Classical and Stringy Properties of Black Holes

Memoria de Tesis Doctoral realizada por

David Pereñiguez Rodriguez

presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para optar al Título de Doctor en Física Teórica

Tesis Doctoral dirigida por el **Prof. D. Tomás Ortín Miguel**[1]
y por el **Dr. Pablo Antonio Cano Molina-Niñirola**[2]

[1]Profesor de Investigación del Instituto de Física Teórica UAM/CSIC
[2]Investigador postdoctoral en Katholieke Universiteit Leuven

Departamento de Física Teórica
Universidad Autónoma de Madrid

Instituto de Física Teórica
UAM/CSIC

Universidad Autónoma de Madrid

Instituto de Física Teórica UAM-CSIC

Madrid, Junio de 2022

“[...] no vaya a ser que a alguien se le pase por la cabeza la idea estúpida de parar los relojes de los campanarios o de quitarle el badajo a las campanas pensando que de esa manera detendría el tiempo y podría contradecir lo que es mi decisión irrevocable [...]”
José Saramago (Las intermitencias de la muerte).

Agradecimientos

Quiero agradecer a mi supervisor, Tomás Ortín, el haberme dado la oportunidad de realizar esta tesis en IFT. Ha sido un privilegio trabajar contigo. También quiero agradecerte la libertad que me has permitido desde el primer día para explorar todo aquello que ha capturado mi interés, estando siempre disponible para discutir y guiarme cuando lo he necesitado. Es también un privilegio ser el primer estudiante de doctorado de Pablo Cano. Ha sido un placer trabajar contigo y estoy muy agradecido por el apoyo que he tenido por tu parte como supervisor.

I am also very grateful to Vitor Cardoso, for giving me the opportunity of joining his group in Lisbon for few months. Being part of the group at IST has been a very nice and stimulating experiences.

Ha sido un placer dar clase, discutir sobre física y tener una gran amistad con Daniel Areán.

During this thesis I also had nice and stimulating discussions with Carlos Shahbazi, Pablo Bueno, Pedro F. Ramírez, Alejandro Ruiperez, Luca Romano, Glenn Barnich, Roberto Emparan, David Mateos, Cristiano Germani, José L.F. Barbón, Maria J. Rodriguez, Jorge Rocha, Andrea Maselli, Laura Bernard, Alexandre Le Tiec, Marc Casals, and Luca Santoni.

También quiero mencionar a los compañeros con quien he tenido el placer de compartir despacho: Gabriel, Pablo, Alejandro, Ángel, Zach y Matteo. Ha sido un placer interaccionar con vosotros y hablar de física y tantas otras cosas.

Por último, quiero agradecer a mis amigos, a mi familia y a Irene su apoyo incondicional.

List of Publications

This thesis is based on the following papers, coauthored by the candidate:

[1] *Love numbers and magnetic susceptibility of charged black holes*
D. Pereñiguez, V. Cardoso
Phys.Rev.D 105 (2022) 4, 044026 (arXiv:2112.08400)

[2] *Komar integral and Smarr formula for axion-dilaton black holes versus S duality*
D. Mitsios, T. Ortín, D. Pereñiguez
JHEP 08 (2021) 019 (arXiv:2106.07495)

[3] *Quasinormal modes of NUT-charged black branes in the AdS/CFT correspondence*
P. A. Cano, D. Pereñiguez
arXiv:2101.10652

[4] *The first law and Wald entropy formula of heterotic stringy black holes at first order in α'*
Z. Elgood, T. Ortín, D. Pereñiguez
JHEP 05 (2021) 110 (arXiv:2012.14892)

[5] *The first law of heterotic stringy black hole mechanics at zeroth order in α'*
Z. Elgood, D. Mitsios, T. Ortín, D. Pereñiguez
JHEP 07 (2021) 007 (arXiv:2012.13323)

[6] *Extremal Rotating Black Holes in Einsteinian Cubic Gravity*
P. A. Cano, D. Pereñiguez
Phys.Rev.D 101 (2020) 4, 044016 (arXiv:1910.10721)

[7] *Magnetic Charges and Wald Entropy*
T. Ortín, D. Pereñiguez
(To appear)

During the realisation of this thesis, the candidate also published the following paper, not included in the present manuscript:

[8] *p-brane Newton–Cartan geometry*
D. Pereñiguez
J.Math.Phys. 60 (2019) 11, 112501 (arXiv:1908.04801)

Abstract

This thesis is devoted to the study of dynamical and thermodynamical properties of black holes. It has two parts.

Part I considers black holes in the context of the low energy effective actions of string theory. The first few higher-derivative corrections induced by finite-size effects in the string length $\ell_s \sim \sqrt{\alpha'}$, where α' is the Regge slope parameter, are well understood for the heterotic superstring (HST). α'-corrected black hole solutions are available and computing their entropy is crucial given its relation to string microstates. However, the Iyer–Wald entropy formula gives a result that is not gauge invariant. This is due to the fact that the original computation assumes that all fields are tensors with no internal gauge freedom. In this thesis, Wald's derivation is revisited using a formalism that accommodates gauge symmetry conveniently. The main result is a gauge- and Lorentz- invariant entropy formula that includes the first order corrections in α'. It is also shown, in some particular theories, how magnetic-type terms can be included in the generic proofs of the laws of black hole thermodynamics, even though magnetic charges are not directly associated to gauge symmetry.

Part II focuses on dynamical aspects of black holes in different contexts. Rotating black holes in higher-derivative theories are poorly understood due to the complexity of the equations of motion. The problem can be simplified by considering the near horizon geometry of an extremal, charged and rotating black hole. A non-perturbative solution of such a class is presented in a cubic theory called Einsteinian Cubic Gravity. It is the first example in which the entropy of a rotating black hole of higher-order gravity has been exactly computed.

In the context of the AdS/CFT correspondence, NUT-charged AdS black holes describe equilibrium states of neutral fluids subject to non-trivial flows at the boundary. Physical transport properties, however, remain largely unexplored. The master equations governing gravitational fluctuations on a class of NUT-charged AdS black holes are derived in this thesis. These exhibit an intriguing relation to Landau quantisation. The gravitational quasinormal mode spectrum of a NUT-charged black hole is computed for the first time, and the spacetime appears to be robustly stable despite the existence of closed causal curves ("time machines"). There is an interesting class of quasi-hydrodynamic modes for which analytic dispersion relations are constructed as a definite holographic prediction for the dual fluid.

The last chapter of this thesis deals with the tidal deformability of black holes. Tidal interactions, encoded linearly in the so-called tidal Love numbers, become significant in the last stages of the inspiral phase of a merger. In the case of vacuum, four-dimensional black holes, the tidal Love numbers are zero. The robustness of such a property is investigated by studying the static deformability of charged black holes. It is shown that tidal response coefficients keep on vanishing, in a very non-trivial way, from neutrality all the way down to extremality. This is true not only for gravity (spin-2), but also for spin-0 and spin-1 deformations. In higher dimensions, however, the tidal response is non-trivial and charging up the hole can excite new polarisation modes. One exception is the static response of spin-0 perturbations, which happens to vanish at extremality in any dimension. These results call for further investigation of the tidal deformability properties of black holes.

Resumen

Esta tesis está dedicada al estudio de propiedades dinámicas y termodinámicas de los agujeros negros. Consta de dos partes.

La parte I considera agujeros negros en el contexto de las acciones efectivas de teoría de cuerdas. Las primeras correcciones en derivadas superiores, inducidas por efectos de tamaño finito en la longitud de la cuerda $\ell_s \sim \sqrt{\alpha'}$, donde α' es el parámetro de Regge, se conocen bien en el caso de la supercuerda heterótica (HST). Además, se dispone de soluciones de agujero negro con correcciones en α' y el cálculo de su entropía es crucial dada su relación con los microestados de cuerdas. Sin embargo, la fórmula de entropía de Iyer-Wald da un resultado que no es invariante gauge. Esto se debe a que el cálculo original supone que todos los campos son tensores sin libertad gauge interna. En esta tesis, se revisa la derivación de Wald utilizando un formalismo que incluye convenientemente la simetría gauge. El resultado principal es una fórmula de entropía invariante gauge y Lorentz que incluye correcciones a primer orden en α'. También se muestra, en algunas teorías particulares, cómo pueden incluirse términos de tipo magnético en las demostraciones genéricas de las leyes de la termodinámica de los agujeros negros.

La Parte II se centra en aspectos dinámicos de los agujeros negros en distintos contextos. Los agujeros negros en rotación de teorías en derivadas superiores son poco conocidos debido a la complejidad de las ecuaciones del movimiento. El problema puede simplificarse considerando la geometría cercana al horizonte de un agujero negro extremo, cargado y en rotación. En esta tesis se da una solución no perturbativa de dicha clase en una teoría cúbica llamada Einsteinian Cubic Gravity. Se trata del primer ejemplo, en gravedades de orden superior, en que la entropía de un agujero negro en rotación puede calcularse de forma exacta.

En el contexto de la correspondencia AdS/CFT, los agujeros negros en AdS con carga NUT describen estados de fluidos neutros en equilibrio sujetos a flujos no triviales en la frontera. Sin embargo, las propiedades físicas de transporte permanecen en gran medida inexploradas. En esta tesis se derivan las ecuaciones maestras que gobiernan las fluctuaciones gravitacionales en una clase de agujero negro en AdS con carga NUT. Esto conduce al primer cálculo del espectro gravitacional cuasinormal de un agujero negro con carga NUT. El espaciotiempo se muestra robustamente estable a pesar de la existencia de curvas causales cerradas ("máquinas del tiempo"). Hay una clase interesante de modos cuasihidrodinámicos para los que se construyen relaciones de dispersión analíticas a modo de predicción holográfica para el fluido dual.

El último capítulo de esta tesis trata sobre la deformabilidad de marea de los agujeros negros. Las interacciones de marea, codificadas linealmente en los llamados números de Love, adquieren importancia en las últimas etapas de la fase espiral de una colisión. En el caso de los agujeros negros en cuatro dimensiones en el vacío, los números de Love se anulan. La solidez de esta propiedad es investigada estudiando la deformabilidad estática de agujeros negros cargados. Se demuestra que los coeficientes de respuesta siguen anulándose, de forma muy no trivial, en todo el rango comprendido entre la neutralidad y la extremalidad. Esto es cierto no sólo para la gravedad (espín-2), sino también para las deformaciones de espín-0 y espín-1. Sin embargo, en mayores dimensiones, la respuesta de marea no es nula y la carga del agujero puede excitar nuevos modos de polarización. Una excepción es la respuesta estática de las perturbaciones de espín-0, que resulta anularse en la extremalidad en cualquier dimensión.

Contents

1
Introduction

The most remarkable theoretical predictions of physics come after enlarging our theories to accommodate newly proposed principles of nature. In Einstein's General Relativity (GR), the Equivalence Principle and Special Relativity are beautifully implemented in the (vacuum) field equation,

$$R_{\mu\nu} = 0 \tag{1.1}$$

the apparent simplicity of which contrasts with its actual complexity as a system of second order, coupled, non-linear partial differential equations. A revolutionary, non-perturbative prediction of (1.1) is the existence of *black holes* which, furthermore, arise as the simplest solutions of the theory. Intuitively, these are regions in spacetime subject to a gravitational pull which is strong enough to let no signal emerge and reach an external observer. Consequently, these are *causal* holes in the structure of spacetime: if something falls inside a black hole, it cannot communicate with the exterior ever again (and has a rather hopeless fate, as we shall argue in this introduction). What is more remarkable, there is strong evidence that such an astonishing prediction of (1.1) takes place in nature. Nowadays it is believed that there is a supermassive black hole ($\sim 10^6 - 10^{10} M$) at the centre of almost every large galaxy. During the writing of this thesis, the Event Horizon Telescope Collaboration published for the first time pictures of the supermassive black holes that lie at the centres of M87 ($\sim 6.5 \times 10^9 M$) [1] and the Milky Way (Sagittarius A*) ($\sim 4 \times 10^6 M$) [2]. Furthermore, the detection of gravitational waves (GW) (yet another prediction of (1.1)) by the LIGO and Virgo collaborations [3] opened up a new channel of observation of the universe: gravitational wave astronomy. Since 2016, this has allowed the observation of mergers of black holes with (initial) masses in the range $\sim 5 - 85 M$ [4]. More speculatively, primordial black holes formed at the early stages of the big bang have been proposed as a candidate for dark matter [5–7] even though robust experimental evidence is still lacking.

From a theoretical perspective, black holes are central in the development of our understanding of the fundamental laws of nature. Being extremely efficient GW sources, black hole mergers will allow us to probe the strong field regime of gravity to exquisite precision with future-planned Earth- and space-based detectors [8, 9], thus testing GR with great accuracy [10–14] and also challenging our beyond-GR theories [15]. At a more fundamental level, black holes constitute physical scenarios in which strong gravitational fields and quantum interactions coexist. Therefore, their features raise questions about the nature of their microscopic structure, that guide theorists towards the construction of a theory of quantum gravity.

In this introduction, we first provide a reasonably self-contained review of the main properties of black holes in GR. Then, in section 1.2 we discuss dynamical aspects of black holes that are relevant for Chapters 5 to 7. Finally, in Section 1.3 we review the proofs of some thermodynamical properties of black holes and introduce some of the motivations of Chapters 2 to 4.

Note on conventions: The introduction and Part II of this thesis follow the conventions of [16]. Part I follows the conventions of [152].

1.1 Black Hole Basics

1.1.1 Definition and Generic Properties

A *spacetime* is a pair (M, g) where M is a real manifold and g a Lorentzian metric on M.[1] We are interested in spacetimes that are solutions of Einstein's equation and represent ideally isolated systems, a star or a compactly supported source, say. Such class of solutions can be obtained by supplementing Einstein's equations with suitable boundary conditions. These should guarantee that, far from the sources, the spacetime becomes flat (i.e. approaches Minkowski's space). This is a fairly intuitive requirement and, therefore, it is possible to provide a precise (although rather technical) notion of *asymptotically flat spacetime*, its main pieces being the future and past null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$. Less familiar to our intuition are black holes, and hence we are forced to define them from what they are not. A *black hole* in an asymptotically flat spacetime is the complement in M of the causal past of future null infinity $J^-(\mathscr{I}^+)$, that is,

$$B = M - \big(M \cap J^-(\mathscr{I}^+)\big) \tag{1.2}$$

In other words, a black hole is a region of spacetime that is causally disconnected from future null infinity. Intuitively, this can be understood as a region subject to a gravitational pull that is strong enough to let no signal escape out of it. Hence, the boundary of B in M defines a no-return surface known as the *event horizon of* B,

$$\mathcal{H} = M \cap \partial J^-(\mathscr{I}^+) \tag{1.3}$$

$\mathcal{H}$ is a null hypersurface with no future end-points, but it may have past end-points [21] (further properties of $\mathcal{H}$ are discussed below and in section 1.1.2). The latter fact indicates that horizons can start forming somewhere, e.g. as a result of gravitational collapse. Indeed, there are several explicit solutions containing black hole regions (some describing the formation of black holes [22] which intend to model the gravitational collapse of stars [23–25]), but exhibiting examples is not enough to show whether the existence of black holes in nature is a *generic* prediction of GR.[2] Hence, before discussing exact solutions, here we revisit few crucial, generic properties (and conjectures) that suggest an answer in the affirmative to the previous question.

The energy and linear and angular momenta of a spacetime are given by the *Arnowitt-*

[1] Most of this section is based on [16–20].

[2] In fact, before the major progress achieved in the 60's, it was a matter of intense debate whether spacetime singularities were general or merely due to symmetry assumptions [26].

Deser-Misner (ADM) charges [27,28], defined as integrals at infinity[3]

$$E_{ADM} = \frac{1}{16\pi} \lim_{r\to\infty} \oint_{S^2_r} dA n_j \left(\partial_i h_{ji} - \partial_j h_{ii}\right) \tag{1.4}$$

$$P_i = \frac{1}{8\pi} \lim_{r\to\infty} \oint_{S^2_r} dA n_j \left(K_{ij} - K\delta_{ij}\right) \tag{1.5}$$

$$J_i = \frac{1}{8\pi}\epsilon_{ilm} \lim_{r\to\infty} \oint_{S^2_r} dA n_j x^l (K_{mj} - K\delta_{mj}) \tag{1.6}$$

E_{ADM} is the *total* energy of the spacetime and, as reviewed in section 1.3.1, it is related to the existence of asymptotic (not necessarily exact) symmetries. The above formula can be derived from various perspectives, e.g. as the on-shell value of the Hamiltonian of GR or as a conserved charge associated to an asymptotic symmetry (we will review the latter derivation in section 1.3.1). Remarkably, the *positive energy theorem* [29,30] establishes that, under minimal physical assumptions,

$$M_{ADM} = \sqrt{E^2_{ADM} - P_i P_i} \geq 0 \tag{1.7}$$

where we introduced the *ADM mass* M_{ADM}.[4] In particular, this means that there is a physically meaningful notion of the total mass and energy of a black hole spacetime. This is in sharp contrast with the well-known ill-definiteness of the local energy density in GR [30]. Besides the ADM quantities, black holes can be additionally charged under gauge fields, but we shall discuss this in some detail in section 1.3.1.

Regarding the black hole region and the event horizon, several generic properties have been found since the 60's. An instrumental result is *Penrose's singularity theorem* [31], which is based on the notion of trapped surface. Consider a codimension-2 spacelike surface S. At each point there are (up to normalisation) precisely two linearly independent, future directed, null vectors $k^\pm$ (it is conventional to set the relative normalisation to $g(k^+, k^-) = -1$). Extend $k^\pm$ geodesically off S. The expansions of both families, defined by

$$\theta^\pm = \nabla^\mu k^\pm_\mu \tag{1.8}$$

can be used to classify S.[5] In particular, we say that S is *future-trapped* if $\theta^\pm < 0$, *marginally future-trapped* if $\theta^\pm \leq 0$ and $\theta^\mp = 0$ and *stationary or minimal* if $\theta^\pm = 0$. The expansion measures the rate of change of the cross-sectional area of the null geodesic congruence [33]. Thus, the area of a future trapped surface decreases locally along the future directed, lightlike geodesic flows of both $k^\pm$. If, furthermore, the trapped surface belongs to a globally hyperbolic spacetime, with non-compact Cauchy slice and matter satisfying the null energy condition, Penrose proved that at least one of the geodesic families is future-inextendible and incomplete [31].[6] However, Penrose's theorem does not establish whether this is because the spacetime is itself extendible, or because it is truly singular.

[3] We set $G = 1$ throughout the introduction.
[4] Equality holds only in Minkowski's space.
[5] See [32] for a more modern description of trapped surfaces, based on the mean curvature vector of S.
[6] A result worth a half of the 2020 Nobel Prize in Physics.

As a classical theory, GR is expected to be deterministic. Then, asymptotically flat initial data that is geodesically complete should fix uniquely the *entire* spacetime if Einstein's equations are satisfied. Furthermore, one expects such a spacetime to be asymptotically flat. This motivated Penrose's *Strong and Weak Cosmic Censorship conjectures* (SCC and WCC, respectively) [34]. The former asserts that the maximal development of generic, asymptotically flat and geodesically complete initial data is inextendible[7] thus preventing the formation of regions lying beyond the causal domain of the initial data. The latter claims that, furthermore, this maximal development is asymptotically flat. Intuitively, the role of the SCC conjecture is to guarantee determinism of GR, while the WCC conjecture implies that singularities are cloaked by event horizons (see below). Notice that these two conjectures are logically independent. Even though general proofs have not been found so far, there are good reasons to believe that both SCC and WCC conjectures are correct.[8]

The formation of trapped surfaces from asymptotically flat, geodesically complete initial data is a generic (i.e. not fine-tuned) feature of GR [40, 41]. If a trapped surface forms, then Penrose's theorem asserts that the maximal development is not geodesically complete. Assuming that the SCC conjecture holds, such incompleteness must be due to a true singularity because the spacetime can not be extended any further (we say the *spacetime is singular*, in the sense that it is inextendible and geodesically incomplete). The singularity (roughly, the "locus" where geodesics terminate) can not be causally connected to any point of the spacetime since that would contradict global hyperbolicity of the maximal development. However, it could extend infinitely (e.g. along a null surface) and intersect $\mathscr{I}^+$, thus preventing null infinity from being geodesically complete. This cannot happen if the WCC conjecture is correct because $\mathscr{I}^+$ is geodesically complete in an asymptotically flat spacetime by definition. Therefore, the singularity must lie entirely in the complement of $J^-(\mathscr{I}^+)$, which is precisely the definition of a black hole region. One concludes that, if both SCC and WCC conjectures are correct, then black holes and singularities are a generic prediction of GR and, furthermore, singularities must lie entirely inside black holes.

1.1.2 Classification of Solutions and Black Hole Uniqueness

Consider a process of gravitational collapse that results in the formation of a black hole. A fraction of the matter will be swallowed by the hole, while the remaining energy will be radiated away e.g. in the form of gravitational waves [42, 43]. At late times, one expects the spacetime outside the event horizon to approach a stationary state. Thus, exactly stationary solutions should be good approximations to the spacetime at late times after collapse.

[7] The word generic in this definition excludes the violation of the conjecture by fine-tuned initial data, such as an asymptotically flat Cauchy slice of the Reissner–Nordstrom solution [35, 36].

[8] In particular, the SCC conjecture has been proven for Minkowski's space [37]. The WCC conjecture, on the other hand, can be tested through the *Penrose inequality* [38]. This can be verified locally (e.g. in numerical simulations), and its failure is believed to be a sign that the WCC conjecture is incorrect. However, no numerical example violating it has been found and, on the contrary, it has been proven for time-symmetric initial data [39].

Stationary Black Holes

We say that an asymptotically flat spacetime is *stationary* if it exhibits a Killing vector field k_t that is timelike in a neighbourhood of $\mathscr{I}^{\pm}$[9]. We say it is *static* if k_t is hypersurface orthogonal (i.e. the distribution of tangent planes orthogonal to k_t is integrable or, equivalently, $k_t \wedge dk_t = 0$). It is easy to show (e.g. by constructing a suitable local chart adapted to k_t) that stationarity implies symmetry under time translations and staticity implies, in addition, symmetry under time reversal.

There are many stationary solutions describing the gravitational field created by an irregular object at rest. Remarkably, though, a stationary spacetime containing a black hole is, in a sense, a rigid state. More precisely, if an asymptotically flat space is stationary, analytic[10] and contains an event horizon $\mathcal{H}$, then it is *stationary and axisymmetric* (with axial Killing vector field k_ϕ, satisfying $[k_t, k_\phi] = 0$) and, furthermore, the event horizon is a *Killing horizon* (these are sometimes called rigidity theorems [44]). In other words, a stationary black hole is necessarily symmetric with respect to an "axis of rotation" (the fixed points of k_ϕ). We recall that a null hypersurface $\mathcal{N}$ is a Killing horizon of a Killing vector field k if k is normal to $\mathcal{N}$[11]. In the case of a stationary black hole, k is a linear combination of k_t and k_ϕ,

$$k = k_t + \Omega_H k_\phi \tag{1.9}$$

Normalising k_t so that $g(k_t, k_t) \to -1$ at infinity, the constant Ω_H is the angular velocity of the horizon relative to an asymptotic observer at rest. Notice that k needs to be null only at $\mathcal{H}$, and it follows that[12]

$$k^\mu \nabla_\mu k_\nu \overset{\mathcal{H}}{=} \kappa k_\nu \tag{1.10}$$

where κ, referred to as the *surface gravity*, is a function on $\mathcal{H}$ that, remarkably, turns out to be constant under minimal assumptions (in particular, no reference to equations of motion is required, see section 1.3.2). Finally, we say that a black hole is *extremal* if $\kappa = 0$, which means that k^μ is tangent to the affinelly-parametrised generators of $\mathcal{H}$.

Spherical Solutions

Einstein's equation is very involved and solving it explicitly requires making some additional assumptions. One possibility consists in obtaining and classifying solutions according to the properties of their isometry groups. This is one of the main approaches to the study of solutions in GR. It is very powerful when the amount of symmetry is large enough, but it is not so useful in less symmetric situations. In such cases, less obvious approaches (based on algebraic properties rather than geometric ones) are necessary in order to obtain and classify solutions, as reviewed below. To illustrate the power of the

[9] Requiring that k_t is everywhere timelike outside the black hole is too strong. In fact, it can be proven that k_t (if it is not hypersurface orthogonal) becomes spacelike in part of the exterior region $J^-(\mathscr{I}^+) \cap J^+(\mathscr{I}^-)$ (the ergoregion) [17].

[10] This is an unsatisfactory assumption that conflicts with causality, but we shall accept it in this discussion for simplicity.

[11] Hypersurfaces have naturally normal 1-forms. "Normal vectors" are a luxury provided by the metric through rising the index to the normal 1-forms. However, if the hypersurface is null, the normal vectors lie along the surface. It is easy to show that they are tangent to the null geodesics that generate the hypersurface (what we refer to as the generators).

[12] $\overset{\mathcal{H}}{=}$ means "evaluated at $\mathcal{H}$" or "pulled-back on $\mathcal{H}$". The context should clarify which of the meanings is assumed.

approach based on the isometry groups, assume that a spacetime satisfying the vacuum Einstein's equation is *spherically symmetric*, that is, the isometry group has a subgroup isomorphic to $SO(3)$ whit orbits through each point homeomorphic to 2-spheres. Then, the Jebsen–Birkhoff theorem [45, 46] establishes that such a spacetime is isometric to (at least part of the maximally extended) Schwarzschild solution [47]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.11}$$

This solution is asymptotically flat with infinity lying at $r \to \infty$ and ADM mass M. Furthermore, ∂_t is a Killing vector field normal to surfaces of constant t and timelike if $r > 2M$. This implies that the exterior of any gravitating body that is spherically symmetric must be static (notice this was not an assumption of the Jebsen–Birkhoff theorem), regardless of the nature of the "interior". In particular, assuming that the solution extends also to $r \leq 2M$ the spacetime (1.11) describes the unique spherically symmetric black hole of vacuum GR, known as the Schwarzschild black hole (this is studied in detail in the next section). Models for stars or processes of spherically-symmetric collapse are constructed by gluing (1.11) to an "interior" solution at some $r_0 > 2M$ or $r_0 = r_0(\tau)$, respectively [22]. The Schwarzschild solution can be obtained easily from Einstein's equation due to the large amount of symmetry it possesses. A similar approach is unfortunately not useful if only stationarity and axial symmetry are assumed. However, this is a relevant situation since the generic final state of a collapse is expected (by the arguments at the beginning of the section and the rigidity theorems) to be a stationary and axisymmetric black hole.

Algebraic Classification

An exact solution describing a stationary, rotating black hole was obtained by Kerr in [48], forty-seven years after Schwarzschild's solution. The title of the original paper, *"Gravitational field of a spinning mass as an example of algebraically special metrics"* indicates that such an achievement was possible due to a description of spacetimes based on their algebraic properties. This approach has led to some of the most remarkable progress in GR, and we shall review it here briefly (since this is relevant for Chapter 6). The algebraic classification is most conveniently presented in its spinorial version [49]. Eventually, this can be translated into tensorial language, more suitable to applications, via de Newman–Penrose formalism [50].[13] The fact underlying the spinorial description of GR is that $SL(2, \mathbb{C})$ is the universal cover of the Lorentz group. This allows the construction of the spinor bundle from the bundle of Lorentz frames and establishes a canonical isomorphism between real spacetime spinors $\psi^{AA'} = \bar{\psi}^{AA'}$ (where $A, A' = 0, 1$ label the linear and antilinear legs of the spinor) and tangent vectors.[14] Thus, any spacetime tensor $T^{\mu\dots}{}_{\nu\dots}$ can be described by its spinor analogue $T^{AA'\dots}{}_{BB'\dots}$. For instance, the spinor analogue of the spacetime metric $g_{AA'BB'}$ is related to the spinor 2-form $\epsilon_{AB} = -\epsilon_{BA}$ (which can be used to rise and lower spinor indices) by $g_{AA'BB'} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$ and one can write

$$g_{\mu\nu} = -\epsilon_{AB}\bar{\epsilon}_{A'B'}\sigma_\mu^{AA'}\sigma_\nu^{BB'} \tag{1.12}$$

where $\sigma_\mu^{AA'} = \bar{\sigma}_\mu^{AA'}$ is a basis of spinor-valued 1-forms $\sigma_\mu^{AA'}$ (this can be compared with the description of a metric in tetrads $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$). Remarkably, the spinor analogue

[13] We follow the conventions of [51], consistent with our choice of signature $(-+++)$ for the introduction. Notice main references on the topic [52] and some text books [16, 18] use the opposite choice.

[14] See e.g. [16] for an introductory discussion.

of the Weyl tensor is described simply by a totally symmetric spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ known as the *Weyl spinor* [49],

$$C_{AA'BB'CC'DD'} = \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \tag{1.13}$$

As a direct application of the fundamental theorem of algebra it follows that Ψ_{ABCD}, due to its total symmetry, admits a *canonical decomposition*[15]

$$\Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(2)}_{B}\kappa^{(3)}_{C}\kappa^{(4)}_{D)} \tag{1.14}$$

where the spinors $\kappa^{(i)}_A$ are uniquely defined up to normalisation and are referred to as *principal spinors*. There are four of them although some may be aligned, i.e. occur with multiplicity larger than one in (1.14). In that case we say the spinor is a *repeated principal spinor*. Noticing that $\psi^A\psi_A = 0$ for any spinor, it follows that a necessary and sufficient condition for κ^A to be a principal spinor with multiplicity m is[16]

$$\Psi_{A_1 \ldots A_{5-m} \ldots}\kappa^{A_1} \ldots \kappa^{A_{5-m}} = 0 \qquad \text{and} \qquad \Psi_{A_1 \ldots A_{4-m} \ldots}\kappa^{A_1} \ldots \kappa^{A_{4-m}} \neq 0 \tag{1.15}$$

It is precisely the structure of repeated principal spinors what defines Petrov's classification of the Weyl tensor at a given point:

$$\begin{aligned}
\text{Type I}: &\quad \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(2)}_{B}\kappa^{(3)}_{C}\kappa^{(4)}_{D)} \\
\text{Type II}: &\quad \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(2)}_{C}\kappa^{(3)}_{D)} \\
\text{Type D}: &\quad \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(2)}_{C}\kappa^{(2)}_{D)} \\
\text{Type III}: &\quad \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(1)}_{C}\kappa^{(2)}_{D)} \\
\text{Type N}: &\quad \Psi_{ABCD} = \kappa^{(1)}_{(A}\kappa^{(1)}_{B}\kappa^{(1)}_{C}\kappa^{(1)}_{D)}
\end{aligned} \tag{1.16}$$

where principal spinors with different upper labels are not aligned. A spacetime is *algebraically general* if its Weyl tensor is everywhere type I, *algebraically special of type II* if its Weyl tensor is everywhere type II, etc.

This rather formal classification becomes very useful in practice when translated into tensor language via the Newman–Penrose formalism [50]. Given a basis (*dyad*) of spinors $\{o^A, \iota^A\}$ satisfying $o_A\iota^A = 1$ we can associate a *complex null tetrad* or *Newman–Penrose frame* $(\bar{\mathbf{m}}, \mathbf{m}, \mathbf{l}, \mathbf{k})$ as

$$\mathbf{k}^\mu = -\sigma^\mu_{AA'}o^A\bar{o}^{A'}, \quad \mathbf{l}^\mu = -\sigma^\mu_{AA'}\iota^A\bar{\iota}^{A'}, \quad \mathbf{m}^\mu = -\sigma^\mu_{AA'}o^A\bar{\iota}^{A'}, \quad \bar{\mathbf{m}}^\mu = -\sigma^\mu_{AA'}\iota^A\bar{o}^{A'} \tag{1.17}$$

$\mathbf{k}$ and $\mathbf{l}$ are real (their right hand sides above are invariant under complex conjugation) while $\mathbf{m}$ is not, $\bar{\mathbf{m}}$ being its complex conjugate. Furthermore, from (1.12) one has

$$\begin{aligned}
&\mathbf{k}^\mu\mathbf{l}_\mu = -\mathbf{m}^\mu\bar{\mathbf{m}}_\mu = -1 \\
&\mathbf{k}^\mu\mathbf{k}_\mu = \mathbf{l}^\mu\mathbf{l}_\mu = \mathbf{m}^\mu\mathbf{m}_\mu = \mathbf{k}^\mu\mathbf{m}_\mu = \mathbf{l}^\mu\mathbf{m}_\mu = 0
\end{aligned} \tag{1.18}$$

One can use a Newmann–Penrose frame to expand any tensor and the connection in components [18, 50, 52]. In particular, the ten independent (real) components of the Weyl

[15] See e.g. Proposition 3.5.18 of [52].
[16] See e.g. Proposition 3.5.26 of [52].

tensor $C_{\mu\nu\rho\sigma}$ are encoded in the five complex scalars,

$$\begin{aligned}
\Psi_0 &= \Psi_{ABCD}o^A o^B o^C o^D = C_{\mu\nu\rho\sigma}k^\mu m^\nu k^\rho m^\sigma \\
\Psi_1 &= \Psi_{ABCD}o^A o^B o^C \iota^D = C_{\mu\nu\rho\sigma}k^\mu l^\nu k^\rho m^\sigma \\
\Psi_2 &= \Psi_{ABCD}o^A o^B \iota^C \iota^D = C_{\mu\nu\rho\sigma}k^\mu m^\nu \bar{m}^\rho l^\sigma \\
\Psi_3 &= \Psi_{ABCD}o^A \iota^B \iota^C \iota^D = C_{\mu\nu\rho\sigma}k^\mu l^\nu \bar{m}^\rho l^\sigma \\
\Psi_4 &= \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D = C_{\mu\nu\rho\sigma}\bar{m}^\mu l^\nu \bar{m}^\rho l^\sigma
\end{aligned} \tag{1.19}$$

The power of the formalism relies on a convenient choice of null frame. If κ^A is a principal spinor, then we say that the vector $\kappa^A\bar{\kappa}^{A'}$ is a *principal null direction* (PND), and a *repeated PND* if κ^A is a repeated principal spinor.[17] Choosing a dyad $\{o^A, \iota^A\}$ aligned with two principal spinors (equivalently, choosing $\mathbf{k}$ and $\mathbf{l}$ aligned with PND's), it follows from (1.19) and (1.15) that $\Psi_0 = \Psi_4 = 0$. If, furthermore, the principal spinors have higher multiplicities, then more Weyl scalars (1.19) vanish (of course, for type N spaces only one spinor of the dyad, say o^A, can be aligned with a principal spinor). In general, choosing the Newman–Penrose frame this way one has, from (1.15) (by convention we align o^A with the principal spinor with higher multiplicity),

$$\begin{aligned}
\text{Type I}: &\quad \Psi_0 = \Psi_4 = 0 \\
\text{Type II}: &\quad \Psi_0 = \Psi_1 = \Psi_4 = 0 \\
\text{Type D}: &\quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \\
\text{Type III}: &\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \\
\text{Type N}: &\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0
\end{aligned} \tag{1.20}$$

That is, the condition of algebraic speciality translates into the vanishing of several components of the Weyl tensor. This is one of the main properties underlying the successfulness of the spinorial approach to applications in GR (see discussion in [53]).

The classification and properties discussed so far hold for all spacetimes since no assumption about the equations of motion has been made. A priori, there is no reason to expect that algebraically special solutions should play an important role in GR. For instance, rotating black hole solutions of some alternative gravity theories are not algebraically special in the Petrov sense [54]. Remarkably, the *Goldberg–Sachs theorems* uncover an intimate relation between Einstein's equation and the algebraic classification. One may think of the general structure of a Goldberg–Sachs theorem as follows: given a condition in the curvature (e.g. the vacuum Einstein's equation) it establishes an equivalence between geometric and algebraic properties of null vector fields. For instance, if a spacetime is Ricci-flat, then a null vector field is a repeated PND if, and only if, it is tangent to a null, shear-free geodesic congruence [18, 51].[18] This instrumental result led to one of the most spectacular results in GR, due to Kinnersley [55]. Assuming that a vacuum space is Petrov type D, the Goldberg–Sachs theorem guarantees the existence of coordinates adapted to congruences of null, shear-free geodesics aligned with PND's.

[17] The necessary and sufficient conditions for a principal spinor to be repeated with multiplicity m (1.15) can be translated into a condition for the corresponding vector contracted with the Weyl tensor [16, 50]. Besides being rather unilluminating conditions, these are remarkably more involved than (1.15).

[18] This last statement is expressed in a remarkably simple form in the Newman–Penrose formalism: $\kappa = 0 = \sigma \Leftrightarrow \Psi_0 = 0 = \Psi_1$, where κ and σ are some connection components (dubbed spin coefficients in [18, 50]).

This simplifies the equations enough to be integrated completely yielding the most general Petrov type D solution. The Kerr black hole, presented below, is a particular case of Kinnersley's solution. However, the importance of the Newman–Penrose approach goes beyond the construction of explicit solutions. As reviewed in Section 1.2, it is also crucial in the study of gravitational waves on top of black hole spacetimes (the analysis in Chapter 6 is based on this approach).

Kerr's Black Hole and Uniqueness

Kerr considered metrics of the Kerr–Schild form,

$$g_{\mu\nu} = \eta_{\mu\nu} - 2S k_\mu k_\nu \tag{1.21}$$

where $\eta_{\mu\nu}$ is Minkowski's metric, S is a function and k_μ a null vector with respect to both metrics, $g_{\mu\nu}k^\mu k^\nu = \eta_{\mu\nu}k^\mu k^\nu = 0$. If the vacuum Einstein's equation holds $R_{\mu\nu} = 0$ it can be shown that k_μ is a repeated PND. Imposing, furthermore, that the space is type D, the equations can be integrated yielding Kerr's solution [48]. In Boyer-Lindquist coordinates it is

$$\begin{aligned} ds^2 = &- \left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)\Big(dt^2 - 2a\sin^2\theta\left(\frac{r^2+a^2-\Delta}{\Sigma}\right)dtd\phi \\ &\left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\Big(\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned} \tag{1.22}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2 \tag{1.23}$$

Kerr's solution describes a stationary, rotating black hole with ADM mass M and ADM angular momentum aM. It exhibits a regular event horizon at $r_+ = M + \sqrt{M^2 - a^2}$ as long as Kerr's bound $|a| \leq M$ is respected (i.e. the black hole does not rotate too fast). Remarkably, Carter [56] and Robinson [57] proved that *Kerr's space is the unique asymptotically flat, stationary and axisymmetric black hole solution of GR.*[19] Therefore, according to GR (and assuming that both SCC and WCC conjectures are true) the final state of gravitational collapse is generically a Kerr black hole. This is a striking result since the initial state may be arbitrarily complicated, while the final state is determined uniquely (outside the hole) by just two numbers M and J. The theorem by Carter and Robinson has been extended to theories with matter, in which the set of parameters that determine completely the solution is enlarged to (M, J, Q) where Q denotes collectively the electric and magnetic charges [17]. This fact supports the so-called "no-hair" conjecture [58], according to which black holes are uniquely determined by M, J and their conserved gauge charges.

Higher Dimensions

Black holes of GR in an arbitrary number of spacetime dimensions are important in high-energy physics. In particular, they play a crucial role in string theory and holography. However, due to the larger number of degrees of freedom, several of the results presented above do not hold in $D > 4$. Heuristically, this can be understood as follows [59, 60]: in

[19] However, the exterior of a stationary and axisymmetric object is not described by Kerr's metric in general. This is unlike the spherically symmetric case (see above).

$D > 4$ black holes can exhibit non-compact horizons (extended black objects) and their dynamics allow arbitrarily large spins [61], unlike in $D = 4$ where Kerr's bound dictates $|a| \leq M$. Combining these two properties one can form black holes with non-spherical topologies like black rings [62] (balanced by centrifugal force due to fast spinning) and black hole uniqueness is broken, even infinitely if black rings are charged [63]. Furthermore, the "rigidity" theorems are not as strong as in $D = 4$ [59] and no useful extension of the Newman–Penrose formalism has been found so far [60], so the classification and obtention of solutions is significantly less successful than in $D = 4$. However, it is precisely this richer dynamics of higher-dimensional GR that has lead to the discovery of several interesting phenomena, some of which are discussed in section 1.2.

1.1.3 Schwarzschild's Black Hole

Schwarzschild's black hole is a good example to illustrate some of the concepts introduced above. The line element is given by (1.11), and a convenient Newman–Penrose frame is[20]

$$\mathbf{k} = dt + \frac{dr}{f(r)} \qquad \mathbf{l} = \frac{1}{2}\left(f(r)dt - dr\right) \qquad \mathbf{m} = \frac{1}{r\sqrt{2}}\left(r^2 d\theta + i r^2 \sin\theta d\phi\right) \tag{1.24}$$

where $f(r) = 1 - 2M/r$. $\mathbf{k}$ and $\mathbf{l}$ are tangent to null, shear-free geodesic congruences and, furthermore, $\mathbf{k}$ is affinely parametrised, $\nabla_{\mathbf{k}}\mathbf{k} = 0$. By the Goldberg–Sachs theorem, $\mathbf{k}$ and $\mathbf{l}$ are aligned with repeated PNDs. Thus, Schwarzschild's space is Petrov type D and, from (1.20), in the frame (1.24) the only non-vanishing Weyl scalar is

$$\Psi_2 = -\frac{M}{r^3} \tag{1.25}$$

Some components of the metric (1.11) and frame (1.24) are pathologic at $r = 2M$. This may be due to a true singularity or to a breakdown of the coordinates at $r = 2M$. One way of investigating this is by working in coordinates adapted to the null congruence generated by $\mathbf{k}$. Since $d\mathbf{k} = 0$ at least locally we can use a coordinate v defined by $dv = \mathbf{k}$. Thus, v labels null hypersurfaces generated by $\mathbf{k}$. In terms of (v, r, θ, ϕ) (known as ingoing Eddington–Finkelstein coordinates) the metric (1.11) reads

$$ds^2 = -f(r)dv^2 + 2drdv + r^2 d\Omega^2 \tag{1.26}$$

and, furthermore, $\mathbf{k}$ is expressed as $\mathbf{k} = \partial_r$ so the area-radius r is an affine parameter of the null congruence generated by $\mathbf{k}$. As a consequence, in these coordinates the metric (1.26) is well behaved at $r \in (0, \infty)$, so in particular, it is regular at $r = 2M$. The timelike Killing vector $k_t = \partial_t$ of (1.11) is now expressed as $k_t = \partial_v$. Since $k_t \cdot k_t = -f(r)$, one has that k_t is null on the hypersurface $r = 2M$, which we shall refer to as $\mathcal{H}$. In addition, $k_t \overset{\mathcal{H}}{=} dr$ so it is also normal to $r = 2M$ and, consequently, $\mathcal{H}$ is a Killing horizon of k_t. Using the Killing equation one has $\nabla k_t = (1/2) dk_t$, so

$$k_t \cdot \nabla k_t = \frac{1}{2} k_t \cdot dk_t = \frac{1}{2} k_t \cdot d\left(-f(r)dv + dr\right) \overset{\mathcal{H}}{=} \frac{f'(r = 2M)}{2} k_t \tag{1.27}$$

where in the last step $\overset{\mathcal{H}}{=}$ means evaluated at $\mathcal{H}$. From (1.27) it follows that the surface gravity is $\kappa = \frac{f'(r=2M)}{2} = 1/4M$, and also that v is not an affine parameter of the

[20] To alleviate notation we denote by the same symbol a vector (or 1-form) and its metric dual, except for the position of the index (if written).

congruence of generators of $\mathcal{H}$. In fact, the geodesics in $\mathcal{H}$ generated by $k_t = \partial_v$ with $v \in (-\infty, \infty)$ are incomplete. This can be seen by using a new coordinate V which at $\mathcal{H}$ is an affine parameter of the generators. Writing $dV = h(v)dv$ then $\partial_V = (1/h(v))\partial_v = (1/h(v))k_t$ and from (1.27) it follows that

$$\partial_V \cdot \nabla \partial_V \overset{\mathcal{H}}{=} \frac{1}{h(v)^3}\left(-h'(v) + \kappa h(v)\right)\partial_v \tag{1.28}$$

The r.h.s. must vanish if V is an affine parameter of the generators of $\mathcal{H}$, so we conclude that $V \sim e^{\kappa v}$. Now it is manifest that the geodesics generated by k_t with $v \in (-\infty, \infty)$ are incomplete, since these correspond to $V \in \mathbb{R}^+$ instead of $V \in \mathbb{R}$. Playing the same game with u, the retarded counterpart of v defined by $du = dt - \frac{dr}{f(r)}$, one is lead to the so-called Kruskal-Szekeres coordinates

$$U = -e^{-\kappa u} \qquad V = e^{\kappa v} \tag{1.29}$$

and in terms of (U, V, θ, ϕ) the metric reads

$$ds^2 = -\frac{32M^3 e^{-2\kappa r}}{r} dU dV + r^2 d\Omega^2 \tag{1.30}$$

where r is given implicitly in terms of U, V by

$$UV = -2\kappa r e^{2\kappa r} f(r) \tag{1.31}$$

Just like (u, v), the coordinates (U, V) label null hypersurfaces and dU and dV are tangent to affine generators[21]. The difference lies in the fact that (U, V) are also affine parameters of the generators of $\mathcal{H}$. Consequently, (1.30) can be used to analytically continue the metric (1.11) to the range $(U, V) \in (-\infty, \infty)$ where $r > 0$ (see (1.31)). In these coordinates the static Killing vector is

$$k_t = \kappa \left(V\partial_V - U\partial_U\right) \tag{1.32}$$

and its Killing horizon $\mathcal{H}$ at $r = 2M$ corresponds to the surfaces $U = 0$ and $V = 0$. That is, $\mathcal{H}$ is the intersection of two null hypersurfaces. This is known as a *bifurcate Killing horizon*, and the bifurcation surface, denoted by $\mathcal{BH}$, is a fixed point of the Killing vector field. This is so because the Killing vector field must vanish in order to be normal to both surfaces simultaneously. In the spacetime (1.30) the bifurcation surface is the 2-sphere at $U = V = 0$, where $k_t \overset{\mathcal{BH}}{=} 0$ (see (1.32)). Furthermore, for any bifurcate Killing horizon one has[22]

$$\nabla_\mu k_\nu \overset{\mathcal{BH}}{=} \kappa n_{\mu\nu} \tag{1.33}$$

where $n_{\mu\nu}$ is the binormal to the $\mathcal{BH}$ with the conventional normalisation $n_{\mu\nu}n^{\mu\nu} = -2$ (i.e. $n_{\mu\nu}$ is the natural volume element in $\mathfrak{X}(\mathcal{BH})^\perp$). In general, the structure of spacetime in the neighbourhood of a bifurcation surface is as sketched in Figure 1.1 and, furthermore, it can be shown that there is always a (time-orientation-reversing) local isometry in the neighbourhood of the $\mathcal{BH}$ which maps region I to region III and region II to region IV [64].[23] One such isometry in the case of (1.30) is simply $(U, V) \to (-U, -V)$.

[21] Indeed $(dU)^\mu \nabla_\mu (dU)_\nu = (dU)^\mu \nabla_\mu \nabla_\nu U = (dU)^\mu \nabla_\nu \nabla_\mu U = (1/2)\nabla_\nu \left[(dU)_\mu (dU)^\mu\right] = 0$, and similarly for dV.

[22] This can be checked easily in explicit examples by using that $\nabla k = (1/2)dk$ for Killing vector fields. We prove it in general in Appendix B.

[23] This is a technical piece that results very useful in deriving the first law of black hole mechanics in general theories.

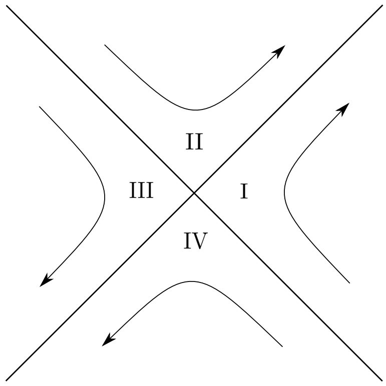

Figure 1.1: General structure of a neighbourhood of the bifurcation surface $\mathcal{BH}$ of a Killing horizon $\mathcal{H}$. Thick lines represent $\mathcal{H}$, with $\mathcal{BH}$ at the intersection. The thin arrows represent the flow of the Killing vector associated to $\mathcal{H}$. The local time-orientation-reversing isometry maps region I to region III and region II to region IV [64].

The global causal structure of spacetime can be represented accurately in a *Penrose–Carter diagram*. Penrose–Carter diagrams are associated to *conformal compactifications*, reductions of spacetime to finite size that preserve the causal structure. Only two coordinates are represented, one for time T and one for space X, with T in the vertical axis and time flowing upwards. Furthermore, these coordinates are chosen so that light rays are straight lines of slope $\pm\pi/4$. For the (maximally extended) Schwarzschild solution one has the diagram shown in Figure 1.2. It consists of two asymptotically flat regions I and III, a black hole region II and a white hole region IV. The four regions are separated by the Killing horizon $\mathcal{H}$, which coincides with the event horizons of both future null infinities $\mathscr{I}^+$. The isometry $(U,V) \to (-U,-V)$ maps I to III and II to IV, so they can be regarded as the time reversal of each other. II and IV contain a spacelike singularity at $r = 0$ (which is discussed below). The spacetime in those regions is not static because k_t becomes spacelike, and the Jebsen–Birkhoff theorem establishes that the metric must be isometric to

$$ds^2 = -\left(\frac{2M}{\tilde{t}} - 1\right)^{-1} d\tilde{t}^2 + \left(\frac{2M}{\tilde{t}} - 1\right) dx^2 + \tilde{t}^2 d\Omega^2 \tag{1.34}$$

with $|\tilde{t}| < 2M$. Hence, spacetime is an homogeneous cylinder $\mathbb{R} \times \mathbb{S}^2$ (surfaces of constant $\tilde{t}$) that shrinks and stretches in the $\mathbb{S}^2$ and $\mathbb{R}$ factors, respectively, as time flows into the future in region II and into the past in region IV. Both III and IV are a consequence of assuming that the Schwarzschild black hole is "eternal" when analytically continuating the solution to the past of $t = -\infty$. In models of gravitational collapse, III and IV are replaced by the collapsing matter in the interior.

We conclude by discussing the trapped surfaces of (1.30). The vector fields

$$k^+ = -(dU)^\mu \partial_\mu = \frac{re^{2\kappa r}}{32M^3} \partial_V \tag{1.35}$$

$$k^- = -\alpha(dV)^\mu \partial_\mu = \alpha \frac{re^{2\kappa r}}{32M^3} \partial_U \qquad (\alpha > 0) \tag{1.36}$$

are future-directed and tangent to a congruence of affinely parametrised null geodesics. Furthermore, they are normal to the 2-spheres at constant (U,V) (the positive constant

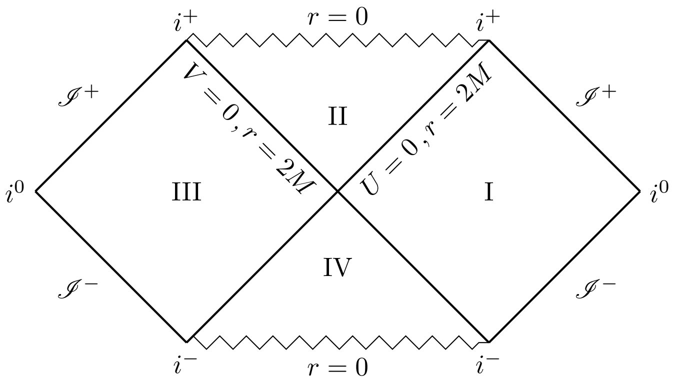

Figure 1.2: Penrose–Carter diagram of the maximally extended Schwarzschild spacetime (1.30). Each point represents a 2-sphere and lines at $\pm\pi/4$ are null hypersurfaces of constant U and V (so time flows to the future as one moves upwards). $\mathscr{I}^+$ and $\mathscr{I}^-$ are future and past null infinity, the points i^+ and i^- are future and past timelike infinity and the points i^0 are spatial infinity.

α can be used to fix the relative normalisation at a specific 2-sphere to $g(k^+, k^-) = -1$). The corresponding expansions are

$$\theta^+ = \nabla_\mu (k^+)^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} (k^+)^\mu \right) = -\frac{\kappa}{r} U \tag{1.37}$$

$$\theta^- = \nabla_\mu (k^-)^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} (k^-)^\mu \right) = -\alpha \frac{\kappa}{r} V \tag{1.38}$$

In regions I, III and IV the 2-spheres are not trapped since at least one of the families has positive expansion. On the portion of $\mathcal{H}$ that bounds II, the 2-spheres are marginally future-trapped since the family that lies along the horizon has vanishing expansion (as it should be for Killing horizons), while it is negative for the family entering II. The only exception is the bifurcation 2-sphere at $U = V = 0$, since both expansions vanish and the surface is stationary. All 2-spheres in region II are future-trapped since both expansions are strictly negative. From Penrose's theorem, at least one of the families emerging from any of these 2-surfaces should be incomplete and inextendible. In our example this is the case for both families, which terminate at $r = 0$. What remains to be clarified is whether the incompleteness is due to extendibility of the maximal Cauchy development (i.e. whether the space (1.30) can be extended beyond $r = 0$), or it is due to a true singularity. Computing the Kretschmann scalar one finds

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = 48 \Psi_2^2 = 48 \frac{M^2}{r^6} \tag{1.39}$$

where in the second step we used the vacuum Einstein equation $R_{\mu\nu} = 0$. Since (1.39) is an invariant quantity, we conclude that the curvature tensor is singular at $r = 0$ and thus the spacetime can not be extended smoothly beyond that hypersurface.

1.2 Dynamical Properties

The study of exact solutions describing stationary black holes is motivated in part by the fact that black holes are generic predictions of GR. However, once an exact solution is found, it is not guaranteed that it describes a stable (and, thus, generic) configuration. Thus, obtaining exact solutions does not suffice and one has to explore the dynamics of neighbouring states. In addition, gravitational wave (GW) astronomy detects routinely merger processes of compact objects, a large portion of which are believed to be black holes [3]. These are highly dynamical processes that can be used to test the strong field regime of gravity [10]. From a different point of view, as explained in Chapter 6, the dynamics of black holes in Anti de Sitter is dual to certain properties of QFTs, which is a striking fact following from the AdS/CFT correspondence [65].

The reasons above motivate the study of the dynamics of black holes. To that end, many symmetry assumptions such as algebraic speciality, a non-trivial isometry group, etc. , need to be dropped in solving Einstein's equations. In that situation one is typically forced to approach the problem numerically. An alternative analytic, yet perturbative, approach is to consider linear fluctuations of an exact black hole solution. While the information that can be obtained this way is partial, it is still physically meaningful and provides very valuable insights on the dynamics of black holes.

Black Hole Mergers

Qualitatively, the waveforms of the typical black hole mergers observed by LIGO and Virgo present three stages [15]. First, there is an inspiral phase in which the black holes orbit each other emitting gravitational waves. As energy is radiated away, the orbits decrease in radius and increase in frequency. The merger phase encompasses the plunge and coalescence of the black holes. Once the black holes have merged into a single one, there is a last stage referred to as the *ringdown* in which the final black hole emits GWs in its "natural modes", with some characteristic frequencies and damping times (see Figure 1.3). Describing the merger phase requires accounting for all the non-linearities in Einstein's equations and, typically, this stage is studied through numerical methods. On the other hand, some aspects of the ringdown and inspiral phases can be approached through black hole perturbation theory.

Vishveshwara [67] and Press [68] noticed that black holes exhibit some "free oscillation modes" that rule the behaviour of perturbations at late times. These are essentially the waves observed during the ringdown, and their frequencies and damping times depend solely on the structure of the final black hole.[24] Such frequencies and damping times can be obtained from black hole perturbation theory by studying a specific class of fluctuations known as quasinormal modes (QNMs). These are defined as perturbations of definite frequency that are regular on the event horizon (which is equivalent to requiring that the waves can only cross the horizon inwards) and purely outgoing waves at infinity (thus encoding the idea that QNMs are free oscillations) [69]. This is a characteristic value problem which has solutions only for a discrete set of complex frequencies, the so-called quasinormal frequencies, whose real and imaginary parts are the (real) frequencies and damping times of the free oscillations of the black hole, respectively. From the uniqueness

[24] In Chandrasekar's words [18], at late times the black hole radiates "*in the manner of a bell sounding its last dying pure notes*".

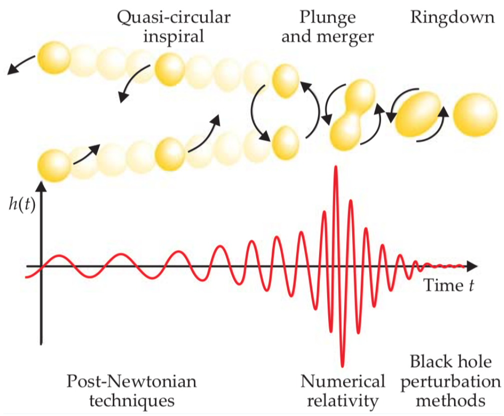

Figure 1.3: Illustration of a typical binary black hole merger, exhibiting the three main phases discussed in the text. Image source: [66].

theorems presented above, it follows that the QNM spectrum observed in the ringdown phase of a binary black hole merger must be fully determined by the mass M and angular momentum J of a Kerr black hole. QNMs of Kerr where computed by Detweiler [70], and, very recently, an exhaustive study has been published about the QNMs of gravito-electromagnetic fluctuations of Kerr–Newman's black hole [71–73]. Then, measuring two (three) quasinormal modes fixes M and J (and Q) assuming the black hole is neutral (charged), but measuring a third (fourth) one is already a test of GR. QNM's can, thus, be used to test the no-hair conjecture which is one of the main goals of *black hole spectroscopy* [74–76]. QNMs are also important in holography, but their definition and physical meaning is significantly different so we shall discuss them in Chapter 6.

Consider now the late stages in the inspiral phase of a binary merger. As the orbits decrease in radius the bodies get closer and tidal interactions become significant. In turn, this manifests itself in the shape and phase of the gravitational waveform emitted by the coalescing binary, which receives corrections at 5th post-Newtonian order [77–80]. Focusing on one of the objects, tidal interactions consist in internal moments induced by external ones caused by the companion [19], and such response is parametrised by the so-called Tidal Love Numbers (TLN). These can be thought of as gravitational susceptibilities that (just like their electric counterparts) depend only on the structure of the deformed object. In the case of neutron stars, it was shown in [77, 78] that the imprint of the TLNs in the waveform can be used to extract information about the equation of state of the star, even above currently understood nuclear densities. In GR, TLNs have a precise definition in the context of perturbation theory as static gravitational fluctuations[25] and, quite remarkably, these are exactly zero for static black holes [80] and exhibit only a dissipative component for the rotating ones [81, 82]. Thus, TLNs are observables that inform about the presence of an horizon in spacetime. In addition, the fact that they vanish for black holes is a very specific signature of $4D$ GR, since the TLNs of black holes are non-vanishing in higher-order theories [83], asymptotically AdS spaces [84] and higher dimensional GR [85–87]. Consequently, in the last few years, much effort has been

[25] This is motivated by the expectation that at the late stages of the inspiral, when tidal interactions are significant, the characteristic time scale is still much larger than the spatial one. Thus, time dependencies can be treated adiabatically.

made towards understanding what protects the vanishing of TLNs in GR and its relation to the "no-hair" conjecture [88], under what (environmental) mechanisms can black hole TLNs be excited effectively [89], and studying the multipolar structure of some horizonless substitutes of black holes (e.g. fuzzballs) [90]. In particular, it was shown in [87] that the vanishing of the tidal response is not specific of gravity in $4D$ GR, since the same is true for spin-0 and spin-1 fluctuations and, as shown in [91] (and discussed in Chapter 7), this fact remains true in a very nontrivial manner even when the black hole is charged.

Onset of Instabilities

Proving that a black hole is stable at the linear level is in general a difficult task (even without considering mode composition), and it is, furthermore, inconclusive, since instabilities could set in at higher orders. However, working linearly one can prove that a black hole is unstable e.g. by exhibiting regular modes that grow with time. Identifying the conditions under which instabilities arise provides very valuable insights on the physics of black holes. One of the most celebrated examples is the so-called Gregory–Laflamme instability [92]. It is a gravitational instability of black branes (black holes extended with flat extra directions) under the propagation of long wavelength modes along the extended directions.[26] Thus, it is reasonable to expect the onset of Gregory–Laflamme-like instabilities whenever a horizon is characterised by two significantly different length scales. This intuition has proven extremely useful in e.g. the interpretation of more complicated black hole solutions [94], the construction of black hole brane-worlds [95] and the study of thermodynamic properties of black hole saddles in Euclidean quantum gravity [96]. Another important instance is the superradiant instability of rotating black holes, intimately related to the Penrose process, and we refer the reader to [97] for a review.

To conclude this section we shall sketch the two main approaches to the study of black hole fluctuations. These are extensively used in Chapters 6 and 7.

1.2.1 Black Hole Perturbation Theory in Higher Dimensions

In black hole perturbation theory one considers a linear deviation $h_{\mu\nu}$ off an exact black hole solution $\bar{g}_{\mu\nu}$, so the spacetime metric is approximated as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1.40}$$

where $h_{\mu\nu}$ satisfies the linearised Einstein equation $G^{(1)}_{\mu\nu}[h] = 0$ (where $G^{(1)}_{\mu\nu}[\cdot]$ denotes Einstein's tensor linearised around $\bar{g}$ acting as an operator on $h_{\mu\nu}$). The main obstacles in the analysis are the large amount of equations and gauge redundancies $h_{\mu\nu} \sim h_{\mu\nu} + \pounds_X \bar{g}_{\mu\nu}$. When the background spacetime has enough symmetry, it is possible to organise the fluctuations in decoupled sets associated to the different kinds of harmonics. This was first considered in $4D$ by Regge and Wheeler [98], Zerilli [99, 100] and Moncrief [101], and was later generalised rigorously to higher dimensions by Kodama and Ishibashi [102]. We shall follow the latter authors here since this includes the four-dimensional case.

Assume that a black hole spacetime (M, g_{AB}) has the structure

$$M = \underbrace{\mathcal{N}^m}_{y^a} \times \underbrace{\mathcal{K}^n}_{z^i}, \qquad ds^2 = g_{AB}dx^A dx^B = g_{ab}(y)dy^a dy^b + r^2(y)\gamma_{ij}(z)dz^i dz^j \tag{1.41}$$

[26] In many senses it can be regarded as a black hole version of the Rayleigh-Plateau instability of fluid dynamics [93].

where $(\mathcal{K}^n, \gamma_{ij})$ is an n-dimensional, Euclidean Einstein manifold, $(\mathcal{N}^m, g_{ab})$ is an m-dimensional Lorentzian manifold and $r^2(y)$ a function on $\mathcal{N}^m$ called the warp factor.[27] The pieces of the metric perturbation with legs on $\mathcal{K}^n$, i.e. (h_{ai}, h_{ij}), can be further decomposed separating the contribution of the transverse (and traceless in the case of h_{ij}) parts.[28] The transverse and traceless pieces with two legs on $\mathcal{K}^n$ constitute the so-called *tensor sector* which we denote collectively by $\mathcal{T}_{ij}$. The transverse pieces with one leg on $\mathcal{K}^n$ conform the *vector sector* $\mathcal{V}_i$ and the remaining pieces, which have no legs on $\mathcal{K}^n$ are the *scalar sector* $\mathcal{S}$. Then, it is possible to expand each sector as

$$\mathcal{T} = T(y)\mathbb{T}_{ij}, \quad \mathcal{V} = V(y)\mathbb{V}_i, \quad \mathcal{S} = S(y)\mathbb{S} \tag{1.42}$$

where $\mathbb{T}_{ij}, \mathbb{V}_i$ and $\mathbb{S}$ are tensor, vector and scalar harmonics[29] of $\mathcal{K}^n$ [102] and $T(y), V(y)$ and $S(y)$ are tensors on $\mathcal{N}^m$. Plugging (1.42) into $G^{(1)}_{\mu\nu}[h] = 0$, the sectors decouple forming three distinct sets of linear, coupled PDEs on $\mathcal{N}^m$. Thus, one can work sector by sector, construct gauge-invariant variables and, hopefully, derive decoupled equations within each sector. In particular, this has been done explicitly for higher-dimensional charged black holes of the form (1.41).

The main drawback of this approach is that it cannot be applied to rotating black holes in general.[30] Exceptionally, in four spacetime dimensions the algebraic description introduced in section 1.1.2 provides an alternative and elegant approach that accounts for a much wider class of spacetimes.

1.2.2 Black Hole Perturbation Theory in 4D

Teukolsky [105] considered a Ricci-flat background space of Petrov type D[31]. The Goldberg-Sachs theorem guarantees that one can choose a Newman–Penrose frame $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$ where $\mathbf{k}, \mathbf{l}$ are PNDs and tangent to shearfree null geodesics. This translates into the vanishing of a remarkably large number Newman–Penrose variables of the background (see e.g. (1.20)). This has two main consequences when considering linear perturbations [53]: first, the linearised Weyl scalars $\Psi^{(1)}_0$ and $\Psi^{(1)}_4$ are gauge-invariant quantities, both with respect to diffeomorphisms and frame rotations. Second, several structure equations are homogeneous in quantities that vanish on the background, so they become already linearised and are remarkably simple. This allows one to derive two identities of the form[32]

$$\mathcal{O}_0(\Psi^{(1)}_0[h]) = S^{\mu\nu}_0 G^{(1)}_{\mu\nu}[h] \qquad \mathcal{O}_4(\Psi^{(1)}_4[h]) \;= S^{\mu\nu}_4 G^{(1)}_{\mu\nu}[h] \tag{1.43}$$

that hold for all (off-shell) metric fluctuations $h_{\mu\nu}$, where Teukolsky's operators $\mathcal{O}_0, \mathcal{O}_4$ are linear, second order differential operators acting on functions [105], $\Psi^{(1)}_0[h], \Psi^{(1)}_4[h]$ are the linearised Weyl scalars induced by $h_{\mu\nu}$, and $S^{\mu\nu}_0, S^{\mu\nu}_4$ are yet another set of linear, second

[27] For example, Schwarzschild's space (1.11) has this form with $\mathcal{K}^n$ being the round 2-sphere and $\mathcal{N}^m$ being the (t, r)-plane.

[28] This is just the Hodge decomposition of h_{ai} and an analogue thing for h_{ij}.

[29] We notice that in four spacetime dimensions there are no tensor harmonics, and the vector harmonics on the 2-sphere are just the Hodge duals of the covariant derivatives of $\mathbb{S}$.

[30] Two exceptions are the tensor perturbations of single spinning [103] and cohomogeneity-1 [104] Myers–Perry black holes, and the latter case is not in the framework of Kodama and Ishibashi.

[31] All concepts regarding the algebraic description of spacetimes used here have been previously introduced in section 1.1.2.

[32] Here we depart slightly from the original derivation by Teukolsky and follow a line closer in spirit to [106] in order prepare the discussion on metric reconstruction.

order differential operators mapping 2-tensors into functions [106]. Assuming that the linearised Einstein equations are satisfied, the r.h.s. of (1.43) vanish thus leading to the celebrated Teukolsky equations: two decoupled, linear, second order PDE's for $\Psi_0^{(1)}[h]$ and $\Psi_4^{(1)}[h]$ that hold on any Ricci-flat, Petrov type D space. These can be easily generalised to solutions with a cosmological constant. Furthermore, when specialised to Kerr's spacetime, both $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ admit separable solutions and, therefore, Teukolsky's equations translate into ODEs. Such separability has also been observed in other spacetimes, an example of which is considered in Chapter 6, and it is ultimately due to the existence of a principal Killing–Yano tensor of the background spacetime [107]. Besides its efficiency in deriving convenient equations, this approach is also physical because $\Psi_0^{(1)}$ and $\Psi_4^{(1)}$ control the gravitational wave energy fluxes at the horizon and infinity, respectively [105]. In particular, the waveform models and templates are constructed for $\Psi_4^{(1)}$ (see e.g. [108] for an explicit computation of the waveform sourced by a particle orbiting a black hole).

Finally, some problems require one to know the actual metric perturbation $h_{\mu\nu}$, and not just the solution for the master variables. For instance, in holography one needs to impose boundary conditions on the physical fields and, then, translate those into boundary conditions for the master variable that we know how to solve [109]. Such metric-reconstruction problem seems intractable due to the large number of Einstein and Newman–Penrose equations. However, in a considerable *tour de force* Chandrasekar did it in the case of Kerr's metric [110, 111] (also reviewed in [18]). Using a different approach and under some non-trivial assumptions, Cohen and Kegeles [112] and Chrzanowski [113] provided a prescription for reconstructing electromagnetic and gravitational perturbations of any space of type D. However, it was finally Wald in [106] who was able to reformulate the problem in a more convenient language giving a remarkably simple proof of Cohen, Kegeles and Chrzanowski's formulas. The argument consists simply in noticing that $G^{(1)}_{\mu\nu}[\cdot]$ is self adjoint $G^{(1)\dagger}_{\mu\nu}[\cdot] = G^{(1)}_{\mu\nu}[\cdot]$ (in the sense specified in [106]). Then, taking the adjoint of (1.43) one has

$$\Psi^\dagger \mathcal{O}^\dagger = G^{(1)\mu\nu} S^\dagger_{\mu\nu} \tag{1.44}$$

Now, if one has a solution φ of $\mathcal{O}^\dagger(\varphi) = 0$ (called Hertz potential), it follows from (1.44) that one can generate a solution for the metric perturbation by just acting with $S^\dagger_{\mu\nu}$ on φ

$$h_{\mu\nu} = S^\dagger_{\mu\nu}(\varphi) \tag{1.45}$$

Furthermore, Hertz potentials φ and solutions of Teukolsky's equation $\mathcal{O}(\Psi) = 0$ are related through a simple rescaling by a function. Hence, metric perturbations can be equally generated from those. This result will be useful in Chapter 6.

1.3 Black Hole Thermodynamics

So far we have discussed black holes as special solutions of GR which constitute a robust theoretical prediction and whose existence in nature is supported by strong experimental evidence. However, GR cannot be a complete theory. From the quantum-mechanical point of view, GR is not renormalisable [114–116], so one expects it to be just an effective description of a UV-complete theory. Therefore, everything we have discussed so far about black holes is only the *classical* picture. Indeed, some properties of black holes indicate that these are actually thermodynamical systems consisting of an extremely large number of microscopic degrees of freedom. However, a precise description of the latter requires the knowledge of a quantum theory of gravity.

In the remaining of this introduction we will review the laws of black hole thermodynamics, discuss the joy and sorrow these have inflicted to fundamental physics, and provide some motivation for Part I of this thesis.

Black Hole Mechanics and Hawking Radiation

Hartle and Hawking [117] and Bardeen, Carter and Hawking [118] showed that perturbations of Kerr's black hole are subject to a set of laws, known as *the four laws of black hole mechanics*. The *second law*, due to Hawking [119], establishes that if Einstein's equation holds and matter satisfies the null energy condition, then the area $\mathcal{A}$ of spatial sections of the horizon does not decreases along the future-directed generators

$$\delta\mathcal{A} \geq 0 \tag{1.46}$$

Underlying this result is the fact that, under the above assumptions, the generators of $\mathcal{H}$ have non-negative expansion $\theta \geq 0$. The *zeroth law* establishes that the surface gravity κ is constant on the future event horizon of a stationary black hole spacetime obeying the dominant energy condition (below we will see that actually the zeroth law is a geometric consequence of the definition of Killing horizons, and does not depend on the equations of motion). The *first law* relates the variations of the mass δM, of the angular momentum δJ and of the horizon area, $\delta\mathcal{A}$, of a Kerr black hole via the equation

$$\frac{\kappa}{8\pi}\delta\mathcal{A} = \delta M - \Omega_H \delta J \tag{1.47}$$

where Ω_H is the horizon's angular velocity introduced in (1.9). Bardeen, Carter and Hawking [118] proved this formula for symmetric perturbations of Kerr, while Hartle and Hawking [117] derived it by reproducing a physical process of accretion. Later on, it was shown that (1.47) holds in situations that are substantially more general than those considered originally, and it was enlarged to include contributions from matter [120]. Finally, in [118] a *third law* was also proposed (and later proven in [121]) stating that reducing the surface gravity to zero is a process that necessarily involves an infinite amount of time.

Remarkably, the four laws of black hole mechanics coincide with the four laws of ordinary thermodynamics for a system of internal energy M, angular momentum J, angular velocity Ω_H, temperature $\gamma\kappa/8\pi$ and entropy $\mathcal{A}/\gamma$, where γ is an unknown constant. Assuming that this analogy is not merely coincidental implies accepting that black holes have physical temperature and entropy. The latter is actually a reasonable assumption. Indeed, if black holes do not have entropy then just by throwing an entropic object into it the entropy of the universe would decrease, thus violating the second law of thermodynamics. Based on this idea, Bekenstein [122] proposed that black holes have an entropy precisely proportional to their area. A non-vanishing temperature, however, is seemingly contradictory. If black holes had a temperature then they would radiate just like any other warm body, but this is in conflict with the classical picture (nothing escapes the black hole region). Insisting in the idea that black holes have a physical temperature implies that they must radiate quantum-mechanically. In a revolutionary work, Hawking [123, 124] showed that treating matter quantum-mechanically (in the so-called semi-classical approximation) then, at late times after collapse, black holes radiate like a black body at the *Hawking temperature*

$$T_H = \frac{\kappa}{2\pi} \tag{1.48}$$

which determines $\gamma = 4$ in the analogy above, and leads to the conclusion that black holes have an entropy given by

$$S_{BH} = \frac{\mathcal{A}}{4} \tag{1.49}$$

which is known as the *Bekenstein–Hawking entropy*. This important result shows that black holes are truly thermodynamical systems and that the four laws governing their mechanics are just the ordinary laws of thermodynamics applied to black holes. Furthermore, the sum of the entropies of the black hole and of the matter outside the horizon does not decrease during Hawking evaporation [125, 126] (the so-called *generalised second law of thermodynamics*) even though the area of the hole may decrease.

The Microscopic Nature of Black Holes and the Information Paradox

Hawking's striking result raised inevitable questions about the nature of black holes. First, if black holes have an entropy S then statistical mechanics implies that there must be $\sim e^S$ microstates compatible with the macroscopic thermodynamic variables. However, the uniqueness theorems establish that associated to those thermodynamic variables (M, J, Q) there is precisely one spacetime geometry describing a black hole. It is, therefore, not clear what the microstates that give raise to the entropy of a black hole are. The second puzzle follows by considering a process of Hawking radiation that leads to a complete evaporation. Since the radiation is exactly thermal, the final state must be a mixed one, but this is in conflict with unitary evolution (essentially, time evolution preserving probability densities), which forbids transitions from pure to mixed states. It follows that some information is just lost in the process of black hole evaporation, a fact that yields the so-called *black hole information paradox*. If quantum gravity is unitary, then it should be possible to refine the approximations in Hawking's calculation and restore unitarity in black hole evaporation.

Understanding the nature of black hole microstates and the information paradox requires working in the context of a potential theory of quantum gravity. String theory is, to date, the most promising candidate. It is in such a framework that Susskind [127] proposed that black holes are effective descriptions of quantum systems consisting of strings and branes, a fact that lead Strominger and Vafa [128] to reproduce, for the first time, the Bekenstein–Hawking entropy of a black hole by counting the associated string microstates. This is one of the most important achievements of string theory and, together with improved techniques for computing the entropy of Hawking radiation [129–135], suggest that black hole evaporation is a unitary process.

These results motivate studying the corrections to the black hole entropy induced by subleading effects of string theory. The effective actions governing the behaviour of superstring theories at low energies enlarge the standard theory of GR by coupling it to light gauge fields and allowing for extra-dimensions. Furthermore, if finite size effects on the string length are accounted for, the effective action includes terms of higher-order in the spacetime curvature. In those cases, the entropy of a black hole is no longer given by the area of the horizon, and the correct identification of the macroscopic entropy is crucial for a meaningful comparison with the result from string microstate counting. In Sections

1.3.1 and 1.3.2 we introduce a framework in which the black hole entropy can be identified in a gauge-invariant guise. This is extensively used in Part I of this thesis.

1.3.1 Charges in Covariant Gauge Theories

Wald [139] understood that, in pure gravity theories, the entropy of a stationary black hole is the Noether charge associated to the Killing vector field that generates the horizon because it satisfies the first law of black hole thermodynamics. This point of view is very useful because it can be applied to general higher-order gravities. However, Wald's argument does not extend trivially to theories with fields that have internal gauge symmetry. If gauge fields are present, then conserved charges are associated to symmetries that act not only on spacetime but also in the internal space. Here we review some general results about charges in covariant gauge theories that will be useful for studying black hole thermodynamics.

Let $\mathbf{L}(\Phi)$ denote the Lagrangian d-form of a generally covariant and gauge invariant theory in d spacetime dimensions. The fields are collectively denoted by Φ. For example, in the Einstein–Maxwell theory Φ contains the metric (or Vielbein) and a vector potential. The first order variation of the Lagrangian reads

$$\delta\mathbf{L} = \mathbf{E}\delta\Phi + d\mathbf{\Theta}(\delta\Phi) \tag{1.50}$$

where the equations of motion $\mathbf{E} = \delta\mathbf{L}/\delta\Phi$ are the Euler–Lagrange derivative of $\mathbf{L}$ and $\mathbf{\Theta}(\delta\Phi)$ is the *symplectic potential form*, which is linear in $\delta\Phi$ and collects the boundary terms picked in the integration by parts. We consider fields with some internal gauge freedom, so the linear action of a general automorphism can be written as

$$\delta_{\xi,\lambda}\Phi = -\pounds_{\xi}\Phi + \delta_{\lambda}\Phi \tag{1.51}$$

where the vector field ξ generates a diffeomorphism, $\delta_\lambda\Phi$ denotes the action of the internal gauge symmetry generated by a (local) parameter λ, and the relative minus sign is purely conventional. Given a solution Φ, the *generalised Noether theorem* [136, 137, 142] establishes that there is a bijection between (certain equivalence classes of) parameters (ξ, λ) satisfying

$$\delta_{\xi,\lambda}\Phi = 0 \tag{1.52}$$

called *reducibility parameters*, and $(d-2)$-forms that are closed on-shell (modulo exact forms). In more physical terms, the latter can be thought of as charges that satisfy a Gauss law.

There are two classes of parameters satisfying (1.52) that will be particularly important. The first are vertical (i.e. $\xi = 0$) gauge transformations λ satisfying

$$\delta_\lambda\Phi = 0 \tag{1.53}$$

Gauge charges arise as the conserved quantities associated to this class of symmetries. Assuming that a solution admits a Killing vector field k, the second important class consists of the gauge parameters (k, λ_k) that satisfy

$$\delta_{k,\lambda_k}\Phi = 0 \tag{1.54}$$

Equation (1.54) can be seen as a covariant generalisation of the Killing equation where λ_k acts as a "compensating" or "induced" gauge transformation.[33] Associated to the symmetry (1.54) is the *Noether–Wald charge*. In general it does not satisfy a Gauss law but, as we will see, some closely related charges do. The latter are crucial in deriving the first law of black hole mechanics and also the Smarr relation.

In order to associate charges to gauge parameters we need *Noether's second theorem* [136, 137]. It asserts that, off-shell and for generic parameters (ξ, λ),

$$\mathbf{E}\delta_{\xi,\lambda}\Phi = d\mathbf{S}_{\xi,\lambda} \tag{1.55}$$

where $\mathbf{S}_{\xi,\lambda}$ is a $(d-1)$-form proportional to the equations of motion and their derivatives. This is equivalent to the statement that there exist certain (off-shell) identities amongst the equations of motion [136], the so called *Noether identities*. Since $\mathbf{S}_{\xi,\lambda}$ vanishes on-shell, it gives a trivial conserved current. However, it is possible to construct non-trivial "lower-degree conserved currents", that is, non-vanishing $(d-2)$-forms that are closed on-shell (which, as mentioned above, may be thought of as charges satisfying a Gauss law).

Under the action of (1.51) the first variation of $\mathbf{L}$ can be written in two different ways. Assuming that the Lagrangian is gauge invariant and generally covariant implies

$$\delta\mathbf{L} = -\pounds_\xi\mathbf{L} = -\left(\iota_\xi d + d\iota_\xi\right)\mathbf{L} = -d\left(\iota_\xi\mathbf{L}\right) \tag{1.56}$$

Using Noether's second theorem,

$$\delta\mathbf{L} = \mathbf{E}\delta_{\xi,\lambda}\Phi + d\mathbf{\Theta}(\delta\Phi) = d\left[\mathbf{S}_{\xi,\lambda} + \mathbf{\Theta}(\delta_{\xi,\lambda}\Phi)\right] \tag{1.57}$$

Therefore, at least locally, there is a $(d-2)$-form $\mathbf{Q}_{\xi,\lambda}$ that satisfies the off-shell identity

$$d\mathbf{Q}_{\xi,\lambda} = \mathbf{\Theta}(\delta_{\xi,\lambda}\Phi) + \iota_\xi\mathbf{L} + \mathbf{S}_{\xi,\lambda} \tag{1.58}$$

Furthermore, we shall assume that $\mathbf{Q}_{\xi,\lambda}$ is a local function of the fields and gauge parameters.[34]

Considering vertical gauge transformations λ satisfying (1.53), it follows from (1.58) that

$$d\mathbf{Q}_\lambda \doteq 0 \tag{1.59}$$

where $\doteq$ means "evaluated on-shell", and we used that $\xi = 0$, $\mathbf{S}_\lambda \doteq 0$ and that $\mathbf{\Theta}(\delta\Phi)$ is linear in $\delta\Phi$ so it vanishes if λ satisfies (1.53). In Chapters 2 to 4, conserved gauge charges are defined as certain integrals of these $\mathbf{Q}_\lambda$.

The form $\mathbf{Q}_{k,\lambda_k}$ associated to (k, λ_k) is the *Noether–Wald charge*. On-shell, it fails to be closed by

$$d\mathbf{Q}_{k,\lambda_k} \doteq \iota_k\mathbf{L} \tag{1.60}$$

where we used (1.58) again. Thus, it satisfies a Gauss law if the Lagrangian vanishes on-shell. This is the case of important examples such as vacuum GR, but it is not true

[33] In principle, one can provide a precise non-linear version of these notions once the geometric structure of the theory is specified, e.g. a principal fibre bundle [138], but this can be quite technical (specially for higher-rank gauge potentials). Here, in order to keep the discussion general and simple we stick to the linear action of symmetries.

