strong since they imply the vanishing of α_2, so one may wonder if imposing the non-negativity of entropy corrections at each order is a well-motivated bound.

Indeed, one must bear in mind that imposing the corrections to the entropy to be positive (or, analogously, requiring the corrections to the extremal mass to be negative) is not equivalent to demanding the coefficients at every order of the expansion to be positive, being the latter a much stronger condition. In fact, the condition $\alpha_2 = 0$ would imply in the context of string theory that the metric does not receive any correction at all at first order in α', but it is known that such corrections at first order in α' do take place[4] [125]. This shows that the WGC may be applied just to the leading-order corrections, not order by order in the expansion.

3.5 Quadratic theories. Exact duality invariance with non-minimal couplings

As we have seen, any duality-invariant modification of Einstein-Maxwell theory requires the introduction of an infinite tower of higher-derivative terms, due to the non-linearity of this symmetry. Finding these terms becomes very hard and there seems to be no simple way of obtaining a formula for the n-th order density. Thus, here we focus on a subset of these theories that have a simpler form: those which are quadratic in the Maxwell field strength. First, we will obtain a necessary and sufficient condition for such linear theories to be exactly duality-invariant and, afterwards, we will present the explicit form of all of them.

Let us consider the action

$$I = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{|g|}\left[R - \chi^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right], \tag{3.132}$$

where $\chi^{\mu\nu\rho\sigma}$ is a tensor built out of the metric, the Riemann tensor and covariant derivatives of it. Such tensor $\chi^{\mu\nu\rho\sigma}$ will be called susceptibility tensor since the relation between the dual field H and the original field strength F will be seen to be linear and governed by such $\chi^{\mu\nu\rho\sigma}$. Without loss of generality, we can assume that it has the symmetries

$$\chi^{\mu\nu\rho\sigma} = -\chi^{\nu\mu\rho\sigma} = -\chi^{\mu\nu\sigma\rho} = \chi^{\rho\sigma\mu\nu}\,. \tag{3.133}$$

Then, we are going to show that the equations of motion of this theory are invariant under duality rotations if and only if

$$(\star\chi)_{\mu\nu}{}^{\alpha\beta}(\star\chi)_{\alpha\beta}{}^{\rho\sigma} = -\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,, \tag{3.134}$$

where

$$(\star\chi)_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\lambda\gamma}\chi^{\lambda\gamma\alpha\beta}\,. \tag{3.135}$$

Let us prove that (3.134) is indeed a necessary condition for duality invariance. For that, we start with the constitutive relation, which can be written as

$$\star H_{\mu\nu} = -\chi_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}\,. \tag{3.136}$$

If we consider the SO(2) transformation which sends $H \to -F$ and $F \to H$, duality invariance then requires:

$$\star H_{\mu\nu} = -\chi_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}\,, \quad \star F_{\mu\nu} = \chi_{\mu\nu}{}^{\alpha\beta}H_{\alpha\beta}\,. \tag{3.137}$$

[4] Although the solution presented in Reference [125] contains more fields, at zeroth order in α' the only active ones are the metric and an Abelian gauge field. Moreover, it can be checked that the subsequent 4-dimensional Einstein equation (in the Einstein frame) at first order in α' is not affected by the new fields that become active at this order, so we conclude that the presence of leading-order corrections in the metric must be due to the existence of non-trivial operators in the action at first order in α'.

Applying the star operator in both sides,

$$H_{\mu\nu} = (\star\chi)_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}\,, \quad F_{\mu\nu} = -(\star\chi)_{\mu\nu}{}^{\alpha\beta}H_{\alpha\beta}\,. \tag{3.138}$$

Substituting the second equation into the first one,

$$H_{\mu\nu} = -(\star\chi)_{\mu\nu}{}^{\alpha\beta}(\star\chi)_{\alpha\beta}{}^{\rho\sigma}H_{\rho\sigma}\,, \tag{3.139}$$

so (3.134) must hold. In order to prove sufficiency, we must ensure that the constitutive relation and the Einstein's equations remain invariant. The constitutive relation is easily seen to be invariant, since the equation (3.136) together with its inverse can be written in a manifestly duality-invariant way as follows:

$$\sigma^A{}_B\mathcal{F}^B_{\mu\nu} = (\star\chi)_{\mu\nu}{}^{\alpha\beta}\mathcal{F}^A_{\alpha\beta} \tag{3.140}$$

where $\mathcal{F}^A$ is the vector of 2-forms

$$\mathcal{F}^A = \begin{pmatrix} F \\ H \end{pmatrix} \tag{3.141}$$

and $\sigma^A{}_B$ is the symplectic matrix

$$\sigma^A{}_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.142}$$

The SO(2) invariance of this equation follows from that of $\sigma^A{}_B$, and also note that the equation is consistent because both operators $\sigma^A{}_B$ and $(\star\chi)_{\mu\nu}{}^{\alpha\beta}$ satisfy that their square is minus the identity.

Let us now show that the Einstein's equations are invariant as well. Adapting the results of Appendix 3, the gravitational equations of motion of (3.132) can be seen to be:

$$\begin{aligned} G_{\mu\nu} = \sum_{n=0}^{n_{\max}} \Bigg[& (-1)^n \nabla_{\alpha_n\ldots\alpha_1} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} - 2(-1)^n\nabla^\sigma\nabla^\beta\nabla_{\alpha_n\ldots\alpha_1}P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|\sigma|\nu)\beta} \\ & - 2P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|}{}^{\lambda\rho\sigma}\nabla_{\alpha_1\ldots\alpha_n}R_{|\nu)\lambda\rho\sigma} - \frac{1}{2}\sum_{i=1}^{n} P_{\nabla^n}{}^{\alpha_1\ldots}{}_{(\hat{\mu}|}{}^{\ldots\alpha_n\lambda\kappa\rho\sigma}\nabla_{\alpha_1\ldots|\hat{\nu})\ldots\alpha_n}R_{\lambda\kappa\rho\sigma}\Bigg] \\ & + 2F_{\langle\mu|\alpha}\chi_{|\nu\rangle}{}^{\alpha\rho\sigma}F_{\rho\sigma}\,, \end{aligned} \tag{3.143}$$

where the hats over the free indices μ and ν denote that they replace the indices α_i in the i-th position, $n_{\max}$ is the maximum number of explicit covariant derivatives appearing in the action and

$$P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} = -\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}F_{\kappa\pi}F_{\gamma\lambda}\,, \tag{3.144}$$

standing $\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}$ for $\nabla_{\alpha_1}\ldots\nabla_{\alpha_n}R_{\mu\nu\rho\sigma}$. The last term of (3.143) can be arranged in a duality-invariant fashion, because

$$2F_{\langle\mu|\alpha}\chi_{|\nu\rangle}{}^{\alpha\rho\sigma}F_{\rho\sigma} = -2F_{\langle\mu|\alpha}\star H_{|\nu\rangle}{}^{\alpha} = -\sigma_{AB}\mathcal{F}^A{}_{\langle\mu|\alpha}\star\mathcal{F}^B{}_{|\nu\rangle}{}^{\alpha}\,, \tag{3.145}$$

where σ_{AB} has the same matrix form as $\sigma^A{}_B$. On the other hand, the tensors $P_{\nabla^n}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma}$ are also invariant. To see this, let us first rewrite them as follows:

$$P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} = \frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}(\star\chi)_{\gamma\lambda}{}^{\tau\epsilon}F_{\kappa\pi}H_{\tau\epsilon} \tag{3.146}$$

where we are using the inverse of (3.136). Now, differentiating (3.134) with respect to the n-th covariant derivative of the curvature, it follows that

$$\frac{\partial(\star\chi)_{\kappa\pi}{}^{\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}(\star\chi)_{\gamma\lambda}{}^{\tau\epsilon}+(\star\chi)_{\kappa\pi}{}^{\gamma\lambda}\frac{\partial(\star\chi)_{\gamma\lambda}{}^{\tau\epsilon}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}=0\,, \tag{3.147}$$

which is equivalent to

$$\frac{\partial\chi^{\alpha\beta\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}(\star\chi)_{\gamma\lambda}{}^{\tau\epsilon}+\frac{\partial\chi^{\tau\epsilon\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}(\star\chi)_{\gamma\lambda}{}^{\alpha\beta}=0\,. \tag{3.148}$$

This allows us to write the tensors $P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma}$ in the manifestly duality-invariant form

$$\begin{aligned}P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma}&=\frac{1}{2}\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}(\star\chi)_{\gamma\lambda}{}^{\tau\epsilon}\sigma_{AB}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^B{}_{\tau\epsilon}\\&=-\frac{1}{2}\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{\mu\nu\rho\sigma}}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^A{}_{\gamma\lambda}\,,\end{aligned} \tag{3.149}$$

where in the last equality we used (3.140) and (3.134) in order to simplify the result. In sum, all the equations can be written as

$$\begin{aligned}G^{\mu\nu}=&\sum_{n=0}^{n_{\rm max}}\Bigg[(-1)^n\nabla_\sigma\nabla_\beta\nabla_{\alpha_n\ldots\alpha_1}\left(\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{(\mu|\sigma|\nu)\beta}}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^A{}_{\gamma\lambda}\right)\\&-\frac{(-1)^n}{2}\nabla_{\alpha_n\ldots\alpha_1}\left(\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{(\mu|\xi\eta\zeta}}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^A{}_{\gamma\lambda}\right)R^{|\nu)}{}_{\xi\eta\zeta}\\&+\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots\alpha_n}R_{(\mu|\xi\eta\zeta}}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^A{}_{\gamma\lambda}\nabla_{\alpha_1\ldots\alpha_n}R^{|\nu)}{}_{\xi\eta\zeta}\\&+\frac{1}{4}\frac{\partial\chi^{\kappa\pi\gamma\lambda}}{\partial\nabla_{\alpha_1\ldots(\hat{\mu}|\ldots\alpha_n}R_{\xi\eta\zeta\epsilon}}\mathcal{F}^A{}_{\kappa\pi}\mathcal{F}^A{}_{\gamma\lambda}\nabla_{\alpha_1\ldots}{}^{|\hat{\nu})}{}_{\ldots\alpha_n}R_{\xi\eta\zeta\epsilon}\Bigg]-\sigma_{AB}\mathcal{F}^{A\langle\mu|\alpha}\star\mathcal{F}^{B|\nu\rangle}{}_\alpha\end{aligned} \tag{3.150}$$

$$\sigma^A{}_B\mathcal{F}^B_{\mu\nu}=(\star\chi)_{\mu\nu}{}^{\alpha\beta}\mathcal{F}^A_{\alpha\beta}\,, \tag{3.151}$$

$$\mathrm{d}\mathcal{F}^A=0\,. \tag{3.152}$$

Thus, we have proven the following Proposition.

Proposition 3.1. *Any theory of gravity and electromagnetism with at most quadratic dependence on $F_{\mu\nu}$ (as in (3.132)) is exactly duality-invariant if and only if*

$$(\star\chi)_{\mu\nu}{}^{\alpha\beta}(\star\chi)_{\alpha\beta}{}^{\rho\sigma}=-\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,. \tag{3.153}$$

Proposition 3.1 is very interesting, since it shows that invariance under the whole group of SO(2) rotations is guaranteed if the equations of motion are invariant under the very particular rotation of electromagnetic fields given by $\frac{\pi}{2}$. This is a very peculiar feature of theories with quadratic dependence on the gauge field strength $F_{\mu\nu}$ which we do not expect for more generic theories, because the constitutive relation relating H and F would no longer be linear.

3.5.1 All exactly duality-invariant quadratic theories

Now we commit ourselves to find the explicit form of all exactly duality-invariant theories with quadratic dependence on the field strength $F_{\mu\nu}$ and including non-minimal couplings to gravity. For that, first we note that, making use of the properties of the Levi-Civita tensor, the equation (3.134) can be rewritten as

$$6\chi_{[\alpha\beta}{}^{\alpha\beta}\chi_{\mu\nu]}{}^{\rho\sigma}=\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,. \tag{3.154}$$

This is a quadratic tensor equation that admits infinitely many solutions. However, since we are interested in theories that reduce to Einstein-Maxwell's theory at low energies, the tensor $\chi_{\mu\nu\rho\sigma}$ should reduce to the indentity when the curvature is small, and we can expand it as

$$\chi_{\mu\nu}{}^{\rho\sigma}=\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}+\sum_{n=1}^{\infty}\alpha^{n}\chi^{(n)}{}_{\mu\nu}{}^{\rho\sigma}\,, \tag{3.155}$$

where α is a parameter with units of length^2 and each $\chi^{(n)}{}_{\mu\nu}{}^{\rho\sigma}$ contains $2n$ derivatives of the metric. Inserting this expansion in the equation above yields the following relation for the n-th order tensor

$$6\chi^{(n)}{}_{[\alpha\beta}{}^{\alpha\beta}\delta_{\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma}+6\chi^{(n)}{}_{[\mu\nu}{}^{\rho\sigma}\delta_{\alpha}{}^{\alpha}\delta_{\beta]}{}^{\beta}+6\sum_{p=1}^{n-1}\chi^{(p)}{}_{[\alpha\beta}{}^{\alpha\beta}\chi^{(n-p)}{}_{\mu\nu]}{}^{\rho\sigma}=0\,, \tag{3.156}$$

and after expanding the antisymmetrization in the first two terms, we can write this as follows,

$$2\chi^{(n)}{}_{\mu\nu}{}^{\rho\sigma}-4\chi^{(n)}{}_{\alpha[\mu}{}^{\alpha[\rho}\delta_{\nu]}{}^{\sigma]}+\chi^{(n)}{}_{\alpha\beta}{}^{\alpha\beta}\delta_{[\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma}=-6\sum_{p=1}^{n-1}\chi^{(p)}{}_{[\alpha\beta}{}^{\alpha\beta}\chi^{(n-p)}{}_{\mu\nu]}{}^{\rho\sigma}\,. \tag{3.157}$$

Now, this is an inhomogeneous linear tensor equation for $\chi^{(n)}{}_{\mu\nu}{}^{\rho\sigma}$, and so, the general solution can be expressed as the sum of a particular solution plus the general solution of the associated homogeneous equation. The latter reads

$$2\chi_{\rm h}{}^{(n)}{}_{\mu\nu}{}^{\rho\sigma}-4\chi_{\rm h}{}^{(n)}{}_{\alpha[\mu}{}^{\alpha[\rho}\delta_{\nu]}{}^{\sigma]}+\chi_{\rm h}{}^{(n)}{}_{\alpha\beta}{}^{\alpha\beta}\delta_{[\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma}=0\,, \tag{3.158}$$

and taking the trace in $\nu\sigma$ we have

$$\frac{1}{2}\chi_{\rm h}{}^{(n)}{}_{\alpha\beta}{}^{\alpha\beta}\delta_{\mu}{}^{\rho}=0\,. \tag{3.159}$$

Therefore, we have $\chi_{\rm h}{}^{(n)}{}_{\alpha\beta}{}^{\alpha\beta}=0$ and the general solution of the homogeneous equation can be expressed as

$$\chi_{\rm h}{}^{(n)}{}_{\mu\nu}{}^{\rho\sigma}=\mathcal{T}^{(n)}{}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,,\quad\text{where}\quad\mathcal{T}^{(n)}{}_{\mu}{}^{\mu}=0\,. \tag{3.160}$$

This is, the solution is characterized by an arbitrary traceless (and symmetric) tensor $\mathcal{T}^{(n)}_{\mu\nu}$. Now, coming back to the inhomogeneous equation, we realize that, since the trace of the left-hand-side of (3.157) is proportional to the identity, a necessary condition in order for a solution to exist is that

$$\sum_{p=1}^{n-1}\chi^{(p)}{}_{[\alpha\beta}{}^{\alpha\beta}\chi^{(n-p)}{}_{\mu\nu]}{}^{\rho\nu}\propto\delta_{\mu}{}^{\rho}\,. \tag{3.161}$$

However, this is guaranteed by the following property satisfied by all tensors ${Q^{(1)}}_{\mu\nu\rho\sigma}$ and ${Q^{(2)}}_{\mu\nu\rho\sigma}$ which are antisymmetric in the indices $\{\mu\nu\}$ and $\{\rho\sigma\}$:

$$ {Q^{(1)}}_{[\alpha\beta}{}^{\alpha\beta}{Q^{(2)}}_{\mu\lambda]\nu}{}^{\lambda}+{Q^{(2)}}_{[\alpha\beta}{}^{\alpha\beta}{Q^{(1)}}_{\mu\lambda]\nu}{}^{\lambda}=\frac{1}{2}{Q^{(1)}}_{[\alpha\beta}{}^{\alpha\beta}{Q^{(2)}}_{\gamma\lambda]}{}^{\gamma\lambda}g_{\mu\nu}\,. \tag{3.162} $$

This property is proven by expanding the antisymmetrization in the identity (valid in four dimensions) ${Q^{(1)}}_{[\alpha\beta}{}^{\alpha\beta}{Q^{(2)}}_{\gamma\lambda}{}^{\gamma\lambda}g_{\mu]\nu}=0$. Since the summation in (3.161) is symmetric in the exchange of $\chi^{(p)}$ and $\chi^{(n-p)}$ and every such $\chi^{(p)}$ and $\chi^{(n-p)}$ have the required symmetries for (3.162) to hold, we conclude that (3.161) is always satisfied.

Hence, taking into account (3.162), one can see that the general solution to (3.157) reads:

$$ {\chi^{(n)}}_{\mu\nu}{}^{\rho\sigma}={\mathcal{T}^{(n)}}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}-3\sum_{p=1}^{n-1}{\chi^{(p)}}_{[\alpha\beta}{}^{\alpha\beta}{\chi^{(n-p)}}_{\mu\nu]}{}^{\rho\sigma}\,, \tag{3.163} $$

and thus it is determined by a set of traceless symmetric tensors $\{\mathcal{T}^{(n)}_{\mu\nu}\}_{n\geq 1}$. Once these tensors are specified, one can compute the tensor $\chi_{\mu\nu}{}^{\rho\sigma}$ at arbitrary orders by using this recursive relation. Note that even when the set of non-vanishing $\mathcal{T}^{(n)}_{\mu\nu}$ tensors is finite, the series contains always an infinite number of terms.

Collect now all tensors $\mathcal{T}^{(n)}_{\mu\nu}$ in a single traceless symmetric tensor:

$$ \mathcal{T}_{\mu\nu}=\sum_{n=1}^{\infty}\alpha^{n-1}\mathcal{T}^{(n)}_{\mu\nu}\,. \tag{3.164} $$

Let us expand the susceptibility tensor $\chi_{\mu\nu}{}^{\rho\sigma}$ in terms of this general $\mathcal{T}_{\mu\nu}$:

$$ \chi_{\mu\nu}{}^{\rho\sigma}=\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}+\alpha\mathcal{T}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}+\sum_{n=2}^{\infty}\alpha^{n}{\hat{\chi}^{(n)}}_{\mu\nu}{}^{\rho\sigma}\,, \tag{3.165} $$

where[5]

$$ {\hat{\chi}^{(n)}}_{\mu\nu}{}^{\rho\sigma}=-3\sum_{p=1}^{n-1}{\hat{\chi}^{(p)}}_{[\alpha\beta}{}^{\alpha\beta}{\hat{\chi}^{(n-p)}}_{\mu\nu]}{}^{\rho\sigma}=\frac{1}{2}\sum_{p=1}^{n-1}\left(\star\hat{\chi}^{(p)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\star\hat{\chi}^{(n-p)}\right)_{\alpha\beta}{}^{\rho\sigma}\,, \tag{3.167} $$

being the first term of the sequence ${\hat{\chi}^{(1)}}_{\mu\nu}{}^{\rho\sigma}=\mathcal{T}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}$. The fact that (3.155) and (3.163) are equivalent to (3.165) and (3.167) can be seen by expanding in powers of α and noticing that the corresponding expressions match order by order.

The recursive relations (3.167) allow us to write the Lagrangian at arbitrary orders in the curvature. In fact, all the previous terms ${\hat{\chi}^{(n)}}_{\mu\nu}{}^{\rho\sigma}$ can be explicitly summed to yield a fully-non perturbative duality-invariant theory. For that we first note the following results, obtained through direct computation:

$$ \epsilon_{\mu\nu\alpha\beta}\hat{\chi}^{(1)\alpha\beta}{}_{\rho\sigma}=-{\hat{\chi}^{(1)}}_{\mu\nu}{}^{\alpha\beta}\epsilon_{\alpha\beta\rho\sigma}\,, \tag{3.168} $$

[5] We have used that for tensors $Q^{(1)}_{\mu\nu\rho\sigma}$ and $Q^{(2)}_{\mu\nu\rho\sigma}$ which are antisymmetric in the indices $\{\mu\nu\}$ and $\{\rho\sigma\}$ but symmetric in the exchange of these pairs of indices, the following holds:

$$ {(\star Q)^{(1)}}_{\mu\nu}{}^{\alpha\beta}{(\star Q)^{(2)}}_{\alpha\beta}{}^{\rho\sigma}=-6{Q^{(1)}}_{[\alpha\beta}{}^{\alpha\beta}{Q^{(2)}}_{\mu\nu]}{}^{\rho\sigma}\,. \tag{3.166} $$

Observe that the different ${\hat{\chi}^{(n)}}_{\mu\nu}{}^{\rho\sigma}$ are symmetric in the exchange of the pairs of indices $\{\mu\nu\}$ and $\{\rho\sigma\}$, as guaranteed by the construction of the complete $\chi_{\mu\nu}{}^{\rho\sigma}$.

$$\hat{\chi}^{(2)}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\hat{\chi}^{(1)}{}_{\mu\nu}{}^{\alpha\beta}\hat{\chi}^{(1)}{}_{\alpha\beta}{}^{\rho\sigma}\,. \tag{3.169}$$

Now let b_n denote the n-th coefficient of the Taylor series $\sqrt{1+x^2} = \sum_{n=0}^{\infty} b_n x^{2n}$. We have that $b_0 = 1$, $b_1 = 1/2$ and

$$b_n = (-1)^{n+1}\frac{(2n-3)!}{n!(n-2)!2^{2n-2}}\,, \quad n > 1\,. \tag{3.170}$$

We note that these coefficients satisfy the property

$$b_{n+1} = -1/2\sum_{p=1}^{n} b_p b_{n+1-p}\,, \tag{3.171}$$

which we will need later. We are going to prove that that for $n > 1$:

$$\hat{\chi}^{(2n)}{}_{\mu\nu}{}^{\rho\sigma} = b_n\left(\hat{\chi}^{(1)}\right)^{2n}{}_{\mu\nu}{}^{\rho\sigma}\,, \tag{3.172}$$

$$\hat{\chi}^{(2n+1)}{}_{\mu\nu}{}^{\rho\sigma} = 0\,, \tag{3.173}$$

where we have defined

$$\begin{aligned}\left(\hat{\chi}^{(1)}\right)^{k}{}_{\mu\nu}{}^{\rho\sigma} = \hat{\chi}^{(1)}{}_{\mu\nu}{}^{\alpha_1\beta_1}\hat{\chi}^{(1)}{}_{\alpha_1\beta_1}{}^{\alpha_2\beta_2}\cdots& \\ \cdots\hat{\chi}^{(1)}{}_{\alpha_{k-1}\beta_{k-1}}{}^{\rho\sigma}\,, \quad k > 1\,.&\end{aligned} \tag{3.174}$$

The proof for (3.172) can be done by induction. First, we notice that (3.169) guarantees that (3.172) is true for $n = 1$. Next assume that it is valid for generic n. For $m, p \in 2\mathbb{N}$ such that $m + p = 2n + 2$ we find that

$$\begin{aligned}\left(\star\hat{\chi}^{(p)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\star\hat{\chi}^{(m)}\right)_{\alpha\beta}{}^{\rho\sigma} &= b_p b_m\left(\star\hat{\chi}^{(1)}\right)_{\mu\nu}{}^{\lambda\gamma}\left(\hat{\chi}^{(1)}\right)^{p-1}{}_{\lambda\gamma}{}^{\alpha\beta}\left(\star\hat{\chi}^{(1)}\right)_{\alpha\beta}{}^{\eta\kappa}\left(\hat{\chi}^{(1)}\right)^{m-1}{}_{\eta\kappa}{}^{\rho\sigma} \\ &= -b_p b_m\left(\hat{\chi}^{(1)}\right)^{2n+2}{}_{\mu\nu}{}^{\rho\sigma}\,,\end{aligned} \tag{3.175}$$

where we have exploited Eq. (3.168) to get rid of the Hodge star operators appropriately. Now, taking into account (3.171), by virtue of (3.167) we observe that (3.172) is indeed satisfied for $n+1$ as well. On the other hand, in order to see that (3.173) holds, it suffices to check that it satisfies the recursive relations (3.167). However, after noticing that

$$\begin{aligned}\left(\star\hat{\chi}^{(1)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\star\hat{\chi}^{(2n)}\right)_{\alpha\beta}{}^{\rho\sigma} &= b_n\left(\star\hat{\chi}^{(1)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\star\hat{\chi}^{(1)}\right)_{\alpha\beta}{}^{\lambda\gamma}\left(\hat{\chi}^{(1)}\right)^{2n-1}{}_{\lambda\gamma}{}^{\rho\sigma} \\ &= -b_n\left(\star\hat{\chi}^{(1)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\hat{\chi}^{(1)}\right)^{2n-1}{}_{\alpha\beta}{}^{\lambda\gamma}\left(\star\hat{\chi}^{(1)}\right)_{\lambda\gamma}{}^{\rho\sigma} \\ &= -\left(\star\hat{\chi}^{(2n)}\right)_{\mu\nu}{}^{\alpha\beta}\left(\star\hat{\chi}^{(1)}\right)_{\alpha\beta}{}^{\rho\sigma}\,,\end{aligned} \tag{3.176}$$

where we have made a wide use of Eq. (3.168), we realize that the recursive relations are identically satisfied. Hence (3.172) and (3.173) are the solution to the recursive relations (3.167). Since the b_n are the coefficients of the Taylor series of $\sqrt{1+x^2}$, one can explicitly sum all tensors $\hat{\chi}^{(n)}{}_{\mu\nu}{}^{\rho\sigma}$ to obtain:

$$\chi_{\mu\nu}{}^{\rho\sigma} = \alpha\chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma} + \sqrt{\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} + \alpha^2\left(\chi^{(1)}\right)^2{}_{\mu\nu}{}^{\rho\sigma}}\,, \tag{3.177}$$

where we have relabelled $\hat{\chi}^{(1)}{}_{\mu\nu}{}^{\rho\sigma} \to \chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma}$ for the sake of simplicity. By taking (3.177) into (3.132), we find a fully non-perturbative expression for all exactly duality-invariant theories with non-minimal couplings between gravity and electromagnetism. These are the very first instances of such theories and, interestingly enough, we observe that it possesses a high grade of resemblance with the usual (Einstein)Born-Infeld theories, as the Lagrangian involves the square root of certain quantity. Altogether, we have shown the following theorem.

Theorem 3.1. *Any theory of gravity and electromagnetism with at most quadratic dependence on $F_{\mu\nu}$ is exactly duality-invariant if and only if it takes the form (3.132) with susceptibility tensor*

$$\chi_{\mu\nu}{}^{\rho\sigma} = \alpha\chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma} + \sqrt{\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} + \alpha^2\left(\chi^{(1)}\right)^2{}_{\mu\nu}{}^{\rho\sigma}}\,, \quad \chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = \mathcal{T}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,, \tag{3.178}$$

where $\mathcal{T}_{\mu\nu}$ is a traceless and symmetric tensor built out from the curvature and covariant derivatives of it.

From now on we will now concentrate on the study of the particular exactly duality-invariant theory given by

$$\chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = \hat{R}_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}\,, \tag{3.179}$$

where $\hat{R}_{\mu\nu}$ is the traceless Ricci tensor. The reason for doing this is twofold. First, because it does not seem possible to say anything concrete about (black hole) solutions of the subsequent theories if we do not even specify the particular theory we are working with. Secondly, while exactly duality-invariant theories with minimal couplings have been previously considered in the literature [315, 317–319, 322, 535, 537–543], this is no longer the case when one adds non-minimal couplings, and therefore we decide to initiate the exploration of this set of theories with the choice given by (3.179).

3.5.2 Static and spherically symmetric configurations

Once we have obtained exactly duality-invariant fully non-perturbative theories with non-minimal couplings, our next objective will be to try to understand some features about its solutions. As explained before, we will concentrate on the theory given by (3.179) and we will focus on static and spherically symmetric (SSS) configurations, since they possess enough symmetry to be amenable to computations but still they are physically meaningful. These can in general be written in terms of the following ansatz:

$$\mathrm{d}s^2 = -N(r)^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,, \tag{3.180}$$

$$F = -A_t'(r)\mathrm{d}t\wedge\mathrm{d}r + P\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\,. \tag{3.181}$$

Here the metric depends on two functions $f(r)$ and $N(r)$, while $A_t(r)$ is the electrostatic potential and P is a constant that represents the magnetic charge in Planck units. In order to compute the explicit form of the susceptibility tensor, note first that the traceless part of the Ricci tensor for an SSS metric reads

$$\hat{R}^\alpha{}_\beta = (X+Y)\tau^\alpha{}_\beta + (X-Y)\rho^\alpha{}_\beta - X\sigma^\alpha{}_\beta\,, \tag{3.182}$$

where

$$X = \frac{N(2f - 2 - r^2 f'') - r(3rf'N' + 2frN'')}{4r^2 N}\,, \quad Y = -\frac{fN'}{rN} \tag{3.183}$$

and where τ, ρ and σ are the orthogonal projectors

$$\tau^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{t}\delta^{t}{}_{\beta}\,, \quad \rho^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{r}\delta^{r}{}_{\beta}\,, \quad \sigma^{\alpha}{}_{\beta} = \sum_{i=\theta,\phi} \delta^{\alpha}{}_{i}\delta^{i}{}_{\beta}\,. \tag{3.184}$$

On the other hand, static and spherical symmetry force χ (and *a fortiori* all the different $\chi^{(n)}$) to take the form

$$\chi_{\mu\nu}{}^{\rho\sigma} = B\tau_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]} + C\tau_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + D\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + E\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,, \tag{3.185}$$

where B, C, D, E are functions of r. Taking into account that

$$\delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]} = 2\tau_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]} + 2\tau_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + 2\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + \sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,, \tag{3.186}$$

$$\chi^{(1)}{}_{\mu\nu}{}^{\rho\sigma} = -2X\tau_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]} - Y\tau_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + Y\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + X\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,. \tag{3.187}$$

and that the projectors τ, ρ and σ are mutually orthogonal, it is not difficult to obtain the coefficients B, C, D, E from (3.177). These take the following simple values:

$$B = -2\alpha X + 2\sqrt{1 + \alpha^2 X^2}\,, \tag{3.188}$$

$$C = -\alpha Y + 2\sqrt{1 + \frac{\alpha^2 Y^2}{4}}\,, \tag{3.189}$$

$$E = \frac{2}{B} = \alpha X + \sqrt{1 + \alpha^2 X^2}\,, \tag{3.190}$$

$$D = \frac{4}{C} = \alpha Y + 2\sqrt{1 + \frac{\alpha^2 Y^2}{4}}\,, \tag{3.191}$$

Consequently, we have been able to find the exact form of the susceptibility tensor. This allows us to evaluate the reduced Lagrangian for the SSS ansatz given by (3.180) and (3.181), which takes the form

$$L = \int \mathrm{d}\theta \mathrm{d}\phi \sqrt{|g|}\mathcal{L}|_{\rm SSS} = \frac{1}{4}\left[Nr^2 R|_{\rm SSS} - 2P^2\frac{NE}{r^2} + 2(A_t')^2\frac{r^2}{NE}\right]\,. \tag{3.192}$$

Then we can find the equations of motion by varying this Lagrangian with respect to A_t, f and N [6]. The variation with respect to A_t yields

$$\frac{\delta L}{\delta A_t} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{A_t' r^2}{NE}\right) = 0\,, \tag{3.193}$$

from where it follows that

$$A_t' = \frac{QNE}{r^2}\,, \tag{3.194}$$

where the integration constant Q represents the electric charge in Planck units. On the other hand, taking the variation with respect to f and N and using the previous result, we find that the equations for the metric functions can be expressed as

$$f - 1 + rf' = -(P^2 + Q^2)\frac{\delta}{\delta N}\left(\frac{NE}{r^2}\right)\,, \tag{3.195}$$

[6]This process is indeed equivalent to computing first the complete Einstein and Maxwell equations and evaluating them on the SSS ansatz, as explained in Chapter 2 and [4].

$$rN' = (P^2+Q^2)\frac{\delta}{\delta f}\left(\frac{NE}{r^2}\right). \tag{3.196}$$

Therefore, they are manifestly invariant under a rotation of the charges Q and P, and it follows that the metric only depends on the combination $P^2+Q^2 \equiv \mathcal{Q}^2$. Due to the complicated form of E in (3.190), these are highly-non linear fourth-order equations for N and f, whose solution cannot be obtained analytically. However, for small α one can obtain the solution as a power series in this parameter. To order α^2 it reads

$$\begin{aligned} f = 1 &- \frac{2M}{r} + \frac{\mathcal{Q}^2}{r^2} - \frac{\left(7\mathcal{Q}^4+5\mathcal{Q}^2 r(2r-3M)\right)\alpha}{10r^6} \\ &+ \frac{\left(5012\mathcal{Q}^6+15\mathcal{Q}^4 r(408r-721M)\right)\alpha^2}{1680r^{10}} + \mathcal{O}(\alpha^3)\,, \end{aligned} \tag{3.197}$$

$$N = 1 + \frac{\mathcal{Q}^2\alpha}{4r^4} - \frac{41\mathcal{Q}^4\alpha^2}{32r^8} + \mathcal{O}(\alpha^3)\,, \tag{3.198}$$

where M is the mass. We can see this solution is a deformation of the Reissner-Nordström one. However, the perturbative expansion in α is only valid as long as the corrections are small and hence we cannot see what happens to black holes when $\alpha \sim \mathcal{Q}^2$. In that regime, one would need to resort to numeric methods to solve the equations of motion.

3.5.3 Extremal black holes and near-horizon geometries

Fortunately, the situation improves if we are interested in extremal black holes. In that case, it is possible to obtain the near-horizon metric as well as the black hole entropy by using the Sen's method [496, 544]. This method essentially consists in evaluating the Lagrangian on an $\mathrm{AdS}_2 \times S^2$ geometry. The near-horizon solution is then obtained by extremizing the action, while the entropy is given by the Legendre transform of the Lagrangian with respect to the electric field. We follow this process in detail next.

We start by considering the following $\mathrm{AdS}_2 \times S^2$ ansatz for the metric and the field strength:

$$\mathrm{d}s^2 = a\left(-\rho^2\mathrm{d}t^2 + \frac{1}{\rho^2}\mathrm{d}\rho^2\right) + b\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\,, \tag{3.199}$$

$$F = -e\mathrm{d}t\wedge\mathrm{d}\rho + P\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi\,. \tag{3.200}$$

Here $a = R^2_{\mathrm{AdS}_2}$, $b = R^2_{S^2}$ are the radii squared of the AdS_2 factor and of the black hole horizon, respectively, P is the magnetic charge and e will be related to the electric charge. This geometry can be obtained from the general SSS ansatz in Eqs. (3.180) and (3.181) by setting $r = \sqrt{b}+\rho$, $f = \rho^2/a$, $N = a$, $A'_t = e$ and keeping the leading terms in the expansion around $\rho \to 0$. Thus, the reduced Lagrangian reads in this case

$$L(a,b,e,P) = \frac{1}{2}\left(a - b - P^2\hat{E} + e^2\frac{1}{\hat{E}}\right)\,, \tag{3.201}$$

where

$$\hat{E} = -\alpha\frac{a+b}{2b^2} + \frac{a}{b}\sqrt{1+\alpha^2\frac{(a+b)^2}{4a^2b^2}}\,. \tag{3.202}$$

The entropy function $\mathcal{E}(a,b,e;Q,P)$ is then defined as:

$$\mathcal{E}(a,b,e;Q,P) = 2\pi(eQ - L(a,b,e,P))\,, \tag{3.203}$$

where Q is the electric charge of the configuration. Extremizing the entropy function with respect to a, b and e yields the equations satisfied by a and b as well as the relation between the electric charge and e. Indeed, the equation $\partial\mathcal{E}/\partial e = 0$ yields

$$e = \hat{E} Q \, . \tag{3.204}$$

On the other hand, by differentiating with respect to a and b and using this result we obtain the following sets of equations for a and b:

$$\frac{1}{\pi}\frac{\partial \mathcal{E}}{\partial a} = -1 + (P^2+Q^2)\frac{\partial \hat{E}}{\partial a} = 0 \, , \tag{3.205}$$

$$\frac{1}{\pi}\frac{\partial \mathcal{E}}{\partial b} = 1 + (P^2+Q^2)\frac{\partial \hat{E}}{\partial b} = 0 \, . \tag{3.206}$$

We observe again that the equations are invariant under a rotation of the electric and magnetic charges. Notice also that these equations are highly nonlinear — in fact, they are not even polynomial — due to the form of $\hat{E}$ given above. In spite of this, these equations can be solved in full generality and we observe that they admit four different solutions. However, there is only one solution with $a, b > 0$, and it is given by

$$a = \frac{1}{2}\left(P^2+Q^2+\alpha+\sqrt{(P^2+Q^2)^2-\alpha^2}\right) \, , \tag{3.207}$$

$$b = \frac{1}{2}\left(P^2+Q^2-\alpha+\sqrt{(P^2+Q^2)^2-\alpha^2}\right) \, . \tag{3.208}$$

Interestingly, this implies that $\hat{E} = 1$ and hence $e = Q$. Finally, substituting these values for a, b, e in the entropy function (3.203) we arrive to the following result for the entropy of these extremal black holes:

$$S = \pi(P^2+Q^2-\alpha) \, . \tag{3.209}$$

Surprisingly enough, we find that there is only a constant correction to the entropy with respect to the Einstein-Maxwell value — we remark that this is the exact value of the entropy and not just an approximation. Notice that, even when one adds only a finite number of higher-order terms in the action, the entropy (and the rest of the quantities) will be typically modified by an infinite tower of α terms. Here we observe the opposite: the action contains an infinite number of higher-order terms as dictated by duality invariance, but in turn the entropy only has a correction of order α.

Let us take a closer look at this solution. While the entropy is finite and real for any value of the charges, we see that this is not the case for a and b. In fact, for any sign of α we see that these extremal geometries only exist for

$$P^2+Q^2 \geq |\alpha| \, . \tag{3.210}$$

Therefore, there is a minimum amount of charge needed to produce an extremal black hole, implying that all black holes with $P^2+Q^2 < |\alpha|$ must be necessarily non-extremal. On the other hand, the properties of these black holes near the minimal charge are quite different depending on the sign of α. When $\alpha > 0$, the radius of the AdS_2 tends to the constant value $a = \alpha$ as $P^2+Q^2 \to \alpha$, while the area of the horizon and the entropy vanish in this limit. In the case of $\alpha < 0$ we observe the contrary: the radius of AdS_2 goes to zero, while both the entropy and the area tend to a constant value, namely, $S = A/2 = 2\pi|\alpha|$, as $P^2 + Q^2 \to |\alpha|$.

In order to determine the sign of α, one may use the mild form [474, 528] of the WGC [416], which (we remind) states that the corrections to the mass of extremal black holes must be non-positive, so that the decay of an extremal black hole into a set of smaller black holes is possible. The near-horizon geometry does not allow one to obtain the mass of the black hole, but we can obtain it from the perturbative solution (3.198). Imposing the extremality condition $f(r_+) = f'(r_+) = 0$, we find that

$$M_{\rm ext} \quad = \quad \mathcal{Q} - \frac{\alpha}{10\mathcal{Q}} - \frac{\alpha^2}{84\mathcal{Q}^3} + \mathcal{O}(\alpha^3)\,, \tag{3.211}$$

while the extremal radius r_+ agrees exactly with the expansion of $\sqrt{b}$ in (3.208). We have checked that the α-expansion of the mass converges very rapidly and the expression above turns out to be very accurate even for $\mathcal{Q} = \sqrt{|\alpha|}$. We see that in order for the corrections to the mass to be non-positive we must have $\alpha \geq 0$. Then, at the minimal charge $\mathcal{Q}^{\rm min} = \sqrt{\alpha}$ the mass becomes $M_{\rm ext}^{\rm min} \approx 0.88\sqrt{\alpha}$ and the entropy and area of extremal black holes vanish.

3.6 Discussion

In this chapter we have studied higher-order extensions of Einstein-Maxwell theory which are invariant under electromagnetic duality rotations. We started by considering a general higher-derivative theory of gravity and electromagnetism and obtained the necessary and sufficient conditions for theories up to eight derivatives to preserve electromagnetic duality at a perturbative level. It would be interesting to extend this result to all orders in the derivative expansion, but we leave this task for the future. Then we used these results to derive the most general parity- and duality-invariant theory up to eight derivatives.

Next we studied the effect of field redefinitions on duality-invariant theories, which led us to a remarkable simplification of those actions. We showed that, up to six derivatives, one can always remove all the higher-order terms with explicit Maxwell field strengths, leaving one with a higher-derivative gravity minimally coupled to the Maxwell Lagrangian. In other words, this result implies that, in a duality-invariant theory, higher-derivative corrections can always be chosen in a scheme such that they do not modify the Maxwell Lagrangian at all. We argued that this phenomenon should take place for any theory with any number of derivatives, but so far this claim remains as a conjecture. Appealing open questions are those of providing a proof for the conjecture or showing that any theory is equivalent via metric redefinitions to Maxwell theory coupled to a higher-order gravity at a non-perturbative level as well.

In this context, we wrote the most general six-derivative duality-invariant action after the use of metric redefinitions, noticing that the number of higher-order operators can be effectively reduced to four, plus a topological one. We studied the charged, static and spherically symmetric black hole solutions of this theory and computed their thermodynamic properties — entropy, temperature and electromagnetic potentials at the horizon — allowing us to check explicitly that the first law of thermodynamics holds.

Using these results, we obtained additional constraints on the higher-derivative couplings by applying the recently proposed mild form of the WGC [474, 475]. According to this conjecture, the corrections to the charge-to-mass ratio of extremal black holes in any consistent theory of quantum gravity should be non-negative, thus allowing the decay of extremal black holes and evading the existence of remnants. This has also been related to the non-negativity of entropy corrections [528, 531], although, as we discussed, the connection is not complete. We determined the constraints coming from these conditions not only for the leading higher-derivative corrections but also for the subleading ones. Demanding that the subleading corrections to the entropy remain non-negative leads to especially strong constraints — see (3.131) —, implying in particular the non-existence of leading-order corrections in ST solutions. However, we observe this is in contradiction to well-established results in the literature [125], so we check that the mild form of the WGC should only be applied to the leading entropy corrections.

Afterwards we focused on higher-order theories with a quadratic dependence on the field strength, being able to write the explicit form of all such theories which are exactly duality-invariant. It would also be interesting to look for more general nonminimal duality-preserving theories, *i.e.*, including as well higher powers of the Maxwell field strength, but this is a challenging problem which will be treated elsewhere.

Focusing on the simplest of these theories, we have studied its static and spherically symmetric solutions. As we have shown, the equations of motion satisfied by the metric in the latter theory are invariant under rotations of the electric and magnetic charges, but due to their complexity they can only be solved analytically in the perturbative regime — see (3.198). However, we found that the near-horizon geometry of extremal black holes can be obtained exactly. A remarkable aspect about these extremal black holes is that their entropy only receives a constant correction, which is striking since the action is modified in a very nonlinear way. A similar result is observed in the case of Einstein-Born-Infeld theory, which suggests that duality somehow simplifies the corrections to the entropy. It would be interesting to explore other theories to understand this possible connection better, but we do not have as of this moment a simple explanation for this observation.

In addition, these extremal black holes possess a minimal charge below which no solutions exist. Thus, it would follow that any black hole with a charge below this minimum value must be non-extremal — no matter how small the mass is. We have also shown that the WGC imposes the coupling constant α to be positive, which led us to the conclusion that, at the minimal charge, the area and entropy of extremal black hole vanish. This is an intriguing behavior, and it is tantalizing to assume that this minimal charge coincides precisely with the elementary electric charge. An extremal black hole with the charge of an electron is trivially the one with the lowest (non-zero) charge, and one could argue that its entropy would vanish because it would contain only one microstate. However, we note that the entropy can always be shifted by the introduction of a topological Gauss-Bonnet term in the action, so the entropy of the minimal extremal black hole can be changed.

These issues could be better understood by trying to embed this theory in ST, in whose case, a precise entropy counting is available, *e.g.* [97, 496, 545–547]. In fact, it is possible to check that our solution (3.198) coincides with the α'-corrected Reissner-Nordström black hole of Ref. [125], upon the identification $\alpha = \alpha'/8$ [7]. This shows that our theory (3.132) captures some of the stringy α'-corrections, at least in the situations where the additional degrees of freedom besides the metric and the electromagnetic field can be neglected.

[7] More precisely, both metrics are related by a redefinition $g^{\mathrm{ST}}_{\mu\nu} = g^{\mathrm{ours}}_{\mu\nu} + 3\alpha T_{\mu\nu}$ to first order in α, but this only means that the theories are written in different frames.

Appendix 3 Invariance of Einstein equation in theories with covariant derivatives

In this appendix we are going to show that the invariance of the constitutive relation (3.7) under duality transformations implies the invariance of the Einstein's equations also for those theories with explicit covariant derivatives of the curvature. For that, we shall follow an analogous reasoning to that of Subsection 3.1.2, where we showed the invariance of the Einstein's equations under duality rotations once that of (3.7) is guaranteed.

If we consider a generic theory with arbitrary dependence on the covariant derivatives of the Riemann curvature tensor (as in (3.1), with no derivatives of $F_{\mu\nu}$), so that $\mathcal{L} = \mathcal{L}(g^{\mu\nu}, F_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \dots)$, the main difficulty we encounter is that the tensor $\mathcal{E}_{\mu\nu}$, defined back in Eq. (3.10), takes a much more complicated form to that given at (3.43). However, we can make use of many computations and results presented in Subsection 3.1.2 for algebraic theories to achieve our goal. In fact, the arguments used for algebraic theories are valid in this general case up to Eq. (3.58). Thus, our task will be to show the validity of the equations (3.59) and (3.60) also for these general theories so that the invariance of the Einstein's equations will be guaranteed. We rewrite here those equations for the benefit of the reader:

$$\mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu}(T) + \frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T) \circ \frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}} = 0\,, \tag{3.212}$$

$$\begin{aligned}&\mathcal{E}^{\mathrm{IH}(8)}_{\mu\nu}(T) + \frac{1}{4}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T) \circ \frac{\partial\mathcal{L}_{(6)}}{\partial F_{\alpha\beta}} - \frac{1}{16}\frac{\delta^2\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}\delta F_{\rho\sigma}}(T) \circ \frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}} \circ \frac{\partial\mathcal{L}_{(4)}}{\partial F^{\rho\sigma}}\\ &+\frac{1}{4}\frac{\delta\mathcal{E}^{\mathrm{H}(6)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T) \circ \frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}} = 0\,.\end{aligned} \tag{3.213}$$

Let us first of all find the form of the equations of motion for theories with dependence on derivatives of the curvature. For that, let us define

$$P_{\nabla^n}{}^{\alpha_1\dots\alpha_n\mu\nu\rho\sigma} = \frac{\partial\mathcal{L}}{\partial\nabla_{\alpha_1\dots\alpha_n}R_{\mu\nu\rho\sigma}}\,, \tag{3.214}$$

which enjoys the same symmetries as $\nabla_{\alpha_1\dots\alpha_n}R_{\mu\nu\rho\sigma}$, which is a short-hand notation for $\nabla_{\alpha_1}\dots\nabla_{\alpha_n}R_{\mu\nu\rho\sigma}$. Consequently, the metric variation of the corresponding action $I[g,A]$ associated to $\mathcal{L}(g^{\mu\nu}, F_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \dots)$ takes the form:

$$\begin{aligned}\delta I[g,A](\delta g^{\mu\nu},0) = \frac{1}{16\pi G}\int_M \mathrm{d}^4x\sqrt{|g|}\Bigg\{&-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}\\ &+\sum_{n=0}^{\infty}P_{\nabla^n}{}^{\alpha_1\dots\alpha_n\mu\nu\rho\sigma}\delta\nabla_{\alpha_1\dots\alpha_n}R_{\mu\nu\rho\sigma}\Bigg\}\,.\end{aligned} \tag{3.215}$$

If $\xi \in \mathfrak{X}(M)$ denotes an arbitrary vector field, we can write that the Lie derivative $L_\xi\mathcal{L}$ in two different ways:

$$L_\xi\mathcal{L} = \xi^\kappa\nabla_\kappa\mathcal{L} = \xi^\kappa\sum_{n=0}^{\infty}P_{\nabla^n}{}^{\alpha_1\dots\alpha_n\mu\nu\rho\sigma}\nabla_\kappa\nabla_{\alpha_1}\dots\nabla_{\alpha_n}R_{\mu\nu\rho\sigma} + \xi^\kappa\frac{\partial\mathcal{L}}{\partial F^{\alpha\beta}}\nabla_\kappa F^{\alpha\beta}\,, \tag{3.216}$$

$$L_\xi \mathcal{L} = \sum_{n=0}^{\infty} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} L_\xi \nabla_{\alpha_1\ldots\alpha_n} R_{\mu\nu\rho\sigma} + \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} L_\xi F_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} L_\xi g_{\alpha\beta} \,. \tag{3.217}$$

On the other hand, we have the following identities:

$$\begin{aligned} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} L_\xi \nabla_{\alpha_1\ldots\alpha_n} R_{\mu\nu\rho\sigma} = {} & \xi^\kappa P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} \nabla_{\kappa\alpha_1\ldots\alpha_n} R_{\mu\nu\rho\sigma} \\ & + 4(\nabla_\kappa \xi_\gamma) P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\kappa\nu\rho\sigma} \nabla_{\alpha_1\ldots\alpha_n} R^\gamma{}_{\nu\rho\sigma} \\ & + \sum_{i=0}^{n} (\nabla_{\alpha_i}\xi_{\beta_i}) P_{\nabla^n}{}^{\alpha_1\ldots\alpha_i\ldots\alpha_n\mu\nu\rho\sigma} \nabla_{\alpha_1\ldots}{}^{\beta_i}{}_{\ldots\alpha_n} R_{\mu\nu\rho\sigma} \,, \end{aligned} \tag{3.218}$$

$$L_\xi g_{\alpha\beta} = 2\nabla_{(\alpha}\xi_{\beta)} \,, \tag{3.219}$$

$$\frac{\partial L}{\partial F_{\alpha\beta}} L_\xi F_{\alpha\beta} = \xi^\mu \nabla_\mu F_{\alpha\beta} \frac{\partial L}{\partial F_{\alpha\beta}} + 2\nabla_\alpha \xi^\mu F_{\mu\beta} \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \,. \tag{3.220}$$

Consequently, we learn that

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = {} & -2P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\lambda\rho\sigma} \nabla_{\alpha_1\ldots\alpha_n} R^\nu{}_{\lambda\rho\sigma} - \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F^{\nu\rho} \\ & - \frac{1}{2} \sum_{i=1}^{n} P_{\nabla^n}{}^{\alpha_1\ldots\hat{\mu}\ldots\alpha_n\lambda\kappa\rho\sigma} \nabla_{\alpha_1\ldots}{}^{\hat{\nu}}{}_{\ldots\alpha_n} R_{\lambda\kappa\rho\sigma} \,, \end{aligned} \tag{3.221}$$

where the hats over the free indices μ and ν denote that they replace the indices α_i in the i-th position. Taking into account that, up to total derivatives,

$$\begin{aligned} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma} \delta\nabla_{\alpha_1\ldots\alpha_n} R_{\mu\nu\rho\sigma} = {} & (-1)^{n+1} \nabla_{\alpha_n\ldots\alpha_1} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_\mu{}^{\nu\rho\sigma} R_{\beta\nu\rho\sigma} \delta g^{\mu\beta} \\ & + 2(-1)^n \nabla^\sigma \nabla^\beta \nabla_{\alpha_n\ldots\alpha_1} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{\mu\sigma\nu\beta} \delta g^{\mu\nu} \,, \end{aligned} \tag{3.222}$$

we find that

$$\begin{aligned} \mathcal{E}_{\mu\nu} = {} & \sum_{n=0}^{n_{\max}} \Big[2(-1)^{n+1} \nabla^\sigma \nabla^\beta \nabla_{\alpha_n\ldots\alpha_1} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|\sigma|\nu)\beta} + (-1)^n \nabla_{\alpha_n\ldots\alpha_1} P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu}{}^{\rho\sigma\gamma} R_{\nu)\rho\sigma\gamma} \\ & - 2P_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|}{}^{\lambda\rho\sigma} \nabla_{\alpha_1\ldots\alpha_n} R_{|\nu)\lambda\rho\sigma} - \frac{1}{2} \sum_{i=1}^{n} P_{\nabla^n}{}^{\alpha_1\ldots}{}_{(\hat{\mu}|}{}^{\ldots\alpha_n\lambda\kappa\rho\sigma} \nabla_{\alpha_1\ldots|\hat{\nu})\ldots\alpha_n} R_{\lambda\kappa\rho\sigma} \Big] \\ & + \frac{1}{2} g_{\mu\nu} \left(\mathcal{L} - \frac{1}{2} F_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \right) \,, \end{aligned} \tag{3.223}$$

where $n_{\max}$ is the maximum number of explicit covariant derivatives appearing in the action.

Let us remark at this point that, as in the case of algebraic theories, the equations $\mathcal{E}^{\mathrm{H}(2n)}_{\mu\nu}$ associated to the homogeneous Lagrangians $\mathcal{L}^{\mathrm{H}}_{(2n)}$ only depend on $F_{\mu\nu}$ through the Maxwell stress tensor $T_{\mu\nu}$. This follows from the fact that for every monomial we have $F_{\alpha\beta}\frac{\partial \mathcal{L}_i}{\partial F_{\alpha\beta}} \propto \mathcal{L}_i$. On the other hand, if $\mathcal{L}$ is a function of $T_{\mu\nu}$ so are the various P_{∇^n} tensors. Thus we can in fact write $\mathcal{E}^{\mathrm{H}(2n)}_{\mu\nu}(T)$, so that we can apply Eq. (3.58).

Finally, in order to show (3.212) and (3.213), let us note the following formula, which generalizes (3.61) for an arbitrary number of covariant derivatives:

$$\frac{\delta \nabla_{\mu_1\ldots\mu_n} \mathcal{Q}_{\nu_1\ldots\nu_n}}{\delta F_{\alpha\beta}} \circ \mathcal{A}_{\alpha\beta} = \nabla_{\mu_1\ldots\mu_n} \left(\frac{\delta \mathcal{Q}_{\nu_1\ldots\nu_n}}{\delta F_{\alpha\beta}} \circ \mathcal{A}_{\alpha\beta} \right) \,, \tag{3.224}$$

where $\mathcal{Q}_{\nu_1\ldots\nu_n}$ is any tensor with dependence on $F_{\alpha\beta}$ and its covariant derivatives and where $\mathcal{A}_{\alpha\beta}$ is an arbitrary antisymmetric tensor. Using this formula, and noting the structure of (3.223), we check after some computations that both (3.212) and (3.213) hold. For the sake of clarity, let us illustrate this fact more explicitly for Eq. (3.212). By use of (3.224), we have that

$$\begin{aligned}\frac{\delta\mathcal{E}^{(4)}_{\mu\nu}}{\delta F_{\alpha\beta}}(T)\circ\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}=\sum_{n=0}^{n_{\max}}\Bigg[&2(-1)^{n+1}\nabla^{\sigma}\nabla^{\beta}\nabla_{\alpha_n\ldots\alpha_1}\hat{P}_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|\sigma|\nu)\beta}\\&+(-1)^n\nabla_{\alpha_n\ldots\alpha_1}\hat{P}_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma}-2\hat{P}_{\nabla^n}{}^{\alpha_1\ldots\alpha_n}{}_{(\mu|}{}^{\lambda\rho\sigma}\nabla_{\alpha_1\ldots\alpha_n}R_{|\nu)\lambda\rho\sigma}\\&-\frac{1}{2}\sum_{i=1}^{n}\hat{P}_{\nabla^n}{}^{\alpha_1\ldots}{}_{(\hat{\mu}|}{}^{\ldots\alpha_n\lambda\kappa\rho\sigma}\nabla_{\alpha_1\ldots|\hat{\nu})\ldots\alpha_n}R_{\lambda\kappa\rho\sigma}\Bigg]\\&+\frac{1}{2}g_{\mu\nu}\frac{\partial\mathcal{L}_{(4)}}{\partial F^{\alpha\beta}}\frac{\partial}{\partial F_{\alpha\beta}}\left(\mathcal{L}_{(4)}-\frac{1}{2}F_{\rho\sigma}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\rho\sigma}}\right)\,,\end{aligned}\tag{3.225}$$

where we have defined

$$\hat{P}_{\nabla^n}{}^{\alpha_1\ldots\alpha_n\mu\nu\rho\sigma}=\frac{\partial P^{(4)\,\alpha_1\ldots\alpha_n\mu\nu\rho\sigma}_{\nabla^n}}{\partial F^{\lambda\kappa}}\frac{\partial\mathcal{L}_{(4)}}{\partial F_{\lambda\kappa}}\,.\tag{3.226}$$

We identify in (3.225) precisely the term $-4\mathcal{E}^{\mathrm{IH}(6)}_{\mu\nu}(T)$, so we conclude. The proof of (3.213) goes along similar lines.

4

Higher-derivative holography with a chemical potential

In previous chapters, we analyzed higher-derivative extensions of Einstein-Maxwell theory. In particular, we were able to identify infinite instances of EQGs in Chapter 2, which are defined as those possessing magnetically- or electrically-charged, static and spherically symmetric solutions completely characterized by a single function with (at most) second-order equation of motion. Such definition was coined in the context of four-dimensional theories, so it might be necessary to adapt it when considering higher-order theories of gravity and electromagnetism in arbitrary spacetime dimensions, what could be specially useful in the holographic context.

Indeed, higher-derivative theories of gravity play a relevant role in the context of the AdS/CFT correspondence [418–420], as they can lead to new insights on the physics of conformal field theories. On the one hand, certain higher-derivative terms capture finite N and finite coupling effects in the boundary CFT, as is the case, for instance, for the corrections that appear explicitly in Type IIB ST [108, 128, 131, 132, 504]. In this situation, one is typically interested in a perturbative treatment of the corrections, as the $1/N$ and $1/\lambda$ effects are supposed to be small. On the other hand, higher-derivative gravities can be used to probe more general universality classes of CFTs than those covered by Einstein gravity [424, 425, 435, 548, 549]. In other words, they allow one to explore a larger region in the space of CFTs via holography. A paradigmatic example of this is provided by the three-point function of the stress-energy tensor $\langle TTT\rangle$, which for a general d-dimensional CFT depends on three parameters. As is well known, for holographic CFTs dual to Einstein gravity only one of these parameters is non-vanishing, but one can achieve a general $\langle TTT\rangle$ structure by considering a higher-curvature theory in the bulk [243, 426, 550]. It is also worth noting that, since for a given CFT all the parameters of this correlator could be of order one, from this point of view it even makes sense to study the higher-derivative theory in a non-perturbative fashion.

The program of studying the holographic aspects of higher-derivative theories as models for more general classes of CFTs has provided many insights into the dynamics of highly-interacting quantum field theories. One of the most impressive applications of this approach consists in unveiling universal properties valid for arbitrary CFTs, whose determination from first principles is sometimes obscure. In this line we can mention the holographic c-theorem established by Refs. [508, 551], the universal behavior of corner contributions to the entanglement entropy found in Refs. [429, 552], and more recently, the universal relationship between the free energy of a CFT in a squashed sphere and the coefficients of $\langle TTT\rangle$ observed in [259, 270] — see also [488, 553–555] for other interesting examples. On broader terms, higher-order gravities allow one to inspect which features of holographic CFTs dual to Einstein gravity are general and which ones can be changed. In this way, it is natural to wonder about the possible effects of higher-derivative terms on the holographic predictions regarding, for example, hydrodynamics, entanglement structure, superconductors, etc. — see *e.g.* [128, 130, 132, 284, 427, 446, 493, 556–565] for an incomplete list of references on these topics.

In this chapter we will be interested in higher-derivative bulk theories that contain not only the metric, but also a vector field, which according to the holographic duality couples to a current operator J^a in the boundary. Similarly to the case of pure gravity, the higher-derivative terms permit us to study more general classes of dual CFTs. An important quantity in this regard is the mixed correlator $\langle TJJ \rangle$, which has a fixed form for holographic Einstein-Maxwell (EM) theory, but which for a general CFT may contain an additional structure. The presence of this extra structure can be encoded in the energy-flux parameter a_2 of Ref. [435], which is zero for EM theory, but which can get a non-vanishing value for higher-derivative theories — in particular, it requires non-minimal couplings.

The presence of a vector field also allows us to explore the effect of a chemical potential in the CFT. It is then interesting to study how the holographic predictions for certain properties of the CFT, such as charged entanglement and Rényi entropies [447], change when we vary the couplings of the higher-order terms. Although some of these questions have already been explored, most of the analyses so far have followed a perturbative approach [497, 566–569], or have either stick to particular models, *e.g.*, [558, 559, 570]. On the other hand, our goal is to perform a non-perturbative analysis of this type of theories taking into account all kinds of interactions between gravity and electromagnetism. This includes, in particular, non-minimal couplings of the form RFF, which, to the best of our knowledge, have not been studied in a non-perturbative fashion in the holographic context yet. As we show, these are actually the most interesting terms to be added to the Einstein-Maxwell action due to their effects on the dual CFT.

A key question in order to carry out an exact exploration rather than a perturbative one is to have a bulk theory which is amenable to analytic computations, which is typically not the case when there are higher derivatives involved. From previous chapters, it is clear that EQGs conform very intriguing candidates for these purposes. Consequently, we will generalize the construction of EQGs to arbitrary dimensions and, afterwards, study the holographic aspects of these theories. Apart from establishing basic entries of the holographic dictionary, these theories will inspire us to discover a new universal relation for the charged entanglement entropy, which we will rigorously prove.

This chapter is arranged as follows. First, we describe the proper generalization of EQGs in arbitrary dimensions and display infinite instances of such theories of arbitrary order in the curvature and the field strength. Then, restricting ourselves to the most generic four-derivative EQG, we study asymptotically AdS black hole solutions. Next, some fundamental entries of the holographic dictionary of these theories are settled, in particular those related to $\langle TTT \rangle$ and $\langle TJJ \rangle$ correlators. Then we study the relationship between constraints imposed by causality in the bulk and unitarity in the boundary, as well as those constraints arising from the WGC for the bulk theory. Later we compute the charged Rényi entropies to quadratic order in the chemical potential and we prove a universal formula for the associated charged entanglement entropy in $d(\geq 3)$-dimensional CFTs. Finally, we conclude with a discussion of the most important results.

4.1 Electromagnetic Quasitopological Gravities in arbitrary dimension

4.1.1 Gravity, $(d-2)$-forms and their electromagnetic dual

In this chapter, we consider $(d+1)$-dimensional theories of gravity and of a $(d-2)$-form B with an action given by

$$I = \int \mathrm{d}^{d+1}x\sqrt{|g|}\,\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\cdots\mu_{d-1}})\,, \tag{4.1}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the metric $g_{\mu\nu}$, and the $(d-1)$-form H is the field strength of B, $H = \mathrm{d}B$. The Lagrangian is supposed to be a scalar function built out of these tensors, and we implicitly assume that it has a polynomial form or that it can be expanded as such. In particular, we are interested in theories that reduce to the standard Einstein-$(d-2)$-form Lagrangian for small curvatures and field strengths,

$$\mathcal{L} = \frac{1}{16\pi G}\left[R + \frac{d(d-1)}{L^2} - \frac{2}{(d-1)!}H_{\mu_1\ldots\mu_{d-1}}H^{\mu_1\cdots\mu_{d-1}} + \ldots\right]. \tag{4.2}$$

These theories are invariant under diffeomorphisms and under gauge transformations $B \to B + \mathrm{d}\Lambda$, where Λ is a $(d-3)$-form, and their equations of motion obtained from the variation of the action read

$$P_{(\mu}{}^{\rho\sigma\gamma}R_{\nu)\rho\sigma\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{L} + 2\nabla^\sigma\nabla^\rho P_{(\mu|\sigma|\nu)\rho} - (d-1)\mathcal{M}_{(\mu}{}^{\alpha_1\ldots\alpha_{d-2}}H_{\nu)\alpha_1\ldots\alpha_{d-2}} = 0\,, \tag{4.3}$$

$$\nabla_\mu\mathcal{M}^{\mu\nu_1\ldots\nu_{d-2}} = 0\,, \tag{4.4}$$

where

$$P^{\alpha\beta\rho\gamma} = \frac{\partial\mathcal{L}}{\partial R_{\alpha\beta\rho\gamma}}\,, \quad \mathcal{M}^{\alpha_1\ldots\alpha_{d-1}} = -\frac{1}{2}\frac{\partial\mathcal{L}}{\partial H_{\alpha_1\ldots\alpha_{d-1}}}\,. \tag{4.5}$$

Our interest in these theories lies on the fact that they allow for black hole solutions magnetically charged under the form B, as we explain below. Furthermore, the $(d-2)$-form can be related to a 1-form (a vector field) by means of a duality transformation, and therefore we can map any of these theories to a higher-derivative extension of Einstein-Maxwell theory, which is the interpretation in which we are most interested.

Let us quickly review the process of dualization, which we already explained in four dimensions in Section 2.1.2. Starting from the theory (4.1), we can dualize the $(d-2)$-form B into a 1-form by introducing the Bianchi identity $dH = 0$ in the action as follows[1]

$$\tilde{I} = \int \mathrm{d}^{d+1}x\sqrt{|g|}\left\{\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\cdots\mu_{d-1}}) + \frac{A_{\alpha_1}\partial_{\alpha_2}H_{\alpha_3\ldots\alpha_D}\epsilon^{\alpha_1\ldots\alpha_D}}{4\pi G(d-1)!}\right\}. \tag{4.6}$$

At this point, A_μ is a Lagrange multiplier whose variation yields the Bianchi identity of H, which is now considered as a fundamental variable instead of B. We can integrate by parts to express the action as

$$\tilde{I} = \int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{|g|}\left\{\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho\sigma}, H_{\mu_1\cdots\mu_{d-1}}) + \frac{1}{4\pi G(d-1)!}(\star F)_{\alpha_1\ldots\alpha_{D-2}}H^{\alpha_1\ldots\alpha_{D-2}}\right\}$$

[1]We introduce the factor $1/(4\pi G)$ bearing in mind that the Lagrangian $\mathcal{L}$ will contain an overall $1/(16\pi G)$ normalization.

$$-\frac{1}{4\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{|h|}n^\mu A^\nu (\star H)_{\mu\nu}\,, \tag{4.7}$$

where we have defined $F = dA$. The variation with respect to A_μ still yields the Bianchi identity of H, but now it becomes clear that the variation with respect to H yields an algebraic relation between this field and F, namely

$$F = 4\pi G(d-1)!\star\frac{\partial\mathcal{L}}{\partial H}\,. \tag{4.8}$$

Then, one should invert this relation in order to get $H(F)$, and inserting this back into the action one would get the dual theory for the vector A_μ. Note that the dualization process also generates a boundary term, which is precisely the term that makes the variational principle for the vector well posed, and that, when computing the Euclidean action, corresponds to working in the canonical ensemble (fixed electric charge).

It is important to notice that the dual Lagrangian $\tilde{\mathcal{L}}$ is the Legendre transform of $\mathcal{L}$ with respect to H. Then, by the properties of the Legendre transform one can write the inverse relation between H and F as follows

$$H = -8\pi G\star\frac{\partial\tilde{\mathcal{L}}}{\partial F}\,. \tag{4.9}$$

This relation is useful because it allows us to identify the electric and magnetic charges in both frames. In fact, in the frame of the $(d-2)$-form we will have solutions with magnetic charge, which in the frame of the vector field correspond to electrically charged solutions. We define this charge in either frame as

$$Q = \frac{1}{4\pi G}\int_{\mathcal{S}_{d-1}} H = -2\int_{\mathcal{S}_{d-1}}\star\frac{\partial\tilde{\mathcal{L}}}{\partial F}\,, \tag{4.10}$$

where the integral is performed over any spacelike co-dimension two hypersurface $\mathcal{S}_{d-1}$ that encloses the charge source. In the case of black hole solutions, $\mathcal{S}_{d-1}$ can be any surface that encloses the black hole horizon.

Inverting (4.8) explicitly in order to obtain the dual Lagrangian is in general not possible. However, an important type of theories that we will consider in this chapter are those quadratic in H, and all of them can be written as

$$I = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left\{\mathcal{L}_{\text{grav}} - \frac{2}{(d-1)!}(H^2)_{\mu\nu}{}^{\rho\sigma}Q^{\mu\nu}{}_{\rho\sigma}\right\}, \tag{4.11}$$

where $\mathcal{L}_{\text{grav}} = R + \ldots$ only depends on the curvature, and where we are introducing the notation[2]

$$\left(H^2\right)^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_n} \equiv H^{\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_{d-1}}H_{\nu_1\cdots\nu_n\mu_{n+1}\cdots\mu_{d-1}}\,. \tag{4.12}$$

In this case, it is possible to find the dual theory explicitly. The relation (4.8) can be written in this case as

$$(\star F)_{\alpha_1\ldots\alpha_{d-1}} = Q^{\mu\nu}{}_{[\alpha_1\alpha_2}H_{\alpha_3\ldots\alpha_{d-1}]\mu\nu}\,. \tag{4.13}$$

[2]That the most general quadratic Lagrangian can be written using only the object $(H^2)_{\mu\nu}{}^{\rho\sigma}$ (*i.e*, with only four free indices) can be proven by writing the Lagrangian in terms of $\star H$ first.

This can be inverted in the following way. Let us first introduce the following tensor,

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \frac{12}{(d-1)(d-2)} Q^{[\alpha\beta}{}_{\alpha\beta}\delta^{\mu}{}_{\rho}\delta^{\nu]}{}_{\sigma}\,, \tag{4.14}$$

and its inverse, that we denote by $(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma}$, and which by definition is determined from the equation

$$(\tilde{Q}^{-1})^{\mu\nu}{}_{\alpha\beta}\tilde{Q}^{\alpha\beta}{}_{\rho\sigma} = \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,. \tag{4.15}$$

Then, one can check that (4.13) is inverted by

$$H_{\alpha_1\ldots\alpha_{d-1}} = \frac{1}{2}\epsilon_{\alpha_1\ldots\alpha_{d-1}\rho\sigma}(\tilde{Q}^{-1})^{\rho\sigma}{}_{\alpha\beta}F^{\alpha\beta}\,. \tag{4.16}$$

and the dual action reads simply

$$\begin{aligned}\tilde{I} =&\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\Big\{\mathcal{L}_{\text{grav}} - F_{\mu\nu}F^{\rho\sigma}(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma}\Big\}\\ &+\frac{1}{4\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^{d}x\sqrt{|h|}n^{\mu}A^{\nu}(\tilde{Q}^{-1})^{\alpha\beta}{}_{\mu\nu}F_{\alpha\beta}\,.\end{aligned} \tag{4.17}$$

When the Lagrangian contains terms beyond quadratic order in H, such as $(H^2)^2$, the equation (4.8) becomes a tensorial polynomial equation, whose resolution is more involved. One could nevertheless solve it by assuming a series expansion in F.

4.1.2 Electromagnetic Quasitopological Gravities: general definition

We are interested in studying the charged static solutions with spherical, planar or hyperbolic sections of the theories (4.1). A general metric ansatz for these configurations reads

$$\mathrm{d}s^2_{N,f} = -N^2(r)f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Sigma^2_{k,(d-1)}\,, \tag{4.18}$$

where the metric $\mathrm{d}\Sigma^2_{k,(d-1)}$ is given by

$$\mathrm{d}\Sigma^2_{k,(d-1)} = \begin{cases}\mathrm{d}\Omega^2_{(d-1)} & \text{for } k=1 \text{ (spherical)},\\ \frac{1}{L^2}\mathrm{d}x^2_{(d-1)} & \text{for } k=0 \text{ (flat)},\\ \mathrm{d}\Xi^2_{(d-1)} & \text{for } k=-1 \text{ (hyperbolic)}.\end{cases} \tag{4.19}$$

In addition, we assume the following magnetic ansatz for the H field,

$$H_q = q\,\omega_{k,(d-1)}\,, \tag{4.20}$$

where q is a constant related to the magnetic charge and $\omega_{k,(d-1)}$ is the volume form of $\mathrm{d}\Sigma^2_{k,(d-1)}$, whose integral yields the volume of this space, that we denote by $V_{k,(d-1)} = \int\omega_{k,(d-1)}$. It is obvious that this H satisfies the Bianchi identity $\mathrm{d}H = 0$, but one can also check that, for any theory of the form (4.1), it also solves its equation of motion (4.4) when we use the metric (4.18). Since we do not have to worry about the "Maxwell equation" anymore, the problem of finding the solutions becomes simpler: one only has to solve the equations for the metric functions N and f, that, as shown in [4], can be obtained by means of the reduced Lagrangian,

$$L_{N,f} = \sqrt{|g|}\mathcal{L}|_{ds^2_{N,f},H_q}\,. \tag{4.21}$$

The equations of motion are obtained simply by varying this Lagrangian with respect to the functions f and N,

$$\mathcal{E}_N = \frac{\delta L_{N,f}}{\delta N}\,, \qquad \mathcal{E}_f = \frac{\delta L_{N,f}}{\delta f}\,. \tag{4.22}$$

One can then prove that $\mathcal{E}_N = \mathcal{E}_f = 0$ imply that the Einstein equations (4.3) are satisfied, taking into account that H_q solved its own equation (4.4),

So far the analysis is completely general, but typically one would not be able to solve these equations for a generic Lagrangian. For this reason, it is interesting to restrict ourselves to a subset of theories, introduced as Electromagnetic Quasitopological Gravities (EQG) in [4] (in $d+1=4$), that make possible to perform analytic computations. These theories are simply characterized by the condition that

$$\left.\frac{\delta L_{N,f}}{\delta f}\right|_{N=\text{const.}} \equiv 0 \quad \forall\ f(r)\,. \tag{4.23}$$

In other words, for these theories the reduced Lagrangian $L_{N,f}$ is a total derivative when $N(r)$ takes a constant value. In the purely gravitational case, this definition gives rise to the Generalized Quasitopological Gravities [78,79,216,249,250], which include Quasitopological [238,239,241,242] and Lovelock gravities [225,226,230] as particular cases. Our construction extends the definition of those theories to include a $(d-2)$-form (or equivalently, a vector field upon dualization), allowing one to study charged black hole solutions. Let us note that the standard two-derivative theory (4.2) satisfies (4.23) and therefore belongs to the EQG class. In general, all the theories in this family satisfy a number of properties, which are the same as for their four-dimensional counterparts studied in [4], and that we summarize here.

1. The degrees of freedom that propagate in maximally symmetric backgrounds are the same as in the two-derivative theory. This is particularly relevant for the gravitational sector of the theory, since general higher-order gravities typically propagate a massive ghost-like graviton and a scalar mode along with the massless graviton. The condition (4.23) guarantees that these modes are absent on the vacuum.

2. The theory allows for charged solutions of the form (4.18), (4.20) with $N(r) = N_k =$ const., *i.e*, characterized by a single function $f(r)$.

3. The equation for the function $f(r)$, which is obtained from $\mathcal{E}_N\big|_{N=N_k} = 0$, can be integrated once, and the integration constant is proportional to the total mass of the spacetime.

4. For some theories the integrated equation for $f(r)$ is algebraic and hence it can be solved trivially: if this happens, the theory is of the "Quasitopological" subclass. Other times the integrated equation is a second order ODE for $f(r)$, and that type of theories is of the "Generalized Quasitopological" subclass.

5. In all cases, the thermodynamic properties of charged black holes can be accessed analytically.

In this Chapter we will only deal with the Quasitopological class of Lagrangians, which already constitute a quite extensive set, as we show below.

4.1.3 Four-derivative EQGs

Let us begin by classifying the theories belonging to the EQG family at the four-derivative level. There are four types of terms one could include in the Lagrangian at that order, namely, those of the types R^2, RH^2, H^4 and $(\nabla H)^2$, although our interest lies mostly on the first two. In the case of quadratic curvature Lagrangians, we know there are three independent densities,

$$\mathcal{L}_{R^2} = \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \,, \tag{4.24}$$

but there is only one combination of these that satisfies the "single-function" condition (4.23): the Gauss-Bonnet density (*i.e.*, the quadratic Lovelock Lagrangian),

$$\mathcal{X}_4 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \,. \tag{4.25}$$

That Lovelock gravity satisfies (4.23) and possesses single-function solutions of the form (4.18) is well known [227, 231–233], so let us turn our attention to the next case.

Regarding the operators of the form RH^2, there are again three of them, that can be written as[3]

$$\mathcal{L}_{RH^2} = \alpha_1 H^2 R + \alpha_2 \left(H^2\right)^{\mu}{}_{\nu} R^{\nu}{}_{\mu} + \alpha_3 \left(H^2\right)^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} \,, \tag{4.26}$$

where we recall that we are using the notation introduced in Eq. (4.12). Evaluating this Lagrangian on (4.18) and (4.20) with $N(r)$ equal to a constant value N_k, we obtain

$$\begin{aligned} r^{d-1}\mathcal{L}_{RH^2}\Big|_{N_k,f} = \frac{q^2(d-1)!}{r^{d+1}}\Big[&\left(-2\alpha_3+\alpha_1(1-d)(d-2)-\alpha_2(d-2)\right)(f-k) \\ &+ f'\left(2\alpha_1(1-d)r-\alpha_2 r\right)-\alpha_1 r^2 f''\Big] \,, \end{aligned} \tag{4.27}$$

where we included the factor r^{d-1} from the volume element $\sqrt{|g|}$. In order for this Lagrangian to belong to the EQG family we apply the condition (4.23) that tells us that the quantity above should be a total derivative. It is straightforward to compute the functional derivative of this Lagrangian with respect to f and we find that there is a single condition in order for it to vanish identically,

$$\alpha_3 = -(2d-1)(d-1)\alpha_1 - (d-1)\alpha_2 \,. \tag{4.28}$$

Therefore, there are two linearly independent contractions of the form H^2R that we can add to the two-derivative Lagrangian and maintain single-function solutions. Moving to the next case, in general dimensions there are two independent operators of the form H^4 that do not violate parity, which can be chosen as[4]

$$\mathcal{L}_{H^4} = \beta_1 \left(H^2\right)^2 + \beta_2 \left(H^2\right)^{\mu}{}_{\nu} \left(H^2\right)^{\nu}{}_{\mu} \,. \tag{4.29}$$

[3] There is a fourth contraction of the form $\left(H^2\right)^{\mu\nu}{}_{\rho\sigma} R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}$, but it can be checked that this is related to the term multiplied by α_3 by means of the Bianchi identity of the Riemann tensor.

[4] In order to see that there are only two independent terms, it is clearer to work in terms of the two-form $G = \star H$. There are only two inequivalent quartic contractions: $(G_{\mu\nu}G^{\mu\nu})^2$ and $G_{\mu}{}^{\nu}G_{\nu}{}^{\alpha}G_{\alpha}{}^{\beta}G_{\beta}{}^{\mu}$.

When evaluated on (4.18) and (4.20) we see that both on-shell densities are independent of $f(r)$ and therefore they both belong to the EQG class straightforwardly. However, it will be enough for our purposes to only keep one of them, as both terms contribute to spherical/planar/hyperbolic black hole solutions in the exactly same way. Thus, we will take for simplicity the $(H^2)^2$ operator. Finally, we find that there are no terms of the form $(\nabla H)^2$ belonging to the EQG class.

Therefore, introducing appropriate normalization factors, we have the following four-derivative EQG theory

$$\begin{aligned}I_{\text{EQG},4} =&\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\Bigg[R+\frac{d(d-1)}{L^2}-\frac{2}{(d-1)!}H^2+\frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4\\&+\frac{2\alpha_1 L^2}{(d-1)!}\left(H^2R-(d-1)(2d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right)+\\&+\frac{2\alpha_2 L^2}{(d-1)!}\left(R^{\mu}{}_{\nu}\left(H^2\right)^{\nu}{}_{\mu}-(d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right)+\frac{\beta L^2}{(d-1)!^2}\left(H^2\right)^2\Bigg].\end{aligned} \tag{4.30}$$

This is the theory in which we are going to focus in the rest of the chapter. Certainly, the most interesting part of it is given by the non-minimally coupled terms RH^2, which have not been considered before in the literature.

Interestingly, having four independent parameters, this theory is general enough from the point of view of Effective Field Theory. As shown by Refs. [497,567], an EFT extension of Einstein-Maxwell theory (or in our case, Einstein-$(d-2)$-form theory) only requires four independent parity-preserving terms, as the rest of higher-derivative operators can be removed via field redefinitions. We have checked that our Lagrangian above indeed spans this basis of four independent operators, which means that we can capture any parity-preserving four-derivative correction to Einstein-Maxwell theory. It could be particularly interesting to use it to capture the corrections arising from supersymmetric theories in $d=4$ [571–573]. Although five-dimensional Supergravity theories with higher-derivative corrections also have parity-breaking Chern-Simons terms, which we are not including, it turns out those terms do not affect most (or none) of the results we are going to discuss in this chapter.

There is a crucial difference between our approach and the EFT one, though, which is the fact that we are going to carry out a fully non-perturbative analysis of our theory (4.30), while in EFT one is usually restricted to the linear perturbative regime. Of course, one can always recover this perturbative regime from our analytic and exact results by expanding linearly in the couplings. However, the exact result is clearly more interesting and it could serve as an educated guess for the behavior of these theories and their holographic duals beyond the limited perturbative approach.

Let us close this section by taking note of the electromagnetic dual theory of (4.30). The fact that we have an H^4 term makes it difficult to invert (4.8) explicitly, so obtaining a closed expression for the dual action is involved (although perhaps not impossible). However, it is easy to obtain the dual action if we perform a derivative expansion. In that case we can write $H(F)=H_0(F)+H_2(F)L^2+\mathcal{O}(L^4)$, and the inversion of (4.8) at each order in L is straightforward. We find that the dual theory, to fourth order in derivatives, reads

$$\tilde{I}_{\text{EQT},4}=\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\Bigg[R+\frac{d(d-1)}{L^2}-F^2+\frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4$$

$$+\frac{L^2}{d-2}RF^2\left(3d\alpha_1+\frac{d\alpha_2}{(d-1)}\right)-\frac{2L^2}{d-2}F_{\mu\alpha}F_{\nu}{}^{\alpha}R^{\mu\nu}\left(4(2d-1)\alpha_1+\frac{(3d-2)\alpha_2}{(d-1)}\right)$$
$$+\frac{2L^2}{d-2}F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma}((2d-1)\alpha_1+\alpha_2)+\frac{\beta}{4}L^2(F^2)^2+\mathcal{O}(L^4)\Bigg]\,, \tag{4.31}$$

and it contains an infinite tower of higher-order terms that we could also compute.

We study the black hole solutions of (4.30) in Section 4.2 next.

4.2 AdS vacua and black hole solutions

In this section we focus on the solutions of the theory (4.30), starting by determining its AdS vacua. As is well known, the higher-derivative terms modify the length scale of AdS, $\tilde{L}$, which no longer coincides with the cosmological-constant scale L. It is customary to denote

$$\tilde{L}=\frac{L}{\sqrt{f_\infty}} \tag{4.32}$$

for a dimensionless constant f_∞, so that for pure AdS space the Riemann tensor takes the form

$$R^{\mu\nu}{}_{\rho\sigma}=-\frac{2f_\infty}{L^2}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,. \tag{4.33}$$

Taking this into the Einstein equations (4.3), one finds that f_∞ must satisfy

$$1-f_\infty+\lambda f_\infty^2=0\,, \tag{4.34}$$

which is the well-known result for Gauss-Bonnet (GB) gravity [243]. This polynomial equation has two real roots if $\lambda\leq 1/4$, but only one is continuously connected to the Einstein gravity vacuum when $\lambda=0$, and this is

$$f_\infty=\frac{1}{2\lambda}\left[1-\sqrt{1-4\lambda}\right]\,. \tag{4.35}$$

When $\lambda>1/4$ there is no AdS solution, so this is the maximum value λ can take. As corresponding to Lovelock gravity, but also to the complete family of Generalized Quasitopological Gravities, the linearized gravitational equations around this vacuum are identical to the linearized Einstein equations, up to the identification of an effective Newton's constant that determines the coupling to matter [88]. In the case of GB gravity, the effective Newton's constant reads

$$G_{\text{eff}}=\frac{G}{1-2\lambda f_\infty}\,. \tag{4.36}$$

Observe that the denominator in this expression is the slope of the AdS vacuum equation (4.34). This is in fact no accident and the same property holds for all theories with an Einstein-like spectrum [89, 259]. We also note that G_{eff} is divergent in the limit $\lambda\to 1/4$, which is known as the critical theory [574, 575].

Let us now obtain the static spherically/plane/hyperbolic-symmetric solutions of (4.30). By construction, this theory belongs to the EQG class, and therefore it allows for solutions of the form (4.18) and (4.20)with $N(r)=N_k=$ constant. As a matter of fact, the equation $\delta L_{N,f}/\delta f=0$ computed from the reduced Lagrangian implies that $N'(r)=0$, so that these are the only solutions. Then, we only have to find the function $f(r)$ by solving the equation $\delta L_{N,f}/\delta N|_{N=N_k} = 0$. This equation takes the form of a total derivative — as it should happen according to the results in [216] — and explicitly it reads

$$\begin{aligned}\frac{\delta L_{N,f}}{\delta N} =&\frac{\mathrm{d}}{\mathrm{d}r}\Bigg[(d-1)\frac{r^d}{L^2}\left(1-\frac{L^2}{r^2}(f(r)-k)+\lambda\frac{L^4}{r^4}(f(r)-k)^2\right) \\ &+\frac{2q^2}{d-2}\frac{1}{r^d}\Big(r^2+(d-1)(d-2)L^2\alpha_1 f(r)+(d-2)kL^2\big(3(d-1)\alpha_1+\alpha_2\big)\Big)\Bigg]=0\,.\end{aligned} \tag{4.37}$$

We note that the integrated equation is algebraic in $f(r)$, not differential, which characterizes this theory as belonging to the proper Quasitopological subclass. Let us also remark that in this equation one should take $\lambda = 0$ in $d = 3$, as in that case the GB invariant does not really contribute to the equations of motion (note that the normalization factor of the GB term in (4.30) diverges for $d = 3$, so the limit $d \to 3$ would seem to give a finite contribution[5]). Equating the argument of the derivative to a constant m, which will be related to the physical mass of the black hole, and introducing

$$X := \frac{L^2}{r^2}(f(r)-k)\,, \tag{4.38}$$

we can rewrite the equation as follows,

$$\lambda X^2 - \Gamma(r)X + 1 + \Upsilon(r) = 0\,, \tag{4.39}$$

where

$$\Gamma(r) = 1 - \frac{2\alpha_1 L^2 q^2}{r^{2(d-1)}}\,, \tag{4.40}$$

$$\begin{aligned}\Upsilon(r) =& -\frac{mL^2}{(d-1)r^d}+\frac{2L^2q^2}{(d-1)(d-2)r^{2(d-1)}}\left(1+k(d-2)\frac{L^2}{r^2}(4(d-1)\alpha_1+\alpha_2)\right)\\ &-\frac{\beta L^4 q^4}{(3d-4)(d-1)r^{4(d-1)}}\,.\end{aligned} \tag{4.41}$$

This is simply a quadratic polynomial in X and thus we can solve it straightforwardly obtaining

$$f(r) = k + \frac{r^2}{2\lambda L^2}\left[\Gamma(r) \pm \sqrt{\Gamma^2(r) - 4\lambda(1+\Upsilon(r))}\right]\,. \tag{4.42}$$

We have two roots, that correspond to two solutions connected to different AdS vacua at $r \to \infty$. We should choose the one that reduces to the Einstein gravity result in the limit $\lambda = 0$, and this is the one with the "$-$" sign. It is worth noting that, when $\lambda = 0$ (which is always the case for $d \leq 3$), this solution simply becomes

$$f(r) = k + \frac{r^2(1+\Upsilon(r))}{L^2\Gamma(r)}\,. \tag{4.43}$$

Let us then identify the physical properties of this solution. For $r \to \infty$, $f(r)$ behaves as

$$f(r) = f_\infty\frac{r^2}{L^2} + k - \frac{m}{(d-1)(1-2\lambda f_\infty)r^{d-2}} + \mathcal{O}\left(\frac{1}{r^{2(d-2)}}\right) + \cdots\,, \tag{4.44}$$

[5]This and similar observations were noted by Ref. [576] to propose a non-trivial $D \to 4$ limit for GB gravity, but the validity of this approach has been contested [577–579].

where f_∞ is given by (4.35). Therefore, it asymptotes to the AdS vacuum that we have determined above. On the other hand, the mass M is identified by looking at the following term in the asymptotic expansion of f [86, 90, 91, 514, 580],

$$-\frac{16\pi G_{\text{eff}} M}{(d-1)N_k V_{k,d-1}}\frac{1}{r^{d-2}} \in f(r)\,. \tag{4.45}$$

where G_{eff} is the effective Newton's constant and the factor N_k takes into account the normalization of the time coordinate at infinity, which is equivalent to a change of units. Also note that, in the cases in which the volume of the transverse sections $V_{k,d-1}$ is infinite, one would instead define an energy density $\rho = M/V_{k,d-1}$.

Using the value of G_{eff} given by (4.36), we get that the physical mass of the black hole is

$$M = \frac{N_k V_{k,d-1}}{16\pi G} m\,, \tag{4.46}$$

which is proportional to m, as mentioned before. On the other hand, we define the magnetic charge of the $(d-2)$-form B by

$$Q = \frac{1}{4\pi G}\int_{\mathcal{S}_{d-1}} H\,, \tag{4.47}$$

where the integral is performed over any spacelike co-dimension two hypersurface $\mathcal{S}_{d-1}$ that encloses $r=0$. Note that, as we discussed around (4.10), this quantity is also the electric charge of the dual theory. It is straightforward to see that

$$Q = \frac{V_{k,d-1}}{4\pi G} q\,, \tag{4.48}$$

and again in the cases $k = 0, -1$ one could define instead a charge density $Q/V_{k,d-1}$.

It will also be important for later purposes to determine the electrostatic potential of the dual theory. The field strength of the dual vector field A_μ is obtained according to (4.8). Evaluating that expression on the metric (4.18) and on the H-field (4.20), we find that it corresponds to a pure electric field,

$$\begin{aligned} F =& \mathrm{d}t\wedge \mathrm{d}r N_k q\Bigg[-\frac{1}{r^{d-1}}-\frac{L^2\alpha_1}{r^{d+1}}\left(3d(d-1)k-3d(d-1)f(r)+2(d-1)rf'(r)+r^2f''(r)\right)\\ &-\frac{L^2\alpha_2}{r^{d+1}}\left(dk-df(r)+rf'(r)\right)+\frac{L^2q^2\beta}{r^{3(d-1)}}\Bigg]\,. \end{aligned} \tag{4.49}$$

Surprisingly, this can be written explicitly as a total derivative, $F_{tr} = -\Phi'(r)$, where

$$\begin{aligned} \Phi(r) =& -N_k q\Bigg[\frac{1}{(d-2)r^{d-2}}+\frac{L^2\alpha_1}{r^d}\left(3(d-1)k-3(d-1)f(r)-rf'(r)\right)\\ &+\frac{L^2\alpha_2}{r^d}\left(k-f(r)\right)-\frac{L^2q^2\beta}{(3d-4)r^{3d-4}}\Bigg]+\Phi_\infty \end{aligned} \tag{4.50}$$

is the electrostatic potential. We are adding an integration constant Φ_∞ that represents the value of the potential at infinity.

The solution given by (4.42) represents a black hole as long as the function $f(r)$ has a zero $f(r_+) = 0$ (which would correspond to a horizon) which is smoothly connected to infinity (this is, there should be no singularities between $r = r_+$ and $r \to \infty$). It is easier to look at the position of the horizon directly from (4.39). In fact, at the horizon we have $X(r_+) = -kL^2/r_+^2$, and hence we get

$$\lambda\frac{k^2L^4}{r_+^4} + \Gamma(r_+)\frac{kL^2}{r_+^2} + 1 + \Upsilon(r_+) = 0\,. \tag{4.51}$$

We cannot obtain the value of r_+ explicitly from this equation, but it is useful to express instead the mass as a function of r_+ and the charge,

$$\begin{aligned} M =&\frac{N_k V_{k,d-1}}{16\pi G}\Bigg[(d-1)\Big(kr_+^{d-2} + \frac{r_+^d}{L^2} + \lambda k^2 L^2 r_+^{d-4}\Big) \\ &+ \frac{2q^2}{(d-2)r_+^{d-2}}\left(1 + k(d-2)\frac{L^2}{r_+^2}(3(d-1)\alpha_1 + \alpha_2)\right) - \frac{\beta L^2 q^4}{(3d-4)r_+^{3d-4}}\Bigg]\,. \end{aligned} \tag{4.52}$$

The Hawking temperature of the black hole is given by $T = N_k f'(r_+)/4\pi$. This can be easily evaluated by differentiating the equation (4.39) with respect to r and evaluating at r_+, which yields

$$\begin{aligned} T =&\frac{N_k}{4\pi r_+\left(1 - 2L^2q^2\alpha_1 r_+^{-2(d-1)} + 2kL^2\lambda r_+^{-2}\right)}\Bigg[\left((d-2)k + d\frac{r_+^2}{L^2} + (d-4)k^2\lambda\frac{L^2}{r_+^2}\right) \\ &- \frac{2q^2}{(d-1)r_+^{2(d-1)}}\left(r_+^2 + dkL^2(3(d-1)\alpha_1 + \alpha_2)\right) + \frac{\beta L^2 q^4}{(d-1)r_+^{2(2d-3)}}\Bigg]\,. \end{aligned} \tag{4.53}$$

On the other hand, we impose the electrostatic potential (4.50) to vanish at the horizon.[6] In this way, the asymptotic value of the potential reads

$$\begin{aligned} \Phi_\infty = N_k q\Bigg[&\frac{1}{(d-2)r_+^{d-2}} + \frac{L^2\alpha_1}{r_+^d}\left(3(d-1)k - r_+\frac{4\pi T}{N_k}\right) \\ &+ \frac{L^2\alpha_2 k}{r_+^d} - \frac{L^2q^2\beta}{(3d-4)r_+^{3d-4}}\Bigg]\,. \end{aligned} \tag{4.54}$$

Finally, let us compute the entropy of the black hole. This is given by the Iyer-Wald's formula [366, 370]

$$S = -2\pi\int_\Sigma \mathrm{d}^{d-1}x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\ , \tag{4.55}$$

where h is the determinant of the induced metric at the horizon, and $\epsilon_{\mu\nu}$ is the binormal, normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. Evaluating this expression, one finds the value of the entropy

$$S = \frac{r_+^{d-1}V_{k,d-1}}{4G}\left(1 + \frac{2L^2q^2\alpha_1}{r_+^{2d-2}} + \frac{2L^2k(d-1)\lambda}{(d-3)r_+^2}\right)\,. \tag{4.56}$$

[6] The motivation for this is clearer if one works in Euclidean signature, $t = i\tau$: the vector $A = A_\tau d\tau$ would be singular at $r = r_+$ unless $A_\tau(r_+) = 0$. In Chapters 2 and 3 we did not need to account for this.

Now we devote ourselves to the computation of the free energy through the Euclidean action. For that, let us work in the frame of the $(d-2)$-form B. We first perform a Wick rotation of our black hole solutions by writing $t = i\tau$. The Euclidean time τ has a periodicity $\tau \sim \beta + \tau$, with $\beta = 1/T$, where the temperature is given by (4.53). The bulk part of the Euclidean action reads

$$\begin{aligned} I_{\rm E}^{\rm bulk} = &-\frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{g}\Bigg[R + \frac{d(d-1)}{L^2} - \frac{2}{(d-1)!}H^2 + \frac{\lambda}{(d-2)(d-3)}L^2\mathcal{X}_4 \\ &+ \frac{2\alpha_1 L^2}{(d-1)!}\left(H^2 R - (d-1)(2d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) + \\ &+ \frac{2\alpha_2 L^2}{(d-1)!}\left(R^{\mu}{}_{\nu}\left(H^2\right)^{\nu}{}_{\mu} - (d-1)R^{\mu\nu}{}_{\rho\sigma}\left(H^2\right)^{\rho\sigma}{}_{\mu\nu}\right) + \frac{\beta L^2}{(d-1)!^2}\left(H^2\right)^2\Bigg]\,. \end{aligned} \tag{4.57}$$

On top of this, we need to include generalized York-Gibbons-Hawking boundary terms to make the variational problem well posed [80,81], as well as counterterms, to make the action finite [511]. The generalized YGH term for the Gauss-Bonnet density is known [523, 524], as well as the appropriate conterterms [581]. However, for the sake of simplicity, we can use instead the effective boundary terms proposed in Ref. [257] (see also [582]),

$$I_{\rm E}^{\rm bdry} = -2C\int_{\partial\mathcal{M}} \mathrm{d}^d x\sqrt{h}\left(K - \frac{d-1}{\tilde{L}} - \frac{\tilde{L}\Theta[d-3]}{2(d-2)}\mathcal{R} + \ldots\right)\,. \tag{4.58}$$

Here, K is the trace of the extrinsic curvature of the boundary, $\mathcal{R}$ is the Ricci scalar of the boundary metric, and $\Theta[d-3] = 1$ for $d \geq 3$ and 0 otherwise. Additional $\mathcal{O}(\mathcal{R}^n)$ terms appear for $d \geq 5$. These are simply the same boundary terms as in Einstein gravity, but with a different proportionality constant, which reads

$$C = -\frac{\tilde{L}^2}{2d}\mathcal{L}\Big|_{\rm AdS}\,, \tag{4.59}$$

where $\mathcal{L}|_{\rm AdS}$ is the Lagrangian evaluated on the AdS vacuum to which the solutions asymptote. This prescription is valid for asymptotically AdS solutions (as in our case) and at least for theories that do not propagate additional degrees of freedom over AdS vacua (as in the case of Generalized Quasitopological theories), although we suspect this method actually works for general theories. For our Lagrangian, we have

$$C = \frac{1}{16\pi G}\left(1 - \frac{2(d-1)}{d-3}\lambda f_\infty\right)\,, \tag{4.60}$$

where we recall that $\tilde{L}^2 = L^2/f_\infty$. On the other hand, the variation of the terms RH^2 with respect to the metric decays very fast at infinity, so one does not need to include boundary terms. Also, they behave at infinity as the H^2 term, so no counterterms are needed either.

In order to compute the Euclidean action, we note that the Lagrangian becomes an explicit total derivative when evaluated on (4.18) with $N(r) = N_k = $ const. (this is actually the defining property of the Electromagnetic Quasitopological Gravities). We find

$$16\pi G\mathcal{L}\Big|_{\rm on-shell} = \frac{1}{r^{d-1}}\frac{\mathrm{d}\mathcal{I}(r)}{\mathrm{d}r}\,, \tag{4.61}$$

where

$$
\begin{aligned}
\mathcal{I}(r) =& - r^{d-1}f'(r) - (d-1)r^{d-2}(f(r)-k) + \frac{(d-1)r^d}{L^2} + \frac{2q^2r^{2-d}}{d-2} \\
&- 2\alpha_1 L^2 q^2 r^{-d}\big(3(d-1)(f(r)-k) + rf'(r)\big) - 2\alpha_2 L^2 q^2 r^{-d}(f(r)-k) \\
&+ \frac{(d-1)\lambda}{d-3} r^{d-4}(f(r)-k)\big((d-3)(f(r)-k) + 2rf'(r)\big) + \frac{\beta L^2 q^4 r^{4-3d}}{4-3d}\,.
\end{aligned}
\tag{4.62}
$$

Therefore, the bulk part of the Euclidean action is given by

$$
I_{\mathrm{E}}^{\mathrm{bulk}} = -\frac{\beta N_k V_{k,(d-1)}}{16\pi G}\int_{r_+}^{\infty} \mathrm{d}r\, r^{d-1}\mathcal{L} = \frac{\beta N_k V_{k,(d-1)}}{16\pi G}\Big[\mathcal{I}(r_+) - \mathcal{I}(r\to\infty)\Big]\,. \tag{4.63}
$$

The evaluation at infinity $\mathcal{I}(r\to\infty)$ is divergent, but one can check all these divergencies are exactly cancelled by the boundary contributions (4.58). Furthermore, the boundary terms do not introduce any meaningful finite terms to the on-shell action.[7] Hence, we get

$$
I_{\mathrm{E}} = I_{\mathrm{E}}^{\mathrm{bulk}} + I_{\mathrm{E}}^{\mathrm{bdry}} = \frac{\beta N_k V_{k,(d-1)}}{16\pi G}\mathcal{I}(r_+)\,. \tag{4.64}
$$

Now, let us note that the fact that we are computing the Euclidean action in the frame of the B-form has a non-trivial effect. As we observed in Section 4.1.1, when we dualize the B-form into a vector field, we generate a boundary term in the Maxwell frame, that in the thermodynamic context corresponds to working in the canonical ensemble. This implies that the Euclidean action we have computed corresponds to the Helmholtz free energy $F = TI_{\mathrm{E}}$, which is a function of the temperature and of the charge. From the result above, we find

$$
\begin{aligned}
F = \frac{N_k V_{k,(d-1)}}{16\pi G}\mathcal{I}(r_+) =& \frac{N_k V_{k,(d-1)}}{16\pi G}\Bigg[\frac{(d-1)r_+^d}{L^2} - r_+^{d-1}\frac{4\pi T}{N_k} + k(d-1)r_+^{d-2} + \frac{2q^2 r_+^{2-d}}{d-2} \\
&- 2\alpha_1 L^2 q^2 r_+^{-d}\left(-3k(d-1) + \frac{4\pi T r_+}{N_k}\right) + 2k\alpha_2 L^2 q^2 r_+^{-d} \\
&+ (d-1)\lambda r_+^{d-4} k\left(k - \frac{8\pi T r_+}{(d-3)N_k}\right) + \frac{\beta L^2 q^4 r_+^{4-3d}}{4-3d}\Bigg]\,,
\end{aligned}
\tag{4.65}
$$

We also introduce the chemical potential μ as $\mu = \lim_{r\to\infty}\ell_*^{-1}A_t$, where ℓ_* is a length scale whose inclusion will be justified in Section 4.3. From (4.54) it takes the form

$$
\mu = \frac{N_k q}{\ell_*}\left[\frac{1}{(d-2)r_+^{d-2}} + \frac{L^2\alpha_1}{r_+^d}\left(3(d-1)k - r_+\frac{4\pi T}{N_k}\right) + \frac{L^2\alpha_2 k}{r_+^d} - \frac{L^2 q^2 \beta}{(3d-4)r_+^{3d-4}}\right]. \tag{4.66}
$$

We then check that this free energy satisfies the usual first law[8],

$$
\mathrm{d}F = -S\mathrm{d}T + \mu\,\mathrm{d}\mathcal{N}\,, \tag{4.67}
$$

[7] In odd d some counterterms can introduce contributions of the form $I_{\mathrm{E}} \to I_{\mathrm{E}} + c\beta$, for a constant c, but this simply represents a global shift in the free energy. We will simply assume that these finite counterterms have been chosen so that pure AdS has zero free energy.

[8] Differently from Chapters 2 and 3, here we demanded the gauge vector to vanish at the horizon. This in turn implies that it is its asymptotic value, which we identify with the chemical potential, what appears in the first law.

where S is Wald's entropy given by (4.56), and where $\mathcal{N} = Q\ell_*$, where Q is the physical charge introduced in (4.47), *i.e.*,

$$\mathcal{N} = \frac{V_{k,(d-1)}\ell_* q}{4\pi G}\,, \tag{4.68}$$

and it represents the number of charged particles under the current J in the boundary theory.

We wish to work in the grand canonical ensemble (*i.e.*, at fixed chemical potential), so instead of F we are interested in the grand potential (or grand free energy), defined as

$$\Omega = F - \mu\mathcal{N}\,. \tag{4.69}$$

This can also be obtained directly from the Euclidean action by adding or removing appropriate boundary terms (depending of whether we are in the Maxwell or B-form frames). By construction, this satisfies

$$\mathrm{d}\Omega = -S\mathrm{d}T - \mathcal{N}\mathrm{d}\mu\,, \tag{4.70}$$

and it is to be understood as a function of T and μ. Explicitly, it reads

$$\begin{aligned}\Omega =&\frac{N_k V_{k,(d-1)}}{16\pi G}\Bigg[\frac{(d-1)r_+^d}{L^2} - \frac{2q^2r_+^{2-d}}{d-2} + r_+^{d-2}\left((d-1)k - \frac{4\pi T r_+}{N_k}\right) + \frac{3\beta L^2 q^4 r_+^{4-3d}}{3d-4}\\ &+(d-1)\lambda k L^2 r_+^{-4+d}\left(k - \frac{8\pi T r_+}{N_k(d-3)}\right) + 2\alpha_1 L^2 q^2 r_+^{-d}\left(-3(d-1)k + \frac{4\pi T r_+}{N_k}\right)\\ &-2k\alpha_2 L^2 q^2 r_+^{-d}\Bigg]\,.\end{aligned} \tag{4.71}$$

4.3 Holographic dictionary

The family of Electromagnetic Quasitopological Gravities introduced in Section 4.1 is most naturally written in terms of a $(d-2)$-form field. However, as we saw in Subsection 4.1.1, this $(d-2)$-form can be dualized into a vector field, and hence these theories are actually equivalent to higher-derivative extensions of Einstein-Maxwell theory. While we will perform many computations in the frame of the $(d-2)$-form, their holographic aspects are better understood in terms of the vector field in the "Maxwell frame".

Vector fields in the bulk of AdS couple to currents in the boundary theory. In our case, we are working with a dimensionless gauge field A_μ, but the holographic dictionary actually requires that the vector has dimensions of energy. Thus, the field that couples to the dual current, J^a, is not A_μ but rather

$$\tilde{A}_\mu = \ell_*^{-1}A_\mu\,, \tag{4.72}$$

where ℓ_* is a length scale that should be fixed by the particular duality in each case. Here we do not know what the dual theory is, so we keep ℓ_* general. This implies that, for instance, the chemical potential in the dual theory is identified as

$$\mu = \lim_{r\to\infty}\tilde{A}_t = \lim_{r\to\infty}\ell_*^{-1}A_t\,. \tag{4.73}$$

In this section we compute other entries of the holographic dictionary of these theories: the two-point function $\langle JJ\rangle$ and the energy flux after an insertion of J^a, which is equivalent to the 3-point function $\langle TJJ\rangle$. We also review the $\langle TT\rangle$ and $\langle TTT\rangle$ correlators.

Our goal is to study the electromagnetic dual of the four-derivative Electromagnetic Quasitopological theory given by (4.30). Observe however that the term H^4 will not play any role in this section, since in order to compute $\langle JJ\rangle$ and $\langle TJJ\rangle$ we only need the quadratic terms. Thus, we can ignore the H^4 term for all practical purposes. In addition, in this section we do not really need to stick to the EQG family, so out of generality we can consider the action

$$I=\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[R+\frac{d(d-1)}{L^2}+\frac{\lambda L^2\mathcal{X}_4}{(d-2)(d-3)}-\frac{2(H^2)_{\mu\nu}{}^{\rho\sigma}Q^{\mu\nu}{}_{\rho\sigma}}{(d-1)!}\right], \tag{4.74}$$

where $Q^{\mu\nu}{}_{\rho\sigma}$ contains the three possible couplings at linear order in the curvature,

$$Q^{\mu\nu}{}_{\rho\sigma}=\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\left(1-\alpha_1 L^2 R\right)-\alpha_2 L^2 R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}-\alpha_3 L^2 R^{\mu\nu}{}_{\rho\sigma}\,. \tag{4.75}$$

Then, the tensor $\tilde{Q}$ defined in (4.14) reads

$$\begin{aligned}\tilde{Q}^{\mu\nu}{}_{\rho\sigma}=&\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\left[1-L^2R\left(\alpha_1+\frac{\alpha_2}{d-1}+\frac{2\alpha_3}{(d-1)(d-2)}\right)\right]\\&+2L^2\left(\frac{\alpha_2}{d-1}+\frac{4\alpha_3}{(d-1)(d-2)}\right)R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}-\frac{2\alpha_3}{(d-1)(d-2)}L^2R^{\mu\nu}{}_{\rho\sigma}\,,\end{aligned} \tag{4.76}$$

and we can write the dual theory using the inverse of this tensor as

$$\tilde{I}=\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[R+\frac{d(d-1)}{L^2}+\frac{\lambda L^2\mathcal{X}_4}{(d-2)(d-3)}-(\tilde{Q}^{-1})^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]. \tag{4.77}$$

The EQG case (4.30) is then recovered by setting

$$\alpha_3=-(2d-1)(d-1)\alpha_1-(d-1)\alpha_2\,. \tag{4.78}$$

4.3.1 Stress tensor 2- and 3-point functions

It is a well-known fact that holographic higher-order gravities give rise to a different correlator structure of the dual stress-energy tensor. For the Gauss-Bonnet correction in (4.77) this effect is well known [426, 550, 583], and thus we only need to quote the results from the literature.

The 2-point function of the stress-energy tensor in any CFT has the form

$$\langle T_{ab}(x)T_{cd}(0)\rangle=\frac{C_T}{|x|^{2d}}\mathcal{I}_{ab,cd}(x)\,, \tag{4.79}$$

where $\mathcal{I}_{ab,cd}(x)$ is a fixed tensorial structure and C_T is the central charge. Holographically, this correlator is determined by studying linearized gravitational fluctuations around the AdS vacuum and evaluating the action on this solution. Now, since the linearized equations of GB gravity are identical to those of Einstein gravity upon a renormalization of Newton's constant, the value of C_T is essentially obtained from the one in GR by replacing G by G_{eff} in Eq. (4.36), this is

$$C_T = \frac{(1-2\lambda f_\infty)\Gamma(d+2)}{8(d-1)\Gamma(d/2)\pi^{(d+2)/2}}\frac{\tilde{L}^{d-1}}{G}\,. \tag{4.80}$$

We recall that $\tilde{L} = L/\sqrt{f_\infty}$ is the AdS radius, where f_∞ is given by (4.35).

On the other hand, the 3-point function $\langle TTT\rangle$ in theories that preserve parity is only characterized by three constants [433]. The Ward identity of the stress tensor provides a relation between these constants and the central charge C_T, so only two additional parameters are necessary to determine the 3-point function. These parameters can be chosen to be the coefficients t_2 and t_4 that measure the energy fluxes at infinity after an insertion of the stress tensor [435]. In fact, the explicit relation between the coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of the 3-point function and the parameters t_2 and t_4 was found in Ref. [426].

In holographic Einstein gravity one finds $t_2 = t_4 = 0$, and thus higher-order gravities allow one to explore more general universality classes of dual CFTs. In particular, in Gauss-Bonnet gravity the coefficient t_2 is non-vanishing for $d > 3$ and it reads [426]

$$t_2 = \frac{4\lambda f_\infty}{1-2\lambda f_\infty}\frac{d(d-1)}{(d-2)(d-3)}\,. \tag{4.81}$$

On the other hand, $t_4 = 0$ for the theory (4.77). A non-vanishing t_4 can be achieved by introducing other higher-derivative terms such as Quasitopological [243] and Generalized Quasitopological Gravity [257,270], or more general theories with an Einstein-like spectrum [489]. However, since our focus in this chapter is the presence of non-minimally coupled gauge fields, it will be enough to stick to the case of the Gauss-Bonnet correction.

4.3.2 Current 2-point function

In a CFT, the two-point function of any pair of operators is constrained by conformal symmetry up to a proportionality constant. In the case of a current J^a, we have

$$\langle J_a(x)J_b(y)\rangle = \frac{C_J}{|x-y|^{2(d-1)}}I_{ab}(x-y)\,, \tag{4.82}$$

where the quantity $I_{ab}(x)$ is defined as

$$I^{ab}(x) = g^{ab} - 2\frac{x^a x^b}{x^2}\,, \tag{4.83}$$

and the constant C_J is the central charge of the current J. As a first example, let us compute this constant for a CFT dual to the following theory,

$$\begin{aligned} I_{\text{example}} = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\bigg\{ & R + \frac{d(d-1)}{L^2} - F^2 + \epsilon_1 L^2 R F^2 + \epsilon_2 L^2 R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} \\ & + \epsilon_3 L^2 R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\bigg\}\,. \end{aligned} \tag{4.84}$$

Notice that, in terms of $\tilde{A}_\mu = \ell_*^{-1}A_\mu$, the Maxwell term in the action can be written as $-\frac{1}{4g^2}\tilde{F}^2$, from where we identify the gauge coupling constant g,

$$g^{-2} = \frac{\ell_*^2}{4\pi G}\,. \tag{4.85}$$

Now, in order to compute C_J, we have to consider a small perturbation of A_μ around pure AdS space and to evaluate the action in the corresponding solution with appropriate boundary conditions. Since in this example we do not have a GB term in the action, the AdS curvature is simply

$$R^{\mu\nu}{}_{\rho\sigma} = -\frac{2}{L^2}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,, \tag{4.86}$$

and we have the following

$$\begin{aligned} F^2 - \epsilon_1 L^2 R F^2 - \epsilon_2 L^2 R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - \epsilon_3 L^2 R_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma}\big|_{\text{AdS}} \\ = \left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3\right)F^2\,. \end{aligned} \tag{4.87}$$

Thus, around pure AdS spacetime, the only effect of the non-minimal couplings is to rescale the gauge coupling constant, so that we get an effective constant that reads

$$g_{\text{eff}}^{-2} = g^{-2}\left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3\right)\,. \tag{4.88}$$

Therefore, it is already clear that the central charge C_J in the theory (4.84) is the same one as in Einstein-Maxwell theory, but replacing g by g_{eff}. This yields

$$C_J^{\text{example}} = \left(1 + d(d+1)\epsilon_1 + d\epsilon_2 + 2\epsilon_3\right)C_J^{\text{EM}}\,, \tag{4.89}$$

where the Einstein-Maxwell central charge C_J^{EM} reads[9]

$$C_J^{\text{EM}} = \frac{\Gamma(d)}{\Gamma(d/2-1)}\frac{\ell_*^2\tilde{L}^{d-3}}{4\pi^{d/2+1}G}\,, \tag{4.90}$$

and in this case $\tilde{L} = L$. Note that unitarity requires that $C_J > 0$, which sets a bound on the couplings ϵ_i.

Let us now turn to the case of interest here, corresponding to the theory for the $(d-2)$-form (4.74), which we expressed in the Maxwell frame in (4.77). The most difficult aspect of this theory is that it involves computing the inverse of a tensor, $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$. However, this can be trivially inverted on an AdS background. On account of the GB term, the AdS radius is in this case is $\tilde{L} = L/\sqrt{f_\infty}$, and when evaluated on the curvature tensor (4.33), both tensors (4.75) and (4.76) take the following value

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\text{eff}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,, \tag{4.91}$$

where

$$\alpha_{\text{eff}} = 1 + f_\infty\alpha_1 d(d+1) + f_\infty\alpha_2 d + 2f_\infty\alpha_3\,. \tag{4.92}$$

Thus, the inverse of this tensor is simply

$$(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma} = \frac{1}{\alpha_{\text{eff}}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,. \tag{4.93}$$

Therefore, around an AdS vacuum, the quadratic term of the field $\tilde{A}_\mu = \ell_*^{-1}A_\mu$ in the action (4.77) is given by

$$\mathcal{L}_{\tilde{F}^2} = -\frac{1}{4g_{\text{eff}}^2}\tilde{F}^2\,, \quad g_{\text{eff}}^2 = \frac{4\pi G}{\ell_*^2}\alpha_{\text{eff}}\,. \tag{4.94}$$

[9]This charge is four times that of [447] to account for the different normalization of the vector field.

Following the same logic as in the previous example, we conclude that the central charge C_J is the same as for Einstein-Maxwell theory, but rescaled by the constant α_{eff},

$$C_J = \frac{C_J^{\text{EM}}}{\alpha_{\text{eff}}} \,. \tag{4.95}$$

Interestingly, since the duality transformation has the effect of inverting the effective gauge coupling, the combination α_{eff} appears in the denominator rather than in the numerator of C_J. Thus, the 2-point function can now diverge for finite values of the couplings α_i while it vanishes if we take any of these couplings to infinity. In any case, due to unitarity we have to impose the constraint

$$\alpha_{\text{eff}} > 0 \,, \tag{4.96}$$

which sets a bound on the α_i parameters. For the Electromagnetic Quasitopological Gravity (4.30), this reduces to the result

$$\alpha_{\text{eff}}^{\text{EQG}} = 1 - f_\infty \alpha_1 (3d^2 - 7d + 2) - f_\infty \alpha_2 (d-2) \,. \tag{4.97}$$

4.3.3 Energy fluxes

We wish now to perform a conformal collider thought experiment as introduced in Ref. [435]. Consider a CFT$_d$ in flat space $ds^2 = -dt^2 + \delta_{ij} dx^i dx^j$ in its vacuum state, that we denote by $|0\rangle$. For future reference, we note that the bulk geometry dual to this CFT in this state is pure AdS in the Poincaré patch, expressed as

$$\mathrm{d}s^2 = \frac{\tilde{L}^2}{z^2} \left[-(\mathrm{d}x^0)^2 + \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{d}z^2 \right] \,, \tag{4.98}$$

with $x^0 = t$. We then want to perform an insertion of a current operator of the form $\epsilon_i J^i$, where ϵ_i is a constant polarization tensor, and we wish to obtain the energy flux measured at infinity. More precisely, we consider an operator of the form

$$\mathcal{O}_E = \int \mathrm{d}^d x \epsilon_i J^i e^{-iEx^0} \psi(x/\sigma) \,, \tag{4.99}$$

where $\psi(x/\sigma)$ is a distribution function that localizes the insertion at $x^a = 0$ for $\sigma \to 0$, and E is the energy. In terms of the cartesian coordinates x^a, the operator for the energy flux in the direction $\vec{n}$ is given by

$$\mathcal{E}(\vec{n}) = \lim_{r \to \infty} r^{d-2} \int_{-\infty}^{\infty} \mathrm{d}x^0 \, T^0{}_i \left(x^0, r\vec{n} \right) n^i \,, \tag{4.100}$$

where $r^2 \equiv \delta_{ij} x^i x^j$. We are interested in the expectation value for the energy flux after the insertion of the operator $\mathcal{O}_E$,

$$\langle \mathcal{E}(\vec{n}) \rangle_{\mathcal{O}_E} = \frac{\langle 0 | \mathcal{O}_E^\dagger \mathcal{E}(\vec{n}) \mathcal{O}_E | 0 \rangle}{\langle 0 | \mathcal{O}_E^\dagger \mathcal{O}_E | 0 \rangle} \,. \tag{4.101}$$

By making use of the $O(d-1)$ symmetry of the problem, one can then see that the expectation value of this energy flux takes the form [435]

$$\langle \mathcal{E}(\vec{n}) \rangle_J = \frac{E}{\Omega_{(d-2)}} \left[1 + a_2 \left(\frac{|\epsilon \cdot n|^2}{|\epsilon|^2} - \frac{1}{d-1} \right) \right] \,, \tag{4.102}$$

where $\Omega_{(d-2)}$ is the volume of the $(d-2)$-sphere of unit radius and a_2 is a theory-dependent constant. By the construction of $\langle\mathcal{E}(\vec{n})\rangle$, it is clear that it involves an integrated $\langle TJJ\rangle$ correlator over an integrated two-point function $\langle JJ\rangle$. As it turns out, the three-point function $\langle TJJ\rangle$ is constrained by conformal symmetry up to two constants. The parameter a_2 is clearly a function of these constants, and the Ward symmetry of the stress-energy tensor provides an additional relation between these and C_J. Therefore, the 3-point function $\langle TJJ\rangle$ is fully determined by the central charge C_J together with the parameter a_2. We show the explicit relation in the next section.

Holographically, the energy fluxes can be obtained by evaluating the gravitational action on the background of a shock wave, given by the metric

$$\mathrm{d}s^2 = \frac{\tilde{L}^2}{u^2}\left[\delta(y^+)\mathcal{W}\left(y^i,u\right)\left(\mathrm{d}y^+\right)^2 - \mathrm{d}y^+\mathrm{d}y^- + \sum_{i=1}^{d-2}\left(\mathrm{d}y^i\right)^2 + \mathrm{d}u^2\right]. \tag{4.103}$$

It is important that the coordinates (y^a,u) are not the same as the original cartesian coordinates (x^a,z) of (4.98), but related to them according to

$$y^+ = -\frac{1}{x^+}\,, \quad y^- \equiv x^- - \frac{\sum_{i=1}^{d-2}(x^i)^2}{x^+} - \frac{z^2}{x^+}\,, \quad y^i \equiv \frac{x^i}{x^+}\,, \quad u = \frac{z}{x^+}\,, \tag{4.104}$$

for $i = 1,2,\ldots,d-2$, and where $x^\pm = x^0 \pm x^{d-1}$. We refer to the Refs. [243, 435] for additional details on this construction. This metric is a solution of the gravitational field equations if $\mathcal{W}$ satisfies the equation

$$\partial_u^2\mathcal{W} - \frac{d-1}{u}\partial_u\mathcal{W} + \sum_{i=1}^{d-2}\partial_i^2\mathcal{W} = 0\,, \tag{4.105}$$

which holds for Einstein gravity and for general higher-derivative extensions of it [584]. We are interested in the following solution of the previous equation,

$$\mathcal{W}(y^i,u) = \frac{\mathcal{W}_0\,u^d}{\left(u^2 + \sum_{i=1}^{d-2}(y^i - y_0^i)^2\right)^{d-1}}\,, \tag{4.106}$$

where $\mathcal{W}_0$ is a normalization constant and $y_0^i = n^i/(1+n^{d-1})$, where n^i are the components of the vector $\vec{n}$ in the frame described by the coordinates x^i, related to y^i, y^+ and y^- as given in (4.104).

Now, since we want to measure energy fluxes of an excited state, we must consider a perturbation of the vector field A_μ on top of this background. In particular, an insertion with the operator (4.100) is dual to a non-normalizable perturbation of the vector field. Choosing for instance a constant polarization in the x^1 direction, this means that we must consider a vector with boundary condition $A_{x^1} \propto z^0 e^{-iEx^0}$ when $z \to 0$. When extended to the bulk and expressed in the (y^a,u) coordinate system, it is known [435] that this kind of perturbation behaves near $y^+ = 0$ as

$$A_{y^1}(y^+ \approx 0, y^-, y^i, u) \sim e^{iEy^-/2}\delta(y^1)\ldots\delta(y^{d-2})\delta(u-1)\,. \tag{4.107}$$

This will be important later, as the shock wave is localized at $y^+ = 0$ and hence we will eventually have to evaluate A_μ at $y^+ = 0$.

Working directly in terms of the (y^a, u) coordinates, we may simply consider a perturbation of the form

$$A = \mathrm{d}y^1 A_{y^1} + \mathrm{d}y^+ A_{y^+} \,. \tag{4.108}$$

The non-vanishing components of its field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are simply

$$F_{\mu\nu} = 2\partial_{[\mu|} A_{y^1} \delta^1_{|\nu]} + 2\partial_{[\mu|} A_{y^+} \delta^+_{|\nu]} \,. \tag{4.109}$$

In principle, the dynamics of the field A is determined by the action with higher-order corrections, in the background (4.103). However, if we ignore contact terms (this is, terms of the form $A\mathcal{W}$) in its equations of motion, they reduce simply to Maxwell's equations,

$$\nabla_\mu F^{\mu\nu} = 0 \,, \tag{4.110}$$

in the same way that the dual Lagrangian on vacuum AdS is equal to the Maxwell Lagrangian with a modified coupling constant. By imposing the following condition,

$$\partial_- A_{y^+} = \frac{1}{2}\partial_{y^1} A_{y^1} \,, \tag{4.111}$$

which ensures that the perturbation is transverse, $\nabla_\mu A^\mu = 0$, the Maxwell equation is reduced to the following equation for A_{y^1}

$$-4\partial_+\partial_- A_{y^1} + \partial_u^2 A_{y^1} - \frac{d-3}{u}\partial_u A_{y^1} + \sum_{i=1}^{d-2}\partial_i^2 A_{y^1} = 0 \,. \tag{4.112}$$

The solution to this equation with the boundary conditions discussed above (note that they are expressed in terms of the x coordinates) then develops the behavior in (4.107).