[34] This can be verified from an explicit computation of $\mathbf{Q}_{\xi,\lambda}$ for a given theory or in general [139, 140].

in general. One way of constructing a conserved charge is to write the r.h.s. above as a suitable total derivative,

$$\iota_k \mathbf{L} \doteq d\mathbf{W}_k \tag{1.61}$$

which can always be done, at least locally, because $0 = \pounds_k \mathbf{L} = d\left(\iota_k \mathbf{L}\right)$. The form

$$\mathbf{K}_k \equiv \mathbf{Q}_{k,\lambda_k} - \mathbf{W}_k \tag{1.62}$$

is the *generalised Komar charge* (see e.g. [141] and also Chapter 4 and references therein). It is closed on-shell and, as shown in Chapter 4, it can be used to derive Smarr relations for black holes that include contributions of both electric and magnetic types combined in a duality invariant fashion. To derive a first law, however, one needs a conserved charge that involves fluctuations of the spacetime. Thus, consider a background solution Φ and take the first variation of (1.58) by a perturbation $\delta\Phi$ that satisfies the linearised equations of motion. Notice that no assumption is made neither on the gauge parameters nor on the symmetries of the fluctuation. Then, one arrives at the so-called *fundamental theorem of covariant phase space* [136, 137, 142], which establishes that[35]

$$d\mathbf{k}_{\xi,\lambda} \doteq \boldsymbol{\omega}\left(\delta\Phi, \delta_{\xi,\lambda}\Phi\right) \tag{1.63}$$

where, up to a total derivative,

$$\mathbf{k}_{\xi,\lambda} = \delta\mathbf{Q}_{\xi,\lambda} + \iota_\xi \boldsymbol{\Theta}(\delta\Phi) \tag{1.64}$$

and $\boldsymbol{\omega}\left(\delta_1\Phi, \delta_2\Phi\right)$ is the *presymplectic potential*,

$$\boldsymbol{\omega}\left(\delta_1\Phi, \delta_2\Phi\right) \equiv \delta_1\boldsymbol{\Theta}(\delta_2\Phi) - \delta_2\boldsymbol{\Theta}(\delta_1\Phi) \tag{1.65}$$

$\boldsymbol{\omega}\left(\delta_1\Phi, \delta_2\Phi\right)$ is an antisymmetric bilinear of the variations $\delta_1\Phi$ and $\delta_2\Phi$.[36] Then, taking $(\xi, \lambda) = (k, \lambda_k)$, one has (1.54) so the r.h.s. of (1.63) vanishes and, consequently, $\mathbf{k}_{k,\lambda_k}$ is closed. The first law of black hole mechanics will follow from this fact, as discussed in the next section.

Finally, notice that, in general, there may exist no (k, λ_k) satisfying (1.54) (e.g. a solution of vacuum GR with no Killing vector fields). However, one can construct asymptotically conserved charges in, say, a gravitational theory, by just requiring the existence of asymptotic Killing vector fields. For example, if k approaches a Killing vector field at infinity and $\mathcal{S}^{d-2}$ is an asymptotic $(d-2)$-sphere, then (upon imposing suitable boundary conditions on the perturbation) the quantity

$$\oint_{\mathcal{S}^{d-2}} \mathbf{k}_k \tag{1.66}$$

does not depend on the choice of $\mathcal{S}^{d-2}$ close to infinity because $d\mathbf{k}_k$ vanishes asymptotically. We say $\mathbf{k}_k$ is integrable if there is a local function of the fields H_k satisfying

$$\int_{\mathcal{S}^{d-2}} \mathbf{k}_k = \oint_{\mathcal{S}^{d-2}} \left(\delta\mathbf{Q}_k + \iota_k\boldsymbol{\Theta}(\delta g)\right) = \delta H_k \tag{1.67}$$

[35] These computations are made in detail for the derivation of Wald's entropy in Section B.3 of Appendix B.

[36] If $\delta_1\Phi$ and $\delta_2\Phi$ do not commute by assuming e.g. that the gauge parameters depend on the fields, one needs to include an extra term in the r.h.s. of (1.65), in order to keep the property of bilinearity, which may introduce an additional contribution to $\mathbf{k}_{\xi,\lambda}$, see e.g. [138]. Some of these follows automatically working in the variational bi-complex, and defining $\boldsymbol{\omega}$ as a $(2, D-1)$-form $\boldsymbol{\omega} = \delta\boldsymbol{\Theta}$ [137].

The criteria of integrability of charges are analogous to those of differential forms [137]. In an asymptotically flat solution of GR, if k converges asymptotically to the time translations ∂_t (rotation generator ∂_ϕ) of Minkowski space, then H_k gives the ADM mass (angular momentum, if $\mathcal{S}^{d-2}$ is taken tangent to ∂_ϕ) [136]. If, furthermore, k_t (k_ϕ) is an exact symmetry approaching ∂_t (∂_ϕ) at infinity then H_{k_t} (H_{k_ϕ}) also coincides with Komar's mass (angular momentum). Moreover, it can be shown that the charges H_k associated to asymptotic symmetries form, with the bracket

$$\{H_{k_1}, H_{k_2}\} \equiv \oint_{\mathcal{S}^{d-2}} \mathbf{k}_{k_1}[\delta_{k_2} g] \tag{1.68}$$

an algebra which is a central extension of the asymptotic symmetry algebra of the spacetime [136, 137].

In the following section we review how these results can be used to derive the first law of black hole mechanics in pure gravity theories. Then, we conclude it by discussing the problems one encounters in extending the proof to theories that include gauge fields, thus motivating the work presented in Part I.

1.3.2 Black Hole Mechanics in Pure Gravity Theories

Consider a pure gravity theory with action[37]

$$S[g] = \int \mathbf{L} = \int L(g_{\mu\nu}, R_{\mu\nu\rho\sigma})\boldsymbol{\epsilon} \tag{1.69}$$

where $L(g_{\mu\nu}, R_{\mu\nu\rho\sigma})$ is an arbitrary function constructed with invariants of the Riemann tensor (for simplicity we do not consider derivatives of the Riemann tensor here), and $\boldsymbol{\epsilon}$ is the metric volume form (we leave the spacetime dimension d arbitrary). Two important tensors in these theories are

$$P^{\mu\nu\rho\sigma} \equiv \left(\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}\right)\Bigg|_{g_{\alpha\beta}} \tag{1.70}$$

$$\mathcal{R}^{\mu\nu} \equiv P^{\mu\alpha\beta\gamma} R^{\nu}{}_{\alpha\beta\gamma} \tag{1.71}$$

$P^{\mu\nu\rho\sigma}$ is uniquely defined if it is assumed to inherit the symmetries of the Riemann tensor, and $\mathcal{R}^{\mu\nu}$ reduces to the Ricci tensor in GR. Several identities that hold generally, such as the symmetry of (1.71), $\mathcal{R}^{\mu\nu} = \mathcal{R}^{\nu\mu}$, will not be proven here (see [143] for a discussion on identities of higher-order gravities).

Consider a stationary, axisymmetric, asymptotically flat black hole solution of our theory (1.69), and assume that the event horizon coincides with a bifurcate Killing horizon of

$$k = k_t + \Omega_H^{(i)} k_{\phi_{(i)}} \tag{1.72}$$

where $k_{\phi_{(i)}}$ are the rotation generators,[38] the constants $\Omega_H^{(i)}$ are the associated angular velocities, and k_t is the stationary Killing vector field normalised to $k_t^2 = -1$ at infinity.

[37] Here we sketch the derivation in [140]. Detailed computations are given in Appendix B.

[38] More precisely, $k_{\phi_{(i)}}$ are at most $N = \lfloor (d-1)/2 \rfloor$ commuting Killing vectors labelled by i, whose orbits are isomorphic to $U(1)$, and at infinity approach the rotation generators in each of the N spatial planes [59].

From the discussion in the previous section, if a fluctuation $\delta g_{\mu\nu}$ satisfies the linearised equations of motion, and k is a Killing vector field of the background, i.e.

$$\delta_k g_{\mu\nu} = -\pounds_k g_{\mu\nu} = 0 \tag{1.73}$$

then

$$d\left[\delta \mathbf{Q}_k + \iota_k \mathbf{\Theta}(\delta g_{\mu\nu})\right] = 0 \tag{1.74}$$

Now, take a spacelike, codimension-1 surface Σ with boundaries at the bifurcation surface $\mathcal{BH}$ and an asymptotic $(d-2)$-sphere $\mathcal{S}_\infty$, see Figure 1.4. From (1.74), one has

$$\int_{\mathcal{BH}} \left[\delta \mathbf{Q}_k + \iota_k \mathbf{\Theta}(\delta g)\right] = \oint_{\mathcal{S}_\infty} \left[\delta \mathbf{Q}_k + \iota_k \mathbf{\Theta}(\delta g)\right] \tag{1.75}$$

The integral at infinity gives [140]

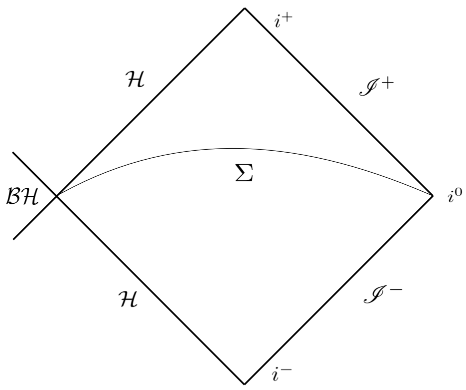

Figure 1.4: Integration surface Σ between the bifurcation surface $\mathcal{BH}$ and a $\mathcal{S}_\infty$ that asymptotes to i^0.

$$\oint_{\mathcal{S}_\infty} \left[\delta \mathbf{Q}_k + \iota_k \mathbf{\Theta}(\delta g)\right] = \delta M - \Omega_H^{(i)} \delta J_{(i)} \tag{1.76}$$

where M and $J_{(i)}$ are the (asymptotically-) conserved charges associated to k_t and $k_{\phi_{(i)}}$ (see the previous section) and correspond to the mass and angular momenta of the spacetime, respectively (the ADM ones in the case of GR [136, 140]). To evaluate the integral at the horizon one needs the zeroth law. On $\mathcal{H}$, the Killing vector field satisfies (1.10) and, furthermore, it vanishes at the bifurcation surface $k \stackrel{\mathcal{BH}}{=} 0$ and its covariant derivative there is[39]

$$\nabla_\mu k_\nu \stackrel{\mathcal{BH}}{=} \kappa n_{\mu\nu} \tag{1.77}$$

with $n_{\mu\nu}$ the binormal to the $\mathcal{BH}$. As shown in Appendix B, using these properties of k one can show that κ, the surface gravity, is constant on $\mathcal{H}$ without using the equations of motion (actually, this can be proven under even more general assumptions [144]). We will see that the generalised zeroth laws, which are the statement that the potentials associated to gauge fields are constant on the horizon, also follow from the properties of

[39] A proof of this can be found in Appendix B.

Killing horizons only, and no equations of motion are used in proving them. With this, at $\mathcal{BH}$ one has [140]

$$\oint_{\mathcal{BH}} [\delta \mathbf{Q}_k + \iota_k \mathbf{\Theta}(\delta g)] = \frac{\kappa}{2\pi} \delta S \tag{1.78}$$

where we introduced the *Wald entropy*[40]

$$S = -2\pi \oint_{\mathcal{BH}} n_{\mu\nu} P^{\mu\nu\alpha\beta} \boldsymbol{\epsilon}_{\alpha\beta} \tag{1.79}$$

Finally, in terms of Hawking's temperature (1.48), one finds the following first law of black hole mechanics,

$$\delta M = T \delta S + \Omega_H^{(i)} \delta J_{(i)} \tag{1.80}$$

The fact that T is a physical temperature allows one to interpret (1.80) as the first law of thermodynamics and conclude that the entropy of the system is given by Wald's formula (1.79). In GR, $P^{\mu\nu\rho\sigma} = g^{\mu[\rho} g^{\sigma]\nu}$ and the Wald and Bekenstein–Hawking entropies coincide, but this is not the case for more general theories. Let us remark that some of the pieces used in deriving the first law (1.80), such as the Noether–Wald charge, are defined only up to the addition of exact forms. However, none of these ambiguities alter the final result (1.80) [140, 145]. Furthermore, in the cases in which the black hole entropy can be computed through the Euclidean gravitational path integral this has been found to coincide with Wald's result (1.79) (see e.g. [146]). Such coincidence with a "first principles" derivation of the entropy constitutes further evidence that (1.79) is a physical entropy.

In the previous derivation we restricted ourselves to pure gravity theories. Consider now coupling (1.69) to, say, a $U(1)$ gauge field $A = A_\mu dx^\mu$ with gauge symmetry

$$A \to A + d\chi \tag{1.81}$$

One may proceed to derive a first law by reproducing the steps in the proof above, treating A just like we treated g. Then one encounters three main problems:

- The Lie derivative of A along a vector field ξ, unlike that of $g_{\mu\nu}$, is not invariant under gauge transformations (1.81). In particular, a statement of the form

$$\pounds_k A = 0 \tag{1.82}$$

  which is crucial in the derivation of the fundamental identity (1.74), is not gauge-invariant (unlike e.g. $\pounds_k F = 0$ where $F = dA$). If one insists in following this approach then it is necessary to assume that there is a particular gauge of the background solution where (1.82) holds. This is indeed the case for stationary solutions. However, in general, this gauge will not extend from the horizon to infinity [147], which is precisely the region we want to integrate on. More importantly, working in a non-generic gauge can lead in some cases to gauge-dependent expressions for the quantities in the first law, in particular for the entropy, which is inadmissible. This is discussed in more detail in Chapters 2 and 3.

[40] If the theory contains derivatives of the curvature, then $P^{\mu\nu\rho\sigma}$ is replaced by the Euler–Lagrange derivative of L with respect to the Riemann tensor, instead of just the partial derivative (1.70).

- Gauge charges are associated to symmetries generated by gauge parameters. The latter do not appear in the proof above if the action on A by ξ is just $\pounds_\xi A$. Thus, it is not clear what gauge charges should appear in the first law amongst all possible notions [148]. Intimately related to this, it is not clear, either, what should play the role of the potentials conjugate to the charges, and whether these satisfy a generalised zeroth law (i.e. they are constant on the horizon). In very simple cases, e.g. in the Einstein–Maxwell theory, one can identify the potential as $k \cdot A$ in a gauge in which $\pounds_k A = 0$ and prove it is constant on the horizon,[41] but it is not clear how this extends to theories with a richer gauge structure, such as those considered in Chapters 2 and 3. Furthermore, in theories where gravitational gauge transformations (Vielbein rotations) also act on matter (through e.g. Nicolai–Townsend transformations), identifying the potentials unambiguously is crucial in order to tell apart which contributions in the first law correspond to the entropy and which to matter (see Chapter 3).

- Not all charges are associated to the action of a gauge symmetry. Therefore, even if one considers gauge transformations acting on A (as we shall do) it is not clear how e.g. magnetic and scalar charges may appear in the derivation of the first law. However, it is known from explicit solutions that black holes can satisfy first laws in which not only variations of magnetic charges appear, but also variations of the scalar moduli weighted by the corresponding scalar charge [149]. It would be desirable to understand how such terms can be included in the generic proof of the first law.

Some of these problems have been noticed and addressed in the literature. The idea is that one should consider the linear action of a general automorphisms of the theory, i.e. the simultaneous action of a diffeomorphism and a gauge transformation. Thus, Jacobson and Mohd [150] considered the Vielbein formulation of GR and extended Wald's proof in a gauge-covariant manner by using the *Lorentz–Lie derivative*, which is a combined action of the Lie derivative and a local Lorentz transformation (see [151] and also [152, 153] for the Lorentz–Lie derivative of arbitrary Lorentz tensors in the context of supergravity, which builds on earlier work by Lichnerowicz, Kosmann and others [154–160]). Prabhu [138] generalised this by coupling the theory to a gauge connection living on a general principal fibre bundle, and Horowitz and Copsey [147] and Compère [161] considered the coupling of GR to a higher-rank gauge potential. However, more general theories in which both structures (i.e. gauge connections and p-form potentials) coexist and are entangled by gauge transformations have not been considered from this perspective. In particular, the effective action of the heterotic superstring at first order in α' is of this kind and the results of [138, 147, 150, 161] do not extend trivially to that theory. Notice, also, that magnetic charges remain absent in the first laws proven in [138, 147, 161].[42]

In sum, there are three things that need to be addressed. First, (1.54) has to be solved for λ_k in a general gauge. Second, one needs a notion of potential that satisfies a zeroth law and appears as a conjugate variable of the gauge charges in the first law. And third, magnetic charges need to be included in some way. All three things are provided by the *momentum maps* [152]. To illustrate how it works, consider a minimally coupled

[41] Notice, for example, that if $k \cdot A$ is a non-zero constant on the bifurcation surface then it follows that the gauge $\pounds_k A = 0$ is singular at $\mathcal{BH}$, because k vanishes there.

[42] In [147] Horowitz and Copsey are able to include a magnetic-type contribution for theories with a Chern-Simons piece, but clearly magnetic charges are also expected in theories with no such term.

$(p+1)$-form potential A, with field strength $F = dA$ and gauge symmetry $A \to A + d\lambda$. The *electric momentum map* $\mathcal{P}_k$ associated to a Killing vector field k that leaves invariant F, i.e. $\pounds_k F = 0$, is defined, up to a total derivative, by the equation [162] [43]

$$d\mathcal{P}_k = -\iota_k F \tag{1.83}$$

Then, a gauge parameter

$$\lambda_k = \iota_k A - \mathcal{P}_k \tag{1.84}$$

solves (1.54), i.e.

$$\delta_{k,\lambda_k} A = -\pounds_k A + d\lambda_k = 0 \tag{1.85}$$

without making any assumption on the gauge of A. In particular, (1.83) is a gauge-invariant equation. Furthermore, $\mathcal{P}_k$ plays the role of the potential conjugate to the electric charge (as already suggested by the very definition (1.83)). Indeed, assuming that k has a bifurcate Killing horizon with bifurcation surface $\mathcal{BH}$, then from $k \overset{\mathcal{BH}}{=} 0$ follows the *restricted generalised zeroth law*

$$d\mathcal{P}_k \overset{\mathcal{BH}}{=} 0 \tag{1.86}$$

This allows a Hodge decomposition of $\mathcal{P}_k$ on $\mathcal{BH}$ of the form [147, 161]

$$\mathcal{P}_k \overset{\mathcal{BH}}{=} de + \Phi^i h_i \tag{1.87}$$

where de is an exact form, h_i are a basis of harmonic forms on $\mathcal{BH}$ and Φ^i are constants. The latter quantities are precisely the potentials at the horizon, and appear in the first law and Smarr relation as variables conjugate to the gauge charges, as shown in Chapters 2 to 4. Finally, since $\star F$ is closed on-shell and k generates a symmetry, then

$$0 = \pounds_k \star F = (\iota_k d + d\iota_k) \star F = d\left(\iota_k \star F\right) \tag{1.88}$$

and one can also introduce a *magnetic momentum map* $\tilde{\mathcal{P}}_k$ associated to k as

$$d\tilde{\mathcal{P}}_k = -\iota_k \star F \tag{1.89}$$

Magnetic potentials at the horizon are constructed, mutatis mutandis, as in the electric case (1.86)-(1.87). As shown in Chapter 4, these enter in the Smarr relation as variables conjugate to the magnetic charges, and the resulting combinations of electric and magnetic pieces are duality invariant (in Section 4.6 of Chapter 4 we briefly discuss, based on an upcoming publication [163], that in a similar manner magnetic charges can also be included in the first law in a duality invariant fashion). Even though we have used the example of a minimally coupled $(p+1)$-form, the approach can be extended in a natural way to more general theories. All these concepts are crucial for Part I of the thesis.

Before moving on, let us remark that in some frameworks one can obtain first laws involving variations of the parameters of the theory (e.g. the cosmological constant [164] in black hole chemistry [165–167]). This can be included in the formalism described above through a suitable dualisation of the dimensionful couplings of the theory into $(d-1)$-form potentials, and has interesting applications to superstring compactifications [168].

[43] This can always be done because $0 = \pounds_k F = (\iota_k d + d\iota_k)F = d\iota_k F$.

The Second Law

We close this section by discussing briefly the second law in higher-order gravity. Hawking's area theorem in GR does not extend naturally to Wald's entropy in a higher-order gravity theory, so in principle there is no guarantee that it satisfies a second law. First, notice that (1.79), as well as its first variation, are defined as quantities on the bifurcation surface. Intuitively, though, the second law should be a statement about the monotonicity of an entropy along the horizon. Jacobson, Kang, and Myers (JKM) [145] observed that there are some ambiguities in extending the entropy formula (1.79) to an arbitrary cross-section of the horizon of a non-stationary black hole. However, such ambiguities vanish on Killing horizons, as well as their first variations at the bifurcation surface [145, 169]. As discussed in a variety of contexts [169–172], the form of the ambiguities can be fixed by requiring that the resulting entropy is indeed non-decreasing along the horizon at linear level, which resembles a second law. Quite interestingly, this seems to fail at non-linear level, what suggests that the higher-derivative couplings are physically sensible only when treated perturbatively, as consistent truncations of a UV-complete theory.

1.3.3 Black Holes in String Theory

According to statistical physics, the entropy of a system is given by the number of microscopic configurations compatible with a given macroscopic state. The laws of black hole mechanics, together with Hawking's radiation, show that black holes are thermodynamic systems with definite macroscopic entropy, but do not reveal the nature of their microscopic structure. String theory is the most promising candidate of a quantum theory of gravity and has succeeded, in some cases, in revealing the microscopic origin of the black hole entropy. To conclude this introduction, in the following we review very briefly how GR emerges from string theory, and discuss the microscopic description that black holes admit in such a framework.

Low Energy Effective Actions

String theory is a quantum theory of interacting relativistic strings.[44] Strings are one-dimensional objects with mass and length scales given by $m_s = \ell_s^{-1} = (\alpha')^{-1/2}$, where α' is the so-called Regge slope and it is the unique dimensionful parameter of the theory. They live in a "target" spacetime, and are described by the embedding function $X^\mu(\sigma)$, where X^μ are the target space coordinates and $\xi^i = (\tau, \sigma)$ the worldsheet coordinates [177].

We are interested in superstring theories, which are string theories endowed with worldsheet spinors ψ^μ and are invariant under local worldsheet supersymmetry [178–181]. Strings can be open or closed, and the different boundary conditions determine completely their spectrum. Open strings have their ends attached to $(p+1)$-dimensional timelike surfaces called *Dp*-branes [182], which play a fundamental role as we will see. Consistency at the quantum level imposes very stringent constraints on the theory. First, cancelling the conformal anomaly requires setting the spacetime dimension to $d = 10$. Requiring, furthermore, that the theory is free of tachyons and has spacetime supersymmetry reduces the possibilities to only five superstring theories: type I, type IIA, type IIB and the $SO(32)$ and $E_8 \times E_8$ heterotic theories. The massless modes in the spectrum govern

[44] See e.g. [173–176] for an introduction to (super-)string theory.

the low energy behaviour of the theory, in which the length of the string goes to zero, $\alpha' \to 0$, and massive states decouple. These modes always contain (with the exception of type I) a common bosonic Neveu–Schwarz (NSNS) sector consisting of a graviton $g_{\mu\nu}$, a dilaton ϕ and the Kalb–Ramond (KR) 2-form $B_{\mu\nu}$. Effective actions can be constructed by looking for theories that match the $\alpha' \to 0$ limit of string amplitudes. However, an alternative approach consists in taking the action for a string coupled to the background fields $g_{\mu\nu}, \phi, B_{\mu\nu}, ...$, and requiring conformal invariance at the quantum level [183–185]. This amounts to requiring the vanishing of some β functionals. At leading order in the worldsheet loop expansion, the actions whose equations of motion are equivalent to the vanishing of the β functionals of each of the five superstring theories are precisely the ten-dimensional supergravities [152]

$$S_{\text{sugra}} = \frac{g_s^2}{16\pi G_N^{(10)}} \int dx^{10} \sqrt{g} e^{-2\phi} \left\{ R - 4\nabla\phi \cdot \nabla\phi + \frac{1}{2 \cdot 3!} H^{(0)} \cdot H^{(0)} + ... \right\} \tag{1.90}$$

where only the common NSNS sector is shown. Here $H^{(0)} = dB$ and g_s is the string coupling, related to the vacuum expectation value of the dilaton by $g_s = \langle e^{\phi} \rangle$. In solutions that asymptote a vacuum, this vacuum expectation value coincides with the asymptotic value of the dilaton. The ten-dimensional Newton's constant is

$$G^{(10)} = 8\pi^6 g_s^2 \ell_s^8 \tag{1.91}$$

These theories may be truncated and dimensionally reduced. In the most usual procedure, the spacetime is assumed to have manifold structure $M_{10} = M_4 \times C_6$ where C_6 is a compact space, such as a 6-torus T^6. Imposing certain symmetry conditions on the field configurations along C_6 (e.g. keeping only the zeroth modes of the KK tower) the compact space can be integrated effectively in (1.90) thus leading to a four-dimensional theory with Newton constant

$$G^{(4)} = \frac{8\pi^6 g_s^2 \ell_s^8}{Vol(C_6)} \tag{1.92}$$

where $Vol(C_6)$ is the volume of the compact space. Then, calling $\ell_c \sim (Vol(C_6))^{(1/6)}$ the typical length scale of C_6, the four-dimensional Planck length is $\ell_P \sim \sqrt{G^{(4)}} \sim g_s \ell_s^4/\ell_c^3$. The gravitational part of the action after dimensional reduction is still governed, at leading order in α', by the Einstein–Hilbert Lagrangian. This is the way in which GR emerges at low energies from string theory, even though it is accompanied by a set of light fields, coming from the original theory and the various pieces of the dimensional reduction.

In general, the low energy effective actions of superstrings are given as a double perturbative expansion in g_s and α' [186, 187], in which the leading order terms are the supergravities (1.90). Focusing on the heterotic case, the supergravity multiplet can be consistently coupled to a Yang–Mills vector multiplet. Requiring that, furthermore, the theory is free of gauge and gravitational anomalies, as well as invariant under local supersymmetry, introduces an infinite series of higher-derivative corrections both to the action and supersymmetry transformations [188, 189]. First, the KR field-strength is modified as

$$H = H^{(0)} + \frac{\alpha'}{4}\left(\omega^{YM} + \omega^{(0)}_{(-)}\right) \tag{1.93}$$

where the Chern–Simons 3-form of the gauge connection A^A is given by

$$\omega^{YM} = dA^A \wedge A^A - \frac{1}{3!} f_{ABC} A^A \wedge A^B \wedge A^C \tag{1.94}$$

and the Lorentz Chern–Simons 3-form is similarly given by

$$\omega^{(0)}_{(\pm)} = R^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{a} + \tfrac{1}{3}\Omega^{(0)}_{(\pm)}{}^{a}{}_{b} \wedge \Omega^{(0)}_{(\pm)}{}^{b}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{c}{}_{a} \tag{1.95}$$

with torsional connection $\Omega^{(0)}_{(\pm)\,ab}$ and curvature $R^{(0)}_{(\pm)}{}^{ab}$ given by

$$\Omega^{(0)}_{(\pm)\,ab} = \omega_{ab} \pm \tfrac{1}{2}\imath_b \imath_a H^{(0)} \tag{1.96}$$

$$R^{(0)}_{(\pm)}{}^{ab} = d\Omega^{(0)}_{(\pm)}{}^{ab} - \Omega^{(0)}_{(\pm)}{}^{a}{}_{c} \wedge \Omega^{(0)}_{(\pm)}{}^{cb} \tag{1.97}$$

The corrections to the action were found in [189], up to eight order in derivatives, by requiring invariance under local supersymmetry. These arise already at first order in α' and read (omitting the fermionic sector)

$$\begin{aligned} S = \frac{g_s^2}{16\pi G_N^{(10)}} \int dx^{10} \sqrt{g} e^{-2\phi} \Big\{ \Big(& R - 4\nabla\phi \cdot \nabla\phi + \frac{1}{2 \cdot 3!} H \cdot H \\ & - \frac{\alpha'}{2} T^{(0)} - \frac{\alpha'^3}{4} \left(T^{(2)}\right)^2 - \frac{\alpha'^3}{48} \left(T^{(4)}\right)^2 + \dots \Big\} \Big(\end{aligned} \tag{1.98}$$

where the T-tensors are

$$T^{(4)} \equiv \frac{1}{4} \left(F^A \wedge F^A - R_{(-)ab} \wedge R_{(-)}{}^{ab} \right)$$

$$T^{(2)}{}_{\mu\nu} \equiv \frac{1}{4} \left(F^A{}_{\mu\rho} F^A{}_{\nu}{}^{\rho} - R_{(-)\mu\rho ab} R_{(-)\nu}{}^{\rho ab} \right)$$

$$T^{(0)} \equiv T^{(2)\mu}{}_{\mu}$$

The laws of black hole mechanics in this theory will be studied in Chapters 2 and 3 at leading and first orders in α', respectively.

Microscopic Origin of Black Hole Entropy

Black hole solutions have been obtained in several supergravity theories which arise after truncations and dimensional reductions of the ten-dimensional supergravities introduced above. The microscopic origin of the black hole entropy is best understood for extremal, supersymmetric black holes. Following a bottom up approach, consider the STU model of $\mathcal{N} = 1$, $d = 5$ supergravity. The *3-charge black hole* is a supersymmetric black hole solution of STU charged under each of the three Abelian vectors of the theory, with electric charges q_{D1}, q_{D5} and q_P. The area of the horizon is given by

$$A_H = 2\pi^2 \sqrt{q_{D1} q_{D5} q_P} \tag{1.99}$$

This black hole can be regarded as a ten-dimensional string background by rewriting it as a solution of Type IIB compactified on a five-torus $\mathbb{T}^5 = \mathbb{T}^4 \times \mathbb{S}^1_z$. From that perspective, it describes a bound state of a system composed by N_{D1} D1-branes wrapping the $\mathbb{S}^1_z$ (equivalently a D1-brane with winding number N_{D1}) and N_{D5} D5-branes wrapped on

$\mathbb{T}^5 = \mathbb{T}^4 \times \mathbb{S}^1_z$, with N_P units of KK momentum along z. These parameters are related to the charges of the original solution by

$$q_{\rm D5} = g_s \alpha' N_{\rm D5}\,, \qquad q_{\rm D1} = \frac{g_s \alpha'^3 N_{\rm D1}}{V}\,, \qquad q_{\rm P} = \frac{g_s^2 \alpha'^4}{R^2 V} N_{\rm P}\,, \tag{1.100}$$

where $2\pi R$ and $(2\pi)^4 V$ are the volumes of $\mathbb{S}^1_z$ and $\mathbb{T}^4$, respectively. Using (1.99) and (1.92) (adapted to a dimensional reduction to $5d$ instead of $4d$) the Bekenstein–Hawking entropy of the black hole reads

$$S = 2\pi\sqrt{N_{D1} N_{D5} N_P} \tag{1.101}$$

The entropy does not depend on g_s nor on any continuous parameter and, furthermore, it is given by the winding and KK numbers, which are natural numbers. This already suggests that a microscopic interpretation in terms of degeneracy of states could be possible.

Strominger and Vafa [128] considered the limit in which the size of the circle is much larger than that of the four-torus. In that limit the low-energy dynamics of the D1-D5 system is described by open strings with ends on the D-branes. These are governed by a (1+1)-CFT on the $\mathbb{S}^1_z$ and Cardy's formula [190] can be used to obtain the degeneracy of states carrying N_P units of momentum. Supersymmetry plays a crucial role here, since it protects the number of microstates in going from the supergravity regime to that in which the system is described by open strings on D-branes. Remarkably, at leading order Cardy's formula gives precisely the Bekenstein–Hawking entropy (1.101). This constitutes a major achievement of string theory.

It is natural to ask whether the agreement between the macroscopic entropy, associated to the black hole, and the microscopic one holds beyond the leading order. In that scenario, as discussed above, the macroscopic black hole entropy receives additional contributions due to the higher-derivative corrections to the supergravity actions. The aim of Chapters 2 and 3 is to study such corrections in the heterotic theory (1.98).

Part I

String Black Hole Thermodynamics

2

The First Law of Heterotic Stringy Black Hole Mechanics at Zeroth Order in α'

This chapter is based on:
The first law of heterotic stringy black hole mechanics at zeroth order in α'
Z. Elgood, D. Mitsios, T. Ortín, D. Pereñiguez
JHEP 07 (2021) 007 (arXiv:2012.13323)

In Ref. [139], Wald showed that, in a theory of gravity invariant under diffeomorphisms, the black hole entropy is essentially the Noether charge associated to that invariance. The proof consists in showing that this charge plays the role of entropy in the first law of black hole mechanics [118]. In presence of matter, though, some terms in the total Noether charge are identified with other terms in the first law and only the "gravitational" part of the Noether charge can be identified with the entropy and, in principle, it is necessary to go through the proof of the first law in order to identify the entropy. In Ref. [140], Iyer and Wald studied theories of gravity coupled to matter and found a prescription (henceforth called the *Iyer-Wald prescription*) to compute directly the entropy. In the derivation of the Iyer-Wald prescription though, it was assumed that all the fields of the theory are tensors, a condition which, in the Standard Model for instance, would only be satisfied by the metric, since the rest of the fields have some kind of gauge freedom, including the Higgs "scalar". If we decide to describe the gravitational field through the Vielbein (as the presence of fermions in the Standard Model demands), not even the gravitational field would be a tensor.

This problem was first noticed by Jacobson and Mohd [150] in the context of the theory of General Relativity described by a Vielbein.[1] They solved the problem by "improving" the standard Lie derivative (in the language of [192, 193]) by adding a local Lorentz transformation that covariantizes it. This Lorentz-covariant Lie derivative, also known as *Lie-Lorentz derivative* occurs naturally in supergravity and it was described in that context for arbitrary Lorentz tensors in Ref. [151][2] building upon earlier work on the Lie derivative of Lorentz spinors by Lichnerowicz, Kosmann and others [154–160]. A recent application to supergravity, including the fermion fields can be found in Ref. [194].

A more general and mathematically rigorous treatment based on the theory of principal bundles was given in Ref. [138] by Prabhu, who was motivated by the problems found by Gao in Ref. [195]. However, String and Supergravity theories have p-form fields with gauge freedom that cannot be described in that framework. Furthermore, the effective action and the field strengths often contain Chern-Simons terms which make the action

[1]Fields with gauge freedom had already been correctly dealt with in Refs. [161, 191], for instance.
[2]See also Ref. [152] and, for a more mathematically rigorous point of view, Ref. [153].

invariant only up to total derivatives and complicate the gauge transformations of the p-form fields. When the Chern-Simons terms depend on the spin (Lorentz) connection, gauge invariance and diffeomorphism invariance become entangled in a very complex form.

One of the simplest theories with a Chern-Simons term in the action is "minimal" ($\mathcal{N}=1$) 5-dimensional supergravity [196], which only contains a 1-form coupled to gravity. In order to deal with the lack of exact gauge invariance one has to take into account the total derivative in the definition of the Noether current [191]. However, the entropy obtained by this method in Ref. [197] in the case of the "gravitational" Chern-Simons terms (both in the action or in the Kalb-Ramond field strength) of the Heterotic Superstring effective action turned out to be gauge-dependent.[3] This problem was dealt with in Ref. [200], albeit in a rather complicated form.

In a recent paper [162] we studied the use of gauge-covariant Lie derivatives in the context of the Einstein-Maxwell theory using momentum maps to construct the derivatives. Momentum maps arise naturally wherever symmetries of a base manifold have to be related to gauge transformations [152, 201] and they are unsurprisingly ubiquitous in gauged supergravity. As a matter of fact, the Lie-Lorentz derivative can be constructed in terms of a Lorentz momentum map and in [162] we also used a *Maxwell momentum map* to construct a *Lie-Maxwell derivative*, covariant under the gauge transformations of the Maxwell field.

This procedure guarantees the gauge-invariance of the results and, as a byproduct, we found a very interesting relation between momentum maps and generalized zeroth laws also observed, in a completely different language by Prabhu in Ref. [138].

In this paper we extend this method to a theory with Abelian Chern-Simons terms in a field strength: the effective action of the Heterotic Superstring compactified on a torus to zeroth order in α'. This theory can be seen as a generalization of the theory considered by Compère in Ref. [161] and as a first step towards dealing with the effective action of the Heterotic Superstring to first order in α', which contains non-Abelian and Lorentz ("gravitational") Chern-Simons terms of the kind considered by Tachikawa [189, 202]. The introduction of momentum maps will allow us to obtain invariant results in a rather simple form, basically because they allow us to determine explicitly the gauge parameters that leave invariant all the fields of a given solution [136]. They also allow us to construct forms which are closed on the bifurcation sphere, from which the definitions of the potentials that appear in the first law will follow [147, 161]. The closedness of those forms, therefore, plays the role of the generalized zeroth law, albeit restricted to the bifurcation sphere. Hence, we will refer to these properties as the *restricted generalized zeroth laws*.

As we are going to see in the proof of the first law, there is a very precise, almost clockwork, relation between the closed forms that satisfy the restricted generalized zeroth laws and the definitions of the conserved charges [136, 203–205]. Only when both have been correctly identified is it possible to find the first law and identify the entropy.

In theories with Chern-Simons terms, several different definitions of charges have been proposed and used in the literature (see, for instance, Ref. [148] and references therein). The proof of the first law demands that we use the so-called *Page charge*, which in this context is conserved, localized and on-shell gauge invariant. Only when we use this charge definition for the 1-forms, the closed 1-form associated to the KR potentials Φ^i over the bifurcation sphere appears [147, 161] and the term $\Phi^i \delta \mathcal{Q}_i$ of the first law associated to

[3] The same happens when one naively uses the Iyer-Wald prescription, as noticed in [198, 199].

the "dipole charges" [63, 147, 161, 206–208] can be identified.

In theories with "gravitational" Chern-Simons terms, such as the effective action of the Heterotic Superstring at first order in α' the same mechanism should play a role in the proof of the first law, but the terms that modify the gravitational charges will contribute to the entropy instead [209]. It is in this precise sense that this work is a first step towards the proof of the first law and the determination of a gauge-invariant entropy formula for that theory. The previous discussion should have made clear that such a formula is not yet available, as we have also explained in Refs. [198, 199]. Even though the calculations of some black-hole entropies using the Iyer-Wald prescription seem to give the right value of the entropy in some cases,[4] it is clear that the results obtained using an entropy formula which is not gauge-invariant cannot be trusted in general. It is also clear that the comparison between entropies computed through macroscopic and microscopic methods [128] only make sense if both computations are reliable, and furthermore, only if the relation between the parameters of the black hole solution and of the microscopic theory is well understood. At first order in α', there is no full-proof entropy formula, as we have explained, and the identification of the parameters of the black-hole solutions (charges) with the numbers of branes and other parameters that appear in the microscopic entropy, has issues that still have not been fully understood [212]. This is one of the main motivations for this work.

This paper is organized as follows: in Section 2.1 we introduce the effective action of the Heterotic Superstring compactified on a torus at leading order in α'. In Section 2.2 we study the action of the symmetries of the theory on the fields, the parameters of the transformations that leave all of them invariant, and compute the associated conserved charges, including the Wald-Noether charge. In Section 2.3 we study the restricted generalized zeroth laws that we will use in the proof of the first law in Section 3.4. In Section 2.5 we consider as an example the charged, non-extremal, 5-dimensional black ring solution of pure $\mathcal{N} = 1, d = 5$ supergravity of Ref. [213] and compute its momentum maps. Section 3.6 contains a brief discussion of our results. In the appendix we show how the Heterotic Superstring effective action compactified on $\mathrm{T}^4\times\mathrm{S}^1$ (trivial compactification on T^4) can be understood as a model $\mathcal{N} = 1, d = 5$ supergravity coupled to two vector supermultiplets, which provides an embedding of this model into the Heterotic Superstring effective action. We also show how this model can be consistently tructated to pure $\mathcal{N} = 1, d = 5$ supergravity. Again, this provides an embedding of pure $\mathcal{N} = 1, d = 5$ supergravity and, in particular of the black ring solution of Ref. [213] into the Heterotic Superstring effective action, so we can apply the formulae and results obtained in the main body of the paper to that solution.

[4] In Ref. [210] it was shown that the entropy of the α'-corrected non-extremal Reissner-Nordström black hole based in the string embedding of Ref. [211], computed with the entropy formula derived in Ref. [198] using the Iyer-Wald prescription satisfies the thermodynamic relation $\partial S/\partial M = T^{-1}$. That entropy formula is not invariant under Lorentz transformations, though. In a general frame it will give wrong values for the entropy and the reason why it gives the right value in that particular case, in the particular frame in which the calculation was carried out, sill needs to be explained [209]. The same entropy formula has been used to compute the entropy of some α'-corrected extremal black holes and the results, although reasonable, cannot be tested using the same relation.

2.1 The Heterotic Superstring effective action on T^n at zeroth order in α'

When the effective action of the Heterotic Superstring at leading order in α' is compactified on a T^n, it describes the dynamics of the $(10-n)$-dimensional (string-frame) metric $g_{\mu\nu}$, Kalb-Ramond 2-form $B_{\mu\nu}$, dilaton field ϕ, Kaluza-Klein (KK) and winding 1-forms $A^m{}_\mu$ and $B_{m\,\mu}$, respectively, and the scalars that parametrize the $O(n,n)/O(n)\times O(n)$ coset space, collected in the symmetric $O(n,n)$ matrix M that we will write with upper $O(n,n)$ indices $I, J, \ldots$ as M^{IJ}. This means that M satisfies

$$M^{IJ}\Omega_{JK}M^{KL}\Omega_{LM} = \delta^I{}_M\,, \tag{2.1}$$

where

$$(\Omega_{IJ}) \equiv \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0 \end{pmatrix}, \tag{2.2}$$

is the off-diagonal form of the $O(n,n)$ metric. Eq. (2.1) implies that

$$M_{IJ} \equiv (M^{-1})_{IJ} = \Omega_{IK}M^{KL}\Omega_{LJ}\,. \tag{2.3}$$

Using the notation and conventions of Refs. [152, 199] (in particular, for differential forms, we use those of Ref. [162]), and calling the physical scalars in M_{IJ} ϕ^x, the action of the $d = (10-n)$-dimensional takes the form

$$\begin{aligned} S[e^a, B, \phi, \mathcal{A}^I, \phi^x] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi} \Big[(-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} - 4d\phi \wedge \star d\phi \\ -\tfrac{1}{8} dM_{IJ} \wedge \star dM^{IJ} + (-1)^d \tfrac{1}{2} M_{IJ}\mathcal{F}^I \wedge \star\mathcal{F}^J + \tfrac{1}{2} H \wedge \star H \Big] \\ \equiv \int \mathbf{L}\,. \end{aligned} \tag{2.4}$$

In this action $e^a = e^a{}_\mu dx^\mu$ are the string-frame Vielbeins, $\star$ stands for the Hodge dual and, therefore

$$\star\,(e^a \wedge e^b) = \frac{1}{(d-2)!}\epsilon_{c_1\cdots c_{d-2}}{}^{ab} e^{c_1} \wedge \cdots \wedge e^{c_{d-2}}\,. \tag{2.5}$$

Furthermore, $\omega^{ab} = \omega_\mu{}^{ab} dx^\mu$ is the Levi-Civita spin connection[5] and $R^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu$ is its field strength (the curvature) 2-form, defined as

$$R^{ab} \equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}\,. \tag{2.6}$$

[5]It is antisymmetric $\omega^{ab} = -\omega^{ba}$ and satisfies $De^a = de^a - \omega^a{}_b \wedge e^b = 0$. We are using the second-order formalism.

$g_s^{(d)}$ and $G_N^{(d)}$ are, respectively, the $d = (10-n)$-dimensional string coupling and Newton constant. [6]

$\mathcal{F}^I$ is the $\mathrm{O}(n,n)$ vector of the 2-form field strengths of the KK and winding vectors

$$\mathcal{F}^I \equiv \begin{pmatrix} F^m \\ G_m \end{pmatrix}, \qquad F^m = dA^m\,, \qquad G_m = dB_m\,, \tag{2.8}$$

which can also be defined in terms of the $\mathrm{O}(n,n)$ vector of 1-forms denoted by $\mathcal{A}^I$

$$\mathcal{A}^I \equiv \begin{pmatrix} A^m \\ B_m \end{pmatrix}, \qquad \mathcal{F}^I = d\mathcal{A}^I\,. \tag{2.9}$$

H is the Kalb-Ramond 3-form field strength, defined by

$$H \equiv dB - \tfrac{1}{2}\mathcal{A}_I \wedge d\mathcal{A}^I\,, \qquad \mathcal{A}_I = \Omega_{IJ}\mathcal{A}^J\,. \tag{2.10}$$

The kinetic term of the scalars ϕ^x that parametrize the $\mathrm{O}(n,n)/(\mathrm{O}(n)\times\mathrm{O}(n))$ coset space can also be written in the form

$$-\tfrac{1}{8}dM_{IJ} \wedge \star dM^{IJ} = \tfrac{1}{2}g_{xy}d\phi^x \wedge \star d\phi^y\,, \tag{2.11}$$

where the metric $g_{xy}(\phi)$ is given by

$$g_{xy} \equiv \tfrac{1}{4}\left(\partial_x M_{IK}M^{KJ}\right)\left(\partial_y M_{JK}M^{KI}\right). \tag{2.12}$$

Under a general variation of the fields, the action varies as

$$\delta S = \int \left\{\mathbf{E}_a \wedge \delta e^a + \mathbf{E}_B \wedge \delta B + \mathbf{E}_\phi \delta\phi + \mathbf{E}_I \wedge \delta\mathcal{A}^I + \mathbf{E}_x \delta\phi^x + d\mathbf{\Theta}(\varphi, \delta\varphi)\right\}, \tag{2.13}$$

where, suppressing the factors of $g^{(d)\,2}(16\pi G_N^{(d)})^1$ for simplicity, the Einstein equations $\mathbf{E}_a$ are given by

[6] They are related to the 10-dimensional constants through the volume of the T^n, V_n, by

$$g_s^2 = V_n/(2\pi\ell_s)^n g_s^{(d)\,2}\,, \tag{2.7a}$$

$$G_N^{(10)} = G_N^{(d)}V_n\,. \tag{2.7b}$$

$$
\begin{aligned}
\mathbf{E}_a = {} & e^{-2\phi}\imath_a \star (e^c \wedge e^d) \wedge R_{cd} - 2\mathcal{D}(\imath_b d e^{-2\phi}) \wedge \star(e^b \wedge e^c) g_{ca} \\
& + (-1)^{d-1} 4 e^{-2\phi} \left(\imath_a d\phi \star d\phi + d\phi \wedge \imath_a \star d\phi\right) \\
& + \frac{(-1)^d}{2} e^{-2\phi} g_{xy} \left(\imath_a d\phi^x \star d\phi^y + d\phi^x \wedge \imath_a \star d\phi^y\right) \\
& + \frac{1}{2} e^{-2\phi} M_{IJ} \left(\imath_a \mathcal{F}^I \wedge \star \mathcal{F}^J - \mathcal{F}^I \wedge \imath_a \star \mathcal{F}^J\right) \\
& + \frac{(-1)^d}{2} e^{-2\phi} \left(\imath_a H \wedge \star H + H \wedge \imath_a \star H\right) ,
\end{aligned}
\tag{2.14}
$$

the equations of motion of the matter fields are given by

$$
\mathbf{E}_B = -d\left(e^{-2\phi} \star H\right) , \tag{2.15a}
$$

$$
\mathbf{E}_\phi = 8d\left(e^{-2\phi} \star d\phi\right) - 2\mathbf{L} , \tag{2.15b}
$$

$$
\mathbf{E}_I = \tilde{\mathbf{E}}_I + \tfrac{1}{2}\mathbf{E}_B \wedge \mathcal{A}_I , \tag{2.15c}
$$

$$
\tilde{\mathbf{E}}_I \equiv -\left\{d\left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J\right) + (-1)^{d-1} e^{-2\phi} \star H \wedge \mathcal{F}_I\right\} , \tag{2.15d}
$$

$$
\mathbf{E}_x = -g_{xy}\left[d\left(e^{-2\phi} \star d\phi^y\right) + e^{-2\phi}\Gamma_{zw}{}^y d\phi^z \wedge \star d\phi^w\right] + \frac{(-1)^d}{2} e^{-2\phi} \partial_x M_{IJ} \mathcal{F}^I \wedge \star \mathcal{F}^J , \tag{2.15e}
$$

and

$$
\begin{aligned}
\mathbf{\Theta}(\varphi, \delta\varphi) = {} & -e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta\omega_{ab} + 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta e_b \\
& - 8 e^{-2\phi} \star d\phi \delta\phi - \tfrac{1}{4} e^{-2\phi} \star dM^{IJ} \delta M_{IJ} \\
& + e^{-2\phi} M_{IJ} \star \mathcal{F}^J \wedge \delta \mathcal{A}^I + e^{-2\phi} \star H \wedge \left(\delta B + \tfrac{1}{2}\mathcal{A}_I \wedge \delta\mathcal{A}^I\right) .
\end{aligned}
\tag{2.16}
$$

The equations of motion of the 1-forms $\mathbf{E}_I$ can be written in the alternative form

$$
\mathbf{E}_I = -d\left\{e^{-2\phi} M_{IJ} \star \mathcal{F}^J + \star H \wedge \mathcal{A}_I\right\} - \tfrac{1}{2}\mathbf{E}_B \wedge \mathcal{A}_I . \tag{2.17}
$$

This form appears naturally in the definition of the electric charges Eq. (2.32).

Here, and in what follows, φ stands for all the fields of the theory. $\mathbf{E}_\varphi$ denotes collectively all their equations of motion.

2.2 Variations of the fields

In this section we are going to study the transformations of the fields under the different symmetries of the action and determine which parameters of the transformations leave a complete field configuration invariant. The conserved charges of those configurations will be associated to those parameters. As a general rule, only if one combines several transformations can one find parameters that simultaneously leave all the fields invariant.

The simplest case in which this happens will involve the gauge transformations of the 1-form fields: the parameters that leave them invariant do not leave the KR field invariant at the same time, unless we perform a KR gauge transformation with a parameter related to that of the other gauge symmetry. As a result, there is an additional term in the formula that gives the electric charges, but it is the presence of this additional term that guarantees the conservation of the charge and the independence of the integration surface (as long as we do not include sources, that is, on-shell).

The transformation of several fields under diffeomorphisms must also be supplemented by "compensating" gauge transformations, including local Lorentz transformations if we want all the fields to be left invariant by those generating isometries (Killing vectors). There are several ways of understanding this need but we believe that the most fundamental is to realize that fields with gauge freedoms (*i.e.* all fields except for the metric and the dilaton field) are not tensors and do not transform as such under diffeomorphisms. The "compensating gauge transformations" can be seen as gauge transformations induced by the diffeomorphisms. Only when they are properly taken into account can one find Killing vector fields that leave all the fields invariant. Furthermore, only then the vanishing of the variations of the fields is invariant under gauge transformations. A more detailed discussion and additional references to this topic can be found in Ref. [162]. The conserved charge associated to diffeomorphisms, the Wald-Noether charge, will therefore include terms related to gauge symmetries and their associated conserved charges, which will ultimately contribute to the first law.

As we will see, only when all these details are properly taken into account can the first law be proven and the entropy identified.

We start by describing the gauge symmetries of the theory (other than diffeomorphisms) and the associated conserved charges.

2.2.1 Gauge transformations

The gauge transformations of the fields are

$$\delta_\sigma e^a = \sigma^a{}_b e^b \,, \tag{2.18a}$$

$$\delta_\chi \mathcal{A}^I = d\chi^I \,, \tag{2.18b}$$

$$\delta B = (\delta_\Lambda + \delta_\chi) B = d\Lambda + \tfrac{1}{2}\chi_I d\mathcal{A}^I \,, \tag{2.18c}$$

where $\sigma^{(ab)}(x) = 0$ are the parameters of local Lorentz transformations, $\chi^I(x)$ is a $\mathrm{O}(n,n)$ vector if scalar gauge parameters and $\Lambda = \Lambda_\mu(x)dx^\mu$ is a 1-form gauge parameter. They leave invariant the field strengths $\mathcal{F}^I$ and H, but they induce the following transformations on the spin connection and curvature

$$\delta_\sigma \omega^{ab} = \mathcal{D}\sigma^{ab} = d\sigma^{ab} - 2\omega^{[a|}{}_c \sigma^{c|b]} \,, \tag{2.19a}$$

$$\delta_\sigma R^{ab} = 2\sigma^{[a|}{}_c R^{c|b]} \,. \tag{2.19b}$$

For the sake of completeness and later use, we quote the Ricci identity in our conventions:

$$\mathcal{D}\mathcal{D}\sigma^{ab} = -2R^{[a|}{}_c \sigma^{c|b]} = \delta_\sigma R^{ab} \,. \tag{2.20}$$

The action is manifestly invariant under these gauge transformations. This leads to the following Noether identities

$$\mathbf{E}^{[a} \wedge e^{b]} = 0 \,, \tag{2.21a}$$

$$d\tilde{\mathbf{E}}_I + (-1)^d \mathbf{E}_B \wedge \mathcal{F}_I = 0 \,, \tag{2.21b}$$

$$d\mathbf{E}_B = 0 \,, \tag{2.21c}$$

2.2.2 Gauge charges

Let us study the conserved charges associated to the gauge transformations δ_χ, δ_Λ and, for the sake of completeness, δ_σ, starting with δ_Λ, which is simpler to deal with.

The variation of the action under δ_Λ transformations follows from Eqs. (2.13) and (2.16)

$$\begin{aligned}
\delta_\Lambda S &= \int \left\{ \mathbf{E}_B \wedge \delta_\Lambda B + d\left(e^{-2\phi} \star H \wedge \delta_\Lambda B \right) \right\} \\
&= \int \left\{ \mathbf{E}_B \wedge d\Lambda + d\left(e^{-2\phi} \star H \wedge d\Lambda \right) \right\} .
\end{aligned} \tag{2.22}$$

Integrating by parts the first term and using the Noether identity Eq. (2.21c)

$$\delta_\Lambda S = \int d\left(\Lambda \wedge \mathbf{E}_B + e^{-2\phi} \star H \wedge d\Lambda\right) = \int d\mathbf{J}[\Lambda]\,. \tag{2.23}$$

The invariance of the action under these gauge transformations indicates that the current $\mathbf{J}[\Lambda]$ must be locally exact, so that, locally, there is a $\mathbf{Q}[\Lambda]$ such that $\mathbf{J}[\Lambda] = d\mathbf{Q}[\Lambda]$. It is easy to see that

$$\mathbf{Q}[\Lambda] = \Lambda \wedge \left(e^{-2\phi} \star H\right). \tag{2.24}$$

The conserved charge is given by the integral of the conserved $(d-2)$-form $\mathcal{Q}[\Lambda]$ over $(d-2)$-dimensional compact surfaces $\mathcal{S}_{d-2}$ for Λs that leave invariant the KR field Bs. These are closed 1-forms. Following [147, 161], using the Hodge decomposition theorem, these closed 1-forms Λ can be written as the sum of an exact and a harmonic form $\Lambda_e = d\lambda$ and Λ_h, respectively. The exact form Λ_e will not contribute to the integral on-shell because

$$Q(\Lambda_e) = \int_{\mathcal{S}_{d-2}} d\lambda \wedge \left(e^{-2\phi} \star H\right) = \int_{\mathcal{S}_{d-2}} d\left[\lambda \wedge \left(e^{-2\phi} \star H\right)\right] - \int_{\mathcal{S}_{d-2}} \lambda \wedge \mathbf{E}_B\,. \tag{2.25}$$

Therefore,

$$Q(\Lambda) = \int_{\mathcal{S}_{d-2}} \Lambda_h \wedge \left(e^{-2\phi} \star H\right). \tag{2.26}$$

Then, using the duality between homology and cohomology, if C_{Λ_h} is the $(d-3)$-cycle dual to Λ_h, we arrive at the charges

$$Q(\Lambda) = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{C_{\Lambda_h}} e^{-2\phi} \star H\,, \tag{2.27}$$

where we have added a conventional sign and recovered the factor of $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$ that we have omitted. From the string theory point of view, these charges are just winding numbers of strings whose transverse space is the cycle C_{Λ_h}. Two homologically equivalent cycles give the same value of the charge on-shell, that is, if there are no sources of the KR field in the $(d-2)$-dimensional volume whose boundary is the union of the two properly oriented $(d-3)$-cycles.

Let us now consider the conserved charges associated to the invariance under δ_χ. This transformation acts on the 1-forms $\mathcal{A}^I$ and on the KR 2-form B. Transformations with constant χ^I (closed 0-forms) leave invariant the 1-forms, but they do not leave invariant B. They only change it by an exact 2-form $d\left(\frac{1}{2}\chi_I \mathcal{A}^I\right)$. Thus, we must add a compensating Λ gauge transformation with parameter $\Lambda_\chi = -\frac{1}{2}\chi_I \mathcal{A}^I$ and consider the transformation of B

$$\delta_\chi B = -\tfrac{1}{2} d\left(\chi_I \mathcal{A}^I\right) + \tfrac{1}{2}\chi_I d\mathcal{A}^I = -\tfrac{1}{2} d\chi_I \wedge \mathcal{A}^I\,. \tag{2.28}$$

Then, from Eqs. (2.13) and (2.16) and the modified transformation rule Eq. (3.35), we get

$$
\begin{aligned}
\delta_\chi S = \int \{ & \mathbf{E}_B \wedge \delta_\chi B + \mathbf{E}_I \wedge \delta_\chi \mathcal{A}^I \\
& + d\left[e^{-2\phi} M_{IJ} \star \mathcal{F}^J \wedge \delta_\chi \mathcal{A}^I + e^{-2\phi} \star H \wedge \left(\delta_\chi B + \tfrac{1}{2} \mathcal{A}_I \wedge \delta_\chi \mathcal{A}^I \right) \right] \Big\} , \\
= \int & \left\{ \left(\mathbf{E}_I + \tfrac{1}{2} \mathbf{E}_B \wedge \mathcal{A}_I \right) \wedge d\chi^I + d\left[\left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J + e^{-2\phi} \star H \wedge \mathcal{A}_I \right) \wedge d\chi^I \right] \right\} .
\end{aligned} \tag{2.29}
$$

Integrating by parts the first term and using the Noether identities Eqs. (2.21b) and (2.21c) we get

$$
\delta_\chi S = \int d\left\{ (-1)^{d-1} \chi^I \left(\mathbf{E}_I + \tfrac{1}{2} \mathbf{E}_B \wedge \mathcal{A}_I \right) + \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J + e^{-2\phi} \star H \wedge \mathcal{A}_I \right) \wedge d\chi^I \right\} . \tag{2.30}
$$

The usual argument leads to the conserved $(d-2)$-form

$$
\mathbf{Q}[\chi] = (-1)^d \chi^I \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J + e^{-2\phi} \star H \wedge \mathcal{A}_I \right) , \tag{2.31}
$$

and the definition of electric charges

$$
\mathcal{Q}_I = \frac{(-1)^{d-1} g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{S}_{(d-2)}} \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J + e^{-2\phi} \star H \wedge \mathcal{A}_I \right) , \tag{2.32}
$$

where we have added a conventional sign. Again, this charge is on-shell invariant under homologically-equivalent deformations of $\mathcal{S}_{(d-2)}$. This follows from the equation of motion written in the alternative form Eq. (2.17). It is also on-shell invariant under the δ_χ transformations, in spite of the explicit occurrence of the vector fields $\mathcal{A}_I$: the second term in the integrand has the same structure as the integrand of the KR charge and, for the same reason, it is invariant on-shell when we add to $\mathcal{A}_I$ exact 1-forms.

This charge is, in the terminology used by Marolf in Ref. [148], a *Page charge* but, as we have explained, apart from localized and conserved, it is also gauge invariant on-shell. The formalism leads us to use precisely this charge, which will be the one occurring in the first law of black hole mechanics.

Finally, let us consider the charge associated to the invariance under local Lorentz transformations δ_σ, which act on the Vielbein and on all the fields derived from it: spin connection and curvature. Let us postpone for the time being the conditions that the parameters that leave all of them invariant have to satisfy and lets study the transformation of the action. From Eqs. (2.13) and (2.16) we find

$$
\delta_\sigma S = \int \left\{ \mathbf{E}_a \wedge \delta_\sigma e^a + d\left[-e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta_\sigma \omega_{ab} + 2\imath_a d e^{-2\phi} \star (e^a \wedge e^b) \wedge \delta_\sigma e_b \right] \right\} , \tag{2.33}
$$

and using Eqs. (2.18a) and (2.19a) and the Noether identity Eq. (2.21a), we find that the integrand immediately reduces to a total derivative,

$$\delta_\sigma S = \int\!\!\!\int d\mathbf{J}[\sigma]\,,$$
$$\mathbf{J}[\sigma] = (-1)^{d-1} e^{-2\phi}\mathcal{D}\sigma_{ab}\wedge\star(e^a\wedge e^b) + 2\sigma_{bc}\imath_a de^{-2\phi}\star(e^a\wedge e^b)\wedge e^c\,. \tag{2.34}$$

The standard argument tells us that $\mathbf{J}[\sigma] = d\mathbf{Q}[\sigma]$. Integrating by parts the first term

$$\mathbf{J}[\sigma] = d\left\{(-1)^{d-1}e^{-2\phi}\sigma_{ab}\star(e^a\wedge e^b)\right\} + 3\left(\sigma_{[bc}\imath_{a]}de^{-2\phi}\right)\star(e^a\wedge e^b)\wedge e^c\,. \tag{2.35}$$

The last term vanishes identically because[7] $\star(e^a\wedge e^b)\wedge e^c = 2\eta^{c[a}\star e^{b]}$ and we arrive at

$$\mathbf{Q}[\sigma] = (-1)^{d-1}e^{-2\phi}\star(e^a\wedge e^b)\wedge\sigma_{ab}\,. \tag{2.37}$$

Now we have to consider Lorentz parameters that leave all the fields invariant. The spin connection and curvature are left invariant by covariantly constant parameters

$$\mathcal{D}\sigma^a{}_b = 0\,, \tag{2.38}$$

but the invariance of the Vielbein $\sigma^a{}_b e^b = 0$ can only be satisfied for $\sigma^a{}_b = 0$, and would automatically imply the vanishing of $\mathbf{Q}[\sigma]$.

The $(d-2)$-form, though, reappears in the proof of the first law for a Lorentz parameter that is covariantly constant over the bifurcation surface. We also notice that terms of higher order in the Lorentz curvature, such as those which arise with α' corrections, lead to a non-vanishing Lorentz charge Ref. [209] .

2.2.3 Diffeomorphisms and covariant Lie derivatives

As we have discussed in the introduction, out of the fundamental fields of our theory, only the dilaton ϕ and the $\mathrm{O}(n,n)/(\mathrm{O}(n)\times\mathrm{O}(n))$ scalars ϕ^x transform as a tensor under diffeomorphisms $\delta_\xi x^\mu = \xi^\mu$, that is[8]

$$\delta_\xi\phi = -\pounds_\xi\phi = -\imath_\xi d\phi\,, \tag{2.40a}$$

$$\delta_\xi\phi^x = -\pounds_\xi\phi^x = -\imath_\xi d\phi^x\,. \tag{2.40b}$$

[7] Here we use the property

$$\star\,\omega^{(p)}\wedge\hat{\xi} = \star\imath_\xi\omega^{(p)}\,, \tag{2.36}$$

which is valid for any p-form $\omega^{(p)}$ and any vector field $\xi = \xi^\mu\partial_\mu$ and its dual 1-form $\hat{\xi} = \xi_\mu dx^\mu$.

[8] The metric $g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\mu$ and the 2- and 3-form field strengths $\mathcal{F}, H$ also transform as tensors:

$$\delta_\xi g_{\mu\nu} = -\pounds_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}\,, \tag{2.39a}$$

$$\delta_\xi\mathcal{F} = -\pounds_\xi\mathcal{F} = -(\imath_\xi d + d\imath_\xi)\mathcal{F}\,, \tag{2.39b}$$

$$\delta_\xi H = -\pounds_\xi H = -(\imath_\xi d + d\imath_\xi)H\,. \tag{2.39c}$$

The Vielbein e^a, the vectors (1-forms), $\mathcal{A}$, and the KR 2-form, B, have gauge freedoms and transform as tensors up to *compensating* gauge transformations. These compensating gauge transformations can be determined by

1. Requiring gauge-covariance of the complete transformation law (which can then be interpreted as a gauge-covariant Lie derivative) and
2. Imposing that, for diffeomorphisms which are symmetries of the field configuration that we are considering (in particular, for isometries), the complete transformation (covariant Lie derivative) vanishes. The first condition ensures that this vanishing is gauge-invariant.

In what follows we will denote by k the vector fields ξ that generate diffeomorphisms that leave invariant the complete field configuration. k is, in particular, a Killing vector of the metric.

In a recent paper [162] we reviewed the construction of a Lie derivative of the Vielbein, spin connection and curvature covariant under local Lorentz transformations (*Lie-Lorentz derivative*) of Refs. [151, 152] that build upon earlier work by Lichnerowicz, Kosmann and others [154–157]. In Ref. [162] we also dealt with Abelian vector fields in similar terms. It is convenient to quickly review these results starting with the Abelian vector case, adapted to the present situation.

The transformation of the Abelian vector fields $\mathcal{A}^I$ under diffeomorphisms can be defined as

$$\delta_\xi \mathcal{A}^I = -\mathbb{L}_\xi \mathcal{A}^I\,, \tag{2.41}$$

where $\mathbb{L}_\xi \mathcal{A}^I$ is the *Lie-Maxwell derivative*, defined by

$$\mathbb{L}_\xi \mathcal{A}^I \equiv \imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\,. \tag{2.42}$$

Here $\mathcal{P}_\xi{}^I$ is a gauge-invariant $\mathrm{O}(n,n)$ vector of functions that depends on $\mathcal{A}^I$ and on the generator of diffeomorphisms ξ and it is assumed to have the property that, when $\xi = k$, it satisfies the equation

$$d\mathcal{P}_k{}^I = -\imath_k \mathcal{F}^I\,. \tag{2.43}$$

The invariance of the 2-form $\mathcal{F}^I$ guarantees the local existence of $\mathcal{P}_k{}^I$, which is known as the *momentum map* associated to k. On the other hand, Eq. (2.43) ensures that the two properties of the variations of the fields under diffeomorphisms that we have demanded are satisfied. Finally, observe that the Lie-Maxwell derivative is just a combination of the standard Lie derivative plus a compensating gauge transformation with parameter

$$\chi_\xi{}^I = \imath_\xi \mathcal{A}^I - \mathcal{P}_\xi{}^I\,. \tag{2.44}$$

For fields with Lorentz indices (Vielbein, spin connection and curvature), the variation under diffeomorphisms is also given by (minus) a Lorentz-covariant generalization of the Lie derivative $\delta_\xi = -\mathbb{L}_\xi$ usually called *Lie-Lorentz derivative* Refs. [151, 152, 154–157]. This derivative can also be constructed by adding to the standard Lie derivative a compensating Lorentz transformation with the parameter

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - \nabla^{[a}\xi^{b]} \,. \tag{2.45}$$

For the Vielbein, the Lie-Lorentz derivative can be expressed in several equivalent and manifestly Lorentz-covariant forms

$$\mathbb{L}_\xi e^a{}_\mu = \tfrac{1}{2} e^{a\,\nu} \left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right) \tag{2.46a}$$

$$\mathbb{L}_\xi e^a = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b \,, \tag{2.46b}$$

where

$$P_\xi{}^{ab} \equiv \nabla^{[a}\xi^{b]} \,, \tag{2.47}$$

satisfies, when $\xi = k$, the equation

$$\imath_k R^{ab} = -\mathcal{D} P_k{}^{ab} \,, \tag{2.48}$$

that shows that we can view $P_k{}^{ab}$ as a momentum map as well.[9]

In the form Eq. (2.46a) we immediately see that the Lie-Lorentz derivative of the Vielbein vanishes when $\xi = k$, a Killing vector. The same is true for the connection and curvature.

Observe that $P_\xi{}^{ab}$ transforms covariantly under local Lorentz transformations.

The above transformation of the Vielbein induce the following transformations of the spin connection and curvature that we quote for later use:

$$\delta_\xi \omega^{ab} = -\mathbb{L}_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D} P_\xi{}^{ab} \right) , \tag{2.49a}$$

$$\delta_\xi R^{ab} = -\mathbb{L}_\xi R^{ab} = -\left(\mathcal{D} \imath_\xi R^{ab} - 2 P_\xi{}^{[a}{}_c R^{b]c} \right) . \tag{2.49b}$$

Observe that the Lie-Lorentz derivative of the spin connection has the same structure as that of the Abelian connection $\mathcal{A}^I$ in Eq. (2.42), *i.e.* the inner product of ξ with the curvature plus the derivative of the momentum map.

In asymptotically-flat stationary black-hole spacetimes with bifurcate horizon, if k is the Killing vector whose Killing horizon coincides with the event horizon and $\mathcal{BH}$ is the bifurcation sphere,

$$P_k{}^{ab} = \nabla^{[a}k^{b]} \stackrel{\mathcal{BH}}{=} \kappa n^{ab} \,, \tag{2.50}$$

where κ is the surface gravity and n^{ab} is the binormal to the event horizon, with the normalization $n^{ab} n_{ab} = -2$. The zeroth law of black-hole mechanics stating that κ is constant over the horizon [118, 144] is associated to the Lorentz momentum map, just as the generalized zeroth law that states that the electric potential is also constant over

[9] Compare this equation to Eq. (2.43).

the horizon in the Einstein-Maxwell theory is associated to the Maxwell momentum map [162].[10] We are going to see that further "generalized zeroth laws" are also associated to momentum maps when we restrict ourselves to the bifurcation surface. We will call them *restricted generalized zeroth laws.*

Let us now consider the KR field. It is convenient to start by considering the transformation of the 3-form field strength H defined in Eq. (2.10) under diffeomorphisms. Since it is gauge invariant, upon use of its Bianchi identity

$$\delta_\xi H = -\pounds_\xi H = -\imath_\xi dH - d\imath_\xi H = \imath_\xi \mathcal{F}_I \wedge \mathcal{F}^I - d\imath_\xi H \,. \tag{2.51}$$

When $\xi = k$, this expression must vanish and we can use Eq. (2.43), which leads to the identity

$$\delta_\xi H = -d\left(\imath_k H + \mathcal{P}_{k\,I}\mathcal{F}^I\right) = 0\,, \tag{2.52}$$

which, in turn, implies the local existence of a gauge-invariant 1-form that we will also call a momentum map, satisfying

$$-\,\imath_k H - \mathcal{P}_{k\,I}\mathcal{F}^I = dP_k \,. \tag{2.53}$$

The KR momentum map plays a fundamental role in the definition of the variation of the KR 2-form B under diffeomorphisms which should be of the general form

$$\delta_\xi B = -\pounds_\xi B + \left(\delta_{\Lambda_\xi} + \delta_{\chi_\xi}\right) B\,, \tag{2.54}$$

where χ_ξ and Λ_ξ are scalar and 1-form parameters of compensating gauge transformations. They will generically depend on $\mathcal{A}^I$ and B as well as on ξ. ${\chi_\xi}^I$ has to be the same parameter used in the definition of the Lie-Maxwell derivative Eq. (2.44) and we just have to determine Λ_ξ. Now, the Maxwell and Lorentz cases suggest that we try

$$\Lambda_\xi = \imath_\xi B - P_\xi\,, \tag{2.55}$$

which leads to

$$\begin{aligned}\delta_\xi B &= -\pounds_\xi B + d(\imath_\xi B - P_\xi) + \tfrac{1}{2}\chi_{\xi\,I} d\mathcal{A}^I \\ &= -\left(\imath_\xi H + \mathcal{P}_{\xi\,I}\mathcal{F}^I + dP_\xi\right) + \tfrac{1}{2}\mathcal{A}_I \wedge \imath_\xi \mathcal{F}^I + \tfrac{1}{2}\mathcal{P}_{\xi\,I}\mathcal{F}^I \,.\end{aligned} \tag{2.56}$$

When $\xi = k$, though,

$$\delta_k B = d\left(\tfrac{1}{2}\mathcal{P}_{k\,I}\mathcal{A}^I\right). \tag{2.57}$$

This is not zero but it can be absorbed into a redefinition of Λ_ξ:

$$\Lambda_\xi = \imath_\xi B - P_\xi - \tfrac{1}{2}\mathcal{P}_{k\,I}\mathcal{A}^I\,, \tag{2.58}$$

[10] This parallelism between zeroth laws was observed in [138], also in the wider context of Einstein-Yang-Mills theories.

which gives the variation

$$\delta_\xi B = -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) - \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I \,. \tag{2.59}$$

This form of the variation makes it evident that $\delta_k B = 0$, because $\delta_k \mathcal{A}^I = 0$ and because of the definition of the KR momentum map 1-form Eq. (2.53).

It remains to check that the vanishing of this variation is a gauge-invariant statement. Indeed, if we perform a gauge transformation in $\delta_\xi B$, taking into account that all the momentum maps and $\delta_\xi \mathcal{A}^I$ are gauge-invariant, we find

$$\delta_{\text{gauge}}\delta_\xi B = -\tfrac{1}{2}\delta_{\text{gauge}}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I \,, \tag{2.60}$$

which vanishes identically for $\xi = k$.

2.2.4 The Wald-Noether charge

The Wald-Noether charge is the conserved $(d-2)$-form associated to the invariance of the action under diffeomorphisms [139]. The transformations that we are going to consider (combinations of standard Lie derivative and gauge transformations, as we have explained) are

$$\delta_\xi \phi = -\imath_\xi d\phi \,, \tag{2.61a}$$

$$\delta_\xi \phi^x = -\imath_\xi d\phi^x \,. \tag{2.61b}$$

$$\delta_\xi \mathcal{A}^I = -\left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right) \,, \tag{2.61c}$$

$$\delta_\xi e^a = -\left(\mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\right) \,, \tag{2.61d}$$

$$\delta_\xi \omega^{ab} = -\left(\imath_\xi R^{ab} + \mathcal{D}P_\xi{}^{ab}\right) \,, \tag{2.61e}$$

$$\delta_\xi B + \tfrac{1}{2}\mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I = -\left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \,. \tag{2.61f}$$

From Eq. (2.13), and using the definition of $\tilde{\mathbf{E}}_I$ in Eqs. (2.15c) and (2.15d) to cancel the terms of the form $\mathbf{E}_B \wedge \mathcal{A}_I \wedge \delta_\xi \mathcal{A}^I$, we get

$$\begin{aligned}\delta_\xi S = -\int \Big\{ & \mathbf{E}_a \wedge \left(\mathcal{D}\imath_\xi e^a + P_\xi{}^a{}_b e^b\right) + \mathbf{E}_B \wedge \left(\imath_\xi H + \mathcal{P}_{\xi\, I}\mathcal{F}^I + dP_\xi\right) \\ & + \tilde{\mathbf{E}}_I \wedge \left(\imath_\xi \mathcal{F}^I + d\mathcal{P}_\xi{}^I\right) + \mathbf{E}_\phi \imath_\xi d\phi + \mathbf{E}_x \imath_\xi d\phi^x \\ & - d\mathbf{\Theta}(\varphi, \delta_\xi\varphi)\Big\} \,,\end{aligned} \tag{2.62}$$

while, from Eq. (2.16), we get

$$\begin{aligned}\mathbf{\Theta}(\varphi,\delta_\xi\varphi) = {} & e^{-2\phi}\star(e^a\wedge e^b)\wedge(\imath_\xi R_{ab}+\mathcal{D}P_{\xi\,ab})\\ & -2\imath_a de^{-2\phi}\star(e^a\wedge e^b)\wedge(\mathcal{D}\xi_b+P_{\xi\,bc}e^c)\\ & +8e^{-2\phi}\star d\phi\imath_\xi d\phi - e^{-2\phi}g_{xy}\star d\phi^y\imath_\xi d\phi^x \\ & -e^{-2\phi}M_{IJ}\star\mathcal{F}^J\wedge\left(\imath_\xi\mathcal{F}^I+d\mathcal{P}_\xi{}^I\right)\\ & -e^{-2\phi}\star H\wedge\left(\imath_\xi H+\mathcal{P}_{\xi\,I}\mathcal{F}^I+dP_\xi\right).\end{aligned} \tag{2.63}$$

Next, we consider the terms in $\delta_\xi S$ that contain momentum maps, integrating by parts those which involve their derivatives:

$$\begin{aligned} & \mathbf{E}_a\wedge P_\xi{}^a{}_b e^b+\tilde{\mathbf{E}}_I\wedge d\mathcal{P}_\xi{}^I+\mathbf{E}_B\wedge\left(\mathcal{P}_{\xi\,I}\mathcal{F}^I+dP_\xi\right)\\ = {} & \mathbf{E}^{[a}\wedge e^{b]}P_{\xi\,ab}+P_\xi d\mathbf{E}_B+(-1)^d\mathcal{P}_{\xi\,I}\left[d\tilde{\mathbf{E}}^I+(-1)^d\mathbf{E}_B\wedge\mathcal{F}^I\right]\\ & +d\left(P_\xi\wedge\mathbf{E}_B+(-1)^{d-1}\mathcal{P}_\xi{}^I\tilde{\mathbf{E}}_I\right).\end{aligned} \tag{2.64}$$

The terms in the first line vanish as a consequence of the Noether identities Eqs. (2.21a)-(2.21c) and we are left with the total derivative which will be added to $\mathbf{\Theta}(\varphi,\delta_\xi\varphi)$. Thus, the variation of the action takes the form

$$\begin{aligned}\delta_\xi S = -\int\Big\{ & \mathbf{E}_a\wedge\mathcal{D}\imath_\xi e^a+\mathbf{E}_B\wedge\imath_\xi H+\tilde{\mathbf{E}}_I\wedge\imath_\xi\mathcal{F}^I+\mathbf{E}_\phi\imath_\xi d\phi+\mathbf{E}_x\imath_\xi d\phi^x\\ & -d\left[\mathbf{\Theta}(\varphi,\delta_\xi\varphi)-P_\xi\wedge\mathbf{E}_B+(-1)^d\mathcal{P}_\xi{}^I\tilde{\mathbf{E}}_I\right]\Big\}.\end{aligned} \tag{2.65}$$

Integrating the first term of Eq. (2.65) by parts we get another total derivative to add to $\mathbf{\Theta}(\varphi,\delta_\xi\varphi)$ and ($\imath_\xi e^a=\xi^a$)

$$(-1)^d\mathcal{D}\mathbf{E}_a\xi^a+\mathbf{E}_B\wedge\imath_\xi H+\tilde{\mathbf{E}}_I\wedge\imath_\xi\mathcal{F}^I+\mathbf{E}_\phi\imath_\xi d\phi+\mathbf{E}_x\imath_\xi d\phi^x=0\,, \tag{2.66}$$

by virtue of the Noether identity associated to the invariance under diffeomorphisms and, therefore,

$$\delta_\xi S=\int d\mathbf{\Theta}'(\varphi,\delta_\xi\varphi)\,, \tag{2.67}$$

where

$$\mathbf{\Theta}'(\varphi,\delta_\xi\varphi)=\mathbf{\Theta}(\varphi,\delta_\xi\varphi)+(-1)^d\mathbf{E}_a\xi^a-P_\xi\wedge\mathbf{E}_B+(-1)^d\mathcal{P}_\xi{}^I\tilde{\mathbf{E}}_I\,. \tag{2.68}$$

Usually, the last three terms, which are proportional to equations of motion and vanish on-shell, are ignored for this very reason. However, we have found that keeping them is actually quite useful for finding the Wald-Noether charge, because they are exactly what is needed to write **J** as a total derivative. Without them, we would have had to guess which combinations of the equations of motion should be added to achieve that goal. Furthermore, the result that we will obtain will be valid off-shell.

Since the action is exactly invariant under the gauge transformations Eq. (2.18), but it is only invariant up to a total derivative under standard infinitesimal diffeomorphisms, under the combined transformations Eqs. (2.61)

$$\delta_\xi S = -\int d\imath_\xi \mathbf{L}\,, \tag{2.69}$$

which, combined with Eq. (3.77), leads to the identity

$$d\mathbf{J} = 0\,, \tag{2.70}$$

which holds off-shell for arbitrary ξ with

$$\mathbf{J} \equiv \mathbf{\Theta}'(\varphi, \delta_\xi \varphi) + \imath_\xi \mathbf{L}\,. \tag{2.71}$$

Eq. (2.70) implies the local existence of a $(d-2)$-form $\mathbf{Q}[\xi]$ such that

$$\mathbf{J} = d\mathbf{Q}[\xi]\,. \tag{2.72}$$

Using the previous results we find that, up to total derivatives and up to the overall factor $(g_s^{(d)\,2} 16\pi G_N^{(d)})^{-1}$ that we are suppressing to get simpler expressions

$$\begin{aligned} \mathbf{Q}[\xi] = {} & (-1)^d \star (e^a \wedge e^b) \left[e^{-2\phi} P_{\xi\,ab} - 2\imath_a d e^{-2\phi} \xi_b\right] \\ & + (-1)^{d-1} \mathcal{P}_\xi{}^I \left(e^{-2\phi} M_{IJ} \star \mathcal{F}^J\right) - P_\xi \wedge \left(e^{-2\phi} \star H\right). \end{aligned} \tag{2.73}$$

2.3 Zeroth laws

The zeroth law and its generalizations, ensuring that the surface gravity and the electrostatic potential are constant over the event (Killing) horizon $\mathcal{H}$ are important ingredients in the standard derivation of the first law of black-hole mechanics in the context of the Einstein-Maxwell theory [118]. In presence of higher-rank p-form fields, it is not clear how these laws should be further generalized. However, it is possible to proof the first law using Wald's formalism working on the bifurcation sphere $\mathcal{BH}$, where the Killing vector k associated to the horizon vanishes. This restricts the validity of the proof to bifurcate horizons but, on the other hand, it makes it possible to carry out the proof using a more restricted form of the (generalized) zeroth laws which states the closedness of the electrostatic potential and its higher-rank generalizations on $\mathcal{BH}$. Since the electrostatic potential is a scalar, its closedness implies that it is constant on $\mathcal{BH}$, which is a restricted version of the generalized zeroth law. For higher-rank potentials closedness is, actually, all we need, as we will see in the next section.

We start by assuming that all the field strengths of the theory are regular on the horizon.[11] This implies that

$$\imath_k \mathcal{F}^I \stackrel{\mathcal{BH}}{=} 0\,, \tag{2.74a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,. \tag{2.74b}$$

The first equation directly implies the closedness of the components of the momentum map $\mathcal{P}_k^I$ on $\mathcal{BH}$ on account of its definition Eq. (2.43), and, hence, its constancy on $\mathcal{BH}$, a statement that we can call *restricted generalized zeroth law* after the natural identification of $\mathcal{P}_k^I$ with the electrostatic black-hole potential Φ^I. Observe that, our gauge-invariant definition of the electrostatic black-hole potential guarantees that it is fully defined up to an additive constant that can be determined by setting the value of the potential at infinity to zero.

Using Eq. (2.74b) and the constancy of $\mathcal{P}_k^I$ on on $\mathcal{BH}$ in the definition of the KR momentum map Eq. (2.53) we find that

$$0 \stackrel{\mathcal{BH}}{=} -\imath_k H = dP_k + \mathcal{P}_{k\,I}\mathcal{F}^I \stackrel{\mathcal{H}}{=} d\left(P_k + \mathcal{P}_{k\,I}\mathcal{A}^I\right). \tag{2.75}$$

We can call the combination $P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$ that is closed on $\mathcal{BH}$ the KR black-hole potential Φ and its closedness can be understood as another restricted generalized zeroth law of black-hole mechanics in this theory. Observe that Φ is not gauge-invariant, but P_k is only defined up to shifts by exact 1-forms anyway and, when we use Φ as the 1-form Λ in the calculation of the KR charge Eq. (2.26), the addition of exact 1-forms does not change the value of the associated KR charge Eq. (3.28). The fact that this Φ occurs in the expressions leading to the first law precisely plays this role is quite a non-trivial check of the consistency of our results.

2.4 The first law

We start by defining the *pre-symplectic* $(d-1)$*-form* [214]

$$\omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1\mathbf{\Theta}(\varphi, \delta_2\varphi) - \delta_2\mathbf{\Theta}(\varphi, \delta_1\varphi)\,, \tag{2.76}$$

and the *symplectic form* relative to the Cauchy surface Σ

$$\Omega(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \int_\Sigma \omega(\varphi, \delta_1\varphi, \delta_2\varphi)\,. \tag{2.77}$$

Now, following Ref. [140], when φ solves the equations of motion $\mathbf{E}_\varphi = 0$ if $\delta_1\varphi = \delta\varphi$ is an arbitrary variation of the fields and $\delta_2\varphi = \delta_\xi\varphi$ is their variation under diffeomorphisms, we have that

$$\omega(\varphi, \delta\varphi, \delta_\xi\varphi) = \delta\mathbf{J} + d\imath_\xi\mathbf{\Theta}' = \delta d\mathbf{Q}[\xi] + d\imath_\xi\mathbf{\Theta}'\,, \tag{2.78}$$

[11] Observe that in this theory in which all the field strengths are gauge-invariant, this is a gauge-invariant statement that should be valid in a regular coordinate patch.

where, in our case, $\mathbf{J} = d\mathbf{Q}$, where $\mathbf{Q}$ is given by Eq. (2.73) and $\mathbf{\Theta}'$ is given in Eq. (3.78). Since, on-shell, $\mathbf{\Theta} = \mathbf{\Theta}'$, we have that, if $\delta\varphi$ satisfies the linearized equations of motion, $\delta d\mathbf{Q} = d\delta\mathbf{Q}$. Furthermore, if the parameter $\xi = k$ generates a transformation that leaves invariant the field configuration, $\delta_k\varphi = 0$,[12] linearity implies that $\omega(\varphi, \delta\varphi, \delta_k\varphi) = 0$, and

$$d\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right) = 0\,. \tag{2.79}$$

Integrating this expression over a hypersurface Σ with boundary $\delta\Sigma$ and using Stokes' theorem we arrive at

$$\oint_{\delta\Sigma}\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right) = 0\,. \tag{2.80}$$

We are interested in asymptotically flat, stationary, black-hole spacetimes and we choose k as the Killing vector whose Killing horizon coincides with the event horizon $\mathcal{H}$, which we assume to be a bifurcate horizon. This Killing vector k is assumed to be linear combination with constant coefficients Ω^n of the timelike Killing vector associated to stationarity, $t^\mu\partial_\mu$ and the $[\frac{1}{2}(d-1)]$ inequivalent rotations $\phi_n^\mu\partial_\mu$

$$k^\mu = t^\mu + \Omega^n\phi_n^\mu\,. \tag{2.81}$$

Furthermore, we choose the hypersurface Σ to be the space between infinity and the bifurcation sphere ($\mathcal{BH}$) on which $k = 0$. Then, its boundary $\delta\Sigma$ has two disconnected pieces: a $(d-2)$-sphere at infinity, S^{d-2}_∞, and the bifurcation sphere $\mathcal{BH}$. Then, taking into account that $k = 0$ on $\mathcal{BH}$, we obtain the relation

$$\delta\int_{\mathcal{BH}}\mathbf{Q}[k] = \oint_{\mathrm{S}^{d-2}_\infty}\left(\delta\mathbf{Q}[k] + \imath_k\mathbf{\Theta}'\right). \tag{2.82}$$

As explained in Ref. [140, 161], the right-hand side can be identified with $\delta M - \Omega^m\delta J_n$, where M is the total mass of the black-hole spacetime and J_n are the independent components of the angular momentum.

Using the explicit form of $\mathbf{Q}[k]$, Eq. (2.73), and restoring the overall factor $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$, we find

$$\begin{aligned}
\delta\oint_{\mathcal{BH}}\mathbf{Q}[k] = {}& \frac{(-1)^{d-1}g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}\mathcal{P}_k{}^I\left(e^{-2\phi}M_{IJ}\star\mathcal{F}^J\right)\\
&-\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}P_k\wedge\left(e^{-2\phi}\star H\right)\\
&+\frac{(-1)^d g_s^{(d)\,2}}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}}\star(e^a\wedge e^b)\left[e^{-2\phi}P_{k\,ab} - 2\imath_a de^{-2\phi}k_b\right].
\end{aligned} \tag{2.83}$$

[12] We have constructed variations of the fields δ_ξ for which this is possible.

The last term vanishes over the bifurcation sphere and will be removed from now on.

As it is, this expression has two problems that make it difficult for us to obtain the kind of terms that occur in the first law. In the first line, we have an expression that we should be able to interpret in terms of the electric charges $\mathcal{Q}_I$. However, when we compare this with Eq. (2.32) we see that the second term in the integrand is missing. Without that term, the charge is not conserved. On the other hand, in the second line, we have an expression that we should be able to interpret in terms of the KR charge using Eq. (2.26). However, the 1-form P_k is not closed on $\mathcal{BH}$.

The solution to these two problems is unique: the addition and subtraction of the term $\mathcal{P}_{k\,I}\mathcal{A}^I \wedge \left(e^{-2\phi} \star H\right)$ in the integrand, so that the integral to evaluate on $\mathcal{BH}$ takes the form

$$
\begin{aligned}
\delta \int_{\mathcal{BH}} \mathbf{Q}[k] = {}& \frac{(-1)^{d-1} g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} \mathcal{P}_k{}^I \left[e^{-2\phi} M_{IJ} \star \mathcal{F}^J + e^{-2\phi} \star H \wedge \mathcal{A}_I\right] \\
& - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} \left(P_k + \mathcal{P}_{k\,I}\mathcal{A}^I\right) \wedge \left(e^{-2\phi} \star H\right) \\
& + \frac{(-1)^d g_s^{(d)\,2}}{16\pi G_N^{(d)}} \delta \int_{\mathcal{BH}} e^{-2\phi} \star (e^a \wedge e^b) P_{k\,ab}\,.
\end{aligned}
\tag{2.84}
$$

Now, using the generalized zeroth law that ensures that $\mathcal{P}_k{}^I \equiv \Phi^I$ is constant over $\mathcal{H}$, in particular on $\mathcal{BH}$, and the definition of electric charge Eq. (2.32), the first term in the right-hand side takes the form

$$
\Phi^I \delta \mathcal{Q}_I\,. \tag{2.85}
$$

Next, from the closedness of the combination $\Phi = P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$ on $\mathcal{BH}$, (the restricted generalized zeroth law) using the Hodge decomposition

$$
P_k + \mathcal{P}_{k\,I}\mathcal{A}^I \stackrel{\mathcal{BH}}{=} de + \Phi^i \Lambda_{h\,i}\,, \tag{2.86}
$$

where the $\Lambda_{h\,i}$ are harmonic 1-forms on $\mathcal{BH}$ and the Φ^1 are constants that have the interpretation of potentials associated to the charge of the KR field (the dipole charge of Ref. [63] in particular), and using the definition Eq. (3.28), we find that the second term in the right-hand side takes the form

$$
\Phi^i \delta Q_i\,, \qquad Q_i \equiv Q[\Lambda_{h\,i}]\,. \tag{2.87}
$$

Observe that the addition and subtraction of the term $\mathcal{P}_{k\,I}\mathcal{A}^I \wedge \left(e^{-2\phi} \star H\right)$ has been crucial to recover the correct definition of the charges which, in particular, demands the occurrence of the closed 1-form $P_k + \mathcal{P}_{k\,I}\mathcal{A}^I$.

Now, let us consider the third integral. Before we compute it explicitly, we notice that the integrand is identical, up to a sign, to the Lorentz charge Eq. (2.37) computed for the Lorentz parameter $P_k{}^a{}_b$ which is covariantly constant over the bifurcation surface. This coincidence is very intriguing and will be further explored in Ref. [209].

Using Eq. (2.50)

$$\begin{aligned}\frac{(-1)^d\kappa}{16\pi G_N^{(d)}}\delta\int_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)}\star(e^a\wedge e^b)n_{ab} = & -\frac{\kappa}{16\pi G_N^{(d)}}\delta\oint_{\mathcal{BH}} e^{-2(\phi-\phi_\infty)}n^{ab}n_{ab} \\ & = T\delta\frac{\mathcal{A}_\mathcal{H}}{4G_N^{(d)}}\,,\end{aligned} \tag{2.88}$$

where we have used the normalization of the binormal $n_{ab}n^{ab} = -2$, $T = \kappa/2\pi$ is the Hawking temperature and

$$\mathcal{A}_\mathcal{H} \equiv \oint_{\mathcal{B}} d^{d-2}S e^{-2(\phi-\phi_\infty)}\,, \tag{2.89}$$

is the area of the horizon measured with the *modified Einstein frame metric* [215] which is obtained from the string one by multiplying by the conformal factor $e^{-4(\phi-\phi_\infty)/(d-2)}$, and computed using the spatial section $\mathcal{BH}$.

We finally get the following expression for the first law of black hole mechanics in the Heterotic Superstring effective action to leading order in α':

$$\delta M = T\delta\frac{\mathcal{A}_\mathcal{H}}{4G_N^{(d)}} + \Omega^m\delta J_m + \Phi^i\delta Q_i + \Phi^I\delta\mathcal{Q}_I\,, \tag{2.90}$$

which leads to the interpretation of the area of the horizon divided by $4G_N^{(d)}$ as the black-hole entropy.

2.5 Momentum Maps for Black Rings in $d = 5$

In this section we are goin to illustrate how the definitions made and the properties proven in the previous sections work in an explicit example. In particular, we are going to determine the values of the momentum maps, checking the restricted generalized zeroth laws.

The solution we are going to consider is a non-extremal, charged, black ring solution of pure $\mathcal{N} = 1, d = 5$ supergravity which can be easily embedded in the toroidally-compactified Heterotic Superstring effective field theory using the results in Appendix C. This embedding is necessary because all the definitions and formulae that we have developed are adapted to that theory. In Appendix C we show how the action Eq. (2.4), for $d = 5$ can be consistently truncated to that of pure $\mathcal{N} = 1, d = 5$ supergravity Eq. C.26 in two steps:

1. A direct truncation of some fields of the Heterotic theory, to obtain a model of $\mathcal{N} = 1, d = 5$ supergravity coupled to two vector multiplets. The Kalb-Ramond 2-form has to be dualized into a 1-form in order to obtain the supergravity theory

in the standard form, with 3 1-forms which can be treated on the same footing and which may be linearly combined.

2. A consistent truncation of the two vector supermultiplets. In this truncation, rather than setting two of the vector fields to zero, they are identified with the surviving vector, up to numerical factors. This allows the scalars in the vecort supermultiplets to take their vacuum values.

Given a solution of pure $\mathcal{N}=1, d=5$ supergravity, one can easily retrace those steps, restoring, first, the two "matter" vector fields so the solution becomes now a solution of $\mathcal{N}=1, d=5$ supergravity coupled to two vector multiplets. Then, dualizing the vector in the supergravity multiplet to recover the Kalb-Ramond 2-form, the solution can immediately be interpreted as a solution of the Heterotic Superstring effective field theory in which many other fields simply take their vacuum values.

The non-extremal, charged, black ring solution that we are going to consider is the one given in Section 4 of Ref. [213]. This solution belongs to a more general family of non-supersymmetric black rings with three charges α_i, three dipoles μ_i, with $i=1,2,3$, and two angular momenta J_φ and J_ψ in the theory with two vector supermultiplets. The solution above corresponds to setting all three charges and three dipoles equal, $\alpha_i=\alpha$ and $\mu_i=\mu$ for all i. This identification of the charges and dipoles coprresponds to the identification between the vector fields that leads from the supergravity theory with matter to the theory of pure supergravity. Let us review the solution and its main features.

The physical fields of the solution (the metric and the Abelian connection A) can be written in terms of the five parameters $(R,\alpha,\mu,\lambda,\nu)$ (all of them dimensionless except for the length scale R) and the three functions, $F(\xi), H(\xi)$ and $G(\xi)$, given by

$$H(\xi)=1-\mu\xi\,,\qquad F(\xi)=1+\lambda\xi\,,\qquad G(\xi)=(1-\xi^2)(1+\nu\xi)\,. \tag{2.91}$$

The line element is

$$\begin{aligned}ds^2=&\frac{U(x,y)}{h_\alpha^2(x,y)}\left(dt+\omega_\psi(y)d\psi+\omega_\varphi(x)d\varphi\right)^2-h_\alpha(x,y)F(x)H(x)H(y)^2\times\\&\times\frac{R^2}{(x-y)^2}\left[-\frac{G(y)}{F(y)H(y)^3}d\psi^2-\frac{dy^2}{G(y)}+\frac{dx^2}{G(x)}+\frac{G(x)}{F(x)H(x)^3}d\varphi^2\right],\end{aligned} \tag{2.92}$$

where we use the shorthand notation $s=\sinh\alpha$ and $c=\cosh\alpha$, the following combinations of the fundamental parameters

$$C_\lambda=\epsilon_\lambda\sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,,\quad C_\mu=\epsilon_\mu\sqrt{\mu(\mu+\nu)\frac{1-\mu}{1+\mu}}\,,\quad \epsilon_{\lambda,\mu}=\pm1\,, \tag{2.93}$$

and the following combinations of the fundamental functions in Eq. (2.91)

$$U(x,y) = \frac{H(x)}{H(y)}\frac{F(y)}{F(x)}\,, \tag{2.94a}$$

$$h_\alpha(x,y) = 1 + \frac{(\lambda+\mu)(x-y)}{F(x)H(y)}s^2\,, \tag{2.94b}$$

$$\omega_\psi(y) = R(1+y)\left[\frac{1}{F(y)}C_\lambda c^3 - \frac{3}{H(y)}C_\mu c s^2\right]\Bigg(, \tag{2.94c}$$

$$\omega_\varphi(x) = -R(1+x)s\left(\frac{1}{F(x)}C_\lambda s^2 - \frac{3}{H(x)}C_\mu c^2\right)\Bigg(. \tag{2.94d}$$

Finally, the gauge field reads

$$-A/\sqrt{3} = \frac{U(x,y)-1}{h_\alpha(x,y)}csdt$$

$$+\frac{R(1+y)}{h_\alpha(x,y)}\left[\frac{U(x,y)}{F(y)}C_\lambda c^2 s - \frac{U(x,y)}{H(y)}C_\mu s^3 - \frac{2}{H(y)}C_\mu c^2 s\right]\Bigg(d\psi$$

$$+\frac{R(1+x)}{h_\alpha(x,y)}\left[2\frac{U(x,y)}{H(x)}C_\mu c s^2 - \frac{1}{F(x)}C_\lambda c s^2 + \frac{1}{H(x)}C_\lambda c^3\right]\Bigg(d\varphi\,. \tag{2.95a}$$

The parameters of the solution must satisfy the constraints

$$0 < \nu \leq \lambda < 1\,, \qquad 0 \leq \mu < 1\,, \tag{2.96}$$

to avoid naked singularities. Additional constraints arise from the codition of absence of Dirac-Misner strings and conical sigularities, as we are going to see.

The coordinates x, y take values in

$$-\infty < y \leq -1\,, \qquad -1 \leq x \leq 1\,. \tag{2.97}$$

The surfaces of constant y have the topology $S^2 \times S^1$. x is a polar coordinate on the S^2 (essentially, $x \sim \cos\theta$), which is also parametrized by φ, which plays the role of azymuthal angle. ψ parametrizes the S^1, see Fig. 2.1. Spatial infinity is approached when both x and y go to -1, although the coordinates are ill-defined in that limit.[13] The orbits of the vector ∂_φ close off at $x = -1$, but do not do the same at $x = 1$ unless $\omega_\varphi(x = +1) = 0$, which can forces us to require

$$\frac{C_\lambda}{1+\lambda}s^2 = \frac{3C_\mu}{1-\mu}c^2\,, \tag{2.98}$$

[13]Good coordinates at infinity can be found in Ref. [213].

which removes any possible Dirac-Misner strings. (The same constraint makes $A_\varphi(x=+1)$ independent of y.) Then, the fixed point sets of ∂_ψ and ∂_φ are, respectively, $y=-1$ (axis of the ring) and $x=1,-1$ (inner and outer axes of the S^2).

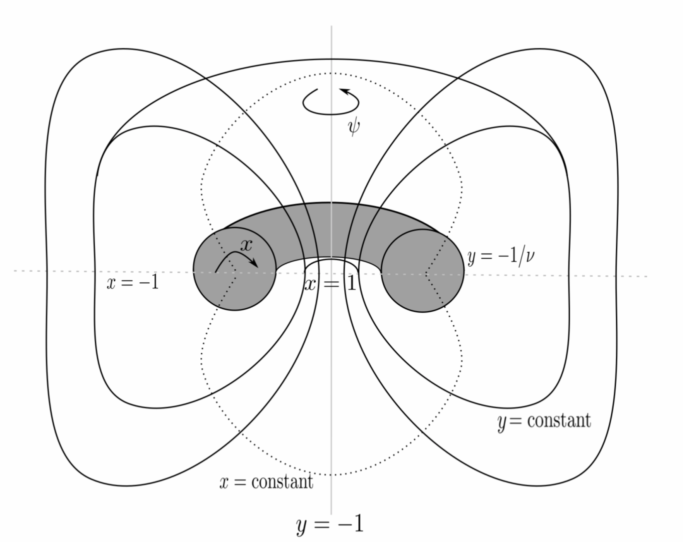

Figure 2.1: Sketch of a section of constant t and φ of the black ring (figure based on Ref. [63]). The disc at $x=1$ and infinite annulus at $x=-1$ are the axes (fixed points) of ∂_φ, while the axis of the ring is at $y=-1$ (fixed points of ∂_ψ). Surfaces of constant y have topology $S^1\times S^2$. $y=-1/\nu$ corresponds to the horizon (shaded surface) while surfaces of constant $y\in(-1/\nu,-1)$ are fatter rings containing the horizon in their interior.

Finally, the periods of ψ and φ must be chosen appropriately so as to avoid conical singularities. The axes $y=-1$ and $x=-1$ (which extend to infinity) are regular for the periods

$$\Delta\psi=\Delta\varphi=2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,(1+\mu)^{3/2}\ . \tag{2.99}$$

For generic values of the parameters, though, the period of φ required by smoothness at the inner axis, $x=1$, differs from the above $\Delta\varphi$. Making both periods coincide ("balancing" the ring) is possible only when the following constraint holds

$$\left(\frac{1-\nu}{1+\nu}\right)^2=\frac{1-\lambda}{1+\lambda}\left(\frac{1+\mu}{1-\mu}\right)^3\ . \tag{2.100}$$

Henceforth we shall assume that Eqs. (2.98) and (2.100) hold, so that, effectively, we will be dealing with a three-parameter family of solutions. As shown in Ref. [213], the mass, the two independent angular momenta and the area of the event horizon of the solution read

$$M = \frac{3\pi R^2}{4G_N^{(5)}} \frac{(\lambda+\mu)(1+\mu)^2}{1-\nu} \cosh 2\alpha\,, \tag{2.101a}$$

$$J_\psi = \frac{\pi R^3}{2G_N^{(5)}} \frac{(1-\lambda)^{3/2}(1+\mu)^{9/2}}{(1-\nu)^2} \left[\frac{C_\lambda}{1-\lambda} c^3 - \frac{3C_\mu}{1+\mu} s^2 c \right] \Big(, \tag{2.101b}$$

$$J_\varphi = -\frac{3\pi R^3}{G_N^{(5)}} \frac{\sqrt{1-\lambda}\,(1+\mu)^{7/2}(\lambda+\mu)}{(1-\nu)^2(1-\mu)} C_\mu c^2 s\,, \tag{2.101c}$$

$$\mathcal{A}_{\mathcal{H}} = 8\pi^2 R^3 \frac{(1-\lambda)(\lambda-\nu)^{1/2}(1+\mu)^3(\nu+\mu)^{3/2}}{(1-\nu)^2(1+\nu)} \quad \frac{C_\lambda}{\lambda-\nu} c^3 + \frac{3C_\mu}{\nu+\mu} s^2 c \quad . \tag{2.101d}$$

There is an ergosurface at $y = -1/\lambda$, where the norm of ∂_t vanishes, and the event horizon lies at $y = -1/\nu$. It is a Killing horizon of

$$k = \partial_t + \Omega \partial_\psi, \tag{2.102}$$

where Ω, the angular velocity of the horizon in the direction ψ, can be conveniently written as $\Omega = -1/\omega_\psi(-1/\nu)$.[14] A rather unusual property of this solution is that the horizon has no angular velocity in the direction φ even though $J_\varphi \neq 0$. Finally, the horizon temperature is

$$T_{\mathcal{H}}^{-1} = 4\pi R \frac{\sqrt{\lambda-\nu}(\mu+\nu)^{3/2}}{\nu(1+\nu)} \quad \frac{C_\lambda}{\lambda-\nu} c^3 + \frac{3C_\mu}{\nu+\mu} s^2 c \quad . \tag{2.103}$$

This solution of pure $\mathcal{N} = 1, d = 5$ supergravity corresponds to a following solution of the Heterotic Superstring effective field theory compactified on $\mathrm{T}^4 \times \mathrm{S}^1$ with the same

[14] Notice we work with coordinates φ, ψ whose periods are not the standard ones, but those given in Eq. (2.99).

metric and the non-trivial matter fields given by[15]

$$\phi = \phi_\infty \,, \tag{2.104a}$$

$$M_{IJ} = \begin{pmatrix} k_\infty^2 & 0 \\ 0 & k_\infty^{-2} \end{pmatrix} \,, \tag{2.104b}$$

$$\mathcal{A}^I = \begin{pmatrix} k_\infty^{-1} \\ k_\infty \end{pmatrix} \mathcal{A} \,, \tag{2.104c}$$

$$H = dB - \tfrac{1}{2}\mathcal{A}_I \wedge \mathcal{F}^I = \star\mathcal{F} \tag{2.104d}$$

where, for convenience, we have introduced $\mathcal{A} = -A/\sqrt{3}$ and its field strength $\mathcal{F} = d\mathcal{A}$. Let us obtain the vector and KR momentum maps asociated to the Killing vector k in Eq. (2.102) for this solution, denoted, respectively, as ${\mathcal{P}_k}^I$ and P_k. In the following we consider a constant t surface Σ defined by which extends from the bifurcate surface (here, a ring) $\mathcal{BH}$ at $y = -1/\nu$ to infinity (analogously to one leaf of the Einstein–Rosen bridge). The vector momentum maps $\mathcal{P}_k^I$ can be written as

$$\mathcal{P}_k^I = \begin{pmatrix} k_\infty^{-1} \\ k_\infty \end{pmatrix} \mathcal{P}_k \,, \tag{2.105}$$

where $\mathcal{P}_k$ satisfies the equation

$$d\mathcal{P}_k = -\imath_k \mathcal{F} \,. \tag{2.106}$$

Since in our gauge $\pounds_k \mathcal{A} = 0$ it is clear that a solution (as a matter of fact, any solution) of the above equation is provided by

$$\mathcal{P}_k = \imath_k \mathcal{A} + C \,, \tag{2.107}$$

for some constant C. Notice, though, that this is not the definition of the momentum map, but rather a particular form of $\mathcal{P}_k$ which is available in the gauge in which the black-ring solution is given. The momentum map is, by definition, gauge invariant. The constant C is determined by demanding $\mathcal{P}_k$ (which will be interpreted as the black ring's electrostatic potential Φ) to vanish at infinity, and it is not difficult to see that $C = 0$.

This solution admits an analytic prolongation to the bifurcate ring $\mathcal{BH}$ at $y = -1/\nu$ (and actually beyond that) and, in agreement with the generalised zeroth law, it is a constant over the whole event horizon $\mathcal{H}$ that we will denote by $\Phi_\mathcal{H}$

[15] The fields that arise in the compactification over T^4 and which are set to their vacuum values (they are trivial) have not been considered. In particular, the index I takes only two values because the fields corresponding to the other values are trivial.

$$
\begin{aligned}
\mathcal{P}_k &\stackrel{\mathcal{H}}{=} \mathcal{P}_k(x, -1/\nu) \\
&= -\frac{\cosh 2\alpha\,[C_\lambda(\mu+\nu) + 3C_\mu(\lambda-\nu)] + C_\lambda(\mu+\nu) + C_\mu(\lambda-\nu)}{\cosh 2\alpha\,[C_\lambda(\mu+\nu) + 3C_\mu(\lambda-\nu)] + C_\lambda(\mu+\nu) - 3C_\mu(\lambda-\nu)} \tanh\alpha \qquad (2.108) \\
&\equiv \Phi_{\mathcal{H}}\,.
\end{aligned}
$$

Observe that, in the gauge in which the solution is given, the potential $\mathcal{A}$ is ill-defined over $\mathcal{BH}$: $\imath_k\mathcal{A}$ is a non-vanishing constant there and k vanishes, which implies that $\mathcal{A}$ must diverge there. It is worth stressing that the momentum map is unaffected by such gauge pathologies since the solution Eq. (2.107) extends from infinity all the way down to $\mathcal{BH}$ (and beyond). This is a consequence of the fact that, although the momentum maps may only exist locally, they are defined by a gauge invariant equation.

The KR momentum map 1-form, P_k, is defined by Eq. (2.53), and, for this particular solution

$$
dP_k = -\left(\imath_k H + \mathcal{P}_k{}^I \mathcal{F}_I\right) = -\left(\imath_k \star \mathcal{F} + 2\mathcal{P}_k\mathcal{F}\right)\,. \qquad (2.109)
$$

If we knew the KR potential B in a gauge in which $\pounds_k B = 0$, using $\mathcal{P}_k = \imath_k\mathcal{A}$, we would obtain the KR momentum map 1-form

$$
P_k = \imath_k B - \mathcal{P}_k\mathcal{A} + \alpha\,, \qquad (2.110)
$$

where α is an arbitrary closed 1-form, $d\alpha = 0$, that could be determined by imposing regularity: smoothness of P_k both at the axis of the ring, $P_\psi(x, y = -1) = 0$, and at the outer axis of the spheres, $P_\varphi(x = -1, y) = 0$, so that it is well defined when approaching infinity). Finding B is, however, as hard as finding P_k directly from Eq. (2.109), which is what we are going to do, taking into account that we are only interested in the pullback of P_k to the constant-t surface Σ, which must be of the form

$$
P_k \stackrel{\Sigma}{=} P^{\Sigma}_{k\,\varphi}(x,y)d\varphi + P^{\Sigma}_{k\,\psi}(x,y)d\psi\,, \qquad (2.111)
$$

because of the general form of the solution.

The two functions $P^{\Sigma}_{k\,\varphi}(x,y)$ and $P^{\Sigma}_{k\,\psi}(x,y)$ are given by

$$P^{\Sigma}_{k\,\varphi}(x,y) = -\int^{y} (\imath_k \star \mathcal{F} + 2\mathcal{P}_k \mathcal{F})_{y\varphi}\, dy + f_{\varphi}(x)$$

$$= -2\mathcal{P}_k \mathcal{A}_{\varphi} + \int^{y} I_{\varphi}(x,y)dy + f_{\varphi}(x)\,, \tag{2.112a}$$

$$P^{\Sigma}_{k\,\psi}(x,y) = -\int^{y} (\imath_k \star \mathcal{F} + 2\mathcal{P}_k \mathcal{F})_{y\psi}\, dy + f_{\psi}(x)$$

$$= -2\mathcal{P}_k \mathcal{A}_{\psi} + \int^{y} I_{\psi}(x,y)dy + f_{\psi}(x)\,, \tag{2.112b}$$

where

$$I_{\varphi}(x,y) = 2\mathcal{A}_{\varphi}\left(\partial_y \mathcal{A}_t + \Omega \partial_y \mathcal{A}_{\psi}\right)$$

$$+ \partial_x \mathcal{A}_t \left(\frac{R^2 \Omega F(x) G(x) H(y) h(x,y)^2}{F(y) H(x) (x-y)^2} + \frac{F(y) G(x) H(y) \omega_{\psi}(y)(\Omega \omega_{\psi}(y)+1)}{F(x) G(y) H(x) h(x,y)}\right)\Bigg($$

$$- \partial_x \mathcal{A}_t \left(\frac{\Omega H(x)^2 \omega_{\varphi}(x)^2}{H(y)^2 h(x,y)}\right)\Bigg($$

$$- \partial_x \mathcal{A}_{\psi} \frac{F(y) G(x) H(y)(\Omega \omega_{\psi}(y)+1)}{F(x) G(y) H(x) h(x,y)} + \partial_x \mathcal{A}_{\varphi} \frac{\Omega H(x)^2 \omega_{\varphi}(x)}{H(y)^2 h(x,y)}\,, \tag{2.113a}$$

$$I_{\psi}(x,y) = \frac{H(x)^2 \left(\omega_{\varphi}(x) \partial_x \mathcal{A}_t - \partial_x \mathcal{A}_{\varphi}\right)}{H(y)^2 h(x,y)} + 2\mathcal{A}_{\psi}\left(\partial_y \mathcal{A}_t + \Omega \partial_y \mathcal{A}_{\psi}\right)\,, \tag{2.113b}$$

for some functions $f_{\varphi}(x)$ and $f_{\psi}(x)$ to be determined.

In this form, the functions are well defined at $y = -1/\nu$ (and beyond), and we can analytically prolongate P_k there.

The functions $f_{\varphi}(x)$ and $f_{\psi}(x)$ can be readily fixed from the fact that the combination $P_k + 2\mathcal{P}_k \mathcal{A}$ is closed on $\mathcal{BH}$ (the restricted generalized zeroth law). Indeed, pulling back on $\mathcal{BH}$ the KR momentum map Eq. (2.109), one has

$$d\left(P_k + 2\Phi_{\mathcal{H}} \mathcal{A}\right) \overset{\mathcal{BH}}{=} 0\,. \tag{2.114}$$

Thus, a solution of the form (2.111) that is well defined at $y = -1/\nu$ must satisfy the boundary condition

$$P_k \overset{\mathcal{BH}}{=} -2\Phi_{\mathcal{H}} \mathcal{A} + C_{\varphi} d\varphi + C_{\psi} d\psi \tag{2.115}$$

for some constants C_φ and C_ψ. This implies that our solution reads

$$P^{\Sigma}_{k\,\varphi}(x,y) = -2\mathcal{P}_k\mathcal{A}_\varphi + \int_{-1/\nu}^{y} I_\varphi(x,y)dy + C_\varphi\,, \tag{2.116a}$$

$$P^{\Sigma}_{\psi}(x,y) = -2\mathcal{P}_k\mathcal{A}_\psi + \int_{-1/\nu}^{y} I_\psi(x,y)dy + C_\psi\,. \tag{2.116b}$$

Remarkably,

$$\int_{-1/\nu}^{y} I_\varphi(-1,y)dy = 0\,, \qquad\qquad \forall y \neq -1\,, \tag{2.117a}$$

$$\int_{-1/\nu}^{-1} I_\psi(x,y)dy = \frac{\cosh 2\alpha\left[C_\lambda(\mu+\nu)+C_\mu(\nu-\lambda)\right]+C_\lambda(\mu+\nu)+C_\mu(\lambda-\nu)}{\cosh 2\alpha\left[C_\lambda(\mu+\nu)+3C_\mu(\lambda-\nu)\right]+C_\lambda(\mu+\nu)-3C_\mu(\lambda-\nu)}\times$$

$$\times\frac{\nu-1}{\mu+\nu}C_\mu R\,\operatorname{sech}\alpha\,, \qquad\qquad \forall x\,, \tag{2.117b}$$

so regularity at $y=-1$ and $x=-1$ is achieved by setting

$$C_\varphi = 0\,, \tag{2.118}$$

$$C_\psi = \frac{\cosh 2\alpha\left[C_\lambda(\mu+\nu)+C_\mu(\nu-\lambda)\right]+C_\lambda(\mu+\nu)+C_\mu(\lambda-\nu)}{\cosh 2\alpha\left[C_\lambda(\mu+\nu)+3C_\mu(\lambda-\nu)\right]+C_\lambda(\mu+\nu)-3C_\mu(\lambda-\nu)}\frac{1-\nu}{\mu+\nu}C_\mu R\,\operatorname{sech}\alpha$$

$$\equiv C(\lambda,\mu,\nu,\alpha)\frac{1-\nu}{\mu+\nu}C_\mu R\,\operatorname{sech}\alpha, \tag{2.119}$$

which completes the solution.

We conclude by noticing that the associated KR potential 1-form at $\mathcal{BH}$ is *purely harmonic* and given by,

$$\Phi_{KR} = P_k + 2\mathcal{P}_k\mathcal{A} \overset{\mathcal{BH}}{=} \Phi_{KR\,\tilde{\psi}}d\tilde{\psi}\,, \tag{2.120}$$

where $\tilde{\psi} = (2\pi/\Delta\psi)\psi$ is the angular coordinate with canonical period $\tilde{\psi}\sim\tilde{\psi}+2\pi$ and

$$\Phi_{KR\,\tilde{\psi}} = C_\psi\frac{\Delta\psi}{2\pi} = C(\lambda,\mu,\nu,\alpha)\frac{\sqrt{1-\lambda}(1+\mu)^{3/2}}{\mu+\nu}C_\mu R\,\operatorname{sech}\alpha\,. \tag{2.121}$$

For $\alpha=0$, Φ_{KR} coincides with the potential given in Ref. [63] up to (parameter-independent) numerical prefactors.

2.6 Discussion

In this paper we have derived the first law of black hole mechanics in the context of the effective action of the Heterotic Superstring compactified on a torus at leading order in α'. The first law includes the variations of the conserved charges of the 1-forms, $\mathcal{Q}_I$, and of the charges associated to the KR field, $\mathcal{Q}_i$, multiplied by the potentials Φ^I and Φ^i which are constants that we have computed on the bifurcation surface.

The main ingredients in this proof are the identification of the parameters of the gauge transformations that generate symmetries of the complete field configurations, the careful definitions of the associated charges and the corresponding potentials through what we have called restricted generalized zeroth laws. Due to the interactions between 1-forms and the KR 2-form induced by the Chern-Simons terms, all the terms involving charges and potentials in the first law are interrelated and all their definitions are either right or wrong simultaneously. This can be seen as a test of our definitions and of the final result.

In the theory considered in this paper we have arrived at the well-known result that the entropy is one quarter of the area. In theories of higher order in the curvature it is known that there are additional contributions from the terms that contain the curvature, as the Iyer-Wald prescription makes manifest. However, as explained in the introduction, in the case of the Heterotic Superstring effective action at first order in α', we also expect that the need to have well-defined charges and, simultaneously, closed forms over the bifurcation sphere will result in the need to include additional terms in the "gravitational charge" that, in the end, will give us the entropy. Work in this direction is well under way [209].

Finally, we would like to comment upon two apparent shortcomings of Wald's formalism: it is not clear how to include the variation of the scalar charges and the moduli [149,216] in the first law. In 5 dimensions, for instance, the KR field is dual to a 1-form and black-hole solutions electrically charged with respect to this dual 1-form exist. If we describe the theory in terms of the KR 2-form, it is not clear how to make the variation of this electric charge appear in the first law following this procedure. In this particular case, the electric charge of the 1-forms would be associated to S5-branes wrapped on T^5 and it would be very interesting to see the precise definition of this kind of charge to try to solve the ambiguities detected in Ref. [212].

3

The First Law and Wald Entropy Formula of Heterotic Stringy Black Holes at First Order in α'

This chapter is based on:
The first law and Wald entropy formula of heterotic stringy black holes at first order in α'
Z. Elgood, T. Ortín, D. Pereñiguez
JHEP 05 (2021) 110 (arXiv:2012.14892)

The interpretation of the black-hole entropy in terms of the degeneracy of string microstates is, beyond any doubt, one of the main achievements of String Theory [128]. This interpretation relies, on the one hand, on the correct identification of the black-hole charges in terms of branes whose presence affects the quantization of the string. On the other, it depends on a correct calculation of the macroscopic entropy. In simple cases, at leading order in α', the identification of the field fluxes with the brane sources that would produce them is straightforward and, also, the macroscopic entropy is given by the Bekenstein-Hawking formula $S = \mathcal{A}_{\mathcal{H}}/(4G_N)$, where $\mathcal{A}_{\mathcal{H}}$ is the area of the horizon. In more complicated cases, the couplings can make the identification of the brane sources through the charges more complicated [212] and, beyond leading order in α', the presence of terms of higher order in the curvature and, in the Heterotic Superstring case, of complicated Yang-Mills (YM) and Lorentz Chern-Simons terms [189] can also make the calculation of the macroscopic entropy very difficult. This is the problem we will deal with in this paper.

The standard method to calculate the black-hole entropy in theories of higher order in the curvature is to use Wald's formalism [139, 214], usually applying directly the Iyer-Wald prescription [140]. As we have recently discussed in Refs. [162, 198, 199] (see also references therein), the Iyer-Wald prescription was derived assuming that all the fields of the theory behave as tensors under diffeomorphisms which, as matter of fact, is only true for the metric and uncharged scalars. All the fields of the Standard Model, except for the metric, have some kind of gauge freedom and do not transform as tensors under diffeomorphisms. Even the gravitational field, if it is described by a Vielbein instead of by a metric, has a gauge freedom, as it transforms under local Lorentz transformations. In theories with fermions, Viebeins are necessary to work with the spinorial fields in curved space time.

This problem was first noticed and solved by Jacobson and Mohd in Ref. [150] for the Einstein-Hilbert action written in terms of the Vielbein. The solution consists in going back to the basic formalism of [139, 214] and deal carefully with the gauge (local Lorentz) symmetry. In practice, this means taking into account the gauge transformations induced by the diffeomorphisms on the Vielbein. This can be done, for instance, by defining a Lorentz-covariant Lie derivative (*Lie-Lorentz derivative*) which can be decomposed into a standard Lie derivative and a local Lorentz transformation and which, apart from being covariant under Local lorentz transformations, vanishes identically when the diffeomorphism is an isometry of the metric (see Refs. [151, 152][1] which build on earlier work by Lichnerowicz, Kosmann and others [154–160]). The Lie-Lorentz derivative has been recently used to extend the proof of the first law of black mechanics to supergravity, including the spinorial fields, in Ref. [194].

A more mathematically rigorous (and complicated) treatment based on the theory of principal bundles, that also applied to Yang-Mills fields, was given by Prabhu in Ref. [138].[2] Apart from the mathematical complexity, this approach cannot be used to handle higher-rank form fields such as the Kalb-Ramond (KR) field. For this reason, in Ref. [162] we proposed a simpler alternative, based on the construction of covariant Lie derivatives of all the fields with gauge freedom (a Maxwell field in the case of Ref. [162]). This construction is based on the introduction of *momentum maps* [152, 201] which play a crucial role in this paper and which we will define later. The Lie-Lorentz derivative can also be seen as based on the definition of a Lorentz momentum map.[3]

In Ref. [218] we have shown how to use momentum maps to construct covariant Lie derivatives in the Heterotic Superstring Effective action compactified in a torus at zeroth order in α'. The KR field of that theory contains Abelian Chern-Simons terms[4] which induce Nicolai-Townsend transformations of the 2-form [219]. These terms modify the definitions of the conserved charges which ultimately appear in the first law of black hole mechanics along the lines of the classical Refs. [136, 203–205].

In this paper we are going to use the same technique quite extensively to deal with the variety of fields and couplings that occur in the Heterotic Superstring effective action at first order in α' and prove the first law of black hole mechanics, identifying the entropy. As we are going to see, the entropy formula obtained is manifestly gauge-invariant and contains only terms which are known and can be computed explicitly. This is the first entropy formula proposed for this theory that satisfies all this properties. It allows us to compute reliably the entropy of black hole solutions to first order in α' and compare the result with the entropy computed through microstate counting. As we will show in the last section, it gives the same results as the non-gauge-invariant formulae used in Refs. [198, 199, 210] in certain basis.[5] This confirms the values of the entropies obtained in those references, and shows why, in spite of the manifest deficiencies of the entropy formulae used, we obtained the right result.

A very interesting aspect of the momentum maps is that they are related to the zeroth law of black hole mechanics and its generalizations.[6] In the simplest case, the momentum map associated to a Maxwell field can be interpreted as the electrostatic potential.[7] The *generalized zeroth law* states that it is constant over the black hole horizon

[1]See also Ref. [153] for a more mathematically rigorous point of view.

[2]See also Ref. [217] for a different point of view on this problem.

[3]In Refs. [192, 193], momentum maps emerge as "improved gauge transformations".

[4]Only the Kaluza-Klein and winding vector fields appear there at zeroth order in α'.

[5]These results differ slightly from the results obtained in Refs. [220,221] using the Iyer-Wald prescription in the higher-dimensional action before dimensional reduction. As pointed out in Ref. [212], the dependence on the Riemann tensor changes after dimensional reduction and the formulae in Refs. [198, 199, 210] have been found using the dimensionally-reduced action. The formula that we give here does not suffer of any of these problems. See the discussion in Section 3.6.

[6]This was first noticed by Prabhu, albeit in a completely different language [138].

[7]The Maxwell momentum map is defined in a gauge invariant form, and so is the electostatic potential.

[118]. The horizon's surface gravity, which is the subject of the zeroth law, is also related to the Lorentz momentum map. For higher-rank fields, Copsey and Horowitz [147] and, afterwards, Compère [161] proved a restricted form of the generalized zeroth law (restricted because it refers only to the bifurcation sphere) which follows from the closedness of certain differential form on it. In Ref. [218] we proved that these closed forms are related to the momentum maps and we will call these statements *restricted generalized zeroth laws*. Here we will extend the results of Ref. [218] to YM and KR fields and to the more complicated couplings of the Heterotic Superstring effective action at first order in α'.[8]

The restricted generalized zeroth laws play a crucial role in the proof of the first law and in the identification of the entropy and they are intimately related to the definitions of conserved charges. In Wald's formalism, the entropy is identified only after the terms $\sim \Phi\delta\mathcal{Q}$ have been identified in the first law. As in Ref. [218], this identification requires the addition and subtraction of several terms as demanded by the definitions of the charges $\mathcal{Q}$ and the potentials Φ on account of the restricted generalized zeroth laws. However, in this case, some of the terms added and subtracted will be shown to contribute to the entropy.

This paper is organized as follows: in Section 3.1 we introduce the effective action of the Heterotic Superstring to first order in α' and find how it changes under an arbitrary variation of the fields, which allows us to determine the equations of motion. In Section 3.2 we study how the fields change under gauge and general coordinate transformations. We construct variations of the fields that vanish when the parameters of the transformations generate a symmetry of the field configuration and we find the integrals that give the associated conserved charges. The conserved charge associated to the invariance under diffeomorphisms is the Wald-Noether charge. As we have discussed, the correct identification of the conserved charges is essential to obtain for the correct identification of the entropy in the first law. In Section 3.3 we discuss the restricted generalized zeroth laws of this theory, which also play an essential role in the proof of the first law. In Section 3.4 we prove the first law using the results obtained in the previous sections, which leads us to identify the Wald entropy formula in Section 3.5. Section 3.6 contains a discussion of our results, comparing them with the existing literature.

3.1 The HST effective action at first order in α'

The Heterotic Superstring effective action can be described at first order in α' as follows [189]:[9] we start by defining the zeroth-order KR field strength $H^{(0)}$ and its components ${H^{(0)}}_{\mu\nu\rho}$ as

$$H^{(0)} \equiv dB = \tfrac{1}{3!}H_{\mu\nu\rho}dx^{\mu}\wedge dx^{\mu}\wedge dx^{\rho}\,, \tag{3.1}$$

where $B = \frac{1}{2}B_{\mu\nu}dx^{\mu}\wedge dx^{\mu}$ is the KR 2-form potential. Then, if $\omega^{ab} = \omega_{\mu}{}^{ab}dx^{\mu}$ is the Levi-Civita spin connection,[10] we define the zeroth-order torsionful spin connections[11]

This is in contrast wit the standard definitions of the electrostatic potential used in the literature.

[8]Some of these couplings have been discussed before in the literature, specially in Ref. [197] (see also references therein). See the discussion in Section 3.6.

[9]We use the conventions of Ref. [152], reviewed for the zeroth-order case in Ref. [218]. In particular, the relation with the fields in Ref. [189] can be found in Ref. [222].

[10]If $e^a = e^a{}_{\mu}dx^{\mu}$ are the Vielbein, the spin connection is defined to satisfy the Cartan structure equation $\mathcal{D}e^a \equiv de^a - \omega^a{}_b \wedge e^b = 0$.

[11]We denote by $\imath_a A$ the inner product of $e_a \equiv e_a{}^{\mu}\partial_{\mu}$ ($e_a{}^{\mu}e^b{}_{\mu} = \delta^a{}_b$) with the differential form A. If A

$$\Omega^{(0)}_{(\pm)\,ab} = \omega_{ab} \pm \tfrac{1}{2}\imath_b \imath_a H^{(0)}\,, \tag{3.2}$$

and their corresponding zeroth-order curvature 2-forms and Chern-Simons 3-forms

$$R^{(0)\,ab}_{(\pm)} \equiv d\Omega^{(0)\,ab}_{(\pm)} - \Omega^{(0)\,a}_{(\pm)\;\;c} \wedge \Omega^{(0)\,cb}_{(\pm)}\,, \tag{3.3a}$$

$$\omega^{(0)}_{(\pm)} = R^{(0)\,a}_{(\pm)\;\;b} \wedge \Omega^{(0)\,b}_{(\pm)\;\;a} + \tfrac{1}{3}\Omega^{(0)\,a}_{(\pm)\;\;b} \wedge \Omega^{(0)\,b}_{(\pm)\;\;c} \wedge \Omega^{(0)\,c}_{(\pm)\;\;a}\,. \tag{3.3b}$$

Next, we define the gauge field strength 2-form and the Chern-Simons 3-forms for the YM field $A^A = A^A{}_\mu dx^\mu$ by

$$F^A = dA^A + \tfrac{1}{2} f_{BC}{}^A A^B \wedge A^C\,, \tag{3.4}$$

$$\omega^{\rm YM} = F_A \wedge A^A - \tfrac{1}{6} f_{ABC} A^A \wedge A^B \wedge A^C\,, \tag{3.5}$$

where we have lowered the adjoint group indices $A, B, C, \ldots$ in the structure constants $f_{AB}{}^C$ and gauge fields using the Killing metric.

Then, we can define the first-order KR field strength 3-form as

$$H^{(1)} \equiv H^{(0)} + \frac{\alpha'}{4}\left(\omega^{\rm YM} + \omega^{(0)}_{(-)}\right)\,. \tag{3.6}$$

Its Bianchi identity takes the well-known form

$$dH^{(1)} = \frac{\alpha'}{4}\left(F_A \wedge F^A + R^{(0)\,a}_{(-)\;\;b} \wedge R^{(0)\,b}_{(-)\;\;a}\right)\,. \tag{3.7}$$

Having made these definitions and adding the dilaton field ϕ, we can write the Heterotic Superstring effective action to first-order in α' as

$$\begin{aligned} S^{(1)}[e^a, B, A^A, \phi] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int e^{-2\phi} \Big[(-1)^{d-1} \star (e^a \wedge e^b) \wedge R_{ab} - 4 d\phi \wedge \star d\phi \\ + \tfrac{1}{2} H^{(1)} \wedge \star H^{(1)} + (-1)^d \frac{\alpha'}{4}\left(F_A \wedge \star F^A + R^{(0)\,a}_{(-)\;\;b} \wedge \star R^{(0)\,b}_{(-)\;\;a}\right)\Big] \\ \equiv \int \mathbf{L}^{(1)}\,. \end{aligned} \tag{3.8}$$

Although this action is defined in 10 dimensions, we have left the dimension arbitrary (d) because that allows us to use the results in other dimensions after trivial dimensional reduction on a torus. In this action, $G_N{}^{(d)}$ is the d-dimensional Newton constant and

is a p-form with components $A_{\mu_1\cdots\mu_p}$, $\imath_a A$ is the $(p-1)$ form with components $e_a{}^\nu A_{\nu\mu_1\cdots\mu_{p-1}}$.

$g_s^{(d)}$ is the d-dimensional string coupling constant, identified with the vacuum expectation value of the exponential of the d-dimensional dilaton field $g_s^{(d)} =< e^{\phi} >$. In solutions such as black holes that asymptote to a vacuum solution at infinity $e^{\phi} \to e^{\phi_\infty} =< e^{\phi} >= g_s^{(d)}$.

This is a very complex action. Due to this complexity and to the lemma proven in Ref. [189] which we will explain later, it is convenient to perform a general variation of the action in two steps: first, we only vary the action with respect to the *explicit* occurrences of the fields, where we define "explicit occurrences" as those which do not take place in the torsionful spin connection $\Omega^{(0)}_{(-)}$. Then, we vary the action with respect to the occurrences of the fields via $\Omega^{(0)}_{(-)}$ using the chain rule. All the occurrences of the dilaton and YM fields are explicit, but those of the Vielbein and KR field are not, because they (and only they) are present in $\Omega^{(0)}_{(-)}$.

Thus, setting $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1} = 1$ for the time being in order to simplify the formulae, we find that under a general variation of the "explicit" occurrences of the fields, the action transforms as follows:

$$\delta_{\exp} S^{(1)} = \int \Big\{ \mathbf{E}^{(1)}_{\exp\, a} \wedge \delta e^a + \mathbf{E}^{(1)}_{\exp\, B} \wedge \delta B + \mathbf{E}^{(1)}_{\phi}\delta\phi + \mathbf{E}^{(1)}_{A}\delta A^A + d\mathbf{\Theta}^{(1)}_{\exp}(\varphi, \delta\varphi) \Big\} , \tag{3.9}$$

where φ stands for all the fields of the theory,

$$\begin{aligned}
\mathbf{E}^{(1)}_{\exp\, a} &= e^{-2\phi}\imath_a \star (e^c \wedge e^d) \wedge R_{cd} - 2\mathcal{D}(\imath_b d e^{-2\phi}) \wedge \star(e^b \wedge e^c) g_{ca} \\
&\quad + (-1)^{d-1} 4 e^{-2\phi} \left(\imath_a d\phi \star d\phi + d\phi \wedge \imath_a \star d\phi\right) \\
&\quad + \frac{(-1)^d}{2} e^{-2\phi} \left(\imath_a H^{(1)} \wedge \star H^{(1)} + H^{(1)} \wedge \imath_a \star H^{(1)}\right) \\
&\quad + \frac{\alpha'}{4} e^{-2\phi} \Big(\imath_a F_A \wedge \star F^A - F_A \wedge \imath_a \star F^A \\
&\quad + \imath_a R_{(-)}^{(0)}{}^{b}{}_{c} \wedge \star R_{(-)}^{(0)}{}^{c}{}_{b} - R_{(-)}^{(0)}{}^{b}{}_{c} \wedge \imath_a \star R_{(-)}^{(0)}{}^{c}{}_{b}\Big)
\end{aligned} \tag{3.10a}$$

$$\mathbf{E}^{(1)}_{\exp\, B} = -d\left(e^{-2\phi} \star H^{(1)}\right), \tag{3.10b}$$

$$\mathbf{E}^{(1)}_{\phi} = 8d\left(e^{-2\phi} \star d\phi\right) - 2\mathbf{L}^{(1)}\,, \tag{3.10c}$$

$$\mathbf{E}^{(1)}_{A} = -\frac{\alpha'}{2}\left\{\mathcal{D}\left(e^{-2\phi} \star F_A\right) + (-1)^d e^{-2\phi} \star H^{(0)} \wedge F_A\right\} - \frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp\, B} \wedge A_A\,, \tag{3.10d}$$

and

$$\mathbf{\Theta}^{(1)}_{\text{exp}}(\varphi,\delta\varphi) = -e^{-2\phi}\star(e^a\wedge e^b)\wedge\delta\omega_{ab} + 2\imath_a de^{-2\phi}\star(e^a\wedge e^b)\wedge\delta e_b - 8e^{-2\phi}\star d\phi\delta\phi$$
$$+ e^{-2\phi}\star H^{(1)}\wedge\delta B + \frac{\alpha'}{2}e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(1)}\wedge A_A\right)\wedge\delta A^A\,. \tag{3.11}$$

An alternative form of the YM equations that arises in the calculations is

$$\mathbf{E}^{(1)}_A = -\frac{\alpha'}{2}\mathcal{D}\left(e^{-2\phi}\star F_A - e^{-2\phi}\star H^{(0)}\wedge A_A\right) + (-1)^{d-1}\frac{\alpha'}{4}e^{-2\phi}\star H^{(0)}\wedge dA_A\,. \tag{3.12}$$

Observe that neither the YM equations of motion transform covariantly nor $\mathbf{\Theta}^{(1)}_{\text{exp}}$ is invariant under YM gauge transformations. For the YM equations this is not a big problem since the troublesome term is proportional to the KR equation of motion, but there is no obvious fix for the pre-symplectic potential. Nevertheless, we will see that, in the end, we will get gauge-invariant charges and, in particular a gauge-invariant Wald-Noether charge.

An important property of the HST effective action is that the YM fields and the torsionful spin connection occur in it exactly on the same footing [202]. The variation of the action with respect to the torsionful spin connection takes exactly the same form as the YM equation, the only difference being the group indices and their contractions. Thus,

$$\delta S^{(1)} = \int\left\{\mathbf{E}^{(1)}_{\text{exp}\,a}\wedge\delta e^a + \mathbf{E}^{(1)}_{\text{exp}\,B}\wedge\delta B + \mathbf{E}^{(1)}_{\phi}\delta\phi + \mathbf{E}^{(1)}_{A}\wedge\delta A^A + \mathbf{E}^{(1)\,b}{}_a\wedge\delta\Omega^{(0)}_{(-)}{}^a{}_b\right.$$
$$\left. + d\mathbf{\Theta}^{(1)}(\varphi,\delta\varphi)\right\}, \tag{3.13}$$

where the variation with respect to the torsionful spin connection is given by

$$\mathbf{E}^{(1)\,b}{}_a = -\frac{\alpha'}{2}\left\{\mathcal{D}_{(-)}\left(e^{-2\phi}\star R^{(0)}_{(-)}{}^b{}_a\right) + (-1)^d e^{-2\phi}\star H^{(0)}\wedge R^{(0)}_{(-)}{}^b{}_a\right\} - \frac{\alpha'}{4}\mathbf{E}^{(1)}_{\text{exp}\,B}\wedge\Omega^{(0)}_{(-)}{}^b{}_a\,, \tag{3.14}$$

or

$$\mathbf{E}^{(1)\,b}{}_a = -\frac{\alpha'}{2}\mathcal{D}_{(-)}\left(e^{-2\phi}\star R^{(0)}_{(-)}{}^b{}_a - e^{-2\phi}\star H^{(0)}\wedge\Omega^{(0)}_{(-)}{}^b{}_a\right) + (-1)^{d-1}\frac{\alpha'}{4}\star H^{(0)}\wedge d\Omega^{(0)}_{(-)}{}^b{}_a\,, \tag{3.15}$$

and the pre-symplectic $(d-1)$-form is given by

$$\mathbf{\Theta}^{(1)}(\varphi,\delta\varphi) = \mathbf{\Theta}^{(1)}_{\text{exp}}(\varphi,\delta\varphi) + \frac{\alpha'}{2}e^{-2\phi}\left(\star R^{(0)}_{(-)}{}^b{}_a - \tfrac{1}{2}\star H^{(1)}\wedge\Omega^{(0)}_{(-)}{}^b{}_a\right)\wedge\delta\Omega^{(0)}_{(-)}{}^a{}_b\,, \tag{3.16}$$

with $\mathbf{\Theta}^{(1)}_{\exp}(\varphi, \delta\varphi)$ given in Eq. (3.11).

The parallelism between the YM and torsionful spin connection terms also leads to the same problems of non-covariance of $\mathbf{E}^{(1)\,b}{}_a$ and non-invariance of the additional term in $\mathbf{\Theta}^{(1)}$.

An important difference between the equations of motion of these two connections is that, according to the lemma proven in Ref. [189], $\mathbf{E}^{(1)\,a}{}_b$ is proportional to α' and to a combination of the zeroth-order equations $\mathbf{E}^{(0)}_a, \mathbf{E}^{(0)}_B$ and $\mathbf{E}^{(0)}_\phi$. This means that field configurations that solve the equations $\mathbf{E}^{(1)}_{\exp\,a} = 0$, $\mathbf{E}^{(1)}_{\exp\,B} = 0$, $\mathbf{E}^{(1)}_\phi = 0$ and $\mathbf{E}^{(1)}_A = 0$ are solutions of the complete first-order equations, to that order in α'. This crucial property effectively reduces the degree of the differential equations to 2, avoiding the problems that arise with dynamical equations that involve derivatives of the fields of higher order.

3.2 Variations of the fields

It is convenient to start by describing the gauge transformations of the fields and the associated Noether identities to be able to compute the associated conserved charges. Afterwards, we will discuss the transformations of the fields under diffeomorphisms and the associated Wald-Noether charge.

3.2.1 Gauge transformations

The fields occurring in the effective action Eq. (3.8) transform under 3 kinds of gauge transformations:

1. KR gauge transformations with 1-form parameter Λ, δ_Λ, which only act on B.
2. YM gauge transformations with parameter χ^A, δ_χ, which act on the YM fields and on B as Nicolai-Townsend transformations.
3. Local Lorentz transformations with parameter σ^{ab}, δ_σ, which act on the Vielbein and induce transformations of spin connections and curvature and which also act on B as Nicolai-Townsend transformations.

The transformation rules are

$$\delta_\sigma e^a = \sigma^a{}_b e^b \,, \tag{3.17a}$$

$$\delta_\chi A^A = \mathcal{D}\chi^A \equiv d\chi^A + f_{BC}{}^A A^B \chi^C \,, \tag{3.17b}$$

$$\delta B = (\delta_\Lambda + \delta_\chi + \delta_\sigma) B = d\Lambda - \frac{\alpha'}{4}\chi_A dA^A - \frac{\alpha'}{4}\sigma^a{}_b d\Omega^{(0)}_{(-)}{}^b{}_a \,. \tag{3.17c}$$

The induced local Lorentz transformations of the connections are

$$\delta_\sigma \omega^{ab} = \mathcal{D}\sigma^{ab} = d\sigma^{ab} - 2\omega^{[a|}{}_c\sigma^{c|b]}\,, \tag{3.18a}$$

$$\delta_\sigma \Omega^{(0)\,ab}_{(-)} = \mathcal{D}^{(0)}_{(-)}\sigma^{ab} = d\sigma^{ab} - 2\Omega^{(0)\,[a|}_{(-)}{}_c\sigma^{c|b]}\,, \tag{3.18b}$$

and the transformations of the curvatures are

$$\delta_\chi F^A = -\chi^B f_{BC}{}^A F^C \tag{3.19a}$$

$$\delta_\sigma R^{ab} = 2\sigma^{[a|}{}_c R^{c|b]}\,. \tag{3.19b}$$

$$\delta_\sigma R^{(0)\,ab}_{(-)} = 2\sigma^{[a|}{}_c R^{(0)\,c|b]}_{(-)}\,. \tag{3.19c}$$

Finally, for the sake of completeness and their later use, we quote the gauge transformations of the Chern-Simons 3-forms

$$\delta_\chi \omega^{\text{YM}} = \frac{\alpha'}{4} d\left(\chi_A dA^A\right), \tag{3.20a}$$

$$\delta_\sigma \omega^{(0)}_{(-)} = +\frac{\alpha'}{4} d\left(\sigma^a{}_b d\Omega^{(0)\,b}_{(-)}{}_a\right)\,, \tag{3.20b}$$

and the Ricci identities

$$\mathcal{D}\mathcal{D}\chi^A = -f_{BC}{}^A\chi^B F^C = \delta_\chi F^A\,, \tag{3.21a}$$

$$\mathcal{D}^{(0)}_{(-)}\mathcal{D}^{(0)}_{(-)}\sigma^{ab} = -2R^{(0)\,[a|}_{(-)}{}_c\sigma^{c|b]} = \delta_\sigma R^{(0)\,ab}_{(-)}\,. \tag{3.21b}$$

The exact invariance of the action $S^{(1)}$ in Eq. (3.8) under the above gauge transformations leads, in a rather trivial way, to the following Noether identities [222]

$$d\mathbf{E}^{(1)}_{\exp B} = 0\,, \tag{3.22a}$$

$$\mathcal{D}\mathbf{E}^{(1)}_A + (-1)^{d-1}\frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge dA_A = 0\,, \tag{3.22b}$$

$$\mathcal{D}^{(0)}_{(-)}\mathbf{E}^{(1)}{}_b{}^a + (-1)^{d-1}\frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge d\Omega^{(0)}_{(-)\,b}{}^a = 0\,, \tag{3.22c}$$

$$\mathbf{E}^{(1)\,[a}_{\exp}\wedge e^{b]} + \frac{\alpha'}{4}\mathbf{E}^{(1)}_{\exp B}\wedge d\Omega^{(0)\,ab} + (-1)^{d-1}\mathcal{D}^{(0)}_{(-)}\mathbf{E}^{(1)\,ab} = 0\,. \tag{3.22d}$$

Eq. (3.22c) is just a particular case of Eq. (3.22b) with adjoint Lorentz indices. Furthermore, the last two identities imply the symmetry of the Einstein equation, which in the language f differential forms and Vielbeins, is expressed in the form

$$\mathbf{E}^{(1)\,[a}_{\text{exp}} \wedge e^{b]} = 0\,. \tag{3.23}$$

3.2.2 Gauge charges

For ths sake of simplicity, we are going to start by the charge associated to the δ_Λ transformations, that we are going to call Kalb-Ramond charge.

Kalb-Ramond charge

Let us consider the transformation of the action Eq. (3.8) under the gauge transformations δ_Λ. Taking into account that this symmetry only acts on B,[12] Eqs. (3.13) and (3.16) we get

$$\delta_\Lambda S^{(1)} = \int \left\{ \mathbf{E}^{(1)}_{\text{exp}\,B} \wedge d\Lambda + d\left[e^{-2\phi} \star H^{(1)} \wedge d\Lambda \right] \right\}. \tag{3.24}$$

Integrating by parts the first term and using the Noether identity Eq. (3.22a)

$$\delta_\Lambda S^{(1)} = \int d\left\{ (-1)^d \mathbf{E}^{(1)}_{\text{exp}\,B} \wedge \Lambda + e^{-2\phi} \star H^{(1)} \wedge d\Lambda \right\} \equiv \int d\mathbf{J}[\Lambda]\,. \tag{3.25}$$

Since $\delta_\Lambda S^{(1)} = 0$, the integrand must vanish, which means that $\mathbf{J}[\Lambda]$ must be locally exact. Indeed,

$$\mathbf{J}[\Lambda] = d\mathbf{Q}[\Lambda]\,, \quad \text{with} \quad \mathbf{Q}[\Lambda] = \Lambda \wedge \left(e^{-2\phi} \star H^{(1)} \right). \tag{3.26}$$

Integrating the $(d-2)$-form $\mathbf{Q}[\Lambda]$ over $(d-2)$-dimensional compact surfaces $\mathcal{S}_{d-2}$ for Λs that leave invariant the KR field B we get conserved charges associated to those Λs. These Λs are simply closed 1-forms.[13] The Hodge decomposition theorem allows us to write each of them as the sum of an exact and a harmonic form that we denote by Λ_e and Λ_h, respectively. On-shell, the exact form $\Lambda_e = d\lambda$ will not contribute to the integral and the charge will be given by

$$\mathcal{Q}(\Lambda_h) = \int_{\mathcal{S}_{d-2}} \Lambda_h \wedge \left(e^{-2\phi} \star H \right). \tag{3.27}$$

Now we can use duality between homology and cohomology: if C_{Λ_h} is the $(d-3)$-cycle dual to Λ_h we arrive at the charges

$$\mathcal{Q}(\Lambda_h) = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{C_{\Lambda_h}} e^{-2\phi} \star H\,, \tag{3.28}$$

[12] We consider the variation of the torsionful spin connection to be zero under this transformation.

[13] Here we follow Refs. [147, 161]. This discussion is identical to the discussion we made for the zeroth-order case in Ref. [218].

where we have recovered the factor of $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$ and added a conventional sign.

Yang-Mills charge

Now, let us consider the charges associated to the YM gauge transformations δ_χ. Again, from Eqs. (3.13) and (3.16), taking into account that this symmetry acts on the YM fields A^A but also on the KR 2-form B, we have

$$\begin{aligned}\delta_\chi S^{(1)} = \int \Big\{ & \mathbf{E}^{(1)}_{\exp B} \wedge \delta_\chi B + \mathbf{E}^{(1)}_A \wedge \delta_\chi A^A \\ & + d\left[e^{-2\phi} \star H^{(1)} \wedge \delta_\chi B + \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(1)} \wedge A_A\right) \wedge \delta_\chi A^A \right] \Big\} \,.\end{aligned} \tag{3.29}$$

The parameters χ^A that we will use are those that preserve the field configuration, leaving A^A and B invariant. The YM fields are left invariant by covariantly constant χ^As, *i.e.* χ^As that we will denote by κ^A satisfying

$$\mathcal{D}\kappa^A = 0\,. \tag{3.30}$$

We can call these parameters *vertical Killing vector fields* from he principal bundle point of view, with the standard Killing vectors of the base manifold playing the rôle of *horizontal Killing vector fields*.

The integrability condition of the vertical Killing vector equation is, according to Eq. (3.21a),

$$\delta_\kappa F^A = -f_{BC}{}^A \kappa^B F^C = 0\,, \tag{3.31}$$

so they also leave invariant the field strengths, as expected.

The vertical Killing vector fields κ^As will not leave B invariant, though, but we can rewrite the transformation in the form

$$\delta_\chi B = -\frac{\alpha'}{4}\kappa_A dA^A = -\frac{\alpha'}{2}\kappa_A F^A + d\left(\frac{\alpha'}{4}\kappa_A A^A\right)\,. \tag{3.32}$$

Now we observe that, due to the YM Bianchi identity $\mathcal{D}F^A = 0$, $\kappa_A F^A$ is a closed 2-form and, locally, there is a 1-form Ψ_κ such that

$$d\Psi_\kappa = -\kappa_A F^A\,, \tag{3.33}$$

and which we will call *vertical YM momentum map*.[14]

Then, we define the parameter of a compensating Λ transformation

$$\Lambda_\chi = -\frac{\alpha'}{2}\Psi_\chi - \frac{\alpha'}{4}\chi_A A^A\,, \tag{3.34}$$

[14] Compare this equation with the equation satisfied by the standard (horizontal) YM momentum map Eq. (3.59).

where Ψ_χ is a 1-form such that, when $\chi^A = \kappa^A$ (*i.e.* when it is a vertical Killing vector field), it satisfies Eq. (3.33). Combining the original δ_χ transformation with the compensating δ_{Λ_χ} transformation we find a new $\delta_\chi B$ that vanishes for covariantly constant χ^As:

$$\delta_\chi B \equiv -\frac{\alpha'}{2}\left(d\Psi_\chi + \chi_A F^A\right) - \frac{\alpha'}{4}\mathcal{D}\chi_A \wedge A^A \,. \tag{3.35}$$

The vanishing of $\delta_\chi B$ for covariantly constant χ^As is gauge invariant because

$$\delta_{\chi'}\delta_\chi \sim \mathcal{D}\chi \,. \tag{3.36}$$

Substituting the transformation Eq. (3.35) and the standard gauge transformation of the YM fields into Eq. (3.29) we get

$$\begin{aligned}
\delta_\chi S^{(1)} = \int \Big\{ & \mathbf{E}^{(1)}_A \wedge \mathcal{D}\chi^A + \mathbf{E}^{(1)}_{\exp B} \wedge \left[-d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) - \frac{\alpha'}{4}\chi_A dA^A\right] \\
& + d\Big\{ e^{-2\phi} \star H^{(1)} \wedge \left[-d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) - \frac{\alpha'}{4}\chi_A dA^A\right] \\
& + \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(1)} \wedge A_A\right) \wedge \mathcal{D}\chi^A \Big\}\Big\} \,.
\end{aligned} \tag{3.37}$$

Integrating by parts the first terms and combining the different terms in an appropriate way we can rewrite the variation in the form

$$\begin{aligned}
\delta_\chi S^{(1)} = \int \Big\{ & (-1)^d \chi^A \left(\mathcal{D}\mathbf{E}^{(1)}_A + (-1)^{d-1}\frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp B} \wedge dA_A\right) \\
& - \left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) \wedge d\mathbf{E}^{(0)}_{\exp B} \\
& + d\Big\{ (-1)^{d-1}\chi^A \left(\mathbf{E}^{(1)}_A + (-1)^d \frac{\alpha'}{4} e^{-2\phi} \star H^{(0)} \wedge dA_A\right) \\
& - \left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) \wedge \mathbf{E}^{(0)}_{\exp B} \\
& + e^{-2\phi} \star H^{(1)} \wedge \left[-d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right)\right] \\
& + \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(1)} \wedge A_A\right) \wedge \mathcal{D}\chi^A \Big\}\Big\} \,.
\end{aligned} \tag{3.38}$$

The terms in the first and second lines vanish identically because of the Noether identities Eqs. (3.22b) and (3.22a), respectively, and we arrive to

$$\begin{aligned}\delta_\chi S^{(1)} = \int d\Big\{ & (-1)^{d-1}\chi^A \left(\mathbf{E}_A^{(1)} + (-1)^d \frac{\alpha'}{4} e^{-2\phi} \star H^{(0)} \wedge dA_A\right) \\ & - \left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) \wedge \mathbf{E}_{\exp B}^{(0)} \\ & - d\left(\frac{\alpha'}{2}\Psi_\chi + \frac{\alpha'}{4}\chi_A A^A\right) \wedge \left(e^{-2\phi} \star H^{(0)}\right) \\ & + \frac{\alpha'}{2} e^{-2\phi}\left(\star F_A - \tfrac{1}{2} \star H^{(1)} \wedge A_A\right) \wedge \mathcal{D}\chi^A \Big\} \\ \equiv \int & d\mathbf{J}[\chi]\,.\end{aligned} \tag{3.39}$$

The same arguments we made in the previous case lead to the existence of a $(d-2)$-form $\mathbf{Q}[\chi]$ such that $\mathbf{J}[\chi] = d\mathbf{Q}[\chi]$. The $(d-2)$-form is given by

$$\mathbf{Q}[\chi] = -(-1)^d \frac{\alpha'}{2}\left\{e^{-2\phi} \star \left(-\chi^A F_A\right) + (-1)^d \Psi_\chi \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right\}\,. \tag{3.40}$$

For Abelian vector fields the κ^As are constant and $\Psi_\kappa = \kappa_A A^A$ (up to a total derivative) and we recover immediately the $\mathbf{Q}[\chi]$ found in Ref. [218]. On the other hand, when we change Ψ_κ by a total derivative, $\mathbf{Q}[\kappa]$ is invariant on-shell up to a total derivative which will not contribute to the charge which is now given by the integral

$$\mathcal{Q}[\kappa] = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{S^{d-2}} (-1)^d \frac{\alpha'}{2}\left\{e^{-2\phi} \star d\Psi_\kappa + (-1)^d \Psi_\kappa \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right\}, \tag{3.41}$$

where we have made use of the definition of the vertical momentum map Ψ_κ in Eq. (3.33).

Lorentz charge

Let us now consider local Lorentz transformations. As we have stressed repeatedly we can treat the local Lorentz transformations and the torsionful spin connection in parallel to the YM gauge transformations and the gauge fields. The only difference is the presence of one additional term in the Lorentz case: the Einstein-Hilbert case. If we follow the same steps as in the YM case we arrive to

$$\mathbf{Q}[\sigma] = (-1)^{d-1} e^{-2\phi} \star (e^a \wedge e^b)\sigma_{ab} - (-1)^d \frac{\alpha'}{2}\left\{e^{-2\phi} \star \left(-\sigma^a{}_b R^{(0)\,b}{}_a\right) + (-1)^d \Pi_\sigma \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right\}, \tag{3.42}$$

where Π_σ is a 1-form that becomes a *vertical Lorentz momentum map* whan the Lorentz parameter $\sigma^a{}_b = \kappa^a{}_b$, a Lorentz parameter that generates a symmetry of the field configuration, *i.e. a vertical Killing vector*. This happens when the Vielbein and the spin connection are left invariant

$$\kappa^a{}_b e^b = 0\,, \tag{3.43a}$$

$$\mathcal{D}\kappa^a{}_b = 0\,. \tag{3.43b}$$

These two conditions imply the invariance of the torsion $\frac{1}{2}\imath_b\imath_a H^{(0)}$ Hence, they also implies the invariance of the torsionful spin connection $\Omega^{(0)}_{(-)}{}^a{}_b$,

$$\mathcal{D}^{(0)}_{(-)}\kappa^a{}_b = 0\,. \tag{3.44}$$

These conditions can be used to modify the transformation of the KR field so that it is also left invariant, as we did in the YM case. We just quote the final form:

$$\delta_\sigma B = -\frac{\alpha'}{2}\left(d\Pi_\sigma + \kappa^a{}_b R^{(0)}_{(-)}{}^b{}_a\right) - \frac{\alpha'}{4}\mathcal{D}^{(0)}_{(-)}\sigma^a{}_b \wedge \Omega^{(0)}_{(-)}{}^b{}_a\,, \tag{3.45}$$

where the vertical Lorentz momentum map Π_σ is such that, when $\sigma^a{}_b = \kappa^a{}_b$

$$d\Pi_\kappa = \kappa^a{}_b R^{(0)}_{(-)}{}^b{}_a\,. \tag{3.46}$$

The conserved charge is the integral of the $(d-2)$-form Eq. (3.42) for vertical Killing vector fields $\kappa^a{}_b$ satisfying Eqs. (3.43) and (3.43b). The first condition annihilates the first term, corresponding to the Einstein-Hilbert term in the action but the rest of the terms survive in this case and we get the non-vanishing Lorentz charge

$$\mathcal{Q}[\kappa] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{S^{d-2}}\left\{(-1)^d\frac{\alpha'}{2}\left[e^{-2\phi}\star d\Pi_\kappa + (-1)^d\Pi_\kappa\wedge\left(e^{-2\phi}\star H^{(0)}\right)\right]\right\}. \tag{3.47}$$

In the proof of the first law we will find the integral of $(d-2)$-form Eq. (3.42) for a Lorentz parameter that satisfies Eq. (3.43b) only. This integral give, precisely, the entropy.

3.2.3 The transformations under diffeomorphisms

Now we turn our attention to the diffeomorphisms. Our treatment is similar to the treatment of the δ_χ gauge transformations, although the use of compensating gauge transformations admits a more general justification in terms of the gauge covariance of the modified transformations (covariant Lie derivatives). Since we have discussed at length these modifications in Refs. [162, 218] we will only discuss the aspects not covered there: torsionful spin connections, non-Abelian gauge fields and the more complicated transformations of the KR 2-form.

In this section k will always be a (horizontal) Killing vector which generates a symmetry of the complete field configuration.

Lie-Lorentz derivatives

The transformations of the Vielbeins, the Levi-Civita spin connection and its curvature 2-form have been discussed in Refs. [162, 218], but it is convenient to adapt some of the formulae to the torsionful spin connection. They are generically given in terms of the Lie-Lorentz (or Lorentz-covariant Lie derivative Refs. [151, 152, 154–157]) by $\delta_\xi = -\mathbb{L}_\xi$. Therefore, we will continue this discussion in terms of the latter.

The parameter of the compensating local Lorentz transformation that appears in the Lie-Lorentz derivative of $\Omega^{(0)\,ab}_{(-)}$ is still given by

$$\sigma_\xi{}^{ab} = \imath_\xi \omega^{ab} - \nabla^{[a}\xi^{b]}\,, \tag{3.48}$$

but it is useful to rewrite it using $\Omega^{(0)\,ab}_{(-)}$ in the covariant derivatives. Due to the complete antisymmetry of the torsion, it takes the simple form

$$\sigma_\xi{}^{ab} = \imath_\xi \Omega^{(0)\,ab}_{(-)} - \mathcal{D}^{(0)\,[a}_{(+)}\xi^{b]}\,. \tag{3.49}$$

Observe that the presence of fully antisymmetric torsion does not modify the Killing equation[15]

$$2\mathcal{D}^{(0)}_{(\pm)\,(a}\xi_{b)} = 0\,. \tag{3.50}$$

Notice that Eqs. (3.49) and (3.50) are completely independent of $H^{(0)}$ even if we have formally rewritten them in terms of the torsionful spin connection $\Omega^{(0)}_{(-)}$.

The Lie-Lorentz derivative of the torsion $\imath_b \imath_a H^{(0)}$ follows the general formula while that of the Levi-Civita connection ω^{ab} is given by

$$\mathbb{L}_\xi \omega^{ab} = \pounds_\xi \omega^{ab} - \mathcal{D}\sigma_\xi{}^{ab}\,, \tag{3.51}$$

and, therefore, it is easy to see that

$$\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)} = \pounds_\xi \Omega^{(0)\,ab}_{(-)} - \mathcal{D}^{(0)}_{(-)}\sigma_\xi{}^{ab}\,, \tag{3.52}$$

and it is equally easy to see that it can be rewritten in the form

$$\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)} = \imath_\xi R^{(0)\,ab}_{(-)} + \mathcal{D}_{(-)}P_{(-)\xi}{}^{ab}\,, \tag{3.53}$$

with

$$P_{(-)\xi}{}^{ab} \equiv \mathcal{D}^{(0)\,[a}_{(+)}\xi^{b]}\,, \tag{3.54}$$

The identity

$$\xi^\nu R^{(0)}_{(-)\,\nu\mu}{}^{ab} + \mathcal{D}^{(0)}_{(-)\,\mu}P_{(-)\xi}{}^{ab} = \mathcal{D}^{(0)\,[a}_{(-)}\left(\nabla^{b]}\xi_\mu + \nabla_\mu \xi^{b]}\right) - \tfrac{3}{2}\nabla_{[\mu|}\left(\xi^\nu H^{(0)}_{\nu|\rho\sigma]}\right)e^{a\rho}e^{b\sigma}\,, \tag{3.55}$$

[15] The presence of generic torsion does modify the Killing equation.

proves that $\delta_\xi \Omega^{(0)\,ab}_{(-)} = -\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)}$ vanishes when $\xi^\mu = k^\mu$, because, in that case,

$$-\imath_k R^{(0)\,ab}_{(-)} = \mathcal{D}^{(0)}_{(-)} P_{(-)k}{}^{ab}\,. \tag{3.56}$$

Because $P_{(-)k}{}^{ab}$ satisfies this equation, we will call it the *horizontal Lorentz momentum map associated to the torsionful spin connection.*

k, then, generates a diffeomorphism that leaves invariant the metric and the KR 3-form field strength.

Again, $P_{(-)\xi}{}^{ab}$ is a Lorentz tensor and $\delta_\xi \Omega^{(0)\,ab}_{(-)} = -\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)}$ is a Lorentz tensor although $\Omega^{(0)\,ab}_{(-)}$ is a connection. When it vanishes, it vanishes in all Lorentz frames.

Lie-Yang-Mills derivatives

Since the spin connection is just the connection of the Lorentz group, this case is very similar to the previous one, the main difference being that the YM fields are fundamental fields while the spin connection is a composite field. Apart from this, in many (but not all, because of the absence of a YM analogue of the Vielbein) instances we may just apply the same formulae with the sole change of the adjoint group indices, as we are going to see.

In order to find the gauge-covariant Lie derivative of YM fields it is convenient to consider the Lie-Lorentz derivative of the curvature tensor first. In this case, since we do not know the form of the parameter of the compensating gauge transformation, we can simply consider the standard Lie derivative of the gauge field strength 2-form defined in Eq. (3.4):

$$\pounds_\xi F^A = (\imath_\xi d + d\imath_\xi) F^A = \mathcal{D}\imath_\xi F^A - f_{BC}{}^A \imath_\xi A^B F^C\,, \tag{3.57}$$

where we have used the Bianchi identity $\mathcal{D}F^A = 0$.

When $\xi = k$ this expression should vanish up to an infinitesimal gauge transformation with some parameter that we denote by $\tilde{\chi}_k{}^A$. Then,

$$\mathcal{D}\imath_\xi F^A = f_{BC}{}^A \left(\imath_\xi A^B + \tilde{\chi}_k{}^B\right) F^C \equiv f_{BC}{}^A P_k{}^B F^C\,, \tag{3.58}$$

which, upon use of the Ricci identity Eq. (3.21a), can be solved by a $P_k{}^A$ that we call the *(horizontal) Yang-Mills momentum map* satisfying the equation

$$-\imath_k F^A = \mathcal{D} P_k{}^A\,. \tag{3.59}$$

Eq. (3.56) is nothing by a particular case of this equation for which the momentum map is explicitly known. This happens because we know how to express the gauge field in terms of a more fundamental field (the Vielbein). In general, the general form of $P_k{}^A$ is not known but is determined up to a covariantly-constant gauge parameter. We will use a $P_\xi{}^A$ which is undetermined except for the fact that it reduces to $P_k{}^A$ satisfying Eq. (3.59) for Killing vectors.

Now, we can use as definition of the Lie-Yang-Mills derivative of F^A the following expression which is guaranteed to vanish when $\xi = k$ on account of Eq. (3.58):

$$\mathbb{L}_\xi F^A = \mathcal{D}\imath_\xi F^A - f_{BC}{}^A P_\xi{}^B F^C = \pounds_\xi F^A - \delta_{\chi_\xi} F^A \,, \tag{3.60}$$

where the gauge compensating parameter $\chi_\xi{}^A$ is given by the (now usual) expression

$$\chi_\xi{}^A = \imath_\xi A^A - P_\xi{}^A \,. \tag{3.61}$$

The Lie-Yang-Mills derivative of the gauge field is, then

$$\mathbb{L}_\xi A^A \equiv \pounds_\xi A^A - \mathcal{D}\chi_\xi{}^A = \imath_\xi F^A + \mathcal{D}P_\xi{}^A \,, \tag{3.62}$$

and, by construction, it vanishes automatically when ξ is a Killing vector field k^μ and $P_k{}^A$ is the momentum map satisfying Eq. (3.59).

The Kalb-Ramond field

The parameters of the compensating YM and local Lorentz transformations of the KR field are the same transformations $\chi_\xi{}^A$ and $\sigma_\xi{}^{ab}$ that we perform on other fields with YM and Lorentz indices, given by Eqs. (3.61) and (3.48). Thus, if we want to construct a transformation of this field under diffeomorphisms that annihilates it when $\xi = k$ by combining its standard Lie derivative with gauge transformations, the only gauge parameter we can still play with is the 1-form Λ because the rest are already completely determined. We have

$$\begin{aligned}\delta_\xi B &= -\,\pounds_\xi B + (\delta_{\Lambda_\xi} + \delta_{\chi_\xi} + \delta_{\sigma_\xi})B \\ &= -\,\pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\chi_{\xi\,A} dA^A - \frac{\alpha'}{4}\sigma_\xi{}^a{}_b d\Omega^{(0)}_{(-)}{}^b{}_a \,.\end{aligned} \tag{3.63}$$

Again, it is convenient to start by considering the transformation of the 3-form field strength $H^{(1)}$ defined in Eq. (3.6) under diffeomorphisms, because it is gauge invariant:

$$\begin{aligned}\delta_\xi H^{(1)} &= -\,\pounds_\xi H^{(1)} \\ &= -\,\imath_\xi dH^{(1)} - d\imath_\xi H^{(1)} \\ &= -\,d\imath_\xi H^{(1)} - \frac{\alpha'}{2}\left(\imath_\xi F_A \wedge F^A + \imath_\xi R^{(0)}_{(-)}{}^a{}_b \wedge R^{(0)}_{(-)}{}^b{}_a\right) \,,\end{aligned} \tag{3.64}$$

where we have used the Bianchi identity Eq. (3.7).