In order to compute the energy flux we have to evaluate the on-shell action and extract the piece proportional to $\mathcal{W}A^2$ (since this is the piece in the action that couples to TJJ). For our theory (4.77), this requires us to evaluate first the tensor $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$, and then compute the components of its inverse $(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma}$ using the relation (4.15). The tensor $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$ is given by (4.76), and taking into account that the shockwave (4.103) is an Einstein space satisfying

$$R_{\mu\nu} = -\frac{df_\infty}{L^2} g_{\mu\nu} \,, \tag{4.113}$$

we find that

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\text{eff}} \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} - \frac{2\alpha_3}{(d-1)(d-2)} L^2 W^{\mu\nu}{}_{\rho\sigma} \,. \tag{4.114}$$

Here the constant α_{eff} is given by (4.92) and $W^{\mu\nu}{}_{\rho\sigma}$ is the Weyl tensor, whose non-vanishing components read

$$\begin{aligned}
W^{-i}{}_{+j} &= \delta(y^+)\frac{f_\infty u}{L^2}\left[u\partial_i\partial_j\mathcal{W} - \delta^i_j\partial_u\mathcal{W}\right] \,, \\
W^{-i}{}_{u+} &= W^{u-}{}_{+i} = -\delta(y^+)\frac{f_\infty}{L^2}u^2\partial_i\partial_u\mathcal{W} \,, \\
W^{u-}{}_{u+} &= \delta(y^+)\frac{f_\infty u}{L^2}\left[u\partial_i\partial_i\mathcal{W} - (d-2)\partial_u\mathcal{W}\right] \,,
\end{aligned} \tag{4.115}$$

plus those obtained interchanging indices. These expressions have been simplified by using the equation of motion (4.105), since we will use them to evaluate the on-shell action. We note that, as corresponding to a wave, the Weyl tensor satisfies

$$W^{\mu\nu}{}_{\rho\sigma} W^{\rho\sigma}{}_{\alpha\beta} = 0 \,, \tag{4.116}$$

and therefore, the inverse of $\tilde{Q}$ simply reads

$$(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma} = \frac{1}{\alpha_{\rm eff}}\delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]} + \frac{2\alpha_3}{\alpha_{\rm eff}^2(d-1)(d-2)}L^2 W^{\mu\nu}{}_{\rho\sigma} \,. \tag{4.117}$$

We are then ready to evaluate the on-shell action (4.77). Since we are only interested in the piece of the form $\mathcal{W}A^2$, we only need to compute the following term,

$$\begin{aligned}\tilde{I}_{WA^2} &= -\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}(\tilde{Q}^{-1})^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\ &= -\frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\left[\frac{1}{\alpha_{\rm eff}}F^2 + \frac{2\alpha_3 L^2}{\alpha_{\rm eff}^2(d-1)(d-2)}W^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right].\end{aligned} \tag{4.118}$$

Since the only component of the inverse metric that depends on $\mathcal{W}$ is g^{--}, we have

$$F^2 = 2(F_{-1})^2 g^{--}g^{11} + \ldots = -\frac{8f_\infty^2 u^4\delta(y^+)\mathcal{W}}{L^4}(\partial_- A_{y^1})^2 + \ldots \,, \tag{4.119}$$

where the ellipsis denote terms that do not depend on $\mathcal{W}$ and therefore are irrelevant for this computation. On the other hand, we have

$$W^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4W^{-1-1}(F_{-1})^2 = -\delta(y^+)\frac{8f_\infty^3 u^6}{L^6}\left[\partial_1^2\mathcal{W} - \frac{1}{u}\partial_u\mathcal{W}\right](\partial_- A_{y^1})^2 \,. \tag{4.120}$$

Then, putting these two contributions together and integrating by parts, we find

$$\tilde{I}_{WA^2} = -\frac{1}{4\pi G\alpha_{\rm eff}}\int \mathrm{d}u\mathrm{d}^d y\frac{\tilde{L}^{d-3}}{u^{d-3}}\delta(y^+)\mathcal{W}A_{y^1}\partial_-^2 A_{y^1}\left[1 + \frac{2f_\infty\alpha_3}{\alpha_{\rm eff}(d-1)(d-2)}T_2\right] \,, \tag{4.121}$$

where we defined

$$T_2 = \frac{u(u\partial_1\partial_1\mathcal{W} - \partial_u\mathcal{W})}{\mathcal{W}} \,. \tag{4.122}$$

Since the shock wave localizes the integral to $y^+ = 0$, and since A_{y^1} behaves as in (4.107), we have to evaluate the integrand at $u = 1$ and $y^i = 0$, which can be done in a straightforward manner by plugging in the solution for $\mathcal{W}$ (4.106). Taking into account that the perturbation in (4.108) has a polarization $\epsilon = (\epsilon_1, 0, \ldots, 0)$, we have the following value of T_2,

$$T_2\Big|_{u=1,y^i=0} = d(d-1)\left(n_1^2 - \frac{1}{d-1}\right) = d(d-1)\left(\frac{|\epsilon\cdot n|^2}{|\epsilon|^2} - \frac{1}{d-1}\right) \,. \tag{4.123}$$

Therefore, comparing the expressions of the energy flux (4.102) and the on-shell action (4.121), we immediately read off the coefficient a_2,

$$a_2 = \frac{2d\alpha_3 f_\infty}{(d-2)\alpha_{\rm eff}} = \frac{2d\alpha_3 f_\infty}{(d-2)(1 + f_\infty\alpha_1 d(d+1) + f_\infty\alpha_2 d + 2f_\infty\alpha_3)} \,, \tag{4.124}$$

where we have made use of (4.92). In the case of EQG, given by the action (4.30), this result reduces to

$$a_2^{\rm EQG} = -\frac{2d(d-1)\left((2d-1)\alpha_1 + \alpha_2\right)f_\infty}{(d-2)(1 - (3d^2 - 7d + 2)f_\infty\alpha_1 - (d-2)f_\infty\alpha_2)} \,. \tag{4.125}$$

4.3.4 Three-point function $\langle TJJ\rangle$

The three point correlator $\langle TJJ\rangle$ in position space in a CFT is constrained by conformal symmetry to have the form [433, 434]

$$\langle T_{ab}(x_1)J_c(x_2)J_d(x_3)\rangle = \frac{t_{abef}(X_{23})g^{eg}g^{fh}I_{cg}(x_{21})I_{dh}(x_{31})}{|x_{12}|^d|x_{13}|^d|x_{23}|^{d-2}}\,, \tag{4.126}$$

where $I_{ab}(x)$ is the structure introduced in Eq. (4.83) and

$$\begin{aligned} t_{abcd}(X^a) &= \hat{a}h^{(1)}_{ab}(\hat{X}^a)g_{cd} + \hat{b}h^{(1)}_{ab}(\hat{X}^a)h^{(1)}_{cd}(\hat{X}^a) + \hat{c}h^{(2)}_{abcd}(\hat{X}^a) + \hat{e}h^{(3)}_{abcd}\,,\\ h^{(1)}_{ab}(\hat{X}^a) &= \hat{X}_a\hat{X}_b - \frac{1}{d}g_{ab}\,,\\ h^{(2)}_{abcd}(\hat{X}^a) &= 4\hat{X}_{(a}g_{b)(d}\hat{X}_{c)} - \frac{4}{d}\hat{X}_a\hat{X}_b g_{cd} - \frac{4}{d}\hat{X}_c\hat{X}_d g_{ab} + \frac{4}{d^2}g_{ab}g_{cd}\,,\\ h^{(3)}_{abcd} &= g_{ac}g_{bd} + g_{ad}g_{bc} - \frac{2}{d}g_{ab}g_{cd}\,, \end{aligned} \tag{4.127}$$

where we also have

$$x^a_{12} = x^a_1 - x^a_2\,, \quad X^a_{23} = \frac{x^a_{13}}{|x_{13}|^2} - \frac{x^a_{23}}{|x_{23}|^2}\,, \quad \hat{X}^a_{12} = \frac{X^a_{12}}{|X_{12}|}\,, \tag{4.128}$$

and so on with their corresponding permutations. This expression depends on four theory-dependent constants $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{e}$. However, only two of them are free parameters because of the following constraints coming from current conservation:

$$d\hat{a} - 2\hat{b} + 2(d-2)\hat{c} = 0, \qquad \hat{b} - d(d-2)\hat{e} = 0\,. \tag{4.129}$$

Following Ref. [447], we will work in terms of $\hat{c}$ and $\hat{e}$. In addition, there is one Ward identity that relates the central charge C_J to these coefficients, namely,

$$C_J = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d+2}{2}\right)}(\hat{c}+\hat{e})\,. \tag{4.130}$$

This reduces the number of independent parameters to just one, and this one can be related to the coefficient a_2 entering into the expectation value of the energy flux (4.102). As is clear from Eq. (4.101), this flux involves an integrated $\langle TJJ\rangle$ correlator, and therefore it is a somewhat straightforward (but tedious) field theory computation to obtain the desired relationship. This was performed in general dimensions by Ref. [585], finding the result[10]

$$a_2 = \frac{(d-1)((d-2)d\hat{e}-\hat{c})}{(d-2)(\hat{c}+\hat{e})}\,. \tag{4.131}$$

In this way, we can fully determine the 3-point function $\langle TJJ\rangle$ from C_J and a_2. Inverting the two equations above we can indeed write

$$\hat{c} = \frac{C_J(d-2)\Gamma\left(\frac{d+2}{2}\right)}{2\pi^{d/2}(d-1)^3}\left(d(d-1)-a_2\right)\,, \tag{4.132}$$

[10]The final result offered by Ref. [585] (their formula (6.14)) has a minus sign with respect to the value we show here. However, we have reviewed their computations and we believe that this sign is a typo. Also, our formula here coincides with the value of a_2 for the $d = 4$ case provided by Ref. [435].

$$\hat{e} = \frac{C_J \Gamma\left(\frac{d+2}{2}\right)}{2\pi^{d/2}(d-1)^3}\left(d-1+(d-2)a_2\right) . \tag{4.133}$$

Finally, using the values of a_2 and C_J found for our EQGs, given by Eqs. (4.125) and (4.95) respectively, we have

$$\hat{c}^{\text{EQG}} = \frac{d(d-2)^2 d! L^{d-3}\ell_*^2\left[1-\frac{(d-1)}{d-2}(3d^2-10d+2)f_\infty\alpha_1 - \frac{d^2-4d+2}{d-2}f_\infty\alpha_2\right]}{2^5(d-1)^2\pi^{d+1}f_\infty^{(d-3)/2}G\left[1-(d-2)f_\infty\left((3d-1)\alpha_1+\alpha_2\right)\right]^2} , \tag{4.134}$$

$$\hat{e}^{\text{EQG}} = \frac{d(d-2)(d-2)! L^{d-3}\ell_*^2\left[1-(d-1)(7d-2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2\right]}{2^5(d+1)\pi^{d+1}f_\infty^{(d-3)/2}G\left[1-(d-2)f_\infty\left((3d-1)\alpha_1+\alpha_2\right)\right]^2} . \tag{4.135}$$

This result will be important for us in Section 4.5.

4.4 Causality, unitarity and constraints from the Weak Gravity Conjecture

The theory (4.30), which is going to be the focus of our holographic explorations in Sections 4.4 and 4.5, depends on four free parameters. As we have seen in the previous section, these parameters modify several entries of the holographic dictionary allowing us to probe more general universality classes of holographic CFTs than those covered by Einstein-Maxwell theory. However, they are not completely free, as one must demand that the hypothetical dual theory satisfies reasonable physical properties, such as unitarity. Thus, we must determine the allowed values of these parameters if we want to obtain any sensible answers from holography.

4.4.1 Unitarity in the boundary

In the boundary theory, several constraints are found by demanding that the different correlators and energy fluxes defined in the previous section respect unitarity.

There is an even more fundamental condition that our theory must satisfy: the existence of an AdS vacuum. From Eq. (4.35), which determines the AdS scale $\tilde{L} = L/\sqrt{f_\infty}$ we see that this happens if

$$\lambda \leq \frac{1}{4} , \quad (d > 3) , \tag{4.136}$$

which we take into account from the start.

Constraints from $\langle TT \rangle$ and $\langle TTT \rangle$

One first condition comes from demanding that the central charge of the stress-tensor two-point function be positive, $C_T > 0$. This is also directly interpreted as a unitarity condition in the bulk, as it is equivalent to imposing $G_{\text{eff}} > 0$, hence preventing the graviton from having a negative energy. In the presence of the Gauss-Bonnet term, the central charge is given by (4.80), and therefore we must impose

$$1 - 2\lambda f_\infty > 0 . \tag{4.137}$$

One can see this is always satisfied for all the allowed values of λ, $\lambda < 1/4$, and therefore this condition does not provide additional constraints.

On the other hand, a stronger bound is achieved by demanding positivity of the energy 1-point function. Analogously to what we saw in Subsection 4.3.3, the expectation value of the energy flux produced after an insertion of the stress-energy tensor $\epsilon_{ij}T^{ij}$ in general reads [435]

$$\langle\mathcal{E}(\vec{n})\rangle_T = \frac{E}{\Omega_{(d-2)}}\left[1 + t_2\left(\frac{\epsilon^*_{ij}\epsilon_{il}n^j n^l}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{1}{d-1}\right) + t_4\left(\frac{|\epsilon_{ij}n^i n^j|}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{2}{d^2-1}\right)\right] . \tag{4.138}$$

For holographic CFTs dual to (4.30), we have $t_4 = 0$ while t_2 is given by (4.81). The energy flux must be positive in any direction n^i and for any choice of polarization ϵ_{ij}. These conditions were analyzed by Ref. [426] in general dimensions, finding that λ is bound to the following interval,

$$-\frac{(3d+2)(d-2)}{4(d+2)^2} \leq \lambda \leq \frac{(d-2)(d-3)(d^2-d+6)}{4(d^2-3d+6)^2} . \tag{4.139}$$

We note that $\lambda = 1/4$ is not allowed by the upper bound in any dimension, while the lower bound prevents λ to become too negative.

Constraints from $\langle JJ\rangle$ and $\langle TJJ\rangle$

The unitarity constraints on the Gauss-Bonnet coupling were known since Refs. [426, 550, 583]. Let us now discuss the novel constraints on the parameters α_1 and α_2 of the non-minimally coupled terms. These work very similarly to the gravitational case and follow from the unitarity of $\langle JJ\rangle$ and the energy one-point function.

The central charge of the current two-point function is given by Eq. (4.95), and, as we already discussed there, its positivity implies that

$$\alpha_{\text{eff}}^{\text{EQG}} = 1 - f_\infty\alpha_1(3d^2-7d+2) - f_\infty\alpha_2(d-2) > 0 . \tag{4.140}$$

Again, since this quantity is, up to a constant, the coupling constant of the Maxwell field, its positivity is equivalent to demanding that photons carry positive energy in the bulk.

We can obtain more interesting bounds from the energy flux created after an insertion of the current operator, given by (4.102). Demanding that the energy flux is positive in any direction, we find that the parameter a_2 must satisfy

$$-\frac{d-1}{d-2} \leq a_2 \leq d-1 , \tag{4.141}$$

where the upper bound comes from $\vec{n} \perp \vec{\epsilon}$ and the lower bound from $\vec{n} \propto \vec{\epsilon}$. Using the value of a_2 for our Electromagnetic Quasitopological Gravities, given by (4.125), this translates into

$$-1 \leq -\frac{2d\left((2d-1)\alpha_1 + \alpha_2\right)f_\infty}{1-(3d^2-7d+2)f_\infty\alpha_1-(d-2)f_\infty\alpha_2} \leq d-2 . \tag{4.142}$$

Now, since the denominator of this expression is precisely $\alpha_{\text{eff}}^{\text{EQG}}$, which is assumed to be positive, by multiplying the whole inequality by $\alpha_{\text{eff}}^{\text{EQG}}$ we can express the two constraints as follows

$$1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2 \geq 0 , \tag{4.143}$$

$$1-\frac{(d-1)(3d^2-14d+4)}{(d-2)}f_\infty\alpha_1-\frac{d^2-6d+4}{(d-2)}f_\infty\alpha_2\geq 0\,. \tag{4.144}$$

One should not forget to impose (4.140) together with these constraints. We note that the last inequality has a different character depending on the dimension: the coefficient of α_1 is positive for $d=3,4$ and negative for $d\geq 5$, while that of α_2 is positive for $d=3,4,5$ and changes sign for $d\geq 6$. For instance, if $\alpha_2=0$ we find that α_1 must lie within the interval

$$-\frac{(d-1)(-3d^2+14d-4)}{(d-2)}\leq f_\infty\alpha_1\leq\frac{1}{7d^2-9d+2}\,,\quad d=3,4\,,\;\alpha_2=0\,, \tag{4.145}$$

but the lower bound disappears for $d\geq 5$.

We note that the bounds are imposed directly on the renormalized couplings $f_\infty\alpha_{1,2}$ rather than on the original couplings. However, observe that the value of f_∞ is always close to one for the allowed values of λ in (4.139) (and it is one in $d=3$). In Fig. 4.1 we show the different constraints and the allowed region in the $(f_\infty\alpha_1, f_\infty\alpha_2)$ plane. We see that the permitted region grows bigger with the dimension. A very interesting property is that, in $d=3,4,5$, there is an absolute upper bound for α_1, regardless of the value of α_2. This value is found at the intersection of the three constraints and it reads

$$f_\infty\alpha_1\leq\frac{1}{d(d-2)}\,,\quad (d=3,4,5)\,. \tag{4.146}$$

Likewise, there is an absolute lower bound for α_2 in $d=3,4$:

$$f_\infty\alpha_2\geq-\frac{2d-1}{d(d-2)}\,,\quad (d=3,4)\,. \tag{4.147}$$

For higher dimensions, these parameters can take values in the full real line, but interestingly they both cannot be too positive. In fact, only very small values are allowed in that case, as follows from the graph (d) in Fig. 4.1.

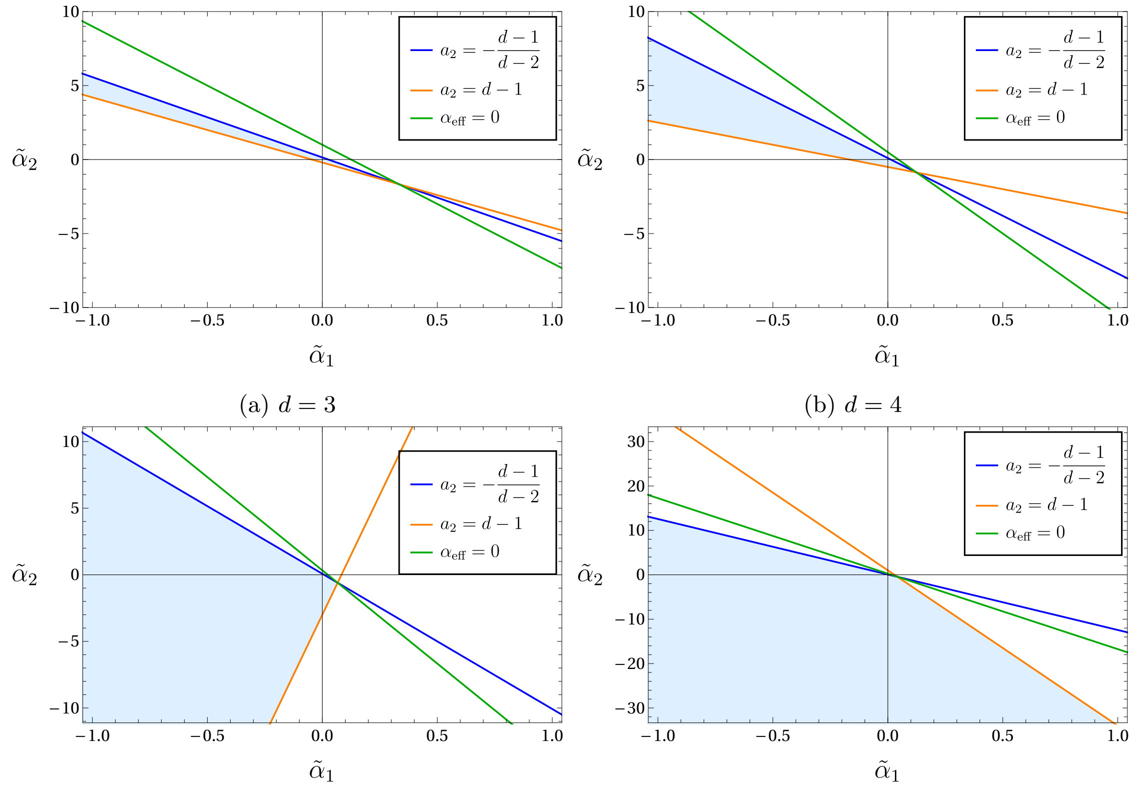
4.4.2 Causality in the bulk

On general grounds, it is to be expected that physically consistent bulk theories give rise to consistent dual CFTs, and vice versa. Hence, the unitarity constraints we have discussed must also have a meaning in the bulk. In the case of the constraints coming from the two-point functions $\langle TT\rangle$ and $\langle JJ\rangle$, the interpretation is direct, as the positivity of the central charges is related to that of the energy of gravitational and electromagnetic waves in the bulk. However, the bulk interpretation of the constraints coming from the positivity of the energy one-point function is more subtle. At least in the case of Lovelock gravity, it is known that demanding $\langle\mathcal{E}(\vec{n})\rangle_T\geq 0$ is equivalent to enforcing the bulk theory to respect causality [436,438,583,586], in the sense that one avoids superluminal propagation of gravitational waves [557,559,586,587].[11] Here we investigate the analogous connection between causality of electromagnetic waves and positivity of $\langle\mathcal{E}(\vec{n})\rangle_J$, given by (4.102).

Our starting point is a neutral planar black hole solution of the theory (4.30), with a metric

$$\mathrm{d}s^2=-\frac{f(r)}{f_\infty}\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f(r)}+\frac{r^2}{L^2}\mathrm{d}x^2_{(d-1)}\,, \tag{4.148}$$

[11] This connection is less clear in other theories outside the Lovelock family [243,425].



(a) $d = 3$ (b) $d = 4$

(c) $d = 5$ (d) $d = 6$

Figure 4.1: Bounds in the constants $\tilde{\alpha}_1 = f_\infty \alpha_1$ and $\tilde{\alpha}_2 = f_\infty \alpha_2$ obtained from unitarity and positivity of energy fluxes, given by (4.140), (4.143) and (4.144), in different dimensions. The allowed region in each case is shaded in blue, and is infinite. For any $d > 6$ the allowing region looks qualitatively similar to that obtained for $d = 6$.

where the function $f(r)$ is given by

$$f(r) = \frac{r^2}{2\lambda L^2}\left(1 - \sqrt{1 - 4\lambda + \frac{4\lambda L^2 m}{(d-1)r^d}}\right) . \tag{4.149}$$

Note that this is the metric (4.18) in which we have set $N_0^2 = 1/f_\infty$, so that the speed of light at the boundary is one. In order to study the speed of electromagnetic waves in the theory (4.30), we can either use its formulation in terms of the $(d-2)$-form B or in terms of the dual vector A — the result will be independent of the frame employed. Let us consider then a perturbation of the $(d-2)$-form in this black hole background. At linear order, the equation for B can be written as

$$\nabla_{\alpha_1}\left(\tilde{Q}^{[\mu\nu}{}_{\mu\nu} H^{\alpha_1 \ldots \alpha_{d-1}]}\right) = 0 \, , \tag{4.150}$$

where $\tilde{Q}^{\mu\nu}{}_{\rho\sigma}$ is the tensor introduced in (4.76). Particularized to the EQG case, this tensor reads

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \delta^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\left[1 + \left(\frac{3d\alpha_1}{d-2} + \frac{d\alpha_2}{(d-1)(d-2)}\right) L^2 R\right] + \frac{2}{d-2}((2d-1)\alpha_1 + \alpha_2) L^2 R^{\mu\nu}{}_{\rho\sigma}$$

$$
-2\left(\frac{4(2d-1)\alpha_1}{d-2}+\frac{(3d-2)\alpha_2}{(d-1)(d-2)}\right)L^2 R^{[\mu}{}_{[\rho}\delta^{\nu]}{}_{\sigma]}\,. \tag{4.151}
$$

When evaluated on the metric (4.148), it takes the form

$$
\tilde{Q}_{\mu\nu}{}^{\rho\sigma}=\gamma_1\rho_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]}+2\gamma_2\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}+\gamma_3\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,, \tag{4.152}
$$

where

$$
\rho^\alpha{}_\beta=\delta^\alpha{}_t\delta^t{}_\beta+\delta^\alpha{}_r\delta^r{}_\beta\,,\quad \sigma^\alpha{}_\beta=\sum_{i=1}^{d-1}\delta^\alpha{}_i\delta^i{}_\beta\,, \tag{4.153}
$$

are the projectors in the (t,r) and transverse directions, and $\gamma_{1,2,3}$ are the following functions,

$$
\gamma_1=1-\frac{\alpha_1 L^2}{r^2}\left[3d(d-1)f-2(d-1)rf'-r^2f''\right]-\frac{\alpha_2L^2}{r^2}\left[df-rf'\right]\,, \tag{4.154}
$$

$$
\begin{aligned}
\gamma_2=1&-\frac{\alpha_1 L^2}{r^2}\left[(3d^2-11d+4)f+2drf'(r)-r^2f''\right]\\
&-\frac{\alpha_2L^2}{2(d-1)r^2}\left[2(d^2-4d+2)f+(d+1)rf'-r^2f''\right]\,,
\end{aligned} \tag{4.155}
$$

$$
\begin{aligned}
\gamma_3=1&-\frac{\alpha_1 L^2}{(d-2)r^2}\left[(d-5)(3d^2-10d+4)f+2(3d^2-11d+4)rf'+3dr^2f''\right]\\
&-\frac{\alpha_2L^2}{(d-1)(d-2)r^2}\left[(d-3)(d^2-6d+4)f+2(d^2-4d+2)rf'+dr^2f''\right]\,.
\end{aligned} \tag{4.156}
$$

Now, let us consider the following fluctuation of B, with a polarization orthogonal to x^1,

$$
B=\psi(r)e^{-i\omega t+ikx^1}\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{d-1}\,. \tag{4.157}
$$

Its field strength $H=\mathrm{d}B$ is given by

$$
\begin{aligned}
H=e^{-i\omega t+ikx^1}\Big(&\psi'(r)\mathrm{d}r\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{d-1}-i\omega\mathrm{d}t\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{d-1}\\
&+ik\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\cdots\wedge\mathrm{d}x^{d-1}\Big)\,,
\end{aligned} \tag{4.158}
$$

and one can see that, with this ansatz, the equations of motion (4.150) are reduced to a single component (corresponding to the indices $\alpha_2\ldots\alpha_{d-1}=x^2\ldots x^{d-1}$), so that we do not need to activate other components of B. Since we want to study the small wavelength limit $\omega,k\to\infty$ we only need to keep the derivatives with respect to t and x^1. Under this approximation, we get

$$
\nabla_\alpha\left(\tilde{Q}^{[\mu\nu}{}_{\mu\nu}H^{\alpha x^2\ldots x^{d-1}]}\right)\propto\frac{L^{2(d-2)}}{r^{2(d-2)}}\left(-\frac{f_\infty}{f(r)}(i\omega)^2\gamma_2+\frac{L^2}{r^2}(ik)^2\gamma_1\right)B^{x^2\ldots x^{d-1}}\,, \tag{4.159}
$$

and hence we get the following dispersion relation

$$
\frac{\omega^2}{k^2}=\frac{\gamma_1L^2f(r)}{\gamma_2f_\infty r^2}\,. \tag{4.160}
$$

If we expand this near infinity, we obtain

$$\frac{\omega^2}{k^2} = 1 - \frac{L^2 m\left[1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2\right]}{(d-1)(2-f_\infty)\alpha_{\rm eff}^{\rm EQG}r^d} + \mathcal{O}\left(\frac{1}{r^{2d}}\right). \tag{4.161}$$

Now, this is the phase velocity (squared) of the wave front, and consistency with causality requires that it be smaller than the speed of light, $\omega/k \leq 1$. Since $f_\infty < 2$ and we take $\alpha_{\rm eff}^{\rm EQG} > 0$, the condition $\omega/k \leq 1$ implies

$$1-(7d^2-9d+2)f_\infty\alpha_1-(3d-2)f_\infty\alpha_2 \geq 0\,, \tag{4.162}$$

which matches precisely the constraint (4.143) computed from the lower bound in the allowed range of values of a_2. Now, playing with several values of the parameters that respect this bound, it appears that no other constraints are necessary: once (4.162) is satisfied, then $\omega^2/k^2 \leq 1$ everywhere inside the bulk. A more thorough of these causality constraints deeper in the bulk interior would be convenient, though.

We can obtain different constraints by choosing inequivalent polarizations for the B field. This means that we have to consider a B field which is polarized along the r direction. However, since the physical constraints on the Maxwell frame are the same, it is simpler to just study a perturbation of the dual vector field A_μ of the form

$$A = \phi(r)e^{-i\omega t + ikx^2}\mathrm{d}x^1\,. \tag{4.163}$$

One can see that the H form obtained by dualizing this vector is not of the form (4.158), and in particular it has a term $\sim k\mathrm{d}t\wedge\mathrm{d}r\wedge\mathrm{d}x^3\wedge\ldots\wedge\mathrm{d}x^{d-1}$, indicating polarization of B along the r direction. The (linearized) modified Maxwell equation for this vector reads

$$\nabla_\mu\left[(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma}F^{\rho\sigma}\right] = 0\,, \tag{4.164}$$

where $(\tilde{Q}^{-1})^{\mu\nu}{}_{\rho\sigma}$ is the inverse of the tensor in Eq. (4.152). One can see the inverse is simply given by

$$(\tilde{Q}^{-1})_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\gamma_1}\rho_{[\mu}{}^{[\rho}\rho_{\nu]}{}^{\sigma]} + \frac{2}{\gamma_2}\rho_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]} + \frac{1}{\gamma_3}\sigma_{[\mu}{}^{[\rho}\sigma_{\nu]}{}^{\sigma]}\,. \tag{4.165}$$

Now, the Maxwell equation for the ansatz in Eq. (4.163) is reduced to the single component $\nu = x^1$, which reads

$$\left(\frac{\omega^2 L^2 f_\infty}{r^2 f\gamma_2} - \frac{k^2L^4}{r^4\gamma_3}\right)\phi + \frac{(d-1)fL^2}{\gamma_2 r^3}\phi' + \frac{d}{dr}\left(\frac{fL^2\phi'}{r^2\gamma_2}\right) = 0\,. \tag{4.166}$$

Thus, in the short-wavelength limit we get

$$\frac{\omega^2}{k^2} = \frac{L^2 f(r)\gamma_2}{r^2 f_\infty\gamma_3}\,, \tag{4.167}$$

and expanding this near infinity we have

$$\frac{\omega^2}{k^2} = 1 - \frac{L^2 m\left[1-\frac{d-1}{d-2}(3d^2-14d+4)f_\infty\alpha_1-\frac{d^2-6d+4}{d-2}f_\infty\alpha_2\right]}{(d-1)(2-f_\infty)\alpha_{\rm eff}^{\rm EQG}r^d} + \mathcal{O}\left(\frac{1}{r^{2d}}\right), \tag{4.168}$$

where we plugged in the values of γ_2 and γ_3 given in (4.155) and (4.156). In order for this perturbation not to violate causality, it is necessary that $\omega^2/k^2 \leq 1$ as we move away from the boundary, and therefore, we obtain the constraint

$$d-2-(d-1)(3d^2-14d+4)f_\infty\alpha_1-(d^2-6d+4)f_\infty\alpha_2 \geq 0\,, \tag{4.169}$$

which is precisely the condition obtained by looking at the upper bound in the value of a_2, given in Eq. (4.144). By looking at the behavior of (4.167) in the bulk for several choices of the parameters, we seem to find that, whenever Eq. (4.169) is satisfied, then $\omega^2/k^2 \leq 1$ everywhere. However, it would again be interesting to perform a more thorough analysis in this regard.

One can be convinced that there are no other inequivalent polarizations by counting the number of them captured by (4.157) and (4.163). If we fix the direction of propagation, (4.157) is the only possible B field orthogonal to the direction of propagation and with no t and r components, while there are $d-2$ polarizations of the type (4.163) for A obtained by exchanging $\mathrm{d}x^1$ with $\mathrm{d}x^i$, $i \neq 2$. In total we have $d-1 = D-2$ different polarizations, which is the number of degrees of freedom of a massless vector field (and of a $(D-3)$-form) in D dimensions. Therefore, we conclude that there are no additional constraints from causality in the background of a neutral black brane.

It would be interesting to study as well the case of charged black branes, which would indeed be relevant if one wishes to perform holography in such backgrounds. In that case, gravitational and electromagnetic perturbations are linearly coupled, making the analysis of the speed of propagation a bit more involved. However, this could perhaps lead to even stronger constraints than the ones we have derived.

Finally, let us note that there are other types of causality violations, like the ones found in Ref. [428] involving the graviton three-point vertex. One of the implications of that work in the holographic context is that the Gauss-Bonnet coupling (in units of the AdS scale) must be very small: $|\lambda| << 1$. These bounds would be applicable in principle to any higher-order gravity that modifies the three-point function structure of Einstein gravity, but let us note that there are non-trivial higher-curvature terms that do not modify this three-point function, and one could not apply these results to them. In any case, we do not know of similar constraints for the RH^2 and H^4 terms in our theory (4.30). As a matter of fact, there are theories, as QCD, that have a large value of a_2, and in order to capture these holographically one needs bulk theories with non-minimal higher-derivative terms with $\sim O(1)$ couplings, as noted in [435].

4.4.3 WGC and positivity of entropy corrections

So far, we have been able to constrain three of the four parameters of our theory (4.30) by imposing unitarity of the boundary theory, which is equivalent to causality in the bulk theory. However, the parameter β is still unconstrained as it does not affect any 2- or 3-point function. Also, the existing constraints basically prevent the couplings from becoming too large, but they do not say anything about the sign of these parameters. Interestingly enough, additional constraints can be found by applying the mild form of the Weak Gravity Conjecture (WGC) [474, 528], which has recently received a lot of attention in the context of higher-derivative theories [6, 119, 125, 126, 475, 529, 530, 532, 533, 588, 589]. In the case of AdS spacetime, the implications of the WGC were recently studied in Ref. [476] — see also [590–592]. One of the heuristic ideas behind the WGC is that extremal black holes should be able to decay. This will happen if there exists a particle whose charge-to-mass ratio is larger than the one of an extremal black hole, which is the standard form of the WGC [416,593]. However, the mild form involves only black holes and essentially it claims that the decay of an extremal black hole into a set of smaller black holes should be possible, at least from the point of view of energy and charge conservation. Since extremal black holes have a fixed mass for a given value of the charge, $M_{\rm ext}(Q)$, such decay process is only possible if

$$M_{\rm ext}(Q_1+Q_2)\geq M_{\rm ext}(Q_1)+M_{\rm ext}(Q_2)\,. \tag{4.170}$$

For asymptotically flat black holes in Einstein-Maxwell theory we have $M_{\rm ext}(Q)\propto|Q|$, so the inequality above is saturated. On the other hand, higher-derivative corrections will modify the charge-mass relation, and by demanding that the deviations respect the property (4.170) one obtains a constraint on the coefficients of the higher-derivative operators. In all cases, one can see that, in order to preserve (4.170), the corrections to the extremal mass must be negative, $\delta M_{\rm ext}<0$ [594].

In Anti-de Sitter space, however, things work differently. As noted in [476], the bound (4.170) is no longer saturated for extremal AdS black holes, and hence perturbative (arbitrarily small) higher-derivative corrections cannot violate it.[12] Instead, that reference makes use of the proposal of Ref. [528] that the corrections to the entropy of black holes of arbitrary charge and mass should be positive as long as those black holes are thermodynamically stable. It is known [531] that, when applied to near-extremal black holes, the positivity of corrections to the entropy is connected to the negativity of the corrections to the extremal mass (see also [595]). Therefore, one can still use the condition $\delta M_{\rm ext}<0$ to bound the higher-order coefficients, just like in the asymptotically flat case. However, the conditions studied in [476] are more ambitious, as they demand $\delta S>0$ for arbitrary charge and mass (as long as the specific heats are positive), not only for near-extremal black holes. Let us work out these conditions for our theory (4.30).

The Wald entropy of static black holes was computed in (4.56), which we reproduce here for convenience,

$$S=\frac{r_+^{d-1}V_{k,d-1}}{4G}\left(1+\frac{2L^2q^2\alpha_1}{r_+^{2d-2}}+\frac{2kL^2(d-1)\lambda}{(d-3)r_+^2}\right)\,. \tag{4.171}$$

This expression together with the relation (4.52) give us the exact value of the entropy $S(M,q)$. However, here we only need the perturbative correction to the entropy at fixed charge and mass. It is useful to introduce the variable

$$x=\frac{r_+^{(0)}}{L}\,, \tag{4.172}$$

where $r_+^{(0)}$ is the zeroth-order value of the radius, which is obtained implicitly from (4.52) by setting to zero the higher-order terms. We also note that the extremal value of the charge in the two-derivative theory reads

$$q^{(0)}_{\rm ext}=(Lx)^{d-2}\sqrt{\frac{d-1}{2}}\sqrt{dx^2+k(d-2)}\,, \tag{4.173}$$

[12]Note however, that in the limit of small size, AdS black holes behave as asymptotically flat ones, and in that limit (4.170) could still be applied to constrain the higher-derivative corrections.

and thus let us introduce the variable

$$\xi = \frac{q}{q_{\rm ext}^{(0)}}\,, \tag{4.174}$$

that ranges from 0 to 1. Since we are working at fixed M and q, the equation (4.52) allows us to obtain the correction to the horizon radius,

$$r_+ = r_+^{(0)} + r_+^{(1)} + \ldots\,, \tag{4.175}$$

where the first-order correction reads

$$\begin{aligned} r_+^{(1)} = &\frac{k^2\lambda L}{(\xi^2-1)\,x\left((d-2)k+dx^2\right)} + \frac{3\alpha_1(d-1)kL\xi^2}{(\xi^2-1)\,x} + \frac{\alpha_2 kL\xi^2}{(\xi^2-1)\,x} \\ &- \frac{\beta(d-1)L\xi^4\left((d-2)k+dx^2\right)}{4(3d-4)\,(\xi^2-1)\,x}\,. \end{aligned} \tag{4.176}$$

Inserting this into our expression for the entropy, we get the following shift at linear order,

$$\begin{aligned} \delta S(M,q) = \frac{(d-1)L^{d-1}x^{d-3}V_{k,d-1}}{4G}\Bigg[&k\lambda\left(\frac{k}{(\xi^2-1)\left((d-2)k+dx^2\right)}+\frac{2}{d-3}\right) \\ &+\alpha_1\xi^2\left(\frac{k\left(\xi^2(d-2)+2d-1\right)}{\xi^2-1}+dx^2\right)+\frac{\alpha_2 k\xi^2}{\xi^2-1} \\ &-\frac{\beta(d-1)\xi^4\left((d-2)k+dx^2\right)}{4(3d-4)\,(\xi^2-1)}\Bigg]\,. \end{aligned} \tag{4.177}$$

According to [476], we should then demand this correction to be positive for any black hole that is thermodynamically stable at zeroth order. Let us focus on spherically symmetric black holes $k=1$. The $k=0$ case is obtained as the limit of large size of spherical black holes, while the $k=-1$ case is somewhat different and we will comment on it below. We can consider first neutral black holes, $\xi=0$, in whose case only the Gauss-Bonnet term is relevant,

$$\delta S(M,q)\big|_{\xi=0} = \frac{(d-1)L^{d-1}x^{d-3}V_{1,d-1}}{4G}\lambda\left(-\frac{1}{(d-2)+dx^2}+\frac{2}{d-3}\right)\,. \tag{4.178}$$

The variable x can range between 0 and infinity, and for any of these values the quantity between parenthesis is positive for $d\geq 3$.[13] Now, neutral large black holes are known to be stable in AdS, and therefore, the WGC would imply that the GB coupling must be non-negative,

$$\lambda \geq 0\,. \tag{4.179}$$

This actually makes sense, as the Gauss-Bonnet density arises explicitly from stringy effective actions and in many instances[14] this indeed has a positive coupling [110, 124, 596, 597] — see also [598] and the discussion in the Appendix B of [424]. Next, we can look at the

[13] For $d=3$ one should redefine $\hat\lambda = \lambda/(d-3)$ and take the limit $d\to 3$ with fixed $\hat\lambda$. The correction to the entropy is topological and identical for any spherical black hole.

[14] However, Ref. [572] showed that a negative λ can also be achieved, indicating that λ can actually have different signs depending on the setup.

case of (near-) extremal black holes, which are also stable in the two-derivative theory. This corresponds to the limit $\xi \to 1$, and hence we get

$$\delta S(M,q)\big|_{\xi\to 1} = \frac{(d-1)L^{d-1}x^{d-3}V_{1,d-1}}{4G(1-\xi^2)}\left[-\frac{\lambda}{(d-2)+dx^2} - 3(d-1)\alpha_1 - \alpha_2 \right. \\ \left. + \frac{\beta(d-1)\left((d-2)+dx^2\right)}{4(3d-4)}\right]. \tag{4.180}$$

This correction has a non-trivial dependence on the radius of the black hole, and therefore imposing that it be positive implies several constraints on the coupling constants. For large black holes, the β correction dominates and $\delta S \geq 0$ implies

$$\beta \geq 0\,. \tag{4.181}$$

On the other hand, in the limit of small black holes $x \to 0$ we have

$$-\frac{\lambda}{d-2} - 3(d-1)\alpha_1 - \alpha_2 + \frac{\beta(d-1)(d-2)}{4(3d-4)} \geq 0\,. \tag{4.182}$$

This is arguably the most reliable constraint we can produce from the WGC, as small black holes behave as asymptotically flat ones, and one recovers the argument of Eq. (4.170). The condition above implies that the shift in the extremal mass is negative hence ensuring that (4.170) is satisfied for black holes much smaller than the AdS scale.

Finally, another interesting condition comes from large black holes $x \to \infty$ (or equivalently, black branes, $k=0$), of arbitrary charge. In that case we have

$$\delta S(M,q)\Big|_{x\to\infty} = \frac{d(d-1)L^{d-1}x^{d-1}V_{1,d-1}}{4G}\left[\alpha_1\xi^2 + \frac{\beta(d-1)\xi^4}{4(3d-4)\left(1-\xi^2\right)}\right], \tag{4.183}$$

and in order for this quantity to remain positive for any value of $\xi \in [0,1)$, we must impose not only $\beta \geq 0$, but also

$$\alpha_1 \geq 0\,. \tag{4.184}$$

This is a very powerful constraint, since, when combined with the unitarity bounds shown in Fig. 4.1, it implies that α_1 and α_2 can only lie in a small compact set of the plane for $d = 3, 4, 5$. The Gauss-Bonnet coupling is also bound to a small interval $0 \leq \lambda \leq \frac{(d-2)(d-3)(d^2-d+6)}{4(d^2-3d+6)^2}$, so only β can take arbitrarily high values with the current constraints. It would be interesting to investigate whether different constraints could impose an upper bound on β. The results from next section suggest indeed that β should not be too large.

Before closing this section, let us discuss what happens if one attempts to enforce the WGC bounds on hyperbolic black holes as well. For simplicity, we can consider neutral black holes, $\xi = 0$. One can check that all of these solutions are thermally stable in the two-derivative theory, and therefore one should impose $\delta S \geq 0$. From (4.177) we obtain

$$\delta S(M,q)\Big|_{k=-1,\xi=0} = \frac{(d-1)L^{d-1}x^{d-3}V_{-1,d-1}}{4G}(-\lambda)\left(\frac{1}{dx^2-(d-2)} + \frac{2}{d-3}\right), \tag{4.185}$$

and since hyperbolic black holes have $dx^2 - (d-2) \geq 0$, the positivity of δS implies in this case that $\lambda \leq 0$, which is the opposite that what we found for spherical and planar black holes.[15] In principle, these constraints should hold at the same time for any choice of boundary geometry, since the dual CFT is always the same. However, this would lead to the conclusion that $\lambda = 0$, which seems an unreasonably strong constraint. Likewise we find similar stringent bounds on the other couplings if we combine the cases $k = 1$ and $k = -1$. We do not know how to resolve this issue, but we feel more inclined to trust the constraints for spherical black holes, and ignore those for $k = -1$. On the one hand, spherical black holes make direct connection with the original motivation of the WGC regarding black hole evaporation, while the evaporation of a hyperbolic black hole is probably a meaningless problem (they are always stable). On the other, as we mentioned above, a positive GB coupling $\lambda > 0$ is actually realized in many explicit string models (in particular, this is the case in the Heterotic string effective action [110,124]). This suggests that the positivity-of-entropy bounds might not be applicable to hyperbolic black holes, but it would be interesting to understand why.

4.5 Charged Rényi entropies and generalized twist operators

Entanglement entropy (EE) [600] and Rényi entropies (RE) [441,442] — as well as their holographic counterparts [446,601,602] — constitute a very useful way to probe the amount of entanglement in quantum field theories [603,604]. Given a biparition of the Hilbert space into two subspaces A and B, Rényi entropies are defined as

$$S_n(A) = \frac{1}{1-n} \log \mathrm{Tr}\rho_A^n \,, \tag{4.186}$$

where $\rho_A = \mathrm{Tr}_B \rho$ is the reduced density matrix of the subsystem A, obtained by taking the partial trace over the subsystem B of the total density matrix. Here, we are interested in the case in which A and B correspond to the subsystems associated to two spatial regions (at a fixed time) separated by an entangling surface Σ. The Rényi index n is usually considered an integer, which allows one to compute these entropies by using the replica trick [600]. However, if one is able to continue n to an arbitrary real number, then one can recover the entanglement entropy as the limit $n \to 1$,

$$S_{\mathrm{EE}}(A) = -\mathrm{Tr}\left[\rho_A \log(\rho_A)\right] = \lim_{n\to 1} S_n(A) \,. \tag{4.187}$$

Now, these entropies can be generalized to the case in which the QFT is charged under a global symmetry. The appropriate generalization, proposed in Ref. [447], reads

$$S_n(\mu) = \frac{1}{1-n} \log \mathrm{Tr}\left[\bar{\rho}_A(\mu)\right]^n \,, \tag{4.188}$$

where

$$\bar{\rho}_A(\mu) = \frac{\rho_A e^{\mu Q_A}}{\mathrm{Tr}\left[\rho_A e^{\mu Q_A}\right]} \tag{4.189}$$

is a new density matrix that depends on the chemical potential μ, conjugate to the charge Q_A enclosed in the region A.

Let us focus on the case in which the quantum theory is defined in flat space and the entanglement surface Σ is a sphere of radius R, namely $\Sigma = S^{d-2}(R)$. For a CFT,

[15] In $d = 3$, Ref. [599] already noticed that the correction to the entropy associated to the GB term cannot have a definite sign, since one can have black holes of different topologies.

one can then prove, by using the Casini-Huerta-Myers map [445], that these charged Rényi entropies are related to the thermal entropy of the same theory placed on the hyperbolic cylinder $S^1 \times \mathbb{H}^{d-1}(R)$. The precise relation reads [446, 447]

$$S_n(\mu) = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S_{\text{thermal}}(T,\mu)\mathrm{d}T\,, \tag{4.190}$$

where

$$T_0 = \frac{1}{2\pi R}\,. \tag{4.191}$$

We remark that this is a formula that applies to a CFT, but from here it is evident how to compute these quantities holographically. In fact, the thermal entropy of a holographic CFT on $S^1 \times \mathbb{H}^{d-1}(R)$ is nothing but the Wald's entropy of a black hole with a hyperbolic horizon. In this section we explore the properties of the holographic RE for the theory (4.30), and afterwards we also analyze a couple of related quantities: the scaling dimension and the magnetic response of generalized twist operators [447].

4.5.1 Rényi entropies

In order to compute charged Rényi entropies for the holographic CFTs dual to (4.30), we have to consider charged black hole solutions with hyperbolic horizons, which for our theories take the form

$$\mathrm{d}s^2 = -N_{-1}^2 f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Xi^2\,, \tag{4.192}$$

where N_{-1} is a constant, $\mathrm{d}\Xi^2$ is the hyperbolic space of unit radius and $f(r)$ is given by Eq. (4.42) with $k=-1$. Since $f(r)$ behaves asymptotically as $f(r) \sim r^2 f_\infty/L^2$, we set the constant N_{-1} to

$$N_{-1} = \frac{L}{\sqrt{f_\infty}R}\,. \tag{4.193}$$

In this way, the boundary metric is conformal to

$$\mathrm{d}s^2_{\text{bdry}} = -\mathrm{d}t^2 + R^2\mathrm{d}\Xi^2\,, \tag{4.194}$$

so that the spatial slices are hyperbolic spaces of radius R. The Rényi entropies across a spherical region are then computed through the integral (4.190), where S_{thermal} is the black hole entropy, given by (4.56). Notice that it is important that S_{thermal} is considered as a function of T and μ, so that the integration is carried out at constant μ. Although at first sight the integration may look tricky, it is nonetheless straightforward, since the first law (4.70) implies

$$S = -\frac{\partial\Omega(T,\mu)}{\partial T}\,. \tag{4.195}$$

We then obtain

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\left(\Omega(T_0/n,\mu) - \Omega(T_0,\mu)\right)\,. \tag{4.196}$$

Back in Eq. (4.71) we already obtained the expression for the grand canonical potential Ω in terms of the horizon radius and the charge for spherical, planar or hyperbolic horizon topologies. Setting $k=-1$, defining $x=r_+/L$ and $q=px^{d-1}L^{d-2}$ and writing $T=T_0/n=(2\pi Rn)^{-1}$, the expression for Ω reduces to

$$\begin{aligned}\Omega =&\frac{L^{d-1}V_{-1,(d-1)}}{16\pi G\sqrt{f_\infty}R}\Bigg[(d-1)x^d-\frac{2p^2x^d}{d-2}-x^{d-2}\left((d-1)+\frac{2x\sqrt{f_\infty}}{n}\right)+\frac{3\beta p^4x^d}{3d-4}\\&-(d-1)\lambda x^{d-4}\left(1+\frac{4x\sqrt{f_\infty}}{n(d-3)}\right)+2\alpha_1p^2x^{d-2}\left(3(d-1)+\frac{2x\sqrt{f_\infty}R}{n}\right)+2\alpha_2p^2x^{d-2}\Bigg].\end{aligned} \tag{4.197}$$

However, on account of (4.196), we need to write Ω in terms of n and μ, so that we have to find the relations $x=x(n,\mu)$ and $p=p(n,\mu)$. For that, it is convenient to present the expressions of n and μ in terms of x and p, which follow after setting $k=-1$ in Eqs. (4.53) and (4.66):

$$\begin{aligned}\frac{1}{n}=&\frac{1}{2x\sqrt{f_\infty}\left(1-2p^2\alpha_1-2\lambda x^{-2}\right)}\Bigg[\left(-(d-2)+dx^2+(d-4)\lambda x^{-2}\right)\\&-\frac{2p^2}{(d-1)}\left(x^2-d(3(d-1)\alpha_1+\alpha_2)\right)+\frac{\beta x^2p^4}{(d-1)}\Bigg],\end{aligned} \tag{4.198}$$

$$\mu=\frac{Lp}{\ell_*\sqrt{f_\infty}R}\left[\frac{x}{(d-2)}-\frac{\alpha_1}{x}\left(3(d-1)+\frac{2x\sqrt{f_\infty}}{n}\right)-\frac{\alpha_2}{x}-\frac{xp^2\beta}{(3d-4)}\right]. \tag{4.199}$$

The equations (4.197), (4.198) and (4.199) allow us to study the Rényi entropies (4.196) exactly. A useful intermediate expression for $\Omega_n(\mu)\equiv\Omega(T_0/n,\mu)$, is the following one,

$$\begin{aligned}\Omega_n(\mu)=&\frac{L^{d-1}V_{-1,(d-1)}}{16\pi G\sqrt{f_\infty}R}\Bigg[(d-1)x^{d-4}(x^4-x^2+\lambda)+\frac{\beta p^4x^d}{(3d-4)}\\&-2\frac{\sqrt{f_\infty}}{n}x^{d-1}\left(1-\frac{2\lambda(d-1)}{(d-3)x^2}\right)-2\frac{\ell_*R\sqrt{f_\infty}}{L}\mu px^{d-1}\Bigg],\end{aligned} \tag{4.200}$$

which is a bit simpler, but it still depends on $x=x(n,\mu)$ and $p=p(n,\mu)$. In practice, it seems extremely challenging (if not impossible) to analytically invert the equations (4.198) and (4.199) to encounter $x=x(n,\mu)$ and $p=p(n,\mu)$ explicitly. To circumvent this impediment, we focus next in two limiting regimes, namely, small μ and $\mu\to\infty$.

4.5.1.1 Small μ

In order to reduce the clutter, let us introduce the notation

$$\bar{\mu}=\frac{\ell_*R\sqrt{f_\infty}}{L}\mu\,, \tag{4.201}$$

as this combination appears everywhere. We consider here the case in which $\bar{\mu}<<1$, so that carrying out the inversion procedure of Eqs. (4.198) and (4.199) in a perturbative expansion in $\bar{\mu}$ suffices. Furthermore, as an attempt to make explicit computations and capture the effects produced by the non-minimal couplings, we are going to set $\lambda=0$ all along this section (so $f_\infty=1$). After all, the effect of the GB coupling on (uncharged)

Rényi entropies is known [446] — see also [130,257,493,605] for other studies of holographic RE in higher-order gravities.

Consequently, we can expand $x(n,\mu)$ and $p(n,\mu)$ as

$$x(n,\mu)=\hat{x}_n+\delta\hat{x}_n\,\bar{\mu}^2+\mathcal{O}\left(\bar{\mu}^4\right),\qquad p(n,\mu)=\delta p_n\bar{\mu}+\mathcal{O}\left(\bar{\mu}^3\right)\,. \tag{4.202}$$

By solving Eqs. (4.198) and (4.199), the coefficients $\hat{x}_n$, $\delta\hat{x}_n$ and $\delta\tilde{p}_n$ can be found to be

$$\hat{x}_n=\frac{n^{-1}+\sqrt{n^{-2}+d(d-2)}}{d}\,, \tag{4.203}$$

$$\delta\hat{x}_n=-\frac{2(d-2)^2\hat{x}_n^3\left(2(d+1)\alpha_1+\hat{x}_n^2\left(d\alpha_1-\frac{1}{d-1}\right)+\frac{d}{d-1}\alpha_2\right)}{(d(\hat{x}_n^2+1)-2)(\hat{x}_n^2(d(d-2)\alpha_1-1)+(d-2)((2d-1)\alpha_1+\alpha_2))^2}\,, \tag{4.204}$$

$$\delta p_n=\frac{(d-2)\hat{x}_n}{\alpha_{\text{eff}}^{\text{EQG}}-(\hat{x}_n^2-1)(d(d-2)\alpha_1-1)}\,. \tag{4.205}$$

We recall that $\alpha_{\text{eff}}^{\text{EQG}}$, given in Eq. (4.97), is the combination that appears in the denominator of the central charge C_J in Eq. (4.95). Taking into account these perturbative expansions, Ω_n can be written in the following explicit form:

$$\begin{aligned}\Omega_n(\mu)=-\frac{L^{d-1}V_{-1,(d-1)}}{16\pi GR}\Bigg[&\hat{x}_n^{d-2}(\hat{x}_n^2+1)\\&+\frac{2(d-2)\hat{x}_n^d}{\alpha_{\text{eff}}^{\text{EQG}}-(\hat{x}_n^2-1)(d(d-2)\alpha_1-1)}\bar{\mu}^2\Bigg]+\mathcal{O}\left(\bar{\mu}^4\right).\end{aligned} \tag{4.206}$$

Now, noting that $\hat{x}_1=1$, we have

$$\Omega_1(\mu)=-\frac{L^{d-1}V_{-1,(d-1)}}{8\pi GR}\left[1+\frac{d-2}{\alpha_{\text{eff}}^{\text{EQG}}}\bar{\mu}^2\right]+\mathcal{O}\left(\bar{\mu}^4\right). \tag{4.207}$$

From here, we can infer the following form for the n-th Rényi entropy:

$$\begin{aligned}S_n=\frac{nL^{d-1}V_{-1,(d-1)}}{4(n-1)G}\Bigg[&\frac{2-\hat{x}_n^{d-2}(\hat{x}_n^2+1)}{2}\\&+\frac{(d-2)}{\alpha_{\text{eff}}^{\text{EQG}}}\left(1-\frac{\hat{x}_n^d}{1-\frac{(\hat{x}_n^2-1)}{\alpha_{\text{eff}}^{\text{EQG}}}(d(d-2)\alpha_1-1)}\right)\bar{\mu}^2\Bigg]+\mathcal{O}\left(\bar{\mu}^4\right).\end{aligned} \tag{4.208}$$

Let us remark at this point that the volume $V_{-1,(d-1)}$ is a (diverging) function of the ratio between the radius of the entangling surface R and a cut-off δ. In fact, the leading term gives an area law,

$$V_{-1,(d-1)}=\frac{V_{S^{d-2}}}{d-2}\frac{R^{d-2}}{\delta^{d-2}}+\ldots\,,\quad\text{where}\quad V_{S^{d-2}}=\frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}\,. \tag{4.209}$$

It is interesting to keep only the universal part in this expansion, which will provide us with the regularized RE. In even d, the series expansion of the volume contains a term $\log(R/\delta)$, and it is clear that the coefficient of this term is universal as it is invariant under rescalings of the cut-off. On the other hand, for odd d the series contains a constant term. The universality of this term is less clear, as it could be shifted by performing a rescaling of R of the form $R \to R(1+c\delta)$, but we will not worry about this issue here.[16] Taking this into account, one can see that the universal part of the volume reads [445]

$$V^{\text{univ.}}_{-1,d-1} = \frac{\nu_{d-1}}{4\pi} V_{S^{d-1}}\,, \quad \text{where} \quad \nu_{d-1} = \begin{cases} (-)^{\frac{d-2}{2}} 4\log(R/\delta) & d \text{ even}\,, \\ \\ (-)^{\frac{d-1}{2}} 2\pi & d \text{ odd}\,. \end{cases} \tag{4.210}$$

We will use this regularized volume from now on. It is also useful to introduce the following quantity,

$$a^* = \frac{L^{d-1}}{8G} \frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\,, \tag{4.211}$$

which represents the universal contribution to the regularized EE in holographic Einstein gravity. This parameter can also be easily computed for higher-curvature gravities [508,551] and in general it coincides with the a-type trace-anomaly charge in the case of even d, while in odd dimensions it is proportional to the free energy of the corresponding theory evaluated on S^d [445]. Using this parameter, we can finally write our holographic REs as

$$\begin{aligned} S_n = \frac{n a^* \nu_{d-1}}{(n-1)} \Bigg[&\frac{2 - \hat{x}_n^{d-2}(\hat{x}_n^2+1)}{2} \\ &+ \frac{(d-2)}{\alpha_{\text{eff}}^{\text{EQG}}} \left(1 - \frac{\hat{x}_n^d}{1 - \frac{(\hat{x}_n^2-1)}{\alpha_{\text{eff}}^{\text{EQG}}}(d(d-2)\alpha_1 - 1)} \right) \bar{\mu}^2 \Bigg] + \mathcal{O}(\bar{\mu}^4)\,. \end{aligned} \tag{4.212}$$

Let us then explore the properties of these entropies, starting with the relevant case of the entanglement entropy $n \to 1$. This limit yields

$$S_{\text{EE}} = \lim_{n\to 1} S_n = a^* \nu_{d-1} \left[1 + \frac{(d-2)^2(1 - 3d(d-1)\alpha_1 - d\alpha_2)}{(d-1)\left(\alpha_{\text{eff}}^{\text{EQG}}\right)^2} \bar{\mu}^2 \right] + \mathcal{O}(\bar{\mu}^4)\,. \tag{4.213}$$

It is interesting to wonder about the sign of the coefficient of μ^2 in (4.213), or more precisely, of the quantity

$$\left. \frac{\partial_{\bar{\mu}}^2 S_{\text{EE}}}{S_{\text{EE}}} \right|_{\mu=0} = \frac{2(d-2)^2(1 - 3d(d-1)\alpha_1 - d\alpha_2)}{(d-1)\left(\alpha_{\text{eff}}^{\text{EQG}}\right)^2}\,. \tag{4.214}$$

In Einstein-Maxwell theory we can see it is positive, so that the holographic entanglement entropy grows when we turn on a chemical potential. Could this be different in other theories? If the parameters α_1 and α_2 were arbitrary, this coefficient could have either sign, but we must take into account the constraints in Sec. 4.4. In fact, it suffices to consider the unitarity constraints in 4.4.1. Let us first note that the unitarity constraint (4.143) can be expressed as

$$-\frac{2}{d-2} + 2d\alpha_1 + \frac{3d-2}{d(d-2)} \alpha_{\text{eff}}^{\text{EQG}} \geq 0\,. \tag{4.215}$$

[16] A natural regulator in that case is provided by the mutual information [606], which is a UV finite quantity.

Then, we have

$$\begin{aligned} 1-3d(d-1)\alpha_1-d\alpha_2 &= -\frac{2}{d-2}+2d\alpha_1+\frac{d}{d-2}\alpha_{\rm eff}^{\rm EQG} \\ &> -\frac{2}{d-2}+2d\alpha_1+\frac{3d-2}{d(d-2)}\alpha_{\rm eff}^{\rm EQG}\geq 0\,, \end{aligned} \tag{4.216}$$

where we simply used that $\alpha_{\rm eff}^{\rm EQG}>0$ and that $\frac{3d-2}{d(d-2)}<\frac{d}{d-2}$ for $d\geq 3$. Note that the result we obtain is a strict inequality, since $\alpha_{\rm eff}^{\rm EQG}=0$ is not allowed. Thus, this result implies that

$$\left.\frac{\partial_{\bar\mu}^2 S_{\rm EE}}{S_{\rm EE}}\right|_{\mu=0}>0 \tag{4.217}$$

for all the (unitary) holographic CFTs dual to our bulk theories. Given the robustness of this result, it is very tempting to conjecture that the entanglement entropy should always grow with the chemical potential for any unitary CFT at zero temperature[17]. In fact, in Section 4.6 we will explicitly prove that, for general d-dimensional CFTs with $d\geq 3$, the leading correction to the uncharged entanglement entropy across a spherical entangling surface is quadratic in the chemical potential, positive definite, and universally controlled (up to fixed d-dependent constants) by the coefficients C_J and a_2.

Remarkably, it is possible to extend this result to prove that the coefficient of μ^2 for all Rényi entropies associated to (4.30) (in Eq. (4.212)) with $n\geq 1$ is strictly positive. For that, let us note that for $n>1$ we have $\sqrt{(d-2)/d}<x_n<1$. On noting the inequality

$$1-\frac{(3d-2)}{2d}(1-\hat{x}_n^2)<\hat{x}_n^d\,,\quad d\geq 3\,,\, n>1 \tag{4.218}$$

we observe that, defining $\xi=d(d-2)\alpha_1-1$, for $n>1$ we have

$$1-\frac{\hat{x}_n^d}{1-\frac{(\hat{x}_n^2-1)}{\alpha_{\rm eff}^{\rm EQG}}\xi}>1-\frac{1-\frac{(3d-2)}{2d}(1-\hat{x}_n^2)}{1-\frac{(\hat{x}_n^2-1)}{\alpha_{\rm eff}^{\rm EQG}}\xi}=\frac{\frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi}{\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi}\geq 0\,. \tag{4.219}$$

The inequalities here follow from the fact that both the numerator and the denominator in the last term are positive:

$$\begin{aligned} \frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi &= \frac{(1-\hat{x}_n^2)(d-2)}{2}\left(\frac{(3d-2)}{d(d-2)}\alpha_{\rm eff}^{\rm EQG}+2d\alpha_1-\frac{2}{d-2}\right)\geq 0\,, \\ \alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi &> \frac{3d-2}{2d}(1-\hat{x}_n^2)\alpha_{\rm eff}^{\rm EQG}+(1-\hat{x}_n^2)\xi\geq 0\,, \end{aligned} \tag{4.220}$$

where have used (4.215) and taken into account that $1>1-\hat{x}_n^2\geq 0$ and that $\frac{(3d-2)}{2d}(1-\hat{x}_n^2)<1$ for every $d\geq 3$. By applying (4.219) in (4.212) and taking into account (4.217), it follows that

$$\left.\frac{\partial_{\bar\mu}^2 S_n}{S_n}\right|_{\mu=0}>0\,,\quad n\geq 1\,. \tag{4.221}$$

Therefore, we have proven that, as long as unitarity is respected, the Rényi entropies (with $n\geq 1$) always grow when a chemical potential is turned on. Again, it is interesting to speculate about the possible validity of this result beyond our current holographic setup.

[17] This is in line with the results of Ref. [607] for the holographic EE of an infinite rectangular strip in the case of $\mu\neq 0$ and $T=0$.

Regarding the case $n < 1$, we can consider the limit $n \to 0$, which yields

$$\lim_{n\to 0} S_n = \frac{2^{d-1} a^* \nu_{d-1}}{n^{d-1} d^d} \left[1 + \frac{(d-1)n^2 d^2}{4} + \frac{(d-2)n^2 d^2}{2(1-d(d-2)\alpha_1)} \bar{\mu}^2\right] + \mathcal{O}(\bar{\mu}^4)\,. \tag{4.222}$$

For $n \to 0$ the effect of the chemical potential becomes irrelevant, as it scales with n^2 relative to the leading term, but we observe that the coefficient of $\bar{\mu}^2$ is necessarily positive in $d = 3, 4, 5$, on account of the bound (4.146), coming again only from unitarity constraints. However, this fails to be true in higher-dimensions, since α_1 can take arbitrarily large values in $d \geq 6$.

We can finally study the dependence of the REs on the index n. It is known that standard (*i.e.*, at zero chemical potential) REs must satisfy the following inequalities [446]:

$$\begin{aligned} \frac{\partial}{\partial n} S_n \leq 0\,, \quad & \frac{\partial}{\partial n}\left(\frac{n-1}{n} S_n\right) \geq 0\,, \\ \frac{\partial}{\partial n}((n-1)S_n) \geq 0\,, \quad & \frac{\partial^2}{\partial n^2}((n-1)S_n) \leq 0\,. \end{aligned} \tag{4.223}$$

It was shown in Ref. [447] that these inequalities are also satisfied by the holographic charged Rényi entropies in Einstein-Maxwell theory. It is therefore interesting to check whether these inequalities still hold for our holographic higher-derivative theories, assuming that the values of the couplings satisfy the unitarity and WGC constraints in Sec. 4.4. Since the uncharged Rényi entropies for holographic Einstein gravity (obtained by setting $\bar{\mu} = 0$ in Eq. (4.212)) already satisfy such inequalities [446], it suffices to check that the coefficient of $\bar{\mu}^2$ in Eq. (4.212) fulfills them. This will guarantee that the charged RE also satisfy those inequalities, at least in the regime where the $\mathcal{O}(\mu^4)$ terms are subleading. To this aim, we show in Fig 4.2 the profile of $\partial^2_{\bar{\mu}} S_n / S_1\big|_{\mu=0}$ for a few values of α_1 and α_2 which are allowed by the physical constraints in $d = 3, 4$. We check that all the previous inequalities indeed hold for our EQG theories.

It is quite impressive that all of the properties one expects to find in Rényi entropies are satisfied whenever the parameters of the bulk theory are taken to satisfy a minimal set of physical constraints. We remark that for arbitrary values of α_1 and α_2 one could obtain very different results, and even divergencies in the RE. In fact, we have been able to observe that choosing values of these couplings that do not satisfy all the constraints obtained from causality/unitarity and the WGC does lead to these problems. Instead, for the physically sensible values of these parameters, the chemical potential always increases the amount of entanglement and the REs have the same qualitative features found for Einstein-Maxwell theory.

4.5.1.2 Large μ

Let us now study the opposite limit, $\mu \to \infty$. First it is convenient to revise this limit in the case of Einstein-Maxwell theory [447], since this will inspire us to properly generalize the study for our EQG (4.30). For that, let us write the temperature $T = (2\pi R n)^{-1}$ and the chemical potential $\bar{\mu}$ in this particular case:

$$\left.\frac{1}{n}\right|_{\text{EM}} = \frac{1}{2x}\left[d x^2 - (d-2) - \frac{2p^2 x^2}{d-1}\right]\,, \qquad \bar{\mu}|_{\text{EM}} = \frac{p x}{d-2}\,. \tag{4.224}$$

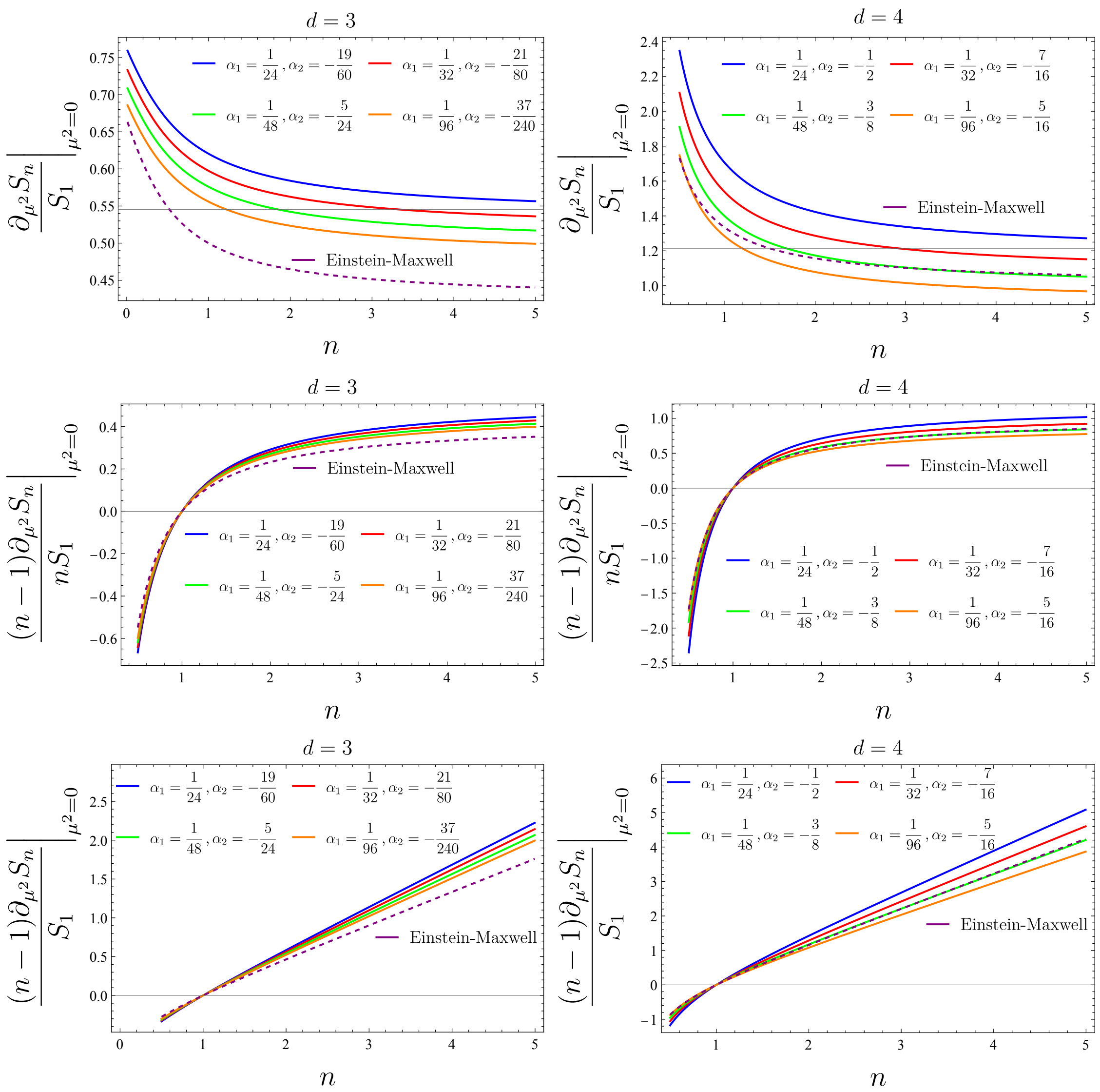


Figure 4.2: The coefficient of $\bar{\mu}^2$ in the Rényi entropies for $d = 3, 4$. We have used different values for the couplings α_1 and α_2 which are compatible with unitarity and the WGC. Remarkably enough, we find that all inequalities (4.223) are satisfied.

From here, one can solve for x to find:

$$x|_{\text{EM}} = \frac{1}{nd}\left[1 + \sqrt{1 + d(d-2)n^2 + \frac{2d(d-2)^2 n^2}{(d-1)}\bar{\mu}^2}\right] . \tag{4.225}$$

In the limit $\bar{\mu} \to \infty$, we infer that

$$\begin{aligned} x|_{\text{EM}} &= (d-2)\sqrt{\frac{2}{d(d-1)}}\bar{\mu} + \frac{1}{nd} + \mathcal{O}\left(\frac{1}{\bar{\mu}}\right) , \\ p|_{\text{EM}} &= \sqrt{\frac{d(d-1)}{2}} - \frac{d-1}{2n(d-2)\bar{\mu}} + \mathcal{O}\left(\frac{1}{\bar{\mu}^2}\right) . \end{aligned} \tag{4.226}$$

Given this structure for the perturbative expansions of x and p as $\bar{\mu} \to \infty$ in the Einstein-Maxwell limit, we expect the corresponding perturbative expansions for our EQG theories to keep the same form:

$$x = x_1\bar{\mu} + x_0 + \mathcal{O}\left(\frac{1}{\bar{\mu}}\right), \qquad p = p_0 + \frac{p_{-1}}{\bar{\mu}} + \mathcal{O}\left(\frac{1}{\bar{\mu}^2}\right). \tag{4.227}$$

In fact, taking these ansätze into Eqs. (4.198) and (4.199), we find:

$$\begin{aligned} p_0 &= \pm\sqrt{\frac{1-\sqrt{1-\beta d(d-1)}}{\beta}}, \qquad x_1 = \frac{(d-2)(3d-4)p_0}{d(p_0^2+(d-1)(d-2))}, \\ p_{-1} &= -\frac{(d-1)\sqrt{f_\infty}(1-2p_0^2\alpha_1)p_0}{2nx_1(d(d-1)-p_0^2)}, \\ x_0 &= \frac{((3d-8)p_0^2-3d(d-1)(d-2))p_{-1}x_1+2/n(3d-4)(d-2)\sqrt{f_\infty}p_0^3\alpha_1}{dp_0(p_0^2+d(d-3)+2)}, \end{aligned} \tag{4.228}$$

where the sign of p_0 coincides with that of $\bar{\mu}$. Taking into account the previous relations and Eqs. (4.196) and (4.200), the Rényi entropy in the limit $\mu \to \infty$ turns out to be:

$$\lim_{\bar{\mu}\to\infty} S_n = \nu_{d-1}\frac{(\ell_* R\mu)^{d-1}\pi^{(d-2)/2}}{8G\Gamma(d/2)}(1+2\alpha_1 p_0^2)\left(\frac{(d-2)(3d-4)p_0\sqrt{f_\infty}}{d(p_0^2+(d-1)(d-2))}\right)^{d-1}. \tag{4.229}$$

In analogous fashion to the Einstein-Maxwell case, we observe that Rényi entropies are independent of n as $\mu \to \infty$ and they scale with μ^{d-1}. Let us note that, since the dependence with n becomes trivial for large μ, it is very likely that the inequalities (4.223), that we showed to hold for small μ, are actually satisfied for every μ. Regarding the sign of the corrections, we note that this is not definite. Since $\alpha_1 > 0$ on account of the WGC, this coupling always has the effect of increasing the value of the RE. On the other hand, by looking at the dependence of x_1 on β (which again must be non-negative) we see that it is a decreasing function for $d = 3, 4$ and a non-monotonic function for $d \geq 5$. Hence, the corrections can either increase or decrease the value of the RE depending on the relative values of the couplings and on the dimension. In spite of this, we notice that this quantity is always positive providing that the WGC is satisfied. If, contrarily, one was not to impose the WGC bounds, then α_1 could get arbitrarily negative, since this behavior is allowed by the unitarity constraints as shown in Fig. 4.1. Therefore, in order to avoid the RE to become negative at large chemical potential, unitarity is not enough, but we also need to impose the WGC.

4.5.1.3 Exact result: an example in connection to the WGC

Let us finally take a look at the exact value of the Rényi entropy as a function of μ and n. Performing a thorough analysis would require a separate work due to the large number of parameters and variables involved, so let us study a quite illustrative example. First, we set $d = 4$, since this is the most interesting case. Then, for a given choice of couplings $\{\lambda, \alpha_1, \alpha_2, \beta\}$ we need to solve (4.198) and (4.199) in order to obtain $S_n(\mu)$ according to (4.196). However, it can happen (and we have observed that this is the case for some values of the couplings) that the equations (4.198) and (4.199) have several admissible solutions for the same n and μ. If this happens, it denotes the existence of multiple phases, and in that case one must choose the one with smallest value of Ω, which is the dominant one. Then, we wish to study the profile of $S_n(\mu)$ when we take into account the physical constraints in Sec. 4.4.

We perform the following experiment: we take a set of random values of the couplings satisfying both WGC and unitarity constraints, and a second set of couplings that satisfy unitarity but not the WGC. We then study the properties of the RE for each set. The details of course depend on the particular values of the couplings, but we show a representative example in Fig. 4.3. In the left column we have represented S_n/S_1 (and also $(n-1)/n(S_n/S_1)$ and $(n-1)S_n/S_1$) as a function of n, for several values of $\bar{\mu}$, for a set of couplings that do not satisfy the WGC (but that do respect unitarity). In the right column, we show the same quantities for a different choice of couplings that now respect both unitarity and the WGC. The differences are stark. While in the right column, the RE is always positive and respects the inequalities (4.223), the RE for the theory that breaks the WGC violates the second and third of them when $\bar{\mu}$ becomes large enough. Furthermore, the RE even become negative in that case.

Certainly, this is only an example, but looking at randomly generated couplings we have not found any instance of a theory that satisfies the WGC and unitarity and behaves as in the left column. In fact, in all those cases we obtain plots similar to those in the right column of Fig. 4.3. Thus, it seems that the WGC bounds are key to produce a sensible dual CFT.

4.5.2 Generalized twist operators

A very interesting notion in the context of Rényi entropies is that of twist operators, which possess a great deal of information about the CFT. Let us remember that in the computation of Rényi entropies for some region A via the replica trick one uses the following result

$$\mathrm{Tr}\rho_A^n = \frac{Z_n}{Z_1^n}\,, \tag{4.230}$$

where Z_n is the partition function of an n-fold cover of Euclidean space in which cuts have been introduced in A. Along these cuts the k-th geometry must be glued to the $(k+1)$-th one by implementing appropriate boundary conditions [600].

However, an alternative route to compute this quantity involves the insertion of dimension-$(d-2)$ operators σ_n (the twist operators) extending over the entangling surface $\Sigma = \partial A$ [446, 564, 600, 608]. Then, the path integral over the replicated geometry can be replaced by a path integral for the symmetric product of n copies of the CFT, with the σ_n inserted, on a single copy of the geometry. One can then obtain the desired trace of ρ_A^n as the expectation value of these twist operators, $\mathrm{Tr}\rho_A^n = \langle\sigma_n\rangle$, computed in the n-fold symmetric product CFT.

It is possible to define a generalized notion of conformal dimension for the twist operators by performing an insertion of the stress-energy tensor T_{ab} at a small distance y from Σ. In particular, the leading singularity of the correlator $\langle T_{ab}\sigma_n\rangle$ takes the form [446, 564],

$$\langle T_{ab}\sigma_n\rangle = -\frac{h_n}{2\pi}\frac{b_{ab}}{y^d}\,, \tag{4.231}$$

where b_{ab} is a fixed tensorial structure and h_n is the conformal dimension of σ_n. In the case of a spherical entangling surface, and with a finite chemical potential, the conformal

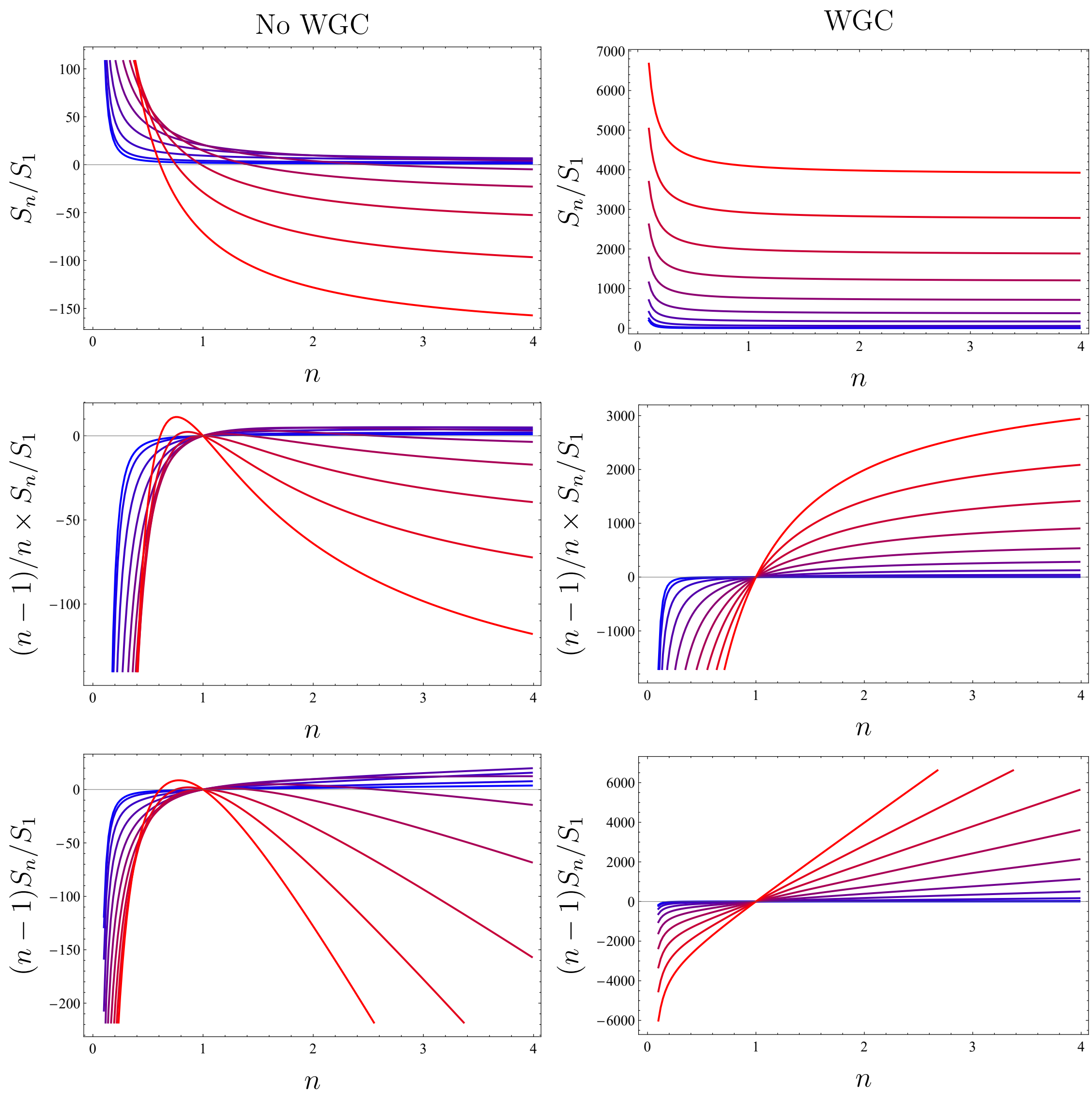


Figure 4.3: Rényi entropies as function of n for two particular choices of couplings in $d = 4$. From blue to red, the different curves correspond to $\bar{\mu}/\sqrt{f_\infty} = 0, 1, \ldots 10$. Left: a theory that satisfies unitarity constraints but not the WGC: $\{\lambda, \alpha_1, \alpha_2, \beta\} = \{0.052, -0.100, 0.875, 0.0049\}$. Right: a theory that satisfies both both unitarity and WGC bounds: $\{\lambda, \alpha_1, \alpha_2, \beta\} = \{0.077, 0.057, -0.596, 0.023\}$. The standard properties of Rényi entropies may be violated if the WGC is not satisfied.

mapping from flat space to the hyperbolic cylinder allows one to show that [446, 447]

$$h_n(\mu) = \frac{2\pi n}{d-1} R^d \left(\mathcal{E}(T_0, \mu = 0) - \mathcal{E}(T_0/n, \mu) \right) , \tag{4.232}$$

where $\mathcal{E}(T, \mu)$ is the thermal energy density of the theory placed on $S^1 \times \mathbb{H}^{d-1}(R)$.

Likewise, when a chemical potential is present, we also have at hand its associated current J^a, and one can also study the correlator $\langle J_a \sigma_n(\mu) \rangle$. In this case, the leading singularity takes the form [447]

$$\langle J_a \sigma_n(\mu)\rangle = \frac{ik_n(\mu)}{2\pi}\frac{\tau_a}{y^{d-1}} \tag{4.233}$$

where τ_a is again a fixed structure determined by the geometry of the setup. The coefficient $k_n(\mu)$ is the magnetic response of the generalized twist operators, and for a spherical entangling surface it can be computed as [447]

$$k_n(\mu) = 2\pi n R^{d-1}\rho(n,\mu)\,, \tag{4.234}$$

where $\rho(n,\mu)$ is the charge density of the theory on $S^1\times\mathbb{H}^{d-1}(R)$ at temperature $T = T_0/n$ and with chemical potential μ.

It is then interesting to consider the expansion of $h_n(\mu)$ and $k_n(\mu)$ around $n = 1$ and $\mu = 0$,

$$\begin{aligned} h_n(\mu) =& \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{l!m!}h_{lm}(n-1)^l\mu^m\,,\\ k_n(\mu) =& \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{l!m!}k_{lm}(n-1)^l\mu^m\,, \end{aligned} \tag{4.235}$$

where

$$h_{lm} = (\partial_n)^l(\partial_\mu)^m h_n(\mu)\Big|_{n=1,\mu=0}\,, \qquad k_{lm} = (\partial_n)^l(\partial_\mu)^m k_n(\mu)\Big|_{n=1,\mu=0}\,. \tag{4.236}$$

As shown by Ref. [447] (and by Refs. [446, 555, 564] in the case of h_n for $\mu = 0$), these coefficients involve integrated correlators of the form $\langle T\ldots TJ\ldots J\rangle$. In particular, the few first coefficients are related to two or three-point functions of T and J, and therefore have a universal form for any CFT. These relations were derived in [446, 447, 564] from first principles, but here we will see that they can be equivalently derived by using holography with higher-derivative terms.

4.5.2.1 Conformal dimension of generalized twist operators

Let us start by studying the conformal dimension of the generalized twist operators, given by Eq. (4.232). Holographically, the energy density $\mathcal{E}$ is nothing but the mass of a hyperbolic black hole over the volume of the hyperbolic boundary, *i.e.*,

$$\mathcal{E}(T,\mu) = \frac{M(T,\mu)}{V_{-1,d-1}R^{d-1}}\,. \tag{4.237}$$

This can be obtained from Eq. (4.52) by setting $k = -1$. Observing that $M(T_0,\mu = 0) = 0$, by virtue of (4.232) we have

$$\begin{aligned} h_n(\mu) =& -\frac{nL^{d-1}}{8(d-1)\sqrt{f_\infty}G}\Bigg[(d-1)\Big(-x^{d-2}+x^d+\lambda x^{d-4}\Big)\\ &+\frac{2p^2x^d}{(d-2)}\left(1-\frac{(d-2)}{x^2}(3(d-1)\alpha_1+\alpha_2)\right)-\frac{\beta p^4x^d}{(3d-4)}\Bigg]\,, \end{aligned} \tag{4.238}$$

where as usual we have introduced $x = r_+/L$, $p = qL/r_+^{d-1}$, which depend on n and μ through the relations (4.198) and (4.199). For $n = 1$ and $\mu = 0$, those equations are solved by $x = 1/\sqrt{f_\infty}$ and $p = 0$. We can then perform an expansion around those values to find

$$\begin{aligned} x =& \frac{d-n}{\sqrt{f_\infty}(d-1)} \\ &+ \left(\frac{\mu \ell_* R}{L}\right)^2 \frac{(d-2)^2 f_\infty^{3/2}\left(1-\alpha_1(3d+2)(d-1)f_\infty - \alpha_2 d f_\infty\right)}{(d-1)^2(2-f_\infty)\left(\alpha_{\text{eff}}^{\text{EQG}}\right)^2} + \ldots\,, \end{aligned} \tag{4.239}$$

$$\begin{aligned} p =& \left(\frac{\mu \ell_* R}{L}\right)\left[\frac{(d-2)f_\infty}{\alpha_{\text{eff}}^{\text{EQG}}}\right. \\ &\left.+(n-1)\frac{(d-2)f_\infty\left(1+\alpha_1(d-2)(d-1)f_\infty+\alpha_2(d-2)f_\infty\right)}{(d-1)\left(\alpha_{\text{eff}}^{\text{EQG}}\right)^2}+\ldots\right]+\ldots\,, \end{aligned} \tag{4.240}$$

where we only show the terms that we will need. From these expressions it is straightforward to obtain the expansion of h_n in (4.238) and to read off the values of the derivatives. In the first place, we find

$$h_{10} = \frac{1-2\lambda f_\infty}{4(d-1)G}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-1}\,, \tag{4.241}$$

and comparing with the value of the central charge C_T four our theory, given by Eq. (4.80), we realize that this relation can be written as

$$h_{10} = 2\pi^{d/2+1}\frac{\Gamma(d/2)}{\Gamma(d+2)}C_T\,. \tag{4.242}$$

This is precisely the relation found in Ref. [446]. In a similar way, the second derivative of h_n at vanishing μ, that is, h_{20}, is completely determined in terms of C_T and the 3-point function coefficients t_2 and t_4 [555, 564]. Those relations have been shown to be identically satisfied for holographic higher-curvature gravities [257, 270, 555]. Thus, let us turn our attention to the derivatives of h_n with respect to μ, which, to the best of our knowledge, have not been studied in detail for higher-derivative theories.

From (4.239), (4.240) and (4.238) we find

$$h_{02} = \frac{(d-2)^2\ell_*^2R^2}{2(d-1)^2G}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3}\frac{f_\infty\left((6d-1)(d-1)\alpha_1+(2d-1)\alpha_2\right)+\frac{3-2d}{d-2}}{\left(\alpha_{\text{eff}}^{\text{EQG}}\right)^2}\,. \tag{4.243}$$

Now, looking at Eqs. (4.95), (4.97) and (4.125), we see that this expression can be written in terms of the central charge C_J and the flux parameter a_2 as

$$h_{02} = -(2\pi R)^2\frac{C_J\pi^{\frac{d}{2}-1}\Gamma\left(\frac{d}{2}\right)}{(d-1)^3\Gamma(d+1)}\left((d-1)d(2d-3)+a_2(d-2)^2\right)\,. \tag{4.244}$$

Finally, we can write it in terms of the $\langle TJJ\rangle$ coefficients $\hat{c}$ and $\hat{e}$ using the relations (4.130) and (4.131), and we get[18]

$$h_{02} = -(2\pi R)^2\frac{4\pi^{d-1}}{\Gamma(d+1)}\left(\frac{2}{d}\hat{c}+\hat{e}\right)\,, \tag{4.245}$$

[18] We could of course derive this relation directly from the values of $\hat{c}$ and $\hat{e}$ for EQG, given by Eqs. (4.134) and (4.135), but we find it interesting to show the intermediate expression in terms of C_J and a_2.

which is precisely the result in Eq. (2.45) of [447] and which applies to any CFT.[19]

4.5.2.2 Magnetic response of generalized twist operators

Let us now take a look at the magnetic response $k_n(\mu)$, which we can compute using the relation (4.234). The charge density in the boundary theory is simply

$$\rho(n,\mu) = \frac{\ell_* Q}{R^{d-1} V_{-1,d-1}}\,, \tag{4.246}$$

where Q is given by Eq. (4.48). Therefore we get

$$k_n(\mu) = \frac{n\ell_* q}{2G} = \frac{\ell_* L^{d-2}}{2G} npx^{d-1}\,. \tag{4.247}$$

By using (4.239) and (4.240) we can easily expand this quantity near $n=1$ and $\mu=0$ and read off its derivatives. For the first derivative with respect to μ we obtain

$$k_{01} = \frac{(d-2)\ell_*^2 R}{2G\alpha_{\rm eff}^{\rm EQG}}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3} = 8\pi^{d/2+1} R \frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma(d+1)} C_J\,, \tag{4.248}$$

where in the second equality we used (4.95). Again, up to a factor of $(2\pi R)$ that arises from different normalization conventions for μ, this coincides with Eq. (2.57) of [447]. We can also compute the mixed partial derivative k_{11}, which yields

$$k_{11} = \frac{(d-2)\ell_*^2 R\left[1+(d-2)f_\infty\left((d-1)\alpha_1+\alpha_2\right)\right]}{4(d-1)G\left(\alpha_{\rm eff}^{\rm EQG}\right)^2}\left(\frac{L}{\sqrt{f_\infty}}\right)^{d-3}\,. \tag{4.249}$$

This can be express in terms of C_J (4.95) and a_2 (4.125) as

$$k_{11} = \frac{4RC_J\pi^{\frac{d}{2}+1}\Gamma\left(\frac{d}{2}\right)}{(d-1)^2\Gamma(d+1)}\left(d(d-1)-a_2(d-2)^2\right)\,, \tag{4.250}$$

or in terms of the $\langle TJJ\rangle$ coefficients (4.134) and (4.135) as

$$k_{11} = \frac{16\pi^{d+1}R}{d\Gamma(d+1)}\left(2\hat{c}-d(d-3)\hat{e}\right)\,. \tag{4.251}$$

Thus, we reproduce in this case Eq. (2.56) of [447]. Let us remark that Ref. [447] checked that these relations held for holographic Einstein-Maxwell theory, but that case is somewhat restricted as the dual theory has $a_2=0$. To the best of our knowledge, this is the first holographic derivation of these universal relationships in a theory with a general $\langle TJJ\rangle$ three-point function.

[19] Note that we have an additional $(2\pi R)^2$ factor with respect to [447], which comes from the fact that they normalize the chemical potential with a factor of $1/(2\pi R)$, that we do not introduce.

4.6 A universal feature of charged entanglement entropy

In this last section of the chapter we devote ourselves to the proof of the following identity for the charged entanglement entropy of a spherical region in d-dimensional CFTs with $d \geq 3$:

$$\frac{S_{\rm EE}}{\nu_{d-1}} = a^* + \frac{\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)}{d(d-1)}a_2\right](\mu R)^2 + \mathcal{O}(\mu^3)\,. \tag{4.252}$$

First, we will motivate this result by showing its validity for an infinite set of higher-derivative theories of gravity coupled to a $(d-2)$ form, which are of arbitrary order in the curvature and in the gauge field strength and generalize those written in Eq. (4.30). Similarly, we will also show that they hold for free fermions and free bosons in $d = 4$ using heat-kernel techniques. Finally, we will present the general proof of (4.252), which will be carried out by employing the notions of twist operators and magnetic response, already coined in Section 4.5.

4.6.1 Charged entanglement entropy in EQGs in any dimension $d \geq 3$

It is possible to construct EQGs in any spacetime dimension $D = d + 1$ at arbitrary order in the curvature tensor and the field strength. In the case of pure gravity theories, Quasitopological and Generalized Quasitopological at all orders were obtained in Ref. [250], so let us focus here in the case of non-minimally coupled theories. In analogy with the four-dimensional theories identified in [4], we have been able to find the following infinite families of EQGs:

$$\begin{aligned} I_{\rm EQG}^{\rm gen} = \frac{1}{16\pi G}\int \mathrm{d}^{d+1}x\sqrt{|g|}\Bigg[R+\frac{d(d-1)}{L^2}-\frac{2H^2}{(d-1)!}+\frac{\lambda L^2\mathcal{X}_4}{(d-2)(d-3)} \\ +\frac{2}{(d-1)!}\sum_{s=0}^{\infty}\sum_{m=1}^{\infty}L^{2(s+m-1)}\left(\alpha_{1,s,m}\mathcal{L}^{(a)}_{d,s,m}+\alpha_{2,s,m}\mathcal{L}^{(b)}_{d,s,m}\right)\Bigg]\,, \end{aligned} \tag{4.253}$$

where

$$\mathcal{X}_4 \equiv +R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \tag{4.254}$$

$$\begin{aligned} \mathcal{L}^{(a)}_{d,s,m} \equiv \Big(sR\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma} + \kappa_{d,s,m}\left(R^s\right)^{\mu\nu}{}_{\rho\sigma} \\ +2s(s-1)R_{\gamma}{}^{\mu}R^{\beta}{}_{\rho}\left(R^{s-2}\right)^{\gamma\nu}{}_{\beta\sigma}\Big)(H^2)_{\mu\nu}{}^{\rho\sigma}(H^2)^{m-1}\,, \end{aligned} \tag{4.255}$$

$$\mathcal{L}^{(b)}_{d,s,m} \equiv \frac{1}{2}\left(2sR_{\mu}{}^{\alpha}\delta_{\nu}{}^{\beta}+g_{d,s,m}R^{\alpha\beta}{}_{\mu\nu}\right)\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma}(H^2)^{\rho\sigma}{}_{\alpha\beta}(H^2)^{m-1}\,, \tag{4.256}$$

and where we used the notation

$$\begin{aligned} \left(H^2\right)^{\rho\sigma}{}_{\mu\nu} \equiv H^{\rho\sigma\alpha_3\alpha_4\ldots\alpha_{d-1}}H_{\mu\nu\alpha_3\alpha_4\ldots\alpha_{d-1}}\,, \\ \left(R^k\right)^{\mu\nu}{}_{\rho\sigma} \equiv R^{\mu\nu}{}_{\alpha_1\beta_1}R^{\alpha_1\beta_1}{}_{\alpha_2\beta_2}\ldots R^{\alpha_{k-1}\beta_{k-1}}{}_{\rho\sigma} \end{aligned} \tag{4.257}$$

and introduce the constants $g_{d,s,m} \equiv -d(s-1)-2(d-1)m$ and $\kappa_{d,s,m} \equiv (1-g_{d,s,m})g_{d,s,m}/2$.

Any members of the infinite family of theories captured by Eq. (4.253) are examples of Electromagnetic Quasitopological Gravities [4]. By taking the only non-vanishing couplings to be $\alpha_{1,1,1} \equiv \alpha_1$ and $\alpha_{2,1,1} \equiv \alpha_2$ we recover the four-derivative theories discussed in previous sections and defined in Eq. (4.30). The theories (4.253) admit charged black-hole solutions with spherical, planar or hyperbolic sections. Their computational treatment is fairly similar and, since here we are interested only in solutions with hyperbolic sections, we restrict ourselves to this case. Such solutions are of the form

$$\mathrm{d}s^2 = \frac{-L^2}{f_\infty R^2}\left[\frac{r^2}{L^2}f - 1\right]\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left[\frac{r^2}{L^2}f - 1\right]} + r^2\mathrm{d}\Xi^2\,, \quad H_q = q\,\omega_{-1,(d-1)}\,, \tag{4.258}$$

where $\mathrm{d}\Xi^2$ is the metric of the unit hyperbolic space, $\omega_{-1,(d-1)}$ the associated volume form and we remind that $f_\infty \equiv L^2/\tilde{L}^2$ —where $\tilde{L}$ is the AdS$_{(d+1)}$ radius— can be written in terms of the Gauss-Bonnet coupling as $2\lambda f_\infty = 1 - \sqrt{1-4\lambda}$. Remarkably, the above solutions are characterized by a single metric function $f(r)$. The full non-linear equations of (4.253) collapse to a single first-order differential equation for $f(r)$ which can be integrated once, yielding the following algebraic equation

$$\begin{aligned} 0 = &+ \frac{r^2}{L^2}(1-f) - \frac{16\pi R\sqrt{f_\infty}GM}{(d-1)LV_{-1,d-1}r^{d-2}} + \frac{2q^2}{(d-2)(d-1)r^{2(d-2)}} + \frac{\lambda r^2}{L^2}f^2 \\ &+ \sum_{s,m}\frac{q^{2m}L^{2m}(-2)^s\Gamma(d)^{m-1}}{r^{2m(d-1)}}f^{s-1}\left[\frac{2s}{d-1}((2m-1)(d-1)-1+ds)\alpha_{1,s,m} + \frac{s\alpha_{2,s,m}}{d-1}\right. \\ &\left. - \left(\left(1-2m-4s+4ms+\frac{2s(ds-1)}{d-1}\right)\alpha_{1,s,m} + \frac{(s-1)\alpha_{2,s,m}}{d-1}\right)\frac{r^2}{L^2}f\right]. \end{aligned} \tag{4.259}$$

Here M is an integration constant to be identified with the mass of the solution and $\sum_{s,m} \equiv \sum_{s=0}^{\infty}\sum_{m=1}^{\infty}$. Assume now that g_{tt} has some zero along the positive real axis and let $r_+ = \max\left\{r \in \mathbb{R}^+ | f(r) = \frac{L^2}{r^2}\right\}$. Defining $x \equiv r_+/L$ and $p \equiv qL^{2-d}x^{1-d}$, and evaluating (4.259) at $r = r_+$, the mass M of the subsequent black hole solution can be seen to to be

$$\begin{aligned} \left[\frac{16\pi R\sqrt{f_\infty}G}{L^{d-1}V_{-1,d-1}}\right]M = &+ (d-1)x^{d-2}\left(x^2-1\right) + \frac{2p^2x^d}{(d-2)} + (d-1)\lambda x^{d-4} \\ &+ \sum_{s,m}\frac{(-2)^s\Gamma(d)^{m-1}p^{2m}}{x^{2s-d}}(-(d-1)(1-2m-2s)\alpha_{1,s,m} + \alpha_{2,s,m})\,. \end{aligned} \tag{4.260}$$

Similarly, taking into account that the temperature T is given by $4\pi RL\sqrt{f_\infty}T = r_+^2 f'(r_+) + \dfrac{2L}{r_+}$, we find the following expression,

$$\begin{aligned} 4\pi R\sqrt{f_\infty}T = &\frac{(d-1)(2+d(x^2-1)+(d-4)\lambda x^{-2}) - 2p^2x^2}{(d-1)(x-2\lambda x^{-1}) + \sum_{s,m}(-2)^s\Gamma(d)^{m-1}p^{2m}x^{3-2s}sd_{s,m}\alpha_{1,s,m}} \\ &- \frac{\sum_{s,m}(-2)^s\Gamma(d)^{m-1}p^{2m}x^{-2(s-1)}t_{s,m}}{x-2\lambda x^{-1} + \sum_{s,m}(-2)^s\Gamma(d)^{m-1}(d-1)^{-1}p^{2m}x^{3-2s}sd_{s,m}\alpha_{1,s,m}}\,, \end{aligned} \tag{4.261}$$

where we have implicitly defined:

$$d_{s,m} = (2s+d(2m-1)-2m-1)\,, \quad t_{s,m} = d_{s,m}\left((2m+2s-1)\alpha_{1,s,m} + \frac{\alpha_{2,s,m}}{d-1}\right). \tag{4.262}$$

The computation of the black hole entropy S is carried out using the Iyer-Wald formula [366,370],

$$S=-2\pi\int_{\Sigma}\mathrm{d}^{d-1}x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,, \tag{4.263}$$

where Σ is a cross section of the black hole horizon and $\epsilon_{\mu\nu}$ is its binormal. This can be straightforwardly applied to the theories (4.253) and further evaluated for the black hole metric (4.258), yielding

$$S=\frac{x^{d-1}L^{d-1}V_{-1,d-1}}{4G}\left[1-\frac{2(d-1)\lambda}{(d-3)x^2}-\sum_{s,m}\frac{(-2)^s\Gamma(d)^{m-1}sp^{2m}}{x^{2(s-1)}}\alpha_{1,s,m}\right]. \tag{4.264}$$

Interestingly, the entropy does not receive any corrections from the density $\mathcal{L}^{(b)}_{d,s,m}$ but only from $\mathcal{X}_4$ and $\mathcal{L}^{(a)}_{d,s,m}$.

As explained in Section 4.3, the chemical potential μ is defined as the asymptotic value of the electrostatic potential $\tilde{A}_t$ after demanding that $\tilde{A}_t\Big|_{r_+}=0$ (see (4.73)). For the any-derivative EQGs given by (4.253), it can be checked that the chemical potential μ reads

$$\left[\frac{L^{d-2}x^{d-1}V_{-1,d-1}\ell_*}{4\pi G}\right]\mu=\frac{\partial M}{\partial p}-T\frac{\partial S}{\partial p}\,. \tag{4.265}$$

Taking into account this expression and the previous presented thermodynamic magnitudes, it is possible to show that the first law of black hole thermodynamics holds, namely,

$$\mathrm{d}M=T\mathrm{d}S+\mu\,\mathrm{d}\mathcal{N}\,,\quad \text{where}\quad \mathcal{N}\equiv q\cdot\left[\frac{V_{-1,d-1}\ell_*}{4\pi G}\right] \tag{4.266}$$

is the total charge in the boundary theory.

Now, our goal is to compute the vacuum charged EE for the boundary theory across a spherical entangling surface of radius R. Such entanglement entropy can be obtained from the thermal entropy of the same theory placed on the hyperbolic cylinder $S^1\times\mathbb{H}^{d-1}(R)$ at temperature $T_0=1/(2\pi R)$ [445–447]. Then, using the holographic dictionary, such thermal entropy turns out to be just the Wald entropy of a black hole with hyperbolic horizon, *i.e.*,

$$S_{\mathrm{EE}}(\mu)=S(T_0,\mu)\,. \tag{4.267}$$

Consequently, for the derivation of the charged entanglement entropy, we need to evaluate the Wald entropy (4.264) at temperature $T=T_0$ and in terms of the chemical potential μ. Above, in Eq. (4.264) we wrote $S=S(x,p)$, so we need to find the inverse functions $x=x(T_0,\mu)$ and $p=p(T_0,\mu)$. We will carry out this procedure in a perturbative fashion in μ and we will restrict ourselves to the leading-order corrections (so that it suffices to keep only the terms quadratic in H). We find

$$x=\hat{x}+\delta x_2(\ell_*\mu)^2+\mathcal{O}(\mu^4)\,,\quad p=\delta p_1(\ell_*\mu)+\mathcal{O}(\mu^3)\,,\quad \hat{x}=\frac{1}{\sqrt{f_\infty}}\,, \tag{4.268}$$

$$\delta p_1=\frac{2f_\infty R}{L\left(\frac{2}{d-2}+\sum_{s=0}^{\infty}(-2f_\infty)^s((d+2ds-1)\alpha_{1,s}+\alpha_{2,s}\right)}\,, \tag{4.269}$$

$$\delta x_2=-\frac{(\delta p_1)^2(2+\sum_{s=0}^{\infty}(-2)^s f_\infty^s\delta x_{2,s})}{2(d-1)^2(f_\infty-2)\sqrt{f_\infty}}\,, \tag{4.270}$$

where $\delta x_{2,s} = 2 - 4s + d(d-3+2(d-1)s+4s^2)\alpha_{1,s} + (d-2+2s)\alpha_{2,s}$. Plugging the (perturbative) expressions found above into Eq. (4.264), we find that the entanglement entropy to quadratic order in μ reads

$$\begin{aligned} S_{\rm EE} = \frac{\tilde{L}^{d-1}V_{-1,d-1}}{4G}\Bigg[1 - \frac{2(d-1)}{d-3}\lambda f_\infty \\ + \left(\frac{\sqrt{f_\infty}R}{L}\right)^2 \frac{(\ell_*\mu)^2}{\alpha_{\rm eff}}\left(\frac{(d-2)^2}{d-1} + \frac{(d-2)^2\beta_{\rm eff}}{(d-1)^2\alpha_{\rm eff}}\right)\Bigg] + \mathcal{O}(\mu^4)\,, \end{aligned} \tag{4.271}$$

where we have defined the parameters

$$\begin{aligned} \alpha_{\rm eff} &\equiv 1 + \sum_{s=0}^{\infty}(-2)^{s-1}f_\infty^s(2-d)\left((d-1+2ds)\alpha_{1,s} + \alpha_{2,s}\right)\,, \\ \beta_{\rm eff} &\equiv \sum_{s=0}^{\infty}(-2f_\infty)^s(d-1)s\left((2ds-1)\alpha_{1,s} + \alpha_{2,s}\right)\,. \end{aligned} \tag{4.272}$$

Our next goal will be trying to express the charged entanglement entropy (up to quadratic order in μ^2) in terms of the charges C_J and a_2 of the CFT dual to the theories (4.253). On the one hand, if F denotes the dual field strength of H, C_J is obtained by working out the effective gauge coupling of F^2 when we evaluate the action on a pure AdS background [447]. Owing the fact that we will restrict ourselves to quadratic terms in μ, it is enough to keep in the action (4.253) those terms up to quadratic order in H. If $I_{\rm dual}^{F^2}$ denotes the dual theory to any theory containing terms of up to second-order in H, then according to Section 4.1 $I_{\rm dual}^{F^2}$ can be shown to be

$$I_{\rm dual}^{F^2} = \int \frac{{\rm d}^{d+1}x\sqrt{|g|}}{16\pi G}\left[R + \frac{d(d-1)}{L^2} + \frac{\lambda L^2 \mathcal{X}_4}{(d-2)(d-3)} - (\tilde{Q}^{-1})_{\mu\nu}{}^{\rho\sigma}F^{\mu\nu}F_{\rho\sigma}\right], \tag{4.273}$$

where

$$\tilde{Q}^{\mu\nu}{}_{\rho\sigma} \equiv \frac{12}{(d-1)(d-2)}Q^{[\alpha\beta}{}_{\alpha\beta}\delta^{\mu\nu]}{}_{\rho\sigma}\,, \tag{4.274}$$

$$\begin{aligned} Q^{\alpha\beta}{}_{\rho\sigma} \equiv \delta^{\alpha\beta}{}_{\rho\sigma} - \sum_{s=0}^{\infty}\Bigg[\frac{1}{2}\left(R^{s-1}\right)^{\mu\nu}{}_{\rho\sigma}\left(2sR_\mu{}^{[\alpha}\delta_\nu{}^{\beta]} + g_{d,s,1}R^{\alpha\beta}{}_{\mu\nu}\right)\alpha_{2,s} \\ + \left(sR\left(R^{s-1}\right)^{\alpha\beta}{}_{\rho\sigma} + \kappa_{d,s,1}\left(R^s\right)^{\alpha\beta}{}_{\rho\sigma} + 2s(s-1)\left(R^{s-2}\right)^{\mu[\alpha}{}_{\nu[\rho|}R_\mu{}^{\beta]}R^\nu{}_{|\sigma]}\right)\alpha_{1,s}\Bigg]\,, \end{aligned} \tag{4.275}$$

and where

$$(\tilde{Q}^{-1})_{\mu\nu}{}^{\rho\sigma}\tilde{Q}_{\rho\sigma}{}^{\alpha\beta} = \delta_{\mu\nu}{}^{\alpha\beta}\,, \tag{4.276}$$

so that $\tilde{Q}^{-1}$ is the inverse tensor of $\tilde{Q}$, as described in the previous equation, and $\delta_{\mu\nu}{}^{\rho\sigma} = \delta_{[\mu}{}^{[\rho}\delta_{\nu]}{}^{\sigma]}$. We are also defining $\alpha_{1,s,1} \equiv \alpha_{1,s}$ and $\alpha_{2,s,1} \equiv \alpha_{2,s}$. Finding such inverse tensor is generically a rather challenging task, but it is a more manageable one when we restrict ourselves to backgrounds with enough symmetry. In the case at hand, since we are considering a pure AdS space with $R^{\mu\nu}{}_{\rho\sigma} = -2/\tilde{L}^2\delta^{\mu\nu}{}_{\rho\sigma}$, $\tilde{L} = L/\sqrt{f_\infty}$, we have

$$Q^{\mu\nu}{}_{\rho\sigma} = \tilde{Q}^{\mu\nu}{}_{\rho\sigma} = \alpha_{\rm eff}\delta^{\mu\nu}{}_{\rho\sigma}\,, \quad (\tilde{Q}^{-1})_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\alpha_{\rm eff}}\delta^{\mu\nu}{}_{\rho\sigma}\,. \tag{4.277}$$

Consequently, the coefficient of F^2 in (4.273) turns out to be $1/\alpha_{\text{eff}}$. This implies that the net effect of the higher-order terms, as in the four-derivative theory, is the renormalization of the gauge coupling constant, producing in turn the central charge

$$C_J^{\text{EQG}} = \frac{C_J^{\text{EM}}}{\alpha_{\text{eff}}}\,, \quad C_J^{\text{EM}} = \frac{\Gamma(d)}{\Gamma(d/2-1)}\frac{\ell_*^2\tilde{L}^{d-3}}{4\pi^{d/2+1}G}\,, \tag{4.278}$$

being C_J^{EM} the Einstein-Maxwell central charge. On the other hand, the computation of a_2 requires the knowledge of the inverse tensor $\tilde{Q}^{-1}$ on a shock-wave background —see [435] and Section 4.3 for more details in this computation— given by the metric

$$\mathrm{d}s^2 = \frac{\tilde{L}^2}{u^2}\left[\delta(y^+)\mathcal{W}(y^i,u)\left(\mathrm{d}y^+\right)^2 - \mathrm{d}y^+\mathrm{d}y^- + \sum_{j=1}^{d-2}\left(\mathrm{d}y^j\right)^2 + \mathrm{d}u^2\right], \tag{4.279}$$

$$\mathcal{W}(y^i,u) = \frac{\mathcal{W}_0 u^d}{\left(u^2 + \sum_{j=1}^{d-2}(y^j - y_0^j)^2\right)^{d-1}}\,, \quad y_0^j \in \mathbb{R}\,. \tag{4.280}$$

This shock-wave background satisfies that $R_{\mu\nu} = -d/\tilde{L}^2 g_{\mu\nu}$ and, being a Brinkmann spacetime, the square of its Weyl tensor vanishes, *i.e.* $W_{\mu\nu\rho\sigma}W^{\rho\sigma\alpha\beta} = 0$. Taking into account this properties, it can be seen that

$$\tilde{Q}_{\mu\nu}{}^{\rho\sigma} = \alpha_{\text{eff}}\delta_{\mu\nu}{}^{\rho\sigma} - \frac{\beta_{\text{eff}}}{f_\infty(d-1)(d-2)}W_{\mu\nu}{}^{\rho\sigma}\,, \tag{4.281}$$

$$(\tilde{Q}^{-1})_{\mu\nu}{}^{\rho\sigma} = \frac{1}{\alpha_{\text{eff}}}\delta_{\mu\nu}{}^{\rho\sigma} + \frac{\beta_{\text{eff}}}{f_\infty(d-1)(d-2)\alpha_{\text{eff}}^2}W_{\mu\nu}{}^{\rho\sigma}\,, \tag{4.282}$$

where β_{eff} is the parameter introduced in (4.272). We identify this result as formally equivalent to that of Eq. (4.14), obtained in the context of the four-derivative theory (4.30), upon exchange of $-(2(2d-1)(d-1)\alpha_1 + 2(d-1)\alpha_2) \mapsto \beta_{\text{eff}}/f_\infty$. Hence the coefficient a_2 associated to (4.253) will be that of Eq. (4.124), after making the aforementioned substitution, namely,

$$a_2^{\text{EQG}} = \frac{d\beta_{\text{eff}}}{(d-2)\alpha_{\text{eff}}}\,. \tag{4.283}$$

Therefore, taking into account Eqs. (4.278) and (4.283), we notice that the entanglement entropy can be rewritten as

$$\begin{aligned} S_{\text{EE}} = {} & \frac{\tilde{L}^{d-1}V_{-1,d-1}}{4G}\left[1 - \frac{2(d-1)}{d-3}\lambda f_\infty\right] \\ & + \frac{\Gamma(d/2-1)\pi^{d/2+1}V_{-1,d-1}}{\Gamma(d)}C_J^{\text{EQG}}\left(\frac{(d-2)^2}{d-1} + \frac{(d-2)^3 a_2^{\text{EQG}}}{d(d-1)^2}\right)(\mu R)^2 + \mathcal{O}(\mu^4)\,. \end{aligned} \tag{4.284}$$

The regularized volume of the unit hyperbolic space is given by the quantity [445] $V_{-1,d-1} = \nu_{d-1}/(4\pi)\Omega^{d-1}$, where Ω^{d-1} is the volume of the unit sphere S^{d-1} and ν_{d-1} is defined as in (4.210). We then arrive at our final result

$$\frac{S_{\text{EE}}(\mu)}{\nu_{d-1}} = a_{\text{GB}}^* + \frac{\pi^d}{(d-1)^2\Gamma(d-2)}C_J^{\text{EQG}}\left[1 + \frac{(d-2)a_2^{\text{EQG}}}{d(d-1)}\right](\mu R)^2 + \mathcal{O}(\mu^4)\,, \tag{4.285}$$

where we have introduced the a^* charge of Gauss-Bonnet theory, given by

$$a^*_{\rm GB} = \frac{\tilde{L}^{d-1}}{8G}\frac{\pi^{(d-2)/2}}{\Gamma(d/2)}\left[1-\frac{2(d-1)}{d-3}\lambda f_\infty\right]\,. \tag{4.286}$$

Eq. (4.285) fits precisely into the form of (4.252). On the other hand, apart from having derived the first correction to entanglement entropy for infinite families of theories of arbitrary order in the curvature and the gauge field strength, note that if we restrict the result to the four-dimensional theory (4.30), we have in fact generalized the results for the charged entanglement entropy obtained in Section 4.5, since here we have not set λ to zero and thus we have been able to capture as well the effects of the Gauss-Bonnet term.

As a final comment, one may wonder about the effect of including arbitrary pure-gravity Quasitopological higher-order terms [238, 239, 241, 242, 249, 250] into the action (4.253). Given the structure and derivation of Eq. (4.285), we expect such pure-gravity terms to simply produce a renormalization of the constant f_∞, while leaving Eq. (4.285) invariant.

4.6.2 Free field calculations

In this subsection we present the calculation of the charged Rényi entropies for free scalars and fermions in $d = 4$ using heat-kernel techniques [609–612]. Our results here closely follow the derivation in [447], but we use the opportunity to correct a few typos that appear in that paper, which include the final expression for $S_n(\mu)$ in the case of the free fermion.

We will compute the charged Rényi entropy from the free energy on $S^1\times\mathbb{H}^3$. In order to do that, we will use the heat-kernel on such space. For product spaces this factorizes, so one has

$$K_{S^1\times\mathbb{H}^3} = K_{S^1}(\theta_1,\theta_2,t)K_{\mathbb{H}^3}(\vec{y}_1,\vec{y}_2,t)\,. \tag{4.287}$$

Following [447], we consider a purely imaginary chemical potential for a global U(1) charge associated to phase rotations of the fields. This is related to the real chemical potential we use throughout the rest of the section by $\mu_{\rm E} = 2\pi i R\mu$. Incorporating the chemical potential in the heat-kernel amounts to requiring this to satisfy an appropriate boundary condition. This reads

$$K_{S^1}(\theta_1+2\pi n,\theta_2,t) = (-)^f e^{-in\mu_{\rm E}}K_{S^1}(\theta_1,\theta_2,t)\,, \tag{4.288}$$

where $f = 1$ for Dirac fermions and $f = 0$ for scalars. This is achieved by a modified disk heat-kernel of the form[20]

$$K_{S^1}(\theta_1,\theta_2,t) = \frac{1}{\sqrt{4\pi t}}\sum_{m\in\mathbb{Z}} e^{-\frac{(\theta_2-\theta_1+2\pi n m)^2}{4t}}e^{-im(n\mu_{\rm E}+\pi f)}\,. \tag{4.289}$$

Indeed, upon substitution of $\theta_1\to\theta_1+2\pi n$, the numerator of the exponent of the first term becomes $(\theta_2-\theta_1+2\pi n(m-1))^2$. Since the sum is over all integers, one can shift the index $m = m'+1$ leaving the first term as it was originally and producing an overall $(-)^f e^{-in\mu_{\rm E}}$ from the second term. The equal-point heat kernel then reads

$$K_{S^1}(0,0,t) = \frac{1}{\sqrt{4\pi t}}\sum_{m\in\mathbb{Z}} e^{-\frac{\pi^2 n^2 m^2}{t}}e^{-im(n\mu_{\rm E}+\pi f)}\,. \tag{4.290}$$

[20] In [447] there is a missing "n" that should be multiplying "$2\pi m$" in Eq. (A.6) and which propagates throughout the whole appendix.

On the other hand, the equal-point heat kernel for the hyperbolic space reads [447]

$$K_{\mathbb{H}^3}(0,0,t) = \frac{(1+3f)}{(4\pi t)^{3/2}}\left[1+\frac{tf}{2}\right] . \tag{4.291}$$

From these, the grand free energy on $S^1_{(2\pi n)} \times \mathbb{H}^3$ can be computed as

$$\begin{aligned}\Omega_n(\mu_{\rm E}) &= \frac{(-)^{f+1}}{2} V_{\mathbb{H}^3}(2\pi n)\int_0^\infty \frac{\mathrm{d}t}{t} K_{S^1}(0,0,t)K_{\mathbb{H}^3}(0,0,t)\,,\\ &= (-)^{f+1}\frac{n(1+3f)}{16\pi}V_{\mathbb{H}^3}\sum_{m\in\mathbb{Z}} e^{-im(n\mu_{\rm E}+\pi f)}\int_0^\infty \frac{\mathrm{d}t}{t^3}\left[1+\frac{tf}{2}\right]e^{-\frac{\pi^2 n^2 m^2}{t}}\,,\end{aligned} \tag{4.292}$$

where we have rewritten $V_{-1,3}$ as $V_{\mathbb{H}^3}$. The zero mode in the disk heat kernel gives rise to a divergence in the grand free energy [447], so we can ignore it and get for the regulated grand free energy

$$\begin{aligned}\Omega_n &= (-)^{f+1}\frac{(1+3f)V_{\mathbb{H}^3}}{16\pi^5 n^3}\sum_{m\in\mathbb{Z}^+}(-)^{mf}\cos[mn\mu_{\rm E}]\frac{(2+fm^2n^2\pi^2)}{m^4}\,,\\ &= (-)^{f+1}\frac{(1+3f)V_{\mathbb{H}^3}}{16\pi^5 n^3}\Bigg[\frac{fn^2\pi^2}{2}\left(\mathrm{Li}_2[(-)^f e^{-in\mu_{\rm E}}]+\mathrm{Li}_2[(-)^f e^{in\mu_{\rm E}}]\right)+\\ &\qquad\qquad \mathrm{Li}_4[(-)^f e^{-in\mu_{\rm E}}]+\mathrm{Li}_4[(-)^f e^{in\mu_{\rm E}}]\Bigg]\,.\end{aligned} \tag{4.293}$$

From this, the charged Rényi entropy can be obtained as

$$S_n(\mu_{\rm E}) = \frac{1}{n-1}\left[\Omega_n(\mu_{\rm E}) - n\Omega_1(\mu_{\rm E})\right] . \tag{4.294}$$

We find, respectively, for the Dirac fermion and the free scalar,

$$S_n^{\rm f}(\mu_{\rm E}) = \frac{V_{\mathbb{H}^3}}{48\pi}\left[\frac{(1+n)(7+37n^2)}{30n^3} - \frac{(1+n)\mu_{\rm E}^2}{n\pi^2}\right] , \tag{4.295}$$

$$S_n^{\rm s}(\mu_{\rm E}) = \frac{V_{\mathbb{H}^3}}{48\pi}\left[\frac{(1+n)(1+n^2)}{15n^3} - \frac{(1+n)\mu_{\rm E}^2}{2n\pi^2} + \frac{|\mu_{\rm E}|^3}{2\pi^3}\right] . \tag{4.296}$$

The scalar formula agrees with the one presented in [447], but the fermion one is different. There seems to be a missing $1/(4\pi^2)$ multiplying $\mu_{\rm E}^2$ in their Eq. (A.25). Finally, writing these in terms of μ and ν_3 (the particularization of the expression for ν_{d-1} in (4.210) for $d=4$), we find

$$S_n^{\rm f} = \frac{\nu_3}{24}\left[\frac{(1+n)(7+37n^2)}{120n^3} + \frac{(1+n)(\mu R)^2}{n}\right] , \tag{4.297}$$

$$S_n^{\rm s} = \frac{\nu_3}{24}\left[\frac{(1+n)(1+n^2)}{60n^3} + \frac{(1+n)(\mu R)^2}{2n} + |\mu R|^3\right] . \tag{4.298}$$

Interestingly, the exact dependence on μ is much simpler than for our holographic theories, for which, as we saw earlier, a completely explicit formula cannot be obtained. Taking the limit $n\to 1$ we obtain the expression for the entanglement entropies, which read

$$\frac{S_{\rm EE}^{\rm f}(\mu)}{\nu_3} = a_{\rm f}^* + \frac{(\mu R)^2}{12}\,, \qquad \frac{S_{\rm EE}^{\rm s}(\mu)}{\nu_3} = a_{\rm s}^* + \frac{(\mu R)^2}{24} + \frac{|\mu R|^3}{24}\,, \tag{4.299}$$

where $a^*_{\rm f} = 11/360$, $a^*_{\rm s} = 1/360$ are the trace-anomaly coefficients corresponding to a Dirac fermion and a real scalar field, respectively [613–615]. The values of the charges C_J and a_2 for these two models are also well known and read [433, 435, 585, 616]

$$\begin{aligned} C_J^{\rm f} &= \frac{1}{\pi^4}\,, \quad C_J^{\rm s} = \frac{1}{4\pi^4}\,,\\ a_2^{\rm f} &= -\frac{3}{2}\,, \quad a_2^{\rm s} = 3\,. \end{aligned} \tag{4.300}$$

Using these values of the charges C_J and a_2, it is straightforward to verify that the expression (4.252)

$$\frac{S_{\rm EE}(\mu)}{\nu_{d-1}} = a^* + \frac{\pi^d C_J}{(d-1)^2\Gamma(d-2)}\left[1+\frac{(d-2)a_2}{d(d-1)}\right](\mu R)^2 + \mathcal{O}(\mu^3)\,, \tag{4.301}$$

restricted to $d = 4$, holds for both theories.