When $\xi = k$ we can use Eqs. (3.56) and (3.59), integrate by parts, and use now the Bianchi identities for the curvatures, getting:

$$\begin{aligned}\delta_k H^{(1)} &= -\,d\imath_k H^{(1)} + \frac{\alpha'}{2}\left(\mathcal{D}P_{k\,A} \wedge F^A + \mathcal{D}_{(-)}P_{(-)\,k}{}^a{}_b \wedge R^{(0)}_{(-)}{}^b{}_a\right) \\ &= -\,d\left[\imath_k H^{(1)} - \frac{\alpha'}{2}\left(P_{k\,A}F^A + P_{(-)\,k}{}^a{}_b R^{(0)}_{(-)}{}^b{}_a\right)\right] .\end{aligned} \tag{3.65}$$

By assumption, the above expression must vanish identically. Therefore, locally, there must exist a gauge-invariant 1-form, the *horizontal Kalb-Ramond momentum map* P_k, satisfying

$$-\imath_k H^{(1)} + \frac{\alpha'}{2}\left(P_{k\,A}F^A + P_{(-)\,k}{}^a{}_b R_{(-)}^{(0)\,b}{}_a\right) = dP_k\,. \tag{3.66}$$

Then, if we apply the rule of thumb that the parameter of the compensating gauge transformation is the inner product of the vector that generates the diffeomorphisms with the "connection" (here B) minus the momentum map (here some 1-form P_ξ that in this case satisfies Eq. (3.66) when $\xi = k$)

$$\Lambda_\xi = \imath_\xi B - P_\xi\,, \tag{3.67}$$

we arrive at the following candidate to $\delta_\xi B$:

$$\begin{aligned}\delta_\xi B =& -\pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\left(\chi_{\xi\,A}dA^A + \sigma_\xi{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right)\\ =& -\imath_\xi H^{(1)} - \frac{\alpha'}{4}\left(A_A\wedge\imath_\xi F^A + \Omega_{(-)}^{(0)\,a}{}_b\wedge\imath_\xi R_{(-)}^{(0)\,b}{}_a\right)\\ & -dP_\xi + \frac{\alpha'}{4}\left(P_{\xi\,A}dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right).\end{aligned} \tag{3.68}$$

Let us see if, with this definition, $\delta_k B = 0$. Using Eqs. (3.66), (3.59) and (3.56) we get, instead of zero, a total derivative

$$\delta_k B = -\frac{\alpha'}{4}d\left(P_{k\,A}A^A + P_{(-)\,k}{}^a{}_b\Omega_{(-)}^{(0)\,b}{}_a\right), \tag{3.69}$$

which we can simple absorb in redefinition of Λ_ξ in Eq. (3.67):

$$\Lambda_\xi \equiv \imath_\xi B - P_\xi + \frac{\alpha'}{4}d\left(P_{\xi\,A}A^A + P_{(-)\,\xi}{}^a{}_b\Omega_{(-)}^{(0)\,b}{}_a\right). \tag{3.70}$$

With this new parameter,

$$\begin{aligned}\delta_\xi B =& -\pounds_\xi B + d\Lambda_\xi - \frac{\alpha'}{4}\chi_{\xi\,A}dA^A - \frac{\alpha'}{4}\sigma_\xi{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\\ =& -\left[\imath_\xi H^{(1)} - \frac{\alpha'}{2}\left(P_{\xi\,A}F^A + P_{(-)\,\xi}{}^a{}_b R_{(-)}^{(0)\,b}{}_a\right) + dP_k\right]\\ & +\frac{\alpha'}{4}\left(A_A\wedge\delta_\xi A^A + \Omega_{(-)}^{(0)\,a}{}_b\wedge\delta_\xi\Omega_{(-)}^{(0)\,b}{}_a\right)\\ \equiv& -\mathbb{L}_\xi B\,,\end{aligned} \tag{3.71}$$

that vanishes identically when $\xi = k$ by virtue of the definition of the KR momentum map Eq. (3.66) and of $\delta_\xi A^A = \delta_\xi \Omega^{(0)\,b}_{(-)\ \ a} = 0$.

The behavior of this variation under gauge transformations is far from obvious. A direct calculation gives

$$\delta_{\text{gauge}}\delta_\xi B = \frac{\alpha'}{4}\left(d\chi_A \wedge \delta_\xi A^A + d\sigma^a{}_b \wedge \delta_\xi \Omega^{(0)\,b}_{(-)\ \ a}\right)\,, \tag{3.72}$$

with $\delta_\xi A^A = -\mathbb{L}_\xi A^A$ with the Lie-Yang-Mills covariant derivative given by Eq. (3.62) and with $\delta_\xi \Omega^{(0)\,ab}_{(-)} = -\mathbb{L}_\xi \Omega^{(0)\,ab}_{(-)}$, with the Lie-Lorentz derivative given by Eq. (3.53). Therefore, although the $\delta_\xi B$ defined above is not gauge-invariant, $\delta_k B$ vanishes in a gauge-invariant way.

3.2.4 The Wald-Noether charge

Now we consider the variation of the action $S^{(1)}$ given in Eq. (3.8) under the transformations $\delta_\xi = -\mathbb{L}_\xi$ for all the fields, where $\mathbb{L}_\xi$ is the gauge-covariant derivative which, for the Vielbein is given by [162]

$$\mathbb{L}_\xi e^a = \mathcal{D}\xi^a + P_\xi{}^a{}_b e^b\,, \tag{3.73}$$

for the torsionful spin connection in Eq. (3.53), for the YM fields in Eq. (3.62) and for the KR field in Eq. (3.71).

From Eq. (3.13)

$$\begin{aligned}
\delta_\xi S^{(1)} = -\int \Big\{ & \mathbf{E}^{(1)}_{\exp a} \wedge \left(\mathcal{D}\imath_\xi e^a + P_\xi{}^a{}_b e^b\right) + \mathbf{E}^{(1)}_\phi \imath_\xi d\phi \\
& + \mathbf{E}^{(1)}_A \wedge \left(\imath_\xi F^A + \mathcal{D}P_\xi{}^A\right) + \mathbf{E}^{(1)\,b}{}_a \wedge \left(\imath_\xi R^{(0)\,a}_{(-)\ \ b} + \mathcal{D}_{(-)}P_{(-)\xi}{}^a{}_b\right) \\
& + \mathbf{E}^{(1)}_{\exp B} \wedge \Big[\imath_\xi H^{(1)} + \frac{\alpha'}{4}\left(A_A \wedge \imath_\xi F^A + \Omega^{(0)\,a}_{(-)\ \ b} \wedge \imath_\xi R^{(0)\,b}_{(-)\ \ a}\right) \\
& - \frac{\alpha'}{4}\left(P_{\xi\,A} dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega^{(0)\,b}_{(-)\ \ a}\right) + d\left[P_\xi - \frac{\alpha'}{4}\left(P_{\xi\,A}A^A + P_{(-)\,\xi}{}^a{}_b \Omega^{(0)\,b}_{(-)\ \ a}\right)\right]\Big] \\
& - d\mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi)\Big\}\,,
\end{aligned} \tag{3.74}$$

where $\mathbf{\Theta}^{(1)}(\varphi, \delta_\xi\varphi)$ is given by

$$
\begin{aligned}
\mathbf{\Theta}^{(1)}(\varphi,\delta_\xi\varphi) =& e^{-2\phi}\star(e^a\wedge e^b)\wedge(\imath_\xi R_{ab}+\mathcal{D}P_{\xi\,ab}) - 2\imath_a d e^{-2\phi}\star(e^a\wedge e^b)\wedge(\mathcal{D}\imath_\xi e_b + P_{\xi\,bc}e^c)\\
&+ 8e^{-2\phi}\star d\phi\,\imath_\xi d\phi\\
&- e^{-2\phi}\star H^{(1)}\wedge\Big\{\imath_\xi H^{(1)} + \frac{\alpha'}{4}\left(A_A\wedge\imath_\xi F^A + \Omega_{(-)}^{(0)\,a}{}_b\wedge\imath_\xi R_{(-)}^{(0)\,b}{}_a\right)\\
&- \frac{\alpha'}{4}\left(P_{\xi\,A}dA^A + P_{(-)\,\xi}{}^a{}_b d\Omega_{(-)}^{(0)\,b}{}_a\right) + d\left[P_\xi - \frac{\alpha'}{4}\left(P_{\xi\,A}A^A + P_{(-)\,\xi}{}^a{}_b\Omega_{(-)}^{(0)\,b}{}_a\right)\right]\Big\}\\
&- \frac{\alpha'}{2}e^{-2\phi}\left(\star F_A - \tfrac{1}{2}\star H^{(0)}\wedge A_A\right)\wedge\left(\imath_\xi F^A + \mathcal{D}P_\xi{}^A\right)\,.\\
&- \frac{\alpha'}{2}e^{-2\phi}\left(\star R_{(-)}^{(0)\,b}{}_a - \tfrac{1}{2}\star H^{(0)}\wedge\Omega_{(-)}^{(0)\,b}{}_a\right)\wedge\left(\imath_\xi R_{(-)}^{(0)\,a}{}_b + \mathcal{D}_{(-)}P_{(-)\xi}{}^a{}_b\right)\,.
\end{aligned}
\tag{3.75}
$$

Integrating by parts and using the Noether identities Eqs. (3.22a), (3.22b), (3.22c), (3.23) and the Noether identity associated to the invariance under diffeomorphisms

$$
\begin{aligned}
&(-1)^d\mathcal{D}\mathbf{E}^{(1)}_{\exp\,a}\imath_\xi e^a + \mathbf{E}^{(1)}_{\exp\,B}\wedge\imath_\xi H^{(1)} + \mathbf{E}^{(1)}_\phi\imath_\xi d\phi\\
&+\left(\mathbf{E}^{(1)}_A + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\,B}\wedge A_A\right)\wedge\imath_\xi F^A + \left(\mathbf{E}^{(1)\,b}{}_a + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\,B}\wedge\Omega_{(-)}^{(0)\,b}{}_a\right)\wedge\imath_\xi R_{(-)}^{(0)\,a}{}_b\\
&= 0\,,
\end{aligned}
\tag{3.76}
$$

we can see that the volume term in the variation of the action Eq. (3.74) reduces to another total derivative

$$
\delta_\xi S^{(1)} = \int d\mathbf{\Theta}^{(1)\,\prime}(\varphi,\delta_\xi\varphi)\,,
\tag{3.77}
$$

with

$$
\begin{aligned}
\mathbf{\Theta}^{(1)\,\prime}(\varphi,\delta_\xi\varphi) = \mathbf{\Theta}^{(1)}(\varphi,\delta_\xi\varphi) \\
&+ (-1)^d \mathbf{E}^{(1)}_{\exp\, a}\imath_\xi e^a + (-1)^{d-1}\mathbf{E}^{(1)}_{\exp\, B}\wedge P_\xi \\
&+ (-1)^d \left(\mathbf{E}^{(1)}_{A} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B}\wedge A_A\right) P_\xi{}^A \\
&+ (-1)^d \left(\mathbf{E}^{(1)\,b}{}_{a} + \frac{\alpha'}{4}\mathbf{E}^{(0)}_{\exp\, B}\wedge \Omega^{(0)\,b}_{(-)\;\;a}\right) P_{(-)\xi}{}^a{}_b\,.
\end{aligned}
\tag{3.78}
$$

The usual reasoning leads us to the off-shell identity

$$
d\mathbf{J}^{(1)}[\xi] = 0\,,
\tag{3.79}
$$

where

$$
\mathbf{J}^{(1)}[\xi] \equiv d\mathbf{\Theta}^{(1)\,\prime}(\varphi,\delta_\xi\varphi) + \imath_\xi\mathbf{L}^{(1)}\,,
\tag{3.80}
$$

and to the local existence of a $(d-2)$-form $\mathbf{Q}^{(1)}[\xi]$ such that $\mathbf{J}^{(1)}[\xi] = d\mathbf{Q}^{(1)}[\xi]$.

A straightforward calculation leads to the fully gauge-invariant Wald-Noether charge

$$
\begin{aligned}
\mathbf{Q}^{(1)}[\xi] =& (-1)^d \star(e^a\wedge e^b)\left[e^{-2\phi}P_{\xi\,ab} - 2\imath_a d e^{-2\phi}\xi_b\right] \\
&+ (-1)^{d-1}\frac{\alpha'}{2}\left[P_{\xi\,A}e^{-2\phi}\star F^A + P_{(-)\xi}{}^a{}_b\left(e^{-2\phi}\star R^{(0)\,b}_{(-)\;\;a}\right)\right] \\
&- P_\xi\wedge\left(e^{-2\phi}\star H^{(1)}\right),
\end{aligned}
\tag{3.81}
$$

which is one of the main results of this paper.

3.3 Restricted generalized zeroth laws

One of the main ingredients in Wald's approach to the first law of black hole mechanics is the zeroth law stating that κ is constant over the horizon [118]. Originally, this law was proved using the Einstein equations and the dominant energy condition (see, for instance, Ref. [223]) but a completely geometrical proof was presented in Ref. [144].

In presence of an electromagnetic field one also needs to use the *generalized zeroth law* that guarantees that the electrostatic potential is also constant over the whole horizon. There is no purely geometrical proof of this law, though, and the standard proof also makes use of the Einstein equations and of the dominant energy condition. In Ref. [218] we have explained how this proof can be extended to a theory containing an arbitrary number of Abelian vector fields and the KR field coupled to them via Chern-Simons terms. Essentially one gets a sum of non-negative terms containing the contribution of each field, and each of them has to vanish. Extending this proof to the non-Abelian case, as long as we restrict ourselves to a gauge group with definite positive Killing metric because one gets sums of non-negative terms. However, the $R^{(0)\,2}_{(-)}$ term of our theory is of YM type, but with non-definite Killing metric because of the non-compactness of the Lorentz group and the proof cannot be extended to this case in a streightforward manner.

It is, however, possible to proof the first law in bifurcate horizons if one can proof generalized zeroth laws for the matter fields restricted to the bifurcation sphere $\mathcal{BH}$ where the Killing vector associated to the event horizon, k, vanishes identically. These *restricted generalized zeroth laws* state the closedness of certain differential forms on $\mathcal{BH}$. The definitions of the potentials as certain constants follow from them as we are going to explain.

Assuming all the fields are regular over the horizon, it is clear that the inner products of their field strengths with k must vanish on $\mathcal{BH}$:

$$\imath_k d\phi \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.82a}$$

$$\imath_k H \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.82b}$$

$$\imath_k F^A \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.82c}$$

$$\imath_k R^{(0)\,a}_{(-)}{}_b \stackrel{\mathcal{BH}}{=} 0\,. \tag{3.82d}$$

(3.82e)

Eq. (3.82a) is actually true over the whole spacetime, by assumption. From Eq. (3.82c) and the definition of the YM momentum map $P_k{}^A$ we find that

$$\mathcal{D}P_k{}^A \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.83}$$

which tells us that the horizontal YM momentum map $P_k{}^A$ is, at the same time, a vertical Killing vector field on $\mathcal{BH}$. This is what we need in order to have an associated conserved charge there (see the discussion in Section 3.2.2).

Analogously, from Eq. (3.82d) and the definition of the momentum map $P_{(-)k}{}^a{}_b$ Eq. (3.56) we get

$$\mathcal{D}^{(0)}_{(-)} P_{(-)k}{}^a{}_b \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.84}$$

which tells us that the horizontal Lorentz momentum map $P_k{}^A$ is, also, a vertical Killing vector field on $\mathcal{BH}$.

Observe that the last two equations have as consequence the existence of the gauge-invariant 1-forms Ψ_{P_k} and Π_{P_k} defined by

$$d\Pi_{P_k} \stackrel{\mathcal{BH}}{=} P_{(-)k}{}^{a}{}_{b} R_{(-)}^{(0)\,b}{}_{a}\,, \tag{3.85a}$$

$$d\Psi_{P_k} \stackrel{\mathcal{BH}}{=} P_{k\,A} F^{A}\,. \tag{3.85b}$$

The closedness of the right-hand sides of these equations on $\mathcal{BH}$, which guarantee the local existence of Ψ_{P_k} and Π_{P_k} there are the restricted generalized zeroth laws for the YM and torsionful spin connecton fields.

Finally, from Eq. (3.82b) and the definition of the KR momentum map Eq. (3.66) plus the above two equations that define Ψ_{P_k} and Π_{P_k} we get

$$d\left[P_k - \frac{\alpha'}{2}\left(\Psi_{P_k} + \Pi_{P_k}\right)\right] \stackrel{\mathcal{BH}}{=} 0\,, \tag{3.86}$$

which is the restricted generalized zeroth law of the KR field.

3.4 The first law

Following Wald [139], we start by defining the *pre-symplectic* $(d-1)$*-form* [214]

$$\omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \delta_1 \mathbf{\Theta}^{(1)}(\varphi, \delta_2\varphi) - \delta_2 \mathbf{\Theta}^{(1)}(\varphi, \delta_1\varphi)\,, \tag{3.87}$$

and the *symplectic form* relative to the Cauchy surface Σ

$$\Omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi) \equiv \int_{\Sigma} \omega^{(1)}(\varphi, \delta_1\varphi, \delta_2\varphi)\,. \tag{3.88}$$

When φ is a solution of the equations of motion $\mathbf{E}_\varphi = 0$, $\delta_1\varphi = \delta\varphi$ is an arbitrary variation of the fields and $\delta_2\varphi = \delta_\xi\varphi$ is their variation under diffeomorphisms [140]

$$\omega^{(1)}(\varphi, \delta\varphi, \delta_\xi\varphi) = \delta\mathbf{J}^{(1)} + d\imath_\xi \mathbf{\Theta}^{(1)\,\prime} = \delta d\mathbf{Q}^{(1)}[\xi] + d\imath_\xi \mathbf{\Theta}^{(1)\,\prime}\,, \tag{3.89}$$

where, in our case, the Noether-Wald $(d-2)$-form charge $\mathbf{Q}^{(1)}$ is given by Eq. (3.81) and $\mathbf{\Theta}'$ is given in Eq. (3.78). Since, on-shell, $\mathbf{\Theta}^{(1)} = \mathbf{\Theta}^{(1)\,\prime}$, we have that, if $\delta\varphi$ satisfies the linearized equations of motion, $\delta d\mathbf{Q}^{(1)} = d\delta\mathbf{Q}^{(1)}$. Furthermore, if the parameter $\xi = k$ generates a transformation that leaves invariant the field configuration, $\delta_k\varphi = 0$,[16] linearity implies that $\omega^{(1)}(\varphi, \delta\varphi, \delta_k\varphi) = 0$, and

$$d\left(\delta\mathbf{Q}^{(1)}[k] + \imath_k \mathbf{\Theta}^{(1)\,\prime}\right) = 0\,. \tag{3.90}$$

Integrating this expression over a hypersurface Σ with boundary $\delta\Sigma$ and using Stokes' theorem we arrive at

[16]Notice that our goal in Section 3.2.3 was, precisely, to construct variations of the fields δ_ξ with that property.

$$\oint_{\delta\Sigma}\left(\delta\mathbf{Q}^{(1)}[k]+\imath_k\mathbf{\Theta}^{(1)\,\prime}\right)=0\,. \tag{3.91}$$

We consider field configurations that describe asymptotically flat, stationary, black-hole spacetimes with bifurcate horizons $\mathcal{H}$ and the Killing vector k is the one whose Killing horizon is the black hole's event horizon. k, then, will be given by a linear combination with constant coefficients Ω^n of the timelike Killing vector associated to stationarity, $t^\mu\partial_\mu$ and the $[\frac{1}{2}(d-1)]$ generators of inequivalent rotations in d spacetime dimensions $\phi_n^\mu\partial_\mu$

$$k^\mu = t^\mu + \Omega^n\phi_n^\mu\,. \tag{3.92}$$

The constant coefficients Ω^n are the angular velocities of the horizon.

The hypersurface Σ to be the space bounded by infinity and the bifurcation sphere $\mathcal{BH}$ on which $k=0$, so $\delta\Sigma$ has two disconnected pieces: a $(d-2)$-sphere at infinity, S^{d-2}_∞, and the bifurcation sphere $\mathcal{BH}$. Then, taking into account that $k=0$ on $\mathcal{BH}$, we obtain the relation

$$\delta\int_{\mathcal{BH}}\mathbf{Q}^{(1)}[k]=\oint_{\mathrm{S}^{d-2}_\infty}\left(\delta\mathbf{Q}^{(1)}[k]+\imath_k\mathbf{\Theta}^{(1)\,\prime}\right). \tag{3.93}$$

As explained in Ref. [140, 161], the right-hand side can be identified with $\delta M - \Omega^m\delta J_n$, where M is the total mass of the black-hole spacetime and J_n are the independent components of the angular momentum.[17]

Using the explicit form of $\mathbf{Q}^{(1)}[k]$, Eq. (3.81), noticing that $-2\imath_a d e^{-2\phi}k_b \stackrel{\mathcal{BH}}{=} 0$ and restoring the overall factor $g_s^{(d)\,2}(16\pi G_N^{(d)})^{-1}$, we find

$$\begin{aligned}\delta\oint_{\mathcal{BH}}\mathbf{Q}^{(1)}[k] ={}& \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\oint_{\mathcal{BH}}(-1)^d e^{-2\phi}\star(e^a\wedge e^b)P_{k\,ab}\\ &+\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\oint_{\mathcal{BH}}(-1)^{d-1}\frac{\alpha'}{2}P_{(-)k}{}^a{}_b\left(e^{-2\phi}\star R_{(-)}^{(0)\,b}{}_a\right)\\ &+\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\oint_{\mathcal{BH}}(-1)^{d-1}\frac{\alpha'}{2}P_{k\,A}e^{-2\phi}\star F^A\\ &-\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}}\int_{\mathcal{BH}}P_k\wedge\left(e^{-2\phi}\star H^{(1)}\right).\end{aligned} \tag{3.94}$$

The right-hand side ot this identity is expected to be of the form $T\delta S+\Phi\delta\mathcal{Q}$ for some charges $\mathcal{Q}$ and potentials Φ. However, when we compare the third and fourth integrals in the right-hand side with the definitions of the YM and KR charges Eqs. (3.41) and (3.28)

[17] When the spacetime has compact dimensions, the d-dimensional mass M is a combination of the lower-dimensional mass and Kaluza-Klein charges. The details depend on the compactification and will be studied elsewhere.

we see that some terms are missing in the integrand of the first and that, in the second, there is no closed or harmonic form in the integrand, since the horizontal KR momentum map is not necessarily closed on $\mathcal{BH}$. We found a similar problem in Ref. [218] and the solution is essentially the same: add and subtract the same term in different integrals in order to complete the integrand of the definition of YM charge and in order to construct a 1-form which is closed in $\mathcal{BH}$.

The 1-form shich is closed on $\mathcal{BH}$ and which contains P_k follows from the restricted generalized zeroth law of the KR field, Eq. (3.86). We must add a term $-\frac{\alpha'}{2}\Psi_{P_k}$ to the fourth integral and substract the same term to the third, which now contains all the terms associated to the YM charge becuase of the restricted generalized zeroth law Eq. (3.83). However, Eq. (3.86) also tells us to add another term $-\frac{\alpha'}{2}\Pi_{P_k}$ to the fourth integral and we can only compensate by subtracting it to the second. This completes the closed 1-form in the fourth integral and completes the integrand of the Lorentz charge according to Eq. (3.47) and thanks to the restricted generalized zeroth law Eq. (3.84).

The result of these additions and subtractions is

$$
\begin{aligned}
\delta \int_{\mathcal{BH}} \mathbf{Q}^{(1)}[k] = {} & \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^d e^{-2\phi} \star (e^a \wedge e^b) P_{k\,ab} \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Pi_{P_k} + (-1)^d \Pi_{P_k} \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right] \\
& + \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Psi_{P_k} + (-1)^d \Psi_{P_k} \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right] \\
& - \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} \left[P_k - \frac{\alpha'}{2}(\Psi_{P_k} + \Pi_{P_k})\right] \wedge \left(e^{-2\phi} \star H^{(1)}\right) .
\end{aligned}
\tag{3.95}
$$

where Ψ_{P_k} and Π_{P_k} satisfy Eqs. (3.85b) and (3.85a), respectively, whose integrability is guaranteed by the fact that the YM and Lorentz momentum maps are covariantly constant on $\mathcal{BH}$ (the restricted generalized zeroth laws).

Now, let us assume that the particular field configuration under consideration admits a set of covariantly constant YM parameters on $\mathcal{BH}$ that we label with an index I, $\kappa_I{}^A$

$$
\mathcal{D}\kappa_I^A \stackrel{\mathcal{BH}}{=} 0\,, \quad \Rightarrow \quad P_k{}^A \stackrel{\mathcal{BH}}{=} \Phi^I \kappa_I^A\,, \tag{3.96}
$$

where the constants Φ^I will be interpreted as the potentials associated to the YM charges $\mathcal{Q}_I$ computed with the parameter $\kappa_I{}^A$ Eq. (3.41)

$$
\mathcal{Q}_I \equiv \mathcal{Q}[\kappa_I] = \frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \int_{\mathcal{BH}} (-1)^{d-1} \frac{\alpha'}{2} \left[e^{-2\phi} \star d\Psi_I + (-1)^d \Psi_I \wedge \left(e^{-2\phi} \star H^{(0)}\right)\right], \tag{3.97}
$$

where

$$d\Psi_I = -\kappa_{I\,A} F^A \,. \tag{3.98}$$

As a result, the third line in Eq. (3.95) becomes $\Phi^I \delta \mathcal{Q}_I$.

Now, following Refs. [147, 161], as a consequence of the KR restricted generalized zeroth law Eq. (3.86), we can write (Hodge decomposition)

$$P_k - \frac{\alpha'}{2}\left(\Psi_{P_k} + \Pi_{P_k}\right) \stackrel{\mathcal{BH}}{=} de + \Phi^i \Lambda_{h\,i} \,, \tag{3.99}$$

where e is some function, the $\Lambda_{h\,i}$ are the harmonic 1-forms of the bifurcation sphere and the Φ^i are constants that can be interpreted as the potentials associated to the KR charges $\mathcal{Q}_i = \mathcal{Q}(\Lambda_{h\,i})$ Eq. (3.28)

$$\mathcal{Q}_i = -\frac{g_s^{(d)\,2}}{16\pi G_N^{(d)}} \oint_{C_{\Lambda_{h\,i}}} e^{-2\phi} \star H \,, \tag{3.100}$$

where $C_{\Lambda_{h\,i}}$ is the $(d-3)$-cycle dual to the harmonic 1-form $\Lambda_{h\,i}$ in $\mathcal{BH}$.

As a result, the fourth line in Eq. (3.95) becomes $\Phi^i \delta \mathcal{Q}_i$ and we are left with the first two, which are linear in the Lorentz momentum map $P_k{}^{ab}$, which, on $\mathcal{BH}$, is given by κn^{ab}, where n^{ab} is the binormal to the horizon. The terms in those two lines must, therefore, be interpreted as those giving rise to the term $T\delta S$ in the first law

$$\delta M = T\delta S + \Phi^I \delta \mathcal{Q}_I + \Phi^i \delta \mathcal{Q}_i + \Omega^n \delta J_n \,. \tag{3.101}$$

3.5 Wald entropy

It follows from the results of the previous section that the entropy is given by

$$S = (-1)^d \frac{g_s^{(d)\,2}}{8G_N^{(d)}} \int_{\mathcal{BH}} e^{-2\phi} \left\{ \left[\star(e^a \wedge e^b) + \frac{\alpha'}{2} e^{-2\phi} \star R_{(-)}^{(0)\,ab} \right] n_{ab} + (-1)^d \frac{\alpha'}{2} \Pi_n \wedge \star H^{(0)} \right\} , \tag{3.102}$$

where we have the defined the 1-form Π_n (vertical Lorentz momentum map associated to the binormal) on the bifurcation sphere

$$d\Pi_n \stackrel{\mathcal{BH}}{=} R_{(-)}^{(0)\,ab} n_{ab} \,. \tag{3.103}$$

This is the main result of this paper, which we will discuss in the next section. It is worth stressing that the term that involves Π_n, and which has been shown to given an important contribution to the entropy of well-known black-hole solutions Refs. [198, 199, 210, 220, 221] occurs in the entropy formula just to cancel an equivalent term that we had to add to get the correct definition of the KR charge and the associated potential. Without a detailed knowledge of the conserved charges, the restricted generalized zeroth laws and the potentials associated, the presence of that term in the entropy formula could not have been guessed.

3.6 Discussion

In this paper we have derived an entropy formula for the black-hole solutions of the Heterotic Superstring effective action to first order in α' using Wald's formalism [139, 214] taking carefully into account all the symmetries of the theory. A a result, our entropy formula Eq. (3.102) is manifestly gauge invariant. In particular, it is manifestly invariant under local Lorentz transformations.

It is interesting to compare this result with the one that would follow form the direct (and naive) application of the Iyer-Wald prescription [140]. The first two terms in Eq. (3.102) can be obtained from Eq. (3.8) by varying the Einstein-Hilbert term and the $R^2_{(-)}$ term with respect to the Riemann curvature tensor, but the third term cannot be obtained in that way from the H^2 term. As stressed in Refs. [198, 199, 210], the variation of this term with respect to the Riemann tensor gives a term of the form

$$\frac{\alpha'}{4}e^{-2\phi}\left(\Omega^{(0)}_{(-)}{}^{ab}n_{ab}\right)\wedge\star H^{(0)}\,, \tag{3.104}$$

which is not Lorentz-covariant. The coefficient of this term differs from the last term in Eq. (3.102) if we associate Π_n to $\Omega^{(0)}_{(-)}{}^{ab}n_{ab}$, which is the right thing to do as we are going to show. But this coefficient changes after dimensional reduction, as observed in Ref. [212]. The explicit calculation in Ref. [210] shows that the right coefficient is the one that arises after dimensional reduction,[18] but, certainly, there are ambiguities in the way in which the Chern-Simons terms are defined in lower dimensions.

It is interesting to observe that because $\mathcal{D}n_{ab}\stackrel{\mathcal{BH}}{=}0$,

$$d\Pi_n\stackrel{\mathcal{BH}}{=}d\left(\Omega^{(0)}_{(-)}{}^{ab}n_{ab}\right)+\Omega^{(0)}_{(-)}{}^{a}{}_{c}\wedge\Omega^{(0)}_{(-)}{}^{cb}n_{ab}\,. \tag{3.106}$$

For the non-extremal Reissner-Nordström black hole of Ref. [211], whose α' corrections were computed in Ref. [210], the second term vanishes identically in the tangent space basis used (see Appendix C). This shows that, in that basis, our entropy formula and the entropy formula obtained via the Iyer-Wald prescription (after dimensional reduction) give the same result. Of course, our formula is valid in any basis.

Our entropy formula seems to differ from the entropy formula obtained in Ref. [224], but a detailed comparison is not possible since that formula contains undetermined parameters that guarantee its invariance under Lorentz transformations. In Ref. [224] it was argued that those undetermined parameters do not contribute to the entropy in certain cases but, without an explicit expression, it is difficult to understand why or when this may happen. Furthermore, as we have shown, the identification of the entropy formula can only be made after the first law of black hole mechanics has been proven and this requires a careful identification of the conserved charges of the theory: some terms (the one involving Π_n) occur in the entropy formula only because they are needed to compensate other terms that have to appear in the correct definition of the KR charge. This analysis was simply not carried out in Ref. [224].

[18] The entropy calculated in this way satisfies the first law or, equivalently, the thermodynamic relation

$$\frac{\partial S}{\partial M}=\frac{1}{T}\,. \tag{3.105}$$

Our entropy formula (the contribution due to the presence of Lorentz- or gravitational Chern-Simons terms in $H^{(1)}$) also differs from the one found in Ref. [197]. Observe that Eq. (40) in Ref. [197], similar to the terms contains in the formulae derived in Refs. [198, 199] and to Eq. (3.104) is not covariant. Thus, it may give the right result in certain basis, if at all.[19] The problems in the derivation of Ref. [197] are having overlooked the KR conserved charge and the determination of the gauge parameters that generate symmetries of the complete field configuration.

Finally, it is interesting to notice that the entropy formula looks like the charge associated to the Lorentz transformations generated by the binormal to the horizon. These transformations preserve the connections ω and $\Omega^{(0)}_{(-)}$ on the bifurcation sphere, but they do not preserve the Vielbein, as we assumed in Section 3.2.2 (Eq. (3.43)), which produces an additional term associated to the Einstein-Hilbert term.

The main use of the entropy formula that we have found is to put in solid ground the calculations of the macroscopic entropies of α'-corrected black holes, an ineluctable condition for a fair comparison with the microscopic ones. More α'-corrected solutions have recently become available to this end [225, 226]. As mentioned in the introduction, another necessary ingredient for this comparison is the correct identification of the relation between the charges of the black hole and the branes in the string background. These results and those of our previous work [218] single out a very precise definition of the conserved charges, which turn out to be of *Page type*, conserved and gauge-invariant under the assumptions made. This fact should shed light on this problem and we intend to pursue this line of research in future work.

[19] the non-covariance of Tachikawa's entropy formula was observed in Ref. [200], where an alternative method was devised to deal with this problem. Nevertheless, the formula obtained in Ref. [200] reduces to Tachikawa's in $\mathcal{BH}$, apparently losing the covariance, while ours does not.

4

Komar Integral and Smarr Formula for Axion-Dilaton Black Holes Versus S Duality

This chapter is based on:
Komar integral and Smarr formula for axion-dilaton black holes versus S duality
D. Mitsios, T. Ortín, D. Pereñiguez
JHEP 08 (2021) 019 (arXiv:2106.07495)

In Refs. [139, 140, 214] Wald and his collaborators Lee and Iyer constructed a powerful formalism that could be used to prove the first law of black-hole mechanics [118] and, through this proof, to find the entropy formula for black-hole solutions of any diffeomorphism-invariant theory. This formalism has been very successful in absence of matter fields but it was not clear how to use it on their presence. It is known that, in many cases, these fields give rise to new terms in the first law, associated to the possible variations of the conserved charges associated to them. It was unclear how these terms could arise in this formalism since it is based in diffeomorphism invariance alone and, apparently, the gauge symmetries that ensure the conservation of the charges that occur in the additional terms of the first law play no rôle whatsoever.

As we have discussed in Refs. [162, 209, 218], diffeomorphisms and gauge transformations are, actually, closely related, because gauge fields are not just tensors. This was one of the main assumptions in the derivation of the well-known Iyer-Wald prescription for the entropy Refs. [140]. The transformation of a gauge field under an isometry which leaves invariant all the fields of a black-hole solution always induces a gauge transformation, which, when correctly taken into account [138] (via *covariant Lie derivatives*, for instance), gives rise to the missing terms in the first law. If one uses a tetrad formulation, although the Vielbein is not a matter field, one must properly take into account that it transforms under local Lorentz transformations as well [150] using the Lie-Lorentz covariant derivative (see Refs. [151–153] and references therein).

Still, terms associated to the variations of charges which are not associated to gauge symmetries, such as magnetic charges, will not appear in these derivations of the first law based on Wald's formalism, while they are known to appear in other derivations of the first law [227]. Terms associated to the variations of the asymptotic values of the scalars (*moduli*) such as those found in Ref. [149] (see, also, Ref. [216]), will not appear, either. This fact does not invalidate the first law, but it is a limitation to its applicability since one cannot study the effects of the variations of the missing charges.

Smarr formulae [228] provide another approach to this problem. They are closely related to the first law: the scaling arguments of Refs. [164, 229] show how the thermodynamical variables (typically, charges) and their conjugate thermodynamical potentials must occur in the Smarr formula. This argument explains why there are no terms associated to the moduli in the first law if one accepts that the black-hole mass does not depend on them when it is expressed in terms of the entropy and the conserved charges.[1]

If the black holes under consideration have magnetic charges, then their Smarr formula must contain a term proportional to them and their associated potentials.

As explained in Refs. [164,229], Smarr formulae can be derived from Komar integrals [232]. In Ref. [233] it was shown how to construct Komar integrals in general theories using Wald's formalism. The integrand contains a surface term which is the Noether-Wald charge and a volume term proportional to the on-shell Lagrangian density. As shown in Ref. [141], the volume term can always be expressed as a surface term. Since the variation of the integral the Noether-Wald charge gives the first law without variations of magnetic charges and since, as we have argued, the Smarr formula must contain terms with magnetic charges and potentials, it is not clear how and if those terms are going to appear. Moreover, electric and magnetic terms must occur in an electric-magnetic symmetric form in the Smarr formula if the equations of motion of the theory have that property.

In this paper we want to study if and how this electric-magnetic duality invariance of the Smarr formula arises from a formalism (Wald's) which is not electric-magnetic symmetric because only the gauge transformations which imply the conservation of the electric charges are taken into account. To this order, in Section 4.1, we are going to study the static black-hole solutions of a 4-dimensional theory whose equations of motion are invariant under the archetype of electric-magnetic (or S-) duality group: "axion-dilaton gravity," which is the bosonic sector of pure, ungauged, $\mathcal{N}=4, d=4$ supergravity [234]. The family of solutions that we are going to study, found in Ref. [235] is invariant, as a family, under the $\mathrm{SL}(2,\mathbb{Z})$ duality group and the results obtained should be automatically invariant under that group. These solutions will be introduced in Section 4.2. In Section 4.3 we will construct the Komar integral as a surface integral in a manifestly gauge and diffeomorphism-covariant form using the momentum maps introduced in Refs. [162, 209, 218]. In Section 4.4 we will use the Komar integral to explicitly test the Smarr formula for the static axion-dilaton black holes under consideration. A general form of the Smarr formula will, then, be given in Section 4.5, where we will discuss its electric-magnetic $\mathrm{SL}(2,\mathbb{R})$ invariance. Finally, Section 3.6 contains our conclusions and some directions for future work.

4.1 Axion-dilaton gravity

The 4-dimensional model known as "axion-dilaton gravity" is nothing but the bosonic sector of pure, ungauged, $\mathcal{N}=4, d=4$ supergravity [234] and describes two scalars: the axion a and the dilaton ϕ combined into the complex *axidilaton* field $\lambda \equiv a + ie^{-2\phi}$ (often denoted by τ) that parametrizes the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, and six 1-form fields $A^m = A^m{}_\mu dx^\mu$ with 2-form field strengths

$$F^m = dA^m \,, \tag{4.1}$$

coupled to gravity, which we will describe through the Vierbein $e^a = e^a{}_\mu dx^\mu$. The number

[1]This fact follows from the independence of the entropy on the moduli, which, to the best of our knowledge, has been proven for static, extremal, asymptotically-flat black holes only [230, 231].

of 1-forms does not play a relevant rôle if it is larger than one, and can be left undetermined although it has to be set to six if one wants to embed the solutions of the theory into the Heterotic Superstring (HST) effective action compactified on a T^6. The model with just two 1-forms can also be viewed as a model of $\mathcal{N}=2, d=4$ supergravity coupled to a single vector multiplet, and one can use the powerful solution-generating techniques developed in that class of models to construct extremal [236,237] and non-extremal [237,238] black-hole solutions.

The action of the theory in the conventions of Ref. [239][2] in differential-form language is (summation over repeated m indices is understood)

$$
\begin{aligned}
S = \frac{1}{16\pi G_N^{(4)}} \int \Big[& -\star(e^a \wedge e^b)\wedge R_{ab} + 2d\phi\wedge\star d\phi + \tfrac{1}{2}e^{4\phi}da\wedge\star da \\
& + 2e^{-2\phi}F^m\wedge\star F^m + 2aF^m\wedge F^m \Big] \\
\equiv \int \mathbf{L}\, .
\end{aligned}
\tag{4.2}
$$

We will set $G_N^{(4)}=1$ and we will ignore the normalization factor $(16\pi)^{-1}$ for the time being.

The equations of motion are defined by

$$
\delta S = \int \{\mathbf{E}_a \wedge \delta e^a + \mathbf{E}_\phi \delta\phi + \mathbf{E}_{(a)}\delta a + \mathbf{E}_m \wedge A^m + d\mathbf{\Theta}(\varphi,\delta\varphi) \quad , \tag{4.3}
$$

and given by

$$
\begin{aligned}
\mathbf{E}_a = {} & \imath_a \star(e^b\wedge e^c)\wedge R_{bc} + 2\left(\imath_a d\phi \star d\phi + d\phi\wedge \imath_a \star d\phi\right) \\
& + \tfrac{1}{2}e^{4\phi}\left(\imath_a da\star da + da\wedge\imath_a\star da\right) + 2e^{-2\phi}\left(\imath_a F^m\wedge\star F^m - F^m\wedge\imath_a\star F^m\right) ,
\end{aligned}
\tag{4.4a}
$$

$$
\mathbf{E}_\phi = -4d\star d\phi + 2e^{4\phi}da\wedge\star da - 4e^{-2\phi}F^m\wedge\star F^m\, , \tag{4.4b}
$$

$$
\mathbf{E}_{(a)} = -d\left(e^{4\phi}\star da\right) + 2F^m\wedge F^m\, , \tag{4.4c}
$$

$$
\mathbf{E}_m = -4dF_m\, , \tag{4.4d}
$$

where we have defined the dual 2-form field strength

$$
F_m \equiv \tfrac{1}{4}\frac{\delta S}{\delta F^m} = e^{-2\phi}\star F^m + aF^m\, . \tag{4.5}
$$

[2] The only difference with the conventions of Refs. [235,240–243] is that no imaginary units are introduced with the Hodge dualization. These conventions are the same used in Refs. [162,209,218].

Furthermore,

$$\mathbf{\Theta}(\varphi, \delta\varphi) = -\star(e^a \wedge e^b) \wedge \delta\omega_{ab} + 4 \star d\phi\delta\phi + e^{4\phi} \star da\delta a + 4F_m \wedge \delta A^m \,. \tag{4.6}$$

Since the Maxwell equations tell us that the F_ms are closed on-shell, we can introduce a dual 1-form field A_m defined by

$$F_m = dA_m \,. \tag{4.7}$$

4.2 Static dilaton-axion black hole solutions

The most general family of non-extremal, static, black holes with non-trivial dilaton, axion and electromagnetic fields was obtained in Ref. [235].[3] In the notation of Ref. [239], these solutions take the form[4]

$$\begin{aligned}
ds^2 &= e^{2U}dt^2 - e^{-2U}dr^2 - R^2 d\Omega^2_{(2)}\,,\\
\lambda &= \frac{\lambda_\infty r + \lambda^*_\infty \Upsilon}{r+\Upsilon}\,,\\
A^m{}_t &= e^{\phi_\infty} R^{-2}[\Gamma^m(r+\Upsilon) + \text{c.c.}]\,,\\
A_{m\,t} &= e^{\phi_\infty} R^{-2}[\Gamma^m(\lambda_\infty r + \lambda^*_\infty\Upsilon) + \text{c.c.}]\,,
\end{aligned} \tag{4.8}$$

where the functions that occur in the metric are

$$\begin{aligned}
e^{2U} &= R^{-2}(r-r_+)(r-r_-)\,, \qquad & r_\pm &= M \pm r_0\,,\\
R^2 &= r^2 - |\Upsilon|^2\,, & r_0^2 &= M^2 + |\Upsilon|^2 - 4\Gamma^m\Gamma^{m\,*}\,.
\end{aligned} \tag{4.9}$$

In these functions, M is the ADM mass, the constants Γ^m are related to the complex electromagnetic charges, $\lambda_\infty = a_\infty + ie^{-2\phi_\infty}$ is the asymptotic value of the axidilaton and

[3] These solutions were obtained by an $SL(2,\mathbb{R})$ rotation of those found in Ref. [241]. The case with a single 1-form had been dealt with in Ref. [244], but it is qualitatively different since these solutions can have electric and magnetic charges and vanishing axion. In their turn, the solutions of Ref. [241] are a generalization of those in Ref. [240], which were originally discovered by Gibbons and Maeda in Refs. [245,246]. The single-vector case was rediscovered by Garfinkle, Horowitz and Strominger in Ref. [247] and it is the solution on which the $SL(2,\mathbb{R})$ rotation was performed in Ref. [244]. Stationary generalizations (inclusion of NUT charge) were constructed in [242] and, for the extremal case, using supersymmetry and spinorial techniques, in Ref. [248] (see also Ref. [243].) Finally, the most general, non-extremal, stationary black-hole solution of the model was constructed in Ref. [239].

[4] This presentation of the solutions uses only the time components of the original and dual vector fields. As we are going to see, this information in enough to fully reconstruct all the components of these vectors.

$\Upsilon = \Sigma + i\Delta$ is the axidilaton charge. All these parameters are defined by the asymptotic expansions

$$g_{tt} \sim 1 - \frac{2M}{r}\,, \tag{4.10a}$$

$$\lambda \sim \lambda_\infty - ie^{-2\phi_\infty}\frac{2\Upsilon}{r}\,, \tag{4.10b}$$

$$\tfrac{1}{2}\left[F^m{}_{tr} + i \star F^m{}_{tr}\right] \sim \frac{e^{+\phi_\infty}\Gamma^m}{r^2} = \frac{e^{+\phi_\infty}(Q^m + iP^m)/2}{r^2}\,. \tag{4.10c}$$

The axidilaton charge is not an independent parameter. In accordance with the no-hair theorem, it is a function of the ADM mass and the electric and magnetic charges

$$\Upsilon = -\frac{2}{M}\Gamma^{m\,*}\Gamma^{m\,*}\,. \tag{4.11}$$

The singularity is hidden under a horizon located at $r = r_+$ if $r_0^2 > 0$, and it is hidden or coincides with it (but still is invisible for external observers) if $r_0 = 0$.

The solution has been expressed, by convenience, using only the electric components of the 1-forms and the dual 1-forms. The magnetic components can be obtained as follows. From the definition of the dual 2-form field strengths Eq. (4.5), we get

$$F_{m\,rt} = \frac{e^{-2\phi}}{R^2\sin\theta}F^m{}_{\theta\varphi} + aF^m{}_{rt}\,, \tag{4.12}$$

so

$$F^m{}_{\theta\varphi} = e^{2\phi}R^2\sin\theta\,(F_{m\,rt} - aF^m{}_{rt}) = 2e^{\phi_\infty}\Im\mathrm{m}(\Gamma^m)\sin\theta\,. \tag{4.13}$$

The gauge field A^m, then, has to be defined in two patches. On the $z \geq -\epsilon$ patch it is given by the 1-form

$$A^{m\,+} = e^{\phi_\infty}R^{-2}[\Gamma^m(r+\Upsilon) + \text{c.c.}]dt + 2e^{\phi_\infty}\Im\mathrm{m}(\Gamma^m)(1-\cos\theta)d\varphi\,, \tag{4.14}$$

which is regular in that region[5] and in the $z \leq +\epsilon$ patch, it is given by the 1-form

$$A^{m\,-} = e^{\phi_\infty}R^{-2}[\Gamma^m(r+\Upsilon) + \text{c.c.}]dt - 2e^{\phi_\infty}\Im\mathrm{m}(\Gamma^m)(1+\cos\theta)d\varphi\,, \tag{4.15}$$

which is also regular in that patch. $A^{m\,+}$ and $A^{m\,-}$ differ by the gauge transformation

$$A^{m\,+} - A^{m\,-} = d\left[4e^{\phi_\infty}\Im\mathrm{m}(\Gamma^m)\varphi\right]. \tag{4.16}$$

We can also compute the complete dual vector fields. From the definition Eq. (4.5) we find that

[5] The Dirac string singularity of this 1-form lies in the negative z axis.

$$F_{m\,\theta\varphi} = e^{-2\phi}R^2\sin\theta F^m{}_{tr} + aF^m{}_{\theta\varphi} = 2e^{\phi_\infty}\left\{e^{-2\phi_\infty}\Re\mathrm{e}(\Gamma^m) + a_\infty\Im\mathrm{m}(\Gamma^m)\right\}\sin\theta\,, \quad (4.17)$$

and

$$\begin{aligned} A_m{}^+ &= e^{\phi_\infty}R^{-2}[\Gamma^m(\lambda_\infty r + \lambda^*_\infty\Upsilon) + \text{c.c.}]dt \\ &\quad + 2e^{\phi_\infty}\left\{e^{-2\phi_\infty}\Re\mathrm{e}(\Gamma^m) + a_\infty\Im\mathrm{m}(\Gamma^m)\right\}(1-\cos\theta)d\varphi\,, \quad (4.18a) \\ A_m{}^- &= e^{\phi_\infty}R^{-2}[\Gamma^m(\lambda_\infty r + \lambda^*_\infty\Upsilon) + \text{c.c.}]dt \\ &\quad - 2e^{\phi_\infty}\left\{e^{-2\phi_\infty}\Re\mathrm{e}(\Gamma^m) + a_\infty\Im\mathrm{m}(\Gamma^m)\right\}(1+\cos\theta)d\varphi\,, \quad (4.18b) \end{aligned}$$

in the same two patches, and

$$A_m{}^+ - A_m{}^- = d\left\{4e^{\phi_\infty}\left[e^{-2\phi_\infty}\Re\mathrm{e}(\Gamma^m) + a_\infty\Im\mathrm{m}(\Gamma^m)\right]\varphi\right\}. \quad (4.19)$$

The Hawking temperature and Bekenstein-Hawking entropy of these black holes are given by

$$T = \frac{1}{4\pi}\partial_r g_{tt}(r_+) = \frac{r_0}{2\pi R^2(r_+)}\,, \quad (4.20a)$$

$$S = \pi R^2(r_+)\,. \quad (4.20b)$$

Observe that, as usual in 4-dimensional, static black holes

$$2ST = r_0\,. \quad (4.21)$$

Then, it is not difficult to find a Smarr-type relation adding the ADM mass to the above relation:

$$\begin{aligned} M &= 2ST + M - r_0 = 2ST + r_- = 2ST + \frac{r_-r_+}{r_+} = 2ST + \frac{M^2 - r_0^2}{r_+} \\ &= 2ST + \frac{4\Gamma^m\Gamma^{m\,*} - |\Upsilon|^2}{r_+} = 2ST + \frac{Q^m}{r_+}Q^m + \frac{P^m}{r_+}P^m - \frac{\Sigma}{r_+}\Sigma - \frac{\Delta}{r_+}\Delta\,. \end{aligned} \quad (4.22)$$

This relation is correct (by construction) and, looking at it, it is tempting to conclude that the $1/r_+$ terms (including those associated to the scalar charges) can immediately be identified with potentials on the horizon. However, as we are going to see, r_- can be rewritten in other ways in which only potentials associated to the electric and magnetic charges occur. Note that the usual scaling argument does not allow for terms including scalar charges or potentials because, by the no-hair theorem, these cannot be independent. Indeed, the Komar charge leaves only room for electric and magnetic potentials and charges, and, as we are going to see, the integral gives the above relation, although in a highly non-trivial way.

4.3 Komar integral

As explained, for instance, in Refs. [164, 229] Smarr formulae [228] can be systematically obtained from Komar integrals [232]. These can be constructed using Wald's formalism following Ref. [233], rewriting the volume integral terms as surface terms as explained in Ref. [141]. In that reference, though, the integrand of the surface integral was determined after explicit evaluation of the Lagrangian density on a particular family of solutions and, here, we are going to show how that integrand can be found in general.[6]

Let us review the construction of the Komar charge and integral in Ref. [141,233]. It is not difficult to see that, on-shell[7] and for a Killing vector k that generates a symmetry of the whole field configuration

$$\mathbf{J}[k] \doteq \imath_k \mathbf{L} \,. \tag{4.23}$$

On the other hand, for any vector field ξ, we have the off-shell (local) identity

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] \,. \tag{4.24}$$

Combining these two relations, we find that, on-shell and for a Killing vector k that generates a symmetry of the whole field configuration

$$d\mathbf{Q}[k] - \imath_k \mathbf{L} \doteq 0 \,. \tag{4.25}$$

However, if k generates a symmetry of the whole field configuration,

$$0 \doteq \pounds_k \mathbf{L} = d\imath_k \mathbf{L} \,, \tag{4.26}$$

which implies the local existence of a $(d-2)$-form ω_k such that

$$d\omega_k \doteq \imath_k \mathbf{L} \,. \tag{4.27}$$

It follows that, under the aforementioned conditions,

$$d\left\{\mathbf{Q}[k] - \omega_k\right\} \doteq 0 \,. \tag{4.28}$$

and we can define the Komar integral over the codimension-2 surface Σ^{d-2} Ref. [141]

[6] It is assumed, though, that we are restricting ourselves to solutions admitting a timelike Killing vector with a Killing horizon.

[7] We are going to use the symbol $\doteq$ for identities that only hold on-shell.

$$\mathcal{K}(\Sigma^{d-2}) = (-1)^{d-1} \oint_{\Sigma^{d-2}} \{\mathbf{Q}[k] - \omega_k\} \,. \tag{4.29}$$

Smarr formulae for black-hole spacetimes are obtained by integrating the identity Eq. (4.28) on hypersurfaces Σ with boundaries at the horizon and spatial infinity $\partial\Sigma_h$ (usually, the bifurcation surface) and $\partial\Sigma_\infty$, respectively upon use of Stokes theorem:

$$\mathcal{K}(\partial\Sigma_\infty) = \mathcal{K}(\partial\Sigma_h) \,. \tag{4.30}$$

Using the techniques developed in Refs. [162,209,218] and some of the results found in them, we can readily find the Noether-Wald charge for axion-dilaton gravity:

$$\mathbf{Q}[\xi] = \star(e^a \wedge e^b) e^{-2\phi} P_{\xi\, ab} - 4P^m{}_\xi F_m \,. \tag{4.31}$$

Here,

$$P_{\xi\, ab} = \nabla_{[a}\xi_{b]} \,. \tag{4.32}$$

Also, the functions $P^m{}_\xi$ can be understood as the parameters of compensating gauge transformations of the 1-forms with the property that, when $\xi = k$, they satisfy the relations

$$dP^m{}_k = -\imath_k F^m \,, \tag{4.33}$$

that define the *momentum maps* associated to the Killing vector k and the gauge fields A^m. Although this is a gauge-invariant definition, these objects are defined up to an additive constant. Since they can be interpreted as electrostatic potentials, the constant can be determined by a sensible boundary condition, such as the vanishing of the potentials at spatial infinity.

In order to compute ω_k, we have to determine the on-shell value of the Lagrangian density $\mathbf{L}$ first, for a generic solution. In this case, it is enough to use the trace of the Einstein equations Eqs. (4.4a). In differential-form language, to take the trace we must compute $e^a \wedge \mathbf{E}_a$, taking into account that, for a p-form $\omega^{(p)}$,

$$e^a \wedge \imath_a \omega^{(p)} = p\omega^{(p)} \,. \tag{4.34}$$

We get

$$\begin{aligned} e^a \wedge \mathbf{E}_a &= -2\left\{-e^{-2\phi} \star (e^c \wedge e^d) \wedge R_{cd} + 2d\phi \wedge \star d\phi + \tfrac{1}{2} e^{4\phi} da \wedge \star da\right\} \\ &= -2\left\{\mathbf{L} - 2e^{-2\phi} F^m \wedge \star F^m - 2aF^m \wedge F^m\right\} , \end{aligned} \tag{4.35}$$

so

$$\mathbf{L} \doteq 2e^{-2\phi} F^m \wedge \star F^m + 2aF^m \wedge F^m = 2F^m \wedge F_m \,, \tag{4.36}$$

and

$$\imath_k \mathbf{L} \doteq 2\imath_k F^m \wedge F_m + 2F^m \wedge \imath_k F_m \,. \tag{4.37}$$

In order to find ω_k for general configurations, we are going to use the definition of the (*electric*) momentum maps Eq. (4.33) but we need to define their magnetic duals. Since, by assumption, the dual field strengths are left invariant by the isometry generated by k,

$$0 = \pounds_k F_m = d\imath_k F_m + \imath_k dF_m \doteq d\imath_k F_m \,, \tag{4.38}$$

where we have used the Maxwell equations. Then, locally, there are functions $P_{m\,k}$ (*magnetic momentum maps*) such that

$$dP_{m\,k} \doteq -\imath_k F_m \,. \tag{4.39}$$

Thus, upon use of the Maxwell equations and Bianchi identities,

$$\imath_k \mathbf{L} \doteq -2dP^m{}_k \wedge F_m - 2F^m \wedge dP_{m\,k} \doteq d\left\{-2P^m{}_k F_m - 2F^m P_{m\,k}\right\} = d\omega_k \,, \tag{4.40}$$

and the Komar charge is given by

$$\mathbf{Q}[k] - \omega_k = \star(e^a \wedge e^b) e^{-2\phi} P_{k\,ab} - 2\left(P^m{}_k F_m - P_{m\,k} F^m\right) \,. \tag{4.41}$$

Observe that the electromagnetic terms occur in a symplectic-invariant combination now. This hints at the electric-magnetic ($\mathrm{SL}(2,\mathbb{R})$) invariance of the Komar charge, a fact that we will study in Section 4.5. Before studying this invariance, we are going to check the validity of this formula in the family of static black holes introduced in Section 4.2 by direct computation of the Komar integral.

4.4 Checking the Smarr formula for static axion-dilaton black holes

Now we want to compute the Komar integrals over the bifurcation sphere on the horizon and over a sphere at spatial infinity for the static axion-dilaton black holes introduced in Section 4.2. Thus, we are interested in the $\theta\varphi$ components of the integrand only. We compute them term by term and we recover the normalization factor $(16\pi)^{-1}$. First,

$$\star\,(e^a \wedge e^b) P_{k\,ab} = \frac{1}{2\sqrt{|g|}} \varepsilon_{\mu\nu\rho\sigma} \nabla^\mu k^\nu dx^\rho \wedge dx^\sigma \,, \tag{4.42}$$

and, for these solutions

$$\nabla^\mu k^\nu = \delta^{t[\mu} \delta^{\nu]r} \partial_r e^{2U} \,, \tag{4.43a}$$

$$\star(e^a \wedge e^b) P_{k\,ab} = -r^2 \partial_r e^{2U} \sin\theta d\theta \wedge d\varphi \,. \tag{4.43b}$$

The electric and magnetic momentum maps can be taken to be

$$P^m{}_k = A^m{}_t\,, \qquad P_{m\,k} = A_{m\,t}\,, \tag{4.44}$$

and, the second term in the Komar charge Eq. (4.41) is (only $\theta\varphi$ components)

$$\begin{aligned} -2\left(P^m{}_k F_{m\,\theta\varphi} - P_{m\,k}F^m{}_{\theta\varphi}\right) &= -2\left\{A^m{}_t\left[e^{-2\phi}\left(\star F^m\right)_{\theta\varphi} + aF^m{}_{\theta\varphi}\right] - A_{m\,t}F^m{}_{\theta\varphi}\right\} \\ &= \left\{2R^2 e^{-2\phi}A^m{}_t\partial_r A^m{}_t + 4e^{\phi_\infty}\left(A_{m\,t} - aA^m{}_t\right)\Im\mathrm{m}(\Gamma^m)\right\}\sin\theta\,. \end{aligned} \tag{4.45}$$

Integrating over a 2-sphere of constant radius r, we get

$$\mathcal{K}(S^2{}_r) = \tfrac{1}{4}r^2\partial_r e^{2U} - \tfrac{1}{2}R^2 e^{-2\phi}A^m{}_t\partial_r A^m{}_t - e^{\phi_\infty}\left(A_{m\,t} - aA^m{}_t\right)\Im\mathrm{m}(\Gamma^m)\,. \tag{4.46}$$

At infinity, only the first term contributes, giving

$$\mathcal{K}(S^2{}_\infty) = M/2\,. \tag{4.47}$$

Over the bifurcation sphere[8], the first term gives $ST = r_0/2$, but we have to evaluate carefully the second and third terms. We introduce some notation:

$$A \equiv \lambda_\infty r + \lambda^*_\infty\Upsilon\,, \qquad B \equiv r + \Upsilon\,, \quad \Rightarrow \quad \lambda = A/B\,. \tag{4.48}$$

The second term in Eq. (4.46) is

$$\begin{aligned} -\tfrac{1}{2}R^2 e^{-2\phi}A^m{}_t\partial_r A^m{}_t &= \frac{1}{2R^2|r+\Upsilon|^2}\left[\Gamma^m B + \text{c.c}\right]\left[-2|r+\Upsilon|^2\Re\mathrm{e}(\Gamma^m) + 4\Im\mathrm{m}(\Gamma^m)\Im\mathrm{m}(\Upsilon)r\right] \\ &= \frac{1}{2R^2}\left[\Gamma^m(\Gamma^m + \Gamma^{m\,*})B + \text{c.c}\right] \\ &\quad + \frac{r}{R^2|r+\Upsilon|^2}\left[i\Gamma^m(\Gamma^m - \Gamma^{m\,*})B + \text{c.c}\right]\Im\mathrm{m}(\Upsilon)\,. \end{aligned} \tag{4.49}$$

Using the relation Eq. (4.11) it is not hard to see that at $r = r_+$

$$i\Gamma^m(\Gamma^m - \Gamma^{m\,*})B(r_+) + \text{c.c} = -\tfrac{1}{2}R^2(r_+)\Im\mathrm{m}(\Upsilon)\,. \tag{4.50}$$

Then,

[8] Actually, it is enough to set $r = r_+$

$$
\begin{aligned}
-\tfrac{1}{2}R^2 e^{-2\phi} A^m{}_t \partial_r A^m{}_t \Big|_{r_+} &= \frac{2|\Gamma|^2 r_+ - M|\Upsilon|^2}{2R^2(r_+)} - \frac{(Mr_+ - 2|\Gamma|^2)}{2R^2(r_+)} \Re\mathrm{e}(\Upsilon) - \frac{r_+[\Im\mathrm{m}(\Upsilon)]^2}{2|r_+ + \Upsilon|^2} \\
&= \frac{r_-}{4} + \frac{\Re\mathrm{e}(\Upsilon)}{4} - \frac{r_+[\Im\mathrm{m}(\Upsilon)]^2}{2|r_+ + \Upsilon|^2}\,.
\end{aligned}
\tag{4.51}
$$

The third term in Eq. (4.46) is

$$
\begin{aligned}
-e^{\phi_\infty}\left(A_{m\,t} - aA^m{}_t\right)\Im\mathrm{m}(\Gamma^m) &= -\frac{e^{2\phi_\infty}}{R^2}\left[\Gamma^m(A - aB) + \text{c.c.}\right]\Im\mathrm{m}(\Gamma^m) \\
&= -\frac{e^{-2(\phi-\phi_\infty)}}{R^2}\left[i\Gamma^m B + \text{c.c.}\right]\Im\mathrm{m}(\Gamma^m) \\
&= \frac{1}{2|r + \Upsilon|^2}\left[\left(M\Re\mathrm{e}(\Upsilon) + 2|\Gamma|^2\right) r + M|\Upsilon|^2 + 2|\Gamma|^2\Re\mathrm{e}(\Upsilon)\right].
\end{aligned}
\tag{4.52}
$$

Combining these two partial results at $r = r_+$ and operating, we get

$$
\begin{aligned}
&\frac{r_-}{4} - \frac{\Re\mathrm{e}(\Upsilon)}{4} + \frac{\left(M\Re\mathrm{e}(\Upsilon) + 2|\Gamma|^2 - [\Im\mathrm{m}(\Upsilon)]^2\right) r_+ + M|\Upsilon|^2 + 2|\Gamma|^2\Re\mathrm{e}(\Upsilon)}{2|r_+ + \Upsilon|^2} \\
&= \frac{r_-}{4} + \frac{\Re\mathrm{e}(\Upsilon)\left(2Mr_+ + 4|\Gamma|^2 - |r_+ + \Upsilon|^2\right) + 2\left(2|\Gamma|^2 - [\Im\mathrm{m}(\Upsilon)]^2\right) r_+ + 2M|\Upsilon|^2}{4|r_+ + \Upsilon|^2} \\
&= \frac{r_-}{4} + \frac{\Re\mathrm{e}(\Upsilon)\left(2Mr_+ - r_+^2 + 4|\Gamma|^2 - |\Upsilon|^2\right) + 2\left(2|\Gamma|^2 - |\Upsilon)|^2\right) r_+ + 2M|\Upsilon|^2}{4|r_+ + \Upsilon|^2} \\
&= \frac{r_-}{4} + \frac{2\Re\mathrm{e}(\Upsilon) r_+ r_- + r_+^2 r_- - |\Upsilon)|^2 r_-}{4|r_+ + \Upsilon|^2} \\
&= \frac{r_-}{2}\,,
\end{aligned}
\tag{4.53}
$$

which gives the Smarr formula proposed in Section 4.2, Eqs. (4.22).

4.5 Charges, potentials and S duality

The static axion-dilaton black holes introduced in Section 4.2 are the most general black holes in that class according to the no-hair theorems because they have the maximum number of independent parameters (moduli λ_∞ and conserved charges M, Γ^m) allowed by it. Hence, we have proven the validity of the Smarr formula in this theory for static black holes. However, we have to rewrite it in terms of the potentials and charges.

The charges which are quantized in this theory are not the components of Γ^m, but

$$p^m \equiv \frac{1}{8\pi G_N^{(4)}} \iint F^m = e^{\phi_\infty} \Im\mathrm{m}(\Gamma^m)/G_N^{(4)}\,, \tag{4.54a}$$

$$q_m \equiv \frac{1}{8\pi G_N^{(4)}} \iint F_m = e^{\phi_\infty} \left[e^{-2\phi_\infty} \Re\mathrm{e}(\Gamma^m) + a_\infty \Im\mathrm{m}(\Gamma^m) \right] / G_N^{(4)}\,. \tag{4.54b}$$

According to the discussions in Refs. [162, 209, 218], the potentials can be identified, up to a normalization factor, with the momentum maps $P^m{}_k$ and $P_{m\,k}$ evaluated over the black-hole horizon:

$$\Phi^m \equiv 2\, P^m{}_k|_{r_h}\,, \tag{4.55a}$$

$$\Phi_m \equiv 2\, P_{m\,k}|_{r_h}\,, \tag{4.55b}$$

and they are guaranteed to be constant at least over the bifurcation sphere $\mathcal{BH}$, according to the *restricted, generalized zeroth laws.*[9] We normalize them to vanish at infinity for the asymptotically-flat solutions we are interested in.

Therefore,

$$\frac{1}{16\pi G_N^{(4)}} \iint_{S^2_\infty} 2\,(P^m{}_k F_m - P_{m\,k} F^m) = 0\,, \tag{4.56a}$$

$$\frac{1}{16\pi G_N^{(4)}} \iint_{\mathcal{BH}} 2\,(P^m{}_k F_m - P_{m\,k} F^m) = \tfrac{1}{2}\,(\Phi^m q_m - \Phi_m p^m)\,. \tag{4.56b}$$

On the other hand, on general grounds and in the static case,

$$-\frac{1}{16\pi G_N^{(4)}} \iint_{S^2_\infty} \star(e^a \wedge e^b) e^{-2\phi} P_{k\,ab} = M/2\,, \tag{4.57a}$$

$$-\frac{1}{16\pi G_N^{(4)}} \iint_{\mathcal{BH}} \star(e^a \wedge e^b) e^{-2\phi} P_{k\,ab} = ST\,, \tag{4.57b}$$

and the Smarr formula takes the general form[10]

[9] This result may be extended to the complete event horizon using the arguments in Ref. [145].

[10] A previous derivation of a Smarr formula in this theory was made in Ref. [249] and our results should be compared with those in that reference.

$$M = 2ST + \Phi^m q_m - \Phi_m p^m\,. \tag{4.58}$$

While our definitions of charges and potentials seem to be identical to those in Refs. [149, 227], we get a different sign for the last term. The scaling arguments explained in Refs. [164, 229] indicate that the sign should be a plus if we define $\Phi^m = \partial M/\partial q_m$. We can always add a sign to our definition of Φ_m to make it coincide with that definition, but we are going to argue that a relative minus sign between the last two terms is the natural sign if we take into account that the Smarr formula should be invariant under the dualities of the theory. These always act on the vector fields of a 4-dimensional theory through a symplectic embedding [250].

In this particular case, it is convenient to define the symplectic vector of field strengths as follows:

$$(\mathcal{F}^M) \equiv \begin{pmatrix} F_m \\ F^m \end{pmatrix}, \tag{4.59}$$

since the action of a SL(2, $\mathbb{R}$) ∼Sp(2, $\mathbb{R}$) duality transformation

$$S \equiv (S^M{}_N) \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{4.60}$$

on them and on the axidilaton takes a simpler form:

$$\mathcal{F}'^M = S^M{}_N \mathcal{F}^N\,, \qquad \lambda' = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}\,, \qquad \alpha\delta - \beta\gamma = 1\,. \tag{4.61}$$

It follows from the definitions that

$$(\mathcal{P}^M{}_k) \equiv \begin{pmatrix} P_{m\,k} \\ P^m{}_k \end{pmatrix}, \qquad (\Phi^M) \equiv \begin{pmatrix} \Phi_m \\ \Phi^m \end{pmatrix}, \qquad (\mathcal{Q}^M) \equiv \begin{pmatrix} q_m \\ p^m \end{pmatrix}, \tag{4.62}$$

transform as the SL(2, $\mathbb{R}$) vector $\mathcal{F}^M$.

An important property of the duality group SL(2, $\mathbb{R}$) is that it is isomorphic to Sp(2, $\mathbb{R}$) since the condition

$$S^M{}_P \Omega_{MN} S^N{}_Q = \Omega_{PQ}\,, \qquad (\Omega_{MN}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.63}$$

also implies $\alpha\delta - \beta\gamma = 1$ for the matrix S. Thus, the combination of potentials and charges occurring in the Smarr formula Eq. (4.58)

$$\Phi^m q_m - \Phi_m p^m = \mathcal{Q}^M \Phi^N \Omega_{MN} \tag{4.64}$$

is manifestly SL(2, $\mathbb{R}$) ∼Sp(2, $\mathbb{R}$)-invariant. The explicit calculation of this term in Section 4.4 is a proof of this invariance.

4.6 Discussion

In this paper we have shown how the momentum maps introduced in Ref. [162, 209, 218] in the context of black-hole thermodynamics can be used to express the Komar integral obtained in the context of Wald's formalism [233] as a surface integral in a manifestly covariant way, generalizing the results of [141, 164, 229]. We have also shown how, in its turn, this integral can be used to derive a Smarr formula which is manifestly symplectic invariant. We have checked this formula explicitly in the most general family of static axidilaton black holes, constructed in Ref. [235]. It is trivial to extend these results to theories with more scalars and more complicated kinetic matrices (period matrices in the language of $\mathcal{N} = 2$ theories).

Symplectic invariance is a property to be expected of a general Smarr formula because this relation is just a relation between physical parameters occurring in the metric, which is symplectic invariant. It is, nevertheless, surprising, how this property of the Smarr formula and of the Komar integral from which it is derived, arises from a combination of the Noether-Wald charge and the on-shell Lagrangian density which are not separately symplectic invariant.

However, the lack of symplectic invariance of the Noether-Wald charge seems to lead to a first law without magnetic charges. One could argue that this is to be expected since Wald's formalism is based on gauge symmetries and there is no gauge symmetry associated to the conservation of magnetic charges (at least in the standard, off-shell, formulation of electromagnetism and its generalizations). But this is somewhat unsatisfactory because it is known, from explicit solutions, that magnetic terms are present in general in the first law [149].[11] In an upcoming publication [163], we show how the variation of magnetic charges can be accounted for in Wald's formalism, and we derive a first law that includes magnetic terms. The crucial observation is that the variational identity that leads to the first law contains terms of the form[12]

$$d\left[... + \iota_k \star F \wedge \delta A + ... \right] = 0 \tag{4.65}$$

where only the relevant piece is shown. Since perturbations δA that probe variations of magnetic charges are not globally defined (they are only defined up to gauge transformations, so that $\delta F = d\delta A$ is regular everywhere), the terms of the form $\sim k \times \delta A$ should not be discarded at the bifurcation surface, since the singularities of δA may compensate the vanishing of k. Instead, one can write

$$\iota_k \star F \wedge \delta A = (-1)^{\tilde{p}} \tilde{P}_k \wedge \delta F - d\left(\tilde{P}_k \wedge \delta A\right) \tag{4.66}$$

where we used the magnetic momentum map $d\tilde{P}_k = -\iota_k \star F$. Substituting (4.66) into (4.65), the second term of (4.66) does not contribute and one has

$$d\left[... + (-1)^{\tilde{p}} \tilde{P}_k \wedge \delta F + ... \right] = 0 \tag{4.67}$$

Integrating the variational identity in the form (4.67) leads to a first law with magnetic terms. Similarly to the case of the Komar charge, the combination of electric and magnetic

[11] It is also problematic, since this is the only formalism that can be applied to theories of higher order in the curvature.

[12] For simplicity here we consider a minimally coupled vector.

pieces is duality invariant. The details and a more extended discussion with explicit examples including black holes and black rings will be given in [163]. Progress regarding the terms that involve variations of the scalar moduli [149] is also underway.

Part II

Dynamical Aspects of Black Holes

5

Extremal Rotating Black Holes in Einsteinian Cubic Gravity

This chapter is based on:
Extremal Rotating Black Holes in Einsteinian Cubic Gravity
P. A. Cano, D. Pereñiguez
Phys.Rev.D 101 (2020) 4, 044016 (arXiv:1910.10721)

General Relativity describes accurately the dynamics of the gravitational field in the regime of relatively low curvature, but modifications of this theory are expected to appear at high energies. The fact that GR is incompatible with quantum mechanics [114, 251, 252] indicates that it should be regarded as an effective theory, presumably arising from an underlying theory of quantum gravity. Independently of what the UV-completion of GR turns out to be, it is broadly accepted that an effective low-energy description of that theory will contain the Einstein-Hilbert action plus an infinite tower of higher-derivative corrections — this is, in particular, a definite prediction of String Theory [186, 187, 189, 253–256]. Such corrections modify the behaviour of the gravitational field when the distances involved are of the order of the length scale of new physics. Thus, they become extremely relevant in the very early universe or near black hole singularities, but also at the level of the horizon of small enough black holes. It is therefore an interesting task to determine the properties of the modified black hole solutions, with particular emphasis on the corrections to the thermodynamic quantities, such as entropy and temperature [139, 140, 145, 257, 258].

From the point of view of Effective Field Theory (EFT), one should treat the higher-derivative corrections as perturbations over the GR geometry. Obtaining the corrected solutions in this perturbative approach is usually an accessible task; however, perturbative solutions give us very little information. In fact, the perturbative corrections are only valid as long as they remain very small, and many potentially interesting phenomena, that would appear at a non-perturbative level, are lost. For this reason, it is interesting to find exact black hole solutions of higher-order gravity.

The problem of obtaining exact black hole solutions is, of course, more complicated. Let us consider first the case of spherically symmetric black holes. Until very recently, the only theories in which exact solutions modifying in a non-trivial way the Schwarzschild geometry had been constructed were Lovelock [258–266] and Quasi-topological gravities [267–270], both types of theories existing only in $D > 4$ dimensions.[1] The gap in $D = 4$

[1]There are theories in which Einstein metrics are exact solutions (*e.g.* if the Lagrangian only contains Ricci curvature [271, 272]), and other that possess "non-Schwarzschild" solutions [273, 274]. We are not including these in our discussion.

has been recently filled thanks to the construction of a new type of theories with very interesting properties. Known as Generalized Quasi-topological gravities (GQTGs) [275], these theories allow for simple spherically symmetric black hole solutions whose thermodynamic properties can be studied analytically [275–277]. Besides, GQTGs exist in all dimensions (including, in particular, $D = 4$) and at all orders in curvature [278], and very likely they provide a basis to construct the most general EFT for gravity [279]. Spherically symmetric solutions in these theories have been studied at all orders in curvature in $D = 4$ [280] and at cubic [275, 281] and quartic order [277, 282] in various dimensions, and this has allowed us to gain substantial information about spherically symmetric black holes in higher-order gravity. In particular, one of the most remarkable features of these theories is that black holes become stable below certain mass [280], hence avoiding the complete evaporation in a finite time and the final explosion of black holes. This is analogous to the behaviour of higher-dimensional Lovelock black holes found long ago in Ref. [257]. In this paper we will consider an extension of Einstein gravity containing the simplest non-trivial Generalized Quasi-topological density in $D = 4$, which is known as Einsteinian cubic gravity (ECG) [283]. This theory was the first member of the GQT class to be discovered and we review some of its properties as well as recent results in Sec. 5.1.

Despite the success in the construction of spherically symmetric black holes, a remaining issue in the world of higher-order gravities is to find rotating black hole geometries.[2] In fact, exact rotating solutions have not even been found in Lovelock gravity, which is the simplest non-trivial extension of GR that one could consider.[3] Thus, the question about what a rotating black hole in higher-derivative gravity is like has not been answered yet. However, this is a primordial question, since, after all, realistic black holes will in general possess angular momentum.

The equations of motion for an axisymmetric and stationary metric are far more complicated than those in the spherically symmetric case. Even though we expect some simplification of the equations taking place for GQTGs — because they do so in the static case —, obtaining a complete rotating black hole solution would necessarily require a laborious numeric computation. However, there are several limits in which the problem is simplified. On the one hand, one might consider slowly-rotating solutions and stay perturbative in the spin. This has been explored in the case of quadratic [287] and cubic [288] Lovelock gravity. The case for $D = 4$ ECG will be reported in a coming publication [289]. On the other hand, it is possible to study the opposite limit, namely, the case of extremal black holes. In this situation, the horizon is placed at an infinite distance and the near-horizon limit is well-defined, giving rise to a new solution of the gravitational equations. This near-horizon geometry has more symmetries than the global solution, and this enormously simplifies the problem of solving the field equations. We will show in this paper that the equations of motion of ECG reduce in this case to a single second-order ODE. This equation has to be solved numerically, but most remarkably, we will see that it is possible to obtain the exact expressions for the area and entropy of these black holes without using any approximation. We are not aware that a similar analysis has been performed for other pure-metric higher-order gravities, but let us mention that Ref. [290] computed the (perturbative) corrections to the near-horizon geometry of extremal Kerr

[2] Let us note that exact rotating black hole solutions have been constructed numerically for some scalar-tensor theories containing higher-curvature terms [284, 285], but not for pure gravity theories.

[3] A honorable exception is the solution found in Ref. [286], corresponding to a rotating black hole in $D = 5$ Gauss-Bonnet gravity at a special point of the parameter space in which there is a unique maximally symmetric solution.

black holes in the case of Einstein-dilaton-Gauss-Bonnet [291–293] and dynamical Chern-Simons [294] gravities.

For generality purposes, we will add as well a Maxwell field into the game, which will allow us to study rotating and charged extremal black holes. This will prove to be useful, as $\mathrm{AdS}_2\times\mathbb{S}^2$ geometries — corresponding to non-rotating charged black holes — are always solutions of higher-order gravities. The rotating black holes can then be studied as a deformation of these geometries, which facilitates the analysis of the solutions. However, we will also show that there are new branches of solutions that do not reduce to $\mathrm{AdS}_2\times\mathbb{S}^2$ geometries in any limit. These solutions do not exist in the Einstein gravity limit and, as we will see, they have somewhat exotic properties.

The paper is organized as follows. We start in Sec. 5.1 by introducing our theory, corresponding to ECG coupled to a Maxwell field. In Sec. 5.2 we write the metric and vector ansätze for a rotating near-horizon geometry possessing an $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$ isometry group, and we evaluate and partially solve the equations of motion. We reduce the field equations to a single second-order ODE for one variable. Then we discuss the boundary conditions that need to be imposed in order to obtain fully regular solutions. In Sec. 5.3 we study in detail the solutions of the previous equation that are smooth deformations of $\mathrm{AdS}_2\times\mathbb{S}^2$ geometries. We construct solutions — both numerically and in the slowly-rotating approximation — which are labeled by the total charge Q and by a parameter x_0 which we argue is related to the spin $a=J/M$. More interestingly, we find that both the area and the Wald's entropy can be obtained exactly, and we study them as functions of Q and x_0. In addition, the physically meaningful relation $S(\mathcal{A},Q)$ is derived and we also study its profile. In Sec. 5.4 we analyze the full space of near-horizon geometries, showing that there exists an important degeneracy of solutions. We discuss the properties of the additional branches and comment on the structure of the diagram $S(\mathcal{A},Q)$. Finally, we draw our conclusions in Sec. 5.5. We also include a number of appendices that support and extend some of the results in the main text.

5.1 Einsteinian cubic gravity

We consider the following theory

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left\{-2\Lambda+R-\frac{\mu L^4}{8}\mathcal{P}-F_{\mu\nu}F^{\mu\nu}\right\}, \tag{5.1}$$

which consists of the (cosmological) Einstein-Maxwell action — where $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$ — plus a cubic curvature correction $\mathcal{P}$, the Einsteinian cubic gravity density [283]

$$\mathcal{P}=12R_{\mu\ \nu}^{\ \rho\ \sigma}R_{\rho\ \sigma}^{\ \alpha\ \beta}R_{\alpha\ \beta}^{\ \mu\ \nu}+R_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \alpha\beta}R_{\alpha\beta}^{\ \ \mu\nu}-12R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}+8R_{\mu}^{\ \nu}R_{\nu}^{\ \rho}R_{\rho}^{\ \mu}. \tag{5.2}$$

Also, μ is a dimensionless coupling while L is a length scale that determines the distance at which the gravitational interaction is modified.

As stated earlier, $\mathcal{P}$ is the lowest-order non-trivial member of the GQT family of theories in $D=4$.[4] On a historical note, this theory was first identified by the special

[4] At cubic order in curvature there is another GQT term that was denoted by $\mathcal{C}$ in Ref. [295]. However, this term makes no contribution to spherically symmetric solutions, and we have checked that it is irrelevant for our present setup, too.

form of its linearized equations on maximally symmetric backgrounds, which turn out to be of second order in any dimension [283]. Afterwards, the simple form of spherically symmetric black hole solutions in this theory was noticed [295, 296], and this triggered the construction of the GQT class of theories [275–277]. By now, many other aspects of ECG have been explored, including the characterization of observational deviations with respect to GR [297, 298], holographic applications [299–301], inflationary cosmologies [302–304][5] and other types of solutions [305–307].

Up to the six-derivative level, $\mathcal{P}$ represents the leading parity-preserving higher-derivative correction to the Einstein-Hilbert action [279]. However, when a Maxwell field is included, there are other terms that we could add at this order. Schematically, these would be of the form F^4, RF^2, F^6, RF^4, R^2F^2. Nevertheless, it is not our intention to study the most general correction to extremal Kerr-Newman geometries. Instead, we focus on the theory above because it will allow as to perform many explicit computations that are practically unaccessible for other higher-derivative theories.

The equations of motion of (5.1) read

$$\mathcal{E}_{\mu\nu} = T_{\mu\nu}\,, \tag{5.3}$$
$$\nabla_\mu F^{\mu\nu} = 0\,, \tag{5.4}$$

where the gravitational tensor $\mathcal{E}_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ are given by

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{\mu L^4}{8}\left(P_{\mu\sigma\rho\lambda}R_{\nu}{}^{\sigma\rho\lambda} - \frac{\mathcal{P}}{2}g_{\mu\nu} + 2\nabla^\alpha\nabla^\beta P_{\mu\alpha\nu\beta}\right)\,, \tag{5.5}$$

$$T_{\mu\nu} = 2F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\,, \tag{5.6}$$

and where

$$\begin{aligned} P_{\mu\nu}{}^{\alpha\beta} =& 36R_{[\mu|\sigma}{}^{[\alpha|}{}_{\rho}R_{|\nu]}{}^{\sigma|\beta]\rho} + 3R_{\mu\nu}{}^{\sigma\rho}R_{\sigma\rho}{}^{\alpha\beta} - 12R_{[\mu}{}^{[\alpha}R_{\nu]}{}^{\beta]} \\ &- 24R^{\sigma\rho}R_{\sigma[\mu|\rho}{}^{[\alpha}\delta_{|\nu]}{}^{\beta]} + 24R_{\sigma}{}^{[\alpha|}R^{\sigma}{}_{[\mu}\delta_{\nu]}{}^{|\beta]}\,. \end{aligned} \tag{5.7}$$

5.2 Near-horizon geometries

Near-horizon geometries of extremal rotating black holes possess an isometry group SL(2, ℝ)× U(1), and a general ansatz for this type of metrics can be written as [308]

$$ds^2 = (x^2+n^2)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \frac{dx^2}{f(x)} + N(x)^2 f(x)(d\psi - 2nrdt)^2\,, \tag{5.8}$$

which depends on two functions $f(x)$ and $N(x)$ and on one constant n. In addition, we consider a vector field of the following form

$$A = h(x)(d\psi - 2nrdt)\,, \tag{5.9}$$

which depends on another function $h(x)$. Then, we have to insert this ansatz in the equations of motion (5.3) and solve them. Due to the symmetries of the ansatz, one

[5]In the cosmological context, the solutions appearing in Refs. [302–304] were constructed in a modified cubic theory that takes the form $\mathcal{P} - 8\mathcal{C}$, where $\mathcal{P}$ is the ECG term — see (5.2) — and $\mathcal{C}$ is the cubic piece that we referred to in footnote 4.

can check that the only independent components of the Einstein's equations are those corresponding to $\mu\nu = xx$ and $\mu\nu = \psi\psi$ — the rest are related to them by the Bianchi identities. Thus, we only need to solve those equations together with the Maxwell equation.

An important observation is that these equations allow for solutions that have $N(x) = 1$. The reason is that, when evaluated on $N(x) = 1$, the components of the gravitational tensor — which we show in Appendix D.1 — become proportional, namely

$$\mathcal{E}_{\psi\psi}\Big|_{N(x)=1} = f(x)^2 \mathcal{E}_{xx}\Big|_{N(x)=1}, \tag{5.10}$$

and the same property holds for the Maxwell energy-momentum tensor $T_{\mu\nu}$. In general, higher-derivative gravities do not satisfy the condition (5.10), meaning that these theories do not allow for solutions with constant $N(x)$. In turn, it is quite remarkable that this property holds for ECG. As we are going to see, this represents a drastic simplification of the equations of motion. Let us also note at this point that, besides the solutions with $N(x) = 1$, there can be other solutions. In fact, Einstein gravity allows for solutions with non-constant $N(x)$, but these turn out to be singular, and only the solutions with $N(x) = 1$ represent the near-horizon geometry of extremal Kerr-Newman black holes. In the same way, ECG will presumably allow as well for this type of singular solutions when $N(x)$ is non-constant. Thus, from now on we set $N(x) = 1$.

Now, we can evaluate Maxwell's equation, which turns out to be independent of $f(x)$:

$$d \star F = \left[\left(h'(x)(x^2+n^2)\right)' + \frac{4n^2 h(x)}{x^2+n^2}\right] dt \wedge dt \wedge dx = 0\,, \tag{5.11}$$

where the prime denotes derivation with respect to x. The general solution of this equation reads

$$h(x) = \frac{a(x^2-n^2)}{x^2+n^2} + \frac{2bnx}{x^2+n^2}\,, \tag{5.12}$$

where a and b are two integration constants that are related to the electric and magnetic charges. Thus, at this point we have reduced the problem to solving one equation for $f(x)$, namely $\mathcal{E}_{xx} = T_{xx}$. However, before going into the resolution of this equation, let us massage a bit the solution in its current form. Let us note that the coordinate x is compact and it will range within two values $x_0 > 0$ and $-x_0$. These values are determined by the vanishing of the function $f(x)$ — which is assumed to be even — at those points: $f(x_0) = f(-x_0) = 0$. Also, let us introduce the quantity

$$\omega \equiv -\frac{f'(x_0)}{2} = \frac{f'(-x_0)}{2} > 0\,. \tag{5.13}$$

Then, observe that in order to avoid a conical singularity at $x = \pm x_0$ — these points will correspond to the poles of the horizon — the coordinate ψ must have period $2\pi/\omega$. Using these results, we can already compute the electric and magnetic charges even if we do not know explicitly the function $f(x)$. In Planck units, these charges read

$$q = \frac{1}{4\pi}\oint\!\!\oint \star F = \frac{2anx_0}{\omega(x_0^2+n^2)}\,, \tag{5.14}$$

$$p = \frac{1}{4\pi}\oint\!\!\oint F = \frac{2bnx_0}{\omega(x_0^2+n^2)}\,, \tag{5.15}$$

where the integration is performed on any surface of constant t and r. We note that these are the actual values of the charges that we would obtain in a global solution containing an asymptotic region. Let us finally exchange x and ψ in terms of two new coordinates

$$x = x_0 y\,, \quad y \in [-1, 1]\,, \tag{5.16}$$

$$\psi = \frac{\phi}{\omega}\,, \quad \phi \in [0, 2\pi)\,, \tag{5.17}$$

and let us introduce the function

$$g(y) = \frac{f(yx_0)}{x_0^2}\,. \tag{5.18}$$

In this way, we rewrite our solution in the following form

$$ds^2 = (y^2x_0^2 + n^2)\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + \frac{dy^2}{g(y)} + \frac{x_0^2}{\omega^2}g(y)(d\phi - 2\omega n r dt)^2\,, \tag{5.19}$$

$$A = \frac{x_0^2 + n^2}{y^2x_0^2 + n^2}\left[q\frac{y^2x_0^2 - n^2}{2nx_0} + py\right](d\phi - 2\omega n r dt)\,, \tag{5.20}$$

and by construction, $g(y)$ satisfies

$$g(1) = 0\,, \quad g'(1) = -\frac{2\omega}{x_0}\,. \tag{5.21}$$

Let us finally evaluate the remaining equation, which in the new coordinates is $\mathcal{E}_{yy} = T_{yy}$. On the one hand, we have

$$T_{yy} = \frac{\omega^2\left(x_0^2 + n^2\right)^2 Q^2}{x_0^2 g(y)\left(n^2 + x_0^2y^2\right)^2}\,, \tag{5.22}$$

where $Q^2 = q^2 + p^2$. On the other hand, $\mathcal{E}_{yy}$ takes the form of a total derivative, namely

$$\mathcal{E}_{yy} = \frac{y^2}{(y^2x_0^2 + n^2)^2 g(y)}\frac{d}{dy}\mathcal{E}(g, g', g''; y)\,, \tag{5.23}$$

where

$$\begin{aligned}
\mathcal{E}(g, g', g''; y) = &-\frac{n^2}{y} + yx_0^2 + g\left(\frac{n^2x_0^2}{y} + yx_0^4\right) + \Lambda\left(-\frac{n^4}{y} + 2n^2yx_0^2 + \frac{1}{3}y^3x_0^4\right)\\
&+ L^4\mu\left[\frac{3g^3n^2x_0^6\left(n^2 - 9y^2x_0^2\right)}{y\left(n^2 + y^2x_0^2\right)^3} + \left(-\frac{3x_0^2}{4y} - \frac{3gn^2x_0^4}{2n^2y + 2y^3x_0^2}\right)(g')^2\right.\\
&+ \frac{1}{4}x_0^4\left(g'\right)^3 + \left(-\frac{3g^2x_0^6\left(-17n^2 + y^2x_0^2\right)}{2\left(n^2 + y^2x_0^2\right)^2}\right)\left(-\frac{3gx_0^4}{2\left(n^2 + y^2x_0^2\right)}\right)g'\\
&\left.+ g\left(\frac{3x_0^2}{2y} + \frac{3gx_0^4\left(-4n^2 + y^2x_0^2\right)}{2y\left(n^2 + y^2x_0^2\right)}\right)\left(-\frac{3}{4}x_0^4g'\right)g''\right]
\end{aligned} \tag{5.24}$$

Hence, integrating both sides of the equation we obtain

$$\mathcal{E}(g, g', g''; y) = -\frac{\omega^2\left(x_0^2 + n^2\right)^2 Q^2}{x_0^2 y} + N\,, \tag{5.25}$$

where N is an integration constant. Thus, we have reduced the equations of motion to a single ODE of second order for $g(y)$.

Our task now is to solve the previous equation in order to obtain near-horizon geometries. So far, we have included a non-vanishing cosmological constant for generality, but for the sake of simplicity we set $\Lambda = 0$ from now on. The case of $\Lambda \neq 0$ is briefly discussed in Appendix D.4.

5.2.1 Einstein gravity

Let us first of all check that we recover the near-horizon geometry of extremal Kerr-Newman black holes when we set $\mu = 0$. In that case, Eq. (5.25) is simply algebraic and we obtain the solution straightforwardly,

$$g(y) = \frac{n^2 - Q^2 \left(n^2 + x_0^2\right)^2 \omega^2/x_0^2 + Ny - x_0^2 y^2}{x_0^2 \left(n^2 + x_0^2 y^2\right)} \,. \tag{5.26}$$

We can see that the parameter N breaks the symmetry $y \leftrightarrow -y$ of the solution that we assumed in identifying the charges q, p. More importantly, when N is present (and $x_0 \neq 0$), there is necessarily a conical singularity at one of the poles of the horizon (where g vanishes), because the slope of g will be different in each one. In fact, N is the NUT charge, and it is known that NUT-charged, rotating black holes present this type of conical singularities at the horizon [309]. In order to avoid these problems, we set $N = 0$. In that case, $g(y)$ is even, and we have to impose the conditions (5.21), which are going to fix several relations between the parameters of the solution. We find

$$n = \sqrt{Q^2 + x_0^2}\,, \quad \omega = \frac{x_0}{Q^2 + 2x_0^2}\,, \tag{5.27}$$

and after simplifying we obtain

$$g(y) = \frac{1 - y^2}{Q^2 + x_0^2(1 + y^2)} \,. \tag{5.28}$$

We see that this is the near-horizon geometry of extremal Kerr-Newman black holes (NHEKN) [310], where x_0 is nothing but the angular momentum per mass $x_0 = a$. Likewise, $n = M$ is the total mass and ω is the angular velocity of the horizon. In addition, we can compute the area, which reads

$$\mathcal{A} = \frac{4\pi x_0}{\omega} = 4\pi(Q^2 + 2x_0^2) \,. \tag{5.29}$$

For $x_0 = 0$ we recover $\mathrm{AdS}_2 \times \mathbb{S}^2$, which is the near-horizon geometry of extremal Reissner-Nordstrom black holes.

5.2.2 Einsteinian cubic gravity

Let us now consider a non-vanishing μ. In analogy to the Einstein gravity case, we set the NUT charge to zero, $N = 0$, in order to avoid conical singularities. Now, once the corrections are included, the equation (5.25) becomes of second order and we need to impose appropriate boundary conditions in order to solve it. We warn that the constraints

(5.21) are not really boundary conditions: they are restrictions to the parameters of the solution. Instead, the boundary conditions we will impose are the following: (1) the solution is even, and this is equivalent to asking $g'(0) = 0$. (2) The solution is regular at $y = \pm 1$, i.e. it is analytic at those points. Therefore, according to (5.21), the solution should have a Taylor expansion near $y = 1$ of the form

$$g(y) = -\frac{2\omega}{x_0}(y-1) + \sum_{k=2}^{\infty} g_k (y-1)^k \,, \tag{5.30}$$

for some coefficients g_k. When this expansion is inserted in (5.25) we can Taylor-expand the equation as well, obtaining the following series

$$y\mathcal{E}(g, g', g''; y) + \frac{\omega^2 \left(x_0^2 + n^2\right)^2 Q^2}{x_0^2} = \sum_{k=0}^{\infty} C_k (y-1)^k \,. \tag{5.31}$$

Thus, all the coefficients C_k must vanish and this gives us a series of equations for the parameters of the solution. Remarkably, the first two equations C_0 and C_1 are independent of the g_k, and instead they provide two relations between x_0, n, ω and Q:

$$\begin{aligned} x_0^2 - n^2 + \frac{Q^2\omega^2\left(n^2 + x_0^2\right)^2}{x_0^2} - \mu L^4 \omega^2 \left(2x_0\omega + 3\right) = 0 \,, \\ \left(n^2 + x_0^2\right)\left(n^2\omega + x_0^2\omega - x_0\right) + \mu L^4 \omega^2 \left(-5n^2\omega + x_0^2\omega + 3x_0\right) = 0 \,. \end{aligned} \tag{5.32}$$

We have seen that in Einstein gravity x_0 is identified with the angular momentum per mass, a, while in turn n is the mass and ω is the angular velocity. We cannot expect that the same identifications work for higher-curvature gravity, and, since we lack the asymptotic region, we cannot correctly identify these quantities. Nevertheless, since x_0 controls the degree of non-sphericity of the solution, we do expect that there will be a monotonous relation between this parameter and the angular momentum — we recall that this parameter enters in the metric as $ds^2 = (x_0^2 y^2 + n^2) ds^2_{\mathrm{AdS}_2} + \ldots$. Hence, it seems reasonable to use x_0 and the charge Q to label our solutions. Then, the equations (5.32) provide us with the values of $n(x_0, Q)$ and $\omega(x_0, Q)$. It is worth emphasizing that such equations are exact; we have implemented no approximation in our approach. Besides, this allows us to compute the area of these black holes even if we do not know g explicitly, since it is given by

$$\mathcal{A} = \frac{4\pi x_0}{\omega} \,. \tag{5.33}$$

Then, once the parameters n and ω (or alternatively $\mathcal{A}$) are determined, we can solve the rest of the equations $C_2 = 0$, $C_3 = 0$, etc. It tuns out that these equations fix all the coefficients $g_{k>3}$ in (5.30) in terms of g_2, which is the only free parameter. Thus, we find that there is only a one-parameter family of solutions that are regular at the pole $y = 1$, which means that regularity is in fact fixing one integration constant. Now, the remaining parameter g_2 is determined by the condition that g be an even function, which is equivalent to asking $g'(0) = 0$. Thus, we have a well-defined boundary problem, which at most will possess a discrete number of solutions.

5.3 The $\mathrm{AdS}_2 \times \mathbb{S}^2$ branch

Let us summarize our findings so far. Our near-horizon geometries are labelled by two parameters which we can choose to be Q and x_0. Imposing regularity of the solution at $y = \pm 1$ yields the equations (5.32), whose solutions give the possible values of n and ω. Finally, the differential equation (5.25) must be solved imposing the regularity condition (5.30) and $g'(0) = 0$. As we will see later, the equations (5.32) have more than one solution for fixed Q and x_0, which leads to an important degeneracy of near-horizon geometries that have the same Q and x_0. However, it turns out that there is only one branch of solutions that are smoothly connected to an $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry in the limit of $x_0 \to 0$. In this section we will focus our attention on those solutions.

Let us first solve the equations (5.32) when $x_0 << Q$ by assuming a series expansion of the form $\omega = \sum_n \omega_k x_0^k$, $n^2 = \sum_k c_k x_0^k$. We find the following solution

$$\begin{aligned} n^2 =& Q^2 + x_0^2 \left(1 + \frac{\mu L^4}{Q^4}\right) + \mathcal{O}(x_0^4)\,, \\ \omega =& \frac{x_0}{Q^2} + \frac{x_0^3}{Q^4}\left(-2 + \frac{\mu L^4}{Q^4}\right) + \mathcal{O}(x_0^5)\,, \end{aligned} \tag{5.34}$$

where the higher-order terms can be easily computed as well and we show few of them in Appendix D.2. Now, let us also assume a series expansion of the metric function $g(y)$, so that

$$g(y) = \sum_{k=0}^{\infty} x_0^{2k} g_k(y)\,. \tag{5.35}$$

Plugging this expansion together with (5.34) in the equation (5.25) we find the equation satisfied by every component $g_k(y)$. The leading term g_0— which is the only one that survives in the limit $x_0 \to 0$ — satisfies the following equation

$$-1 + y^2 + g_0 Q^2 + \frac{3L^4\mu}{4}\left(\frac{4}{Q^4} - \left(g_0'\right)^2 + 2 g_0 g_0''\right) = 0\,. \tag{5.36}$$

We can see that a solution of this equation fulfilling the appropriate boundary conditions is given by

$$g_0(y) = \frac{1-y^2}{Q^2}\,. \tag{5.37}$$

Thus, in the limit $x_0 \to 0$ the metric (5.19) becomes

$$ds^2 = Q^2\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + Q^2\left(\frac{dy^2}{1-y^2} + (1-y^2)d\phi^2\right), \tag{5.38}$$

which corresponds to an $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry. In fact, this is the near-horizon geometry of extremal Reissner-Nordstrom black holes, and, as we can see, it possesses no corrections. Thus, this is an exact solution of ECG for any value of μ. Let us then consider the effect of rotation by assuming a finite x_0. Analyzing the equations for the following terms, $g_k(y)$, we see that they all allow for a solution which is a polynomial in y, and that this solution is the only one that satisfies the boundary conditions. For instance, up to quadratic order in x_0 we have

$$g(y) = (1-y^2)\left[\frac{1}{Q^2} - x_0^2 \frac{\left(Q^8 - 3\mu L^4 Q^4 + 9\mu^2 L^8 + y^2\left(Q^8 - 16\mu L^4 Q^4\right)\right)}{Q^8\left(Q^4 - 9L^4\mu\right)} + \ldots\right] \tag{5.39}$$

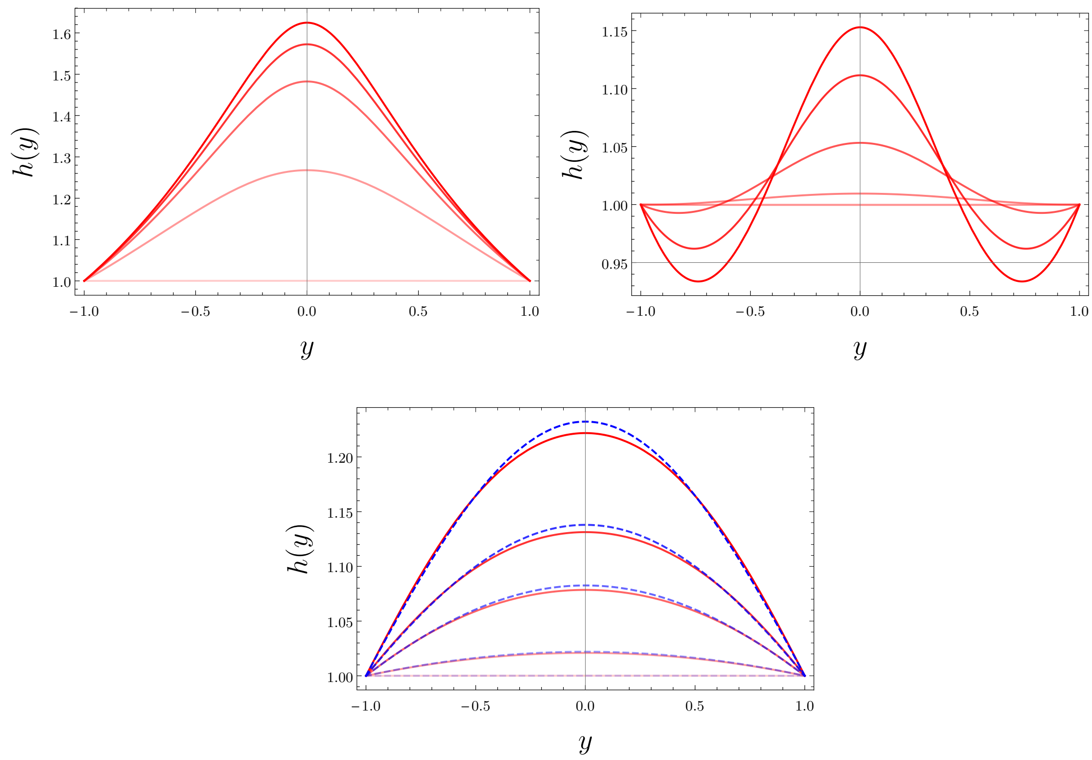

Figure 5.1: Profile of the solution for various values of Q and x_0. We show the quantity $h(y) \equiv \frac{x_0 g(y)}{\omega(1-y^2)}$, which measures the non-sphericity of the solution (for $\mathbb{S}^2$ this quantity is constant). Top left: we show the solution for $Q^4 = 3\mu L^4$ and $x_0 = 0, 0.3Q, 0.35Q, 0.36Q, 0.364Q$. Top right: $Q^4 = 15\mu L^4$ and $x_0 = 0, 0.1Q, 0.2Q, 0.3Q, 0.34Q$. We observe that the profile is very different in both cases, because we have passed the critical value of $Q^4 = 9\mu L^4$. Bottom: $Q^4 = 150\mu L^4$ and $x_0 = 0, 0.15Q, 0.3Q, 0.4Q, 0.55Q$. The size of the black hole is larger and the solution becomes more similar to the NHEKN one, shown in blue dashed lines for comparison.

and more terms are shown in the Appendix D.3. A few comments are in order. First, let us remark that this is a perturbative expansion in x_0, but it is exact in μ. Second, we observe that if we put $\mu = 0$ in the expression above we get $g(y) = (1-y^2)(Q^{-2} - x_0^2 Q^{-4}(1+y^2) + \ldots)$, which coincides with the perturbative expansion of the NHEKN solution (5.28), and the same holds for the higher-order terms that we show in the appendix. Therefore, these solutions in principle approach the NHEKN one when $\mu \to 0$, or more precisely, when $Q >> \mu^{1/4} L$, *i.e.*, when the size of the black hole is much larger than the length scale of the corrections. However, there is a subtlety: we observe that the $\mathcal{O}(x_0^2)$ term (and also all the higher-order ones) diverges for $Q^4 = 9\mu L^4$. In general, we observe that all the terms of order greater or equal than $2n$ diverge for $Q^4 = 3((n+1)^2 - 1)\mu L^4$. This implies that, when Q crosses one of these values, the solution changes discontinuously, and near those critical values we seem to find no solution. Therefore, as we increase Q and x_0, the solution will approach the NHEKN one, but it will make it in a non-continuous way. This is best understood by constructing the non-perturbative numerical solutions. We show several of them in Fig. 5.1, where we represent the function $h(y) \equiv \frac{x_0 g(y)}{\omega(1-y^2)}$, which allows for a more direct comparison between the different curves. We have checked that, when x_0 is small enough, the slowly-rotating expansion (5.39) gives a very good approximation to the

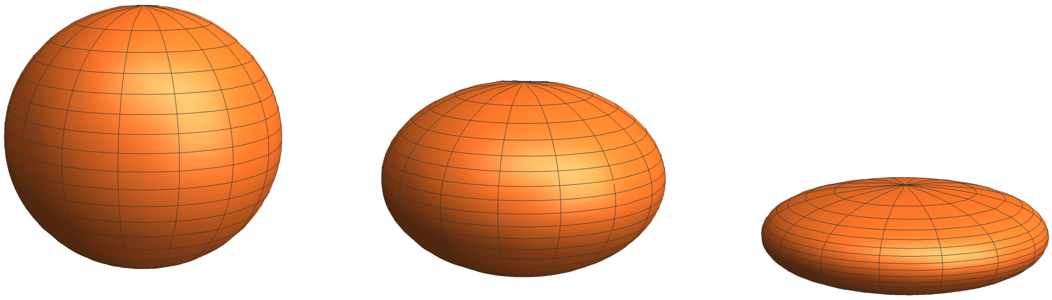

Figure 5.2: Isometric embedding of the horizon in $\mathbb{E}^3$ for the black holes with $Q^4 = 3\mu L^4$ and $x_0 = 0$, $x_0 = 0.3Q$ and $x_0 = 0.364Q$.

numerical curves. Looking at Fig. 5.1 we observe that, indeed, the profile of the solution is quite different for distinct values of Q, but eventually it becomes similar to the NHEKN one for large black holes. In addition, in Fig. 5.2 we show the embedding of the black horizon in Euclidean space for some of these solutions.

One important drawback, though, is that we do not seem to find solutions when x_0/Q is large. As we can see, Eq. (5.25) becomes singular at the points in which $\frac{3x_0^2}{2} + \frac{3gx_0^4\left(-4n^2+y^2x_0^2\right)}{2\left(n^2+y^2x_0^2\right)} - \frac{3}{4}x_0^4 y g' = 0$, which implies that the coefficient of g'' vanishes. This only happens when the ratio x_0/Q is large enough. For example, if we evaluate the previous expression for NHEKN geometries and we ask that it does not vanish at any point, we must impose $x_0/Q < 1/\sqrt{3}$. Now, if that quantity vanishes, the solution will typically become singular at that point, unless we fix a regularity boundary condition there. But in that case, we cannot impose the boundary conditions of regularity at $y = \pm 1$ and that $g'(0) = 0$. Hence, we find that, even in the regime where the corrections are small, the equation (5.25) has no regular solutions correcting the NHEKN geometry for x_0/Q large.

In addition, our numerical exploration indicates the existence of an important multiplicity of solutions even when the boundary conditions are fixed. This is, once we have solved (5.32) and found $n(x_0, Q)$, $\omega(x_0, Q)$, the equation (5.25) seems to have different solutions that differ on the profile of $g(y)$. This already happens in the $x_0 \to 0$ equation (5.36), which possesses other solutions than (5.37) satisfying $g(\pm 1) = 0$, $g'(\pm 1) = \mp 2/Q^2$. These do not need to be similar to the NHEKN geometry even when μ is small, and in general they will possess a different domain of existence from the solutions considered in the preceding paragraph. In any case, all of these solutions are characterized by the same set of parameters x_0, Q, n, ω, so they share a number of common properties.

In order to simplify the discussion, in the next subsection we will remain agnostic about the existence or non-existence of solutions of Eq. (5.25). Providing some solution exist, we are going to see that the area and entropy can be obtained exactly without knowing the profile of $g(y)$.

5.3.1 Area and entropy

As we have seen, it is possible to solve the equation (5.25) either perturbatively in x_0 or numerically. Nevertheless, there are some properties of these near-horizon geometries that we can compute exactly. One of them is the area, which is given by (5.33). Then, using Eqs. (5.32) one is able to obtain the area as a function of x_0 and Q. The relation $\mathcal{A}(x_0, Q)$ for several values of Q is shown in Fig. 5.3. Near $x_0 = 0$, one can use the expansions (5.34) in order to obtain the approximation

$$\frac{\mathcal{A}}{4\pi} = Q^2 + x_0^2 \left(2 - \frac{\mu L^4}{Q^4}\right) \Bigg(+ \frac{12\mu L^4 x_0^4}{Q^{10}} \left(\mu L^4 - Q^4\right) + \ldots \,, \tag{5.40}$$

which is valid as long as $x_0 << Q$. Thus, for $x_0 \to 0$, the area reduces to the corresponding value of extremal Reissner-Nordstrom black holes, but looking at Fig. 5.3 we see that an interesting behaviour takes place when we increase x_0. If the charge is large enough, the corresponding curve differs slightly from the value in Einstein gravity for intermediate values of x_0, but for large x_0 one recovers again the extremal Kerr-Newman result $\mathcal{A} \sim 4\pi(Q^2 + 2x_0^2)$. On the other hand, if the charge is too small — the threshold value is

$$Q_{\text{thr}} \approx 1.13\mu^{1/4} L \tag{5.41}$$

— the curve does not approach the Einstein gravity result, and instead we see that $\mathcal{A}$ tends to a constant for $x_0 \to \infty$. This represents an exotic solution that does not exist in Einstein gravity, and it satisfies

$$\mathcal{A} = 4\pi\alpha, \quad n^2 = \frac{x_0^2 \left(2\alpha + Q^2\right)}{5Q^2} \Bigg(, \quad \text{when } \; x_0 \to \infty \,, \tag{5.42}$$

where α is a constant determined from the equation

$$2\alpha^3 + 12\alpha^2 Q^2 + 18\alpha Q^4 - 25\mu L^4 Q^2 = 0 \,. \tag{5.43}$$

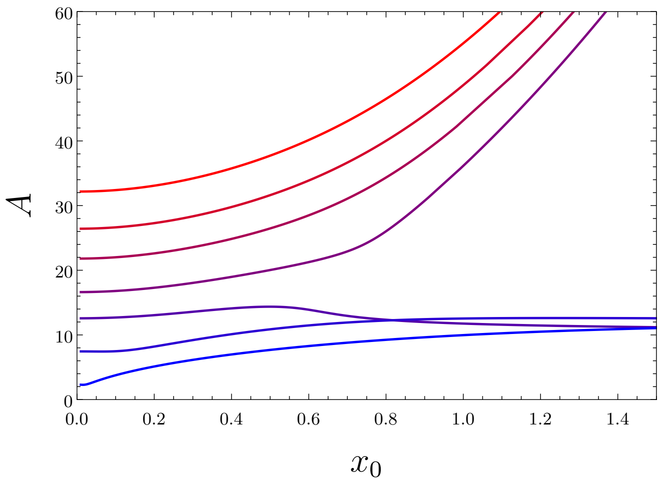

Figure 5.3: Area of black holes that are smooth deformations of $AdS_2 \times \mathbb{S}^2$ geometries as a function of x_0 for various values of Q. From blue to red we have $Q = 0.43, 0.77, 1, 1.15, 3^{1/4}, 1.45, 1.6$. We work in units such that $\mu L^4 = 1$. For large enough Q, the curves tend to the Einstein gravity values in both limits $x_0 \to 0, \infty$, but when Q is too small the area tends to a constant value for $x_0 \to \infty$.

On the other hand, near-horizon geometries allow us to compute the entropy of black holes, even if we do not know the behavior in the asymptotic region, thanks to Wald's entropy formula [139, 140, 145], which reads[6]

$$S = -2\pi \int_{\mathcal{H}} d^2x\sqrt{h}\frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}\,. \tag{5.45}$$

In this expression, the integral is taken over the horizon $\mathcal{H}$, h is the determinant of the induced metric on $\mathcal{H}$ and $\epsilon_{\mu\nu}$ is the binormal of the horizon, normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$.

Applying Wald's formula (5.45) to our theory (5.1), we get

$$S = \frac{1}{4G}\int_{\mathcal{H}} d^2x\sqrt{h}\left[1 + \frac{\mu L^4}{16}P_{\mu\nu\alpha\beta}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\right], \tag{5.46}$$

where $P_{\mu\nu\alpha\beta}$ is the tensor defined in (5.7). The horizon of the metric (5.19) is placed at $r = 0$, but the integration can be equivalently performed on any slice of constant t and r. The non-vanishing components of the binormal read $\epsilon_{tr} = -\epsilon_{rt} = (y^2x_0^2 + n^2)$, so that $P^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = 4(y^2x_0^2 + n^2)^2P^{trtr}$. Remarkably, we find that this quantity takes the form of a total derivative,

$$P^{\mu\nu\alpha\beta}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} = 12x_0^2\frac{d}{dy}\left(-\frac{4g^2n^2yx_0^2}{(n^2+y^2x_0^2)^3} + \frac{4gn^2g'}{(n^2+y^2x_0^2)^2} + \frac{y\,(g')^2}{n^2+y^2x_0^2}\right) \tag{5.47}$$

Therefore, the integral can be performed without knowing the details of $g(y)$ — we only require the conditions (5.21) — and the entropy reads

$$S = \frac{\pi x_0}{G\omega}\left[1 + \frac{3\mu L^4\omega^2}{n^2+x_0^2}\right]. \tag{5.48}$$

Now, using again Eqs. (5.32) we can study the entropy as a function of x_0 and Q. For instance, in the limit $x_0 << Q$, we obtain the following approximate value,

$$S = \frac{\pi}{G}\left[Q^2 + 2x_0^2\left(1+\frac{\mu L^4}{Q^4}\right) + \frac{12\mu L^4x_0^4}{Q^{10}}\left(\mu L^4 - 2Q^4\right)\right], \tag{5.49}$$

while for large x_0 we have to distinguish between the two different possibilities,

$$S(x_0 \to \infty) = \begin{cases} \frac{\pi}{G}\left(Q^2 + 2x_0^2\right) & \text{if} \quad Q > Q_{\text{thr}} \\ \frac{\pi\alpha}{G}\left(1 + \frac{15\mu L^4}{\alpha^2(2\alpha+6Q^2)}\right) & \text{if} \quad Q < Q_{\text{thr}}\,, \end{cases} \tag{5.50}$$

where α is the parameter that we introduced in (5.42). The complete profile of $S(x_0)$ for various values of the charge is shown in Fig. 5.4.

One disadvantage of this analysis is that, as we mentioned earlier, the parameter x_0 cannot be identified with the angular momentum, and therefore, the relation $S(x_0, Q)$ does not have a direct physical interpretation. Nevertheless, we can also study the entropy

[6] For Lagrangians containing covariant derivatives of the Riemann tensor, the partial derivative of the Lagrangian should be replaced by the Euler-Lagrange derivative of the gravitational Lagrangian as if the Riemann tensor were an independent variable, this is

$$\frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} - \nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial\nabla_\alpha R_{\mu\nu\rho\sigma}}\right) + \ldots \tag{5.44}$$

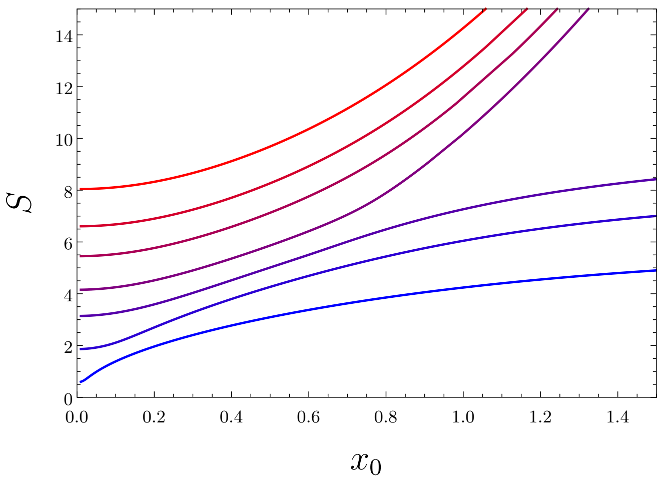

Figure 5.4: Entropy of black holes that are smooth deformations of $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometries as a function of x_0 for various values of Q. From blue to red we have $Q = 0.43, 0.77, 1, 1.15, 3^{1/4}, 1.45, 1.6$. We work in units such that $\mu L^4 = 1$. For large enough Q, the curves tend to the Einstein gravity values in both limits $x_0 \to 0, \infty$, but when Q is too small the area tends to a constant value for $x_0 \to \infty$.

as a function of the area and of the charge, *i.e*, $S(\mathcal{A}, Q)$, and in this case the relation is meaningful since it involves three physically relevant quantities. In fact, it is interesting to check that the entropy is not only a function of the area, since it depends also on the relative amount of charge and angular momentum of the black hole. Manipulating the equations in (5.32), we can write the entropy (5.48) in the following form,

$$S = \frac{\mathcal{A}}{4G}\left[1 + \frac{48\pi^2 \mu L^4 \lambda(\mathcal{A}, Q)}{\mathcal{A}^2}\right], \tag{5.51}$$

where $\lambda(\mathcal{A}, Q)$ is a function obtained as a solution of the equation

$$\begin{aligned} 0 = &\left(\frac{\mathcal{A}}{4\pi}\right)^3 \left[Q^2 - \left(\frac{\mathcal{A}}{4\pi}\right)\right] + \lambda\left(\frac{\mathcal{A}}{4\pi}\right)\left[2\left(\frac{\mathcal{A}}{4\pi}\right)^3 + 2\left(\frac{\mathcal{A}}{4\pi}\right)\mu L^4 - 3\mu L^4 Q^2\right] \\ &+ 12\lambda^3 \mu L^4 \left[\left(\frac{\mathcal{A}}{4\pi}\right)^2 - \mu L^4\right] + 3\lambda^2 \mu L^4 \left[5\mu L^4 - 6\left(\frac{\mathcal{A}}{4\pi}\right)^2\right] \end{aligned} \tag{5.52}$$

On general grounds, for a fixed value of the area, the charge can vary from $Q = 0$, which would correspond to a neutral rotating black hole, to $Q^2_{\max} = \mathcal{A}/(4\pi)$, in whose case there is no rotation and the solution is $\mathrm{AdS}_2 \times \mathbb{S}^2$.[7] It is then an interesting exercise to determine for which of these black holes of fixed area the entropy is maximal. In Fig. 5.5 we show the ratio $\frac{S}{\mathcal{A}/(4G)}$ as a function of the charge for several fixed values of the area. First, we observe that, indeed, the entropy does not only depend on the area, but also on the charge. For $Q = Q_{\max}$ we get $S = \mathcal{A}/(4G)$, since in that case the solution has no corrections. Nevertheless, when we decrease the charge leaving the area fixed — which implies that we turn on the angular momentum — the ratio between entropy and area

[7] When the area is sufficiently small we obtain solutions that have $Q > Q_{\max}$ — see Fig. 5.8 — but here we focus only in the case in which Q ranges from 0 to $Q_{\max}$ for simplicity.

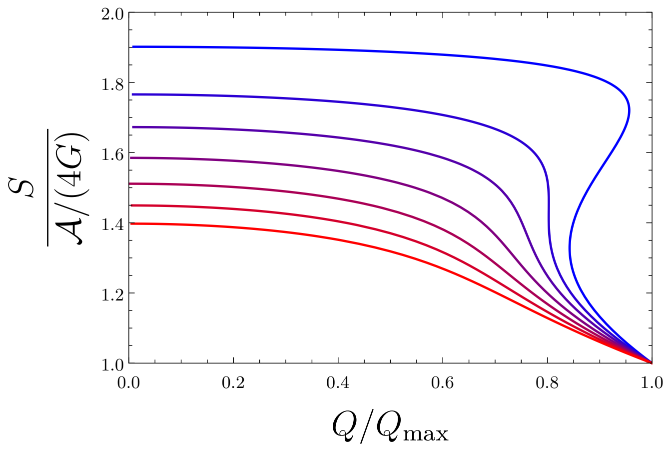

Figure 5.5: Entropy of black holes that are smooth deformations of $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometries as a function of the charge for fixed values of the area. We plot the ratio $S/(\mathcal{A}/(4G))$ in order to facilitate the comparison between the different curves, while the charge is normalized by $Q_{\max} = \mathcal{A}/(4\pi)$. From blue to red we have $\mathcal{A}/(4\pi\sqrt{\mu}L^2) = 1.8, 1.91, 2, 2.1, 2.2, 2.3, 2.4$. We observe the presence of a critical point where the curve starts being multivalued.

increases. In all cases shown we see that, for a given area, a purely rotating black hole is the one that stores more information. We also observe an interesting phenomenon taking place when the area is small enough: if $\mathcal{A} < 1.91 \times 4\pi\sqrt{\mu}L^2$ the corresponding curve becomes multivalued, indicating the existence of several black holes with same area and charge, but different entropy. This suggests the presence of a phase transition from the black hole of smaller entropy to the one of larger entropy. In that case, we see that the phase space would contain a critical point at $\mathcal{A}_{\rm cr} \approx 1.91 \times 4\pi\sqrt{\mu}L^2$, $Q_{\rm cr} \approx 1.11\mu^{1/4}L$, $S_{\rm cr} \approx 8.63\frac{\sqrt{\mu}L^2}{G}$. This picture is not completely accurate, though, because one should fix the angular momentum instead of the area in order to compare different solutions, and also because at zero temperature one cannot speak of phase transitions. Nevertheless, this result does suggest that some sort of decay could take place from one type of solution to another.

5.4 Additional solutions

In the previous section we focused on the branch of solutions that are smoothly connected to an $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry, since these are particularly relevant — and the only ones that exist in Einstein gravity. However, when we solve the system of equations (5.32) we observe that other solutions for $n(x_0, Q)$ and $\omega(x_0, Q)$ exist. A useful way of visualizing the space of solutions is to study the relation $\mathcal{A}(x_0)$ for fixed values of the charge, which we show in Fig. 5.6. This plot contains the curves that we showed in Fig. 5.3, but we see that new branches appear. In fact, for fixed values of Q and x_0 there can be up to four different solutions, which represent black holes with very different properties.

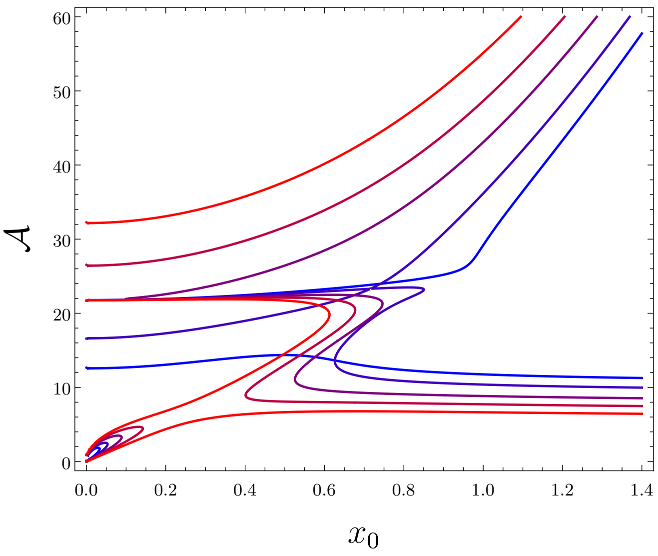

Figure 5.6: Black hole area as a function of x_0 for various values of Q. We include all the branches of solutions. From blue to red we have $Q = 1, 1.15, 3^{1/4}, 1.45, 1.6$. We work in units such that $\mu L^4 = 1$.

5.4.1 Branches of solutions

In the limit of $x_0 \to \infty$, we observe that there are only two possible solutions; one which recovers the properties of extremal Kerr-Newman black holes — in particular, $\mathcal{A} \to 4\pi(Q^2 + 2x_0^2)$ — and another one whose area tends to a constant — see Eq. (5.42). On the other hand, near $x_0 = 0$ we have in general four different solutions, which can be obtained by assuming different expansions of the parameters n and ω, as we show in the Appendix D.2. One of them belongs to the $\mathrm{AdS}_2 \times \mathbb{S}^2$ branch that we studied in the previous section, so we will now analyze the additional solutions.

Branch A

One possible solution of the equations (5.32) yields

$$n^2 = \frac{x_0^4 \left(2\sqrt{3\mu}L^2 + 3Q^2\right)}{18\mu L^4} + \mathcal{O}(x_0^6) \tag{5.53}$$

$$\mathcal{A} = 4\pi \left[\sqrt{3\mu}L^2 + x_0^2\left(\frac{1}{2} - \frac{Q^2}{4\sqrt{3\mu}L^2}\right) + \mathcal{O}(x_0^4)\right] \tag{5.54}$$

where we recall that $\mathcal{A} = 4\pi x_0/\omega$. It is important to note that the near-horizon geometry corresponding to this choice of parameters exists for arbitrarily small values of x_0, but not for $x_0 = 0$. One remarkable fact about this solution is that in the limit of $x_0 \to 0$ the area tends to a constant value which is independent of the charge. On the other hand, the entropy can be computed using (5.48), and we obtain

$$S = \frac{2\pi}{G}\left[\sqrt{3\mu}L^2 - x_0^2\left(\frac{1}{6} + \frac{Q^2}{4\sqrt{3\mu}L^2}\right) + \mathcal{O}(x_0^4)\right] \tag{5.55}$$

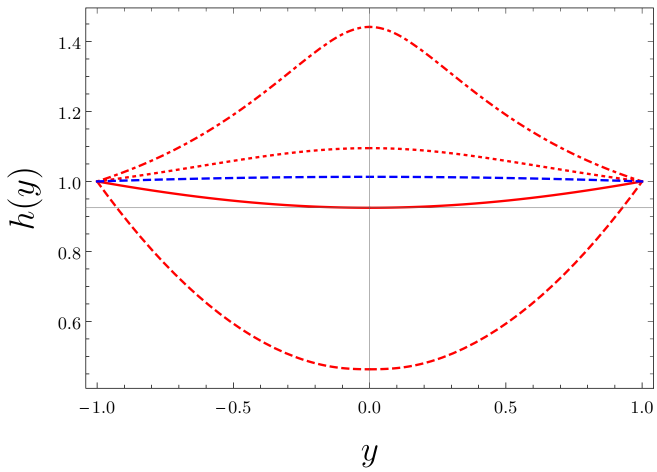

Figure 5.7: Different near-horizon geometries with $Q = (9\mu)^{1/4}L$ and $x_0 = 0.2\mu^{1/4}L$. In each case, we show the quantity $h(y) \equiv \frac{x_0 g(y)}{\omega(1-y^2)}$, which allows for a simpler comparison between the several curves. Solid red line: AdS branch. Red dashed line: branch A. Red dot-dashed line: branch B. Red dotted line: branch C. Blue dashed line: Kerr-Newman case.

Thus, the entropy also tends to a universal constant value in the limit of vanishing x_0, which interestingly enough corresponds to $\mathcal{A}/(2G)$. Observe that, for fixed values of Q and x_0, this solution can be entropically favoured with respect to the one belonging to the $\mathrm{AdS}_2 \times \mathbb{S}2$ branch. In fact, we get the following condition for small values of x_0:

$$S_A > S_{\mathrm{AdS}_2 \times \mathbb{S}_2} \Leftrightarrow Q^2 < 2\sqrt{3\mu}L^2 - \frac{7x_0^2}{2} + \dots . \tag{5.56}$$

However, this is not enough in order to argue that a transition from one solution to another will take place when that bound is saturated, since the angular momentum could depend differently on x_0 in both solutions, and hence, we would be comparing black holes with different conserved charges. In fact, this solution has $x_0/n \sim 1/x_0$ when $x_0 \to 0$, which implies that the geometry departs largely from $\mathrm{AdS}_2 \times \mathbb{S}^2$, and this suggests that it actually could have a large angular momentum.

Finally, let us comment on how the black holes in this branch behave as we increase the angular momentum. Looking at Fig. 5.6 we observe three possibilities. If the charge is large enough, there is a maximum value of x_0 for which we can extend the branch, and at this point it merges with branch C. If the charge is smaller, the branch is connected to the solutions that have a finite area in the limit $x_0 \to \infty$, and if it is small enough ($Q < Q_{\mathrm{thr}} \approx 1.13\mu^{1/4}L$), it is connected to the Kerr-Newman branch. In other words, this implies that if we take an initial black hole with little charge but large area and angular momentum, the black hole will approach one of the solutions in this branch as it losses angular momentum, instead of an $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry.

Branch B

The second additional solution has the following values of n^2 and $\mathcal{A}$

$$n^2 = x_0 \frac{\sqrt{3\mu}L^2}{Q} + x_0^2 \left(\frac{25\mu L^4}{6Q^4} - \frac{4}{5} \right) + \mathcal{O}(x_0^3) \tag{5.57}$$

$$\mathcal{A} = 4\pi \left[x_0 \frac{5\sqrt{\mu/3}L^2}{Q} + x_0^2 \left(\frac{25\mu L^4}{2Q^4} - \frac{5}{3} \right) + \mathcal{O}(x_0^3) \right] . \tag{5.58}$$

In this case, the area tends to zero independently of the charge when $x_0 \to 0$. Also, unlike in the previous case, we have $x_0/n \to 0$, which we can interpret as a sign that the geometry is indeed slowly rotating. Now, the most interesting fact about this branch of solutions is that, even though the area vanishes in the limit of $x_0 \to 0$, their entropy remains finite, namely

$$S = \frac{3\pi Q^2}{5G} + x_0 \, 2\pi \frac{6Q^4 - 25\mu L^4}{25G\sqrt{3\mu}L^2 Q} + \mathcal{O}(x_0^2) \, . \tag{5.59}$$

Thus, the entropy per unit area in these black holes becomes arbitrarily large.

Branch C

The third and last additional solution allows for a series expansion in powers of $x_0^{1/2}$, and the leading terms for n^2, area and entropy read

$$n^2 = x_0 \frac{\sqrt{6\mu}L^2}{Q} - (\sqrt{x_0})^3 \, \frac{(2/3)^{1/4} \, \mu^{3/4} L^3}{Q^{5/2}} \, , \tag{5.60}$$

$$\mathcal{A} = \sqrt{x_0} \; 2^{3/4} 3^{1/4} 2\pi \mu^{1/4} L \sqrt{Q} - x_0 \frac{\pi\sqrt{6\mu}L^2}{Q} \, , \tag{5.61}$$

$$S = \sqrt{x_0} \; \frac{2^{3/4}3^{1/4}\pi \; \mu^{1/4} L \sqrt{Q}}{G} + x_0 \; \frac{\pi\sqrt{\mu/6}L^2}{GQ} \, , \tag{5.62}$$

$$\tag{5.63}$$

Note that, again, $x_0/n \to 0$, so that this solution can actually be slowly rotating, while the entropy tends to $S \to \mathcal{A}/(2G)$.

Once the desired branch is chosen, it is possible to solve the equation (5.25) numerically in order to obtain the profile of $g(y)$, as we explained previously. A comparison between these solutions is shown in Fig. 5.7.

5.4.2 Entropy as a function of area and charge

The preceding analysis is useful in order to characterize the space of near-horizon geometries of ECG, but it has the disadvantage that we cannot interpret x_0 as the spin parameter a. Thus, it is more meaningful to study the relation $S(\mathcal{A}, Q)$, which we can find exactly by using Eqs. (5.51) and (5.52). In Fig. 5.5 we only plotted part of this relation. The complete structure of $S(\mathcal{A}, Q)$ including all the solutions is quite involved and we show it as a 3-dimensional plot in Fig. 5.8. In obtaining this surface we have taken into account that the solutions of Eq. (5.52) must be such that $n^2 > 0$ and $x_0^2 > 0$. The red line corresponds to the $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometries, and interestingly these are the only ones for which $S = \mathcal{A}/(4G)$ — any other solution has $S > \mathcal{A}/(4G)$. We also represent the various $x_0 \to 0$ limits, which correspond the yellow, black and blue curves (for branches A, B and C respectively).

As we can see, for large enough horizon area, the surface in Fig. 5.8 has only one branch, which recovers the Einstein gravity behaviour when $\mathcal{A} \to \infty$. Now, imagine that

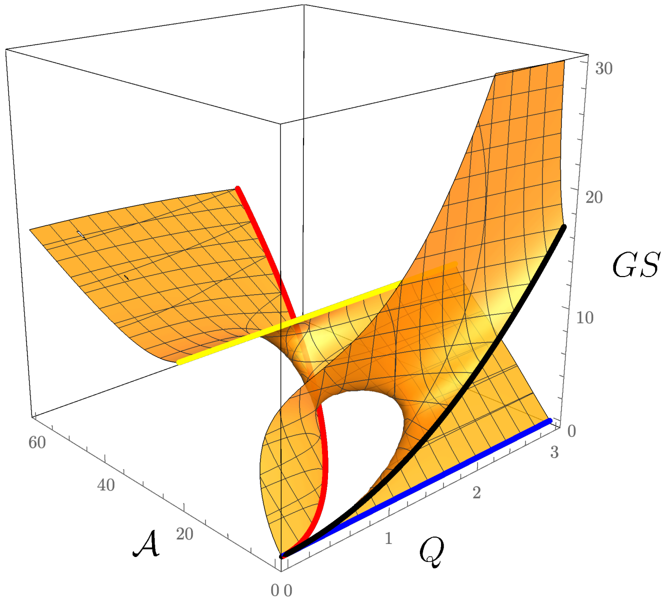

Figure 5.8: Black hole entropy as a function of the area and charge. The thick color lines represent the different $x_0 \to 0$ limits: the red line corresponds to the $\mathrm{AdS}_2 \times \mathbb{S}^2$ solutions, while yellow, black and blue lines correspond to branches A, B and C, respectively. We work in units such that $\mu L^4 = 1$.

we take one of these large black holes and we start decreasing the area leaving the charge fixed — this could be interpreted as the black hole radiating away the angular momentum.[8] We find that there are two possible endpoints of this process: if the charge is large enough, then at some point we hit an $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometry and the black hole has radiated all the angular momentum. In order to continue evaporating it must now lose charge. On the other hand, if the charge is too small (as we saw earlier, $Q < Q_{\rm thr} \approx 1.13\mu^{1/4}L$), we approach the yellow line, which corresponds to the $x_0 \to 0$ limit of branch A, and for which $\mathcal{A} = 4\pi\sqrt{3\mu}L^2$. Interestingly, in this situation the area and the entropy of the black hole remain constant even if it loses (or gains) charge. Thus, the final product of black hole evaporation is quite different depending on which path we follow in the phase space. We also observe that for small $\mathcal{A}$ the surface $S(\mathcal{A}, Q)$ is multivalued, hence transitions or decays between solutions might occur. This illustrates that the phase space of (extremal) black hole solutions may become quite complicated in higher-derivative gravity.

5.5 Discussion

In this paper we have provided the first non-perturbative examples of near-horizon geometries of rotating black holes in higher-order gravity. This has been possible thanks to the special form of the equations of motion of Einsteinian cubic gravity — the density given by (5.2) — which can be reduced to a single second-order differential equation for one variable. Even more striking, we have been able to obtain the area and the entropy exactly in terms of the parameters of the solution, and in particular, we found the rela-

[8]This picture is not completely accurate because we are moving in the space of extremal black holes. Thus, one should imagine that energy is emitted along with angular momentum, so that we keep the black hole extremal, or near-extremal, during the process.

tion between black hole area, charge and entropy, $S(\mathcal{A}, Q)$ — see Eqs. (5.51) and (5.52). It must be noted that obtaining these quantities analytically is not possible in general higher-order theories, where the simplification of the equations that we reported does not take place. However, we do expect that there is a subset of Generalized Quasi-topological theories for which the same simplification takes place. This subset will correspond to the same type of theories admitting taub-NUT solutions that was studied in [306], where, in particular, a quartic four-dimensional density of this kind was constructed. We expect that higher-order versions of these densities exist as well, and it would be interesting to study extremal near-horizon geometries in this family of theories, thus generalizing the results presented here. In fact, we believe that the higher-order generalizations could solve some of the difficulties that we have found in our analysis and that we discuss next.

5.5.1 Large angular momentum?

Perhaps the most worrisome problem we have found is that the equation (5.25) seems to have no smooth solutions when the angular momentum is large compared to the charge. In particular, purely rotating (regular) black holes do not exist even in the regime where the corrections are supposed to be small. The reason, as we explained, is the vanishing of the coefficient of g'' in Eq. (5.25) at some point, which implies that the solution will not be smooth there. This issue could go away for higher-order densities, or for some appropriate combination of those, and it would be interesting to explore this possibility. On the other hand, this problem could be related to the fact that we are dealing with extremal black holes. It is known that there are certain difficulties associated with extremality (*e.g.*, the instability of horizons [311]), and these arise explicitly in the case of higher-derivative theories — we further comment on this below. Therefore, it might happen that the problem of non-existence only affects to extremal black holes, but that (arbitrarily) near-extremal ones are fine. Despite this drawback, we believe the values found for the entropy and area of these black holes are meaningful even in the region of parameter space where no solution seems to exist. Indeed, from the point of view of EFT one should assume a perturbative expansion of the solution, and in this scheme the issue in the differential equation (5.25) disappears. Thus, at least the perturbative corrections to the entropy,

$$S = \frac{\mathcal{A}}{4G}\left[1 + \frac{24\pi^2\mu L^4(\mathcal{A} - 4\pi Q^2)}{\mathcal{A}^3} + \mathcal{O}(L^8)\right], \tag{5.64}$$

should be meaningful in the full parameter space.

5.5.2 Multiplicity of solutions

Paradoxically enough, when Eq. (5.25) allows for solutions, it has many. We have seen that for fixed values of x_0 and Q, we have usually several branches of solutions with different values of the area and the entropy. But we also observed that, even when the corresponding branch has been chosen, the equation (5.25) can have several solutions. This is, we can have different near-horizon geometries with the same values of the charge, x_0, area and entropy, which only differ in the shape of the horizon. Thus, in Sec. 5.3 we only constructed numerically the solutions that are smooth deformations of $\mathrm{AdS}_2 \times \mathbb{S}^2$ geometries, but in general there are more solutions which are characterized by the same set of integration constants. In particular, the equation (5.36) corresponding to the limit

$x_0 \to 0$ seems to have an increasing number of solutions as Q grows. This means that there are solutions of the form $\mathrm{AdS}_2 \times \mathcal{M}_2$, where $\mathcal{M}_2$ is not a sphere, but nonetheless all of these solutions have the same area and entropy. A similar situation occurs for finite x_0. While this is an interesting phenomenon, a thorough classification of these solutions would considerable enlarge the present manuscript, and thus these additional solutions could be studied elsewhere. The degeneracy of solutions seems to be related to the sign of the higher-order coupling μ, and it would have not appeared had we taken $\mu < 0$. The reason for taking $\mu > 0$ is that this is required in order for asymptotically flat/AdS black holes to exist [296]. However, it is possible that for other higher-order densities the sign that allows for black holes is the same that would yield unicity of near-horizon geometries.[9]

5.5.3 Global solutions?

Another relevant question is whether there exist global black hole solutions (containing an asymptotic region) of which the solutions we have constructed are the near-horizon limit. Although it may appear shocking at first, we do not expect those solutions to exist. The reason is that the boundary problem in higher-derivative gravity is not well-posed in the presence of a degenerate horizon. This is more easily understood in the case of static, charged black holes, which allow for a simple description in ECG. Those solutions were briefly discussed in [296], where, similarly to the case here, it was shown that the equations of motion reduce to a second-order equation for one variable. Then, one has to impose a boundary condition at infinity and another one at the horizon, and this fixes the solution. But when the horizon is degenerate, the condition at the horizon turns out to fix two integration constants and it is not possible to demand the asymptotic condition. Hence, no black hole solutions exist in that case. Nevertheless, arbitrarily near-extremal ones exist, and we expect that the same behaviour will be found in the rotating case. Hence, the near-horizon geometries we have constructed make sense as a limit that non-extremal black holes can approach, but never reach. In particular, the area and entropy (and also the shape) of non-extremal black holes will tend to those found here when they approach extremality.

5.5.4 Asymptotic charges

Finally, one limitation of the near-horizon analysis is that we lose the information about the mass and the angular momentum of these black holes. We argued that the variables x_0 and n would be related, respectively, to the spin a and to the mass M but we lack a precise relation. Knowing the values of a and M would be very interesting in order to study corrections to the extremality bound and to determine the relation between the entropy and the physical charges, $S(a, Q)$. A possible direction to achieve this goal would entail finding a generalization of Komar charge for higher-order gravities that would allow us to write the asymptotic charges as an integral over the horizon [164].

[9] A similar phenomenon has been observed in the cosmological context, where the appropriate sign for black holes in ECG is the opposite to the one required in order to produce inflation. However, for quartic densities (and in general, densities containing even powers of the curvature), both signs agree [304].

6

Quasinormal Modes of NUT-charged Black Branes in the AdS/CFT Correspondence

This chapter is based on:
Quasinormal modes of NUT-charged black branes in the AdS/CFT correspondence
P. A. Cano, D. Pereñiguez
arXiv:2101.10652

Motivated by the AdS/CFT correspondence [312–314], the study of asymptotically anti-de Sitter (AdS) black holes has been a major field of research in the last two decades. According to this correspondence, black hole solutions in the bulk of AdS are dual to a thermal quantum field theory living in the boundary of the spacetime and whose temperature is given by the Hawking's temperature of the black hole. In this context, the holographic dictionary can be applied to gain a great deal of information about the hydrodynamics of strongly-coupled plasmas[1] by studying the properties of the black hole solutions [315–318]. In particular, perturbations of different fields in the background of a black hole geometry can be used to compute transport coefficients and correlators in the dual theory, and thus providing us with valuable results that can be difficult to obtain by first principles in the quantum theory. In the case of metric perturbations, these couple to the stress-energy tensor of the boundary theory, and hence they capture density and pressure fluctuations.

In a black hole, the late-time behaviour of perturbations is ruled by the quasinormal modes (QNMs), which are solutions satisfying an outgoing boundary condition at the horizon (*i.e.*, absence of waves coming from the horizon) plus — in the context of AdS/CFT — Dirichlet boundary conditions at infinity— see the reviews [69,319]. Quasinormal modes only exist for a discrete set of complex frequencies, called the QNM frequencies, and whose imaginary part determines the damping time. The QNMs of black holes defined in this way correspond to the poles of the retarded Green functions of the dual theory and therefore they characterize the response of the dual plasma under perturbations [320–327].

A large part of the literature on this topic has focused on AdS_5 solutions — see the previous references — and especially on black holes with a planar horizon, since these are dual to a 4-dimensional CFT in flat space. In this paper, nonetheless, we are interested in AdS_4 geometries. As a matter of fact, the AdS_4/CFT_3 correspondence is well-motivated [328] and it can indeed be relevant for certain condensed-matter systems that behave effectively as $2+1$ dimensional [317,318]. The quasinormal modes of 4-dimensional Schwarzschild-AdS black holes were studied in Refs. [329–332], while those of black holes

[1] Since these can be studied in an appropriate regime under the hydrodynamic approximation, we sometimes refer to these plasmas as "fluids".

with planar, toroidal and cylindrical topologies were first computed in Refs. [333, 334]. The results on the latter were later revised and extended in Ref. [335] by implementing the boundary conditions required by holography. On the other hand, the quasinormal modes of large Kerr-AdS black holes were analyzed in [336].

In addition to these cases, there is a family of gravitational solutions that has not been yet fully exploited in holography: black holes with NUT charge [337–340]. Taub-NUT[2] solutions have the distinct property of being only locally asymptotically AdS, which translates into the fact that the boundary is no longer (locally) conformally flat. Thus, NUT charge breaks conformal invariance of the dual theory, and this may allow us to probe non-trivial aspects of the CFT. For instance, Euclidean AdS-Taub-NUT solutions describe CFTs placed on squashed spheres [339, 340], and studying how the free energy depends on the NUT charge has led to interesting results both in supersymmetric [341,342] and non-supersymmetric [301, 343–345] setups.

Lorentzian Taub-NUT solutions, on the other hand, have been less studied in the context of holography due to their seemingly pathological properties. Indeed, these solutions contain Misner strings and closed time-like curves [346, 347], and they give rise to an apparent failure of the first law of thermodynamics [348]. However, there is a renewed interest in "rehabilitating" these spacetimes. On the one hand, Ref. [349] has shown that freely falling observers do not experience any of these pathologies, since there are no closed time-like geodesics and Misner strings are invisible to those observers — see also [350]. On the other hand, the thermodynamic description of Taub-NUT solutions has been finally understood on the basis that Misner strings are acceptable and that, accordingly, the NUT charge should be regarded as an independent thermodynamic variable [351–353] — see also [354].

Lorentzian AdS-Taub-NUT solutions give indeed rise to interesting boundary theories. In Refs. [355, 356] it was noted that, unlike the Kerr-AdS solution, NUT-charged solutions describe fluids with vorticity, and hence explore a qualitatively different aspect of the dual theory. More recently, Ref. [357] initiated the study of scalar perturbations of spherical Taub-NUTs in connection to holography, finding that the result is dramatically dependent on whether the Misner string is regarded as physical or not. In this work, we will consider instead the case of planar Taub-NUT black holes [358] — we recall that, just like in the case of AdS black holes, NUT-charged solutions can have either spherical, planar or hyperbolic transverse sections. We consider this case to be particularly interesting for two main reasons. First, the planar NUT solutions are free of Misner strings, so that one gets rid of all the difficulties and ambiguities introduced by these objects. Second, these solutions are a generalization of the planar black holes, and hence the boundary metric can be considered as a continuous deformation of flat space. More precisely, the boundary of these geometries is similar to a Gödel universe [358], where the NUT charge controls the rotation. In this sense, it is interesting to see how the properties of the dual strongly-coupled plasma change as we increase the NUT charge.

In this paper, we explore this question by computing the quasinormal mode spectrum of planar Taub-NUT black holes. We shall perform an analysis of (massless) scalar, electromagnetic and gravitational perturbations, providing — to the best of our knowledge — the first complete calculation of quasinormal modes of black holes with NUT charge.

The paper is organized as follows

[2]We use the term "Taub-NUT" to refer indistinctly to both NUT-type and bolt-type solutions.

- In Section 6.1 we review the planar Taub-NUT geometries, establishing their basic properties, their thermodynamics description and introducing the Newman-Penrose formalism that we use in the next section.

- In Section 6.2 we perform perturbation theory on these geometries. The case of a scalar field is considered first and we note an interesting analogy between the angular separation of the QNMs and Landau quantization. We then use the Newman-Penrose formalism to derive separable equations for the master electromagnetic and gravitational variables.

- In Section 6.3 we study the boundary conditions for QNMs. Imposing Dirichlet boundary conditions on the electromagnetic and gravitational perturbations, we derive the form of the boundary conditions on the master Newmann-Penrose variables. We find that, besides the QNM frequency, the QNMs depend on another parameter related to the polarization, and which has to be determined by solving simultaneously the equations for both NP variables. In the gravitational case we determine analytically this polarization parameter by using the Teukolsy-Starobinsky identities, and hence we reduce the problem to solving only one equation with fixed boundary conditions. On the other hand, we find that the electromagnetic NP variables satisfy degenerate equations, and therefore the polarization parameter cannot be determined.

- We compute the QNM frequencies of scalar and gravitational perturbations in Section 6.4. Despite the breaking of parity, the spectra of both types of perturbations is symmetric under the change of sign of the NUT charge. We obtain an analytic approximation for a special family of gravitational QNMs, that we call pseudo-hydrodynamic modes, whose frequency vanishes in the zero NUT charge limit. In addition, we provide strong evidence that no unstable mode exists.

- We present our conclusions in Section 6.5.

6.1 Planar Taub-NUT black holes and their holographic dual

We consider Einstein gravity with a negative cosmological constant,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left[R+\frac{6}{L^2}\right], \tag{6.1}$$

In this paper, we are interested in the following solution of Einstein's theory, corresponding to a Taub-NUT black hole with planar topology [340],

$$ds^2 = -V(r)\left(dt+\frac{2n}{L^2}xdy\right)^2+\frac{dr^2}{V(r)}+\frac{r^2+n^2}{L^2}\left(dx^2+dy^2\right), \tag{6.2}$$

where n is the NUT charge, the function $V(r)$ is given by

$$V(r) = \frac{(r-r_+)\left(3n^4+6n^2rr_+ + rr_+\left(r^2+rr_+ + r_+^2\right)\right)}{L^2 r_+\left(n^2+r^2\right)}, \tag{6.3}$$

and the coordinates (x, y) span $\mathbb{R}^2$. For $n = 0$ this solution reduces to the AdS black brane, but nevertheless it has some remarkable properties that we review next. First of all, this solution conserves all the symmetries of the black brane, corresponding to time translations and the symmetries of $\mathbb{R}^2$, with the difference that the latter now act non-trivially in the time variable. The corresponding four Killing vectors read

$$\begin{aligned}\xi_{(t)} &= \partial_t\,,\\ \xi_{(1)} &= -\frac{2n}{L^2}y\partial_t + \partial_x\,,\\ \xi_{(2)} &= \partial_y\,,\\ \xi_{(3)} &= \frac{n}{L^2}(x^2-y^2)\partial_t + y\partial_x - x\partial_y\,.\end{aligned} \tag{6.4}$$

Note that these symmetries allow one to consider quotients of this solution by discrete groups. For instance one may take y to be periodic, in which case the black hole would have cylindrical topology. We will restrict to the case of (x, y) spanning the plane.

The event horizon of the black hole is located at $r = r_+ > 0$, which is a Killing horizon for $\xi_{(t)}$. The corresponding surface gravity reads

$$\kappa = \frac{1}{2}V'(r_+) = \frac{3(n^2+r_+^2)}{2L^2 r_+}\,. \tag{6.5}$$

One can see that the function $V(r)$ is strictly positive for $r_+ < r < \infty$ and hence there are no other horizons for ∂_t. There are, however, horizons for the other Killing vectors, which indicate the presence of closed timelike curves (CTCs). For instance, the norm of $\xi_{(2)}$ reads

$$\xi_{(2)}^2 = \frac{r^2+n^2}{L^2} - V(r)\left(\frac{2nx}{L^2}\right)^2\,, \tag{6.6}$$

and hence it becomes timelike if x is large. A quite representative CTC is given by $x = R\cos\phi$, $y = R\sin\phi$, $t = -nxy/L^2$, with $\phi \in [0, 2\pi)$, whose tangent vector $u^\mu = \dot{x}^\mu$ has a norm

$$u^2 = -V(r)\frac{n^2R^4}{L^4} + \frac{(r^2+n^2)R^2}{L^2}\,, \tag{6.7}$$

so that it is everywhere timelike if R is large enough. The symmetries of this spacetime imply that there are CTCs of this type around any point (in the region $r > r_+$), but, however, there are no closed timelike geodesics [349, 359], so that the solution is possibly less pathological than one would expect. More importantly, as the example above shows, CTCs only appear at large distances, which means that, around any point there is an open set in which no CTCs exist. In particular, there are no CTCs contained in regions with a radius $R < L^2/n$ in the (x, y) plane. Within these regions one can define a Cauchy surface and make sense of dynamics [360, 361]. On the other hand, unlike the spherical Taub-NUT solutions, these NUT black branes do not possess Misner singularities.

At infinity, the metric function $V(r)$ behaves as $V(r) = r^2/L^2 + \mathcal{O}(1)$, and hence the boundary metric at $r \to \infty$ is conformally equivalent to

$$d\hat{s}^2 = -\left(dt + \frac{2n}{L^2}xdy\right)^2 + dx^2 + dy^2\,. \tag{6.8}$$

This metric is not conformally flat, and therefore the solution is only asymptotically locally AdS. In the boundary theory, this means that conformal invariance is broken. However, the boundary still has many symmetries — given by (6.4) — and one can see that it is a homogeneous space corresponding to a Lorentzian continuation of Nil space — the group manifold of Heisenberg's group. Indeed, note that the translational Killing vectors satisfy the Heisenberg's algebra

$$\left[\xi_{(t)}, \xi_{(1)}\right] = \left[\xi_{(t)}, \xi_{(2)}\right] = 0\,, \quad \left[\xi_{(1)}, \xi_{(2)}\right] = \frac{2n}{L^2}\xi_{(t)}\,. \tag{6.9}$$

On a more physical perspective, the metric (6.8) can be interpreted as a rotating universe, very similar to the non-trivial $(2+1)$-dimensional section of the famous Gödel solution [362], the paradigmatic example of a universe with closed timelike curves.[3] Hence, when one applies the holographic dictionary to these solutions, one is probing the dynamics of a quantum theory placed in this exotic spacetime. Although the existence of a globally defined timelike Killing vector allows one to define a Hamiltonian, performing quantum field theory in this background is challenging due to its unusual causal structure [359,364–366].

In order to answer if these metrics are appropriate to perform QFT, one can also study the "acceptability" criterion of Refs. [367–369], obtained after Wick rotation of the metric. Following the analysis in section 4.5 of [368], it is easy to see that the quasi-Euclidean metric obtained by the replacement $t = i\tau$ in (6.2) is allowable only in the regions with $(x-x_0)^2 + (y-y_0)^2 < L^2/n$ around any point (x_0, y_0), which coincide with the causally regular regions.[4] Extending a QFT outside these regions thus must involve non-standard methods. In this sense, holography can be used to gain some insight about the behavior of a quantum theory in such spacetime. Besides, the dual CFT would be in a thermal state whose properties are determined by the thermodynamic quantities of the black hole, that we review next.

6.1.1 Thermodynamics

The temperature of the NUT-charged black branes is given by Hawking's result $T = \kappa/(2\pi)$, so that

$$T = \frac{3(n^2 + r_+^2)}{4\pi L^2 r_+}\,. \tag{6.10}$$

One can see that, for a given n, the temperature reaches its minimum value for $r_+ = |n|$, in whose case we have $T = T_*$, where

$$T_* = \frac{3|n|}{2\pi L^2}\,. \tag{6.11}$$

On the other hand, the temperature diverges both for $r_+ \to 0$ and $r_+ \to \infty$. Hence, when $T > T_*$, there are two different black hole solutions with the same T and n. This allows us to distinguish three different families of solutions, corresponding to $n < -r_+$, $-r_+ < n < r_+$ or $n > r_+$. We can also identify the mass of the solution by analyzing the

[3] More precisely, the metric (6.8) is the equatorial section of the Som-Raychaudhuri solution [363], as originally noted in [358]. Both metrics have qualitatively similar properties.

[4] As observed by Witten [368], in the case of rotating black holes, considering an imaginary angular momentum gives rise to Euclidean metrics but it is also problematic, ultimately because the horizon is generated by the vector $\partial_t + \Omega\partial_\phi$, where Ω is the angular velocity. However, in our case the generator of the horizon is only ∂_t, so there seems to be no reason not to consider also a imaginary NUT charge $n = i\hat{n}$, in which case one obtains a Euclidean and perfectly allowable metric — see (6.15) below.

behaviour near infinity. In fact, one can just apply the usual the ADM result which tells us that the total energy E can be identified by looking at the $1/r$ term in the asymptotic expansion of V. In particular, the coefficient of that term should be equal to $-8\pi GL^2E/V_2$, where V_2 is the volume of the transverse space, $V_2 = \int dxdy$. Note that in this case V_2 is infinite, and hence it is more appropriate to talk about energy density $\rho = E/V_2$, rather than total energy. This quantity, in fact, can be interpreted as the energy density in the boundary CFT. The expansion of $V(r)$ reads

$$V(r) = \frac{r^2}{L^2} + \frac{5n^2}{L^2} - \frac{r_+^4 + 6n^2r_+^2 - 3n^4}{L^2rr_+} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{6.12}$$

and therefore, we get

$$\rho = \frac{r_+^4 + 6n^2r_+^2 - 3n^4}{8\pi GL^4r_+}\,. \tag{6.13}$$

On the other hand, the entropy of the black hole is given by $S = A/(4G)$, but since this area of the horizon is divergent, it is again convenient to work in terms if the entropy density, $s = S/V_2$, which reads

$$s = \frac{r_+^2 + n^2}{4GL^2} \tag{6.14}$$

Now, an apparent puzzle in the case of these solutions is that the first law of thermodynamics does not seem to hold, *i.e.*, we get $d\rho \neq Tds$ when varying the previous expressions with respect to r_+. However, the reason is that the NUT charge should also be interpreted as a thermodynamical variable which will modify the first law. For a long time this was a source of confusion in the case of spherical Taub-NUT black holes, since regularity of the Euclidean geometry imposes a restriction between NUT charge and temperature [348]. Only recently it was realized that one can achieve a full-cohomegeneity first law for spherical NUTs by allowing the NUT charge to vary independently. In the case of planar NUT black holes, however, there is no restriction between n and T, and it is natural to treat the NUT charge as an additional thermodynamic variable. To the best of our knowledge, the existence of a first law in the case of planar Taub-NUT solutions was first reported in [306].

In order to complete the thermodynamic characterization of these planar NUT black holes, we must compute the free energy from the Euclidean on-shell action. The Euclidean solution is obtained, not only by Wick-rotating the time coordinate, $t = i\tau$, but also the NUT charge, $\hat{n} = in$. In that case the metric reads

$$ds_E^2 = V(r)\left(d\tau + \frac{2\hat{n}}{L^2}xdy\right)^2 + \frac{dr^2}{V(r)} + \frac{r^2 - \hat{n}^2}{L^2}\left(dx^2 + dy^2\right) \tag{6.15}$$

It is important to note that, in Euclidean signature, only the solutions with $r_+^2 \geq \hat{n}^2$ are regular, which means that the Lorentzian solutions with $n^2 > r_+^2$ do not have an Euclidean description. This suggests that for a given $T > T_*$ only the solution with $r_+^2 \geq n^2$ should be taken into account in the path integral, and hence that it is the dominant saddle. Let us also mention that, in the literature, the Euclidean solutions with $r_+^2 = \hat{n}^2$ are called Taub-NUT, while the rest are Taub-bolt. However, we shall make no distinctions since the former can be considered as a limit of the latter.

The free energy can be computed from the following well-posed and regularized Euclidean action

$$I_E = -\frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left[R+\frac{6}{L^2}\right] - \frac{1}{8\pi G}\int d^3x\sqrt{h}\left[K-\frac{2}{L}-\frac{L}{2}\mathcal{R}\right], \tag{6.16}$$

where K is the extrinsic curvature of the boundary and $\mathcal{R}$ is the Ricci scalar of the boundary's intrinsic. The free energy $F = TI_E$ reads

$$F = -\frac{V_2(r_+^4+3\hat{n}^4)}{16\pi G L^4 r_+}\,. \tag{6.17}$$

Let us then define the free-energy density $\varepsilon = F/V_2$ and express this result in terms of the Lorentzian NUT charge n,

$$\varepsilon = -\frac{(r_+^4+3n^4)}{16\pi G L^4 r_+}\,. \tag{6.18}$$

Now, it turns out that, instead of n, the thermodynamic relations are most naturally written in terms of the variable $\theta = \frac{1}{n}$. Then, using the chain rule one can compute the derivatives of the free energy at constant θ and T, which read

$$s = -\left(\frac{\partial\varepsilon}{\partial T}\right)_\theta = \frac{r_+^2+n^2}{4GL^2}\,, \tag{6.19}$$

$$\psi = -\left(\frac{\partial\varepsilon}{\partial\theta}\right)_T = \frac{3n^3(r_+^2-n^2)}{8\pi G L^4 r_+}\,. \tag{6.20}$$

We check that s indeed coincides with the Bekenstein-Hawking entropy density. On the other hand, ψ is a new thermodynamic potential conjugate to θ. Making use of these results, one observes that the energy ρ computed according to the ADM prescription, coincides with the double Legendre transform of the free energy with respect to T and θ.

$$\rho = \varepsilon + Ts + \theta\psi\,. \tag{6.21}$$

This is is not the standard definition of internal energy, which suggests that, in the presence of NUT charge, the potentials ε and ρ probably have a different thermodynamic interpretation. In any case, this result implies that ρ satisfies the following first law,

$$d\rho = Tds + \theta d\psi\,. \tag{6.22}$$

Finally, we can study the thermodynamic stability of these solutions. One can first can compute the specific heat at constant θ,

$$C_\theta = T\left(\frac{\partial s}{\partial T}\right)_\theta = \frac{r_+^2(n^2+r_+^2)}{2GL^2(r_+^2-n^2)}\,, \tag{6.23}$$

and one can see that $C_\theta > 0$ as long as $r_+^2 > n^2$, implying thus stability when n is held fixed. More generally, one can study the concavity of the free energy, for which one may compute the second variation of ε, which reads

$$\delta^2\varepsilon = -\frac{2\pi r_+^3}{3G\left(r_+^2-n^2\right)}\delta T^2 - \frac{2n^3\left(n^2+r_+^2\right)}{2GL^2\left(r_+^2-n^2\right)}\delta T\delta\theta + \frac{3n^4\left(3r_+^4-10n^2r_+^2+3n^4\right)}{8\pi G L^4 r_+\left(r_+^2-n^2\right)}\delta\theta^2\,. \tag{6.24}$$

The solution will be thermally stable if this is a negative-definite quadratic from, but we can see that this never happens because the term with $\delta\theta^2$ is positive for $r_+^2 > n^2$, while the one of δT^2 is only negative in that region. Therefore, these planar Taub-NUT black holes are only thermodynamically stable under changes of the temperature but not under changes of n.

6.1.2 Newman–Penrose formalism

The description of perturbations on a black hole spacetime is a task of extraordinary complexity. The linearized equations governing first order field components on local coordinates are considerably involved already in the simplest backgrounds such as Schwarzschild's black hole, and almost intractable in more realistic cases like Kerr's spacetime. In addition, it is far from obvious how the large amount of gauge symmetry should be fixed. Teukolsky's seminal work [105] constituted a major breakthrough in the clarification of these issues. Considering an algebraically special background space, of Petrov type D (e.g. Schwarzschild and Kerr spacetimes), he derived decoupled equations for perturbations of several kinds and, furthermore, these admit solutions in separable form. One of the elements underlying such a remarkable success is the Newman–Penrose (NP) formalism [50]. In particular, it provides a very natural formulation of Petrov's classification, as well as the Goldberg–Sachs theorem, and this translates into the vanishing of several NP variables of the background. It is in this situation that the equations decouple and, in addition, become gauge invariant.

The study of perturbations on the background (6.2) can be conveniently performed in the NP formalism. A Newman–Penrose frame on a pseudo–Riemannian space[5] is a complex tetrad $\mathbf{e}_a$,

$$\mathbf{e}_1 = \mathbf{m}, \quad \mathbf{e}_2 = \overline{\mathbf{m}}, \quad \mathbf{e}_3 = \mathbf{l}, \quad \mathbf{e}_4 = \mathbf{k}, \tag{6.25}$$

composed of two real, null vectors $\mathbf{k}$ and $\mathbf{l}$, and one complex, null vector $\mathbf{m}$ together with its conjugate $\overline{\mathbf{m}}$, so that

$$\mathbf{k}\cdot\mathbf{k} = \mathbf{l}\cdot\mathbf{l} = \mathbf{m}\cdot\mathbf{m} = 0, \tag{6.26}$$

and these are further subject to the normalization conditions

$$\mathbf{k}\cdot\mathbf{l} = -\mathbf{m}\cdot\overline{\mathbf{m}} = -1, \quad \mathbf{k}\cdot\mathbf{m} = \mathbf{l}\cdot\mathbf{m} = 0. \tag{6.27}$$

When acting as operators on functions φ, it is customary to give particular names to the vectors of the NP basis,

$$D\varphi := k^\mu \partial_\mu \varphi, \quad \Delta\varphi := l^\mu \partial_\mu \varphi, \quad \delta\varphi := m^\mu \partial_\mu \varphi, \quad \delta^*\varphi := \overline{m}^\mu \partial_\mu \varphi. \tag{6.28}$$

A convenient choice for the space (6.2) is

$$\mathbf{k} = k^\mu \partial_\mu = \frac{1}{V}\left(\partial_t + V\partial_r\right), \quad \mathbf{l} = l^\mu \partial_\mu = \frac{1}{2}\left(\partial_t - V\partial_r\right), \tag{6.29}$$

$$\mathbf{m} = m^\mu \partial_\mu = iL\frac{e^{-i\arctan(r/n)}}{\sqrt{2(n^2+r^2)}}\left(\partial_x + i\partial_y - i\frac{2nx}{L^2}\partial_t\right). \tag{6.30}$$

The vectors $\mathbf{k}$ and $\mathbf{l}$ are geodesic and shear-free so that the following spin coefficients vanish

$$\kappa = \sigma = \nu = \lambda = 0\,. \tag{6.31}$$

[5]We will be following the conventions in [51].

By the Goldberg–Sachs theorem it follows that the space must be of Petrov type D, so four out of the five Weyl scalars vanish

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0. \tag{6.32}$$

In addition, the frame has been chosen to be parallelly propagated along $\mathbf{k}$, *i.e.*

$$\nabla_{\mathbf{k}}\mathbf{k} = \nabla_{\mathbf{k}}\mathbf{l} = \nabla_{\mathbf{k}}\mathbf{m} = 0, \tag{6.33}$$

a property that implies the vanishing of additional spin coefficients.