4.6.3 Proof of the universal relation for charged entanglement entropies for general CFTs

The previous results strongly suggest that Eq. (4.252) holds for general CFTs. As it turns out, a proof of such universality can be easily achieved using a combination of the results presented in Ref. [447] along with some thermodynamic identities. In order to do this, we need to depart momentarily from the vacuum temperature T_0 and consider a CFT on the hyperbolic cylinder at an arbitrary temperature T. The thermal entropy of a given CFT in such state can be used to compute the Rényi entropy $S_n(\mu)$ across a spherical entangling region [446, 447], the Rényi index being related to the temperature by $n = T_0/T$.

In order to proceed, we need to consider a set of related quantities: the twist operators $\sigma_n(\mu)$. In the replica trick approach to the evaluation of Rényi/entanglement entropy, the entangling region is cut from each of the spacetime copies and consecutive copies are sewn together along the entangling surface. Such boundary conditions can be understood as produced by the insertion of $(d-2)$-dimensional operators along the entangling surface [446,564,600,617]. In the charged Rényi/EE case, the entangling surface carries a "magnetic flux" $-in\mu$ which can be understood as attaching a Dirac sheet to the twist operators [447].

The leading divergence in the correlator of $\sigma_n(\mu)$ with the current operator defines the so called "magnetic response" $k_n(\mu)$ as [447]

$$\langle J_a \sigma_n(\mu)\rangle = \frac{ik_n(\mu)}{2\pi}\frac{\epsilon_{ab}n^b}{y^{d-1}}\,, \tag{4.302}$$

where y is the distance between the insertions, n^b is a unit vector normal to J_a from the twist operator insertion and ϵ_{ab} is the volume form of the two-dimensional space orthogonal to the entangling surface. In the case of a spherical entangling surface, the magnetic response is given by (4.234), and we rewrite here for the sake of convenience:

$$k_n(\mu) = 2\pi n R^{d-1}\rho(n,\mu)\,, \tag{4.303}$$

where $\rho(n,\mu)$ is the charge density of the CFT on the hyperbolic cylinder at temperature $T = T_0/n$. As it turns out, this quantity has a universal expansion around $n = 1$ and

$\mu = 0$ whose leading terms can be expressed in terms of the coefficients characterizing the $\langle TJJ\rangle$ correlator. Collecting results from [447] and Subsection 4.5.2, we have

$$\begin{aligned} k_n\Big|_{n=1,\mu=0} &= \partial_n k_n\Big|_{n=1,\mu=0} = 0\,, \\ \partial_\mu k_n\Big|_{n=1,\mu=0} &= \frac{16R\pi^{d+1}}{\Gamma(d+1)}[\hat{c}+\hat{e}]\,, \\ \partial_n\partial_\mu k_n\Big|_{n=1,\mu=0} &= \frac{16R\pi^{d+1}}{d\Gamma(d+1)}\left[2\hat{c}-d(d-3)\hat{e}\right]\,, \end{aligned} \tag{4.304}$$

where the charges $\hat{c}, \hat{e}$ are related to C_J, a_2 by [447, 585]

$$\begin{aligned} \hat{c} &= \frac{C_J(d-2)\Gamma\left(\frac{d+2}{2}\right)}{2\pi^{d/2}(d-1)^3}\left[d(d-1)-a_2\right]\,, \\ \hat{e} &= \frac{C_J\Gamma\left(\frac{d+2}{2}\right)}{2\pi^{d/2}(d-1)^3}\left[d-1+(d-2)a_2\right]\,. \end{aligned} \tag{4.305}$$

Let us now consider the thermal entropy S of the CFT on the hyperbolic cylinder. In the grand canonical ensemble, the first law of thermodynamics reads

$$\mathrm{d}\Omega = -S\mathrm{d}T - \mathcal{N}\mathrm{d}\mu\,, \tag{4.306}$$

where Ω is the grand potential and $\mathcal{N} = V_{-1,d-1}R^{d-1}\rho$ is the total charge. From the first law the following thermodynamic relation can be obtained

$$\partial_\mu S = -\partial_\mu\partial_T\Omega = -\partial_T\partial_\mu\Omega = \partial_T\mathcal{N}\,. \tag{4.307}$$

Writing $\mathcal{N}$ in terms of the magnetic response $k_n(\mu)$, and using that $\partial_T = -\frac{T_0}{T^2}\partial_n$, we have

$$\partial_\mu S = -\frac{T_0 V_{-1,d-1}}{2\pi T^2}\partial_n\left(\frac{k_n(\mu)}{n}\right)\,. \tag{4.308}$$

Expanding the derivatives, evaluating for $n = 1$ ($T = T_0$) and $\mu = 0$ and using Eqs. (4.248), it immediately follows that the first derivative term vanishes, *i.e.,*

$$\partial_\mu S_{\mathrm{EE}}\big|_{\mu=0} = 0\,. \tag{4.309}$$

Taking a second derivative with respect to μ in Eq. (4.308), we have

$$\partial^2_\mu S = -\frac{T_0 V_{-1,d-1}}{2\pi T^2}\partial_\mu\partial_n\left(\frac{k_n(\mu)}{n}\right)\,. \tag{4.310}$$

Evaluating again for $n = 1$ ($T = T_0$) and $\mu = 0$, we have

$$\partial^2_\mu S_{\mathrm{EE}}\big|_{\mu=0} = RV_{-1,d-1}\left[\partial_\mu k_n - \partial_\mu\partial_n k_n\right]\Big|_{n=1,\mu=0}\,. \tag{4.311}$$

Using then Eq. (4.248), we can rewrite this as

$$\partial^2_\mu S_{\mathrm{EE}}\big|_{\mu=0} = V_{-1,d-1}\frac{16(d-2)R^2\pi^{d+1}}{d\Gamma(d+1)}\left[\hat{c}+d\hat{e}\right]\,, \tag{4.312}$$

which, via Eq. (4.305) reduces to Eq. (4.252). This therefore completes the proof that such relation is universally valid for arbitrary CFTs.

4.7 Discussion

We have carried out a general analysis of the holographic aspects of the EQG given by (4.30). This is a theory containing a $(d-2)$-form, but as we have seen it can be dualized into a theory with a vector field, and we use this formulation to make contact with holography. One of the most interesting aspects of this theory is that it contains non-minimal couplings, which affect the central charge of the two-point function $\langle JJ \rangle$, and more importantly, give rise to a non-vanishing parameter a_2 (see (4.125)) that controls the angular distribution of the energy one-point function (4.102). This in turn means that the boundary theory has a general $\langle TJJ \rangle$ correlator. Therefore, we can probe holographic CFTs beyond the universality class given by Einstein-Maxwell theory. In addition, the special properties of the EQG theories allow us to carry out a fully analytic and exact study of many of their holographic aspects, so we do not have to restrict ourselves to the perturbative regime. Regarding the any-order EQGs constructed in Subsection 4.6.1, it is clear that they provide interesting holographic models for future endeavors.

One of the main questions we tried to answer is that of how the physics of the CFT can change while satisfying physically reasonable conditions. Thus, we have constrained the couplings of our bulk theory by demanding that the boundary CFT respects unitarity. This means that the central charges C_T and C_J, as well as the energy fluxes $\langle \mathcal{E}(\vec{n}) \rangle_J$ and $\langle \mathcal{E}(\vec{n}) \rangle_T$ (see resp. (4.102) and (4.138)), have to remain positive. We also studied the constraints coming from demanding causality in the bulk in the background of a planar neutral black hole. In the case of gravitational fluctuations, it is known that these causality constraints imply the positivity of the energy flux $\langle \mathcal{E}(\vec{n}) \rangle_T$ [436,438,583,586]. Here we have shown that demanding that the electromagnetic waves do not propagate faster than light is equivalent to the constraints obtained from the positivity of $\langle \mathcal{E}(\vec{n}) \rangle_J$. These causality bounds follow from looking at the phase velocity of electromagnetic waves close to the boundary of AdS. We have not observed additional causality constraints when extending these conditions deeper into the bulk, but our analysis in this regard is limited due to the number of parameters involved, so it would be interesting to do a more thorough search of causality constraints in the bulk interior, as in Ref. [437]. Likewise, it would be desirable to extend these bounds to the case of charged black hole backgrounds.

One of the novelties in our analysis was the inclusion of constraints from the Weak Gravity Conjecture. As proposed by Ref. [528] and recently explored by Ref. [476] in the case of AdS, the so-called mild form of the WGC demands that the corrections to the entropy of thermally stable black holes be positive in the microcanonical ensemble. This implies in particular that the charge-to-mass ratio of extremal black holes is corrected positively [531], which is the most familiar form of the WGC for asymptotically flat black holes [474, 594]. However, demanding the entropy corrections to be positive is a more general condition than that and is amenable to the AdS case. When applied to spherical (and planar) black holes, we obtain constraints on the signs of (certain combinations of) the couplings, and these become very powerful when combined with unitarity/causality bounds. In fact, we obtain that the couplings α_1, α_2 and λ of (4.30) can only lie in a very small compact set of $\mathbb{R}^3$ in $d = 3, 4, 5$. The only parameter that can be unbounded is β, which is simply required to satisfy $\beta \geq 0$ by the WGC. However, we suspect that additional causality/unitarity conditions should provide an upper limit for β. In another vein, it is necessary on general grounds to get a better understanding of the WGC in AdS space; in particular, to understand what are the implications of this conjecture for the dual

CFT.

Now, when the positivity-of-entropy bounds are implemented instead for hyperbolic black holes we find something quite remarkable that was not noticed in Ref. [476]: some of the constraints become incompatible with those coming from spherical black holes. For instance, demanding that the corrections to the entropy for large spherical black holes and for hyperbolic black holes (both of which are stable) be non-negative can only be achieved if the GB coupling is vanishing, $\lambda = 0$. This looks like an unreasonably strong constraint, even more taking into account that a positive GB coupling (which is the sign imposed by $\delta S > 0$ in spherical black holes) is explicitly realized in string theory effective actions [110, 124], which should be compatible with the WGC. This calls into question the validity of the WGC bounds for hyperbolic black holes, and thus we decided to trust only the conditions imposed by the spherical case. As we see later, this leads to quite reasonable physics even when hyperbolic black holes are concerned, *e.g.*, for Rényi entropies — we comment on this below. However, it is also worth mentioning that Ref. [572] recently provided examples of string theory realizations in which $\lambda < 0$. If the sign of λ can be arbitrary within string theory, this would indeed contradict the positive-entropy bounds of [528] as well as the results of [598]. Hence, one would conclude that these bounds are too strong, and perhaps only a weaker version of them holds true — for instance, one could think of applying these bounds only to spherical near-extremal black holes. Clearly, all of this deserves further attention.

Next, we studied holographic charged Rényi entropies and their associated generalized twist operators, both of which are related to the thermodynamic properties of black holes with hyperbolic horizons. We observed that, providing that the dual CFT respects unitarity, the chemical potential always increases Rényi entropies with $n \geq 1$. Furthermore, standard Rényi entropies are known to satisfy a number of inequalities as a function of the index n — see (4.223) — so we wondered if these held in our higher-derivative theories as well. As it turns out, these seem to be always satisfied if one assumes all of the constraints we have studied. However, if one gives up the WGC bounds, it is found that the RE can violate some of these inequalities, and they could even become negative — see Fig. 4.3. It is quite remarkable that the WGC avoids this behavior, which points in the direction that the WGC bounds in AdS are necessary in order to give rise to a sensible theory in the boundary.

We then computed the scaling dimension $h_n(\mu)$ and magnetic response $k_n(\mu)$ of generalized twist operators, as introduced by Ref. [447]. By using the entries for the holographic dictionary of the EQG (4.30), we have obtained a series of relationships between the derivatives of $h_n(\mu)$ and $k_n(\mu)$ at $n = 1$, $\mu = 0$ and C_T, C_J and the coefficients of $\langle TJJ\rangle$ (see Eqs. (4.242), (4.245), (4.304) and (4.251)). These are actually universal relations that hold for any CFT, and they were first derived from first principles in Refs. [446, 447]. The fact that one can independently derive these results by using holographic higher-derivative theories is a proof of the power of this approach to learn about universality in CFTs. It is remarkable how everything comes together taking into account the number of computations involved in obtaining these formulas from two completely different approaches.

Finally, inspired by the results for the four-derivative EQG (4.30), we have explicitly proven that the universal formula (4.252) holds for general CFTs in $d \geq 3$. In $d = 2$, there are various reasons to expect a different situation. On the one hand, observe that the coefficient a_2 is not even defined in that case. Similarly, from Eq. (4.95) it is clear that C_J for our holographic calculations is divergent for $d = 2$ and therefore meaningless. The free-field results reported in [447] also suggest a different structure in that case, including possible linear terms in μ or jumps in $S_n(\mu)$ as n and μ vary. Additional two-dimensional counterexamples to the subleading quadratic behavior in μ have appeared in [618]. It would be interesting to investigate these features further —natural candidates would be three-dimensional holographic EQGs [619].

On a different front, it would also be interesting to rederive Eq. (4.252) using the techniques developed in [620]. In the case of a small perturbation by a relevant operator $\mathcal{O}$, the leading correction to the EE across a sphere was shown to be quadratic in the perturbation and proportional to a double integral of $\langle K\mathcal{O}\mathcal{O}\rangle - \langle \mathcal{O}\mathcal{O}\rangle$, where K is the modular Hamiltonian of ρ_A —which for spheres involves an integral of the stress tensor. In the present context, it would be natural to relate $\mathcal{O}$ to the charge operator, which would bring about integrals of $\langle TJJ\rangle$ and $\langle JJ\rangle$, precisely as expected from Eq. (4.252).

In [621], a somewhat similar universal relation for charged Rényi entropies —involving the uncharged result plus an extra term— was obtained in the case of discrete symmetry groups. It would be nice to study the connection between Eq. (4.252) and the approach developed in that paper and [622] in the case of continuous groups.

A particularly interesting application of our formula is to the case of supersymmetric (S) CFTs, which come with a global R-symmetry group. For instance, for $d=4$, $\mathcal{N}=1$ SCFTs one has a $\mathrm{U}(1)_R$ current with [435, 623, 624]

$$C_J^{\mathcal{N}=1,\,\mathrm{U}(1)_R} = \frac{4c}{\pi^4}\,, \quad a_2^{\mathcal{N}=1,\,\mathrm{U}(1)_R} = 3\left(1-\frac{a}{c}\right)\,, \tag{4.313}$$

and therefore, our formula (4.252) yields the prediction

$$S_{\mathrm{EE}}^{\mathcal{N}=1,\,\mathrm{U}(1)_R} = \nu_3\left[a + \frac{2}{3}\left(c-\frac{a}{3}\right)(\mu R)^2 + \dots\right]\,, \tag{4.314}$$

where we used $a^* = a$ and c is the other trace-anomaly coefficient. Similarly, for $\mathcal{N}=2$ SCFTs, the R-symmetry group is $\mathrm{U}(1)_R\times$ $\mathrm{SU}(2)_R$. Using the corresponding values of C_J and a_2 [425, 625], one finds[21]

$$\begin{aligned} S_{\mathrm{EE}}^{\mathcal{N}=2,\,\mathrm{U}(1)_R} &= \nu_3\left[a + 2\left(c-\frac{a}{3}\right)(\mu R)^2 + \dots\right]\,, \\ S_{\mathrm{EE}}^{\mathcal{N}=2,\,\mathrm{SU}(2)_R} &= \nu_3\left[a + \frac{1}{6}\left(2c-a\right)(\mu R)^2 + \dots\right]\,. \end{aligned} \tag{4.315}$$

It would be interesting to verify these predictions using alternative methods.

Finally, it is natural to wonder what additional relations connecting quantum information measures and universal CFT quantities may still remain to be discovered.

[21] In the $\mathrm{SU}(2)_R$ case one should understand that μ couples to a $\mathrm{U}(1)$ subgroup of it.

Second Part

Geometric aspects of Supergravity and String Theory

5

Spinor flows on three-dimensional Cauchy hypersurfaces

This chapter initiates the Second Part of the thesis, devoted to the study of geometric aspects of Supergravity and ST. It consists of Chapters 5, 6 and 7 and covers some mathematical topics of high relevance in the context of theoretical physics. More concretely, in this part we will study real parallel spinors on globally hyperbolic manifolds, self-dual Einstein four-manifolds admitting a principal isometric action of the three-dimensional Heisenberg group and the use of contact metric structures for the construction of Supergravity solutions. Among these problems, in the present chapter we will be interested in the first of them.

Globally hyperbolic four-dimensional Lorentzian manifolds play a fundamental role in Lorentzian geometry and mathematical physics, especially in mathematical General Relativity, where they provide a natural class of four-dimensional spacetimes for which the initial value problem of Einstein field equations is well posed [329, 335]. A natural geometric condition to impose on a globally hyperbolic *spin* four-manifold (M, g) is the existence of a (real) spinor parallel with respect to the Levi-Civita associated to g. In particular, in the context of $\mathcal{N} = 1$ pure Supergravity in four dimensions, a spacetime is said to be supersymmetric if it is endowed with a parallel spinor [106]. Despite the fact that the local structure of Lorentzian four-manifolds admitting a parallel spinor is well known since the early days of mathematical GR and Supergravity [462], see also [626], the more refined global differential geometric and topological aspects of such Lorentzian manifolds have been addressed in the literature only recently [477, 478, 627–629], see also [630–634] for related global problems in Lorentzian geometry.

The main goal of this chapter is to investigate the differential geometry and topology of connected, oriented and time-oriented globally hyperbolic Lorentzian four-manifolds (M, g) carrying a real parallel spinor field $\varepsilon \in \Gamma(\mathrm{S}_g)$, understood as a section of a bundle of irreducible real Clifford modules over the bundle of Clifford algebras of (M, g) (see Section I.8). In order to do this, we will exploit the theory of *parabolic pairs*, recently developed in [473], which provides an equivalent description of a parallel spinor as a pair of certain one-forms satisfying a specific system of first order partial differential equations. The theory of parabolic pairs is a particular case of a general framework developed in that reference to study irreducible real spinors satisfying a generalized Killing spinor equation.

Using this formalism, we will be able to reformulate the evolution problem for a parallel spinor as a system of flow (partial differential) equations for a family of functions $\{\beta_t\}_{t\in\mathcal{I}}$ and a family of coframes $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ on an appropriately chosen Cauchy hypersurface $\Sigma \subset M$. We will say that a family $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ is a *parallel spinor flow* if it satisfies the aforementioned system of differential equations. Using the notion of parallel spinor flow, the corresponding constraint equations of the initial value problem of a parallel spinor can be shown to be equivalent to a differential system, the *parallel Cauchy differential system*, for a pair $(\mathfrak{e}, \Theta)$, where $\mathfrak{e}$ is a coframe and Θ is a symmetric two-tensor on Σ. One of the salient features of this formulation is that the Riemannian h metric induced by (M, g) on Σ is given by the canonical metric for which $\mathfrak{e} = (e_u, e_l, e_n)$ becomes an orthonormal coframe, that is, $h_{\mathfrak{e}} = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n$.

However, interestingly enough, (h, Θ) are precisely the variables of the configuration space of the vacuum *Hamiltonian* and *momentum* constraint equations [335]. This allows to define the notion of initial data *admissible* to both the parallel spinor flow and the vacuum Einstein flow, which immediately leads to the natural question of the compatibility of both flows when starting on common admissible data. We will solve this question in the affirmative and prove that a parallel spinor flow whose initial data are admissible to both problems preserves the Hamiltonian and momentum constraints and yields a Ricci flat Lorentzian four-manifold (M, g). We find this result non-trivial due to the fact that a Lorentzian manifold admitting a parallel spinor need not be Ricci flat [626]. As a corollary, we obtain the *initial data* characterization of a parallel spinor on a Ricci flat Lorentzian four-manifold which, to the best of our knowledge, is the first of its type in the literature.

Another important feature of parallel spinorial flows is that they admit a canonical notion of *left-invariance* in terms of which we can define the notion of left-invariance of parallel spinors. Similarly, we also introduce the notion of left-invariant parallel Cauchy pairs, being able to provide a classification of all left-invariant Cauchy pairs on connected and simply connected Lie groups. We use this result to obtain the classification of all associated left-invariant parallel spinor flows. As expected, the result strongly depends on the initial data $(\mathfrak{e}, \Theta)$.

The outline of the chapter is as follows. First we present the theory of parallel spinors in terms of parabolic pairs and we use it to characterize all standard Brinkmann space-times admitting parallel spinors. Then we provide the description of parallel spinors on globally hyperbolic Lorentzian four-manifolds as parallel spinor flows on an appropriately chosen Cauchy surface and prove the compatibility of the parallel spinor flow with the vacuum Einstein flow. Afterwards we characterize parallel Cauchy pairs on simply connected Cauchy surfaces and we characterize all parallel Cauchy pairs and associated foliations in the compact case. Next we classify all left-invariant parallel Cauchy pairs on simply connected Lie groups, which allows us to classify all left-invariant parallel spinor flows on simply connected three-dimensional Lie groups, elaborating on some of their properties. Later, we study a particular class of parallel spinor flows which we characterize geometrically and solve explicitly in particular cases. Finally, we conclude with a brief discussion of our results.

5.1 Parallel real spinors on Lorentzian four-manifolds

In this section we develop the theory of parallel spinors on four-dimensional Lorentzian manifolds, assuming as the starting point of our investigation one of the main results of [473], which characterizes parallel spinors in terms of a certain type of distribution satisfying a prescribed system of partial differential equations.

5.1.1 General theory

Let (M, g) be a four-dimensional *spacetime*, that is, a connected, oriented and time oriented Lorentzian four-manifold equipped with a Lorentzian metric g. We assume that (M, g) is equipped with a bundle of irreducible real spinors S_g. This is by definition a bundle of irreducible real Clifford modules over the bundle of Clifford algebras of (M, g). Existence of such S_g is in general obstructed. The obstruction was shown in [459, 635] to be equivalent to the existence of a spin structure Q_g, in which case S_g can be considered to be a vector bundle associated to Q_g through the tautological representation induced by the natural embedding $\mathrm{Spin}_+(3,1) \subset \mathrm{Cl}(3,1)$, where $\mathrm{Spin}_+(3,1)$ denotes the connected component of the identity of the spin group in signature $(3,1) = -+++$ and $\mathrm{Cl}(3,1)$ denotes the real Clifford algebra in signature $(3,1)$.

Remark 5.1. The *tautological representation* of $\mathrm{Spin}_+(3,1) \subset \mathrm{Cl}(3,1)$ is the representation obtained by restriction of the unique irreducible real Clifford representation $\gamma\colon \mathrm{Cl}(3,1) \to \mathrm{End}(\mathbb{R}^4)$ of $\mathrm{Cl}(3,1)$. This representation is real of real type (the commutant of the image of γ in $\mathrm{End}(\mathbb{R}^4)$ is trivial) and γ is in fact an isomorphism of unital and associative algebras. In particular $\mathbb{R}^4$ admits a skew-symmetric non-degenerate bilinear pairing which is invariant under $\mathrm{Spin}_+(3,1)$ transformations [636, 637] (note that this bilinear *cannot* be chosen to be symmetric).

We will assume, without loss of generality, that (M, g) is spin and equipped with a fixed spin structure Q_g. Then, the Levi-Civita connection ∇^g on (M, g) induces canonically a connection on S_g, the *spinorial* Levi-Civita connection, which we denote for simplicity by the same symbol.

Definition 5.1. A *spinor field* ε on (M, g, S_g) is a smooth section $\varepsilon \in \Gamma(\mathrm{S}_g)$ of S_g. A spinor field ε is said to be *parallel* if $\nabla^g \varepsilon = 0$.

For every lightlike one-form $u \in \Omega^1(M)$ we define an equivalence relation $\sim_u$ on the vector space of one-forms as follows. Given $l_1, l_2 \in \Omega^1(M)$ the equivalence relation $\sim_u$ declares $l_1 \sim_u l_2$ to be equivalent if and only if $l_1 = l_2 + f u$ for a function $f \in C^\infty(M)$. We denote by:

$$\Omega^1_u(M) \stackrel{\text{def.}}{=} \frac{\Omega^1(M)}{\sim_u}\,, \tag{5.1}$$

the $C^\infty(M)$-module of equivalence classes given by $\sim_u$.

Definition 5.2. A parabolic pair $(u, [l])$ on (M, g) consists of a nowhere vanishing null one-form $u \in \Omega^1(M)$ and an equivalence class of one-forms:

$$[l] \in \Omega^1_u(M)\,, \tag{5.2}$$

such that the following equations hold:

$$g(l, u) = 0\,, \qquad g(l, l) = 1\,, \tag{5.3}$$

for some, and hence for all, representatives $l \in [l]$.

The starting point of our analysis is the following result, which follows from [473, Theorems 4.26 and 4.32] and gives the characterization of parallel spinors on (M, g) that will be most convenient for our purposes.

Proposition 5.1. *A spacetime four-manifold (M,g) admits a real parallel spinor field $\varepsilon \in \Gamma(\mathrm{S}_g)$ for some bundle of irreducible spinors S_g over (M,g) if and only if there exists a parabolic pair $(u,[l])$ on (M,g) satisfying:*

$$\nabla^g u = 0\,, \qquad \nabla^g l = \kappa \otimes u\,, \tag{5.4}$$

for some representative (and hence for all) $l \in [l]$ and a one-form $\kappa \in \Omega^1(M)$. Unless additional emphasis is needed, we will just write parallel spinors for real parallel spinors.

Remark 5.2. More precisely, Reference [473] proves that a nowhere vanishing spinor $\varepsilon \in \Gamma(\mathrm{S}_g)$ on (M,g) defines a unique distribution of co-oriented parabolic two-planes in M, which in turn determines uniquely both u and the equivalence class of one-forms $[l]$. Conversely, any such distribution determines a nowhere vanishing spinor on (M,g), unique up to a global sign, with respect to a spin structure on (M,g). Moreover, [473, Theorem 4.26] establishes a correspondence between a certain type of first-order partial differential equations for ε and their equivalent as systems of partial differential equations for $(u,[l])$, of which equations (5.4) constitute the simplest case. The reader is referred to [473] for further details.

Remark 5.3. Given a parabolic pair $(u,[l])$, constructing its associated spinor field $\varepsilon \in \Gamma(\mathrm{S}_g)$ can be difficult, since it requires computing the preimage of the polyform $u + u \wedge l$ through the *square spinor map* [455, §IV]. This is however not problematic for our purposes, since we are not interested in the parallel spinor $\varepsilon \in \Gamma(\mathrm{S}_g)$ *per se* but only in the geometric and topological consequences of its existence. In this context, the main point of equations (5.4) and the general formalism presented in [473] is to provide a framework to study spinorial differential equations without having to consider the spinorial geometry of the underlying pseudo-Riemannian manifold (M,g). This point of view is motivated by the study of supersymmetric solutions of Supergravity, where ε corresponds to the *supersymmetry parameter*, an auxiliary object that *a priori* bears no physical meaning and is only used to define mathematically the notion of *supersymmetric solution*.

Remark 5.4. Recall that if a pair (u,l), with $l \in [l]$, satisfies equations (5.4) with respect to a given $\kappa \in \Omega^1(M)$ then any other representative $l' = l + fu$ satisfies again equation (5.4) with respect to the same null one-form u and a possibly different one-form κ' given by:

$$\kappa' = \kappa + \mathrm{d}f\,. \tag{5.5}$$

We will say that a parabolic pair $(u,[l])$ is *parallel* if it corresponds to a parallel spinor field, that is, if it satisfies (5.4) for a representative $l \in [l]$. The dual $u^\sharp \in \mathfrak{X}(M)$ of u is a parallel vector field on M which is usually referred to as the *Dirac current* of ε in the literature. The fact that the Dirac current of ε is always null is specific (although not exclusive) of the type of irreducible real representation $\gamma\colon \mathrm{Cl}(3,1) \to \mathrm{End}(\mathbb{R}^4)$ that we have used to construct the spinor bundle S_g. Indeed, it can be seen (see for instance [473, Proposition 3.22]) that the pseudo-norm of the Dirac current $u^\sharp$ is given by the pseudo-norm of ε computed with respect to the *admissible bilinear pairing* $\mathcal{B}$ used to construct $u^\sharp$. Admissible bilinear pairings were classified in [636, 637], from which it follows that in our case there exist two admissible pairings, both of them skew-symmetric. Therefore, $\mathcal{B}(\varepsilon,\varepsilon) = 0$ automatically and $u^\sharp$ is always null.

Proposition 5.1 immediately implies that four-dimensional spacetimes admitting a parallel spinor field whose Dirac current is complete are particular instances of *Brinkmann* *manifolds*, which are precisely defined as spacetimes equipped with a complete and parallel null vector field [638]. Other well-known properties of spacetimes admitting a parallel spinor field, such as the special form of their Ricci tensor, are also immediate consequences of Proposition 5.1, which provides an adequate global and coordinate-independent framework to study the geometry and topology of four-dimensional spacetimes admitting parallel spinors. In particular, such framework seems to be specially well-adapted to prove *splitting theorems* in the spirit of [639], where the global geometry of Brinkmann spacetimes was investigated.

5.1.2 Standard Brinkmann spacetimes

In order to illustrate the various uses of Proposition 5.1 and make contact with the existing literature, in this subsection we recover the well-known local characterization of a Lorentzian four-manifold (M, g) admitting a parallel spinor, obtaining along the way the global characterization of standard Brinkmann spacetimes that admit a parallel spinor, which seems to be new in the literature. Recall that by definition a Brinkmann spacetime [638,640] is a Lorentzian four-manifold equipped with a complete parallel null vector. Let $(u, [l])$ be a parallel parabolic pair on (M, g), which by Proposition 5.1 is equivalent to the existence of a parallel spinor. Since u is parallel, (M, g) is locally isometric to a Brinkmann spacetime, whence it suffices to consider (M, g) to be *standard*, namely $M = \mathbb{R}^2 \times X$ in terms of an oriented two-dimensional manifold X, equipped with the metric:

$$g = H_{x_u} \mathrm{d}x_u \otimes \mathrm{d}x_u + \mathrm{d}x_u \odot \alpha_{x_u} + \mathrm{d}x_u \odot \mathrm{d}x_v + q_{x_u} \,. \tag{5.6}$$

where (x_u, x_v) denotes the Cartesian coordinates of $\mathbb{R}^2$, and

$$\{H_{x_u}\}_{x_u \in \mathbb{R}} \,, \quad \{\alpha_{x_u}\}_{x_u \in \mathbb{R}} \,, \quad \{q_{x_u}\}_{x_u \in \mathbb{R}} \,, \tag{5.7}$$

respectively denote a family of functions, a family of one-forms and a family of complete Riemannian metrics on X parametrized by $x_u \in \mathbb{R}$. The vector field ∂_{x_v} is null and parallel, so $g(\partial_{x_v}) = \mathrm{d}x_u$ is a null parallel one-form which we identify with u. We will refer to a parallel spinor on a standard Brinkmann spacetime as adapted if its Dirac current is proportional to ∂_{x_v}. In such case, the first equation in (5.4) is automatically satisfied and we only need to be concerned with the second equation in (5.4), namely:

$$\nabla^g l = \kappa \otimes \mathrm{d}x_u \,, \qquad l \in [l] \,, \qquad \kappa \in \Omega^1(M) \,, \tag{5.8}$$

which needs to be satisfied for a representative in $[l]$. This equation is equivalent to

$$\mathrm{d}l = \kappa \wedge \mathrm{d}x_u \,, \qquad \mathcal{L}_{l^\sharp} g = \kappa \odot \mathrm{d}x_u \,, \tag{5.9}$$

where $l^\sharp$ denotes the metric dual of l with respect to g. Using that $u = \mathrm{d}x_u$ and $g^{-1}(l, u) = l(\partial_{x_v}) = 0$, it follows that there exists a representative $l \in [l]$ of the form:

$$l = l^\perp \,, \tag{5.10}$$

where $l^\perp$ denotes a bi-parametric family of unit-norm one-forms on X parametrized by $(x_u, x_v) \in \mathbb{R}^2$. The first equation in (5.9) is equivalent to

$$\kappa_v \, \mathrm{d}x_u \wedge \mathrm{d}x_v - \partial_{x_v} l^\perp \wedge \mathrm{d}x_v - (\kappa^\perp + \partial_{x_u} l^\perp) \wedge \mathrm{d}x_u + \mathrm{d}_X l^\perp = 0 \,, \tag{5.11}$$

where we have used the splitting $\kappa = \kappa_u \mathrm{d}x_u + \kappa_v \mathrm{d}x_v + \kappa^\perp$ and d_X denotes the exterior derivative operator on X. Hence:

$$\kappa_v = 0\,, \qquad \partial_{x_v} l^\perp = 0\,, \qquad \kappa^\perp = -\partial_{x_u} l^\perp\,, \qquad \mathrm{d}_X l^\perp = 0\,. \tag{5.12}$$

On the other hand, recall that the dual $l^\sharp$ with respect to g is given by the following expression:

$$l^\sharp = -\alpha_{x_u}(l^\perp)\,\partial_{x_v} + q^{-1}_{x_u}(l^\perp) \tag{5.13}$$

which we use to compute the Lie derivative of g along $l^\sharp$:

$$\begin{aligned}\mathcal{L}_{l^\sharp} g = \mathrm{d}H_{x_u}(l^\perp)\,\mathrm{d}x_u \otimes \mathrm{d}x_u + \mathrm{d}x_u \odot (\alpha_{x_u}(\partial_{x_u} l^\perp)\mathrm{d}x_u + \mathcal{L}^X_{l^\perp}\alpha_{x_u}) \\ - \mathrm{d}x_u \odot \mathrm{d}(\alpha_{x_u}(l^\perp)) + \mathcal{L}^X_{l^\perp} q_{x_u}\,,\end{aligned} \tag{5.14}$$

where $\mathcal{L}^X$ denotes the Lie derivative on the surface X and where we have used:

$$\mathcal{L}_{l^\sharp}\mathrm{d}x_u = 0\,, \quad \mathcal{L}_{l^\sharp}\mathrm{d}x_v = -\mathrm{d}(\alpha_{x_u}(l^\perp)) = -\partial_{x_u}(\alpha_{x_u}(l^\perp))\mathrm{d}x_u - \mathrm{d}_X(\alpha_{x_u}(l^\perp))\,, \tag{5.15}$$

$$\mathcal{L}_{l^\sharp}\alpha_{x_u} = \alpha_{x_u}(\partial_{x_u} l^\perp)\mathrm{d}x_u + \mathcal{L}^X_{l^\perp}\alpha_{x_u}\,, \tag{5.16}$$

$$\mathcal{L}_{l^\sharp} q_{x_u} = \mathcal{L}^X_{l^\perp} q_{x_u} + \mathrm{d}x_u \odot \partial_{x_u} l^\perp - \mathrm{d}x_u \odot (\partial_{x_u} q_{x_u})((l^\perp)^{\sharp_q})\,. \tag{5.17}$$

Hence, the second equation in (5.9) is equivalent to

$$\kappa_u = \frac{1}{2}\mathrm{d}H_{x_u}(l^\perp) - (\partial_{x_u}\alpha_{x_u})(l^\perp)\,, \tag{5.18}$$

$$\nabla^{q_{x_u}} l^\perp = 0\,, \quad 2\partial_{x_u} l^\perp - 2(\partial_{x_u} q_{x_u})((l^\perp)^{\sharp_q}) + l^\perp \lrcorner\, \mathrm{d}_X \alpha_{x_u} = 0\,, \tag{5.19}$$

where we have used that $\mathrm{d}_X l^\perp = 0$ and where $\nabla^{q_{x_u}}$ denotes the Levi-Civita connection of the Riemannian metric q_{x_u} on X. Altogether we obtain the following result.

Proposition 5.2. *A standard Brinkmann spacetime admits an adapted parallel spinor if and only if it is isometric to the following model:*

$$(M, g) = (\mathbb{R}^2 \times X, H_{x_u}\mathrm{d}x_u \otimes \mathrm{d}x_u + \mathrm{d}x_u \odot \alpha_{x_u} + \mathrm{d}x_u \odot \mathrm{d}x_v + q_{x_u})\,, \tag{5.20}$$

where $\{q_{x_u}\}_{x_u \in \mathbb{R}}$ is a family of complete flat metrics on X, and there exists a family of unit-norm one-forms $\{l^\perp_{x_u}\}_{x_u \in \mathbb{R}}$ such that

$$\partial_{x_u} l^\perp - (\partial_{x_u} q_{x_u})((l^\perp)^{\sharp_q}) + \frac{1}{2} l^\perp \lrcorner\, \mathrm{d}_X \alpha_{x_u} = 0\,, \qquad \nabla^{q_{x_u}} l^\perp = 0\,. \tag{5.21}$$

In particular, $\partial_{x_u} l^\perp = \mu_{x_u} \star_{q_{x_u}} l^\perp$ for a family of constants $\{\mu_{x_u}\}_{x_u \in \mathbb{R}}$, $\star_{q_{x_u}}$ being the Hodge dual with respect to q_{x_u}.

By uniformization, we conclude that X is diffeomorphic to either $\mathbb{R}^2$, $\mathbb{R}^2 \backslash \{0\}$ or T^2. Appropriately choosing local coordinates the previous result directly implies that a four-dimensional spacetime admitting parallel spinors is locally isometric to a *pp-wave* [640–642], defined as a Brinkmann space for which the Riemann curvature tensor R^g satisfies that[1] $|\mathrm{R}^g|^2_g = 0$.

[1] Using some local coordinates, $|\mathrm{R}^g|^2_g = \frac{1}{4}\mathrm{R}^g_{\mu\nu\rho\sigma}(\mathrm{R}^g)^{\mu\nu\rho\sigma}$.

5.2 Globally hyperbolic case

Rather than investigating the global geometry and topology of general spacetimes admitting parallel spinors, exploiting for instance the refined screen bundle construction that can be developed in the presence of a parabolic pair, we restrict the *causality* of (M, g) and we assume in the following that (M, g) is a globally hyperbolic four-dimensional spacetime, as proposed in [478, 627]. A celebrated theorem of Bernal and Sánchez [345, 346] states that in this case (M, g) has the following isometry type:

$$(M, g) = (\mathbb{R} \times \Sigma, -\beta_t^2 \mathrm{d}t \otimes \mathrm{d}t + h_t)\,, \tag{5.22}$$

where t is the canonical coordinate on $\mathbb{R}$, $\{\beta_t\}_{t\in\mathbb{R}}$ is a smooth family of nowhere vanishing functions on Σ and $\{h_t\}_{t\in\mathbb{R}}$ is a family of complete Riemannian metrics[2] on Σ. From now on we consider the identification (5.22) to be fixed. We set

$$\Sigma_t \stackrel{\text{def.}}{=} \{t\} \times \Sigma \hookrightarrow M\,, \qquad \Sigma \stackrel{\text{def.}}{=} \{0\} \times \Sigma \hookrightarrow M\,, \tag{5.23}$$

and define

$$\mathfrak{t}_t = \beta_t\, \mathrm{d}t\,, \tag{5.24}$$

to be the outward-pointing unit time-like one-form orthogonal to Σ_t for every $t \in \mathbb{R}$. We will consider $\Sigma \hookrightarrow M$, endowed with the induced Riemannian metric

$$h \stackrel{\text{def.}}{=} h_0|_{T\Sigma\times T\Sigma}\,, \tag{5.25}$$

to be the Cauchy hypersurface of (M, g). The *shape operator* or scalar second fundamental form Θ_t of the embedded manifold $\Sigma_t \hookrightarrow M$ is defined in the usual way as follows:

$$\Theta_t \stackrel{\text{def.}}{=} \nabla^g \mathfrak{t}_t|_{T\Sigma_t\times T\Sigma_t}\,, \tag{5.26}$$

This definition can be seen to be equivalent to

$$\Theta_t = -\frac{1}{2\beta_t}\partial_t h_t \in \Gamma(T^*\Sigma_t \odot T^*\Sigma_t)\,. \tag{5.27}$$

Moreover, it can be seen that

$$\nabla^g \alpha|_{T\Sigma_t\times TM} = \nabla^{h_t}\alpha + \Theta_t(\alpha) \otimes \mathfrak{t}_t\,, \qquad \forall\, \alpha \in \Omega^1(\Sigma_t)\,, \tag{5.28}$$

where ∇^{h_t} denotes the Levi-Civita connection on (Σ_t, h_t) and $\Theta_t(\alpha) := \Theta_t(\alpha^{\sharp_{h_t}})$ is by definition the evaluation of Θ_t on the metric dual of α. Given a parabolic pair $(u, [l])$, we write

$$u = u_t^0\, \mathfrak{t}_t + u_t^\perp\,, \qquad l = l_t^0\, \mathfrak{t}_t + l_t^\perp \in [l]\,, \tag{5.29}$$

where the superscript $\perp$ denotes orthogonal projection to $T^*\Sigma_t$ and where we have defined

$$u_t^0 = -g(u, \mathfrak{t}_t)\,, \qquad l_t^0 = -g(l, \mathfrak{t}_t)\,, \tag{5.30}$$

Using the previous orthogonal splitting of u and l we can obtain an equivalent characterization of parallel spinors on a globally hyperbolic spacetime in terms of *flow equations* on Σ. First we prove the following two lemmas.

[2] Note that these are not the most general globally hyperbolic manifolds one may consider, since examples with non-complete Cauchy slices are known [643].

Lemma 5.1. *[627, Lemma 3.1] Let $u \in \Omega^1(M)$ be a null one-form on the globally hyperbolic manifold (5.22). Then, $\nabla^g u = 0$ if and only if*

$$(\nabla^g_{v_1} u)(v_2) = 0\,, \tag{5.31}$$

for every $v_1 \in \mathfrak{X}(M)$ and every $v_2 \in \mathfrak{X}(\Sigma_t)$.

Proof. We compute:

$$0 = g(\nabla^g u, u) = u^0_t\, g(\nabla^g u, \mathfrak{t}_t) + g(\nabla^g u, u^\perp_t) = u^0_t\, g(\nabla^g u, \mathfrak{t}_t)\,, \tag{5.32}$$

where we have used that the spatial projection of $\nabla^g u$ is zero by assumption. □

Lemma 5.2. *A globally hyperbolic four-manifold $(M, g) = (\mathbb{R}\times\Sigma, -\beta_t^2 \mathrm{d}t\otimes\mathrm{d}t + h_t)$ admits a parabolic pair, and hence a parallel spinor field, if and only if there exists a family of orthogonal one-forms $\{u^\perp_t, l^\perp_t\}_{t\in\mathbb{R}}$ on Σ satisfying the following equations:*

$$\partial_t u^\perp_t + \beta_t\Theta_t(u^\perp_t) = u^0_t \mathrm{d}\beta_t\,, \qquad u^0_t\partial_t l^\perp_t + \beta_t u^0_t\Theta_t(l^\perp_t) + \mathrm{d}\beta_t(l^\perp_t)u^\perp_t = 0\,, \tag{5.33}$$

$$\nabla^{h_t} u^\perp_t + u^0_t\Theta_t = 0\,, \qquad u^0_t\nabla^{h_t} l^\perp_t = \Theta_t(l^\perp_t)\otimes u^\perp_t\,, \tag{5.34}$$

as well as

$$(u^0_t)^2 = |u^\perp_t|^2_{h_t}\,, \qquad |l^\perp_t|^2_{h_t} = 1\,, \tag{5.35}$$

In particular, $\partial_t u^0_t = \mathrm{d}\beta_t(u^\perp_t)$ and $\mathrm{d}u^0_t + \Theta(u^\perp_t) = 0$. If equations (5.33) and (5.34) are satisfied, the corresponding parabolic pair $(u, [l])$ is given by:

$$u = u^0_t\mathfrak{t}_t + u^\perp_t\,, \qquad [l] = [l^\perp_t]\,. \tag{5.36}$$

where $|u^\perp_t|^2_{h_t} = h_t(u^\perp_t, u^\perp_t)$ and $|l^\perp_t|^2_{h_t} = h_t(l^\perp_t, l^\perp_t)$.

Proof. Let $(u, [l])$ be a parabolic pair satisfying equations (5.4). Write $u = u^0_t\,\mathfrak{t}_t + u^\perp_t$. We can find a representative $l \in [l]$ such that

$$l = l^\perp_t \in \Omega^1(\Sigma_t)\,, \qquad t \in \mathbb{R}\,, \tag{5.37}$$

that is, with l purely spatial. Using this representative together with Lemma 5.1, it follows that equations (5.4) are equivalent to

$$\nabla^g_{\partial_t} u|_{T\Sigma_t} = 0\,, \quad \nabla^g_{v_t} u|_{T\Sigma_t} = 0\,, \tag{5.38}$$

$$\nabla^g_{\partial_t} l^\perp_t = \kappa(\partial_t)\, u\,, \quad \nabla^g_{v_t} l^\perp_t = \kappa(v_t)\, u\,, \quad \forall\ v_t \in T\Sigma_t\,. \tag{5.39}$$

Denote by $\kappa^\perp_t$ the spatial projection of $\kappa \in \Omega^1(M)$. We compute:

$$\nabla^g_{\partial_t} u|_{T\Sigma_t} = \partial_t u^\perp_t + \beta_t\Theta_t(u^\perp_t) - u^0_t\mathrm{d}\beta_t\,, \quad \nabla^g u|_{T\Sigma_t\times T\Sigma_t} = \nabla^{h_t}u^\perp_t + u^0_t\Theta_t\,, \tag{5.40}$$

$$\nabla^g_{\partial_t} l^\perp_t = \partial_t l^\perp_t - \mathrm{d}\beta_t(l^\perp_t)\mathfrak{t}_t + \beta_t\Theta_t(l^\perp_t) = \kappa(\partial_t)(u^0_t\mathfrak{t}_t + u^\perp_t)\,, \tag{5.41}$$

$$\nabla^g l^\perp_t|_{T\Sigma_t\times TM} = \nabla^{h_t} l^\perp_t + \Theta_t(l^\perp_t)\otimes\mathfrak{t}_t = \kappa^\perp_t\otimes(u^0_t\mathfrak{t}_t + u^\perp_t)\,. \tag{5.42}$$

Isolating κ in the previous equations we obtain:

$$\kappa(\partial_t) = -\frac{1}{u^0_t}\mathrm{d}\beta_t(l^\perp_t)\,, \qquad \kappa^\perp_t = \frac{1}{u^0_t}\Theta_t(l^\perp_t)\,, \tag{5.43}$$

Plugging these equations back into the expressions for the covariant derivatives of $l^\perp_t$ we obtain all equations in (5.33) and (5.34). The fact that these equations imply $\partial_t u^0_t = \mathrm{d}\beta_t(u^\perp_t)$ and $\mathrm{d}u^0_t + \Theta(u^\perp_t) = 0$ follows now by respectively manipulating the time and exterior derivatives of $(u^0_t)^2 = |u^\perp_t|^2_{h_t}$. The converse follows directly by construction and hence we conclude. $\square$

Remark 5.5. In the previous discussion we have used informally the notion of *family of tensors parametrized by* $\mathbb{R}$. This notion can be given a rigorous meaning as follows. A family of, say, one-forms $\{\alpha_t\}_{t\in\mathbb{R}}$ on Σ is by definition a smooth section $\alpha\colon \mathbb{R}\times\Sigma \to \mathrm{p}^*(T^*\Sigma)$ of the pull-back of $T^*\Sigma$ by the canonical projection $\mathrm{p}\colon \mathbb{R}\times\Sigma\to\Sigma$. Families of other types of tensors are defined similarly.

The previous lemma gives the necessary and sufficient conditions for a globally hyperbolic Lorentzian four-manifold (M,g) to admit a real parallel spinor field. We may consider as variables of these equations tuples of the form:

$$\left(\{\beta_t\}_{t\in\mathcal{I}}, \{h_t\}_{t\in\mathcal{I}}, \{u^0_t\}_{t\in\mathcal{I}}, \left\{u^\perp_t\right\}_{t\in\mathcal{I}}, \left\{l^\perp_t\right\}_{t\in\mathcal{I}}\right) . \tag{5.44}$$

These tuples contain the information about both the spinor and the underlying globally hyperbolic Lorentzian metric. However, we can actually reformulate the problem of a real parallel spinor on a globally hyperbolic manifold in terms of a family of functions and coframes on Σ satisfying some prescribed system of partial differential equations, as the following theorem shows.

Theorem 5.1. *An oriented globally hyperbolic Lorentzian four-manifold (M,g) admits a parallel spinor field if and only if there exists an orientation preserving diffeomorphism identifying $M=\mathcal{I}\times\Sigma$, where Σ is an oriented three-manifold equipped with a family of strictly positive functions $\{\beta_t\}_{t\in\mathcal{I}}$ on Σ and a family $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ of sections of $\mathrm{F}(\Sigma)$ satisfying the following system of differential equations:*

$$\partial_t e^t_a + \mathrm{d}\beta_t(e^t_a)e^t_u + \beta_t\Theta_t(e^t_a) = \delta_{au}\mathrm{d}\beta_t\,, \qquad \mathrm{d}\mathfrak{e}^t = \Theta_t(\mathfrak{e}^t)\wedge e^t_u\,, \tag{5.45}$$

$$\partial_t(\Theta_t(e^t_u)) + \mathrm{d}(\mathrm{d}\beta_t(e^t_u)) = 0\,, \qquad [\Theta_t(e^t_u)] = 0\in H^1(\Sigma,\mathbb{R}) \tag{5.46}$$

where $\mathfrak{e}^t = (e^t_u, e^t_l, e^t_n)\colon \Sigma\to\mathrm{F}(\Sigma)$ and

$$h_{\mathfrak{e}^t} = e^t_u\otimes e^t_u + e^t_l\otimes e^t_l + e^t_n\otimes e^t_n\,, \qquad \Theta_t = -\frac{1}{2\beta_t}\partial_t h_{\mathfrak{e}^t}\,. \tag{5.47}$$

In this case, the globally hyperbolic metric g is given by:

$$g = -\beta_t^2\mathrm{d}t\otimes\mathrm{d}t + h_{\mathfrak{e}^t}\,, \tag{5.48}$$

where t is the Cartesian coordinate in the splitting $M=\mathbb{R}\times\Sigma$.

Proof. By Lemma 5.2, a globally hyperbolic Lorentzian four-manifold (M,g) admits a real parallel spinor if and only if there exists a Cauchy surface $\Sigma\hookrightarrow M$ equipped with a tuple (5.44) satisfying equations (5.33), (5.34) and (5.35). Let

$$\left(\{\beta_t\}_{t\in\mathcal{I}}, \{h_t\}_{t\in\mathcal{I}}, \{u^0_t\}_{t\in\mathcal{I}}, \left\{u^\perp_t\right\}_{t\in\mathcal{I}}, \left\{l^\perp_t\right\}_{t\in\mathcal{I}}\right) , \tag{5.49}$$

be such a solution and define

$$e^t_u = \frac{u^\perp_t}{u^0_t}\,, \qquad e^t_l = l^\perp_t\,. \tag{5.50}$$

Then (e^t_u, e^t_l) is a family of nowhere vanishing and orthonormal one-forms on Σ, which can be canonically completed to a family of orthonormal coframes $\left\{\mathfrak{e}^t = (e^t_u, e^t_l, e^t_n)\right\}$ by defining the family of one-forms $\left\{e^t_n\right\}_{t\in\mathcal{I}}$ as follows:

$$e^t_n := \star_{h_t}(e^t_u \wedge e^t_l)\,, \tag{5.51}$$

where $\star_{h_t}$ denotes the Hodge dual associated to the family of metrics $\{h_t\}_{t\in\mathcal{I}}$. Plugging equations (5.50) into the first and second equations in (5.33) and manipulating the time derivative of $\left\{u^\perp_t\right\}_{t\in\mathcal{I}}$ we obtain the first equation in (5.33) for $a = u$ and $a = l$. For $\left\{e^t_n\right\}_{t\in\mathcal{I}}$ we compute as follows:

$$\begin{aligned}
0 &= \partial_t(h^{-1}_t(e^t_n, e^t_u)) = (\partial_t h^{-1}_t)(e^t_n, e^t_u) + h^{-1}_t(\partial_t e^t_n, e^t_u) + h^{-1}_t(e^t_n, \partial_t e^t_u)\\
&= 2\beta_t\Theta_t(e^t_n, e^t_u) + h^{-1}_t(\partial_t e^t_n, e^t_u) + h^{-1}_t(e^t_n, \mathrm{d}\beta_t - \beta_t\Theta_t(e^t_u))\\
&= \beta_t\Theta_t(e^t_n, e^t_u) + h^{-1}_t(\partial_t e^t_n, e^t_u) + \mathrm{d}\beta_t(e^t_n)\,, \qquad (5.52)\\
0 &= \partial_t(h^{-1}_t(e^t_n, e^t_l)) = (\partial_t h^{-1}_t)(e^t_n, e^t_l) + h^{-1}_t(\partial_t e^t_n, e^t_l) + h^{-1}_t(e^t_n, \partial_t e^t_l) = 2\beta_t\Theta_t(e^t_n, e^t_u)\\
&\quad + h^{-1}_t(\partial_t e^t_n, e^t_l) + h^{-1}_t(e^t_n, \beta_t\Theta_t(e^t_l)) = \beta_t\Theta_t(e^t_n, e^t_u) + h^{-1}_t(\partial_t e^t_n, e^t_l)\,, \qquad (5.53)
\end{aligned}$$

which immediately implies the first equation in (5.45) for the remaining case $a = n$. On the other hand, by Lemma 5.2, we have:

$$\frac{\mathrm{d}u^0_t}{u^0_t} + \Theta_t(e^t_u) = 0\,, \tag{5.54}$$

whence $[\Theta_t(e^t_u)] = 0 \in H^1(\Sigma, \mathbb{R})$, which yields the second equation in (5.46). Taking the time derivative of the previous equations we obtain:

$$\mathrm{d}\partial_t \log|u^0_t| + \partial_t(\Theta_t(e^t_u)) = 0\,. \tag{5.55}$$

Since by Lemma 5.2 we have $\partial_t u^0_t = \mathrm{d}\beta_t(u^\perp_t)$, the previous equation implies the first equation in (5.46). We compute:

$$\nabla^{h_t} e^t_n = \nabla^h \star_{h_t} (e^t_u \wedge e^t_l) = \star_{h_t}(\nabla^{h_t} e^t_u \wedge e^t_l) + \star_{h_t}(e^t_u \wedge \nabla^{h_t} e^t_l) = \Theta_t(e^t_n) \otimes e^t_u\,. \tag{5.56}$$

The skew-symmetrization of the previous equation together with the skew-symmetrization of equations (5.34) yields the second equation in (5.45). Conversely, suppose that $\left\{\mathfrak{e}^t, \beta_t\right\}_{t\in\mathcal{I}}$ is a solution of equations (5.45) and (5.46), and set

$$h_{\mathfrak{e}^t} = e^t_u \otimes e^t_u + e^t_l \otimes e^t_l + e^t_n \otimes e^t_n\,, \qquad \Theta_t = -\frac{1}{2\beta_t}\partial_t h_{\mathfrak{e}^t}\,. \tag{5.57}$$

Since $[\Theta_t(e^t_u)] = 0$ in $H^1(\Sigma, \mathbb{R}) = 0$, there exists a smooth family of functions $\left\{\bar{\mathfrak{f}}_t\right\}_{t\in\mathbb{R}}$ such that

$$\mathrm{d}\bar{\mathfrak{f}}_t = -\Theta_t(e^t_u)\,. \tag{5.58}$$

Taking the time-derivative of the previous expression we obtain:

$$\mathrm{d}\partial_t\bar{\mathfrak{f}}_t = -\partial_t(\Theta_t(e^t_u))\,. \tag{5.59}$$

Hence, comparing with the first equation in (5.46) we conclude:

$$\mathrm{d}\partial_t\bar{\mathfrak{f}}_t = \mathrm{d}(\mathrm{d}\beta_t(e^t_u))\,. \tag{5.60}$$

implying $\partial_t\bar{\mathfrak{f}}_t = \mathrm{d}\beta_t(e^t_u) + c(t)$ for a certain function $c(t)$ depending exclusively on t. Set $\mathfrak{f}_t := \bar{\mathfrak{f}}_t - \int c(\tau)\mathrm{d}\tau$, By construction we have $\partial_t\mathfrak{f}_t = \mathrm{d}\beta_t(e^t_u)$. Furthermore,

$$\mathrm{d}\partial_t\mathfrak{f}_t = -\partial_t\Theta_t(e^t_u)\,. \tag{5.61}$$

Define now $u^\perp_t := e^{\mathfrak{f}_t}\, e^t_u$ and $l^\perp_t := e^t_l$. The fact that both e^t_u and e^t_l satisfy the first equation in (5.45) implies:

$$\partial_t u^\perp_t + \beta_t\Theta_t(u^\perp_t) + (\mathrm{d}\beta_t(e^t_u) - \partial_t\mathfrak{f}_t)u^\perp_t = u^0_t\mathrm{d}\beta_t\,, \tag{5.62}$$

as well as:

$$u^0_t\partial_t l^\perp_t + \beta_t u^0_t\Theta_t(l^\perp_t) + \mathrm{d}\beta_t(l^\perp_t)u^\perp_t = 0\,, \tag{5.63}$$

where we have identified $u^0_t := e^{\mathfrak{f}_t}$. Using the fact that $\partial_t\mathfrak{f}_t = \mathrm{d}\beta_t(e^t_u)$ we obtain equations (5.33). Equations (5.34) follow directly by interpreting the second equation in (5.45) as the first Cartan structure equations for the coframe $\mathfrak{e}^t$, considered as orthonormal with respect to the metric $h_{\mathfrak{e}^t} = e^t_u\otimes e^t_u + e^t_l\otimes e^t_l + e^t_n\otimes e^t_n$. Finally, equations (5.35) hold by construction and hence we conclude. □

Definition 5.3. Equations (5.45) and (5.46) are the *(real) parallel spinor flow equations.* A *real parallel spinor flow* is a family $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ of functions and coframes on Σ satisfying the real parallel spinor flow equations.

Therefore, a globally hyperbolic Lorentzian four-manifold admits a parallel spinor if and only if it admits a Cauchy surface carrying a parallel spinor flow $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$. In particular, the corresponding parallel spinor ε can be fully reconstructed from $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$. We remark that for our purposes the explicit expression of the parallel spinor associated to a given parallel spinor flow $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ is of no relevance in itself. Instead, we are interested in the geometric and topological consequences associated to the existence of a parallel spinor ε, rather than on its specific expression.

5.2.1 The constraint equations

The parallel spinor flow equations pose an evolution problem whose associated constraint equations are equivalent to the constraint equations of the evolution problem posed by a parallel spinor on a globally hyperbolic Lorentzian four-manifold. Take $\Sigma := \Sigma_0$ as the Cauchy hypersurface of (M, g) and set

$$\mathfrak{e} := \mathfrak{e}^0\,, \quad \Theta := \Theta_0\,. \tag{5.64}$$

Proposition 5.3. *A globally hyperbolic four-manifold (M, g) with Cauchy surface $\Sigma \hookrightarrow M$ and second fundamental form $\Theta \in \Gamma(T^*\Sigma \odot T^*\Sigma)$ admits a parallel spinor $\varepsilon \in \Gamma(\mathrm{S}_g)$ if and only if Σ admits an orthonormal frame $\mathfrak{e} = (e_u, e_l, e_n)$ such that*

$$\mathrm{d}e_u = \Theta(e_u) \wedge e_u \,, \quad \mathrm{d}e_l = \Theta(e_l) \wedge e_u \,, \quad \mathrm{d}e_n = \Theta(e_n) \wedge e_u \,, \tag{5.65}$$

$$[\Theta(e_u)] = 0 \in H^1(\Sigma, \mathbb{R}) \,. \tag{5.66}$$

Proof. Equation (5.65) and (5.66) are the restriction of the second set of equations in (5.45) and of the second equation in (5.46) to Σ, so it is clear that they are necessary conditions. Regarding sufficiency, note that the initial value problem of a parallel null spinor is well posed by the results of [477, 478, 627] and a parallel spinor is equivalent to a parallel parabolic pair (see Proposition 5.1). Hence every solution to (5.65) and (5.66) admits a Lorentzian development carrying a parallel spinor and containing as Cauchy surface the submanifold (Σ, h) with associated second fundamental form Θ and we conclude. □

Remark 5.6. The constraint equations corresponding to a parallel spinor on a globally hyperbolic Lorentzian manifold are well known to correspond to the *imaginary* generalized Killing spinor equation with respect to the shape operator of the Cauchy hypersurface [478, 628, 644]. Such type of characterization also applies to our problem, however we do not need to consider it thanks to the description of parallel spinors as parabolic pairs provided in Proposition 5.1.

Remark 5.7. Equation (5.66) is to be interpreted as a *cohomological* condition, which is equivalent to $\mathrm{d}(\Theta(e_u)) = 0$ if $H^1(\Sigma, \mathbb{R}) = 0$. However, it may restrict the discrete quotients to which a given solution on the universal cover descends, since an exact one-form on Σ may descend to a closed non-exact one-form on certain quotients of Σ.

We will consider equations (5.65) and (5.66) as the constraint equations of the parallel spinor flow, whose solutions $(\mathfrak{e}, \Theta)$ are by definition the allowed initial data of the parallel spinor flow. We will refer to equations (5.65) and (5.66) as the *parallel Cauchy differential system.*

Definition 5.4. A *parallel Cauchy pair* $(\mathfrak{e}, \Theta)$ is a solution of the parallel Cauchy differential system.

For further reference, we also introduce the notion of Codazzi spinors. According to standard usage in the literature, if the shape operator of a given solution $(\mathfrak{e}, \Theta)$ satisfies

$$\nabla^h \Theta \in \Gamma(\mathrm{S}^3 T^*\Sigma) \,, \tag{5.67}$$

then we say that Θ is a *Codazzi tensor* on Σ. More explicitly, a shape operator Θ is a Codazzi tensor if and only if

$$(\nabla^h_{v_1} \Theta)(v_2, v_3) = (\nabla^h_{v_2} \Theta)(v_1, v_3) \,, \tag{5.68}$$

for every $v_1, v_2, v_3 \in \mathfrak{X}(\Sigma)$. Such (imaginary) *Codazzi spinors* were studied [628] and correspond to the constraint equations of a parallel spinor on a globally hyperbolic Lorentzian manifold of constant curvature. More recently, Reference [478] determines the local isometry type of the Cauchy surface of any Lorentzian manifold carrying a parallel spinor, showing that, in the four-dimensional case, corresponds to a certain warped product involving a family of two-dimensional flat metrics. Therefore, the results of this article can be considered as a continuation of those of [478] in the specific case of four Lorentzian dimensions.

Let $\mathrm{Conf}(\Sigma)$ denote the configuration space of the parallel Cauchy differential system, that is, its space of variables $(\mathfrak{e},\Theta)$, and let $\mathrm{Sol}(\Sigma)$ be the space of parallel Cauchy pairs. Note that the function β_0 does not occur in the parallel Cauchy differential system, exactly as it happens with the initial value problem posed by the Ricci flat condition of a Lorentzian metric [329, 335]. Given a pair $(\mathfrak{e},\Theta)\in\mathrm{Conf}(\Sigma)$, we denote by $h_{\mathfrak{e}}$ the Riemannian metric on Σ defined as:

$$h_{\mathfrak{e}} = e_u\otimes e_u + e_l\otimes e_l + e_n\otimes e_n\,, \tag{5.69}$$

where $\mathfrak{e}=(e_u,e_l,e_n)$. We say that $(\mathfrak{e},\Theta)$ is complete if $h_{\mathfrak{e}}$ is a complete Riemannian metric on Σ. Denote by $\mathrm{Met}(\Sigma)\times\Gamma(T^*\Sigma\odot T^*\Sigma)$ the set of pairs consisting of Riemannian metrics and symmetric two-tensors on Σ. We obtain a canonical map

$$\Psi\colon \mathrm{Conf}(\Sigma)\to\mathrm{Met}(\Sigma)\times\Gamma(T^*M\odot T^*M)\,,\qquad (\mathfrak{e},\Theta)\mapsto(h_{\mathfrak{e}},\Theta)\,. \tag{5.70}$$

The set $\mathrm{Met}(\Sigma)\times\Gamma(T^*M\odot T^*M)$ is in fact the configuration space of the constraint equations associated to the Cauchy problem posed by the Ricci flat condition on a globally hyperbolic Lorentzian four-manifold with Cauchy surface Σ, which are given by [329, 335]:

$$\mathrm{R}_h = |\Theta|^2_h - \mathrm{Tr}_h(\Theta)^2\,,\qquad \mathrm{d}\mathrm{Tr}_h(\Theta)=\mathrm{div}_h(\Theta)\,, \tag{5.71}$$

for pairs $(h,\mathfrak{e})\in\mathrm{Met}(\Sigma)\times\Gamma(T^*M\odot T^*M)$.

Remark 5.8. The first equation in (5.71) is usually called the *Hamiltonian constraint* whereas the second equation in (5.71) is usually called the *momentum constraint*.

Therefore, the map Ψ provides a natural link between the initial value problem associated to a parallel spinor and the initial value problem associated to the Ricci-flatness condition. In particular, it allows introducing a natural notion of *admissible* initial data to both evolution problems.

Definition 5.5. A parallel Cauchy pair $(\mathfrak{e},\Theta)$ is *constrained Ricci flat* if $(h_{\mathfrak{e}},\Theta)$ satisfies the momentum and Hamiltonian constraints (5.71).

Lemma 5.3. *Let $(\mathfrak{e},\Theta)$ be a parallel Cauchy pair and write $\mathfrak{e}=(e_u,e_l,e_n)$. Then*

$$\mathrm{div}_h(\Theta)\wedge e_u = \mathrm{d}\mathrm{Tr}(\Theta)\wedge e_u\,. \tag{5.72}$$

Proof. The statement is equivalent to

$$\mathrm{div}_{h_{\mathfrak{e}}}(\Theta)(e_l)=\mathrm{d}\mathrm{Tr}(\Theta)(e_l)\,,\qquad \mathrm{div}_{h_{\mathfrak{e}}}(\Theta)(e_n)=\mathrm{d}\mathrm{Tr}_{h_{\mathfrak{e}}}(\Theta)(e_n)\,. \tag{5.73}$$

Note that we indistinctly denote with the same symbol one-forms and their duals by the metric wherever no possible confusion may arise. Now we write:

$$\Theta=\Theta_{ab}\,e_a\otimes e_b\,,\qquad \Theta_{ab}\in C^\infty(\Sigma)\,,\qquad a,b=u,l,n\,, \tag{5.74}$$

where $\Theta_{ab} \in C^\infty(\Sigma)$ are smooth functions. Also, recall that by the definition of parallel Cauchy coframe, $\mathfrak{e} = (e_u, e_l, e_n)$ satisfies

$$\nabla^{h_\mathfrak{e}}_{e_b} e_a = -\delta_{au}\Theta(e_b) + \Theta(e_a, e_b) e_u \,. \tag{5.75}$$

Using the previous equation, we compute:

$$\mathrm{div}_h(\Theta)(e_l) = \sum_a (\nabla^{h_\mathfrak{e}}_{e_a}\Theta)(e_a, e_l) = \sum_a e_a(\Theta_{al}) - \sum_a \Theta_{ul}\Theta_{aa} \,, \tag{5.76}$$

as well as:

$$\mathrm{dTr}(\Theta)(e_l) = \sum_a e_l(\Theta_{aa}) \,. \tag{5.77}$$

Hence,

$$\mathrm{dTr}_{h_\mathfrak{e}}(\Theta)(e_l) - \mathrm{div}_h(\Theta)(e_l) = -e_u(\Theta_{ul}) - e_n(\Theta_{ln}) + e_l(\Theta_{uu}) + e_l(\Theta_{nn}) + \Theta_{ul}\mathrm{Tr}_{h_\mathfrak{e}}(\Theta) \,. \tag{5.78}$$

Using now that $\mathrm{d}^2 e_n = 0$ we obtain $e_l(\Theta_{nn}) - e_n(\Theta_{ln}) = \Theta_{ln}\Theta_{un} - \Theta_{nn}\Theta_{ul}$, which in turn implies $\mathrm{dTr}(\Theta)(e_l) = \mathrm{div}_h(\Theta)(e_l)$. Similarly $\mathrm{div}_h(\Theta)(e_n) = \mathrm{dTr}(\Theta)(e_n)$ and we conclude. □

For further reference, we obtain the Ricci tensor and scalar curvature of the Riemannian metric $h_\mathfrak{e}$ associated to a parallel Cauchy pair $(\mathfrak{e}, \Theta)$.

Proposition 5.4. *Let $(\mathfrak{e}, \Theta)$ be a parallel Cauchy pair. The Ricci curvature of $h_\mathfrak{e}$ is given by:*

$$\mathrm{Ric}^\mathfrak{e} = \Theta \circ \Theta - \mathrm{Tr}_\mathfrak{e}(\Theta)\Theta + (\mathrm{dTr}_\mathfrak{e}(\Theta) - \mathrm{div}_\mathfrak{e}(\Theta)) \otimes e_u + \nabla^\mathfrak{e}_{e_u}\Theta - (\nabla^\mathfrak{e}\Theta)(e_u) \,, \tag{5.79}$$

whereas the scalar curvature of $h_\mathfrak{e}$ reads

$$\mathrm{R}^\mathfrak{e} = |\Theta|^2_\mathfrak{e} - \mathrm{Tr}_\mathfrak{e}(\Theta)^2 - 2(\mathrm{div}_\mathfrak{e}(\Theta)(e_u^\sharp) - \mathrm{dTr}_\mathfrak{e}(\Theta)(e_u^\sharp)) \,, \tag{5.80}$$

where d_{∇^h} denotes the exterior covariant derivative associated to ∇^h.

Proof. The result is proven through a direct computation using the fact that for a parallel Cauchy pair $(\mathfrak{e}, \Theta)$ we have:

$$\nabla^{h_\mathfrak{e}} e_a^\sharp = \Theta(e_a^\sharp) \otimes e_u^\sharp - \delta_{ua}\Theta^\sharp \,, \qquad a = u, l, n \tag{5.81}$$

as well as:

$$\nabla^{h_\mathfrak{e}}(\Theta(e_a^\sharp)) = \textstyle\sum_b (\mathrm{d}\Theta_{ab} \otimes e_b + \Theta_{ab}\Theta_b \otimes e_u) - \Theta_{ua}\Theta \,, \tag{5.82}$$

where we have written $\Theta = \Theta_{ab} e_a \otimes e_b$, $a, b = u, l, n$. In our conventions the Ricci curvature reads

$$\mathrm{Ric}^\mathfrak{e} = \sum_c e_u^\sharp \lrcorner \mathrm{d}_{\nabla^\mathfrak{e}}(e_c \otimes \Theta_c) - \sum_a e_a^\sharp \lrcorner \mathrm{d}_{\nabla^\mathfrak{e}}(\Theta_a \otimes e_u) \,, \tag{5.83}$$

where $\mathrm{d}_{\nabla^\mathfrak{e}}$ denotes the exterior covariant derivative for one-forms taking values on one-forms. Expanding the desired result for the Ricci tensor follows. Taking the trace of Equation (5.79), we obtain:

$$\mathrm{R}^h = |\Theta|^2 - \mathrm{Tr}(\Theta)^2 - \mathrm{div}_h(\Theta)(e_u^\sharp) + \mathrm{dTr}(\Theta)(e_u^\sharp) + \mathrm{Tr}_\mathfrak{e}(\nabla_{e_u}\Theta - (\nabla\Theta)(e_u)) \,. \tag{5.84}$$

The last term can be written as follows:

$$\sum_a ((\nabla_{e_u^\sharp}\Theta)(e_a^\sharp, e_a^\sharp) - (\nabla_{e_a^\sharp}\Theta)(e_u^\sharp, e_a^\sharp)) = \mathrm{dTr}_{\mathfrak{e}}(\Theta)(e_u^\sharp) - \mathrm{div}_{\mathfrak{e}}(\Theta)(e_u^\sharp)\,, \tag{5.85}$$

whence

$$\mathrm{R}^{\mathfrak{e}} = |\Theta|^2 - \mathrm{Tr}(\Theta)^2 - 2(\mathrm{div}_h(\Theta)(e_u^\sharp) - \mathrm{dTr}_{\mathfrak{e}}(\Theta)(e_u^\sharp))\,, \tag{5.86}$$

and we conclude. □

Remark 5.9. If Θ is Codazzi then Equation (5.79) simplifies to:

$$\mathrm{Ric}^{\mathfrak{e}} = \Theta \circ \Theta - \mathrm{Tr}_{\mathfrak{e}}(\Theta)\Theta\,, \tag{5.87}$$

which matches [628, Proposition 5] modulo an unimportant constant factor.

Proposition 5.5. *A Cauchy pair $(\mathfrak{e}, \Theta)$ satisfies the Hamiltonian constraint, that is, the first equation in (5.71), if and only if $(\mathfrak{e}, \Theta)$ satisfies the momentum constraint, that is, the second equation in (5.71).*

Proof. Follows from the explicit expression (5.80) for the scalar curvature of $h_{\mathfrak{e}}$ upon use of Lemma 5.3. □

Proposition 5.6. *A pair $(\mathfrak{e}, \Theta) \in \mathrm{Conf}(\Sigma)$ is a constrained Ricci flat parallel Cauchy pair if and only if*

$$\mathrm{d}e_u = \Theta(e_u) \wedge e_u\,, \quad \mathrm{d}e_l = \Theta(e_l) \wedge e_u\,, \quad \mathrm{d}e_n = \Theta(e_n) \wedge e_u\,, \tag{5.88}$$

$$[\Theta(e_u)] = 0 \in H^1(\Sigma, \mathbb{R})\,, \quad \mathrm{R}^{h_{\mathfrak{e}}} = |\Theta|^2 - \mathrm{Tr}(\Theta)^2\,. \tag{5.89}$$

where $h_{\mathfrak{e}}$ is the Riemannian metric associated to $(\mathfrak{e}, \Theta)$. In particular, every Cauchy pair $(\mathfrak{e}, \Theta)$ whose shape operator Θ is Codazzi is constrained Ricci flat.

Proof. By Proposition 5.5 we only need to prove that if $(\mathfrak{e}, \Theta)$ is a parallel Cauchy pair and Θ is a Codazzi shape operator then the momentum constraint is automatically satisfied. Fix a point $p \in \Sigma$ and an orthonormal (with respect to $h_{\mathfrak{e}}$) frame $\{e_a\}$, $a = 1, 2, 3$, such that $\nabla^{h_{\mathfrak{e}}} e_a|_p = 0$. We compute at $p \in \Sigma$:

$$\begin{aligned} \mathrm{dTr}(\Theta)|_p &= \sum_a \mathrm{d}(\Theta(e_a, e_a))|_p = \sum_a (\nabla^{h_{\mathfrak{e}}}\Theta)(e_a, e_a)|_p + 2\sum_a \Theta(\nabla^{h_{\mathfrak{e}}} e_a, e_a)|_p \\ &= \sum_a (\nabla^{h_{\mathfrak{e}}}_{e_a}\Theta)(e_a)|_p = \mathrm{div}_{h_{\mathfrak{e}}}(\Theta)|_p\,, \end{aligned} \tag{5.90}$$

and hence we conclude. □

Remark 5.10. We will refer to a parallel Cauchy pair $(\mathfrak{e}, \Theta)$ whose shape operator is Codazzi as a *Codazzi parallel Cauchy pair.*

Proposition 5.6 summarizes necessary conditions that a pair $(\mathfrak{e}, \Theta)$ needs to satisfy in order for the Lorentzian development of $(\Sigma, h_{\mathfrak{e}})$ to be a Ricci flat Lorentzian four-manifold admitting a parallel spinor field. These conditions are satisfied by all examples in [628].

Example 5.1. Take $\Sigma = \tau_{3,\mu}$ to be the simply-connected non-unimodular Lie group $\tau_{3,\mu}$ where $-1 < \mu \leq 1$, $\mu \neq 0$, is a constant, see [645, Chapter 7] for its precise definition. On $\tau_{3,\mu}$ there exists a left-invariant coframe (e^1, e^2, e^3) satisfying

$$\mathrm{d}e^1 = 0\,, \qquad \mathrm{d}e^2 = \mu\, e^2 \wedge e^1\,, \qquad \mathrm{d}e^3 = e^3 \wedge e^1\,. \tag{5.91}$$

Set

$$\begin{gathered}\mathfrak{e} = (e_u, e_l, e_n) := (e^1, e^2, e^3)\,, \quad h_{\mathfrak{e}} = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n\,,\\ \Theta := h + (\mu - 1)\, e_l \otimes e_l\,.\end{gathered} \tag{5.92}$$

A direct computation shows that $(\mathfrak{e}, \Theta)$ defines a parallel Cauchy pair on $\tau_{3,\mu}$, that is, $(\mathfrak{e}, \Theta)$ is a solution of equations (5.65) and (5.66). Note that since $\mathrm{d}e_u = 0$ and $\tau_{3,\mu}$ is simply connected, the one-form $e_u = \Theta(e_u)$ is automatically exact. In particular, we have:

$$\nabla^h e_u = -\mu\, e_l \otimes e_l - e_n \otimes e_n\,, \qquad \nabla^h e_l = \mu\, e_l \otimes e_u\,, \qquad \nabla^h e_n = e_n \otimes e_u\,, \tag{5.93}$$

conditions which are equivalent to equations (5.65). More explicitly, write $e_u = \mathrm{d}\mathfrak{f}$ for a real function $\mathfrak{f} \in C^\infty(\Sigma)$. Then $(\hat{e}_l = e^{\mu\mathfrak{f}} e_l, \hat{e}_n = e^{\mathfrak{f}} e_n)$ defines a pair of closed nowhere vanishing one-forms. In particular, $\hat{\mathfrak{e}} = (e_u, \hat{e}_l, \hat{e}_n)$ is a closed global coframe on Σ. Set

$$h_{\hat{\mathfrak{e}}} \stackrel{\text{def.}}{=} e_u \otimes e_u + \hat{e}_l \otimes \hat{e}_l + \hat{e}_n \otimes \hat{e}_n\,, \tag{5.94}$$

to be the Riemannian metric defined as $\hat{\mathfrak{e}} \stackrel{\text{def.}}{=} (e_u, \hat{e}_l, \hat{e}_n)$. Since $\mathrm{d}\hat{\mathfrak{e}} = 0$, the metric $h_{\hat{\mathfrak{e}}}$ is flat and therefore

$$h_{\mathfrak{e}} = e_u \otimes e_u + e^{-2\mu\mathfrak{f}} \hat{e}_l \otimes \hat{e}_l + e^{-2\mathfrak{f}} \hat{e}_n \otimes \hat{e}_n\,, \tag{5.95}$$

is a warped product of flat metrics. Even more, since $\hat{\mathfrak{e}} = (e_u, \hat{e}_l, \hat{e}_n)$ is a closed coframe there exist local coordinates (z, x, y) (global, if $\hat{\mathfrak{e}}$ is complete) such that:

$$e_u = \mathrm{d}\mathfrak{f} = \mathrm{d}z\,, \qquad \hat{e}_l = \mathrm{d}x\,, \qquad \hat{e}_n = \mathrm{d}y\,. \tag{5.96}$$

Therefore, the metric can be written as follows:

$$h_{\mathfrak{e}} = \mathrm{d}z \otimes \mathrm{d}z + e^{-2\mu z} \mathrm{d}x \otimes \mathrm{d}x + e^{-2z} \mathrm{d}y \otimes \mathrm{d}y\,. \tag{5.97}$$

The scalar curvature of $h_{\mathfrak{e}}$ can be computed to be:

$$\mathrm{R}_h = -2(1 + \mu + \mu^2)\,. \tag{5.98}$$

which, together with the fact that $|\Theta|^2_h = 2 + \mu^2$ and $\mathrm{Tr}_h(\Theta)^2 = (2 + \mu)^2$ shows that the Hamiltonian constraint is satisfied if and only if[3]:

$$\mu = 1\,. \tag{5.99}$$

Since the momentum constraint is clearly satisfied if and only if $\mu = 1$, we conclude that if $\mu \neq 1$ we obtain a solution to the constraint equations (5.33) and (5.34) whose Lorentzian development yields a non Ricci flat Lorentzian four manifold. On the other hand, if $\mu = 1$, the Riemannian three-manifold $(\Sigma, h_{\mathfrak{e}})$ admits a Lorentzian development which is Ricci flat and admits a parallel spinor, by virtue of Theorem 5.2 (which we will state and prove afterwards). Finally, when $\mu = 1$, the parallel Cauchy pair turns out to be additionally Codazzi and $(\Sigma, h_{\mathfrak{e}}) \hookrightarrow (M, g)$ is a totally umbilical submanifold of (M, g).

[3] Recall that $-1 < \mu \leq 1$ and $\mu \neq 0$.

A Cauchy pair is said to be *complete* if $(\Sigma, h_{\mathfrak{e}})$ is a complete Riemannian three-manifold. When necessary, the dual of a Cauchy coframe $\mathfrak{e}$ will be denoted by $\mathfrak{e}^\sharp = (e^\sharp_u, e^\sharp_l, e^\sharp_n)$. Denote by

$$\mathrm{Conf}(\Sigma) \stackrel{\text{def.}}{=} \Gamma(\mathrm{S}^2 T^*\Sigma) \times \Gamma(\mathrm{F}(\Sigma))\,, \tag{5.100}$$

the configuration space of the Cauchy differential system that is, its space of variables. Likewise, denote by

$$\mathrm{Sol}(\Sigma) \subset \mathrm{Conf}(\Sigma)\,, \tag{5.101}$$

the subspace of solutions of the Cauchy differential system. We have a canonical map

$$\mathrm{Sol}(\Sigma) \to \mathrm{Met}_c(\Sigma)\,, \qquad (\mathfrak{e}, \Theta) \mapsto h_{\mathfrak{e}}\,, \tag{5.102}$$

where $\mathrm{Met}_c(\Sigma)$ denotes the space of complete Riemannian metrics on Σ. The image of the previous map, which we denote by $\mathrm{Met}^s_c(\Sigma)$, is by definition the space of complete Riemannian metrics on Σ that admit a solution to the Cauchy differential system for a shape operator $\Theta \in \Gamma(\mathrm{S}^2 T^*\Sigma)$. The group of orientation preserving diffeomorphisms $\mathrm{Diff}(\Sigma)$ has a natural left action on $\mathrm{Conf}(\Sigma)$ given by push-forward:

$$\mathbb{A}\colon \mathrm{Diff}(\Sigma) \times \mathrm{Conf}(\Sigma) \to \mathrm{Conf}(\Sigma)\,, \quad (\mathfrak{u}, (\mathfrak{e}, \Theta)) \mapsto (\mathfrak{u}_*\mathfrak{e}, \mathfrak{u}_*\Theta)\,. \tag{5.103}$$

For every $\mathfrak{u} \in \mathrm{Diff}(\Sigma)$, define

$$\mathbb{A}_{\mathfrak{u}}\colon \mathrm{Conf}(\Sigma) \to \mathrm{Conf}(\Sigma)\,, \quad (\mathfrak{e}, \Theta) \mapsto (\mathfrak{u}_*\mathfrak{e}, \mathfrak{u}_*\Theta)\,. \tag{5.104}$$

Lemma 5.4. *Let $\mathfrak{u} \in \mathrm{Diff}(\Sigma)$. Then, $(\mathfrak{e}, \Theta) \in \mathrm{Sol}(\Sigma)$ if and only if $\mathbb{A}_{\mathfrak{u}}(\mathfrak{e}, \Theta) \in \mathrm{Sol}(\Sigma)$.*

Proof. We compute:

$$\mathrm{d}e_u = \Theta(e_u) \wedge e_u \Leftrightarrow \mathfrak{u}_*\mathrm{d}e_u = \mathfrak{u}_*(\Theta(e_u) \wedge e_u) \Leftrightarrow \mathrm{d}\mathfrak{u}_* e_u = (\mathfrak{u}_*\Theta)(\mathfrak{u}_* e_u) \wedge \mathfrak{u}_* e_u\,, \tag{5.105}$$

and similarly for the remaining equations of the Cauchy differential system (5.65). □

Therefore, the orientation-preserving diffeomorphism group of Σ has a well-defined action on the space of parallel Cauchy pairs and we can consider the quotient

$$\mathfrak{M}(\Sigma) \stackrel{\text{def.}}{=} \mathrm{Sol}(\Sigma)/\mathrm{Diff}(\Sigma)\,, \tag{5.106}$$

defined through the action $\mathbb{A}$. We call $\mathfrak{M}(\Sigma)$ the *moduli space* of parallel Cauchy pairs on Σ, which we will hopefully investigate in the future. In the following we will consider two parallel Cauchy pairs to be isomorphic if they are related by an orientation preserving diffeomorphism of Σ as prescribed by the action $\mathbb{A}$.

5.2.2 Initial data characterization

Denote by $\mathcal{P}(\Sigma)$ the set of parallel spinor flows on Σ, that is, the set of families $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ satisfying the parallel spinor flow equations (5.45) and (5.46). We have a canonical map

$$\Phi\colon \mathcal{P}(\Sigma) \to \mathrm{Lor}_\circ(M)\,, \qquad \left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}} \mapsto g = -\beta_t^2 \mathrm{d}t^2 + h_{\mathfrak{e}^t}\,, \tag{5.107}$$

from $\mathcal{P}(\Sigma)$ to the set $\mathrm{Lor}_\circ(M)$ of globally hyperbolic Lorentzian metrics on $M = \mathcal{I} \times \Sigma$. For simplicity in the exposition, we will refer to $\Phi(\{\beta_t, \mathfrak{e}^t\})$ as the globally hyperbolic metric determined by $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$. Given a left-invariant parallel spinor flow $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$, there exists a smooth family functions $\{\mathfrak{f}_t\}_{t\in\mathcal{I}}$ such that

$$\mathrm{d}\mathfrak{f}_t = -\Theta_t(e^t_u)\,, \qquad \partial_t \mathfrak{f}_t = \mathrm{d}\beta_t(e^t_u)\,, \tag{5.108}$$

which is unique modulo the addition of a real constant. Using this family of functions, we obtain a canonical map:

$$\Xi\colon \mathcal{P}(\Sigma) \to \mathcal{B}(M)\,, \qquad \{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}} \mapsto (u = e^{\mathfrak{f}_t}(\beta_t \mathrm{d}t + e^t_u), [l = e^t_l])\,, \tag{5.109}$$

from the set of parallel spinor flows on Σ to the set $\mathcal{B}(M)$ of parabolic pairs on M with respect to the globally hyperbolic metric determined by the given parallel spinor flow. The previous maps provide a construction which is essentially *inverse* to the splitting and reduction implemented at the beginning of this Section 5.2 and which allows us to relate properties of a given parallel spinor flow to properties of its associated globally hyperbolic four-dimensional Lorentzian metric. For further reference, we introduce the *Hamiltonian function* of a parallel spinor flow $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ as follows:

$$\mathcal{H}\colon M = \mathbb{R}\times\Sigma \to \mathbb{R}\,, \qquad (t,p) \mapsto (\mathrm{R}_{h_t} - |\Theta_t|^2_{h_t} + \mathrm{Tr}_{h_t}(\Theta_t)^2)|_p\,, \tag{5.110}$$

where $h_t := h_{\mathfrak{e}^t}$ denotes the three-dimensional metric restricted to the Cauchy surface Σ_t and $\mathrm{R}_{h_{\mathfrak{e}^t}}$ its scalar curvature.

Proposition 5.7. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a parallel spinor flow on Σ. The Ricci curvature of $g = \Phi(\{\beta_t, \mathfrak{e}^t\})$ reads*

$$\mathrm{Ric}^g = \frac{1}{2}\mathcal{H}e^{-2\mathfrak{f}_t} u\otimes u\,, \tag{5.111}$$

where $\Xi(\{\beta_t, \mathfrak{e}^t\}) = (u, [l])$ and $u = e^{\mathfrak{f}_t}(\beta_t \mathrm{d}t + e^t_u)$.

Proof. Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a parallel spinor flow on Σ and let $g = \Phi(\{\beta_t, \mathfrak{e}^t\}) = -\beta_t^2 \mathrm{d}t\otimes\mathrm{d}t + h_t$ its associated globally hyperbolic metric on $M = \mathbb{R}\times\Sigma$. The pair $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ defines a global orthonormal coframe (e_0, e_1, e_2, e_3) on (M, g) given by:

$$e_0|_{(t,p)} := \beta_t|_p \mathrm{d}t\,, \qquad e_1|_{(t,p)} := e^t_u|_p\,, \qquad e_2|_{(t,p)} := e^t_l|_p\,, \qquad e_3|_{(t,p)} := e^t_n|_p\,. \tag{5.112}$$

The fact that $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is a parallel spinor flow implies that the exterior derivatives of the coframe (e_0, e_1, e_2, e_3) on M are prescribed as follows:

$$\mathrm{d}e_0 = \mathrm{d}\log(\beta_t)\wedge e_0\,, \tag{5.113}$$

$$\mathrm{d}e_a = (\mathrm{d}\log(\beta_t)(e_a)\, e_1 + \Theta_t(e_a) - \delta_{a1}\mathrm{d}\log(\beta_t))\wedge e_0 + \Theta_t(e_a)\wedge e_1\,, \tag{5.114}$$

where $a = 1, 2, 3$. Interpreting the previous expression as the first Cartan structure equations for ∇^g with respect to the orthonormal coframe (e_0, e_1, e_2, e_3) and using repeatedly equations (5.45) and (5.46), a tedious calculation yields (5.111) and hence we conclude. □

Remark 5.11. It is well known that the Ricci curvature Ric^g of a Lorentzian four-manifold admitting parallel spinors is of the form $\mathrm{Ric}^g = f u \otimes u$ for some function $f \in C^\infty(M)$ [640]. Nonetheless, and to the best of our knowledge, equation (5.111) is the first precise characterization of such function f in the case of globally hyperbolic Lorentzian four-manifolds.

Theorem 5.2. *The parallel spinor flow preserves the vacuum momentum and Hamiltonian constraints.*

Proof. Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a parallel spinor flow. Taking the divergence of Equation (5.111) we obtain:

$$\mathrm{d}(\mathcal{H} e^{-2\mathfrak{f}_t})(u^\sharp) = 0\,, \tag{5.115}$$

which can be equivalently written as follows:

$$\mathfrak{D}(\mathcal{H}) = \rho\mathcal{H}\,, \tag{5.116}$$

where $\mathfrak{D}$ is a first-order symmetric hyperbolic differential operator and ρ is a function completely determined by $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$. Given such $\mathfrak{D}$ and ρ associated to $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$, consider now the initial value problem:

$$\mathfrak{D}(F) = \rho\, F\,, \qquad F|_\Sigma = 0\,, \tag{5.117}$$

for an arbitrary function F on M. By the existence and uniqueness theorem for this type of equations, see [646, Theorem 19] and references therein, every solution must be zero on a neighborhood of Σ. Since $\mathcal{H}$ is in particular a solution of this equation, it must vanish on a neighborhood of Σ. Therefore, there exists a subinterval $\mathcal{I}' = (a,b) \subseteq \mathcal{I}$ containing zero such that $\mathcal{H}|_t = 0$ for every $t \in \mathcal{I}'$. By Proposition 5.5 this implies that the momentum constraint is also satisfied for every $t \in \mathcal{I}'$ and hence the parallel spinor flow preserves the Hamiltonian and momentum constraints on $\mathcal{I}'$. If $\mathcal{I}' = \mathcal{I}$ we conclude, so assume that $\mathcal{I}' = (a,b) \subset \mathcal{I}$ is the proper maximal subinterval of $\mathcal{I}$ for which the result holds. Since the parallel spinor flow $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is well defined in $\mathcal{I}$, then both ρ and $\mathcal{H}$ must be well defined on $\mathcal{I}\times\Sigma$. Hence, by point-wise continuity on Σ we must have that $\mathcal{H}|_b = 0$ and therefore we can apply the previous argument to the initial value problem *starting* at $b \in \mathcal{I}$. Hence there exists an $\varepsilon > 0$ for which the result holds on $(a, b+\varepsilon)$, in contradiction with (a,b) being maximal. Therefore, $\mathcal{I}' = \mathcal{I}$ and we conclude. □

Call a triple (Σ, h, Θ) an *initial vacuum* data if (h,Θ) satisfies the Hamiltonian and momentum constraints. The previous theorem can be applied to prove an *initial data* characterization of parallel spinors on Ricci flat Lorentzian four-manifolds.

Corollary 5.1. *An initial vacuum data (Σ, h, Θ) admits a Ricci flat Lorentzian development carrying a parallel spinor if and only if there exists a global orthonormal coframe $\mathfrak{e}$ on Σ such that $(\mathfrak{e}, \Theta)$ is a parallel Cauchy pair.*

Proof. The *only if* condition follows from Theorem 5.1. For the *if* condition, let (Σ, h, Θ) be an initial vacuum data. If in addition there exists a global orthonormal coframe $\mathfrak{e}$ on Σ such that $(\mathfrak{e}, \Theta)$ is a parallel Cauchy pair, then the constraint equations of the initial value problem of a parallel spinor are satisfied. By [477, 478], the initial value problem is well posed and there exists a Lorentzian development of Σ carrying a parallel spinor.

By Theorem 5.2, this Lorentzian development satisfies the Hamiltonian and momentum constraint for every $t \in \mathcal{I}$, and by equation (5.111) we conclude that this Lorentzian development is Ricci flat. $\square$

Additionally, we obtain the following corollary.

Corollary 5.2. *A globally hyperbolic Lorentzian four-manifold (M, g) admitting a parallel spinor is Ricci flat if and only if there exists an adapted Cauchy hypersurface $\Sigma \subset M$ whose Hamiltonian constraint vanishes.*

Here we say that a Cauchy surface Σ in (M, g) is adapted if (M, g) has the isometry type (5.22) with h_0 given by the pull-back of g to Σ.

5.3 The topology and geometry of Cauchy pairs

In this section we investigate the diffeomorphism and isometry type of oriented three-manifolds Σ admitting a complete Cauchy pair $(\mathfrak{e}, \Theta) \in \mathrm{Sol}(\Sigma)$.

5.3.1 General considerations

Lemma 5.5. *Let $(\mathfrak{e}, \Theta)$ be a complete Cauchy pair on Σ. The frame $\mathfrak{e}^\sharp = (e_u^\sharp, e_l^\sharp, e_n^\sharp)$ dual of $\mathfrak{e}$ is complete, that is, each of its elements is a complete vector field on Σ.*

Proof. Follows from the fact that $h_\mathfrak{e}$ is by assumption a complete Riemannian metric on Σ respect to which each of the elements of $\mathfrak{e}$ has unit norm, see[4] [647, Page 154, Exercise 11]. $\square$

Lemma 5.6. *Let $\mathfrak{e} = (e_u, e_l, e_n)$ be a complete Cauchy coframe. The distribution $\ker(e_u) \subset T\Sigma$ is integrable and defines a codimension one transversely orientable foliation in $(\Sigma, h_\mathfrak{e})$ whose leaves are complete and flat Riemann surfaces with respect to the metric induced by $h_\mathfrak{e}$.*

Proof. The first equation in the Cauchy differential system (5.65) immediately implies

$$e_u \wedge \mathrm{d}e_u = 0\,, \tag{5.118}$$

and thus Cartan's criterion implies in turn that $\ker(e_u) \subset T\Sigma$ defines an integrable transversely orientable codimension one distribution, whose associated foliation we denote by $\mathcal{F}_\mathfrak{e}$. Let $p \in \Sigma$ and denote by $\mathcal{F}_{\mathfrak{e},p} \subset \Sigma$ the maximal leaf of $\mathcal{F}_\mathfrak{e}$ passing through p. The cotangent space of $\mathcal{F}_{\mathfrak{e},p}$ is spanned over $C^\infty(\mathcal{F}_{\mathfrak{e},p})$ by the restriction of e_l and e_n:

$$T^*\mathcal{F}_{\mathfrak{e},p} = \mathrm{Span}_{C^\infty(\mathcal{F}_{\mathfrak{e},p})}(e_l|_{T\mathcal{F}_{\mathfrak{e},p}}, e_n|_{T\mathcal{F}_{\mathfrak{e},p}})\,. \tag{5.119}$$

Furthermore:

$$h_\mathfrak{e}|_{\mathcal{F}_{\mathfrak{e},p}} = e_l|_{T\mathcal{F}_{\mathfrak{e},p}} \otimes e_l|_{T\mathcal{F}_{\mathfrak{e},p}} + e_n|_{T\mathcal{F}_{\mathfrak{e},p}} \otimes e_n|_{T\mathcal{F}_{\mathfrak{e},p}}\,. \tag{5.120}$$

[4] Note however that there is a typo in Exercise 11, the correct condition being, using the notation of the exercise, $|X(p)| < c$ rather than $|X(p)| > c$.

A direct computation, using the fact that $\mathfrak{e}$ is a parallel Cauchy coframe, shows that $(e_l|_{T\mathcal{F}_{\mathfrak{e},p}}, e_n|_{T\mathcal{F}_{\mathfrak{e},p}})$ is a flat coframe with respect to the Levi-Civita connection of the metric induced by $h_{\mathfrak{e}}$ whence $h_{\mathfrak{e}}|_{\mathcal{F}_{\mathfrak{e},p}}$ is flat. The fact that the leaves of $\mathcal{F}_{\mathfrak{e}}$ equipped with the metric induced by $h_{\mathfrak{e}}$ are complete manifolds follows from completeness of $h_{\mathfrak{e}}$ and is proved explicitly in [648, Proposition 1.26]. □

Since the leaves of $\mathcal{F}_{\mathfrak{e}}$ are complete and flat they must be isometric to either the Euclidean plane, the Euclidean cylinder or a flat torus. As we will see momentarily, this poses strong constraints on the differentiable topology of Σ. Given a Cauchy pair $(\mathfrak{e}, \Theta)$, the cohomological condition (5.66) guarantees that there exists a function $\mathfrak{f} \in C^\infty(\Sigma)$ such that

$$\Theta(e_u) = -\mathrm{d}\mathfrak{f} \,. \tag{5.121}$$

Therefore, by the first equation in (5.65), the one-form $\hat{e}_u := e^{\mathfrak{f}} e_u \in \Omega^1(\Sigma)$ is closed and satisfies $\ker(\hat{e}_u) = \ker(e_u)$, implying that we can consider $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ as a foliation given by the kernel of the nowhere vanishing closed one-form $\hat{e}_u$, a type of foliation that has been extensively studied in the literature, see for example [649, 650]. It can be easily seen that the metric $h_{\mathfrak{e}}$ will not be, in general, bundle-like with respect to $\mathcal{F}_{\mathfrak{e}}$. On the other hand, given a Cauchy pair $(\mathfrak{e}, \Theta)$, the following *modified* Riemannian metric:

$$h_{\hat{\mathfrak{e}}} = \hat{e}_u \otimes \hat{e}_u + e_l \otimes e_l + e_n \otimes e_n \,, \qquad \hat{\mathfrak{e}} = (\hat{e}_u, e_l, e_n) \,, \tag{5.122}$$

is indeed bundle-like, that is, it satisfies the following condition:

$$\mathcal{L}_v h_{\hat{\mathfrak{e}}}|_{T\mathcal{F}_{\mathfrak{e}}^{\perp_{h_{\mathfrak{e}}}}} = 0 \,, \qquad \forall\, v \in \Gamma(T\mathcal{F}_{\mathfrak{e}}) \,. \tag{5.123}$$

In other words, $h_{\hat{\mathfrak{e}}}|_{T\mathcal{F}_{\mathfrak{e}}^{\perp_{h_{\mathfrak{e}}}}}$ is a holonomy invariant transversal metric.

Remark 5.12. By Lemma 5.5, $e_u^\sharp \in \mathfrak{X}(\Sigma)$ is a complete vector field on Σ. However, the same statement may not hold for $\hat{e}_u^\sharp \in \mathfrak{X}(\Sigma)$, the metric dual of $\hat{e}_u$ with respect to $\hat{h}_{\mathfrak{e}}$.

Definition 5.6. A Cauchy pair $(\mathfrak{e}, \Theta)$ is *fully complete* if it is complete and in addition $\hat{e}_u^\sharp \in \mathfrak{X}(\Sigma)$ is complete.

The notion of fully complete Cauchy pair is convenient to obtain global results about Cauchy pairs by using completeness of $\hat{e}_u$ to identify the leaves of $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$.

Proposition 5.8. *Let $(\mathfrak{e}, \Theta)$ be a fully complete Cauchy pair on Σ with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$. The following holds:*

1. *All leaves are diffeomorphic to a model leaf given by either the plane $\mathbb{R}^2$, the cylinder or the torus.*

2. *Either all leaves are closed or all leaves are dense in Σ.*

3. *The Riemannian universal cover of $(\Sigma, h_{\mathfrak{e}})$ is isometric to $(\mathbb{R}^3, \bar{h}_{\mathfrak{e}})$ with metric $\bar{h}_{\mathfrak{e}}$ given by:*

$$\bar{h}_{\mathfrak{e}} \overset{\text{def.}}{=} e^{2\mathfrak{u}} \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{h}_x \,, \tag{5.124}$$

where x is the first Cartesian coordinate of $\mathbb{R}^3$, $\mathfrak{u} \in C^\infty(\mathbb{R}^3)$ is a smooth function and, for every $x \in \mathbb{R}$, h_x is a flat Euclidean metric on $\{x\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. If $(\mathfrak{e}, \Theta)$ is not fully complete the previous characterization is only guaranteed to hold locally.

Remark 5.13. Item (3) in the previous proposition recovers, in the specific case of four Lorentzian dimensions, items (1) and (2) in [478, Theorem 4].

Proof. *Bar* over a symbol will denote lift to the universal cover of Σ, denoted by $\bar{\Sigma}$. We prove the proposition point by point:

1. Since $\mathcal{F}_{\mathfrak{e}}$ is determined by a closed nowhere-vanishing one-form the fact that all its leaves must be diffeomorphic is classical, see [649, 651]. To motivate it, recall that the Lie derivative of $\hat{e}_u$ along $\hat{e}_u^\sharp$ is zero (note that this is not true in general for e_u and $e_u^\sharp$). Hence, the flow $(\psi_t)_{t\in\mathbb{R}}$ given by the complete vector field $\hat{e}_u^\sharp$ preserves the leaves of $\mathcal{F}_{\mathfrak{e}}$, that is, maps leaves to leaves diffeomorphically. Furthermore, for every $p, q \in \Sigma$ there exists a $t_0 \in \mathbb{R}$ such that
$$\psi_{t_0}|_{\mathcal{F}_{\mathfrak{e},p}} \colon \mathcal{F}_{\mathfrak{e},p} \to \mathcal{F}_{\mathfrak{e},q} \,. \tag{5.125}$$
Hence, all leaves of $\mathcal{F}_{\mathfrak{e}}$ are diffeomorphic and by Lemma 5.6 they must be all diffeomorphic to either the plane, the cylinder or the torus.

2. Follows from [652, Proposition 5.1].

3. The fact that the universal cover $\bar{\Sigma}$ is diffeomorphic to $\mathbb{R} \times \bar{\mathcal{F}}_{\mathfrak{e}}$, where $\bar{\mathcal{F}}_{\mathfrak{e}}$ denotes the universal cover of the typical leaf of $\mathcal{F}_{\mathfrak{e}}$ is proven in detail in [629, Proposition 8]. Furthermore, the foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ lifts to the foliation whose leaves are given by $\{x\} \times \bar{\mathcal{F}}_{\mathfrak{e}} \subset \mathbb{R} \times \bar{\mathcal{F}}_{\mathfrak{e}}$ for $x \in \mathbb{R}$. Since the typical leaf of $\mathcal{F}_{\mathfrak{e}}$ is either the plane, the cylinder or the torus, then $\bar{\mathcal{F}}_{\mathfrak{e}} = \mathbb{R}^2$ and therefore $\bar{\Sigma} = \mathbb{R}^3$. The lift $\bar{e}_u$ of e_u to $\bar{\Sigma}$ is orthogonal to $T^*\bar{\mathcal{F}}_{\mathfrak{e}} \subset T^*\bar{\Sigma}$, whence:
$$\bar{e}_u = e^{\mathfrak{u}} \mathrm{d}x \,, \qquad \mathfrak{u} \in C^\infty(\mathbb{R}^3) \,, \tag{5.126}$$
where $\mathfrak{u}$ is a function on $\mathbb{R}^3$ satisfying $\bar{\Theta}(\bar{e}_u) = -\mathrm{d}\mathfrak{u}$. Since the distribution $T\bar{\mathcal{F}}_{\mathfrak{e}} \subset T\bar{\Sigma}$ is determined by the kernel of $\bar{e}_u$ we conclude that the lift of $h_{\mathfrak{e}}$ to $\bar{\Sigma}$ can be written as follows:
$$\bar{h}_{\mathfrak{e}} \stackrel{\text{def.}}{=} e^{2\mathfrak{u}} \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{h}_x \,, \tag{5.127}$$
for a family $\{\mathfrak{h}_x\}_{x\in\mathbb{R}}$ of two-dimensional metrics on $\mathbb{R}^2$, which must be flat by Lemma 5.6. □

The leaves of the foliation $\mathcal{F}_{\mathfrak{e}}$ are all mutually diffeomorphic but a priori may not be mutually isometric since (the dual of) $\hat{e}_u$ which generates the flow that allows to identify different leaves of $\mathcal{F}_{\mathfrak{e}}$ may not be an isometry of $h_{\mathfrak{e}}$. We will refer to the type of any leaf of $\mathcal{F}_{\mathfrak{e}}$ as the *typical leaf* of $\mathcal{F}_{\mathfrak{e}}$, considered as a Riemann surface with the induced orientation. If the typical leaf of $\mathcal{F}_{\mathfrak{e}}$ is compact we obtain the following result.

Proposition 5.9. *Let $(\mathfrak{e}, \Theta)$ be a fully complete Cauchy pair on Σ with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$. If the typical leaf of $\mathcal{F}_{\mathfrak{e}}$ is a flat torus then either $\Sigma = \mathbb{R} \times T^2$ or Σ admits the structure of a fiber bundle $\pi_{\mathfrak{e}} \colon \Sigma \to S^1$ inducing $\mathcal{F}_{\mathfrak{e}}$.*

Proof. Follows directly from [653, Corollary 8.6] by using the fact that every locally trivial fibration over $\mathbb{R}$ is trivial as well as the fact that if the leaves of $\mathcal{F}_{\mathfrak{e}}$ are compact then they must be diffeomorphic to the torus. □

Lemma 5.7. *Let $(\mathfrak{e}, \Theta)$ be a complete Cauchy pair on Σ with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$. Then, Σ admits a canonical locally free action of $\mathbb{R}^2$ whose orbits are the leaves of $\mathcal{F}_{\mathfrak{e}}$.*

Proof. Consider a Cauchy pair $(\mathfrak{e}, \Theta)$ and define the map

$$\Psi\colon \mathbb{R}^2 \times \Sigma \to \Sigma\,, \qquad (t_1, t_2, p) \mapsto \Phi^{t_1}_{e_l} \circ \Phi^{t_2}_{e_n}(p)\,, \tag{5.128}$$

where $\Phi^{t_1}_{e_l}$ (respectively $\Phi^{t_2}_{e_n}$) denotes the flow generated by $e_l^\sharp$ (respectively $e_n^\sharp$) at the *time* t_1 (respectively t_2). Using that $\mathfrak{e}$ is a solution of the Cauchy differential system, we obtain:

$$[e_l^\sharp, e_n^\sharp] = \nabla^{h_\mathfrak{e}}_{e_l^\sharp} e_n^\sharp - \nabla^{h_\mathfrak{e}}_{e_n^\sharp} e_l^\sharp = 0\,, \tag{5.129}$$

hence Ψ defines a smooth action of $\mathbb{R}^2$ on Σ, which, since both e_l and e_n are nowhere vanishing, is locally free. Furthermore, the fact that $e_l^\sharp$ and $e_n^\sharp$ are complete and span $T\mathcal{F}_\mathfrak{e} \subset T\Sigma$ implies that the orbits of Φ correspond to the leaves of $\mathcal{F}_\mathfrak{e}$. □

Locally free actions of the group $\mathbb{R}^2$ on three-manifolds have been extensively studied extensively in the literature, see [654–657] and references therein, especially in relation with the problem of finding the number of nowhere vanishing and everywhere linearly independent commuting vector fields on a compact three-manifold.

Proposition 5.10. *Let $(\mathfrak{e}, \Theta)$ be a fully complete Cauchy pair on Σ such that the restriction of Θ to $T\mathcal{F}_\mathfrak{e} \subset T\Sigma$ vanishes, that is, $\Theta|_{T\mathcal{F}_\mathfrak{e}\times T\mathcal{F}_\mathfrak{e}} = 0$. Then, Σ is diffeomorphic to $T^k \times \mathbb{R}^{3-k}$ for some integer $k \in \{0, 1, 2, 3\}$.*

Proof. Let $(\mathfrak{e}, \Theta)$ be a Cauchy pair such that $\Theta|_{T\mathcal{F}_\mathfrak{e}\times T\mathcal{F}_\mathfrak{e}} = 0$. Then, $\hat{\mathfrak{e}}^\sharp$ is a global frame of commuting vector fields, which can be used to define a smooth action of $\mathbb{R}^3$ on Σ exactly as it occurred in the proof of Lemma 5.7 to define an action of $\mathbb{R}^2$. Since $\hat{\mathfrak{e}}$ is assumed to be complete, this action is transitive. The final step of the proof consist in showing that the stabilizer of the action is of the form $\mathbb{Z}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{3-k}$ acting naturally on $\mathbb{R}^3$. This is explicitly proven in [649, Chapter 4]. □

5.3.2 Complete Cauchy pairs on the universal Riemannian cover

Let $(\mathfrak{e}, \Theta)$ be a fully complete Cauchy pair on Σ. Proposition 5.8 states that the universal Riemannian cover of $(\Sigma, h_\mathfrak{e})$ is isometric to $\mathbb{R}^3$ when the latter is equipped with the metric:

$$\bar{h}_\mathfrak{e} \stackrel{\text{def.}}{=} e^{2\mathfrak{u}} \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{h}_x\,, \tag{5.130}$$

where $\mathfrak{h}_x$ is a flat metric on $\{x\} \times \mathbb{R}^2 \subset \mathbb{R}^3$ for every $x \in \mathbb{R}$. The corresponding Cauchy coframe reads

$$e_u = e^{\mathfrak{u}} \mathrm{d}x\,, \qquad e_l = e_l(x)\,, \qquad e_n = e_n(x)\,, \tag{5.131}$$

where e_l and e_n depend only on the coordinate x. A quick computation shows that the exterior derivative of this frame is given by:

$$\mathrm{d}e_u = \mathrm{d}\mathfrak{u} \wedge e_u\,, \qquad \mathrm{d}e_l = e^{-\mathfrak{u}} e_u \wedge \mathcal{L}_x e_l\,, \qquad \mathrm{d}e_n = e^{-\mathfrak{u}} e_u \wedge \mathcal{L}_x e_n\,, \tag{5.132}$$

where the symbol $\mathcal{L}_x$ denotes Lie derivative with respect to ∂_x. Plugging the previous equations into the Cauchy differential system (5.65) we obtain the following lemma.

Lemma 5.8. *A pair $(\mathfrak{e}, \Theta) \in \mathrm{Conf}(\mathbb{R}^3)$, where $\mathfrak{e}$ is given by the coframe (5.131), is a Cauchy pair if and only if the following equations are satisfied:*

$$(\mathrm{d}\mathfrak{u} - \Theta(e_u)) \wedge e_u = 0\,, \quad (\Theta(e_l) + e^{-\mathfrak{u}}\mathcal{L}_x e_l) \wedge e_u = 0\,, \quad (\Theta(e_n) + e^{-\mathfrak{u}}\mathcal{L}_x e_n) \wedge e_u = 0\,. \tag{5.133}$$

The previous lemma is used in the following theorem to *solve* the shape operator of a parallel Cauchy pair $(\mathfrak{e}, \Theta)$ defined on a connected and simply connected three-manifold Σ in terms of the Cauchy coframe $\mathfrak{e}$.

Theorem 5.3. *A pair $(\mathfrak{e}, \Theta) \in \mathrm{Conf}(\Sigma)$ is a parallel and fully complete Cauchy pair on a connected and simply connected three-manifold Σ if and only if there exist global coordinates (x, y, z) identifying $\Sigma = \mathbb{R}^3$ such that $\mathfrak{e}$ satisfies:*

$$\mathfrak{e} = (e^{\mathfrak{u}}\mathrm{d}x, e_l(x), e_n(x))\,, \quad (\mathcal{L}_x e_l)(e_n^\sharp) = (\mathcal{L}_x e_n)(e_l^\sharp)\,, \tag{5.134}$$

and in addition:

$$\Theta = (\mathfrak{F}(x)\, e^{-\mathfrak{u}} + \partial_x e^{-\mathfrak{u}})\, e_u \otimes e_u + e_u \otimes \mathrm{d}\mathfrak{u} + \mathrm{d}\mathfrak{u} \otimes e_u - \frac{1}{2} e^{-\mathfrak{u}} \mathcal{L}_x h_x\,, \tag{5.135}$$

where $\mathfrak{F} \in C^\infty(\mathbb{R})$ is a function of x.

Remark 5.14. The second equation in (5.134) is non-trivial in general and hence restricts the type of coframes that can occur as part of a parallel Cauchy pair.

Proof. Let $(\mathfrak{e}, \Theta)$ be a Cauchy pair on a connected and simply connected three-manifold Σ. The fact that there exist global coordinates (x, y, z) identifying Σ with $\mathbb{R}^3$ respect to which $\mathfrak{e}$ is given by:

$$\mathfrak{e} = (e^{\mathfrak{u}}\mathrm{d}x, e_l(x), e_n(x))\,, \tag{5.136}$$

follows directly from Proposition 5.8. On the other hand, Lemma 5.8 implies

$$\Theta(e_u) = \mathrm{d}\mathfrak{u} + f_u\, e_u\,, \quad \Theta(e_l) = f_l\, e_u - e^{-\mathfrak{u}}\mathcal{L}_x e_l\,, \quad \Theta(e_n) = f_n\, e_u - e^{-\mathfrak{u}}\mathcal{L}_x e_n\,, \tag{5.137}$$

for functions $f_u, f_l, f_n \in C^\infty(\mathbb{R}^3)$. Symmetry of Θ is equivalent to the following equations:

$$f_l = \mathrm{d}\mathfrak{u}(e_l^\sharp)\,, \qquad f_n = \mathrm{d}\mathfrak{u}(e_n^\sharp)\,, \qquad (\mathcal{L}_x e_l)(e_n^\sharp) = (\mathcal{L}_x e_n)(e_l^\sharp)\,. \tag{5.138}$$

These conditions imply that Θ must be of the form:

$$\Theta = (f_u - \mathrm{d}\mathfrak{u}(e_u^\sharp))\, e_u \otimes e_u + e_u \otimes \mathrm{d}\mathfrak{u} + \mathrm{d}\mathfrak{u} \otimes e_u - \frac{1}{2} e^{-\mathfrak{u}} \mathcal{L}_x h_x\,, \tag{5.139}$$

Furthermore, the fact that $\Theta(e_u)$ must be closed, whence exact, is equivalent to

$$\mathrm{d}(f_u\, e_u) = \mathrm{d}(f_u e^{\mathfrak{u}}) \wedge \mathrm{d}x = 0\,. \tag{5.140}$$

Therefore, $f_u e^{\mathfrak{u}} = \mathfrak{F}(x)$ for a smooth function $\mathfrak{F}$ depending exclusively on the coordinate x. Plugging this expression back in (5.139) we obtain (5.135). The converse follows by construction and can be verified explicitly by inserting (5.135) in the parallel Cauchy differential system (5.65) and (5.66). □

Remark 5.15. Theorem 5.3 recovers [478, Theorem 4] in the language of parallel Cauchy pairs and in the specific case of four Lorentzian dimensions, refining it and providing an alternative proof of the result. The refinement is contained in the *extra* information provided by the Cauchy coframe $\mathfrak{e}$, which needs to satisfy equations (5.134). On the other hand, equation (5.135) does not specify uniquely Θ but allows the freedom of choosing the arbitrary function $\mathcal{F}(x)$. This arbitrary function seems to be absent in [478, Theorem 4].

Example 5.2. Using the notation and framework established by Theorem 5.3, assume that

$$\mathfrak{h}_x = e^{2\mathfrak{w}(x)}(\mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z)\,, \tag{5.141}$$

where (x, y, z) are the Cartesian coordinates of $\mathbb{R}^3$ and $\mathfrak{w}(x)$ is a function on $\mathbb{R}^3$ depending only on the coordinate x. As defined above, $\mathfrak{h}_x$ is clearly a family of flat metrics on $\mathbb{R}^2$ parametrized by $x \in \mathbb{R}$. The corresponding parallel Cauchy coframe reads

$$\mathfrak{e} = (e^{\mathfrak{u}}\mathrm{d}x, e^{\mathfrak{w}(x)}\mathrm{d}y, e^{\mathfrak{w}(x)}\mathrm{d}z)\,, \tag{5.142}$$

One easily checks that the second equation in (5.134) is automatically satisfied. On the other hand, the corresponding parallel shape operator is given by:

$$\Theta = (\mathfrak{F}\, e^{-\mathfrak{u}} + \partial_x e^{-\mathfrak{u}})\, e_u \otimes e_u + e_u \otimes \mathrm{d}\mathfrak{u} + \mathrm{d}\mathfrak{u} \otimes e_u - \partial_x \mathfrak{w}(x)\, e^{-\mathfrak{u}} \mathfrak{h}_x\,. \tag{5.143}$$

Using the previous expression, we compute:

$$\mathrm{Tr}_{\mathfrak{e}}(\Theta) = e^{-\mathfrak{u}}(\mathfrak{F} + \partial_x \mathfrak{u} - 2\partial_x \mathfrak{w})\,, \tag{5.144}$$

$$|\Theta|^2_{\mathfrak{e}} = e^{-2\mathfrak{u}}((\mathfrak{F} + \partial_x\mathfrak{u})^2 + 2(\partial_x\mathfrak{w})^2) + 2e^{-2\mathfrak{w}}((\partial_y\mathfrak{u})^2 + (\partial_z\mathfrak{u})^2)\,. \tag{5.145}$$

In particular,

$$|\Theta|^2_{\mathfrak{e}} - \mathrm{Tr}_{\mathfrak{e}}(\Theta)^2 = 2e^{-2\mathfrak{u}}\partial_x\mathfrak{w}(x)(2(\mathfrak{F}(x) + \partial_x\mathfrak{u}) - \partial_x\mathfrak{w}(x)) + 2e^{-2\mathfrak{w}}((\partial_y\mathfrak{u})^2 + (\partial_z\mathfrak{u})^2)\,, \tag{5.146}$$

and since the scalar curvature of $h_{\mathfrak{e}}$ is given by:

$$\mathrm{R}^{\mathfrak{e}} = e^{-2\mathfrak{u}}\left(4\partial_x\mathfrak{w}\partial_x\mathfrak{u} - 4\partial_x^2\mathfrak{w} - 6(\partial_x\mathfrak{w})^2\right) - 2e^{-2\mathfrak{w}}\left((\partial_y\mathfrak{u})^2 + (\partial_z\mathfrak{u})^2 + \partial_y^2\mathfrak{u} + \partial_z^2\mathfrak{u}\right)\,, \tag{5.147}$$

we conclude that such parallel Cauchy pair $(\mathfrak{e}, \Theta)$ is constrained Ricci flat if and only if

$$2e^{2\mathfrak{w}}\left(\mathfrak{F}\partial_x\mathfrak{w} + \partial_x^2\mathfrak{w} + (\partial_x\mathfrak{w})^2\right) + e^{2\mathfrak{u}}\left(2(\partial_y\mathfrak{u})^2 + 2(\partial_z\mathfrak{u})^2 + \partial_y^2\mathfrak{u} + \partial_z^2\mathfrak{u}\right) = 0\,. \tag{5.148}$$

If the second term in the previous equation only depends on x and $\partial_x\mathfrak{w} \neq 0$ everywhere, then we can always solve it by choosing $\mathfrak{F}$ as follows:

$$\mathfrak{F} = -\frac{1}{\partial_x\mathfrak{w}}\left(\partial_x^2\mathfrak{w} + (\partial_x\mathfrak{w})^2\right) - \frac{e^{2(\mathfrak{u}-\mathfrak{w})}}{2\partial_x\mathfrak{w}}\left(2(\partial_y\mathfrak{u})^2 + 2(\partial_z\mathfrak{u})^2 + \partial_y^2\mathfrak{u} + \partial_z^2\mathfrak{u}\right)\,. \tag{5.149}$$

5.3.3 Parallel Cauchy pairs on compact three-manifolds

In this section we consider the isometry type of Cauchy pairs on closed three-manifolds, commenting briefly on the compact case with boundary.

Proposition 5.11. *Let Σ be an oriented closed three-manifold admitting a Cauchy pair $(\mathfrak{e}, \Theta)$. Then Σ is diffeomorphic to a torus bundle over S^1, that is, it is diffeomorphic to the suspension $\mathcal{X}_{\mathfrak{k}}$ of T^2 by an element $\mathfrak{k} \in \mathrm{SL}(2,\mathbb{Z})$.*

Proof. Let $(\mathfrak{e}, \Theta)$ be a Cauchy pair on Σ. By Lemma 5.7 Σ admits locally free action of $\mathbb{R}^2$. Reference [657] proves that Σ admits such an action if and only if Σ is diffeomorphic to a locally trivial torus bundle over S^1, which can always be constructed as a suspension of T^2 by an element $\mathfrak{k} \in \mathrm{SL}(2,\mathbb{Z})$ acting linearly on T^2. □

Since it will be of importance in the following, we briefly recall the suspension construction of a torus bundle over S^1, which depends on a choice of orientation preserving diffeomorphism of T^2 modulo homotopy equivalence. Since $\mathrm{Diff}(T^2)$ is homotopy equivalent to $\mathrm{SL}(2,\mathbb{Z})$ acting *linearly* on T^2, it is enough to consider elements in $\mathrm{SL}(2,\mathbb{Z})$. Let $\mathfrak{k} \in \mathrm{SL}(2,\mathbb{Z})$ and denote by $\langle \mathfrak{k} \rangle \subset \mathrm{SL}(2,\mathbb{Z})$ the cyclic group generated by the element $\mathfrak{k}$. There exists a natural properly discontinuous fixed point free action of $\langle \mathfrak{k} \rangle$ on $\mathbb{R} \times T^2$ given by:

$$\mathfrak{k} \cdot (z, v) = (z+1, \mathfrak{k}(v)) \,, \qquad (z, v) \in \mathbb{R} \times T^2 \,, \tag{5.150}$$

where $\mathfrak{k}$ acts linearly on $\mathbb{R}^2/\mathbb{Z}^2$. The suspension of $\mathbb{R} \times T^2$ by $\mathfrak{k} \in \mathrm{SL}(2,\mathbb{Z})$ is by definition the quotient:

$$\mathcal{X}_{\mathfrak{k}} = \frac{\mathbb{R} \times T^2}{\langle \mathfrak{k} \rangle} \,, \tag{5.151}$$

equipped with the projection:

$$\pi \colon \mathcal{X}_{\mathfrak{k}} \to S^1 = \mathbb{R}/\mathbb{Z} \,, \qquad [z, v] \mapsto [z] \,. \tag{5.152}$$

Equivalently, $\mathcal{X}_{\mathfrak{k}}$ can be constructed by gluing $\{0\} \times T^2$ and $\{1\} \times T^2$ in $[0,1] \times T^2$ through the diffeomorphism $\mathfrak{k} \colon T^2 \to T^2$. The element $\mathfrak{k} \in \mathrm{SL}(2,\mathbb{Z})$ determines completely the topology of $\mathcal{X}_{\mathfrak{k}}$ and in particular determines if a given foliation of $\mathcal{X}_{\mathfrak{k}}$ admits a bundle-like metric. Note that, given a Cauchy pair $(\mathfrak{e}, \Theta)$ on $\Sigma = \mathcal{X}_{\mathfrak{k}}$, the leaves of the foliation $\mathcal{F}_{\mathfrak{e}} \subset \mathcal{X}_{\mathfrak{k}}$ will not coincide in general with the fibers of $\mathcal{X}_{\mathfrak{k}}$. We summarize now two important methods for constructing foliations in $\mathcal{X}_{\mathfrak{k}}$.