As shown by Teukolsky [105], the vanishing of these quantities makes perturbation theory tractable on such a background — we will make use of those results in next section. For that, we need the spin connection, whose non-vanishing components read

$$\rho = -\Gamma_{142} = \frac{-1}{r+in}, \quad \mu = \Gamma_{231} = -\frac{V}{2(r+in)}, \quad \gamma = \frac{1}{2}\left(\Gamma_{433} - \Gamma_{123}\right) = \frac{1}{4}\left(V' + \frac{2ni}{n^2+r^2}V\right) \tag{6.34}$$

On the other hand, the Ricci tensor has the same components as the metric $R_{12} = -R_{34} = -6/L^2$, and the only non-vanishing Weyl scalar, Ψ_2, reads

$$\Psi_2 = -C_{\mu\nu\rho\sigma}k^\mu m^\nu l^\rho \overline{m}^\sigma = -C_{4132} = -\frac{1-3\epsilon i}{2L^2}\left(\frac{1+i\epsilon}{(r/r_+)+i\epsilon}\right)^3, \tag{6.35}$$

where $\epsilon = n/r_+$. This completes the enumeration of non-vanishing NP variables of the space (6.15), in the frame (6.29).

6.2 Perturbation theory

In this section we study scalar, electromagnetic and gravitational perturbations around the planar NUT black holes introduced in the previous section. By using the Newman-Penrose formalism, we will show that in all cases the perturbations can be analyzed through a few master variables that satisfy decoupled equations. Once the problem is reduced to a decoupled equation for a scalar variable Ψ, one can try to separate variables. Now, an important difference with respect to the NUT-neutral case is that the translational Killing vectors $\xi_{(1)}$, $\xi_{(2)}$ do not commute, $[\xi_{(1)}, \xi_{(2)}] \neq 0$, and as usual these do not commute with the rotational vector $\xi_{(3)}$. Hence, one cannot fully separate the equations a priori, and at best one can choose the variable Ψ to be an eigenfunction for one of the sets of commuting Killing vectors

$$\{\xi_{(t)}, \xi_{(1)}\}, \quad \{\xi_{(t)}, \xi_{(2)}\}, \quad \{\xi_{(t)}, \xi_{(3)}\}. \tag{6.36}$$

In the coordinates in which (6.2) is expressed, the vector $\xi_{(2)} = \partial_y$ is a coordinate vector and hence it is appropriate to choose the set $\{\xi_{(t)}, \xi_{(2)}\}$. Due to the symmetries of the metric, this is completely equivalent to choosing the set $\{\xi_{(t)}, \xi_{(1)}\}$. To see this, notice that the transformation $t' = t + \frac{2n}{L^2}xy$, $x' = -y$, $y' = x$ leaves the metric invariant while setting $\xi_{(1)} = \partial'_y$. On the other hand, the analysis of quasinormal modes using the set $\{\xi_{(t)}, \xi_{(3)}\}$ is more obscure, but one expects again that the results would be equivalent. From now on we assume that our perturbations are eigenfunctions of $\xi_{(t)}$ and $\xi_{(2)}$, and hence we have

$$\left.\begin{array}{rcl}\xi_{(t)}\Psi &=& -i\omega\Psi \\ \xi_{(2)}\Psi &=& ik\Psi\end{array}\right\} \quad \Rightarrow \quad \Psi = e^{-i(\omega t - ky)} h(r,x)\,. \tag{6.37}$$

In addition, the dependence on the x and r coordinates can be further separated as we show below.

6.2.1 Scalar perturbations

Let us consider first the case of a massless scalar field ϕ in the background of (6.2) satisfying the wave equation,

$$\nabla^\mu \nabla_\mu \phi = 0\,. \tag{6.38}$$

As we just discussed above, we separate the t and y coordinates according to

$$\phi = e^{-i(\omega t - ky)} \frac{\psi(r,x)}{\sqrt{r^2+n^2}}\,, \tag{6.39}$$

where the factor of $1/\sqrt{r^2+n^2}$ is conventional. Then, we find that ψ satisfies the following equation:

$$-V\frac{\partial}{\partial r}\left(V\frac{\partial \psi}{\partial r}\right) + \psi\left[-\omega^2 + \frac{V}{r^2+n^2}\left(V' r + \frac{n^2 V}{(r^2+n^2)}\right)\right] + \frac{VL^2}{(n^2+r^2)}\left[-\frac{\partial^2\psi}{\partial x^2} + \left(k + \frac{2nx\omega}{L^2}\right)^2\psi\right] = 0\,. \tag{6.40}$$

We then note that this equation admits for separable solutions

$$\psi(r,x) = Y(r)\mathcal{H}(x)\,, \tag{6.41}$$

where $Y(r)$ and $\mathcal{H}(x)$ satisfy respectively the following equations

$$-V\frac{d}{dr}\left(V\frac{dY}{dr}\right) + Y\left[-\omega^2 + \frac{V}{r^2+n^2}\left(2L^2\mathcal{E} + V' r + \frac{n^2 V}{(r^2+n^2)}\right)\right] = 0\,, \tag{6.42}$$

$$-\frac{1}{2}\frac{d^2\mathcal{H}}{dx^2} + \frac{1}{2}\left(k + \frac{2nx\omega}{L^2}\right)^2\mathcal{H} = \mathcal{E}\mathcal{H}\,, \tag{6.43}$$

where $\mathcal{E}$ is a constant. In the case of vanishing NUT charge, this constant can take any value, as it is related to the wavenumber in the x direction, which can be chosen freely. This situation changes dramatically in the presence of NUT charge. Indeed, we observe a quite remarkable fact: the equation (6.43) is identical to that of a quantum harmonic oscillator, where the point of equilibrium is located at $x_0 = -kL^2/(2n\omega)$, and where the corresponding mass and frequency are $m = 1$, $\omega_{os}^2 = (2n\omega/L^2)^2$. There is an even more accurate analogy with Landau quantization that we explore below. Now we search for regular solutions such that $\mathcal{H}(x) \to 0$ at $x \to \pm\infty$, and this leads to the familiar results for the eigenfunctions and eigenvalues of the harmonic oscillator. There is a catch, though, since we have to take into account that ω is complex and that n can have either sign. Thus, we must distinguish between the cases $\mathrm{Re}(n\omega) > 0$ and $\mathrm{Re}(n\omega) < 0$. Introducing

$$s = \mathrm{sign}\left[\mathrm{Re}(n\omega)\right] \,, \tag{6.44}$$

we have that the physically relevant solution of (6.43) reads

$$\mathcal{H}_q(x) = e^{-s\frac{n\omega}{L^2}\left(x+\frac{kL^2}{2n\omega}\right)^2} H_q\left(\sqrt{\frac{sn\omega}{L^2}}\left(x+\frac{kL^2}{2n\omega}\right)\right), \quad q = 0, 1, \ldots, \tag{6.45}$$

where $H_q(z)$ are the Hermite's polynomials. The eigenvalues $\mathcal{E}_q$ read in turn

$$\mathcal{E}_q = \frac{sn\omega}{L^2}(1+2q)\,. \tag{6.46}$$

Thus, unlike the NUT-neutral case, we obtain a quantization condition on the angular part of the perturbations, and hence the spectrum of quasinormal modes will be discrete. Also note that the eigenvalues $\mathcal{E}_q$ are independent from the wavenumber in the y direction, k, and hence the quasinormal modes will be degenerate.

Now we can bring this result to the radial equation (6.48), and it also proves useful to perform the following redefinitions

$$z = \frac{r_+}{r}\,, \quad \epsilon = \frac{n}{r_+}\,, \quad \hat{\omega} = \frac{L^2\omega}{r_+}\,, \quad \hat{V} = \frac{L^2 V}{r_+^2}\,. \tag{6.47}$$

Then, the radial equation reads

$$\hat{V}z^2\frac{d}{dz}\left(\hat{V}z^2\frac{dY}{dz}\right) + Y\left[\left(\hat{\omega}^2 - \frac{z^2\hat{V}}{1+z^2\epsilon^2} \quad 2s\epsilon\hat{\omega}(1+2q) - \partial_z\hat{V}z + \frac{\epsilon^2 z^2\hat{V}}{(1+\epsilon^2)}\right)\right] = 0 \tag{6.48}$$

Notice that the only free parameters in this equation are ϵ, the dimensionless frequency $\hat{\omega}$ and the index q.

*

Relation to Landau quantization Interestingly, the perturbations in the Taub-NUT backgrounds organize in an analogous way to the Landau levels of a charged particle moving in a uniform magnetic field. In order to establish this analogy, let us first note that we can write the metric (6.2) in a gauge-invariant form as

$$ds^2 = -V(r)\left(dt + A\right)^2 + \frac{dr^2}{V(r)} + \frac{r^2+n^2}{L^2}\left(dx^2+dy^2\right)\,, \tag{6.49}$$

where A is a 1-form satisfying $dA = \frac{2n}{L^2}dx \wedge dy$. Thus, coordinate transformations of the form $t \to t + f(x,y)$ can be reabsorbed as gauge transformations $A \to A - df$. Then, one can in fact interpret this A as a uniform magnetic field with magnitude $B = 2n/L^2$. Let us then consider a particle of charge e moving in the (x,y) plane (in flat space) in the background of this field. The Hamiltonian is given by

$$\mathbf{H} = \frac{1}{2}\pi_x^2 + \frac{1}{2}\pi_y^2\,, \tag{6.50}$$

where, in the gauge $A = Bxdy$, the momenta read

$$\pi_x = -i\partial_x \,, \quad \pi_y = -i\partial_y - exB \,. \tag{6.51}$$

Then, the Schördinger equation $\mathbf{H}\psi = E\psi$ yields

$$\left[-\frac{1}{2}\partial_x^2 + \frac{1}{2}\left(-i\partial_y - exB\right)^2\right]\psi = E\psi \,, \tag{6.52}$$

and by using $-i\partial_y\psi = k\psi$ we get the same equation (6.43) we got for the angular part of the perturbations in the Taub-NUT geometry, provided one identifies the charge of the particle with the frequency of the perturbation as $e = -\omega$. Thus, the transverse (x, y) part of the quasinormal modes of the Taub-NUT background are eigenfunctions of this Hamiltonian and have the same quantization, which is given by the Landau levels $q = 0, 1, \ldots$. As we show next, electromagnetic and gravitational perturbations organize in a similiar fashion. Clearly, this analogy can be traced back to the fact that NUT charge is the gravitational equivalent of magnetic charge.

6.2.2 Electromagnetic and Gravitational perturbations

Let us now address the study of perturbations electromagnetic and gravitational perturbations. Thus, we consider a vector field A_μ satisfying Maxwell equations in the background of (6.2)

$$\nabla_\mu F^{\mu\nu} = 0 \,, \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \,, \tag{6.53}$$

and a metric perturbation $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ satisfying the linearized Einstein's equations

$$G^L_{\mu\nu}\left[h_{\alpha\beta}\right] - \frac{3}{L^2}h_{\mu\nu} = 0 \,. \tag{6.54}$$

While the symmetries of the (6.2) may still allow one to perform a complete decomposition of A_μ and $h_{\mu\nu}$ — see [370–372] for the case of SU(2) symmetry and [373] for electromagnetic perturbations in the Kerr-NUT-(A)dS spacetime — we find that the Newman-Penrose formalism offers a possibly clearer way to compute perturbations.

In the NP frame, the field strength of the Maxwell field is described by three independent (complex) components that are customarily denoted as,

$$\phi_0 = F_{\mu\nu}k^\mu m^\nu = F_{41}, \quad \phi_1 = \frac{1}{2}F_{\mu\nu}\left(k^\mu l^\nu + \overline{m}^\mu m^\nu\right) = \frac{1}{2}(F_{43}+F_{21}), \quad \phi_2 = F_{\mu\nu}\overline{m}^\mu l^\nu = F_{23}. \tag{6.55}$$

Since the background considered here is neutral, the ϕ_i correspond to linear electromagnetic perturbations. In addition, since the metric (6.2) is of Petrov type D and our NP frame (6.29) has $\mathbf{k}$ and $\mathbf{l}$ aligned with the repeated principal null directions, we can apply directly the results from Teukolsky [105]. These imply that ϕ_0 and ϕ_2 satisfy two decoupled equations that read

$$\begin{aligned} \left[\left(D - 2\rho - \rho^*\right)\left(\Delta + \mu - 2\gamma\right) - \delta\delta^*\right]\phi_0 =& 0 \,, \\ \left[\left(\Delta + \gamma - \gamma^* + 2\mu + \mu^*\right)\left(D - \rho\right) - \delta^*\delta\right]\phi_2 =& 0 \end{aligned} \tag{6.56}$$

On the other hand, gravitational perturbations are described by small changes in the NP frame, $\mathbf{e}_a \to \mathbf{e}_a + \mathbf{e}_a^{(1)} + ...$, which, in turn, induce changes in the NP variables introduced above, *e.g.* $\Psi_a \to \Psi_a + \Psi_a^{(1)} + ...$ for the Weyl scalars Ψ_a. Since Ψ_0, Ψ_1, Ψ_3 and Ψ_4 all have a vanishing background value, it follows that they are already linearized. In particular the scalars Ψ_0 and Ψ_4, which are defined as

$$\Psi_0 = C_{\mu\nu\rho\sigma} k^\mu m^\nu k^\rho m^\sigma \,, \tag{6.57}$$

$$\Psi_4 = C_{\mu\nu\rho\sigma} l^\mu \overline{m}^\nu l^\rho \overline{m}^\sigma \,, \tag{6.58}$$

satisfy two decoupled equations,

$$\begin{aligned} \left[(D - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) - \delta\delta^* - 3\Psi_2\right]\Psi_0 &= 0\,, \\ \left[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D - \rho) - \delta^*\delta - 3\Psi_2\right]\Psi_4 &= 0 \end{aligned} \tag{6.59}$$

We now search for separable solutions of the variables (ϕ_0, ϕ_2) and (Ψ_0, Ψ_4). For any of these — call it ψ — we can separate the dependence on t and y as in (6.37), so that we write $\psi = e^{-i(\omega t - ky)}\mathcal{H}(x)R(r)$. On the other hand, the dependence on the coordinate x in (6.56) and (6.59) only appears in the operator $\delta\delta^*$, which reads

$$\delta\delta^* = \frac{L^2}{2(n^2+r^2)}\left[\partial_x^2 + \left(\partial_y - \frac{2nx}{L^2}\partial_t\right)^2 + i\frac{2n}{L^2}\partial_t\right]. \tag{6.60}$$

Thus, demanding that $\delta\delta^*\psi = \lambda(r)\psi$ leads to the same equation for $\mathcal{H}$ as in the scalar case, given by (6.43). Likewise, by imposing regularity of $\mathcal{H}$ at infinity we obtain the Hermite functions (6.45) and therefore we get

$$\delta\delta^*\psi = -\frac{L^2}{n^2+r^2}\left(\mathcal{E}_q - \frac{n\omega}{L^2}\right)\psi\,, \quad \delta^*\delta\psi = -\frac{L^2}{n^2+r^2}\left(\mathcal{E}_q + \frac{n\omega}{L^2}\right)\psi\,, \tag{6.61}$$

where the eigenvalues $\mathcal{E}_q$ are those in (6.46). Finally, it is possible to write (6.56) and (6.59) in a symmetric form [374] by introducing the radial functions $Y_{\pm 1}$ and $Y_{\pm 2}$ as,

$$\phi_0 = \frac{e^{-2i\arctan(r/n)}}{V\sqrt{n^2+r^2}} e^{-i\omega t + iky}\mathcal{H}_q(x)Y_{+1}(r), \quad \phi_2 = \frac{1}{\sqrt{n^2+r^2}} e^{-i\omega t + iky}\mathcal{H}_q(x)Y_{-1}(r)\,, \tag{6.62}$$

and

$$\Psi_0 = \frac{e^{-4i\arctan(r/n)}}{V^2\sqrt{n^2+r^2}} e^{-i\omega t + iky}\mathcal{H}_q(x)Y_{+2}(r), \quad \Psi_4 = \frac{1}{\sqrt{n^2+r^2}} e^{-i\omega t + iky}\mathcal{H}_q(x)Y_{-2}(r)\,. \tag{6.63}$$

Then Eqs. (6.56) and (6.59) yield the following master equations for the radial variables

$$\Lambda^2 Y_{\pm S}(r) + SP(r)\Lambda_\pm Y_{\pm S}(r) - \left(\frac{2L^2\mathcal{E}_q}{r^2+n^2} + Q_S(r)\right)V(r)Y_{\pm S}(r) = 0\,, \tag{6.64}$$

where $S = 1$ for electromagnetic perturbations and $S = 2$ for gravitational ones. Here we have introduced the differential operators

$$\Lambda_\pm = \frac{d}{dr_*} \pm i\omega\,, \qquad \Lambda^2 = \frac{d^2}{dr_*^2} + \omega^2\,, \quad \text{where} \quad \frac{d}{dr^*} = V\frac{d}{dr}\,, \tag{6.65}$$

and $P(r)$ and $Q_S(r)$ are functions given by

$$P = -V' + \frac{2(r-2in)V}{n^2+r^2}\,, \tag{6.66}$$

and

$$Q_S = \begin{cases} \dfrac{3n^2 V}{(n^2+r^2)^2} & (S=1) \\ \dfrac{\left(12n^2+8inr-r^2\right)V}{(n^2+r^2)^2} - \dfrac{(4in-r)V'}{n^2+r^2} - \dfrac{V''}{2} & (S=2) \end{cases} \tag{6.67}$$

These equations can be written in a dimensionless way by introducing z, ϵ and $\hat{\omega}$ as in (6.47), which implies that the dimensionless QNM frequencies $\hat{\omega}$ will only depend on ϵ and the level q. In order to obtain these frequencies, the radial equations (6.64) must be supplemented with suitable boundary conditions, which we determine in the following section.

6.3 Boundary conditions

Once we have determined the master equations governing the perturbations of scalar, electromagnetic and gravitational fields, we are interested in studying the corresponding quasinormal modes, which are determined by a specific choice of boundary conditions. At the horizon of the black holes, these modes satisfy the condition of behaving as outgoing waves, while the conditions at the boundary of AdS can be chosen in different ways. For instance, one might impose the master variables to vanish at infinity [333, 334]. However, we are interested in making contact with the AdS/CFT correspondence, and in that case the boundary conditions are uniquely determined [109, 375, 376]. The bulk perturbations must be such that the non-normalisable modes do not fluctuate at infinity. Only in that case the quasinormal frequencies correspond to the poles of thermal correlators at the boundary. One expects that such boundary condition, together with regularity at the horizon, makes the ODE's over-determined thus allowing a discrete set of frequencies only (the quasinormal modes). This is obviously the case for spin-0 fields, but it is less clear for higher spins since the boundary conditions are imposed on the actual fields and not on the master variables. Rather surprisingly, we find that in the electromagnetic case the boundary conditions are degenerate and do not fix the polarisability (the relative amplitude between independent modes of the master variables) in terms of the frequency. This suggests that the spectrum of thermal poles depends continuously on such parameter. On the other hand, the boundary conditions are unproblematic in the gravitational case and quasinormal frequencies can be obtained nicely. For this reason, below we discuss the scalar and gravitational perturbations in detail and relegate to Appendix E.1 the discussion of the electromagnetic case.

In order to study the boundary conditions it is useful to introduce first the coordinate

$$z = \frac{r_+}{r}\,, \tag{6.68}$$

so that the metric can be written as

$$ds^2 = \frac{1}{z^2}\left[-\frac{r_+^2 f(z)}{L^2}\left(dt + \frac{2n}{L^2}xdy\right)^2 + \frac{dz^2L^2}{f(z)} + \frac{r_+^2 + z^2n^2}{L^2}\left(dx^2 + dy^2\right)\right] \tag{6.69}$$

$$f(z) = \frac{L^2}{r_+^2}z^2V(r_+/z)\,. \tag{6.70}$$

In this way, infinity corresponds to $z = 0$, while the horizon is placed at $z = 1$. On the other hand, the tortoise coordinate r_* is defined by

$$r_* = -\int\frac{dzL^2}{r_+f(z)}\,, \tag{6.71}$$

and we note that near the horizon $z = 1$ it reads

$$r_* \approx \frac{1}{4\pi T}\log(1-z)\,, \tag{6.72}$$

where T is the Hawking temperature (6.10).

6.3.1 Scalar field

In the near-horizon region $z = 1$, the solution to the radial scalar equation (6.48) can be expanded in a Frobenius series

$$Y(z) = (1-z)^\alpha\left[c_0 + +c_1(1-z) + c_2(1-z)^2 + \ldots\right] \tag{6.73}$$

The indicial equation has the following two solutions for α,

$$\alpha_\pm = \pm\frac{i\hat{\omega}}{3(1+\epsilon^2)}\,, \tag{6.74}$$

and taking into account (6.72) and that $\hat{\omega} = L^2\omega/r_+$, we get that the solution behaves as $Y \sim e^{4\pi T\alpha_\pm r_*} = e^{\pm i\omega r_*}$. Since the solution must behave as an outgoing wave at the horizon, we must choose the root α_-.

On the other hand, near the AdS boundary $z = 0$ we find that there are two independent modes:

$$Y(z) = az^2 + bz^{-1}\left(1 + \mathcal{O}(z)\right) \quad \text{when} \quad z \to 0\,. \tag{6.75}$$

We keep the normalizable mode, which is the one that couples to a scalar field in the dual theory, and hence we have to impose that $Y(0) = 0$. The conditions at infinity and at the horizon can only be satisfied simultaneously by a discrete set of complex frequencies ω: the quasinormal mode frequencies.

6.3.2 Gravitational field

Let us finally turn to the case of the boundary conditions for gravitational perturbations. In the near-horizon region we find that the outgoing-wave condition leads to the following form of the radial functions $Y_{\pm 2}$,

$$Y_{\pm 2} \sim (1-z)^{\alpha_{\pm 2}} \quad \text{when} \quad z \to 1, \tag{6.76}$$

where

$$\alpha_{+2} = -\frac{i\hat{\omega}}{3(1+\epsilon^2)}\,, \quad \alpha_{-2} = 2 - \frac{i\hat{\omega}}{3(1+\epsilon^2)}\,. \tag{6.77}$$

On the other hand, the discussion on boundary conditions at infinity proceeds analogously to the electromagnetic case. First, by analyzing the solutions of the Newman-Penrose variables $Y_{\pm 2}$, one can see that near the boundary they behave as

$$Y_{\pm 2}(z) = a_{\pm 2} + b_{\pm 2} z \quad \text{when} \quad z \to 0\,. \tag{6.78}$$

The integration constants $a_{\pm 2}$ and $b_{\pm 2}$ will be then ultimately related to the boundary conditions imposed on the metric perturbation. Let us consider a metric perturbation $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ in the geometry of these NUT black branes. Due to gauge freedom, we can always choose a gauge in which $h_{\mu z} = 0$, so that the non-vanishing components are those transverse to the z direction, h_{ab}. Then, near $z = 0$, the metric perturbation h_{ab} has two modes,

$$h_{ab} = z h^{(1)}_{ab} + z^{-2}\left(h^{(2)}_{ab} + \mathcal{O}(z)\right) \quad \text{when} \quad z \to 0\,. \tag{6.79}$$

The holographic dictionary tells us that the renormalizable mode is the one coupled to the dual stress-energy tensor, T^{ab}, and therefore we set $h^{(2)}_{ab} = 0$. Now we can use the fact that ∂_t and ∂_y are Killing vectors in order to separate variables, so that we have

$$h_{ab} = z e^{-i(\omega t - k y)} \gamma_{ab}(x) + \mathcal{O}(z^3) \quad \text{when} \quad z \to 0\,. \tag{6.80}$$

However, just like in the case of electromagnetic perturbations, we can always set $k = 0$ by performing the isometric transformation (E.14). For the sake of completeness let us point out that the transformed metric perturbation reads

$$\hat{h}_{ab} = z e^{-i\left(\omega \hat{t} - \hat{k} y\right)} \hat{\gamma}_{ab}(\hat{x})\,, \quad \hat{k} = k + \frac{2n\omega\sigma}{L^2} \tag{6.81}$$

where

$$\hat{\gamma}_{\hat{t}\hat{t}} = \gamma_{tt}\,, \quad \hat{\gamma}_{\hat{t}\hat{x}} = \gamma_{tx}\,, \quad \hat{\gamma}_{\hat{x}\hat{x}} = \gamma_{xx} \tag{6.82}$$

$$\hat{\gamma}_{\hat{t}y} = \gamma_{ty} - \frac{2n\sigma}{L^2}\gamma_{tt}\,, \quad \hat{\gamma}_{\hat{x}y} = \gamma_{xy} - \frac{2n\sigma}{L^2}\gamma_{tx}\,, \quad \hat{\gamma}_{yy} = \gamma_{yy} - \frac{4n\sigma}{L^2}\gamma_{ty} + \left(\frac{2n\sigma}{L^2}\right)^2 \gamma_{tt}\,. \tag{6.83}$$

so that, by choosing $\sigma = -kL^2/(2n\omega)$ we get $\hat{k} = 0$. Thus, let us set $k = 0$ from now on.

Next, we have to determine the equations satisfied by the "polarization matrix" γ_{ab}. By expanding the linearized Einstein equations around $z = 0$, we find that the components of this matrix satisfy four equations, corresponding to to $G_{\mu z} + 3/L^2 g_{\mu z} = 0$. These yield

$$\frac{2n\omega}{L^2}x\gamma_{tx} + i\gamma'_{tx} - \omega(\gamma_{xx} + \gamma_{yy}) = 0\,,$$

$$\left(1 - \frac{4n^2x^2}{L^4}\right)\gamma_{tt} + \frac{4nx}{L^2}\gamma_{ty} - \gamma_{xx} - \gamma_{yy} = 0\,,$$

$$\frac{4n^2x}{L^4}\gamma_{tt} - \left(1 - \frac{4n^2x^2}{L^4}\right)\gamma'_{tt} - i\omega\left(1 - \frac{4n^2x^2}{L^4}\right)\gamma_{tx} - \frac{2n}{L^2}\gamma_{ty} - \frac{4nx}{L^2}\gamma'_{ty} - \frac{2in\omega x}{L^2}\gamma_{xy} + \gamma'_{yy} = 0\,,$$

$$\omega\left(1 - \frac{4n^2x^2}{L^4}\right)\gamma_{ty} - i\gamma xy' + \frac{2nx\omega}{L^2}\gamma_{yy} = 0\,,$$

where a prime denotes a derivative with respect to x. Let us now leave these equations for a moment to consider the NP variables Ψ_0 and Ψ_4. These can be computed from the metric perturbation $h_{\mu\nu}$ according to their definition in (6.57) and (6.58). In doing this, one has to be careful to take into account not only the variation of the Weyl tensor, but also the variation in the frame, *i.e*,

$$\Psi_0 = \delta C_{\mu\nu\rho\sigma}k^\mu m^\nu k^\rho m^\sigma + C_{\mu\nu\rho\sigma}\delta(k^\mu m^\nu k^\rho m^\sigma)\,. \tag{6.84}$$

However, since the only non-vanishing Weyl scalar in the background is $\Psi_2 = -C_{\mu\nu\rho\sigma}k^\mu m^\nu l^\rho \overline{m}^\sigma$, which is obtained from the contraction with the four different frame vectors, it is clear that $C_{\mu\nu\rho\sigma}\delta(k^\mu m^\nu k^\rho m^\sigma) = 0$, because in this expression the Weyl tensor is always contracted twice either with k or with m, and therefore no combination involving Ψ_2 appears. A similar argument holds for Ψ_4, and hence it is enough to keep only the variation of Weyl curvature when computing these scalars in the perturbed metric. Using (6.80) and expanding near $z = 0$ we find that

$$\hat{\Psi}_0 = e^{-i\omega t}\left[A_{+2} + B_{+2}z + \mathcal{O}(z^2)\right]\,, \tag{6.85}$$

$$\hat{\Psi}_4 = e^{-i\omega t}\left[A_{-2} + B_{-2}z + \mathcal{O}(z^2)\right]\,, \tag{6.86}$$

where $\hat{\Psi}_0$ and $\hat{\Psi}_4$ are the rescaled variables

$$\hat{\Psi}_0 = V^2\sqrt{n^2 + r^2}e^{+4i\arctan(r/n)}\Psi_0\,, \qquad \hat{\Psi}_4 = \sqrt{n^2 + r^2}\Psi_4\,, \tag{6.87}$$

and the coefficients $A_{\pm 2}$, $B_{\pm 2}$ read

$$A_{\pm 2} = -\frac{3\cdot 2^{\pm 1}}{L^2}\left[\frac{n^2x^2}{L^4}\gamma_{tt} \pm \frac{inx}{L^2}\gamma_{tx} - \frac{nx}{L^2}\gamma_{ty} - \frac{1}{4}(\gamma_{xx} - \gamma_{yy}) \mp \frac{i}{2}\gamma_{xy}\right]\Bigg(\tag{6.88}$$

$$B_{\pm 2} = \mp\frac{3\cdot 2^{\pm 1}}{L^2 r_+}\left[\left(\pm\frac{in}{2}\left(\left(1-\frac{2n^2x^2}{L^4}\right)\gamma_{tt}+\frac{nx}{2}\left(\frac{2n}{L^2}\mp\omega\right)\right)(\gamma_{tx}\pm i\gamma_{ty})\right.\right.$$
$$\left.\left.+\frac{1}{4}\left(n\mp L^2\omega\right)(\pm i\gamma_{xx}-2\gamma_{xy}\mp i\gamma_{yy})+\frac{1}{4}\left(\mp 2inx\gamma'_{tt}-L^2\left(\gamma'_{tx}+\pm i\gamma'_{ty}\right)\right)\right]\right) \tag{6.89}$$

Now, when searching for quasinormal modes, we demand that the variables $\Psi_{0,4}$ be separable, and this gives us additional equations for the metric perturbation. If these are separable, then we have seen that they have the form

$$\hat{\Psi}_0 = e^{-i\omega t}\mathcal{H}_{q_{+2}}(x)\left[a_{+2}+b_{+2}z+\mathcal{O}(z^2)\right]\,, \tag{6.90}$$
$$\hat{\Psi}_4 = e^{-i\omega t}\mathcal{H}_{q_{-2}}(x)\left[a_{-2}+b_{-2}z+\mathcal{O}(z^2)\right]\,, \tag{6.91}$$

where we are using (6.78), and the levels $q_{\pm 2}$ are allowed to be different. Thus, consistency with separability demands the following constraints

$$A_{\pm 2} = a_{\pm 2}\mathcal{H}_{q_{\pm 2}}\,,\quad B_{\pm 2} = b_{\pm 2}\mathcal{H}_{q_{\pm 2}}\,. \tag{6.92}$$

In total, (6.92) and (6.84) form a system of eight equations for the six variables γ_{ab}, and therefore it is an overdetermined system; in order for a solution to exist, the parameters $a_{\pm 2}$, $b_{\pm 2}$ and the levels $q_{\pm 2}$ cannot be arbitrary. By analyzing those equations, one can see that a solution exists only if the levels $q_{\pm 2}$ are related according to

$$q_{+2} = q_{-2} + 4s\,, \tag{6.93}$$

so that q_{-2} takes the values $q_{-2} = 0, 1, 2, \ldots$ for $s = 1$ and $q_{-2} = 4, 5, 6, \ldots$ for $s = -1$. In addition, the ratios

$$\lambda_{\pm 2} = \frac{b_{\pm 2}}{a_{\pm 2}}\,, \tag{6.94}$$

must be related according to

$$\lambda_{-2} = \frac{M_q + P_q\lambda_{+2}}{Q_q + S_q\lambda_{+2}}\,, \tag{6.95}$$

where

$$M_q = -\,i\left(\left(8q^2+40q+41\right)\hat{\omega}^2\epsilon^2 - 3(2q+5)\hat{\omega}^3\epsilon + 7(2q+5)\hat{\omega}\epsilon^3 + \hat{\omega}^4 + 2\epsilon^4\right), \tag{6.96}$$

$$P_q = \left(2q^2+2q-5\right)\hat{\omega}\epsilon^2 - 2(2q+3)\hat{\omega}^2\epsilon + \hat{\omega}^3 - 2\epsilon^3\,, \tag{6.97}$$

$$Q_q = \left(2q^2+18q+35\right)\hat{\omega}\epsilon^2 - 2(2q+7)\hat{\omega}^2\epsilon + \hat{\omega}^3 + 2\epsilon^3\,, \tag{6.98}$$

$$S_q = i\left(-(2q+5)\hat{\omega}\epsilon + \hat{\omega}^2 - 2\epsilon^2\right), \tag{6.99}$$

and where

$$q = \begin{cases} q_{-2} & \text{if} \quad s = 1\,, \\ -1 - q_{-2} & \text{if} \quad s = -1 \end{cases} \tag{6.100}$$

There is also a relation between the normalizations of Y_{+2} and Y_{-2}, which reads

$$2\epsilon^2\hat{\omega}\frac{a_{-2}}{a_{+2}} = \left(2q_{-2}^2 + 18q_{-2} + 35\right)\hat{\omega}\epsilon^2 - 2(2q_{-2}+7)\hat{\omega}^2\epsilon + \hat{\omega}^3 + 2\epsilon^3 \\ - i\lambda_{+2}\left((2q_{-2}+5)\hat{\omega}\epsilon - \hat{\omega}^2 + 2\epsilon^2\right) \tag{6.101}$$

for $s = 1$ and

$$2\epsilon^2\hat{\omega}\frac{a_{-2}}{a_{+2}} = \frac{\left(2q_{-2}^2 - 14q_{-2} + 19\right)\hat{\omega}\epsilon^2 + 2(2q_{-2}-5)\hat{\omega}^2\epsilon + \hat{\omega}^3 + 2\epsilon^3}{16(q_{-2}-3)(q_{-2}-2)(q_{-2}-1)q_{-2}} \\ + \frac{i\lambda_{+2}\left((2q_{-2}-3)\hat{\omega}\epsilon + \hat{\omega}^2 - 2\epsilon^2\right)}{16(q_{-2}-3)(q_{-2}-2)(q_{-2}-1)q_{-2}} \tag{6.102}$$

for $s = -1$, but this is irrelevant for the computation of quasinormal modes. Finally, one can obtain an explicit solution for the γ_{ab} in terms of Hermite functions $\mathcal{H}_p$, which we show in Appendix E.2.

These results fix the boundary conditions up to the choice of the complex constant λ_{+2} (and up to trivial rescalings of $Y_{\pm 2}$). In the case of vanishing NUT charge, there are two admissible values of λ_{+2} that give rise to quasinormal modes, and these correspond to choosing either parity odd or parity even polarizations. However, in the case at hands the background breaks parity, and hence one cannot determine a priori the value of λ_{+2}. Then, in order to find the quasinormal modes, one has to solve simultaneously the equations (6.64) for Y_{+2} and Y_{-2} with the boundary conditions discussed above. Unlike in the electromagnetic case, these equations are not degenerate, and the problem will only have solutions for a discrete set of values of ω (the quasinormal frequencies) and λ_{+2} (which determine the polarization).

Fortunately, it is possible to find an analytic result for the polarization parameter λ_{+2} by using the so-called Teukolsky-Starobinsky identities (see e.g. [377] and [374] for a detailed analysis of those in Kerr's space, and [109] for Kerr-(A)dS in the context of holography). These relate solutions of the Y_{+2} variable with those of Y_{-2}, and vice-versa. In order to find these identities, it is useful to introduce two new radial functions defined by

$$R_{(+2)} = \frac{(r^2+n^2)^{3/2}}{\Delta}e^{-4i\arctan(r/n)}Y_{+2}\,, \quad R_{(-2)} = \frac{(r+in)^4}{(r^2+n^2)^{1/2}\Delta}Y_{-2}\,, \tag{6.103}$$

where $\Delta = (r^2+n^2)V(r)$. In terms of these, the radial equations for each level (q_{+2} or q_{-2} in each case) read

$$\left[\mathcal{D}_{-1}\Delta\mathcal{D}_1^\dagger + 6\left(\frac{r^2+n^2}{L^2} + i\omega r\right) - 4\omega n\left(s(q_{+2}+1/2)-2\right)\right]R_{(+2)}^{q_{+2}} = 0 \tag{6.104}$$

$$\left[\mathcal{D}_{-1}^\dagger\Delta\mathcal{D}_1 + 6\left(\frac{r^2+n^2}{L^2} - i\omega r\right) - 4\omega n\left(s(q_{-2}+1/2)+2\right)\right]R_{(-2)}^{q_{-2}} = 0\,, \tag{6.105}$$

where we have introduced the operators

$$\mathcal{D}_m = \partial_r - i\omega\frac{r^2+n^2}{\Delta} + m\frac{\Delta'}{\Delta}, \qquad \mathcal{D}_m^\dagger = \partial_r + i\omega\frac{r^2+n^2}{\Delta} + m\frac{\Delta'}{\Delta}, \tag{6.106}$$

which satisfy the properties

$$\mathcal{D}_m\Delta = \Delta\mathcal{D}_{m+1}, \qquad \mathcal{D}_m^\dagger\Delta = \Delta\mathcal{D}_{m+1}^\dagger \tag{6.107}$$

We see that the variables $R^{q+2}_{(+2)}$ and $R^{q-2}_{(-2)}$ satisfy conjugate equations when the levels q_{+2} and q_{-2} are related as in (6.93), and in that case it is possible to show the following relations (the TS identities):

$$\mathcal{D}_{-1}^\dagger\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\mathcal{D}_1^\dagger R^{\omega,q_{+2}}_{(+2)} = C_{(-2)}R^{\omega,q_{+2}-4s}_{(-2)}\,, \tag{6.108}$$

$$\mathcal{D}_{-1}\Delta\mathcal{D}_0\mathcal{D}_0\Delta\mathcal{D}_1 R^{\omega,q_{-2}}_{(-2)} = C_{(+2)}R^{\omega,q_{-2}+4s}_{(+2)}\,, \tag{6.109}$$

where $C_{(\pm 2)}$ are certain complex constants that can always be chosen as complex-conjugates of each other by an appropriate choice of normalization of the radial functions. These relations mean that given a solution $R^{q+2}_{(+2)}$ of the $(+2)$ equation, then $\mathcal{D}_{-1}^\dagger\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\mathcal{D}_1^\dagger R^{\omega,q_+}_{(+2)}$ is a solution of the (-2)-equation with same frequency but Landau level $q_{-2} = q_{+2} - 4s$, and similarly for the second identity. We remark that these relations map the solutions of the radial equations into each other, but this does not necessarily mean that these relations are actually realized for generic perturbations — proving that is much harder. However, in the case of quasinormal modes, it is not difficult to see that the TS identities map the solutions with the correct boundary conditions at the horizon (6.76) into each other. This means that, at least when searching for quasinormal modes, the TS identities do hold. These identities allow us to obtain the value of λ_{+2} and to reduce the problem of finding QNMs to solving one equation for one variable.

To show this, consider the asymptotic behavior of Y_{+2} with generic Robin boundary conditions,

$$Y_{+2} = a_{+2}(1 + \lambda_{+2}z + \mathcal{O}(z^2))\,. \tag{6.110}$$

Using the relations (6.103) and (6.108) one finds that the variable Y_{-2} then satisfies[6]

$$Y_{-2} = a_{-2}(1 + \lambda_{-2}z + \mathcal{O}(z^2))\,, \tag{6.111}$$

for certain a_{-2}, and where λ_{-2} reads

$$\lambda_{-2} = \frac{\hat{M}_{\hat{q}} + \hat{P}_{\hat{q}}\lambda_{+2}}{\hat{Q}_{\hat{q}} + \hat{S}_{\hat{q}}\lambda_{+2}}\,, \tag{6.112}$$

where

$$\hat{M}_{\hat{q}} = -4i\Big(\big(8\hat{q}^2 - 24\hat{q} + 9\big)\hat{\omega}^2\epsilon^2 + 7(2\hat{q}-3)\hat{\omega}\epsilon^3 + (9-6\hat{q})\hat{\omega}^3\epsilon + \hat{\omega}^4$$

[6] In performing this map one has to be careful to include the $\mathcal{O}(z^2)$ terms (not shown above) in Y_{+2}.

$$+ 2\epsilon \left(9i\epsilon^4 + 25\epsilon^3 - 18i\epsilon^2 - 3i\right)\Big) \tag{6.113}$$

$$\hat{P}_{\hat{q}} = 4\left(2\hat{q}^2 - 14\hat{q} + 19\right)\hat{\omega}\epsilon^2 - 8(2\hat{q}-5)\hat{\omega}^2\epsilon + 4\hat{\omega}^3 - 9i\epsilon^4 - 32\epsilon^3 + 18i\epsilon^2 + 3i\,, \tag{6.114}$$

$$\hat{Q}_{\hat{q}} = 4\left(2\hat{q}^2 + 2\hat{q} - 5\right)\hat{\omega}\epsilon^2 + 8(1-2\hat{q})\hat{\omega}^2\epsilon + 4\hat{\omega}^3 + 9i\epsilon^4 + 32\epsilon^3 - 18i\epsilon^2 - 3i\,, \tag{6.115}$$

$$\hat{S}_{\hat{q}} = 4i\left((3-2\hat{q})\hat{\omega}\epsilon + \hat{\omega}^2 - 2\epsilon^2\right)\,, \tag{6.116}$$

and where in this case we are defining

$$\hat{q} = \begin{cases} q_{+2} & \text{if} \quad s = 1\,, \\ 1 - q_{+2} & \text{if} \quad s = -1 \end{cases} \tag{6.117}$$

Comparing with (6.95) we have two relations between λ_{+2} and λ_{-2}, so we can determine both parameters. We get two different solutions, which read

$$\lambda^{(\pm)}_{+2} = \frac{i}{(3-2\hat{q})\hat{\omega}\epsilon + \hat{\omega}^2 - 2\epsilon^2}\Big(2\hat{q}^2\hat{\omega}\epsilon^2 + 2\hat{q}\hat{\omega}\epsilon^2 - 4\hat{q}\hat{\omega}^2\epsilon + \hat{\omega}^3 - 5\hat{\omega}\epsilon^2 + 8\epsilon^3 + 2\hat{\omega}^2\epsilon$$
$$\mp 2\epsilon^2\sqrt{(\hat{q}-3)(\hat{q}-2)(\hat{q}-1)\hat{q}\hat{\omega}^2 + 9\epsilon^2}\Big)\,, \tag{6.118}$$

$$\lambda^{(\pm)}_{-2} = -\lambda^{(\pm)}_{+2} - 8i\epsilon\,. \tag{6.119}$$

Each of these solutions corresponds to one of the two possible polarization modes of gravitational waves.

As a check of our computations we may consider the limit of vanishing NUT charge. In order to recover the perturbations for the planar black hole with momentum $\vec{k}$ one should take the limit $n \to 0$ and $q_{\pm 2} \to \infty$ in a way in which $4sn\omega q_{\pm 2}/L^2 \to \vec{k}^2$. By doing so, we get the following limiting values of $\lambda_{\pm 2}$,

$$\lambda^{(-)}_{+2} = -\lambda^{(-)}_{-2} = -\frac{i\left(\hat{k}^2 - 2\hat{\omega}^2\right)}{2\hat{\omega}}\,, \tag{6.120}$$

$$\lambda^{(+)}_{+2} = -\lambda^{(+)}_{-2} = \frac{2i\hat{\omega}\left(\hat{k}^2 - \hat{\omega}^2\right)}{\hat{k}^2 - 2\hat{\omega}^2}\,, \tag{6.121}$$

where $\hat{k} = L^2\vec{k}/r_+$ is the dimensionless momentum. It is not difficult to check that these coefficients precisely correspond the appropriate boundary conditions for the variables $Y_{\pm 2}$ for odd $(-)$ and even $(+)$ parity perturbations of the black brane, respectively — see [335].

Let us also mention that, instead of establishing a boundary condition for the radial Teukolsky variables by computing them from the metric perturbation, it is possible to go the other way around by using the so-called Hertz potentials. These are related to the

Teukoslky variables and allow one to reconstruct the metric perturbation from them. In this way, one can determine what boundary conditions for the radial variables give rise to Dirichlet boundary conditions for the metric perturbation. We study this alternative method in Appendix E.3, finding perfect agreement with our results above.

In sum, we have found that, in order to find the gravitational QNMs, we have to solve the radial equation (6.64) for Y_{+2} with the boundary conditions given by (6.76), (6.110) and (6.118) (or equivalently, the equation for Y_{-2} with the conditions (6.76), (6.111) and (6.119)).[7] This problem only has solutions for a discrete set of complex frequencies, which are the quasinormal-mode frequencies.

6.4 Quasinormal modes

Having reduced the study of perturbations to a one-dimensional problem given by the radial equations (6.48) and (6.64) and having determined the boundary conditions that the corresponding variables must satisfy, we are now ready to compute the quasinormal modes. Before showing the explicit results, we can first determine some general properties of the quasinormal mode frequencies ω. First, note that the dimensionless frequencies $\hat{\omega}$ will only depend on ϵ and on the level q (plus on the overtone number, which we omit). Therefore, the actual frequencies scale linearly with the size of the black brane for fixed ϵ,

$$\omega = \frac{\hat{\omega}_q(\epsilon)}{L^2} r_+ . \tag{6.122}$$

In other words, since $\epsilon = n/r_+$, we conclude that the QNM frequencies are homogeneous functions of degree 1 of r_+ and n. From the point of view of the dual CFT, however, the quantities r_+ and n do not have a direct interpretation, and instead the physically relevant quantities in the boundary theory are the ratio n/L^2 — see (6.8) — and the temperature T given by (6.10). The QNM frequencies are then homogeneous functions of T and n/L^2, and they can be conveniently expressed in terms of the dimensionless ratio

$$\xi = \frac{3n}{2\pi T L^2}, \tag{6.123}$$

which satisfies $-1 \leq \xi \leq 1$. Then, the QNM frequencies read

$$\omega = 2\pi \frac{\xi^2 \hat{\omega}_q\left(\epsilon(\xi)\right)}{3\left(1-\sqrt{1-\xi^2}\right)} T, \tag{6.124}$$

where ϵ and ξ are related by[8]

$$\epsilon(\xi) = \frac{1}{\xi}\left(1-\sqrt{1-\xi^2}\right) \tag{6.125}$$

Thus, we shall study ω/T as a function of ξ. The frequencies feature in addition a symmetry under the exchange of sign of n (or ξ, equivalently). Namely, we have

$$\omega_q(-n) = -\omega_q^*(n), \tag{6.126}$$

[7] The normalization constants $a_{\pm 2}$ in (6.110) and (6.111) are irrelevant for the definition of the QNMs.

[8] Notice that given ξ, there are actually two compatible values of ϵ, given by $\epsilon_\pm(\xi) = \frac{1}{\xi}\left(1 \pm \sqrt{1-\xi^2}\right)$. However, only the $(-)$ branch contributes to the Euclidean saddle point and thus we will focus on this case.

meaning that given a QNM frequency $\omega_q(n)$ of the solution with NUT charge n, then $-\omega_q^*(n)$ is a frequency of the solution with charge $-n$. This result can be obtained by noticing that the complex-conjugate variables $Y_{\pm S}^*$ satisfy the same equations and boundary conditions as $Y_{\pm S}$ with $\omega \to -\omega^*$ and $n \to -n$. Thus, there is a correspondence between the QNMs with $\mathrm{Re}(\omega) > 0$ and NUT charge n and those with $\mathrm{Re}(\omega) < 0$ and NUT charge $-n$, and vice-versa. Hence, it is sufficient to focus on studying the QNMs with $\mathrm{Re}(\omega) > 0$ for both positive and negative n. In the case of scalar QNMs, one can also see that the frequencies are actually symmetric under the change of sign of n, $\omega_{\rm scalar}(n) = \omega_{\rm scalar}(-n)$, because the radial equation is invariant under the change of sign of n. For the gravitational perturbations, however, one can see that the replacement $n \to -n$ is not a symmetry of the master equations (6.64), nor of the boundary conditions (6.118). Thus, in principle one should not expect the spectrum of quasinormal modes to be identical for positive and negative n.

In order to compute the QNM frequencies, we use the following method. Taking into account the boundary conditions we have determined, we first expand the corresponding variables Y_S near the horizon using a Frobenius series and asymptotically using a Taylor expansion. This gives us two approximate solutions $Y_S^+(z)$ and $Y_S^\infty(z)$ valid in the regions $z \sim 1$ and $z \sim 0$, respectively. One must then try to glue both solutions, but this only will be possible if $\hat{\omega}$ is a QNM frequency. One may use directly the asymptotic expansions $Y_S^+(z)$ and $Y_S^\infty(z)$ to find the QNM frequencies by imposing the glueing condition $Y_S^\infty \partial_z Y_S^+ - \partial_z Y_S^\infty Y_S^+\big|_{z_{\rm joint}} = 0$, for some intermediate $z_{\rm joint}$. This yields an algebraic equation for ω, whose solutions should converge to the QNM frequencies when the number of terms in the asymptotic expansions tend to infinity, However, we have found that the convergence is not very good as we increase the NUT charge, and in order to improve the accuracy of our results we use a numerical integration.[9] Thus, we use the near-horizon expansion $Y_S^+(z)$ to initialize the numerical method at some $z_{\rm ini}$ close to $z = 1$, and we numerically integrate the solution up to some $z_{\rm end}$ close to $z = 0$. Then, we compute the Wronskian

$$W_S = Y_S^\infty \partial_z Y_S^{\rm num} - \partial_z Y_S^\infty Y_S^{\rm num}\Big|_{z_{\rm end}}, \tag{6.127}$$

and we search for solutions of $W_S = 0$. In the case of the scalar field, $S = 0$, we have a single equation, $W_0 = 0$, that determines the QNM frequencies. In the gravitational case, $S = \pm 2$, we can use any of the two equations $W_2 = 0$ or $W_{-2} = 0$. As discussed in the previous section, the two variables $Y_{\pm 2}$ are isospectral when provided with their own set of boundary conditions, (6.118) and (6.119), respectively, so it is enough to work with only one of them; for instance, Y_{+2}. In the electromagnetic case, as discussed in Appendix E.1, both variables $Y_{\pm 1}$ are isospectral for *any* choice of the free polarization parameter λ_{+1}, which seems to indicate that the QNMs depend continuously on this parameter. We will leave this case for future developments, and we will focus here on the scalar and gravitational QNMs.

In order to understand the structure of these quasinormal modes, it is important to see how they relate to those of the black brane. One can see that, in the limit of vanishing

[9] For the numerical integration we use the `ImplicitRungeKutta` method implemented in Mathematica and we set `Precisiongoal`$\to$ 10. We estimate that the largest source of error in the computation of the QNMs comes from the initial conditions obtained by the series expansions, and thus the error in the numerical integration is negligible. By changing the order of these expansions and the values of $z_{\rm ini}$ and $z_{\rm end}$ we estimate that the relative error in the results that we present below is typically of the order of 10^{-3}.

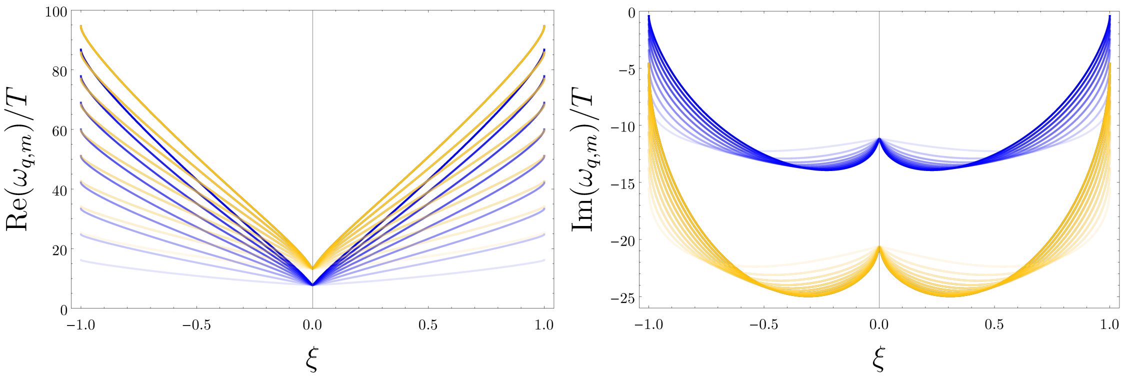

Figure 6.1: Real and imaginary parts of the scalar QMN frequencies $\omega_{q,m}/T$ as a function of $\xi = \frac{3n}{2\pi T L^2}$. In order of increasing opacity the curves correspond to the levels $q = 0, 1, ..., 8$. The fundamental mode ($m = 0$) is shown in blue and the first overtone ($m = 1$) in red.

NUT charge, we should recover the quasinormal modes of vanishing momentum ($\hat{k} = 0$) of the black brane. Also, note that the spectrum becomes independent of the level q in that limit, and hence an infinite number of modes ω_q of different q collapse to the same mode. On the other hand, it is not clear that one can recover the QNMs of black branes with arbitrary momentum in the limit of $n \to 0$. Note that this momentum can be identified as

$$\hat{k}^2 = \lim_{n\to 0} 2\mathcal{E}_q = \lim_{n\to 0} 2sn\omega(1+2q)/L^2\,, \tag{6.128}$$

thus, in order to get a non-vanishing value one must take simultaneously $n \to 0$ and $q \to \infty$ in a way that qn remains finite in that limit. However, the resulting value of $\hat{k}^2$ would be in general complex unless one chooses q to be complex as well, but in that case the connection with the QNMs of Taub-NUT black holes is broken. Hence, one should not expect to recover all the QNMs of the planar black holes in a continuous way. In any case, as a test for our method, we have checked that in this limit we reproduce the correct values for the axial and polar gravitational QNM frequencies, as shown in tables 3 and 2 of Refs [333] and [335], respectively. Let us now present our results.

6.4.1 Scalar

We start with the simple case of a massless scalar field. For every value of ξ and the level q, there is an infinite family of QNMs $\omega_{q,m}$, where, for decreasing order of the imaginary part we label these modes by $m = 0, 1, \ldots$. The one with the largest imaginary part is the fundamental mode ($m = 0$) and the rest are overtones.

In figure 6.1 we show the fundamental mode and the first overtone for the scalar QNM frequencies for the levels $q = 0, 1, \ldots 8$. As discussed above, we see that in the limit $\xi \to 0$ all the modes with different q collapse to the same corresponding mode of the black brane. As a check, we get that

$$\omega_{q,0}(0) \approx (7.75 - 11.2i)T \approx (1.85 - 2.66i)r_+/L^2\,, \tag{6.129}$$

which agrees with the fundamental mode of the black brane when $r_+^2 >> \vec{k}^2$ [333]. As we can see in figure 6.1, the real part of ω grows almost linearly with ξ (or n), while the imaginary part has a non-monotonic dependence. Also, note that these frequencies are symmetric for $\xi \to -\xi$. For $\xi \sim \pm 1$ we see that $\text{Im}(\omega_{q,m}) \sim 0$ for large q, but our numeric results suggest that it never becomes positive, and therefore, scalar perturbations are stable for the whole range of ξ. We recall that the results in Fig. 6.1 refer to the branch of black holes with positive specific heat, $r_+^2 > n^2$. We have briefly looked to case of $r_+^2 < n^2$, and for those black holes our results indicate that all the quasinormal modes have very small imaginary parts $\text{Im}(\omega_{q,m}) \sim 0$, but that still do not cross 0. In fact, it is even possible to prove analytically that $\text{Im}[\omega] < 0$ in the scalar sector for $\frac{n}{r_+} \lesssim 24$. Introducing the new scalar variable

$$\Psi := e^{i\omega r_*} Y(r) \tag{6.130}$$

and performing the usual trick of multiplying by the complex conjugate and integrating (6.42) from r_+ to ∞ [329, 374] one gets the equation

$$\int_{r_+}^{\infty} dr \left[V|\Psi'|^2 + \mathbf{V}|\Psi|^2 \right] = -\frac{|\omega|^2 |\Psi|^2(r_+)}{\text{Im}[\omega]} \tag{6.131}$$

where

$$\mathbf{V} = \frac{rV'}{n^2 + r^2} + \frac{n^2 V}{(n^2 + r^2)^2}. \tag{6.132}$$

It is easy to check that each term in the left hand side of (6.131) is positive definite in $r_+ < r < \infty$ for $\epsilon \lesssim 24$. In particular, this proves that $\text{Im}[\omega] < 0$ for all scalar perturbations in the physical backgrounds, which lie at $\epsilon < 1$.

6.4.2 Gravitational

Let us now turn to the most interesting case of gravitational modes. We recall that these come in two different classes, $\omega_q^{(\pm)}$, corresponding to the two different polarization modes given in (6.118). In addition, in analogy with the case of the black brane, we may distinguish two families of modes according to the their behaviour in the limit $n \to 0$.

Pseudo-hydrodynamic mode

We find that for every level q there is a special mode such that $\omega_q \to 0$ in the limit $n \to 0$. We recall that, in the case of the black brane, both axial and polar perturbations contain a hydrodynamic mode, *i.e*, one whose frequency vanishes when $\vec{k} \to 0$ [335]. In the presence of NUT charge, one cannot talk about hydrodynamic modes because the spectrum of quasinormal modes is discrete, and thus we refer to the modes ω_q that vanish for $n \to 0$ as "pseudo-hydrodynamic". These must be in fact related to the hydrodynamic modes of the black brane.

We find that these pseudo-hydrodynamic modes only exist for the $(+)$ polarization — we comment on the absence of these modes for $(-)$ polarization below — and we show their corresponding quasinormal frequencies in Fig. 6.2 for the levels $q = 0, \ldots 8$ (where $q = q_{+2} - 4$ if $\text{Re}(n\omega) > 0$ and $q = q_{+2}$ if $\text{Re}(n\omega) < 0$). As we can see, the real part behaves linearly with ξ near $\xi = 0$, while the imaginary part is quadratic in that region. For larger values of ξ the real part of ω_q transitions to a different linear dependence, while the imaginary part has a non-monotonic behaviour. Indeed, after reaching a minimum

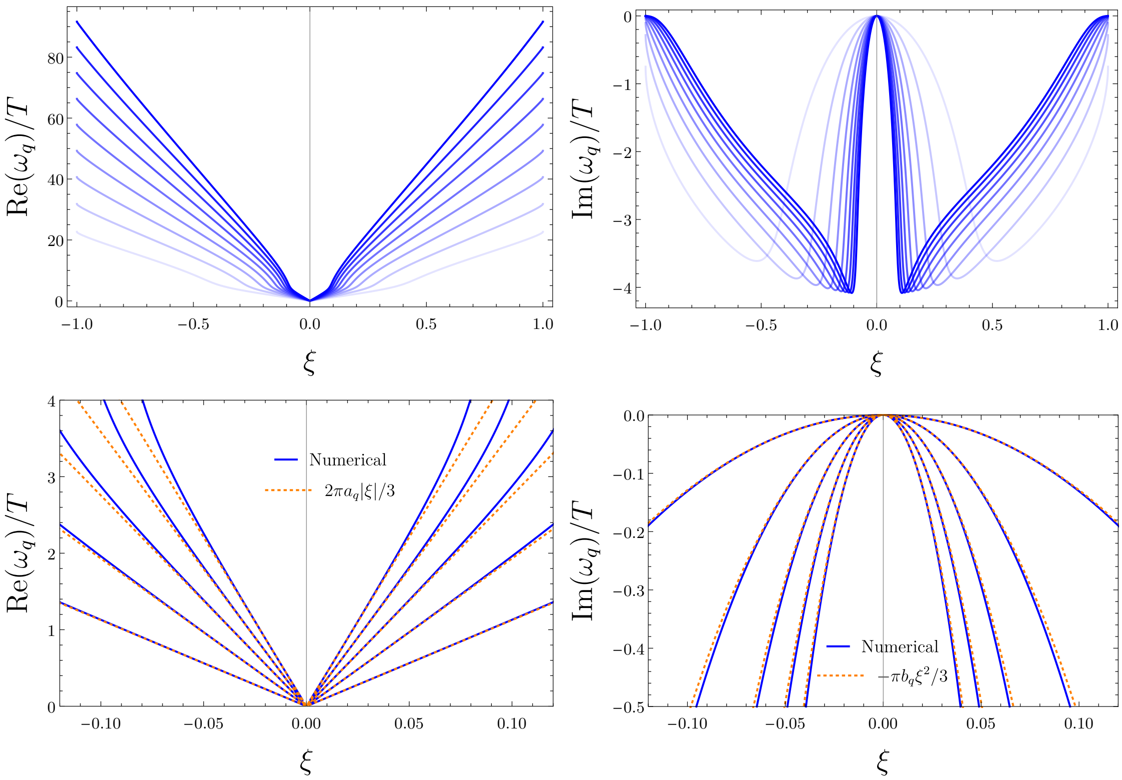

Figure 6.2: Pseudo-hydrodynamic QNM frequencies of gravitational perturbations as a function of $\xi = \frac{3n}{2\pi T L^2}$. Top row: in order of increasing opacity we show the levels $q = 0, 1, ..., 8$, where $q = q_{+2} - 4$ if $\text{Re}(n\omega) > 0$ and $q = q_{+2}$ if $\text{Re}(n\omega) < 0$. Bottom row: behaviour near $\xi = 0$ and comparison with the analytic result (6.139). In order to facilitate the visualization in that case we only show the modes with $q = 0, 2, 4, 6, 8$.

value, the imaginary part grows and becomes close to 0 for $\xi = \pm 1$. We observe that for larger q, the imaginary part becomes even smaller near $\xi = \pm 1$, but interestingly it does not become positive, which indicates that there are no unstable modes — we study the stability of these solutions below. Another property of these QNM frequencies that is worth remarking is that they are symmetric under the exchange $\xi \to -\xi$. This indicates that the exchange of sign of the NUT charge must be indeed a hidden symmetry of the equations (6.64) with the boundary conditions (6.118) and (6.119).

Let us now focus on the region $\xi << 1$. We can actually obtain analytic approximations for the pseudo-hydrodynamic QNMs in this limit. Recalling first the boundary conditions (6.76), we can expand the function Y_{+2} near $z = 1$ as

$$Y_{+2}(z) = (1-z)^{-\frac{i\hat{\omega}}{3(1+\epsilon^2)}} \sum_{i=0}^{\infty} c_i (1-z)^i \,. \tag{6.133}$$

Using the master equation (6.64) one can then find explicitly the values of all the coefficients c_i in terms of the first one up to a given order $i = i_{\text{max}}$. We can then implement a method similar the one of Horowitz and Hubeny [329] and glue this expansion with the one in (6.111) at $z = 0$, which yields the equation

$$\lambda_{+2}^{(-)} Y_{+2}(0) - Y'_{+2}(0) = 0 \,, \tag{6.134}$$

where we recall that $\lambda_{+2}^{(-)}$ is given by (6.118). In general, this method can be used to obtain an approximate solution to the QNM frequencies, but in the limit $|\xi| << 1$ we can obtain an analytic result. Taking into account the input from the numerical result, we will have

$$\hat{\omega}_q = \epsilon a_q - i b_q \epsilon^2 + \mathcal{O}(\epsilon^3)\,, \tag{6.135}$$

for some coefficients a_q and b_q (near $\xi = 0$ the relation between this variable and ϵ is simply $\xi \approx 2\epsilon$). Without loss of generality, let us consider $\epsilon > 0$ and $\mathrm{Re}(\omega) > 0$. Then, in the limit $\epsilon \to 0$, one can see that the expansion (6.133) collapses to a polynomial, and we have $\lim_\epsilon Y'_{+2}(0) = \frac{3}{2} \lim_\epsilon Y_{+2}(0)$. On the other hand, for generic values of a_q, the coefficient $\lambda_{+2}^{(+)}$ is $\mathcal{O}(\epsilon)$ in that limit, and hence the equation (6.134) is not satisfied. The only way in which $\lambda_{+2}^{(+)}$ does not vanish at $\epsilon = 0$ is when the denominator in (6.118) is of order ϵ^3, and it is easy to see that this happens when

$$a_q^2 - (5 + 2q)a_q - 2 = 0\,, \tag{6.136}$$

where we recall that for $\mathrm{Re}(n\omega) > 0$ we are defining $q = q_{+2} - 4$, which takes the values $q = 0, 1, 2, \ldots$. The positive root of this equation yields the following value for a_q,

$$a_q = \frac{1}{2}\left(2q + 5 + \sqrt{4q^2 + 20q + 33}\right)\,. \tag{6.137}$$

With this choice, one can solve the equation (6.134) order by order in the ϵ expansion. For, say, $i_{\max} = 3$, one finds

$$b_q = \frac{4}{3}\left(q^2 + 5q + 4 + \frac{(q+2)(q+3)(2q+5)}{\sqrt{4q^2 + 20q + 33}}\right)\,. \tag{6.138}$$

Then it is easy to check that this result does not change for larger values of $i_{\max}$, and thus this value of b_q is exact. This leads to the following expression for the physical frequency ω

$$\omega_q = \frac{2\pi T}{3}\left[a_q \xi - \frac{i b_q}{2}\xi^2\right] = \frac{n a_q}{L^2} - i\frac{3 b_q n^2}{4\pi T L^4}\,, \tag{6.139}$$

which is valid when $q|\xi| << 1$. As we show in the second row of Fig. 6.2, these expressions match the numeric results with great accuracy. Finally, it is interesting to analyze what happens in the limit $n \to 0$ and $q \to \infty$ such that qn remains finite. In that case we have

$$\omega \approx \frac{2nq}{L^2} - i\frac{2(nq)^2}{\pi T L^4}\,. \tag{6.140}$$

On the other hand, we recall that in this limit we can identify a momentum for the perturbations $\hat{k}$ according to (6.128), which yields

$$\hat{k}^2 = \frac{4nq\omega}{L^2}\,. \tag{6.141}$$

Moreover, this is a real momentum when $|n|q << 1$. Combining this expression with (6.140) we obtain the following effective dispersion relation for small $\hat{k}$

$$\omega \approx \frac{\hat{k}}{\sqrt{2}} - i\frac{\hat{k}^2}{8\pi T}\,. \tag{6.142}$$

This is precisely the dispersion relation for the hydrodynamic mode of polar perturbations in the absence of NUT charge [335]. Hence, the pseudo-hydrodynamic modes of the NUT-charged black holes are analogous to that mode of the black brane. One may wonder why we do not obtain other modes similar to the hydrodynamic mode of axial perturbations (which correspond to the $(-)$ family in (6.118)). The reason is that such mode is purely damped, and according to the identification (6.128) we would need to choose q to be imaginary in order to recover a solution of the black brane with real momentum. Thus, that mode is simply not present in the Taub-NUT planar black holes.

When ξ becomes larger, we cannot obtain an analytic result for the frequencies, but we can obtain a reasonable good approximation for the real part. In fact, we observe that the real part of the dimensionless frequencies $\hat{\omega}_q$ is a linear function of ϵ, and a fit to the numerical data shows that the slope is proportional to q. Namely, we get

$$\mathrm{Re}(\hat{\omega}_q) \approx 4.04(q+3)\epsilon\,, \tag{6.143}$$

plus a constant term that is much smaller. Interestingly, this seems to work not only for $\epsilon \leq 1$, but for arbitrarily large ϵ. Now, when we take into account (6.124), we deduce that the dimensionful frequencies ω_q are also a linear function of ξ

$$\mathrm{Re}(\omega_q) \approx \frac{2\pi T\xi}{3} 4.04(q+3) \approx \frac{4.04n}{L^2}(q+3)\,. \tag{6.144}$$

Ordinary quasinormal modes

The rest of the gravitational quasinormal modes have frequencies that tend to a constant, non-vanishing value in the limit $n \to 0$. These are labeled by the polarization type $\pm$ defined in (6.118), the Landau level q and the overtone number $m = 0, 1, 2, ...$, and we denote them $\omega^{\pm}_{q,m}$. As already remarked before, the values of these frequencies for $n \to 0$ will correspond to the black brane's QNM frequencies at vanishing momentum. It is known that the polar and axial QNMs of the black brane become degenerate when the momentum tends to zero [335], which means that, in our case, both classes of modes $\omega^{+}_{q,m}$ and $\omega^{-}_{q,m}$ also become degenerate. For the same m, the frequencies of all the modes in the two families collapse to the same value,

$$\lim_{n\to 0} \omega^{+}_{q,m}(n) = \lim_{n\to 0} \omega^{-}_{q',m}(n) \equiv \omega_m(0) \quad \forall\, q, q' \tag{6.145}$$

In Fig. 6.3 we show the lowest $(m = 0)$ QNMs for a few levels q, where the first thing we notice is that the spectrum is again symmetric for $\xi > 0$ and $\xi < 0$. The structure of the QNM frequencies as a function of ξ is somewhat similar to the one of the pseudo-hydrodynamic modes, with the real part scaling almost linearly with q for most of the range of ξ. In particular, we have the following fits to the real parts of the dimensionless frequencies

$$\mathrm{Re}(\hat{\omega}^{+}_q) \approx (16.6 + 4.37q)\epsilon + c^{+}_q\,, \quad \mathrm{Re}(\hat{\omega}^{-}_q) \approx (14.9 + 4.27q)\epsilon\, + c^{-}_q\,, \tag{6.146}$$

where the constant terms are small. When we use (6.124), this produces an almost linear relation between ω_q and ξ, although the non-vanishing constant terms introduce non-linearities near $\xi = \pm 1$. On the other hand, the imaginary part becomes very small as

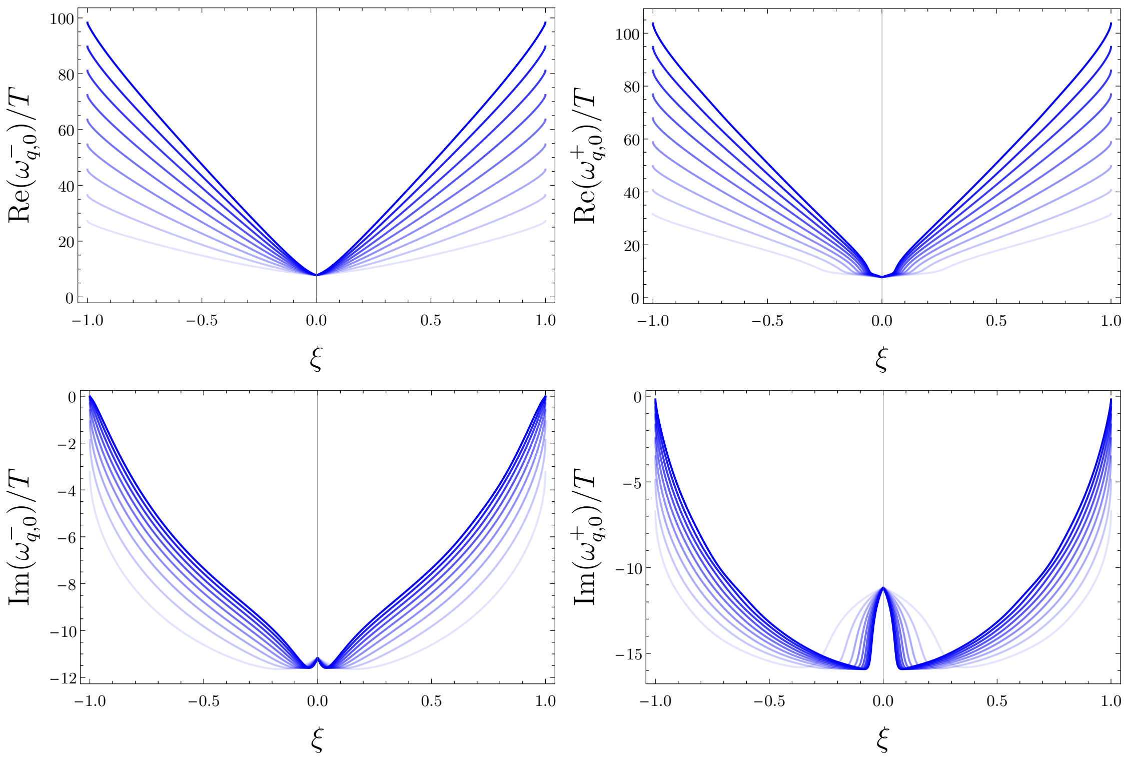

Figure 6.3: Ordinary gravitational quasinormal modes: we show the lowest overtones $m = 0$ of both families $\omega^-_{q,0}$ (left) and $\omega^+_{q,0}$ (right) as a function of $\xi = \frac{3n}{2\pi T L^2}$. In order of increasing opacity the curves correspond to the levels $q = 0, 1, \ldots, 8$.

$\xi \to \pm 1$, but as before, we do not observe any mode becoming unstable. In addition, for every value of ξ and q, the imaginary parts of these modes are larger (in absolute value) than those of the pseudo-hydrodynamic modes, and hence there is no level crossing. In the opposite limit, at $\xi = 0$, all the modes collapse to $\omega^{\pm}_{q,0}(0) \approx (1.849 - 2.664i) r_+/L^2$, which agrees with the first ordinary mode of the black brane in the limit of vanishing momentum [333–335].

Stability

So far, all the modes we have found are stable, meaning that their associated frequencies lie in the lower half of the complex plane. In order to show that the Taub-NUT solution is (linearly) stable one must prove that this property holds for every quasinormal modes. Here we provide evidence that this is indeed the case, but for future analyses it would be important to provide a solid proof of this fact.

As we have seen, the quasinormal modes with the lowest imaginary part are the pseudo-hydrodynamic ones, and the imaginary part becomes smaller as we increase q. Therefore, we should analyze the behaviour of these modes when $q \to \infty$. In Fig. 6.4 we have plotted the trajectories in the complex plane of these modes for many values of q and a some selected values of $\epsilon = n/r_+$. Thanks to the logarithmic scale in the vertical axis, we can see clearly that the imaginary part tends to zero exponentially with q and that it also decreases when ϵ grows. Indeed, a fit to the numerical data reveals that the

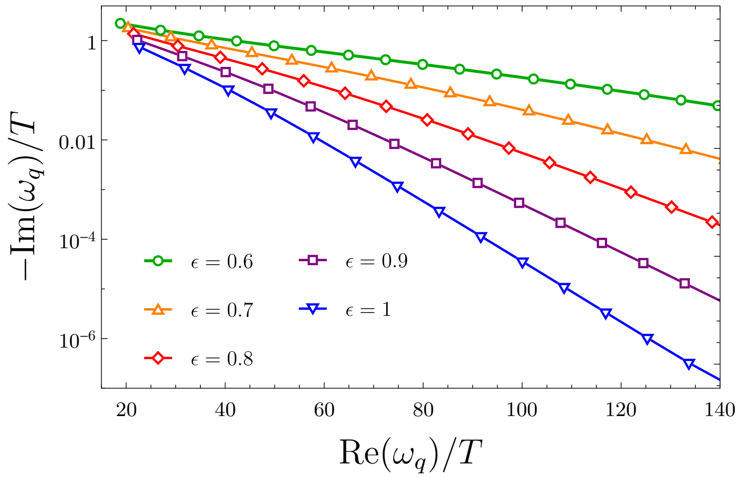

Figure 6.4: Trajectories in the complex plane of the QNM frequencies ω_q of lowest imaginary part (corresponding to the pseudo-hydrodynamic modes) for a few values of ϵ. For large q the imaginary part tends to zero exponentially, but it never becomes positive.

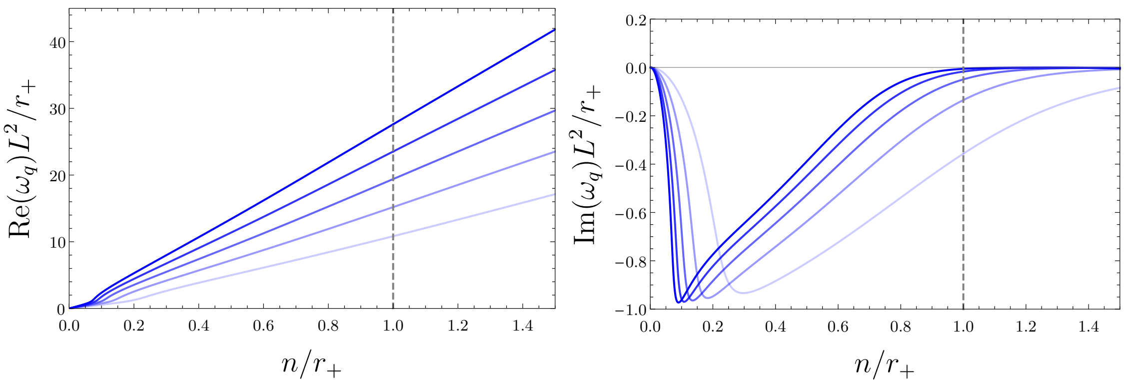

Figure 6.5: Pseudo-hydrodynamic quasinormal modes extended for $n/r_+ > 1$ for the levels $q = 0, ..., 4$. There is nothing special at the point $n = r_+$ and the modes keep on being stable beyond it. However, their imaginary parts become exponentially small as we increase n.

imaginary part of the QNM frequencies ω_q for large q is well approximated by

$$\mathrm{Im}(\omega_q(\epsilon)) \approx -TA(\epsilon)e^{-(2.2\epsilon-1.05)q}\,, \tag{6.147}$$

which is valid as long as ϵ is not far from 1. For smaller values of ϵ, the imaginary part is larger (in absolute value) and therefore, the negativity of $\mathrm{Im}(\omega_q(\epsilon))$ for $\epsilon = 1$ implies the stability of all the modes with $\epsilon \leq 1$. However, the asymptotic behaviour for $q \to \infty$ is difficult to access for small ϵ, since it requires going to larger and larger q, in which case our numeric method becomes less accurate. In any case, our data suggests that the imaginary part of ω_q ultimately decays exponentially with q for any value of ϵ. Thus, the conclusion is that the lowest-lying modes for every q are stable for every $|\epsilon| \leq 1$, and by extension all the modes are. This signals that, despite the apparent pathological properties of the NUT-charged spacetimes, they actually give rise to stable and well-defined dynamics.

Finally, although we have focused on the case $|\epsilon| \leq 1$ because it is the relevant one for holography, one may wonder what happens if we take even larger values of the NUT charge $|\epsilon| \geq 1$. In fact, since those solutions do not posses an Euclidean continuation, one may think that they could be unstable. In Fig. 6.5, we show the lowest gravitational QNM for a few values of q as a function of $\epsilon = n/r_+$, extended beyond $\epsilon = 1$. We observe nothing special going on at that point, and in fact, the modes keep on being stable as we increase ϵ. Nevertheless, Fig. 6.5 shows that Taub-NUT solutions with increasingly large NUT charge have more quasinormal modes with extremely small imaginary parts, and it would be interesting to study if this could eventually give rise to a non-trivial instability when nonlinearities are taken into account.

6.5 Conclusions

We have performed a thorough analysis of the quasinormal modes of the planar Taub-NUT spacetimes given by (6.2). As we discussed, these describe the linear response to perturbations of a strongly-coupled plasma placed in the geometry (6.8), corresponding to a Gödel-type universe with closed time-like curves.

Our analysis revealed that QNMs in this background organize analogously to the Landau levels of a charged particle in a uniform magnetic field. Thus, unlike in the case of planar black holes, the spectrum of QNM frequencies is discrete and labeled by a unique quantum number q (the Landau level). On the other hand, the QNMs are infinitely degenerate in the momentum k along the isometric direction, which we chose to be y. Another novel aspect introduced by the NUT charge is that all the reflection symmetries of the spacetime are broken, which implies that one cannot decompose the perturbations of fields with spin into modes of definite parity. This leads to the appearance of an additional "polarization parameter" λ_{+2} characterizing the gravitational QNMs. This parameter has to be determined together with the corresponding QNM frequency ω by solving simultaneously the equations for the two NP variables Ψ_0 and Ψ_4. By using the Teukolsky-Starobinsky identities, we have been able to determine this parameter, which has two admissible values $\lambda_{+2}^{(\pm)}$ – see Eq. (6.118). In the limit of vanishing NUT charge, these values give rise to modes with odd and even parity in the background of the black brane. Then, the boundary conditions for each of the NP variables are fully determined and it is enough to solve the radial equation for one of them to find the QNMs. Finally, despite parity violation, we found that the spectrum of gravitational QNM frequencies is symmetric under the change of sign of the NUT charge. In addition, there is a conjugation symmetry that relates the positive-frequency modes of the solution with charge n to the negative-frequency ones of the solution with charge $-n$, and vice-versa — see Eq. (6.126).

In the case of electromagnetic perturbations we have shown that a similar method does not work, since the boundary conditions for the NP variables ϕ_0 and ϕ_2 are degenerate. Thus the corresponding polarization parameter λ_{+1} cannot be determined in this way or by parity arguments. This may lead to the conclusion that the spectrum of QNMs depends continuously on this parameter, but this issue certainly deserves further research. Perhaps analyzing the perturbations in terms of the vector field rather than in terms of the Newman-Penrose variables could shed light on this problem.

Our numerical results on the scalar and gravitational QNM frequencies show that all of them lie in the lower half of the complex plane, and hence no instabilities are found despite the exotic causal structure of these spacetimes. Thus, this constitutes yet another step into the rehabilitation of Lorentzian spacetimes with NUT charge, in line with Refs. [349–353]. If we now apply the AdS/CFT correspondence, this result tells us not only that one should be able to perform quantum field theory in the background of the causality-violating metric (6.8), but that it should be possible to obtain sensible answers. Hence, it would now be interesting to perform a direct QFT computation in (6.8) to try to reproduce the results obtained from holography. In particular, we managed to obtain an analytic expression for the pseudo-hydrodynamic modes in the limit of small NUT charge — see Eq. (6.139). As we have shown, that result generalizes the standard dispersion relation for the sound mode in flat space to the case of the background (6.8) when $n/L^2 << T$. It would be extremely interesting to attempt a derivation of that relation by studying the perturbations of a fluid in such background.

Let us close our paper by commenting on other directions that should be considered. As we already mentioned, one should try to understand better the properties of electromagnetic QNMs. On the other hand, we have focused mainly on the scalar and gravitational modes with lowest imaginary part, but it would be interesting to complete the classification of QNMs by analyzing the overtone structure and the highly damped modes. In addition, even though we have provided compelling numerical evidence that no unstable QNMs exist, it would be important to offer a mathematical proof of this fact. Finally, it would also be worth extending these results to the case of Taub-NUT solutions of different topologies — the spherical case is particularly interesting due to the interplay with the Misner string [357] — or to higher dimensions. Hopefully these will offer further insight on the role of NUT charge in the AdS/CFT correspondence.

7
Love Numbers and Magnetic Susceptibility of Charged Black Holes

This chapter is based on:
Love numbers and magnetic susceptibility of charged black holes
D. Pereñiguez, V. Cardoso
Phys.Rev.D 105 (2022) 4, 044026 (arXiv:2112.08400)

The advent of gravitational-wave (GW) astronomy [378, 379] and of very long baseline interferometry [1, 380] allows access to the hitherto invisible Universe [15, 97, 381–383]. In this vast landscape, compact objects such as black holes (BHs) hold a tremendous discovery potential, allowing for unprecedented tests of General Relativity (GR) in the strong-field regime [15, 97, 381, 382, 384, 385]: are BHs described by classical General Relativity [386] in vacuum, and up to which extent are matter effects important and measurable [387–389]? Do BHs exist and how can we *quantify* the presence of horizons in the spacetime [381, 390]?

The answer to the above questions requires an understanding of the dynamics of BH spacetimes in general setups, a notoriously difficult task. A key component in how BHs respond dynamically lies in their deformability properties, encoded in so-called tidal Love numbers (TLNs) [391, 392]. These leave a detectable imprint in the GW signal emitted by compact binaries in the late stages of their orbital evolution. An intriguing result in classical, vacuum GR concerns the *vanishing* of the TLNs of BHs [79–82, 393, 394]. The precise cancellation of the TLNs of BHs within Einstein's theory may pose a problem of "naturalness" [395–397], which can be argued to be as puzzling as the strong CP and the hierarchy problem in particle physics, or as the cosmological constant problem. The resolution of this issue in BH physics could lead to – testable, since they would be encoded in GW data –smoking-gun effects of new physics.