- *Linear plane foliations on T^3.* Denote by $\mathrm{Diff}(S^1)$ the group of orientation preserving diffeomorphisms of S^1 and consider the three-manifold $\mathbb{R}^2 \times S^1$. Fix a representation

$$\rho = (\rho_a, \rho_b) \colon \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Diff}(S^1) \,, \tag{5.153}$$

  such that the rotational numbers $r_a \in S^1$ and $r_b \in S^1$ of $\rho_a(1)$ and $\rho_b(1)$ are both irrational and rationally independent. Then, $\rho_a(1)$ and $\rho_b(1)$ generate a subgroup of the orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$, which we denote by

$$\langle \rho_a(1), \rho_b(1) \rangle \subset \mathrm{Diff}(S^1) \,. \tag{5.154}$$

  There is a canonical fixed point free action of $\langle \rho_a(1), \rho_b(1) \rangle$ on $\mathbb{R}^2 \times S^1$ given by:

$$\rho_a(1) \cdot (x_1, x_2, \theta) = (x_1 + 1, x_2, \rho_a(1)(\theta)) \,, \tag{5.155}$$

$$\rho_b(1) \cdot (x_1, x_2, \theta) = (x_1, x_2 + 1, \rho_b(1)(\theta)) \,, \tag{5.156}$$

on the generators $\rho_a(1)$ and $\rho_b(1)$. The quotient

$$\mathcal{X}_\rho := \mathbb{R}^2 \times S^1/\langle\rho_a(1), \rho_b(1)\rangle\,, \tag{5.157}$$

of $\mathbb{R}^2 \times S^1$ by the previous action is diffeomorphic to T^3 and the plane foliation of $\mathbb{R}^2 \times S^1$ whose leaves are embedded planes $\mathbb{R}^2 \times \{\theta\} \subset \mathbb{R}^2 \times S^1$, $\theta \in S^1$, descends to a foliation by planes of $\mathbb{R}^2 \times S^1/\langle\rho_a(1), \rho_b(1)\rangle$, which is called the *suspension foliation* defined by ρ and it is denoted by

$$\mathcal{F}_\rho \subset \mathcal{X}_\rho = \mathbb{R}^2 \times S^1/\langle\rho_a(1), \rho_b(1)\rangle\,. \tag{5.158}$$

In particular, $\mathcal{X}_\rho$ admits the structure of a S^1 bundle over T^2 transverse to $\mathcal{F}_\rho$, which is obtained by the standard associated bundle construction. Note that $\rho_a(1)$ and $\rho_b(1)$ may not be rotations of S^1 by a constant angle. In general, the foliation $\mathcal{F}_\rho$ is only C^0 isomorphic to a foliation for which $\rho_a(1)$ and $\rho_b(1)$ are rotations, see [658] for a explicit counterexample. However, if $\mathcal{F}_\rho$ is defined by a non-singular closed one-form then $\mathcal{F}_\rho$ is at least C^1 isomorphic to a foliation for which $\rho_a(1)$ and $\rho_b(1)$ are rotations [658].

- *Cylinder foliations of circle bundles.* Consider the foliation $\mathcal{F}_0 \subset T^2 \times \mathbb{R}$ whose leaves are defined to be the embedded submanifolds $\{\theta_1\} \times S^1 \times \mathbb{R} \subset T^2 \times \mathbb{R} = S^1 \times S^1 \times \mathbb{R}$ for $\theta_1 \in S^1$. For every diffeomorphism $\mathfrak{f}\colon T^2 \to T^2$ preserving the foliation by standard circles $\{\theta_1\} \times S^1 \subset T^2 = S^1 \times S^1$ and such that its restriction to the first circle factor $\mathfrak{f}|_{S^1\times\{\theta_2\}} : S^1 \to S^1$ has an irrational rotation number, we define a diffeomorphism of $T^2 \times \mathbb{R}$ as follows:

$$T^2 \times \mathbb{R} \to T^2 \times \mathbb{R}\,, \qquad (\theta_1, \theta_2, x) \mapsto (\mathfrak{f}(\theta_1, \theta_2), x + 1)\,. \tag{5.159}$$

By [655, Theorem 2] and [659, Page 254 Théorème 1] $\mathfrak{f} \in \mathrm{SL}(2, \mathbb{Z})$ is conjugate to an element of the form:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})\,, \tag{5.160}$$

where $n \in \mathbb{Z}$ is an integer. Denote by $\langle\mathfrak{f}\rangle \subset \mathrm{Diff}(T^2 \times \mathbb{R})$ the cyclic subgroup of $\mathrm{Diff}(T^2 \times \mathbb{R})$ generated by the previous action, and define:

$$\mathcal{X}_\mathfrak{f} := \frac{\mathbb{R} \times T^2}{\langle\mathfrak{f}\rangle}\,, \tag{5.161}$$

to be the quotient of $\mathbb{R} \times T^2$ by $\langle\mathfrak{f}\rangle$, which defines a fiber bundle $\pi_\mathfrak{f}\colon \mathcal{X}_\mathfrak{f} \to S^1$ with projection:

$$\pi_\mathfrak{f}([\theta_1, \theta_2, x]) = [x] \in S^1\,. \tag{5.162}$$

We see that the action of $\langle\mathfrak{f}\rangle$ preserves by construction $\mathcal{F}_0$, whence $\mathcal{F}_0$ descends to a foliation $\mathcal{F}_\mathfrak{f} \subset \mathcal{X}_\mathfrak{f}$ whose fibers are all diffeomorphic to the cylinder. More explicitly, the leaves of the foliation are given by:

$$\mathrm{p}_\mathfrak{f}(\{\theta\} \times S^1 \times \mathbb{R}) \subset \mathcal{X}_\mathfrak{f}\,, \quad \theta \in S^1\,, \tag{5.163}$$

where $\mathrm{p}_\mathfrak{f}\colon T^2 \times \mathbb{R} \to \mathcal{X}_\mathfrak{f}$ denotes the canonical projection.

Proposition 5.12. *Every codimension-one foliation of $\mathcal{X}_{\mathfrak{k}}$ defined by the kernel of a nowhere vanishing closed one-form whose leaves are all diffeomorphic to either the plane $\mathbb{R}^2$ or the cylinder $\mathbb{R}^2 \backslash \{0\}$ is isomorphic to one of the foliations defined above.*

Remark 5.16. By *isomorphic foliations* we mean foliations for which there exists a C^1 diffeomorphism between their total spaces of the foliations mapping leaves to leaves diffeomorphically.

Proof. The result is proven in [659] for the case of cylinder leaves and in [658] for the case of plane leaves. □

Theorem 5.4. *Let $(\mathfrak{e}, \Theta)$ be a Cauchy pair on an oriented closed three-manifold Σ with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ and Riemannian metric $h_{\mathfrak{e}}$. Then, one and only one of the following cases occur:*

1. *$\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ is a foliation by plane leaves and there exists an isometry*

$$(\Sigma, h_{\mathfrak{e}}) = (\mathbb{R}^2 \times S^1, \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + e^{2\mathfrak{u}}\mathrm{d}\theta \otimes \mathrm{d}\theta)/\langle \rho_a(1), \rho_b(1) \rangle\,, \tag{5.164}$$

where $\rho_a(1), \rho_b(1) \in \mathrm{Diff}(S^1)$ are rotations of rationally independent constant irrational angle, respectively, and $\mathfrak{u} \in C^\infty(\mathbb{R}^2)$ is a function depending only on x_1 and x_2. In particular, Σ is diffeomorphic to T^3 and $\mathcal{F}_{\mathfrak{e}}$ is isomorphic to the foliation $\mathcal{F}_\rho$ described above.

2. *$\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ is a foliation by cylinder leaves and there exists an isometry*

$$(\Sigma, h_{\mathfrak{e}}) = (S^1 \times S^1 \times \mathbb{R}, e^{2\mathfrak{u}}\mathrm{d}\theta \otimes \mathrm{d}\theta + h)/\langle \mathfrak{f} \rangle\,, \tag{5.165}$$

where $\mathfrak{f} \in \mathrm{Diff}(T^2 \times \mathbb{R})$ is as prescribed in (5.159) and (5.160), $\mathfrak{u} \in C^\infty(S^1 \times \mathbb{R})$ is a function depending only on the second factor $S^1 \times \mathbb{R}$ above and h is a flat metric on $S^1 \times \mathbb{R}$. In particular, $\mathcal{F}_{\mathfrak{e}}$ is isomorphic to the foliation $\mathcal{F}_{\mathfrak{f}}$ previously described.

3. *$\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ is a foliation by torus leaves and $(\Sigma, h_{\mathfrak{e}})$ is a conformal Riemannian submersion over S^1 with flat fibers and whose conformal factor is determined, modulo constant multiplicative factors, by*

$$\Theta(e_u) = -\mathrm{d}\mathfrak{f}\,. \tag{5.166}$$

In particular, Σ is diffeomorphic to a torus suspension by an element $\mathfrak{t} \in \mathrm{SL}(2, \mathbb{Z})$.

Proof. We prove the statement point by point.

1. Let $(\mathfrak{e}, \Theta)$ be a parallel Cauchy pair with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ by planes. Then, and as explained above, Σ is diffeomorphic to T^3 (any compact connected 3-manifold with a foliation by planes is diffeomorphic to T^3), $(\Sigma, h_{\mathfrak{e}})$ is covered by $S^1 \times \mathbb{R}^2$ and $\mathcal{F}_{\mathfrak{e}}$ lifts to the plane foliation of $S^1 \times \mathbb{R}^2$ whose leaves are embedded planes $\{\theta\} \times \mathbb{R}^2 \subset S^1 \times \mathbb{R}^2$, $\theta \in S^1$. Hence, the lift of $h_{\mathfrak{e}}$ to $S^1 \times \mathbb{R}^2$ reads

$$(S^1 \times \mathbb{R}^2, \hat{h}_{\mathfrak{e}} = e^{2\mathfrak{u}}\mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta)\,, \tag{5.167}$$

where $\mathfrak{u}$ is a function on $S^1 \times \mathbb{R}^2$, θ is an angular coordinate on S^1 and h_θ is a family of flat metrics on $\mathbb{R}^2$ parametrized by $\theta \in S^1$. Consequently, $(\Sigma, h_{\mathfrak{e}})$ has the following isometry type:

$$(\Sigma, h_{\mathfrak{e}}) = (S^1 \times \mathbb{R}^2, e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta)/\langle \rho_a(1), \rho_b(1) \rangle \,, \tag{5.168}$$

For the metric $e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta$ to descend to Σ through the previous quotient we must have:

$$\begin{aligned} \rho_a(1)^*(e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta) = e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta \,, \\ \rho_b(1)^*(e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta) = e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_\theta \,, \end{aligned} \tag{5.169}$$

which immediately implies:

$$\mathfrak{u} \circ \rho_o(1) = \mathfrak{u} \,, \quad h_{\theta \circ \rho_o(1)} = h_\theta \,, \tag{5.170}$$

for $o = a, b$. Since $\langle \rho_a(1), \rho_b(1) \rangle$ generates a dense subgroup (recall that the action of any diffeomorphism $\chi : S^1 \to S^1$ with constant irrational rotation number has dense orbits) of S^1 this implies in turn that h_θ and $\mathfrak{u}$ are constant along S^1.

2. Let $(\mathfrak{e}, \Theta)$ be a parallel Cauchy pair with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ by cylinder leaves. Then, and as explained above, $(\Sigma, h_{\mathfrak{e}})$ is covered by $T^2 \times \mathbb{R}$ and $\mathcal{F}_{\mathfrak{e}}$ lifts to the cylinder foliation of $T^2 \times \mathbb{R}$ whose leaves are the embedded cylinders $\mathbb{R} \times S^1 \times \{\theta\} \subset T^2 \times \mathbb{R}$, $\theta \in S^1$. Hence, the lift of $h_{\mathfrak{e}}$ to $S^1 \times S^1 \times \mathbb{R}$ is given by:

$$(S^1 \times S^1 \times \mathbb{R}, \hat{h}_{\mathfrak{e}} = e^{2\mathfrak{u}} \mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + h_{\theta_1}) \,, \tag{5.171}$$

where $\mathfrak{u}$ is a function on $S^1 \times S^1 \times \mathbb{R}$, (θ_1, θ_2) are angular coordinates on $S^1 \times S^1$ and h_{θ_1} is a family of flat metrics on $S^1 \times \mathbb{R}$ parametrized by $\theta_1 \in S^1$. Then:

$$(\Sigma, h_{\mathfrak{e}}) = (S^1 \times S^1 \times \mathbb{R}, e^{2\mathfrak{u}} \mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + h_{\theta_1})/\langle \mathfrak{f} \rangle \,, \tag{5.172}$$

For the metric $e^{2\mathfrak{u}} \mathrm{d}\theta \otimes \mathrm{d}\theta + h_{\theta_1}$ to descend to Σ the group we must have:

$$\mathfrak{f}^*(e^{2\mathfrak{u}} \mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + h_{\theta_1}) = e^{2\mathfrak{u}} \mathrm{d}\theta_1 \otimes \mathrm{d}\theta_1 + h_{\theta_1} \,, \tag{5.173}$$

which, since the rotation number of $\mathfrak{f}$ is irrational, immediately implies, as in the previous case, that neither h_{θ_1} nor $\mathfrak{u}$ depend on θ_1.

3. Let $(\mathfrak{e}, \Theta)$ be a parallel Cauchy pair with associated foliation $\mathcal{F}_{\mathfrak{e}} \subset \Sigma$ by torus leaves. Since $\mathcal{F}_{\mathfrak{e}}$ has trivial holonomy and Σ is connected and compact, [653, Corollary 8.6] implies that $\mathcal{F}_{\mathfrak{e}}$ arises as the fibers of a fibration $\pi \colon \Sigma \to S^1$ and

$$T\Sigma = H \oplus V \,, \tag{5.174}$$

where $V := \ker(\mathrm{d}\pi)$ and H is spanned by $e_u^\sharp$. In particular, the vertical bundle V is spanned by $e_l^\sharp$ and $e_n^\sharp$, so the fibers of π are flat and we obtain a conformal submersion over S^1. The fact that the conformal factor $e^{\mathfrak{f}}$ is as described in the statement follows from the first equation of the parallel Cauchy differential system, namely:

$$\mathrm{d}e_u = -\mathrm{d}\mathfrak{f} \wedge e_u \,, \tag{5.175}$$

which implies $\mathrm{d}(e^{\mathfrak{f}} e_u) = 0$. Hence $e^{\mathfrak{f}} e_u$ is locally the exterior derivative of a coordinate $\hat{x}$ and the horizontal metric is locally $\mathrm{d}\hat{x} \otimes \mathrm{d}\hat{x}$. □

5.4 Left-invariant parallel Cauchy pairs on Lie groups

In this section we investigate left-invariant parallel Cauchy pairs on connected and simply connected three-dimensional Lie groups. In order to do this, we will exploit the classification of connected and simply connected three-dimensional Riemannian Lie groups developed in [660], together with the fact that every left-invariant Cauchy pair $(\mathfrak{e}, \Theta)$ defines a left-invariant metric $h_{\mathfrak{e}}$.

Let $(\mathfrak{e}, \Theta) \in \mathrm{Conf}(\Sigma)$ be a left-invariant Cauchy pair on a three-dimensional connected and simply connected Lie group $\Sigma = \mathrm{G}$, that is, $\mathfrak{e}$ is a left-invariant coframe and Θ is a left-invariant shape operator on G. Write:

$$\Theta = \sum_{a,b} \Theta_{ab}\, e_a \otimes e_b\,, \qquad \Theta_{ab} \in \mathbb{R}\,, \qquad a, b = u, l, n\,. \tag{5.176}$$

in terms of the left-invariant Cauchy coframe $\mathfrak{e} = (e_u, e_l, e_n)$. Using the previous expression for Θ, the Cauchy differential system (5.65) evaluated on $(\mathfrak{e}, \Theta)$ is equivalent to

$$\begin{gathered} \mathrm{d}e_u = (\Theta_{ul} e_l + \Theta_{un} e_n) \wedge e_u\,, \quad \mathrm{d}e_l = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u\,, \\ \mathrm{d}e_n = (\Theta_{nl} e_l + \Theta_{nn} e_n) \wedge e_u\,. \end{gathered} \tag{5.177}$$

Taking the exterior derivative of the previous equations, we obtain the corresponding *integrability conditions*:

$$\Theta_{ll}\Theta_{un} - \Theta_{ln}\Theta_{ul} = 0\,, \qquad \Theta_{ln}\Theta_{un} - \Theta_{nn}\Theta_{ul} = 0\,. \tag{5.178}$$

For further reference, we define the following quantities:

$$T := \Theta_{ll} + \Theta_{nn}\,, \qquad \Delta := \Theta_{ll}\Theta_{nn} - \Theta_{ln}^2\,, \tag{5.179}$$

which respectively correspond to the trace and determinant of Θ restricted to the distribution defined by the kernel of e_u.

Proposition 5.13. *A left-invariant Cauchy pair $(\mathfrak{e}, \Theta)$ satisfies the cohomological condition $[\Theta(e_u)] = 0$ if and only if:*

$$(\Theta_{ul}^2 + \Theta_{un}^2)\mathrm{Tr}_{\mathfrak{e}}(\Theta) = 0\,. \tag{5.180}$$

Proof. Since Σ is by assumption simply connected we have $H^1(\Sigma) = 0$ and it suffices to prove that $\Theta(e_u)$ is closed. We impose:

$$\mathrm{d}\Theta(e_u) = \Theta_{uu}\mathrm{d}e_u + \Theta_{ul}\mathrm{d}e_l + \Theta_{un}\mathrm{d}e_n = 0\,. \tag{5.181}$$

Using the parallel Cauchy differential system (5.65), the previous condition is equivalent to the following equations:

$$\Theta_{uu}\Theta_{ul} + \Theta_{ul}\Theta_{ll} + \Theta_{un}\Theta_{ln} = 0\,, \quad \Theta_{uu}\Theta_{un} + \Theta_{ul}\Theta_{ln} + \Theta_{un}\Theta_{nn} = 0\,, \tag{5.182}$$

which, upon the use of the integrability condition (5.178) of $(\mathfrak{e}, \Theta)$, are in turn equivalent to

$$\Theta_{ul}\mathrm{Tr}_{\mathfrak{e}}(\Theta) = 0\,, \qquad \Theta_{un}\mathrm{Tr}_{\mathfrak{e}}(\Theta) = 0\,. \tag{5.183}$$

These equations are satisfied if and only if $\Theta_{ul} = \Theta_{un} = 0$ or $\mathrm{Tr}_{\mathfrak{e}}(\Theta) = 0$ (or both) hold. $\square$

We consider now the case in which G is unimodular.

Lemma 5.9. *Let* $(\mathfrak{e}, \Theta) \in \mathrm{Sol}(\mathrm{G})$ *be a parallel Cauchy pair. Then, the simply connected three-dimensional group* G *is unimodular if and only if:*

$$T = \Theta_{ll} + \Theta_{nn} = 0\,, \qquad \Theta_{un} = \Theta_{ul} = 0\,. \tag{5.184}$$

Proof. A Lie group G is unimodular if and only if the adjoint map of the associated Lie algebra has vanishing trace. Since the parallel Cauchy coframe $\mathfrak{e} = (e_u, e_l, e_n)$ is left-invariant, unimodularity of G is equivalent to:

$$\begin{gathered} \mathrm{d}e_l(e_u^\sharp, e_l^\sharp) + \mathrm{d}e_n(e_u^\sharp, e_n^\sharp) = 0\,, \quad \mathrm{d}e_u(e_l^\sharp, e_u^\sharp) + \mathrm{d}e_n(e_l^\sharp, e_n^\sharp) = 0\,, \\ \mathrm{d}e_u(e_n^\sharp, e_u^\sharp) + \mathrm{d}e_l(e_n^\sharp, e_l^\sharp) = 0\,, \end{gathered} \tag{5.185}$$

which in turn is equivalent to:

$$\Theta_{ll} + \Theta_{nn} = 0\,, \qquad \Theta_{ul} = 0\,, \qquad \Theta_{un} = 0\,, \tag{5.186}$$

upon the use of the parallel Cauchy differential system (5.177). □

Proposition 5.14. *Let* $(\mathfrak{e}, \Theta)$ *be a left-invariant Cauchy pair on an unimodular Lie group* G*. Then, one and only one of the following holds:*

- $\Delta = 0$ *and* $(\mathrm{G}, h_{\mathfrak{e}})$ *is isometric to the additive abelian Lie group* $\mathbb{R}^3$ *equipped with its standard invariant flat Riemannian metric.*
- $\Delta \neq 0$ *and* Σ *is isometric to the group* $\mathrm{E}(1,1)$ *of rigid motions of two-dimensional Minkowski space equipped with a left-invariant Riemannian metric.*

Proof. We distinguish between the cases $\Delta = 0$ and $\Delta \neq 0$.

- $\Delta = 0$. Since $\Theta_{ll}\Theta_{nn} = \Theta_{ln}^2$ and we have $\Theta_{ll} + \Theta_{nn} = 0$ by unimodularity, we obtain that $\Theta_{ll} = \Theta_{nn} = \Theta_{ln} = 0$. Also, again by unimodularity, $\Theta_{ul} = \Theta_{un} = 0$, so we conclude that $\mathrm{d}\mathfrak{e} = 0$ and Σ is isomorphic to the abelian Lie group $\mathbb{R}^3$.
- $\Delta \neq 0$. By unimodularity, see equation (5.184), we have $\Theta_{ll} = -\Theta_{nn}$ and hence $\Delta < 0$. The exterior derivative of the Cauchy coframe $\mathfrak{e}$ can be then written as follows:

  $$\mathrm{d}e_u = 0\,, \quad \mathrm{d}e_l = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u\,, \quad \mathrm{d}e_n = (\Theta_{ln} e_l - \Theta_{ll} e_n) \wedge e_u\,. \tag{5.187}$$

  If $\Theta_{ll} = 0$ the previous equations reduce to

  $$\mathrm{d}e_u = 0\,, \quad \mathrm{d}e_l = \Theta_{ln} e_n \wedge e_u\,, \quad \mathrm{d}e_n = \Theta_{ln} e_l \wedge e_u\,. \tag{5.188}$$

  Since $\Delta < 0$, we have $\Theta_{ln} \neq 0$ and after rescaling e_u by Θ_{ln} we obtain:

  $$\mathrm{d}e'_u = 0\,, \quad \mathrm{d}e_l = e_n \wedge e'_u\,, \quad \mathrm{d}e_n = e_l \wedge e'_u\,. \tag{5.189}$$

  Comparing with the classification of unimodular Riemannian Lie groups [660], see also Appendix A of [661] for a concise summary, existence of such left-invariant coframe implies that G is isomorphic to the Lie group $\mathrm{E}(1,1)$. If $\Theta_{ll} \neq 0$ we consider following change of coframes:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & \sqrt{|\Delta|} \end{pmatrix} \begin{pmatrix} e_l \\ e_n \\ e_u \end{pmatrix} , \tag{5.190}$$

where

$$\sin\beta = \frac{\sqrt{2}}{2}\sqrt{1 - \frac{\Theta_{ln}}{\sqrt{|\Delta|}}} , \quad \cos\beta = \frac{\sqrt{2}}{2}\frac{\Theta_{ll}}{\sqrt{|\Delta|}}\frac{1}{\sqrt{1-\frac{\Theta_{ln}}{\sqrt{|\Delta|}}}} . \tag{5.191}$$

The exterior derivative of the transformed coframe (e_1, e_2, e_3) reads

$$\mathrm{d}e_1 = e_2 \wedge e_3 , \quad \mathrm{d}e_2 = e_1 \wedge e_3 , \quad \mathrm{d}e_3 = 0 . \tag{5.192}$$

By the classification of unimodular Riemannian Lie groups [660], existence of such left-invariant coframe implies that G is again isomorphic to the Lie group $\mathrm{E}(1,1)$, and hence we conclude. □

We consider now the case in which G is non-unimodular.

Proposition 5.15. *Let $(\mathfrak{e}, \Theta)$ be a left-invariant Cauchy pair on a non-unimodular Lie group G. Then, one and only one of the following holds:*

- *$\Delta = 0$ and $(\mathrm{G}, h_{\mathfrak{e}})$ is isometric to the Lie group $\tau_2 \oplus \mathbb{R}$ equipped with a left-invariant Riemannian metric.*
- *$\Delta \neq 0$ and $(\mathrm{G}, h_{\mathfrak{e}})$ is isometric to $\tau_{3,\mu}$ equipped with a left-invariant Riemannian metric, where μ is given by one of the following possibilities:*

  1. *If $\Theta_{ln} \neq 0$, by:*
  $$\mu = \frac{T - \mathrm{sign}(T)\sqrt{T^2 - 4\Delta}}{T + \mathrm{sign}(T)\sqrt{T^2 - 4\Delta}} . \tag{5.193}$$
  2. *If $\Theta_{ln} = 0$ and $|\Theta_{ll}| \geq |\Theta_{nn}|$, by:*
  $$\mu = \frac{\Theta_{nn}}{\Theta_{ll}} . \tag{5.194}$$
  3. *If $\Theta_{ln} = 0$ and $|\Theta_{nn}| \geq |\Theta_{ll}|$, by:*
  $$\mu = \frac{\Theta_{ll}}{\Theta_{nn}} . \tag{5.195}$$

  Recall that the possible values of μ satisfy $-1 < \mu \leq 1$, $\mu \neq 0$.

Proof. We distinguish between the cases $\Delta = 0$ and $\Delta \neq 0$.

- $\Delta = 0$. Assume first that $T = \Theta_{ll} + \Theta_{nn} = 0$. Conditions $T = 0$ and $\Delta = 0$ can hold simultaneously if and only if $\Theta_{ll} = \Theta_{nn} = \Theta_{ln} = 0$. Hence:

$$\mathrm{d}e_u = \Theta_{ul} e_l \wedge e_u + \Theta_{un} e_n \wedge e_u \,, \quad \mathrm{d}e_l = 0 \,, \quad \mathrm{d}e_n = 0 \,, \quad \Theta_{uu} = 0 \,, \tag{5.196}$$

where the last equation is equivalent to the one-form $\Theta(e_u)$ being exact. Since the coefficients Θ_{ul} and Θ_{un} cannot simultaneously vanish (otherwise G would be unimodular) defining $e_1 = e_u$, $e_2 = \Theta_{un} e_l - \Theta_{ul} e_n$, $e_3 = \Theta_{ul} e_l + \Theta_{un} e_n$ we conclude that G is isomorphic to $\tau_2 \oplus \mathbb{R}$.

If $T \neq 0$, then either $\Theta_{ll} \neq 0$ or $\Theta_{nn} \neq 0$ or both are non-vanishing. Assume $\Theta_{ll} \neq 0$ (completely analogous results hold if we consider $\Theta_{nn} \neq 0$). In this case, the integrability conditions (5.178) demand

$$\Theta_{un} = \frac{\Theta_{ln}}{\Theta_{ll}} \Theta_{ul} \,. \tag{5.197}$$

This equation, together with condition $\Delta = 0$, implies

$$\begin{aligned} \mathrm{d}e_u = \Theta_{ul}(e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n) \wedge e_u \,, \quad \mathrm{d}e_l = \Theta_{ll}(e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n) \wedge e_u \,, \\ \mathrm{d}e_n = \Theta_{ln}(e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n) \wedge e_u \,, \end{aligned} \tag{5.198}$$

which must be considered together with equation $(\Theta_{ul}^2 + \Theta_{un}^2)\mathrm{Tr}_{\mathfrak{c}}(\Theta) = 0$ to guarantee that $\Theta(e_u)$ is closed. We distinguish the following possibilities:

1. $\Theta_{ul} = \Theta_{ln} = 0$. In this case, it can be easily seen that G is isomorphic to $\tau_2 \oplus \mathbb{R}$.
2. $\Theta_{ul} = 0$ and $\Theta_{ln} \neq 0$. In this case, we obtain:

$$\mathrm{d}e_u = 0 \,, \quad \mathrm{d}e_l = \Theta_{ll}(e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n) \wedge e_u \,, \quad \mathrm{d}e_n = \Theta_{ln}(e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n) \wedge e_u \,. \tag{5.199}$$

   Defining $e_1 := e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n$, $e_2 := e_l - \frac{\Theta_{ll}}{\Theta_{ln}} e_n$ and $e_3 := T e_u$, we get:

$$\mathrm{d}e_1 = e_1 \wedge e_3 \,, \quad \mathrm{d}e_2 = \mathrm{d}e_3 = 0 \,, \tag{5.200}$$

   Hence G is isomorphic to $\tau_2 \oplus \mathbb{R}$.
3. $\Theta_{ln} = 0$, but $\Theta_{ul} \neq 0$. In this case, we find:

$$\mathrm{d}e_u = \Theta_{ul} e_l \wedge e_u \,, \quad \mathrm{d}e_l = \Theta_{ll} e_l \wedge e_u \,, \quad \mathrm{d}e_n = 0 \,. \tag{5.201}$$

   Defining $e_1 := e_l + \frac{\Theta_{ll}}{\Theta_{ul}} e_u$, $e_2 := e_l - \frac{\Theta_{ll}}{\Theta_{ul}} e_u$ and $e_3 := e_n$, we conclude that G is isomorphic to $\tau_2 \oplus \mathbb{R}$ once we impose $\Theta_{uu} = -T$ in order to satisfy $[\Theta(e_u)] = 0$.
4. $\Theta_{ln} \neq 0$ and $\Theta_{ul} \neq 0$. Define $e_2 := e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n$ and $e_3 := e_l - \frac{\Theta_{ll}}{\Theta_{ln}} e_n$. We obtain:

$$\mathrm{d}e_u = \Theta_{ul} e_2 \wedge e_u \,, \quad \mathrm{d}e_2 = T e_2 \wedge e_u \,, \quad \mathrm{d}e_3 = 0 \,. \tag{5.202}$$

   We redefine $\tilde{e}_2 = e_2 - \frac{T}{\Theta_{ul}} e_u$ and $e_1 = e_u$, we finally obtain:

$$\mathrm{d}e_1 = -\Theta_{ul} e_1 \wedge \tilde{e}_2 \,, \quad \mathrm{d}\tilde{e}_2 = 0 \,, \quad \mathrm{d}e_3 = 0 \,, \tag{5.203}$$

   so G is isomorphic to $\tau_2 \oplus \mathbb{R}$ on observing that $\Theta_{uu} = -T$ for $\Theta(e_u)$ to be closed.

- $\Delta \neq 0$. Since $\Delta \neq 0$, the only possible solution to the integrability conditions (5.178) is $\Theta_{ul} = \Theta_{un} = 0$. Hence, non-unimodularity necessarily requires that $T = \Theta_{ll} + \Theta_{nn} \neq 0$ and the parallel Cauchy differential system reduces to

$$\mathrm{d}e_u = 0\,, \quad \mathrm{d}e_l = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u\,, \quad \mathrm{d}e_n = (\Theta_{ln} e_l + \Theta_{nn} e_n) \wedge e_u\,. \tag{5.204}$$

Assume $\Theta_{ln} \neq 0$ and define a global coframe (e_1, e_2, e_3) as follows:

$$\begin{pmatrix} e_1 \\ \\ e_2 \\ \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\lambda - \Theta_{ll}}{\Theta_{ln}} & 0 \\ \\ 1 & \frac{\mu - \Theta_{ll}}{\Theta_{ln}} & 0 \\ \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} e_l \\ \\ e_n \\ \\ e_u \end{pmatrix}, \tag{5.205}$$

where:

$$\lambda = \frac{1}{2}(T + \mathrm{sign}(T)\sqrt{T^2 - 4\Delta})\,, \quad \mu = \frac{1}{2}(T - \mathrm{sign}(T)\sqrt{T^2 - 4\Delta})\,. \tag{5.206}$$

Note that $\lambda = \mu$ if and only if $\Theta_{ll} = \Theta_{nn}$ and $\Theta_{ln} = 0$, which is not possible since we are assuming $\Theta_{ln} \neq 0$. The exterior derivative of (e_1, e_2, e_3) can be shown to be given by:

$$\mathrm{d}e_1 = e_1 \wedge e_3\,, \qquad \mathrm{d}e_2 = \tilde{\mu} e_2 \wedge e_3\,, \qquad \mathrm{d}e_3 = 0\,, \tag{5.207}$$

where we defined $\tilde{\mu} = \frac{\mu}{\lambda}$. Note that $1 > |\tilde{\mu}| > 0$, since $\Theta_{ln} \neq 0$ and $\Delta \neq 0$. Hence, G is isomorphic to $\tau_{3,\tilde{\mu}}$.

If $\Theta_{ln} = 0$, the exterior derivative of the Cauchy coframe $\mathfrak{e}$ reads

$$\mathrm{d}e_u = 0\,, \quad \mathrm{d}e_l = \Theta_{ll} e_l \wedge e_u\,, \quad \mathrm{d}e_n = \Theta_{nn} e_n \wedge e_u\,. \tag{5.208}$$

Assume first that $|\Theta_{ll}| \geq |\Theta_{nn}|$. Note that $\Theta_{ll} \neq -\Theta_{nn}$ by non-unimodularity. By rescaling e_u, we obtain:

$$\mathrm{d}e_u = 0\,, \quad \mathrm{d}e_l = e_l \wedge e_u\,, \quad \mathrm{d}e_n = \frac{\Theta_{nn}}{\Theta_{ll}} e_n \wedge e_u\,. \tag{5.209}$$

Since $1 \geq \frac{\Theta_{nn}}{\Theta_{ll}} > -1$ and $\Theta_{nn} \neq 0$ (otherwise $\Delta = 0$), we conclude Σ is isomorphic to $\tau_{3,\frac{\Theta_{nn}}{\Theta_{ll}}}$. An analogous conclusion holds if $|\Theta_{nn}| \geq |\Theta_{ll}|$.

□

Proposition 5.16. *The shape operator Θ of a parallel Cauchy pair $(\mathfrak{e}, \Theta)$ on* G *is Codazzi if and only if:*

$$C_a \overset{\text{def.}}{=} e_u \otimes \Theta \circ \Theta(e_a) - \Theta(e_u) \otimes \Theta(e_a) - \delta_{ua} \Theta \circ \Theta + \Theta_{ua} \Theta = 0 \tag{5.210}$$

for every $a = u, l, n$.

Proof. We compute:

$$\nabla^{\mathfrak{e}}_{e_a} \Theta = -\Theta(e_u) \otimes \Theta(e_a) - \Theta(e_a) \otimes \Theta(e_u) + \Theta \circ \Theta(e_a) \otimes e_u + e_u \otimes \Theta \circ \Theta(e_a)\,, \tag{5.211}$$

Similarly,

$$(\nabla^{\mathfrak{e}} \Theta)(e_a) = -\Theta_{ua} \Theta + \delta_{ua} \Theta \circ \Theta + \Theta \circ \Theta(e_a) \otimes e_u - \Theta(e_a) \otimes \Theta(e_u)\,. \tag{5.212}$$

Since Θ is Codazzi if and only if $\nabla^{\mathfrak{e}}_{e_a} \Theta = (\nabla^{\mathfrak{e}} \Theta)(e_a)$ for all $a = u, l, n$, matching the previous pair of equations we obtain (5.210) and we conclude. □

Remark 5.17. It is not hard to see that:

$$C_a(e_b, e_d) = -C_b(e_a, e_d)\,, \tag{5.213}$$

for every $a, b, c = u, l, n$. We will use this identity momentarily.

Proposition 5.17. *A parallel Cauchy pair $(\mathfrak{e}, \Theta)$ on G is constrained Ricci flat if and only if:*

$$\Theta_{uu}\mathrm{Tr}_{\mathfrak{e}}\Theta = |\Theta|^2_{\mathfrak{e}}\,. \tag{5.214}$$

Proof. By Proposition 5.5, the Hamiltonian and momentum constraints for a Cauchy pair are equivalent. We consider the momentum constraint. We have $\mathrm{dTr}_{h_{\mathfrak{e}}}(\Theta) = 0$. Hence, by Lemma 5.3 the constraint Ricci-flatness condition for $(\Theta, \mathfrak{e})$ is equivalent to

$$\mathrm{div}_{\mathfrak{e}}(\Theta)(e_u) = 0\,. \tag{5.215}$$

Using Equation (5.211) we compute:

$$\mathrm{div}_{\mathfrak{e}}\Theta = \sum_a (\nabla^{\mathfrak{e}}_{e_a}\Theta)(e_a) = -\mathrm{Tr}_{\mathfrak{e}}(\Theta)\Theta(e_u) + |\Theta|^2_{\mathfrak{e}} e_u\,, \tag{5.216}$$

and therefore we conclude. □

Lemma 5.10. *The shape operator Θ of a parallel Cauchy pair $(\mathfrak{e}, \Theta)$ is Codazzi if and only if it satisfies one of the following conditions:*

- $\Theta_{ul} = \Theta_{un} = \Theta_{ln} = 0$, $\Theta^2_{ll} = \Theta_{ll}\Theta_{uu}$, $\Theta^2_{nn} = \Theta_{nn}\Theta_{uu}$.
- $\Theta(e_u) = Te_u$, $\Delta = 0$.

Proof. Let $C_a \in \Gamma(T^*\mathrm{G} \otimes T^*\mathrm{G})$ denote the tensor defined in Proposition 5.16. Remark 5.17 states that the only non-trivial and independent components are those corresponding to $C_a(e_u, e_l)$, $C_a(e_u, e_n)$ and $C(e_l, e_n)$. Imposing these components to vanish we obtain:

$$2\Theta^2_{ul} + \Theta^2_{ll} + \Theta^2_{ln} - \Theta_{uu}\Theta_{ll} = 0\,, \quad 2\Theta^2_{un} + \Theta^2_{nn} + \Theta^2_{ln} - \Theta_{uu}\Theta_{nn} = 0\,,$$
$$2\Theta_{ul}\Theta_{un} + \Theta_{nn}\Theta_{ln} + \Theta_{ln}\Theta_{ll} - \Theta_{uu}\Theta_{ln} = 0\,, \tag{5.217}$$

In order to solve them we impose the cohomological condition as stated in Proposition 5.13. Since the cohomological condition is satisfied if either $\Theta_{ul} = \Theta_{un} = 0$ or $\Theta_{uu} = -T$, we distinguish between these two cases:

- $\Theta_{ul} = \Theta_{un} = 0$. Let us split this case into two subcategories:
  - $\Theta_{ln} = 0$. One notices that the equations reduce directly to $\Theta_{uu}\Theta_{nn} = \Theta^2_{nn}$ and $\Theta_{uu}\Theta_{ll} = \Theta^2_{ll}$.
  - $\Theta_{ln} \neq 0$. In such a case, from the last equation of (5.217) one finds $\Theta_{uu} = T$ and, upon substitution in the remaining equations they become linearly dependent and equivalent to the condition $\Delta = 0$.

- $\Theta_{uu} = -T$. In such a case, by summing the first and the second equations of (5.217) and performing explicitly the substitution $\Theta_{uu} = -T$, we find

$$2\Theta_{ul}^2 + 2\Theta_{un}^2 + (\Theta_{ll} + \Theta_{nn})^2 + 2\Theta_{ln}^2 + \Theta_{ll}^2 + \Theta_{nn}^2 = 0\,. \tag{5.218}$$

  This implies $\Theta_{ul} = \Theta_{un} = \Theta_{ll} = \Theta_{nn} = \Theta_{ln} = \Theta_{uu} = 0$, which brings us to the previous bullet-point. □

We elaborate now on the results of the previous discussion in order to obtain a full classification result about left-invariant parallel Cauchy pairs $(\mathfrak{e}, \Theta)$ on connected and simply connected three-dimensional Lie groups, characterizing those which are in addition Codazzi or constrained Ricci flat. Collecting all results from Propositions 5.14 and 5.15 and bearing in mind Proposition 5.17 and Lemma 5.10, we obtain the following result.

Theorem 5.5. *A connected and simply-connected Lie group* G *admits left-invariant parallel Cauchy pairs (respectively constrained Ricci flat parallel Cauchy pairs or a Codazzi parallel Cauchy pairs) if and only if* G *is isomorphic to one of the Lie groups listed in the Table 5.1. If that is the case, a left-invariant shape operator* Θ *belongs to a Cauchy pair* $(\mathfrak{e}, \Theta)$ *for certain left-invariant coframe* $\mathfrak{e}$ *if and only if* Θ *is of the form listed at Table 5.1 when written in terms of* $\mathfrak{e} = (e_u, e_l, e_n)$.

The previous theorem will be used extensively in the next section. We have the following corollary.

Corollary 5.3. *Let* G *be a connected and simply connected Lie group with a left-invariant Cauchy pair. Then the isomorphism type of* G *is prescribed by* T, Δ *and* $\lambda = \sqrt{\Theta_{ul}^2 + \Theta_{un}^2}$ *as follows:*

- *If* $T = \Delta = \lambda = 0$, *then* $\mathrm{G} \simeq \mathbb{R}^3$.
- *If* $T = \lambda = 0$ *but* $\Delta \neq 0$, *then* $\mathrm{G} \simeq \mathrm{E}(1,1)$.
- *If* $\Delta = 0$ *but* $\lambda^2 + T^2 \neq 0$, *then* $\mathrm{G} \simeq \tau_2 \oplus \mathbb{R}$.
- *If* $T, \Delta \neq 0$ *and* $\lambda = 0$, *then* $\mathrm{G} \simeq \tau_{3,\mu}$.

Observe that the case $\lambda \neq 0$ *and* $\Delta \neq 0$ *is not allowed.*

Noe that we are using standard notation for the groups G as explained for example in [661, Appendix A].

5.5 Left-invariant parallel spinor flows

In this section we introduce the notion of left-invariant parallel spinor flow and solve it explicitly.

G	Cauchy parallel pair	Constrained Ricci flat	Codazzi
$\mathbb{R}^3$	$\Theta = \Theta_{uu} e_u \otimes e_u$	$\Theta = \Theta_{uu} e_u \otimes e_u$	$\Theta = \Theta_{uu} e_u \otimes e_u$
$\mathrm{E}(1,1)$	$\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n, \quad \Theta_{ll} = -\Theta_{nn}$	Not allowed	Not allowed
$\tau_2 \oplus \mathbb{R}$	$\Theta = (\Theta_{ul} e_l + \Theta_{un} e_n) \odot e_u$ $\Theta_{ul}^2 + \Theta_{un}^2 \neq 0$	Not allowed	Not allowed
	$\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n,$ $T \neq 0\,, \Delta = 0$	$\Theta = T e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n,$ $T \neq 0\,, \Delta = 0$	$\Theta = T e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n,$ $T \neq 0\,, \Delta = 0$
	$\Theta = -T e_u \otimes e_u + \Theta_{ul} e_u \odot e_l + \Theta_{ll} e_l \otimes e_l\,, \quad \Theta_{ul}, \Theta_{nn} \neq 0$	Not allowed	Not allowed
	$\Theta = -T e_u \otimes e_u + \Theta_{un} e_u \odot e_n + \Theta_{nn} e_n \otimes e_n\,, \quad \Theta_{un}, \Theta_{ll} \neq 0$	Not allowed	Not allowed
	$\Theta = -T e_u \otimes e_u + \Theta_{ul} e_u \odot e_l + \Theta_{un} e_u \odot e_n + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n,\ \Theta_{ln}(\Theta_{ul}^2 + \Theta_{un}^2) \neq 0\,,$ $\Theta_{nn} = \dfrac{\Theta_{un}}{\Theta_{ul}} \Theta_{ln}\,, \Theta_{ll} = \dfrac{\Theta_{ul}}{\Theta_{un}} \Theta_{ln}$	Not allowed	Not allowed
$\tau_{3,\mu}$	$\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n, \quad T, \Delta \neq 0$	$\Theta = \left(\frac{T^2 - 2\Delta}{T}\right) e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j$ $i,j = l,n, \quad T, \Delta \neq 0$	Not allowed

Table 5.1: Classification of left-invariant Cauchy pairs, indicating if they are constrained Ricci flat and Codazzi. In the case $\mathrm{G} \simeq \tau_{3,\mu}$, μ is equal to (5.193) (if $\Theta_{ln} \neq 0$), (5.194) (if $\Theta_{ln} = 0$ and $|\Theta_{ll}| \geq \Theta_{nn}$) or (5.195) (if $\Theta_{ln} = 0$ and $|\Theta_{nn}| \geq \Theta_{ll}$).

5.5.1 Reformulation

Let G be a simply connected three-dimensional Lie-group. We say that a parallel spinor flow $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ defined on G is *left-invariant* if both β_t and $\mathfrak{e}^t$ are left-invariant for every $t \in \mathcal{I}$. The latter condition immediately implies that $h_{\mathfrak{e}^t}$ is a left-invariant Riemannian metric and β_t is constant for every $t \in \mathcal{I}$. Let $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ be a family of left-invariant coframes on G. Any square matrix $\mathcal{A} \in \mathrm{Mat}(3,\mathbb{R})$ acts naturally on $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ as follows:

$$\mathcal{A}(\mathfrak{e}^t) := \begin{pmatrix} \sum_b \mathcal{A}_{ub} e^t_b \\ \sum_b \mathcal{A}_{lb} e^t_b \\ \sum_b \mathcal{A}_{nb} e^t_b \end{pmatrix} \tag{5.219}$$

where we label the entries $\mathcal{A}_{ab}$ of $\mathcal{A}$ by the indices $a, b = u, l, n$. As a direct consequence of Theorem 5.1 we have the following result.

Proposition 5.18. *A simply connected three-dimensional Lie group* G *admits a left-invariant parallel spinor flow if and only if there exists a smooth family of non-zero constants* $\{\beta_t\}_{t\in\mathcal{I}}$ *and a family* $\{\mathfrak{e}^t\}_{t\in\mathcal{I}}$ *of left-invariant coframes on* G *satisfying the following differential system:*

$$\partial_t\mathfrak{e}^t + \beta_t\Theta_t(\mathfrak{e}^t) = 0\,, \quad \mathrm{d}\mathfrak{e}^t = \Theta_t(\mathfrak{e}^t)\wedge e^t_u\,, \quad \partial_t(\Theta_t(e^t_u)) = 0\,, \quad \mathrm{d}\Theta_t(e^t_u) = 0\,, \tag{5.220}$$

to which we will refer as the left-invariant (real) parallel spinor flow equations.

We will refer to solutions $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ of the left-invariant parallel spinor flow equations as *left-invariant parallel spinor flows*. Given a left-invariant parallel spinor flow $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$, we write:

$$\Theta^t = \sum_{a,b}\Theta^t_{ab}e^t_a\otimes e^t_n\,, \qquad a,b = u,l,n\,, \tag{5.221}$$

in terms of uniquely defined functions (Θ^t_{ab}) on $\mathcal{I}$.

Lemma 5.11. *Let* $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ *be a left-invariant parallel spinor flow. The following equations hold:*

$$\partial_t\Theta^t_{uu} = \beta_t((\Theta^t_{uu})^2 + (\Theta^t_{ul})^2 + (\Theta^t_{un})^2)\,, \quad \partial_t\Theta^t_{ul} = \partial_t\Theta^t_{un} = 0\,, \tag{5.222}$$

$$\partial_t\Theta^t_{ll} = \beta_t\Theta^t_{ll}\Theta^t_{uu} - \beta_t(\Theta^t_{ul})^2\,, \quad \partial_t\Theta^t_{ln} = \beta_t\Theta^t_{ln}\Theta^t_{uu} - \beta_t\Theta^t_{un}\Theta^t_{ul}\,, \tag{5.223}$$

$$\partial_t\Theta^t_{nn} = \beta_t\Theta^t_{nn}\Theta^t_{uu} - \beta_t(\Theta^t_{un})^2\,, \quad \Theta^t_{ln}\Theta^t_{ul} = \Theta^t_{ll}\Theta^t_{un}\,, \quad \Theta^t_{ln}\Theta^t_{un} = \Theta^t_{nn}\Theta^t_{ul}\,, \tag{5.224}$$

$$\Theta^t_{ll}\Theta^t_{ul} + \Theta^t_{ln}\Theta^t_{un} + \Theta^t_{ul}\Theta^t_{uu} = 0\,, \quad \Theta^t_{ln}\Theta^t_{ul} + \Theta^t_{nn}\Theta^t_{un} + \Theta^t_{un}\Theta^t_{uu} = 0\,. \tag{5.225}$$

In particular, $\Theta^t_{ul} = \Theta_{ul}$ *and* $\Theta^t_{un} = \Theta_{un}$ *for some constants* $\Theta_{ul}, \Theta_{un}\in\mathbb{R}$.

Proof. A direct computation shows that equation $\partial_t(\Theta_t(e^t_u)) = 0$ is equivalent to

$$\partial_t\Theta^t_{ub} = \beta_t\Theta^t_{ua}\Theta^t_{ab}\,. \tag{5.226}$$

On the other hand, equation $\mathrm{d}\Theta_t(e^t_u) = 0$ is equivalent to

$$\Theta^t_{ua}\Theta^t_{al} = 0\,, \qquad \Theta^t_{ua}\Theta^t_{an} = 0\,. \tag{5.227}$$

The previous equations can be combined into the following equivalent conditions:

$$\partial_t\Theta^t_{uu} = \beta_t((\Theta^t_{uu})^2 + (\Theta^t_{ul})^2 + (\Theta^t_{un})^2)\,, \quad \partial_t\Theta^t_{ul} = \partial_t\Theta^t_{un} = 0\,, \tag{5.228}$$

$$\Theta^t_{ll}\Theta^t_{ul} + \Theta^t_{ln}\Theta^t_{un} + \Theta^t_{ul}\Theta^t_{uu} = 0\,, \quad \Theta^t_{ln}\Theta^t_{ul} + \Theta^t_{nn}\Theta^t_{un} + \Theta^t_{un}\Theta^t_{uu} = 0\,, \tag{5.229}$$

which recover (5.222) and (5.225). Similarly, equation $\mathrm{d}(\Theta_t(\mathfrak{e}^t)\wedge e^t_u) = 0$ is equivalent to

$$\Theta^t_{ln}\Theta^t_{ul} = \Theta^t_{ll}\Theta^t_{un}\,, \qquad \Theta^t_{ln}\Theta^t_{un} = \Theta^t_{nn}\Theta^t_{ul}\,, \tag{5.230}$$

which yields the two rightmost equations in (5.224). We take now the exterior derivative of the first equation in (5.220) and combine the result with the second equation in (5.220):

$$\begin{aligned}\mathrm{d}(\partial_t e^t_a + \beta_t\Theta_t(e^t_a)) &= \partial_t(\Theta^t_{ab}e^t_b\wedge e^t_u) + \beta_t\Theta^t_{ab}\Theta^t_{bc}e^t_c\wedge e^t_u\\ &= (\partial_t\Theta^t_{ab}\delta_{uc} - \beta_t\Theta^t_{ab}\Theta^t_{uc})e^t_b\wedge e^t_c = 0\,.\end{aligned} \tag{5.231}$$

Expanding the previous equation we obtain (5.223) and the first equation in (5.224) and we conclude. □

Remark 5.18. We will refer to the equations of Lemma 5.11 as the *integrability conditions* of the left-invariant parallel spinor flow.

The following observation is crucial in order to *decouple* the left-invariant parallel spinor flow equations.

Lemma 5.12. *A pair $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is a left-invariant parallel spinor flow if and only if there exists a family of left-invariant two-tensors $\{\mathcal{K}_t\}_{t\in\mathcal{I}}$ such that the following equations are satisfied:*

$$\partial_t\mathfrak{e}^t + \beta_t\mathcal{K}_t(\mathfrak{e}^t) = 0\,, \quad \mathrm{d}\mathfrak{e}^t = \mathcal{K}_t(\mathfrak{e}^t)\wedge e^t_u\,, \quad \partial_t(\mathcal{K}_t(e^t_u)) = 0\,, \quad \mathrm{d}(\mathcal{K}_t(e^t_u)) = 0\,. \tag{5.232}$$

Proof. The *only if* direction follows immediately from the definition of left-invariant parallel Cauchy pair by taking $\{\mathcal{K}_t\}_{t\in\mathcal{I}} = \{\Theta_t\}_{t\in\mathcal{I}}$. For the *if* direction we simply compute:

$$\Theta_t = -\frac{1}{2\beta_t}\partial_t h_{\mathfrak{e}^t} = -\frac{1}{2\beta_t}((\partial_t e^t_a)\otimes e^t_a + e^t_a\otimes(\partial_t e^t_a)) = \mathcal{K}_t\,, \tag{5.233}$$

hence equations (5.220) are satisfied and $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is a left-invariant parallel spinor flow. □

By the previous Lemma we promote the components of $\{\Theta_t\}_{t\in\mathcal{I}}$ with respect to the basis $\{\mathfrak{e}^t\}_{t\in\mathcal{I}}$ to be independent variables of the left-invariant parallel spinor flow equations (5.220). Within this interpretation, the variables of left-invariant parallel spinor flow equations consist of triples $\{\beta_t, \mathfrak{e}^t, \Theta^t_{ab}\}_{t\in\mathcal{I}}$, where $\{\Theta^t_{ab}\}_{t\in\mathcal{I}}$ is a family of symmetric matrices. On the other hand, the integrability conditions of Lemma 5.11 are interpreted as a system of equations for a pair $\{\beta_t, \Theta^t_{ab}\}_{t\in\mathcal{I}}$. In particular, the first equation in (5.220) is linear in the variable $\mathfrak{e}^t$ and can be conveniently rewritten as follows. For any family of coframes $\{\mathfrak{e}^t\}_{t\in\mathcal{I}}$, set $\mathfrak{e} = \mathfrak{e}^0$ and consider the unique smooth path

$$\mathcal{U}^t\colon \mathcal{I}\to \mathrm{Gl}_+(3,\mathbb{R})\,, \qquad t\mapsto \mathcal{U}^t\,, \tag{5.234}$$

such that $\mathfrak{e}^t = \mathcal{U}^t(\mathfrak{e})$, where $\mathrm{Gl}_+(3,\mathbb{R})$ denotes the identity component in the general linear group $\mathrm{Gl}(3,\mathbb{R})$. More explicitly,

$$\mathfrak{e}^t_a = \sum_b \mathcal{U}^t_{ab}\mathfrak{e}_b\,, \qquad a,b = u,l,n\,, \tag{5.235}$$

where $\mathcal{U}^t_{ab}\in C^\infty(\mathrm{G})$ are the components of $\mathcal{U}^t$. Plugging $\mathfrak{e}^t = \mathcal{U}^t(\mathfrak{e})$ in the first equation in (5.220) we obtain the following equivalent equation:

$$\partial_t\mathcal{U}^t_{ac} + \beta_t\Theta^t_{ab}\mathcal{U}^t_{bc} = 0\,, \quad a,b,c = u,l,n\,, \tag{5.236}$$

with initial condition $\mathcal{U}^0 = \mathrm{Id}$.

A necessary condition for a solution $\{\beta_t, \Theta^t_{ab}\}_{t\in\mathcal{I}}$ of the integrability conditions to arise from an honest left-invariant parallel spinor pair is the existence of a left-invariant coframe $\mathfrak{e}$ on Σ such that $(\mathfrak{e},\Theta)$ is a Cauchy pair, where $\Theta = \Theta^0_{ab}e_a\otimes e_b$. Consequently we define the set $\mathbb{I}(\Sigma)$ of *admissible* solutions to the integrability equations as the set of pairs $(\{\beta_t, \Theta^t_{ab}\}_{t\in\mathcal{I}}, \mathfrak{e})$ such that $\{\beta_t, \Theta^t_{ab}\}_{t\in\mathcal{I}}$ is a solution to the integrability equations and $(\mathfrak{e},\Theta)$ is a left-invariant parallel Cauchy pair.

Proposition 5.19. *There exists a natural bijection $\varphi\colon \mathbb{I}(\Sigma) \to \mathcal{P}(\Sigma)$ which maps every pair:*

$$(\left\{\beta_t, \Theta^t_{ab}\right\}_{t\in\mathcal{I}}, \mathfrak{e}) \in \mathbb{I}(\Sigma)\,, \tag{5.237}$$

to the pair $\left\{\beta_t, \mathfrak{e}^t = \mathcal{U}^t(\mathfrak{e})\right\}_{t\in\mathcal{I}} \in \mathcal{P}(\Sigma)$, where $\left\{\mathcal{U}^t\right\}_{t\in\mathcal{I}}$ is the unique solution of (5.236) with initial condition $\mathcal{U}^0 = \mathrm{Id}$.

Remark 5.19. The inverse of φ maps every left-invariant parallel spinor flow $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ to the pair $(\left\{\beta_t, \Theta^t_{ab}\right\}, \mathfrak{e})$, where Θ^t_{ab} are the components of the shape operator associated to $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ in the basis $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ and $\mathfrak{e} = \mathfrak{e}^0$.

Proof. Let $(\left\{\beta_t, \Theta^t_{ab}\right\}, \mathfrak{e}) \in \mathbb{I}(\Sigma)$ and let $\left\{\mathcal{U}^t\right\}_{t\in\mathcal{I}}$ be the solution of (5.236) with initial condition $\mathcal{U}^0 = \mathrm{Id}$, which exists and is unique on $\mathcal{I}$ by standard ODE theory [662, Theorem 5.2]. We need to prove that $\left\{\beta_t, \mathfrak{e}^t = \mathcal{U}^t(\mathfrak{e})\right\}_{t\in\mathcal{I}}$ is a left-invariant parallel spinor flow. Since $\left\{\mathcal{U}^t\right\}_{t\in\mathcal{I}}$ satisfies (5.236) for the given $\left\{\beta_t, \Theta^t_{ab}\right\}$, it follows that $\Theta^t = \Theta^t_{ab} e^t_a \otimes e^t_b$ is the shape operator associated to $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ whence the first equation in (5.220) is satisfied. On the other hand, the third and fourth equations in (5.220) are immediately implied by the integrability conditions satisfied by $\left\{\beta_t, \Theta^t_{ab}\right\}$. Regarding the second equation in (5.220), we observe that the integrability conditions contain the equation $\mathrm{d}(\Theta_t(\mathfrak{e}^t) \wedge e^t_u) = 0$ and thus

$$\mathrm{d}\mathfrak{e}^t = \Theta^t(\mathfrak{e}^t) \wedge e^t_u + \mathfrak{w}^t\,, \tag{5.238}$$

where $\{\mathfrak{w}^t\}_{t\in\mathcal{I}}$ is a family of triplets of closed two-forms on Σ. Taking the time derivative of the previous equations, plugging the exterior derivative of the first equation in (5.220) and using again the integrability conditions, we obtain that $\mathfrak{w}^t$ satisfies the following differential equation:

$$\partial_t \mathfrak{w}^t_a = -\beta_t \Theta^t_{ad} \mathfrak{w}^t_d\,, \tag{5.239}$$

with initial condition $\mathfrak{w}^0 = \mathfrak{w}$. Restricting equation (5.238) to $t = 0$ it follows that $\mathfrak{w}$ satisfies

$$\mathrm{d}\mathfrak{e} = \Theta(\mathfrak{e}) \wedge e_u + \mathfrak{w}\,, \tag{5.240}$$

Since by assumption $(\mathfrak{e}, \Theta)$ is left-invariant Cauchy pair, the previous equation is satisfied if and only if $\mathfrak{w} = 0$ whence $\mathfrak{w}^t = 0$ by uniqueness of solutions of the linear differential equation (5.239). Therefore, the second equation in (5.220) follows and φ is well defined. The fact that φ is in addition a bijection follows directly by Remark 5.19 and hence we conclude. □

Corollary 5.4. *A pair $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ is a parallel spinor flow if and only if $(\left\{\beta_t, \Theta^t_{ab}\right\}, \mathfrak{e})$ is an admissible solution to the integrability equations.*

Therefore, solving the left-invariant parallel spinor flow is equivalent to solving the integrability conditions with initial condition Θ_{ab} being part of a left-invariant parallel Cauchy pair $(\mathfrak{e}, \Theta)$. We remark that $\{\beta_t\}_{t\in\mathcal{I}}$ is of no relevance locally since it can be eliminated through a reparametrization of time after possibly shrinking $\mathcal{I}$. However, regarding the long time existence of the flow as well as for applications to the construction of four-dimensional Lorentzian metrics it is convenient to keep track of $\mathcal{I}$, whence we maintain $\{\beta_t\}_{t\in\mathcal{I}}$ in the equations.

For further reference we define a *quasi-diagonal* left-invariant parallel spinor flow as one for which $\lambda = \sqrt{\Theta_{ul}^2 + \Theta_{un}^2} = 0$. Since the function $t \to \int_0^t \beta_\tau \mathrm{d}\tau$ is going to be a common occurrence in the following, we define

$$\mathcal{B}_t := \int_0^t \beta_\tau \mathrm{d}\tau\,. \tag{5.241}$$

We distinguish now between the cases $\lambda = 0$ and $\lambda \neq 0$.

Lemma 5.13. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a quasi-diagonal left-invariant parallel spinor flow. Then, the only non-zero components of Θ^t are:*

$$\Theta^t_{uu} = \frac{\Theta_{uu}}{1 - \Theta_{uu}\mathcal{B}_t}\,, \quad \Theta^t_{ij} = \frac{\Theta_{ij}}{1 - \Theta_{uu}\mathcal{B}_t}\,, \qquad i,j = l,n\,, \tag{5.242}$$

where Θ^t is the shape operator associated to $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ and $\Theta = \Theta^0$. Furthermore, every such Θ^t satisfies the integrability equations with quasi-diagonal initial data.

Proof. Setting $\Theta_{ul} = \Theta_{un} = 0$ in the integrability conditions we obtain the following equations:

$$\partial_t \Theta^t_{uu} = \beta_t (\Theta^t_{uu})^2\,, \quad \partial_t \Theta^t_{ij} = \beta_t \Theta^t_{ij}\Theta^t_{uu}\,, \qquad i,j = l,n\,, \tag{5.243}$$

whose general solution is given in the statement of the lemma. □

Remark 5.20. Let $\Theta_{uu} \neq 0$ and define t_0 to be the real number (in case it exists) with the smallest absolute value such that

$$\int_0^{t_0} \beta_\tau \mathrm{d}\tau = \Theta_{uu}^{-1}\,. \tag{5.244}$$

Then the maximal interval on which Θ^t is defined is $\mathcal{I} = (-\infty, t_0)$ if $\Theta_{uu} > 0$ and $\mathcal{I} = (t_0, \infty)$ if $\Theta_{uu} < 0$. This is also the maximal interval on which the left-invariant parallel spinor flow in the quasi-diagonal case can be defined. If such t_0 does not exist, then $\mathcal{I} = \mathbb{R}$.

We consider now the non-quasi-diagonal case $\lambda \neq 0$. Given a pair $\left\{\beta_t, \Theta^t_{ab}\right\}_{t\in\mathcal{I}}$, we introduce for convenience the following function:

$$\mathcal{I} \ni t \mapsto y_t = \lambda\,\mathcal{B}_t + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right]\,, \tag{5.245}$$

where Θ_{ab} are the components of Θ in the basis $\mathfrak{e}$.

Lemma 5.14. *A pair $\left\{\beta_t, \Theta^t_{ab}\right\}_{t\in\mathcal{I}}$ satisfies the integrability equations with non-quasi-diagonal initial value Θ_{ab} if and only if:*

$$\Theta^t_{uu} = \lambda \tan\left[y_t\right]\,, \quad \Theta^t_{ul} = \Theta_{ul}\,, \quad \Theta^t_{un} = \Theta_{un}\,, \tag{5.246}$$

$$\Theta^t_{ij} = c_{ij}\sec\left[y_t\right] - \tfrac{\Theta_{ui}\Theta_{uj}}{\lambda}\tan\left[y_t\right]\,, \qquad i,j = l,n\,, \tag{5.247}$$

where $c_{ll}, c_{nn}, c_{ln} \in \mathbb{R}$ are real constants given by:

$$c_{ll} = \frac{\Theta_{ll}\lambda^2 + \Theta_{ul}^2\Theta_{uu}}{\lambda\sqrt{\lambda^2 + \Theta_{uu}^2}}\,, \quad c_{nn} = \frac{\Theta_{nn}\lambda^2 + \Theta_{un}^2\Theta_{uu}}{\lambda\sqrt{\lambda^2 + \Theta_{uu}^2}}\,, \quad c_{ln} = \frac{\Theta_{ln}\lambda^2 + \Theta_{ul}\Theta_{un}\Theta_{uu}}{\lambda\sqrt{\lambda^2 + \Theta_{uu}^2}}\,, \tag{5.248}$$

such that the following algebraic equations are satisfied:

$$\Theta_{ln}\Theta_{ul} = \Theta_{ll}\Theta_{un}\,, \quad \Theta_{nn}\Theta_{ul} = \Theta_{ln}\Theta_{un}\,,$$
$$\Theta_{ln}\Theta_{un} + \Theta_{ul}(\Theta_{ll} + \Theta_{uu}) = 0\,, \quad \Theta_{ln}\Theta_{ul} + \Theta_{un}(\Theta_{nn} + \Theta_{uu}) = 0\,. \tag{5.249}$$

Remark 5.21. Note that equations (5.249) form an algebraic system for the entries of the initial condition Θ, therefore restricting the allowed initial data that can be used to solve the integrability conditions. This is a manifestation of the fact that the initial data of the parallel spinor flow is constrained by the parallel Cauchy equations. The latter were solved and classified in the left-invariant case in Theorem 5.5 (see Table 5.1), and its solutions can be easily verified to satisfy equations (5.249) automatically.

Proof. By Lemma 5.11 we have $\partial_t\Theta^t_{ul} = \partial_t\Theta^t_{un} = 0$ whence $\Theta^t_{ul} = \Theta_{ul}, \Theta^t_{un} = \Theta_{un}$ for some real constants $\Theta_{ul}, \Theta_{un} \in \mathbb{R}$. Plugging these constants into the first equation of Lemma 5.11 it becomes immediately integrable with solution:

$$\Theta^t_{uu} = \lambda\tan\left[\lambda(\mathcal{B}_t + k_1)\right]\,, \tag{5.250}$$

for a certain constant $k_1 \in \mathbb{R}$. Imposing $\Theta^0_{uu} = \Theta_{uu}$ we obtain:

$$k_1 = \frac{1}{\lambda}(\arctan\left[\frac{\Theta_{uu}}{\lambda}\right] + n\pi)\,, \qquad n \in \mathbb{Z}\,, \tag{5.251}$$

and the expression for Θ^t_{uu} follows. Plugging now $\Theta^t_{uu} = \lambda\tan[y_t]$ in the remaining differential equations of Lemma 5.11 they can be directly integrated, yielding the expressions in the statement after imposing $\Theta^0_{ab} = \Theta_{ab}$. Plugging the explicit expressions for Θ^t_{ab} in the algebraic equations of Lemma 5.11, these can be equivalently reformulated as the algebraic system (5.249) for Θ^t_{ab} at $t = 0$ and we conclude. □

Remark 5.22. Let $t_- < 0$ denote the largest value for which $\lambda\mathcal{B}_{t_-} + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right] = -\frac{\pi}{2}$ and let $t_+ > 0$ denote the smallest value for which $\lambda\mathcal{B}_{t_+} + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right] = \frac{\pi}{2}$ (if t_-, t_+ or both do not exist, we take by convention $t_\pm = \pm\infty$). Then, the maximal interval of definition on which Θ^t is defined is $\mathcal{I} = (t_-, t_+)$.

5.5.2 Classification of left-invariant spinor flows

Proposition 5.11 states that $\Theta^t_{ul} = \Theta_{ul}$ and $\Theta^t_{un} = \Theta_{un}$ for constants $\Theta_{ul}, \Theta_{un} \in \mathbb{R}$. Therefore, we proceed to classify left-invariant parallel spinor flows in terms of the possible values of Θ_{ul} and Θ_{un}. We begin with the classification of quasi-diagonal left-invariant parallel spinor flows, characterized by the condition $\Theta_{ul} = \Theta_{un} = 0$, that is, $\lambda = 0$.

Proposition 5.20. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a quasi-diagonal left-invariant parallel spinor flow with initial data $(\mathfrak{e}, \Theta)$ satisfying $\Theta_{uu} \neq 0$. Define Q to be the orthogonal two by two matrix diagonalizing θ/Θ_{uu} as follows:*

$$\frac{\theta}{\Theta_{uu}} = Q\begin{pmatrix}\rho_+ & 0\\ 0 & \rho_-\end{pmatrix}Q^* \tag{5.252}$$

with eigenvalues ρ_+ and ρ_- and where Q^* denotes the matrix transpose of Q. Then:

$$e^t_u = (1 - \Theta_{uu}\mathcal{B}_t)e_u\,, \quad \begin{pmatrix}e^t_l\\ e^t_n\end{pmatrix} = Q\begin{pmatrix}[1 - \Theta_{uu}\mathcal{B}_t]^{\rho_+} & 0\\ 0 & [1 - \Theta_{uu}\mathcal{B}_t]^{\rho_-}\end{pmatrix}Q^*\begin{pmatrix}e_l\\ e_n\end{pmatrix} \tag{5.253}$$

Conversely, for every family of functions $\{\beta_t\}_{t\in\mathcal{I}}$ the previous expression defines a parallel spinor flow on G. *The case $\Theta_{uu}=0$ is recovered by taking the formal limit $\Theta_{uu}\to 0$.*

Proof. Define momentarily the function $\mathcal{I}\ni t\to x_t := \mathrm{Log}\,[1-\Theta_{uu}\mathcal{B}_t]$. By Proposition 5.19 and Corollary 5.4 it suffices to use the explicit expression for Θ^t obtained in Lemma 5.13 to solve Equation (5.236) with initial condition $\mathcal{U}^0=\mathrm{Id}$ on a simply connected Lie group admitting quasi-diagonal parallel Cauchy pairs. Plugging the explicit expression of Θ^t in (5.236) we obtain:

$$\partial_t\,\mathcal{U}^t_{uc}=\partial_t x_t\,\mathcal{U}^t_{uc}\,,\qquad \begin{pmatrix}\partial_t\mathcal{U}^t_{lc}\\ \partial_t\mathcal{U}^t_{nc}\end{pmatrix}=\frac{\partial_t x_t}{\Theta_{uu}}\begin{pmatrix}\Theta_{ll} & \Theta_{ln}\\ \Theta_{ln} & \Theta_{nn}\end{pmatrix}\begin{pmatrix}\mathcal{U}^t_{lc}\\ \mathcal{U}^t_{nc}\end{pmatrix}\,,\qquad c=u,l,n\,. \tag{5.254}$$

The general solution to the equations for $\mathcal{U}^t_{uc}$ with initial condition $\mathcal{U}^0=\mathrm{Id}$ is given by:

$$\mathcal{U}^t_{uu}=1-\Theta_{uu}\mathcal{B}_t\,,\quad \mathcal{U}^t_{ul}=\mathcal{U}^t_{un}=0\,. \tag{5.255}$$

Consider now the diagonalization of the constant matrix occurring in the differential equations for $\mathcal{U}^t_{ic}$:

$$\frac{1}{\Theta_{uu}}\begin{pmatrix}\Theta_{ll} & \Theta_{ln}\\ \Theta_{ln} & \Theta_{nn}\end{pmatrix}=Q\begin{pmatrix}\rho_+ & 0\\ 0 & \rho_-\end{pmatrix}Q^*\,, \tag{5.256}$$

where Q is a two by two orthogonal matrix and Q^* is its transpose. The eigenvalues are explicitly given by:

$$\rho_\pm=\frac{T\pm\sqrt{T^2-4\Delta}}{2\Theta_{uu}}\,. \tag{5.257}$$

We obtain:

$$Q^*\begin{pmatrix}\partial_t\mathcal{U}^t_{lc}\\ \partial_t\mathcal{U}^t_{nc}\end{pmatrix}=\partial_t x_t\begin{pmatrix}\rho_+ & 0\\ 0 & \rho_-\end{pmatrix}Q^*\begin{pmatrix}\mathcal{U}^t_{lc}\\ \mathcal{U}^t_{nc}\end{pmatrix}\,,\qquad c=u,l,n\,, \tag{5.258}$$

whose general solution is given by:

$$\begin{pmatrix}\mathcal{U}^t_{lc}\\ \mathcal{U}^t_{nc}\end{pmatrix}=Q\begin{pmatrix}k^+_c e^{\rho_+ x_t}\\ k^-_c e^{\rho_- x_t}\end{pmatrix}=Q\begin{pmatrix}k^+_c\,[1-\Theta_{uu}\mathcal{B}_t]^{\rho_+}\\ k^-_c\,[1-\Theta_{uu}\mathcal{B}_t]^{\rho_-}\end{pmatrix}\,,\qquad c=u,l,n\,, \tag{5.259}$$

for constants $k^+_c, k^-_c\in\mathbb{R}$. Imposing the initial condition $\mathcal{U}^0=\mathrm{Id}$ we obtain the following expression for k^+_c and k^-_c:

$$\begin{pmatrix}k^+_c\\ k^-_c\end{pmatrix}=Q^*\begin{pmatrix}\delta_{lc}\\ \delta_{nc}\end{pmatrix}\,,\qquad c=u,l,n\,, \tag{5.260}$$

whence:

$$\begin{pmatrix}k^+_u\\ k^-_u\end{pmatrix}=0\,,\quad \begin{pmatrix}k^+_l\\ k^-_l\end{pmatrix}=Q^*\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}Q^*_{ll}\\ Q^*_{nl}\end{pmatrix}\,,\quad \begin{pmatrix}k^+_n\\ k^-_n\end{pmatrix}=Q^*\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}Q^*_{ln}\\ Q^*_{nn}\end{pmatrix}\,. \tag{5.261}$$

We infer that

$$\begin{pmatrix}\mathcal{U}^t_{ll} & \mathcal{U}^t_{ln}\\ \mathcal{U}^t_{nl} & \mathcal{U}^t_{nn}\end{pmatrix}=Q\begin{pmatrix}e^{\rho_+ x_t} & 0\\ 0 & e^{\rho_- x_t}\end{pmatrix}Q^*=Q\begin{pmatrix}[1-\Theta_{uu}\mathcal{B}_t]^{\rho_+} & 0\\ 0 & [1-\Theta_{uu}\mathcal{B}_t]^{\rho_-}\end{pmatrix}Q^* \tag{5.262}$$

and the statement is proven. The converse follows by construction upon use of Lemma 5.12 and Proposition 5.19. It can be easily seen that the case $\Theta_{uu}=0$ is obtained by taking the formal limit $\Theta_{uu}\to 0$ and we conclude. □

Remark 5.23. The Ricci tensor of the family of Riemannian metrics $\{h_{\mathfrak{e}^t}\}_{t\in\mathcal{I}}$ associated to a a left-invariant quasi-diagonal parallel spinor flow $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ is given by:

$$\mathrm{Ric}^{h_{\mathfrak{e}^t}} = -T^t\Theta_t + \frac{\mathcal{H}_t}{2} e^t_u \otimes e^t_u \,, \tag{5.263}$$

where $T^t = \Theta^t_{ll} + \Theta^t_{nn}$. If $\mathcal{H}_t = 0$ for every $t \in \mathcal{I}$, that is, if the parallel Cauchy pair given by $\left\{\beta_t, \mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ is constrained Ricci flat, then:

$$\mathrm{Ric}^{h_{\mathfrak{e}^t}} = \frac{T^t}{2\beta_t}\partial_t h_{\mathfrak{e}^t} \,, \tag{5.264}$$

which, after a reparametrization of the time coordinate can be brought into the form $\mathrm{Ric}^{h_{\mathfrak{e}^\tau}} = -2\partial_\tau h_{\mathfrak{e}^\tau}$ after possibly shrinking $\mathcal{I}$. Hence, this gives a particular example of a left-invariant Ricci flow on G.

We consider now $\Theta_{ul}\Theta_{un} = 0$ but $\Theta^2_{ul} + \Theta^2_{un} \neq 0$. This case necessarily corresponds to $\mathrm{G} = \tau_2 \oplus \mathbb{R}$.

Proposition 5.21. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a left-invariant parallel spinor flow with initial parallel Cauchy pair $(\mathfrak{e}, \Theta)$ satisfying $\Theta_{ul}\Theta_{un} = 0$ and $\lambda \neq 0$. Then:*

- *If $\Theta_{ul} = 0$ the following holds:*

$$e^t_u = (1 - \Theta_{uu}\mathcal{B}_t)\; e_u - \Theta_{un}\mathcal{B}_t\; e_n \,, \quad e^t_l = e_l \,, \tag{5.265}$$

$$e^t_n = \left(\tfrac{\Theta_{uu}}{\Theta_{un}} - \tfrac{\lambda}{\Theta_{un}}(1 - \Theta_{uu}\mathcal{B}_t)\tan[y_t]\right) e_u + (1 + \lambda\mathcal{B}_t \tan[y_t])\, e_n \,. \tag{5.266}$$

- *If $\Theta_{un} = 0$ the following holds:*

$$e^t_u = (1 - \Theta_{uu}\mathcal{B}_t)\; e_u - \Theta_{ul}\mathcal{B}_t\; e_l \,, \quad e^t_n = e_n \,, \tag{5.267}$$

$$e^t_l = \left(\tfrac{\Theta_{uu}}{\Theta_{ul}} - \tfrac{\lambda}{\Theta_{ul}}(1 - \Theta_{uu}\mathcal{B}_t)\tan[y_t]\right) e_u + (1 + \lambda\mathcal{B}_t \tan[y_t])\, e_l \,. \tag{5.268}$$

Conversely, every such family $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is a left-invariant parallel spinor flow for every $\{\beta_t\}_{t\in\mathcal{I}}$.

Proof. We prove the case $\Theta_{ul} = 0$ and $\Theta_{un} \neq 0$ since the case $\Theta_{un} = 0$ and $\Theta_{ul} \neq 0$ follows similarly. Setting $\Theta_{ul} = 0$ and assuming $\Theta_{un} \neq 0$ in Lemma 5.14 we immediately obtain:

$$\Theta^t_{uu} = -\Theta^t_{nn} = \lambda\tan[y_t] \,, \qquad \Theta^t_{ll} = \Theta^t_{ln} = 0 \,. \tag{5.269}$$

where we have also used that, in this case, $\Theta_{ln} = \Theta_{ll} = 0$ and $\Theta_{uu} = -\Theta_{nn}$ as summarized in Theorem 5.5. Hence:

$$\Theta^t = \Theta_{un}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda\tan[y_t]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \,, \tag{5.270}$$

and equation (5.236) reduces to

$$\partial_t\mathcal{U}^t + \beta_t\Theta_{un}\begin{pmatrix} \mathcal{U}^t_{nu} & \mathcal{U}^t_{nl} & \mathcal{U}^t_{nn} \\ 0 & 0 & 0 \\ \mathcal{U}^t_{uu} & \mathcal{U}^t_{ul} & \mathcal{U}^t_{un} \end{pmatrix} + \lambda\beta_t\tan[y_t]\begin{pmatrix} \mathcal{U}^t_{uu} & \mathcal{U}^t_{ul} & \mathcal{U}^t_{un} \\ 0 & 0 & 0 \\ -\mathcal{U}^t_{nu} & -\mathcal{U}^t_{nl} & -\mathcal{U}^t_{nn} \end{pmatrix} = 0 \,, \tag{5.271}$$

or, equivalently:

$$\partial_t \mathcal{U}^t_{uc} + \beta_t \lambda \tan[y_t]\, \mathcal{U}^t_{uc} + \beta_t \Theta_{un} \mathcal{U}^t_{nc} = 0\,, \quad \partial_t \mathcal{U}^t_{lc} = 0\,, \tag{5.272}$$

$$\partial_t \mathcal{U}^t_{nc} - \beta_t \lambda \tan[y_t]\, \mathcal{U}_{nc} + \beta_t \Theta_{un} \mathcal{U}^t_{uc} = 0\,. \tag{5.273}$$

The general solution to this system with initial condition $\mathcal{U}^0 = \mathrm{Id}$ is given by:

$$\mathcal{U}^t_{uu} = 1 - \Theta_{uu}\mathcal{B}_t\,, \quad \mathcal{U}^t_{un} = -\Theta_{un}\mathcal{B}_t\,, \quad \mathcal{U}^t_{ul} = \mathcal{U}^t_{lu} = \mathcal{U}^t_{ln} = \mathcal{U}^t_{nl} = 0\,, \quad \mathcal{U}^t_{ll} = 1\,, \tag{5.274}$$

$$\mathcal{U}^t_{nn} = 1 + \lambda \mathcal{B}_t \tan[y_t]\;, \quad \mathcal{U}^t_{nu} = \tfrac{\Theta_{uu}}{\Theta_{un}} - \tfrac{\lambda}{\Theta_{un}}(1 - \Theta_{uu}\mathcal{B}_t)\tan[y_t]\;, \tag{5.275}$$

which implies the statement. The converse follows by construction upon use of Lemma 5.12 and Proposition 5.19. □

Remark 5.24. The Ricci tensor of the family of metrics $\{h_{\mathfrak{e}^t}\}_{t\in\mathcal{I}}$ associated to a left-invariant parallel spinor with if $\Theta_{ul} = 0$ but $\Theta_{un} \neq 0$ reads

$$\mathrm{Ric}^{h_{\mathfrak{e}^t}} = -\Theta_t \circ \Theta_t = \frac{\mathcal{H}_t}{4}(h_{\mathfrak{e}^t} - e^t_n \otimes e^t_n)\,. \tag{5.276}$$

Recall that $\nabla^{h_{\mathfrak{e}^t}} e^t_n = 0$ and thus $\left\{h_{\mathfrak{e}^t}, e^t_n\right\}_{t\in\mathcal{I}}$ defines a family of η-Einstein cosymplectic structures [663,664]. On the other hand, if $\Theta_{un} = 0$ but $\Theta_{ul} \neq 0$ the curvature of $\{h_{\mathfrak{e}^t}\}_{t\in\mathcal{I}}$ is given by:

$$\mathrm{Ric}^{h_{\mathfrak{e}^t}} = -\Theta_t \circ \Theta_t = \frac{\mathcal{H}_t}{4}(h_{\mathfrak{e}^t} - e^t_l \otimes e^t_l)\,. \tag{5.277}$$

whence $\{h_{\mathfrak{e}^t}, e^t_l\}_{t\in\mathcal{I}}$ defines as well a family of η-Einstein cosymplectic structures on G.

Finally we consider $\Theta_{ul}\Theta_{un} \neq 0$, a case that again corresponds to $\mathrm{G} = \tau_2 \oplus \mathbb{R}$.

Proposition 5.22. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a left-invariant parallel spinor flow with initial parallel Cauchy pair $(\mathfrak{e}, \Theta)$ satisfying $\Theta_{ul}\Theta_{un} \neq 0$. Then:*

$$e^t_u = e_u + \mathcal{B}_t(T e_u - \Theta_{ul} e_l - \Theta_{un} e_n)\,, \tag{5.278}$$

$$e^t_l = -\tfrac{\Theta_{ul}}{\lambda}\Upsilon\, e_u + \left(1 + \tfrac{\Theta^2_{ul}\mathcal{B}_t}{\lambda}\tan[y_t]\right) e_l + \tfrac{\Theta_{ul}\Theta_{un}\mathcal{B}_t}{\lambda}\tan[y_t]\; e_n\,, \tag{5.279}$$

$$e^t_n = -\tfrac{\Theta_{un}}{\lambda}\Upsilon\, e_u + \tfrac{\Theta_{ul}\Theta_{un}\mathcal{B}_t}{\lambda}\tan[y_t]\, e_l + \left(1 + \tfrac{\Theta^2_{un}\mathcal{B}_t}{\lambda}\;\tan[y_t]\right) e_n\,, \tag{5.280}$$

where $\Upsilon = \frac{T}{\lambda} + (1 + T\mathcal{B}_t)\tan[y_t]$. Conversely, every such family $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ is a left-invariant parallel spinor flow for every $\{\beta_t\}_{t\in\mathcal{I}}$.

Proof. Assuming $\Theta_{ul}, \Theta_{un} \neq 0$ in Lemma 5.14 we obtain:

$$\Theta^t_{uu} = \lambda \tan[y_t]\;, \quad \Theta^t_{ll} = \frac{\Theta_{ul}}{\Theta_{un}}\Theta^t_{ln}\,, \quad \Theta^t_{nn} = \frac{\Theta_{un}}{\Theta_{ul}}\Theta^t_{ln}\,, \quad \Theta^t_{ln} = -\frac{\Theta_{ul}\Theta_{un}}{\Theta^2_{ul} + \Theta^2_{un}}\Theta^t_{uu}\,. \tag{5.281}$$

Note that $\Theta^t_{uu} = -\Theta^t_{ll} - \Theta^t_{nn}$. Hence:

$$\Theta^t = \begin{pmatrix} 0 & \Theta_{ul} & \Theta_{un} \\ \Theta_{ul} & 0 & 0 \\ \Theta_{un} & 0 & 0 \end{pmatrix} - \frac{\tan[y_t]}{\lambda}\begin{pmatrix} -\lambda^2 & 0 & 0 \\ 0 & \Theta^2_{ul} & \Theta_{ul}\Theta_{un} \\ 0 & \Theta_{ul}\Theta_{un} & \Theta^2_{un} \end{pmatrix}\,, \tag{5.282}$$

and Equation (5.236) reduces to

$$\frac{1}{\beta_t}\partial_t\mathcal{U}^t_{uc} + \mathcal{U}^t_{lc}\Theta_{ul} + \mathcal{U}^t_{nc}\Theta_{un} + \mathcal{U}^t_{uc}\Theta^t_{uu} = 0\,, \tag{5.283}$$

$$\frac{1}{\beta_t}\partial_t\mathcal{U}^t_{lc} + \Theta_{ul}\left(\mathcal{U}^t_{uc} - \lambda^{-1}(\Theta_{ul}\mathcal{U}^t_{lc} + \Theta_{un}\mathcal{U}^t_{nc})\tan[y_t]\right) = 0\,, \tag{5.284}$$

$$\frac{1}{\beta_t}\partial_t\mathcal{U}^t_{nc} + \Theta_{un}\left(\mathcal{U}^t_{uc} - \lambda^{-1}(\Theta_{ul}\mathcal{U}^t_{lc} + \Theta_{un}\mathcal{U}^t_{nc})\tan[y_t]\right) = 0\,. \tag{5.285}$$

The general solution to this system with initial condition $\mathcal{U}^0 = \mathrm{Id}$ is given by:

$$\mathcal{U}^t_{uu} = 1 - \Theta_{uu}\mathcal{B}_t\,, \quad \mathcal{U}^t_{ul} = -\Theta_{ul}\mathcal{B}_t\,, \quad \mathcal{U}^t_{un} = -\Theta_{un}\mathcal{B}_t\,, \tag{5.286}$$

$$\mathcal{U}^t_{lu} = -\frac{\Theta_{ul}}{\lambda}\Upsilon\,, \quad \mathcal{U}^t_{ll} = 1 + \frac{\Theta^2_{ul}\mathcal{B}_t}{\lambda}\tan[y_t]\,, \quad \mathcal{U}^t_{ln} = \frac{\Theta_{ul}\Theta_{un}\mathcal{B}_t}{\lambda}\tan[y_t]\,, \tag{5.287}$$

$$\mathcal{U}^t_{nu} = -\frac{\Theta_{un}}{\lambda}\Upsilon\,, \quad \mathcal{U}^t_{nl} = \frac{\Theta_{ul}\Theta_{un}\mathcal{B}_t}{\lambda}\tan[y_t]\,, \quad \mathcal{U}^t_{nn} = 1 + \frac{\Theta^2_{un}\mathcal{B}_t}{\lambda}\tan[y_t]\,, \tag{5.288}$$

where $\Upsilon = \frac{T}{\lambda} + (1 + T\mathcal{B}_t)\tan[y_t]$ and we conclude. □

Remark 5.25. The three-dimensional Ricci tensor of the family of Riemannian metrics $\{h_{\mathfrak{e}^t}\}_{t\in\mathcal{I}}$ associated to a left-invariant parallel spinor flow with $\Theta_{ul}\Theta_{un} \neq 0$ reads

$$\mathrm{Ric}^{h_{\mathfrak{e}^t}} = -\Theta_t \circ \Theta_t = \frac{\mathcal{H}_t}{4}(h_{\mathfrak{e}^t} - \eta_t \otimes \eta_t)\,, \quad \eta_t = \frac{1}{\sqrt{\Theta^2_{ul} + \Theta^2_{un}}}(\Theta_{un}e^t_l - \Theta_{ul}e^t_n)\,. \tag{5.289}$$

Note that $\nabla^{h_{\mathfrak{e}^t}}\eta_t = 0$, so $\{h_{\mathfrak{e}^t}, \eta_t\}_{t\in\mathcal{I}}$ defines a family of η-Einstein cosymplectic Riemannian structures on G.

As a corollary to the classification of left-invariant parallel spinor flows presented in Propositions 5.20, 5.21 and 5.22 we can explicitly obtain the evolution of the Hamiltonian constraint in each case.

Corollary 5.5. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a left-invariant parallel spinor in (M, g).*

- *If $\Theta_{ul} = \Theta_{un} = 0$, then $\mathcal{H}_t = \frac{\mathcal{H}_0}{(1-\Theta_{uu}\mathcal{B}_t)^2}$.*
- *If $\Theta_{ul} = 0$ but $\Theta_{un} \neq 0$ then $\mathcal{H}_t = \frac{\Theta^2_{un}\mathcal{H}_0}{\Theta^2_{uu}+\Theta^2_{un}}\sec^2\left[\lambda\mathcal{B}_t + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right]\right]$.*
- *If $\Theta_{un} = 0$ but $\Theta_{ul} \neq 0$ then $\mathcal{H}_t = \frac{\Theta^2_{ul}\mathcal{H}_0}{\Theta^2_{uu}+\Theta^2_{ul}}\sec^2\left[\lambda\mathcal{B}_t + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right]\right]$.*
- *If $\Theta_{ul}, \Theta_{un} \neq 0$ then $\mathcal{H}_t = \frac{\lambda^2\mathcal{H}_0}{\lambda^2+\Theta^2_{uu}}\sec^2\left[\lambda\mathcal{B}_t + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right]\right]$.*

where $\mathcal{H}_0$ is the Hamiltonian constraint at time $t = 0$.

Since the secant function has no zeroes, the Hamiltonian constraint vanishes for a given $t \in \mathcal{I}$, and hence for every $t \in \mathcal{I}$, if and only if it vanish at $t = 0$, consistently with Theorem 5.2. Theorem 5.5 implies that only quasi-diagonal left-invariant parallel spinor flows admit constrained Ricci flat initial data. Therefore the Hamiltonian constraint of left-invariant parallel spinor flows with $\lambda \neq 0$ is non-vanishing for every $t \in \mathcal{I}$ and such left-invariant parallel spinor flows cannot produce four-dimensional Ricci flat Lorentzian metrics.

Altogether, we prove the following theorem.

Theorem 5.6. *Let $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ be a left-invariant parallel spinor on a simply-connected Lie group G. Denote by $\{h_{\mathfrak{e}^t}\}_{t\in\mathcal{I}}$ the family of Riemannian metrics associated to $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathcal{I}}$ and by $(\mathfrak{e}, \Theta)$ its initial parallel Cauchy pair.*

- *If $\Theta_{ul}^2 + \Theta_{un}^2 = 0$ and $\Theta_{uu} \neq 0$ then:*

$$h_{\mathfrak{e}^t} = (1-\Theta_{uu}\mathcal{B}_t)^2\, e_u\otimes e_u + (e_l, e_n)\, Q\begin{pmatrix}[1-\Theta_{uu}\mathcal{B}_t]^{2\rho_+} & 0\\ 0 & [1-\Theta_{uu}\mathcal{B}_t]^{2\rho_-}\end{pmatrix} Q^*\begin{pmatrix}e_l\\ e_n\end{pmatrix}. \tag{5.290}$$

  where $\rho_\pm = \frac{T\pm\sqrt{T^2-4\Delta}}{2\Theta_{uu}}$ are the eigenvalues of θ/Θ_{uu} and Q is its orthogonal diagonalization matrix. In particular, $\mathrm{G} = \mathbb{R}^3$ if $\theta = 0$, $\mathrm{G} = \mathrm{E}(1,1)$ if $T = 0$ and $\theta \neq 0$, $\mathrm{G} = \tau_2\oplus\mathbb{R}$ if $T \neq 0$ and $\Delta = 0$ and $\mathrm{G} = \tau_{3,\mu}$ if $T\neq 0$ and $\Delta \neq 0$. The case $\Theta_{uu} = 0$ is obtained by taking the formal limit $\Theta_{uu}\to 0$ in the previous expressions.

- *If $\Theta_{ul} = 0$ but $\Theta_{un} \neq 0$, we have:*

$$\begin{aligned} h_{\mathfrak{e}^t} = &\Big((1-\Theta_{uu}\mathcal{B}_t)^2\sec^2[y_t] + \tfrac{\Theta_{uu}^2}{\lambda^2} - \tfrac{2\Theta_{uu}}{\lambda}(1-\Theta_{uu}\mathcal{B}_t)\tan[y_t]\Big)\, e_u\otimes e_u\\ &\Big(\tfrac{\Theta_{uu}}{\Theta_{un}} - \Theta_{un}\mathcal{B}_t\sec^2[y_t](1-\Theta_{uu}\mathcal{B}_t) - \tfrac{\lambda}{\Theta_{un}}\tan[y_t](1-2\Theta_{uu}\mathcal{B}_t)\Big)\, e_u\odot e_n\\ &+e_l\otimes e_l + \big(1+\lambda^2\mathcal{B}_t^2\sec^2[y_t] + 2\,\mathcal{B}_t\lambda\tan[y_t]\big)\, e_n\otimes e_n\,, \end{aligned} \tag{5.291}$$

  where $y_t = \lambda\,\mathcal{B}_t + \arctan\left[\frac{\Theta_{uu}}{\lambda}\right]$. In particular $\mathrm{G} = \tau_2\oplus\mathbb{R}$. The case $\Theta_{ul}\neq 0$ but $\Theta_{un} = 0$ is obtained by just exchanging the subindices l and n in the previous expression.

- *If $\Theta_{ul}\Theta_{un} \neq 0$ then:*

$$\begin{aligned} h_{\mathfrak{e}^t} = &\Big((1+T\,\mathcal{B}_t)^2 + \big(\tan[y_t](1+T\mathcal{B}_t) + \tfrac{T}{\lambda}\big)^2\Big)\, e_u\otimes e_u\\ &-\Big(\tfrac{T}{\lambda^2} + \tfrac{\tan[y_t]}{\lambda}(1+2T\mathcal{B}_t) + \mathcal{B}_t(1+T\mathcal{B}_t)\big)\sec^2[y_t]\Big)\,(\Theta_{ul}e_u\odot e_l + \Theta_{un}e_u\odot e_n)\\ &+\Big(1+\Theta_{ul}^2\mathcal{B}_t\Big(\mathcal{B}_t\sec^2[y_t] + \tfrac{2\tan[y_t]}{\lambda}\Big)\Big)\, e_l\otimes e_l + \Theta_{ul}\Theta_{un}\mathcal{B}_t\sec^2[y_t]\Big(\mathcal{B}_t + \tfrac{\sin[2y_t]}{\lambda}\Big)\, e_l\odot e_n\\ &+\Big(1+\Theta_{un}^2\mathcal{B}_t\Big(\mathcal{B}_t\sec^2[y_t] + \tfrac{2\tan[y_t]}{\lambda}\Big)\Big)\, e_n\otimes e_n\,, \end{aligned} \tag{5.292}$$

  In particular, $\mathrm{G} = \tau_2\oplus\mathbb{R}$.

Furthermore, if $\lambda = 0$ the flow is globally defined (namely $\mathcal{I} = \mathbb{R}$) if and only if $\int_0^\infty \beta_\tau \mathrm{d}\tau < |\Theta_{uu}^{-1}|$, whereas if $\lambda \neq 0$ the flow is globally defined if and only if $|y_t| < \frac{\pi}{2}$ $\forall t\in\mathbb{R}$.

Proof. Theorem 5.6 follows through a direct computation by using the explicit form of the left-invariant parallel spinor flow obtained in Propositions 5.20, 5.21 and 5.22 for each of the possible cases, after using Theorem 5.5 to identify the underlying Lie group in each case. □

5.6 Comoving parallel spinor flows

Finally, in this section we consider a specific type of parallel spinor flow which admits a particularly neat geometric description, with the goal of obtaining explicit non-left-invariant *time-dependent* Lorentzian four-manifolds admitting parallel spinors.

5.6.1 Globally hyperbolic comoving spacetimes

We consider a particular class of parallel spinor flows given by imposing the condition $\beta_t = 1$ for all $t \in \mathcal{I}$.

Definition 5.7. A parallel spinor flow $\{\beta_t, \mathfrak{e}^t\}_{t\in\mathbb{R}})$ is *comoving* if $\beta_t = 1$ for every $t \in \mathcal{I}$.

A comoving parallel spinor flow on a manifold of the form $M = \mathcal{I} \times \Sigma$, where Σ is an oriented three-manifold, will be always understood as a parallel spinor flow on Σ with respect to the cartesian coordinate of the open interval $\mathcal{I} \subset \mathbb{R}$.

Definition 5.8. A four-dimensional spacetime (M, g) is a *comoving globally hyperbolic spacetime* if it is isometric to a model of the form:

$$(M, g) = (\mathcal{I} \times \Sigma, -\mathrm{d}t \otimes \mathrm{d}t + h_t)\,, \tag{5.293}$$

for a family $\{h_t\}_{t\in\mathcal{I}}$ of complete Riemannian metrics on Σ, where $\mathcal{I} \subset \mathbb{R}$ is an interval.

A metric of the type $g = -\mathrm{d}t^2 + h_t$ will be called a *comoving globally hyperbolic.*

Remark 5.26. The term *comoving* is motivated by the fact that the local metric of a comoving observer in a cosmological background is of comoving globally hyperbolic type. In particular, the time factor of the metric is constant.

Proposition 5.23. *An oriented four-manifold (M, g) admits a comoving parallel spinor flow if and only if the associated family of coframes $\{\mathfrak{e}^t\}_{t\in\mathcal{I}}$ satisfies the following system of partial differential equations:*

$$\partial_t \mathfrak{e}^t + \Theta_t(\mathfrak{e}^t) = 0\,, \quad \mathrm{d}\mathfrak{e}^t = \Theta_t(\mathfrak{e}^t) \wedge e^t_u\,, \quad [\Theta_t(e^t_u)] = 0 \in H^1(\Sigma, \mathbb{R})\,, \quad \partial_t \Theta_t(e^t_u) = 0\,, \tag{5.294}$$

If this is the case, the corresponding comoving globally hyperbolic metric is given by:

$$g = -\mathrm{d}t \otimes \mathrm{d}t + h_{\mathfrak{e}^t}\,, \qquad h_{\mathfrak{e}^t} = e^t_u \otimes e^t_u + e^t_l \otimes e^t_l + e^t_n \otimes e^t_n\,. \tag{5.295}$$

Proof. Just by setting $\beta_t = 1$ in Theorem 5.1. □

We will refer to equations (5.294) as the *comoving parallel-spinor flow equations*, and we will refer to its solutions as *comoving parallel spinor flows.* The general investigation of comoving parallel spinor flows is beyond the scope of this article and will be considered elsewhere. Instead, we consider two particular important cases in detail.

5.6.2 A diagonal example on $\mathbb{R}^3$.

Set $\Sigma = \mathbb{R}^3$ with Cartesian coordinates (x, y, z) and consider comoving parallel spinor flows $\left\{\mathfrak{e}^t\right\}_{t\in\mathcal{I}}$ of the form:

$$\mathfrak{e}^t = (\mathfrak{f}^t_u \mathrm{d}x, \mathfrak{f}^t_l \mathrm{d}y, \mathfrak{f}^t_n \mathrm{d}z)\,, \tag{5.296}$$

for families of functions $\left\{\mathfrak{f}^t_u\right\}_{t\in\mathbb{R}}$, $\left\{\mathfrak{f}^t_l\right\}_{t\in\mathbb{R}}$ and $\left\{\mathfrak{f}^t_n\right\}_{t\in\mathbb{R}}$ on $\mathbb{R}^3$. Hence,

$$h_{\mathfrak{e}^t} = (\mathfrak{f}^t_u)^2 \mathrm{d}x \otimes \mathrm{d}x + (\mathfrak{f}^t_l)^2 \mathrm{d}y \otimes \mathrm{d}y + (\mathfrak{f}^t_n)^2 \mathrm{d}z \otimes \mathrm{d}z\,, \quad (\mathfrak{e}^t)^\sharp = \left(\frac{1}{\mathfrak{f}^t_u}\partial_x, \frac{1}{\mathfrak{f}^t_l}\partial_y, \frac{1}{\mathfrak{f}^t_n}\partial_z\right)\,, \tag{5.297}$$

We compute:

$$\Theta_t = -(\mathfrak{f}^t_u \partial_t \mathfrak{f}^t_u \, \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{f}^t_l \partial_t \mathfrak{f}^t_l \, \mathrm{d}y \otimes \mathrm{d}y + \mathfrak{f}^t_n \partial_t \mathfrak{f}^t_n \, \mathrm{d}z \otimes \mathrm{d}z)\,. \tag{5.298}$$

Therefore, equation $\partial_t \mathfrak{e}^t + \Theta_t(\mathfrak{e}^t) = 0$ is automatically satisfied. On the other hand, equations $[\Theta_t(e^t_u)] = 0$ and $\partial_t \Theta_t(e^t_u) = 0$ are equivalent to

$$\partial_t \mathrm{d}\mathfrak{f}_u \wedge \mathrm{d}x = 0\,, \qquad \partial^2_t \mathfrak{f}_u = 0\,, \tag{5.299}$$

implying that

$$\mathfrak{f}^t_u = \mathfrak{a} + \mathfrak{b}\, t\,, \tag{5.300}$$

where $\mathfrak{b} = \mathfrak{b}(x)$ is a function of the coordinate x and $\mathfrak{a} = \mathfrak{a}(x, y, z)$ is a function of all coordinates of $\mathbb{R}^3$. Note that, in order to have a well-defined comoving parallel spinor flow, we must impose the constraint

$$\mathfrak{f}^t_u(t, x, y, z) = \mathfrak{a}(x, y, z) + \mathfrak{b}(x)\, t \neq 0\,, \tag{5.301}$$

for every $t \in \mathcal{I}$ and $(x, y, z) \in \mathbb{R}^3$, which translates into a constraint in the allowed domain of definition $\mathcal{I} \subset \mathbb{R}$ of t. The only equations that remain to be solved for

$$\mathfrak{e}^t = ((\mathfrak{a} + \mathfrak{b}\, t)\mathrm{d}x, \mathfrak{f}^t_l \mathrm{d}y, \mathfrak{f}^t_n \mathrm{d}z) \tag{5.302}$$

to be a comoving parallel spinor flow are $\mathrm{d}\mathfrak{e}^t = \Theta(\mathfrak{e}^t) \wedge e^t_u$, which can be shown to be equivalent to

$$\mathrm{d}\mathfrak{a} \wedge \mathrm{d}x = 0\,, \qquad (\mathrm{d}\mathfrak{f}^t_l - \partial_t \mathfrak{f}^t_l \mathfrak{f}^t_u \mathrm{d}x) \wedge \mathrm{d}y = 0\,, \qquad (\mathrm{d}\mathfrak{f}^t_n - \partial_t \mathfrak{f}^t_n \mathfrak{f}^t_u \mathrm{d}x) \wedge \mathrm{d}z = 0\,. \tag{5.303}$$

These equations are in turn equivalent to:

$$\mathfrak{a} = \mathfrak{a}(x)\,, \quad \partial_x \mathfrak{f}^t_l = \mathfrak{f}^t_u \partial_t \mathfrak{f}^t_l\,, \quad \partial_z \mathfrak{f}^t_l = 0\,, \quad \partial_x \mathfrak{f}^t_n = \mathfrak{f}^t_u \partial_t \mathfrak{f}^t_n\,, \quad \partial_y \mathfrak{f}^t_n = 0\,, \tag{5.304}$$

which do have explicit solutions, as we will show later in particular examples. On the other hand a direct computation shows that the Ricci curvature of the comoving globally hyperbolic Lorentzian metric $g = -\mathrm{d}t \otimes \mathrm{d}t + h_{\mathfrak{e}^t}$ associated to such $\mathfrak{e}^t$ vanishes if and only if the following condition holds:

$$\mathfrak{b}\left(\frac{\partial_t \mathfrak{f}^t_l}{\mathfrak{f}_l} + \frac{\partial_t \mathfrak{f}^t_n}{\mathfrak{f}_n}\right) - \frac{\partial_t \partial_x \mathfrak{f}^t_l}{\mathfrak{f}_l} - \frac{\partial_t \partial_x \mathfrak{f}^t_n}{\mathfrak{f}_n} = 0\,. \tag{5.305}$$

This condition will be explored in the examples below.

Example 5.3. Suppose that both $\mathfrak{a}$ and $\mathfrak{b}$ are constants, with $\mathfrak{b} \neq 0$. With this assumption, a general solution of equations (5.304) is of the form:

$$\mathfrak{f}^t_l = \mathfrak{L}_l(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, y)\,, \quad \mathfrak{f}^t_n = \mathfrak{L}_n(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, z)\,, \tag{5.306}$$

for nowhere vanishing smooth functions $\mathfrak{L}_l, \mathfrak{L}_n \in C^\infty(\mathbb{R}^2)$. The corresponding coframe $\mathfrak{e}^t$ reads

$$\mathfrak{e}^t = ((\mathfrak{a} + \mathfrak{b}t)\mathrm{d}x, \mathfrak{L}_l(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, y)\mathrm{d}y, \mathfrak{L}_n(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, z)\mathrm{d}z)\,, \tag{5.307}$$

which is well defined in the intervals $t \in \mathcal{I}_1 = (-\infty, -\frac{\mathfrak{a}}{\mathfrak{b}})$ or $t \in \mathcal{I}_2 = (-\frac{\mathfrak{a}}{\mathfrak{b}}, \infty)$. The metric associated to the previous global coframe is given by:

$$g = -\mathrm{d}t \otimes \mathrm{d}t + (\mathfrak{a} + \mathfrak{b}\, t)^2 \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{L}_l(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, y)^2 \mathrm{d}y \otimes \mathrm{d}y \\ + \mathfrak{L}_n(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, z)^2 \mathrm{d}z \otimes \mathrm{d}z\,, \tag{5.308}$$

which provides a large family of four-dimensional Lorentzian metrics admitting a parallel spinor. If the induced Riemannian spatial metric:

$$h_{\mathfrak{e}^t} = (\mathfrak{a} + \mathfrak{b}\, t)^2 \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{L}_l(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, y)^2 \mathrm{d}y \otimes \mathrm{d}y \\ + \mathfrak{L}_n(x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}, z)^2 \mathrm{d}z \otimes \mathrm{d}z\,, \tag{5.309}$$

on $\{t\} \times \mathbb{R}^3 \subset \mathcal{I}_i \times \mathbb{R}^3$ (for $i = 1, 2$) is complete for all $t \in \mathcal{I}_i$ we obtain a family of comoving globally hyperbolic metrics on $\mathcal{I}_i \times \mathbb{R}^3$. Equation (5.305), implies now that g is Ricci flat if and only if

$$\frac{\mathfrak{b}\partial_\zeta \mathfrak{L}_l - \partial_\zeta\partial_\zeta \mathfrak{L}_l}{\mathfrak{L}_l} + \frac{\mathfrak{b}\partial_\zeta \mathfrak{L}_n - \partial_\zeta\partial_\zeta \mathfrak{L}_n}{\mathfrak{L}_n} = 0\,, \tag{5.310}$$

where we have defined $\zeta(t, x) := x + \log|\mathfrak{a} + \mathfrak{b}\, t|/\mathfrak{b}$. This Ricci-flatness condition is satisfied if the functions $\mathfrak{L}_l(\zeta, y)$ and $\mathfrak{L}_n(\zeta, z)$ take the form:

$$\mathfrak{L}_l(\zeta, y) = w_1(y) e^{\mathfrak{b}\zeta} + w_2(y)\,, \quad \mathfrak{L}_n(\zeta, z) = w_3(z) e^{\mathfrak{b}\zeta} + w_4(z)\,, \tag{5.311}$$

where w_1, w_2, w_3, w_4 are arbitrary smooth functions.

Example 5.4. Assume that $\mathfrak{a}$ is a possibly non-constant strictly positive function and $\mathfrak{b} = 0$. With this assumption, the general solution of equations (5.304) is of the form:

$$\mathfrak{f}^t_l = \mathfrak{L}_l\left(t + \int_0^x \mathfrak{a}(\tau)\mathrm{d}\tau, y\right)\,, \quad \mathfrak{f}^t_n = \mathfrak{L}_n\left(t + \int_0^x \mathfrak{a}(\tau)\mathrm{d}\tau, z\right)\,, \tag{5.312}$$

for nowhere vanishing smooth functions $\mathfrak{L}_l, \mathfrak{L}_n \in C^\infty(\mathbb{R}^2)$. The corresponding coframe $\mathfrak{e}^t$ reads:

$$\mathfrak{e}^t = \left(\mathfrak{a}(x)\mathrm{d}x, \mathfrak{L}_l\left(t + \int_0^x \mathfrak{a}(\tau)\mathrm{d}\tau, y\right)\mathrm{d}y, \mathfrak{L}_n\left(t + \int_0^x \mathfrak{a}(\tau)\mathrm{d}\tau, z\right)\mathrm{d}z\right)\,, \tag{5.313}$$

which is well defined for $t \in \mathcal{I} = \mathbb{R}$. The metric associated to the previous global coframe is given, after a change and relabeling of coordinates, by the following expression:

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{L}_l(t + x, y)^2 \mathrm{d}y \otimes \mathrm{d}y + \mathfrak{L}_n(t + x, z)^2 \mathrm{d}z \otimes \mathrm{d}z\,, \tag{5.314}$$

which provides a large family of four-dimensional Lorentzian metrics admitting a parallel spinor. If the induced Riemannian spatial metric:

$$h_{\mathfrak{e}^t} = \mathrm{d}x \otimes \mathrm{d}x + \mathfrak{L}_l(t + x, y)^2 \mathrm{d}y \otimes \mathrm{d}y + \mathfrak{L}_n(t + x, z)^2 \mathrm{d}z \otimes \mathrm{d}z\,, \tag{5.315}$$

on $\{t\} \times \mathbb{R}^3 \subset \mathcal{I} \times \mathbb{R}^3$ is complete for all $t \in \mathcal{I}$ we obtain a family of comoving globally hyperbolic metrics on $\mathcal{I} \times \mathbb{R}^3$. Implementing the change of coordinates

$$x^+ = \frac{t + x}{\sqrt{2}}\,, \qquad x^- = \frac{-t + x}{\sqrt{2}}\,, \tag{5.316}$$

the metric g is given by:

$$g = \mathrm{d}x^+ \odot \mathrm{d}x^- + \mathfrak{L}_l(x^+, y)^2 \mathrm{d}y \otimes \mathrm{d}y + \mathfrak{L}_n(x^+, z)^2 \mathrm{d}z \otimes \mathrm{d}z \,, \tag{5.317}$$

after a suitable redefinition of the functions $\mathfrak{L}_l$ and $\mathfrak{L}_n$. This metric is a particular case of a Lorentzian metric expressed in Schimming coordiantes [665], which exists in every Lorentzian manifold admitting a parallel null vector field. Equation (5.305) implies now that g is Ricci flat if and only if

$$\frac{\partial_{x^+}\partial_{x^+}\mathcal{L}_l}{\mathcal{L}_l} + \frac{\partial_{x^+}\partial_{x^+}\mathcal{L}_n}{\mathcal{L}_n} = 0 \,, \tag{5.318}$$

Some simple solutions can be found just by setting $\partial_{x^+}\partial_{x^+}\mathcal{L}_l = \partial_{x^+}\partial_{x^+}\mathcal{L}_n = 0$:

$$\mathcal{L}_l = w_1(y)x^+ + w_2(y) \,, \quad \mathcal{L}_n = w_3(z)x^+ + w_4(z) \,, \tag{5.319}$$

where w_1, w_2, w_3, w_4 are arbitrary smooth functions.

5.6.3 An example in Schimming coordinates.

In Reference [665] it was proven that any four-dimensional spacetime (M, g) equipped with a parallel lightlike vector field $u^\sharp \in \mathfrak{X}(M)$ admits local coordinates (x^+, x^-, y_1, y_2) in which the metric g and the vector field $u^\sharp$ are written as follows:

$$g = \mathrm{d}x^+ \odot \mathrm{d}x^- + k_{x^+} \,, \qquad u^\sharp = \frac{\partial}{\partial x^-} \,. \tag{5.320}$$

where

$$k_{x^+}(y_1, y_2) = k_{x^+\, ij} \mathrm{d}y_i \otimes \mathrm{d}y_j \,, \qquad i, j = 1, 2 \,. \tag{5.321}$$

is a family of two-dimensional metrics parametrized by the coordinate x^+. A simple change of coordinates $\sqrt{2}x^+ = t + x$ and $\sqrt{2}x^- = x - t$ allows to write the previous metric g as

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + k_{t+x} \,, \tag{5.322}$$

whence we obtain a particular type of comoving globally hyperbolic spacetimes. Therefore, it is natural to study comoving parallel spinor flows adapted to the structure of the metric (5.322). Assume that the previous coordinate system is globally defined. Then, the Cauchy surface is given by $\Sigma = \mathbb{R} \times X$, with X an oriented two-dimensional manifold, and the metric takes the form $h_t = \mathrm{d}x \otimes \mathrm{d}x + k_{t+x}$. Consequently, we assume that our comoving parallel spinor flow is of the form:

$$\mathfrak{e}^t = (\mathrm{d}x, e^t_l(x), e^t_n(x)) \,, \tag{5.323}$$

where $k_{t+x} = e^t_l \otimes e^t_l + e^t_n \otimes e^t_n$. The comoving parallel spinor flow equations (5.294) reduce to

$$\partial_t e^t_i(x) + \Theta_t(e^t_i(x)) = 0 \,, \quad \partial_x e^t_i(x) + \Theta_t(e^t_i(x)) = 0 \,, \quad \mathrm{d}e^t_i(x)|_X = 0 \,, \quad \Theta_t(\partial_x) = 0 \,. \tag{5.324}$$

Hence, the comoving parallel spinor flow can be considered as a bi-parametric flow which is parametrized by t and x, for a family of closed oriented frames on X. In particular

(X, k_{t+x}) is flat for every $(t,x) \in \mathbb{R}^2$ and therefore isometric to Euclidean space, the flat cylinder or a flat torus. Equations (5.324) immediately imply

$$\partial_t e^t_i(x) = \partial_x e^t_i(x)\,, \quad i = l, n\,. \tag{5.325}$$

Therefore, choosing coordinates (y_1, y_2) on X, global for the plane and local for the torus and cylinder cases, every such family of solutions can be written as follows:

$$e^t_l(x) = \mathfrak{f}^l_1(t+x)\mathrm{d}y_1 + \mathfrak{f}^l_2(t+x)\mathrm{d}y_2\,, \quad e^t_n(x) = \mathfrak{f}^n_1(t+x)\mathrm{d}y_1 + \mathfrak{f}^n_2(t+x)\mathrm{d}y_2\,, \tag{5.326}$$

for functions $\mathfrak{f}^l_1, \mathfrak{f}^l_2, \mathfrak{f}^n_1, \mathfrak{f}^n_2 \in C^\infty(\mathbb{R})$ satisfying the following condition everywhere:

$$\delta = \mathfrak{f}^l_1\mathfrak{f}^n_2 - \mathfrak{f}^n_1\mathfrak{f}^l_2 \neq 0\,. \tag{5.327}$$

If this condition is satisfied, the dual frame is given by:

$$e^t_l(x)^\sharp = \frac{1}{\delta}(\mathfrak{f}^n_2\partial_{y_1} - \mathfrak{f}^n_1\partial_{y_2})\,, \quad e^t_n(x)^\sharp = \frac{1}{\delta}(-\mathfrak{f}^l_2\partial_{y_1} + \mathfrak{f}^l_1\partial_{y_2})\,. \tag{5.328}$$

In order to guarantee that (5.327) is satisfied, we assume the following ansatz:

$$f^l_1 := e^{f_l} \quad \mathfrak{p} := f^l_2 = -f^n_1\,, \quad f^n_2 := e^{f_n} \tag{5.329}$$

in terms of functions $f_l\, f_n, \mathfrak{p} \in C^\infty(\mathbb{R})$. This implies $\delta = e^{f_l+f_n} + \mathfrak{p}^2 > 0$ and therefore equations (5.324) further reduce to

$$\partial_{x^+}e^{x^+}_l = (\partial_{x^+}e^{x^+}_l)((e^{x^+}_l)^\sharp)e^{x^+}_l + (\partial_{x^+}e^{x^+}_n)((e^{x^+}_l)^\sharp)e^{x^+}_n\,, \tag{5.330}$$
$$\partial_{x^+}e^{x^+}_n = (\partial_{x^+}e^{x^+}_l)((e^{x^+}_n)^\sharp)e^{x^+}_l + (\partial_{x^+}e^{x^+}_n)((e^{x^+}_n)^\sharp)e^{x^+}_n\,, \tag{5.331}$$

where we have gone back to the coordinate $x^+ = t + x$ and written $e^{x^+}_i := e^t_i(x)$, $i = l, n$. In particular, note that the condition $(\partial_{x^+}e^{x^+}_n)((e^{x^+}_l)^\sharp) = (\partial_{x^+}e^{x^+}_l)((e^{x^+}_n)^\sharp)$ is necessarily satisfied, as required by Theorem 5.3. By direct computation one finds that equations (5.330) and equations (5.331) turn out to yield a single linearly independent equation which takes the form:

$$\mathfrak{p}(\partial_{x^+}e^{f_l} + \partial_{x^+}e^{f_n}) = (e^{f_l} + e^{f_n})\partial_{x^+}\mathfrak{p}\,. \tag{5.332}$$

The general solution of the previous equation is given by:

$$\mathfrak{p} = c\,(e^{f_l} + e^{f_n})\,, \tag{5.333}$$

for any real constant c. Therefore we are led to the following Lorentzian metric, which by construction admits a parallel lightlike vector field given by ∂_{x^-}:

$$\begin{aligned} g = \mathrm{d}x^+ \odot \mathrm{d}x^- + (e^{2f_l} + c^2(e^{f_l} + e^{f_n})^2)\mathrm{d}y_1 \otimes \mathrm{d}y_1 + c(e^{2f_l} - e^{2f_n})\mathrm{d}y_1 \odot \mathrm{d}y_2 \\ + (e^{2f_n} + c^2(e^{f_l} + e^{f_n})^2)\mathrm{d}y_2 \otimes \mathrm{d}y_2\,. \end{aligned} \tag{5.334}$$

The Ricci tensor of the previous metric is given by:

$$\mathrm{Ric}^g = \Big[2c^2e^{f_l}((\partial_{x^+}f_l)^2 + \partial_{x^+}\partial_{x^+}f_l) + 2c^2e^{f_n}((\partial_{x^+}f_n)^2 + \partial_{x^+}\partial_{x^+}f_n) \tag{5.335}$$

$$+ (1+2c^2)e^{f_l+f_n}((\partial_{x^+}f_l)^2+\partial_{x^+}\partial_{x^+}f_l+(\partial_{x^+}f_n)^2+\partial_{x^+}\partial_{x^+}f_n)\Big]\mathrm{d}x^+\otimes\mathrm{d}x^+\,,$$

which vanishes if the following conditions are satisfied:

$$(\partial_{x^+}f_l)^2+\partial_{x^+}\partial_{x^+}f_l=0\,,\quad (\partial_{x^+}f_n)^2+\partial_{x^+}\partial_{x^+}f_n=0\,. \tag{5.336}$$

These ODEs are solved by:

$$f_l(x^+)=a+\log|x^+-b|\,,\quad f_n(x^+)=c+\log|x^+-d|\,, \tag{5.337}$$

for real constants $a,b,c,d\in\mathbb{R}$. These solutions are well defined if $x^+\in(-\infty,\min(b,d))$ or if $x^+\in(\max(b,d),+\infty)$. However, although f_l and f_n present divergences whenever the argument of the logarithm vanishes, this is not problematic for the metric (5.334) as long as $b\neq d$ (otherwise the metric would be degenerate at $x^+=b=d$), since both functions f_l and f_n appear exponentiated. In addition, it can be checked that the space time $(\mathbb{R}^2\times X,g)$ is a plane wave, since the Riemann curvature tensor $\mathrm{R}^g:\Lambda^2T(\mathbb{R}^2\times X)\to\Lambda^2T(\mathbb{R}^2\times X)$ satisfies $\mathrm{R}^g|_{(\partial_{x^-})^\perp\wedge(\partial_{x^-})^\perp}=0$ and $\nabla_V\mathrm{R}^g=0$ for all $V\in(\partial_{x^-})^\perp$.

5.7 Discussion

In this chapter we have introduced the parallel spinor flow, defined as the evolution flow prescribed by a parallel spinor on a globally hyperbolic Lorentzian four-manifold. It consists of a system of partial differential equations for a family of functions and coframes on an appropriate Cauchy surface $\Sigma\subset M$. We have proved such parallel spinor flow to be equivalent to the existence of a real parallel spinor, thanks to the description of a real parallel spinor in terms of a pair of one-forms satisfying a certain system of partial differential equations. This way, we have obtained a reformulation of the *a priori* more abstruse spinorial problem, which could facilitate, among other aspects that we have explicitly shown, the proof of the well-posedness of the initial value problem of a real parallel spinor. It would be highly interesting to extend these results in the case of real Killing spinor [106].

We also examined all standard Brinkmann spacetimes allowing for real parallel spinors, rederiving along the way the well-known result [626, 642] that Lorentzian manifolds with real parallel spinors are examples of pp-waves. Beyond its intrinsic mathematical definition, these spacetimes are extremely relevant for gravitational wave physics, since they are exact solutions to Einstein's field equations that can be used to model gravitational radiation, satisfy in some cases a (linear) superposition principle and around null geodesics every spacetime looks like a (plane) pp-wave [666]. Consequently, given this novel reformulation of a real parallel spinor in terms of the parallel spinor flow, it would be very interesting to reinterpret the physical properties of the associated pp-waves in terms of geometrical features of the corresponding family of coframes (for instance, it could be intriguing to understand what geometric structures front waves may correspond to within the context of parallel spinor flows).

In the case of globally hyperbolic spacetimes, we were able to show a very powerful result: although Lorentzian metrics admitting parallel spinors are not necessarily Ricci flat, the parallel spinor flow preserves the vacuum momentum and Hamiltonian constraints and therefore the Einstein and parallel spinor flows coincide on common initial data. Using this result, we provide an initial data characterization of real parallel spinors on Ricci flat Lorentzian four-manifolds. This is highly remarkable, since we are solving the Einstein flow, which is second-order, through the parallel spinor flow, which first-order in derivatives, suggesting thus the intriguing possibility of using first-order hyperbolic spinorial flows to construct special solutions of curvature flows and GR.

Afterwards, we concentrated on the study of the topology and geometry of the subsequent parallel Cauchy pairs. We managed to classify all compact three-manifolds admitting parallel Cauchy pairs, proving that they are canonically equipped with a locally free action of $\mathbb{R}^2$ and are isomorphic to certain torus bundles over S^1, whose Riemannian structure we characterized in detail. Since the constraint equations of a parallel spinor correspond to a certain type of imaginary generalized Killing spinor equations [628], these results can be interpreted as classification results for three-manifolds admitting imaginary generalized Killing spinors.

Next, we committed ourselves to the study of left-invariant parallel spinor flows. We carried out first the classification of the associated left-invariant parallel Cauchy pairs on simply connected Lie groups in order to obtain, in a second stage, the classification of left-invariant parallel spinor flows on simply connected Lie groups, deriving the corresponding necessary and sufficient conditions for such flows to be immortal. These are, to the best of our knowledge, the first non-trivial examples of evolution flows of parallel spinors. Also, we used some of these examples to construct families of η-Einstein cosymplectic structures and to produce solutions to the left-invariant Ricci flow in three dimensions. This is surprising since the hyperbolic type of flow we have considered has a priori no relation with any parabolic type of curvature flows. It would be interesting to explore this potential relation in more generality and for more complicated types of spinorial equations, especially for those appearing as Killing spinor equations in four-dimensional Supergravity theories.

Finally, we defined the so-called comoving parallel spinor flows, which are special cases of parallel spinor flows which admit a neat and novel geometric interpretation. Restricting to this class, we have solved the corresponding equations in several examples, obtaining explicit families of four-dimensional Lorentzian manifolds carrying parallel spinors. As previously mentioned, the general investigation of comoving parallel spinor flows deserves further exploration and will be hopefully treated elsewhere.

6

Heisenberg-invariant self-dual Einstein four-manifolds

In the previous chapter, we studied the geometric properties of globally hyperbolic four-manifolds which are endowed with a real parallel spinor through its equivalent formulation in terms of a parallel spinor flow. This was of interest because these types of equations appear within the study of supersymmetric solutions of $\mathcal{N} = 1$ pure Supergravity in four dimensions.

Now we concentrate on the geometry of certain scalar manifolds appearing in Supergravity and ST. Although they do not describe the spacetime dynamics, they are fundamental for the understanding of the ST moduli spaces. In the case of four-dimensional scalar manifolds, it is known that particularly relevant instances of such four-manifolds are those with a principal and isometric action of the three-dimensional Heisenberg group (see the explanation below), so we will commit ourselves in this chapter to their investigation.

Indeed, principal group actions on pseudo-Riemannian manifolds play a prominent role in differential geometry. Many fundamental concepts, such as principal bundles or homogeneous spaces, are based on this notion. Among them, it is particularly interesting to consider cohomogeneity one principal actions, in which the corresponding orbits are of codimension one. This allows to reduce interesting systems of partial differential equations, such as the Einstein equation, to systems of ordinary differential equations. This is extremely relevant in theoretical physics as well, since the above principle allows to solve the field equations of GR in many important cases, for instance, within the context of cosmological models [667].