The above properties only hold in vacuum, while astrophysical BHs are surrounded by matter, even if dilute. Indeed, it was shown that such environmental effects can conspire to produce small but nonvanishing TLNs [89]. Other, light matter fields could arise in extensions of the Standard Model, or in higher dimensional theories [398–400]. While their abundance could be negligible, it is unclear if their very existence contributes to nontrivial TLNs, but extra degrees of freedom, particularly scalar fields, can contribute with nonvanishing TLNs in some specific theories [401].

Here, we address the following main question: what is the effect of charge and electromagnetic fields on the static polarisability of BHs? In particular, can charge excite new modes of static polarisation? Furthermore, we consider this in an arbitrary number of spacetime dimensions $D \geq 4$. This is a well-motivated setup for different reasons. First, the physics of higher-dimensional, charged BHs is a matter of interest *per se*. In particular, these play a central role in the microscopic derivations of the Bekenstein–Hawking entropy [128, 215] as well as in the computation of its stringy corrections [209, 218]. Upon dimensional reduction, such BHs can also be relevant in astrophysics. While KK excitations do not seem reachable in astrophysical processes [86], in brane-world type reductions the extra-dimensions induce a definite signature in the BH frequency spectrum [402]. It is important to revisit these scenarios with focus on the static response. However, first one needs to understand the higher-dimensional degrees of freedom in more natural settings (e.g. n-dimensional spherical symmetry). Finally, from a more technical viewpoint, spacetime dimensionality D can be seen as a regularisation parameter to obtain four-dimensional TLNs by taking $D \to 4$, hence also understanding how special such parameter is in the space of possible values [85, 87].

7.1 Charged black holes in D Dimensions

We are interested in the static response of D-dimensional, asymptotically flat BHs which are charged under matter gauge fields. One of the simplest theories containing BHs fulfilling such requirements is Einstein–Maxwell theory in arbitrary spacetime dimension D. The field content is the metric g_{AB} and a $U(1)$ gauge field $\mathcal{A}_A$, both subject to the action

$$S[g, \mathcal{A}] = \frac{1}{2\kappa^2} \int d^D x \sqrt{g}\, R - \frac{1}{4} \int d^D x \sqrt{g}\, \mathcal{F}^2 \,, \tag{7.1}$$

where $\mathcal{F} = d\mathcal{A}$ is the field-strength and κ^2 the D-dimensional gravitational coupling. The equations of motion take the familiar form

$$\begin{aligned} G_{AB} &= \kappa^2 T_{AB}\,, \quad d \star \mathcal{F} = 0 \\ T_{AB} &= \mathcal{F}_{AC}\mathcal{F}_B^{\ C} - \frac{1}{4} g_{AB}\mathcal{F}^2 \,. \end{aligned} \tag{7.2}$$

There is a large set of black objects solving these equations that are of interest in several contexts [59]. Here we are concerned with linear fluctuations on such spaces, which is a problem of significant complexity and hard to approach in various cases. An analysis of the perturbations based on harmonic decomposition is possible as long as the BH solutions enjoy enough structure [102, 403], and this is the only situation in which a complete description of the perturbations in arbitrary D is known. Here we shall restrict to the BH solutions of (7.1) in which such analysis holds.

Static, spherically-symmetric BHs of (7.1) carrying electric charge are described by the Reissner–Nordström–Tangherlini solutions [35, 36, 404]. The metric and field strength read

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_n^2, \qquad \mathcal{F} = E_0 dt \wedge dr \,, \tag{7.3}$$

where $f = f(r)$, $E_0 = E_0(r)$ and we find it convenient to define the dimension parameter

$$n = D - 2 \,, \tag{7.4}$$

and

$$f = 1 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}}, \qquad E_0 = \frac{q}{r^n}, \qquad Q^2 = \frac{\kappa^2 q^2}{n(n-1)} \,, \tag{7.5}$$

with M and Q the BH mass and charge (up to factors) respectively. The metric (7.3) has Killing horizons relative to $k = \partial_t$ at

$$r_{\pm}^{n-1} = M \pm \sqrt{M^2 - Q^2}\,. \tag{7.6}$$

Consequently, the solution exhibits a regular event horizon at $r = r_+$ as long as the extremality bound $|Q| \leq M$ is preserved. In that case, the Hawking temperature of the BH is

$$T_H = \frac{n-1}{4\pi r_+^n}\left(r_+^{n-1} - r_-^{n-1}\right). \tag{7.7}$$

When the extremality bound is saturated, the event and Cauchy horizons merge and $T_H = 0$. On the other hand, as one approaches the neutral limit $Q = 0$ the Cauchy horizon r_- coalesces with the curvature singularity at $r = 0$ and the solution reduces to Schwarzschild–Tangherlini [404]. We will see that this plays a crucial role for the master equations governing static perturbations. Whenever any of these two limits takes place, i.e. $Q = 0$ or $T_H = 0$, the equations become hypergeometric and Love numbers and magnetic susceptibilities are exactly solvable. For intermediate values of the BH charge the equations pick an extra pole (the Cauchy horizon) and are, therefore, less amenable. Nevertheless, we still manage to get exact results in most cases. In the following we derive the master equations governing static perturbations of (7.3) for both the tensor and vector sectors.

7.2 Perturbation theory

A large class of BH spacetimes can be written as a warped product of an n-dimensional euclidean Einstein manifold $(\mathcal{K}^n, \gamma_{ij})$ and an m-dimensional Lorentzian manifold $(\mathcal{N}^m, g_{ab})$ $(i, j = 1, .., n$ and $a, b = 1, ..., m)$. The spacetime is $(n+m)$-dimensional with manifold structure $M = \mathcal{N}^m \times \mathcal{K}^n$ and, in adapted coordinates $x^A = (y^a, z^i)$, the metric takes the form

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)\gamma_{ij}(z)dz^i dz^j\,, \tag{7.8}$$

where $r(y)$ is the warping factor defined as a function on $\mathcal{N}^m$. A metric with structure (7.8) is only compatible with energy-momentum tensors of the form

$$T_{ai} = 0, \quad T^i{}_j = P\delta^i{}_j\,, \tag{7.9}$$

where P is a function on $\mathcal{N}^m$. Although such a spacetime is notably general, the fact that $\mathcal{K}^n$ is Einstein still allows an analysis of fluctuations based on harmonic decomposition. This is due to Kodama and Ishibashi (KI) who established a completely covariant and gauge-invariant approach to perturbation theory on these spaces [102, 403].

In the KI formalism, taking advantage of the structure of Eq. (7.8) one decomposes a general perturbation in tensor, vector and scalar sectors. After projection on the corresponding harmonics, Einstein's equations decouple in three sets of partial differential equations (PDEs) on $\mathcal{N}^m$, one for each sector. This holds for a general energy-momentum tensor, and the equations may be simplified by assuming its covariant conservation. Once a specific field content has been chosen, Einstein's equations are supplemented with the matter equations of motion. In the case of Einstein–Maxwell theory, the vector potential can already be decomposed in scalar $(\delta A_a, a)$ and vector $A_i^{(1)}$ components[1]

$$\delta\mathcal{A} = \delta A_a dy^a + \left(A_i^{(1)} + \hat{D}_i a\right) dz^i \,, \text{ with } \hat{D}^i A_i^{(1)} = 0 \,, \tag{7.10}$$

from which it follows that the matter tensor sector is empty in this theory. The final form of the equations is given in terms of a gauge-invariant basis of variables that can be constructed for each sector. The BHs of Einstein–Maxwell theory considered in this work, described by Eq. (7.3), fall in the class of Eq. (7.8) with $m = 2$ and $\mathcal{K}^n = \mathbb{S}^n$. In the remaining of this work we adopt the KI formalism [102, 403] and focus on tensor and vector fluctuations on the background space (7.3). This is convenient because, on the one hand, it suffices to understand the behaviour of test fields as well as interacting gravitational and electromagnetic perturbations. In addition, the equations turn out to be simple enough so as to admit analytical results in several instances. A more thorough analysis including the scalar sector will be considered elsewhere.

7.2.1 Master equations and their static limit: tensor sector

A general tensor perturbation is generated by just two gauge-invariant variables (H_T, τ_T) [102, 403],

$$h_{ij} = 2r^2 H_T \mathbb{T}_{ij}, \quad \delta T_{ij} = r^2\left(\tau_T + 2PH_T\right)\mathbb{T}_{ij} \,, \tag{7.11}$$

where $\mathbb{T}_{ij}$ are the tensor harmonics on $\mathbb{S}^n$ satisfying

$$\left(\hat{D}^k\hat{D}_k + k_t^2\right)\mathbb{T}_{ij} = 0, \quad \mathbb{T}^i{}_i = 0 = \hat{D}^j\mathbb{T}_{ji} \,, \tag{7.12}$$

with spectrum

$$k_t^2 = L(L+n-1) - 2 \,, \qquad L = 2, \ldots \tag{7.13}$$

Furthermore, the Maxwell field-strength δF does not contribute to the tensor part of the energy-momentum tensor,

$$\tau_T = 0 \,, \tag{7.14}$$

and the Einstein–Maxwell equations reduce to a single PDE on $\mathcal{N}^2$ for H_T,

$$\Box H_T + \frac{n}{r} Dr \cdot DH_T - \frac{k_t^2 + 2}{r^2} H_T = 0 \,. \tag{7.15}$$

As noted by KI [102, 403], Eq. (7.15) turns out to be the same as that satisfied by a test, massless scalar field on our background if k_t^2 is appropriately identified with the angular momentum number. Therefore, the tensor sector can also be used to infer properties of test fields on (7.3).

After a field redefinition $H_T = r^{-n/2}\phi$ to get rid of the term $\sim Dr \cdot DH_T$ the master equation becomes

$$\left(\Box + V\right)\phi = 0 \,, \tag{7.16}$$

with

$$V = \frac{n(3n-2)}{4}\frac{Q^2}{r^{2n}} - \frac{4k_t^2 + 8 + n^2 - 2n}{4r^2} - \frac{n^2}{2}\frac{M}{r^{n+1}} \,. \tag{7.17}$$

[1] Equivalently, one may regard $\delta\mathcal{F} = d\delta\mathcal{A}$ as the basic variable and decompose it with respect to $\mathcal{K}^n$. This seems the most natural approach for matter fields of higher ranks.

We are interested in the static solutions of this equation, that is, solutions satisfying $\pounds_k \phi = 0$ where k is the static time-like Killing vector of (7.3). Either in Schwarzschild or Eddington–Finkelstein coordinates, this translates into the requirement that ϕ is a function of r only, $\phi = \phi(r)$. When specialised for a static perturbation, Eq. (7.16) becomes an ODE of Fuchsian type with four regular singular points: infinity, the event horizon, the Cauchy horizon and the singularity. Therefore, it can be cast in Heun's form [405, 406]. To see this, we first introduce the dimensionless variable

$$z = \left(\frac{r_+}{r}\right)^{n-1} . \tag{7.18}$$

Then, after a field redefinition

$$H_T(z) = r(z)^{-n/2} z^{\frac{2l(n-1)+n-2}{2(n-1)}} \Psi(z) , \tag{7.19}$$

the master equation becomes of Heun's type,

$$\Psi'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\eta}{z-z_c}\right)\Psi' + \frac{\alpha\beta\,(z-h)}{z(z-1)\,(z-z_c)}\Psi = 0 , \tag{7.20}$$

where primes stand for derivatives with respect to z and with coefficients

$$\begin{aligned} z_c &= \cot^2\left(\frac{\epsilon}{2}\right) \quad \gamma = 2(l+1), \quad \delta = 1, \quad \eta = 1 , \\ \alpha &= 2+l, \quad \beta = 1+l, \quad h = \frac{(l+1)^2\left(\frac{\epsilon}{2}\right)}{l+2} . \end{aligned} \tag{7.21}$$

This equation depends on two dimensionless parameters, ϵ and l, defined as

$$l = \frac{L}{n-1}, \qquad \sin\epsilon = \frac{Q}{M} , \tag{7.22}$$

where L is the harmonic number defined by (7.13). The extremality bound dictates $|Q| \leq M$, and without loss of generality we can restrict to $\epsilon \in [0, \pi/2]$ with neturality and extremality lying at 0 and $\pi/2$, respectively. Equation (7.20) has regular poles at $z = 0, 1, z_c, \infty$, corresponding respectively to infinity, the event and Cauchy horizons and the singularity.

It is interesting to specialize the general equation (7.20) to neutral and extremal cases. The regular singularity at the Cauchy horizon z_c collides with that on the event horizon as $T_H \to 0$, while in the neutral limit $Q \to 0$ it merges with the spacetime curvature singularity (see Figure 7.1). Quite interestingly, in none of these limits the merging produces an irregular singularity. Instead, one has three regular singularities at infinity $z = 0$, the horizon $z = 1$ and the curvature singularity $z = \infty$. Consequently, the equation becomes of hypergeometric type and in such cases one can use the theory of hypergeometric functions to obtain analytically the response parameters, as discussed in [85, 87] for the neutral limit. We will find the same pole structure in the vector sector.

Explicitly, in the neutral case equation (7.20) can be immediately evaluated at $\epsilon = 0$ giving the hypergeometric equation

$$z(1-z)\Psi'' + \left[c - (a+b+1)z\right]\Psi' - ab\Psi = 0 , \tag{7.23}$$

with coefficients

$$a = b = l+1, \quad c = 2(l+1) . \tag{7.24}$$

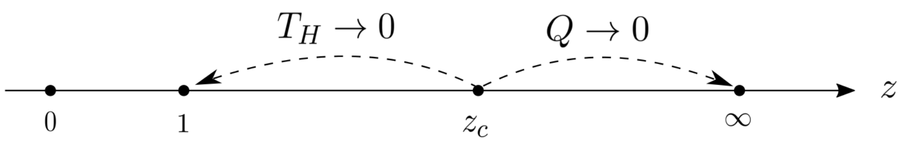

Figure 7.1: Singularity structure of the master equations. The regular singular point at the Cauchy horizon $z = z_c$ coalesces with those at the event horizon $z = 1$ and spacetime singularity $z = \infty$ in the extremal and neutral limits, respectively.

This coincides with the equation obtained in Ref. [87] for the tensor degree of freedom. In the extremal case $\epsilon = \pi/2$, after a field redefinition

$$\Psi(z) = (1-z)^l \psi(z) \,, \tag{7.25}$$

one obtains again an hypergeometric equation (7.23) for $\psi(z)$, now with parameters

$$a = c = 2(l+1) \,, \; b = 2l + 1 \,. \tag{7.26}$$

In sum, we have found that a static tensor perturbation is governed by Heun's equation (7.20) with coefficients given in (7.21). In the neutral and extremal limits, it reduces to an hypergeometric equation (7.23) with coefficients given in (7.24) and (7.26), respectively. In the following section we will discuss solutions to these equations and obtain the associated response parameters.

7.2.2 Master equations and their static limit: vector sector

The vector sector of a general perturbation is composed of [102, 403]

$$h_{ai} = h_a^{(1)} \mathbb{V}_i \,, \quad h_{ij} = -2k_v h_T^{(1)} \mathbb{V}_{ij} \,, \tag{7.27}$$

$$\delta T_{ai} = T_a^{(1)} \mathbb{V}_i \,, \quad \delta T_{ij} = -2k_v T_T^{(1)} \mathbb{V}_{ij} \,, \tag{7.28}$$

where the vector harmonics $\mathbb{V}_i$ satisfy

$$\left(\hat{D}^j \hat{D}_j + k_v^2\right) \mathbb{V}_i = 0 \,, \; \hat{D}^i \mathbb{V}_i = 0 \,, \; \mathbb{V}_{ij} := -\frac{1}{k_v} \hat{D}_{(i} \mathbb{V}_{j)} \,,$$
$$k_v^2 = L(L+n-1) - 1 \,, \quad L = 1, 2, \ldots \tag{7.29}$$

Excluding the special harmonic case $L = 1$, a basis of gauge-invariant variables in $\mathcal{N}^2$ is

$$F_a^{(1)} = \frac{1}{r}\left(h_a^{(1)} - r^2 D_a \left(\frac{h_T^{(1)}}{r^2}\right)\right) , \tag{7.30}$$

$$\tau_a^{(1)} = \frac{1}{r}\left(T_a^{(1)} - P h_a^{(1)}\right) , \tag{7.31}$$

$$\tau_T = \frac{2k_v}{r^2}\left(-T_T^{(1)} + P h_T^{(1)}\right) . \tag{7.32}$$

There are two Einstein equations for this sector plus one coming from conservation of T_{AB}, $\delta(\nabla^M T_{MA}) = 0$. The latter can be combined with one of the Einstein equations to give an integrability condition, which allows one to trade F_a by a function Ω satisfying

$$D_a \Omega = \epsilon_{ac}\left(r^{n-1} F^c - 2\frac{\kappa^2}{m_V} r^{n+1} \tau^c\right) , \tag{7.33}$$

$$m_V = k_v^2 - (n-1) = (L-1)(L+n)\,. \tag{7.34}$$

Notice that $m_V = 0$ only for the special harmonic $L = 1$ that we consider separately. In addition, the vector sector of Maxwell's field is generated by a single gauge-invariant function A on $\mathcal{N}^2$,

$$\delta\mathcal{A} = A\mathbb{V}_i dz^i, \quad \tau_a = -\frac{q}{r^{n+1}}\epsilon_{ab}D^b A, \quad \tau_T = 0\,. \tag{7.35}$$

In terms of the gauge-invariant functions (Ω, A) on $\mathcal{N}^2$, the Einstein and Maxwell equations are reduced to a pair of coupled PDEs

$$r^n D^a\left(\frac{D_a\Omega}{r^n}\right) - \frac{m_V}{r^2}\Omega = -\frac{2\kappa^2}{m_V}r^n\epsilon^{ab}D_a\left(r\tau_b\right)\,, \tag{7.36}$$

$$\frac{1}{r^{n-2}}D_a\left(r^{n-2}D^a A\right) - \frac{k_v^2+n-1}{r^2}A = \frac{qm_V}{r^{2n}}\Omega\,. \tag{7.37}$$

Introducing the field redefinitions

$$\phi_\pm = a_\pm r^{-n/2}\left(\Omega - \frac{2\kappa^2 q}{m_V}A\right) + b_\pm r^{\frac{n-2}{2}}A\,, \tag{7.38}$$

we find that equations (7.36) and (7.37) decouple if

$$(a_+, b_+) = \left(\frac{Qm_V}{\Delta + M\left(n^2-1\right)}\sigma_{(+)}, \frac{Q}{q}\sigma_{(+)}\right), \tag{7.39}$$

$$(a_-, b_-) = \left(\sigma_{(-)}, -\frac{2\kappa^2 q}{\Delta + M\left(n^2-1\right)}\sigma_{(-)}\right), \tag{7.40}$$

where $\sigma_{(\pm)}$ are any two (non-zero) constants and the positive constant Δ satisfies

$$\Delta^2 = M^2\left(n^2-1\right)^2 + 2n(n-1)m_V Q^2\,. \tag{7.41}$$

With this, $\phi_\pm$ satisfy master equations of the form

$$\left(\Box + V_\pm\right)\phi_\pm = 0\,, \tag{7.42}$$

with

$$V_\pm = -\frac{k_v^2+1+n^2/4-n/2}{r^2} - \frac{n(5n-2)Q^2/4}{r^{2n}} - \frac{-(n^2+2)M/2 \pm \Delta}{r^{n+1}}\,. \tag{7.43}$$

A comment here is in order. This derivation of Eq. (7.42) reproduces that in Ref. [102, 403] with the difference that the decoupling parameters (7.39) and (7.40) are defined only up to their global factors. This is due to the fact that the general solution must depend on two independent amplitudes. In the neutral background these are clearly associated to the gravitational and electromagnetic fluctuations. However, when the BH is charged, such fluctuations couple and the independent amplitudes refer to the modes $\phi_\pm$ that contain fixed proportions of gravitational and electromagnetic contributions. This fact will be important for the definition of vector Love numbers and magnetic susceptibilities. Lastly, notice that on the neutral background ϕ_- and ϕ_+ reduce to the standard gravitational and electromagnetic master variables respectively, so it may still be sensible to regard them as the gravitational and electromagnetic degrees of freedom even in the charged case.

We are interested in static solutions of (7.42). In terms of the new variables $\Psi_\pm$,

$$\phi_\pm = z^{\frac{2l(n-1)+n-2}{2(n-1)}} \Psi_\pm \,, \tag{7.44}$$

we obtain once again Heun's differential equation (7.20), but now with parameters given by

$$\begin{aligned} z_c &= \cot^2\left(\frac{\epsilon}{2}\right) \,, \quad \alpha = l + 3 + \frac{1}{n-1} \,, \; \beta = l - \frac{1}{n-1} \,, \\ \gamma &= 2(l+1) \,, \; \delta = \eta = 1 \,, \\ h_\pm &= \frac{2\left(\frac{\epsilon}{2}\right)\left(1 + 2l(n-1)^2(l+2) + n^2 - 4n \pm \tilde{\Delta}\right)}{2\left(2 + l^2(n-1)^2 + 3l(n-1)^2 - 3n\right)} \,. \end{aligned} \tag{7.45}$$

The dimensionless variable z and parameters l and ϵ are given by (7.18) and (7.22), respectively, while $\tilde{\Delta} := \Delta/M$.

This equation has the same pole structure as the master equation of the tensor sector, with regular singularities at infinity $z = 0$, event horizon $z = 1$, Cauchy horizon $z = z_c$ and curvature singularity $z = \infty$. Just as in the tensor case, the Cauchy horizon z_c merges with the singular points at the event horizon and the curvature singularity in the extremal and neutral limits, respectively, leading in both cases to an hypergeometric equation (see Figure 7.1). The equations for the neutral case are obtained just by evaluating (7.20) at $\epsilon = 0$, and have hypergeometric form

$$z(1-z)\Psi_\pm'' + \left[c - (a_\pm + b_\pm + 1)z\right]\Psi_\pm' - a_\pm b_\pm \Psi_\pm = 0 \,, \tag{7.46}$$

with parameters $c = 2(l+1)$ and

$$a_+ = a_- + 1 = l - \frac{1}{n-1} + 1 \,, \tag{7.47}$$

$$b_+ = b_- - 1 = l + \frac{1}{n-1} + 1 \,. \tag{7.48}$$

Again, we find agreement with previous results for the neutral case [87]. The extremal limit is a bit more involved. After a field redefinition

$$\Psi_\pm = (1-z)^{\frac{1}{2}\left(\frac{\Sigma_\pm}{n-1} - 1\right)} \psi_\pm \,, \tag{7.49}$$

where

$$\Sigma_\pm = \sqrt{(n-1)(5n+3) + 4\left(m_V \pm \tilde{\Delta}\right)} \tag{7.50}$$

and introducing the symbol

$$S_{(\rho,\sigma)} = \frac{\Sigma_\rho + \sigma(3n-1)}{2(n-1)} \quad \text{with} \quad \rho, \sigma = \pm \,, \tag{7.51}$$

the equations for $\psi_\pm$ take hypergeometric form with $c = 2(l+1)$ and

$$a_\pm = 1 + l + S_{(\pm,+)}, \quad b_\pm = 1 + l + S_{(\pm,-)} \,. \tag{7.52}$$

Lastly, we consider the special harmonic mode. This corresponds to the case that $\mathbb{V}_i$ is a Killing vector field in $\mathcal{K}^n$, i.e. $\mathbb{V}_{ij} = 0$. For $\mathcal{K}^n = \mathbb{S}^n$ this happens only if $m_V = (L-1)(L+n) = 0$, i.e. $L = 1$ [102, 403]. The projection of the perturbation into this harmonic, unlike the general one (7.27) and (7.28), is composed of just

$$h_{ai} = h_a^{(1)} \mathbb{V}_i \,, \, \delta T_{ai} = T_a^{(1)} \mathbb{V}_i \,. \tag{7.53}$$

The gauge-invariant variables in this case are

$$F_{ab} = r D_a \left(\frac{h_b^{(1)}}{r^2} \right) - r D_b \left(\frac{h_a^{(1)}}{r^2} \right) , \tag{7.54}$$

$$\tau_a = \frac{1}{r} \left(T_a^{(1)} - P h_a^{(1)} \right) = \frac{-q}{r^{n+1}} \epsilon_{ab} D^b A \,, \tag{7.55}$$

and F_{ab} can be solved exactly as [102, 403]

$$F = q \frac{2\kappa^2}{r^{n+1}} A - \frac{2\kappa^2}{r^{n+1}} \tau_0 \,, \tag{7.56}$$

where $F = (1/2)\epsilon^{ab} F_{ab}$ and τ_0 is an arbitrary integration constant. It follows that the gravitational special mode is non-dynamical. In particular, τ_0 generates a small rotation so restricting to a static background requires setting $\tau_0 = 0$. In terms of $\phi_+ = r^{\frac{n-2}{2}} A$, Maxwell's equation reduces precisely to the "+" equation in (7.42) with $L = 1$.

In sum, we have found that static vector perturbations are governed by equations of Heun's type (7.20) with parameters (7.45). In the neutral and extremal limits these become hypergeometric, with parameters (7.47)-(7.48) and (7.52), respectively. The special harmonic is recovered by just setting $L = 1$ in the electromagnetic mode (+) and disregarding the gravitational one (−). In the following section we discuss static solutions to these equations and obtain the associated response parameters.

7.3 Static response

The original works that established the vanishing of BH Love numbers in four dimensions, both in neutral [80] and charged [401] cases, followed an approach based on a full GR computation. Recently, the authors in Ref. [87] considered also this point of view to compute the static response of fields with integer spin, 0, 1 and 2, fluctuating on a neutral Schwarzschild–Tangherlini background. Along the lines of [85], they also showed that response parameters obtained in that way can be regarded as coefficients in a worldline effective action associated to the BH, thus clarifying some concerns about ambiguities in the definition of Love numbers [407, 408]. All these motivates us to adopt a full GR approach to study the static response of charged BHs in arbitrary D.

7.3.1 Tensor Love numbers

The parameters governing the static response of a system to a tidal field can be obtained by inspection of the solutions at infinity. Consider first a tensor perturbation on (7.3), which is described by (7.20). From the standard theory Fuchsian equations, in a neighbourhood of $z = 0$ the general solution has the form [405]

$$\Psi(z) = A\Psi_{\text{resp}}(z) + B\left(z^{-2l-1}\Psi_{\text{tidal}}(z) + R\Psi_{\text{resp}}(z)\ln z\right). \tag{7.57}$$

Here, A and B are arbitrary constants multiplying two linearly independent solutions. The first one, $\Psi_{\text{resp}}(z)$, is analytic at $z = 0$ and without loss of generality we choose to normalise it as $\Psi_{\text{resp}}(z) = 1+O(z)$. The second solution contains, in general, a logarithmic term where R is some constant and $\Psi_{\text{tidal}}(z)$ is another analytic function at $z = 0$ that we chose to normalise as $\Psi_{\text{tidal}}(z) = 1 + O(z)$. Of course, the indices of our equation at $z = 0$ are 0 and $-(2l+1)$, and the latter quantity serves as a discriminant between qualitatively different cases:

• $2l + 1 \notin \mathbb{N}$: In this case the Frobenius solutions associated to each index at $z = 0$ are linearly independent and one has $R = 0$. After imposing regularity at the horizon $z = 1$ the relative normalisation between A and B gets fixed,

$$\Psi(z) = B\left(k\Psi_{\text{resp}}(z) + z^{-2l-1}\Psi_{\text{tidal}}(z)\right). \tag{7.58}$$

The growing mode at infinity $\sim z^{-2l-1}$ has the interpretation of an external tidal field while $\Psi_{\text{resp}}(z)$, which is regular at $z = 0$, is the response of the system. The parameter k is the (dimensionless) tidal Love number, which is precisely the quantity controlling the fall-off induced by the tidal field. Since it is completely determined by the requirement of regularity at the horizon and does not depend on the amplitude of the tidal field, the Love number k is an intrinsic property of the BH.

• $2l + 1 \in \mathbb{N}$: In general, the second solution exhibits a logarithmic term, so the constant R may not vanish. Again, regularity at the horizon $z = 1$ fixes the relative normalisation between A and B,

$$\Psi(z) = B\left(k\Psi_{\text{resp}}(z) + z^{-2l-1}\Psi_{\text{tidal}}(z) + R\Psi_{\text{resp}}(z)\ln z\right). \tag{7.59}$$

However, unlike the case where $2l + 1 \notin \mathbb{N}$, now the quantity k is ambiguous due to power mixing. From the regular solution (7.59), there is no natural way of telling apart which contribution to the power series comes from the response and which from the tidal field. In particular, $k\Psi_{\text{resp}}(z)$ can be completely absorbed order by order in the term $z^{-2l-1}\Psi_{\text{tidal}}(z)$. Similar observations where noted in [87]. The invariant piece of information here is R. Furthermore, as shown in [85] and discussed in [87] the logarithmic term corresponds to a classical RG running of the induced response which is characterised by R, so we shall take $R\ln z$ as the response "parameter" in this case. Nevertheless, there is a remarkable exception within the case $2l+1 \in \mathbb{N}$. It may be that (7.20) admits a second solution where $R = 0$ and $\Psi_{\text{tidal}}(z)$ is a polynomial of degree $\leq 2l+1$. This purely growing mode is a tidal field and, furthermore, being just a terminating series in z it is precisely the solution that is regular on the horizon $z = 1$. It follows that the Love number is zero in this case[2]. As shown below, this is exactly what happens in $D = 4$.

Notice that this definition of Love numbers is in complete analogy with those in the literature in several contexts [80, 84, 85, 87, 401] and, in particular, it reduces exactly to that of [87] for the neutral BH. In the following we compute the tensor Love numbers for neutral and extremal limits separately, and then consider the case of finite charge and temperature.

[2]There is some discussion on whether this argument can be applied to the rotating case [82, 394].

Neutral and extremal limits

For vanishing BH charge $Q = 0$ static tensor perturbations are governed by the hypergeometric equation (7.23) with parameters (7.24). Writing the general solution in terms of hypergeometric functions and using the connection formulas between Kummer's solutions, the authors in [87] computed the response parameters defined as in the previous section. We list them here for completeness,

$$k_{\text{tensor}}^{(\text{neut})} = \begin{cases} \frac{2l+1}{2\pi} \frac{\Gamma(l+1)^4}{\Gamma(2l+2)^2} \tan(\pi l) & l \notin \mathbb{N}, \frac{1}{2}\mathbb{N} \\ \frac{(-1)^{2l}\Gamma(l+1)^2}{(2l)!(2l+1)!\Gamma(-l)^2} \ln z & l \in \frac{1}{2}\mathbb{N} \\ 0 & l \in \mathbb{N} \end{cases} \tag{7.60}$$

and notice that the only relevant case in $D = 4$ is $l \in \mathbb{N}$. In the extremal case the static tensor perturbation ψ in (7.25) is likewise subject to an hypergeometric equation, but now with parameters (7.26). Such equation turns out to admit a remarkably simple general solution for all l,

$$\psi(z) = \frac{A}{(1-z)^{2l+1}} + \frac{B}{z^{2l+1}}, \tag{7.61}$$

with A and B arbitrary constants. Clearly, imposing regularity at the horizon $z = 1$ fixes $A = 0$, thus leaving just a pure tidal field $\psi(z) \sim z^{-2l-1}$. This leads to the interesting result that tensor Love numbers vanish at extremality in any number of spacetime dimensions,

$$k_{\text{tensor}}^{(\text{ext})} = 0. \tag{7.62}$$

Finite charge and temperature

For intermediate charges $0 < Q < M$, the Cauchy horizon introduces an additional pole in the master equation, which becomes of Heun's type (7.20). Unfortunately, the latter is not as symmetric as the hypergeometric equation, so no analogue of Kummer's solutions exist and connection formulas are not available in general [405, 406, 409]. Thus, it is not clear how to write suitably the analytic prolongation of a solution, say, from a neighbourhood of $z = 0$ to a neighbourhood of $z = 1$[3]. This makes it difficult to obtain the response parameters proceeding as in the neutral and extremal limits. Rather remarkably, though, for tensor perturbations it is possible to obtain analytical results for all l. Consider first the degenerate case $l \in \frac{1}{2}\mathbb{N}$. After choosing the normalisation of $\Psi_{tidal}(z)$ as $\Psi_{tidal}(z) = 1 + O(z)$, equation (7.20) applied to the second solution of (7.57) fixes R completely and it is possible to obtain its exact value after solving just a few orders. Furthermore, the result can be written in closed form

$$R_{\text{tensor}} = R_{\text{tensor}}^{(\text{neut})} \left(\frac{\cos\epsilon}{\cos^2(\epsilon/2)}\right)^{2l+1} = R_{\text{tensor}}^{(\text{neut})} \left(\frac{4\pi r_+}{n-1} T_H\right)^{2l+1} \quad \left(l \in \frac{1}{2}\mathbb{N}\right), \tag{7.63}$$

where r_+ and T_H are the radius and temperature of the BH, and $R_{\text{tensor}}^{(\text{neut})}$ is the coefficient in front of the logarithm in the neutral case (7.60). Notice that (7.63) vanish at extremality, $T_H = 0$, as expected from the result in (7.62). For $l \in \mathbb{N}$ we find that the second solution

[3]See [410, 411] for recent progress in tackling this issue for Heun's confluent equation, which is the relevant ODE for oscillating perturbations in Kerr's black hole.

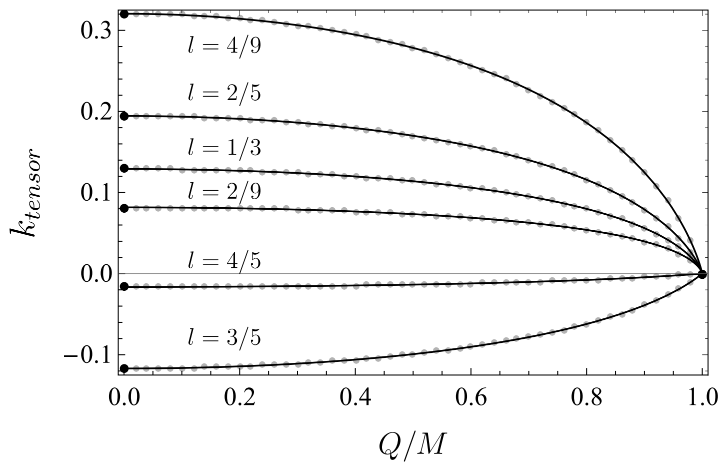

Figure 7.2: Tensor TLNs for some values of $l = L/(n-1)$ in the generic case $2l+1 \notin \mathbb{N}$. For $n = 6, 10$, we represent $L = 2, 3, 4$. Gray dots are the numerical values, solid black dots at the edges are the analytic predictions at neutrality $Q = 0$ and extremality $Q = M$, and the solid black lines correspond to the analytic formula (7.65). We observe that (7.65) is indeed in perfect agreement with both analytic and numeric results.

is just $z^{-2l-1}\Psi_{\text{tidal}}(z)$, with no logarithmic term, where $\Psi_{\text{tidal}}(z)$ is a polynomial of degree l, so

$$k_{\text{tensor}} = 0 \quad (l \in \mathbb{N}) \ . \tag{7.64}$$

This is the only relevant case for $D = 4$, where l takes values just in $\mathbb{N}$. Tensor perturbations do not exist in four dimensions, but due to the close relation between the tensor sector and (massless) scalar fields, the result (7.64) shows that $4D$, electrically charged BHs do not polarise under tidal fields of scalar type. Finally, for $l \notin \mathbb{N}, \frac{1}{2}\mathbb{N}$ with no connection formulas available it is most likely that the only way of obtaining the Love numbers at finite Q and T_H is numerically. However, in views of the results (7.63) and (7.64) it is very tempting to try with

$$k_{\text{tensor}} = k_{\text{tensor}}^{(\text{neut})} \left(\frac{4\pi r_+}{n-1} T_H\right)^{2l+1} \qquad \left(l \notin \mathbb{N}, \frac{1}{2}\mathbb{N}\right) . \tag{7.65}$$

where $k_{\text{tensor}}^{(\text{neut})}$ is the neutral Love number shown in (7.60). We compared this expression with the numerical results obtained for k_{tensor} and have found exact agreement. In Figure 7.2 we illustrate this for various values of l. This confirms the validity of (7.65), although a rigorous proof is still desirable.

We conclude that the tensor Love numbers of a charged BH of radius r_+ at temperature T_H are

$$\begin{aligned} k_{\text{tensor}} &= k_{\text{tensor}}^{(\text{neut})} \left(\frac{4\pi r_+}{n-1} T_H\right)^{2l+1} \\ &= \begin{cases} \frac{2l+1}{2\pi} \frac{\Gamma(l+1)^4}{\Gamma(2l+2)^2} \tan(\pi l) \left(\frac{4\pi r_+}{n-1} T_H\right)^{2l+1} & l \notin \mathbb{N}, \frac{1}{2}\mathbb{N} \\ \frac{(-1)^{2l}\Gamma(l+1)^2}{(2l)!(2l+1)!\Gamma(-l)^2} \left(\frac{4\pi r_+}{n-1} T_H\right)^{2l+1} \ln z & l \in \frac{1}{2}\mathbb{N} \\ 0 & l \in \mathbb{N} \end{cases} \end{aligned} \tag{7.66}$$

It is clear that these vanish at extremality, $T_H = 0$, thus recovering (7.62), and reduce to those obtained in [87] for $Q = 0$ (see (7.60)). At this point it is natural to wonder how general the vanishing of Love numbers at extremality is. In the following section we show that vector Love numbers and magnetic susceptibilities do not vanish at $T_H = 0$. Instead, BHs become significantly more polarised as one approaches the extremality bound.

7.3.2 Vector Love numbers and magnetic susceptibility

Response parameters $k_\pm$ can be defined for the master variables of the vector sector $\Psi_\pm$ just as we did for the tensor master variable. Recall that such $k_\pm$ may be just numbers or could contain a logarithm in the degenerate cases. The notions of vector Love number and magnetic susceptibility, though, are defined relative to the original fields, that is, the metric perturbation and Maxwell's vector potential [401]. More precisely, vector Love numbers (magnetic susceptibility) measure the response of the BH when there is no electromagnetic (gravitational) tidal field at infinity. Physically, this can be thought of as the BH being perturbed by the presence of a massive yet neutral (light yet highly charged) companion.

The decoupled degrees of freedom (7.38) are defined up to their respective independent amplitudes, $\sigma_{(\pm)}$. These modulate the intensity with which each mode contributes to the total perturbation. Vanishing tidal fields at infinity are achieved for particular choices of such amplitudes. To see this, it is more convenient to trade the absolute amplitudes $\sigma_{(\pm)}$ by a relative amplitude $\mathbf{\Theta}$ and a global amplitude $\mathbf{A}$ defined as

$$\mathbf{\Theta} := \frac{\sigma_{(-)}}{\sigma_{(+)}}\,, \qquad \mathbf{A} := \frac{\left(\tilde{\Delta} + n^2 - 1\right)^2}{2m_V(n-1)n\sin^2(\epsilon) + \left(\tilde{\Delta} + n^2 - 1\right)^2}\frac{1}{\sigma_{(-)}} \tag{7.67}$$

In terms of these, the original fields[4] take the form

$$\begin{aligned} h_{ai} =& \mathbf{A}\frac{\epsilon_{ab}}{r^{n-2}}D^b\left[r^{n/2}z(r)^{\frac{2l(n-1)+n-2}{2(n-1)}}\right. \qquad (7.68)\\ &\left.\times\left(k_{\text{vector}}(\mathbf{\Theta}) + \left(1 + \frac{2(n-1)n\sin\epsilon}{\tilde{\Delta}+n^2-1}\mathbf{\Theta}\right)z^{-2l-1} + \ldots\right)\right]V_i \\ \delta A_i =& \mathbf{A}\mathbf{\Theta}\frac{\sqrt{n(n-1)}}{\kappa}r^{-\frac{n-2}{2}}z(r)^{\frac{2l(n-1)+n-2}{2(n-1)}} \qquad (7.69)\\ &\times\left(k_{\text{magnetic}}(\mathbf{\Theta}) + \left(1 - \frac{m_V\sin\epsilon}{\tilde{\Delta}+n^2-1}\frac{1}{\mathbf{\Theta}}\right)z^{-2l-1} + \ldots\right)V_i \end{aligned}$$

where

$$k_{\text{vector}}(\mathbf{\Theta}) = k_- + \frac{2(n-1)n\sin\epsilon}{\tilde{\Delta}+n^2-1}\mathbf{\Theta}k_+\,, \tag{7.70}$$

$$k_{\text{magnetic}}(\mathbf{\Theta}) = k_+ - \frac{m_V\sin\epsilon}{\tilde{\Delta}+n^2-1}\frac{k_-}{\mathbf{\Theta}}\,, \tag{7.71}$$

[4] The condition on the gravitational perturbation is actually imposed on the gauge invariant variable F_a in (7.30). For clarity here we give it in terms of the metric variable h_{ai}, but this is implicitly evaluated in the gauge $h_T^{(1)} = 0$, where metric perturbation and gauge invariant variable coincide.

and we are keeping only the relevant terms of the master variables $\Psi_\pm$, that is, the tidal mode and the response fall-off. The quantities $k_{\rm vector}(\mathbf{\Theta})$ and $k_{\rm magnetic}(\mathbf{\Theta})$ are a measure of the response of the BH to a gravito-magnetic tidal field characterised by $\mathbf{\Theta}$, the relative intensity between the gravitational and magnetic contributions. The vector Love numbers and the magnetic susceptibility are precisely these quantities evaluated at the $\mathbf{\Theta}$'s in which there is no magnetic or no gravitational tidal fields respectively, that is, when no term $\sim z^{-2l-1}$ is present in the expansion of (7.69) or (7.68) [401],

$$k_{\rm vector} = k_- + 2\sin^2\epsilon \frac{m_V(n-1)n}{\left(\tilde{\Delta}+n^2-1\right)^2} k_+ \, , \tag{7.72}$$

$$k_{\rm magnetic} = k_+ + 2\sin^2\epsilon \frac{m_V(n-1)n}{\left(\tilde{\Delta}+n^2-1\right)^2} k_- \, . \tag{7.73}$$

With this, vector Love numbers $k_{\rm vector}$ and magnetic susceptibility $k_{\rm magnetic}$ are related simply by $+ \leftrightarrow -$, and $k_\pm$ can be obtained from the master equations of $\Psi_\pm$ proceeding as we did for the tensor variable.

Neutral and extremal limits

It is convenient to deal first with neutral and extremal limits since the equations undergo a significant simplification. The response parameters for $Q = 0$ were found in [87] by solving (7.46) with parameters (7.47)-(7.48)[5]. Let us consider a maximally charged BH $Q = M$. Static perturbations are described by the master variable $\psi_\pm$ (see (7.49)) subject to an hypergeometric equation with coefficients (7.52). The response parameters $k_\pm$ in degenerate and non-degenerate cases are obtained as follows

First case: $2l+1 \notin \mathbb{N}$: The general solution can be written as [405, 406, 409]

$$\psi_\pm(z) = AF\left[a_\pm, b_\pm; c|z\right] + Bz^{1-c}F\left[a_\pm - c+1, b_\pm - c+1; 2-c|z\right] \, , \tag{7.75}$$

where A and B are arbitrary constants, $a_\pm, b_\pm$ and c are given in (7.52) and $F[a, b; c|z]$ denotes the hypergeometric function. Since the latter are normalised according to $F[a, b; c|0] = 1$, the response parameter $k_\pm$ enters the solution as (see Section 7.3.1)

$$\psi_\pm(z) \; = Bk_\pm F\left[a_\pm, b_\pm; c|z\right] + Bz^{1-c}F\left[a_\pm - c+1, b_\pm - c+1; 2-c|z\right] \, . \tag{7.76}$$

Using the connection formula [406, 409]

$$\begin{aligned}\frac{\sin\left[\pi\left(c-a-b\right)\right]}{\pi\Gamma\left(c\right)} F\left[a,b;c|z\right] =& \frac{F\left[a,b;a+b-c+1|1-z\right]}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)\Gamma\left(a+b-c+1\right)} \\ &- \left(1-z\right)^{c-a-b}\frac{F\left[c-a,c-b;c-a-b+1|1-z\right]}{\Gamma\left(a\right)\Gamma\left(b\right)\Gamma\left(c-a-b+1\right)} \, ,\end{aligned} \tag{7.77}$$

[5] We obtain exact agreement with the results of [87] with the exception of the magnetic susceptibility in the case that l is a generic number. This may well be a typo and we take the opportunity to provide the corrected result:

$$k_V \;=\; (2\hat{L}+1)\frac{\Gamma\left(\hat{L}+1+\frac{1}{D-3}\right)^2\Gamma\left(1+\hat{L}-\frac{1}{D-3}\right)^2}{\Gamma\left(2\hat{L}+2\right)^2}\frac{\sin\left[\pi\left(\hat{L}+\frac{1}{D-3}\right)\right]\sin\left[\pi\left(\hat{L}-\frac{1}{D-3}\right)\right]}{\pi\sin\left(2\pi\hat{L}\right)} , \tag{7.74}$$

where $\hat{L}$ and k_V stand for our l and $k_{\rm magnetic}$, respectively, in their notation.

one can write explicitly the analytic continuation of each hypergeometric function in (7.76) to a neighbourhood of $z = 1$. In our case,

$$\begin{aligned}\frac{\psi_\pm(z)}{B} =& -k_\pm \frac{\pi\Gamma(c)\,(1-z)^{c-a_\pm-b_\pm}\,F\left[c-a_\pm, c-b_\pm; c-a_\pm-b_\pm+1|1-z\right]}{\sin\left[\pi\left(c-a_\pm-b_\pm\right)\right]\Gamma(a_\pm)\Gamma(b_\pm)\Gamma(c-a_\pm-b_\pm+1)} \\ &- \frac{\pi\Gamma(2-c)(1-z)^{c-a_\pm-b_\pm}z^{1-c}F\left[1-a_\pm, 1-b_\pm; c-a_\pm-b_\pm+1|1-z\right]}{\sin\left[\pi\left(c-a_\pm-b_\pm\right)\right]\Gamma(a_\pm-c+1)\Gamma(b_\pm-c+1)\Gamma(c-a_\pm-b_\pm+1)} \\ &+ \left(\text{Terms Regular at } z=1\right),\end{aligned} \tag{7.78}$$

and using the further index displacement

$$z^{1-c}F\left[1-a_\pm, 1-b_\pm; c-a_\pm-b_\pm+1|1-z\right] = F\left[c-a_\pm, c-b_\pm; c-a_\pm-b_\pm+1|1-z\right] \tag{7.79}$$

equation (7.78) reads

$$\begin{aligned}\frac{\psi_\pm(z)}{B} =& -\left[\frac{k_\pm\Gamma(c)}{\Gamma(a_\pm)\Gamma(b_\pm)} + \frac{\Gamma(2-c)}{\Gamma(a_\pm-c+1)\Gamma(b_\pm-c+1)}\right] \\ &\times \pi\frac{(1-z)^{c-a_\pm-b_\pm}\,F\left[c-a_\pm, c-b_\pm; c-a_\pm-b_\pm+1|1-z\right]}{\sin\left[\pi\left(c-a_\pm-b_\pm\right)\right]\Gamma(c-a_\pm-b_\pm+1)} \\ &+ \left(\text{Terms Regular at } z=1\right).\end{aligned} \tag{7.80, 7.81}$$

The coefficients (7.52) of the extremal master equation satisfy

$$c - a_\pm - b_\pm = -\frac{\Sigma_\pm}{n-1}\,, \tag{7.82}$$

so the first term in (7.81) is singular at the horizon $z = 1$ unless $k_\pm$ are chosen to make the prefactor vanish, that is, in terms of l and the symbol $S_{(\pm,\pm)}$ (see (7.51)),

$$k_\pm = -\frac{\left(S_{(\pm,+)}+l\right)\left(S_{(\pm,-)}+l\right)}{2l(2l+1)}\frac{\Gamma(-2l)}{\Gamma(2l)}\frac{\Gamma\left(S_{(\pm,+)}+l\right)}{\Gamma\left(S_{(\pm,+)}-l\right)}\frac{\Gamma\left(S_{(\pm,-)}+l\right)}{\Gamma\left(S_{(\pm,-)}-l\right)}. \tag{7.83}$$

Second case: $2l + 1 \in \mathbb{N}$: Here we shall additionally distinguish between $D \neq 4$ and $D = 4$. In the former case the general solution takes the form

$$\psi_\pm(z) = AF\left[a_\pm, b_\pm; c|z\right] + BF\left[a_\pm, b_\pm; a_\pm+b_\pm-c+1|1-z\right]\,, \tag{7.84}$$

and only the second solution is regular at $z = 1$, which implies $A = 0$. Again using appropriate connection formulas in the degenerate cases it is easy to show that [405, 406, 409]

$$F\left[a_\pm, b_\pm; a_\pm+b_\pm-c+1|1-z\right] \sim \left(z^{-2l-1} + \ldots + R_\pm F\left[a_\pm, b_\pm; c|z\right]\ln z\right), \tag{7.85}$$

where the ellipsis denotes subleading terms in z and, in terms of l and $S_{(\pm,\pm)}$, $R_\pm$ reads

$$R_\pm = (-1)^{2l}\frac{\left(S_{(\pm,+)}+l\right)\left(S_{(\pm,-)}+l\right)}{(2l+1)!(2l)!}\frac{\Gamma\left(S_{(\pm,+)}+l\right)}{\Gamma\left(S_{(\pm,+)}-l\right)}\frac{\Gamma\left(S_{(\pm,-)}+l\right)}{\Gamma\left(S_{(\pm,-)}-l\right)}. \tag{7.86}$$

If $D = 4$, however, the coefficients in (7.52) become highly degenerate,

$$a_+ = a_- + 2 = c + 3, \quad b_+ = b_- + 2 = c - 4, \quad c = 2(l+1)\,. \tag{7.87}$$

In particular, all of them are integers and the general solution is

$$\psi_{\pm}(z) = AF\left[a_{\pm}, b_{\pm}; c|z\right] + Bz^{-2l-1}\begin{cases} F\left[4, -1; -2l|z\right] & (+) \\ F\left[2, -3; -2l|z\right] & (-) \end{cases}. \tag{7.88}$$

Regularity at the horizon $z = 1$ sets $A = 0$ and the functions in the braces are just polynomials in z (we recall that $L = 1$ has no gravitational mode $(-)$). This is a purely tidal field and, thus, we conclude that in $D = 4$

$$k_{\pm} = 0\,. \tag{7.89}$$

To summarise, we have found that the response parameters $k_{\pm}$ of the extremal BHs are given by

$$k_{\pm} = \begin{cases} \frac{\left(S_{(\pm,+)}+l\right)\left(S_{(\pm,-)}+l\right)}{2l(2l+1)}\frac{\Gamma(-2l)}{\Gamma(2l)}\frac{\Gamma\left(S_{(\pm,+)}+l\right)}{\Gamma\left(S_{(\pm,+)}-l\right)}\frac{\Gamma\left(S_{(\pm,-)}+l\right)}{\Gamma\left(S_{(\pm,-)}-l\right)} & 2l+1 \notin \mathbb{Z} \\ (-1)^{2l}\frac{\left(S_{(\pm,+)}+l\right)\left(S_{(\pm,-)}+l\right)}{(2l+1)!2l!}\frac{\Gamma\left(S_{(\pm,+)}+l\right)}{\Gamma\left(S_{(\pm,+)}-l\right)}\frac{\Gamma\left(S_{(\pm,-)}+l\right)}{\Gamma\left(S_{(\pm,-)}-l\right)}\ln z & 2l+1 \in \mathbb{Z}, \quad D \neq 4 \\ 0 & D = 4 \end{cases} \tag{7.90}$$

where the symbol $S_{(\pm,\pm)}$ is defined in (7.51). The vector Love numbers and the magnetic susceptibility are obtained by plugging such $k_{\pm}$'s into (7.72) and (7.73), respectively.

It is worth making a remark here before considering the BH with finite Q and T_H. We have found that vector Love numbers and magnetic susceptibilities do not vanish at extremality unless $D = 4$. This is in contrast with the tensor sector, where Love numbers are $\sim T_H^{2l+1}$ and thus vanish at zero temperature. Quite the opposite, for vector perturbations the charge triggers polarisations in modes that are otherwise not excited in the neutral case. Indeed, Ref. [87] found that some special modes in the vector sectors (both of gravitational and electromagnetic types) do not exhibit a static response to external fields when the BH is not charged. These have $l \in \frac{1}{2}\mathbb{N}$ in $D = 5$ or $L = N(D-3) \pm 1$ in $D > 5$ with $N \in \mathbb{N}$ (notice these always include the special mode $L = 1$). Such special modes seem to be a property of magnetic-like perturbations since they have no analogue in the corresponding scalar sectors. However, when the BH is maximally charged we have found that such harmonics do not fall within any special class and, therefore, exhibit some polarisation (of both gravitational and magnetic types) according to (7.90). Therefore, a non-trivial static response in these harmonics is a signature of non-vanishing charge. In the following section we show that, indeed, charging up the BH has the effect (in the vector sector) of increasing the intensity of the response and even turning on new modes of polarisation.

Finite charge and temperature

For intermediate values of the BH charge the equation governing static perturbations in the vector sector (7.20) has an extra pole due to the Cauchy horizon. Thus, the treatment in terms of hypergeometric functions considered in the neutral and extremal cases does not apply. While in the degenerate case $2l + 1 \in \mathbb{N}$ it is still possible to obtain exact analytic results, for general l with $2l + 1 \notin \mathbb{N}$ we proceed numerically.

First case: $2l + 1 \in \mathbb{N}$: Again we shall distinguish the cases $D \neq 4$ and $D = 4$. Consider first $D \neq 4$ and let $L_{\alpha,\beta,\gamma,\delta,\eta,h,z_c}[\cdot]$ be Heun's operator, so that Heun's equation

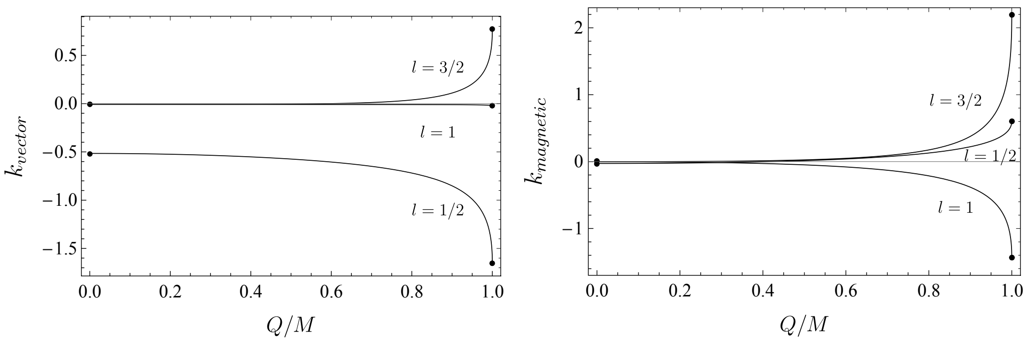

Figure 7.3: k_{vector} (top) and k_{magnetic} (bottom), in the degenerate case $2l+1 \in \mathbb{N}$ (omitting the factor $\ln z$). We show $L = 4, 8, 12$ in $D = 11$. Solid black lines are the analytic results obtained as explained in the main text. These interpolate exactly between the analytic predictions at $Q = 0$ and $Q = M$, represented with solid black dots.

for a function $f(z)$ reads $L_{\alpha,\beta,\gamma,\delta,\eta,h,z_c}[f(z)] = 0$. Much like in the tensor case, after choosing the normalisation of $\Psi_{\text{tidal}(\pm)}(z)$ as $\Psi_{\text{tidal}(\pm)}(z) = 1 + O(z)$, imposing Heun's equation (7.20) on the second solution of (7.57) fixes $R_\pm$ completely. In particular, using that $L_{\alpha,\beta,\gamma,\delta,\eta,h,z_c}[\Psi_{\text{resp}(\pm)}(z)] = 0$ and expanding $L_{\alpha,\beta,\gamma,\delta,\eta,h,z_c}[z^{-2l-1}\Psi_{\text{tidal}(\pm)}(z)] = z^{-2l-2}\sum_{i=0} a_i^{(\pm)} z^i$ it follows that $R_\pm$ is formally given by

$$R_\pm = -\frac{a_{2l}^{(\pm)}}{2l+1}\,. \tag{7.91}$$

The coefficient $a_{2l}^{(\pm)}$ depends on the coefficients at all previous orders $a_{i<2l}^{(\pm)}$, and it is not clear whether it is possible to give the general result for any l, n, and ϵ (as it was in the tensor sector). However, given a particular value of l one can just solve all previous orders $a_{i<2l}$ and get, through (7.91), the exact result of $R_\pm$ in terms of n, ϵ. For example, for $l = 1$ we find

$$R_\pm = \Bigg[\Big(-2n^4 + 3n^3 - 7n^2 + 11n - 13 \pm \left(-2n^2 + 3n + 11\right)\tilde{\Delta} \\ + \left(2n^2 - 3n + 1\right)\left(n^2 \pm \tilde{\Delta} - 7\right)\cos(2\epsilon)\Big)\Bigg]\frac{n^2 \sec^6\left(\frac{\epsilon}{2}\right)}{96(n-1)^6}\,, \tag{7.92}$$

and it is easy to check that this interpolates between the neutral result in [87] and the extremal one in (7.90) (as ϵ goes from 0 to $\pi/2$, respectively). Love numbers and magnetic susceptibilities are finally obtained by plugging these results into (7.72) and (7.73). In Figure 7.3 we show k_{vector} and k_{magnetic} in $D = 11$ for several harmonics l. Next we consider $D = 4$. Once again we find a second solution with $R_\pm = 0$ and $\Psi_{\text{tidal}(\pm)}(z)$ a polynomial of degree $< 2l+1$. This is the solution that is regular at the horizon $z = 1$ and consists solely of a tidal field, so once more $k_{\text{vector}} = 0$ and $k_{\text{magnetic}} = 0$ in four dimensions, now for any value of the BH temperature T_H.

Second case: $2l + 1 \notin \mathbb{N}$: In this case, there seems to be no clear way of guessing the results for $k_\pm$ out of those for $R_\pm$ in (7.91) as we did for the tensor sector. Thus, we proceed numerically by implementing a standard shooting method (similar to that used in [412]) which matches the regular solution at the horizon $z = 1$ with one at infinity of the

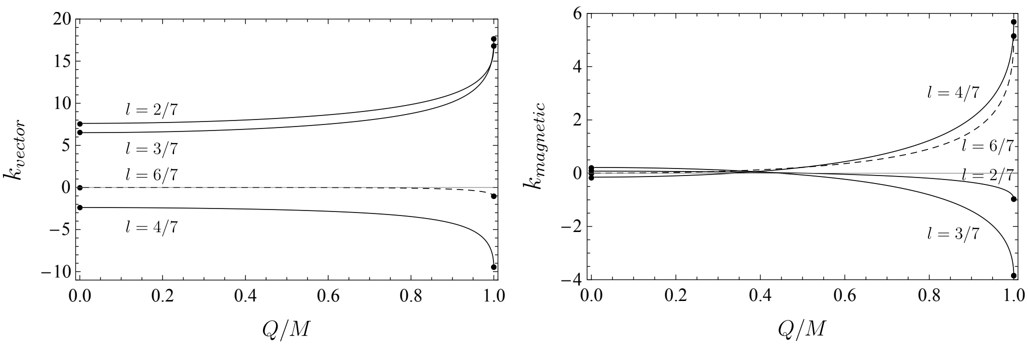

Figure 7.4: k_{vector} (top) and k_{magnetic} (bottom), in the general case $2l + 1 \notin \mathbb{N}$. We show $L = 2, 3, 4, 6$ in $D = 10$. Solid black lines are the results obtained numerically and solid black dots are the analytical results at $Q = 0$ and $Q = M$. The harmonic $L = 6$, represented with a dashed line, is an example of the special modes that do not polarise at $Q = 0$, but exhibit a non-trivial response as Q grows.

form (7.58), thus obtaining the values of $k_{\pm}$. Then k_{vector} and k_{magnetic} follow from (7.72) and (7.73). In Figure 7.4 we show k_{vector} and k_{magnetic} in $D = 10$ for several harmonics l.

These results confirm the analytical predictions at $Q = 0$ and $T_H = 0$. We can conclude that charged BHs exhibit a stronger response to gravitational and electromagnetic tidal fields, relative to their neutral counterparts. Even more, a non-vanishing charge turns on new modes of gravitational and magnetic polarisation that are otherwise not responsive for $Q = 0$. A non-trivial static response in such harmonics is, therefore, a definite signature of charge. We can also confirm that in four dimensions, for all T_H, both tidal Love numbers and magnetic susceptibilities vanish. This property is strongly related to the fact that, in $D = 4$, the equations become degenerate enough so as to admit purely-growing polynomial solutions.

7.4 Discussion

We have studied the effect of charge on the static polarizability of BHs in $D \geq 4$ spacetime dimensions. While the four-dimensional setup remains intriguingly special, with all response parameters vanishing, TLNs and magnetic susceptibilities exhibit a rich structure in $D > 4$. In particular, charging up the BH turns on new vector-type modes of polarisation, while tensor Love numbers (encoding also the response to scalar tidal fields) decrease and eventually vanish at extremality. More precisely, our results can be summarised as follows.

(i) The relevant differential equations are of Fuchsian type with 4 poles (Heun) at infinity, the event and Cauchy horizons and the curvature singularity. In the neutral ($Q = 0$) and extremal ($T_H = 0$) limits, the equations become hypergeometric and TLNs are exactly solvable.

(ii) For the tensor sector of gravitational perturbations we showed that all TLNs (equivalently, the response to scalar tidal fields) vanish at extremality, $T_H = 0$, and confirmed the results in the literature for $Q = 0$ [87]. Even for arbitrary (subextremal) values of the BH charge we are able to obtain the exact result analytically, finding that tensor TLNs follow a power law in the BH temperature, $k_{\text{tensor}} \sim T_H^{2l+1}$.

(iii) For the so-called vector sector, we find analytical expressions for the Love numbers and magnetic susceptibilities at extremality, $T_H = 0$. We also recover results in the liter-

ature at zero charge [87], correcting the reported result for the magnetic susceptibilities. For intermediate Q's we find some results analytically and some numerically, in all cases confirming our analytic predictions at $T_H = 0$ and those in the literature for the neutral case, $Q = 0$. In contrast to the tensor sector, we found that charged BHs exhibit a stronger response to gravitational and electromagnetic tidal fields (of vector type), relative to their neutral counterparts. In addition, we showed that the BH charge excites new modes of gravitational and magnetic polarisation that are otherwise not responsive for $Q = 0$. A non-trivial static response in such harmonics is, therefore, a definite signature of charge.
(iv) Our results show that in four dimensions and for all values of the charge, all response parameters vanish. This property is strongly related to the fact that, in $D = 4$, the equations become degenerate enough so as to admit purely-growing polynomial solutions.

Our results raise interesting questions in various directions. First, it is desirable to understand and explore further the special properties of tensor modes (scalar fields) at extremality, possibly including black hole rotation in a suitable spin configuration. In parallel, it would also be interesting to consider BHs carrying a more general charge configuration and study whether these excite new modes of polarisation, similarly to what we found in the vector sector. These and more aspects about tidal deformability of charged BHs will be addressed in future work.

Part III

Conclusions and Appendices

A
Conclusions

A.1 English Version

In Chapters 2 and 3, we studied the laws of black hole mechanics in the context of the low energy effective actions of the heterotic superstring. Momentum maps allowed us to deal systematically with the Nicolai–Townsend transformations when constructing conserved charges. They were also crucial for deriving the zeroth laws of matter fields and identifying the potentials conjugate to the gauge charges. Using the latter we were able to derive a first law of black hole mechanics in which all matter terms have the form of a potential times the variation of a charge. We obtained that, at first order in α', the black hole entropy is given by a gauge- and Lorentz- invariant expression in which all terms can be computed explicitly. Such a formula was still lacking in the literature and is one of the main results of this thesis. We argued that, in general, it does not coincide with Wald's entropy formula which, for this theory, is not gauge invariant.

Chapter 4 was devoted to study the role of magnetic charges in the laws of black hole mechanics. These charges are not associated to a gauge symmetry, so it is not clear how to include them in Wald's formalism. We considered axion-dilaton gravity, whose equations are invariant under the archetype of electric-magnetic (or S-) duality group $SL(2,\mathbb{R})$. Introducing the magnetic momentum maps, we constructed a generalised Komar charge that is duality invariant. This was used to derive a duality invariant Smarr formula for the asymptotically flat, static and spherically symmetric black holes of the theory. The most general solution describing such class of black holes is known, and the formula was verified explicitly. We also discussed how magnetic charges can be included in the first law.

In Chapter 5 we constructed a non-perturbative solution describing the near horizon geometry of an extremal, charged and rotating black hole of Einsteinian Cubic Gravity. Several families of solutions were found, one of them being smoothly connected to the near horizon geometry of the Kerr–Newman black hole. For all of them, the Wald entropy could be computed exactly. This is the first example in which the entropy of a rotating black hole can be evaluated exactly in a higher-order gravity.

In Chapter 6 we derived the master equations governing spin-0, spin-1 and spin-2 fluctuations of a NUT-charged black brane in AdS. We computed, for the first time, the gravitational quasinormal mode spectrum of a NUT-charged spacetime. No unstable gravitational mode was found, and mode-stability was proven for scalar fluctuations. Our boundary conditions were chosen following AdS/CFT in order to interpret the quasinormal frequencies as poles in the thermal correlators of the boundary theory. We found a family of pseudo-hydrodynamic modes in the spectrum and obtained their dispersion relation. It would be interesting to recover this holographic prediction by studying the hydrodynamic regime of a neutral fluid in the boundary geometry.

The last chapter was devoted to study the tidal deformability of charged black holes. It is well known that tidal Love numbers of four-dimensional, asymptotically flat vacuum black holes vanish. In Chapter 7, we showed that this is still true if the black hole is charged, and that scalar and electromagnetic response coefficients are also vanishing. For higher-dimensional black holes, however, such response coefficients are nonvanishing and typically increase as the black hole is charged. The scalar response is an exception, since it goes to zero as extremality is approached in any number of dimensions.

A.2 Spanish Version

En los Capítulos 2 y 3, estudiamos las leyes de la mecánica de agujeros negros en el contexto de las acciones efectivas de la supercuerda heterótica. Los momentum maps nos permitieron tratar sistemáticamente las transformaciones de Nicolai–Townsend al construir cargas conservadas. También fueron cruciales para derivar las leyes zero de los campos de materia e identificar los potenciales conjugados a las cargas gauge. Utilizando estos últimos, pudimos derivar una primera ley de la mecánica de agujeros negros en que todos los términos de materia tienen la forma de un potencial multiplicando la variación de una carga. Obtuvimos que, a primer orden en α', la entropía de los agujeros negros viene dada por una expresión invariante de gauge y Lorentz en la que todos los términos pueden calcularse explícitamente. Dicha fórmula aún no existía en la literatura y es uno de los principales resultados de esta tesis. Argumentamos que, en general, no coincide con la fórmula de entropía de Iyer–Wald que, para esta teoría, no es invariante gauge.

El Capítulo 4 se dedicó a estudiar el papel de las cargas magnéticas en las leyes de la mecánica de los agujeros negros. Estas cargas no están asociadas a una simetría gauge, por lo que no está claro cómo incluirlas en el formalismo de Wald. Consideramos la gravedad axión-dilatón, cuyas ecuaciones son invariantes bajo el arquetipo de grupo de S-dualidad, $SL(2,\mathbb{R})$. Introduciendo los momentum maps magnéticos, construimos una carga de Komar generalizada que es invariante bajo S-dualidad. Esto se utilizó para derivar una fórmula de Smarr, también invariante, para los agujeros negros asintóticamente planos, estáticos y esféricamente simétricos de la teoría. La solución más general que describe esta clase de agujeros negros es conocida, y la fórmula pudo verificarse explícitamente. También discutimos cómo se pueden incluir las cargas magnéticas en la primera ley.

En el Capítulo 5 construimos una solución no perturbativa que describe la geometría cercana al horizonte de un agujero negro extremo, cargado y en rotación en la teoría cúbica Einsteinian Cubic Gravity. Se encontraron varias familias de soluciones, una de ellas conectada de forma suave a la geometría cercana al horizonte del agujero negro de Kerr–Newman. Para todas ellas, la entropía de Wald pudo calcularse exactamente. Este es el primer ejemplo en que la entropía de un agujero negro en rotación puede evaluarse de forma exacta en una gravedad de orden superior.

En el Capítulo 6 derivamos las ecuaciones maestras que gobiernan las fluctuaciones de espín-0, espín-1 y espín-2 de una brana negra con carga NUT en AdS. Calculamos, por primera vez, el espectro de modos gravitacionales cuasinormales de un espaciotiempo con carga NUT. No hallamos ningún modo gravitacional inestable y, para fluctuaciones escalares, pudimos demostrar la estabilidad del espaciotiempo. Nuestras condiciones de contorno se eligieron de acuerdo con la correspondencia AdS/CFT con tal de interpretar las frecuencias cuasinormales como polos en los correladores térmicos de la teoría de la frontera. Encontramos una familia de modos pseudohidrodinámicos en el espectro y obtuvimos su relación de dispersión. Sería interesante recuperar esta predicción holográfica estudiando el régimen hidrodinámico de un fluido neutro en la geometría de la frontera.

El último capítulo se dedicó a estudiar la deformabilidad de marea de agujeros negros cargados. Es bien sabido que, en cuatro dimensiones, los números de Love de agujeros negros asintóticamente planos en el vacío son zero. En el Capítulo 7, demostramos que esto sigue siendo cierto si el agujero negro está cargado, y que además los coeficientes de respuesta escalares y electromagnéticos también son nulos. Sin embargo, para agujeros negros en mayores dimensiones, estos coeficientes de respuesta no son zero y suelen aumentar a medida que el agujero negro es cargado. La respuesta de marea escalar es una excepción, ya que se desvanece a medida que la carga se aproxima a su valor extremo en cualquier número de dimensiones.

B
Some Proofs

B.1 The Binormal to $\mathcal{BH}$

Let U be the tangent to a congruence of affinely-parametrised null geodesics containing a geodesically complete Killing horizon $\mathcal{H}$ of a Killing vector field k. Let S be a spacelike section of $\mathcal{H}$ and let N be the unique vector on S satisfying $N \in \mathfrak{X}(S)^{\perp}$, $N^2 \overset{S}{=} 0$ and $N \cdot U \overset{S}{=} -1$. Then,

$$dk \overset{S}{=} 2\kappa U \wedge N + P\left(N \cdot dk\right) \wedge U \tag{B.1}$$

where $\overset{S}{=}$ means evaluated at S, κ is the surface gravity of $\mathcal{H}$ and P is the projector on S relative to N. In components, $P^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + U^{\mu} N_{\nu} + N^{\mu} U_{\nu}$. The proof follows by noticing that $k \wedge dk \overset{\mathcal{H}}{=} 0$ and $k \overset{\mathcal{H}}{=} hU$ for some function h, so

$$U \wedge dk \overset{\mathcal{H}}{=} 0 \tag{B.2}$$

where $h \neq 0$. However, (B.2) still holds if $\mathcal{H}$ has a regular bifurcation surface $\mathcal{BH} \subset \mathcal{H}$, where $h \overset{\mathcal{BH}}{=} 0$. Indeed, regularity of $\mathcal{BH}$ and geodesic completeness of $\mathcal{H}$ implies continuity of the l.h.s. of (B.2) at $\mathcal{BH}$. Equation (B.1) follows straightforwardly contracting (B.2) with N and using $k \cdot \nabla k \overset{\mathcal{H}}{=} \kappa k$. If $S = \mathcal{BH}$, then

$$\begin{aligned} P\left(N \cdot dk\right)_{\nu} &= -2P^{\mu}_{\ \nu} N^{\alpha} \nabla_{\mu} k_{\alpha} \\ &= -2P^{\mu}_{\ \nu}\left(\nabla_{\mu}\left(N^{\alpha} k_{\alpha}\right) - k_{\alpha} \nabla_{\mu} N^{\alpha}\right) \\ &\overset{\mathcal{BH}}{=} 0 \end{aligned} \tag{B.3}$$

where in the last step we used that each term vanishes at $\mathcal{BH}$ because $N^{\alpha} k_{\alpha} \overset{\mathcal{BH}}{=} 0 \overset{\mathcal{BH}}{=} k$. Thus, from (B.1) it follows that

$$\nabla k \overset{\mathcal{BH}}{=} \kappa U \wedge N \overset{\mathcal{BH}}{=} \kappa n \tag{B.4}$$

where we used that $U \wedge N$ is precisely the binormal n to $\mathcal{BH}$.

B.2 Geometric Proof of the Zeroth Law

The zeroth law is a purely geometric result following from the definition of Killing horizon, and thus its proof does not require specifying a theory. Simply assume that a spacetime (g, M) exhibits a Killing horizon $\mathcal{H}$ relative to a Killing vector k, so

$$k^{\mu} \nabla_{\mu} k_{\nu} \overset{\mathcal{H}}{=} \kappa k_{\nu} \tag{B.5}$$

Since k^μ is tangent to $\mathcal{H}$ we can act at both sides of the above equation with $\pounds_k$. The r.h.s. just yields $(\pounds_k \kappa) k_\nu$ by virtue of $\pounds_k g = 0$ and $\pounds_k k = [k, k] = 0$. Rewriting the l.h.s. as $k \cdot \nabla k = (-1/2) d(k^2)$ and using Cartan's formula $\pounds_k = d\iota_k + \iota_k d$ one has

$$\pounds_k \left(k \cdot \nabla k\right) = -\frac{1}{2} \left(d\iota_k + \iota_k d\right) \left[d(k^2)\right] = -\frac{1}{2} d \left[\iota_k d(k^2)\right] = d \left(k^\mu k^\nu \nabla_\mu k_\nu\right) = 0 \tag{B.6}$$

where in the last step we used Killing's equation in the form $\nabla_{(\mu} k_{\nu)} = 0$. It follows that

$$\pounds_k \kappa \stackrel{\mathcal{H}}{=} 0 \tag{B.7}$$

and, consequently, κ is constant along the generators of $\mathcal{H}$.[1] To prove that κ does not change from generator to generator let us assume that $\mathcal{H}$ is bifurcate, with bifurcation surface $\mathcal{BH}$. There one has $k \stackrel{\mathcal{BH}}{=} 0$ by definition, and furthermore (see Section B.1),

$$\nabla_\mu k_\nu \stackrel{\mathcal{BH}}{=} \kappa n_{\mu\nu} \tag{B.8}$$

where $n_{\mu\nu} = -n_{\nu\mu}$ is the binormal to $\mathcal{BH}$ with $n^{\mu\nu} n_{\mu\nu} \stackrel{\mathcal{BH}}{=} -2$. Now take T^μ tangent to $\mathcal{BH}$ and hit both sides of the equation above with $T^\gamma \nabla_\gamma$. The l.h.s. vanishes at $\mathcal{BH}$,

$$T^\gamma \nabla_\gamma \nabla_\mu k_\nu = T^\gamma R_{\nu\mu\gamma\sigma} k^\sigma \stackrel{\mathcal{BH}}{=} 0 \tag{B.9}$$

where in the second step we used Ricci's identity specialised to a Killing field. Thus, one is left with

$$0 \stackrel{\mathcal{BH}}{=} n_{\mu\nu} T^\gamma \nabla_\gamma \kappa + \kappa T^\gamma \nabla_\gamma n_{\mu\nu} \tag{B.10}$$

and contracting this equation with $n^{\mu\nu}$ one has

$$0 \stackrel{\mathcal{BH}}{=} -2 T^\gamma \nabla_\gamma \kappa + n^{\mu\nu} \kappa T^\gamma \nabla_\gamma n_{\mu\nu} \stackrel{\mathcal{BH}}{=} -2 T^\gamma \nabla_\gamma \kappa + \frac{1}{2} \kappa T^\gamma \nabla_\gamma \left(n^{\mu\nu} n_{\mu\nu}\right) \stackrel{\mathcal{BH}}{=} -2 T^\gamma \nabla_\gamma \kappa \tag{B.11}$$

Since T^μ is any vector tangent to $\mathcal{BH}$, it follows that κ is constant on $\mathcal{BH}$. Thus, κ is constant along each generator and its value does not depend on the choice of generator because it is constant on $\mathcal{BH}$. Then, assuming that $\mathcal{H}$ is geodesically complete, it follows that κ is constant everywhere in $\mathcal{H}$ thus establishing the zeroth law of black hole mechanics. The assumptions considered here about $\mathcal{H}$ are enough for the purpose of this thesis, but one can prove more general results [144]. The interesting point is that, throughout the proof, no use of the equations of motion has been made. In particular, this result remains true when gravity is coupled to matter. In Chapters 2, 3 and 4 we will see that the zeroth laws associated to matter are also independent of the equations of motion.

B.3 Proof of the First Law in Pure Gravity Theories

Here we reproduce in some detail the derivations in [139] and [140].[2] Consider a pure gravity theory in d spacetime dimensions with Lagrangian

$$\mathbf{L} = L(g_{\mu\nu}, R_{\mu\nu\rho\sigma}) \boldsymbol{\epsilon} \tag{B.12}$$

[1]In particular, along the ones that are affinely-parametrised, with tangent U, since $0 = \pounds_k \kappa = k(\kappa) = hU(\kappa)$ where h is a function that is non-vanishing where $k \neq 0$.

[2]However, we make use of the Noether identities in order to write off-shell identities that were only given on-shell in the original references.

The general, off-shell, first variation of the identity (1.58) in the main text reads explicitly

$$
\begin{aligned}
d\delta\mathbf{Q}_\xi &= \delta\boldsymbol{\Theta}(\delta_\xi g) + \iota_\xi \delta\mathbf{L} + \delta\mathbf{S}_\xi \\
&= \delta\boldsymbol{\Theta}(\delta_\xi g) + \iota_\xi d\boldsymbol{\Theta}(\delta g) + \iota_\xi \mathbf{E}^{\mu\nu}\delta g_{\mu\nu} + \delta\mathbf{S}_\xi \\
&= \delta\boldsymbol{\Theta}(\delta_\xi g) + \pounds_\xi \boldsymbol{\Theta}(\delta g) - d\left(\iota_\xi \boldsymbol{\Theta}(\delta g)\right) + \iota_\xi \mathbf{E}^{\mu\nu}\delta g_{\mu\nu} + \delta\mathbf{S}_\xi \\
&= \delta\boldsymbol{\Theta}(\delta_\xi g) - \delta_\xi \boldsymbol{\Theta}(\delta g) - d\left(\iota_\xi \boldsymbol{\Theta}(\delta g)\right) + \iota_\xi \mathbf{E}^{\mu\nu}\delta g_{\mu\nu} + \delta\mathbf{S}_\xi
\end{aligned}
\tag{B.13}
$$

where we used (1.50) in the first step, Cartan's formula $\pounds_\xi = \iota_\xi d + d\iota_\xi$ in the second, and $\delta_\xi \delta g = -\pounds_\xi \delta g$ in the third, following our sign convention (1.51).[3] The first two terms in the last line of (B.13) combine into the presymplectic potential introduced in the main text,

$$\boldsymbol{\omega}(\delta g, \delta_\xi g) \equiv \delta\boldsymbol{\Theta}(\delta_\xi g) - \delta_\xi \boldsymbol{\Theta}(\delta g) \tag{B.14}$$

If $g_{\mu\nu}$ is a solution then $\mathbf{E}_{\mu\nu} = 0$, and if $\delta g_{\mu\nu}$ satisfies the linearised equations of motion then $\delta\mathbf{S}_\xi = 0$ because $\mathbf{S}_\xi$ is proportional to the equations of motion and their derivatives. Explicit expressions for the relevant quantities are[4]

$$\mathbf{E}_{\mu\nu} = -\left[\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} L + 2\nabla^\alpha \nabla^\beta P_{\mu\alpha\nu\beta}\right] \boldsymbol{\epsilon} = -\mathcal{E}_{\mu\nu}\boldsymbol{\epsilon} \tag{B.15}$$

$$\boldsymbol{\Theta}(\delta g) = \left[2P^{\mu\alpha\beta\gamma}\nabla_\gamma \delta g_{\alpha\beta} - 2\left(\nabla_\gamma P^{\gamma\alpha\beta\mu}\right)\delta g_{\alpha\beta}\right]\boldsymbol{\epsilon}_\mu = \theta^\mu \boldsymbol{\epsilon}_\mu \tag{B.16}$$

$$\mathbf{S}_\xi = 2\xi_\mu \mathcal{E}^{\mu\nu}\boldsymbol{\epsilon}_\nu \tag{B.17}$$

$$\mathbf{Q}_\xi = \left[P^{\mu\nu\alpha\beta}\nabla_\alpha \xi_\beta + 2\xi_\alpha \nabla_\beta P^{\mu\nu\alpha\beta}\right]\boldsymbol{\epsilon}_{\mu\nu} = (1/2)\Omega_\xi^{\mu\nu}\boldsymbol{\epsilon}_{\mu\nu} = \star\Omega_\xi \tag{B.18}$$

Finally, using these expressions and the properties of k at $\mathcal{H}$, proven in Sections B.1 and B.2, one can evaluate each side of (1.75) as explained in [140] and obtain the first law presented in the main text.

[3] Strictly speaking we have also assumed that $\boldsymbol{\Theta}(\delta g)$ is generally covariant, but this is always possible in these theories [140] and will also be the case in all theories considered in Chapters 2 to 4.

[4] The notation $\boldsymbol{\epsilon}_{\mu\nu\dots}X^{\mu\nu\dots}$ means contraction with the first indices of the volume form.

C

A Truncation of HST on $\mathbb{T}^5$ to a $\mathcal{N}=1, d=5$ Supergravity

A very useful, almost algorithmic, procedure has been developed in Refs. [416–420] to construct supersymmetric solutions (black holes and black rings, in particular) of $\mathcal{N}=1, d=5$ supergravity coupled to vector supermultiplets.[1] We can use this procedure in the context of the Heterotic Superstring Effective action compactified on a T^5 if we find a consistent truncation that produces a model $\mathcal{N}=1, d=5$ supergravity. A very simple truncation with this property has been used, for instance, in Ref. [220]. It can be described more conveniently as a trivial dimensional reduction on a T^4 (with all the fields that arise in the reduction set to their vacuum values) followed by a non-trivial compactification on a circle. The only fields that survive are the KR 2-form (which can be dualized into a vector field), the KK and winding vectors and the dilaton and KK scalars. This field content fits into $\mathcal{N}=1, d=5$ supergravity (metric and graviphoton vector field) coupled to two vector multiplets (one vector and one real scalar field each).

In order to profit from the solution-generating techniques developed for $\mathcal{N}=1, d=5$ supergravity theories, we need to rewrite this truncated version of the Heterotic Superstring effective action in the appropriate form: first, we rewrite the action in the Einstein frame and then we will dualize the KR field into a vector. After that, we will identify the scalar manifold etc.