On the other hand, the homogeneous quaternionic Kähler manifolds of negative scalar curvature (except the quaternionic hyperbolic spaces) have been shown to admit a canonical deformation to a complete quaternionic Kähler manifold with a cohomogeneity-one isometric action [159]. This deformation is a particular case of what is called *one-loop deformation* [120, 157], which appears in the study of scalar manifolds in ST and Supergravity with one-loop corrections. In four dimensions it is found [159] that the isometry group of the deformed quaternionic Kähler manifold is $\mathrm{O}(2) \ltimes \mathrm{H}$, with H the three-dimensional Heisenberg group, and this motivated to carry out the classification of all Riemannian Einstein metrics of non-positive scalar curvature which are invariant under the action of $\mathrm{SO}(2) \ltimes \mathrm{H}$ in $\mathbb{R}^4$ [160]. Apart of a wealth of incomplete metrics, the authors of [160] showed that the only complete manifolds in the above class are the complex hyperbolic plane (also know as universal hypermultiplet in the physics literature) and its complete one-loop deformation.

Recall that a four-dimensional Riemannian metric is called quaternionic-Kähler if it is Einstein and self-dual (for appropriate choice of orientation). The notion of self-duality is also meaningful for metrics of neutral signature[1] and self-dual Einstein metrics of neutral signature are also called quaternionic paraKähler.

In this context, the objective of this work is the classification of all self-dual pseudo-Riemannian Einstein four-manifolds which admit a principal (cohomogeneity-one) isometric action of the Heisenberg group. The hypotheses are, on the one hand, more general than those of [160] in that we allow indefinite metrics and assume only the symmetry group H rather than $\mathrm{SO}(2) \ltimes \mathrm{H}$ but, on the other hand, more specific as we restrict to self-dual metrics.

Up to an overall sign in the metric, the manifolds considered can be decomposed as $(\mathcal{I} \times \mathrm{H}, \varepsilon \mathrm{d}t^2 + \chi_t)$, being $\mathcal{I} \subset \mathbb{R}$ an open interval parametrized by t (that we call time), $\varepsilon = \pm 1$ and $\{\chi_t\}_{t \in \mathcal{I}}$ a family of Riemannian or Lorentzian metrics on H, respectively. The key feature that will allow us to perform such classification is the use of a family of orthonormal or Witt frames on H, which are interpreted as the time evolution of an initial (orthonormal or Witt) frame on H.

More concretely, on the one hand we first consider the proper (i.e. $\mathrm{Scal} \neq 0$) quaternionic (para)Kähler four-manifolds with a Heisenberg principal group action and, attending to the causal character of the center of the Heisenberg group (for neutral-signature manifolds), we determine completely their isometry type. In the Riemannian case, apart from encountering the complex hyperbolic metric and the complete and incomplete one-loop deformed universal hypermultiplet metrics of negative scalar curvature as reported in [160], we also find counterparts positive scalar curvature.

For neutral-signature metrics, we are able to identify the solutions of positive scalar curvature as quaternionic paraKähler geometries arising from the so-called temporal and Euclidean Supergravity (one-loop deformed) c-maps[2] while those with negative scalar curvature do not seem to have been previously considered. Furthermore, we study when such metrics are complete.

On the other hand, we also investigate (para)hyperKähler four-manifolds endowed with a principal action of the Heisenberg group and provide a classification of all of them in terms of their isometry type, in similar lines to the quaternionic case. It turns out that the Ricci flat examples are incomplete with exception of a class of flat examples of neutral signature.

The outline of the chapter is as follows[3]. First we derive some preliminary results that will be key for the subsequent classifications we intend to make, such as proving that Einstein Heisenberg four-manifolds preserve the causal character of the Heisenberg center. Afterwards, we determine all quaternionic (para)Kähler four-manifolds which admit an isometric principal action of the Heisenberg group with non-degenerate orbits, distinguishing between Riemannian and neutral-signature signatures and, in the latter case, splitting the

[1] But not for Lorentzian signature, cf. Remark 6.22.

[2] We remind that the temporal (respectively, Euclidean) Supergravity c-map is induced by the reduction of four-dimensional Minkowskian (respectively, Euclidean) $\mathcal{N} = 2$ Supergravity coupled to vector multiplets over a timelike (respectively, spacelike) dimension, while the the usual Supergravity c-map (also called spatial c-map), from which the one-loop deformed universal hypermultiplet metric arises, is induced by the reduction of four-dimensional $\mathcal{N} = 2$ Supergravity coupled to vector multiplets over a spacelike dimension [137, 149].

[3] Most of the computations presented in this chapter have been carried out with the help of Mathematica 12.3, using the license of the University of Hamburg. The input and output of all computer-based computations can be downloaded at http://www.math.uni-hamburg.de/home/cortes/cortes_murcia.nb.

study between *timelike*, *spacelike* or *lightlike* quaternionic (para)Kähler Heisenberg four-manifolds, depending on the causal character of the Heisenberg center. Next, we classify similarly all (para)hyperKähler four-manifolds with an isometric principal action of the Heisenberg group with non-degenerate orbits. Finally, we conclude with a discussion of our findings.

6.1 The Heisenberg group and Heisenberg four-manifolds

In this section we revise basic features concerning left-invariant pseudo-Riemannian metrics on the Heisenberg group, introduce the concept of a Heisenberg four-manifold and prove some preliminary results about Einstein Heisenberg manifolds.

6.1.1 Heisenberg group

Recall that the three-dimensional Heisenberg group H is the unique, up to isomorphism, connected and simply connected non-abelian nilpotent three-dimensional Lie group. Its Lie algebra $\mathfrak{h}$ is called the Heisenberg algebra. As for any Lie group, there is a natural bijection between left-invariant pseudo-Riemannian metrics on H and pseudo-Euclidean scalar products on $\mathfrak{h}$. We will thus often refer to such a scalar product χ as a pseudo-Riemannian metric on $\mathfrak{h}$ and to the pair $(\mathfrak{h}, \chi)$ as a pseudo-Riemannian Heisenberg algebra.

Let χ be a pseudo-Riemannian metric on $\mathfrak{h}$. Then there always exists an orthonormal or a Witt basis[4] $v_1, v_2, v_3 \in \mathfrak{h}$ such that the Lie brackets are given by:

$$[v_1, v_2] = 0\,, \quad [v_1, v_3] = 0\,, \quad [v_2, v_3] = -2kv_1\,, \quad k \in \mathbb{R}^{>0}\,. \tag{6.1}$$

Note that $k \neq 0$ cannot be absorbed into a redefinition of the basis $\{v_1, v_2, v_3\}$. (However, it can be assumed positive, since the sign can be always switched by multiplying the vectors of the basis by minus one.) We observe that the center of the Heisenberg algebra, to which we will refer as the Heisenberg center, is spanned by v_1, which can be timelike, spacelike or lightlike. Given a pseudo-Riemannian Heisenberg algebra $(\mathfrak{h}, \chi)$, the isometry type of the corresponding pseudo-Riemannian metric on H is uniquely fixed after the specification of an orthonormal or Witt basis $\{v_1, v_2, v_3\}$ satisfying (6.1).

The three-dimensional Heisenberg group can be realized as $\mathbb{R}^3$ together with the following product:

$$(x, y, z) \cdot (a, b, c) = (a + x, b + y, c + z + ya - xb)\,, \quad (x, y, z), (a, b, c) \in \mathbb{R}^3\,. \tag{6.2}$$

From (6.2) one can readily get a basis[5] $\{w_1, w_2, w_3\}$ of $\mathfrak{h}$, given by the left-invariant vector fields

$$w_1 = \partial_z\,, \quad w_2 = \partial_x + ky\partial_z\,, \quad w_3 = \partial_y - kx\partial_z\,, \tag{6.3}$$

where (x, y, z) are standard coordinates on $\mathbb{R}^3$. In particular, note that the only non-vanishing Lie bracket of these vectors is that of $[w_2, w_3] = -2kw_1$. The dual basis of one-forms $\{w^1, w^2, w^3\}$ is given by:

$$w^1 = \mathrm{d}z + kx\mathrm{d}y - ky\mathrm{d}x\,, \quad w^2 = \mathrm{d}x\,, \quad w^3 = \mathrm{d}y\,. \tag{6.4}$$

[4] If (V, χ) is a three-dimensional Lorentzian vector space, we define a Witt basis $\{e_u, e_v, e_3\}$ as one which satisfies $\chi(e_u, e_v) = \chi(e_3, e_3) = 1$ and $\chi(e_u, e_u) = \chi(e_v, e_v) = \chi(e_u, e_3) = \chi(e_v, e_3) = 0$.

[5] Not necessarily orthonormal.

6.1.2 Heisenberg four-manifolds

Definition 6.1. A four-dimensional pseudo-Riemannian manifold (M, g) is said to be a *Heisenberg four-manifold* if it is foliated by the orbits of a principal and isometric action of the three-dimensional Heisenberg group.

Note that a Heisenberg four-manifold (M, g) admits an H-equivariant diffeomorphism identifying M with $\mathcal{I} \times \mathrm{H}$, where $\mathcal{I} \subset \mathbb{R}$ is either an open interval or a circle. Replacing M by its universal covering, if necessary, we can assume the former.

Within the class of Heisenberg four-manifolds (M, g), we shall restrict ourselves to those for which the restriction of g to the leaves is non-degenerate. In such a case, the metric g can be written in the form:

$$g = \varepsilon \mathrm{d}t^2 + \chi_t, \tag{6.5}$$

where $\varepsilon = \pm 1$ and $\{\chi_t\}_{t \in \mathcal{I}}$ is a family of left-invariant metrics on H parametrized by the time coordinate t. The different H-orbits are identified by means of the normal geodesic flow[6] generated by ∂_t, such that $\{(H, \chi_t)\}_{t \in \mathcal{I}}$ defines a family of pseudo-Riemannian Heisenberg groups. Up to orthogonal transformations, we can associate it to a family $\{(\mathfrak{e}^t_i)\}_{t \in \mathcal{I}}$ of left-invariant sections $(\mathfrak{e}^t_i)$ of the frame bundle F(H) such that $(\mathfrak{e}^t_i) \in \mathrm{F(H)}$ is an orthonormal $(i = 1, 2, 3)$ or Witt $(i = u, v, 3)$ frame for χ_t. Then $\{\partial_t, (\mathfrak{e}^t_i)\}$ conforms an orthonormal or Witt frame for (M, g). For ease of notation we may denote the triplet $(\mathfrak{e}^t_i)$ simply by $\mathfrak{e}^t_i$, thus writing $\{\mathfrak{e}^t_i\}_{t \in \mathcal{I}}$ and $\{\partial_t, \mathfrak{e}^t_i\}$ instead of $\{(\mathfrak{e}^t_i)\}_{t \in \mathcal{I}}$ and $\{\partial_t, (\mathfrak{e}^t_i)\}$. Analogously, we denote the corresponding family of dual orthonormal or Witt coframes on $\{(\mathrm{H}, \chi_t)\}_{t \in \mathcal{I}}$ by $\{\mathfrak{e}^i_t\}_{t \in \mathcal{I}}$ and the corresponding dual orthonormal or Witt coframe on (M, g) by $\{\mathrm{d}t, \mathfrak{e}^i_t\}$.
Having said this, we consider g to have the form:

$$g = \varepsilon \mathrm{d}t^2 + \eta_{ij} \mathfrak{e}^i_t \otimes \mathfrak{e}^j_t \,, \qquad \varepsilon = \pm 1 \,, \tag{6.6}$$

where:

$$\eta = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for orthonormal bases} \;, \quad \eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for Witt bases} \,. \tag{6.7}$$

Given the equivalent description of metrics (6.5) in terms of families of orthonormal or Witt frames $\{\mathfrak{e}^t_i\}_{t \in \mathcal{I}}$ in H, we may think of metrics on M as time evolutions of frames on H. (The evolution will be determined later from the self-dual Einstein equations.) In fact, let $t_0 \in \mathcal{I}$ be an initial time and $\mathfrak{e}^{t_0}_i$ an *initial* orthonormal or Witt frame for the initial metric χ_{t_0}. Then the time evolution $\{\mathfrak{e}^t_i\}_{t \in \mathcal{I}}$ of such initial frame determines the metric $\{\chi_t\}_{t \in \mathcal{I}}$ and, in turn, the four-dimensional pseudo-Riemannian manifold (M, g). We can write

$$\mathfrak{e}^t_i = U^t_{ij} \mathfrak{e}^{t_0}_j \,, \quad \mathfrak{e}^i_t = \mathfrak{e}^j_{t_0} (U^t)^{-1}_{ji} \,, \quad U^t \in \mathrm{GL}(3, \mathbb{R}) \,, \quad U^{t_0} = \mathrm{Id} \,, \tag{6.8}$$

where $U^{t_0} = \mathrm{Id}$ is the initial condition for the time evolution now encoded in $t \mapsto U^t$.
We denote by $\mathcal{Z} \subset \mathfrak{h} = T_e H$ the Heisenberg center and by $\mathcal{Z}_t \subset T_{(t,e)}(\{t\} \times H) \cong \mathfrak{h}$ the (constant) line which corresponds to $\mathcal{Z}$ under the canonical identification $\{t\} \times H \cong H$.

[6] We have also properly reparametrized t in order to correspond to the arc length parameter along a normal geodesic.

Definition 6.2. Let (M, g) be a Heisenberg four-manifold of neutral signature. We say it is *timelike, spacelike* or *lightlike* if $\mathcal{Z}_t$ is timelike, spacelike or lightlike respectively for every $t \in \mathcal{I}$.

It will be shown later that the causal character of $\mathcal{Z}_t$ is constant for Einstein Heisenberg four-manifolds.

6.1.3 Choice of adapted frames for Einstein Heisenberg four-manifolds

On studying Riemannian Heisenberg four-manifolds and neutral-signature timelike, spacelike or lightlike Heisenberg four-manifolds (M, g), it is convenient to use the following special frames $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ to describe the four-dimensional metric g.

Proposition 6.1. *Let (M, g) be a Heisenberg four-manifold.*

- *If $\varepsilon = 1$, then (M, g) is Riemannian and there exists an orthonormal frame $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ such that $\mathfrak{e}_1^t$ generates $\mathcal{Z}_t$ and $\mathfrak{e}_2^t$ is a linear combination of $\mathfrak{e}_2^{t_0}$ and $\mathfrak{e}_3^{t_0}$ for all $t \in \mathcal{I}$. This implies:*
$$U^t = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & b(t) & f(t) \\ j(t) & h(t) & c(t) \end{pmatrix}, \quad [\mathfrak{e}_2^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_1^{t_0}, \quad k > 0\,, \tag{6.9}$$
where $a, b, c, f, j, h \in C^\infty(\mathcal{I})$.
- *If $\varepsilon = -1$, then (M, g) is of neutral signature and:*
  - *If (M, g) is a timelike Heisenberg four-manifold, then there exists an orthonormal frame $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ such that $\mathfrak{e}_1^t$ spans $\mathcal{Z}_t$ and $\mathfrak{e}_2^t$ is a linear combination of $\mathfrak{e}_2^{t_0}$ and $\mathfrak{e}_3^{t_0}$ for all $t \in \mathcal{I}$. The conclusion is again (6.9).*
  - *For spacelike Heisenberg four-manifolds (M, g), we may choose[7] the center $\mathcal{Z}_t$ to be spanned by $\mathfrak{e}_3^t$ for every $t \in \mathcal{I}$ and thus we may use the ansatz:*
  $$U^t = \begin{pmatrix} c(t) & h(t) & j(t) \\ f(t) & b(t) & p(t) \\ 0 & 0 & a(t) \end{pmatrix}, [\mathfrak{e}_1^{t_0}, \mathfrak{e}_2^{t_0}] = -2k\mathfrak{e}_3^{t_0}\,, \quad k > 0\,, \tag{6.10}$$
  where $a, b, c, f, j, h, p \in C^\infty(\mathcal{I})$.
  - *If (M, g) is a lightlike Heisenberg four-manifold, we define $\mathfrak{e}_u^t = \frac{1}{\sqrt{2}}(\mathfrak{e}_2^t - \mathfrak{e}_1^t)$ and $\mathfrak{e}_v^t = \frac{1}{\sqrt{2}}(\mathfrak{e}_1^t + \mathfrak{e}_2^t)$, where $(\mathfrak{e}_1^t, \mathfrak{e}_2^t, \mathfrak{e}_3^t)$ is an orthonormal frame such that $\mathfrak{e}_u^{t_0} \in \mathcal{Z}_{t_0}$. For a certain interval $\mathcal{I}_l' \subset \mathcal{I}$ containing t_0, we may choose $\mathfrak{e}_3^t$ to be parallel to $\mathfrak{e}_3^{t_0}$ for every $t \in \mathcal{I}_l'$. Then $\{\mathfrak{e}_u^t, \mathfrak{e}_v^t, \mathfrak{e}_3^t\}$ is a (local) Witt basis such that[8]:*
  $$U_W^t = \begin{pmatrix} c(t) & h(t) & j(t) \\ f(t) & b(t) & p(t) \\ 0 & 0 & a(t) \end{pmatrix}, [\mathfrak{e}_v^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_u^{t_0}\,, \quad k > 0\,, \tag{6.11}$$

[7]The reason to choose $\mathfrak{e}_3^t$ rather than $\mathfrak{e}_1^t$ is (6.7), where we fixed the first vector of the orthonormal basis to be timelike.

[8]Differently from the timelike and spacelike cases, where we impose a vector of the time-dependent orthonormal basis to be parallel to the Heisenberg center, here we opt to fix the spacelike direction in the time-evolving Witt basis $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}_l'}$, so that $\mathfrak{e}_3^t$ remains parallel to $\mathfrak{e}_3^{t_0}$ and $\mathfrak{e}_u^t$ freely changes. As shown in Proposition 6.2, if we impose the lightlike Heisenberg four-manifold to be Einstein, it occurs additionally that $\mathfrak{e}_u^t$ stays parallel to the direction of the center given by $\mathfrak{e}_u^{t_0}$.

where we write U^t_W to indicate that this matrix is related to the Witt basis and $a, b, c, f, j, h, p \in C^\infty(\mathcal{I}'_l)$.

Proof. In the case $\varepsilon = 1$ and in the case $\varepsilon = -1$ with timelike center the plane $E = \mathrm{span}\{\mathfrak{e}^{t_0}_2, \mathfrak{e}^{t_0}_3\}$ is spacelike and the line $\mathcal{Z}_t$ is definite with respect to χ_t. Therefore, it suffices to choose a unit vector $\mathfrak{e}^t_1 \in \mathcal{Z}_t$, a unit vector $\mathfrak{e}^t_2 \in E$ perpendicular to $\mathfrak{e}^t_1$ with respect to χ_t and to complement this pair to an orthonormal frame. When $\varepsilon = -1$ with spacelike center, we can still choose a unit vector $\mathfrak{e}^t_3 \in \mathcal{Z}_t$, which suffices for the claim. Note that we cannot specialize further our choice of ansatz, since we do not know *a priori* the spacetime character of the intersection of $\mathrm{span}\{\mathfrak{e}^{t_0}_1, \mathfrak{e}^{t_0}_2\}$ with $\mathcal{Z}^{\perp_t}_t$, the orthogonal complement to the center with respect to χ_t. Finally, in the lightlike case there exists a subinterval $\mathcal{I}'_l \subset \mathcal{I}$ containing t_0 in which the line generated by $\mathfrak{e}^{t_0}_3$ is spacelike, since this is an open condition. Choosing $\mathfrak{e}^t_3$ to be parallel to $\mathfrak{e}^{t_0}_3$ for every $t \in \mathcal{I}'_l$ we conclude. □

Proposition 6.2. *Let $(M, g) = (\mathcal{I} \times \mathrm{H}, \varepsilon \mathrm{d}t^2 + \chi_t)$ be an Einstein Heisenberg four-manifold of neutral signature. Assume that $\mathcal{Z}_{t_0}$ for some $t_0 \in \mathcal{I}$ is timelike (resp. spacelike or lightlike). Then $\mathcal{Z}_t$ remains timelike (resp. spacelike or lightlike) for all $t \in \mathcal{I}$.*

Proof. First we will show that if the causal or spacetime character of $\mathcal{Z}_{t_0}$ is lightlike, then it remains invariant in an open subinterval of $\mathcal{I}$ containing t_0. For that, assuming that $\mathcal{Z}_{t_0}$ is lightlike, in an open subinterval $\mathcal{I}'_l$ we pick up the ansatz (6.11) for U^t_W and the Lie brackets at t_0. If Ric^g denotes the Riemann tensor of (M, g), we have:

$$\begin{aligned}
\mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_v) &= k\frac{-2ab(bc-fh)(bj-hp)a' + f'(bj-hp)^3}{a^2(bc-fh)^2} \\
&+ k\frac{(bj-hp)^2(c(pb'-bp')) + f(-jb'+hp'))}{a^2(bc-fh)^2} \\
&+ k\frac{-2fh^2pb' + b^3cj' - b^2(h(2jf'+fj')) + c(-jb'+ph'+hp')}{(bc-fh)^2} \\
&+ k\frac{bh(2hpf' + f(jb'+ph'+hp'))}{(bc-fh)^2}\,,
\end{aligned} \tag{6.12}$$

$$\begin{aligned}
\mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_3) &= -k\frac{b^3j(jc'-cj') + b^2(chpj' + 2a^2(-hc'+ch') + j^2(hf'-fh'))}{a^2(bc-fh)^2} \\
&- k\frac{b^2j(cph' + h(-2pc' + fj' - cp') + h^2(2a^2fb' + p^2(-cb'+hf'))}{a^2(bc-fh)^2} \\
&- k\frac{pfh^2(jb'-hp') + bh^2(p^2c' - 2a^2f' - 2jpf')}{a^2(bc-fh)^2} \\
&- k\frac{bhcp(jb'-ph'+hp') + bhf(-j^2b' - hpj' + j(ph'+hp'))}{a^2(bc-fh)^2}\,.
\end{aligned} \tag{6.13}$$

The Einstein condition implies that $\mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_v) = \mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_3) = 0$. We can solve for $j'(t)$ and $h'(t)$ and we get:

$$j'(t) = \frac{q_1(a, b, c, f, h, j, p, a', b', c', f', p')}{a^2b(bc-fh)(2a^2b^2c - f(bj-hp)^2)}\,, \tag{6.14}$$

$$h'(t) = \frac{q_2(a,b,c,f,h,j,p,a',b',c',f',p')}{a^2 b(2a^2b^2c - f(bj-hp)^2)}\,, \tag{6.15}$$

where $q_1, q_2 : \mathbb{R}^{12} \to \mathbb{R}$ are polynomials that vanish on the subspace $V \subset \mathbb{R}^{12}$ given by $j = h = 0$. (Note that the denominators are non-zero in a neighborhood of V, by non-degeneracy of U^t_W.) By a continuity argument, we can guarantee that $h(t) = j(t) = 0$ in $\mathcal{I}'_l$ and therefore $\mathcal{Z}_t$ is lightlike for all $t \in \mathcal{I}'_l$.

Similarly, if $\mathcal{Z}_{t_0}$ is timelike (resp. spacelike), then $\mathcal{Z}_t$ remains timelike (resp. spacelike) in a local subset of $\mathcal{I}$ around t_0, since these are open conditions. Therefore, the three subsets $\mathcal{I}_{\text{time}}$, $\mathcal{I}_{\text{space}}$, $\mathcal{I}_{\text{light}}$ of $\mathcal{I}$ consisting of t at which $\mathcal{Z}_t$ is respectively timelike, spacelike or lightlike are open. By connectedness of $\mathcal{I}$ this proves that $\mathcal{I}$ coincides with one of these three subsets. □

Remark 6.1. Proposition 6.2 implies that timelike, spacelike and lightlike Einstein Heisenberg four-manifolds cover all possibilities of neutral-signature Einstein Heisenberg four-manifolds.

Remark 6.2. Observe that the Einstein condition in the previous proposition is too strong, and one may actually require much weaker conditions for the proposition to equally hold, such as requiring that $\mathrm{Ric}^g(\partial_t)|_{\{t\times H\}} = 0$.

Proposition 6.3. *Let (M,g) be a Riemannian or neutral-signature Einstein Heisenberg four-manifold. Then:*

- *If (M,g) is either Riemannian or of neutral-signature and timelike, then we can choose:*

$$U^t = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & b(t) & 0 \\ 0 & h(t) & c(t) \end{pmatrix}, \quad [\mathfrak{e}^{t_0}_2, \mathfrak{e}^{t_0}_3] = -2k\mathfrak{e}^{t_0}_1, \quad k > 0\,. \tag{6.16}$$

- *If (M,g) is neutral-signature and spacelike, we can choose the following ansatz in an open subinterval $\mathcal{I}'_s \subset \mathcal{I}$ containing t_0:*

$$U^t = \begin{pmatrix} c(t) & h(t) & 0 \\ -h(t) & b(t) & 0 \\ 0 & 0 & a(t) \end{pmatrix}, \quad [\mathfrak{e}^{t_0}_1, \mathfrak{e}^{t_0}_2] = -2k\mathfrak{e}^{t_0}_3, \quad k > 0\,. \tag{6.17}$$

- *If (M,g) is neutral-signature and lightlike, we can pick up in an open interval $\mathcal{I}'_l \subset \mathcal{I}$:*

$$U^t_W = \begin{pmatrix} 1 & 0 & 0 \\ f(t) & b(t) & p(t) \\ 0 & 0 & a(t) \end{pmatrix}, [\mathfrak{e}^{t_0}_v, \mathfrak{e}^{t_0}_3] = -2k\mathfrak{e}^{t_0}_u\,, \quad k > 0\,. \tag{6.18}$$

Proof. Regarding the neutral-signature case:

- If (M,g) is timelike, we can choose for every $t \in \mathcal{I}$ the ansatz (6.9) for the matrix U^t and for the Lie brackets at t_0. If Ric^g denotes the Ricci tensor of (M,g), we find:

$$\mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_2) = \frac{k\,(b\,(jc' - cj') - f\,(jh' - hj'))}{a^2}\,, \quad \mathrm{Ric}^g(\partial_t, \mathfrak{e}^t_3) = \frac{kj\,(fb' - bf')}{a^2}\,. \tag{6.19}$$

If (M, g) is in addition Einstein, then the previous components must identically vanish. Assume $j(t) \neq 0$. Then we would find that $f(t) = f_0 b(t)$ in an interval $\mathcal{I}_0 \subset \mathcal{I}$ in which $b(t) \neq 0$ (which always exists given that $b(t_0) = 1$). Since $f(t_0) = 0$, then $f(t) = 0$ in $\mathcal{I}_0$. Assume now there exists a lower (or upper) bound $t_1 \in \mathcal{I}$ for $\mathcal{I}_0$ such that $b(t_1) = 0$. By continuity we would have that $f(t_1) = 0$ and the matrix U^t would be degenerate at t_1, what contradicts the fact that $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ is an orthonormal basis. Then we learn that such t_1 does not exist and $f(t) = 0$ in the entire interval in $\mathcal{I}$, which in turn implies that $j(t) = j_0 c(t)$ in some open subinterval of $\mathcal{I}$. Owing the fact that $j(t_0) = 0$, by the same reasoning as above we conclude that $j(t) = 0$ in the whole $\mathcal{I}$.

- If (M, g) is spacelike we can choose for every $t \in \mathcal{I}$ the ansatz (6.10) for U^t and the Lie brackets at t_0. If Ric^g denotes the Ricci tensor of (M, g), we have:

$$\mathrm{Ric}^g(\partial_t, \mathfrak{e}_1^t) = k\frac{f'(bj - hp) + c(pb' - bp') + f(hp' - jb')}{a^2}\,, \tag{6.20}$$

$$\mathrm{Ric}^g(\partial_t, \mathfrak{e}_2^t) = k\frac{b(jc' - cj') + h(fj' - pc') + h'(cp - fj)}{a^2}\,. \tag{6.21}$$

In order to have $\mathrm{Ric}^g(\partial_t, \mathfrak{e}_1^t) = \mathrm{Ric}^g(\partial_t, \mathfrak{e}_2^t) = 0$ we must demand:

$$p'(t) = \frac{cpb' - fjb' + f'(bj - hp)}{bc - fh}\,, \tag{6.22}$$

$$j'(t) = \frac{bjc' - hpc' + h'(cp - fj)}{bc - fh}\,. \tag{6.23}$$

However, taking into account the initial conditions $a(t_0) = b(t_0) = c(t_0) = 1$, $f(t_0) = h(t_0) = j(t_0) = p(t_0) = 0$ and $bc - fh \neq 0$ (from the non-degeneracy of U^t), through the use of the uniqueness and existence theorem of ODEs we infer that $p(t) = j(t) = 0$ for all $t \in \mathcal{I}$. Performing now appropriate $\mathrm{SO}(1,1)$ rotations, we can finally impose U_t to have the same form as in (6.17) in an open subinterval $\mathcal{I}_s' \subset \mathcal{I}$.

- If (M, g) is lightlike, then (6.18) just follows from the results obtained in the proof of Proposition 6.2 and by absorbing[9] the factor $c(t)$ in the functions $f(t), b(t)$ and $p(t)$, which yields the same metric.

Regarding the Riemannian case, using (6.9) we compute:

$$\mathrm{Ric}^g(\partial_t, \mathfrak{e}_2^t) = \frac{k\,(b\,(cj' - jc') + f\,(jh' - hj'))}{a^2}\,, \quad \mathrm{Ric}^g(\partial_t, \mathfrak{e}_3^t) = \frac{kj\,(bf' - fb')}{a^2}\,. \tag{6.24}$$

We observe that, up to a global sign, this is exactly the same as the result obtained in (6.19) for timelike Einstein Heisenberg four-manifolds. Then we equivalently conclude that $j(t) = 0$, and by an appropriate $\mathrm{SO}(2)$ rotation, we arrive to (6.16). □

[9]Another way to see this is by noticing that $\mathfrak{e}_t^u = \frac{1}{c(t)}\mathfrak{e}_{t_0}^u$, so that $g = -\mathrm{d}t^2 + \frac{1}{c(t)}\mathfrak{e}_{t_0}^u \odot \mathfrak{e}_t^v + \mathfrak{e}_t^3 \otimes \mathfrak{e}_t^3$. By redefining $\frac{1}{c(t)}\mathfrak{e}_t^v \to \mathfrak{e}_t^v$, we observe that we can set $c(t) = 1$.

6.2 Quaternionic (para)Kähler Heisenberg four-manifolds

In this section we classify all Heisenberg four-manifolds which satisfy the condition of being quaternionic (para)Kähler. For that, we revise first the definition of a quaternionic (para)Kähler four-manifold. Recall first that an orientable pseudo-Riemannian four-manifold of Riemannian or neutral signature is called *half-conformally flat* if its Weyl tensor is self-dual for one of the two orientations.

Definition 6.3. Let (M, g) be a Riemannian or neutral-signature orientable four-manifold. It is said to be *quaternionic Kähler* (resp. *quaternionic paraKähler*) if and only if it is Einstein with non-zero Einstein constant and half-conformally flat. We shall refer to them jointly as *quaternionic (para)Kähler* four-manifolds.

Remark 6.3. We observe that the definition of a quaternionic (para)Kähler four-manifold cannot be applied for larger dimensions, since the notion of self-duality is restricted to four-dimensions. Actually, for dimensions $D = 4m$ with $m > 1$ the definition of quaternionic (para)Kähler manifolds is that of pseudo-Riemannian manifolds admitting a parallel skew-symmetric (para)quaternionic structure Q. We recall that such a structure Q is locally spanned by three anticommuting endomorphism fields $I, J, K = IJ$ such that $I^2 = J^2 = \pm\mathrm{Id}$. However, in four dimensions this definition is too weak, since every orientable four-manifold satisfies it, and it turns out that the natural definition of quaternionic (para)Kähler four-manifold is that of Definition 6.3 [143].

Let W^g denote the Weyl tensor of (M, g). We define the *Weyl self-duality tensor* $\mathcal{W}^g$ as the $(0, 4)$-tensor given by:

$$\mathcal{W}^g(X, Y, U, V) = g((\star\mathrm{W}^g)(X, Y)U, V) - g(\mathrm{W}^g(X, Y)U, V)\,, \quad X, Y, U, V \in \mathfrak{X}(M)\,, \tag{6.25}$$

where $\star$ denotes the Hodge star map with respect to a given orientation on (M, g). Up to a factor, it is the antiself-dual part of the Weyl tensor and, hence, the obstruction to $(M, g, \star)$ being self-dual. Then we have that a four-manifold will be quaternionic (para)Kähler if and only if

$$\mathcal{W}^g = 0\,, \quad \mathrm{Ric}^g = \Lambda g\,, \quad \Lambda \in \mathbb{R} \setminus \{0\}\,, \tag{6.26}$$

for one of the two orientations, where Ric^g denotes the Ricci tensor of (M, g). (We will always consider the orientation such that $\mathcal{W}^g = 0$.)

Now we proceed to the classification of quaternionic (para)Kähler Heisenberg four-manifolds attending to the signature of the metric and the spacetime character of $\mathcal{Z}_{t_0}$.

6.2.1 Quaternionic Kähler and timelike quaternionic paraKähler Heisenberg four-manifolds

We start by classifying all (Riemannian) quaternionic Kähler Heisenberg four-manifolds and all timelike quaternionic paraKähler Heisenberg four-manifolds. Since we will carry out such classification simultaneously, it is convenient to coin the term (timelike) quaternionic (para)Kähler Heisenberg four-manifolds to refer to both of them at once. To particularize our results to one of these cases, we just have to set $\varepsilon = \pm 1$ correspondingly.

The reason for treating them at the same time is that we can use the same identical ansatz to describe the three-dimensional metric χ_t for both quaternionic Kähler and timelike quaternionic paraKähler Heisenberg four-manifolds. Indeed, such ansatz is given by (6.16):

$$U^t = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & b(t) & 0 \\ 0 & h(t) & c(t) \end{pmatrix}, \quad [\mathfrak{e}_2^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_1^{t_0}, \quad k > 0\,. \tag{6.27}$$

Proposition 6.4. *The non-zero components of the Ricci tensor* Ric^g *of the metric obtained from* (6.27) *are:*

$$\mathrm{Ric}^g(\partial_t, \partial_t) = -\frac{2\,(a')^2}{a^2} + \frac{a''}{a} - \frac{\left(4b^2 + h^2\right)(c')^2 + c^2\left(4\,(b')^2 + (h')^2 - 2bb''\right)}{2b^2c^2} + \frac{\left(hc'h' + b^2c''\right)}{b^2c}\,, \tag{6.28}$$

$$\mathrm{Ric}^g(\mathfrak{e}_1^t, \mathfrak{e}_1^t) = \frac{2b^3c^3k^2 - 2bc\,(a')^2 - aa'\,(cb' + bc') + abca''}{a^2bc}\,, \tag{6.29}$$

$$\mathrm{Ric}^g(\mathfrak{e}_2^t, \mathfrak{e}_2^t) = \frac{4b^4c^4k^2 + a^2\left(4c^2\,(b')^2 - (hc' - ch')^2\right) + 2abc\,(ca'b' + ab'c' - acb'')}{-2a^2b^2c^2\varepsilon}\,, \tag{6.30}$$

$$\mathrm{Ric}^g(\mathfrak{e}_2^t, \mathfrak{e}_3^t) = \frac{c\,(hc' - ch')\,(3ab' + ba') + ab\left(h\,(c')^2 - cc'h' + c\,(-hc'' + ch'')\right)}{2ab^2c^2\varepsilon}\,, \tag{6.31}$$

$$\mathrm{Ric}^g(\mathfrak{e}_3^t, \mathfrak{e}_3^t) = \frac{2b^4c^4k^2 + a^2bcb'c' + \frac{a^2}{2}\,(hc' - ch')^2 + ab^2\left(ca'c' + 2a\,(c')^2 - acc''\right)}{-a^2b^2c^2\varepsilon}\,. \tag{6.32}$$

Proof. Just by direct computation. □

Proposition 6.5. *Let* (M, g) *be a (timelike) quaternionic (para)Kähler Heisenberg four-manifold. Then:*

$$a'(t) = -\frac{k^2b^6 + a^2(\varepsilon\Lambda b^2 + (b')^2)}{2abb'}, \quad \varepsilon\Lambda = \frac{3(kb^3 - ab')^3}{a^2b^2(-3kb^3 + ab')}\,, \quad c = b\,, \quad h = 0\,. \tag{6.33}$$

Proof. Remember that a (timelike) quaternionic (para)Kähler Heisenberg four-manifold is Einstein (with non-zero Einstein constant) and is half-conformally flat. Therefore, we have to impose conditions (6.26).

Firstly, we observe that the Einstein condition imposes that $\mathrm{Ric}^g(\mathfrak{e}_2^t, \mathfrak{e}_3^t) = 0$. By Proposition 6.4, one can solve for h'' and get:

$$h'' = \frac{-h(c')^2 + cc'h' + chc''}{c^2} + \frac{ca'(-hc' + ch')}{ac^2} + \frac{3cb'(-hc' + ch')}{bc^2}\,. \tag{6.34}$$

Now we move into the Weyl self-duality tensor $\mathcal{W}^g$ defined back at (6.25). The component $\mathcal{W}^g(\mathfrak{e}_1^t, \mathfrak{e}_2^t, \mathfrak{e}_1^t, \mathfrak{e}_3^t)$ reads

$$\mathcal{W}^g(\mathfrak{e}_1^t, \mathfrak{e}_2^t, \mathfrak{e}_1^t, \mathfrak{e}_3^t) = -\frac{(kbc + a')(-hc' + ch')}{2abc}\,. \tag{6.35}$$

For the latter to be zero, either $kbc + a' = 0$ or $-hc' + ch' = 0$. If $kbc + a' = 0$, we would find that the Einstein constant has to vanish, so if we assume that $\Lambda \neq 0$, we must have

$-hc' + ch' = 0$, which implies in turn that $h(t) = h_0 c(t)$. However, since $h(t_0) = 0$, then $h(t) = 0$.

Setting $h = 0$ in Proposition 6.4, on imposing the Einstein condition $\mathrm{Ric}^g = \Lambda g$ we can solve for $a''(t), b''(t), c''(t)$ and $a'(t)$ and obtain:

$$a'(t) = -\frac{k^2 b^3 c^3 + a^2(\varepsilon\Lambda bc + b'c')}{a(cb' + bc')}\,, \tag{6.36}$$

$$a''(t) = \frac{2k^4 b^7 c^7 + k^2 a^2 b^3 c^3(-3c^2(b')^2 - 2bcb'c' + b^2(4\varepsilon\Lambda c^2 - 3(c')^2)}{a^3 bc(cb' + c'b)^2} + \frac{a^4(2\Lambda^2 b^3 c^3 - c^2(b')^3 c' - b^2 b'c'(-4\varepsilon\Lambda c^2 + (c')^2))}{a^3 bc(cb' + c'b)^2}\,, \tag{6.37}$$

$$b''(t) = \frac{k^2 c^3 b^4(2bc' + cb') + a^2(b^2(c')^2 b' + 2c^2(b')^3 + bcc'(\varepsilon\Lambda b^2 + 2(b')^2))}{a^2 bc(cb' + bc')}\,, \tag{6.38}$$

$$c''(t) = \frac{k^2 b^3 c^4(2cb' + bc') + a^2(c^2(b')^2 c' + 2b^2(c')^3 + bcb'(\varepsilon\Lambda c^2 + 2(c')^2))}{a^2 bc(cb' + bc')}\,. \tag{6.39}$$

More precisely, these equations should be written with the $(cb' + bc')$-factors on the left-hand side, to avoid possible zeros of the denominator, which we will not do to keep the formulas simple. The same comments apply to the formulas below. On substituting these results into the Weyl self-duality tensor, we encounter

$$\mathcal{W}^g(\mathfrak{e}_1^t, \mathfrak{e}_2^t, \mathfrak{e}_1^t, \mathfrak{e}_2^t) = \frac{-3k^3 b^5 c^4 + 3k^2 ab^3 c^2(2cb' + bc') - 3ka^2 bc(\varepsilon\Lambda b^2 c + c(b')^2 + 2bb'c')}{3a^3 b(bc' + cb')} + \frac{2\varepsilon\Lambda bcb' - \varepsilon\Lambda b^2 c' + 3(b')^2 c'}{3b(bc' + cb')}\,. \tag{6.40}$$

From here one can solve for Λ and obtain

$$\Lambda = 3\frac{(kb^2 c - ab')^2(-kbc^2 + ac')}{\varepsilon a^2 b(3kb^2 c^2 - 2acb' + abc')}\,. \tag{6.41}$$

Taking this result into the rest of the components of $\mathcal{W}^g$, we find in particular:

$$\mathcal{W}^g(\partial_t, \mathfrak{e}_1^t, \partial_t, \mathfrak{e}_1^t) = \frac{\varepsilon(cb' - bc')(-kb^2 c + ab')(-kc^2 b + ac')}{abc(3kb^2 c^2 - 2acb' + abc')}\,. \tag{6.42}$$

It follows that the first term, the second or the third term in brackets vanishes. If the second or third one is identically zero, then we find that $\Lambda = 0$, upon substitution in (6.41). Since we are assuming non-zero Einstein constant, we discard this possibility and then $c'b = b'c$, which in turn implies that $b = c$ since $b(t_0) = c(t_0) = 1$. Imposing this condition, we find that all components of $\mathcal{W}^g$ vanish identically. Finally, we observe that (6.41) can be simplified to take the form:

$$\Lambda = \frac{3(kb^3 - ab')^3}{\varepsilon a^2 b^2(-3kb^3 + ab')}\,. \tag{6.43}$$

Upon use of this expression, its first time derivative as well as equation (6.36) we check that (6.37), (6.38) and (6.39) are satisfied as well and we conclude. □

Proposition 6.6. *Let (M, g) be a (timelike) quaternionic (para)Kähler Heisenberg four-manifold. Then:*

- *The eigenvalues of the Weyl tensor, understood as a symmetric endomorphism of the bundle of two-forms, are given by $(-2\nu, \nu, \nu)$, where:*

$$\nu = \frac{16k^3b^7}{\varepsilon a^3(-3kb^3 + ab')} \,. \tag{6.44}$$

- *(M, g) is conformally Kähler for two complex structures with opposite orientations.*

Proof. Define the following triplet of self-dual two-forms:

$$\omega_i = \mathrm{d}t \wedge \mathfrak{e}^i_t + \star(\mathrm{d}t \wedge \mathfrak{e}^i_t) \,, \tag{6.45}$$

where $\star$ denotes the Hodge dual operation. Interpreting the Weyl tensor as a symmetric endomorphism of the bundle of two-forms in the canonical way[10], we observe that:

$$\mathrm{W}^g(\omega_1) = -2\nu\omega_1 \,, \quad \mathrm{W}^g(\omega_2) = \nu\omega_2 \,, \quad \mathrm{W}^g(\omega_3) = \nu\omega_3 \,. \tag{6.46}$$

This proves the first part of the proposition. Regarding the second one, we first note that the rescaled two-form

$$\tilde{\omega}_1 = |\mathrm{W}^g|_g^{2/3}\omega_1 \tag{6.47}$$

is closed and satisfies that $|\tilde{\omega}_1|^2_{\tilde{g}} = 4$ with respect to the rescaled metric $\tilde{g} = |\mathrm{W}^g|_g^{2/3} g$, in agreement with [668, 669] in the Riemannian case. We claim that $\tilde{g}$ is pseudo-Kähler with the Kähler form $\tilde{\omega}_1$. To prove the integrability of the almost complex structure $J_1 = \tilde{g}^{-1}\tilde{\omega}_1 = g^{-1}\omega_1$ we use that the following rescaled two-forms

$$\tilde{\omega}_2 = ab\, \omega_2 \,, \quad \tilde{\omega}_3 = ab\, \omega_3 \tag{6.48}$$

are closed. This is a consequence of Lemma 6.1, a simple generalization of the Hitchin lemma [139]. By introducing a relative sign in (6.45) we can likewise obtain a conformally Kähler structure (M, g, J_1') for the opposite orientation. (So contrary to J_1 the complex structure J_1' is not subordinate to the (para)quaternionic structure $Q = \mathrm{span}\{J_1, J_2, J_3\}$.) □

Lemma 6.1. *Let (M, g) be a pseudo-Riemannian manifold endowed with two anticommuting skew-symmetric endomorphism fields J_2, J_3 such that $J_2^2 = J_3^2 = -\varepsilon\mathrm{Id}$, where $\varepsilon = \pm 1$. Assume that the two-forms $\omega_i = g \circ J_i$ are closed for $i = 2, 3$. Then $J_1 := J_2 J_3$ is an integrable skew-symmetric complex structure.*

Proof. It is trivial to check that J_1 is a skew-symmetric almost complex structure. To prove the integrability we use that $\Omega = \omega_2 + i\varepsilon\omega_3$ is of type $(2, 0)$ with respect to J_1 and non-degenerate (as a complex bilinear form). Due to these properties, it suffices to show that $\Omega([\bar{X}, \bar{Y}], Z) = 0$, for all vector fields X, Y, Z of type $(1, 0)$, since this implies the involutivity of the $(-i)$-eigendistribution of J_1. This is an immediate consequence of $\mathrm{d}\Omega = 0$, since $(\mathrm{d}\Omega)(\bar{X}, \bar{Y}, Z) = -\Omega([\bar{X}, \bar{Y}], Z)$. □

[10] In components, $(\mathrm{W}^g(\omega_i))_{\mu\nu} = (\mathrm{W}^g)_{\mu\nu\rho\sigma}\omega_i^{\rho\sigma}$.

Remark 6.4. We have shown in the proof of Proposition 6.6 that (M, g) is not only conformally Kähler but admits two almost (para)Kähler structures $\tilde{\omega}_2$ and $\tilde{\omega}_3$ compatible with a second conformally rescaled metric $g' := ab\, g$, such that $|\tilde{\omega}_2|^2_{g'} = |\tilde{\omega}_3|^2_{g'} = 4\varepsilon$.

Remark 6.5. Note that Lemma 6.1 can be easily adapted to include the case $J_2^2 = -J_3^2 = \mathrm{Id}$. The conclusion is then that J_1 is an integrable skew-symmetric paracomplex structure. In fact, in that case one can consider $\Omega = \omega_2 + e\omega_3$, which takes values in the ring $\mathbb{R}[e] \cong \mathbb{R} \oplus \mathbb{R}$ generated by e with the relation $e^2 = 1$ (the ring of paracomplex numbers). Note that Ω has type $(2, 0)$ in the sense that $\Omega(J_1\cdot, \cdot) = \Omega(\cdot, J_1\cdot) = e\Omega$ and is non-degenerate in the sense that a real vector X satisfies $\Omega(X + eJ_1X, Y + eJ_1Y) = 0$ for all real vectors Y if and only if $X = 0$.

The second equation in (6.33) is a cubic equation for $b'(t)$, and depending on the values of k and Λ, we may have one or more real solutions. Define[11]:

$$\mathcal{B}_l = \frac{1}{3a^3}\left(3ka^2b^3 - \frac{\varepsilon\Lambda a^6 b^2 e^{-2i\pi(l-1)/3}}{(9k\varepsilon\Lambda a^8 b^5 + \sqrt{\Lambda^2 a^{16} b^6(\varepsilon\Lambda a^2 + 81k^2b^4)})^{1/3}}\right.$$
$$\left. + e^{2i\pi(l-1)/3}(9k\varepsilon\Lambda a^8 b^5 + \sqrt{\Lambda^2 a^{16} b^6(\varepsilon\Lambda a^2 + 81k^2b^4)})^{1/3}\right), \quad l = 1, 2, 3\,. \tag{6.49}$$

Proposition 6.7. *Let (M, g) be a (timelike) quaternionic (para)Kähler Heisenberg four-manifold. Then $a' = -\frac{k^2b^6 + a^2(\varepsilon\Lambda b^2 + (b')^2)}{2abb'}$ and:*

- *If $\varepsilon\Lambda > -81k^2$, $b' = \mathcal{B}_1$.*
- *If $\varepsilon\Lambda \leq -81k^2$, there are three solutions for b' obtained by setting $l = 1, 2, 3$ in (6.49), $b'_l = \mathcal{B}_l$.*

Proof. The result for a' was derived in Proposition 6.5. If we define now $\beta = kb^3 - ab'$, the second equation in (6.33) is equivalent to

$$\beta^3 + \frac{\varepsilon a^2 b^2 \Lambda}{3}\beta + \frac{2k\varepsilon}{3}a^2b^5\Lambda = 0\,. \tag{6.50}$$

At t_0, this equation reads

$$\beta^3 + \frac{\varepsilon\Lambda}{3}\beta + \frac{2k\varepsilon}{3}\Lambda = 0\,. \tag{6.51}$$

The discriminant Δ of this equation takes the form:

$$\Delta = -\frac{4}{27}\Lambda^2\left(\varepsilon\Lambda + 81k^2\right)\,. \tag{6.52}$$

By standard theory of cubic equations, if $\Delta < 0$ then there is just one real solution to (6.51) and if $\Delta \geq 0$ there exist three (maybe multiple) real roots. Thus if $\varepsilon\Lambda > -81k^2$, there is only a unique real solution to (6.51) (given by[12] (6.49) with $l = 1$) and therefore there is a unique solution for $b'(t_0)$, which in turn produces one real solution for $(a(t), b(t))$ defined on an appropriate interval $\mathcal{I}$. If $\varepsilon\Lambda \leq -81k^2$, there are three real roots (with at least two identical roots when the equality holds) to (6.51) and therefore there are three real solutions for $b'(t_0)$ (given by (6.49), $l = 1, 2, 3$). These yield three real solutions for $(a(t), b(t))$, each defined on intervals $\mathcal{I}_l$. □

[11] Given a complex number $z \in \mathbb{C}$, it has always three cubic roots. When we write $z^{1/3}$ we will mean by convention the cubic root $z^{1/3} = |z|^{1/3}e^{i/3\arg(z)}$.

[12] It can be checked that the expression (6.49) with $l = 1$ takes real values for all $\varepsilon\Lambda \neq 0$.

Proposition 6.7 provides a system of ordinary differential equations for $(a(t), b(t))$ with the initial condition $a(t_0) = b(t_0) = 1$. By virtue of the theorem of existence and uniqueness of ordinary differential equations, it is enough to find one solution for each possible value of k and $\varepsilon\Lambda$, since it will be unique[13]. In fact, we find it is convenient to split our study into three different possibilities according to the value of $\varepsilon\Lambda$, and we will distinguish between stationary, negative and positive (timelike) quaternionic (para)Kähler Heisenberg four-manifolds (to be defined below).

6.2.1.1 Stationary solutions

Definition 6.4. A (timelike) quaternionic (para)Kähler Heisenberg four-manifold is said to be stationary if $\varepsilon\Lambda = -6k^2$.

The name stationary comes from the fact that if we set $b = c$ and $h = 0$ in the expression for the Ricci tensor given at Proposition 6.4, then the Einstein condition reads:

$$\begin{aligned} \varepsilon\Lambda &= \mu' + 2\lambda' - \mu^2 - 2\lambda^2\,, && (6.53)\\ \varepsilon\Lambda &= 2\frac{k^2 b^4}{a^2} - (-\mu' + 2\mu\lambda + \mu^2)\,, && (6.54)\\ \Lambda &= -2\varepsilon\frac{k^2 b^4}{a^2} - \varepsilon(-\lambda' + \lambda\mu + 2\lambda^2)\,, && (6.55) \end{aligned}$$

where $\mu = \log(a)'$ and $\lambda = \log(b)'$. If we set $\mu' = \lambda' = 0$ we find that $\varepsilon\Lambda = -6k^2$, hence the name stationary.

Theorem 6.1. *If (M, g) is a stationary (timelike) quaternionic (para)Kähler Heisenberg four-manifold, then*

$$g = \varepsilon \mathrm{d}t^2 + \varepsilon e^{4k(t-t_0)} \mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + e^{2k(t-t_0)} \left(\mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0} + \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\right)\,, \tag{6.56}$$

where $t \in \mathbb{R}$. They are isometric to an open orbit of the solvable Iwasawa subgroup of $\mathrm{SU}(1,2) \cong \mathrm{SU}(2,1)$ on the symmetric space

$$\frac{\mathrm{SU}\left(\frac{3+\varepsilon}{2}, \frac{3-\varepsilon}{2}\right)}{\mathrm{S}\left(\mathrm{U}(1) \times \mathrm{U}\left(\frac{3+\varepsilon}{2}, \frac{1-\varepsilon}{2}\right)\right)}, \tag{6.57}$$

where $\mathrm{U}(p,q)$ denotes the (pseudo-)unitary group of the Hermitian sesquilinear form of index q. Moreover, when (M, g) is Riemannian (resp. neutral signature) it is complete (resp. incomplete).

Proof. Since $\mu' = \lambda' = 0$ implies $\varepsilon\Lambda = -6k^2$, let us start by assuming $\mu' = \lambda' = 0$. Consistency then requires

$$\left(\frac{b^4}{a^2}\right)' = 0\,. \tag{6.58}$$

[13] It can be seen that the conditions for the existence and uniqueness theorem to hold are satisfied, at least locally around the initial condition. Note that the derivatives are already solved at one side of the equations, which facilitates this check.

This implies that $\mu = 2\lambda$. On substituting in (6.53) we encounter:

$$-6\lambda^2 = \varepsilon\Lambda\,, \tag{6.59}$$

and hence $\lambda = \pm\sqrt{\frac{-\varepsilon\Lambda}{6}} = \pm k$ (remember that $k > 0$). By the above, $\mu = \pm 2k$. We observe in turn that the other equations (6.54), (6.55) are satisfied and therefore the solution is:

$$a = e^{\pm 2k(t-t_0)}\,, \quad b = e^{\pm k(t-t_0)}\,, \tag{6.60}$$

where we have already imposed that $a(t_0) = b(t_0) = 1$. From here we directly derive (6.56). Finally, we find that that Weyl tensor is self-dual only if we pick up the minus[14] sign above, so we check that these solutions are indeed stationary (timelike) quaternionic (para)Kähler four-manifolds.

In fact, solution (6.56) exhausts the list of possible stationary (timelike) quaternionic (para)Kähler four-manifolds. First, we note that Proposition 6.7 guarantees that if $\varepsilon\Lambda > -81k^2$, then the cubic equation for b' in (6.33) has a unique real solution given by $b' = \mathcal{B}_0$. Consequently, as explained before, the existence and uniqueness theorem for ODEs guarantees that the static solution (6.56) represents the unique solution for $\varepsilon\Lambda = -6k^2$, completing thus the classification of stationary (timelike) quaternionic (para)Kähler four-manifolds.

On the other hand, after some computations we find that these configurations satisfy that $\nabla \mathrm{R}^g = 0$, where R^g is the Riemann curvature tensor of g. Comparing to the classification of pseudo-Riemannian symmetric spaces of quaternionic Kähler type, see [670] and [671,672], we conclude (comparing curvature tensors) that the resulting spaces are locally isometric to the symmetric spaces (6.57). More precisely, the solutions are locally isometric to a left-invariant metric on the simply transitive solvable Iwasawa subgroup of $\mathrm{SU}(1,2)$ and $\mathrm{SU}(2,1)$ when $\varepsilon = 1$ and $\varepsilon = -1$, respectively. To see this it suffices to observe that the Heisenberg group is included in a four-dimensional group of isometries, which is precisely the above-mentioned Iwasawa group. In fact, the one-parameter group $t \mapsto t+t_0$, $x \mapsto e^{t_0}x$, $y \mapsto e^{t_0}y$, $z \mapsto e^{2t_0}z$ acts by isometries, enlarging the Heisenberg group by a one-parametric group of automorphisms to the aforementioned solvable group. Finally, in the Riemannian case the metric is complete, since the interval $\mathcal{I}$ of definition of g can be extended to the whole real line $\mathbb{R}$, while in the neutral-signature case the metric is (geodesically) incomplete, because $\dfrac{\mathrm{SU}\,(1,2)}{\mathrm{S}\,(\mathrm{U}(1)\times\mathrm{U}\,(1,1))}$ is homotopically equivalent to S^2 and hence it cannot be diffeomorphic to $\mathbb{R}^4$. □

Remark 6.6. We have not used the second equation in (6.33) together with equation (6.49) because it was easier to directly obtain the stationary solutions from the Ricci tensor given at Proposition (6.4). However, it is worth noting that the solution $a = e^{-2k(t-t_0)}$, $b = e^{-k(t-t_0)}$ does indeed solve the second equation in (6.33) and (6.49).

Remark 6.7. In the Riemannian case, it is possible to see that the solution (6.56) is completely equivalent to that obtained in [160, Proposition 4.1]. Using the subindex CS to make reference to quantities of that article, by performing the identifications $e^i_{\mathrm{CS}} = k\mathfrak{e}^i_{t_0}$, $a_{\mathrm{CS}}(t) = k^{-2}a^{-2}(t)$, $b_{\mathrm{CS}}(t) = k^{-2}b^{-2}(t)$, $\mu_{\mathrm{CS}} = 2k$ and $C_{\mathrm{CS}} = k^{-2}e^{-2kt_0}$, we conclude that our stationary quaternionic Kähler Heisenberg four-manifolds are equivalent to the stationary solutions of [160].

[14] Choosing the plus sign, the Weyl tensor is antiself-dual.

6.2.1.2 Negative solutions

Definition 6.5. A (timelike) quaternionic (para)Kähler Heisenberg four-manifold is said to be negative if $\varepsilon\Lambda < 0$ with $\varepsilon\Lambda \neq -6k^2$.

Remark 6.8. Observe that negative quaternionic Kähler Heisenberg four-manifolds have $\Lambda < 0$ while the negative timelike quaternionic paraKähler ones have $\Lambda > 0$.

Let $\gamma \in \mathbb{R}$ and let I be a connected component of the set

$$\{\rho \in \mathbb{R} \,|\, \rho \neq 0, \rho + \gamma > 0 \text{ and } \rho + 2\gamma > 0\}\,. \tag{6.61}$$

Let $\rho : J \xrightarrow{\sim} I$, $t \mapsto \rho(t)$ be a maximal solution of the ordinary differential equation

$$\rho'(t) = \sqrt{-\frac{2\varepsilon\Lambda}{3}}\rho(t)\sqrt{\frac{\rho(t)+\gamma}{\rho(t)+2\gamma}}\,, \tag{6.62}$$

with the initial condition $\rho(0) = \rho_0$. Define

$$A_s(\rho_0,\gamma) = sk\sqrt{-\frac{2\varepsilon\Lambda}{3}}\rho(t)\sqrt{\frac{\rho(t)+2\gamma}{\rho(t)+\gamma}}\,, \quad B_s(\rho_0,\gamma) = s\sqrt{-\frac{\varepsilon\Lambda}{3}}\frac{\rho(t)}{\sqrt{\rho(t)+2\gamma}}\,, \tag{6.63}$$

with $s \in \mathbb{Z}_2$ a sign.

Proposition 6.8. *Let $(A_s(\rho_0,\gamma), B_s(\rho_0,\gamma))$ as in (6.63). On the one hand, if $s = 1$ and $\varepsilon\Lambda < 0$, then there exists a unique pair (ρ_1,γ_1) such that:*

$$A_s(\rho_1,\gamma_1) = B_s(\rho_1,\gamma_1) = 1\,. \tag{6.64}$$

On the other hand, if $s = -1$ and $\varepsilon\Lambda \leq -81k^2$ there exist two pairs of solutions (ρ_2,γ_2) (ρ_3,γ_3) such that:

$$A_s(\rho_2,\gamma_2) = B_s(\rho_2,\gamma_2) = 1\,, \quad A_s(\rho_3,\gamma_3) = B_s(\rho_3,\gamma_3) = 1\,. \tag{6.65}$$

Such initial conditions (ρ_l,γ_l) with $l = 1,2,3$ are given by:

$$\rho_l = -\frac{e^{-(4-2l)i\pi/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}} + e^{(4-2l)i\pi/3}\frac{(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}}{2k(\varepsilon\Lambda)^{2/3}}\,, \tag{6.66}$$

$$2\gamma_l = -\rho_l\left(1+\frac{\varepsilon\Lambda}{3}\rho_l\right)\,. \tag{6.67}$$

Proof. Demanding $A_s(\rho_0,\gamma) = B_s(\rho_0,\gamma) = 1$, we can solve for γ in the last equation obtaining:

$$2\gamma = -\rho_0\left(1+\frac{\varepsilon\Lambda}{3}\rho_0\right)\,. \tag{6.68}$$

Squaring the equation $A_s(\rho_0,\gamma) = 1$ and substituting this result for γ, we arrive at the following cubic polynomial for ρ_0:

$$\rho_0^3 + \frac{3}{4\varepsilon\Lambda k^2}\rho_0 - \frac{9}{4k^2\Lambda^2} = 0\,. \tag{6.69}$$

The (maybe complex) solutions to this cubic equations are

$$\rho_l = -\frac{e^{-(4-2l)i\pi/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}} + e^{(4-2l)i\pi/3}\frac{(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}}{2k(\varepsilon\Lambda)^{2/3}}\,, \quad l=1,2,3\,. \tag{6.70}$$

At least one of the previous solution is real. However, not all real solutions of these equations need to satisfy a posteriori $A_s(\rho_0,\gamma) = B_s(\rho_0,\gamma) = 1$, since in the process of arriving (6.69) one has squared some expressions. Let us split this analysis between the cases $s=\pm1$:

- If $s=1$ we find that there is a unique real solution of (6.70) which in turn satisfies (6.64) for all values of $\varepsilon\Lambda < 0$. This solution is the one in (6.70) for $l=1$, that we denote as ρ_1. Substituting this expression of ρ_1 in (6.68) we obtain the unique solution (ρ_1,γ_1). Also, *a posteriori* we check that $\rho_1+2\gamma_1>0$, $\rho_1+\gamma_1>0$ and $\rho_1\neq 0$ for all $\varepsilon\Lambda<0$. On varying the value of $\varepsilon\Lambda$, we observe by direct inspection that $\rho_0\in(0,+\infty)$, while $\gamma\in(-\infty,(8k^2)^{-1})$. Interestingly, γ is negative when $0>\varepsilon\Lambda>-6k^2$ and positive if $\varepsilon\Lambda<-6k^2$.

- If $s=-1$, we find interestingly enough that no real solutions exist (after checking if they satisfy (6.65)) for $\varepsilon\Lambda>-81k^2$, while for $\varepsilon\Lambda\leq-81k^2$ we have two possible solutions[15]. These solutions are the ones in (6.70) with $l=2$ and $l=3$, that we denote respectively as ρ_2 and ρ_3. Substituting them in (6.68) we obtain two solutions (ρ_2,γ_2) and (ρ_3,γ_3). We check *a posteriori* that both satisfy $\rho_l+2\gamma_l>0$, $\rho_l+\gamma_l>0$ and $\rho_l\neq0$ with $l=2,3$ for all $\varepsilon\Lambda\leq 81k^2$. On the other hand, for the different values of $\varepsilon\Lambda\leq-81k^2$, by direct inspection we see that $\rho_2\in(-(18k^2)^{-1},0)$ and $\rho_3\in\left(-\frac{5}{80k^2},0\right)$, while $\gamma_2\in\left(0,\frac{5}{72k^2}\right)$ and $\gamma_3\in\left(\frac{5}{72k^2},(8k^2)^{-1}\right)$.

□

According to Proposition 6.7, there exists a unique solution for the pair $(a(t),b(t))$ if $\varepsilon\Lambda>-81k^2$, while there are three (some of them identical for $\varepsilon\Lambda=-81k^2$) if $\varepsilon\Lambda\leq-81k^2$. By use of Proposition 6.8, it is possible to find such solutions, which we write next.

Proposition 6.9. *Let (ρ_l,γ_l) for $l=1,2,3$ denote the pairs of Proposition 6.8 and let $(A_s(\rho_l,\gamma_l),B_s(\rho_l,\gamma_l))$ as in (6.63). Set:*

$$(a_1(t),b_1(t)) \;=\; (A_1(\rho_1,\gamma_1),B_1(\rho_1,\gamma_1))\,, \tag{6.71}$$

$$(a_2(t),b_2(t)) \;=\; (A_{-1}(\rho_2,\gamma_2),B_{-1}(\rho_2,\gamma_2))\,, \tag{6.72}$$

$$(a_3(t),b_3(t)) \;=\; (A_{-1}(\rho_3,\gamma_3),B_{-1}(\rho_3,\gamma_3))\,. \tag{6.73}$$

These are all the solutions to (6.33) (or (6.49)) and, consequently, all negative (timelike) quaternionic (para)Kähler Heisenberg four-manifolds. In particular, $(a_1(t),b_1(t))$ is defined for all $\varepsilon\Lambda<0$ while $(a_2(t),b_2(t))$ and $(a_3(t),b_3(t))$ are defined for all $\varepsilon\Lambda\leq-81k^2$. The corresponding pseudo-Riemannian manifolds arising from (6.72) and (6.73), together with those stemming from (6.71) for $\varepsilon\Lambda>-6k^2$, are incomplete, while in the case (6.71) for $\varepsilon=1$ and $\Lambda<-6k^2$ the solution is complete.

[15] They actually coincide for $\varepsilon\Lambda=-81k^2$.

Proof. The fact that (6.71), (6.72) and (6.73) solve equations (6.33) (or (6.49)) with the initial condition $a(t_0) = b(t_0) = 1$ follows from direct computation and by Proposition 6.8. Regarding completeness, we go on a case by case fashion:

- Let us begin by analyzing solutions (6.72) and (6.73). In these cases, we have $\rho_0 < 0$, $\gamma > 0$ but $\rho_0 + \gamma > 0$. We consider the canonical geodesic determined by the coordinate ρ (related to the time coordinate as in (6.62)), with ρ defined between $(-\gamma, \rho_0)$. We compute its length:

$$\sqrt{\left|\frac{3}{2\varepsilon\Lambda}\right|}\int_{-\gamma}^{\rho_0} \frac{1}{|\rho|}\sqrt{\left|\frac{\rho+2\gamma}{\rho+\gamma}\right|}\mathrm{d}\rho \leq C\int_0^{\rho_0+\gamma} \frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau < \infty\,, \tag{6.74}$$

  where $C > 0$ is given by $C = \sqrt{\left|\frac{3}{2\varepsilon\Lambda}\right|}(\rho_0+\gamma)\zeta$, being ζ the maximum of the function $\frac{\sqrt{\rho+2\gamma}}{|\rho|}$ on the compact interval $[-\gamma, \rho_0]$. Since the length of this curve, which arrives to the boundary of the domain definition of the parameter ρ, is finite, we conclude that solutions (6.72) and (6.73) are incomplete.

- For solutions (6.71) with $-6k^2 < \varepsilon\Lambda < 0$, by virtue of Proposition 6.8 and its proof we realize that $\gamma < 0$ (but again, $\rho_0 > -\gamma$). Therefore, through a completely equivalent proof to that provided in the previous bullet-point, we observe that these solutions are incomplete too.

- As explained in Remark 6.10 below, solutions (6.71), (6.72) and (6.73) with $\varepsilon = 1$ are identified with the one-loop deformed universal hypermultiplet metrics (see Remark 6.10) described in [120, 157]. They are known to be complete if and only if γ and the initial condition ρ_0 are positive [148]. For $\varepsilon = 1$ and $\Lambda < -6k^2$, we observe that Proposition 6.8 and its proof ensure that γ and ρ_0 are positive for (6.71), and consequently we infer that they are complete.

□

Remark 6.9. We strongly believe the case (6.71) with $\varepsilon = -1$ and $\Lambda > 6k^2$ to be incomplete as well. Indeed, let us use the coordinates (6.3) to describe the Heisenberg group H. Let us consider a geodesic $\Gamma : J \to (\mathcal{I} \times \mathrm{H})$ for $J \subset \mathbb{R}$ whose coordinates are given by $(\rho(\tau), x(\tau), y(\tau), z(\tau))$ with $\tau \in J$ an affine parameter and with the initial conditions $\rho(0) = \rho_0 > 0$, $x(0) = y(0) = z(0) = 0$, $y'(0) = z'(0) = 0$ and $x'(0) = v_0$. On the one hand, we find that $y(\tau) = z(\tau) = 0$. On the other hand, the solution for $x(\tau)$ can be seen to be:

$$x(\tau) = \kappa\int_{t_0}^{\tau} \frac{\rho^2(\sigma)}{\rho(\sigma)+2\gamma}\mathrm{d}\sigma\,, \tag{6.75}$$

where $\kappa \in \mathbb{R}$ is some constant ensuring that $x'(0) = v_0$. Using this result, the equation for $\rho = \rho(\tau)$ turns out to be:

$$\rho'' = \frac{v_0^2\rho^3(\rho+\gamma)(\rho+4\gamma)}{(\rho+2\gamma)^3} + \frac{(\rho')^2(4\gamma^2+7\gamma\rho+2\rho^2)}{2\rho(\rho+\gamma)(\rho+2\gamma)}\,. \tag{6.76}$$

By numerical analysis, it can be seen that the solutions for the previous second-order differential equation equation are, typically, only defined in a finite interval J. Hence this provides a robust argument in favour of the incompleteness of the solutions.

Remark 6.10. Let us show that the solution (6.71) for $\varepsilon = 1$ is equivalent up to homothety to the one-loop deformed universal hypermultiplet described in [160]. For that, let us denote the quantities of that work by a superscript or a subindex CS. Following their notation, we define $\mathrm{d}t_{\mathrm{CS}} = \sqrt{-\frac{2\Lambda}{3}}\mathrm{d}t$, rescale their metric with the factor $-\frac{2\Lambda}{3}$ (remember $\Lambda < 0$) and define $e^3_{\mathrm{CS}} = \frac{1}{kw}\mathfrak{e}^1_{t_0}$, $e^1_{\mathrm{CS}} = \frac{1}{k\sqrt{w}}\mathfrak{e}^2_{t_0}$ and $e^2_{\mathrm{CS}} = \frac{1}{k\sqrt{w}}\mathfrak{e}^3_{t_0}$ with $w > 0$ such that $c_{\mathrm{CS}} = w\gamma$. Note that the corresponding dual vectors satisfy[16] $[e^{\mathrm{CS}}_2, e^{\mathrm{CS}}_3] = -2e^{\mathrm{CS}}_1$. We also set $a_{\mathrm{CS}} = -2\Lambda k^2 w^2 a^{-2}/3$ and $b_{\mathrm{CS}} = -2\Lambda k^2 w b^{-2}/3$ and rescale $\rho = w\rho_{CS}$. After all these identifications, we still have the freedom to perform time shifts on t_{CS}. In order to match $c_{\mathrm{CS}} > 0$ and the initial condition for ρ_{CS}, ρ^0_{CS}:

- If $c_{\mathrm{CS}} > 0$ and $\rho^0_{CS} > 0$, then we recover our solution (6.71) with $\Lambda < -6k^2$. To see this, we note that in this case $0 < \gamma < (8k^2)^{-1}$ (as Proposition 6.8 and its proof reveal). Consequently after an appropriate time shift and choosing $w = \tan\left(4\pi k^2\gamma\right)$, we observe that we may obtain all possible $c_{\mathrm{CS}} > 0$ and $\rho^0_{CS} > 0$.

- If $c_{\mathrm{CS}} < 0$ and $\rho^0_{CS} > 0$, then they correspond to the solution (6.71) with $0 > \Lambda > -6k^2$. The identification with $(c_{\mathrm{CS}}, \rho^0_{CS})$ is obtained by setting $w = 1$ and performing a suitable time shift on t_{CS}.

- If $c_{\mathrm{CS}} > 0$ and $\rho^0_{CS} < 0$, we distinguish two different cases. Set $w = \tan\left(4\pi k^2\gamma\right)$ as before. Then we identify (Riemannian) solutions (6.72) with $c_{\mathrm{CS}} \in \left(0, \frac{5}{72k^2}\tan\left(\frac{5\pi}{18}\right)\right)$ and (6.73) with $c_{\mathrm{CS}} \in \left(\frac{5}{72k^2}\tan\left(\frac{5\pi}{18}\right), +\infty\right)$ after performing time translations, if necessary. If $c_{\mathrm{CS}} = \frac{5}{72k^2}\tan\left(\frac{5\pi}{18}\right)$, then the corresponding γ is the one for which (6.72) and (6.73) coincide and we can take any of them.

Remark 6.11. Regarding the $\varepsilon = -1$ case, it can be seen that they correspond to a neutral-signature version of the one-loop deformed universal hypermultiplet. More concretely, these negative timelike quaternionic paraKähler Heisenberg four-manifolds can be obtained through the local temporal Supergravity c-map [149]. In order to see this, we just have to start from the trivial zero-dimensional manifold $\bar{M}$ given by a point and set in Equation (59) of Reference [673] (following their notation) $\epsilon_1 = -\epsilon_2 = -1$, $I = 0$, $z^0 = 1$ and $F_0 = \frac{i\epsilon_1}{2}(X^0)^2$, what implies in turn that $e^{\mathcal{K}} = 1/2$ and $(\hat{H}^{ab}) = \mathrm{diag}(1,1)$. Finally, by a completely analogous procedure to that described in Remark 6.10, we observe that we get, up to a global sign, our negative timelike quaternionic paraKähler Heisenberg four-manifolds.

6.2.1.3 Positive solutions

Definition 6.6. A (timelike) quaternionic (para)Kähler Heisenberg four-manifold is said to be positive if $\varepsilon\Lambda > 0$.

Remark 6.12. Note that positive quaternionic Kähler Heisenberg four-manifolds have $\Lambda > 0$ while the positive timelike quaternionic paraKähler ones have $\Lambda < 0$.

Let $\gamma \in \mathbb{R}$ and let I be a connected component of the set

$$\{\rho \in \mathbb{R} \,|\, \rho \neq 0, \rho + \gamma > 0 \text{ and } \rho + 2\gamma < 0\}\,. \tag{6.77}$$

[16] Therefore we may use the coordinates (6.3) and their dual basis of one-forms (6.4) for e^i_{CS} by setting $k = 1$.

Clearly, $\rho > 0$ and $\gamma < 0$. Let $\rho : J \xrightarrow{\sim} I$, $t \mapsto \rho(t)$ be a maximal solution of the ordinary differential equation

$$\rho'(t) = \sqrt{\frac{2\varepsilon\Lambda}{3}}\rho(t)\sqrt{-\frac{\rho(t)+\gamma}{\rho(t)+2\gamma}}\,, \tag{6.78}$$

with initial condition $\rho(0) = \rho_0$. Define

$$A(\rho_0,\gamma) = k\sqrt{\frac{2\varepsilon\Lambda}{3}}\rho(t)\sqrt{-\frac{\rho(t)+2\gamma}{\rho(t)+\gamma}}\,, \quad B(\rho_0,\gamma) = \sqrt{\frac{\varepsilon\Lambda}{3}}\frac{\rho(t)}{\sqrt{-\rho(t)-2\gamma}}\,. \tag{6.79}$$

Proposition 6.10. *Let $(A(\rho_0,\gamma), B(\rho_0,\gamma))$ as in (6.79). If $\varepsilon\Lambda > 0$, there exists a unique pair (ρ_0,γ) such that:*

$$A(\rho_0,\gamma) = B(\rho_0,\gamma) = 1\,. \tag{6.80}$$

Such initial condition (ρ_0,γ) is given by:

$$\rho_0 = \frac{-1 + (\varepsilon\Lambda)^{-1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{2/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}}\,, \quad 2\gamma = -\rho_0\left(1+\frac{\varepsilon\Lambda}{3}\rho_0\right)\,. \tag{6.81}$$

Proof. If we impose $A(\rho_0,\gamma) = B(\rho_0,\gamma) = 1$, we can solve for γ in the equation $B(\rho_0,\gamma) = 1$ and get

$$2\gamma = -\rho_0\left(1+\frac{\varepsilon\Lambda}{3}\rho_0\right)\,. \tag{6.82}$$

Now substituting this result in $A(\rho_0,\gamma) = 1$ we find the following cubic equation:

$$\rho_0^3 + \frac{3}{4\varepsilon\Lambda k^2}\rho_0 - \frac{9}{4k^2\Lambda^2} = 0\,. \tag{6.83}$$

This is formally equivalent to that found for negative (timelike) quaternionic (para)Kähler Heisenberg four-manifolds, although now $\varepsilon\Lambda > 0$. The previous cubic equation has a unique real root when $\varepsilon\Lambda > 0$ and it turns out to be

$$\rho_0 = \frac{-1 + (\varepsilon\Lambda)^{-1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{2/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}} > 0\,. \tag{6.84}$$

Taking this expression into that of γ given at (6.82), we find the unique solution (ρ_0,γ). Finally, after a careful study, in addition to $\rho_0 \neq 0$ we may learn that $\rho_0 + \gamma > 0$ and $\rho_0 + 2\gamma < 0$ for all $\varepsilon\Lambda > 0$ and we conclude. □

According to Proposition 6.7, there exists a unique solution $(a(t), b(t))$ for $\varepsilon\Lambda > 0$, which we proceed to present now.

Proposition 6.11. *Let (ρ_0,γ) denote the pair of Proposition 6.10 and $(A(\rho_0,\gamma), B(\rho_0,\gamma))$ as in (6.79). Set:*

$$(a(t), b(t)) \quad = \quad (A(\rho_0,\gamma), B(\rho_0,\gamma))\,, \tag{6.85}$$

These are all the solutions to (6.33) with $\varepsilon\Lambda > 0$ and, consequently, all positive (timelike) quaternionic (para)Kähler Heisenberg four-manifolds. They all are incomplete.

Proof. The fact that they all are solutions to (6.33) with $\varepsilon\Lambda > 0$ follows by direct computations and by use of Proposition 6.10. Regarding the incompleteness, since $\rho_0 > 0$ and $\gamma < 0$, let us consider the geodesic determined by the coordinate ρ (related to the temporal coordinate as in (6.62)), with ρ defined between $(-\gamma, -2\gamma)$. We calculate its length:

$$\sqrt{\frac{3}{2\varepsilon\Lambda}}\int_{-\gamma}^{-2\gamma}\frac{1}{|\rho|}\sqrt{\left|\frac{\rho+2\gamma}{\rho+\gamma}\right|}\mathrm{d}\rho \leq C\int_0^{-\gamma}\frac{1}{\sqrt{\tau}}\,\mathrm{d}\tau < \infty\,, \tag{6.86}$$

where $C > 0$ is given by $C = -\sqrt{\frac{3}{2\varepsilon\Lambda}}\gamma\zeta$, being ζ the maximum of the function $\frac{\sqrt{|\rho+2\gamma|}}{\rho}$ on the compact interval $[-\gamma, -2\gamma]$. Since the length of this curve, which reaches the boundary of the domain of definition of the parameter ρ, is finite, we conclude that the solution (6.85) is incomplete.

□

Remark 6.13. The non-stationary solutions with $\varepsilon\Lambda > 0$ are obtained from those with $\varepsilon\Lambda < 0$ basically by replacing $\varepsilon\Lambda \to -\varepsilon\Lambda$ and changing suitably the domain of ρ to ensure the reality of the coordinate.

Remark 6.14. We may interpret positive (timelike) quaternionic (para)Kähler Heisenberg four-manifolds as positively (negatively) curved versions of the one-loop deformed universal hypermultiplet solution.

Collecting the results given in Propositions 6.8, 6.9, 6.10 and 6.11 and expressing the metric in terms of the coordinate ρ, as given in (6.61) and (6.62) in the negative case and by (6.77) and (6.78) in the positive case, we can state the following theorem.

Theorem 6.2. *If (M, g) is a (timelike) quaternionic (para)Kähler Heisenberg four-manifold, then*

$$g = -\frac{3(\rho+2\gamma)}{2\varepsilon\Lambda\rho^2}\left(\frac{\varepsilon}{\rho+\gamma}\mathrm{d}\rho^2 + \frac{\varepsilon}{k^2}\frac{\rho+\gamma}{(\rho+2\gamma)^2}\mathfrak{e}^1_{t_0}\otimes\mathfrak{e}^1_{t_0} + 2\left(\mathfrak{e}^2_{t_0}\otimes\mathfrak{e}^2_{t_0} + \mathfrak{e}^3_{t_0}\otimes\mathfrak{e}^3_{t_0}\right)\right)\,, \tag{6.87}$$

where $(\mathfrak{e}^i_{t_0})$ denotes a left-invariant coframe of the Heisenberg group H such that $[\mathfrak{e}^{t_0}_2, \mathfrak{e}^{t_0}_3] = -2k\mathfrak{e}^{t_0}_1$ with $k \neq 0$ and where $\rho \in \mathcal{I}$ for certain open interval $\mathcal{I} \subseteq \mathbb{R}$ specified as follows. For $l = 1, 2, 3$, define the real numbers:

$$\rho_l = -\frac{e^{-(4-2l)i\pi/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}} + e^{(4-2l)i\pi/3}\frac{(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}}{2k(\varepsilon\Lambda)^{2/3}}\,, \tag{6.88}$$

$$\rho_4 = \frac{-1+(\varepsilon\Lambda)^{-1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{2/3}}{2k(\varepsilon\Lambda)^{1/3}(9k+\sqrt{81k^2+\varepsilon\Lambda})^{1/3}}\,. \tag{6.89}$$

For any element ρ_0 of the set

$$\{\rho \in \mathbb{R} \,|\, \rho \neq 0\,, \rho+\gamma > 0 \text{ and } (-1)^{j+1}(\rho+2\gamma) > 0\}\,,$$

we denote by $I^j_{\rho_0}$ the connected component containing ρ_0, where $2\gamma = -\rho_0\left(1+\frac{\varepsilon\Lambda}{3}\rho_0\right)$ and where $\varepsilon\Lambda \neq 0$ is the Einstein constant. Then:

1. *Setting $\mathcal{I} = I^1_{\rho_l}$, $l = 1, 2, 3$, we obtain all (timelike) quaternionic (para)Kähler Heisenberg four-manifolds with $\varepsilon\Lambda < 0$. In particular, the solution given by ρ_1 is defined for all $\varepsilon\Lambda < 0$ while the other two are defined for all $\varepsilon\Lambda \leq -81k^2$. The corresponding pseudo-Riemannian manifolds arising from these last two cases, together with those stemming from the first one for $\varepsilon\Lambda > -6k^2$, are incomplete while the first for $\varepsilon = 1$ and $\Lambda \leq -6k^2$ is complete.*

2. *Setting $\mathcal{I} = I^2_{\rho_4}$, we find all (timelike) quaternionic (para)Kähler Heisenberg four-manifolds with $\varepsilon\Lambda > 0$. They all are incomplete.*

Remark 6.15. Note that Theorem 6.2 includes the positive and negative cases, but also the stationary ones. Indeed, regarding this last case, observe that the proof of Proposition 6.8 shows that $\gamma = 0$ when $\varepsilon\Lambda = -6k^2$. Hence (6.62) becomes $\rho'(t) = 2k\rho(t)$ and we easily recover the results of Theorem 6.1 (choosing the correct initial condition for ρ from Proposition (6.8)), which we decided to state separately for the benefit of the reader.

6.2.2 Spacelike quaternionic paraKähler Heisenberg four-manifolds

Now we continue with the classification of all spacelike quaternionic paraKähler Heisenberg four-manifolds. For that, we shall use the ansatz described in (6.17), valid in an open subinterval $\mathcal{I}'_s \subset \mathcal{I}$, which we rewrite here for the benefit of the reader:

$$U^t = \begin{pmatrix} c(t) & h(t) & 0 \\ -h(t) & b(t) & 0 \\ 0 & 0 & a(t) \end{pmatrix} , \quad [\mathfrak{e}_1^{t_0}, \mathfrak{e}_2^{t_0}] = -2k\mathfrak{e}_3^{t_0} , \quad k > 0 . \tag{6.90}$$

Proposition 6.12. *The non-zero components of the Ricci curvature tensor Ric^g of the metric obtained from (6.90) are:*

$$\begin{aligned}
\mathrm{Ric}^g(\partial_t, \partial_t) &= \frac{(-2a^2ch(7b' + c')h' + a^2c^2(-4(b')^2 + (h')^2) + h^4(-4(a')^2 + 2aa''))}{2a^2(bc + h^2)^2} \\
&+ \frac{a^2h^2((b')^2 + 6b'c' + (c')^2 - 8(h')^2 + 2cb'') + b^2a^2(-4(c')^2 + (h')^2)}{2a^2(bc + h^2)^2} \\
&\frac{+b^2(c^2(-4(a')^2 + 2aa'') + 2a^2cc'')) + 4a^2h^3h'' + 2ba^2c^2b'' - 2ba^2hb'h'}{2a^2(bc + h^2)^2} \\
&+ \frac{2b(a^2h(hc'' - 7c'h') + c(3a^2(h')^2 + h^2(-4(a')^2 + 2aa'') + 2a^2hh''))}{2a^2(bc + h^2)^2} ,
\end{aligned} \tag{6.91}$$

$$\begin{aligned}
\mathrm{Ric}^g(\mathfrak{e}_1^t, \mathfrak{e}_1^t) &= -2k^2\frac{(bc + h^2)^2}{a^2} - \frac{2bcb'c' + 4b^2(c')^2 + b^2(h')^2 - 2bc(h')^2 - c^2(h')^2}{2(bc + h^2)^2} \\
&- \frac{a'(bc' + hh')}{a(bc + h^2)} + \frac{2b^2cc'' + h^2(-(b')^2 + 2b'c' + (c')^2 - 6(h')^2 + 2bc''))}{2(bc + h^2)^2} \\
&+ \frac{2h^3h'' + 2h((b'(b - 2c) - c'(6b + c))h' + bch'')}{2(bc + h^2)^2} ,
\end{aligned} \tag{6.92}$$

$$\mathrm{Ric}^g(\mathfrak{e}_1^t, \mathfrak{e}_2^t) = \frac{3b^2c'h'}{2(bc + h^2)^2} + \frac{a'(-h(b' + c') + (b + c)h')}{2a(bc + h^2)} + \frac{h^2(h(b'' + c'') - 4(b' + c')h')}{2(bc + h^2)^2}$$

$$
\begin{aligned}
&+\frac{c^2(3b'h'-bh'')-bh(3b'c'+(c')^2-4(h')^2+hh'')-chb'(b'+3c')}{2(bc+h^2)^2}\\
&+\frac{ch(4(h')^2+b(b''+c''))-c(h^2h''+b((b'+c')h'-bh''))}{2(bc+h^2)^2}\,, \qquad (6.93)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}^g(\mathfrak{e}_2^t,\mathfrak{e}_2^t) &= 2k^2\frac{(bc+h^2)^2}{a^2}+\frac{a'(cb'+hh')}{a(bc+h^2)}+\frac{b'(4c^2b'-h^2b'-2h^2c'+2bcc')+h^2(c')^2}{2(bc+h^2)^2}\\
&+\frac{h'((c^2-b^2+6h^2-2bc)h'+12chb'-2cc'h+2bh(b'+2c'))}{2(bc+h^2)^2}\\
&-\frac{b''(ch^2+bc^2)+h''h(h^2+bc)}{(bc+h^2)^2}\,, \qquad (6.94)
\end{aligned}
$$

$$
\mathrm{Ric}^g(\mathfrak{e}_3^t,\mathfrak{e}_3^t) = -2k^2\frac{(bc+h^2)^2}{a^2}+\frac{2(a')^2}{a^2}+\frac{a'(cb'+bc'+2hh')}{a(bc+h^2)}-\frac{a''}{a}\,. \qquad (6.95)
$$

Proof. By direct computation. □

Proposition 6.13. *Let (M,g) be a spacelike quaternionic paraKähler Heisenberg four-manifold. Then:*

$$
a'(t)=-\frac{k^2b^6+a^2(-\Lambda b^2+(b')^2)}{2abb'}\,,\quad \Lambda=\frac{3(kb^3+ab')^3}{a^2b^2(3kb^3+ab')}\,,\quad c=b\,,\quad h=0\,. \qquad (6.96)
$$

Furthermore, $\mathcal{I}_s'=\mathcal{I}$.

Proof. We remind that a spacelike quaternionic para Kähler Heisenberg four-manifold is Einstein and its Weyl tensor is self-dual. On the one hand, by setting that $\mathrm{Ric}^g(\mathfrak{e}_1^t,\mathfrak{e}_2^t)=0$, one infers:

$$
\begin{aligned}
h'' \;=\; &\frac{3ac^2b'h'+b^2(ca'+3ac')h'+h^2(ca'-4a(b'+c'))h'-ach((b')^2+3b'c'-4(h')^2)}{a(b+c)(bc+h^2)}\\
&+\frac{h^3(a(b''+c'')-a'(b'+c'))+b(c^2a'h'-hac'(3b'+c')+h^2a'h'+4ah(h')^2)}{a(b+c)(bc+h^2)}\\
&+\frac{bc(a(b'+c')h'+h(a(b''+c'')-a'(b'+c')))}{a(b+c)(bc+h^2)}\,. \qquad (6.97)
\end{aligned}
$$

Computing the Weyl self-duality tensor $\mathcal{W}^g$ for the ansatz (6.90) and substituting the previous result, we have that

$$
\mathcal{W}(\mathfrak{e}_1^t,\partial_t,\mathfrak{e}_3^t,\mathfrak{e}_1^t)=\frac{(kbc+kh^2-a')(-h(b'+c')+(b+c)h')}{2a(bc+h^2)}=0\,. \qquad (6.98)
$$

If $a'=k(bc+h^2)$, this would in turn imply that $\Lambda=0$, but we are imposing the condition that $\Lambda\neq 0$ to study proper quaternionic paraKähler four-manifolds, so we conclude that necessarily $-h(b'+c')+(b+c)h'=0$. Since $h(t_0)=0$ and $b(t_0)+c(t_0)=2$, there exists an open subinterval of $\mathcal{I}_s'$ in which $b(t)+c(t)\neq 0$ and $h(t)=0$. In fact, $b(t)+c(t)\neq 0$ for all $t\in\mathcal{I}_s'$, because otherwise there would be $t_1\in\mathcal{I}_s'$ satisfying $b(t_1)=-c(t_1)$ and $h(t_1)=0$. This would imply in turn that $\det U^{t_1}=-a(t_1)b(t_1)^2$. If $a(t_1)<0$, we see that there was a time $t_2\in\mathcal{I}_s'$ in which $a(t_2)=\det U^{t_2}=0$ (remember that $a(t_0)=1$) and if $a(t_1)>0$, then $\det U^{t_1}<0$ and there was a time t_3 in which $\det U^{t_3}=0$ (note that $\det U^{t_0}=1$).

Since U^t is non-degenerate by hypothesis for all $t \in \mathcal{I}'_s$, we conclude that $h(t) = 0$ and $b(t) + c(t) \neq 0$ in the entire $\mathcal{I}'_s$.

Now we can solve for a'', b'', c'' and a' from the Einstein condition $\mathrm{Ric}^g = \Lambda g$ and we get:

$$a'(t) = -\frac{k^2 b^3 c^3 + a^2(-\Lambda bc + b'c')}{a(cb' + bc')}\,, \tag{6.99}$$

$$\begin{aligned} a''(t) = &\frac{-k^2 a^2 b^3 c^3 \left(2bcb'c' + 3c^2(b')^2 + b^2\left(3(c')^2 + 4\Lambda c^2\right)\right)}{a^3 bc\,(cb' + bc')^2} \\ &+ \frac{a^4\left(-b^2 b'c'\left((c')^2 + 4\Lambda c^2\right) - c^2(b')^3 c' + 2\Lambda^2 b^3 c^3\right) + 2k^4 b^7 c^7}{a^3 bc\,(cb' + bc')^2}\,, \end{aligned} \tag{6.100}$$

$$b''(t) = \frac{a^2 b^2 b'(c')^2 + 2a^2 bc(b')^2 c' + 2a^2 c^2 (b')^3 - \Lambda a^2 b^3 cc' + k^2 b^4 c^4 b' + 2k^2 b^5 c^3 c'}{a^2 bc\,(cb' + bc')}\,, \tag{6.101}$$

$$c''(t) = \frac{a^2\left(bcb'\left(2(c')^2 - \Lambda c^2\right) + c^2(b')^2 c' + 2b^2(c')^3\right) + k^2 b^3 c^4\,(2cb' + bc')}{a^2 bc\,(cb' + bc')}\,. \tag{6.102}$$

Taking these results into the remaining components of the Weyl self-duality tensor, we find in particular that

$$\begin{aligned} \mathcal{W}^g(\mathfrak{e}^t_2, \mathfrak{e}^t_3, \mathfrak{e}^t_2, \mathfrak{e}^t_3) \;=\; & -k^3\frac{b^4 c^4}{a^3(cb' + bc')} - k^2\frac{b^2 c^2(2cb' + bc')}{a^2(cb' + bc')} - kc\frac{2bb'c' + c(b')^2 - \Lambda b^2 c}{a(cb' + bc')} \\ & - \frac{c'(\Lambda b^2 + 3(b')^2) - 2\Lambda bcb'}{3b(cb' + bc')}\,. \end{aligned} \tag{6.103}$$

Equating the last expression to zero, one solves for Λ and finds

$$\Lambda = -\frac{3(kb^2 c + ab')^2(kbc^2 + ac')}{a^2 b(-3kb^2 c^2 - 2acb' + abc')}\,. \tag{6.104}$$

Plugging this last result into the other components of $\mathcal{W}^g$, we encounter in particular:

$$\mathcal{W}^g(\mathfrak{e}^t_1, \mathfrak{e}^t_2, \partial_t, \mathfrak{e}^t_3) = \frac{(kb^2 c + ab')(kbc^2 + ac')(cb' - bc')}{(abc(-3kb^2 c^2 - 2acb' + abc'))}\,. \tag{6.105}$$

For the latter to be zero, either one of the three terms in brackets must be zero. However, if any of the two first terms in brackets is zero, then $\Lambda = 0$, so we discard this possibility and hence $cb' - bc' = 0$, which in turn implies that $c(t) = b(t)$ since $c(t_0) = b(t_0) = 1$. Imposing then that $c(t) = b(t)$ we find that all the components of $\mathcal{W}^g$ vanish identically and, collecting all the results derived up to this point, we arrive to (6.96). We check as well that equations (6.96) are consistent with (6.100),(6.101) and (6.102). Finally, we observe that $h(t) = 0$ and $c(t) = b(t)$ is equivalent to the metric adopting the form $g = -\mathrm{d}t^2 - b^{-2}(t)(e^1_{t_0} \otimes e^1_{t_0} - e^2_{t_0} \otimes e^2_{t_0}) + a^{-2}(t) e^3_{t_0} \otimes e^3_{t_0}$, so it is clear that the metric has a singularity whenever any of the functions $a(t)$ or $b(t)$ converges to zero or diverges. Hence $\mathcal{I}'_s$, the interval of definition of the ansatz (6.90), must coincide with $\mathcal{I}$ and we conclude. □

Proposition 6.14. *Let (M, g) be a spacelike quaternionic paraKähler Heisenberg four-manifold. Then:*

- *The eigenvalues of the Weyl tensor, understood as a symmetric endomorphism of the bundle of two-forms, are given by* $(-2\nu', \nu', \nu')$*, where:*
$$\nu' = \frac{16k^3b^7}{\varepsilon a^3(3kb^3 + ab')} \,. \tag{6.106}$$
- (M, g) *is conformally paraKähler for two paracomplex structures with opposite orientations.*

Proof. The proof is analogous to the timelike case. We define:
$$\omega_i = \mathrm{d}t \wedge \mathfrak{e}^i_t + \star(\mathrm{d}t \wedge \mathfrak{e}^i_t) \,. \tag{6.107}$$
Interpreting the Weyl tensor as a symmetric endomorphism of the bundle of two-forms in the canonical way, we have:
$$\mathrm{W}^g(\omega_1) = \nu'\omega_1 \,, \quad \mathrm{W}^g(\omega_2) = \nu'\omega_2 \,, \quad \mathrm{W}^g(\omega_3) = -2\nu'\omega_3 \,. \tag{6.108}$$
This proves the first bullet-point of the proposition. With regard to the second one, observe that the rescaled two-form
$$\tilde{\omega}_3 = |\mathrm{W}^g|_g^{2/3}\omega_3 \tag{6.109}$$
is closed and satisfies that $|\tilde{\omega}_3|^2_{\tilde{g}} = -4$ with respect to the rescaled metric $\tilde{g} = |\mathrm{W}^g|_g^{2/3} g$. Then, using that the following two-forms
$$\tilde{\omega}_1 = ab\,\omega_2 \,, \quad \tilde{\omega}_2 = ab\,\omega_3 \tag{6.110}$$
are closed, direct application of the para-version of Lemma 6.1, see Remark 6.5, proves that $\tilde{g}$ is paraKähler with paraKähler form $\tilde{\omega}_3$ and we conclude. A conformally paraKähler structure for the opposite orientation can be obtained by introducing a relative sign in (6.107). □

Remark 6.16. In complete analogy to the situation in the timelike case, in the proof of Proposition 6.14 we have shown that (M, g) not only is conformally paraKähler but admits as well two almost (para)Kähler structures $\tilde{\omega}_1$ and $\tilde{\omega}_2$ compatible with a second conformally rescaled metric $g' := ab\, g$, such that $|\tilde{\omega}_1|^2_{g'} = -|\tilde{\omega}_2|^2_{g'} = 4$.

We can construct all solutions to equations (6.96) from solutions to the equations (6.33) for timelike quaternionic paraKähler Heisenberg four-manifolds, as the following proposition shows.

Proposition 6.15. *Let* $(a(t), b(t))$ *solve equations* (6.33) *for* $\varepsilon = -1$ *and a given value of* Λ*. Then* $(a(2t_0 - t), b(2t_0 - t))$ *solve* (6.96) *for the same value of* Λ*.*

Proof. Defining $a_{\mathrm{S}}(t) = a(2t_0 - t)$ and $b_{\mathrm{S}}(t) = b(2t_0 - t)$, we observe that $a'_{\mathrm{S}}(t) = -a'(2t_0 - t)$ and $b'_{\mathrm{S}}(t) = -b'(2t_0 - t)$. If $a(t)$ and $b(t)$ solve (6.33) for a given Λ, then $a_{\mathrm{S}}(t)$ and $b_{\mathrm{S}}(t)$ solve (6.96). Furthermore, $a_{\mathrm{S}}(t_0) = a(t_0) = b_{\mathrm{S}}(t_0) = b(t_0) = 1$ and we conclude. □

Remark 6.17. If $\mathcal{I}^{\text{timelike}}$ denotes the interval in which a timelike solution $(a(t), b(t))$ is defined, then the spacelike counterpart $(a(2t_0 - t), b(2t_0 - t))$ is defined in the interval $\mathcal{I}^{\text{spacelike}} = \{t \in \mathbb{R} \,|\, 2t_0 - t \in \mathcal{I}^{\text{timelike}}\}$.

Given this correspondence between timelike and spacelike quaternionic paraKähler Heisenberg four-manifolds, it is natural to split the study into stationary, negative and positive spacelike quaternionic paraKähler Heisenberg four-manifolds, which are obtained from the associated stationary, negative and positive timelike counterparts.

6.2.2.1 Stationary solutions

Definition 6.7. A spacelike quaternionic (para)Kähler Heisenberg four-manifold is said to be stationary if $\Lambda = 6k^2$.

As in the timelike case, the name stationary comes from the fact that if we set $b = c$ and $h = 0$ in the expression for the Ricci tensor in Proposition 6.12:

$$\mathrm{Ric}^g(\partial_t, \partial_t) = -2\lambda^2 - \mu^2 + 2\lambda' + \mu' \,, \tag{6.111}$$

$$\mathrm{Ric}^g(\mathfrak{e}^t_1, \mathfrak{e}^t_1) = -\frac{2k^2 b^4}{a^2} - 2\lambda^2 - \mu\lambda + \lambda' \,, \tag{6.112}$$

$$\mathrm{Ric}^g(\mathfrak{e}^t_2, \mathfrak{e}^t_2) = \frac{2k^2 b^4}{a^2} + 2\lambda^2 + \lambda\mu - \lambda' \,, \tag{6.113}$$

$$\mathrm{Ric}^g(\mathfrak{e}^t_3, \mathfrak{e}^t_3) = -\frac{2k^2 b^4}{a^2} + 2\lambda\mu + \mu^2 - \mu' \,, \tag{6.114}$$

where $\mu = (\log a(t))'$ and $\lambda = (\log b(t))'$, then if we set $\mu' = \lambda' = 0$ we have that $\Lambda = 6k^2$ and hence the name stationary.

Theorem 6.3. *All stationary spacelike quaternionic paraKähler Heisenberg four-manifolds are given by:*

$$g = -\mathrm{d}t^2 + e^{-2k(t-t_0)} \left(-\mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + \mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0}\right) + e^{-4k(t-t_0)} \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0} \,, \tag{6.115}$$

where $t \in \mathbb{R}$. These solutions are isometric to an open orbit of a four-dimensional solvable subgroup (which contains a Heisenberg subgroup) of $\mathrm{SL}(3,\mathbb{R})$ on the symmetric space

$$\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{S}(\mathrm{GL}(1,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}))} \,. \tag{6.116}$$

Furthermore, they all are incomplete.

Proof. These stationary solutions are just obtained by using Proposition 6.15 and carrying out the change $t \to 2t_0 - t$ in the stationary solutions found in (6.56). Also, the fact that these solutions are the unique stationary spacelike quaternionic paraKähler Heisenberg four-manifolds follows by the use of Proposition 6.15 and Theorem 6.1.

In complete analogy with the timelike case, we encounter that $\nabla^g \mathrm{R}^g = 0$, where R^g is Riemann curvature tensor of g. Upon use of the classification of pseudo-Riemannian symmetric spaces of quaternionic paraKähler type [671,672], we conclude (by comparison of curvature tensors) that the resulting space is locally isometric to the symmetric space (6.116). In particular, the solutions turn out to be isometric to a left-invariant metric on a simply transitive solvable subgroup of $\mathrm{SL}(3,\mathbb{R})$. This can be seen by following analogous arguments to that of the proof of Theorem 6.1. Finally, we infer that the underlying pseudo-Riemannian manifold is incomplete since the timelike quaternionic paraKähler solution, from which the spacelike one is obtained, is incomplete (and this property does not change under a change of coordinate $t \to 2t_0 - t$). □

6.2.2.2 Negative and positive solutions

Definition 6.8. A spacelike quaternionic paraKähler Heisenberg four-manifold is said to be negative if $\Lambda < 0$. Similarly, it is said to be positive if $\Lambda > 0$ but $\Lambda \neq 6k^2$.

By the use of the Proposition 6.15 together with the classification of timelike quaternionic paraKähler Heisenberg four-manifolds, we may actually obtain all negative and positive spacelike counterparts.

Proposition 6.16. *Let $(a(t), b(t))$ as in (6.85). Then $(a(2t_0 - t), b(2t_0 - t))$ are all solutions to (6.96) with $\Lambda < 0$ and, consequently, all the negative spacelike quaternionic paraKähler Heisenberg four-manifolds. They all are incomplete.*

Proposition 6.17. *Let $(a_1(t), b_1(t))$, $(a_2(t), b_2(t))$ and $(a_3(t), b_3(t))$ be as in (6.71), (6.72) and (6.73), respectively. Then $(a_1(2t_0 - t), b_1(2t_0 - t))$ (defined for all $\Lambda > 0$), $(a_2(2t_0 - t), b_2(2t_0 - t))$ and $(a_3(2t_0 - t), b_3(2t_0 - t))$ (the last two defined for all $\Lambda \geq 81k^2$) are all solutions to (6.96) with $\Lambda > 0$ and $\Lambda \neq 6k^2$ and, consequently, all the positive spacelike quaternionic paraKähler Heisenberg four-manifolds. Solutions $(a_2(2t_0 - t), b_2(2t_0 - t))$ and $(a_3(2t_0 - t), b_3(2t_0 - t))$, as well as those arising from $(a_1(2t_0 - t), b_1(2t_0 - t))$ for $0 < \Lambda < 6k^2$, are incomplete[17].*

Proof. The previous two Propositions are shown by direct use of Proposition 6.15 and by the fact that all timelike quaternionic paraKähler Heisenberg four-manifolds are incomplete, a geometric property that does not change after the trivial change of coordinate $t \to 2t_0 - t$. □

Remark 6.18. Negative (resp. positive) timelike quaternionic paraKähler Heisenberg four-manifolds are in correspondence with their positive (resp. negative) spacelike counterparts. We must bear in mind that, in the negative timelike case, $\Lambda > 0$, while $\Lambda < 0$ for positive timelike quaternionic paraKähler Heisenberg four-manifolds, so the results are consistent.

Remark 6.19. In complete analogy with Remark 6.11, positive spacelike quaternionic paraKähler Heisenberg four-manifolds correspond to neutral-signature analogues of the one-loop deformed universal hypermultiplet, since they can be obtained from the so-called Euclidean Supergravity c-map [149]. In fact, if in Equation (59) of Reference [673] we choose $\bar{M}$ to be a point, $\epsilon_1 = 1$, $\epsilon_2 = \pm 1$, $I = 0$, $z^0 = 1$ and $F_0 = \frac{i_{\epsilon_1}}{2}(X^0)^2$, $e^{\mathcal{K}} = -1/2$ and $(\hat{H})^{ab} = \mathrm{diag}(1, -1)$, we observe that after an equivalent procedure to that of Remark 6.10 we have, up to a global sign, the positive spacelike quaternionic paraKähler Heisenberg four-manifolds derived before. Regarding their negative counterparts, we may interpret them as negatively-curved versions of the neutral-signature one-loop deformed universal hypermultiplet.

By taking into account Propositions 6.15, 6.16 and 6.17, we prove the following theorem.

Theorem 6.4. *There exists a one-to-one correspondence between spacelike and timelike quaternionic paraKähler Heisenberg four-manifolds. Any spacelike quaternionic paraKähler Heisenberg four-manifold is isometric to:*

$$g = \frac{3(\rho + 2\gamma)}{2\Lambda\rho^2}\left(-\frac{1}{\rho+\gamma}\mathrm{d}\rho^2 + 2\left(-\mathfrak{e}^1_{t_0}\otimes\mathfrak{e}^1_{t_0} + \mathfrak{e}^2_{t_0}\otimes\mathfrak{e}^2_{t_0}\right) + \frac{\rho+\gamma}{k^2(\rho+2\gamma)^2}\mathfrak{e}^3_{t_0}\otimes\mathfrak{e}^3_{t_0}\right), \quad (6.117)$$

where ρ is defined in the appropriate intervals $I^1_{\rho_l}$, $l = 1, 2, 3$ or $I^2_{\rho_4}$, like in the timelike case presented in Theorem 6.2, and where Λ is the Einstein constant. Here $(\mathfrak{e}^i_{t_0})$ denotes

[17] We expect, in the light of Remark 6.9, solutions $(a_1(2t_0 - t), b_1(2t_0 - t))$ with $\Lambda > 6k^2$ to be incomplete too.

a left-invariant coframe of H *such that* $[\mathfrak{e}_1^{t_0}, \mathfrak{e}_2^{t_0}] = -2k\mathfrak{e}_3^{t_0}$, $k \neq 0$. *The (in)completeness properties are the same as those of their timelike analogues.*

Remark 6.20. By a similar argument to that of Remark 6.15, one observes that the Theorem 6.4 also contains the stationary case presented in Theorem 6.3.

6.2.3 Lightlike quaternionic paraKähler Heisenberg four-manifolds

Finally we classify all lightlike quaternionic paraKähler Heisenberg four-manifolds. We will make use of the ansatz (6.18), which gives us a simple way to describe, through a suitable Witt basis $\{\mathfrak{e}_u^t, \mathfrak{e}_v^t, \mathfrak{e}_3^t\}$, the corresponding metrics of lightlike quaternionic paraKähler Heisenberg four-manifolds. We rewrite here this ansatz, valid in principle in a subinterval $\mathcal{I}_l' \subset \mathcal{I}$:

$$U_W^t = \begin{pmatrix} 1 & 0 & 0 \\ f(t) & b(t) & p(t) \\ 0 & 0 & a(t) \end{pmatrix}, \quad [\mathfrak{e}_v^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_u^{t_0}, \quad k > 0, \tag{6.118}$$

We write first the Ricci curvature tensor Ric^g arising from (6.118).

Proposition 6.18. *Let* (M, g) *denote a lightlike quaternionic paraKähler Heisenberg four-manifold. The non-zero components of the Ricci curvature tensor* Ric^g *read:*

$$\mathrm{Ric}^g(\partial_t, \partial_t) = \frac{aa'' - 2(a')^2}{a^2} - \frac{3(b')^2 - 2bb''}{2b^2}, \tag{6.119}$$

$$\mathrm{Ric}^g(\mathfrak{e}_u^t, \mathfrak{e}_v^t) = \frac{(b')^2}{b^2} + \frac{a'b' - ab''}{2ab}, \tag{6.120}$$

$$\mathrm{Ric}^g(\mathfrak{e}_v^t, \mathfrak{e}_v^t) = \frac{a'(bf' - fb')}{ab} + \frac{(pb' - bp')^2}{2a^2b^2} - \frac{3f(b')^2 - b(3b'f' + fb'')}{b^2} - \frac{f''}{a^2}, \tag{6.121}$$

$$\mathrm{Ric}^g(\mathfrak{e}_v^t, \mathfrak{e}_3^t) = \frac{-2ap(b')^2 + b(2ab'p' + p(-3a'b' + ab'')) + b^2(3a'p' - ap'')}{2a^2b^2}, \tag{6.122}$$

$$\mathrm{Ric}^g(\mathfrak{e}_3^t, \mathfrak{e}_3^t) = \frac{2(a')^2 - aa''}{a^2} + \frac{a'b'}{ab}. \tag{6.123}$$

Proof. By direct computation. □

Proposition 6.19. *Let* (M, g) *be a lightlike Heisenberg four-manifold. It is Einstein with Einstein constant* $\Lambda \neq 0$ *if and only if:*

$$a'' = -\Lambda a + \frac{2(a')^2}{a} + a'\frac{b'}{b}, \tag{6.124}$$

$$b'' = -\Lambda b + \frac{7(b')^2}{4b}, \tag{6.125}$$

$$f'' = \frac{a'(-fb' + bf')}{ab} + \frac{(pb' - bp')^2}{2a^2b^2} + \frac{(-3f(b')^2 + 3bb'f' + bfb''))}{b^2}, \tag{6.126}$$

$$p'' = \frac{-2p(b')^2}{b^2} + \frac{3a'p'}{a} + \frac{2ab'p' + p(-3a'b' + ab'')}{ab}, \tag{6.127}$$

$$a' = a\left(\frac{\Lambda b}{b'} - \frac{b'}{4b}\right). \tag{6.128}$$

Proof. Assume $\mathrm{Ric}^g = \Lambda g$. Using Proposition 6.18, from (6.123) we may solve for a'':

$$a'' = -\Lambda a + \frac{2(a')^2}{a} + a'\frac{b'}{b}\,. \tag{6.129}$$

Now from (6.121) and (6.122) we may solve for f'' and p'':

$$f'' = \frac{a'(-fb' + bf')}{ab} + \frac{(pb' - bp')^2}{2a^2b^2} + \frac{(-3f(b')^2 + 3bb'f' + bfb''))}{b^2}\,, \tag{6.130}$$

$$p'' = \frac{-2p(b')^2}{b^2} + \frac{3a'p'}{a} + \frac{2ab'p' + p(-3a'b' + ab'')}{ab}\,. \tag{6.131}$$

Substituting these results we have obtained on the rest of the components of the Ricci tensor, the Einstein condition is fulfilled if:

$$a' = a\left(\frac{\Lambda b}{b'} - \frac{b'}{4b}\right)\,, \qquad b'' = -\Lambda b + \frac{7(b')^2}{4b}\,. \tag{6.132}$$

After imposing this last condition, we observe that $\mathrm{Ric}^g = \Lambda g$. Checking that the derivative of previous equation is consistent with (6.129), we collect now all the results we have obtained and we conclude. □

In order to classify all lightlike quaternionic paraKähler Heisenberg four-manifolds, we need to know the conditions for an Einstein lightlike Heisenberg four-manifold to be half-conformally flat, which is equivalent to $\mathcal{W}^g = 0$.

Proposition 6.20. *Let (M, g) be a lightlike Einstein Heisenberg four-manifold with Einstein constant $\Lambda \neq 0$. The Weyl self-duality tensor $\mathcal{W}^g$ vanishes identically if and only if:*

$$b(t) = e^{\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\,, \quad a(t) = e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}\,, \quad f(t) = p(t) = 0\,, \quad \Lambda > 0\,. \tag{6.133}$$

In particular, if (M, g) is half-conformally flat, it is actually conformally flat.

Proof. Using the results of Proposition 6.19, we find that

$$\mathcal{W}(\mathfrak{e}^t_u, \partial_t, \partial_t, \mathfrak{e}^t_v) = \frac{\Lambda}{6} - \frac{1}{8}\left(\frac{b'}{b}\right)^2\,. \tag{6.134}$$

Hence,

$$\Lambda = \frac{3}{4}\left(\frac{b'}{b}\right)^2 > 0\,. \tag{6.135}$$

Note that this equation is consistent, since by differentiating with respect to t and using the expression for b'' found in Proposition 6.19, we indeed get that the right-hand side of the previous equation is indeed zero. Similarly, we have that

$$\mathcal{W}(\partial_t, \mathfrak{e}^t_3, \mathfrak{e}^t_v, \mathfrak{e}^t_3) = \frac{b'(-pb' + bp')}{4ab^2} = 0\,. \tag{6.136}$$

If $b' = 0$, this would imply in turn that $\Lambda = 0$. Since we are assuming that $\Lambda \neq 0$, we find that $-pb' + bp' = 0$, which is equivalent to $p(t) = 0$ on taking into account that $p(t_0) = 0$

and that $b(t) \neq 0$ since otherwise U^t_W would be degenerate. Finally, on substituting these results, we also see that

$$\mathcal{W}(\mathfrak{e}^t_v, \partial_t, \mathfrak{e}^t_v, \partial_t) = \frac{b'(-fb' + bf')}{2b^2} = 0\,. \tag{6.137}$$

From here we find that $f(t) = 0$ and then we encounter that not only the self-duality tensor vanishes, but also the Weyl tensor itself, so the subsequent metric is conformally flat. Simplifying the result for a' obtained in Proposition 6.19 by using (6.135), we have that the remaining differential equations to solve are:

$$a' = a\frac{b'}{2b}\,, \quad \Lambda = \frac{3}{4}\left(\frac{b'}{b}\right)^2\,. \tag{6.138}$$

The solution to the previous system of ordinary ODEs with the initial conditions $a(t_0) = b(t_0) = 1$ is:

$$b(t) = e^{\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\,, \quad a(t) = e^{\sqrt{\frac{\Lambda}{3}}(t-t_0)}\,, \tag{6.139}$$

and we conclude. □

Remark 6.21. The metric g, in terms of the coframe $\{\mathrm{d}t, \mathfrak{e}^1_{t_0}, \mathfrak{e}^2_{t_0}, \mathfrak{e}^3_{t_0}\}$ reads:

$$\begin{aligned} g &= -\mathrm{d}t^2 + \frac{1}{b(t)}\mathfrak{e}^u_{t_0} \odot \mathfrak{e}^v_{t_0} + \frac{1}{(a(t))^2}\mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0} \\ &= -\mathrm{d}t^2 + e^{-\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\mathfrak{e}^u_{t_0} \odot \mathfrak{e}^v_{t_0} + e^{-\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\,. \end{aligned} \tag{6.140}$$

We observe from the previous expression that the metric is indeed conformally flat: by defining a new coordinate $\mathrm{d}t = e^{-\frac{\sqrt{\Lambda}}{\sqrt{3}}(t(\tilde{t})-t_0)}\mathrm{d}\tilde{t}$, it is clear that the metric is conformally flat with conformal factor $e^{-\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t(\tilde{t})-t_0)}$.

Theorem 6.5. *All lightlike quaternionic paraKähler Heisenberg four-manifolds are conformally flat and isometric to the solution given by:*

$$g = -\mathrm{d}t^2 + e^{-\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\mathfrak{e}^u_{t_0} \odot \mathfrak{e}^v_{t_0} + e^{-\frac{2\sqrt{\Lambda}}{\sqrt{3}}(t-t_0)}\mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\,, \quad \Lambda > 0\,, \tag{6.141}$$

where $t \in \mathcal{I}'_l = \mathcal{I} = \mathbb{R}$. The subsequent pseudo-Riemannian manifolds are incomplete.

Proof. The fact that $t \in \mathcal{I}'_l = \mathcal{I} = \mathbb{R}$ follows by seeing that (6.141) is defined in the entire real line. Consequently, we only have to show that the corresponding metrics are incomplete. For that, let us set $t_0 = 0$ for the sake of simplicity (we can always achieve it by shifting the time coordinate) and let us use the coordinates (6.3) for H. We consider the geodesic $\Gamma : J \to (\mathcal{I} \times \mathrm{H})$ with $J \subset \mathbb{R}$ (and affine parameter τ) whose coordinates are given by:

$$\left(-\frac{\sqrt{3}}{2\sqrt{\Lambda}}\log\left(\frac{4\Lambda\sinh^2(\tau/2+B)}{3}\right), 0, \int_0^\tau \frac{3}{4\Lambda\sinh^2(\sigma/2+B)}\mathrm{d}\sigma, 0\right)\,, \tag{6.142}$$

where we choose a certain $B \in \mathbb{R}^{>0}$. This geodesic is not defined $\forall\tau \in \mathbb{R}$ and we conclude that the underlying pseudo-Riemannian manifold cannot be complete. □

Remark 6.22. Differently from what happens in the previous cases, when the Heisenberg center is lightlike it is possible to find Lorentzian conformally flat[18] Einstein metrics. Setting the metric to be

$$g^{\text{Lor}} = \mathrm{d}t^2 + \mathfrak{e}^u_t \odot \mathfrak{e}^v_t + \mathfrak{e}^3_t \otimes \mathfrak{e}^3_t \,, \tag{6.143}$$

we can find, in a completely analogous manner to that presented in the study of lightlike quaternionic paraKähler Heisenberg four-manifolds, that the following choice for the functions in U^t_W:

$$b(t) = e^{\frac{2\sqrt{-\Lambda}}{\sqrt{3}}(t-t_0)}\,, \quad a(t) = e^{\sqrt{\frac{-\Lambda}{3}}(t-t_0)}\,, \quad f(t) = p(t) = 0\,, \quad \Lambda < 0 \tag{6.144}$$

yields Lorentzian conformally flat Einstein metrics. As for the neutral-signature metrics, these metrics are incomplete.

6.3 (Para)HyperKähler Heisenberg four-manifolds

In this section we are going to classify all Heisenberg four-manifolds which are furthermore (para)hyperKähler. For that, we begin by providing the most adequate definition of (para)hyperKähler four-manifold for our purposes.

Definition 6.9. Let (M, g) be a neutral-signature or Riemannian orientable four-manifold. It is said to be (para)hyperKähler if there exist three closed self-dual two-forms (called Kähler forms) ω_i on M, with $i = 1, 2, 3$ or $i = u, v, 3$ which satisfy the condition:

$$\omega_i \wedge \omega_j = 2\varepsilon\eta_{ij}\mathrm{dvol}_M\,, \quad i, j = 1, 2, 3\,, \quad \text{or} \quad i, j = u, v, 3\,, \tag{6.145}$$

where $\varepsilon = 1$ if (M, g) is Riemannian and $\varepsilon = -1$ if (M, g) is of neutral signature, η is given by (6.7) and where dvol_M denotes the canonical volume form given by the metric and the fixed orientation on (M, g).

Remark 6.23. The usual definition of (para)hyperKähler four-manifold (M, g) is that of a Riemannian (resp. neutral-signature) four-manifold whose holonomy group is contained in the compact symplectic group $\mathrm{Sp}(1)$ (resp. in the pseudo-symplectic group $\mathrm{SL}(2, \mathbb{R})$). This is equivalent to the existence of three parallel (with respect to the Levi-Civita connection of (M, g)) (para)complex structures $J_i \in \mathrm{End}(TM)$ which are antisymmetric and satisfy[19] $J_i \circ J_j + J_j \circ J_i = -2\varepsilon\eta_{ij}\mathrm{Id}_{TM}$. The relation between this definition and ours is given by the Hitchin lemma [139], which can be rephrased by saying that the existence of three closed and self-dual two-forms $\omega_i \in \Omega^2(M)$ on (M, g) satisfying $\omega_i \wedge \omega_j = 2\varepsilon\eta_{ij}\mathrm{dvol}_M$ is equivalent to the existence of three (integrable) parallel (para)complex structures J_i satisfying the aforementioned relations.

Remark 6.24. We would like to remind the reader that (para)hyperKähler manifolds are Ricci flat and antiself-dual[20].

[18] Note that for a Lorentzian metric the two components of the Weyl tensor ($\pm i$-eigenvectors of the Hodge operator) are related by complex conjugation. So half-conformal flatness in the Lorentzian setting implies conformal flatness.

[19] We allow the possibility of having two (integrable) nilpotent endomorphisms and a paracomplex structure. Note however that these can be obtained from linear combinations of two paracomplex structures and a complex one satisfying the relations $J_i \circ J_j + J_j \circ J_i = -2\varepsilon\eta_{ij}\mathrm{Id}_{TM}$.

[20] Note that we are fixing the orientation for which the Kähler forms are self-dual.

From the previous remark and Proposition 6.2, we have that timelike, spacelike and lightlike parahyperKähler Heisenberg four-manifolds comprise all possible types of parahyperKähler Heisenberg four-manifolds. We introduce the following notation.

Definition 6.10. Let (M, g) be a Riemannian or neutral-signature Heisenberg four-manifold and let $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ be a family of orthonormal or Witt frames on H. For $t_0 \in \mathcal{I}$, we say $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ is a t_0-canonical frame if the only non-vanishing Lie bracket at t_0 is:

- For (M, g) a Riemannian or a timelike Heisenberg four-manifold:

$$[\mathfrak{e}_2^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_1^{t_0}\,, \quad k > 0\,. \tag{6.146}$$

- For spacelike Heisenberg four-manifolds:

$$[\mathfrak{e}_1^{t_0}, \mathfrak{e}_2^{t_0}] = -2k\mathfrak{e}_3^{t_0}\,, \quad k > 0\,. \tag{6.147}$$

- For lightlike Heisenberg four-manifolds:

$$[\mathfrak{e}_v^{t_0}, \mathfrak{e}_3^{t_0}] = -2k\mathfrak{e}_u^{t_0}\,, \quad k > 0\,. \tag{6.148}$$

Lemma 6.2. *Let (M, g) be a Riemannian or neutral-signature Heisenberg four-manifold. It is (para)hyperKähler if and only if, for $t_0 \in \mathcal{I}$, there exists a t_0-canonical frame $\{\mathfrak{e}_i^t\}_{t\in\mathcal{I}}$ such that the following self-dual two-forms:*

$$\omega_i = \sigma_{ij}\mathrm{d}t \wedge \mathfrak{e}_t^j + \sigma_{ij} \star (\mathrm{d}t \wedge \mathfrak{e}_t^j)\,. \tag{6.149}$$

are closed, where $\{\mathfrak{e}_t^i\}_{t\in\mathcal{I}}$ denotes the associated family of coframes dual to the t_0-canonical frame and $\sigma_{i_1 j} \in C^\infty(\mathcal{I})$, $\sigma_{i_2 j}, \sigma_{i_3 j} \in C^\infty(M)$ such that $(\sigma_{ij}(p)) \in \mathrm{SO}(\mathbb{R}^3, \eta)$ for all $p \in M$ with:

- *$i_1 = 1$, $i_2 = 2$ and $i_3 = 3$ if (M, g) is a Riemannian or a timelike Heisenberg four-manifold,*
- *$i_1 = 3$, $i_2 = 1$ and $i_3 = 2$ if (M, g) is a spacelike Heisenberg four-manifold,*
- *$i_1 = u$, $i_2 = v$ and $i_3 = 3$ if (M, g) is a lightlike Heisenberg four-manifold.*

Proof. Assume first that the self-dual two-forms ω_i in Equation (6.149) are closed. By direct computation, we check that $\omega_i \wedge \omega_j = 2\varepsilon\eta_{ij}\mathrm{dvol}_M$ whence (M, g) is (para)hyperKähler.

Conversely, assume that (M, g) is a (para)hyperKähler Heisenberg four-manifold. Let ω_i be the corresponding Kähler forms. Note first that the (para)quaternionic structure Q_p at any point $p \in M$ is given by one of the two simple ideals of $\mathfrak{so}(T_pM) = Q_p \oplus Q'_p$, where $Q_p \cong Q'_p \cong \mathfrak{su}(2)$ or $Q_p \cong Q'_p \cong \mathfrak{sl}(2, \mathbb{R})$. Therefore it is invariant under any orientation-preserving isometry. Consider the vector space $V \cong \mathbb{R}^3$ consisting of all parallel sections of Q endowed with the Euclidean or Lorentzian scalar product $\langle A, B\rangle = -\frac{\varepsilon}{4}\operatorname{Tr} AB$. Since the Heisenberg group acts through orientation-preserving isometries, we obtain a representation $\rho : \mathrm{H} \to \mathrm{SO}(V)$, whose image is a nilpotent subgroup of $\mathrm{SO}(V)$. In the Riemannian case, this leads us to the conclusion that the image of ρ is contained in an SO(2)-subgroup and, therefore, preserves a non-zero vector in V. In the neutral-signature case, the image is contained in a one-dimensional subgroup conjugate to SO(2), $\mathrm{SO}_0(1, 1)$

or to a unipotent group that preserves a lightlike vector. Again, we can conclude that the representation ρ, independently of the signature, always leaves invariant a non-zero vector of V. This implies there is an orthonormal or Witt basis (J_1, J_2, J_3) of V which contains an invariant element. (We recall that according to our conventions $\langle J_i, J_j\rangle = \eta_{ij}$ with η as in 6.7.) If the element has stabilizer $\mathrm{SO}(2)$ in $\mathrm{SO}(3)$, we can assume it is J_1. If it has stabilizer $\mathrm{SO}(1,1)$ we can take it to be J_3. If it is lightlike, the basis (J_1, J_2, J_3) is a Witt basis (J_u, J_v, J_3) and we can assume that the invariant element is J_u. Then the corresponding left-invariant Kähler forms ω_{i_1} can be chosen so that $\sigma_{i_1 j} \in C^\infty(\mathcal{I})$ with $i_1 \in \{1, 3, u\}$ as in the the statement of the lemma and we conclude.

□

After these preliminary results, now we continue with the classification of Riemannian and neutral-signature (para)hyperKähler Heisenberg four-manifolds.

6.3.1 (Timelike) (para)HyperKähler Heisenberg four-manifolds

We carry out the classification of hyperKähler and timelike parahyperKähler Heisenberg four-manifolds at once, since we will see that the procedure is strictly analogous. We fix a t_0-canonical frame $\{\mathfrak{e}^t_i\}_{t\in\mathcal{I}}$ which satisfies (6.146) and we set the ansatz (6.16) for the matrix U^t:

$$U^t = \begin{pmatrix} a(t) & 0 & 0 \\ 0 & b(t) & 0 \\ 0 & h(t) & c(t) \end{pmatrix}, \quad a, b, c, h \in C^\infty(\mathcal{I}) . \tag{6.150}$$

Proposition 6.21. *Let (M, g) be a (timelike) (para)hyperKähler Heisenberg four-manifold. Then it is isometric to*

$$a = (1 + 3k(t - t_0))^{1/3} , \quad b = c = (1 + 3k(t - t_0))^{-1/3} , \quad h = 0 . \tag{6.151}$$

The maximal domain of definition of these incomplete metrics is $(t_0 - (3k)^{-1}, +\infty) \times \mathrm{H}$.

Proof. According to Lemma 6.2, there exists (at least) a Kähler form ω_1 belonging to the (para)hyperKähler structure which is additionally invariant under the Heisenberg group action. If $\omega_1 = \sigma_{1j}\mathrm{d}t \wedge \mathfrak{e}^j_t + \sigma_{1j} \star (\mathrm{d}t \wedge \mathfrak{e}^j_t)$, then it can be seen that the equation $\nabla\omega_1 = 0$ is equivalent to

$$\sigma'_{11} = 0 , \quad \frac{\sigma'_{12}}{\sigma_{13}} = -\frac{\sigma'_{13}}{\sigma_{12}} = \frac{hc' - h'c}{2bc} , \quad a' = kbc , \quad h' = \frac{c'}{c}h , \quad \frac{b'}{b} = \frac{c'}{c} = -k\frac{bc}{a} . \tag{6.152}$$

The unique solution to the previous system of ordinary differential equations with the initial conditions $a(t_0) = b(t_0) = c(t_0) = 1$, $h(t_0) = 0$ and $\sigma_{1j}|_{t_0} = \sigma^0_{1j}$, for $\sigma^0_{1j} \in \mathbb{R}$, turns out to be

$$a = (1 + 3k(t - t_0))^{1/3} , \quad b = c = (1 + 3k(t - t_0))^{-1/3} , \quad h = 0 , \quad \sigma_{1j} = \sigma^0_{1j} . \tag{6.153}$$

This solution is defined in the interval $\mathcal{I} = (t_0 - (3k)^{-1}, +\infty)$ and we observe that the isometry type of (M, g) is completely fixed. Now, using (6.153), we note that the following two-forms:

$$\omega_i = \mathrm{d}t \wedge \mathfrak{e}^i_t + \star(\mathrm{d}t \wedge \mathfrak{e}^i_t) , \quad i = 1, 2, 3 \tag{6.154}$$

are self-dual, closed and satisfy $\omega_i \wedge \omega_j = 2\varepsilon\eta_{ij}\mathrm{dvol}_M$. Hence we conclude. □

Remark 6.25. In terms of the coframe $\{\mathrm{d}t, \mathfrak{e}^i_{t_0}\}$, the metric of a (timelike) (para)hyperKähler Heisenberg four-manifold (M, g) reads

$$g = \varepsilon \mathrm{d}t^2 + \frac{\varepsilon}{(1+3k(t-t_0))^{2/3}} \mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + (1+3k(t-t_0))^{2/3} \left(\mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0} + \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0} \right) . \quad (6.155)$$

We find that (M, g) is Ricci flat and that the Weyl tensor is antiself-dual.

Remark 6.26. Redefine the time coordinate as $e^{3k\tilde{t}} = 1 + 3k(t - t_0)$. Then the metric g reads

$$g = \varepsilon e^{6k\tilde{t}} \mathrm{d}\tilde{t}^2 + \varepsilon e^{-2k\tilde{t}} \mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + e^{2k\tilde{t}} (\mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0} + \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}) . \quad (6.156)$$

Now if we consider the rescaled metric $\hat{g} = e^{-6k\tilde{t}} g$, it takes the form:

$$\hat{g} = \varepsilon \mathrm{d}\tilde{t}^2 + \varepsilon \mathfrak{e}^1_{\tilde{t}} \otimes \mathfrak{e}^1_{\tilde{t}} + \mathfrak{e}^2_{\tilde{t}} \otimes \mathfrak{e}^2_{\tilde{t}} + \mathfrak{e}^3_{\tilde{t}} \otimes \mathfrak{e}^3_{\tilde{t}} , \quad (6.157)$$

with $\mathfrak{e}^1_{\tilde{t}} = e^{-4k\tilde{t}} \mathfrak{e}^1_{t_0}$, $\mathfrak{e}^2_{\tilde{t}} = e^{-2k\tilde{t}} \mathfrak{e}^2_{t_0}$ and $\mathfrak{e}^3_{\tilde{t}} = e^{-2k\tilde{t}} \mathfrak{e}^3_{t_0}$. We compute:

$$\mathrm{d}\mathfrak{e}^1_{\tilde{t}} = 4k \mathfrak{e}^1_{\tilde{t}} \wedge \mathrm{d}\tilde{t} + 2k \mathfrak{e}^2_{\tilde{t}} \wedge \mathfrak{e}^3_{\tilde{t}} , \quad \mathrm{d}\mathfrak{e}^2_{\tilde{t}} = 2k \mathfrak{e}^2_{\tilde{t}} \wedge \mathrm{d}\tilde{t} , \quad \mathrm{d}\mathfrak{e}^3_{\tilde{t}} = 2k \mathfrak{e}^3_{\tilde{t}} \wedge \mathrm{d}\tilde{t} . \quad (6.158)$$

So we observe that $\{\mathrm{d}t, \mathfrak{e}^1_{\tilde{t}}, \mathfrak{e}^2_{\tilde{t}}, \mathfrak{e}^3_{\tilde{t}}\}$ defines a left-invariant coframe on $\mathbb{R}^+ \times H$. Therefore the singularity in the metric (6.155) is just present up to a conformal factor, compare with [674].

6.3.2 Spacelike parahyperKähler Heisenberg four-manifolds

Now we continue with the classification of spacelike parahyperKähler Heisenberg four-manifolds. We set a t_0-canonical frame $\{\mathfrak{e}^t_i\}_{t \in \mathcal{I}}$ which satisfies (6.147) and we use the ansatz (6.17) for the matrix U^t (valid for a subinterval $\mathcal{I}'_s \subset \mathcal{I}$ containing t_0), which we rewrite here for the sake of clarity:

$$U^t = \begin{pmatrix} c(t) & h(t) & 0 \\ -h(t) & b(t) & 0 \\ 0 & 0 & a(t) \end{pmatrix} , \quad a, b, c, h \in C^\infty(\mathcal{I}'_s) . \quad (6.159)$$

Proposition 6.22. *Let (M, g) be a spacelike parahyperKähler Heisenberg four-manifold. Then it is isometric to*

$$a = (1 - 3k(t - t_0))^{1/3} , \quad b = c = (1 - 3k(t - t_0))^{-1/3} , \quad h = 0 . \quad (6.160)$$

The maximal domain of definition of these incomplete metrics is $(-\infty, t_0 + (3k)^{-1}) \times \mathrm{H}$.

Proof. In an analogous fashion to the proof of Proposition 6.21, we consider a Kähler form ω_3 invariant under the Heisenberg group, which we know to exist by virtue of Lemma 6.2. Writing $\omega_3 = \sigma_{3j} \mathrm{d}t \wedge \mathfrak{e}^j_t + \sigma_{3j} \star (\mathrm{d}t \wedge \mathfrak{e}^j_t)$, the parallel condition $\nabla \omega_3 = 0$ implies that:

$$\frac{\sigma'_{31}}{\sigma_{32}} = \frac{\sigma'_{32}}{\sigma_{31}} = \frac{h(c' - b') + h'(b - c)}{2(bc + h^2)} , \quad (6.161)$$

$$h' = h \frac{kb^2c^2 + 2kbch^2 + kh^4 + acc'}{a(bc + c^2 + h^2)} = h \frac{kb^2c^2 + kh^4 + 2kbch^2 + abb'}{a(bc + b^2 + h^2)} . \quad (6.162)$$

$$b' = \frac{kb^3c^2 + kch^4 + kbh^2(2c^2 + h^2) + kb^2c(c^2 + 2h^2) - ah^2c'}{a(bc + c^2 + h^2)}\,, \tag{6.163}$$

$$c' = \frac{kb^3c^2 + kch^4 + kbh^2(2c^2 + h^2) + kb^2c(c^2 + 2h^2) - ah^2b'}{a(bc + b^2 + h^2)}\,, \tag{6.164}$$

$$a' = -k(bc + h^2)\,. \tag{6.165}$$

This is a system of first-order ordinary differential equations with the initial condition $a(t_0) = b(t_0) = c(t_0) = 1$, $h(t_0) = 0$ and $\sigma_{3j}|_{t_0} = \sigma^0_{3j}$, for $\sigma^0_{3j} \in \mathbb{R}$. It can be see to admit a unique solution, which turns out to be

$$a = (1 - 3k(t - t_0))^{1/3}\,, \quad b = c = (1 - 3k(t - t_0))^{-1/3}\,, \quad h = 0\,, \quad \sigma_{3j} = \sigma^0_{3j}\,. \tag{6.166}$$

This solution is defined in the interval $\mathcal{I}'_s = (-\infty, t_0 + (3k)^{-1}))$. Ansatz (6.159) was in principle valid only for a subinterval $\mathcal{I}'_s \subset \mathcal{I}$ containing t_0, but we note that actually $\mathcal{I}'_s = \mathcal{I}$, since at $t = t_0 + (3k)^{-1}$ the metric has a singularity and cannot be extended for larger values of t. Using now (6.166), we observe that the following two-forms:

$$\omega_i = \mathrm{d}t \wedge \mathfrak{e}^i_t + \star(\mathrm{d}t \wedge \mathfrak{e}^i_t)\,, \quad i = 1, 2, 3 \tag{6.167}$$

are self-dual, closed and satisfy $\omega_i \wedge \omega_j = -2\eta_{ij}\mathrm{dvol}_M$. Hence we conclude. □

Remark 6.27. We observe that the subsequent pseudo-Riemannian manifold is Ricci flat and the Weyl tensor is antiself-dual. The metric turns out to be

$$g = -\mathrm{d}t^2 + (1 - 3k(t - t_0))^{2/3}\left(-\mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + \mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0}\right) + \frac{1}{(1 - 3k(t - t_0))^{2/3}}\mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\,. \tag{6.168}$$

6.3.3 Lightlike parahyperKähler Heisenberg four-manifolds

Finally we carry out the classification of all lightlike parahyperKähler Heisenberg four-manifolds. In analogy with the previous cases, we pick a t_0-canonical frame $\{\mathfrak{e}^t_i\}_{t \in \mathcal{I}}$ which satisfies (6.148) at $t_0 \in \mathcal{I}$ and we choose the ansatz (6.18) for U^t_W (valid for a subinterval $\mathcal{I}'_l \subset \mathcal{I}$ containing t_0), which we present here again:

$$U^t_W = \begin{pmatrix} 1 & 0 & 0 \\ f(t) & b(t) & p(t) \\ 0 & 0 & a(t) \end{pmatrix}\,, \quad a, b, c, f, p \in C^\infty(\mathcal{I}'_l) \tag{6.169}$$

Proposition 6.23. *All lightlike parahyperKähler Heisenberg four-manifolds are isometric to $(\mathbb{R} \times \mathrm{H}, g)$, where g is the metric (6.5) constructed from a t_0-canonical frame $\{\mathfrak{e}^t_i\}_{t \in \mathbb{R}}$ such that*

$$a = b = 1\,, \quad f = -2k(t - t_0)\,, \quad p = 0 \tag{6.170}$$

Furthermore, such metric is flat and isometric to $(\mathbb{R}^4, \eta)$ (and therefore, complete).

Proof. Following Lemma 6.2 and its notation, let ω_u denote the corresponding Kähler form that is additionally invariant under the Heisenberg group, which it is guaranteed to exist. If $\omega_u = \sigma_{uj}\mathrm{d}t \wedge \mathfrak{e}^j_t + \sigma_{uj} \star (\mathrm{d}t \wedge \mathfrak{e}^j_t)$, then ω_u being parallel implies:

$$\sigma_{uu} = \frac{1}{2ab}\left(ab'\sigma_{uu} + (bp' - pb')\sigma_{u3}\right)\,, \quad \sigma'_{uv} = -\frac{\sigma_{uv}b'}{2b}\,, \quad \sigma_{u3} = \frac{\sigma_{uv}(pb' - bp')}{2ab} \tag{6.171}$$

$$f' = -2kab + \frac{\sigma_{uu} + 2f\sigma_{uv}}{2b\sigma_{uv}}b'\,, \quad p' = \frac{p\sigma_{uv} + a\sigma_{u3}}{b\sigma_{uv}}b' = \frac{pb'}{b}\,, \quad a' = b' = 0\,. \tag{6.172}$$

The unique solution to the previous system of ODEs with the initial conditions $a(t_0) = b(t_0) = 1$ and $p(t_0) = f(t_0) = 0$ turns out to be[21]

$$a = b = 1\,, \quad f = -2k(t - t_0)\,, \quad p = 0\,, \quad \sigma_{uj} = \sigma^0_{uj}\,, \tag{6.173}$$

where $\sigma^0_{uj} \in \mathbb{R}$ are constants. This solution is trivially defined for $t \in \mathbb{R}$, so we can actually extend $\mathcal{I}'_l$ to be the entire $\mathcal{I}$ and $\mathcal{I}'_l = \mathcal{I} = \mathbb{R}$. Using now (6.173), we observe that the two-forms

$$\omega_i = \mathrm{d}t \wedge \mathfrak{e}^i_t + \star(\mathrm{d}t \wedge \mathfrak{e}^i_t)\,, \quad i = u, v, 3 \tag{6.174}$$

are self-dual, closed and satisfy $\omega_i \wedge \omega_j = 2\varepsilon\eta_{ij}\mathrm{dvol}_M$. This way we obtain all lightlike parahyperKähler Heisenberg four-manifolds, which we easily see to be flat. Finally, after using the coordinates (6.3), it is possible to see that all geodesics with coordinates $(t(\tau), x(\tau), y(\tau), z(\tau))$ and affine parameter τ take the form:

$$\begin{aligned} t(\tau) = A_1 + A_2\tau - kA_3^2\tau^2\,, \quad x(\tau) = A_4 + A_3\tau\,, \quad y(\tau) = A_5 + A_6\tau - kA_3^2\tau^2\,, \\ z(\tau) = A_7 + A_8\tau + kA_3\tau^2(kA_3x(\tau) - 2A_2 + A_6)\,, \end{aligned} \tag{6.175}$$

with $A_l \in \mathbb{R}$ for $l = 1, 2, \dots, 8$. Since these geodesics are defined $\forall\tau \in \mathbb{R}$ we conclude that the subsequent pseudo-Riemannian manifolds are complete and therefore all lightlike parahyperKähler Heisenberg four-manifolds are isometric to four-dimensional flat space $(\mathbb{R}^4, \eta)$.

□

Remark 6.28. The metric of any lightlike parahyperKähler Heisenberg four-manifold in terms is isometric to

$$g = -\mathrm{d}t^2 + \mathfrak{e}^u_{t_0} \odot \mathfrak{e}^v_{t_0} + 4k(t - t_0)\mathfrak{e}^v_{t_0} \otimes \mathfrak{e}^v_{t_0} + \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\,. \tag{6.176}$$

We check by direct inspection that it is indeed flat.

Gathering the results given in Propositions 6.21, 6.22 and 6.23, we prove the following theorem.

Theorem 6.6. *All (timelike) (para)hyperKähler Heisenberg four-manifolds are incomplete and isometric to*

$$g = \varepsilon\mathrm{d}t^2 + \frac{\varepsilon}{(1 + 3k(t - t_0))^{2/3}}\mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + (1 + 3k(t - t_0))^{2/3}(\mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0} + \mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0})\,, \tag{6.177}$$

where $(\mathfrak{e}^i_{t_0})$ is a left-invariant coframe of H such that $[\mathfrak{e}^{t_0}_2, \mathfrak{e}^{t_0}_3] = -2k\mathfrak{e}^{t_0}_1$ for $k \neq 0$. All spacelike parahyperKähler Heisenberg four-manifolds are incomplete and isometric to:

$$g = -\mathrm{d}t^2 + (1 - 3k(t - t_0))^{2/3}(-\mathfrak{e}^1_{t_0} \otimes \mathfrak{e}^1_{t_0} + \mathfrak{e}^2_{t_0} \otimes \mathfrak{e}^2_{t_0}) + \frac{1}{(1 - 3k(t - t_0))^{2/3}}\mathfrak{e}^3_{t_0} \otimes \mathfrak{e}^3_{t_0}\,, \tag{6.178}$$

where now $[\mathfrak{e}^{t_0}_1, \mathfrak{e}^{t_0}_2] = -2k\mathfrak{e}^{t_0}_3$. Finally, all lightlike parahyperKähler Heisenberg four-manifolds are isometric to flat space $(\mathbb{R}^4, \eta)$.

[21] After using $b' = 0$ in the rest of equations, the possible divergences arising from the possibility that $\sigma_{uv} = 0$ disappear.

6.4 Discussion

We have managed to classify all Riemannian and neutral-signature four-manifolds admitting a cohomogeneity one principal action of the Heisenberg group with non-degenerate orbits and whose Weyl tensor is (anti)self-dual. They are either quaternionic (para)Kähler (if the Einstein constant is non-zero) or (para)hyperKähler (if they are Ricci flat).

We began with the study of the quaternionic (para)Kähler Heisenberg four-manifolds. In the Riemannian case, as well as in the case of timelike or spacelike Heisenberg center, we reduced the problem to a system ordinary differential equations of first order for two functions a and b. The resulting quaternionic Kähler manifolds proved to be conformally Kähler whereas the resulting quaternionic paraKähler manifolds were conformally Kähler or conformally paraKähler, depending on the causal character of the Heisenberg center

It turned out that the ODE system took a particularly nice form when $\varepsilon\Lambda = -6k^2$, where k is the structure constant of the adapted frame at initial time, up to a numerical factor. We referred to this case as the stationary case, as its solutions were stationary in the sense that the logarithmic derivatives of the unknown functions were constant. We determined all maximal stationary solutions and showed that they define homogeneous spaces. More precisely, each of these homogeneous spaces could be realized as an open orbit of a four-dimensional solvable Lie group acting by isometries on a symmetric space.

In the non-stationary case, we showed that the above ODE system could be explicitly and completely solved. We found that the solutions occur in one-parametric families and that the parameter could be identified with the one-loop parameter in the perturbative quantum correction of the Supergravity c-map and its temporal and Euclidean versions. Setting the parameter to zero (whenever possible on the considered branch) resulted in one of the stationary solutions. Geometrically this corresponds to a deformation of the locally symmetric space in the class of quaternionic (para)Kähler manifolds. In the case of lightlike center, we found analogously all solutions to the aforementioned system of ODEs and observed that they are conformally flat.

Regarding the (para)hyperKähler counterparts, we were also able to carry out the complete classification result. In the Riemannian case and in the case of timelike and spacelike Heisenberg center, we wrote the explicit form of the (Ricci flat) metrics and proved that they are incomplete. This is in contradistinction to the lightlike case, which we found to be just isometric to flat space (and therefore, complete).

These conclusions suggest many questions and future directions. In particular, could it be possible to provide (partial) classification results if we remove the condition of self-duality for the Weyl tensor? Indeed, it would be very natural in the Ricci flat case to investigate the more general setup of (para)Kähler Heisenberg four-manifolds. Also, it could help us identifying more Lorentzian Einstein Heisenberg four-manifolds, apart from the one given by Remark 6.22. Analogously, we may proceed in the opposite direction: what happens if we keep the self-duality condition, but remove the Einstein condition?

Another interesting question to pose is about the possibility of providing an interpretation of all quaternionic (para)Kähler self-dual Einstein Heisenberg four-manifolds in the context of Supergravity and ST. Indeed, we explained that positively-curved (resp. negatively-curved) Riemannian (resp. timelike and spacelike) quaternionic (para)Kähler Heisenberg four-manifolds can be understood as scalar manifolds appearing in Supergravity and ST. Thus, it would be interesting to analyze if there exists any interpretation for the negatively-curved (resp. positively-curved) ones (as well as for lightlike quaternionic paraKähler Heisenberg four-manifolds). In this direction, it is known that the quaternionic (para)Kähler manifolds constructed through the spatial, temporal and Euclidean Supergravity c-maps and their one-loop deformations can be obtained through the rigid versions of the previous c-maps and the (para)hyperKähler/quaternionic (para)Kähler correspondence [153, 156, 675]. This triggers the question: is it possible that the (para)hyperKähler Heisenberg four-manifolds we have obtained in this chapter can be mapped, through the aforementioned correspondence, to the quaternionic (para)Kähler solutions we have derived? This would strongly suggest that the (para)hyperKähler manifolds we have found arise from the rigid Supergravity c-maps.

Finally, it could be intriguing as well to investigate the possibility of having principal and isometric actions with degenerate orbits or work in higher dimensions $D = 4m$ with $m > 1$.

7

Contact structures in six-dimensional Supergravity

In the last chapter of the thesis we will continue with the study of selected topics belonging to the realm of geometry of Supergravity and ST. More concretely, we will now concentrate on the construction of particular solutions of minimal six-dimensional Supergravity coupled to a tensor multiplet, which can be obtained by intertwining special classes of contact structures.

Indeed, as we have already observed, the ongoing mathematical study and development of Supergravity [106, 133, 134] poses novel and occasionally striking mathematical problems in diverse areas of differential geometry and topology. Some of these mathematical problems are specifically concerned with the classification of Supergravity solutions and the study of the associated moduli spaces of solutions. In this context, the classification of all simply connected Lorentzian manifolds admitting bosonic solutions of Supergravity in the considered dimension remains as an outstanding open problem in the mathematical theory of Supergravity. The purpose of this chapter is to propose a contribution to this problem in the specific case of minimal Supergravity in six Lorentzian dimensions [188, 189] coupled to a tensor multiplet with constant dilaton, a theory that can be neatly rephrased as a natural geometric problem in the realm of six-dimensional Lorentzian geometry with torsion. More concretely, the bosonic configuration space of this six-dimensional Supergravity on a six-manifold M consists on pairs (g, H), where g is a Lorentzian metric on M and H is a three-form.

Rather than intending to provide a full classification result, which seems currently out of reach, we shall try to make some simplifying assumptions, with the aim of developing a method to construct families of solutions as a first step towards the understanding of the general classification problem. In particular, we will assume that the six-dimensional Lorentzian manifold splits as $(M, g) = (N \times X, \chi \oplus h)$, where (N, χ) and (X, h) are three-dimensional Lorentzian and Riemannian three-manifolds, respectively.

In this context, we will be able to prove that appropriate combinations of so-called $\varepsilon\eta$-Einstein contact structures in (N, χ) and (X, h) yield in turn solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton, which can be rephrased in the mathematical setting as producing Lorentzian six-manifolds (M, g) with a Ricci flat metric-compatible connection with isotropic, totally skew-symmetric, closed and co-closed torsion. Such $\varepsilon\eta$-Einstein contact structures are particular cases of ε-contact structures, which encompass usual (three-dimensional) contact Riemannian, contact Lorentzian and para-contact metric structures, but which allow for the Reeb vector field to be null (lightlike). In particular, in the non-null cases they are restricted versions of the usual notion of η-Einstein strutures [161, 164].

The case with null Reeb vector field seems not to have been previously studied in the literature and we have called it *null contact structure*. We show that the Sasaki and K-contact notions can be extended in these cases but are however not equivalent conditions, in contrast to the situation occurring when the Reeb vector field is not lightlike. We define a notion of left-invariant null contact structure and provide a classification of all of them in simply connected three-dimensional Lie groups.

Afterwards, we focus on $\varepsilon\eta$-Einstein contact structures, deriving a classification result for such (left-invariant) structures on three-dimensional simply connected Lie groups for any causal character of the Reeb vector field. This is used later for the construction of novel families of six-dimensional Supergravity solutions, some of which can be interpreted as continuous deformations of the maximally supersymmetric solution on $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times S^3$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ denoting the universal cover of $\mathrm{SL}(2,\mathbb{R})$.

This chapter is organized is as follows. First we introduce the notion of ε-contact structure and explore some of its generic properties, including its formulation on a globally hyperbolic Lorentzian three-manifold. Next we focus on ε-contact structures with lightlike Reeb vector field, introducing the concepts of Sasakian and null K-contact contact structures and classifying those on simply connected three-dimensional Lie groups which are left-invariant. Then we introduce the notion of $\varepsilon\eta$-Einstein ε-contact structure and we classify all left-invariant $\varepsilon\eta$-Einstein contact structures on simply connected three-dimensional Lie groups. Afterwards we present the main result of our work in which we establish the link between $\varepsilon\eta$-Einstein contact structures and solutions of Supergravity in six dimensions. Later we illustrate the type of solutions of six-dimensional Supergravity that are obtained through the combination of $\varepsilon\eta$-Einstein contact structures and conclude with a discussion of our most relevant findings.

7.1 ε-contact metric three-manifolds

In this section we introduce the notion of ε*-contact metric structure*, which encompasses as particular cases the standard definition of contact Riemannian metric structure, contact Lorentzian structure and para-contact metric structure in three dimensions, but which also allows for the *Reeb vector field* to be null.

Definition 7.1. Let M be an oriented three-manifold. An ε-contact metric structure (or ε-contact structure, in short) on M consists of a triple (g, α, ε), with $\varepsilon \in \{-1, 0, 1\}$, g a Riemannian or pseudo-Riemannian metric on M and α a one-form $\alpha \in \Omega^1(M)$, satisfying:

$$\alpha = \star \mathrm{d}\alpha\,, \qquad |\alpha|_g^2 = \varepsilon, \tag{7.1}$$

where $\star\colon \Omega^r(M) \to \Omega^{3-r}(M)$ $(r = 0, 1, 2, 3)$ denotes the Hodge-dual with respect to g and the fixed orientation on M, which is then said to be an ε-contact metric three-manifold (or ε-contact three-manifold, in short). When g is Lorentzian, we will assume that (M, g) is oriented and time-oriented.

Remark 7.1. Note that equation $\alpha = \star\mathrm{d}\alpha$ is equivalent to

$$\star\, \alpha = \sigma_g\, \mathrm{d}\alpha\,, \tag{7.2}$$

where $\sigma_g = +1$ if g is Riemannian and $\sigma_g = -1$ if g is Lorentzian.

Remark 7.2. The motivation to introduce the previous definition will be apparent in Section 7.4, see Theorem 7.5.

Let (g, α, ε) be an ε-contact metric structure on M and let ν_g be the pseudo-Riemannian volume form associated to g and the fixed orientation on M. If g is Riemannian then necessarily $\varepsilon = 1$ (whence it can be omitted) and (g, α) defines in this case a standard contact metric structure on M [161]. To see this, note that the kernel of α defines an oriented non-integrable rank-two distribution,

$$\mathcal{D} \stackrel{\text{def.}}{=} \ker(\alpha) \subset TM \,. \tag{7.3}$$

Denote by $g_{\mathcal{D}}$ the restriction of g to $\mathcal{D}$. Being oriented, the Riemannian volume form of $(\mathcal{D}, g_{\mathcal{D}})$ defines a canonical almost complex structure $J_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D}$ with respect to which $\nu_{\mathcal{D}}$ is of type $(1,1)$. Furthermore, the vector field $\xi = \alpha^\sharp$ dual to α satisfies by definition $\alpha(\xi) = 1$. Equation (7.1) implies

$$\mathrm{d}\alpha(v_1, v_2) = \star\alpha(v_1, v_2) = \nu_{\mathcal{D}}(v_1, v_2) = g_{\mathcal{D}}(v_1, J_{\mathcal{D}} v_2) \,, \qquad v_1, v_2 \in \mathcal{D} \tag{7.4}$$

Therefore, the tuple $(\mathcal{D}, g, \xi, \phi)$, where $\phi|_{\mathcal{D}} = J_{\mathcal{D}}$ and $\phi(\xi) = 0$, defines a standard contact metric structure on M. Conversely, any such Riemannian contact metric structure gives rise to a canonical Riemannian ε-contact metric structure on M. Similar remarks apply when g is Lorentzian and $\varepsilon = -1$, in which case we recover the usual notion of Lorentzian contact metric structure, and when g is Lorentzian and $\varepsilon = 1$, in which case we recover the usual notion of para-contact metric structure. Hence the following holds.

Proposition 7.1. *Let M be an oriented three-manifold. An ε-contact metric structure (g, α, ε) on M defines a canonical Riemannian contact metric structure if g is Riemannian, a canonical Lorentzian contact metric structure if g is Lorentzian and $\varepsilon = -1$, and a canonical para-contact metric structure if g is Lorentzian and $\varepsilon = 1$. The converse also holds for the three previous cases.*

Remark 7.3. By the previous argument, if (g, α, ε) is an ε-contact metric structure on a Riemannian manifold (M, g), we shall just denote it as a Riemannian contact metric structure in order to keep the usual nomenclature in the literature.

Given an ε-contact structure (g, α, ε), we will refer to $\xi = \alpha^\sharp$ (the metric dual of α) as the Reeb vector field of (g, α, ε). The notion of morphism of ε-contact manifolds we consider is the expected one.

Definition 7.2. Let $(M_a, g_a, \alpha_a, \varepsilon_a)$, $a = 1, 2$, be ε-contact three-manifolds. A morphism F from $(M_1, g_1, \alpha_1, \varepsilon_1)$ to $(M_2, g_2, \alpha_2, \varepsilon_2)$ with $\varepsilon_1 = \varepsilon_2$ is an orientation preserving smooth map $F \colon M_1 \to M_2$ such that

$$g_1 = F^* g_2 \,, \qquad \alpha_1 = F^* \alpha_2 \,. \tag{7.5}$$

If such F is not orientation preserving, then we will say that it is an orientation-reversing morphism.

Remark 7.4. We denote by PCont the category whose objects are ε-contact three-manifolds and whose morphisms are defined as above. Relevant subcategories of PCont are the subcategory of contact Riemannian three-manifolds PCont_R and the category $\mathrm{PCont}_L(\varepsilon)$ of ε-contact Lorentzian three-manifolds with Reeb vector field of norm $\varepsilon \in \{-1, 0, 1\}$.

In analogy with the standard theory of Riemannian contact structures, we introduce, associated to every ε-contact metric structure (g,α,ε), two endomorphisms $\phi\colon TM\to TM$ and $\mathfrak{h}\colon TM\to TM$:

$$\phi(v) = -\sigma_g\,(\iota_v\star\alpha)^\sharp\,,\qquad \mathfrak{h}(v)=(\mathcal{L}_\xi\phi)(v)\qquad \forall\, v\in TM\,, \tag{7.6}$$

where $\xi=\alpha^\sharp$ is the Reeb vector field of (g,α,ε) and the symbol $\mathcal{L}$ denotes Lie derivative. We will refer to $\phi\in\Gamma(TM\otimes T^*M)$ as the *characteristic endomorphism* of (g,α,ε). Furthermore, from ϕ and $\mathfrak{h}$ we define

$$\tau \overset{\text{def.}}{=} \mathfrak{h}\circ\phi\colon TM\to TM\,. \tag{7.7}$$

These endomorphisms will play an important role later on. The following lemma summarizes some of the properties enjoyed by ϕ.

Lemma 7.1. *Let $(M,g,\alpha,\varepsilon)\in\mathrm{PCont}$ be an ε-contact metric manifold. The characteristic endomorphism ϕ satisfies:*

$$g(\mathrm{Id}\otimes\phi)=\mathrm{d}\alpha\,,\qquad \phi(\xi)=0\,,\qquad \alpha\circ\phi=0\,, \tag{7.8}$$

$$\phi^2=\sigma_g\,(-\varepsilon\,\mathrm{Id}+\xi\otimes\alpha)\,,\qquad g\circ\phi\otimes\phi=\sigma_g(\varepsilon\,g-\alpha\otimes\alpha)\,. \tag{7.9}$$

where $\xi\overset{\text{def.}}{=}\alpha^\sharp$ denotes the Reeb vector field of (g,α,ε).

Proof. Using the definition of ϕ given in Equation (7.6), we compute:

$$g(v_1,\phi(v_2)) = -\sigma_g g(v_1,(\iota_{v_2}\star\alpha)^\sharp) = -\sigma_g\star\alpha(v_2,v_1)=\mathrm{d}\alpha(v_1,v_2)\,,\quad v_1,v_2\in\mathfrak{X}(M)\,, \tag{7.10}$$

whence the first equation of the lemma holds. The second and third equations of the lemma follow directly from the definition of ϕ. On the other hand, the square of the endomorphism ϕ can be computed to be:

$$\begin{aligned}\phi(\phi(v)) &= [\iota_{(\iota_v\star\alpha)^\sharp}(\star\alpha)]^\sharp = [(\star\alpha)((\iota_v\star\alpha)^\sharp)]^\sharp = [\star(\alpha\wedge(\iota_v\star\alpha))]^\sharp\\ &= [-\star(\iota_v(\alpha\wedge\star\alpha))+\alpha(v)(\star\star\alpha)]^\sharp = -\sigma_g\varepsilon\, v+\sigma_g\alpha(v)\xi\,,\qquad v\in\mathfrak{X}(M)\,,\end{aligned} \tag{7.11}$$

which implies the fourth equation of the lemma. Finally, this last equation follows from

$$\begin{aligned}g(\phi(v_1),\phi(v_2)) &= \mathrm{d}\alpha(\phi(v_1),v_2) = -\mathrm{d}\alpha(v_2,\phi(v_1)) = -g(v_2,\phi^2(v_1))\\ &= \sigma_g\varepsilon g(v_1,v_2)-\sigma_g\alpha(v_1)\alpha(v_2)\,.\end{aligned} \tag{7.12}$$

□

Remark 7.5. Lemma 7.1 recovers key identities satisfied by ε-contact structures which in classical references are taken as part of the definition of Riemannian contact structures [161] ($\varepsilon=\mathfrak{s}_\mathfrak{g}=1$), Lorentzian contact structures [164] ($\varepsilon=\mathfrak{s}_\mathfrak{g}=-1$) or para-contact structures [676] ($\varepsilon=-\mathfrak{s}_\mathfrak{g}=1$).

Remark 7.6. Note that the characteristic endomorphism of an ε-contact metric structure (g,α,ε) is always skew-symmetric with respect to g, that is,

$$g(\phi(v_1),v_2)+g(v_1,\phi(v_2))=0\,,\qquad \forall\, v_1,v_2\in TM\,. \tag{7.13}$$

Given an ε-contact structure (g, α, ε) we define yet another endomorphism, denoted by $\mathfrak{l} \in \mathrm{End}(TM)$, as follows:

$$\mathfrak{l}(v) = \mathrm{R}^g(v, \xi)\xi\,, \qquad \forall\, v \in \mathfrak{X}(M)\,, \tag{7.14}$$

where R^g denotes the Riemann curvature tensor of g. The endomorphism $\mathfrak{l}\colon TM \to TM$ should not be confused with the endomorphism determined by the Ricci curvature, which we denote by Q^g. This makes five the endomorphisms $(\phi, \mathfrak{h}, \tau, \mathfrak{l}, \mathrm{Q}^g)$ canonically associated to every ε-contact structure (g, α, ε). Manifolds equipped with an ε-contact structure (g, α, ε) admit a special frame which is very convenient for computations.

Definition 7.3. Let (g, α, ε) be an ε-contact metric structure on M with Reeb vector field $\xi \in \mathfrak{X}(M)$. An ε-contact frame is a local frame $\{\xi, u, \phi(u)\}$, where $u \in \mathfrak{X}(M)$ is a nowhere vanishing vector field satisfying:

$$g(u, u) = \sigma_g\, \varepsilon\,, \qquad g(u, \xi) = 1 - \varepsilon^2\,. \tag{7.15}$$

When $\varepsilon = 0$ we will refer to $\{\xi, u, \phi(u)\}$ as a light-cone frame for $(g, \alpha, 0)$, following standard usage in Lorentzian geometry.

Remark 7.7. It is a direct calculation to verify that $\{\xi, u, \phi(u)\}$ is indeed a local frame. We have:

$$g(\xi, \xi) = \varepsilon\,, \qquad g(u, \phi(u)) = 0\,, \qquad g(\xi, \phi(u)) = 0\,, \qquad g(\phi(u), \phi(u)) = 1\,, \tag{7.16}$$

whence $\phi(u)$ is locally nowhere vanishing and point-wise orthogonal to the real span of ξ and u.

Proposition 7.2. *Let (g, α, ε) be an ε-contact metric structure. The following equations hold:*

$$\begin{gathered}\nabla_\xi \xi = 0\,, \qquad \nabla_\xi \phi = 0\,, \qquad \mathfrak{h}(\xi) = 0\,, \qquad \mathfrak{l}(\xi) = 0\,, \qquad \mathrm{Tr}(\mathfrak{h}) = 0\,, \\ \mathcal{L}_\xi \alpha = 0\,, \qquad \mathrm{Tr}(\tau) = 0\,, \qquad \mathfrak{h}\circ\phi + \phi\circ\mathfrak{h} = 0\,,\end{gathered} \tag{7.17}$$

where ∇ denotes the Levi-Civita connection with respect to g. Furthermore, both $\mathfrak{h}$ and τ are symmetric with respect to g.

Proof. The proof of Equations (7.17) follows by direct computation on an ε-contact frame. Thus, we prove only the symmetry properties of $\mathfrak{h}$ and τ. We compute:

$$\begin{aligned} g(\mathfrak{h}(v_1), v_2) &= g((\mathcal{L}_\xi\phi)(v_1), v_2) = g(\mathcal{L}_\xi\phi(v_1) - \phi(\mathcal{L}_\xi v_1), v_2) \\ &= g(-\nabla_{\phi(v_1)}\xi + \phi(\nabla_{v_1}\xi), v_2)\,. \end{aligned} \tag{7.18}$$

This expression vanishes whenever v_1 or v_2 are equal to ξ. Given an ε-contact frame $\{\xi, u, \phi(u)\}$, assume now that both v_1 and v_2 belong to the span of of u and $\phi(u)$. We obtain that

$$g(\mathfrak{h}(v_1), v_2) = \alpha(\nabla_{\phi(v_1)}v_2 + \nabla_{v_1}\phi(v_2)) = \alpha(\nabla_{\phi(v_2)}v_1 + \nabla_{v_2}\phi(v_1)) = g(v_1, \mathfrak{h}(v_2))\,, \tag{7.19}$$

where we have used that $\alpha([\phi(v_1), v_2]) + \alpha([v_1, \phi(v_2)]) = 0$, since

$$0 = g(\phi(v_1), \phi(v_2)) = \mathrm{d}\alpha(\phi(v_1), v_2) - \mathrm{d}\alpha(\phi(v_2), v_1) = (\mathcal{L}_{\phi(v_1)}\alpha)(v_2) - (\mathcal{L}_{\phi(v_2)}\alpha)(v_1)$$

$$= \alpha([\phi(v_1), v_2]) + \alpha([v_1, \phi(v_2)])\,. \tag{7.20}$$

The symmetry of τ follows now directly:

$$g(\tau(v_1), v_2) = g(\mathfrak{h}(\phi(v_1)), v_2) = g(\phi(v_1), \mathfrak{h}(v_2)) = -g(v_1, \phi(\mathfrak{h}(v_2))) = g(v_1, \tau(v_2))\,, \tag{7.21}$$

and we conclude. $\square$

An ε-contact structure (g, α, ε) also satisfies the following identities, which play a key role in the classification of $\varepsilon\eta$-Einstein contact structures.

Proposition 7.3. *Let (g, α, ε) be an ε-contact metric structure. The following equation holds:*

$$2\phi(\nabla\xi) = \mathfrak{h} + \sigma_g(\varepsilon - \xi \otimes \alpha)\,, \tag{7.22}$$

Remark 7.8. We can apply ϕ to the second equation in the previous proposition, obtaining

$$2\phi^2(\nabla\xi) = -2\sigma_g\varepsilon\,\nabla^g\xi = \sigma_g\varepsilon\,\phi + \phi \circ \mathfrak{h}\,. \tag{7.23}$$

For $\varepsilon \neq 0$, that is, for ε-contact structures with non-null Reeb vector field, this equation gives the well-known formula for the covariant derivative of ξ [161, 677]. For $\varepsilon = 0$ this equation reduces to $\tau = 0$, which we will prove independently in Lemma 7.3. Therefore, for ε-contact structures with null Reeb vector field, the covariant derivative of the Reeb vector is not prescribed in terms of ϕ and $\mathfrak{h}$. This has important consequences in the classification of $\varepsilon\eta$-Einstein ε-contact structures with null Reeb vector field.

Proof. We use Koszul's formula for the Levi-Civita connection:

$$\begin{aligned}
&-2g(\phi(\nabla_{v_1}\xi), v_2) = 2g(\nabla_{v_1}\xi, \phi(v_2)) = \xi \cdot g(v_1, \phi(v_2)) - \phi(v_2)\cdot\alpha(v_1) - g(\mathcal{L}_\xi v_1, \phi(v_2))\\
&+ g([\phi(v_2), v_1], \xi) - g(\mathcal{L}_\xi(\phi(v_2)), v_1) = -g(\mathfrak{h}(v_1), v_2) - \phi(v_2)\cdot\alpha(v_1) - \alpha([v_1, \phi(v_2)])\\
&= -g(\mathfrak{h}(v_1), v_2) + g(\phi^2(v_1), v_2) = -g(\mathfrak{h}(v_1), v_2) + g(\sigma_g(-\varepsilon v_1 + \alpha(v_1)\xi), v_2)\,,
\end{aligned} \tag{7.24}$$

for every $v_1, v_2 \in \mathfrak{X}(M)$, where we have used $\phi^2 = \sigma_g\,(-\varepsilon\,\mathrm{Id} + \xi \otimes \alpha)$ and the equation

$$\begin{aligned}
\mathrm{d}\alpha(v_1, \phi(v_2)) &= v_1 \cdot \alpha(\phi(v_2)) - \phi(v_2)\cdot\alpha(v_1) - \alpha([v_1, \phi(v_2)])\\
&= -\phi(v_2)\cdot\alpha(v_1) - \alpha([v_1, \phi(v_2)])\,.
\end{aligned} \tag{7.25}$$

Also, observe that we have used the notation $v \cdot f$ for $v \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ to denote $v(f) \in C^\infty(M)$. $\square$

Remark 7.9. Note that the equation for $\nabla\xi$ given by Proposition 7.3 differs from the one usually found in the literature by a factor of $\frac{1}{2}$. This discrepancy is due to the different convention we are using for the exterior derivative. Given any p-form ω, we define its exterior derivative $\mathrm{d}\omega$ as the $(p+1)$-form given by:

$$\mathrm{d}\omega(X_0, \dots, X_p) = \sum_i (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_i)) \tag{7.26}$$

$$+ \sum_{i<j} (-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p)\,. \tag{7.27}$$

Much of the literature on contact geometry, see for example [161], uses the conventions of Kobayashi and Nomizu [479], in which the formula of the exterior derivative differs by a factor of $\frac{1}{p+1}$ from the one stated above.

Proceeding by analogy with the theory of Riemannian contact metric structures we introduce the notions of Sasakian and K-contact ε-contact metric structures.

Definition 7.4. An ε-contact metric structure (g, α, ε) is said to be Sasakian if $\mathfrak{h} = 0$. It is said to be K-contact if the Reeb vector field is Killing, that is, if $\mathcal{L}_\xi g = 0$.

Remark 7.10. The Sasakian and K-contact conditions are well known to be equivalent for ε-contact structures with $\varepsilon \neq 0$ in three dimensions, see [161, 164, 676]. However, as we will see in Section 7.2, this fails to be the case for null contact metric structures. Indeed, Example 7.4 shows that the theory of Sasakian null contact metric structures is strictly richer that the theory of null K-contact structures.

Sasakian ε-contact metric structures with $\varepsilon \neq 0$ have been extensively studied in the literature, see for instance [161], [164] or [678] for more details and an exhaustive list of references. In particular, it is well known that the Sasakian condition can sometimes be equivalently formulated as a curvature condition involving the Ricci curvature tensor Ric^g. Since this will be of use later, we briefly review this result using our conventions.

Proposition 7.4. *Let (g, α, ε) be an ε-contact structure with $\varepsilon \neq 0$. Then*

$$\mathrm{Ric}^g(\xi, \xi) = \varepsilon\, \sigma_g \left(\frac{1}{2} - \frac{1}{4}\mathrm{Tr}(\mathfrak{h}^2)\right) . \tag{7.28}$$

where Ric^g is the Ricci curvature of g.

Proof. The proofs of the proposition in the Riemannian and Lorentzian $\varepsilon = -1$ cases are presented in detail in [161] and [164], respectively. Hence we focus on the para-contact case.

Proposition 7.3 adapted to the para-contact case yields $\nabla\xi = -\frac{1}{2}\phi + \frac{1}{2}\phi\circ\mathfrak{h}$. Hence, denoting by R^g the Riemann curvature tensor, we have that

$$\begin{aligned} \mathrm{R}^g(\xi, v)\xi &= \nabla_\xi\nabla_v\xi - \nabla_v\nabla_\xi\xi - \nabla_{[\xi,v]}\xi \\ &= -\frac{1}{2}\nabla_\xi(\phi(v)) + \frac{1}{2}\nabla_\xi(\phi(\mathfrak{h}(v))) + \frac{1}{2}\phi([\xi, v]) - \frac{1}{2}\phi(\mathfrak{h}([\xi, v]))\,, \end{aligned} \tag{7.29}$$

for $v \in \ker(\alpha)$ and where we used that $\nabla_\xi\xi = 0$. Applying ϕ to the previous equation, taking into account that $\phi^2 = \mathrm{Id} - \xi\otimes\alpha$ and that $\nabla_\xi\phi = 0$, we obtain:

$$\phi(\mathrm{R}^g(\xi, v)\xi) = \frac{1}{2}\nabla_\xi(-v + \mathfrak{h}(v)) + \frac{1}{2}[\xi, v] - \frac{1}{2}\mathfrak{h}([\xi, v]))\,, \tag{7.30}$$

where we have used that $\alpha\circ\phi = \alpha\circ\mathfrak{h} = 0$ and that $\alpha([\xi, v]) = 0$ (the latter equation follows from the para-contact condition $\mathrm{d}\alpha = -\star\alpha$). Applying now Proposition 7.3, we conclude:

$$\phi(\mathrm{R}^g(\xi, v)\xi) = \frac{1}{2}(\nabla_\xi\mathfrak{h})(v) + \frac{1}{4}\phi(v) - \frac{1}{4}\mathfrak{h}^2(\phi(v))\,. \tag{7.31}$$

On the other hand, $\phi^2(\mathrm{R}^g(\xi, v)\xi) = \mathrm{R}^g(\xi, v)\xi$ since $\alpha(\mathrm{R}^g(\xi, v)\xi) = 0$ vanishes identically. Consequently,

$$\mathrm{R}^g(\xi, v)\xi = \frac{1}{2}\phi(\nabla_\xi\mathfrak{h})(v) + \frac{1}{4}\phi^2(v) - \frac{1}{4}\mathfrak{h}^2(v)\,. \tag{7.32}$$

Equation (7.31) implies that

$$\phi(\mathrm{R}^g(\xi,\phi(v))\xi) = -\frac{1}{2}\phi(\nabla_\xi\mathfrak{h})(v) + \frac{1}{4}\phi^2(v) - \frac{1}{4}\mathfrak{h}^2(v)\,. \tag{7.33}$$

Using now the previous formulae for v of unit norm we obtain:

$$g(\mathrm{R}^g(\xi,v)\xi,v) + g(\phi(\mathrm{R}^g(\xi,\phi(v))\xi),v) = \frac{1}{2} - \frac{1}{4}\mathrm{Tr}(\mathfrak{h}^2) = -\mathrm{Ric}^g(\xi,\xi)\,, \tag{7.34}$$

where we used the fact that $g(\mathfrak{h}^2\phi(v),v) = -g(\mathfrak{h}^2(v),v)$, so $g(\mathfrak{h}^2(v),v) = \frac{1}{2}\mathrm{Tr}(\mathfrak{h}^2)$. Hence,

$$\mathrm{Ric}^g(\xi,\xi) = -\frac{1}{2} + \frac{1}{4}\mathrm{Tr}(\mathfrak{h}^2)\,, \tag{7.35}$$

and we conclude. □

Remark 7.11. The equation proved in Proposition 7.4 differs from the one found usually in the literature by a factor of $\frac{1}{4}$. This difference can be traced back to the different convention used in this work for the exterior derivative and $\mathfrak{h}$ with respect to the conventions used in References [161] and [164].

Proposition 7.5. *An ε-contact structure (g,α,ε) with $\varepsilon\,\sigma_g = 1$ is Sasakian if and only if*

$$\mathrm{Ric}^g(\xi,\xi) = \sigma_g\frac{\varepsilon}{2}\,. \tag{7.36}$$

Proof. The *only if* direction follows by setting $\mathfrak{h} = 0$ in Proposition 7.4. Conversely, if $\mathrm{Ric}^g(\xi,\xi) = \sigma_g\frac{\varepsilon}{2}$ then Proposition 7.4 implies that $\mathrm{Tr}(\mathfrak{h}^2) = 0$. Taking into account that ϕ is skew-adjoint with respect to g and that $\mathfrak{h}$ is self-adjoint with respect to g, we obtain $g(\mathfrak{h}^2(X),\phi(X)) = 0$ for every vector field X. Since $\mathfrak{h}(\xi) = 0$ and g is positive definite when restricted to $\ker(\alpha)$ (assuming $\varepsilon\sigma_g = 1$), then condition $\mathrm{Tr}(\mathfrak{h}^2) = 0$ implies that $\mathfrak{h}^2 = 0$. Since $\mathfrak{h}$ is self-adjoint, this yields $\mathfrak{h} = 0$. □

7.1.1 Globally hyperbolic ε-contact metric three-manifolds

In this section we describe ε-contact metric structures on globally hyperoblic Lorentzian manifolds. This class of ε-contact metric three-manifolds is specially relevant for our purposes, since they can be used to construct globally hyperbolic Lorentzian six-manifolds equipped with a Ricci flat metric connection with totally skew-symmetric and closed torsion, as described in Section 7.4 and Theorem 7.5. At the same time, globally hyperbolic solutions of six-dimensional Supergravity play a prominent role in the celebrated fuzzball proposal to describe the microscopic entropy of a black hole, see [42, 197] and references therein for more details.

Let (M,g) be a globally hyperbolic Lorentzian three-manifold. A celebrated theorem of Bernal and Sánchez [345] states that (M,g) admits the following presentation:

$$(M,g) = (\mathbb{R}\times \mathrm{X},\ -\beta_t^2\,\mathrm{d}t\otimes\mathrm{d}t + q_t)\,, \tag{7.37}$$

where t is a coordinate on $\mathbb{R}$, $\{\beta_t\}_{t\in\mathbb{R}}$ is a family of nowhere vanishing functions on $\mathrm{X} \stackrel{\text{def.}}{=} \{0\}\times\mathrm{X}$ and $\{q_t\}$ is a family of complete Riemannian metrics on X. In this presentation,

(M, g) is oriented and time-oriented, which immediately fixes an orientation on X. We denote by ν_{q_t} the Riemannian volume form associated to q_t. Let α be a one form on M. Set $e^0_t = \beta_t\, \mathrm{d}t$. We write

$$\alpha = F_t\, e^0 + \alpha^\perp_t\,, \tag{7.38}$$

where $\{F_t\}_{t\in\mathbb{R}}$ is a unique family of nowhere vanishing functions on X and $\{\alpha_t\}_{t\in\mathbb{R}}$ is a unique family of one-forms on X. With these provisos in mind, the dual of α can be computed to be:

$$\star\,\alpha = -F_t\,\nu_{q_t} - e^0\wedge \star_{q_t}\alpha^\perp_t\,, \tag{7.39}$$

where $\star_{q_t}$ denotes the Hodge dual of (X, q_t). On the other hand, the exterior derivative of α reads

$$\mathrm{d}\alpha = \mathrm{d}_{\mathrm{X}}F_t\wedge e^0_t + \frac{F_t}{\beta_t}\mathrm{d}_{\mathrm{X}}\beta_t\wedge e^0_t + \frac{1}{\beta_t}e^0_t\wedge\partial_t\alpha^\perp_t + \mathrm{d}_{\mathrm{X}}\alpha^\perp_t\,, \tag{7.40}$$

where d_{X} is the exterior derivative on X. Define $n_t \overset{\text{def.}}{=} \frac{1}{\beta_t}\partial_t$. Altogether, the previous discussion implies the following characterization of ε-contact structures on (M, g).

Proposition 7.6. *Let $(M, g) = (\mathbb{R}\times\mathrm{X}, -\beta^2_t\,\mathrm{d}t\otimes\mathrm{d}t + q_t)$ be a globally hyperbolic Lorentzian three-manifold. A one-form $\alpha\in\Omega^1(M)$ defines an ε-contact metric structure on (M, g) if and only if:*

$$\mathrm{d}_{\mathrm{X}}\alpha^\perp_t = F_t\,\nu_{q_t}\,,\qquad \star_{q_t}\alpha^\perp_t + \frac{1}{\beta_t}\mathrm{d}_{\mathrm{X}}(\beta_t F_t) = \mathcal{L}_{n_t}\alpha^\perp_t\,,\qquad |\alpha^\perp_t|^2_{q_t} = \varepsilon + F^2_t\,, \tag{7.41}$$

where $\{F_t\}_{t\in\mathbb{R}}$, $\{\beta_t\}$ and $\left\{\alpha^\perp_t\right\}_{t\in\mathbb{R}}$ are defined above and $\alpha = F_t\,e^0 + \alpha^\perp_t$.

Remark 7.12. The previous proposition yields a system of flow equations for a pair of families of functions $\{F_t\}_{t\in\mathbb{R}}$, $\{\beta_t\}_{t\in\mathbb{R}}$, a family of one-forms $\left\{\alpha^\perp_t\right\}_{t\in\mathbb{R}}$ and a family of complete Riemannian metrics $\{q_t\}_{t\in\mathbb{R}}$ on an oriented two-manifold X. To the best of our knowledge, this system has not been studied in the literature. We hope to study it in more detail elsewhere.

Definition 7.5. Given a three-manifold $M = \mathbb{R}\times X$, a globally hyperbolic ε-contact structure on M is a tuple

$$(\{\beta_t\}_{t\in\mathbb{R}}\,, \{F_t\}_{t\in\mathbb{R}}\,, \left\{\alpha^\perp_t\right\}_{t\in\mathbb{R}}\,, \{q_t\}_{t\in\mathbb{R}})\,, \tag{7.42}$$

defined as specified in Remark 7.12, which satisfies Equations (7.41).

Given a globally hyperbolic ε-contact structure $(\{\beta_t\}_{t\in\mathbb{R}}\,, \{F_t\}_{t\in\mathbb{R}}\,, \left\{\alpha^\perp_t\right\}_{t\in\mathbb{R}}\,, \{q_t\}_{t\in\mathbb{R}})$, it is clear from the previous discussion how to reconstruct (g, α, ε). We consider now a particular example in the case $\varepsilon = 1$.

Example 7.1. In the conditions of Proposition 7.6 set

$$F_t = 0\,,\qquad \beta_t = 1\,. \tag{7.43}$$

With these choices for $\{F_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$, Equations (7.41) read

$$\mathrm{d}_{\mathrm{X}}\alpha^\perp_t = 0\,,\qquad \star_{q_t}\alpha^\perp_t = \partial_t\alpha^\perp_t\,,\qquad |\alpha^\perp_t|^2_{q_t} = \varepsilon\,, \tag{7.44}$$

which immediately implies $\varepsilon = 1$, corresponding to the case in which the Reeb vector field is spacelike and the associated pair $(g, \alpha, \varepsilon = 1)$ defines a para-contact metric structure. Consider now

$$(\mathrm{X}, q_t) = (\mathbb{R}^2, e^{2U_t}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)) \,, \tag{7.45}$$

where $\{U_t\}_{t\in\mathbb{R}}$ is a family of constant functions and (x, y) are coordinates on $\mathbb{R}^2$. Let us write $\alpha^\perp = e^{U_t}(\alpha_1^\perp \mathrm{d}x + \alpha_2^\perp \mathrm{d}y)$. With these assumptions, Equations (7.44) are equivalent to

$$\partial_t(\alpha_1^\perp e^{U_t}) = -\alpha_2^\perp e^{U_t} \,, \qquad \partial_t(\alpha_2^\perp e^{U_t}) = \alpha_1^\perp e^{U_t} \,, \qquad (\alpha_1^\perp)^2 + (\alpha_2^\perp)^2 = 1 \,. \tag{7.46}$$

These equations are solved by:

$$\alpha_1^\perp e^{U_t} = l_2 \cos(t) - l_1 \sin(t) \,, \quad \alpha_2^\perp e^{U_t} = l_1 \cos(t) + l_2 \sin(t) \,, \quad e^{2U_t} = l_1^2 + l_2^2 \,, \tag{7.47}$$

for real constants l_1 and l_2 such that $l_1^2 + l_2^2 \neq 0$.

7.2 Null contact metric structures

The case of ε-contact metric structures with null Reeb vector field ($\varepsilon = 0$) seems to be new in the literature and thus deserves further attention. For simplicity in the exposition, we will refer to Lorentzian ε-contact metric structures with $\varepsilon = 0$ simply as null contact structures. The definition of null contact metric structure allows for the Reeb vector field to be identically zero. We will refer to this case as being trivial. Unless otherwise specified, we will always consider non-trivial null contact metric structures.

Remark 7.13. Let (g, α) be a null contact structure. We have:

$$\alpha \wedge \mathrm{d}\alpha = -\alpha \wedge \star\alpha = 0 \,, \tag{7.48}$$

where we have used that $|\alpha|_g^2 = 0$. Therefore, the previous computation implies that the one-form α of a null contact structure is not a *contact form* since it does not satisfy $\alpha \wedge \mathrm{d}\alpha \neq 0$. Nevertheless, given that the definition of ε-contact structure encompasses Riemannian contact ($\varepsilon = 1$), Lorentzian contact ($\varepsilon = -1$) and para-contact ($\varepsilon = 1$) metric structures, we interpret the remaining case $\varepsilon = 0$ as a natural generalization of the formers but in which the Reeb vector field is null. Hence, we will continue referring to the case $\varepsilon = 0$ as a null contact structure. Moreover, we will show later in this section that null contact structures admit meaningful notions of Sasakianity and K-contactness, analogously to the $\varepsilon \neq 0$ cases.

Having shed some light onto the nature of null contact structures, we proceed to investigate their most relevant properties. We recall that Lemma 7.1 applied to a null contact structure (g, α) implies:

$$\phi^2 = -\xi \otimes \alpha \,, \tag{7.49}$$

from which the following lemma follows.

Lemma 7.2. *The characteristic endomorphism ϕ of a null contact metric structure (g, α) satisfies*

$$\phi^3 = 0 \,. \tag{7.50}$$

Therefore, ϕ is nilpotent.

In the null contact case, the tensor field $\mathfrak{h} \overset{\text{def.}}{=} \mathcal{L}_\xi \phi$ satisfies additional properties not listed in Proposition 7.3 (compare also with Remark 7.8).

Lemma 7.3. *Let (g, α) be a null contact structure. Then $\phi \circ \mathfrak{h} = \mathfrak{h} \circ \phi = 0$.*

Proof. Let $v_1, v_2 \in TM$. Combining the first and fifth equations in Lemma 7.1, we have

$$\mathrm{d}\alpha(v_1, \phi(v_2)) = -\alpha(v_1)\alpha(v_2)\,. \tag{7.51}$$

Applying the Lie derivative along the Reeb vector field ξ to both sides of the previous equation, we obtain

$$\mathrm{d}\alpha(\mathcal{L}_\xi v_1, \phi(v_2)) + \mathrm{d}\alpha(v_1, \mathfrak{h}(v_2)) + \mathrm{d}\alpha(v_1, \phi(\mathcal{L}_\xi v_2)) = -\alpha(\mathcal{L}_\xi v_1)\alpha(v_2) - \alpha(v_1)\alpha(\mathcal{L}_\xi v_2)\,, \tag{7.52}$$

whence

$$\mathrm{d}\alpha(v_1, \mathfrak{h}(v_2)) = 0\,, \tag{7.53}$$

which by the first equation in Lemma 7.1 is equivalent to

$$g(v_1, \phi \circ \mathfrak{h}(v_2)) = 0\,, \tag{7.54}$$

for every $v_1, v_2 \in TM$. Hence, $\phi \circ \mathfrak{h} = 0$, which, combined with Proposition 7.2 implies $\mathfrak{h} \circ \phi = 0$ and we conclude. $\square$

Proposition 7.7. *The tensor field $\mathfrak{h}$ associated to a null contact structure (g, α) over M can always be written as*

$$\mathfrak{h} = \mu\, \xi \otimes \alpha\,, \tag{7.55}$$

for a function $\mu \in C^\infty(M)$.

Remark 7.14. The previous proposition implies that $\mathfrak{h}$ cannot have non-zero eigenvalues, in notable contrast with the situation occurring when $\varepsilon \neq 0$ [161, 162, 164], where certain eigenbundles of $\mathfrak{h}$ with non-zero eigenvalue play a crucial role in the classification of $\varepsilon\eta$-Einstein and so-called (κ, μ) contact metric three-manifolds [161, 679].

Proof. Choose a light-cone frame $\{\xi, u, \phi(u)\}$. By Lemma 7.3, $\phi(\xi) = 0$ hence $\xi \in \ker(\phi)$. By Remark 7.7 we have $g(\phi(u), \phi(u)) = 1$, whence $u \notin \ker(\phi)$. Furthermore, $\phi^2(u) = -\xi$, so $\phi(u) \notin \ker(\phi)$ and we conclude that $\ker(\phi) = \mathrm{Span}_{C^\infty}\{\xi\}$. By Lemma 7.3 we have $\phi \circ \mathfrak{h} = 0$, and thus $\mathrm{Im}(\mathfrak{h}) \subseteq \ker(\phi)$, whence

$$\mathfrak{h} = \gamma \otimes \xi \tag{7.56}$$

for a certain one-form γ. Since $\mathfrak{h}(\xi) = 0$ (see Proposition 7.3) we conclude that $\gamma = \mu\, \alpha + c\, \phi(u)^\flat$ for some functions μ and c. On the other hand, by Lemma 7.3 we have $\mathfrak{h} \circ \phi = 0$, which is equivalent to $\gamma \circ \phi = 0$. This implies that $c = 0$ after evaluating at u, that is, imposing $\gamma(\phi(u)) = 0$. $\square$

Remark 7.15. Expressed in a light-cone frame $\{\xi, u, \phi(u)\}$, the endomorphism $\mathfrak{h}$ has the following point-wise matrix form:

$$\mathfrak{h} = \begin{bmatrix} 0 & \mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

whence it has a unique type as an endomorphism of the tangent space. This shall be compared with the results of [677] where, in the para-contact case, the possible types of $\mathfrak{h}$ in an orthonormal special basis were classified.

The additional properties satisfied by the tensor field $\mathfrak{h}$ allows us to obtain an explicit expression for the Lie brackets of a light-cone frame $\{\xi, u, \phi(u)\}$.

Lemma 7.4. *Let (g, α) be a null contact structure over M. For a light-cone frame $\{\xi, u, \phi(u)\}$ we have:*

$$[\xi, u] = b\,\xi + c\,\phi(u)\,, \quad [\xi, \phi(u)] = (\mu - c)\,\xi\,, \quad [u, \phi(u)] = e\,\xi + u + f\,\phi(u), \tag{7.57}$$

where $\mu = g(u, \mathfrak{h}(u))$ and b, c, e and f are local functions.

Proof. Equation $\mathcal{L}_\xi \alpha = 0$ implies that $\alpha([\xi, u]) = 0$, whence $[\xi, u] = b\,\xi + c\,\phi(u)$. Therefore,

$$[\xi, \phi(u)] = \phi([\xi, u]) + \mathfrak{h}(u) = -c\,\xi + \mu\,\xi = (\mu - c)\,\xi\,. \tag{7.58}$$

Furthermore, by the ε-contact structure condition $\star\alpha = -\mathrm{d}\alpha$, we deduce that $\alpha([u, \phi(u)]) = 1$, so necessarily $[u, \phi(u)] = e\,\xi + u + f\,\phi(u)$. □

The following examples show that null contact metric structures are abundant.

Example 7.2. Take $M = \mathbb{R}^3$ and fix $g = \delta$ to be the standard Minkowski metric of signature $(-, +, +)$ in $\mathbb{R}^3$ with orthonormal coordinates (t, x, y), t being timelike. Then

$$\delta = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y\,, \qquad \alpha = e^y\,(\mathrm{d}t - \mathrm{d}x) \tag{7.59}$$

is a null contact metric structure. The characteristic endomorphism of this null contact structure can be found to be

$$\phi = e^y\,\partial_y \otimes \mathrm{d}t - e^y\,\partial_y \otimes \mathrm{d}x + e^y\,(\partial_t + \partial_x) \otimes \mathrm{d}y\,. \tag{7.60}$$

A direct computation shows that

$$\phi^2 = e^{2y}(\partial_t + \partial_x) \otimes \mathrm{d}t - e^{2y}(\partial_t + \partial_x) \otimes \mathrm{d}x = -\xi \otimes \alpha\,, \qquad \phi^3 = 0\,, \tag{7.61}$$

as expected. The Reeb vector field is given by $\xi = \alpha^\sharp = -e^y\,(\partial_t + \partial_x)$ and we have

$$\mathcal{L}_\xi \delta = e^y\,(\mathrm{d}t \odot \mathrm{d}y - \mathrm{d}x \odot \mathrm{d}y)\,. \tag{7.62}$$

Hence (δ, α) is not K-contact, and by Proposition 7.11 below, it cannot be Sasakian either. On the other hand, the endomorphism $\mathfrak{h}$ can be found to be

$$\mathfrak{h} \overset{\text{def.}}{=} \mathcal{L}_\xi \phi = -\xi \otimes \alpha\,, \tag{7.63}$$

in agreement with Proposition 7.7 with $\mu = -1$.

Example 7.3. Consider the $M = \widetilde{\mathrm{SL}}(2,\mathbb{R})$, where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ denotes the universal cover group of $\mathrm{SL}(2,\mathbb{R})$. There exists a Lorentzian metric g with left-invariant orthonormal global frame $\left\{e^0, e^1, e^2\right\}$ on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ satisfying [680]:

$$\mathrm{d}e^0 = e^1 \wedge e^2\,, \qquad \mathrm{d}e^1 = e^0 \wedge e^2\,, \qquad \mathrm{d}e^2 = -e^0 \wedge e^1 \tag{7.64}$$

and whose associated Lorentzian metric g is given by:

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2\,. \tag{7.65}$$

Hence $\star e^0 = -e^1 \wedge e^2$, $\star e^1 = -e^0 \wedge e^2$ and $\star e^2 = e^0 \wedge e^1$. We take α to be a null and left-invariant one-form on M and we expand it the basis $\left\{e^0, e^1, e^2\right\}$,

$$\alpha = \alpha_0 e^0 + \alpha_1 e^1 + \alpha_2 e^2\,, \tag{7.66}$$

for some real constant coefficients $\{\alpha_a\}$, $a = 0,1,2$, satisfying $\alpha_0^2 = \alpha_1^2 + \alpha_2^2$. We compute:

$$\mathrm{d}\alpha = \alpha_0 e^1 \wedge e^2 + \alpha_1 e^0 \wedge e^2 - \alpha_2 e^0 \wedge e^1 = -\star\alpha\,. \tag{7.67}$$

Therefore, every null left-invariant one-form on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ equipped with the Lorentzian metric g defines a null contact structure (g,α). We obtain now the characteristic endomorphism ϕ of (g,α) as well as its square. A direct computation yields the following matrix representation for ϕ in the basis $\left\{e^0, e^1, e^2\right\}$:

$$\phi = \begin{bmatrix} 0 & \alpha_2 & -\alpha_1 \\ \alpha_2 & 0 & \alpha_0 \\ -\alpha_1 & -\alpha_0 & 0 \end{bmatrix}$$

For the square and cubic powers of ϕ we obtain:

$$\phi^2 = \begin{bmatrix} \alpha_0^2 & \alpha_0\alpha_1 & \alpha_0\alpha_2 \\ -\alpha_0\alpha_1 & -\alpha_1^2 & -\alpha_1\alpha_2 \\ -\alpha_0\alpha_2 & -\alpha_1\alpha_2 & -\alpha_2^2 \end{bmatrix} = -\left[\xi \otimes \alpha\right]\,, \qquad \phi^3 = 0\,,$$

as expected from Lemma 7.2.

Remark 7.16. By the results of [681, 682], see in particular Theorem A in [682], Examples 7.2 and 7.3 show that all closed Lorentzian three-manifolds admitting a non-compact isometry group carry null contact structures.

We generalize now Example 7.3 by classifying all simply connected, Lorentzian, three-dimensional Lie groups admitting left-invariant null contact structures. For this, we will exploit the classification of this type of groups available in the literature [679, 680, 683], which we summarize for completeness in Appendix 7.A.

Proposition 7.8. *A three-dimensional connected and simply connected Lie group* G *admits a left-invariant null contact structure* (g,α) *if and only if* (G, g, α) *is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame* $\{e_0, e_1, e_2\}$ *appearing in Theorem 7.6:*

<table>
<tr><th>$\mathfrak{g}$</th><th>Structure constants ($s \in \mathbb{Z}_2$)</th><th>α</th><th>G</th></tr>
<tr><td>$\mathfrak{g}_1$</td><td>$a \neq 0\,,\, b = s$</td><td>$\alpha_1 = 0\,,\ \alpha_0 = -\alpha_2$</td><td>$\widetilde{\mathrm{SL}}(2,\mathbb{R})$</td></tr>
<tr><td rowspan="4">$\mathfrak{g}_3$</td><td>$a \neq 0\,,\, b = c = s$</td><td>$\alpha_1 = 0\,,\ \alpha_0^2 = \alpha_2^2$</td><td>$\widetilde{\mathrm{SL}}(2,\mathbb{R})$</td></tr>
<tr><td>$a = c = s\,, b \neq 0$</td><td rowspan="2">$\alpha_2 = 0\,,\ \alpha_0^2 = \alpha_1^2$</td><td>$\widetilde{\mathrm{SL}}(2,\mathbb{R})$</td></tr>
<tr><td>$a = s\,,\, b = 0\,,\, c = s$</td><td>$\widetilde{\mathrm{E}}(1,1)$</td></tr>
<tr><td>$a = b = c = s$</td><td>$\alpha_0^2 = \alpha_1^2 + \alpha_2^2\,, \alpha_1, \alpha_2 \neq 0$</td><td>$\widetilde{\mathrm{SL}}(2,\mathbb{R})$</td></tr>
<tr><td rowspan="2">$\mathfrak{g}_4$</td><td>$b = s + \mu\,,\, a \neq 0\,,\, \mu \in \mathbb{Z}_2$</td><td rowspan="2">$\alpha_1 = 0\,,\ \alpha_0 = \mu\alpha_2$</td><td>$\widetilde{\mathrm{SL}}(2,\mathbb{R})$</td></tr>
<tr><td>$b = s + \mu\,, a = 0\,, \mu \in \mathbb{Z}_2$</td><td>$\widetilde{\mathrm{E}}(1,1)$</td></tr>
<tr><td>$\mathfrak{g}_6$</td><td>$\mu a = (b + s)\,,\, \mu d = (c + s)$
$b = c\,,\, a = d \neq 0$</td><td>$\alpha_1 = 0\,,\ \alpha_0 = -\mu\alpha_2$</td><td>$\mathfrak{G}_6$</td></tr>
</table>

Proof. Every connected and simply connected three-dimensional Lorentzian Lie group is isometric to one of the items listed in Theorem 7.6 through an isometry which may be orientation-reversing. Therefore we proceed on a case by case basis by solving the equation

$$\star\,\alpha = -s\,\mathrm{d}\alpha\,, \qquad s \in \{-1, +1\} \tag{7.68}$$

in each of the cases listed in the table appearing in Theorem 7.6. The sign s is introduced because the ε-contact condition is not invariant under orientation reversing morphisms. If a solution is found with $s = -1$ then an orientation reversing isometry yields an isomorphic Lorentzian Lie algebra admitting a null-contact structure satisfying equation $\star\alpha = -\mathrm{d}\alpha$. Let $\{e_0, e_1, e_2\}$ be the frame appearing in Theorem 7.6 and let $\{e^0, e^1, e^2\}$ be its dual frame. We fix the volume form to be $\nu = e^0 \wedge e^1 \wedge e^2$. For the rest of the proof we will write

$$\alpha = \alpha_0\, e^0 + \alpha_1\, e^1 + \alpha_2\, e^2\,, \tag{7.69}$$

where α_0, α_1 and α_2 are real coefficients satisfying $\alpha_0^2 = \alpha_1^2 + \alpha_2^2$ (hence $\alpha_0 \neq 0$ in order for α to be non-zero). If $\{e^0, e^1, e^2\}$ denotes the dual coframe, we compute:

$$\star\,\alpha = -\alpha_0 e^1 \wedge e^2 - \alpha_1 e^0 \wedge e^2 + \alpha_2 e^0 \wedge e^1\,. \tag{7.70}$$

- **Case** $\mathfrak{g}_1$. In the orthonormal frame for $\mathfrak{g}_1$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned}&(-a\alpha_1 - b\alpha_2)\, e^0 \wedge e^1 + (a\alpha_0 + b\alpha_1 + a\alpha_2)\, e^0 \wedge e^2 + (b\alpha_0 - a\alpha_1)\, e^1 \wedge e^2\\ &= s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1)\,,\end{aligned} \tag{7.71}$$

  A solution (a, b) exists for a non-zero α if and only if $b = s$. Using that $a \neq 0$ by Theorem 7.6, if $b = s$ we obtain $\alpha_0 = -\alpha_2$, so $\alpha_1 = 0$.

- **Case** $\mathfrak{g}_2$. In the orthonormal frame for $\mathfrak{g}_2$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned}&a\alpha_1 e^0 \wedge e^2 + (c\alpha_0 - b\alpha_2)\, e^0 \wedge e^1 + (c\alpha_2 + b\alpha_0)\, e^1 \wedge e^2\\ &= s\alpha_0\, e^1 \wedge e^2 + s\alpha_1\, e^0 \wedge e^2 - s\alpha_2\, e^0 \wedge e^1\,.\end{aligned} \tag{7.72}$$

The previous equations imply the condition $c^2 + (b - s)^2 = 0$, whose unique solution is $c = 0$ and $b = s$. However the value $c = 0$ is forbidden for $\mathfrak{g}_2$ algebras, so no null contact structures exist on this type of algebras.

- **Case** $\mathfrak{g}_3$. In the orthonormal frame for $\mathfrak{g}_3$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$c\alpha_0\, e^1 \wedge e^2 + a\alpha_1\, e^0 \wedge e^2 - b\alpha_2\, e^0 \wedge e^1 = s\alpha_0 e^1 \wedge e^2 + s\alpha_1 e^0 \wedge e^2 - s\alpha_2 e^0 \wedge e^1 \,. \quad (7.73)$$

Since $\alpha_0 \neq 0$ if α is non-vanishing, we have $c = s$. If $\alpha_1 = 0$, then $\alpha_0^2 = \alpha_2^2$ and $b = s$. If $\alpha_2 = 0$, then $\alpha_0^2 = \alpha_1^2$ and $a = s$. If $\alpha_1, \alpha_2 \neq 0$, then $a = b = s$.

- **Case** $\mathfrak{g}_4$. In the orthonormal frame for $\mathfrak{g}_4$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned} &a\alpha_1\, e^0 \wedge e^2 + (-(2\mu - b)\alpha_0 + \alpha_2) e^1 \wedge e^2 + (\alpha_0 - b\alpha_2)\, e^0 \wedge e^1 \\ &= s\alpha_0\, e^1 \wedge e^2 + s\alpha_1\, e^0 \wedge e^2 - s\alpha_2\, e^0 \wedge e^1 \,. \end{aligned} \quad (7.74)$$

This equation admits non-trivial solutions only if $\alpha_2 \neq 0$, $b \neq s$ (which is required in order to have $\alpha_0 \neq 0$) and

$$b^2 - 2b(s + \mu) + 2(s\mu + 1) = 0 \,. \quad (7.75)$$

The previous equation has the unique solution $b = \mu + s$, which satisfies $b \neq s$. Setting $b = \mu + s$ we obtain $\alpha_0 = \mu\alpha_2$, whence $\alpha_1 = 0$.

- **Case** $\mathfrak{g}_5$. In the orthonormal frame for $\mathfrak{g}_5$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$(a\alpha_1 + b\alpha_2)\, e^0 \wedge e^1 + (c\alpha_1 + d\alpha_2)\, e^0 \wedge e^2 = s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1) \,. \quad (7.76)$$

This equation implies $\alpha_0 = 0$, which in turn yields $\alpha = 0$. Therefore, $\mathfrak{g}_5$ does not admit null contact structures.

- **Case** $\mathfrak{g}_6$. In the orthonormal frame for $\mathfrak{g}_6$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$(d\alpha_0 + c\alpha_2)\, e^0 \wedge e^1 + (-b\alpha_0 - a\alpha_2)\, e^1 \wedge e^2 = s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1) \,. \quad (7.77)$$

We obtain $\alpha_1 = 0$, so that $\alpha_0 = -\mu\alpha_2$ for $\mu \in \mathbb{Z}_2$. Consequently, $\mu a = s + b$ and $\mu d = s + c$. The remaining conditions follow from equations $ac = bd$ and $a + d \neq 0$, which must hold by Theorem 7.6.

- **Case** $\mathfrak{g}_7$. In the orthonormal frame for $\mathfrak{g}_7$ given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned} (b\alpha_0 + a\alpha_1 + b\alpha_2)\, e^0 \wedge e^1 + (d\alpha_0 + c\alpha_1 + d\alpha_2)\, e^0 \wedge e^2 + (b\alpha_0 + a\alpha_1 + b\alpha_2)\, e^1 \wedge e^2 \\ = s(\alpha_0 e^1 \wedge e^2 + \alpha_1 e^0 \wedge e^2 - \alpha_2 e^0 \wedge e^1) \,. \end{aligned} \quad (7.78)$$

with $a + d \neq 0$ and $ac = 0$. Imposing $ac = 0$ implies $\alpha_0 = 0$, whence a Lorentzian Lie group with Lie algebra isomorphic to $\mathfrak{g}_7$ does not admit non-trivial null contact structures.

□

7.2.1 Sasakian null contact structures

The Sasakian condition of a Riemannian or Lorentzian (para-)contact metric structure with non-null Reeb vector field can be defined in terms of the integrability of certain endomorphism defined from ϕ on $T(M \times \mathbb{R})$ [161, 169, 679]. In the null case $\varepsilon = 0$, we proceed in analogous way and we introduce the following endomorphism, which mimics the definition occurring in the case $\varepsilon \neq 0$:

$$J : T(M \times \mathbb{R}) \to T(M \times \mathbb{R}) \,, \qquad (v, c\,\partial_q) \mapsto (\phi(v) + c\,\xi, \alpha(v)\partial_q) \,, \tag{7.79}$$

where q is the fixed canonical coordinate on $\mathbb{R}$ and $c \in \mathbb{R}$. A direct computation shows that $J^2 = 0$. We clarify first the relevant notion of *integrability* of a field of endomorphisms.

Definition 7.6. Let $E \in \Gamma(TN \otimes T^*N)$ be a field of endomorphisms on a manifold N. Then, E is said to be integrable if around every point $n \in N$ there exists a coordinate system on which the local matrix representation of E has constant coefficients.

For almost complex structures the previous definition is equivalent to the standard definition of integrability in terms of existence of holomorphic charts. Necessary and sufficient conditions for a nilpotent endomorphism (such as J) to be integrable have been studied in the literature, see [684–688] and [689] for a thorough exposition of this and related topics. By a result of Thomson [687, Theorem 2], we have that $J \in \Gamma(\mathrm{End}(T(M \times \mathbb{R})))$ is integrable if and only if the following three conditions hold simultaneously:

- The Nijenhuis torsion tensor of J vanishes.
- J is a zero-deformable field of endomorphisms.
- The distribution $\mathrm{Ker}(J) \subset T(M \times \mathbb{R})$ is involutive.

Recall that the Nijenhuis torsion tensor associated to J is defined as

$$\mathcal{N}_J(v_1, v_2) = [J(v_1), J(v_2)] - J[v_1, J(v_2)] - J[J(v_1), v_2] + J^2[v_1, v_2] \,. \tag{7.80}$$

Note that since $J^2 = 0$, the last term in the right hand side identically vanishes. Also, we remind the reader that a certain field of endomorphisms is said to be zero-deformable if around every point of M there exists a frame relative to which the Jordan form of this endomorphism field is constant.

Lemma 7.5. *The endomorphism $J \in \Gamma(End(T(M \times \mathbb{R})))$ is zero-deformable.*

Proof. Fix a point in $M \times \mathbb{R}$ and consider the tangent-space basis $\{\xi, u, \phi(u), \partial_q\}$, where $\{\xi, u, \phi(u)\}$ is a light-cone basis. In this basis J has the following matrix representation:

$$J = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

Since the basis $(\xi, u, \phi(u), \partial_q)$ exists at every point in $M \times \mathbb{R}$, we conclude that J is always locally conjugate to the same constant Jordan form. □

Lemma 7.6. *The distribution* $\mathrm{Ker}(J) \subset T(M\times\mathbb{R})$ *is involutive.*

Proof. Fix a local frame $\{\xi, u, \phi(u), \partial_q\}$ on $M\times\mathbb{R}$, where $\{\xi, u, \phi(u)\}$ is a light-cone frame for M. The kernel of J is locally spanned by

$$\mathrm{Ker}(J) = \mathrm{Span}_{C^\infty}(\xi, \phi(u)+\partial_q)\,. \tag{7.81}$$

Lemma 7.4 implies now:

$$[\xi, \phi(u)+\partial_q] = [\xi, \phi(u)] = (\mu - c)\,\xi\,, \tag{7.82}$$

and we conclude. □

Proposition 7.9. *A null contact metric structure* (g,α) *is Sasakian if and only if its associated endomorphism* $J\colon T(M\times\mathbb{R}) \to T(M\times\mathbb{R})$ *is integrable.*

Proof. Assume first that (g,α) is Sasakian, that is, $\mathfrak{h}=0$. By Lemma 7.5 the endomorphism J is zero-deformable and by Lemma 7.6 its kernel $\mathrm{Ker}(J)$ is involutive. Therefore by Theorem [687, Theorem 2] we only need to show that $\mathcal{N}_J$ vanishes to prove that J is integrable. Since $\mathcal{N}_J$ is a tensor, it is enough to prove that it vanishes on a light cone frame $\{\xi, u, \phi(u), \partial_q\}$. We compute:

$$\mathcal{N}_J(\xi,u) = -J[\xi,\phi(u)] = 0\,,\quad \mathcal{N}_J(\xi,\phi(u)) = -J[\xi, J(\phi(u))] = 0\,, \tag{7.83}$$

$$\mathcal{N}_J(u,\phi(u)) = -J[J(\phi(u)), J(\phi(u))] = 0\,,\quad \mathcal{N}_J(\xi,\partial_q) = -J[\xi, J(\partial_q)] = 0\,, \tag{7.84}$$

$$\begin{aligned}\mathcal{N}_J(u,\partial_q) &= [\phi(u),\xi] - J[u,\xi] = -\mathcal{L}_\xi(\phi(u)) + J(\mathcal{L}_\xi u)\\ &= -\mathcal{L}_\xi(\phi(u)) + \phi(\mathcal{L}_\xi u) = -\mathfrak{h}(u) = 0\,,\end{aligned} \tag{7.85}$$

$$\begin{aligned}\mathcal{N}_J(\phi(u),\partial_q) &= [J(\phi(u)), J(\partial_q)] - J[\phi(u), J(\partial_q)]\\ &= [\phi^2(u),\xi] - J[\phi(u),\xi] = \phi^2(\mathcal{L}_\xi u) = 0\,,\end{aligned} \tag{7.86}$$

whence J is integrable. The converse follows now directly by applying the previous formulae, upon use of Lemmas 7.5 and 7.6 and the fact that $\mathfrak{h}(u)=0$ if and only if $\mathfrak{h}=0$. □

We proceed now to classify all left-invariant Sasakian null contact structures on simply connected, Lorentzian, three-dimensional Lie groups by exploiting Proposition 7.8.

Proposition 7.10. *A three-dimensional connected and simply connected Lie group* G *admits a left-invariant Sasakian null contact structure* (g,α) *if and only if* (G, g, α) *is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame* $\{e_0, e_1, e_2\}$ *appearing in Theorem 7.6:*

$\mathfrak{g}$	Structure constants ($s\in\mathbb{Z}_2$)	α	G
$\mathfrak{g}_1$	$a\neq 0\,,\ b=s$	$\alpha_1=0\,,\ \alpha_0=-\alpha_2$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$
$\mathfrak{g}_3$	$a=b=c=s$	$\alpha_0^2=\alpha_1^2+\alpha_2^2$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$
$\mathfrak{g}_4$	$b=s+\mu\,,\ a=s\,,\ \mu\in\mathbb{Z}_2$	$\alpha_1=0\,,\ \alpha_0=\mu\alpha_2$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$
$\mathfrak{g}_6$	$b=c=-\frac{s}{2}\,, a=d=\frac{\mu s}{2}\,,$	$\alpha_1=0\,,\ \alpha_0=-\mu\alpha_2$	$\mathfrak{G}_6$

Proof. We proceed by verifying which cases in Proposition 7.8 satisfy $\mathfrak{h} = 0$. By Proposition 7.7, it is enough to prove that $\mathfrak{h}(u) = 0$, where $u \in \{\xi, u, \phi(u)\}$ belongs to a light-cone frame. In the following computations it will be very convenient to use that the matrix expression of the characteristic endomorphism ϕ in the orthonormal basis $\{e_0, e_1, e_2\}$ used at Theorem 7.6 is given by

$$\phi = \begin{bmatrix} 0 & \alpha_2 & -\alpha_1 \\ \alpha_2 & 0 & \alpha_0 \\ -\alpha_1 & -\alpha_0 & 0 \end{bmatrix} . \tag{7.87}$$

Furthermore, the Reeb vector field ξ is given by $\xi = -\alpha_0 e_0 + \alpha_1 e_1 + \alpha_2 e_2$.

- **Case** $\mathfrak{g}_1$. Since $\alpha_1 = 0$ by Proposition 7.8, then the Reeb vector field reads $\xi = -\alpha_0(e_0 + e_2)$. A direct computation shows that $(\xi, u = \frac{1}{2\alpha_0}(e_0 - e_2), -e_1)$ is a light-cone frame (where $\phi(u) = -e_1$). Hence,

$$\mathfrak{h}(u) = \alpha_0[e_0, e_1] + \alpha_0[e_2, e_1] + \phi([e_2, e_0]) = 0\,, \tag{7.88}$$

  implying that every null contact structure on $\mathfrak{g}_1$ is Sasakian.

- **Case** $\mathfrak{g}_3$. According to Proposition 7.8 we distinguish between the cases $\alpha_1 = 0$ with $\alpha_2 \neq 0$, $\alpha_2 = 0$ with $\alpha_1 \neq 0$ and $\alpha_1, \alpha_2 \neq 0$. If $\alpha_1 = 0$, then $\xi = \alpha_0(-e_0 + \mu e_2)$ and $(\xi, u = \frac{1}{2\alpha_0}(e_0 + \mu e_2), \mu e_1)$ is a light-cone frame. We obtain, after using that $b = c = s$:

$$\mathfrak{h}(u) = \alpha_0(s - a)e_0 + \mu\alpha_0(a - s)e_2. \tag{7.89}$$

  Hence, $\mathfrak{h} = 0$ if and only if $a = s$. Similarly, if $\alpha_2 = 0$, then $\xi = \alpha_0(-e_0 + \mu e_1)$ and $\{\xi, u = \frac{1}{2\alpha_0}(e_0 + \mu e_1), \mu e_2\}$ is a light-cone frame (with $\phi(u) = \mu e_2$). Since $a = c = s$, we compute:

$$\mathfrak{h}(u) = \alpha_0(s - b)e_0 + \mu\alpha_0(b - s)e_1\,. \tag{7.90}$$

  Hence $\mathfrak{h} = 0$ if and only if $b = s$. The case $\alpha_1, \alpha_2 \neq 0$ follows along similar lines.

- **Case** $\mathfrak{g}_4$. In this case we have $\xi = \alpha_0(-e_0 + \mu e_2)$ and get a light-cone frame $(\xi, u = \frac{1}{2\alpha_0}(e_0 + \mu e_2), \mu e_1)$. We obtain:

$$\mathfrak{h}(u) = \alpha_0(-(2\mu - b) + \mu - a)e_0 + \alpha_0(1 - \mu b + \mu a)e_2\,, \tag{7.91}$$

  which vanishes if and only if

$$1 - \mu b + \mu a = 0\,. \tag{7.92}$$

  This is in turn equivalent to the constraint $a = s$.

- **Case** $\mathfrak{g}_6$. In this case we have $\xi = -\alpha_0(e_0 + \mu e_2)$ and the light-cone frame $\{\xi, u = \frac{1}{2\alpha_0}(e_0 - \mu e_2), -\mu e_1\}$, where $\phi(u) = -\mu e_1$. We find:

$$\mathfrak{h}(u) = -\alpha_0\mu(d + \mu b)e_0 - \alpha_0\mu(c + \mu a)e_2\,, \tag{7.93}$$

  whence $c = -\mu a$ and $d = -\mu b$. Taking into account now the constraints stated in Proposition 7.8 we conclude.

□

7.2.2 Null K-contact structures

For three-dimensional Riemannian contact structures, Lorentzian contact structures and para-contact structures, the K-contact and Sasakian conditions are equivalent. However, this fails to be the case for null contact structures, as the following example shows.

Example 7.4. Take M to be a connected and simply connected Lie group admitting a left-invariant global coframe $\{e^+, e^-, e^2\}$ satisfying:

$$\mathrm{d}e^+ = -a\, e^+ \wedge e^- - e^+ \wedge e^2\,, \quad \mathrm{d}e^- = e^- \wedge e^2\,, \quad \mathrm{d}e^2 = e^+ \wedge e^- - a\, e^- \wedge e^2\,, \tag{7.94}$$

where $a \in \mathbb{R}\setminus\{0\}$. We denote by $\{e_+, e_-, e_2\}$ the frame dual to $\{e^+, e^-, e^2\}$. We equip M with the Lorentzian metric

$$g = e^+ \odot e^- + e^2 \otimes e^2 \tag{7.95}$$

and we fix the volume form to be $\nu = e^- \wedge e^+ \wedge e^2$. Set $\alpha \overset{\text{def.}}{=} e^-$, whence $\xi = e_+$. Then, (g, α) defines a null contact structure on M since $g(\xi, \xi) = 0$ and

$$\star\, \alpha = -\alpha \wedge e^2 = -\mathrm{d}\alpha\,. \tag{7.96}$$

The characteristic endomorphism ϕ is given by:

$$\phi(\xi) = 0\,, \quad \phi(e_-) = (\iota_{e_-} \star \alpha)^\sharp = -e_2\,, \quad \phi(e_2) = (\iota_{e_2} \star \alpha)^\sharp = \xi\,. \tag{7.97}$$

Therefore, $\{\xi, e_-, -e_2\}$ yields a light-cone frame. Similarly, ϕ^2 can be shown to satisfy:

$$\phi^2(\xi) = 0\,, \quad \phi^2(e_-) = -\phi(e_2) = -\xi\,, \quad \phi^2(e_2) = 0\,. \tag{7.98}$$

Hence we obtain $\phi^2 = -\xi \otimes \alpha$ and $\phi^3 = 0$, as required. The fact that (g, α) is Sasakian follows now from the following computation:

$$\mathfrak{h}(e_-) = [\xi, \phi(e_-)] - \phi([\xi, e_-]) = -\xi + \xi = 0\,. \tag{7.99}$$

which, together with Proposition 7.7 implies that $\mathfrak{h}$ vanishes identically. On the other hand, ξ cannot be a Killing vector field, because

$$(\mathcal{L}_\xi g)(e_-, e_-) = -2\, g([\xi, e_-], e_-) = -2a\, g(\xi, e_-) = -2a \neq 0\,, \tag{7.100}$$

and $a \neq 0$ by assumption. Therefore, (g, α) is a Sasakian null contact structure which fails to be K-contact.

Hence, the Sasakian condition does not imply the K-contact condition.

Proposition 7.11. *Every three-dimensional null K-contact structure (g, α) is Sasakian.*

Proof. Let ξ denote the Reeb vector field associated to (g, α) and choose a light-cone frame $\{\xi, u, \phi(u)\}$. From the K-contact condition we have:

$$0 = -(\mathcal{L}_\xi g)(u, \phi(u)) = g(\mathcal{L}_\xi u, \phi(u)) + g(u, \phi(\mathcal{L}_\xi u)) + g(u, \mathfrak{h}(u)) = g(u, \mathfrak{h}(u))\,, \tag{7.101}$$

where we have used that ϕ is skew-adjoint with respect to g. On the other hand, Corollary 7.7 implies that $\mathfrak{h}(u) = \mu\, \xi$ for some function $\mu \in C^\infty(M)$. Since $g(u, \xi) = 1$, equation $g(u, \mathfrak{h}(u)) = 0$ is equivalent to $\mu = 0$ and thus $\mathfrak{h} = 0$. □

Proposition 7.12. *Let ξ denote the Reeb vector field associated to a Sasakian null-contact structure (g, α), and let $\{\xi, u, \phi(u)\}$ be a light-cone frame. Then, (g, α) is K-contact if and only if:*

$$g(\mathcal{L}_\xi u, u) = 0\,, \tag{7.102}$$

Proof. We evaluate $\mathcal{L}_\xi g$ on a light-cone frame $\{\xi, u, \phi(u)\}$. A direct computation using Lemma 7.4 shows that the only non-trivial term is

$$(\mathcal{L}_\xi g)(u, u) = -2g(\mathcal{L}_\xi u, u) \tag{7.103}$$

and we conclude. □

Remark 7.17. Indeed, not every Sasakian null contact structure satisfies that $g(\mathcal{L}_\xi u, u) = 0$. For instance, in Example 7.4, we have that:

$$g(\mathcal{L}_\xi u, u) = g([e_+, e_-], e_-) = a \neq 0\,, \tag{7.104}$$

as expected. Therefore we learn that the Sasakian condition for null contact structures is weaker than the K-contact condition. Interestingly enough, this is in sharp contrast to what occurs for non-null contact structures: for such structures these conditions are equivalent in three dimensions whereas in higher dimensions the Sasakian condition is stronger than the K-contact condition [161, 164].

Using the classification of simply connected, Lorentzian, three-dimensional Lie groups admitting left-invariant Sasakian null contact structures presented in Proposition 7.10, we obtain in the following an analogous classification for null K-contact structures.

Proposition 7.13. *A three-dimensional connected and simply connected Lie group G admits a left-invariant null K-contact contact structure (g, α) if and only if (G, g, α) is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame $\{e_0, e_1, e_2\}$ appearing in Theorem 7.6:*

$\mathfrak{g}$	Structure constants ($s \in \mathbb{Z}_2$)	α	G
$\mathfrak{g}_3$	$a = b = c = s$	$\alpha_0^2 = \alpha_1^2 + \alpha_2^2$	$\widetilde{\mathrm{SL}}(2, \mathbb{R})$
$\mathfrak{g}_4$	$b = s + \mu\,, a = s\,,\ \mu \in \mathbb{Z}_2$	$\alpha_1 = 0\,, \alpha_0 = \mu\alpha_2$	$\widetilde{\mathrm{SL}}(2, \mathbb{R})$
$\mathfrak{g}_6$	$b = c = -\frac{s}{2}\,,\ a = d = \frac{\mu s}{2}$	$\alpha_1 = 0\,,\ \alpha_0 = -\mu\alpha_2$	$\mathfrak{G}_6$

Proof. Proposition 7.11 states that every null K-contact structure is Sasakian. Therefore, we proceed by checking which cases in Proposition 7.10 are in fact K-contact. By Proposition 7.12, we will test K-contactness by verifying if $g([\xi, u], u) = 0$. We use the terminology introduced in Propositions 7.10 and 7.11:

- **Case** $\mathfrak{g}_1$. We obtain $[\xi, u] = [e_0, e_2] = -se_1 - ae_2 - ae_0 \neq 0$, where $a \neq 0$ by Theorem 7.6. Hence:

$$g([\xi, u], u) = \frac{a}{\alpha_0} \neq 0\,, \tag{7.105}$$

  whence $\mathfrak{g}_1$ does not admit left-invariant null K-contact structures. In fact, Example 7.4 corresponds to a Lie group of type $\mathfrak{g}_1$.

- **Case $\mathfrak{g}_3$.** We distinguish between the cases $\alpha_1 = 0$ or $\alpha_2 = 0$ and $\alpha_1, \alpha_2 \neq 0$. If $\alpha_1 = 0$, then $[\xi, u] = \mu[e_2, e_0] = \mu e_1 = \phi(u)$. Hence $g([\xi, u], u) = 0$. Similarly, if $\alpha_2 = 0$, then $[\xi, u] = \mu[e_1, e_0] = -\mu e_2 = \phi(u)$ and $g([\xi, u], u) = 0$. For $\alpha_1, \alpha_2 \neq 0$ a direct computation shows that $(\mathcal{L}_\xi g)(e_i, e_j) = 0$ for every $i, j \in \{1, 2, 3\}$, whence $\mathcal{L}_\xi g = 0$ and every left-invariant Sasakian null contact structure on $\mathfrak{g}_3$ is also K-contact.

- **Case $\mathfrak{g}_4$.** We have $[\xi, u] = \mu[e_2, e_0] = \mu a e_1 = a\phi(u)$, so $g([\xi, u], u) = 0$ and the Sasakian and K-contact conditions are equivalent on $\mathfrak{g}_4$.

- **Case $\mathfrak{g}_6$.** We have $[\xi, u] = -\mu[e_2, e_0] = 0$, so $g([\xi, u], u) = 0$. Hence the Sasakian and K-contact conditions are equivalent on $\mathfrak{g}_6$.

□

The previous proposition and Example 7.4 show that the theory of null K-contact structures and Sasakian null contact structures in three dimensions has the potential to be richer than its $\varepsilon \neq 0$ counterpart, where the Sasakian and K-contact conditions are equivalent. Further investigation of this issue is beyond the scope of this thesis.

7.3 $\varepsilon\eta$-Einstein ε-contact metric manifolds

We introduce in this section the notion of $\varepsilon\eta$-Einstein ε-contact metric structure on an oriented three-manifold M, which is a particular case of the standard notion of η-Einstein Riemannian/Lorentzian contact metric structure when the Reeb vector field has non-vanishing norm. The definition is motivated by the structure of six-dimensional Supergravity (coupled to a tensor multiplet) and its solutions, see Section 7.4 and Theorem 7.5 for details and applications.

Definition 7.7. An ε-contact metric structure (g, α, ε) on M is said to be $\varepsilon\eta$-Einstein if the Ricci curvature tensor Ric^g of g satisfies

$$\mathrm{Ric}^g = \frac{\mathrm{s}_g}{2}(\lambda^2 + \kappa\,\varepsilon)\, g - \mathrm{s}_g\, \kappa\, \alpha \otimes \alpha\,, \tag{7.106}$$

where $\mathrm{s}_g = 1$ if g is Riemannian, $\mathrm{s}_g = -1$ if g is Lorentzian and $\lambda, \kappa \in \mathbb{R}$ are real constants such that $\kappa \geq 0$ if $\mathrm{s}_g = -1$.

Remark 7.18. Whenever there is no possible confusion, we may abbreviate notation and denote an $\varepsilon\eta$-Einstein ε-contact metric structure (g, α, ε) just by $\varepsilon\eta$-Einstein contact structure.

Remark 7.19. Recall that we denote by $\mathrm{Q}^g \in \Gamma(TM \otimes T^*M)$ the endomorphism associated to Ric^g. Then, (g, α, ε) is $\varepsilon\eta$-Einstein if and only if

$$\mathrm{Q}^g = \frac{\mathrm{s}_g}{2}(\lambda^2 + \kappa\,\varepsilon)\,\mathrm{Id} - \mathrm{s}_g\, \kappa\, \xi \otimes \alpha\,, \tag{7.107}$$

in which case

$$\mathrm{Q}^g(\xi) = \frac{\mathrm{s}_g}{2}(\lambda^2 - \kappa\,\varepsilon)\,\xi\ . \tag{7.108}$$

Therefore ξ is an eigenvector of Q^g with eigenvalue $\rho_\xi = \frac{\mathrm{s}_g}{2}(\lambda^2 - \kappa\,\varepsilon)$.

Definition 7.8. We denote by $\mathrm{PCont}^{\varepsilon\eta}(\varepsilon, \lambda^2, \kappa)$ the category of $\varepsilon\eta$-Einstein ε-contact structures with respect to λ^2 and κ and whose Reeb vector field is of norm ε. Likewise, denote by $\mathrm{PCont}^{\varepsilon\eta}_L(\varepsilon, \lambda^2, \kappa)$ $(\mathrm{PCont}^{\varepsilon\eta}_R(\lambda^2, \kappa))$ the category of Lorentzian (Riemannian) $\varepsilon\eta$-Einstein ε-contact structures with respect to λ^2, κ and whose Reeb vector field is of norm ε.

Remark 7.20. Let $(g, \alpha, \varepsilon) \in \mathrm{PCont}^{\varepsilon\eta}(\varepsilon, \lambda^2, \kappa)$ with $\varepsilon\,\sigma_g = 1$. By Definition 7.7, we have

$$\mathrm{Ric}^g(\xi, \xi) = \frac{\sigma_g}{2}(\lambda^2\varepsilon - \kappa)\,. \tag{7.109}$$

On the other hand, by Proposition 7.5 (g, α, ε) is Sasakian if and only if $\mathrm{Ric}^g(\xi, \xi) = \sigma_g \frac{\varepsilon}{2}$. This is equivalent to

$$\lambda^2 = 1 + \kappa\varepsilon\,, \tag{7.110}$$

In other words, an $\varepsilon\eta$-Einstein ε-contact structure (g, α, ε) such that $\sigma_g\varepsilon = 1$ is Sasakian if and only if $(g, \alpha, \varepsilon) \in \mathrm{PCont}^{\varepsilon\eta}(\varepsilon, 1 + \varepsilon\kappa, \kappa)$.

Lemma 7.7. *Let $(M, g, \alpha, \varepsilon) \in \mathrm{PCont}^{\varepsilon\eta}(\varepsilon, \lambda^2, \kappa)$. Then, the following equation holds:*

$$\mathrm{R}^g(v_1, v_2)(\xi) = \mathcal{K}\,(\alpha(v_2)\,v_1 - \alpha(v_1)\,v_2)\,, \tag{7.111}$$

where

$$\mathcal{K} \stackrel{\text{def.}}{=} \frac{\sigma_g(\lambda^2 - \varepsilon\kappa)}{4}\,, \tag{7.112}$$

and R^g denotes the Riemann tensor on (M, g).

Proof. The results follows directly by plugging the $\varepsilon\eta$-Einstein condition in the expression for the Riemann tensor of a (pseudo-)Riemannian three-manifold in terms of its Ricci curvature. □

Remark 7.21. The previous proposition implies that a Riemannian $\varepsilon\eta$-Einstein contact manifold $(M, g, \alpha, \varepsilon) \in \mathrm{PCont}^{\varepsilon\eta}_R(\lambda^2, \kappa)$ is a (κ, μ) manifold or, alternatively, a Riemannian contact three-manifold whose Reeb vector field belongs to the nullity distribution [690].

In connection to Section 7.1.1, where the ε-contact condition for a one-form on a globally hyperbolic Lorentzian three-manifold was examined, now we consider the $\varepsilon\eta$-Einstein condition on globally hyperbolic Lorentzian three-manifolds. Using the notation of Section 7.1.1, we set

$$(M, g) = (\mathbb{R} \times \mathrm{X}_t,\, g = -\beta_t^2\,\mathrm{d}t \otimes \mathrm{d}t + q_t)\,, \qquad \alpha = F_t\,e^0 + \alpha_t^\perp\,, \tag{7.113}$$

where $\{\beta_t\}_{t\in\mathbb{R}}$, $\{F_t\}_{t\in\mathbb{R}}$, $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{q_t\}_{t\in\mathbb{R}}$ are parametric families on X_t, $e^0 = \beta_t\,\mathrm{d}t$ is the normalized timelike one-form induced by the globally hyperbolic presentation of (M, g) and where we have considered, for every $t \in \mathbb{R}$,

$$\mathrm{X}_t \stackrel{\text{def.}}{=} \{t\} \times \mathrm{X} \subset M \tag{7.114}$$

as an embedded manifold. We introduce the familiar Weingarten tensor W_t and second fundamental form Θ_t:

$$W_t = -\nabla n_t|_{T\mathrm{X}_t} \in \Omega^1(T\mathrm{X}_t)\,, \qquad \Theta_t = -\frac{1}{2\beta_t}\mathcal{L}_{\partial_t} g|_{T\mathrm{X}_t \times T\mathrm{X}_t} \in \Gamma(T^*\mathrm{X}_t \odot T^*\mathrm{X}_t)\,, \tag{7.115}$$

associated to the embedding $X_t \subset M$. The trace of Θ_t with respect to q_t, which we denote by $\mathrm{Tr}_{q_t}(\Theta_t)$, is the *mean curvature* of the embedded surface $X_t \subset M$.

Proposition 7.14. *A tuple* $(\{\beta_t\}_{t\in\mathbb{R}}, \{F_t\}_{t\in\mathbb{R}}, \{\alpha_t^\perp\}_{t\in\mathbb{R}}, \{q_t\}_{t\in\mathbb{R}})$ *on* X_t *is an* $\varepsilon\eta$*-Einstein contact structure as prescribed in* (7.113) *if and only if:*

$$\mathrm{d}_{\mathrm{X}_t}\alpha_t^\perp = F_t\,\nu_{q_t}\,, \quad \star_{q_t}\alpha_t^\perp + \frac{1}{\beta_t}\mathrm{d}_{\mathrm{X}}(\beta_t F_t) = \mathcal{L}_{n_t}\alpha_t^\perp\,, \quad |\alpha_t^\perp|^2_{q_t} = \varepsilon + F_t^2\,, \tag{7.116}$$

$$\mathrm{R}^{q_t} - |\Theta_t|^2_{q_t} + (\mathrm{Tr}_{q_t}\Theta)^2 + \mathfrak{c} = 2\kappa\,F_t^2\,, \quad \mathrm{d}_{\mathrm{X}_t}\mathrm{Tr}_{q_t}(\Theta_t) + \mathrm{div}_{q_t}(\Theta_t) = \kappa\,F_t\,\alpha_t^\perp\,, \tag{7.117}$$

$$\begin{aligned}\mathrm{Ric}^{q_t} + \mathrm{Tr}_{q_t}(\Theta_t)\,\Theta_t - 2\,\Theta_t(\mathrm{Id}\otimes W_t) - \frac{1}{\beta_t}(\dot{\Theta}_t + \nabla^{q_t}\mathrm{d}_{\mathrm{X}_t}\beta_t)\\ = \kappa\,\alpha_t^\perp\otimes\alpha_t^\perp - \frac{1}{2}(\lambda^2 + \kappa\varepsilon)q_t\,,\end{aligned} \tag{7.118}$$

where $\mathcal{L}$ *denotes Lie derivative,* $\mathrm{d}_{\mathrm{X}_t}$ *denotes the exterior derivative on* X_t*,* Tr_{q_t} *denotes trace with respect to* q_t*,* ∇^{q_t} *represents the Levi-Civita connection of* q_t *and* $\mathfrak{c} = \frac{1}{2}(5\lambda^2 + 3\kappa\varepsilon)$ *is a constant.*

Proof. The first line of equations in (7.116) is proven in Proposition 7.6. A direct computation shows that:

$$\mathrm{Ric}^g(n,n) = |\Theta_t|^2_{q_t} + \frac{1}{\beta_t}((\mathrm{Tr}_{q_t}\dot{\Theta})^2 + \Delta_{q_t}\beta_t)\,, \tag{7.119}$$

$$\mathrm{Ric}^g(n)|_{T\mathrm{X}_t} = \mathrm{d}_{\mathrm{X}_t}\mathrm{Tr}_{q_t}(\Theta_t) + \mathrm{div}_{q_t}(\Theta_t)\,, \tag{7.120}$$

$$\mathrm{Ric}^g|_{T\mathrm{X}_t\otimes T\mathrm{X}_t} = \mathrm{Ric}^{q_t} + \mathrm{Tr}(\Theta_t)\,\Theta_t - 2\,\Theta_t(\mathrm{Id}\otimes W_t) - \frac{1}{\beta_t}(\dot{\Theta}_t + \nabla^{q_t}\mathrm{d}_{\mathrm{X}_t}\beta_t)\,. \tag{7.121}$$

Plugging these equations in the $\varepsilon\eta$-Einstein condition (7.106) and combining them with the trace of (7.106) we obtain the second and third lines in (7.116). For more details the reader is referred to [627] and references therein. □

Therefore, the $\varepsilon\eta$-Einstein condition on a globally hyperbolic Lorentian three manifold becomes, as expected, a dynamical equation on the evolution of a pair of functions, a one-form and a metric on a two-dimensional oriented manifold. Note that the only equations in (7.116) that contain *time derivatives* correspond to the second equation in the first line and the equation in the third line. These become the evolution equations for the tuple $(\{\beta_t\}_{t\in\mathbb{R}}, \{F_t\}_{t\in\mathbb{R}}, \{\alpha_t^\perp\}_{t\in\mathbb{R}}, \{q_t\}_{t\in\mathbb{R}})$, while the rest of equations can be considered as *constraint equations* in order to formulate a Cauchy or initial value problem for Lorentzian $\varepsilon\eta$-Einstein ε-contact structures. Set

$$\mathrm{X} \stackrel{\text{def.}}{=} \mathrm{X}_0\,, \quad \Theta \stackrel{\text{def.}}{=} \Theta_0\,, \quad q \stackrel{\text{def.}}{=} q_0\,, \quad \alpha^\perp \stackrel{\text{def.}}{=} \alpha_0^\perp\,, \quad \beta \stackrel{\text{def.}}{=} \beta_0\,, \quad F \stackrel{\text{def.}}{=} F_0\,, \tag{7.122}$$

and consider X as the Cauchy surface for the zero-time intial values $(q, \Theta, F, \alpha^\perp)$ associated to a given tuple

$$(\{\beta_t\}_{t\in\mathbb{R}}, \{F_t\}_{t\in\mathbb{R}}, \left\{\alpha_t^\perp\right\}_{t\in\mathbb{R}}, \{q_t\}_{t\in\mathbb{R}}) \tag{7.123}$$

satisfying equations (7.116). The following result is a direct consequence Proposition 7.14.

Proposition 7.15. *If* $(\{\beta_t\}_{t\in\mathbb{R}}, \{F_t\}_{t\in\mathbb{R}}, \{\alpha_t^\perp\}_{t\in\mathbb{R}}, \{q_t\}_{t\in\mathbb{R}})$ *is a solution of* (7.116) *then* $(q, \Theta, F, \alpha^\perp)$ *satisfies:*

$$\mathrm{d}_{\mathrm{X}}\alpha^\perp = F\,\nu_q\,, \qquad |\alpha^\perp|^2_q = \varepsilon + F^2\,, \tag{7.124}$$

$$\mathrm{R}^q - |\Theta|_q^2 + (\mathrm{Tr}_q\Theta)^2 + \mathfrak{c} = 2\,\kappa\, F^2\,, \qquad \mathrm{d_X}\mathrm{Tr}_q(\Theta) + \mathrm{div}_q(\Theta) = \kappa\, F\,\alpha^\perp\,, \tag{7.125}$$

where Θ *is a symmetric bilinear form on* X.

Remark 7.22. Equations (7.124), to which we will refer as the *constraint equations* of an $\varepsilon\eta$-Einstein structure, generalize the well-known constraint equations of General Relativity coupled to a cosmological constant in $(2+1)$ dimensions via the coupling of an ε-contact structure. If we take $F=0$ then, the second equation in the first line of (7.124) forces $\varepsilon = 1$ and the whole system decouples. The second line in (7.124) corresponds in this case to the constraint equations of General Relativity coupled to a cosmological constant. This system has been studied in the literature, see [691, 692] and references therein. In fact, it would be interesting to explore if a Hamiltonian formulation on (the cotangent space of) Teichmüller space, in the lines of the one presented in [691, 692], can be developed also for $\varepsilon\eta$-Einstein structures. Note that, under the assumption $F=0$ the first line reduces to the condition of $\alpha^\perp$ being closed and of constant norm, a condition which if X is compact can only be satisfied on the torus.

Proving the converse of Proposition 7.15, that is, proving that for every solution of (7.124) there exists a tuple $(\{\beta_t\}_{t\in\mathbb{R}}\,, \{F_t\}_{t\in\mathbb{R}}\,, \{\alpha_t^\perp\}_{t\in\mathbb{R}}\,, \{q_t\}_{t\in\mathbb{R}})$ fulfilling (7.116), would solve the initial value problem of $\varepsilon\eta$-Einstein ε-contact structures. This problem will be considered elsewhere. Having said this, we expect the converse to hold due to the fact that $\varepsilon\eta$-Einstein ε-contact condition arises in a Supergravity theory, which is expected to pose consistent initial value problems due to their supersymmetric structure [693].

We distinguish the cases $\kappa = 0$ and $\kappa \neq 0$. For the sake of simplicity, we focus on the case $\kappa = 0$ in the following, with the goal of showing the existence of particular solutions. If $\kappa = 0$ then Equations (7.124) decouple again (see Remark 7.22) and the second line in (7.124) corresponds to the constraint equations of General Relativity coupled to a cosmological constant. Hence, we focus on the first line in (7.124), which we rewrite as follows:

$$\star\,\mathrm{d}_X\alpha^\perp + \sqrt{|\alpha^\perp|_q^2 - \varepsilon} = 0\,, \tag{7.126}$$

with variable given by a one-form $\alpha^\perp \in \Omega^1(\mathrm{X})$ on a complete Riemann surface (X, q). We focus on the null contact case $\varepsilon = 0$, since it seems to be new in the literature. In this case, the previous equation reduces simply to

$$\star\,\mathrm{d}_X\alpha^\perp + |\alpha^\perp|_q = 0\,. \tag{7.127}$$

We introduce now local isothermal coordinates (x, y) on X, in which the metric q reads $q = F^2\,(\mathrm{d}x^2 + \mathrm{d}y^2)$ for a local function F. In this coordinates, Equation (7.127) reads

$$\partial_x\alpha_y - \partial_y\alpha_x = F\,(\alpha_x^2 + \alpha_y^2)^{\frac{1}{2}}\,, \tag{7.128}$$

where we have written $\alpha^\perp = \alpha_x\mathrm{d}x + \alpha_y\mathrm{d}y$. Assuming $\alpha_x = 0$ the previous equation admits the solution

$$\alpha_y = e^{\int F\,\mathrm{d}x + f(y)}\,, \tag{7.129}$$

where $f(y)$ is a local function depending only on y. Assuming on the other hand $\alpha_y = 0$ the previous equation admits the solution

$$\alpha_x = e^{-\int F\,\mathrm{d}y + f(x)}\,, \tag{7.130}$$

where $f(x)$ is a local function depending only on x. Hence, we obtain the following result.

Proposition 7.16. *Every solution to the constraint equations of General Relativity coupled to a cosmological constant is, at least locally, a solution to the constraint equations of an $\varepsilon\eta$-Einstein null contact structure with $\kappa = 0$.*

We study now the $\varepsilon\eta$-Einstein condition on a case by case basis by distinguishing the signature of g and the causal character of the Reeb vector field.

7.3.1 Riemannian $\varepsilon\eta$-Einstein contact structures

We briefly review in this section the classification $\varepsilon\eta$-Einstein Riemannian contact structures, with the goal of constructing solutions of Supergravity coupled to a tensor multiplet, as explained in Sections 7.4 and 7.5. Riemannian contact structures on three-manifolds have been extensively studied in the literature, see [690, 694, 695] and references therein. Adapting and refining the results of [690, 694, 695] to our situation and conventions we obtain the following theorem.

Theorem 7.1. *[690, 694, 695] Let (M, g, α) be a three-dimensional complete and simply connected $\varepsilon\eta$-Einstein Riemannian contact metric manifold, $(M, g, \alpha) \in \mathrm{PCont}^{\varepsilon\eta}_R(\lambda^2, \kappa)$. Then one of the following holds:*

- *(M, g, α) is Sasakian. If (M, g, α) is in addition a Lie group equipped with a left-invariant $\varepsilon\eta$-Einstein Sasakian structure, then it is isomorphic[1] to a left-invariant $\varepsilon\eta$-Einstein structure on:*
  1. $\mathrm{SU}(2)$ *if* $\lambda^2 = 1 + \kappa\,,\ \kappa > -1$.
  2. H_3 *if* $\lambda^2 = 0,\ \kappa = -1$.
- *(M, g, α) is non-Sasakian and isomorphic to a left-invariant $\varepsilon\eta$-Einstein structure on:*
  1. $\mathrm{SU}(2)$ *if* $\lambda^2 = -\kappa = \frac{1}{2} - \frac{1}{2}\mu^2$, *with* $0 < \mu < 1$ *the positive eigenvalue of the tensor field* $\mathfrak{h}$.
  2. $\widetilde{\mathrm{E}}(2)$ *if* $\lambda^2 = \kappa = 0$.

Proof. If (M, g, α) is Sasakian and not isomorphic to a Lie group equipped with a left-invariant structure (g, α) then we are done. On the other hand, Reference [694] proves that a complete and simply connected $\varepsilon\eta$-Einstein non-Sasakian Riemannian contact three-manifold has a Lie group structure respect to which its $\varepsilon\eta$-Einstein structure is left-invariant. Therefore, we use the classification [660] of three-dimensional Riemannian Lie algebras to proceed on a case by case basis evaluating the $\varepsilon\eta$-Einstein condition. We distinguish between the Sasakian and the non-Sasakian cases.

- Sasakian case. First, if (M, g, α) is a Sasakian unimodular Lie group then there exists an orthonormal left-invariant frame $\{e_1, e_2, e_3\}$ whose associated Lie brackets satisfy [660]:

$$[e_2, e_3] = \mu_1 e_1\,, \quad [e_3, e_1] = \mu_2 e_2\,, \quad [e_1, e_2] = \mu_3 e_3\,, \tag{7.131}$$

[1] Two ε-contact structures are isomorphic in the sense given at Definition 7.2.

for some real constants $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$. This immediately implies that

$$\mathrm{d}e^1 = -\mu_1 e^2 \wedge e^3 \,, \qquad \mathrm{d}e^2 = -\mu_2 e^3 \wedge e^1 \,, \qquad \mathrm{d}e^3 = -\mu_3 e^1 \wedge e^2 \,, \tag{7.132}$$

where $\{e^1, e^2, e^3\}$ is the coframe dual to $\{e_1, e_2, e_3\}$. Expressing $\alpha = \alpha_0 e^0 + \alpha_1 e^1 + \alpha_2 e^2$, the contact condition (we include the sign $s \in \mathbb{Z}_2$ in order to take into account orientation-reversing isometries) reads

$$\star\, \alpha = s\mathrm{d}\alpha \,, \tag{7.133}$$

what is equivalent to:

$$\alpha_1 = -s\mu_1\alpha_1 \,, \quad \alpha_2 = -s\mu_2\alpha_2 \,, \quad \alpha_3 = -s\mu_3\alpha_3 \,. \tag{7.134}$$

Furthermore, the non-zero components of the Ricci curvature tensor Ric^g read

$$\mathrm{Ric}^g(e_1, e_1) = \frac{1}{2}(\mu_1^2 - \mu_2^2 - \mu_3^2 + 2\mu_2\mu_3) \,, \tag{7.135}$$

$$\mathrm{Ric}^g(e_2, e_2) = \frac{1}{2}(-\mu_1^2 + \mu_2^2 - \mu_3^2 + 2\mu_1\mu_3) \,, \tag{7.136}$$

$$\mathrm{Ric}^g(e_3, e_3) = \frac{1}{2}(-\mu_1^2 - \mu_2^2 + \mu_3^2 + 2\mu_1\mu_2) \,, \tag{7.137}$$

$$\mathrm{Ric}^g(e_1, e_2) = \mathrm{Ric}^g(e_1, e_3) = \mathrm{Ric}^g(e_2, e_3) = 0 \,. \tag{7.138}$$

We distinguish now the following subcases of the Sasakian unimodular case:

- Assume $\alpha_1, \alpha_2, \alpha_3 \neq 0$. Then $\mu_1 = \mu_2 = \mu_3 = -s$ and $\mathrm{Ric}^g = \frac{1}{2}g$, which follows from $\lambda^2 = 1$ and $\kappa = 0$. Therefore, choosing $s = -1$ we conclude that (M, g) is isometric to SU(2) equipped with a left-invariant metric.
- Assume $\alpha_1, \alpha_2 \neq 0$ and $\alpha_3 = 0$. Again, $\mu_1, \mu_2 = -s$. Since $\mathrm{Ric}^g(e_1, e_2) = 0$, we obtain $\kappa = 0$. Consequently, $\lambda^2 = 1$ which in turn implies, by equating $\mathrm{Ric}^g(e_1, e_1) = \mathrm{Ric}^g(e_2, e_2) = \mathrm{Ric}^g(e_3, e_3)$, that $\mu_3 = -s$. Taking $s = -1$, we recover the previous case and we conclude that (M, g) is isometric to SU(2). A similar analysis holds when $\alpha_1, \alpha_3 \neq 0$ and $\alpha_2 = 0$ and when $\alpha_2, \alpha_3 \neq 0$ and $\alpha_1 = 0$.
- Assume $\alpha_2 = \alpha_3 = 0$ and $\alpha_1^2 = 1$. Then, $\xi = \pm e_1$ and we obtain $\mathrm{Ric}^g(e_1, e_1) = \frac{1}{2}$ by the Sasaki condition. However, owing to the fact that $\alpha_1 \neq 0$ implies that $\mu_1 = -s$, then we have that

  $$\mathrm{Ric}^g(e_1, e_1) = \frac{1}{2} - \frac{1}{2}(\mu_2 - \mu_3)^2 = \frac{1}{2} \,. \tag{7.139}$$

  Therefore $\mu_2 = \mu_3 = \mu \in \mathbb{R}$. Now, since $\mathrm{Ric}^g(e_2, e_2) = \mathrm{Ric}^g(e_3, e_3) = \frac{1}{2}(2\lambda^2 - 1)$, we get the constraint

  $$\lambda^2 = -s\mu \,. \tag{7.140}$$

  Taking $s = -1$, we conclude that for $\mu > 0$ (M, g) is isometric to SU(2) and that for $\mu = 0$ (M, g) is isometric to H_3. A completely similar analysis holds when $\alpha_2^2 = 1$ and $\alpha_3^2 = 1$.

Secondly, if (M, g, α) is a Sasakian non-unimodular Lie group then there exists an orthonormal left-invariant frame $\{e_1, e_2, e_3\}$ whose associated Lie brackets satisfy [660]:

$$[e_1, e_2] = a e_2 + b e_3\,, \quad [e_1, e_3] = c e_2 + f e_3\,, \quad [e_2, e_3] = 0\,, \tag{7.141}$$

for real numbers $a, b, c, f \in \mathbb{R}$ satisfying that $a + f \neq 0$. These Lie brackets immediately imply:

$$\mathrm{d}e^1 = 0\,, \qquad \mathrm{d}e^2 = -a e^1 \wedge e^2 - c e^1 \wedge e^3\,, \qquad \mathrm{d}e^3 = -b e^1 \wedge e^2 - f e^1 \wedge e^3\,. \tag{7.142}$$

Therefore the contact condition imposes the following constraints:

$$c\alpha_2 + f\alpha_3 = s\alpha_2\,, \quad -a\alpha_2 - b\alpha_3 = s\alpha_3\,, \quad \alpha_1 = 0\,. \tag{7.143}$$

The components of the Ricci curvature tensor in this orthonormal basis read:

$$\mathrm{Ric}^g(e_1, e_1) = -a^2 - f^2 - \frac{(b+c)^2}{2}\,, \quad \mathrm{Ric}^g(e_2, e_2) = \frac{c^2}{2} - a^2 - \frac{b^2}{2} - fa\,, \tag{7.144}$$

$$\mathrm{Ric}^g(e_3, e_3) = -f^2 + \frac{b^2}{2} - \frac{c^2}{2} - af\,, \quad \mathrm{Ric}^g(e_2, e_3) = -ac - bf\,, \tag{7.145}$$

$$\mathrm{Ric}^g(e_1, e_2) = \mathrm{Ric}^g(e_1, e_3) = 0\,. \tag{7.146}$$

In particular, we have:

$$\mathrm{Ric}^g(e_2, e_2) + \mathrm{Ric}^g(e_3, e_3) = -a^2 - 2af - f^2 = -(a+f)^2\,. \tag{7.147}$$

Imposing the $\varepsilon\eta$-Einstein condition, and taking into account that $\alpha_2^2 + \alpha_3^2 = 1$ by the contact condition, we obtain $\mathrm{Ric}^g(e_2, e_2) + \mathrm{Ric}^g(e_3, e_3) = \lambda^2 + \kappa - \kappa\alpha_2^2 - \kappa\alpha_3^2 = \lambda^2$, which in turn yields $\lambda^2 = -(a+f)^2 < 0$. Hence, non-unimodular Riemannian groups admit no left-invariant $\varepsilon\eta$-Einstein structures.

- If (M, g, α) is non-Sasakian, then [694] shows that the $\varepsilon\eta$-Einstein condition implies $\lambda^2 = -\kappa$. Furthermore, [690, 694] prove that there exists a frame $\{\xi, X, \phi(X)\}$, where X is an eigenvector of $\mathfrak{h}$ with positive eigenvalue, whose Lie brackets satisfy:

$$[\xi, X] = \frac{1}{2}(1+\mu)\phi(X)\,, \quad [\xi, \phi(X)] = -\frac{1}{2}(1-\mu)X\,, \quad [X, \phi(X)] = \xi\,, \tag{7.148}$$

where μ is the positive eigenvalue of $\mathfrak{h}$. Comparing the previous Lie brackets to Milnor's classification [660], we obtain that M must be isometric to $\mathrm{SU}(2)$ when $\mu < 1$ and isometric to $\widetilde{\mathrm{E}}(2)$ when $\mu = 1$. Finally, using Proposition 7.4, we find the relation $\mu^2 = 1 - 2\lambda^2$.

□

7.3.2 Lorentzian $\varepsilon\eta$-Einstein structures with timelike Reeb vector field

Let $(M, g, \alpha, -1) \in \mathrm{PCont}_L^{\varepsilon\eta}(-1, \lambda^2, \kappa)$ be an ε-contact metric manifold M with timelike vector field. Note that, by Definition 7.7, (g, α, ε) is $\varepsilon\eta$-Einstein if and only if it satisfies

$$\mathrm{Ric}^g = \frac{1}{2}(-\lambda^2 + \kappa)\,\chi + \kappa\,\alpha \otimes \alpha\,, \tag{7.149}$$

where $\lambda \in \mathbb{R}$ and $\kappa \geq 0$.

Remark 7.23. The Lorentzian $\varepsilon\eta$-Einstein condition with timelike Reeb vector field has a natural physical interpretation in the context of General Relativity. Indeed, it is an *Einstein-like equation*, that is, it can be written as follows:

$$\mathrm{G}(g) = \mathrm{T}(g, \alpha)\,, \tag{7.150}$$

where $\mathrm{G}(g) = \mathrm{Ric}^g - \frac{1}{2}\mathrm{R}^g g$ denotes the Einstein tensor and $\mathrm{T}(g,\alpha) \in \Gamma(T^*M \odot T^*M)$ is (up to an innocent constant factor) the *stress-energy tensor* of Einstein's equations, given in this case by

$$\mathrm{T}(g,\alpha) = \frac{1}{4}(\lambda^2 + \kappa)\,\chi + \kappa\,\alpha \otimes \alpha\,. \tag{7.151}$$

Interestingly enough, $\mathrm{T}(g,\alpha)$ corresponds with the stress-energy tensor of a perfect fluid whose speed is prescribed by ξ and whose pressure p and rest-frame mass density o are constant and given by:

$$p = \frac{\lambda^2 + \kappa}{4}\,, \qquad o = \frac{3\,\kappa - \lambda^2}{4}\,. \tag{7.152}$$

Hence, the $\varepsilon\eta$-Einstein condition in Lorentzian signature with $\varepsilon = -1$ corresponds with the Einstein's General Relativity equations for a Lorentzian metric coupled to a perfect fluid with velocity prescribed by the Reeb vector field. This interpretation allows to apply the extensive literature dedicated to the study of perfect-fluid spacetimes, see for instance [696] and references therein, to the study of Lorentzian $\varepsilon\eta$-Einstein ε-contact structures.

We prove now that a Lorentzian $\varepsilon\eta$-Einstein contact structure $(g, \alpha, -1)$ on a connected and simply connected complete[2] three-dimensional Lorentzian manifold (M, g) is either Sasakian or isometric to a Lie group equipped with a Lorentzian left-invariant contact structure $(g, \alpha, -1)$. We proceed analogously to the Riemannian case [690]. We prove first the existence of a special type of ε-contact frame, particularly convenient for computations.

Lemma 7.8. *Let $(M, g, \alpha, -1) \in \mathrm{PCont}^{\varepsilon\eta}_L(-1, \lambda^2, \kappa)$ be a simply connected Lorentzian $\varepsilon\eta$-Einstein ε-contact metric three-manifold. Then, there exists a global orthonormal frame $\{\xi, X, \phi(X)\}$ such that $\mathfrak{h}(X) = \mu X$ and $\mathfrak{h}(\phi(X)) = -\mu\phi(X)$, where*

$$\mu = \sqrt{1 - (\lambda^2 + \kappa)}\,. \tag{7.153}$$

In particular, $\lambda^2 + \kappa \leq 1$.

Proof. First notice that since M is simply connected and three-dimensional it is parallelizable, whence it admits nowhere vanishing vector fields. If (g, α) is Sasakian then $\mathfrak{h} = 0$ and $\lambda^2 + \kappa = 1$, see Remark 7.20, and the statement is trivial. We consider thus the non-Sasakian case. For every nowhere-vanishing spacelike vector field $Y \in \mathfrak{X}(M)$ of unit norm, $\{\xi, Y, \phi(Y)\}$ is a global ε-contact frame. Furthermore,

$$g(\mathfrak{h}^2(Y), \xi) = 0\,, \qquad g(\mathfrak{h}^2(Y), \phi(Y)) = 0\,, \tag{7.154}$$

since $g(\mathfrak{h}^2(Y), \phi(Y)) = g(Y, \mathfrak{h}^2(\phi(Y))) = g(Y, \phi(\mathfrak{h}^2(Y))) = -g(\mathfrak{h}^2(Y), \phi(Y))$. Therefore,

$$\mathfrak{h}^2(Y) = \sigma^2\,Y = -\sigma^2\phi^2(Y)\,, \tag{7.155}$$

[2] We say that an ε-contact metric three-manifold $(M, \chi, \alpha, \varepsilon = -1)$ is *complete* if all elements of any ε-contact frame are complete on M.

for some $\sigma \in C^\infty(M)$. Therefore, $\mathrm{Tr}(\mathfrak{h}^2) = 2\sigma^2$. Since $(M, g, \alpha, -1) \in \mathrm{PCont}^{\varepsilon\eta}_L(-1, \lambda^2, \kappa)$, combining Proposition 7.4 and Remark 7.20 we obtain that $\sigma = \sqrt{1 - (\lambda^2 + \kappa)}$. Since $(M, g, \alpha, -1)$ is not Sasakian we have $\lambda^2 + \kappa < 1$ whence μ is strictly positive and corresponds with the positive eigenvalue of $\mathfrak{h}$. □

Working in the special global ε-contact frame introduced in Lemma 7.8 we prove that every non-Sasakian $\varepsilon\eta$-Einstein connected and simply connected Lorentzian three-dimensional manifold is a Lie group equipped with a left-invariant $\varepsilon\eta$-Einstein structure.

Proposition 7.17. *Let $(M, g, \alpha, -1) \in \mathrm{PCont}^{\varepsilon\eta}_L(-1, \lambda^2, \kappa)$ be simply connected with $\lambda^2 + \kappa < 1$ (that is, (M, g, α) is not Sasakian). Then, the following Lie brackets hold in the special ε-contact frame $\{\xi, X, \phi(X)\}$ described in Lemma 7.8 :*

$$[\xi, X] = \frac{1}{2}(1 + \sqrt{1 - 2\lambda^2})\phi(X)\,, \quad [\xi, \phi(X)] = \frac{1}{2}(-1 + \sqrt{1 - 2\lambda^2})X\,, \quad [X, \phi(X)] = -\xi\,. \tag{7.156}$$

Furthermore, $\lambda^2 = \kappa$ and $0 \leq \lambda^2 = \kappa < \frac{1}{2}$ since $\kappa \geq 0$.

Proof. Propositions 7.2 and 7.3 imply that

$$\nabla_\xi \xi = 0\,, \quad \nabla_X \xi = -\frac{1}{2}(1 + \mu)\phi(X)\,, \quad \nabla_{\phi(X)} \xi = \frac{1}{2}(1 - \mu)X\,. \tag{7.157}$$

On the other hand, we have that $g(\nabla_X X, \xi) = -g(X, \nabla_X \xi) = 0$ and

$$g(\nabla_{\phi(X)}\phi(X), \xi) = -g(\phi(X), \nabla_{\phi(X)}\xi) = 0\,. \tag{7.158}$$

Therefore, $\nabla_X X = c\phi(X)$ and $\nabla_{\phi(X)}\phi(X) = eX$ for some functions $c, e \in C^\infty(M)$. Similarly, $g(\nabla_\xi X, \xi) = g(\nabla_\xi(\phi(X)), \xi) = 0$, whence $\nabla_\xi X = \beta\phi(X)$ and $\nabla_\xi \phi(X) = -\beta X$ for a function $\beta \in C^\infty(M)$. Furthermore, we have:

$$g(\nabla_X \phi(X), \xi) = -g(\phi(X), \nabla_X \xi) = \frac{1}{2}(1 + \mu)\,, \tag{7.159}$$

$$g(\nabla_{\phi(X)} X, \xi) = -g(X, \nabla_{\phi(X)}\xi) = \frac{1}{2}(\mu - 1)\,, \tag{7.160}$$

$$g(\nabla_X \phi(X), X) = -g(\phi(X), \nabla_X X) = -c\,, \tag{7.161}$$

$$g(\nabla_{\phi(X)} X, \phi(X)) = -g(X, \nabla_{\phi(X)}\phi(X)) = -e\,. \tag{7.162}$$

Summarizing,

$$\nabla_\xi \xi = 0\,, \quad \nabla_X \xi = -\tfrac{1}{2}(1 + \mu)\phi(X)\,, \quad \nabla_{\phi(X)}\xi = \tfrac{1}{2}(1 - \mu)X\,, \tag{7.163}$$

$$\nabla_\xi X = \beta\phi(X)\,, \quad \nabla_X X = c\phi(X)\,, \quad \nabla_{\phi(X)} X = \tfrac{1}{2}(1 - \mu)\xi - e\phi(X)\,, \tag{7.164}$$

$$\nabla_\xi \phi(X) = -\beta X\,, \quad \nabla_X \phi(X) = -\tfrac{1}{2}(1 + \mu)\xi - cX\,, \nabla_{\phi(X)}\phi(X) = eX\,, \tag{7.165}$$

which implies:

$$[\xi, X] = \left(\beta + \frac{1}{2} + \frac{\mu}{2}\right)\phi(X)\,, \quad [\xi, \phi(X)] = \left(-\beta - \frac{1}{2} + \frac{\mu}{2}\right)X\,, \quad [X, \phi(X)] = -\xi - cX + e\phi(X)\,, \tag{7.166}$$

by using the torsion-free property of ∇. Making use of the previous equations, we compute:

$$\mathrm{R}^g(\xi,X)\xi=\left(\mu\beta-\frac{1}{4}+\frac{\mu^2}{4}\right)X\,,\quad \mathrm{R}^g(X,\phi(X))\xi=e\mu X-c\mu\phi(X)\,. \tag{7.167}$$

By comparing with Lemma 7.7, we conclude that $\mu\beta=\mu c=\mu e=0$. Since $\mu\neq 0$ by assumption, then $\beta=c=e=0$. Finally, by $\mathrm{Ric}^g(X,X)=\mathrm{Ric}^g(\phi(X),\phi(X))=0$ we get $\lambda^2=\kappa$. Hence, $\mu=\sqrt{1-2\lambda^2}$ and we conclude. □

Remark 7.24. From the proof of Proposition 7.17 we extract the following covariant derivatives of the ε-contact frame of special type described in Lemma 7.8:

$$\nabla_\xi\xi=0\,,\quad \nabla_X\xi=-\tfrac{1}{2}(1+\sqrt{1-2\lambda^2})\phi(X)\,,\quad \nabla_{\phi(X)}\xi=\tfrac{1}{2}(1-\sqrt{1-2\lambda^2})X\,, \tag{7.168}$$

$$\nabla_\xi X=0\,,\quad \nabla_X X=0\,,\quad \nabla_{\phi(X)}X=\tfrac{1}{2}(1-\sqrt{1-2\lambda^2})\xi\,, \tag{7.169}$$

$$\nabla_\xi\phi(X)=0\,,\quad \nabla_X\phi(X)=-\tfrac{1}{2}(1+\sqrt{1-2\lambda^2})\xi\,,\quad \nabla_{\phi(X)}\phi(X)=0\,. \tag{7.170}$$

Proposition 7.18. *Let $(M,g,\alpha,-1)\in\mathrm{PCont}^{\varepsilon\eta}_L(-1,\lambda^2,\kappa)$ be a complete and simply connected three-dimensional Lorentz $\varepsilon\eta$-Einstein contact manifold. Then, one of the following holds:*

- *$(M,g,\alpha,-1)$ is Sasakian.*
- *$(M,g,\alpha,-1)$ is non-Sasakian and isomorphic to one of the following Lie groups equipped with a left-invariant $\varepsilon\eta$-Einstein structure $(g,\alpha,-1)$:*
  1. *$\widetilde{\mathrm{SL}}(2,\mathbb{R})$ when $\frac{1}{2}>\lambda^2=\kappa>0$.*
  2. *$\widetilde{\mathrm{E}}(1,1)$ when $\lambda^2=\kappa=0$.*

Proof. Assume $(M,g,\alpha,-1)$ is non-Sasakian. By Proposition 7.17, there exists a global orthonormal frame $\{\xi,X,\phi(X)\}$ such that

$$[\xi,X]=\frac{1}{2}(1+\sqrt{1-2\lambda^2})\phi(X)\,,\quad [\xi,\phi(X)]=\frac{1}{2}(-1+\sqrt{1-2\lambda^2})X\,,\\ [X,\phi(X)]=-\xi\,, \tag{7.171}$$

such that $\lambda^2=\kappa$ and $0\leq\lambda^2=\kappa<\frac{1}{2}$. Using [697, Proposition 1.9], the fact that M admits a global frame (its three vector fields being complete, by assumption) with constant structure functions implies that (M,g,α) has a Lie group structure (canonical after fixing an identity point) respect to which $\{\xi,X,\phi(X)\}$ is left-invariant. Using the classification of connected and simply connected three-dimensional Lie groups summarized in Appendix 7.A we conclude that (M,g) is of type $\mathfrak{g}_3$ when $\frac{1}{2}>\lambda^2=\kappa>0$ by identifying

$$\xi=e_0\,,\qquad e_1=X\,,\qquad e_2=\phi(X)\,. \tag{7.172}$$

In particular, we have:

$$a=\frac{1}{2}(1-\sqrt{1-2\lambda^2})\,,\qquad b=\frac{1}{2}(1+\sqrt{1-2\lambda^2})\,,\qquad c=1\,. \tag{7.173}$$

where a, b and c are the real parameters appearing in case $\mathfrak{g}_3$ of Theorem 7.6. Hence, (M, g) is isometric to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ equipped with a left-invariant metric. On the other hand, when $\lambda^2 = \kappa = 0$, identifying:

$$e_0 = -\xi\,, \qquad e_1 = -\phi(X)\,, \qquad e_2 = -X\,, \tag{7.174}$$

we obtain that (M, g) is isometric to $\widetilde{\mathrm{E}}(1,1)$ endowed with a left-invariant metric. □

For Sasakian structures, we obtain the following result.

Lemma 7.9. *Let $(\mathrm{G}, g, \alpha, -1) \in \mathrm{PCont}^{\varepsilon\eta}_{L}(-1, \lambda^2, 1-\lambda^2)$ be a left-invariant Sasakian Lorentzian $\varepsilon\eta$-Einstein contact structure on a simply connected Lie group G. Then, according to the classification of connected and simply connected 3-dimensional Lie groups given in Theorem 7.6, one of the following holds:*

- *G is of type $\mathfrak{g}_3$. In particular, we have that $(\mathrm{G}, g, \alpha, -1)$ is isomorphic to a left-invariant $\varepsilon\eta$-Einstein structure on:*
  1. *$\widetilde{\mathrm{SL}}(2,\mathbb{R})$ if $1 \geq \lambda^2 > 0$.*
  2. *H_3 if $\lambda^2 = 0$.*
- *G is of type $\mathfrak{g}_6$ and $1 \geq \lambda^2 > 0$.*

Proof. Let $\{\xi, X, \phi(X)\}$ be a left-invariant ε-contact frame on $(\mathrm{G}, \xi, \alpha)$. The proof of Proposition 7.17 shows that the following holds:

$$\nabla_\xi \xi = 0\,, \quad \nabla_X \xi = -\tfrac{1}{2}\phi(X)\,, \quad \nabla_{\phi(X)}\xi = \tfrac{1}{2}X\,, \tag{7.175}$$

$$\nabla_\xi X = \beta\phi(X)\,, \quad \nabla_X X = c\phi(X)\,, \quad \nabla_{\phi(X)} X = \tfrac{1}{2}\xi - e\phi(X)\,, \tag{7.176}$$

$$\nabla_\xi \phi(X) = -\beta X\,, \quad \nabla_X \phi(X) = -\tfrac{1}{2}\xi - cX\,, \quad \nabla_{\phi(X)}\phi(X) = eX\,, \tag{7.177}$$

$$\begin{gathered}[\xi, X] = \left(\beta + \frac{1}{2}\right)\phi(X)\,, \quad [\xi, \phi(X)] = -\left(\beta + \frac{1}{2}\right)X \\ [X, \phi(X)] = -\xi - cX + e\phi(X)\,,\end{gathered} \tag{7.178}$$

where $\beta, c, e \in \mathbb{R}$. Imposing the Jacobi identity, we obtain the constraint

$$e\left(\beta + \frac{1}{2}\right)X + c\left(\beta + \frac{1}{2}\right)\phi(X) = 0\,. \tag{7.179}$$

Hence, either $c = e = 0$ or $\beta = -\frac{1}{2}$. We consider these cases separately.

- Assume $c = e = 0$. Imposing that $(\mathrm{G}, g, \alpha) \in \mathrm{PCont}^{\varepsilon\eta}_{L}(-1, \lambda^2, 1-\lambda^2)$, we obtain the condition $\mathrm{Ric}^g(X, X) = \mathrm{Ric}^g(\phi(X), \phi(X)) = \frac{1}{2} - \lambda^2$, which is equivalent to

$$\beta = \lambda^2 - \frac{1}{2}\,. \tag{7.180}$$

  Hence, the Lie brackets of $\{\xi, X, \phi(X)\}$ reduce to

$$[\xi, X] = \lambda^2\phi(X)\,, \quad [\xi, \phi(X)] = -\lambda^2 X\,, \quad [X, \phi(X)] = -\xi\,. \tag{7.181}$$

For $\lambda^2 > 0$, if $\{e_0, e_1, e_2\}$ denotes the orthonormal basis used at Theorem 7.6, identifying $e_0 = \xi$, $e_1 = X$ and $e_2 = \phi(X)$, we conclude that (G, g) is isometric to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ endowed with a left-invariant metric. Similarly, for $\lambda^2 = 0$, setting $e_0 = -\xi$, $e_1 = X$, $e_2 = \phi(X)$ we conclude that (G, g) is isometric to H_3 equipped with a left-invariant metric.

- If $\beta = -\frac{1}{2}$, the only non-trivial constraint from the $\varepsilon\eta$-Einstein condition is given by $\mathrm{Ric}^g(X, X) = \mathrm{Ric}^g(\phi(X), \phi(X)) = \frac{1}{2} - \lambda^2$, which is equivalent to

$$c^2 + e^2 = \lambda^2 \,. \tag{7.182}$$

Consequently, we obtain:

$$[\xi, X] = 0\,, \quad [\xi, \phi(X)] = 0\,, \quad [X, \phi(X)] = -\xi - c\phi(X) + eX\,, \tag{7.183}$$

with $c^2 + e^2 = \lambda^2$. We assume $\lambda^2 \neq 0$, since otherwise we return to the previous bullet point. Parametrizing $e = -|\lambda| \cos\theta$ and $c = |\lambda| \sin\theta$ for some angle $\theta \in \mathbb{R}$, we consider the following orthogonal change of basis:

$$[\xi, \bar{X}, \phi(\bar{X})]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} [\xi, X, \phi(X)]^T\,, \tag{7.184}$$

finding:

$$[\xi, \bar{X}] = 0\,, \quad [\xi, \phi(\bar{X})] = 0\,, \quad [\bar{X}, \phi(\bar{X})] = -\xi - |\lambda|\bar{X}\,. \tag{7.185}$$

Defining $e_0 = \xi$, $e_1 = \phi(\bar{X})$ and $e_2 = \bar{X}$ we conclude, via Theorem 7.6, that the previous Lie brackets correspond to those of a Lie algebra type $\mathfrak{g}_6$.

□

We may summarize the information provided in Proposition 7.18 and Lemma 7.9 in the following theorem.

Theorem 7.2. *A three-dimensional connected and simply connected Lie group G admits a left- invariant $\varepsilon\eta$-Einstein contact structure (g, α) with timelike Reeb vector field if and only if (G, g, α) is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame $\{e_0, e_1, e_2\}$ appearing in Theorem 7.6:*

$\mathfrak{g}$	Structure constants	α	η-Einstein constants	G	Sasakian
$\mathfrak{g}_3$	$\frac{1}{2} > a = 1-b > 0\,, c = 1$	$\alpha = e^0$	$\lambda^2 = \kappa = 2b(1-b)$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	No
	$a = c = 1\,, b = 0$	$\alpha = -e^0$	$\lambda^2 = \kappa = 0$	$\widetilde{\mathrm{E}}(1,1)$	No
	$1 \geq a = b > 0\,, c = 1$	$\alpha = e^0$	$\lambda^2 = 1 - \kappa = a$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes
	$a = b = 0\,, c = -1$	$\alpha = -e^0$	$\lambda^2 = 0\,, \kappa = 1$	H_3	Yes
$\mathfrak{g}_6$	$b = 1\,, c = d = 0\,,$ $1 \geq a^2 > 0$	$\alpha = e^0$	$\lambda^2 = 1 - \kappa = a^2$	$\mathfrak{G}_6$	Yes

Furthermore, if $(M, g, \alpha, -1) \in \mathrm{PCont}_L^{\varepsilon\eta}(-1, \lambda^2, \kappa)$ *is complete and not Sasakian then it is a Lie group equipped with a left-invariant Lorentzian contact structure and isomorphic to a left-invariant* $\varepsilon\eta$*-Einstein structure on either* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *when* $\frac{1}{2} > \lambda^2 = \kappa > 0$ *or on* $\widetilde{\mathrm{E}}(1,1)$ *when* $\lambda^2 = \kappa = 0$.

Proof. Follows from Proposition 7.18 and Lemma 7.9 upon use of Theorem 7.6. □

7.3.3 Lorentzian case with spacelike Reeb vector field

In this subsection we classify all left-invariant $\varepsilon\eta$-Einstein para-contact structures on three-dimensional simply connected Lie groups. Let $(M, g, \alpha, \varepsilon = 1) \in \mathrm{PCont}_L^{\varepsilon\eta}(\varepsilon = 1, \lambda, \kappa)$ be an oriented and time-oriented Lorentzian ε-contact metric manifold. By Definition 7.7, $(g, \alpha, \varepsilon = 1)$ is $\varepsilon\eta$-Einstein if and only if it satisfies:

$$\mathrm{Ric}^g = -\frac{1}{2}(\lambda^2 + \kappa)\, g + \kappa\, \alpha \otimes \alpha\,, \tag{7.186}$$

for real constants $\lambda \in \mathbb{R}$ and $\kappa \geq 0$.

Remark 7.25. Note that, in contrast to the case $\sigma_g\varepsilon = 1$, the endomorphism $\mathfrak{h}$ associated to a para-contact metric structure may not be diagonalizable, whence the techniques usted to classify $\varepsilon\eta$-Einstein ε-contact metric three manifolds with $\sigma_g\varepsilon = 1$ are a priori not applicable here.

Theorem 7.3. *A three-dimensional connected and simply connected Lie group* G *admits a left- invariant* $\varepsilon\eta$*-Einstein para-contact structure* (g, α) *if and only if* (G, g, α) *is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame* $\{e_0, e_1, e_2\}$ *appearing in Theorem* 7.6*:*

$\mathfrak{g}$	Structure constants ($s, \mu \in \mathbb{Z}_2$)	α	η-Einstein constants	G	Sasakian
$\mathfrak{g}_3$	$a = s\,,\ b = c\,,\ sc \geq 1$	$\alpha = \pm e^1$	$\lambda^2 = sc\,,\ \kappa = sc-1$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes
	$b = s\,,\ a = c\,,\ sc \geq 1$	$\alpha = \pm e^2$	$\lambda^2 = sc\,,\ \kappa = sc-1$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes
	$b = 0\,, a = c = s$	$-\alpha_0^2 + \alpha_1^2 = 1$	$\lambda^2 = 0\,, \kappa = 0$	$\widetilde{\mathrm{E}}(1,1)$	No
	$c = 0\,,\ a = b = s$	$\alpha_1^2 + \alpha_2^2 = 1$	$\lambda^2 = 0\,, \kappa = 0$	$\widetilde{\mathrm{E}}(2)$	No
	$a = b = c = s$	$-\alpha_0^2+\alpha_1^2+\alpha_2^2 = 1$	$\lambda^2 = 1\,, \kappa = 0$	$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes
$\mathfrak{g}_6$	$a = b = 0\,,$ $d^2 \geq 1\,, c = -s$	$\alpha_2^2 = 1$	$\lambda^2 = d^2\,, \kappa = d^2 - 1$	$\mathfrak{G}_6$	Yes
	$b = -\mu a \neq 0\,,$ $d = -\mu c = \mu s - a\,,$	$a\alpha_2 = (\mu a - s)\alpha_0$ $\alpha_2^2 = 1 + \alpha_0^2$	$\lambda^2 = 1\,,\ \kappa = 0$		Yes

Proof. We proceed on a case by case basis by checking which of the items appearing in Theorem 7.6 admits an $\varepsilon\eta$-Einstein para-contact metric structure (g, α). For this, we will exploit the formulae presented in Appendix 7.B.

- **Case $\mathfrak{g}_1$.** In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned}(-a\alpha_1 - b\alpha_2)\, e^0\wedge e^1 + (a\alpha_0 + b\alpha_1 + a\alpha_2)\, e^0\wedge e^2 + (b\alpha_0 - a\alpha_1)\, e^1\wedge e^2 \\ = s\,(\alpha_0\, e^1\wedge e^2 + \alpha_1\, e^0\wedge e^2 - \alpha_2\, e^0\wedge e^1)\,,\end{aligned} \tag{7.187}$$

Non-trivial solutions for α exist only if $b = s$, which implies $\alpha_1 = 0$, since $a \neq 0$. This implies in turn $\alpha_0 + \alpha_2 = 0$, which is incompatible with the constraint $-\alpha_0^2 + \alpha_2^2 = 1$. Hence $\mathfrak{g}_1$ does not admit para-contact metric structures.

- **Case $\mathfrak{g}_2$.** In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned}a\alpha_1 e^0\wedge e^2 + (c\alpha_0 - b\alpha_2)\, e^0\wedge e^1 + (c\alpha_2 + b\alpha_0)\, e^1\wedge e^2 \\ = s\,(\alpha_0\, e^1\wedge e^2 + \alpha_1\, e^0\wedge e^2 - \alpha_2\, e^0\wedge e^1)\,.\end{aligned} \tag{7.188}$$

This system of equations implies $c^2 + (b-s)^2 = 0$, which is equivalent to $c = 0$ and $b = s$. Since the value $c = 0$ is forbidden for $\mathfrak{g}_2$ Lie algebras, we conclude that there are no para-contact metric structures on this type of Lie algebras.

- **Case $\mathfrak{g}_3$.** In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$c\alpha_0\, e^1\wedge e^2 + a\alpha_1\, e^0\wedge e^2 - b\alpha_2\, e^0\wedge e^1 = s\,(\alpha_0\, e^1\wedge e^2 + \alpha_1 e^0\wedge e^2 - \alpha_2 e^0\wedge e^1)\,. \tag{7.189}$$

If $\alpha_0, \alpha_1, \alpha_2 \neq 0$, then we have $a = b = c = s$ and $-\alpha_0^2 + \alpha_1^2 + \alpha_2^2 = 1$. If $\alpha_0 = 0$ and $\alpha_1, \alpha_2 \neq 0$, then $\alpha_1^2 + \alpha_2^2 = 1$, $a = b = s$ and c is unconstrained. If $\alpha_1 = 0, \alpha_0, \alpha_2 \neq 0$ (resp. $\alpha_2 = 0, \alpha_0, \alpha_1 \neq 0$) we have $-\alpha_0^2 + \alpha_2^2 = 1$, $b = c = s$ and a unconstrained (resp. $-\alpha_0^2 + \alpha_1^2 = 1$, $a = c = s$ and b unconstrained). If $\alpha_0 = \alpha_1 = 0$ (resp. $\alpha_0 = \alpha_2 = 0$), then $\alpha_2^2 = 1$, $b = s$ and a, c are unconstrained (resp. $\alpha_1^2 = 1$, $a = s$ and b, c unconstrained). Finally, we remark that $a = b = 0$ is never allowed, since it implies $\alpha_1 = \alpha_2 = 0$, whence $\alpha_0^2 = -1$.

We compute now the Ricci curvature. We obtain:

$$\mathrm{Ric}^g(e_0, e_0) = \frac{c^2}{2} + ba - \frac{b^2}{2} - \frac{a^2}{2}\,, \quad \mathrm{Ric}^g(e_0, e_1) = 0\,, \tag{7.190}$$

$$\mathrm{Ric}^g(e_1, e_1) = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{2} - cb\,, \quad \mathrm{Ric}^g(e_0, e_2) = 0\,, \tag{7.191}$$

$$\mathrm{Ric}^g(e_2, e_2) = \frac{a^2}{2} - ac + \frac{c^2}{2} - \frac{b^2}{2}\,, \quad \mathrm{Ric}^g(e_1, e_2) = 0\,. \tag{7.192}$$

We proceed now on a case by case basis:

1. If $\alpha_0, \alpha_1, \alpha_2 \neq 0$, and hence $a = b = c = s$, the $\varepsilon\eta$-Einstein condition reduces to

$$\lambda^2 = 1\,, \qquad \kappa = 0\,. \tag{7.193}$$

which follows by direct computation from (7.190).

2. If $\alpha_0 = 0$ and $\alpha_1, \alpha_2 \neq 0$, we have $a = b = s$ and c unconstrained. Then $\mathrm{Ric}^g(e_1, e_2) = 0$ implies $\kappa = 0$, which solves all off-diagonal $\varepsilon\eta$-Einstein equations. The diagonal components of the $\varepsilon\eta$-Einstein equations are equivalent to
$$\lambda^2 = c^2 \qquad c^2 = sc\,. \tag{7.194}$$
Hence, either $c \neq 0$, which implies $c = s$ and we are back to point (1), or $c = 0$, in which case $\lambda = 0$. The cases $\alpha_1 = 0$ ($\alpha_2 = 0$) and $\alpha_0, \alpha_2 \neq 0$ ($\alpha_0, \alpha_1 \neq 0$) follow analogously.

3. We consider now $\alpha_0 = \alpha_1 = 0$ and $\alpha_2^2 = 1$, so that $b = s$ and a, c unconstrained. In this case, using the formulae of Appendix 7.B the $\varepsilon\eta$-Einstein condition can be found to imply:
$$(c - a)(s - (c + a)) = 0\,, \tag{7.195}$$
whence either $c = a$ or $a + c = s$. If $a = c$ then the $\varepsilon\eta$-Einstein equations are equivalent to $a = s(1+\kappa) = c$ and $\lambda^2 = 1+\kappa$. Therefore, $\lambda^2 = sc$. On the other hand, if $a + c = s$ then the $\varepsilon\eta$-Einstein condition implies $\kappa = -\lambda^2$. However, $\kappa \geq 0$ by the definition of $\varepsilon\eta$-Einstein structure, whence $\lambda = \kappa = 0$ and the $\varepsilon\eta$-Einstein condition reduces to $a(a - s) = 0$. Since $a = 0$ is not allowed if $b = s$ by the type of algebra $\mathfrak{g}_3$, we conclude that $a = s$. The case $\alpha_0 = \alpha_2 = 0$ and $\alpha_1^2 = 1$ follows now along similar lines.

- **Case $\mathfrak{g}_4$.** In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned} &a\alpha_1\, e^0 \wedge e^2 + (-(2\mu - b)\alpha_0 + \alpha_2)\, e^1 \wedge e^2 + (\alpha_0 - b\alpha_2)\, e^0 \wedge e^1 \\ &\quad = s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1)\,. \end{aligned} \tag{7.196}$$

We distinguish the cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$. If $\alpha_2 = 0$ then we must have $\alpha_0 = 0$ and $\alpha_1^2 = 1$ and $a = s$ is the unique solution. If $\alpha_2 \neq 0$ and $b = s$ then $\alpha_0 = 0$ and $\alpha_2 = 0$, a contradiction. Assume then that $\alpha_2 \neq 0$ and $b \neq s$. It follows that $\alpha_0 \neq 0$ and
$$b^2 - 2b(s + \mu) + 2(s\mu + 1) = 0\,. \tag{7.197}$$

The previous equation has the unique solution $b = \mu + s$, which indeed satisfies $b \neq s$. Then, the solutions in this case are given by:

$$\alpha_1^2 = 1\,, \qquad a = s\,, \qquad \alpha_0 = \mu\alpha_2\,. \tag{7.198}$$

To classify which para-contact structures are also $\varepsilon\eta$-Einstein, we proceed as in the previous cases by direct computation. We obtain (imposing $a = s$):

$$\mathrm{Ric}^g(e_0, e_0) = \tfrac{(2\mu - b)^2}{2} - \tfrac{(b - s)^2}{2}\,, \qquad \mathrm{Ric}^g(e_0, e_1) = 0\,, \tag{7.199}$$
$$\mathrm{Ric}^g(e_1, e_1) = -\tfrac{1}{2}\,, \qquad \mathrm{Ric}^g(e_0, e_2) = s + 2(\mu - b)\,, \tag{7.200}$$
$$\mathrm{Ric}^g(e_2, e_2) = \tfrac{(2\mu - b)^2}{2} + \tfrac{1}{2} + s(2\mu - b) - \tfrac{b^2}{2}\,, \qquad \mathrm{Ric}^g(e_1, e_2) = 0\,. \tag{7.201}$$

We distinguish again between the cases $\alpha_2 = 0$ and $\alpha_2 \neq 0$ with $b \neq s$.

1. If $\alpha_2 = 0$ then $\alpha_0 = 0$ and $a = s$. The only non-trivial off-diagonal component of the $\varepsilon\eta$-Einstein condition is $b = \mu + \frac{s}{2}$. Imposing this condition in the timelike diagonal component we obtain

$$\mathrm{Ric}^g(e_0, e_0) = 0 = \frac{1}{2}(\lambda^2 + \kappa)\,, \tag{7.202}$$

whence $\kappa = -\lambda^2$. Since $\kappa \geq 0$ we conclude that $\lambda = \kappa = 0$. However, this is incompatible with $\mathrm{Ric}^g(e_1, e_1) = -\frac{1}{2}$.

2. If $\alpha_2 \neq 0$ and $b = \mu + s$ then $\mathrm{Ric}^g(e_1, e_2) = 0$ implies $\kappa = 0$ since $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. However, then we would need $\mathrm{Ric}^g(e_0, e_2) = -s$ to vanish, which is not possible.

- **Case** $\mathfrak{g}_5$. In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$(a\alpha_1 + b\alpha_2)\, e^0 \wedge e^1 + (c\alpha_1 + d\alpha_2)\, e^0 \wedge e^2 = s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1)\,, \tag{7.203}$$

which immediately implies $\alpha_0 = 0$. The conditions for a para-contact structure to exist are $\alpha_1^2 + \alpha_2^2 = 1$ and $ad = (b+s)(c-s)$, together with the conditions on the coefficients required by the algebra type $\mathfrak{g}_5$, which are $ac + bd = 0$ and $a + d \neq 0$.

We verify now which para-contact structures on $\mathfrak{g}_5$ are $\varepsilon\eta$-Einstein. Using Appendix 7.B we obtain the following components for the Ricci tensor:

$$\mathrm{Ric}^g(e_0, e_0) = -a^2 - d^2 - cb - \tfrac{b^2}{2} - \tfrac{c^2}{2}\,, \quad \mathrm{Ric}^g(e_0, e_1) = 0\,, \tag{7.204}$$

$$\mathrm{Ric}^g(e_1, e_1) = a^2 + ad - \tfrac{c^2}{2} + \tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(e_0, e_2) = 0\,, \tag{7.205}$$

$$\mathrm{Ric}^g(e_2, e_2) = d^2 + da + \tfrac{c^2}{2} - \tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(e_1, e_2) = ac + bd\,. \tag{7.206}$$

We have, $\mathrm{Ric}^g(e_1, e_2) = 0$ automatically by the definition of algebra of type $\mathfrak{g}_5$. Hence, the $\varepsilon\eta$-Einstein condition evaluated in e_1 and e_2 implies:

$$\alpha_1 \alpha_2 \kappa = 0\,. \tag{7.207}$$

On the other hand, since $\alpha_1^2 + \alpha_2^2 = 1$, the $\varepsilon\eta$-Einstein equation implies:

$$\mathrm{Ric}^g(e_1, e_1) + \mathrm{Ric}^g(e_2, e_2) = (a+d)^2 = -\lambda^2\,. \tag{7.208}$$

Since $a + d \neq 0$ by the coefficient conditions of the algebra of type $\mathfrak{g}_5$, the previous equation admits no solutions and therefore an algebra of type $\mathfrak{g}_5$ does not admit $\varepsilon\eta$-Einstein para-contact structures.

- **Case** $\mathfrak{g}_6$. In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$(d\alpha_0 + c\alpha_2)\, e^0 \wedge e^1 + (-b\alpha_0 - a\alpha_2)\, e^1 \wedge e^2 = s\,(\alpha_0\, e^1 \wedge e^2 + \alpha_1\, e^0 \wedge e^2 - \alpha_2\, e^0 \wedge e^1)\,, \tag{7.209}$$

which immediately implies $\alpha_1 = 0$. From the previous (linear) equations we obtain that the conditions to have a non-trivial para-contact structures are $-\alpha_0^2 + \alpha_2^2 = 1$

and $ad = (b+s)(c+s)$, together with the conditions $ac - bd = 0$ and $a + d \neq 0$ required by the algebra type $\mathfrak{g}_6$. The components of the Ricci tensor read:

$$\mathrm{Ric}^g(e_0, e_0) = d^2 + ad + \tfrac{b^2}{2} - \tfrac{c^2}{2}\,, \quad \mathrm{Ric}^g(e_0, e_1) = 0\,, \tag{7.210}$$

$$\mathrm{Ric}^g(e_1, e_1) = -a^2 - d^2 - bc + \tfrac{c^2}{2} + \tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(e_0, e_2) = -ac + bd = 0\,, \tag{7.211}$$

$$\mathrm{Ric}^g(e_2, e_2) = -a^2 - da - \tfrac{c^2}{2} + \tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(e_1, e_2) = 0\,. \tag{7.212}$$

We have $\mathrm{Ric}^g(e_0, e_2) = 0$ identically by the conditions on the coefficients of an algebra of type $\mathfrak{g}_6$. Evaluating the $\varepsilon\eta$-Einstein condition on e_0 and e_2 we obtain

$$\kappa\, \alpha_0 \alpha_2 = 0\,. \tag{7.213}$$

Hence, either $\alpha_0 = 0$ or $\kappa = 0$ since $\alpha_2 = 0$ is not allowed by the para-contact condition. If $\alpha_0 = 0$ then $\alpha_2^2 = 1$, which implies $a = 0$ and $c = -s$. Taking into account that $ac - bd = 0$ and $a + d \neq 0$, we further obtain that $b = 0$, which in turn implies that

$$\lambda^2 = d^2 = \kappa + 1\,, \tag{7.214}$$

whence $d^2 \geq 1$ since we must have $\kappa \geq 0$. Altogether these conditions solve the $\varepsilon\eta$-Einstein equations of a para-contact structure on $\mathfrak{g}_6$. On other hand, if $\kappa = 0$, a combination of the $\varepsilon\eta$-Einstein equations implies:

$$ad = (b+s)(c+s)\,, \quad ac - bd = 0\,, \quad a^2 - d^2 = b^2 - c^2\,, \quad (a-d)^2 = (b-c)^2\,. \tag{7.215}$$

The last equation above reduces to

$$(c - b) = \mu(a - d) \tag{7.216}$$

where $\mu \in \mathbb{Z}_2$. We distinguish now two cases:

1. If $a = d \neq 0$ (using that $a + d \neq 0$) we obtain $c = b$ and the $\varepsilon\eta$-Einstein equations are equivalent to
$$a^2 = \frac{\lambda^2}{4}\,. \tag{7.217}$$
Likewise, the conditions on the coefficients required by the algebra of type $\mathfrak{g}_6$ are
$$a^2 = (b+s)^2\,, \tag{7.218}$$
implying $a = \sigma(b+s)$ for a sign $\sigma \in \mathbb{Z}_2$. Note that $b \neq s$ since $a + d = 2a \neq 0$. Plugging $a = \sigma(b+s)$ in Equation (7.209) we obtain:
$$\alpha_0^2 = \alpha_2^2\,, \tag{7.219}$$
which is incompatible with the para-contact condition $-\alpha_0^2 + \alpha_2^2 = 1$.

2. If $a \neq d$, equation $a^2 - d^2 = b^2 - c^2$ is equivalent to

$$a + d = -\mu(b + c)\,. \tag{7.220}$$

Combining this equation with $(c-b) = \mu(a-d)$ we obtain $a = -\mu b$ and $d = -\mu c$. The constraints on the coefficients required by the algebra of type $\mathfrak{g}_6$ reduce to

$$1 + s(b + c) = 0\,, \tag{7.221}$$

whereas the $\varepsilon\eta$-Einstein equations are tantamount to

$$\lambda^2 = (b + c)^2\,. \tag{7.222}$$

Therefore, using that $1 + s(b + c) = 0$ we obtain $\lambda^2 = 1$. Equation (7.209) is solved by:

$$b = -\mu a\,, \quad c = -s{+}\mu a\,, \quad d = \mu s - a\,, \tag{7.223}$$

$$(-s + \mu a)\alpha_0 = a\alpha_2\,, \quad \alpha_2^2 = 1 + \alpha_0^2\,. \tag{7.224}$$

- **Case $\mathfrak{g}_7$.** In the orthonormal frame given in Appendix 7.A, Equation $\star\alpha = -s\,\mathrm{d}\alpha$ is equivalent to

$$\begin{aligned}&(b\alpha_0 + a\alpha_1 + b\alpha_2)\, e^0 \wedge e^1 + (d\alpha_0 + c\alpha_1 + d\alpha_2)\, e^0 \wedge e^2 \\ &\quad + (b\alpha_0 + a\alpha_1 + b\alpha_2)\, e^1 \wedge e^2 = s\,(\alpha_0 e^1 \wedge e^2 + \alpha_1 e^0 \wedge e^2 - \alpha_2 e^0 \wedge e^1)\,.\end{aligned} \tag{7.225}$$

which immediately implies $\alpha_0 + \alpha_2 = 0$, whence $\alpha_1^2 = 1$. Write $\alpha_1 = \sigma$, with $\sigma \in \mathbb{Z}_2$. With these assumptions, the previous equations boil down to

$$\alpha_0 = s\sigma a\,, \qquad c = s\,, \qquad \alpha_2^2 = a^2\,, \tag{7.226}$$

where $\mu \in \mathbb{Z}_2$. Since $c \neq 0$, then $a = 0$ from the condition $ac = 0$ required by the algebra of type $\mathfrak{g}_7$. Hence $\alpha_0 = \alpha_2 = 0$. With these provisos in mind, the Ricci curvature reads:

$$\mathrm{Ric}^g(e_0, e_0) = -bs - \tfrac{1}{2}\,, \quad \mathrm{Ric}^g(e_0, e_1) = 0\,, \tag{7.227}$$

$$\mathrm{Ric}^g(e_1, e_1) = -\tfrac{1}{2}\,, \quad \mathrm{Ric}^g(e_0, e_2) = sb\,, \tag{7.228}$$

$$\mathrm{Ric}^g(e_2, e_2) = \tfrac{1}{2} - sb\,, \quad \mathrm{Ric}^g(e_1, e_2) = 0\,. \tag{7.229}$$

However, since $\alpha = \sigma\, e^1$, we obtain $b = 0$, which implies $\kappa = -1$, a value that is not permitted by Definition 7.7.

Finally, to verify which of the $\varepsilon\eta$-Einstein para-contact structures are Sasakian we apply first Proposition 7.4, which states that $\varepsilon\eta$-Einstein para-contact structures on $\widetilde{\mathrm{E}}(1,1)$ or $\widetilde{\mathrm{E}}(2)$ can never be Sasakian. Also, a direct computation shows that $\mathfrak{h}$ vanishes for every $\varepsilon\eta$-Einstein para-contact structure on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $\mathfrak{g}_6$. $\square$

7.3.4 Lorentzian case with null Reeb vector field

When $\varepsilon = 0$ the $\varepsilon\eta$-Einstein condition for an ε-contact metric structure (g, α) reduces to

$$\mathrm{Ric}^g = -\frac{\lambda^2}{2}\, g + \kappa\, \alpha \otimes \alpha\,, \tag{7.230}$$

with $\kappa \geq 0$.

Remark 7.26. To the best of our knowledge, this equation has not been considered in the literature. In particular, the methods and techniques used in [690, 694] to classify (κ, μ) and $\varepsilon\eta$-Einstein contact three manifolds do not seem to apply in this case, due to the fact that ϕ^2 is not an isomorphism when restricted to the kernel of α and that $\mathfrak{h}$ cannot have non-zero eigenvalues, see Remarks 7.8 and 7.14.

The goal of this subsection is to classify all left-invariant $\varepsilon\eta$-Einstein null contact structures on a simply connected three-dimensional Lie group G. In order to do this, we will make use of the following lemma.

Lemma 7.10. *Let (g, α) be a null contact metric structure and let $\{\xi, u, \phi(u)\}$ be a light-cone frame. Then (ξ, α) is $\varepsilon\eta$-Einstein if and only if*

$$\mathrm{Ric}^g(\xi, \xi) = \mathrm{Ric}^g(\xi, \phi(u)) = \mathrm{Ric}^g(u, \phi(u)) = 0\,, \tag{7.231}$$

$$\mathrm{Ric}^g(\xi, u) = \mathrm{Ric}^g(\phi(u), \phi(u)) = -\tfrac{\lambda^2}{2}\,, \quad \mathrm{Ric}^g(u, u) = \kappa\,. \tag{7.232}$$

Proof. Follows by direct computation. □

Theorem 7.4. *A three-dimensional connected and simply connected Lie group G admits a left-invariant $\varepsilon\eta$-Einstein null contact structure (g, α) if and only if (G, g, α) is isomorphic, through a possibly orientation-reversing isometry, to one of the items listed in the following table in terms of the orthonormal frame $\{e_0, e_1, e_2\}$ appearing in Theorem 7.6:*

<table>
<tr><th>$\mathfrak{g}$</th><th>Structure constants ($s \in \mathbb{Z}_2$)</th><th>α</th><th>η-Einstein constants</th><th>G</th><th>Sasakian</th></tr>
<tr><td rowspan="2">$\mathfrak{g}_3$</td><td>$a = b = c = s$</td><td>$\alpha_0^2 = \alpha_1^2 + \alpha_2^2$</td><td>$\lambda^2 = 1\,,\ \kappa = 0$</td><td>$\widetilde{\mathrm{SL}}(2, \mathbb{R})$</td><td>Yes</td></tr>
<tr><td>$a = c = s\,,\ b = 0$</td><td>$\alpha_2 = 0\,,$ $\alpha_0^2 = \alpha_1^2$</td><td>$\lambda^2 = 0\,,\ \kappa = 0$</td><td>$\widetilde{\mathrm{E}}(1,1)$</td><td>No</td></tr>
<tr><td rowspan="2">$\mathfrak{g}_4$</td><td>$b = 0\,,\ a = s$</td><td rowspan="2">$\alpha_1 = 0\,,$ $\alpha_0 = \mu\alpha_2$</td><td>$\lambda^2 = 1\,,\ \alpha_0^2\kappa = 1$</td><td>$\widetilde{\mathrm{SL}}(2, \mathbb{R})$</td><td>Yes</td></tr>
<tr><td>$b = 0\,, a = 0\,,$</td><td>$\lambda^2 = 0\,,\ \alpha_0^2\kappa = 2$</td><td>$\widetilde{\mathrm{E}}(1,1)$</td><td>No</td></tr>
<tr><td>$\mathfrak{g}_6$</td><td>$a = d \neq 0\,, b = c$ $a = \mu(b + s)\,, \mu \in \mathbb{Z}_2$</td><td>$\alpha_1 = 0\,,$ $\alpha_0 = -\mu\alpha_2$</td><td>$\lambda^2 = 4a^2\,,$ $\kappa = 0$</td><td>$\mathfrak{G}_6$</td><td>If $a = \frac{\mu s}{2}$</td></tr>
</table>

Proof. Proposition 7.8 classifies all simply connected Lorentzian Lie groups admitting left-invariant null contact structures. Hence, we will proceed by verifying which of the cases appearing in Proposition 7.8 satisfy the $\varepsilon\eta$-Einstein equation. For this, we will use the formulae presented in Appendix 7.B, where the Ricci tensor is computed on a global orthonormal frame.

- **Case** $\mathfrak{g}_1$. The Reeb vector is given by $\xi = -\alpha_0(e_0 + e_2)$. A direct computation using Appendix 7.B shows that $\mathrm{Ric}^g(u, \phi(u)) = -\frac{sa}{\alpha_0}$. Since $a \neq 0$ by the definition of algebra of type $\mathfrak{g}_1$, Lemma 7.10 implies that $\mathfrak{g}_1$ does not admit null $\varepsilon\eta$-Einstein structures.

- **Case** $\mathfrak{g}_3$. We distinguish between the cases $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_1, \alpha_2 \neq 0$. If $\alpha_1 = 0$, a light-cone frame is given by $\xi = \alpha_0(-e_0 + \mu e_2), u = \frac{1}{2\alpha_0}(e_0 + \mu e_2)$ and $\phi(u) = \mu e_1$. We obtain:

$$\mathrm{Ric}^g(\xi,\xi) = \mathrm{Ric}^g(\xi,\phi(u)) = \mathrm{Ric}^g(u,\phi(u)) = 0 \tag{7.233}$$
$$\mathrm{Ric}^g(\xi,u) = -a + \tfrac{a^2}{2}\,, \quad \mathrm{Ric}^g(\phi(u),\phi(u)) = -\tfrac{a^2}{2}\,, \quad \mathrm{Ric}^g(u,u) = 0\,. \tag{7.234}$$

  Hence, the $\varepsilon\eta$-Einstein implies $a = 0$ or $a = s$. Since $a = 0$ is not allowed, we conclude $a = s$, which in turn implies $\lambda^2 = 1$ and $\kappa = 0$.

  If $\alpha_2 = 0$, a similar analysis follows. In this case, a light-cone frame is given by $\xi = \alpha_0(-e_0 + \mu e_1), u = \frac{1}{2\alpha_0}(e_0 + \mu e_1)$ and $\phi(u) = -\mu e_2$. We obtain:

$$\mathrm{Ric}^g(\xi,\xi) = \mathrm{Ric}^g(\xi,\phi(u)) = \mathrm{Ric}^g(u,\phi(u)) = 0 \tag{7.235}$$
$$\mathrm{Ric}^g(\xi,u) = -bs + \tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(\phi(u),\phi(u)) = -\tfrac{b^2}{2}\,, \quad \mathrm{Ric}^g(u,u) = 0\,, \tag{7.236}$$

  which implies either $b = s$ or $b = 0$. For $b = s$, $\lambda^2 = 1$ and $\kappa = 0$, and for $b = 0$, $\lambda^2 = \kappa = 0$.

  Finally, if $\alpha_1, \alpha_2 \neq 0$, we directly obtain $\mathrm{Ric}^g = -\frac{1}{2} g$ and every null contact structure of this type is $\varepsilon\eta$-Einstein (with $\lambda^2 = 1$ and $\kappa = 0$).

- **Case** $\mathfrak{g}_4$. We choose a light-cone frame $\{\xi, u, \phi(u)\}$ with $\xi = \alpha_0(-e_0 + \mu e_2), u = \frac{1}{2\alpha_0}(e_0 + \mu e_2)$ and $\phi(u) = \mu e_1$. We compute:

$$\mathrm{Ric}^g(\xi,\xi) = \mathrm{Ric}^g(\xi,\phi(u)) = \mathrm{Ric}^g(u,\phi(u)) = 0\,, \quad \mathrm{Ric}^g(\xi,u) = \tfrac{a^2}{2} - as\,, \tag{7.237}$$
$$\mathrm{Ric}^g(\phi(u),\phi(u)) = -\tfrac{a^2}{2}\,, \quad \mathrm{Ric}^g(u,u) = \tfrac{\mu}{\alpha_0^2}(a - 2s)\,. \tag{7.238}$$

  These conditions are satisfied if and only if $a = 0$ or $a = s$. For $a = s$, we have that $\lambda^2 = 1$ and $\kappa = -\frac{s\mu}{\alpha_0^2}$, and for $a = 0$, we find that $\lambda^2 = 0$ and $\kappa = -2\frac{s\mu}{\alpha_0^2}$. Since κ must be non-negative by definition, we must require that $s\mu = -1$, which implies that $b = 0$.

- **Case** $\mathfrak{g}_6$. We can choose the light-cone frame $\{\xi, u, \phi(u)\}$ with $\xi = -\alpha_0(e_0 + \mu e_2), u = \frac{1}{2\alpha_0}(e_0 - \mu e_2)$ and $\phi(u) = -\mu e_1$. Imposing the constraints found in Proposition 7.8 for null contact structures on $\mathfrak{g}_6$, we get the following components for the Ricci curvature:

$$\mathrm{Ric}^g(\xi,\phi(u)) = \mathrm{Ric}^g(u,\phi(u)) = 0\,, \quad \mathrm{Ric}^g(\xi,\xi) = 0\,, \tag{7.239}$$
$$\mathrm{Ric}^g(\xi,u) = -2a^2\,, \quad \mathrm{Ric}^g(\phi(u),\phi(u)) = -2a^2\,, \quad \mathrm{Ric}^g(u,u) = 0\,. \tag{7.240}$$

  These equations yield an $\varepsilon\eta$-Einstein structure with $\lambda^2 = 4a^2 \neq 0$ and $\kappa = 0$.

Finally, in order to determine when the different $\varepsilon\eta$-Einstein structures obtained are Sasakian, we just have to make use of Proposition 7.10. □

7.4 Six-dimensional Supergravity and ε-contact structures

In the following, let M be an oriented and spin six-dimensional manifold.

Definition 7.9. The bosonic configuration space of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton on M is defined as the following set:

$$\mathrm{Conf}(M) \stackrel{\text{def.}}{=} \left\{(g,H) \in \mathrm{Lor}(M) \times \Omega^3(M)\right\} , \tag{7.241}$$

where $\mathrm{Lor}(M)$ denotes the set of Lorentzian metrics on M.

Given $(g,H) \in \mathrm{Conf}(M)$ we define ∇^H to be the unique metric-compatible connection on (M,g) with totally skew-symmetric torsion given by $H \in \Omega^3(M)$. In more explicit terms we have

$$\nabla^H = \nabla + \frac{1}{2} g^{-1} H \,, \tag{7.242}$$

where ∇ denotes the Levi-Civita connection associated to g.

Definition 7.10. A pair $(g,H) \in \mathrm{Conf}(M)$ is a bosonic solution of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton on M if:

$$\mathrm{Ric}(\nabla^H) = 0 \,, \qquad \mathrm{d}H = 0 \,, \qquad \mathrm{d} \star_g H = 0 \,, \qquad |H|^2_g = 0 \,, \tag{7.243}$$

where $\mathrm{Ric}(\nabla^H) \in \Gamma(\mathrm{Sym}^2(T^*M))$ is the Ricci curvature tensor of ∇^H and $\star_g \colon \Omega^3(M) \to \Omega^3(M)$ denotes the Hodge dual associated to g. We denote by $\mathrm{Sol}(M) \subset \mathrm{Conf}(M)$ the set of solutions on M.

Remark 7.27. In six Lorentzian dimensions the Hodge dual $\star_g$ on three-forms squares to the identity. Hence, we obtain the splitting

$$\Lambda^3(M) = \Lambda^3_+(M) \oplus \Lambda^3_-(M) \,, \tag{7.244}$$

in terms of self dual $\Lambda^3_+(M)$ and antiself dual $\Lambda^3_-(M)$ three-forms. Using this decomposition, a particular class of solutions of Equations (7.243) is obtained by requiring H to be self-dual, that is:

$$\mathrm{Ric}(\nabla^H) = 0 \,, \qquad \mathrm{d}H = 0 \,, \qquad \star_g H = H \,, \tag{7.245}$$

This set of equations define the bosonic equations of six-dimensional *minimal Supergravity*. Equations (7.243) are more general and allow, for instance, H to be a section of a fixed lagrangian distribution of $\Lambda^3(M)$. The possibility of generalizing minimal Supergravity to this situation was proposed in [698] and it remains, to the best of our knowledge, up for debate.

Theorem 7.5. *Let:*

$$\begin{aligned} (N,\chi,\alpha_N,\varepsilon_N) &\in \mathrm{PCont}^{\varepsilon\eta}_L(\varepsilon_N, \lambda^2, \kappa_N = l^2) \,, \\ (X,h,\alpha_X) &\in \mathrm{PCont}^{\varepsilon\eta}_R(\lambda^2, \kappa_X = |\alpha_N|^2\, l^2) \,. \end{aligned} \tag{7.246}$$

Then, the oriented Cartesian product manifold

$$M = N \times X \tag{7.247}$$

carries a family of solutions $(g, H_{\lambda,l}) \in \mathrm{Sol}(M)$ *of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton given by:*

$$g = \chi \oplus h\,, \qquad H_{\lambda,l} = \lambda\,\nu_\chi + \frac{l}{3}\,(\star_\chi \alpha_N) \wedge \alpha_X + \frac{l}{3}\,\alpha_N \wedge (\star_h \alpha_X) + \lambda\,\nu_h \tag{7.248}$$

and parametrized by $(\lambda, l) \in \mathbb{R}^2$*, where* ν_χ *and* ν_h *are the corresponding pseudo-Riemannian volume forms. Equivalently, the oriented Cartesian Lorentzian product of* $(N, \chi, \alpha_N, \varepsilon_N) \in \mathrm{PCont}^{\varepsilon\eta}_L(\varepsilon_N, \lambda, l)$ *and* $(X, h, \alpha_X) \in \mathrm{PCont}^{\varepsilon\eta}_R(\lambda, \kappa = \varepsilon_N, l)$ *carries a bi-parametric family of metric-compatible, Ricci flat, connections with totally skew-symmetric, isotropic, closed and co-closed torsion prescribed by* $H_{\lambda,l}$.

Proof. We first compute that $H_{\lambda,l}$, as prescribed in the statement of the theorem, is closed:

$$\begin{aligned} \mathrm{d}H_{\lambda,l} &= \frac{l}{3}\,\mathrm{d}(\star_\chi \alpha_N) \wedge \alpha_X + \frac{l}{3}(\star_\chi \alpha_N) \wedge \mathrm{d}\alpha_X + \frac{l}{3}\,\mathrm{d}\alpha_N \wedge (\star_h \alpha_X) - \frac{l}{3}\,\alpha_N \wedge \mathrm{d}(\star_h \alpha_X) \\ &= \frac{l}{3}(\star_\chi \alpha_N) \wedge \mathrm{d}\alpha_X + \frac{l}{3}\,\mathrm{d}\alpha_N \wedge (\star_h \alpha_X) = \frac{l}{3} \star_\chi \alpha_N \wedge \star_h \alpha_X - \frac{l}{3} \star_\chi \alpha_N \wedge \star_h \alpha_X = 0\,, \end{aligned} \tag{7.249}$$

where we have used that $\mathrm{d}\alpha_N = - \star_\chi \alpha_N$ and $\mathrm{d}\alpha_X = \star_h \alpha_X$ by the ε-contact condition. In addition, $H_{\lambda,l}$ is co-closed:

$$\begin{aligned} \mathrm{d} \star H_{\lambda,l} &= \lambda\, \mathrm{d} \star \nu_\chi + \frac{l}{3}\mathrm{d} \star (\star_\chi \alpha_N \wedge \alpha_X) + \frac{l}{3}\mathrm{d} \star (\alpha_N \wedge \star_h \alpha_X) + \lambda\, \mathrm{d} \star \nu_h \\ &= -\lambda\, \mathrm{d}\nu_h + \frac{l}{3}\,\mathrm{d}(\alpha_N \wedge \star_h \alpha_X) + \frac{l}{3}\mathrm{d}(\star_\chi \alpha_N \wedge \alpha_X) - \lambda\, \mathrm{d}\nu_\chi = 0\,, \end{aligned} \tag{7.250}$$

where we have used again the ε-contact condition and the fact that, for $\rho \in \Omega^q(N)$ and $\sigma \in \Omega^r(X)$, we have $\star_g(\rho \wedge \sigma) = (-1)^{r(3-q)} \star_\chi \rho \wedge \star_h \sigma$. We verify now the norm of $H_{\lambda,l}$ indeed vanishes:

$$\begin{aligned} |H_{\lambda,l}|^2_g &= \lambda^2 g(\nu_\chi, \nu_\chi) + \frac{l^2}{9} g((\star_\chi \alpha_N) \wedge \alpha_X, (\star_\chi \alpha_N) \wedge \alpha_X) \\ &+ \frac{l^2}{9} g(\alpha_N \wedge (\star_h \alpha_X), \alpha_N \wedge (\star_h \alpha_X)) + \lambda^2 g(\nu_h, \nu_h) = \lambda^2 \chi(\nu_\chi, \nu_\chi) \\ &+ \frac{l^2}{3} \chi(\star_\chi \alpha_N, \star_\chi \alpha_N) h(\alpha_X, \alpha_X) + \frac{l^2}{3} \chi(\alpha_N, \alpha_N) h(\star_h \alpha_X, \star_h \alpha_X) + \lambda^2 h(\nu_h, \nu_h) \\ &= -6\lambda^2 - \frac{2l^2}{3} \chi(\alpha_N, \alpha_N) h(\alpha_X, \alpha_X) + \frac{2l^2}{3} \chi(\alpha_N, \alpha_N) h(\alpha_X, \alpha_X) + 6\lambda^2 = 0\,. \end{aligned} \tag{7.251}$$

We check next that the Einstein equations are satisfied for the specific choices of constants specified in the statement. We have that

$$\begin{aligned} H_{\lambda,l} \circ H_{\lambda,l}|_{TN \otimes TN} &= \lambda^2 \nu_N \circ \nu_N + \frac{l^2}{9} (\star_\chi \alpha_N \wedge \alpha_X) \circ (\star_\chi \alpha_N \wedge \alpha_X)|_{TN \otimes TN} \\ &+ \frac{l^2}{9} (\alpha_N \wedge \star_h \alpha_X) \circ (\alpha_N \wedge \star_h \alpha_X)|_{TN \otimes TN}\,. \end{aligned} \tag{7.252}$$

We compute also the following:

$$\lambda^2\nu_N\circ\nu_N = -2\,\lambda^2\chi\,,\quad (\alpha_N\wedge\star_h\alpha_X)\circ(\alpha_N\wedge\star_h\alpha_X)|_{TN\otimes TN} = 18\,\alpha_N\otimes\alpha_N\,,\tag{7.253}$$
$$(\star_\chi\alpha_N\wedge\alpha_X)\circ(\star_\chi\alpha_N\wedge\alpha_X)|_{TN\otimes TN} = 18(\alpha_N\otimes\alpha_N - |\alpha_N|^2_\chi\,\chi)\,.\tag{7.254}$$

This implies that

$$\frac{1}{4}H_{\lambda,l}\circ H_{\lambda,l}|_{TN\otimes TN} = -\frac{\lambda^2}{2}\,\chi - \frac{l^2}{2}|\alpha_N|^2_\chi\,\chi + l^2\alpha_N\otimes\alpha_N\,.\tag{7.255}$$

Likewise we have

$$\begin{aligned}H_{\lambda,l}\circ H_{\lambda,l}|_{TX\otimes TX} = {}&\lambda^2\nu_X\circ\nu_X + \tfrac{l^2}{9}(\star_\chi\alpha_N\wedge\alpha_X)\circ(\star_\chi\alpha_N\wedge\alpha_X)|_{TX\otimes TX}\\ &+\tfrac{l^2}{9}(\alpha_N\wedge\star_h\alpha_X)\circ(\alpha_N\wedge\star_h\alpha_X)|_{TX\otimes TX}\,,\end{aligned}\tag{7.256}$$

which in turn implies

$$\frac{1}{4}H_{\lambda,l}\circ H_{\lambda,l}|_{TX\otimes TX} = \frac{\lambda^2}{2}\,h + \frac{l^2}{2}|\alpha_N|^2_\chi\,h - l^2|\alpha_N|^2_\chi\alpha_X\otimes\alpha_X\,.\tag{7.257}$$

Finally, it can be checked the mixed components vanish identically,

$$H_{\lambda,l}\circ H_{\lambda,l}|_{TN\otimes TX} = H_{\lambda,l}\circ H_{\lambda,l}|_{TX\otimes TN} = 0\,.\tag{7.258}$$

From Definition 7.7, we obtain that $(N,\chi,\alpha_N,\varepsilon_N)\in\mathrm{PCont}^{\varepsilon\eta}_L(\varepsilon_N,\lambda^2,\kappa_N = l^2)$ and $(M,h,\alpha_X)\in\mathrm{PCont}^{\varepsilon\eta}_R(\lambda^2,\kappa_X = |\alpha_N|^2\,l^2)$ satisfy, respectively:

$$\mathrm{Ric}^\chi = -\frac{1}{2}(\lambda^2+l^2\varepsilon)\,\chi + l^2\alpha_N\otimes\alpha_N\,,\qquad \mathrm{Ric}^h = \frac{1}{2}(\lambda^2+l^2\,\varepsilon)\,h - l^2\,\varepsilon\,\alpha_X\otimes\alpha_X\,.\tag{7.259}$$

Since by definition $\varepsilon = |\alpha_N|^2_\chi$, the right hand sides of the equations appearing in (7.259) coincide with the right hand sides of equations (7.255) and (7.257), respectively. Hence, the tuple $(g,H_{\lambda,l})$ as defined in the statement of the theorem satisfies:

$$\mathrm{Ric}(\nabla^{H_{\lambda,l}}) = 0\,,\qquad \mathrm{d}H_{\lambda,l} = 0\,,\qquad \mathrm{d}\star_g H_{\lambda,l} = 0\,,\qquad |H_{\lambda,l}|^2_g = 0\,,\tag{7.260}$$

whence it is a solution of six-dimensional Supergravity coupled to a tensor multiplet with constant dilaton. □

Remark 7.28. For ease of reference, we will refer to the solutions (g,H) constructed in Theorem 7.5 as ε*-contact Supergravity solutions* of type $(\varepsilon_N,\lambda,l)$, where $\varepsilon_N\in\{-1,0,1\}$ is the norm of the Reeb vector field of the Lorentzian ε-contact structure occurring in the given solution.

The holonomy of the Levi-Civita connection ∇ of an ε-contact Supergravity solution (g,H) is clearly reducible, since the sub-bundles $TN\subset TM$ and $TX\subset TM$ are preserved by ∇ by construction. In particular, ∇ is a product connection on $TM = TN\times TX$. However, the connection with torsion ∇^H is in general not a product connection. In particular neither $TN\subset TM$ nor $TX\subset TM$ are preserved by ∇^H if $l\neq 0$. Therefore Supergravity ε-contact solutions are in general not the direct product of a pair of three-dimensional pseudo-Riemannian manifolds with torsion.

Theorem 7.5 allows to construct large classes of explicit solutions of six-dimensional Supergravity coupled to a tensor multiplet with constant dilaton by exploiting the extensive literature on $\varepsilon\eta$-Einstein Riemannian and Lorentzian (para-)contact metric three-manifolds, as discussed in Section 7.3, and by employing the new null contact metric structures, as discussed in Sections 7.2 and 7.3. In particular, the previous theorem implies that the construction of examples and development of classification results on Riemannian and Lorentzian $\varepsilon\eta$-Einstein (para-)contact metric three-manifolds (and $\varepsilon\eta$-Einstein null contact structures) can automatically be used to construct new Lorentzian six-manifolds equipped with a Ricci flat metric-compatible connection with totally skew-symmetric, isotropic, closed and co-closed torsion.

7.5 Ricci flat Lorentzian six-manifolds with closed self-dual torsion

In this section we apply the results of the previous sections to the construction of new six-dimensional Lorentzian manifolds equipped with a Ricci flat and metric-compatible connection with totally skew-symmetric, isotropic, closed and co-closed torsion, which in turn yields new solutions of minimal Supergravity coupled to a tensor multiplet with constant dilaton in six dimensions. Excluding the Ricci flat case, the simplest scenario where Theorem 7.5 applies is obtained by taking $l = 0$ and $\lambda \neq 0$. In this situation, the corresponding six-dimensional ε-contact solution is given by:

$$g = \chi \oplus h \,, \qquad H_{\lambda,l} = \lambda\,(\nu_\chi + \nu_h)\,, \quad \lambda \neq 0\,, \tag{7.261}$$

on $M = N \times X$, where χ and h are Einstein with negative and positive Einstein constant, respectively. Assuming that both (N, χ) and (X, h) are connected, simply connected and geodesically complete we conclude that (X, h) is isometric to the round sphere and (N, χ) is isometric to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ equipped with its Einstein metric. In particular:

$$(M, g) = (\widetilde{\mathrm{SL}}(2, \mathbb{R}) \times S^3, \chi \oplus h)\,, \tag{7.262}$$

is a solution of six-dimensional minimal Supergravity, which corresponds with the well-known $\mathrm{AdS}_3 \times S^3$ maximally supersymmetric solution of the theory [193]. For $l \neq 0$, ε-contact Supergravity solutions are not isomorphic to the previous solution[3]. Hence, intuitively speaking we can think of ε-contact Supergravity solutions with $l \neq 0$ as being generically non-supersymmetric *geometric and topological deformations* of the supersymmetric $\mathrm{AdS}_3 \times S^3$ solution, with deformations parametrized by $l \in \mathbb{R}$, $\alpha_N \in \Omega^1(N)$ and $\alpha_X \in \Omega^1(X)$. In the following we consider ε-contact Supergravity solutions of type $\varepsilon_N = -1$, $\varepsilon_N = 0$ and $\varepsilon_N = 1$ separately. We emphasize that in general the values of λ^2 and κ do not uniquely determine the diffeomorphism type of N or X and that the same diffeomorphism type may admite several non-isometric ε-contact Supergravity solutions. In this direction, it is *a priori* possible that there exist, at least when $\varepsilon_N \neq 1$ or in the Sasakian case for $\varepsilon_N = 1$, ε-contact Supergravity solutions for which M is not diffeomorphic to a Lie group.

[3] Another solution fitting the case $l = 0$ is the non-supersymmetric embedding of the Reissner-Nordström black hole presented in [125].

7.5.1 Timelike case: $\varepsilon_N = -1$.

Let

$$(N, \chi, \alpha_N, -1) \in \mathrm{PCont}_L^{\varepsilon\eta}(\lambda^2, l^2, -1)\,, \qquad (X, g, \alpha_X) \in \mathrm{PCont}_R^{\varepsilon\eta}(\lambda^2, -l^2)\,. \tag{7.263}$$

Then, the product of the following pairs of three-manifolds carry ε-contact Supergravity solutions as prescribed in Theorem 7.5 for the specified parameters:

Lorentzian factor		Riemannian factor		
N	Sasakian	X	Sasakian	(λ^2, l^2)
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1 - l^2\,, 1 > l^2 \geq 0$
$\mathfrak{G}_6$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1 - l^2\,, 1 > l^2 \geq 0$
H_3	Yes	H_3	Yes	$\lambda^2 = 0\,, l^2 = 1$
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	No	$\mathrm{SU}(2)$	No	$\lambda^2 = l^2\,, \frac{1}{2} > l^2 > 0$
$\widetilde{\mathrm{E}}(1,1)$	No	$\widetilde{\mathrm{E}}(2)$	No	$\lambda^2 = 0\,, l^2 = 0$

7.5.2 Spacelike case: $\varepsilon_N = 1$.

Let

$$(N, \chi, \alpha_N, 1) \in \mathrm{PCont}_L^{\varepsilon\eta}(\lambda^2, l^2, 1)\,, \qquad (X, g, \alpha_X) \in \mathrm{PCont}_R^{\varepsilon\eta}(\lambda^2, l^2)\,. \tag{7.264}$$

The following direct products of three-manifolds can be endowed with ε-contact Supergravity solutions as indicated in Theorem 7.5 for the values of the parameters specified below:

Lorentzian factor		Riemannian factor		
N	Sasakian	X	Sasakian	(λ^2, l^2)
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1 + l^2\,, l^2 \geq 0$
$\mathfrak{G}_6$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1 + l^2\,, l^2 \geq 0$
$\widetilde{\mathrm{E}}(1,1)$	No	$\widetilde{\mathrm{E}}(2)$	No	$\lambda^2 = 0\,, l^2 = 0$
$\widetilde{\mathrm{E}}(2)$	No	$\widetilde{\mathrm{E}}(2)$	No	$\lambda^2 = 0\,, l^2 = 0$

7.5.3 Null case: $\varepsilon_N = 0$.

Let

$$(N, \chi, \alpha_N, 0) \in \mathrm{PCont}_L^{\varepsilon\eta}(\lambda^2, l^2, 0)\,, \qquad (X, g, \alpha_X) \in \mathrm{PCont}_R^{\varepsilon\eta}(\lambda^2, 0)\,. \tag{7.265}$$

The product of the following Lorentzian and Riemannian three-manifolds carry ε-contact Supergravity solutions as described in Theorem 7.5 for the values of the parameters indicated below:

Lorentzian factor		Riemannian factor		
N	Sasakian	X	Sasakian	(λ^2, l^2)
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1\,, l^2 \geq 0$
$\mathfrak{G}_6$	Yes	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1\,, l^2 = 0$
$\mathfrak{G}_6$	No	$\mathrm{SU}(2)$	Yes	$\lambda^2 = 1\,, l^2 = 0$
$\widetilde{\mathrm{E}}(1,1)$	No	$\widetilde{\mathrm{E}}(2)$	No	$\lambda^2 = 0\,, l^2 \geq 0$

Remark 7.29. Note that both Sasakian and non-Sasakian $\varepsilon\eta$-Einstein null-contact structures on $\mathfrak{G}_6$ can be combined with Riemannian Sasakian structures on SU(2) to yield ε-contact Supergravity solutions. This is possible due to the fact that the $\varepsilon\eta$-Einstein condition imposes in this case a quadratic constraint on the structure constants. Solutions to this quadratic constraint produce either a Sasakian or a non-Sasakian null contact structure, depending on the particular solution chosen.

7.6 Discussion

In this last chapter of the thesis we have proved that the appropriate combination of $\varepsilon\eta$-Einstein contact structures provides solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton. Since the configuration space of this Supergravity is given by a Lorentzian metric g and a three-form H, this naturally poses the question about the possibility of finding analogous solutions in the context of ten-dimensional Type IIB Supergravity. In fact, the latter can be consistently truncated to obtain a theory whose configuration space consists solely of a ten-dimensional Lorentzian metric and a (self-dual) five-form [106, 699], which motivates the generalization of the results of this chapter to the realm of ten-dimensional Type IIB Supergravity.

A key role was played by ε-contact structures, which encompass the usual notions of three-dimensional contact Riemannian, contact Lorentzian and para-contact metric structures, but which also admit the possibility of a null Reeb vector field. This last case was very intriguing. Indeed, while our formalism canonically suggested the definition of null contact structures as those ε-contact structures with $\varepsilon = 0$, they are very special, since strictly speaking they are not contact structures (because $\alpha \wedge \mathrm{d}\alpha = 0$). We encountered infinite instances of such structures and defined the corresponding notions of Sasakian and K-contact null contact structures, noting a striking fact: they are not equivalent conditions, being Sasakianity weaker than K-contactness. It would be very interesting to explore the notion of null contact structures in higher-dimensions or, more generally, in arbitrary dimensions. Which would be the most suitable definition of null contact structures in any dimension, which reduces to the one provided in this chapter for three-dimensions? In particular, it might be interesting to start this investigation in the context of the aforementioned study of ten-dimensional Type IIB Supergravity.

Within the class of ε-contact structures, we were interested in the so-called $\varepsilon\eta$-Einstein contact structures, which for non-null Reeb vector fields correspond to particular cases of the notions of (three-dimensional) η-Einstein Riemannian, Lorentzian and para-contact metric structures. In particular, the $\varepsilon\eta$-Einstein condition can be seen to be equivalent to imposing an Einstein-like equation with the stress-energy tensor of a perfect fluid. This is specially intriguing in the case of null contact structures, which deserves further exploration.

We also classified all left-invariant $\varepsilon\eta$-Einstein contact metric structures on three-dimensional simply connected Lie groups, indicating when they were additionally Sasakian. This permitted us to obtain a plethora of solutions of minimal six-dimensional Supergravity coupled to a tensor multiplet, but still it would be interesting to investigate the possibility of finding different kinds of solutions, perhaps through the study of the Cauchy initial value problem of $\varepsilon\eta$-Einstein contact structures, which we formulated. Another future direction to examine is that of extending the notion of ε-contact structures (and $\varepsilon\eta$-Einstein contact structures) to the realm of Generalized Geometry[4] [702]. It would be appealing to elaborate on the potential notion of left-invariant generalized ε-contact structures on Lie groups, in the spirit of the recently developed theory of left-invariant generalized pseudo-Riemannian metrics on Lie groups [703].

Then we presented the main result of our work, which depicts the exact procedure by which to construct solutions of six-dimensional minimal Supergravity coupled to a tensor multiplet with constant dilaton through the combination of $\varepsilon\eta$-Einstein contact structures. As explained, this allowed us to produce Lorentzian six-manifolds (M, g) with a metric-compatible Ricci flat connection with isotropic, totally antisymmetric, closed and co-closed torsion. Finally, using the classification of left-invariant $\varepsilon\eta$-Einstein contact structures, we obtained infinite families of solutions of six-dimensional minimal Supergravity which we interpreted as (generically non-supersymmetric) geometric and topological deformations of the supersymmetric $\mathrm{AdS}_3 \times S^3$ solution. In particular, it would be interesting to have a better physical understanding of such solutions, as well as knowing explicitly which of them are supersymmetric or not.

[4]Note that the concept of generalized contact structures has already been introduced, see [700, 701].

Appendix 7.A Simply connected three-dimensional Lorentzian Lie groups

For the benefit of the reader, we summarize in the following the classification of all three-dimensional Lorentzian simply connected Lie groups. The table below is extracted from [679, Theorem 4.1]. In the table below $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ denotes the universal cover of the special linear group $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SU}(2)$ stands for the special unitary group in two dimensions, $\widetilde{\mathrm{E}}(2)$ is the universal cover of the group of rigid motions of the Euclidean plane, whereas $\widetilde{\mathrm{E}}(1,1)$ denotes the universal cover of the group of rigid motions of the Minkowski plane and H_3 stands for the three-dimensional Heisenberg group.

Theorem 7.6. *[679, 680, 683] Let (G, g) be a three-dimensional connected, simply connected, Lorentzian Lie group G with left-invariant metric g. Then, precisely, one of the following cases occurs:*

- *G is unimodular and there exists an orthonormal frame $\{e_0, e_1, e_2\}$, with e_0 time-like, such that the Lie algebra of G is one of the following:*

  1. *$\mathfrak{g}_1$:*
  $$\begin{aligned} [e_1, e_2] = a\, e_1 - b\, e_0\,, \quad [e_1, e_0] = -a\, e_1 - b\, e_2\,, \\ [e_2, e_0] = b\, e_1 + a\, e_2 + a\, e_0\,, \quad a \neq 0\,. \end{aligned} \tag{7.266}$$
  In this case $\mathrm{G} \simeq \widetilde{\mathrm{SL}}(2,\mathbb{R})$ if $b \neq 0$, while $\mathrm{G} \simeq \widetilde{\mathrm{E}}(1,1)$ if $b = 0$.

  2. *$\mathfrak{g}_2$:*
  $$\begin{aligned} [e_1, e_2] = -c\, e_2 - b\, e_0\,, \quad [e_1, e_0] = -b\, e_2 + c\, e_0\,, \\ [e_2, e_0] = a\, e_1\,, \quad c \neq 0\,. \end{aligned} \tag{7.267}$$
  In this case $\mathrm{G} \simeq \widetilde{\mathrm{SL}}(2,\mathbb{R})$ if $a \neq 0$, while $\mathrm{G} \simeq \widetilde{\mathrm{E}}(1,1)$ if $a = 0$.

  3. *$\mathfrak{g}_3$:*
  $$[e_1, e_2] = -c\, e_0\,, \quad [e_1, e_0] = -b\, e_2\,, \quad [e_2, e_0] = a\, e_1\,, \tag{7.268}$$
  In this case the isomorphism type of G is listed in the following table.

Simply connected unimodular groups with Lie algebra $\mathfrak{g}_3$			
Lie group G	a	b	c
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	+	+	+
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	+	−	−
$\mathrm{SU}(2)$	+	+	−
$\widetilde{\mathrm{E}}(2)$	+	+	0
$\widetilde{\mathrm{E}}(2)$	+	0	−
$\widetilde{\mathrm{E}}(1,1)$	+	−	0
$\widetilde{\mathrm{E}}(1,1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

4. $\mathfrak{g}_4$:

$$[e_1, e_2] = -e_2 + (2\mu - b)e_0\,, \quad [e_1, e_0] = -b\, e_2 + e_0\,, \\ [e_2, e_0] = a\, e_1\,, \quad \mu \in \mathbb{Z}_2\,. \tag{7.269}$$

In this case the isomorphism type of G *is listed in the following tables.*

Simply connected unimodular groups with Lie algebra $\mathfrak{g}_4$ and $\mu = 1$		
Lie group G	a	b
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	$\neq 0$	$\neq 1$
$\widetilde{\mathrm{E}}(1,1)$	0	$\neq 1$
$\widetilde{\mathrm{E}}(1,1)$	< 0	1
$\widetilde{\mathrm{E}}(2)$	> 0	1
H_3	0	1

Simply connected unimodular groups with Lie algebra $\mathfrak{g}_4$ and $\mu = -1$		
Lie group G	a	b
$\widetilde{\mathrm{SL}}(2,\mathbb{R})$	$\neq 0$	$\neq -1$
$\widetilde{\mathrm{E}}(1,1)$	0	$\neq -1$
$\widetilde{\mathrm{E}}(1,1)$	> 0	-1
$\widetilde{\mathrm{E}}(2)$	< 0	-1
H_3	0	-1

- G *is non-unimodular and there exists an orthonormal frame* $\{e_0, e_1, e_2\}$*, with* e_0 *time-like, such that the Lie algebra of* G *is one of the following:*

1. $\mathfrak{g}_5$:

$$[e_1, e_2] = 0\,, \quad [e_1, e_0] = a\, e_1 + b\, e_2\,, \quad [e_2, e_0] = c\, e_1 + d\, e_2\,, \\ a + d \neq 0\,, \quad a\, c + b\, d = 0\,. \tag{7.270}$$

We denote the unique connected and simply connected Lie group with Lie algebra $\mathfrak{g}_5$ *as* $\mathfrak{G}_5$.

2. $\mathfrak{g}_6$:

$$[e_1, e_2] = a\, e_2 + b\, e_0\,, \quad [e_1, e_0] = c\, e_2 + d\, e_0\,, \quad [e_2, e_0] = 0\,, \\ a + d \neq 0\,, \quad a\, c - b\, d = 0\,. \tag{7.271}$$

We denote the unique connected and simply connected Lie group with Lie algebra $\mathfrak{g}_6$ *as* $\mathfrak{G}_6$.

3. $\mathfrak{g}_7$:

$$[e_1, e_2] = -a\, e_1 - b\, e_2 - b\, e_0\,, \quad [e_1, e_0] = a\, e_1 + b\, e_2 + b\, e_0\,, \\ [e_2, e_0] = c\, e_1 + d\, e_2 + d\, e_0\,, a + d \neq 0\,, \quad ac = 0\,. \tag{7.272}$$

We denote the unique connected and simply connected Lie group with Lie algebra $\mathfrak{g}_7$ as $\mathfrak{G}_7$.

Non-unimodular Lie algebras can be interpreted as deformations of the unimodular ones, with deformation parameter given by:

$$\beta \overset{\text{def.}}{=} a + d\,. \tag{7.273}$$

This is due to the fact that in the limit $\beta \to 0$ all the previous non-unimodular Lie algebras reduce to unimodular Lie algebras [660]. Consider as an example the Lie algebra $\mathfrak{g}_6$ in the limit $\beta \to 0$. In such limit the Lie brackets of $\mathfrak{g}_6$ reduce to:

$$[e_1, e_2] = a\, e_2 + b\, e_0\,, \quad [e_1, e_0] = c\, e_2 - a\, e_0\,, \quad [e_2, e_0] = 0\,, \quad a\,(c+b) = 0\,. \tag{7.274}$$

The specific unimodular algebras occurring in the limit $\beta \to 0$ of $\mathfrak{g}_6$ can be easily identified in some particular cases:

- If $a = 0$, we obtain:

  $$[e_1, e_2] = b\, e_0\,, \quad [e_1, e_0] = c\, e_2\,, \quad [e_2, e_0] = 0\,. \tag{7.275}$$

  Comparing with the classification of unimodular Lie algebras, we conclude that the case $cb > 0$ corresponds to the Lie algebra of $\widetilde{\mathrm{E}}(1,1)$, the case $cb < 0$ corresponds to the Lie algebra of $\widetilde{\mathrm{E}}(2)$, the case $c \neq 0\,, b = 0$ or $c = 0\,, b \neq 0$ corresponds to the Lie algebra of H_3 and the case $c = b = 0$ corresponds to $\mathbb{R}^3$.

- If $a \neq 0$ then we must have $c = -b$. If in addition we set $c = 0$, the brackets take the form:

  $$[e_1, e_2] = a\, e_2\,, \quad [e_1, e_0] = -a\, e_0\,, \quad [e_2, e_0] = 0\,. \tag{7.276}$$

  Comparing with the classification of unimodular Lie algebras, we conclude that this algebra is isomorphic to $\widetilde{\mathrm{E}}(1,1)$.

- Also, if $a = b = 1$ and $c = -b = -1$ the brackets read:

  $$[e_1, e_2] = e_0 + e_2\,, \quad [e_1, e_0] = -e_0 - e_2\,, \quad [e_2, e_0] = 0\,. \tag{7.277}$$

  which defines a Lie algebra isomorphic to H_3.

Therefore, the non-unimodular Lie group $\mathfrak{G}_6$ can be understood as a deformation of various unimodular groups such as $\widetilde{\mathrm{E}}(1,1)$, $\widetilde{\mathrm{E}}(2)$ or H_3. Similar remarks hold for $\mathfrak{G}_5$ and $\mathfrak{G}_7$.

Appendix 7.B Curvature of left-invariant metrics on Lorentzian Lie groups

Let (G, g) be a three-dimensional Lorentzian Lie group with Lie algebra $\mathfrak{g}$. Fix a global left-invariant frame $\{e_0, e_1, e_2\}$ on G. We have:

$$[e_0, e_1] = ae_0 + be_1 + ce_2\,, \quad [e_1, e_2] = de_0 + fe_1 + he_2 \quad [e_0, e_2] = ge_0 + je_1 + ke_2\,, \tag{7.278}$$

where $a, b, c, d, f, h, g, j, k \in \mathbb{R}$ such that the Jacobi identity is satisfied. In this case, imposing $[[e_0, e_1], e_2] + [[e_2, e_0], e_1] + [[e_1, e_2], e_0] = 0$ yields the constraints

$$bd + kd - fa - hg = 0\,, \quad ja - gb + kf - hj = 0\,, \quad ak + hb - gc - fc = 0\,. \tag{7.279}$$

Using Koszul formula, we compute the following covariant derivatives:

$$\nabla_{e_0} e_0 = a e_1 + g e_2\,,\ \nabla_{e_0} e_1 = a e_0 + \frac{c - j + d}{2} e_2\,,\ \nabla_{e_0} e_2 = g e_0 + \frac{j - c - d}{2} e_1\,, \tag{7.280}$$

$$\nabla_{e_1} e_0 = -b e_1 + \frac{d - c - j}{2} e_2\,,\ \nabla_{e_1} e_1 = -b e_0 - f e_2\,,\ \nabla_{e_1} e_2 = \frac{d - c - j}{2} e_0 + f e_1\,, \tag{7.281}$$

$$\nabla_{e_2} e_0 = -\frac{j + d + c}{2} e_1 - k e_2\,,\ \nabla_{e_2} e_1 = -\frac{j + d + c}{2} e_0 - h e_2\,,\ \nabla_{e_2} e_2 = -k e_0 + h e_1\,. \tag{7.282}$$

Using the previous covariant derivatives we compute the Riemann curvature tensor (with the convention $\mathrm{R}^g(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$):

$$\begin{aligned}
\mathrm{R}^g(e_0, e_1)e_0 &= \left[- a^2 + b^2 - gf + \frac{1}{4}(-c + d - j)(j - c - d) + \frac{c}{2}(j + d + c) \right] e_1 \qquad (7.283)\\
&\quad + [-bd + af - ag + ck + bj] e_2\,,\\
\mathrm{R}^g(e_0, e_1)e_1 &= \left[- a^2 + b^2 - gf + \frac{1}{4}(-c + d - j)(j - c - d) + \frac{c}{2}(j + d + c) \right] e_0 \qquad (7.284)\\
&\quad + [-bg + bf - ad + hc + aj] e_2\,,\\
\mathrm{R}^g(e_0, e_1)e_2 &= [fa - ag + bj - bd + ck] e_0 + [ad - aj + gb - bf - ch] e_1\,, \qquad (7.285)\\
\mathrm{R}^g(e_1, e_2)e_0 &= [-kf + fb - da + hc + hj] e_1 + [-bh + hk - dg + fj + fc] e_2\,, \qquad (7.286)\\
\mathrm{R}^g(e_1, e_2)e_1 &= [hc + hj - fk - da + fb] e_0 \qquad (7.287)\\
&\quad + \left[- bk + f^2 + h^2 - \frac{d}{2}(c - j + d) - \frac{1}{4}(d + j + c)(-c + d - j) \right] e_2\,,\\
\mathrm{R}^g(e_1, e_2)e_2 &= [-bh + fj + fc - dg + hk] e_0 \qquad (7.288)\\
&\quad + \left[bk - f^2 - h^2 + \frac{d}{2}(c - j + d) + \frac{1}{4}(d + j + c)(-c + d - j) \right] e_1\,,\\
\mathrm{R}^g(e_0, e_2)e_0 &= [kc + kd - gh - ga + jb] e_1 \qquad (7.289)\\
&\quad + \left[ah - g^2 + k^2 - \frac{j}{2}(-c + d - j) - \frac{1}{4}(j + d + c)(c - j + d) \right] e_2\,,\\
\mathrm{R}^g(e_0, e_2)e_1 &= [-hg - ga + jb + kc + kd] e_0 + [ak + jf + kh - gc - gd] e_2\,, \qquad (7.290)\\
\mathrm{R}^g(e_0, e_2)e_2 &= \left[\frac{1}{4}(j - c - d)(d + j + c) - g^2 + k^2 + \frac{j}{2}(-d + c + j) + ah \right] e_0 \qquad (7.291)\\
&\quad + [-ka + gd + gc - fj - kh] e_1\,,
\end{aligned}$$

From here, we obtain the Ricci curvature tensor Ric^g, which we remind it is defined as $\mathrm{Ric}^g(u, v) = \mathrm{Tr}\,(w \to \mathrm{R}^g(w, u)v)$ (following the conventions of the Second Part of the thesis):

$$\mathrm{Ric}^g(e_0, e_0) = a^2 - b^2 + gf + g^2 - k^2 - ah + \frac{d^2}{2} - cj - \frac{c^2}{2} - \frac{j^2}{2}\,, \tag{7.292}$$

$$\mathrm{Ric}^g(e_0, e_1) = bh - fj - fc + dg - hk\,, \tag{7.293}$$

$$\mathrm{Ric}^g(e_0, e_2) = hc + hj - fk - da + fb\,, \tag{7.294}$$

$$\mathrm{Ric}^g(e_1, e_1) = -a^2 + b^2 - f^2 - h^2 - fg + bk - \frac{j^2}{2} + dc + \frac{c^2}{2} + \frac{d^2}{2}\,, \tag{7.295}$$

$$\mathrm{Ric}^g(e_1, e_2) = fa - ag + bj - bd + ck\,, \tag{7.296}$$

$$\mathrm{Ric}^g(e_2, e_2) = -g^2 + k^2 + ah + kb - f^2 - h^2 + \frac{j^2}{2} - jd + \frac{d^2}{2} - \frac{c^2}{2}\,. \tag{7.297}$$

Finally, the scalar curvature Scal^g reads

$$\begin{aligned}\mathrm{Scal}^g &= -2a^2 + 2b^2 - 2gf - 2g^2 + 2k^2 + 2ah + \\ &\frac{d^2}{2} + cj + \frac{c^2}{2} + \frac{j^2}{2} - 2f^2 - 2h^2 + 2bk + dc - jd\,.\end{aligned} \tag{7.298}$$

Conclusions/Conclusiones

Conclusions and Future Directions

Conclusions and Executive Summary

In this thesis we have investigated the physics and geometry of gravity at high energies. Firstly, we have explored its physical behavior in this regime through the study of higher-order gravities, with a special emphasis in those with non-minimal couplings to electromagnetism. Secondly, we have probed geometric aspects of gravity at high energies via the examination of mathematical structures of interest arising in the context of Supergravity and String Theory (ST).

Higher-order gravities

In the First Part of the thesis, we carried out a detailed analysis of a special class of higher-order gravities. In particular, we studied both purely gravitational higher-curvature theories and higher-order gravities with a non-minimally coupled U(1) gauge vector field of the (Generalized) Quasitopological type, defined as those admitting static and spherically symmetric (SSS) solutions characterized by a single function $f(r) = -g_{tt} = 1/g_{rr}$ with (at most) second-order equation of motion.

We began with the study of Generalized Quasitopological Gravities (GQGs), which are purely gravitational theories. We reviewed other intriguing properties of these theories, such as having second-order linearized equations around maximally symmetric backgrounds, possessing a continuous and well-defined Einstein limit or allowing for the exact computation of the thermodynamic properties of the subsequent (SSS) black holes. Given these interesting features, together the (recently proven) fact that GQGs exist at every order and spacetime dimension, we embarked on a highly non-trivial mission: showing that GQGs span all higher-curvature gravities once field redefinitions of the metric are considered. This would provide GQGs a fundamental importance within the set of higher-order gravities and justify once and for all their relevance.

Interestingly enough, we succeeded in this assignment, being able to prove rigorously that all higher-curvature gravities composed of terms which either contain no covariant derivatives of the curvature or, otherwise, possess two and only two covariant derivatives or belong to the eight-derivative level at most can be mapped via perturbative field redefinitions to a GQG. Claiming the latter to hold for any higher-order theory of gravity remains as a conjecture. In any case, this suggests that the physics of black holes in generic higher-curvature gravities is captured by their GQG counterparts, dramatically easier to deal with. We illustrate this fact with the gravity sector of Type IIB ST in $\mathrm{AdS}_5 \times S^5$ at order $\mathcal{O}(\alpha'^3)$, building the explicit (perturbative) map into a GQG and showing that the thermodynamic properties of black holes in both frames match.

Next we were interested in extrapolating, somehow, the notion of GQGs into the realm of higher-order gravities with a non-minimally coupled vector field. Indeed, higher-derivative effective actions including gravity and electromagnetism are expected to incorporate all types of couplings between the curvature and the gauge field strength and, furthermore, non-minimal couplings could trigger new effects and phenomena, so they are worth exploring. We showed that such generalization is possible, calling the subsequent theories Electromagnetic (Generalized) Quasitopological Gravities (E(G)QGs).

They can be divided into two subclasses: those with algebraic equation of motion for $f(r)$

(EQGs) and those which have a second-order equation (EGQGs). Working in four dimensions, we were able to identify two infinite families of EQGs of arbitrary order in the curvature and the gauge field strength. We studied magnetically-charged black hole solutions whose thermodynamics we accessed analytically. We observed that, quite generally, the singularity at the core of the black hole is regularized by the higher-derivative corrections, producing in turn a globally regular geometry. On the other hand, we managed to find an infinite family of proper EGQGs as well. Although not deriving the explicit charged black hole solutions, we could explore their thermodynamic properties analytically. In both cases, we focused on extremal black holes and studied, among other physical aspects, the corrections to the extremal charge-to-mass ratio at a non-perturbative level.

Afterwards, through the dualization of EQGs possessing magnetic solutions with globally regular geometry, we were able to discover a non-minimal higher-derivative extension of Einstein-Maxwell theory in which electrically-charged black holes and point charges have completely regular gravitational and electromagnetic fields. In particular, we obtained an exact SSS solution of this theory reducing to the Reissner-Nordström one at weak coupling, but regularizing the singularity at $r = 0$ for arbitrary mass and non-vanishing charge. To the best of our knowledge, this was the first explicit example of a theory that fully regularizes both gravitational and electromagnetic fields, thus showing that higher-order corrections are capable of resolving GR singularities.

Later we concentrated on the study of higher-order theories of gravity and electromagnetism which are invariant under electromagnetic duality rotations, admitting the presence of non-minimal couplings. Indeed, symmetries conform perhaps the most fundamental guiding principle in theoretical physics, providing the natural fields and variables to consider within the study of a physical system as well as constraining the type of terms that may appear in the (effective) action. Therefore, since Einstein-Maxwell theory is already invariant under SO(2) duality rotations, it is reasonable to seek for higher-derivative extensions which respect this invariance.

In this context, first we worked in a derivative expansion of the action and classified all Lagrangians coming from the truncation of an exactly duality-invariant theory up to the eight-derivative level. Then we investigated the effect of field redefinitions and showed that, to six derivatives, the most general duality-invariant theory can be mapped to Maxwell theory minimally coupled to a purely gravitational higher-order gravity, what motivated us to conjecture that this occurs at all orders. We also studied charged black hole solutions in this special six-derivative theory and explored additional constraints on the couplings imposed by the Weak Gravity Conjecture (WGC).

In an attempt to derive exact results, we decided to restrict ourselves to the study of higher-order gravities with quadratic dependence on a non-minimally coupled Maxwell field strength. Remarkably enough, we managed to obtain a closed form for the action of all such theories which are duality-invariant, observing a highly degree of (formal) resemblance with Born-Infeld Lagrangians. We examined the SSS black hole solutions of the simplest of these exactly duality-invariant theories and, focusing on extremal black holes, determined analytically both the near-horizon geometry and the entropy, which is given by the Einstein-Maxwell value plus a constant correction. We theorized this very simple expression for the entropy was due to duality invariance.

The First Part of the thesis concluded with the study of holographic aspects of any-dimensional EQGs, finding infinite examples of these theories to every order in the curvature and the field strength. Their very special structure allowed us to obtain analytic and fully non-perturbative results, thus paving the way to access different universality classes of dual Conformal Field Theories (CFTs).

For the sake of concreteness, we restricted ourselves to the more generic four-derivative EQG (in arbitrary dimension). We computed the coefficients of the correlators $\langle JJ \rangle$ and $\langle TJJ \rangle$, which are given in terms of the couplings of EQG densities. Similarly, we investigated the constraints coming from CFT unitarity and positivity of energy (which we found to be equivalent to those arising from demanding causality in the bulk) and from the WGC. Later we focused on charged Rényi entropies (RE), which are appropriate generalizations of the standard RE in the presence of global symmetries, and observed that the usual properties of RE are preserved if the aforementioned physical constraints are fulfilled.

As in the uncharged case, charged RE contain naturally a notion of charged entanglement entropy. We explored this magnitude further in more generality and discovered that for a general $d(\geq 3)$-dimensional CFT, the first correction with respect to the uncharged entanglement entropy across a spherical entangling surface appears at quadratic order in the chemical potential and is positive definite and universally controlled by the coefficients determining the correlators $\langle JJ \rangle$ and $\langle TJJ \rangle$ of the theory. This result was motivated by analytic holographic calculations for any-dimensional and any-derivative EQGs and for free fields in $d = 4$.

The key concepts to remember from the First Part of the thesis are shown below.

Executive Summary of the First Part

- GQGs provide a spanning set of the space of all higher-curvature gravities $\mathcal{L}(g^{\mu\nu}, R_{\mu\nu\rho\sigma})$.
- E(G)QGs have been discovered and the first examples identified.
- The first theory regularizing the Reissner-Nordström singularities has been found.
- Duality-invariant theories, up to six-derivatives, are equivalent to a gravitational theory with a minimally coupled U(1) vector field.
- All exactly duality-invariant higher-order theories quadratic in the Maxwell field strength have been classified.
- Holographic aspects of any-dimensional EQGs have been derived in a fully non-perturbative fashion.
- A universal relation regarding the (charged) entanglement entropy in general $d(\geq 3)$-dimensional CFTs has been discovered.

Geometric aspects of Supergravity and String Theory

In the Second Part of the thesis, we presented a study of the geometry of Supergravity and ST. Evidently, geometric structures are ubiquitous in realm of high-energy gravity, so it is an unrealizable task that of trying to cover all their appearances in a single work. Instead, we considered more intriguing to focus on various aspects of the interplay between gravity and geometry at high energies which could help us acquire a global picture of this connection. With this in mind, we started with the investigation of parallel spinors on globally hyperbolic four-manifolds, then we obtained classification results of self-dual Einstein four-manifolds which admit an isometric and principal action of the Heisenberg group and finally we showed how special classes of contact structures can be used for the construction of (novel) Supergravity solutions.

We studied first real parallel spinors on globally hyperbolic four-manifolds, motivated by several reasons. Firstly, one could argue a purely mathematical interest. Secondly, they are GR solutions, either in vacuum or, more generally, in presence of a pressureless null dust. And thirdly, they provide supersymmetric solutions of pure $\mathcal{N} = 1$, $D = 4$ Supergravity, allowing us to explore the global geometric properties of supersymmetric spacetimes.

We began by exposing the general theory of real parallel spinors on globally hyperbolic four-manifolds and next we reformulated the associated initial value problem in terms of a first-order system of partial differential equations for a family of functions and coframes on an appropriate (three-dimensional) Cauchy surface, defining the so-called parallel spinor flow. This is extremely useful since it allowed us to show that the parallel spinor flow preserves the vacuum momentum and Hamiltonian constraints, meaning that the parallel spinor and the Einstein flows coincide on common initial data. In turn, this provided an initial data characterization of real parallel spinors on Ricci flat Lorentzian four-manifolds.

Afterwards we investigated the subsequent constraint equations of the parallel spinor flow, which define the parallel Cauchy differential system. More concretely, we studied the topology and geometry of the Cauchy surfaces admitting solutions to this system and characterized all left-invariant solutions on three-dimensional simply connected Lie groups. We used this result to classify all left-invariant parallel spinor flows on simply connected Cauchy surfaces allowing for a Lie group structure and finally we examined a special class of parallel spinor flows in which the family of functions is just a constant.

Secondly, we moved to the investigation of self-dual Einstein four-manifolds with an isometric and principal action of the three-dimensional Heisenberg group H. This was motivated by the discovery that the isometry group of the one-loop deformed universal hypermultiplet, arising in the context of scalar manifolds of four-dimensional Supergravity (or in compactifications of Type II strings), is given by $\mathrm{O}(2) \ltimes \mathrm{H}$. However, if one reduces the isometry group down to H, the problem becomes too general to obtain explicit results, so we decided to add the requirement of self-duality of the Weyl tensor and focus on the classification of all such (pseudo-)Riemannian manifolds.

We started with the case of non-vanishing Einstein constant, which corresponds to quaternionic (para)Kähler manifolds. We carried out the complete classification of all of them and, apart from finding the one-loop deformed universal hypermultiplet metrics (as well as neutral-signature versions of it), we encountered positively-curved (resp. negatively-curved) Riemannian (resp. neutral-signature) counterparts. Additionally, we established when the subsequent metrics were geodesically (in)complete.

Then we analyzed the case of Ricci flat metrics, which we identified with (para)hyperKähler geometries. We classified all such manifolds and proved that they are all incomplete except in the case in which the Heisenberg center is lightlike, which is isometric to flat space.

Finally, we concluded with a study of particular classes of contact structures and of their utility for the construction of solutions of Supergravity. This work should be contextualized within the program of classifying Supergravity solutions and the corresponding moduli spaces of solutions in a given spacetime dimension, which stands as an open problem in the (mathematical) theory of Supergravity. We opted to consider the specific case of minimal six-dimensional Supergravity coupled to a tensor multiplet with constant dilaton. Moreover, as a first approach to the general classification problem, which currently seems inaccessible, we restricted ourselves to the special class of solutions corresponding to the direct product of three-dimensional Lorentzian and Riemannian manifolds.

With this in mind, we first defined the concept of ε-contact structures, which encompasses the usual three-dimensional contact Riemannian, contact Lorentzian and para-contact metric structures, but which also includes the possibility of a null Reeb vector field. We called this latter case null contact structure and introduced the notions of Sasakian and K-contact null contact structures, showing that the associated K-contact condition is stronger than Sasakianity.

Later we defined the $\varepsilon\eta$-Einstein contact structures, which include particular cases of the standard (three-dimensional) η-Einstein Riemannian, Lorentzian and para-contact structures, as well as a subclass of null contact structures. We provided a complete classification of all of them in three dimensional simply connected Lie groups. Next we explicitly showed the procedure by which the appropriate combination of $\varepsilon\eta$-Einstein contact structures produces solutions of minimal six-dimensional Supergravity coupled to a tensor multiplet with constant dilaton, which can be equivalently understood as six-dimensional Lorentzian manifolds with a Ricci flat metric-compatible connection with isotropic, totally skew-symmetric, closed and co-closed torsion. We concluded by illustrating particular instances of solutions, which we interpreted as (generically non-supersymmetric) deformations of the usual supersymmetric solution on $\mathrm{AdS}_3 \times S^3$.

In a nutshell, the essential contents of the Second Part of the thesis are the following.

Executive Summary of the Second Part

- The Cauchy problem for a real parallel spinor on a globally hyperbolic four-manifold can be reformulated as a system of differential equations for a family of functions and coframes on a Cauchy surface.
- At least in four spacetime dimensions, the parallel spinor and the (vacuum) Einstein flows coincide on common initial data.
- All self-dual Einstein four-manifolds invariant under a principal and isometric action of the three-dimensional Heisenberg group with non-degenerate orbits have been classified.
- Null contact structures have been identified, defining the associated notions of Sasakianity and K-contactness.
- Appropriate combinations of $\varepsilon\eta$-Einstein contact structures produce solutions of six-dimensional minimal Supergravity.

Future directions

It is a well-known paradoxical phenomenon in physics (and science in general) that the more we think we know, the more we realize we do not know. This thesis has not circumvented this (fortunate[1]) problem, leaving behind a plethora of open questions. Some of them have been already mentioned along the text, but we have found convenient to (re)write explicitly the most intriguing future directions we may think of as of this moment:

1. Prove (or disprove) Conjectures 1.1 and 1.2.
2. Characterize more accurately the set of all GQGs at every order and dimension, in the light of [251]. In particular, could we find the explicit covariant form of all inequivalent algebraic GQGs at every order and dimension?
3. Study explicit black hole solutions of proper GQGs.
4. Would it be possible to show the existence of GQGs (or other purely gravitational theories) admitting simple enough rotating black hole solutions whose expression we may determine analytically (or through a certain differential equation which is numerically manageable)?
5. Find purely gravitational higher-order theories with fully regular SSS solutions (*i.e.,* regularizing the Schwarzschild singularity).
6. Examine the possibility of defining duality rotations for purely gravitational theories (perhaps as suggested in the recent work [704]) and study which theories might be (exactly) invariant under this duality.
7. Determine the most general structure of E(G)QG Lagrangians.
8. Explore the applicability of E(G)QGs to the cosmological setting or the existence of E(G)QGs with Taub-NUT solutions.
9. Investigate the effects of field redefinitions in E(G)QGs. Could any higher-order gravity and electromagnetism be mapped, through appropriate field redefinitions, to an E(G)QG, in analogy with the situation for purely gravitational theories?
10. Find the necessary and sufficient conditions for any higher-order theory of gravity and electromagnetism to be (perturbatively) duality-invariant (that is, extend the result obtained in Chapter 3 at the eight-derivative level).
11. Prove (or disprove) Conjecture 3.1.
12. Obtain higher-order theories not necessarily quadratic in the Maxwell field strength which are exactly invariant under (electromagnetic) duality rotations.
13. Even restricting ourselves to exactly duality-invariant theories which are quadratic in the Maxwell field strength, study the subsequent (extremal) black holes and their entropy.
14. Why had the entropy (3.209) such a simple expression? We suspected it was due to duality invariance, but it would be gratifying to prove this fact from first principles.

[1] Otherwise, no more PhD theses would be written.

15. Find exact SSS solutions of exactly duality-invariant theories. Could we encounter instances of these theories with regular SSS solutions?

16. Following the previous question, is it possible to identify E(G)QGs which are furthermore (exactly) duality-invariant?

17. Study the holographic aspects of the any-dimensional and any-order EQGs constructed in (4.255) and (4.256).

18. Identify examples of proper EGQGs in arbitrary dimensions and analyze their holographic properties. Could this be done analytically, as for the EQGs given by (4.30)?

19. Explore in more detail the relation between causality constraints in the bulk and unitarity constraints in the boundary CFT (in particular, for more general theories than (4.30)).

20. Understand better the WGC in AdS space, especially the implications of this conjecture for the dual CFT.

21. Regarding the universal formula (4.252), it would be interesting to rederive it using the formalism developed in [620] or find additional two-dimensional counterexamples to (4.252) through the investigation of three-dimensional EQGs [619].

22. Extend the results of Chapter 5 for the case of a real Killing spinor or more generic ones.

23. Using the connection between parallel spinors and pp-waves, interpret physical and geometric aspects of the associated pp-waves in terms of the parallel spinor flow formalism.

24. Investigate further the possibility of using first-order hyperbolic spinorial flows to construct special solutions of GR.

25. In connection with the previous point, examine in more detail the precise type of three-dimensional curvature flows one obtains on the Cauchy surface.

26. Construct generic non left-invariant parallel spinor flows, perhaps focusing on the subclass of comoving parallel spinor flows.

27. Is it possible to provide a (partial) classification result for all Einstein four-manifolds admitting an isometric and principal action of the three-dimensional Heisenberg group with non-degenerate orbits (that is, removing the self-duality condition)? And if we impose further the (para)Kähler condition?

28. Derive explicit Lorentzian examples of (Einstein) Heisenberg four-manifolds.

29. Can we give an interpretation of all self-dual Einstein Heisenberg four-manifolds within ST and Supergravity?

30. In the framework of Chapter 6, consider other principal and isometric actions provided by different Lie groups. Would they have any ST and Supergravity embedding?

31. Again, within the context of Chapter 6, analyze the possibility of orbits with induced degenerate metrics or classify (para)quaternionic Kähler and (para)hyperKähler manifolds with Heisenberg symmetry in dimensions $D = 4n$ with $n > 1$.

32. Study the possibility of generalizing the results of Chapter 7 to ten-dimensional Type IIB Supergravity, when truncating all fields but the metric and the (self-dual) five-form.

33. Determine explicitly if the solutions of minimal six-dimensional Supergravity we obtained, interpreted as geometric and topological deformations of the canonical $\mathrm{AdS}_3 \times S^3$ supersymmetric solution, are indeed supersymmetric or not.

34. Continue the investigation of null contact structures and generalize this notion for arbitrary (odd) dimension.

35. Go ahead with the classification of all $\varepsilon\eta$-Einstein contact structures, not restricting ourselves to left-invariant ones. In particular, this could be intriguing in the case of null contact structures, because of the very peculiar form of the subsequent stress-energy tensor.

36. In the context of (minimal) six-dimensional Supergravity, explore different ansätze of solutions, like admitting warped products or, modifying more drastically the starting point, considering direct products of Lorentzian four-dimensional manifolds and (two-dimensional) Riemannian surfaces.

The previous proposals are just some natural future directions arising from the research carried out in this thesis, but there are of course many other open questions[2] which could be interesting as well. We hope to address at least some of them in the near future.

[2]Perhaps not even identified by the author as of the moment of writing this document.

Conclusiones y Direcciones Futuras

Conclusiones y Compendio de Resultados

En esta tesis doctoral se ha realizado un estudio de la física y geometría de la gravedad a altas energías. Primero, hemos explorado su comportamiento físico en este régimen mediante las gravedades de orden superior, con especial énfasis en aquellas con acoplamientos no mínimos a electromagnetismo. En segundo lugar, hemos inspeccionado aspectos geométricos de la gravedad a altas energías mediante el análisis de estructuras matemáticas de interés que surgen en el contexto de Supergravedad y Teoría de Cuerdas (TC).

Gravedades de orden superior

En la Primera Parte de la tesis, realizamos un detallado estudio de una clase particular de gravedades de orden superior. En concreto, investigamos tanto teorías de orden superior puramente gravitacionales como gravedades de orden superior con un vector gauge U(1) no mínimamente acoplado que pertenecen a la tipología Cuasitopológica (Generalizada). Se caracterizan por admitir soluciones estáticas esféricamente simétricas (EES) determinadas por una única función $f(r) = -g_{tt} = 1/g_{rr}$ con ecuación de movimiento de segundo orden (como máximo).

Comenzamos con el estudio de las correspondientes Gravedades Cuasitopológicas Generalizadas (GCGs). Revisamos varias propiedades atractivas que estas teorías poseen por definición, tales como ecuaciones linealizadas de segundo orden en fondos máximamente simétricos, un límite a Relatividad General continuo y bien definido o permitir el cálculo exacto de la termodinámica de los subsiguientes agujeros negros (EES). Dadas estas propiedades sumamente interesantes, junto con el hecho (recientemente probado) de que existen GCGs a todo orden y en todas dimensiones, decidimos embarcarnos en una difícil misión: demostrar que las GCGs generan todas las gravedades de orden superior si se tienen en cuenta redefiniciones de la métrica. Tal peculiaridad otorgaría a las GCGs una importancia fundamental y justificaría de una vez por todas su relevancia.

Logramos llevar a cabo dicha empresa y fuimos capaces de mostrar que todas las gravedades de orden superior compuestas de términos que o bien no contienen derivadas covariantes de la curvatura o, en caso contrario, poseen dos y solo dos derivadas covariantes o tienen ocho derivadas como máximo se pueden reescribir mediante redefiniciones de campo perturbativas como GCGs. La extensión de dicha aseveración para cualquier teoría de orden superior gravitatoria permanece a día de hoy en el ámbito conjetural. De todas formas, la anterior observación sugiere que la física de agujeros negros en teorías genéricas de orden superior viene capturada por las GCGs asociadas, mucho más manejables. Ilustramos esta propiedad con el sector gravitacional de la TC Tipo IIB en $\mathrm{AdS}_5 \times S^5$ a orden $\mathcal{O}(\alpha'^3)$, construyendo asimismo el mecanismo explícito para reformularlo como una GCG y mostrando la equivalencia de la termodinámica de agujeros negros en ambos contextos.

A continuación, nos interesamos en extrapolar, de algún modo, la noción de GCGs al ámbito de las teorías de orden superior dotadas de un vector acoplado no mínimamente. En efecto, se espera que las acciones efectivas con derivadas superiores que incluyen gravedad y electromagnetismo incorporen todos los tipos posibles de acoplamientos entre la curvatura y el campo gauge, que podrían generar nuevos efectos y fenómenos interesantes. Por lo tanto, queda patente la necesidad de explorar esta dirección. Afortunadamente, demostramos que tal generalización es posible, hallando así los primeros ejemplos no triviales de Gravedades Electromagnéticas Cuasitopológicas (Generalizadas) (GEC(G)s).

Se pueden dividir en dos grupos: aquellas con ecuación de movimiento algebraica para $f(r)$ (GECs) y aquellas con ecuación de segundo orden (GECGs). Trabajando en cuatro dimensiones, identificamos dos familias infinitas de GECs a orden arbitrario en la curvatura y en el campo gauge. Estudiamos agujeros negros con carga magnética y obtuvimos soluciones analíticas cuyas propiedades termodinámicas pudimos calcular de forma exacta. Asimismo, observamos que, en bastante generalidad, las correcciones de orden superior regularizan la singularidad en el centro del agujero negro, dando lugar a una geometría completamente regular. Por otra parte, también encontramos una familia infinita de GECGs propias. Aunque no derivamos los soluciones de tipo agujero negro con carga magnética explícitamente, fuimos capaces de explorar sus propiedades termodinámicas analíticamente. En ambos casos, nos centramos en agujeros negros extremos y exploramos, entre otros aspectos, las correcciones a la relación carga/masa extrema a nivel no perturbativo.

A continuación, mediante la dualización de GECs que poseen soluciones magnéticas con geometría completamente regular, descubrimos una extensión con derivadas superiores y acoplamientos no mínimos de la teoría de Einstein-Maxwell en la que tanto agujeros negros con carga eléctrica como cargas puntuales tienen campos electromagnéticos y gravitacionales completamente regulares. En particular, obtuvimos una solución EES de esta teoría que se reduce a la de Reissner-Nordström para acoplamiento débil, pero que además regulariza la singularidad en $r = 0$ para toda masa y carga no nula. Creemos que se trata del primer ejemplo explícito de teoría que regulariza totalmente el campo gravitacional y electromagnético, probando así que las correcciones de orden superior son capaces de regularizar singularidades de la Relatividad General (RG).

Después nos concentramos en el estudio de teorías de orden superior de gravedad y electromagnetismo con acoplamientos no mínimos que son invariantes bajo rotaciones de dualidad electromagnética. En efecto, las simetrías conforman quizá uno de los principios básicos más fundamentales de la Física Teórica, proporcionando las variables y campos naturales a considerar en el estudio de un sistema físico y restringiendo el tipo de términos que pueden aparecer en la acción (efectiva). Por lo tanto, puesto que la teoría de Einstein-Maxwell ya es invariante bajo rotaciones de dualidad SO(2), es razonable buscar extensiones con derivadas superiores que respeten dicha invariancia.

En este contexto, primero escribimos un desarrollo en derivadas superiores de la acción y clasificamos todos los Lagrangianos que provienen de la truncación hasta ocho derivadas de una teoría exactamente invariante bajo dualidad. Luego, investigamos el efecto de las redefiniciones de campo y mostramos que, hasta seis derivadas, la teoría más general invariante bajo dualidad se puede reformular como la teoría de Maxwell mínimamente acoplada a una teoría de orden superior de gravedad pura, lo que nos llevó a conjeturar que este fenómeno tiene lugar a todos los órdenes. También estudiamos soluciones cargadas de tipo agujero negro en esta teoría especial de seis derivadas y exploramos condiciones adicionales impuestas por la Conjetura de Gravedad Débil (CGD).

En un intento por obtener resultados exactos, optamos por restringirnos al estudio de gravedades de orden superior con dependencia cuadrática en el campo de Maxwell. Notablemente, logramos deducir una forma cerrada para la acción de todas tales teorías invariantes bajo dualidad, observando un alto grado de similitud (formal) con los Lagrangianos de Born-Infeld. Examinamos las soluciones EES de tipo agujero negro de la más simple de estas teorías con acoplamientos no mínimos y, centrándonos en agujeros negros extremos, determinamos analíticamente tanto la geometría próxima al horizonte como la entropía, que viene dada por el valor de Einstein-Maxwell más una corrección constante. Teorizamos que dicha expresión tan simple para la entropía se debía a invariancia bajo dualidad.

Concluimos la Primera Parte de la tesis con el estudio de aspectos holográficos de GECs en cualquier dimensión, las cuales pudimos identificar a todo orden en la curvatura y en el campo de Maxwell. La estructura especial de dichas teorías nos permitió obtener resultados analíticos y completamente no perturbativos, lo que nos facilitó el acceso a clases de universalidad diferentes de Teorías Conformes de Campos (TCCs).

Para fijar ideas, nos restringimos a la GEC más general con términos de cuatro derivadas (pero en dimensión arbitraria). Logramos calcular los coeficientes de los correladores $\langle JJ \rangle$ y $\langle TJJ \rangle$ en términos de los acoplamientos de las densidades GEC. Asimismo, investigamos las ligaduras provenientes de unitariedad de la TCC y de la positividad de la energía (que mostramos que son equivalentes a imponer causalidad en el espaciotiempo) y de la CGD. Luego nos centramos en la entropías de Rényi (ER) cargadas, que son generalizaciones de las ER estándar en presencia de simetrías globales. Observamos que se respetan las propiedades usuales de las ER si se satisfacen las ligaduras mencionadas.

Como en el caso sin carga, las ER cargadas incluyen una noción de entropía de entrelazamiento cargada. Indagamos acerca de esta magnitud en mayor generalidad y descubrimos que, para cualquier TCC en dimensión $d \geq 3$, la primera corrección con respecto a la entropía de entrelazamiento sin carga a través de una superficie de entrelazamiento esférica surge a segundo orden en el potencial químico, es definido positivo y está controlado de forma universal por los coeficientes que determinan los correladores $\langle JJ \rangle$ y $\langle TJJ \rangle$. Este resultado fue motivado por cálculos analíticos holográficos en GECs a todo orden y para toda dimensión y en teorías de campos libres en $d = 4$.

Las ideas fundamentales de la Primera Parte de la tesis se muestran a continuación.

Ideas clave de la Primera Parte

- Las GCGs forman un conjunto generador del espacio de todas las gravedades de orden superior $\mathcal{L}(g^{\mu\nu}, R_{\mu\nu\rho\sigma})$.
- Se han descubierto las GEC(G)s e identificado los primeros ejemplos.
- Se ha hallado la primera teoría que regulariza las singularidades de Reissner-Nordström.
- Hasta sexto orden, las teorías invariantes bajo dualidad son equivalentes a una teoría de gravedad con un vector U(1) mínimamente acoplado.
- Se han clasificado todas las teorías de orden superior exactamente invariantes bajo dualidad y cuadráticas en el campo de Maxwell.
- Se han estudiado aspectos holográficos de GECs en dimensión arbitraria y de forma completamente no perturbativa.
- Se ha descubierto una relación universal que concierne la entropía de entrelazamiento cargada y válida para toda TCC en dimensión $d \geq 3$.

Aspectos geométricos de la Supergravedad y la Teoría de Cuerdas

En la Segunda Parte de la tesis, presentamos un estudio de la geometría de Supergravedad y TC. Evidentemente, las estructuras geométricas en el ámbito de la gravedad a altas energías son omnipresentes, por lo que se trata de una tarea prácticamente irrealizable el intentar cubrir todas ellas en un solo trabajo. En cambio, consideramos más interesante centrarse en aspectos concretos de la relación entre geometría y gravedad a altas energías, con el objetivo de construir una imagen global de esta conexión. Por ello, comenzamos investigando espinores paralelos en variedades globalmente hiperbólicas, luego obtuvimos resultados de clasificación de cuatro-variedades Einstein autoduales que admiten una acción isométrica y principal del grupo de Heisenberg y finalmente mostramos cómo se pueden construir (nuevas) soluciones de Supergravedad seis-dimensional mediante clases especiales de estructuras de contacto.

Primero, estudiamos la existencia de espinores reales paralelos en variedades globalmente hiperbólicas por varios motivos. En primer lugar, se puede argüir un interés puramente matemático. En segundo lugar, resultan ser soluciones de RG en el vacío o, en mayor generalidad, en presencia de polvo de tipo luz sin presión. Y, en tercer lugar, proporcionan soluciones supersimétricas de Supergravedad pura $\mathcal{N} = 1$ en $D = 4$, lo que nos permite explorar las propiedades geométricas globales de espaciotiempos supersimétricos.

Empezamos con una exposición de la teoría general de espinores reales paralelos en cuatro-variedades globalmente hiperbólicas y, a continuación, reescribimos el problema asociado de valores iniciales en términos de un sistema de ecuaciones en derivadas parciales de primer orden para una familia de funciones y bases ortonormales del espacio cotangente en una superficie de Cauchy (tres-dimensional) apropiada. Dicha reformulación resulta muy útil, ya que nos permite demostrar que el flujo de espinor paralelo preserva tanto la ligadura Hamiltoniana como la de momento en el vacío, lo que implica que el flujo de Einstein y el de espinor paralelo coinciden en datos iniciales comunes. En consecuencia, obtuvimos una caracterización de datos iniciales de espinores reales paralelos en cuatro-variedades lorentzianas Ricci planas.

Después, investigamos las ecuaciones de ligadura correspondientes al flujo de espinor paralelo, que definen el sistema diferencial paralelo de Cauchy. En concreto, estudiamos la topología y geometría de superficies de Cauchy que admiten soluciones a este sistema y caracterizamos todas las soluciones invariantes por la izquierda en grupos de Lie tres-dimensionales simplemente conexos. Más tarde, usamos este resultado para clasificar todos los flujos de espinor paralelo invariantes por la izquierda en superficies de Cauchy con estructura de grupo de Lie y finalmente examinamos una clase especial de flujos de espinor paralelo en la que la familia de funciones viene dada por una constante.

En segundo lugar, investigamos cuatro-variedades Einstein autoduales con una acción isométrica y principal del grupo de Heisenberg tres-dimensional H. Dicha indagación está motivada por el descubrimiento de que el grupo de isometría del hipermultiplete universal a un bucle, que surge en el contexto de variedades escalares de Supergravedad en cuatro dimensiones (o en compactificaciones de cuerdas de Tipo II), viene dado por $\mathrm{O}(2) \ltimes \mathrm{H}$. En consecuencia, es relevante explorar qué tipo de variedades (pseudo-)riemannianas encontramos si reducimos el grupo de isometría a tan solo H. No obstante, el problema resulta demasiado general, por lo que estimamos conveniente añadir la condición de autodualidad del tensor de Weyl y centrarnos en la clasificación de todas tales variedades.

Comenzamos con el caso de constante de Einstein no nula, que se corresponde con variedades cuaterniónicas (para)Kähler. Llevamos a cabo una clasificación completa de todas ellas y, además de encontrar las métricas del hipermultiplete universal a un bucle (así como versiones de signatura neutra), hallamos homólogas riemannianas (resp. de signatura neutra) con curvatura positiva (resp. negativa). Además, establecimos cuándo las métricas asociadas eran geodésicamente (in)completas.

Posteriormente analizamos el caso de métricas Ricci planas, que de forma natural identificamos con geometrías (para)hiperKähler. Clasificamos todas tales variedades y demostramos que todas ellas son incompletas salvo en el caso en que el centro de Heisenberg es de tipo luz, que resulta ser isométrico al espacio plano.

Finalmente, concluimos con un estudio de clases particulares de estructuras de contacto y de su utilidad para la construcción de soluciones de Supergravedad. Este trabajo se debe entender dentro del programa de clasificación de soluciones de Supergravedad y sus espacios de módulos asociados para una dimensión espaciotemporal dada, que continúa siendo un problema abierto en la teoría (matemática) de la Supergravedad. Decidimos considerar el caso específico de Supergravedad mínima en seis dimensiones acoplada a un multiplete tensorial con dilatón constante. Además, como primer acercamiento al problema de clasificación general, cuya resolución todavía parece a día de hoy inaccesible, nos restringimos a la clase especial de soluciones correspondiente al producto directo de tres-variedades lorentzianas y riemannianas.

Con estas ideas en mente, primero definimos las estructuras de ε-contacto, que engloban las habituales estructuras de contacto riemannianas, lorentzianas y estructuras métricas de para-contacto, pero que también admiten la opción de un vector de Reeb de tipo luz. Denominamos este último caso estructura de contacto nula e introdujimos las nociones de estructuras de contacto nulas Sasakianas y de K-contacto, demostrando que la condición de Sasaki asociada es más débil que la de K-contacto.

Después, definimos las estructuras de contacto $\varepsilon\eta$-Einstein, que incluyen casos particulares de estructuras riemannianas, lorentzianas y de para-contacto η-Einstein tres-dimensionales así como un subconjunto de estructuras de contacto nulas, y dimos una clasificación completa de todas ellas en grupos de Lie tres-dimensionales simplemente conexos. Luego, demostramos explícitamente el proceso por el que ciertas combinaciones apropiadas de estructuras de contacto $\varepsilon\eta$-Einstein producen soluciones de Supergravedad mínima en seis dimensiones acoplada a un multiplete tensorial con dilatón constante, que se pueden interpretar equivalentemente como seis-variedades lorentzianas con una conexión métrica Ricci plana con torsión isótropa, totalmente antisimétrica, cerrada y co-cerrada. Finalizamos ilustrando ejemplos particulares de soluciones, que concebimos como deformaciones (genéricamente no supersimétricas) de la solución supersimétrica estándar en $\mathrm{AdS}_3 \times S^3$.

En resumen, los contenidos esenciales de la Segunda Parte de la tesis son los siguientes.

Ideas clave de la Segunda Parte

- El problema de Cauchy para un espinor real paralelo en una cuatro-variedad globalmente hiperbólica se puede reformular como un sistema de ecuaciones diferenciales para una familia de funciones y bases ortonormales del espacio cotangente en una superficie de Cauchy.
- Al menos en cuatro dimensiones, el flujo de espinor paralelo y el de Einstein (en el vacío) coinciden para datos iniciales comunes.
- Se han clasificado todas las cuatro-variedades Einstein autoduales invariantes bajo una acción principal e isométrica del grupo de Heisenberg con órbitas no degeneradas.
- Se han definido las estructuras nulas de contacto, introduciendo las nociones correspondientes de Sasakianidad y K-contacto.
- Ciertas combinaciones de estructuras de contacto $\varepsilon\eta$-Einstein producen soluciones de Supergravedad mínima seis-dimensional.

Direcciones futuras

Es una paradoja bien conocida en física (y en la ciencia en general) que cuanto más creemos saber, más nos percatamos que desconocemos. Esta tesis no ha supuesto una excepción a dicho (afortunado[3]) fenómeno, dejando a su paso miríadas de problemas abiertos. Varios de ellos han sido ya mencionados en el texto, pero hemos visto conveniente (re)escribir explícitamente algunas de las direcciones futuras más interesantes que hemos concebido hasta el momento:

1. Demostrar (o refutar) las Conjeturas 1.1 y 1.2.
2. Caracterizar de forma más precisa el conjunto de todas las GCGs a todo orden y dimensión, a la luz de [251]. En particular, ¿podríamos encontrar explícitamente la forma covariante de todas las GCGs algebraicas inequivalentes a todo orden y dimensión?
3. Estudiar soluciones explícitas de tipo agujero negro de GCGs propias.
4. ¿Podríamos demostrar la existencia de GCGs (o de otras gravedades de orden superior) que admitan soluciones de tipo agujero negro en rotación que sean lo suficientemente simples como para determinar su expresión analíticamente (o mediante una ecuación diferencial tratable numéricamente?
5. Hallar teorías de orden superior de gravedad pura con soluciones EES completamente regulares (es decir, regularizando la singularidad de Schwarzschild).

[3] En caso contrario, no se escribirían más tesis doctorales.

6. Examinar la posibilidad de definir rotaciones de dualidad en teorías de gravedad puras (quizá como se ha sugerido en el reciente trabajo [704]) y estudiar qué teorías de orden superior podrían ser (exactamente) invariantes bajo esta dualidad.

7. Determinar la estructura más general de los Lagrangianos de tipo GEC(G).

8. Indagar sobre la aplicabilidad de las GEC(G)s en el contexto cosmológico o explorar si existen GEC(G)s con soluciones Taub-NUT.

9. Investigar los efectos de redefiniciones de campo en GEC(G)s. ¿Podría reescribirse cualquier teoría (de orden superior) de gravedad y electromagnetismo, a través de ciertas redefiniciones de campo, como una GEC(G), en analogía con la situación para las teorías de gravedad pura?

10. Hallar las condiciones necesarias y suficientes para que cualquier teoría de orden superior de gravedad y electromagnetismo sea (perturbativamente) invariante bajo dualidad (es decir, extender los resultados obtenidos en el Capítulo 4 para ocho derivadas).

11. Demostrar (o refutar) la Conjetura 3.1.

12. Obtener teorías de orden superior exactamente invariantes bajo dualidad (electromagnética) no necesariamente cuadráticas en el campo de Maxwell.

13. Incluso restringiéndonos a teorías exactamente invariantes bajo dualidad que son cuadráticas en el campo de Maxwell, estudiar los agujeros negros (extremos) correspondientes y su entropía.

14. ¿Por qué la entropía (3.209) tiene una expresión tan simple? Sospechamos que se debía a invariancia bajo dualidad, pero sería gratificante probar este hecho desde primeros principios.

15. Construir soluciones EES exactas de teorías exactamente invariantes bajo dualidad. ¿Podríamos hallar ejemplos de tales teorías con soluciones EES regulares?

16. En relación con la pregunta anterior, ¿es posible identificar GEC(G)s que sean además (exactamente) invariantes bajo dualidad?

17. Estudiar los aspectos holográficos de las GECs a todo orden y dimensión arbitraria construidas en (4.255) y (4.256).

18. Encontrar ejemplos de GECGs propias en dimensión arbitraria y analizar sus propiedades holográficas. ¿Sería posible estudiar estas propiedades de forma analítica, como ocurre para las GECs (4.30)?

19. Explorar en más detalle la relación entre imponer causalidad en el espaciotiempo y unitariedad en la TCC fronteriza (en particular, para teorías más generales que (4.30)).

20. Entender mejor la CGD en AdS, especialmente las consecuencias de esta conjetura para la TCC dual.

21. En referencia a la fórmula universal (4.252), sería interesante reobtenerla usando el formalismo desarrollado en [620] o hallar contraejemplos adicionales en dos dimensiones a (4.252) mediante la investigación de GECs tres-dimensionales [619].

22. Extender los resultados del Capítulo 5 para el caso de un espinor real de Killing u otros más genéricos.

23. Usando la conexión entre espinores paralelos y ondas pp, interpretar los aspectos físicos y geométricos de las ondas pp asociadas en términos del formalismo del flujo de espinor paralelo.

24. Investigar la posibilidad de usar flujos espinoriales hiperbólicos de primer orden para construir soluciones especiales de RG.

25. En línea con lo anterior, examinar en mayor profundidad el tipo preciso de flujos de curvatura tres-dimensionales que se obtienen en la superficie de Cauchy.

26. Construir flujos de espinor paralelo no invariantes por la izquierda, quizá centrándonos en la clase particular de flujos de espinor paralelo comóviles.

27. ¿Es posible hallar resultados (parciales) de clasificación de cuatro-variedades Einstein que admiten una acción isométrica y principal del grupo de Heisenberg tres-dimensional con órbitas no degeneradas (es decir, quitando la condición de autodualidad)? ¿Y si imponemos la condición (para)Kähler?

28. Encontrar ejemplos lorentzianos explícitos de cuatro-variedades Heisenberg (Einstein).

29. ¿Podríamos interpretar todas las cuatro-variedades Heisenberg Einstein autoduales en el contexto de TC y Supergravedad?

30. En el ámbito del Capítulo 6, considerar otras acciones isométricas y principales de grupos de Lie diferentes. ¿Tendrían alguna posibilidad de surgir en el estudio de TC y Supergravedad?

31. De nuevo, en el contexto del Capítulo 6, analizar la posibilidad de órbitas con métricas inducidas degeneradas o clasificar variedades (para)cuaterniónicas Kähler o (para)hiperKähler con simetría Heisenberg en dimensiones $D = 4n$ con $n > 1$.

32. Estudiar la posibilidad de generalizar los resultados del Capítulo 7 en el caso de Supergravedad diez-dimensional Tipo IIB, cuando se truncan todos los campos excepto la métrica y la cinco-forma (autodual).

33. Determinar explícitamente si las soluciones de Supergravedad mínima en seis dimensiones que obtuvimos, interpretadas como deformaciones topológicas y geométricas de la solución canónica supersimétrica $\mathrm{AdS}_3 \times S^3$, son realmente supersimétricas o no.

34. Proseguir con la investigación de estructuras nulas de contacto y generalizar esta noción para dimensión (impar) arbitraria.

35. Avanzar con la clasificación de todas las estructuras de contacto $\varepsilon\eta$-Einstein, sin restringirse a las invariantes por la izquierda. Podría ser realmente relevante en el caso de estructuras de contacto nulas, ya que el tensor energía-momento correspondiente adquiere una estructura muy interesante.

36. En el contexto de Supergravedad (mínima) seis-dimensional, explorar diferentes tipos de soluciones, como productos deformados o, modificando de forma más drástica el punto de partida, productos directos de cuatro-variedades lorentzianas y superficies de Riemann (dos-dimensionales).

Las propuestas anteriores son algunas direcciones futuras naturales que surgen de la investigación realizada en esta tesis, pero hay muchísimos más problemas abiertos[4] que podrían resultar de gran interés. Anhelamos afrontar algunos de ellos en el futuro próximo.

[4]Quizá todavía no identificados por el autor en el momento de escribir este documento.

Bibliography

[1] P. Bueno, P. A. Cano, J. Moreno and Á. Murcia, *All higher-curvature gravities as Generalized quasi-topological gravities*, *JHEP* **11** (2019) 062, [1906.00987].

[2] Á. Murcia and C. S. Shahbazi, *Contact metric three manifolds and Lorentzian geometry with torsion in six-dimensional supergravity*, *J. Geom. Phys.* **158** (2020) 103868, [1912.08723].

[3] P. A. Cano and Á. Murcia, *Resolution of Reissner-Nordström singularities by higher-derivative corrections*, *Class. Quant. Grav.* **38** (2021) 075014, [2006.15149].

[4] P. A. Cano and Á. Murcia, *Electromagnetic Quasitopological Gravities*, *JHEP* **10** (2020) 125, [2007.04331].

[5] Á. Murcia and C. Shahbazi, *Parallel spinors on globally hyperbolic lorentzian four-manifolds*, *Annals of Global Analysis and Geometry* **61** (2022) 253–292.

[6] P. A. Cano and Á. Murcia, *Duality-invariant extensions of Einstein-Maxwell theory*, *JHEP* **08** (2021) 042, [2104.07674].

[7] P. A. Cano and Á. Murcia, *Exact electromagnetic duality with nonminimal couplings*, *Phys. Rev. D* **104** (2021) L101501, [2105.09868].

[8] Á. Murcia and C. S. Shahbazi, *Parallel spinor flows on three-dimensional Cauchy hypersurfaces*, 2109.13906.

[9] V. Cortés and Á. Murcia, *Heisenberg-invariant self-dual Einstein manifolds*, 2111.11361.

[10] P. A. Cano, Á. Murcia, A. R. Sánchez and X. Zhang, *Higher-derivative holography with a chemical potential*, Accepted for publication in JHEP (2022) , [2202.10473].

[11] P. Bueno, P. A. Cano, Á. Murcia and A. Rivadulla Sánchez, *A universal feature of charged entanglement entropy*, 2203.04325.

[12] A. Moroianu, Á. Murcia and C. S. Shahbazi, *Heterotic solitons on four-manifolds*, 2101.10309.

[13] P. A. Cano, Á. Murcia, P. F. Ramírez and A. Ruipérez, *On small black holes, KK monopoles and solitonic 5-branes*, *JHEP* **05** (2021) 272, [2102.04476].

[14] Á. Murcia, *ε-contact structures and six-dimensional supergravity*, Accepted for publication in the Proceedings of the X International Meeting on Lorentzian Geometry (2021) , [2106.07602].

[15] S. Weinberg, *The quantum theory of fields*, vol. 1. Cambridge university press, 1995.

[16] S. Weinberg, *The quantum theory of fields*, vol. 2. Cambridge university press, 1995.

[17] M. Srednicki, *Quantum field theory*. Cambridge University Press, 2007.

[18] M. D. Schwartz, *Quantum field theory and the standard model*. Cambridge University Press, 2014.

[19] G. 't Hooft and M. J. G. Veltman, *One loop divergencies in the theory of gravitation*, *Ann. Inst. H. Poincare Phys. Theor. A* **20** (1974) 69–94.

[20] S. Deser and P. van Nieuwenhuizen, *One Loop Divergences of Quantized Einstein-Maxwell Fields*, *Phys. Rev. D* **10** (1974) 401.

[21] S. Deser, H.-S. Tsao and P. van Nieuwenhuizen, *One Loop Divergences of the Einstein Yang-Mills System*, *Phys. Rev. D* **10** (1974) 3337.

[22] M. H. Goroff and A. Sagnotti, *Quantum Gravity at two loops*, *Phys. Lett. B* **160** (1985) 81–86.

[23] M. H. Goroff and A. Sagnotti, *The Ultraviolet Behavior of Einstein Gravity*, *Nucl. Phys. B* **266** (1986) 709–736.

[24] A. E. M. van de Ven, *Two loop quantum gravity*, *Nucl. Phys. B* **378** (1992) 309–366.

[25] G. Bertone, *Particle dark matter: observations, models and searches*. Cambridge University Press, 2010.

[26] L. Roszkowski, E. M. Sessolo and S. Trojanowski, *WIMP dark matter candidates and searches—current status and future prospects*, *Rept. Prog. Phys.* **81** (2018) 066201, [1707.06277].

[27] L. Hui, J. P. Ostriker, S. Tremaine and E. Witten, *Ultralight scalars as cosmological dark matter*, *Phys. Rev. D* **95** (2017) 043541, [1610.08297].

[28] B. J. Carr and S. W. Hawking, *Black holes in the early Universe*, *Mon. Not. Roy. Astron. Soc.* **168** (1974) 399–415.

[29] S. Clesse and J. García-Bellido, *Massive Primordial Black Holes from Hybrid Inflation as Dark Matter and the seeds of Galaxies*, *Phys. Rev. D* **92** (2015) 023524, [1501.07565].

[30] S. Clesse and J. García-Bellido, *The clustering of massive Primordial Black Holes as Dark Matter: measuring their mass distribution with Advanced LIGO*, *Phys. Dark Univ.* **15** (2017) 142–147, [1603.05234].

[31] J. García-Bellido and E. Ruiz Morales, *Primordial black holes from single field models of inflation*, *Phys. Dark Univ.* **18** (2017) 47–54, [1702.03901].

[32] J. M. Ezquiaga Bravo, *Probing the Dark Universe with Gravitational Waves*. PhD thesis, U. Autonoma, Madrid (main), 2019.

[33] Planck collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6, [1807.06209].

[34] G. Gubitosi, F. Piazza and F. Vernizzi, *The Effective Field Theory of Dark Energy*, *JCAP* **02** (2013) 032, [1210.0201].

[35] J. K. Bloomfield, E. E. Flanagan, M. Park and S. Watson, *Dark energy or modified gravity? An effective field theory approach*, *JCAP* **08** (2013) 010, [1211.7054].

[36] S. W. Hawking and R. Penrose, *The Singularities of gravitational collapse and cosmology*, *Proc. Roy. Soc. Lond. A* **314** (1970) 529–548.

[37] J. M. Bardeen, B. Carter and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.

[38] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[39] J. D. Bekenstein, *Black holes and the second law*, *Lett. Nuovo Cim.* **4** (1972) 737–740.

[40] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (1973) 2333–2346.

[41] J. D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev. D* **9** (1974) 3292–3300.

[42] S. D. Mathur, *The Information paradox: A Pedagogical introduction*, *Class. Quant. Grav.* **26** (2009) 224001, [0909.1038].

[43] D. Harlow, *Jerusalem Lectures on Black Holes and Quantum Information*, *Rev. Mod. Phys.* **88** (2016) 015002, [1409.1231].

[44] P. Chen, Y. C. Ong and D.-h. Yeom, *Black Hole Remnants and the Information Loss Paradox*, *Phys. Rept.* **603** (2015) 1–45, [1412.8366].

[45] A. Ashtekar, *New Variables for Classical and Quantum Gravity*, *Phys. Rev. Lett.* **57** (1986) 2244–2247.

[46] C. Rovelli and L. Smolin, *Knot Theory and Quantum Gravity*, *Phys. Rev. Lett.* **61** (1988) 1155.

[47] J. F. Barbero G., *Real Ashtekar variables for Lorentzian signature space times*, *Phys. Rev. D* **51** (1995) 5507–5510, [gr-qc/9410014].

[48] J. F. Barbero G., *Reality conditions and Ashtekar variables: A Different perspective*, *Phys. Rev. D* **51** (1995) 5498–5506, [gr-qc/9410013].

[49] A. Einstein, *Zur allgemeinen Relativitätstheorie*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915) 778–786.

[50] A. Einstein, *Zur allgemeinen Relativitätstheorie (Nachtrag)*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915) 799–801.

[51] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915) 831–839.

[52] A. Einstein, *Die feldgleichungen der gravitation*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften* (1915) 844–847.

[53] D. Hilbert, *Die Grundlagen der Physik. (Erste Mitteilung)*, *Gött. Nachr.* (1915) 395–407.

[54] K. Nordtvedt, *Equivalence Principle for Massive Bodies. 2. Theory*, *Phys. Rev.* **169** (1968) 1017–1025.

[55] C. M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4, [1403.7377].

[56] C. M. Will, *On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics*, *Proc. Nat. Acad. Sci.* **108** (2011) 5938, [1102.5192].

[57] A. Einstein, *Über den einfluß der schwerkraft auf die ausbreitung des lichtes*, *Annalen der Physik* **340** (1911) 898–908.

[58] A. Einstein, *Lens-Like Action of a Star by the Deviation of Light in the Gravitational Field*, *Science* **84** (1936) 506–507.

[59] I. I. Shapiro, *Fourth Test of General Relativity*, *Phys. Rev. Lett.* **13** (1964) 789–791.

[60] I. I. Shapiro, M. E. Ash, R. P. Ingalls, W. B. Smith, D. B. Campbell, R. B. Dyce et al., *Fourth test of general relativity - new radar result*, *Phys. Rev. Lett.* **26** (1971) 1132–1135.

[61] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [1602.03837].

[62] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence*, *Phys. Rev. Lett.* **116** (2016) 241103, [1606.04855].

[63] LIGO Scientific, VIRGO collaboration, B. P. Abbott et al., *GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2*, *Phys. Rev. Lett.* **118** (2017) 221101, [1706.01812].

[64] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence*, *Phys. Rev. Lett.* **119** (2017) 141101, [1709.09660].

[65] LIGO Scientific, Virgo collaboration, B. . P. . Abbott et al., *GW170608: Observation of a 19-solar-mass Binary Black Hole Coalescence*, *Astrophys. J. Lett.* **851** (2017) L35, [1711.05578].

[66] LIGO Scientific, Virgo collaboration, R. Abbott et al., *GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object*, *Astrophys. J. Lett.* **896** (2020) L44, [2006.12611].

[67] LIGO Scientific, Virgo collaboration, R. Abbott et al., *GW190521: A Binary Black Hole Merger with a Total Mass of* $150M_\odot$, *Phys. Rev. Lett.* **125** (2020) 101102, [2009.01075].

[68] Event Horizon Telescope collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, *Astrophys. J. Lett.* **875** (2019) L1, [1906.11238].

[69] Event Horizon Telescope collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole*, *Astrophys. J. Lett.* **875** (2019) L6, [1906.11243].

[70] Event Horizon Telescope collaboration, K. Akiyama et al., *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, *Astrophys. J. Lett.* **930** (2022) L12.

[71] Event Horizon Telescope collaboration, K. Akiyama et al., *First Sagittarius A* Event Horizon Telescope Results. VI. Testing the Black Hole Metric*, *Astrophys. J. Lett.* **930** (2022) L17.

[72] LISA collaboration, P. Amaro-Seoane et al., *Laser Interferometer Space Antenna*, 1702.00786.

[73] H. Georgi, *Effective field theory*, *Annual review of nuclear and particle science* **43** (1993) 209–252.

[74] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, *Phys. Rev. D* **50** (1994) 3874–3888, [gr-qc/9405057].

[75] K. S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, *Phys. Rev. D* **16** (1977) 953–969.

[76] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, *Phys. Lett. B* **91** (1980) 99–102.

[77] R. P. Woodard, *Ostrogradsky's theorem on Hamiltonian instability*, *Scholarpedia* **10** (2015) 32243, [1506.02210].

[78] P. Bueno and P. A. Cano, *Einsteinian cubic gravity*, *Phys. Rev. D* **94** (2016) 104005, [1607.06463].

[79] R. A. Hennigar, D. Kubizňák and R. B. Mann, *Generalized quasitopological gravity*, *Phys. Rev. D* **95** (2017) 104042, [1703.01631].

[80] J. W. York, Jr., *Role of conformal three geometry in the dynamics of gravitation*, *Phys. Rev. Lett.* **28** (1972) 1082–1085.

[81] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev. D* **15** (1977) 2752–2756.

[82] R. L. Arnowitt, S. Deser and C. W. Misner, *Canonical variables for general relativity*, *Phys. Rev.* **117** (1960) 1595–1602.

[83] R. Arnowitt, S. Deser and C. W. Misner, *Energy and the criteria for radiation in general relativity*, *Physical Review* **118** (1960) 1100.

[84] R. Arnowitt, S. Deser and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, *Physical Review* **122** (1961) 997.

[85] R. L. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997–2027, [gr-qc/0405109].

[86] L. F. Abbott and S. Deser, *Stability of Gravity with a Cosmological Constant*, *Nucl. Phys.* **B195** (1982) 76–96.

[87] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, 6, 2001. hep-th/0106109.

[88] P. Bueno, P. A. Cano, V. S. Min and M. R. Visser, *Aspects of general higher-order gravities*, *Phys. Rev. D* **95** (2017) 044010, [1610.08519].

[89] P. A. Cano Molina-Niñirola, *Higher-Curvature Gravity, Black Holes and Holography*. PhD thesis, Universidad Autónoma de Madrid, 2019. 1912.07035.

[90] S. Deser and B. Tekin, *Energy in generic higher curvature gravity theories*, *Phys. Rev.* **D67** (2003) 084009, [hep-th/0212292].

[91] H. Adami, M. R. Setare, T. C. Sisman and B. Tekin, *Conserved Charges in Extended Theories of Gravity*, 1710.07252.

[92] L. Álvarez-Gaumé and E. Witten, *Gravitational Anomalies*, *Nucl. Phys. B* **234** (1984) 269.

[93] M. B. Green and J. H. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, *Phys. Lett. B* **149** (1984) 117–122.

[94] L. Álvarez-Gaumé and P. H. Ginsparg, *The Structure of Gauge and Gravitational Anomalies*, *Annals Phys.* **161** (1985) 423.

[95] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007, 10.1017/CBO9780511816079.

[96] L. E. Ibáñez and A. M. Uranga, *String theory and particle physics: An introduction to string phenomenology*. Cambridge University Press, 2, 2012.

[97] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99–104, [hep-th/9601029].

[98] C. G. Callan and J. M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys. B* **472** (1996) 591–610, [hep-th/9602043].

[99] J. H. Schwarz, *Superstring Theory*, *Phys. Rept.* **89** (1982) 223–322.

[100] M. Green, M. Green, J. Schwarz and E. Witten, *Superstring Theory: Volume 1, Introduction*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1988.

[101] M. B. Green, J. H. Schwarz and E. Witten, *Superstring theory: volume 2, loop amplitudes, anomalies and phenomenology*. Cambridge university press, 1988.

[102] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007, 10.1017/CBO9780511618123.

[103] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in Background Fields*, Nucl. Phys. B **262** (1985) 593–609.

[104] E. S. Fradkin and A. A. Tseytlin, *Effective Field Theory from Quantized Strings*, Phys. Lett. B **158** (1985) 316–322.

[105] R. Blumenhagen, A. Font, A. Kumar and D. Lust, *Aspects of type 0 string theory*, Class. Quant. Grav. **17** (2000) 989–996, [hep-th/9908155].

[106] T. Ortín, *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2nd ed. ed., 7, 2015, 10.1017/CBO9781139019750.

[107] J. Scherk and J. H. Schwarz, *Dual Models and the Geometry of Space-Time*, Phys. Lett. B **52** (1974) 347–350.

[108] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, *Four Loop beta Function for the N=1 and N=2 Supersymmetric Nonlinear Sigma Model in Two-Dimensions*, Phys. Lett. B **173** (1986) 423–428.

[109] M. T. Grisaru and D. Zanon, *σ Model Superstring Corrections to the Einstein-hilbert Action*, Phys. Lett. B **177** (1986) 347–351.

[110] R. R. Metsaev and A. A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, Nucl. Phys. B **293** (1987) 385–419.

[111] W. A. Chemissany, M. de Roo and S. Panda, *alpha'-Corrections to Heterotic Superstring Effective Action Revisited*, JHEP **08** (2007) 037, [0706.3636].

[112] C. Vafa, *The String landscape and the swampland*, hep-th/0509212.

[113] E. Palti, *The Swampland: Introduction and Review*, Fortsch. Phys. **67** (2019) 1900037, [1903.06239].

[114] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, 2102.01111.

[115] P. A. Cano, P. Meessen, T. Ortín and P. F. Ramírez, *α'-corrected black holes in String Theory*, JHEP **05** (2018) 110, [1803.01919].

[116] S. Chimento, P. Meessen, T. Ortín, P. F. Ramírez and A. Ruipérez, *On a family of α'-corrected solutions of the Heterotic Superstring effective action*, JHEP **07** (2018) 080, [1803.04463].

[117] A. Ruipérez Vicente, *Black holes in string theory with higher-derivative corrections*. PhD thesis, U. Autónoma, Madrid (main), 2020.

[118] C. Eloy, *Supergravity : AdS$_3$ vacua and higher-derivative corrections*. PhD thesis, Lyon, Ecole Normale Superieure, Université de Lyon, 2021.

[119] P. A. Cano, T. Ortín, A. Ruipérez and M. Zatti, *Non-supersymmetric black holes with α' corrections*, JHEP **03** (2022) 103, [2111.15579].

[120] I. Antoniadis, R. Minasian, S. Theisen and P. Vanhove, *String loop corrections to the universal hypermultiplet*, *Class. Quant. Grav.* **20** (2003) 5079–5102, [hep-th/0307268].

[121] M. Abe, H. Kubota and N. Sakai, *Loop Corrections to the Heterotic String Effective Lagrangian*, *Phys. Lett. B* **200** (1988) 461–465.

[122] M. de Roo, H. Suelmann and A. Wiedemann, *The Supersymmetric effective action of the heterotic string in ten-dimensions*, *Nucl. Phys. B* **405** (1993) 326–366, [hep-th/9210099].

[123] D. J. Gross and J. H. Sloan, *The Quartic Effective Action for the Heterotic String*, *Nucl. Phys. B* **291** (1987) 41–89.

[124] E. A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, *Nucl. Phys. B* **328** (1989) 439–468.

[125] P. A. Cano, S. Chimento, R. Linares, T. Ortín and P. F. Ramírez, *α' corrections of Reissner-Nordström black holes*, *JHEP* **02** (2020) 031, [1910.14324].

[126] P. A. Cano, T. Ortín and P. F. Ramírez, *On the extremality bound of stringy black holes*, *JHEP* **02** (2020) 175, [1909.08530].

[127] F. Moura, *Type II and heterotic one loop string effective actions in four dimensions*, *JHEP* **06** (2007) 052, [hep-th/0703026].

[128] R. C. Myers, M. F. Paulos and A. Sinha, *Quantum corrections to eta/s*, *Phys. Rev.* **D79** (2009) 041901, [0806.2156].

[129] A. Buchel, R. C. Myers, M. F. Paulos and A. Sinha, *Universal holographic hydrodynamics at finite coupling*, *Phys. Lett.* **B669** (2008) 364–370, [0808.1837].

[130] D. A. Galante and R. C. Myers, *Holographic Renyi entropies at finite coupling*, *JHEP* **08** (2013) 063, [1305.7191].

[131] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, *Coupling constant dependence in the thermodynamics of N=4 supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B534** (1998) 202–222, [hep-th/9805156].

[132] A. Buchel, J. T. Liu and A. O. Starinets, *Coupling constant dependence of the shear viscosity in N=4 supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B707** (2005) 56–68, [hep-th/0406264].

[133] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge University Press, 2012.

[134] S. Cecotti, *Supersymmetric field theories*. Cambridge University Press, 2015.

[135] A. Tomasiello, *Geometry of String Theory Compactifications*. Cambridge University Press, 1, 2022, 10.1017/9781108635745.

[136] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of Euclidean supersymmetry. 1. Vector multiplets*, *JHEP* **03** (2004) 028, [hep-th/0312001].

[137] V. Cortés, C. Mayer, T. Mohaupt and F. Saueressig, *Special geometry of euclidean supersymmetry ii: hypermultiplets and the c-map*, *JHEP* **2005** (2005) 025.

[138] A. Moroianu, *Lectures on Kähler geometry*, vol. 69. Cambridge University Press, 2007.

[139] N. Hitchin, *Hyper-kähler manifolds*, *Séminaire Bourbaki* **1991** (1992) 137–166.

[140] H. Kamada, *Neutral hyperkähler structures on primary kodaira surfaces*, *Tsukuba journal of mathematics* **23** (1999) 321–332.

[141] M. Berger, *Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes*, *Bulletin de la Société Mathématique de France* **83** (1955) 279–330.

[142] T. Leistner, *On the classification of lorentzian holonomy groups*, *Journal of Differential Geometry* **76** (2007) 423–484.

[143] A. Besse, *Einstein Manifolds.* Classics in Mathematics. Springer Berlin Heidelberg, 1987.

[144] A. Swann, *HyperKähler and quaternionic Kähler geometry*, *Math. Ann.* **289** (1991) 421–450, [1804.09624].

[145] N. Seiberg, *Observations on the Moduli Space of Superconformal Field Theories*, *Nucl. Phys. B* **303** (1988) 286–304.

[146] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys. A* **4** (1989) 2475.

[147] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type ii superstring vacua of calabi-yau spaces*, *Nuclear Physics B* **332** (1990) 317–332.

[148] D. V. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt, *Quaternionic kähler metrics associated with special kähler manifolds*, *J. Geom. Phys.* **92** (2015) 271–287.

[149] V. Cortés, P. Dempster, T. Mohaupt and O. Vaughan, *Special geometry of euclidean supersymmetry iv: the local c-map*, *JHEP* **2015** (2015) 1–61.

[150] M. Rocek, C. Vafa and S. Vandoren, *Hypermultiplets and topological strings*, *JHEP* **02** (2006) 062, [hep-th/0512206].

[151] B. de Wit and F. Saueressig, *Off-shell N=2 tensor supermultiplets*, *JHEP* **09** (2006) 062, [hep-th/0606148].

[152] A. Haydys, *Hyperkähler and quaternionic kähler manifolds with s1-symmetries*, *Journal of Geometry and Physics* **58** (2008) 293–306.

[153] S. Alexandrov, D. Persson and B. Pioline, *Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence*, *JHEP* **12** (2011) 027, [1110.0466].

[154] D. V. Alekseevsky, V. Cortés and T. Mohaupt, *Conification of Kähler and hyper-Kähler manifolds*, *Commun. Math. Phys.* **324** (2013) 637–655, [1205.2964].

[155] N. Hitchin, *On the hyperkähler/quaternion kähler correspondence*, *Communications in Mathematical Physics* **324** (2013) 77–106.

[156] M. Dyckmanns and O. Vaughan, *The para-HK/QK correspondence*, *J. Geom. Phys.* **116** (2017) 244–257, [1601.05001].

[157] D. Robles-Llana, F. Saueressig and S. Vandoren, *String loop corrected hypermultiplet moduli spaces*, *JHEP* **03** (2006) 081, [hep-th/0602164].

[158] V. Cortés and A. Saha, *Quarter-pinched einstein metrics interpolating between real and complex hyperbolic metrics*, *Mathematische Zeitschrift* **290** (2018) 155–166.

[159] V. Cortés, A. Saha and D. Thung, *Symmetries of quaternionic kähler manifolds with s^1-symmetry*, *Trans. London Math. Soc.* **8** (2021) 95–119.

[160] V. Cortés and A. Saha, *Four-dimensional einstein manifolds with heisenberg symmetry*, *Ann. di Mat. Pura ed Appl.* (2021) 1–21.

[161] D. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds.* Progress in Mathematics. Birkhäuser Boston, 2002.

[162] S. Zamkovoy, *Canonical connections on paracontact manifolds*, *Annals of Global Analysis and Geometry* **36** (2009) 37–60.

[163] G. Calvaruso and D. Perrone, *Contact pseudo-metric manifolds*, *Differential Geometry and its Applications* **28** (2010) 615–634.

[164] G. Calvaruso, *Contact lorentzian manifolds*, *Differential Geometry and its Applications* **29** (2011) S41–S51.

[165] Y. Hatakeyama, Y. Ogawa and S. Tanno, *Some properties of manifolds with contact metric structure*, *Tohoku Mathematical Journal, Second Series* **15** (1963) 42–48.

[166] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure i*, *Tohoku Mathematical Journal, Second Series* **12** (1960) 459–476.

[167] S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with contact metric structures*, *Journal of the Mathematical Society of Japan* **14** (1962) 249–271.

[168] C. Boyer and K. Galicki, *Sasakian Geometry.* Oxford mathematical monographs. Oxford University Press, 2008.

[169] D. V. Alekseevsky, V. Cortés, A. Galaev and T. Leistner, *Cones over pseudo-Riemannian manifolds and their holonomy*, *J. Reine Angew. Math.* **635** (2009) 23 – 69, [0707.3063].

[170] C. P. Boyer, K. Galicki and P. Matzeu, *On eta-Einstein Sasakian geometry*, *Commun. Math. Phys.* **262** (2006) 177–208, [math/0406627].

[171] J. Sparks, *Sasaki-Einstein Manifolds*, *Surveys Diff. Geom.* **16** (2011) 265–324, [1004.2461].

[172] V. I. Arnol'd, *Mathematical methods of classical mechanics*, vol. 60. Springer Science & Business Media, 1978.

[173] M. de León and C. Sardón, *Cosymplectic and contact structures for time-dependent and dissipative hamiltonian systems*, *Journal of Physics A: Mathematical and Theoretical* **50** (2017) 255205.

[174] A. Bravetti, H. Cruz and D. Tapias, *Contact hamiltonian mechanics*, *Annals of Physics* **376** (2017) 17–39.

[175] R. Hermann, *Geometry, Physics, and Systems*, vol. 18. M. Dekker, 1973.

[176] R. Mrugala, *Geometrical formulation of equilibrium phenomenological thermodynamics*, *Reports on Mathematical Physics* **14** (1978) 419–427.

[177] R. Mrugala, J. D. Nulton, J. C. Schön and P. Salamon, *Contact structure in thermodynamic theory*, *Reports on mathematical physics* **29** (1991) 109–121.

[178] H. Quevedo, *Geometrothermodynamics*, *J. Math. Phys.* **48** (2007) 013506, [physics/0604164].

[179] A. Bravetti, *Contact geometry and thermodynamics*, *International Journal of Geometric Methods in Modern Physics* **16** (2019) 1940003.

[180] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Sasaki-Einstein metrics on* $S^2 \times S^3$, *Adv. Theor. Math. Phys.* **8** (2004) 711–734, [hep-th/0403002].

[181] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, *A New infinite class of Sasaki-Einstein manifolds*, Adv. Theor. Math. Phys. **8** (2004) 987–1000, [hep-th/0403038].

[182] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Commun. Math. Phys. **262** (2006) 51–89, [hep-th/0411238].

[183] D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, Commun. Math. Phys. **280** (2008) 611–673, [hep-th/0603021].

[184] D. Cassani, G. Dall'Agata and A. F. Faedo, *Type IIB supergravity on squashed Sasaki-Einstein manifolds*, JHEP **05** (2010) 094, [1003.4283].

[185] J. Figueroa-O'Farrill and A. Santi, *Sasakian manifolds and M-theory*, Class. Quant. Grav. **33** (2016) 095004, [1511.03460].

[186] D. Flores-Alfonso, C. S. Lopez-Monsalvo and M. Maceda, *Contact Geometry in Superconductors and New Massive Gravity*, Phys. Lett. B **815** (2021) 136143, [2011.13499].

[187] N. Marcus and J. H. Schwarz, *Field Theories That Have No Manifestly Lorentz Invariant Formulation*, Phys. Lett. B **115** (1982) 111.

[188] H. Nishino and E. Sezgin, *Matter and Gauge Couplings of N=2 Supergravity in Six-Dimensions*, Phys. Lett. B **144** (1984) 187–192.

[189] H. Nishino and E. Sezgin, *The Complete $N = 2$, $d = 6$ Supergravity With Matter and Yang-Mills Couplings*, Nucl. Phys. B **278** (1986) 353–379.

[190] L. J. Romans, *Selfduality for Interacting Fields: Covariant Field Equations for Six-dimensional Chiral Supergravities*, Nucl. Phys. B **276** (1986) 71.

[191] S. Ferrara, F. Riccioni and A. Sagnotti, *Tensor and vector multiplets in six-dimensional supergravity*, Nucl. Phys. B **519** (1998) 115–140, [hep-th/9711059].

[192] J. B. Gutowski, D. Martelli and H. S. Reall, *All Supersymmetric solutions of minimal supergravity in six-dimensions*, Class. Quant. Grav. **20** (2003) 5049–5078, [hep-th/0306235].

[193] A. Chamseddine, J. M. Figueroa-O'Farrill and W. Sabra, *Supergravity vacua and Lorentzian Lie groups*, hep-th/0306278.

[194] D. Martelli and J. F. Morales, *Bubbling AdS(3)*, JHEP **02** (2005) 048, [hep-th/0412136].

[195] H. Lin, O. Lunin and J. M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, JHEP **10** (2004) 025, [hep-th/0409174].

[196] J. T. Liu and D. Vaman, *Bubbling 1/2 BPS solutions of minimal six-dimensional supergravity*, Phys. Lett. B **642** (2006) 411–419, [hep-th/0412242].

[197] I. Bena and N. P. Warner, *Black holes, black rings and their microstates*, Lect. Notes Phys. **755** (2008) 1–92, [hep-th/0701216].

[198] M. Akyol and G. Papadopoulos, *Spinorial geometry and Killing spinor equations of 6-D supergravity*, Class. Quant. Grav. **28** (2011) 105001, [1010.2632].

[199] M. Akyol and G. Papadopoulos, *Topology and geometry of 6-dimensional (1,0) supergravity black hole horizons*, Class. Quant. Grav. **29** (2012) 055002, [1109.4254].

[200] F. Apruzzi, M. Fazzi, D. Rosa and A. Tomasiello, *All AdS$_7$ solutions of type II supergravity*, *JHEP* **04** (2014) 064, [1309.2949].

[201] J. Figueroa-O'Farrill and N. Hustler, *The homogeneity theorem for supergravity backgrounds II: the six-dimensional theories*, *JHEP* **04** (2014) 131, [1312.7509].

[202] P. de Medeiros, J. Figueroa-O'Farrill and A. Santi, *Killing superalgebras for Lorentzian six-manifolds*, *J. Geom. Phys.* **132** (2018) 13–44, [1804.00319].

[203] H. Het Lam and S. Vandoren, *BPS solutions of six-dimensional (1, 0) supergravity coupled to tensor multiplets*, *JHEP* **06** (2018) 021, [1804.04681].

[204] P. A. Cano and T. Ortín, *The structure of all the supersymmetric solutions of ungauged $\mathcal{N} = (1, 0), d = 6$ supergravity*, *Class. Quant. Grav.* **36** (2019) 125007, [1804.04945].

[205] S. Ivanov, *Heterotic supersymmetry, anomaly cancellation and equations of motion*, *Phys. Lett. B* **685** (2010) 190–196, [0908.2927].

[206] H. A. Buchdahl, *Non-linear Lagrangians and cosmological theory*, *Mon. Not. Roy. Astron. Soc.* **150** (1970) 1.

[207] T. P. Sotiriou and V. Faraoni, *f(R) Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451–497, [0805.1726].

[208] A. De Felice and S. Tsujikawa, *f(R) theories*, *Living Rev. Rel.* **13** (2010) 3, [1002.4928].

[209] P. Teyssandier and P. Tourrenc, *The Cauchy problem for the $R + R^2$ theories of gravity without torsion*, *J. Math. Phys.* **24** (1983) 2793.

[210] J. D. Barrow and S. Cotsakis, *Inflation and the Conformal Structure of Higher Order Gravity Theories*, *Phys. Lett. B* **214** (1988) 515–518.

[211] R. P. Woodard, *Avoiding dark energy with 1/r modifications of gravity*, *Lect. Notes Phys.* **720** (2007) 403–433, [astro-ph/0601672].

[212] S. Nojiri and S. D. Odintsov, *Modified gravity with negative and positive powers of the curvature: Unification of the inflation and of the cosmic acceleration*, *Phys. Rev. D* **68** (2003) 123512, [hep-th/0307288].

[213] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, *On the evolution of density perturbations in f(R) theories of gravity*, *Phys. Rev. D* **77** (2008) 123515, [0802.2999].

[214] S. A. Appleby, R. A. Battye and A. A. Starobinsky, *Curing singularities in cosmological evolution of F(R) gravity*, *JCAP* **06** (2010) 005, [0909.1737].

[215] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, *Black Holes in f(R) theories*, *Phys. Rev. D* **80** (2009) 124011, [0907.3872].

[216] P. Bueno and P. A. Cano, *On black holes in higher-derivative gravities*, *Class. Quant. Grav.* **34** (2017) 175008, [1703.04625].

[217] E. Pechlaner and R. Sexl, *On quadratic lagrangians in general relativity*, *Communications in Mathematical Physics* **2** (1966) 165–175.

[218] F. C. Michel, *Gravitational collapse and higher-order gravitational lagrangians*, *Annals of Physics* **76** (1973) 281–298.

[219] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, *Black Holes in Higher-Derivative Gravity*, *Phys. Rev. Lett.* **114** (2015) 171601, [1502.01028].

[220] A. Casado-Turrión, A. de la Cruz-Dombriz and A. Dobado, *Is gravitational collapse possible in f(R) gravity?*, *Phys. Rev. D* **105** (2022) 084060, [2202.04439].

[221] A. Einstein, *Einheitliche feldtheorie yon gravitation und elektrizität*, *Pruess. Akad. Wiss.* **414** (1925) .

[222] G. J. Olmo, *Palatini Approach to Modified Gravity: f(R) Theories and Beyond*, *Int. J. Mod. Phys. D* **20** (2011) 413–462, [1101.3864].

[223] C. Lanczos, *Elektromagnetismus als natürliche eigenschaft der riemannschen geometrie*, *Zeitschrift für Physik* **73** (1932) 147–168.

[224] C. Lanczos, *A remarkable property of the riemann-christoffel tensor in four dimensions*, *Annals of Mathematics* (1938) 842–850.

[225] D. Lovelock, *Divergence-free tensorial concomitants*, *aequationes mathematicae* **4** (1970) 127–138.

[226] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498–501.

[227] D. G. Boulware and S. Deser, *String Generated Gravity Models*, *Phys. Rev. Lett.* **55** (1985) 2656.

[228] P. A. Cano and A. Ruipérez, *String gravity in D=4*, *Phys. Rev. D* **105** (2022) 044022, [2111.04750].

[229] S.-s. Chern, *A simple intrinsic proof of the gauss-bonnet formula for closed riemannian manifolds*, *Annals of mathematics* (1944) 747–752.

[230] T. Padmanabhan and D. Kothawala, *Lanczos-Lovelock models of gravity*, *Phys. Rept.* **531** (2013) 115–171, [arXiv:1302.2151].

[231] R.-G. Cai, *Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev.* **D65** (2002) 084014, [hep-th/0109133].

[232] X. O. Camanho and J. D. Edelstein, *A Lovelock black hole bestiary*, *Class. Quant. Grav.* **30** (2013) 035009, [1103.3669].

[233] J. T. Wheeler, *Symmetric Solutions to the Gauss-Bonnet Extended Einstein Equations*, *Nucl. Phys.* **B268** (1986) 737–746.

[234] J. T. Wheeler, *Symmetric Solutions to the Maximally Gauss-Bonnet Extended Einstein Equations*, *Nucl. Phys. B* **273** (1986) 732.

[235] Q. Exirifard and M. M. Sheikh-Jabbari, *Lovelock gravity at the crossroads of Palatini and metric formulations*, *Phys. Lett. B* **661** (2008) 158–161, [0705.1879].

[236] M. Borunda, B. Janssen and M. Bastero-Gil, *Palatini versus metric formulation in higher curvature gravity*, *JCAP* **11** (2008) 008, [0804.4440].

[237] N. Dadhich and J. M. Pons, *Consistent Levi Civita truncation uniquely characterizes the Lovelock Lagrangians*, *Phys. Lett. B* **705** (2011) 139–142, [1012.1692].

[238] J. Oliva and S. Ray, *A new cubic theory of gravity in five dimensions: Black hole, Birkhoff's theorem and C-function*, *Class. Quant. Grav.* **27** (2010) 225002, [1003.4773].

[239] R. C. Myers and B. Robinson, *Black Holes in Quasi-topological Gravity*, *JHEP* **08** (2010) 067, [1003.5357].

[240] J. Oliva and S. Ray, *Classification of Six Derivative Lagrangians of Gravity and Static Spherically Symmetric Solutions*, Phys. Rev. **D82** (2010) 124030, [1004.0737].

[241] M. H. Dehghani, A. Bazrafshan, R. B. Mann, M. R. Mehdizadeh, M. Ghanaatian and M. H. Vahidinia, *Black Holes in Quartic Quasitopological Gravity*, Phys. Rev. D **85** (2012) 104009, [1109.4708].

[242] A. Cisterna, L. Guajardo, M. Hassaine and J. Oliva, *Quintic quasi-topological gravity*, JHEP **04** (2017) 066, [1702.04676].

[243] R. C. Myers, M. F. Paulos and A. Sinha, *Holographic studies of quasi-topological gravity*, JHEP **08** (2010) 035, [1004.2055].

[244] J. Oliva and S. Ray, *Birkhoff's Theorem in Higher Derivative Theories of Gravity*, Class. Quant. Grav. **28** (2011) 175007, [1104.1205].

[245] J. Oliva and S. Ray, *Birkhoff's Theorem in Higher Derivative Theories of Gravity II*, Phys. Rev. D **86** (2012) 084014, [1201.5601].

[246] R. A. Hennigar, W. G. Brenna and R. B. Mann, *$P - v$ criticality in quasitopological gravity*, JHEP **07** (2015) 077, [1505.05517].

[247] O. Fierro, N. Mora and J. Oliva, *Slowly rotating black holes in quasitopological gravity*, Phys. Rev. D **103** (2021) 064004, [2012.06618].

[248] P. Bueno and P. A. Cano, *Four-dimensional black holes in Einsteinian cubic gravity*, Phys. Rev. D **94** (2016) 124051, [1610.08019].

[249] J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, *Quintessential Quartic Quasi-topological Quartet*, JHEP **05** (2017) 134, [1703.11007].

[250] P. Bueno, P. A. Cano and R. A. Hennigar, *(Generalized) quasi-topological gravities at all orders*, Class. Quant. Grav. **37** (2020) 015002, [1909.07983].

[251] P. Bueno, P. A. Cano, R. A. Hennigar, M. Lu and J. Moreno, *Generalized quasi-topological gravities: the whole shebang*, 2203.05589.

[252] R. A. Hennigar and R. B. Mann, *Black holes in Einsteinian cubic gravity*, Phys. Rev. D **95** (2017) 064055, [1610.06675].

[253] P. Bueno and P. A. Cano, *Universal black hole stability in four dimensions*, Phys. Rev. **D96** (2017) 024034, [1704.02967].

[254] R. A. Hennigar, *Criticality for charged black branes*, JHEP **09** (2017) 082, [1705.07094].

[255] X.-H. Feng, H. Huang, Z.-F. Mai and H. Lu, *Bounce Universe and Black Holes from Critical Einsteinian Cubic Gravity*, Phys. Rev. **D96** (2017) 104034, [1707.06308].

[256] R. A. Hennigar, M. B. J. Poshteh and R. B. Mann, *Shadows, Signals, and Stability in Einsteinian Cubic Gravity*, Phys. Rev. **D97** (2018) 064041, [1801.03223].

[257] P. Bueno, P. A. Cano and A. Ruipérez, *Holographic studies of Einsteinian cubic gravity*, JHEP **03** (2018) 150, [1802.00018].

[258] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *NUTs and bolts beyond Lovelock*, JHEP **10** (2018) 095, [1808.01671].

[259] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *Universality of Squashed-Sphere Partition Functions*, Phys. Rev. Lett. **122** (2019) 071602, [1808.02052].

[260] G. Arciniega, J. D. Edelstein and L. G. Jaime, *Towards geometric inflation: the cubic case*, Phys. Lett. B **802** (2020) 135272, [1810.08166].

[261] M. B. J. Poshteh and R. B. Mann, *Gravitational Lensing by Black Holes in Einsteinian Cubic Gravity*, Phys. Rev. **D99** (2019) 024035, [1810.10657].

[262] A. Cisterna, N. Grandi and J. Oliva, *On four-dimensional Einsteinian gravity, quasitopological gravity, cosmology and black holes*, Phys. Lett. B **805** (2020) 135435, [1811.06523].

[263] G. Arciniega, P. Bueno, P. A. Cano, J. D. Edelstein, R. A. Hennigar and L. G. Jaime, *Geometric Inflation*, Phys. Lett. B **802** (2020) 135242, [1812.11187].

[264] M. R. Mehdizadeh and A. H. Ziaie, *Traversable wormholes in Einsteinian cubic gravity*, Mod. Phys. Lett. A **35** (2019) 2050017, [1903.10907].

[265] M. Mir, R. A. Hennigar, J. Ahmed and R. B. Mann, *Black hole chemistry and holography in generalized quasi-topological gravity*, JHEP **08** (2019) 068, [1902.02005].

[266] M. Mir and R. B. Mann, *On generalized quasi-topological cubic-quartic gravity: thermodynamics and holography*, JHEP **07** (2019) 012, [1902.10906].

[267] C. Erices, E. Papantonopoulos and E. N. Saridakis, *Cosmology in cubic and $f(P)$ gravity*, Phys. Rev. D **99** (2019) 123527, [1903.11128].

[268] P. A. Cano and D. Pereñíguez, *Extremal Rotating Black Holes in Einsteinian Cubic Gravity*, Phys. Rev. D **101** (2020) 044016, [1910.10721].

[269] D. J. Burger, W. T. Emond and N. Moynihan, *Rotating Black Holes in Cubic Gravity*, Phys. Rev. D **101** (2020) 084009, [1910.11618].

[270] P. Bueno, P. A. Cano, R. A. Hennigar, V. A. Penas and A. Ruipérez, *Partition functions on slightly squashed spheres and flux parameters*, JHEP **04** (2020) 123, [2001.10020].

[271] A. M. Frassino and J. V. Rocha, *Charged black holes in Einsteinian cubic gravity and nonuniqueness*, Phys. Rev. D **102** (2020) 024035, [2002.04071].

[272] I. Quiros, R. García-Salcedo, T. Gonzalez, J. L. M. Martínez and U. Nucamendi, *Global asymptotic dynamics of cosmological Einsteinian cubic gravity*, Phys. Rev. D **102** (2020) 044018, [2003.10516].

[273] M. C. Pookkillath, A. De Felice and A. A. Starobinsky, *Anisotropic instability in a higher order gravity theory*, JCAP **07** (2020) 041, [2004.03912].

[274] C. Adair, P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *Slowly rotating black holes in Einsteinian cubic gravity*, Phys. Rev. D **102** (2020) 084001, [2004.09598].

[275] H. Khodabakhshi, A. Giaimo and R. B. Mann, *Einstein Quartic Gravity: Shadows, Signals, and Stability*, Phys. Rev. D **102** (2020) 044038, [2006.02237].

[276] J. D. Edelstein, D. Vázquez Rodríguez and A. Vilar López, *Aspects of Geometric Inflation*, JCAP **12** (2020) 040, [2006.10007].

[277] R. A. Konoplya, A. F. Zinhailo and Z. Stuchlik, *Quasinormal modes and Hawking radiation of black holes in cubic gravity*, Phys. Rev. D **102** (2020) 044023, [2006.10462].

[278] J. D. Edelstein, R. B. Mann, D. V. Rodríguez and A. Vilar López, *Small free field inflation in higher curvature gravity*, *JHEP* **01** (2021) 029, [2007.07651].

[279] J. Beltrán Jiménez and A. Jiménez-Cano, *On the strong coupling of Einsteinian Cubic Gravity and its generalisations*, *JCAP* **01** (2021) 069, [2009.08197].

[280] P. A. Cano, K. Fransen and T. Hertog, *Novel higher-curvature variations of R^2 inflation*, *Phys. Rev. D* **103** (2021) 103531, [2011.13933].

[281] P. Bueno, J. Camps and A. V. López, *Holographic entanglement entropy for perturbative higher-curvature gravities*, *JHEP* **04** (2021) 145, [2012.14033].

[282] H. R. Bakhtiarizadeh, *Charged rotating black strings in Einsteinian cubic gravity*, *Phys. Rev. D* **105** (2022) 064037, [2111.02663].

[283] P. Bueno, P. A. Cano, Q. Llorens, J. Moreno and G. van der Velde, *Aspects of three-dimensional higher-curvature gravities*, 2201.07266.

[284] J. D. Edelstein, N. Grandi and A. R. Sánchez, *Holographic superconductivity in Einsteinian Cubic Gravity*, 2202.05781.

[285] M. Salgado, *A Simple theorem to generate exact black hole solutions*, *Class. Quant. Grav.* **20** (2003) 4551–4566, [gr-qc/0304010].

[286] M. Kord Zangeneh and A. Kazemi, *Topological Born–Infeld charged black holes in Einsteinian cubic gravity*, *Eur. Phys. J. C* **80** (2020) 794, [2003.04458].

[287] H.-S. Liu, Z.-F. Mai, Y.-Z. Li and H. Lü, *Quasi-topological Electromagnetism: Dark Energy, Dyonic Black Holes, Stable Photon Spheres and Hidden Electromagnetic Duality*, *Sci. China Phys. Mech. Astron.* **63** (2020) 240411, [1907.10876].

[288] A. Cisterna, G. Giribet, J. Oliva and K. Pallikaris, *Quasitopological electromagnetism and black holes*, *Phys. Rev. D* **101** (2020) 124041, [2004.05474].

[289] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, *Nucl. Phys. B* **193** (1981) 221–244.

[290] M. Henneaux and C. Teitelboim, *Dynamics of Chiral (Selfdual) P Forms*, *Phys. Lett. B* **206** (1988) 650–654.

[291] M. J. Duff, *Duality Rotations in String Theory*, *Nucl. Phys. B* **335** (1990) 610.

[292] M. J. Duff and J. X. Lu, *Duality Rotations in Membrane Theory*, *Nucl. Phys. B* **347** (1990) 394–419.

[293] A. Sen, *Electric magnetic duality in string theory*, *Nucl. Phys. B* **404** (1993) 109–126, [hep-th/9207053].

[294] J. H. Schwarz and A. Sen, *Duality symmetric actions*, *Nucl. Phys. B* **411** (1994) 35–63, [hep-th/9304154].

[295] B. de Wit, *Electric magnetic dualities in supergravity*, *Nucl. Phys. B Proc. Suppl.* **101** (2001) 154–171, [hep-th/0103086].

[296] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, *Nucl. Phys. B* **438** (1995) 109–137, [hep-th/9410167].

[297] C. Hull and A. Van Proeyen, *Pseudoduality*, *Phys. Lett. B* **351** (1995) 188–193, [hep-th/9503022].

[298] C. I. Lazaroiu and C. S. Shahbazi, *Generalized Einstein-Scalar-Maxwell theories and locally geometric U-folds*, *Rev. Math. Phys.* **30** (2018) 1850012, [1609.05872].

[299] C. I. Lazaroiu and C. S. Shahbazi, *Four-dimensional geometric supergravity and electromagnetic duality: a brief guide for mathematicians*, *Rev. Roumaine Math. Pures Appl.* **66** (2021) 265–306, [2006.16157].

[300] A. Sen, $\mathrm{O}(d) \otimes \mathrm{O}(d)$ *symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes*, *Phys. Lett. B* **271** (1991) 295–300.

[301] E. Bergshoeff, B. Janssen and T. Ortín, *Solution generating transformations and the string effective action*, *Class. Quant. Grav.* **13** (1996) 321–343, [hep-th/9506156].

[302] K. A. Meissner, *Symmetries of higher order string gravity actions*, *Phys. Lett. B* **392** (1997) 298–304, [hep-th/9610131].

[303] C. Eloy, O. Hohm and H. Samtleben, *Duality Invariance and Higher Derivatives*, *Phys. Rev. D* **101** (2020) 126018, [2004.13140].

[304] Z. Elgood and T. Ortín, *T duality and Wald entropy formula in the Heterotic Superstring effective action at first-order in* α', *JHEP* **10** (2020) 097, [2005.11272].

[305] T. Ortín, *O(n, n) invariance and Wald entropy formula in the Heterotic Superstring effective action at first order in* α', *JHEP* **01** (2021) 187, [2005.14618].

[306] T. Codina, O. Hohm and D. Marques, *String Dualities at Order* α'^3, *Phys. Rev. Lett.* **126** (2021) 171602, [2012.15677].

[307] O. Hohm and B. Zwiebach, *Duality invariant cosmology to all orders in* α*'*, *Phys. Rev. D* **100** (2019) 126011, [1905.06963].

[308] O. Hohm and B. Zwiebach, *Non-perturbative de Sitter vacua via* α' *corrections*, *Int. J. Mod. Phys. D* **28** (2019) 1943002, [1905.06583].

[309] C. Krishnan, *de Sitter,* α'*-Corrections & Duality Invariant Cosmology*, *JCAP* **10** (2019) 009, [1906.09257].

[310] H. Bernardo, R. Brandenberger and G. Franzmann, $O(d, d)$ *covariant string cosmology to all orders in* α', *JHEP* **02** (2020) 178, [1911.00088].

[311] C. A. Núñez and F. E. Rost, *New non-perturbative de Sitter vacua in* α'*-complete cosmology*, *JHEP* **03** (2021) 007, [2011.10091].

[312] D. Marqués and C. A. Núñez, *T-duality and* α*'-corrections*, *JHEP* **10** (2015) 084, [1507.00652].

[313] W. H. Baron, J. J. Fernandez-Melgarejo, D. Marques and C. Núñez, *The Odd story of* α*'-corrections*, *JHEP* **04** (2017) 078, [1702.05489].

[314] H. Razaghian and M. R. Garousi, R^4 *terms in supergravities via T-duality constraint*, *Phys. Rev. D* **97** (2018) 106013, [1801.06834].

[315] M. Born and L. Infeld, *Foundations of the new field theory*, *Proc. Roy. Soc. Lond. A* **144** (1934) 425–451.

[316] I. Bialynicki-Birula, *Nonlinear electrodynamics: Variations on a theme by born and infeld*, *Quantum theory of particles and fields* (1983) 31–48.

[317] I. Bialynicki-Birula, *Field theory of photon dust*, *Acta Phys. Polon. B* **23** (1992) 553–559.

[318] E. Schrödinger, *Contributions to Born's new theory of the electromagnetic field*, *Proc. Roy. Soc. Lond. A* **150** (1935) 465–477.

[319] G. W. Gibbons and D. A. Rasheed, *Electric - magnetic duality rotations in nonlinear electrodynamics*, *Nucl. Phys. B* **454** (1995) 185–206, [hep-th/9506035].

[320] E. S. Fradkin and A. A. Tseytlin, *Nonlinear Electrodynamics from Quantized Strings*, *Phys. Lett. B* **163** (1985) 123–130.

[321] R. G. Leigh, *Dirac-Born-Infeld Action from Dirichlet Sigma Model*, *Mod. Phys. Lett. A* **4** (1989) 2767.

[322] I. Bandos, K. Lechner, D. Sorokin and P. K. Townsend, *A non-linear duality-invariant conformal extension of Maxwell's equations*, *Phys. Rev. D* **102** (2020) 121703, [2007.09092].

[323] B. P. Kosyakov, *Nonlinear electrodynamics with the maximum allowable symmetries*, *Phys. Lett. B* **810** (2020) 135840, [2007.13878].

[324] D. Flores-Alfonso, B. A. González-Morales, R. Linares and M. Maceda, *Black holes and gravitational waves sourced by non-linear duality rotation-invariant conformal electromagnetic matter*, *Phys. Lett. B* **812** (2021) 136011, [2011.10836].

[325] I. Bandos, K. Lechner, D. Sorokin and P. K. Townsend, *On p-form gauge theories and their conformal limits*, *JHEP* **03** (2021) 022, [2012.09286].

[326] A. Bokulić, T. Jurić and I. Smolić, *Black hole thermodynamics in the presence of nonlinear electromagnetic fields*, *Phys. Rev. D* **103** (2021) 124059, [2102.06213].

[327] I. Bandos, K. Lechner, D. Sorokin and P. K. Townsend, *ModMax meets Susy*, *JHEP* **10** (2021) 031, [2106.07547].

[328] D. P. Sorokin, *Introductory Notes on Non-linear Electrodynamics and its Applications*, 2112.12118.

[329] Y. Fourés-Bruhat, *Théoréme d'existence pour certains systémes d'éequations aux derivées partielles non linéaires*, *Acta Mat.* **88** (1952) 141–225.

[330] Y. Choquet-Bruhat and R. Geroch, *Global aspects of the cauchy problem in general relativity*, *Communications in Mathematical Physics* **14** (1969) 329–335.

[331] F. Pirant, A. Schild and R. Skinner, *Quantization of einstein's gravitational field equations. ii*, *Physical Review* **87** (1952) 452.

[332] P. A. M. Dirac, *Generalized hamiltonian dynamics*, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **246** (1958) 326–332.

[333] P. A. M. Dirac, *The theory of gravitation in hamiltonian form*, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **246** (1958) 333–343.

[334] R. Arnowitt, S. Deser and C. W. Misner, *Dynamical structure and definition of energy in general relativity*, *Physical Review* **116** (1959) 1322.

[335] Y. Choquet-Bruhat, *General relativity and the Einstein equations*. OUP Oxford, 2008.

[336] P. Chruściel, G. Galloway and D. Pollack, *Mathematical general relativity: a sampler*, *Bulletin of the American Mathematical Society* **47** (2010) 567–638.

[337] S. Klainerman and F. Nicolò, *The evolution problem in general relativity*, vol. 25. Springer Science & Business Media, 2012.

[338] L. Lehner, *Numerical relativity: a review*, *Classical and Quantum Gravity* **18** (2001) R25.

[339] M. Alcubierre, *Introduction to 3+ 1 numerical relativity*, vol. 140. OUP Oxford, 2008.

[340] T. W. Baumgarte and S. L. Shapiro, *Numerical relativity: solving Einstein's equations on the computer*. Cambridge University Press, 2010.

[341] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

[342] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[343] H. Ringström, *The Cauchy problem in general relativity*, vol. 6. European Mathematical Society, 2009.

[344] R. Geroch, *Domain of dependence*, *Journal of Mathematical Physics* **11** (1970) 437–449.

[345] A. N. Bernal and M. Sánchez, *On Smooth Cauchy hypersurfaces and Geroch's splitting theorem*, *Commun. Math. Phys.* **243** (2003) 461–470, [gr-qc/0306108].

[346] A. N. Bernal and M. Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic space-times*, *Commun. Math. Phys.* **257** (2005) 43–50, [gr-qc/0401112].

[347] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)* (Jan., 1916) 189–196.

[348] Jebsen, J.T., *Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum*, *Ark. f Mat. Astron. och Fys* **15** (1921) 18.

[349] G. D. Birkhoff and R. E. Langer, *Relativity and modern physics*, vol. 1. Harvard University Press Cambridge, 1923.

[350] K. Thorne, *Black Holes & Time Warps: Einstein's Outrageous Legacy (Commonwealth Fund Book Program)*. WW Norton & Company, 1995.

[351] J. M. M. Senovilla, *Hoyos negros y sus misteriosos interiores (Black holes and their mysterious interiors)*, 2102.10112.

[352] J. M. M. Senovilla, *A critical appraisal of the singularity theorems*, *Phil. Trans. A. Math. Phys. Eng. Sci.* **380** (2022) 20210174, [2108.07296].

[353] H. O. Silva, A. Ghosh and A. Buonanno, *Black-hole ringdown as a probe of higher-curvature gravity theories*, 2205.05132.

[354] R. Penrose, *Conformal treatment of infinity*, *Relativity, Groups and Topology* (1963) 565–586.

[355] R. Penrose, *Structure of space-time*, in *Battelle Rencontres – 1967 lectures in mathematics and physics* (C. M. de Witt and J. A. Wheeler, eds.), Battelle rencontres – 1967 lectures in mathematics and physics, (New York), pp. 121–235, W. A. Benjamin, 1968.

[356] S. W. Hawking, *Black holes in general relativity*, *Communications in Mathematical Physics* **25** (1972) 152–166.

[357] A. Ashtekar and A. Magnon, *Asymptotically anti-de Sitter space-times*, *Class. Quant. Grav.* **1** (1984) L39–L44.

[358] B. Carter, *Black holes equilibrium states*, in *Les Houches Summer School of Theoretical Physics: Black Holes*, pp. 57–214, 1973.

[359] H. Friedrich, I. Racz and R. M. Wald, *On the rigidity theorem for space-times with a stationary event horizon or a compact Cauchy horizon*, *Commun. Math. Phys.* **204** (1999) 691–707, [gr-qc/9811021].

[360] I. Racz and R. M. Wald, *Global extensions of space-times describing asymptotic final states of black holes*, *Class. Quant. Grav.* **13** (1996) 539–553, [gr-qc/9507055].

[361] R. M. Wald, *General relativity*. University of Chicago press, 2010.

[362] R. Boyer, *Geodesic killing orbits and bifurcate killing horizons*, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **311** (1969) 245–252.

[363] T. Ortín, *Komar integrals for theories of higher order in the Riemann curvature and black-hole chemistry*, *JHEP* **08** (2021) 023, [2104.10717].

[364] R. M. Wald, *The thermodynamics of black holes*, *Living Rev. Rel.* **4** (2001) 6, [gr-qc/9912119].

[365] W. Israel, *Third Law of Black-Hole Dynamics: A Formulation and Proof*, *Phys. Rev. Lett.* **57** (1986) 397.

[366] R. M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev.* **D48** (1993) 3427–3431, [gr-qc/9307038].

[367] K. Prabhu, *The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom*, *Class. Quant. Grav.* **34** (2017) 035011, [1511.00388].

[368] Z. Elgood, P. Meessen and T. Ortín, *The first law of black hole mechanics in the Einstein-Maxwell theory revisited*, 2006.02792.

[369] Z. Elgood, T. Ortín and D. Pereñíguez, *The first law and Wald entropy formula of heterotic stringy black holes at first order in α'*, *JHEP* **05** (2021) 110, [2012.14892].

[370] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846–864, [gr-qc/9403028].

[371] T. Jacobson, G. Kang and R. C. Myers, *On black hole entropy*, *Phys. Rev.* **D49** (1994) 6587–6598, [gr-qc/9312023].

[372] S. Chatterjee and M. Parikh, *The second law in four-dimensional Einstein-Gauss-Bonnet gravity*, *Class. Quant. Grav.* **31** (2014) 155007, [1312.1323].

[373] X. Dong, *Holographic Entanglement Entropy for General Higher Derivative Gravity*, *JHEP* **01** (2014) 044, [1310.5713].

[374] S. Sarkar and A. C. Wall, *Generalized second law at linear order for actions that are functions of Lovelock densities*, *Phys. Rev. D* **88** (2013) 044017, [1306.1623].

[375] A. C. Wall, *A Second Law for Higher Curvature Gravity*, *Int. J. Mod. Phys. D* **24** (2015) 1544014, [1504.08040].

[376] T. Torii and H. Maeda, *Spacetime structure of static solutions in Gauss-Bonnet gravity: Neutral case*, *Phys. Rev. D* **71** (2005) 124002, [hep-th/0504127].

[377] X. O. Camanho and J. D. Edelstein, *Cosmic censorship in Lovelock theory*, *JHEP* **11** (2013) 151, [1308.0304].

[378] X. O. Camanho, *Lovelock gravity, black holes and holography*. PhD thesis, Santiago de Compostela U., 2013. 1509.08129.

[379] R. Penrose, *Gravitational collapse and space-time singularities*, *Phys. Rev. Lett.* **14** (Jan, 1965) 57–59.

[380] R. P. Geroch, *What is a singularity in general relativity?*, *Annals Phys.* **48** (1968) 526–540.

[381] R. Penrose, *Techniques in differential topology in relativity.* SIAM, 1972.

[382] D. Gannon, *Singularities in nonsimply connected space–times*, *Journal of Mathematical Physics* **16** (1975) 2364–2367.

[383] D. Gannon, *On the topology of spacelike hypersurfaces, singularities, and black holes*, *General Relativity and Gravitation* **7** (1976) 219–232.

[384] R. Penrose, *Gravitational collapse: The role of general relativity*, *Riv. Nuovo Cim.* **1** (1969) 252–276.

[385] J. M. Bardeen, *Non-singular general-relativistic gravitational collapse*, in *Proc. Int. Conf. GR5, Tbilisi*, vol. 174, p. 174, 1968.

[386] I. Dymnikova, *Vacuum nonsingular black hole*, *Gen. Rel. Grav.* **24** (1992) 235–242.

[387] A. Borde, *Regular black holes and topology change*, *Phys. Rev. D* **55** (1997) 7615–7617, [gr-qc/9612057].

[388] S. A. Hayward, *Formation and evaporation of regular black holes*, *Phys. Rev. Lett.* **96** (2006) 031103, [gr-qc/0506126].

[389] V. P. Frolov, *Information loss problem and a 'black hole' model with a closed apparent horizon*, *JHEP* **05** (2014) 049, [1402.5446].

[390] V. P. Frolov, *Notes on nonsingular models of black holes*, *Phys. Rev.* **D94** (2016) 104056, [1609.01758].

[391] R. Carballo-Rubio, F. Di Filippo, S. Liberati, C. Pacilio and M. Visser, *On the viability of regular black holes*, *JHEP* **07** (2018) 023, [1805.02675].

[392] A. Simpson and M. Visser, *Black-bounce to traversable wormhole*, *JCAP* **02** (2019) 042, [1812.07114].

[393] A. Simpson and M. Visser, *Regular black holes with asymptotically Minkowski cores*, *Universe* **6** (2019) 8, [1911.01020].

[394] E. Ayon-Beato and A. García, *Regular black hole in general relativity coupled to nonlinear electrodynamics*, *Phys. Rev. Lett.* **80** (1998) 5056–5059, [gr-qc/9911046].

[395] E. Ayon-Beato and A. García, *Nonsingular charged black hole solution for nonlinear source*, *Gen. Rel. Grav.* **31** (1999) 629–633, [gr-qc/9911084].

[396] E. Ayon-Beato and A. García, *New regular black hole solution from nonlinear electrodynamics*, *Phys. Lett. B* **464** (1999) 25, [hep-th/9911174].

[397] A. B. Balakin, V. V. Bochkarev and J. P. S. Lemos, *Non-minimal coupling for the gravitational and electromagnetic fields: Black hole solutions and solitons*, *Phys. Rev. D* **77** (2008) 084013, [0712.4066].

[398] J. P. S. Lemos and V. T. Zanchin, *Regular black holes: Electrically charged solutions, Reissner-Nordström outside a de Sitter core*, *Phys. Rev. D* **83** (2011) 124005, [1104.4790].

[399] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, *Towards singularity and ghost free theories of gravity*, *Phys. Rev. Lett.* **108** (2012) 031101, [1110.5249].

[400] G. J. Olmo and D. Rubiera-García, *Reissner-Nordström black holes in extended Palatini theories*, *Phys. Rev. D* **86** (2012) 044014, [1207.6004].

[401] A. B. Balakin, J. P. S. Lemos and A. E. Zayats, *Regular nonminimal magnetic black holes in spacetimes with a cosmological constant*, *Phys. Rev. D* **93** (2016) 024008, [1512.02653].

[402] G. J. Olmo and D. Rubiera-García, *Nonsingular Black Holes in $f(R)$ Theories*, *Universe* **1** (2015) 173–185, [1509.02430].

[403] Z.-Y. Fan and X. Wang, *Construction of Regular Black Holes in General Relativity*, *Phys. Rev. D* **94** (2016) 124027, [1610.02636].

[404] A. H. Chamseddine and V. Mukhanov, *Nonsingular Black Hole*, *Eur. Phys. J. C* **77** (2017) 183, [1612.05861].

[405] C. Menchon, G. J. Olmo and D. Rubiera-García, *Nonsingular black holes, wormholes, and de Sitter cores from anisotropic fluids*, *Phys. Rev. D* **96** (2017) 104028, [1709.09592].

[406] O. Sert, *Regular black hole solutions of the non-minimally coupled $Y(R)F^2$ gravity*, *J. Math. Phys.* **57** (2016) 032501, [1512.01172].

[407] L. Buoninfante, A. S. Koshelev, G. Lambiase and A. Mazumdar, *Classical properties of non-local, ghost- and singularity-free gravity*, *JCAP* **09** (2018) 034, [1802.00399].

[408] L. Buoninfante, G. Harmsen, S. Maheshwari and A. Mazumdar, *Nonsingular metric for an electrically charged point-source in ghost-free infinite derivative gravity*, *Phys. Rev. D* **98** (2018) 084009, [1804.09624].

[409] B. L. Giacchini and T. de Paula Netto, *Weak-field limit and regular solutions in polynomial higher-derivative gravities*, *Eur. Phys. J. C* **79** (2019) 217, [1806.05664].

[410] P. A. Cano, S. Chimento, T. Ortín and A. Ruipérez, *Regular Stringy Black Holes?*, *Phys. Rev. D* **99** (2019) 046014, [1806.08377].

[411] M. D. Kruskal, *Maximal extension of Schwarzschild metric*, *Phys. Rev.* **119** (1960) 1743–1745.

[412] G. Szekeres, *On the singularities of a Riemannian manifold*, *Publ. Math. Debrecen* **7** (1960) 285–301.

[413] H. Reissner, *Über die eigengravitation des elektrischen feldes nach der einsteinschen theorie*, *Annalen der Physik* **355** (1916) 106–120.

[414] H. Weyl, *Zur gravitationstheorie*, *Annalen der Physik* **359** (1917) 117–145.

[415] G. Nordström, *On the energy of the gravitational field in Einstein's theory*, *Proc. Kon. Ned. Akad. Wet.* **209** (1918) 1238.

[416] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060, [hep-th/0601001].

[417] B. Carter, *The complete analytic extension of the Reissner-Nordström metric in the special case $e^2 = m^2$*, *Physics Letters* **21** (1966) 423–424.

[418] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[419] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105–114, [hep-th/9802109].

[420] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[421] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308** (1993) 284–296, [gr-qc/9310026].

[422] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [hep-th/9409089].

[423] R. Bousso, *The Holographic principle*, *Rev. Mod. Phys.* **74** (2002) 825–874, [hep-th/0203101].

[424] A. Buchel, R. C. Myers and A. Sinha, *Beyond eta/s = 1/4 pi*, *JHEP* **03** (2009) 084, [0812.2521].

[425] D. M. Hofman, *Higher Derivative Gravity, Causality and Positivity of Energy in a UV complete QFT*, *Nucl. Phys. B* **823** (2009) 174–194, [0907.1625].

[426] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha and M. Smolkin, *Holographic GB gravity in arbitrary dimensions*, *JHEP* **03** (2010) 111, [0911.4257].

[427] L.-Y. Hung, R. C. Myers and M. Smolkin, *On Holographic Entanglement Entropy and Higher Curvature Gravity*, *JHEP* **04** (2011) 025, [1101.5813].

[428] X. O. Camanho, J. D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020, [1407.5597].

[429] P. Bueno, R. C. Myers and W. Witczak-Krempa, *Universality of corner entanglement in conformal field theories*, *Phys. Rev. Lett.* **115** (2015) 021602, [1505.04804].

[430] P. Bueno, *Supergravity, black holes and holography.* PhD thesis, U. Autonoma, Madrid (main), 2015.

[431] H. Osborn and G. M. Shore, *Correlation functions of the energy momentum tensor on spaces of constant curvature*, *Nucl. Phys. B* **571** (2000) 287–357, [hep-th/9909043].

[432] M. Serino, *The four-point correlation function of the energy-momentum tensor in the free conformal field theory of a scalar field*, *Eur. Phys. J. C* **80** (2020) 686, [2004.08668].

[433] H. Osborn and A. C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311–362, [hep-th/9307010].

[434] J. Erdmenger and H. Osborn, *Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions*, *Nucl. Phys. B* **483** (1997) 431–474, [hep-th/9605009].

[435] D. M. Hofman and J. Maldacena, *Conformal collider physics: Energy and charge correlations*, *JHEP* **05** (2008) 012, [0803.1467].

[436] X. O. Camanho and J. D. Edelstein, *Causality in AdS/CFT and Lovelock theory*, *JHEP* **06** (2010) 099, [0912.1944].

[437] X. O. Camanho, J. D. Edelstein and M. F. Paulos, *Lovelock theories, holography and the fate of the viscosity bound*, *JHEP* **05** (2011) 127, [1010.1682].

[438] X. O. Camanho, J. D. Edelstein and J. M. Sánchez De Santos, *Lovelock theory and the AdS/CFT correspondence*, *Gen. Rel. Grav.* **46** (2014) 1637, [1309.6483].

[439] J. Von Neumann, *Mathematical foundations of quantum mechanics*, in *Mathematical Foundations of Quantum Mechanics*. Princeton university press, 1955.

[440] C. E. Shannon, *A mathematical theory of communication*, *The Bell system technical journal* **27** (1948) 379–423.

[441] A. Rényi, *On measures of entropy and information*, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, vol. 4, pp. 547–562, University of California Press, 1961.

[442] A. Rényi, *On the foundations of information theory*, *Revue de l'Institut International de Statistique* (1965) 1–14.

[443] C. Beck and F. Schögl, *Thermodynamics of chaotic systems*. Cambridge University Press, 1995.

[444] K. Życzkowski, *Rényi extrapolation of shannon entropy*, *Open Systems & Information Dynamics* **10** (2003) 297–310.

[445] H. Casini, M. Huerta and R. C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036, [1102.0440].

[446] L.-Y. Hung, R. C. Myers, M. Smolkin and A. Yale, *Holographic Calculations of Renyi Entropy*, *JHEP* **12** (2011) 047, [1110.1084].

[447] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R. C. Myers and T. Sierens, *Holographic Charged Renyi Entropies*, *JHEP* **12** (2013) 059, [1310.4180].

[448] E. Schrödinger, *An undulatory theory of the mechanics of atoms and molecules*, *Phys. Rev.* **28** (Dec, 1926) 1049–1070.

[449] W. Gordon, *Der Comptoneffekt nach der Schrödingerschen Theorie*, *Zeitschrift fur Physik* **40** (Jan., 1926) 117–133.

[450] O. Klein, *Elektrodynamik und Wellenmechanik vom Standpunkt des Korrespondenzprinzips*, *Zeitschrift fur Physik* **41** (Oct., 1927) 407–442.

[451] W. Greiner et al., *Relativistic quantum mechanics*, vol. 2. Springer, 2000.

[452] T. Ohlsson, *Relativistic quantum physics: from advanced quantum mechanics to introductory quantum field theory*. Cambridge University Press, 2011.

[453] P. A. M. Dirac, *The quantum theory of the electron*, *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* **117** (1928) 610–624.

[454] P. A. M. Dirac, *A theory of electrons and protons*, *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character* **126** (1930) 360–365.

[455] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry (PMS-38)*, vol. 38. Princeton university press, 1989.

[456] T. Friedrich, *Dirac operators in Riemannian geometry*, vol. 25. American Mathematical Soc., 2000.

[457] M. J. Hamilton, *Mathematical gauge theory*. Springer, 2017.

[458] B. Booss, B. Booß-Bavnbek, K. P. Wojciechhowski and K. Wojciechowski, *Elliptic boundary problems for Dirac operators*, vol. 4. Springer Science & Business Media, 1993.

[459] C. Lazaroiu and C. Shahbazi, *Real pinor bundles and real lipschitz structures*, *Asian Journal of Mathematics* **23** (01, 2019) 749–836.

[460] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*. Teubner Verlagsgesellschaft, Leipzig, 1981.

[461] M. Karoubi, *Algèbres de clifford et k-théorie*, in *Annales scientifiques de l'École normale supérieure*, vol. 1, pp. 161–270, 1968.

[462] K. P. Tod, *All Metrics Admitting Supercovariantly Constant Spinors*, *Phys. Lett. B* **121** (1983) 241–244.

[463] K. P. Tod, *More on supercovariantly constant spinors*, *Class. Quant. Grav.* **12** (1995) 1801–1820.

[464] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, *Class. Quant. Grav.* **20** (2003) 4587–4634, [hep-th/0209114].

[465] J. P. Gauntlett and S. Pakis, *The Geometry of $D = 11$ killing spinors*, *JHEP* **04** (2003) 039, [hep-th/0212008].

[466] J. P. Gauntlett and J. B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five-dimensions*, *Phys. Rev. D* **68** (2003) 105009, [hep-th/0304064].

[467] J. Bellorín and T. Ortín, *All the supersymmetric configurations of N=4, d=4 supergravity*, *Nucl. Phys. B* **726** (2005) 171–209, [hep-th/0506056].

[468] J. Bellorín, P. Meessen and T. Ortín, *All the supersymmetric solutions of N=1,d=5 ungauged supergravity*, *JHEP* **01** (2007) 020, [hep-th/0610196].

[469] T. Ortín, *The Supersymmetric solutions and extensions of ungauged matter-coupled N=1, d=4 supergravity*, *JHEP* **05** (2008) 034, [0802.1799].

[470] G. Papadopoulos, *New half supersymmetric solutions of the heterotic string*, *Class. Quant. Grav.* **26** (2009) 135001, [0809.1156].

[471] A. Fontanella and T. Ortín, *On the supersymmetric solutions of the Heterotic Superstring effective action*, *JHEP* **06** (2020) 106, [1910.08496].

[472] A. Ruipérez, *Higher-derivative corrections to small black rings*, *Class. Quant. Grav.* **38** (2021) 145011, [2003.02269].

[473] V. Cortés, C. Lazaroiu and C. Shahbazi, *Spinors of real type as polyforms and the generalized killing equation*, *Mathematische Zeitschrift* **299** (2021) 1351–1419.

[474] Y. Hamada, T. Noumi and G. Shiu, *Weak Gravity Conjecture from Unitarity and Causality*, *Phys. Rev. Lett.* **123** (2019) 051601, [1810.03637].

[475] B. Bellazzini, M. Lewandowski and J. Serra, *Positivity of Amplitudes, Weak Gravity Conjecture, and Modified Gravity*, *Phys. Rev. Lett.* **123** (2019) 251103, [1902.03250].

[476] S. Cremonini, C. R. T. Jones, J. T. Liu and B. McPeak, *Higher-Derivative Corrections to Entropy and the Weak Gravity Conjecture in Anti-de Sitter Space*, *JHEP* **09** (2020) 003, [1912.11161].

[477] A. Lischewski, *The Cauchy problem for parallel spinors as first-order symmetric hyperbolic system*, 1503.04946.

[478] T. Leistner and A. Lischewski, *Hyperbolic evolution equations, lorentzian holonomy, and riemannian generalised killing spinors*, *J. Geom. Anal.* **29** (2019) 33–82.

[479] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1. John Wiley and Sons, 1963.

[480] J. Moreno, *Higher-Curvature Gravity and Entanglement Entropy*. PhD thesis, Pontificia Universidad Católica de Valparaíso-Universidad Técnica Federico Santa María, 2022.

[481] A. Karasu, E. Kenar and B. Tekin, *Minimal extension of Einstein's theory: The quartic gravity*, Phys. Rev. D **93** (2016) 084040, [1602.02567].

[482] T. C. Sisman, I. Gullu and B. Tekin, *All unitary cubic curvature gravities in D dimensions*, Class. Quant. Grav. **28** (2011) 195004, [1103.2307].

[483] B. Tekin, *Particle Content of Quadratic and $f(R_{\mu\nu\sigma\rho})$ Theories in $(A)dS$*, Phys. Rev. D **93** (2016) 101502, [1604.00891].

[484] P. Bueno, P. A. Cano, A. O. Lasso and P. F. Ramírez, *f(Lovelock) theories of gravity*, JHEP **04** (2016) 028, [1602.07310].

[485] A. Ghodsi and F. Najafi, *Ricci cubic gravity in d dimensions, gravitons and SAdS/Lifshitz black holes*, Eur. Phys. J. C **77** (2017) 559, [1702.06798].

[486] Y.-Z. Li, H.-S. Liu and H. Lu, *Quasi-Topological Ricci Polynomial Gravities*, JHEP **02** (2018) 166, [1708.07198].

[487] Y.-Z. Li, H. Lu and J.-B. Wu, *Causality and a-theorem Constraints on Ricci Polynomial and Riemann Cubic Gravities*, Phys. Rev. D **97** (2018) 024023, [1711.03650].

[488] Y.-Z. Li, H. Lü and Z.-F. Mai, *Universal Structure of Covariant Holographic Two-Point Functions In Massless Higher-Order Gravities*, JHEP **10** (2018) 063, [1808.00494].

[489] Y.-Z. Li, *Holographic Studies of The Generic Massless Cubic Gravities*, Phys. Rev. D **99** (2019) 066014, [1901.03349].

[490] H. Lü and R. Wen, *Holographic (a,c)-charges and Their Universal Relation in d=6 from Massless Higher-order Gravities*, Phys. Rev. D **99** (2019) 126003, [1901.11037].

[491] R. Arnowitt, S. Deser and C. W. Misner, *Energy and the Criteria for Radiation in General Relativity*, Phys. Rev. **118** (1960) 1100–1104.

[492] R. L. Arnowitt, S. Deser and C. W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. **122** (1961) 997.

[493] A. Dey, P. Roy and T. Sarkar, *On holographic Rényi entropy in some modified theories of gravity*, JHEP **04** (2018) 098, [1609.02290].

[494] W. T. Emond and N. Moynihan, *Scattering Amplitudes, Black Holes and Leading Singularities in Cubic Theories of Gravity*, JHEP **12** (2019) 019, [1905.08213].

[495] G. Exirifard, *The alpha-prime stretched horizon in Heterotic string*, JHEP **10** (2006) 070, [hep-th/0604021].

[496] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, Gen. Rel. Grav. **40** (2008) 2249–2431, [0708.1270].

[497] J. T. Liu and P. Szepietowski, *Higher derivative corrections to R-charged AdS(5) black holes and field redefinitions*, Phys. Rev. **D79** (2009) 084042, [0806.1026].

[498] A. Castro, N. Dehmami, G. Giribet and D. Kastor, *On the Universality of Inner Black Hole Mechanics and Higher Curvature Gravity*, *JHEP* **07** (2013) 164, [1304.1696].

[499] M. R. Mohammadi Mozaffar, A. Mollabashi, M. M. Sheikh-Jabbari and M. H. Vahidinia, *Holographic Entanglement Entropy, Field Redefinition Invariance and Higher Derivative Gravity Theories*, *Phys. Rev.* **D94** (2016) 046002, [1603.05713].

[500] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, *Normal forms for tensor polynomials. 1: The Riemann tensor*, *Class. Quant. Grav.* **9** (1992) 1151–1197.

[501] S. Endlich, V. Gorbenko, J. Huang and L. Senatore, *An effective formalism for testing extensions to General Relativity with gravitational waves*, *JHEP* **09** (2017) 122, [1704.01590].

[502] S. Deser and A. V. Ryzhov, *Curvature invariants of static spherically symmetric geometries*, *Class. Quant. Grav.* **22** (2005) 3315–3324, [gr-qc/0505039].

[503] B. Andrews and C. Hopper, *The Ricci Flow in Riemannian Geometry*. Springer, 2011, 10.1007/978-3-642-16286-2.

[504] D. J. Gross and E. Witten, *Superstring Modifications of Einstein's Equations*, *Nucl. Phys. B* **277** (1986) 1.

[505] P. S. Howe and P. C. West, *The Complete N=2, D=10 Supergravity*, *Nucl. Phys.* **B238** (1984) 181–220.

[506] R. S. Palais, *The principle of symmetric criticality*, *Commun. Math. Phys.* **69** (1979) 19–30.

[507] S. Deser and B. Tekin, *Shortcuts to high symmetry solutions in gravitational theories*, *Class. Quant. Grav.* **20** (2003) 4877–4884, [gr-qc/0306114].

[508] R. C. Myers and A. Sinha, *Holographic c-theorems in arbitrary dimensions*, *JHEP* **01** (2011) 125, [1011.5819].

[509] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023, [hep-th/9806087].

[510] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413–428, [hep-th/9902121].

[511] R. Emparan, C. V. Johnson and R. C. Myers, *Surface terms as counterterms in the AdS / CFT correspondence*, *Phys. Rev.* **D60** (1999) 104001, [hep-th/9903238].

[512] J. D. Edelstein, A. Sánchez Rivadulla and D. V. Rodríguez, *Are there Einsteinian gravities involving covariant derivatives of the Riemann tensor?*, 2204.13567.

[513] T. Padmanabhan, *Some aspects of field equations in generalised theories of gravity*, *Phys. Rev. D* **84** (2011) 124041, [arXiv:1109.3846].

[514] C. Senturk, T. C. Sisman and B. Tekin, *Energy and Angular Momentum in Generic F(Riemann) Theories*, *Phys. Rev.* **D86** (2012) 124030, [1209.2056].

[515] S. Gao, *The First law of black hole mechanics in Einstein-Maxwell and Einstein-Yang-Mills theories*, *Phys. Rev. D* **68** (2003) 044016, [gr-qc/0304094].

[516] D. Rasheed, *Nonlinear electrodynamics: Zeroth and first laws of black hole mechanics*, hep-th/9702087.

[517] V. Iyer and R. M. Wald, *A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes*, Phys. Rev. **D52** (1995) 4430–4439, [gr-qc/9503052].

[518] X.-H. Feng and H. Lu, *Higher-Derivative Gravity with Non-minimally Coupled Maxwell Field*, Eur. Phys. J. C **76** (2016) 178, [1512.09153].

[519] G. Horndeski, *Conservation of Charge and the Einstein-Maxwell Field Equations*, J. Math. Phys. **17** (1976) 1980–1987.

[520] K. Yoshida and K. Shiraishi, *A new vector-tensor theory and higher-dimensional cosmology*, Physica Scripta **43** (Feb, 1991) 129?132.

[521] F. Mueller-Hoissen and R. Sippel, *Spherically Symmetric Solutions of the Nonminimally Coupled Einstein-maxwell Equations*, Class. Quant. Grav. **5** (1988) 1473.

[522] A. B. Balakin, J. P. S. Lemos and A. E. Zayats, *Magnetic black holes and monopoles in a nonminimal Einstein-Yang-Mills theory with a cosmological constant: Exact solutions*, Phys. Rev. D **93** (2016) 084004, [1603.02676].

[523] C. Teitelboim and J. Zanelli, *Dimensionally continued topological gravitation theory in Hamiltonian form*, Class. Quant. Grav. **4** (1987) L125.

[524] R. C. Myers, *Higher Derivative Gravity, Surface Terms and String Theory*, Phys. Rev. D **36** (1987) 392.

[525] N. Deruelle, M. Sasaki, Y. Sendouda and D. Yamauchi, *Hamiltonian formulation of f(Riemann) theories of gravity*, Prog. Theor. Phys. **123** (2010) 169–185, [0908.0679].

[526] J. Smolic and M. Taylor, *Higher derivative effects for 4d AdS gravity*, JHEP **06** (2013) 096, [1301.5205].

[527] A. Teimouri, S. Talaganis, J. Edholm and A. Mazumdar, *Generalised Boundary Terms for Higher Derivative Theories of Gravity*, JHEP **08** (2016) 144, [1606.01911].

[528] C. Cheung, J. Liu and G. N. Remmen, *Proof of the Weak Gravity Conjecture from Black Hole Entropy*, JHEP **10** (2018) 004, [1801.08546].

[529] A. M. Charles, *The Weak Gravity Conjecture, RG Flows, and Supersymmetry*, 1906.07734.

[530] G. J. Loges, T. Noumi and G. Shiu, *Thermodynamics of 4D Dilatonic Black Holes and the Weak Gravity Conjecture*, 1909.01352.

[531] G. Goon and R. Penco, *Universal Relation between Corrections to Entropy and Extremality*, Phys. Rev. Lett. **124** (2020) 101103, [1909.05254].

[532] S. Andriolo, T.-C. Huang, T. Noumi, H. Ooguri and G. Shiu, *Duality and Axionic Weak Gravity*, 2004.13721.

[533] G. J. Loges, T. Noumi and G. Shiu, *Duality and Supersymmetry Constraints on the Weak Gravity Conjecture*, 2006.06696.

[534] K. Bamba and S. D. Odintsov, *Inflation and late-time cosmic acceleration in non-minimal Maxwell-$F(R)$ gravity and the generation of large-scale magnetic fields*, JCAP **04** (2008) 024, [0801.0954].

[535] W. Chemissany, R. Kallosh and T. Ortín, *Born-Infeld with Higher Derivatives*, *Phys. Rev. D* **85** (2012) 046002, [1112.0332].

[536] S. Deser, M. Henneaux and C. Teitelboim, *Electric - magnetic black hole duality*, *Phys. Rev. D* **55** (1997) 826–828, [hep-th/9607182].

[537] G. W. Gibbons, *Aspects of Born-Infeld theory and string / M theory*, *AIP Conf. Proc.* **589** (2001) 324–350, [hep-th/0106059].

[538] G. W. Gibbons and D. A. Rasheed, *Sl(2,R) invariance of nonlinear electrodynamics coupled to an axion and a dilaton*, *Phys. Lett. B* **365** (1996) 46–50, [hep-th/9509141].

[539] M. K. Gaillard and B. Zumino, *Selfduality in nonlinear electromagnetism*, *Lect. Notes Phys.* **509** (1998) 121, [hep-th/9705226].

[540] M. K. Gaillard and B. Zumino, *Nonlinear electromagnetic selfduality and Legendre transformations*, in *A Newton Institute Euroconference on Duality and Supersymmetric Theories*, 12, 1997. hep-th/9712103.

[541] M. Hatsuda, K. Kamimura and S. Sekiya, *Electric magnetic duality invariant Lagrangians*, *Nucl. Phys. B* **561** (1999) 341–353, [hep-th/9906103].

[542] D. Brace, B. Morariu and B. Zumino, *Duality invariant Born-Infeld theory*, hep-th/9905218.

[543] K. Babaei Velni and H. Babaei-Aghbolagh, *On SL (2,R) symmetry in nonlinear electrodynamics theories*, *Nucl. Phys. B* **913** (2016) 987–1000, [1610.07790].

[544] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008, [hep-th/0508042].

[545] D. Kutasov, F. Larsen and R. G. Leigh, *String theory in magnetic monopole backgrounds*, *Nucl. Phys. B* **550** (1999) 183–213, [hep-th/9812027].

[546] A. Dabholkar, *Exact counting of black hole microstates*, *Phys. Rev. Lett.* **94** (2005) 241301, [hep-th/0409148].

[547] P. Dominis Prester and T. Terzic, *α'-exact entropies for BPS and non-BPS extremal dyonic black holes in heterotic string theory from ten-dimensional supersymmetry*, *JHEP* **12** (2008) 088, [0809.4954].

[548] S. Nojiri and S. D. Odintsov, *On the conformal anomaly from higher derivative gravity in AdS / CFT correspondence*, *Int. J. Mod. Phys. A* **15** (2000) 413–428, [hep-th/9903033].

[549] M. Blau, K. S. Narain and E. Gava, *On subleading contributions to the AdS / CFT trace anomaly*, *JHEP* **09** (1999) 018, [hep-th/9904179].

[550] J. de Boer, M. Kulaxizi and A. Parnachev, *AdS(7)/CFT(6), Gauss-Bonnet Gravity, and Viscosity Bound*, *JHEP* **03** (2010) 087, [0910.5347].

[551] R. C. Myers and A. Sinha, *Seeing a c-theorem with holography*, *Phys. Rev. D* **82** (2010) 046006, [1006.1263].

[552] P. Bueno and R. C. Myers, *Corner contributions to holographic entanglement entropy*, *JHEP* **08** (2015) 068, [1505.07842].

[553] E. Perlmutter, *A universal feature of CFT Rényi entropy*, *JHEP* **03** (2014) 117, [1308.1083].

[554] M. Mezei, *Entanglement entropy across a deformed sphere*, *Phys. Rev. D* **91** (2015) 045038, [1411.7011].

[555] C.-S. Chu and R.-X. Miao, *Universality in the shape dependence of holographic Rényi entropy for general higher derivative gravity*, *JHEP* **12** (2016) 036, [1608.00328].

[556] Y. Kats and P. Petrov, *Effect of curvature squared corrections in AdS on the viscosity of the dual gauge theory*, *JHEP* **01** (2009) 044, [0712.0743].

[557] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, *Viscosity Bound Violation in Higher Derivative Gravity*, *Phys. Rev. D* **77** (2008) 126006, [0712.0805].

[558] R.-G. Cai, Z.-Y. Nie and Y.-W. Sun, *Shear Viscosity from Effective Couplings of Gravitons*, *Phys. Rev. D* **78** (2008) 126007, [0811.1665].

[559] X.-H. Ge, Y. Matsuo, F.-W. Shu, S.-J. Sin and T. Tsukioka, *Viscosity Bound, Causality Violation and Instability with Stringy Correction and Charge*, *JHEP* **10** (2008) 009, [0808.2354].

[560] R. Gregory, S. Kanno and J. Soda, *Holographic Superconductors with Higher Curvature Corrections*, *JHEP* **10** (2009) 010, [0907.3203].

[561] J. Jing and S. Chen, *Holographic superconductors in the Born-Infeld electrodynamics*, *Phys. Lett. B* **686** (2010) 68–71, [1001.4227].

[562] R.-G. Cai, Z.-Y. Nie and H.-Q. Zhang, *Holographic p-wave superconductors from Gauss-Bonnet gravity*, *Phys. Rev. D* **82** (2010) 066007, [1007.3321].

[563] J. de Boer, M. Kulaxizi and A. Parnachev, *Holographic Entanglement Entropy in Lovelock Gravities*, *JHEP* **07** (2011) 109, [1101.5781].

[564] L.-Y. Hung, R. C. Myers and M. Smolkin, *Twist operators in higher dimensions*, *JHEP* **10** (2014) 178, [1407.6429].

[565] L. Bianchi, S. Chapman, X. Dong, D. A. Galante, M. Meineri and R. C. Myers, *Shape dependence of holographic Rényi entropy in general dimensions*, *JHEP* **11** (2016) 180, [1607.07418].

[566] S. Cremonini, K. Hanaki, J. T. Liu and P. Szepietowski, *Black holes in five-dimensional gauged supergravity with higher derivatives*, *JHEP* **12** (2009) 045, [0812.3572].

[567] R. C. Myers, M. F. Paulos and A. Sinha, *Holographic Hydrodynamics with a Chemical Potential*, *JHEP* **06** (2009) 006, [0903.2834].

[568] R. C. Myers, S. Sachdev and A. Singh, *Holographic Quantum Critical Transport without Self-Duality*, *Phys. Rev. D* **83** (2011) 066017, [1010.0443].

[569] R.-G. Cai and D.-W. Pang, *Holography of Charged Black Holes with RF^2 Corrections*, *Phys. Rev. D* **84** (2011) 066004, [1104.4453].

[570] X.-H. Ge, S.-J. Sin, S.-F. Wu and G.-H. Yang, *Shear viscosity and instability from third order Lovelock gravity*, *Phys. Rev. D* **80** (2009) 104019, [0905.2675].

[571] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric Completion of an R^2 term in Five-dimensional Supergravity*, *Prog. Theor. Phys.* **117** (2007) 533, [hep-th/0611329].

[572] N. Bobev, K. Hristov and V. Reys, *AdS_5 Holography and Higher-Derivative Supergravity*, 2112.06961.

[573] J. T. Liu and R. J. Saskowski, *Four-derivative Corrections to Minimal Gauged Supergravity in Five Dimensions*, 2201.04690.

[574] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev. D* **62** (2000) 084013, [hep-th/0003271].

[575] Z.-Y. Fan, B. Chen and H. Lu, *Criticality in Einstein–Gauss–Bonnet gravity: gravity without graviton*, *Eur. Phys. J. C* **76** (2016) 542, [1606.02728].

[576] D. Glavan and C. Lin, *Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime*, *Phys. Rev. Lett.* **124** (2020) 081301, [1905.03601].

[577] H. Lu and Y. Pang, *Horndeski gravity as $D \to 4$ limit of Gauss-Bonnet*, *Phys. Lett. B* **809** (2020) 135717, [2003.11552].

[578] M. Gürses, T. c. cSicsman and B. Tekin, *Is there a novel Einstein–Gauss–Bonnet theory in four dimensions?*, *Eur. Phys. J. C* **80** (2020) 647, [2004.03390].

[579] R. A. Hennigar, D. Kubizňák, R. B. Mann and C. Pollack, *On taking the $D \to 4$ limit of Gauss-Bonnet gravity: theory and solutions*, *JHEP* **07** (2020) 027, [2004.09472].

[580] E. Altas and B. Tekin, *New approach to conserved charges of generic gravity in AdS spacetimes*, *Phys. Rev. D* **99** (2019) 044016, [1811.11525].

[581] A. Yale, *Simple counterterms for asymptotically AdS spacetimes in Lovelock gravity*, *Phys. Rev. D* **84** (2011) 104036, [1107.1250].

[582] I. J. Araya, J. D. Edelstein, A. R. Sánchez, D. V. Rodríguez and A. V. Lopez, *Universal renormalization procedure for higher curvature gravities in $D \leq 5$*, *JHEP* **09** (2021) 142, [2108.01126].

[583] X. O. Camanho and J. D. Edelstein, *Causality constraints in AdS/CFT from conformal collider physics and Gauss-Bonnet gravity*, *JHEP* **04** (2010) 007, [0911.3160].

[584] G. T. Horowitz and N. Itzhaki, *Black holes, shock waves, and causality in the AdS / CFT correspondence*, *JHEP* **02** (1999) 010, [hep-th/9901012].

[585] D. Chowdhury, S. Raju, S. Sachdev, A. Singh and P. Strack, *Multipoint correlators of conformal field theories: implications for quantum critical transport*, *Phys. Rev. B* **87** (2013) 085138, [1210.5247].

[586] A. Buchel and R. C. Myers, *Causality of Holographic Hydrodynamics*, *JHEP* **08** (2009) 016, [0906.2922].

[587] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, *The Viscosity Bound and Causality Violation*, *Phys. Rev. Lett.* **100** (2008) 191601, [0802.3318].

[588] N. Arkani-Hamed, Y.-t. Huang, J.-Y. Liu and G. N. Remmen, *Causality, unitarity, and the weak gravity conjecture*, *JHEP* **03** (2022) 083, [2109.13937].

[589] L. Aalsma, *Corrections to extremal black holes from Iyer-Wald formalism*, *Phys. Rev. D* **105** (2022) 066022, [2111.04201].

[590] B. McInnes, *Holographic Dual of The Weak Gravity Conjecture*, *Nucl. Phys. B* **961** (2020) 115270, [2007.05193].

[591] B. McInnes, *The weak gravity conjecture requires the existence of exotic AdS black holes*, *Nucl. Phys. B* **971** (2021) 115525, [2104.07373].

[592] B. McInnes, *Planar Black Holes as a Route to Understanding the Weak Gravity Conjecture*, 2201.01939.

[593] D. Harlow, B. Heidenreich, M. Reece and T. Rudelius, *The Weak Gravity Conjecture: A Review*, 2201.08380.

[594] Y. Kats, L. Motl and M. Padi, *Higher-order corrections to mass-charge relation of extremal black holes*, *JHEP* **12** (2007) 068, [hep-th/0606100].

[595] B. McPeak, *The Entropy Shift and Extremality Shift at Zero Temperature*, 2112.13433.

[596] C. P. Bachas, P. Bain and M. B. Green, *Curvature terms in D-brane actions and their M theory origin*, *JHEP* **05** (1999) 011, [hep-th/9903210].

[597] H. J. Schnitzer and N. Wyllard, *An Orientifold of AdS(5)×T^{11} with D7-branes, the associated α'^2-corrections and their role in the dual $\mathcal{N} = 1$ $\mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N)$ gauge theory*, *JHEP* **08** (2002) 012, [hep-th/0206071].

[598] C. Cheung and G. N. Remmen, *Positivity of Curvature-Squared Corrections in Gravity*, *Phys. Rev. Lett.* **118** (2017) 051601, [1608.02942].

[599] N. Bobev, A. M. Charles, K. Hristov and V. Reys, *Higher-derivative supergravity, AdS$_4$ holography, and black holes*, *JHEP* **08** (2021) 173, [2106.04581].

[600] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, *J. Stat. Mech.* **0406** (2004) P06002, [hep-th/0405152].

[601] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602, [hep-th/0603001].

[602] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, *JHEP* **08** (2006) 045, [hep-th/0605073].

[603] I. R. Klebanov, S. S. Pufu, S. Sachdev and B. R. Safdi, *Renyi Entropies for Free Field Theories*, *JHEP* **04** (2012) 074, [1111.6290].

[604] N. Laflorencie, *Quantum entanglement in condensed matter systems*, *Phys. Rept.* **646** (2016) 1–59, [1512.03388].

[605] V. G. M. Puletti and R. Pourhasan, *Non-analyticity of holographic Rényi entropy in Lovelock gravity*, *JHEP* **08** (2017) 002, [1704.08731].

[606] H. Casini, M. Huerta, R. C. Myers and A. Yale, *Mutual information and the F-theorem*, *JHEP* **10** (2015) 003, [1506.06195].

[607] S. Kundu and J. F. Pedraza, *Aspects of Holographic Entanglement at Finite Temperature and Chemical Potential*, *JHEP* **08** (2016) 177, [1602.07353].

[608] B. Swingle, *Mutual information and the structure of entanglement in quantum field theory*, 1010.4038.

[609] D. V. Vassilevich, *Heat kernel expansion: User's manual*, *Phys. Rept.* **388** (2003) 279–360, [hep-th/0306138].

[610] A. Grigor'yan and M. Noguchi, *The heat kernel on hyperbolic space*, *Bulletin of the London Mathematical Society* **30** (1998) 643–650.

[611] A. Grigor'yan, *Heat kernel upper bounds on a complete non-compact manifold*, *Revista Matemática Iberoamericana* **10** (1994) 395–452.

[612] R. Camporesi and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, *J. Math. Phys.* **35** (1994) 4217–4246.

[613] H. Casini and M. Huerta, *Entanglement entropy for the n-sphere*, *Phys. Lett. B* **694** (2011) 167–171, [1007.1813].

[614] J. S. Dowker, *Entanglement entropy for even spheres*, 1009.3854.

[615] J. Lee, A. Lewkowycz, E. Perlmutter and B. R. Safdi, *Rényi entropy, stationarity, and entanglement of the conformal scalar*, *JHEP* **03** (2015) 075, [1407.7816].

[616] A. Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant O(N) vector model*, *Annals Phys.* **249** (1996) 180–221, [hep-th/9410093].

[617] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, *J. Phys. A* **42** (2009) 504005, [0905.4013].

[618] S. Zhao, C. Northe, K. Weisenberger and R. Meyer, *Charged Moments in W_3 Higher Spin Holography*, 2202.11111.

[619] P. Bueno, P. A. Cano, J. Moreno and G. van der Velde, *Regular black holes in three dimensions*, *Phys. Rev. D* **104** (2021) L021501, [2104.10172].

[620] V. Rosenhaus and M. Smolkin, *Entanglement Entropy: A Perturbative Calculation*, *JHEP* **12** (2014) 179, [1403.3733].

[621] J. M. Magan, *Proof of the universal density of charged states in QFT*, *JHEP* **12** (2021) 100, [2111.02418].

[622] H. Casini, M. Huerta, J. M. Magán and D. Pontello, *Entanglement entropy and superselection sectors. Part I. Global symmetries*, *JHEP* **02** (2020) 014, [1905.10487].

[623] H. Osborn, *N=1 superconformal symmetry in four-dimensional quantum field theory*, *Annals Phys.* **272** (1999) 243–294, [hep-th/9808041].

[624] E. Barnes, E. Gorbatov, K. A. Intriligator, M. Sudano and J. Wright, *The Exact superconformal R-symmetry minimizes tau(RR)*, *Nucl. Phys. B* **730** (2005) 210–222, [hep-th/0507137].

[625] A. D. Shapere and Y. Tachikawa, *Central charges of N=2 superconformal field theories in four dimensions*, *JHEP* **09** (2008) 109, [0804.1957].

[626] R. L. Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor*, *Sémin. Congr., Soc. Math. France, Paris* **4** (2000) 53–94.

[627] H. Baum, T. Leistner and A. Lischewski, *Cauchy problems for lorentzian manifolds with special holonomy*, *Differential Geometry and its Applications* **45** (2016) 43–66.

[628] H. Baum and O. Müller, *Codazzi spinors and globally hyperbolic manifolds with special holonomy*, *Mathematische Zeitschrift* **258** (2008) 185–211.

[629] T. Leistner and D. Schliebner, *Completeness of compact lorentzian manifolds with abelian holonomy*, *Mathematische Annalen* **364** (2016) 1469–1503.

[630] H. Baum, K. Lärz and T. Leistner, *On the full holonomy group of lorentzian manifolds*, *Mathematische Zeitschrift* **277** (2014) 797–828.

[631] Y. V. Bazaĭkin, *Globally hyperbolic lorentzian spaces with special holonomy groups*, *Siberian Mathematical Journal* **50** (2009) 567–579.

[632] A. M. Candela, J. L. Flores and M. Sánchez, *On general plane fronted waves: Geodesics*, *Gen. Rel. Grav.* **35** (2003) 631–649, [gr-qc/0211017].

[633] W. Globke and T. Leistner, *Locally homogeneous pp-waves*, *Journal of Geometry and Physics* **108** (2016) 83–101.

[634] D. Schliebner, *On lorentzian manifolds with highest first betti number*, in *Annales de l'Institut Fourier*, vol. 65, pp. 1423–1436, 2015.

[635] C. I. Lazaroiu and C. S. Shahbazi, *Complex Lipschitz structures and bundles of complex Clifford modules*, *Differ. Geom. Appl.* **61** (2018) 147–169, [1711.07765].

[636] D. V. Alekseevsky, V. Cortés, C. Devchand and A. Van Proeyen, *Polyvector superPoincare algebras*, *Commun. Math. Phys.* **253** (2004) 385–422, [hep-th/0311107].

[637] D. V. Alekseevsky and V. Cortés, *Classification of n-(super)-extended poincaré algebras and bilinear invariants of the spinor representation of spin (p, q)*, *Commun. Math. Phys.* **183** (1997) 477–510.

[638] H. W. Brinkmann, *Einstein spapces which are mapped conformally on each other*, *Math. Ann.* **94** (1925) 119–145.

[639] I. P. Costa e Silva and J. L. Flores, *On the Splitting Problem for Lorentzian Manifolds with an $\mathbb{R}$ -Action with Causal Orbits*, *Annales Henri Poincare* **18** (2017) 1635–1670.

[640] M. Mars, *On local characterization results in geometry and gravitation*, in *From Riemann to Differential Geometry and Relativity*, pp. 541–570. Springer, 2017.

[641] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, .

[642] B. Araneda, *Parallel spinors, pp-waves, and gravitational perturbations*, 2204.13673.

[643] M. Sánchez, *Globally hyperbolic spacetimes: slicings, boundaries and counterexamples*, 2110.13672.

[644] C. Bär, P. Gauduchon and A. Moroianu, *Generalized cylinders in semi-riemannian and spin geometry*, *Mathematische Zeitschrift* **249** (2005) 545–580.

[645] V. Gorbatsevich, O. AL and V. EB, *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras*, vol. 41. Springer Science & Business Media, 1994.

[646] V. Cortés, K. Kröncke and J. Louis, *Geometric Flows and the Geometry of Space-time.* Springer, 2018.

[647] M. P. Do Carmo and J. Flaherty Francis, *Riemannian geometry*, vol. 6. Springer, 1992.

[648] D. Schliebner, *Contributions to the geometry of Lorentzian manifolds with special holonomy.* PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2015.

[649] L. Conlon, *Differentiable manifolds*, vol. 2. Springer, 2001.

[650] P. Tondeur, *Geometry of foliations.* No. 90. Springer Science & Business Media, 1997.

[651] J. Milnor, *Morse Theory.(AM-51), Volume 51.* Princeton University Press, 2016.

[652] L. Conlon, *Transversally parallelizable foliations of codimension two*, *Trans. American Math. Soc.* **194** (1974) 79–102.

[653] R. W. Sharpe, *Differential geometry: Cartan's generalization of Klein's Erlangen program*, vol. 166. Springer Science & Business Media, 2000.

[654] J. L. Arraut and M. Craizer, *Foliations of M^3 defined by $\mathbb{R}^2$-actions*, in *Annales de l'institut Fourier*, vol. 45, pp. 1091–1118, 1995.

[655] G. Chatelet, H. Rosenberg and D. Weil, *A classification of the topological types of $\mathbb{R}^2$-actions on closed orientable 3-manifolds*, *Publications Mathématiques de l'IHÉS* **43** (1974) 261–272.

[656] H. Rosenberg, *Foliations by planes*, *Topology* **7** (1968) 131–138.

[657] H. Rosenberg, R. Roussarie and D. Weil, *A classification of closed orientable 3-manifolds of rank two*, *Annals of Mathematics* (1970) 449–464.

[658] M. R. Herman, *The godbillon-vey invariant of foliations by planes of t^3*, in *Geometry and Topology*, pp. 294–307. Springer, 1977.

[659] G. Hector, *Feuilletages en cylindres*, in *Geometry and topology*, pp. 252–270. Springer, 1977.

[660] J. Milnor, *Curvatures of left invariant metrics on lie groups*, *Advances in mathematics* **21** (1976) 293–329.

[661] M. Freibert, *Cocalibrated g_2-structures on products of four-and three-dimensional lie groups*, *Diff. Geom. Appl.* **31** (2013) 349–373.

[662] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations.* Tata McGraw-Hill Education, 1955.

[663] B. Cappelletti-Montano, A. De Nicola and I. Yudin, *A survey on cosymplectic geometry*, *Reviews in Mathematical Physics* **25** (2013) 1343002.

[664] Z. Olszak, *On almost cosymplectic manifolds*, *Kodai Mathematical Journal* **4** (1981) 239–250.

[665] R. Schimming, *Riemannsche räume mit ebenfrontiger und mit ebener symmetrie*, *Math. Nachr.* **59** (1974) 129–162.

[666] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity*, pp. 271–275. Springer, 1976.

[667] G. F. Ellis and M. A. MacCallum, *A class of homogeneous cosmological models*, *Commun. Math. Phys.* **12** (1969) 108–141.

[668] A. Derdziński, *Self-dual kähler manifolds and einstein manifolds of dimension four*, *Compositio Math.* **49** (1983) 405–433.

[669] V. Apostolov and P. Gauduchon, *Selfdual einstein hermitian four-manifolds*, *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze* **1** (2002) 203–243.

[670] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, *J. Math. Mech.* **14** (1965) 1033–1047.

[671] D. V. Alekseevsky and V. Cortés, *Classification of pseudo-riemannian symmetric spaces of quaternionic kähler type*, *Lie groups and invariant theory* **213** (2005) 33–62.

[672] M. Krahe, *Para-pluriharmonic maps and twistor spaces*, *Handbook of pseudo-Riemannian geometry and supersymmetry* **16** (2010) 497–557.

[673] M. Dyckmanns and O. Vaughan, *The para-hk/qk correspondence*, *J. Geom. Phys.* **116** (2017) 244–257.

[674] M. Dunajski and M. Hoegner, *SU(2) solutions to self-duality equations in eight dimensions*, J. Geom. Phys. **62** (2012) 1747–1759, [1109.4537].

[675] D. V. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt, *Quaternionic Kähler metrics associated with special Kähler manifolds*, J. Geom. Phys. **92** (2015) 271–287, [1305.3549].

[676] G. Calvaruso, *Homogeneous paracontact metric three-manifolds*, *Illinois Journal of Mathematics* **55** (2011) 697–718.

[677] G. Calvaruso, *Geometry of h-paracontact metric manifolds*, *Publicationes Mathematicae* **86** **(3-4)** (2013) .

[678] S. Zamkovoy and A. Bojilov, *A Classification of 3-dimensional paracontact metric manifolds with* $Q\varphi = \varphi Q$, 1910.04593.

[679] G. Calvaruso, *Homogeneous structures on three-dimensional lorentzian manifolds*, *J. Geom. Phys.* **57** (2007) 1279–1291.

[680] S. Rahmani, *Métriques de lorentz sur les groupes de lie unimodulaires, de dimension trois*, *Journal of Geometry and Physics* **9** (1992) 295–302.

[681] S. Dumitrescu and A. Zeghib, *Géométries lorentziennes de dimension 3: classification et complétude*, *Geometriae Dedicata* **149** (2010) 243–273.

[682] C. Frances, *Lorentz dynamics on closed* 3*-manifolds*, *Annales Henri Lebesgue* **3** (2020) 407–471.

[683] L. A. Cordero and P. E. Parker, *Left-invariant lorentzian metrics on 3-dimensional lie groups*, *Rend. Mat. Appl.(7)* **17** (1997) 129–155.

[684] E. T. Kobayashi, *A remark on the nijenhuis tensor.*, *Pacific J. Math.* **12** (1962) 963–977.

[685] J. Lehmann-Lejeune, *Intégrabilité des g-structures définies par une 1-forme* 0*-déformable à valeurs dans le fibré tangent*, in *Ann Inst. Fourier*, vol. 16, pp. 329–387, 1966.

[686] F. Turiel, *Intégrabilité d'un tenseur de type (1, 1) et structure symplectique du fibré cotangent*, *CR Acad. Sci., Paris, Sér. I* **301** (1985) 923–925.

[687] G. Thompson, *The integrability of a field of endomorphisms*, *Mathematica Bohemica* **127** (2002) 605–611.

[688] C. Boubel, *An integrability condition for fields of nilpotent endomorphisms*, 1003.0979.

[689] A. V. Bolsinov, A. Y. Konyaev and V. S. Matveev, *Nijenhuis geometry*, *Advances in Mathematics* **394** (2022) 108001.

[690] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, *Israel Journal of Mathematics* **91** (1995) 189–214.

[691] V. Moncrief, *Reduction of the Einstein equations in (2+1)-dimensions to a Hamiltonian system over Teichmuller space*, J. Math. Phys. **30** (1989) 2907–2914.

[692] L. Andersson, V. Moncrief and A. J. Tromba, *On the global evolution problem in (2+1) gravity*, J. Geom. Phys. **23** (1997) 191, [gr-qc/9610013].

[693] Y. Choquet-Bruhat, *The cauchy problem in extended supergravity,* $n = 1, d = 11$, *Communications in mathematical physics* **97** (1985) 541–552.

[694] T. Koufogiorgos, *On a class of contact riemannian 3-manifolds*, *Results in Mathematics* **27** (1995) 51–62.

[695] H. Geiges, *Normal contact structures on 3-manifolds*, *Tohoku Mathematical Journal, Second Series* **49** (1997) 415–422.

[696] A. A. García-Díaz, *Exact solutions in three-dimensional gravity*. Cambridge University Press, 2017.

[697] F. Tricerri, G. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, vol. 83. Cambridge University Press, 1983.

[698] M. García-Fernández and C. S. Shahbazi, *Self-dual generalized metrics for pure $\mathcal{N} = 1$ six-dimensional Supergravity*, 1505.03088.

[699] M. J. D. Hamilton, *The field and Killing spinor equations of M-theory and type IIA/IIB supergravity in coordinate-free notation*, 1607.00327.

[700] I. Vaisman, *From generalized Kähler to generalized Sasakian structures*, *Journal of Geometry and Symmetry in Physics* **18** (2010) 63–86.

[701] Y. S. Poon and A. Wade, *Generalized contact structures*, *Journal of the London Mathematical Society* **83** (2011) 333–352.

[702] M. Gualtieri, *Generalized complex geometry*. PhD thesis, Oxford University, 2003. 0401221.

[703] V. Cortés and D. Krusche, *Classification of generalized Einstein metrics on 3-dimensional Lie groups*, 2206.01157.

[704] U. Kol, *Duality in Einstein's Gravity*, 2205.05752.