The action of the truncated theory is

$$
\begin{aligned}
S[e^a, B, \phi, k, A, B] = \frac{g_s^{(5)\,2}}{16\pi G_N^{(5)}} \int e^{-2\phi} \Big[& \star(e^a \wedge e^b) \wedge R_{ab} - 4d\phi \wedge \star d\phi \\
& + \tfrac{1}{2}k^{-2}dk \wedge \star dk - \tfrac{1}{2}k^2 F \wedge \star F - \tfrac{1}{2}k^{-2} G \wedge \star G + \tfrac{1}{2} H \wedge \star H \Big] \,,
\end{aligned}
\tag{C.1}
$$

where H is simply

$$
H = dB - \tfrac{1}{2} A \wedge G - \tfrac{1}{2} B \wedge F \,. \tag{C.2}
$$

The string-frame Vielbein e^a is related to the (modified) Einstein-frame Vielbein $\tilde{e}^a$ by

[1] These are supergravities invariant under 8 independent supersymmetry transformations, which are combined in a minimal 5-dimensional spinor. Often, they are referred to as $\mathcal{N}=2, d=5$ supergravities.

$$e^a = e^{2(\phi-\phi_\infty)/3}\tilde{e}^a\,, \qquad g_s = e^{\phi_\infty}\,, \tag{C.3}$$

and the action in the (modified) Einstein frame takes the form (removing the tildes for simplicity)

$$\begin{aligned} S[e^a, B, \phi, k, A, B] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2} k^{-2} dk \wedge \star dk \\ &-\tfrac{1}{2} k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2} k^{-2} e^{-4\phi/3} G \wedge \star G + \tfrac{1}{2} e^{-8\phi/3} H \wedge \star H \Big]\,. \end{aligned} \tag{C.4}$$

The next step is the dualization of the KR 2-form. As usual, we consider the above action as a functional of the 3-form field strength H and add a Lagrange-multiplier term to enforce its Bianchi identity $dH = -\frac{1}{2}\mathcal{F}_I \wedge \mathcal{F}^I$

$$\begin{aligned} S[e^a, H, \phi, k, A, B] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2} k^{-2} dk \wedge \star dk \\ &- \tfrac{1}{2} k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2} k^{-2} e^{-4\phi/3} G \wedge \star G + \tfrac{1}{2} e^{-8\phi/3} H \wedge \star H \\ &-C \wedge (dH + F \wedge G) \Big]\,, \end{aligned} \tag{C.5}$$

where C is the 1-form dual to the 2-form B. Varying this action with respect to H, we get

$$\frac{\delta S}{\delta H} = e^{-8\phi/3} \star H - dC = 0\,, \tag{C.6}$$

which is solved by

$$H = e^{8\phi/3} \star K\,, \qquad K \equiv dC\,. \tag{C.7}$$

Substituting this solution into the action Eq. (C.5) we find the dual action

$$\begin{aligned} S[e^a, \phi, k, A, B, C] = \frac{1}{16\pi G_N^{(5)}} \int \Big[&\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{4}{3} d\phi \wedge \star d\phi + \tfrac{1}{2} k^{-2} dk \wedge \star dk \\ &- \tfrac{1}{2} k^2 e^{-4\phi/3} F \wedge \star F - \tfrac{1}{2} k^{-2} e^{-4\phi/3} G \wedge \star G - \tfrac{1}{2} e^{8\phi/3} K \wedge \star K \\ &-F \wedge G \wedge C \Big]\,. \end{aligned} \tag{C.8}$$

The final step consists in finding the relation between the fields of this action and those of a $\mathcal{N}=1, d=5$ theory with two vector supermultiplets written in the standard form[2]

$$S[e^a, \phi^x, A^I] = \frac{1}{16\pi G_N^{(5)}} \int \Big[\star(e^a \wedge e^b) \wedge R_{ab} + \tfrac{1}{2} g_{xy} d\phi^x \wedge \star d\phi^y - \tfrac{1}{2} a_{IJ} F^I \wedge \star F^J + \tfrac{1}{3^{3/2}} C_{IJK} F^I \wedge F^J \wedge A^K\Big], \tag{C.9}$$

where the indices $I, J, \ldots = 0, 1, 2$ and the indices $x, y, \ldots = 1, 2$. The metrics $g_{xy}(\phi), a_{IJ}(\phi)$ are defined in terms of the symmetric, constant tensor C_{IJK} which fully characterizes the theory and the *real special geometry* of the scalar manifold as follows: we start by defining 3 combinations of the 2 scalars $h^I(\phi)$ that satisfy the constraint

$$C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1\,. \tag{C.10}$$

Next, we define

$$h_I \equiv C_{IJK} h^J h^K, \quad \Rightarrow \quad h^I h_I = 1, \tag{C.11}$$

and

$$h^I_x \equiv -\sqrt{3} h^I{}_{,x} \equiv -\sqrt{3} \frac{\partial h^I}{\partial \phi^x}, \qquad h_{Ix} \equiv +\sqrt{3} h_{I,x}, \quad \Rightarrow \quad h_I h^I_x = h^I h_{Ix} = 0. \tag{C.12}$$

Then, a_{IJ} is defined implicitly by the relations

$$h_I = a_{IJ} h^I, \qquad h_{Ix} = a_{IJ} h^J{}_x. \tag{C.13}$$

It can be checked that

$$a_{IJ} = -2 C_{IJK} h^K + 3 h_I h_J. \tag{C.14}$$

The metric of the scalar manifold $g_{xy}(\phi)$, which we will use to raise and lower x, y indices is (proportional to) the pullback of a_{IJ}

$$g_{xy} \equiv a_{IJ} h^I{}_x h^J{}_y = -2 C_{IJK} h^I_x h^J_y h^K. \tag{C.15}$$

If we make the identifications

$$A^0 = -\sqrt{3} C\,, \qquad A^1 = -\sqrt{3} A\,, \qquad A^2 = -\sqrt{3} B\,, \tag{C.16}$$

we find that

$$C_{012} = 1/6\,, \quad a_{00} = e^{8\phi/3}/3\,, \quad a_{11} = k^2 e^{-4\phi/3}/3\,, \quad a_{22} = k^{-2} e^{-4\phi/3}/3\,. \tag{C.17}$$

[2] Here we are using the notation and conventions of Ref. [421] with minor changes explained in Appendix A of Ref. [422]. See also Ref. [152].

Since, for this C_{IJK}, the only non-vanishing components of a_{IJ} are the diagonal ones with $a_{II} = 3(h_I)^2$ we find that

$$h_0 = e^{4\phi/3}/3\,, \qquad h_1 = k e^{-2\phi/3}/3\,, \qquad h_2 = k^{-1} e^{-2\phi/3}/3\,, \tag{C.18}$$

which, in its turn, implies that

$$h^0 = e^{-4\phi/3}\,, \qquad h^1 = k^{-1} e^{2\phi/3}\,, \qquad h^2 = k e^{2\phi/3}\,. \tag{C.19}$$

Finally, the non-vanishing components of the scalar metric are

$$g_{\phi\phi} = 8/3\,, \qquad g_{kk} = k^{-2}\,. \tag{C.20}$$

The equations of motion of a general $\mathcal{N}=1, d=5$ theory are (up to a global factor of $(16\pi G_N^{(5)})^{-1}$ that we omit for simplicity)

$$\begin{aligned}\mathbf{E}_a = {} & \imath_a \star (e^c \wedge e^d) \wedge R_{cd} - \tfrac{1}{2} g_{xy} \left(\imath_a d\phi^x \star d\phi^y + d\phi^x \wedge \imath_a \star d\phi^y\right) \\ & + \tfrac{1}{2} a_{IJ} \left(\imath_a F^I \wedge \star F^J - F^I \wedge \imath_a \star F^J\right)\,,\end{aligned} \tag{C.21a}$$

$$\mathbf{E}_x = -g_{xy} \left\{ d \star d\phi^y + \Gamma_{zw}{}^y d\phi^z \wedge \star d\phi^w + \tfrac{1}{2} \partial^y a_{IJ} F^I \wedge \star F^J \right\}\,, \tag{C.21b}$$

$$\mathbf{E}_I = -d\left(a_{IJ} \star F^J\right) + \tfrac{1}{\sqrt{3}} C_{IJK} F^J \wedge F^K\,. \tag{C.21c}$$

In this action, ϕ stands, actually, for $\phi - \phi_\infty$. In other words: the field ϕ is constrained to vanish at infinity.

For the particular model that we have obtained as a truncation of the compactified Heterotic Superstring effective action in $d=5$ dimensions, these equations take the particular form

$$\begin{aligned}\mathbf{E}_a = {} & \imath_a \star (e^c \wedge e^d) \wedge R_{cd} - \tfrac{4}{3} \left(\imath_a d\phi \star d\phi + d\phi \wedge \imath_a \star d\phi\right) \\ & - \tfrac{1}{2} k^{-2} \left(\imath_a dk \star dk + dk \wedge \imath_a \star dk\right) + \tfrac{1}{6} e^{8\phi/3} \left(\imath_a F^0 \wedge \star F^0 - F^0 \wedge \imath_a \star F^0\right) \\ & + \tfrac{1}{6} e^{-4\phi/3} k^2 \left(\imath_a F^1 \wedge \star F^1 - F^1 \wedge \imath_a \star F^1\right) + \tfrac{1}{6} e^{-4\phi/3} k^{-2} \left(\imath_a F^2 \wedge \star F^2 - F^2 \wedge \imath_a \star F^2\right)\,,\end{aligned} \tag{C.22a}$$

$$\mathbf{E}_\phi = -\tfrac{8}{3} \left\{ d \star d\phi + \tfrac{1}{6} e^{8\phi/3} F^0 \wedge \star F^0 - \tfrac{1}{12} e^{-4\phi/3} k^2 F^1 \wedge \star F^1 - \tfrac{1}{12} e^{-4\phi/3} k^{-2} F^2 \wedge \star F^2 \right\}\,, \tag{C.22b}$$

$$\mathbf{E}_k = -k^{-2} \left\{ d \star dk - k^{-1} dk \wedge \star k + e^{-4\phi/3} k^3 F^1 \wedge \star F^1 - k^{-1} e^{-4\phi/3} F^2 \wedge \star F^2 \right\}\,, \tag{C.22c}$$

$$\mathbf{E}_0 = -\tfrac{1}{3} d \left(e^{8\phi/3} \star F^0 \right) + \frac{1}{3^{3/2}} F^1 \wedge F^2 \,, \tag{C.22d}$$

$$\mathbf{E}_1 = -\tfrac{1}{3} d \left(e^{-4\phi/3} k^2 \star F^1 \right) + \frac{1}{3^{3/2}} F^0 \wedge F^2 \,, \tag{C.22e}$$

$$\mathbf{E}_2 = -\tfrac{1}{3} d \left(e^{-4\phi/3} k^{-2} \star F^2 \right) + \frac{1}{3^{3/2}} F^0 \wedge F^1 \,. \tag{C.22f}$$

C.1 Further Truncation to Pure $\mathcal{N}=1, d=5$ Supergravity

We can truncate this theory further, to minimal (*pure*) supergravity as follows: if the two scalars are constant, taking into account that for ϕ this constant value must be $\phi = 0$, (we call k_∞ the constant value of k) their equations become the constraints

$$0 = F^0 \wedge \star F^0 - \tfrac{1}{2} k_\infty^2 F^1 \wedge \star F^1 - \tfrac{1}{2} k_\infty^{-2} F^2 \wedge \star F^2 \,, \tag{C.23a}$$

$$0 = k_\infty^3 F^1 \wedge \star F^1 - k_\infty^{-1} F^2 \wedge \star F^2 \,, \tag{C.23b}$$

whose simplest solution is this relation between vector field strengths

$$F^0 = k_\infty F^1 = k_\infty^{-1} F^2 \equiv F \,. \tag{C.24}$$

Substituting this solution into the Einstein and vector equations we get only these two independent equations

$$\mathbf{E}_a = \imath_a \star (e^c \wedge e^d) \wedge R_{cd} + \tfrac{1}{2} \left(\imath_a F \wedge \star F - F \wedge \imath_a \star F \right) \tag{C.25a}$$

$$-\tfrac{2}{3} \mathbf{E} = -\tfrac{1}{3} d \star F + \frac{1}{3^{3/2}} F \wedge F \,, \tag{C.25b}$$

which follow from the action of minimal $d=5$ supergravity [?]

$$S[e^a, A] = \frac{1}{16\pi G_N^{(5)}} \int \left[\star (e^a \wedge e^b) \wedge R_{ab} - \tfrac{1}{2} F \wedge \star F + \tfrac{1}{6\sqrt{3}} F \wedge F \wedge A \right] . \tag{C.26}$$

The truncation procedure we have followed to arrive to this action starting from the 10-dimensional Heterotic Superstring effective action can be easily reversed to embed solutions of pure $\mathcal{N}=1, d=5$ supergravity into the 10-dimensional Heterotic Superstring effective theory. In particular, we apply this recipe to the charged, non-extremal, black ring solution of Ref. [213] in Section 2.5.

D
Explicit Expressions NHEKN-ECG

D.1 Equations of motion

When evaluated on $N(x) = 1$, the gravitational tensor in (5.5) has the following non-vanishing components:

$$\mathcal{E}_{\psi\psi} = f(x)^2 \mathcal{E}_{xx}\,, \tag{D.1}$$
$$\mathcal{E}_{\psi t} = -2nrf(x)^2 \mathcal{E}_{xx}\,, \tag{D.2}$$
$$\mathcal{E}_{tt} = -r^4 \mathcal{E}_{rr} + 4n^2 r^2 f(x)^2 \mathcal{E}_{xx}\,. \tag{D.3}$$

In addition, we can relate $\mathcal{E}_{rr}$ to $\mathcal{E}_{xx}$ thanks to the Bianchi identity $\nabla^{\mu}\mathcal{E}_{\mu\nu} = 0$,

$$\mathcal{E}_{rr} = \frac{\left(n^2+x^2\right)}{2xr^2}\left(f(x)\left(n^2+x^2\right)\frac{d\mathcal{E}_{xx}}{dx} + \mathcal{E}_{xx}\left(\left(n^2+x^2\right)f'(x) + 2xf(x)\right)\right). \tag{D.4}$$

Thus, everything is determined by the component $\mathcal{E}_{xx}$, which reads

$$\begin{aligned} f\mathcal{E}_{xx} =&\Lambda + \frac{f\left(-n^2+x^2\right)+\left(n^2+x^2\right)\left(1+xf'\right)}{\left(n^2+x^2\right)^2} + L^4\mu\Bigg[-\frac{3f^3\left(n^6+16n^4x^2-45n^2x^4\right)}{\left(n^2+x^2\right)^6} \\ &+\left(\frac{3fx^3}{\left(n^2+x^2\right)^4}+\frac{3f^2\left(3n^4x-62n^2x^3+x^5\right)}{\left(n^2+x^2\right)^5}\right)f' - \frac{3n^2xf'^3}{2\left(n^2+x^2\right)^3} \\ &+\left(\frac{3\left(n^2-x^2\right)}{4\left(n^2+x^2\right)^3}+\frac{3f\left(n^4+37n^2x^2-2x^4\right)}{2\left(n^2+x^2\right)^4}\right)f'^2 - \frac{3fx^2\left(f''\right)^2}{4\left(n^2+x^2\right)^2} \\ &+\left(-\frac{3f\left(n^2+2x^2\right)}{2\left(n^2+x^2\right)^3}+\frac{f^2\left(6n^4+45n^2x^2-3x^4\right)}{\left(n^2+x^2\right)^4}+\frac{3f\left(-5n^2x+x^3\right)f'}{\left(n^2+x^2\right)^3}\right)f'' \\ &+\left(\frac{3f^2x\left(-4n^2+x^2\right)}{2\left(n^2+x^2\right)^3}+\frac{3fx}{2\left(n^2+x^2\right)^2}-\frac{3fx^2f'}{4\left(n^2+x^2\right)^2}\right)f^{(3)}\Bigg] \end{aligned} \tag{D.5}$$

The electromagnetic energy-momentum tensor has the same structure and hence the equations of motion are reduced to $\mathcal{E}_{xx} = T_{xx}$.

D.2 Solution of the Thermodynamic Quantities

Three of the four branches of solutions of the constraint equations (5.32) belong to the following class,

$$n(x_0, Q) = x_0^{\alpha} \sum_{k=0}^{\infty} n_k(Q) x_0^k \,, \quad \omega(x_0, Q) = x_0^{\beta} \sum_{k=0}^{\infty} \omega_k(Q) x_0^k, \tag{D.6}$$

where (α, β) are real parameters and we assume $n_0(Q) \neq 0$ and $\omega_0(Q) \neq 0$. The choice $(\alpha, \beta) = (0, 1)$ corresponds to the $\mathrm{AdS}_2 \times \mathbb{S}^2$ branch, while the choices $(\alpha, \beta) = (2, 1)$ and $(\alpha, \beta) = (1/2, 0)$ lead to other two solutions. The remaining branch belongs to the class

$$n(x_0, Q) = \sum_{k=0}^{\infty} n_k(Q) \left(\sqrt{x_0}\right)^k \,, \quad \omega(x_0, Q) = \sum_{k=0}^{\infty} \omega_k(Q) \left(\sqrt{x_0}\right)^k . \tag{D.7}$$

The solution for all coefficients in each of the expansions can be found explicitly. In the following we exhibit the first four terms of the solutions for n^2 and ω, as well as four terms of the corresponding expansions of the area $\mathcal{A}$, Wald entropy S, and relative entropy S/S_{BH}, where $S_{\mathrm{BH}} = \mathcal{A}/(4G)$.

For the branch corresponding to $\mathrm{AdS}_2 \times \mathbb{S}^2$, determined by the coefficients $(\alpha, \beta) = (0, 1)$,

$$\begin{aligned} n^2 =& Q^2 + x_0^2 \left(\frac{\mu L^4}{Q^4} + 1\right) + x_0^4 \, \frac{2\mu L^4 \left(11\mu L^4 - 13 Q^4\right)}{Q^{10}} \\ &+ x_0^6 \, \frac{3\mu L^4 \left(25\mu^2 L^8 - 90\mu L^4 Q^4 + 56 Q^8\right)}{Q^{16}} \end{aligned} \tag{D.8}$$

$$\begin{aligned} \omega =& x_0 \frac{1}{Q^2} - x_0^3 \, \frac{4Q^4 - 2\mu L^4}{2Q^8} + x_0^5 \, \frac{-11\mu^2 L^8 + 8\mu L^4 Q^4 + 4Q^8}{Q^{14}} \\ &- x_0^7 \, \frac{4\left(26\mu^3 L^{12} - 69\mu^2 L^8 Q^4 + 36\mu L^4 Q^8 + 2Q^{12}\right)}{Q^{20}} \end{aligned} \tag{D.9}$$

$$\begin{aligned} \mathcal{A} =& 4\pi Q^2 + \pi x_0^2 \left(8 - \frac{4\mu L^4}{Q^4}\right) + x_0^4 \, \frac{48\pi\mu L^4 \left(\mu L^4 - Q^4\right)}{Q^{10}} \\ &+ x_0^6 \, \frac{12\pi\mu L^4 \left(27\mu^2 L^8 - 70\mu L^4 Q^4 + 36 Q^8\right)}{Q^{16}} \end{aligned} \tag{D.10}$$

$$\begin{aligned} S =& \frac{\pi Q^2}{G} + x_0^2 \, \frac{2\pi \left(\mu L^4 + Q^4\right)}{G Q^4} + x_0^4 \, \frac{12\pi\mu L^4 \left(\mu L^4 - 2Q^4\right)}{G Q^{10}} \\ &- x_0^6 \, \frac{6\pi\mu L^4 \left(3\mu^2 L^8 + 16\mu L^4 Q^4 - 24 Q^8\right)}{G Q^{16}} \end{aligned} \tag{D.11}$$

$$\begin{aligned} S/S_{\mathrm{BH}} =& 1 + x_0^2 \, \frac{3\mu L^4}{Q^6} + x_0^4 \, \frac{3\mu L^4 \left(\mu L^4 - 6Q^4\right)}{Q^{12}} \\ &+ x_0^6 \, \frac{6\mu L^4 \left(-22\mu^2 L^8 + 21\mu L^4 Q^4 + 12 Q^8\right)}{Q^{18}} . \end{aligned} \tag{D.12}$$

For the branch determined by the coefficients $(\alpha, \beta) = (2, 1)$,

$$\begin{aligned} n^2 =& x_0^4 \, \frac{2\sqrt{3\mu}L^2 + 3Q^2}{18\mu L^4} + x_0^6 \, \frac{2\mu L^4 - \sqrt{3\mu}L^2Q^2 + 9Q^4}{108\mu^2 L^8} \\ &+ x_0^8 \, \frac{328\mu^2 L^8 + 612\sqrt{3}\mu^{3/2}L^6Q^2 + 330\mu L^4 Q^4 + 117\sqrt{3\mu}L^2Q^6 + 432Q^8}{5184\sqrt{3}\mu^{7/2}L^{14} + 7776\mu^3 L^{12}Q^2} \end{aligned} \tag{D.13}$$

$$\begin{aligned} \omega =& x_0 \, \frac{1}{\sqrt{3\mu}L^2} + x_0^3 \, \frac{\sqrt{3}Q^2 - 6\sqrt{\mu}L^2}{36\mu^{3/2}L^6} - x_0^5 \, \frac{-100\sqrt{3}\mu L^4 + 72\sqrt{\mu}L^2Q^2 - 33\sqrt{3}Q^4}{2592\mu^{5/2}L^{10}} \\ &- x_0^7 \, \frac{3584\sqrt{3}\mu^2 L^8 + 4344\mu^{3/2}L^6Q^2 + 636\sqrt{3}\mu L^4Q^4 + 198\sqrt{\mu}L^2Q^6 - 765\sqrt{3}Q^8}{31104\mu^{7/2}L^{14}\left(2\sqrt{3\mu}L^2 + 3Q^2\right)} \end{aligned} \tag{D.14}$$

$$\begin{aligned} \mathcal{A} =& 4\pi\sqrt{3\mu}L^2 + x_0^2 \, \pi\left(2 - \frac{Q^2}{\sqrt{3\mu}L^2}\right) - x_0^4 \, \pi\frac{28\mu L^4 + 27Q^4}{72\sqrt{3}\mu^{3/2}L^6} \\ &+ x_0^6 \, \pi\frac{2048\mu^2L^8 + 1224\sqrt{3}\mu^{3/2}L^6Q^2 + 588\mu L^4Q^4 - 246\sqrt{3\mu}L^2Q^6 - 585Q^8}{2592\left(2\mu^3L^{12} + \sqrt{3}\mu^{5/2}L^{10}Q^2\right)} \end{aligned} \tag{D.15}$$

$$\begin{aligned} S =& \frac{2\pi\sqrt{3\mu}L^2}{G} - x_0^2 \, \frac{2\pi L^2 + \pi\sqrt{3/\mu}Q^2}{6GL^2} + x_0^4 \, \frac{\pi\left(68\sqrt{3}\mu L^4 + 48\sqrt{\mu}L^2Q^2 - 27\sqrt{3}Q^4\right)}{432G\mu^{3/2}L^6} \\ &- x_0^6 \, \frac{\pi\left(640\sqrt{3}\mu^2L^8 + 2104\mu^{3/2}L^6Q^2 + 284\sqrt{3}\mu L^4Q^4 - 42\sqrt{\mu}L^2Q^6 + 195\sqrt{3}Q^8\right)}{1728G\mu^{5/2}L^{10}\left(2\sqrt{3\mu}L^2 + 3Q^2\right)} \end{aligned} \tag{D.16}$$

$$S/S_{\mathrm{BH}} = 2 - x_0^2 \, \frac{4}{3\sqrt{3\mu}L^2} + x_0^4 \, \frac{4}{9\mu L^4} - x_0^6 \, \frac{280\mu^{3/2}L^6 + 180\sqrt{3}\mu L^4Q^2 + 54\sqrt{\mu}L^2Q^4 - 3\sqrt{3}Q^6}{216\left(2\sqrt{3}\mu^3L^{12} + 3\mu^{5/2}L^{10}Q^2\right)} \tag{D.17}$$

For the branch determined by the coefficients $(\alpha, \beta) = (1/2, 0)$

$$\begin{aligned} n^2 =& x_0\frac{\sqrt{3\mu}L^2}{Q} + x_0^2\left(\frac{25\mu L^4}{6Q^4} - \frac{4}{5}\right) + x_0^3\frac{128125\mu^2L^8 - 32700\mu L^4Q^4 + 324Q^8}{1800\sqrt{3\mu}L^2Q^7} \\ &+ x_0^4\left(\frac{53125\mu^2L^8}{81Q^{10}} - \frac{4745\mu L^4}{27Q^6} + \frac{16Q^2}{375\mu L^4} + \frac{28}{3Q^2}\right) \end{aligned} \tag{D.18}$$

$$\begin{aligned} \omega =& \frac{\sqrt{3/\mu}Q}{5L^2} - x_0\left(\frac{3}{2Q^2} - \frac{Q^2}{5\mu L^4}\right) + x_0^2 \, \frac{-216875\mu^2L^8 + 33300\mu L^4Q^4 + 1476Q^8}{9000\sqrt{3}\mu^{3/2}L^6Q^5} \\ &- x_0^3\left(-\frac{64Q^4}{1875\mu^2L^8} + \frac{2}{15\mu L^4} + \frac{12475\mu L^4}{54Q^8} - \frac{1141}{27Q^4}\right) \end{aligned} \tag{D.19}$$

$$\mathcal{A} = x_0\ \pi \frac{20\sqrt{\mu/3}L^2}{Q} + x_0^2\ \pi\left(\frac{50\mu L^4}{Q^4} - \frac{20}{3}\right) + x_0^3\ \pi \frac{318125\mu^2L^8 - 60300\mu L^4Q^4 + 324Q^8}{270\sqrt{3\mu}L^2Q^7} \tag{D.20}$$
$$+ x_0^4\ 8\pi \frac{3203125\mu^2L^8 + 72\left(Q^{12}/\mu L^4\right) - 743125\mu L^4Q^4 + 27675Q^8}{2025Q^{10}}$$

$$S = \frac{3\pi Q^2}{5G} + x_0\ 2\pi\frac{6Q^4 - 25\mu L^4}{25G\sqrt{3\mu}L^2Q} + x_0^2\ \pi\frac{-18125\left(\mu L^4/Q^4\right) + 108\left(Q^4/\mu L^4\right) + 6150}{1125G} \tag{D.21}$$
$$+ x_0^3\ \pi\frac{\left(-34203125\mu^3L^{12} + 6656250\mu^2L^8Q^4 - 283500\mu L^4Q^8 + 1944Q^{12}\right)}{67500\sqrt{3}G\mu^{3/2}L^6Q^7}$$

$$S/S_{\mathrm{BH}} = \frac{1}{x_0}\frac{3\sqrt{3/\mu}Q^3}{25L^2} + \frac{27Q^4}{125\mu L^4} - \frac{13}{10} + x_0\frac{-21125\mu^2L^8 + 4380\mu L^4Q^4 + 252Q^8}{1000\sqrt{3}\mu^{3/2}L^6Q^3} \tag{D.22}$$
$$+ x_0^2\left(\frac{224Q^6}{3125\mu^2L^8} - \frac{3595\mu L^4}{18Q^6} + \frac{64Q^2}{125\mu L^4} + \frac{82}{3Q^2}\right)$$

Finally, for the class of solutions defined by (D.7)

$$n^2 = x_0\frac{\sqrt{6\mu}L^2}{Q} - \left(\sqrt{x_0}\right)^3\frac{(2/3)^{1/4}\mu^{3/4}L^3}{Q^{5/2}} - \left(\sqrt{x_0}\right)^4\left(\frac{7\mu L^4}{3Q^4} + \frac{3}{2}\right) \tag{D.23}$$
$$+ \left(\sqrt{x_0}\right)^5\frac{\mu^{1/4}L\left(138Q^4 - 439\mu L^4\right)}{2^{1/4}3^{3/4}24Q^4\left(Q^2\right)^{3/4}}$$

$$\omega = \sqrt{x_0}\frac{(2/3)^{1/4}}{\mu^{1/4}L\sqrt{Q}} + x_0\frac{1}{2Q^2} - \left(\sqrt{x_0}\right)^3\frac{6Q^4 - 115\mu L^4}{2^{1/4}3^{3/4}24\mu^{3/4}L^3Q^{7/2}} \tag{D.24}$$
$$- \left(\sqrt{x_0}\right)^4\frac{21Q^4 - 197\mu L^4}{12\sqrt{6\mu}L^2Q^5}$$

$$\mathcal{A} = \sqrt{x_0}\ 2^{3/4}3^{1/4}2\pi\mu^{1/4}L\sqrt{Q} - x_0\frac{\pi\sqrt{6\mu}L^2}{Q} \tag{D.25}$$
$$+ \left(\sqrt{x_0}\right)^3\frac{\pi\left(6Q^4 - 97\mu L^4\right)}{6\ 2^{3/4}3^{1/4}\mu^{1/4}LQ^{5/2}} + \left(\sqrt{x_0}\right)^4\ 3\pi\left(1 - \frac{8\mu L^4}{Q^4}\right)$$

$$S = \sqrt{x_0}\ \frac{2^{3/4}3^{1/4}\pi\ \mu^{1/4}L\sqrt{Q}}{G} + x_0\ \frac{\pi\sqrt{\mu/6}L^2}{GQ} \tag{D.26}$$
$$+ \left(\sqrt{x_0}\right)^3\frac{\pi\left(47\mu L^4 + 6Q^4\right)}{12\ 2^{3/4}3^{1/4}G\mu^{1/4}LQ^{5/2}} + \left(\sqrt{x_0}\right)^4\ \frac{\pi\left(136\left(\mu L^4/Q^4\right) - 15\right)}{18G}$$

$$S/S_{\rm BH} = 2 + \sqrt{x_0}\,\frac{2\,(2/3)^{3/4}\,\mu^{1/4}L}{Q^{3/2}} + x_0\,\frac{7\sqrt{2\mu/3}L^2}{Q^3} + (\sqrt{x_0})^3\,\frac{103\mu L^4 - 10Q^4}{2\;2^{3/4}3^{1/4}\mu^{1/4}LQ^{9/2}}\,. \tag{D.27}$$

D.3 Solutions for $g_k(y)$

Expanding $g(y)$ as in (5.35) and choosing the parameter configuration of the $\mathrm{AdS}_2 \times \mathbb{S}^2$ branch, we see that the solutions for all $g_k(y)$ that satisfy the boundary conditions are polynomial in y. The first terms read as follows

$$\frac{g_0(y)}{1-y^2} = \frac{1}{Q^2} \tag{D.28}$$

$$\frac{g_1(y)}{1-y^2} = \frac{-Q^8\left(y^2+1\right) + Q^4\mu L^4\left(16y^2+3\right) - 9\mu^2L^8}{Q^{12} - 9Q^8\mu L^4} \tag{D.29}$$

$$\begin{aligned}\frac{g_2(y)}{1-y^2} =& \frac{Q^{24}\left(y^2+1\right)^2 + Q^{20}\mu L^4\left(-232y^4+39y^2-27\right) + Q^{16}\mu^2L^8\left(6555y^4-2218y^2-592\right)}{Q^{14}\left(Q^4-30\mu L^4\right)\left(Q^4-9\mu L^4\right)^3} \\ &+ \frac{3Q^{12}\mu^3L^{12}\left(-21412y^4+9939y^2+3798\right) + 27Q^8\mu^4L^{16}\left(7768y^4-4248y^2-3085\right)}{Q^{14}\left(Q^4-30\mu L^4\right)\left(Q^4-9\mu L^4\right)^3} \\ &+ \frac{-4617Q^4\mu^5L^{20}\left(20y^2-77\right) - 240570\mu^6L^{24}}{Q^{14}\left(Q^4-30\mu L^4\right)\left(Q^4-9\mu L^4\right)^3}\end{aligned} \tag{D.30}$$

$$\begin{aligned}\frac{g_3(y)}{1-y^2} =& \frac{-Q^{44}\left(y^2+1\right)^3 + Q^{40}\mu L^4\left(1273y^6+87y^4-145y^2-15\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{2Q^{36}\mu^2L^8\left(-85596y^6+26710y^4+15907y^2+11113\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{Q^{32}\mu^3L^{12}\left(8837031y^6-4893135y^4-730200y^2-1854008\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{-6Q^{28}\mu^4L^{16}\left(37929171y^6-27161595y^4+80490y^2-10643708\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{18Q^{24}\mu^5L^{20}\left(180123972y^6-151235249y^4+10041658y^2-65511537\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{-81Q^{20}\mu^6L^{24}\left(321947241y^6-302005267y^4+27567259y^2-165642213\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{729Q^{16}\mu^7L^{28}\left(152540040y^6-152997410y^4+9998425y^2-132397703\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{-13122Q^{12}\mu^8L^{32}\left(14926950y^6-14295073y^4-3145538y^2-31632593\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{708588Q^8\mu^9L^{36}\left(130865y^4-373185y^2-1354486\right) + 127545840Q^4\mu^{10}L^{40}\left(875y^2+8112\right)}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5} \\ &+ \frac{-348200143200\mu^{11}L^{44}}{Q^{20}\left(Q^4-63\mu L^4\right)\left(Q^4-30\mu L^4\right)^2\left(Q^4-9\mu L^4\right)^5}\end{aligned} \tag{D.31}$$

D.4 Adding a cosmological constant Λ

For the sake of completeness, here we shall comment on the case of a non-vanishing cosmological constant, $\Lambda \neq 0$. The regularity constraints, analogue to (5.32), for a non-vanishing Λ read

$$
\begin{aligned}
0 = & - n^2 + x_0^2 + \frac{Q^2\omega^2\left(n^2+x_0^2\right)^2}{x_0^2} - \mu L^4 \omega^2 (2x_0\omega + 3) \\
& + \frac{\Lambda}{3}\left(-3n^4 + 6n^2x_0^2 + x_0^4\right),
\end{aligned} \tag{D.32}
$$

$$
\begin{aligned}
0 = & \left(n^2+x_0^2\right)\left(\omega\left(n^2+x_0^2\right) - x_0\right) + \mu L^4\omega^2\left(\omega\left(x_0^2 - 5n^2\right) + 3x_0\right) \\
& - \frac{\Lambda}{3} x_0 \left(n^2 + x_0^2\right)\left(6n^2 + 2x_0^2\right).
\end{aligned} \tag{D.33}
$$

We shall focus on the neighbourhood of solutions for which $\omega = 0$. Such solutions are non-rotating if $x_0 = 0$ in such a way that $\lim_{\omega\to 0}\omega/x_0 \neq 0$, while $\lim_{\omega\to 0} x_0/n = 0$. On the other hand, if $\lim_{\omega\to 0}\omega/x_0 = 0$ and $\lim_{\omega\to 0} x_0/n \neq 0$, the solutions correspond to an ultra-spinning limit and exhibit a non-compact horizon — this only happens for $\Lambda < 0$. Let us first consider the slowly rotating case. Imposing $\omega = 0$, one solution is $x_0 = 0$ and $n^2 = \left(1 - 4\Lambda Q^2 - \sqrt{1-4\Lambda Q^2}\right)/\left(-2\Lambda\left(1-4\Lambda Q^2\right)\right)$. Then, in a neighbourhood of this solution, ω and n^2 read, in powers of x_0,

$$
\begin{aligned}
n^2 = & \frac{1-4\Lambda Q^2 - \sqrt{1-4\Lambda Q^2}}{-2\Lambda\left(1-4\Lambda Q^2\right)} + x_0^2 \left(\frac{Q^4\left(72\Lambda^2\mu L^4 - 6\right) - 6\Lambda\mu L^4 Q^2\left(7\sqrt{1-4\Lambda Q^2}+6\right) - 3\mu L^4\left(\sqrt{1-4\Lambda Q^2}+1\right) - 8\Lambda Q^6}{6Q^4\left(4\Lambda Q^2 - 1\right)}\right) \\
& + \mathcal{O}\left(x_0^4\right)
\end{aligned} \tag{D.34}
$$

$$
\begin{aligned}
\omega = & x_0 \frac{1+\sqrt{1-4\Lambda Q^2}}{2Q^2} \\
& + x_0^3\Lambda^2 \frac{6\Lambda\mu L^4 Q^2\left(7\sqrt{1-4\Lambda Q^2}+8\right) - 3\mu L^4\left(\sqrt{1-4\Lambda Q^2}+1\right) + 4Q^4\left(-36\Lambda^2\mu L^4 + 2\sqrt{1-4\Lambda Q^2}+1\right) - 16\Lambda Q^6}{3Q^4\left(2\Lambda Q^2\left(\sqrt{1-4\Lambda Q^2}-2\right) - \sqrt{1-4\Lambda Q^2}+1\right)} \\
& + \mathcal{O}\left(x_0^5\right).
\end{aligned} \tag{D.35}
$$

On the other hand, performing an expansion of $g(y)$ around $x_0 = 0$ as in (5.35), the first non-vanishing term reads

$$
\frac{g(y)}{1-y^2} = \frac{-\Lambda\left(2\Lambda Q^2 + \sqrt{1-4\Lambda Q^2} - 1\right)}{\Lambda Q^2\left(\sqrt{1-4\Lambda Q^2} - 3\right) - \sqrt{1-4\Lambda Q^2} + 1} + \mathcal{O}\left(x_0^2\right). \tag{D.36}
$$

Thus, we conclude that $\mathrm{AdS}_2 \times \mathbb{S}^2$ is not corrected by ECG also when $\Lambda \neq 0$. However, it admits smooth corrections when the spin is turned on. This is analogous to the $\mathrm{AdS}_2 \times \mathbb{S}^2$

branch for $\Lambda = 0$ discussed above and, in fact, taking the limit $\Lambda \to 0$, the RHS of (D.36) goes to $1/Q^2$ as expected. The expansions for the area and the entropy are

$$\mathcal{A} = \frac{8\pi Q^2}{\sqrt{1 - 4\Lambda Q^2} + 1} + x_0^2 \frac{2\pi\left(6\Lambda\mu L^4 Q^2\left(7\sqrt{1-4\Lambda Q^2}+8\right)-3\mu L^4\left(\sqrt{1-4\Lambda Q^2}+1\right)+4Q^4\left(-36\Lambda^2\mu L^4+2\sqrt{1-4\Lambda Q^2}+1\right)-16\Lambda Q^6\right)}{3Q^4\sqrt{1-4\Lambda Q^2}} + \mathcal{O}\left(x_0^4\right) \tag{D.37}$$

$$S = \frac{2\pi Q^2}{G(\sqrt{1 - 4\Lambda Q^2} + 1)} - x_0^2 \frac{2\pi\left(Q^2\left(18\Lambda^2\mu L^4-\sqrt{1-4\Lambda Q^2}+1\right)+3\Lambda\mu L^4\left(\sqrt{1-4\Lambda Q^2}+1\right)+4\Lambda Q^4\right)}{3GQ^2\left(\sqrt{1-4\Lambda Q^2}-1\right)} + \mathcal{O}(x_0^4)\,. \tag{D.38}$$

Let us now focus our attention on the ultra-spinning case. When $\omega = 0$ another solution is $x_0 = \sqrt{3}/(2\sqrt{-\Lambda})$ and $n^2 = -1/4\Lambda$, which of course is only valid for a negative cosmological constant, $\Lambda < 0$. For simplicity, we shall restrict to the neutral case $Q = 0$. In a neighbourhood of this solution, ω and n read, in powers of $x_0 - \sqrt{3}/(2\sqrt{-\Lambda})$,

$$n^2 = -\frac{1}{4\Lambda}+\frac{1}{2}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^2(-3\Lambda^2\mu L^4-1)+\frac{1}{6}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^3(\sqrt{3}\sqrt{-\Lambda}+21\sqrt{3}\Lambda^2\sqrt{-\Lambda}\mu L^4) + \mathcal{O}\left(\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^4\right) \tag{D.39}$$

$$\omega = \Lambda\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)-\frac{1}{2}\sqrt{3}\sqrt{-\Lambda}\Lambda\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^2-\frac{1}{6}7\Lambda^2\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^3(3\Lambda^2\mu L^4+1) + \mathcal{O}\left(\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^4\right). \tag{D.40}$$

On the other hand, the area and the entropy are

$$\mathcal{A} = -\frac{2\sqrt{3}\pi}{(-\Lambda)^{3/2}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)}+\frac{7\pi}{\Lambda}+\frac{7\pi\left(6\Lambda^2\mu L^4-1\right)}{2\sqrt{3}\sqrt{-\Lambda}}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)+\mathcal{O}\left(\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^2\right) \tag{D.41}$$

$$S = -\frac{\pi\sqrt{3}}{2G(-\Lambda)^{3/2}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)}+\frac{7\pi}{4G\Lambda}+\frac{\pi\left(6\Lambda^2\mu L^4-7\right)}{8\sqrt{3}G\sqrt{-\Lambda}}\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)+\mathcal{O}\left(\left(x_0-\frac{\sqrt{3}}{2\sqrt{-\Lambda}}\right)^2\right). \tag{D.42}$$

The latter pair of quantities are only well defined on an anular neighbourhood centred at $x_0 = \sqrt{3}/2\sqrt{-\Lambda}$. However, this does not mean that there is no solution when $x_0 = \sqrt{3}/2\sqrt{-\Lambda}$. Let us recall that, as long as ω/x_0 remains finite, so does $g'(1)$, according to (5.21). Then, the coordinates can be chosen to parametrize a manifold of topology $\mathrm{AdS}_2 \times \mathbb{S}^2$ by identifying canonically the coordinate ϕ, *i.e.* $\phi \sim \phi + 2\pi$ (see equation (5.19)), and the metric becomes regular everywhere. However, if $\omega/x_0 = 0$ then $g'(1)$ also vanishes and the metric does not describe a regular geometry on $\mathrm{AdS}_2 \times \mathbb{S}^2$. Thus, in order to obtain a solution also at the parameter configuration

$$x_0 = \sqrt{3}/2\sqrt{-\Lambda}, \quad \omega = 0, \quad n^2 = -1/4\Lambda, \tag{D.43}$$

let us rewrite (5.19) in terms of a new angular coordinate

$$\varphi = \frac{x_0}{\omega}\phi \tag{D.44}$$

and identify it with arbitrary period, $\varphi \sim \varphi + \Delta\varphi$. The equations (D.32) and (D.33) are unchanged by this coordinate transformation, so (D.43) constitute a solution. Since

$g'(1) = 0$, the topology our coordinates parametrize is that of $\mathrm{AdS}_2 \times \mathbb{S}^1 \times \mathbb{R}$, and the metric is regular everywhere. The horizon has become non-compact, with topology $\mathbb{S}^1 \times \mathbb{R}$, and is infinitely large, in the sense that the proper length of coordinate curves tangent to ∂_y (which extend from $y = -1$ to $y = 1$) is infinite. However, the horizon has a finite area, $\mathcal{A} = 2\Delta\varphi$, and Wald's correction to the Bekenstein–Hawking entropy vanishes, as can be deduced from (5.47), because now both $g(1)$ and $g'(1)$ are zero. Nevertheless, one can check that the profile of the solution is not going to be the same as in Einstein gravity. This solution of ECG is analogous to the super-entropic black holes of Ref. [413], in the sense that both have non-compact horizons with finite area and can be understood as an entropy-divergent limit of a rotating solution. Thus, it would be interesting to study whether this solutions do or do not respect the Isoperimetric Inequality in the context of extended black hole thermodynamics. However, further investigation in these lines is left for future work.

E

More on Boundary Conditions of AdS-Taub-NUT Black Branes

E.1 Electromagnetic boundary conditions

In the case of the electromagnetic field, the analysis of the boundary conditions in the near-horizon are analogous to the scalar case. Again one finds that the NP variables ϕ_0, ϕ_2 can be expanded in a Frobenius series near $z = 1$, and imposing the condition of outgoing waves one finds the following solutions for the radial functions $Y_{\pm1}$:

$$Y_{\pm1} \sim (1 - z)^{\alpha_{\pm1}} \quad \text{when} \quad z \to 1, \tag{E.1}$$

where

$$\alpha_{+1} = -\frac{i\hat{\omega}}{3(1 + \epsilon^2)}, \quad \alpha_{-1} = 1 - \frac{i\hat{\omega}}{3(1 + \epsilon^2)}. \tag{E.2}$$

The analysis of boundary conditions at infinity, on the other hand, is much involved than in the case of a scalar field. By analyzing the solutions of the radial equations (6.64) for $Y_{\pm1}$, we see that the two independent solutions behave near $z = 0$ as

$$Y_{\pm1}(z) = a_{\pm1} + b_{\pm1} z \quad \text{when} \quad z \to 0, \tag{E.3}$$

where $a_{\pm1}$ and $b_{\pm1}$ are constants. Now, the boundary conditions are not imposed directly on the NP variables but on the perturbation of the Maxwell field A_μ, so we must study how these relate. Let us for into account that we can always choose a gauge in which the z-component the vector vanishes $A_z = 0$. Then, the solutions to Maxwell equations near $z = 0$ behave as $A_a \sim A_a^{(1)} + z A_a^{(2)}$, where a denotes the boundary indices $a = t, x, y$. Therefore, Dirichlet boundary conditions imply that $A_a^{(1)} = 0$, and we only keep the mode that decays at infinity. Separating variables, this means that we can write the vector asymptotically as

$$A_a = z e^{-i(\omega t - k y)} \gamma_a(x) + \mathcal{O}(z^3), \tag{E.4}$$

where γ_a are certain functions and one can check that the following term in the z-expansion is indeed $\mathcal{O}(z^3)$. Now, the functions γ_a are not arbitrary, but we find that Maxwell equations impose the following constraint,

$$\left(\omega - \frac{2nx}{L^2}\left(k + \frac{2nx\omega}{L^2}\right)\right)\gamma_t - i\gamma_x' + \left(k + \frac{2nx\omega}{L^2}\right)\gamma_y = 0. \tag{E.5}$$

On the other hand, we are searching for solutions such that the NP variables ϕ_0 and ϕ_2 are separated, and this will impose, too, conditions on the γ_a. Computing ϕ_0 and ϕ_2 from the vector perturbation (E.4) we find that

$$\hat{\phi}_0 = e^{-i(\omega t - ky)}\left[A_{+1} + B_{+1}z + \mathcal{O}(z^2)\right], \tag{E.6}$$

$$\hat{\phi}_2 = e^{-i(\omega t - ky)}\left[A_{-1} + B_{-1}z + \mathcal{O}(z^2)\right], \tag{E.7}$$

where $\hat{\phi}_{0,2}$ are defined as

$$\hat{\phi}_0 = V\sqrt{n^2+r^2}e^{2i\arctan(r/n)}\phi_0\,, \qquad \hat{\phi}_2 = \sqrt{n^2+r^2}\phi_2\,, \tag{E.8}$$

and the coefficients $A_{\pm 1}$, $B_{\pm 1}$ read

$$A_{\pm 1} = \frac{2^{\pm 1/2}}{4L}\left(-\frac{2inx\gamma_t}{L^2} \pm \gamma_x + i\gamma_y\right), \tag{E.9}$$

$$B_{\pm 1} = \frac{2^{\pm 1/2}}{4Lr_+}\left[L^2\gamma_t' + \left(\mp k - \frac{2n^2x}{L^4}\right)\left(L^2\gamma_t + i\left(\mp n + L^2\omega\right)\gamma_x + \left(n \mp L^2\omega\right)\gamma_y\right)\right]. \tag{E.10}$$

Now, on the other hand, if both ϕ_0 and ϕ_2 can be separated, then the result should read

$$\hat{\phi}_0 = e^{-i(\omega t - ky)}\mathcal{H}_{q_{+1}}(x)\left[a_{+1} + b_{+1}z + \mathcal{O}(z^2)\right], \tag{E.11}$$

$$\hat{\phi}_2 = e^{-i(\omega t - ky)}\mathcal{H}_{q_{-1}}(x)\left[a_{-1} + b_{-1}z + \mathcal{O}(z^2)\right], \tag{E.12}$$

where we have taken into account (E.3) and where $\mathcal{H}_{q_{\pm 1}}(x)$ are the eigenfunctions in (6.45), with two possibly different levels q_{+1} and q_{-1} for each of the variables. Thus, we obtain a system of four equations for the variables γ_a and the four constants $a_{\pm 1}$, $b_{\pm 1}$,

$$A_{\pm 1} = a_{\pm 1}\mathcal{H}_{q_{\pm 1}}\,, \quad B_{\pm 1} = b_{\pm 1}\mathcal{H}_{q_{\pm 1}}\,. \tag{E.13}$$

Together with (E.5), we have to solve a system of five equations which is not guaranteed to have solutions. In order to simplify the computations, at this point it is interesting to note that we can set $k = 0$ without loss of generality. In fact, the change of variables

$$\hat{x} = x - \sigma\,, \qquad \hat{t} = t + \frac{2n}{L^2}\sigma y \tag{E.14}$$

leaves invariant the background metric and therefore is a symmetry of the linearized equations. On the other hand it transforms the perturbation A_a as follows

$$\hat{A}_a = ze^{-i(\omega\hat{t} - \hat{k}y)}\hat{\gamma}_a(\hat{x})\,, \quad \text{where} \quad \hat{k} = k + \frac{2n\omega\sigma}{L^2}\,, \tag{E.15}$$

and

$$\hat{\gamma}_{\hat{t}} = \gamma_t\,, \quad \hat{\gamma}_{\hat{x}} = \gamma_x\,, \quad \hat{\gamma}_y = \gamma_y - \frac{2n\sigma}{L^2}\gamma_t\,. \tag{E.16}$$

Therefore, by choosing $\sigma = -kL^2/(2n\omega)$ we get $\hat{k} = 0$. Equivalently, we can always work with the solution with $k = 0$ and generate another solution with $k \neq 0$ by applying the isometric transformation (E.14). Thus, from now on we set $k = 0$.

One can see that from the five equations in (E.5) and (E.13) it is possible to obtain explicitly the values of γ_t, γ_t', γ_x, γ_x' and γ_y, but of course, in order for this to be an actual solution, γ_t' and γ_x' should in fact be the derivatives of γ_t and γ_x. As it turns out, this only happens when the following constraints meet. First, the two levels q_{+1} and q_{-1} must be related according to

$$q_{+1} = q_{-1} + 2s\,, \tag{E.17}$$

where we recall that $s = \text{sign}\,[\text{Re}(n\omega)]$. Thus we have $q_{-1} = 0, 1, 2, ...$ for $s = 1$ and $q_{-1} = 2, 3, 4, ...$ for $s = -1$. On the other hand, the ratios of the constants $a_\pm$, $b_\pm$,

$$\lambda_{\pm 1} = \frac{b_{\pm 1}}{a_{\pm 1}} \tag{E.18}$$

must be related according to

$$\lambda_{-1} = \frac{\lambda_{+1}\left(2q\epsilon - \hat{\omega} + \epsilon\right) - i\left(2(2q+3)\hat{\omega}\epsilon - \hat{\omega}^2 + 3\epsilon^2\right)}{-i\lambda_{+1} + (2q+5)\epsilon - \hat{\omega}}\,. \tag{E.19}$$

where q is

$$q = \begin{cases} q_{-1} & \text{if} \quad s = 1\,, \\ -1 - q_{-1} & \text{if} \quad s = -1 \end{cases} \tag{E.20}$$

Note that this is all we need in order to characterize the boundary conditions, since the overall normalization of $Y_{\pm 1}$ is not relevant when searching for quasinormal modes. Now, consistency of the system of equations requires an additional constraint that involves such overall normalization,

$$2\epsilon\frac{a_{-1}}{a_{+1}} = \begin{cases} i\lambda_{+1} - (2q_{-1}+5)\epsilon + \hat{\omega} & \text{if} \quad s = 1\,, \\ -\frac{(i\lambda_{+1} + (2q_{-1}-3)\epsilon + \hat{\omega})}{4(q_{-1}-1)q_{-1}} & \text{if} \quad s = -1\,. \end{cases} \tag{E.21}$$

In that case, the explicit solution for the γ_a reads

$$\gamma_t = -\frac{ia_{+1}L^3\left(-i\lambda_{+1} - \hat{\omega} + \epsilon\right)}{\sqrt{2}r_+ x\hat{\omega}\epsilon}\left[2(1+q_{-1})\mathcal{H}_{q_-} + \mathcal{H}_{2+q_-}\right], \tag{E.22}$$

$$\gamma_x = \frac{\sqrt{2}a_{+1}L}{\epsilon}\left[\left(-i\lambda_{+1} + \epsilon(2q_{-1}+5) - \hat{\omega}\right)\mathcal{H}_{q_-} + \epsilon\mathcal{H}_{2+q_-}\right]\,, \tag{E.23}$$

$$\gamma_y = -\frac{i\sqrt{2}a_{+1}L}{\hat{\omega}\epsilon}\Big[\left(2(q_{-1}+1)\epsilon^2 + \lambda_{+1}\left(i\hat{\omega} - 2i(q_{-1}+1)\epsilon\right) - (4q_{-1}+7)\hat{\omega}\epsilon + \hat{\omega}^2\right)\mathcal{H}_{q_-}$$

$$+\,\epsilon(\epsilon-i\lambda_{+1})\mathcal{H}_{2+q_-}\Big] \tag{E.24}$$

for $s=1$, and there is a similar solution for $s=-1$.

Then, in order to find the electromagnetic quasinormal modes, the idea would be to simultaneously solve the radial equations (6.64) for Y_{+1} and Y_{-1} with the levels $q_{\pm 1}$ related according to (E.17) and with the boundary conditions given by (E.1), (E.3) and (E.19). Note that, once ϵ and q_{-1} are specified, the problem only contains two parameters, $\hat{\omega}$ and λ_+, and the hope is a solution exists only for discrete values of these quantities. Unfortunately, this is not the case, since the boundary conditions are degenerate. Indeed, they are equivalent to requiring that ϕ_0 and ϕ_2 emerge from the same vector field. In order to see this, we first note the following Maxwell equations in the NP formalism

$$(D-2\rho)\phi_1=\delta^*\phi_0\,,\quad (D-\rho)\phi_2=\delta^*\phi_1\,. \tag{E.25}$$

Combining these it is possible to derive the following relation between ϕ_0 and ϕ_2,

$$\hat{\delta}^*\hat{\delta}^*\phi_0=R(D-\rho)R(D-2\rho)\phi_2\,, \tag{E.26}$$

where

$$R=\frac{i\sqrt{2(r^2+n^2)}}{L}e^{-i\arctan(r/n)}\,,\quad \hat{\delta}^*=R\delta^*=\partial_x-i\partial_y+i\frac{2nx}{L^2}\partial_t\,. \tag{E.27}$$

Then, by using the decomposition (6.62) one first derives the relation between the levels $q_{\pm 1}$ given in (E.17)[1], and one also obtains a relation between Y_{+1} and Y_{-1},

$$\begin{aligned}Y_{+1}=&-\frac{Y_{-1}\left((2q_{-1}+1)\epsilon\left(3z^4\epsilon^4-6z^2\epsilon^2+z^3\left(3\epsilon^4+6\epsilon^2+1\right)-1\right)+\hat{\omega}\left(z^2\epsilon^2+1\right)^2\right)}{2(q_{-1}+1)(q_{-1}+2)(z-1)\epsilon\left(3z^3\epsilon^4+z^2\left(6\epsilon^2+1\right)+z+1\right)}\\&-\frac{i\left(z^2\epsilon^2+1\right)Y'_{-1}}{2(q_{-1}+1)(q_{-1}+2)\epsilon}\,,\end{aligned} \tag{E.28}$$

$$\begin{aligned}Y_{-1}=&-\frac{Y_{+1}\left((2q_{-1}+5)\epsilon\left(3z^4\epsilon^4-6z^2\epsilon^2+z^3\left(3\epsilon^4+6\epsilon^2+1\right)-1\right)+\hat{\omega}\left(z^2\epsilon^2+1\right)^2\right)}{2(z-1)\epsilon\left(3z^3\epsilon^4+z^2\left(6\epsilon^2+1\right)+z+1\right)}\\&+\frac{i\left(z^2\epsilon^2+1\right)Y'_{+1}}{2\epsilon}\,,\end{aligned} \tag{E.29}$$

where we have used the master equations (6.64). One can see that these relations map the solutions of $Y_{\pm 1}$ with the boundary conditions (E.1) into each other and they imply that the asymptotic behaviour of these functions is always related according to (E.19) — independently of the boundary conditions imposed on the vector A_μ. Therefore, both equations are degenerate and the value of λ_{+1} (or λ_{-1}) cannot be found in this way. In the case of vanishing NUT charge, one can decouple the electromagnetic perturbations in modes of definite parity, which are achieved only for two specific values of λ_{+1} (λ_{-1}).

[1] Interestingly, the operators $\hat{\delta}^*$ and $\hat{\delta}$ act as the ladder operators of the harmonic oscillator, so they raise and lower the Landau level q.

However, NUT charge breaks all reflection symmetries of the background, and therefore we do not have a similar decomposition of the perturbations. Hence, we seem to be unable to determine the polarization parameter $\lambda_{\pm 1}$, which would suggest that the spectrum of QNMs depends continuously on this parameter. Clearly, more research in this direction is needed in order to understand the puzzling properties of electromagnetic perturbations in these geometries.

E.2 Asymptotic form of the metric perturbation

As we have seen, the metric perturbation satisfying Dirichlet boundary conditions can be written near the boundary as

$$h_{ab} = z e^{-i\omega t}\gamma_{ab}(x) + \mathcal{O}(z^3)\,, \tag{E.30}$$

where we are already setting $k = 0$ without loss of generality. The equations of motion allow one to express the component γ_{xx} in terms of the rest as

$$\gamma_{xx} = \left(1 - \frac{4n^2x^2}{L^4}\right)\gamma_{tt} + \frac{4nx}{L^2}\gamma_{ty} - \gamma_{yy}. \tag{E.31}$$

Then, it is convenient to introduce a new matrix σ_{ab} as follows $\gamma_{ab} = e^{-sn\omega x^2/L^2}\sigma_{ab}$. One finds that the equations of motion together with the separability conditions on the NP variables imply that σ_{ab} is given by a finite sum of Hermite polynomials. In the case $s = 1$ it reads

$$\sigma_{tt} = -\frac{10a_{+2}L^2}{3r_+\hat{\omega}\epsilon}H_{q+2}(\hat{x})\left(-(2q+7)\hat{\omega}\epsilon + \hat{\omega}^2 + i\lambda_{+2}\hat{\omega} - 2\epsilon(\epsilon - i\lambda_{+2})\right), \tag{E.32}$$

$$\begin{aligned}\sigma_{tx} = &-\frac{5ia_{+2}L^2}{3\sqrt{2}r_+\left(\hat{\omega}\epsilon^3\right)^{1/2}}\Big[\epsilon H_{q_{-2}+3}\left(\hat{x}\right)\left(-i\lambda_{+2} - \hat{\omega} + \epsilon\right)\\ &+ 2H_{q_{-2}+1}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2} + \left(q_{-2}+1\right)\epsilon\left(\epsilon - i\lambda_{+2}\right) - \left(3q_{-2}+10\right)\hat{\omega}\epsilon + \hat{\omega}^2\right)\Big],\end{aligned} \tag{E.33}$$

$$\begin{aligned}\sigma_{ty} = &\frac{5a_{+2}L^2}{3\sqrt{2}r_+\left(\hat{\omega}\epsilon\right)^{3/2}}\Big[2H_{q_{-2}+1}\left(\hat{x}\right)\left(4\left(q_{-2}+2\right)\epsilon^2\left(\epsilon - i\lambda_{+2}\right) + \hat{\omega}^2\left(\left(-5q_{-2}-14\right)\epsilon + i\lambda_{+2}\right)\right.\\ &\left.+\hat{\omega}\epsilon\left(\left(4q_{-2}^2 + 23q_{-2} + 29\right)\epsilon - i\left(3q_{-2}+5\right)\lambda_{+2}\right) + \hat{\omega}^3\right)\\ &+ \epsilon H_{q_{-2}+3}\left(\hat{x}\right)\left(-i\hat{\omega}\lambda_{+2} + 4\epsilon\left(\epsilon - i\lambda_{+2}\right) + \left(4q_{-2}+13\right)\hat{\omega}\epsilon - \hat{\omega}^2\right)\Big],\end{aligned} \tag{E.34}$$

$$\begin{aligned}\sigma_{xy} = &\frac{5ia_{+2}L^2}{6r_+\hat{\omega}\epsilon^2}\Big[2H_{q_{-2}}\left(\hat{x}\right)\left(-2q_{-2}\left(q_{-2}+2\right)\epsilon^2\left(\epsilon - i\lambda_{+2}\right) + \hat{\omega}^2\left(-2\left(3q_{-2}+8\right)\epsilon + i\lambda_{+2}\right)\right.\\ &\left.+\hat{\omega}\epsilon\left(\left(8q_{-2}^2 + 44q_{-2} + 55\right)\epsilon - i\left(4q_{-2}+7\right)\lambda_{+2}\right) + \hat{\omega}^3\right) - \epsilon^2\left(\epsilon - i\lambda_{+2}\right)H_{q_{-2}+4}\left(\hat{x}\right)\\ &- 2\epsilon H_{q_{-2}+2}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2} + 2\left(q_{-2}+2\right)\epsilon\left(\epsilon - i\lambda_{+2}\right) - \left(4q_{-2}+13\right)\hat{\omega}\epsilon + \hat{\omega}^2\right)\Big],\end{aligned} \tag{E.35}$$

(E.36)

$$
\begin{aligned}
\sigma_{yy} = &-\frac{5a_{+2}L^2}{6r_+\hat{\omega}^2\epsilon^2}\Big[2H_{q_{-2}}(x)\Big(-2\hat{\omega}\epsilon^2\Big(\left(4q_{-2}^3+28q_{-2}^2+54q_{-2}+29\right)\epsilon-i\left(4q_{-2}^2+10q_{-2}+5\right)\lambda_{+2}\Big)\\
&-8\left(q_{-2}^2+3q_{-2}+2\right)\epsilon^3\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^3\left(-2\left(4q_{-2}+9\right)\epsilon+i\lambda_{+2}\right)\\
&+\hat{\omega}^2\epsilon\left(\left(18q_{-2}^2+82q_{-2}+83\right)\epsilon-3i\left(2q_{-2}+3\right)\lambda_{+2}\right)+\hat{\omega}^4\Big)\\
&+\epsilon^2H_{q_{-2}+4}(x)\left(-4\epsilon\left(\epsilon-i\lambda_{+2}\right)-4\left(q_{-2}+3\right)\hat{\omega}\epsilon+\hat{\omega}^2\right)\\
&-2\epsilon H_{q_{-2}+2}(x)\left(\hat{\omega}-2\left(2q_{-2}+5\right)\epsilon\right)\left(i\hat{\omega}\lambda_{+2}-2\epsilon\left(\epsilon-i\lambda_{+2}\right)-\left(2q_{-2}+7\right)\hat{\omega}\epsilon+\hat{\omega}^2\right)\Big]
\end{aligned}
\tag{E.37}
$$

while for $s=-1$ the solution is

$$
\sigma_{tt}=\frac{17a_{+2}L^2H_{q_{+2}+2}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2}+\hat{\omega}\epsilon\left(2q_{+2}+3\right)-2\epsilon\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^2\right)}{6r_+\hat{\omega}\epsilon\left(q_{+2}+1\right)\left(q_{+2}+2\right)},
\tag{E.38}
$$

$$
\begin{aligned}
\sigma_{tx}=&\frac{17ia_{+2}L^2\hat{\omega}}{12\sqrt{2}r_+\left(q_{+2}+1\right)\left(q_{+2}+2\right)\left(q_{+2}+3\right)\left(-\hat{\omega}\epsilon\right)^{3/2}}\Big[2\epsilon\left(q_{+2}^2+5q_{+2}+6\right)\left(i\lambda_{+2}+\hat{\omega}-\epsilon\right)\\
&\times H_{q_{+2}+1}\left(\hat{x}\right)+H_{q_{+2}+3}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2}-\epsilon\left(q_{+2}+4\right)\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}\epsilon\left(3q_{+2}+5\right)+\hat{\omega}^2\right)\Big]
\end{aligned}
\tag{E.39}
$$

$$
\begin{aligned}
\sigma_{ty}=&-\frac{17a_{+2}L^2}{12\sqrt{2}r_+\left(q_{+2}+1\right)\left(q_{+2}+2\right)\left(q_{+2}+3\right)\left(-\hat{\omega}\epsilon\right)^{3/2}}\Big[H_{q_{+2}+3}\left(\hat{x}\right)\Big(-4\epsilon^2\left(q_{+2}+3\right)\left(\epsilon-i\lambda_{+2}\right)\\
&+\hat{\omega}^2\left(\epsilon\left(5q_{+2}+11\right)+i\lambda_{+2}\right)+\hat{\omega}\epsilon\left(\epsilon\left(4q_{+2}^2+17q_{+2}+14\right)+i\lambda_{+2}\left(3q_{+2}+10\right)\right)+\hat{\omega}^3\Big)\\
&+2\epsilon\left(q_{+2}^2+5q_{+2}+6\right)H_{q_{+2}+1}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2}+\hat{\omega}\epsilon\left(4q_{+2}+7\right)-4\epsilon\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^2\right)\Big]
\end{aligned}
\tag{E.40}
$$

$$
\begin{aligned}
\sigma_{xy}=&\frac{17ia_{+2}L^2}{48r_+\hat{\omega}\epsilon^2\left(q_{+2}+1\right)\left(q_{+2}+2\right)\left(q_{+2}+3\right)\left(q_{+2}+4\right)}\Big[H_{q_{+2}+4}\left(\hat{x}\right)\Big(-2\epsilon^2\left(q_{+2}^2+8q_{+2}+15\right)\\
&\times\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^2\left(2\epsilon\left(3q_{+2}+7\right)+i\lambda_{+2}\right)+\hat{\omega}\epsilon\left(\epsilon\left(8q_{+2}^2+36q_{+2}+35\right)+i\lambda_{+2}\left(4q_{+2}+13\right)\right)\\
&+\hat{\omega}^3\Big)-8\epsilon^2\left(q_{+2}^4+10q_{+2}^3+35q_{+2}^2+50q_{+2}+24\right)\left(\epsilon-i\lambda_{+2}\right)H_{q_{+2}}\left(\hat{x}\right)\\
&+4\epsilon\left(q_{+2}^2+7q_{+2}+12\right)H_{q_{+2}+2}\left(\hat{x}\right)\left(i\hat{\omega}\lambda_{+2}-2\epsilon\left(q_{+2}+3\right)\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}\epsilon\left(4q_{+2}+7\right)+\hat{\omega}^2\right)\Big]
\end{aligned}
\tag{E.41}
$$

$$
\begin{aligned}
\sigma_{yy}=&\frac{17a_{+2}L^2}{48r_+\hat{\omega}^2}\Bigg[\Bigg(-\frac{H_{q_{+2}+4}\left(\hat{x}\right)}{\epsilon^2\left(q_{+2}+1\right)\left(q_{+2}+2\right)\left(q_{+2}+3\right)\left(q_{+2}+4\right)}\Big(\\
&2\hat{\omega}\epsilon^2\left(\epsilon\left(4q_{+2}^3+32q_{+2}^2+74q_{+2}+41\right)+i\lambda_{+2}\left(4q_{+2}^2+30q_{+2}+55\right)\right)\\
&-8\epsilon^3\left(q_{+2}^2+7q_{+2}+12\right)\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^3\left(\epsilon\left(8q_{+2}+22\right)+i\lambda_{+2}\right)\\
&+\hat{\omega}^2\epsilon\left(\epsilon\left(18q_{+2}^2+98q_{+2}+123\right)+3i\lambda_{+2}\left(2q_{+2}+7\right)\right)+\hat{\omega}^4\Big)\\
&-\frac{4H_{q_{+2}+2}\left(\hat{x}\right)\left(2\epsilon\left(2q_{+2}+5\right)+\hat{\omega}\right)\left(i\hat{\omega}\lambda_{+2}+\hat{\omega}\epsilon\left(2q_{+2}+3\right)-2\epsilon\left(\epsilon-i\lambda_{+2}\right)+\hat{\omega}^2\right)}{\epsilon\left(q_{+2}+1\right)\left(q_{+2}+2\right)}
\end{aligned}
$$

$$- 8H_{q_{+2}}(\hat{x})\left(4\hat{\omega}\epsilon\left(q_{+2}+2\right) - 4\epsilon\left(\epsilon - i\lambda_{+2}\right) + \hat{\omega}^2\right)\Big)\Big)\Big(\tag{E.42}$$

where in each case $\hat{x} = x\sqrt{\frac{2sn\omega}{L^2}}$.

E.3 Boundary conditions from Hertz's reconstruction map

A priori, it is not clear to which extent the Weyl scalars Ψ_0 and Ψ_4 encode all the information of a metric perturbation. Rather remarkably, though, once solutions for certain decoupled equations (in a specific sense) are known, there is an elegant procedure to reconstruct the whole perturbation. The "master variables" satisfying such equations are referred to as the *Hertz potentials*. This was applied to perturbations of vacuum type-D spaces in [414] and [415]. The results in those references were proven in a more systematic and surprisingly simple form in [106]. In the context of holography, this has proven to be very useful, particularly in the derivation of physical boundary conditions for perturbations in AdS space [109] (see also [375, 376]). In this appendix we rederive our boundary conditions by explicit application of Hertz's reconstruction map.

In our type-D space a complex metric perturbation in a general polarisation state can be written as

$$\begin{aligned} h_{\mu\nu} = &\{-k_\mu k_\nu \bar{\delta}\bar{\delta} - \bar{m}_\mu \bar{m}_\nu (D-\bar{\rho})(D+3\bar{\rho}) + k_{(\mu}\bar{m}_{\nu)}\left[(D-\bar{\rho}+\rho)\bar{\delta} + \bar{\delta}(D+3\bar{\rho})\right]\}\,\bar{\varphi}^{IRG} \\ &+\{-l_\mu l_\nu \delta\delta - m_\mu m_\nu(\Delta - 3\bar{\gamma}+\gamma+\bar{\mu})(\Delta - 4\bar{\gamma} - 3\bar{\mu}) \\ &+ l_{(\mu}m_{\nu)}\left[\delta(\Delta - 4\bar{\gamma} - 3\bar{\mu}) + (\Delta - 3\bar{\gamma} - \gamma + \bar{\mu} - \mu)\delta\right]\}\bar{\varphi}^{ORG} \end{aligned} \tag{E.43}$$

where φ^{IRG} and φ^{ORG} are the Hertz potentials of perturbations in traceless, ingoing ($h^{IRG}_{\mu\nu}k^\mu = 0$) and outgoing ($h^{ORG}_{\mu\nu}l^\mu = 0$) radiation gauge respectively, and satisfy the equations $\mathcal{O}_0^\dagger(\varphi^{IRG}) = 0$ and $\mathcal{O}_4^\dagger(\varphi^{ORG}) = 0$, where $\mathcal{O}_0$ and $\mathcal{O}_4$ are Teukolsky's operators and $\dagger$ denotes the operation of taking the adjoint, as defined in [106]. Following the lines of [415], we have taken $h^{IRG}_{\mu\nu} = 2\,\overline{\left[S_0{}^\dagger_{\mu\nu}\varphi^{IRG}\right]}$ and $h^{ORG}_{\mu\nu} = 2\,\overline{\left[S_4{}^\dagger_{\mu\nu}\varphi^{ORG}\right]}$. Here, S_0 and S_4 are defined by the identities $\mathcal{O}_0 T_0(h) = S_0^{\mu\nu}\mathcal{E}_{\mu\nu}(h)$ and $\mathcal{O}_4 T_4(h) = S_4^{\mu\nu}\mathcal{E}_{\mu\nu}(h)$ where $\mathcal{E}_{\mu\nu}$ is the linearised Einstein equation and T_0 and T_4 the operators that compute Ψ_0 and Ψ_4 out of $h_{\mu\nu}$, respectively (it is now clear, by the property $(AB)^\dagger = B^\dagger A^\dagger$ of composition of adjoints and the self-adjoint property $\mathcal{E}^\dagger_{\mu\nu} = \mathcal{E}_{\mu\nu}$, that a solution φ of $\mathcal{O}_0^\dagger(\varphi) = 0$ generates a solution $h_{\mu\nu} = S_0{}^\dagger_{\mu\nu}\varphi$ of $\mathcal{E}_{\mu\nu}(h) = 0$, and similarly for $\mathcal{O}_4^\dagger$ and $S_4{}^\dagger_{\mu\nu}$). Solutions for φ^{IRG} and φ^{ORG} can be readily obtained by noticing the properties $\mathcal{O}_0^\dagger(\varphi) = \Psi_2^{-4/3}\mathcal{O}_4(\Psi_2^{4/3}\varphi)$ and $\mathcal{O}_4^\dagger(\varphi) = \Psi_2^{-4/3}\mathcal{O}_0(\Psi_2^{4/3}\varphi)$, and take the form

$$\bar{\varphi}^{IRG} = \Delta R^{\omega,q-4}_{(-2)}(r)\mathcal{H}_q(x)e^{-i\omega t}e^{iky} \tag{E.44}$$

$$\bar{\varphi}^{ORG} = \bar{\Psi}_2^{-4/3}\frac{R^{\omega,q}_{(+2)}(r)}{\Delta}\mathcal{H}_{q-4}(x)e^{-i\omega t}e^{iky} \tag{E.45}$$

The radial functions $R^{\omega,q}_{(+2)}$ and $R^{\omega,q-4}_{(-2)}$ are solutions of (6.104) and (6.105), respectively, and we chose them to be related by the Teukolsky–Starobinsky identities (6.108) and (6.109). Also, we recall that these radial functions are related to the $Y_{\pm 2}$ variables according to (6.103). In addition, the angular functions $\mathcal{H}_q(x)$ are the solutions given in (6.45).

With this, it can be verified by direct application of T_0 and T_4 on (E.43) that

$$\Psi_0 = A_{(+2)} \frac{R^{\omega,q}_{(+2)}}{\Delta} \mathcal{H}_q(x) e^{-i\omega t} e^{iky} \tag{E.46}$$

$$\Psi_4 = A_{(-2)} \frac{\Delta R^{\omega,q-4}_{(-2)}}{(r+in)^4} \mathcal{H}_{q-4}(x) e^{-i\omega t} e^{iky} \tag{E.47}$$

where the constants $A_{(\pm 2)}$ are not important for this discussion.

In order to determine the boundary conditions, we perform an asymptotic expansion of $R^{\omega,q}_{(+2)}$ and $R^{\omega,q-4}_{(-2)}$ near infinity, which follows that of the $Y_{\pm 2}$ functions in (6.78) and is determined by the constants $a_{\pm 2}$ and $b_{\pm 2}$. The boundary conditions are most conveniently identified by working in a gauge

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)} \tag{E.48}$$

with $\tilde{h}_{r\mu} = 0$. This can be achieved by expanding the gauge parameter as

$$\xi_\mu = e^{-i\omega t + iky} r^2 \sum_{i=0}^{\infty} r^{-i} \left(f_t^{(i)}(x), f_r^{(i)}(x)/r^2, f_x^{(i)}(x), f_y^{(i)}(x) \right), \tag{E.49}$$

which allows us to cancel as many $1/r^i$ terms in $\tilde{h}_{r\mu}$ as we want by choosing the functions $f_\mu^{(i)}(x)$ appropriately. Then, the resulting metric perturbation $\tilde{h}_{ab}$ typically contains terms that diverge as r^2, which should be removed according to the holographic boundary conditions in (6.80). Some of these can be canceled with additional gauge transformations, but ultimately we find a constraint between the asymptotic expansions of $R^{\omega,q}_{(+2)}$ and $R^{\omega,q-4}_{(-2)}$ at $r \to \infty$, which establishes a relation between the constants (a_{+2}, b_{+2}) and (a_{-2}, b_{-2}). This, in turn, translates into a relation between the ratios of these quantities. λ_{+2} and λ_{-2} as defined in (6.94). On the other hand, the Teukolsky–Starobinsky identities (6.108) and (6.109) provide an additional relation involving λ_{+2} and λ_{-2} — see (6.112). The solutions for $(\lambda_{+2}, \lambda_{-2})$ of this pair of equations are precisely those given in (6.118) and (6.119).

Bibliography

[1] Event Horizon Telescope collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, Astrophys. J. Lett. **875** (2019) L1, [1906.11238].

[2] E. Akiyama, *First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way*, Astrophys. J. Lett. **930** (May, 2022) L12.

[3] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs*, Phys. Rev. X **9** (2019) 031040, [1811.12907].

[4] LIGO Scientific Collaboration and Virgo Collaboration collaboration, A. et.al., *Gw190521: A binary black hole merger with a total mass of* $150\ M_\odot$, Phys. Rev. Lett. **125** (Sep, 2020) 101102.

[5] S. Hawking, *Gravitationally Collapsed Objects of Very Low Mass*, Monthly Notices of the Royal Astronomical Society **152** (04, 1971) 75–78, [https://academic.oup.com/mnras/article-pdf/152/1/75/9360899/mnras152-0075.pdf].

[6] B. J. Carr and S. W. Hawking, *Black holes in the early Universe*, Monthly Notices of the Royal Astronomical Society **168** (Aug., 1974) 399–416.

[7] B. Carr, F. Kuhnel and M. Sandstad, *Primordial Black Holes as Dark Matter*, Phys. Rev. D **94** (2016) 083504, [1607.06077].

[8] M. Maggiore et al., *Science Case for the Einstein Telescope*, JCAP **03** (2020) 050, [1912.02622].

[9] LISA collaboration, K. G. Arun et al., *New Horizons for Fundamental Physics with LISA*, 2205.01597.

[10] LIGO Scientific and Virgo Collaborations collaboration, B. P. e. a. Abbott, *Tests of general relativity with gw150914*, Phys. Rev. Lett. **116** (May, 2016) 221101.

[11] N. Yunes, K. Yagi and F. Pretorius, *Theoretical physics implications of the binary black-hole mergers gw150914 and gw151226*, Phys. Rev. D **94** (Oct, 2016) 084002.

[12] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, Phys. Rev. Lett. **119** (2017) 161101, [1710.05832].

[13] E. Berti, K. Yagi and N. Yunes, *Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (I) Inspiral-Merger*, Gen. Rel. Grav. **50** (2018) 46, [1801.03208].

[14] E. Berti, K. Yagi, H. Yang and N. Yunes, *Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (II) Ringdown*, Gen. Rel. Grav. **50** (2018) 49, [1801.03587].

[15] L. Barack et al., *Black holes, gravitational waves and fundamental physics: a roadmap*, *Class. Quant. Grav.* **36** (2019) 143001, [1806.05195].

[16] R. M. Wald, *General relativity.* Chicago Univ. Press, Chicago, IL, 1984.

[17] S. W. Hawking and G. F. R. Ellis, *The large-scale structure of space-time.* 1973.

[18] S. Chandrasekhar, *The mathematical theory of black holes.* Oxford classic texts in the physical sciences. Oxford Univ. Press, Oxford, 2002.

[19] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation.* W. H. Freeman, San Francisco, 1973.

[20] H. Reall, "Black Holes." http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html.

[21] R. Penrose, *Structure of space–time., pp 121-235 of Battelle Rencontres. /DeWitt, Cecile M. (ed.). New York W. A. Benjamin, Inc. (1968). .*

[22] J. R. Oppenheimer and H. Snyder, *On continued gravitational contraction*, *Phys. Rev.* **56** (Sep, 1939) 455–459.

[23] S. Chandrasekhar, *The maximum mass of ideal white dwarfs*, *Astrophys. J.* **74** (1931) 81–82.

[24] R. C. Tolman, *Static solutions of einstein's field equations for spheres of fluid*, *Phys. Rev.* **55** (Feb, 1939) 364–373.

[25] J. R. Oppenheimer and G. M. Volkoff, *On massive neutron cores*, *Phys. Rev.* **55** (Feb, 1939) 374–381.

[26] E. M. Lifshitz and I. M. Khalatnikov, *Investigations in relativistic cosmology*, *Adv. Phys.* **12** (1963) 185–249.

[27] R. Arnowitt, S. Deser and C. W. Misner, *Dynamical structure and definition of energy in general relativity*, *Phys. Rev.* **116** (Dec, 1959) 1322–1330.

[28] R. L. Arnowitt, S. Deser and C. W. Misner, *The Dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997–2027, [gr-qc/0405109].

[29] R. Schoen and S.-T. Yau, *Positivity of the total mass of a general space-time*, *Phys. Rev. Lett.* **43** (Nov, 1979) 1457–1459.

[30] E. Witten, *A new proof of the positive energy theorem*, *Communications in Mathematical Physics* **80** (1981) 381 – 402.

[31] R. Penrose, *Gravitational collapse and space-time singularities*, *Phys. Rev. Lett.* **14** (Jan, 1965) 57–59.

[32] J. M. M. Senovilla, *Trapped surfaces*, *Int. J. Mod. Phys. D* **20** (2011) 2139, [1107.1344].

[33] M. Kriele, *Spacetime: Foundations of general relativity and differential geometry*, *Spacetime: Foundations of General Relativity and Differential Geometry, by Marcus Kriele. Lecture Notes in Physics, Monographs Series, v.59. Published by Springer-Verlag (Berlin Heidelberg), 1999.* (01, 2001) .

[34] R. Penrose, *Gravitational Collapse: the Role of General Relativity*, *Nuovo Cimento Rivista Serie* **1** (Jan., 1969) 252.

[35] H. Reissner, *Uber die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie*, *Ann. Phys.* **50** (1916) 636–651.

[36] G. Nordstrom*Proc. Kon. Ned. Akad. Wet.* **20** (1918) 1238.

[37] D. Christodoulou and S. Klainerman, *The Global nonlinear stability of the Minkowski space*, .

[38] M. Mars, *Present status of the Penrose inequality*, *Class. Quant. Grav.* **26** (2009) 193001, [0906.5566].

[39] G. Huisken and T. Ilmanen, *The Riemannian Penrose inequality*, *International Mathematics Research Notices* **1997** (01, 1997) 1045–1058, [https://academic.oup.com/imrn/article-pdf/1997/20/1045/2180821/1997-20-1045.pdf].

[40] R. M. Schoen and S.-T. Yau, *The existence of a black hole due to condensation of matter*, *Communications in Mathematical Physics* **90** (1983) 575–579.

[41] D. Christodoulou, *The Formation of Black Holes in General Relativity*, in *12th Marcel Grossmann Meeting on General Relativity*, pp. 24–34, 5, 2008. 0805.3880. DOI.

[42] R. H. Price, *Nonspherical perturbations of relativistic gravitational collapse. i. scalar and gravitational perturbations*, *Phys. Rev. D* **5** (May, 1972) 2419–2438.

[43] R. H. Price, *Nonspherical perturbations of relativistic gravitational collapse. ii. integer-spin, zero-rest-mass fields*, *Phys. Rev. D* **5** (May, 1972) 2439–2454.

[44] S. W. Hawking, *Black holes in general relativity*, *Commun. Math. Phys.* **25** (1972) 152–166.

[45] J. Jebsen.*Ark. Mat. Ast. Fys. (Stockholm)* **15** (1921) nr. 18.

[46] G. D. Birkhoff and R. E. Langer, *Relativity and modern physics*. 1923.

[47] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie*, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin* (Jan., 1916) 189–196.

[48] R. P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, *Phys. Rev. Lett.* **11** (Sep, 1963) 237–238.

[49] R. Penrose, *A Spinor approach to general relativity*, *Annals Phys.* **10** (1960) 171–201.

[50] E. Newman and R. Penrose, *An Approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962) 566–578.

[51] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions of Einstein's field equations*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, 2003, 10.1017/CBO9780511535185.

[52] R. Penrose and W. Rindler, *Spinors and Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 4, 2011, 10.1017/CBO9780511564048.

[53] R. Wald, *On perturbations of a Kerr black hole*, J. Math. Phys. **14** (1973) 1453.

[54] C. B. Owen, N. Yunes and H. Witek, *Petrov type, principal null directions, and Killing tensors of slowly rotating black holes in quadratic gravity*, Phys. Rev. D **103** (2021) 124057, [2103.15891].

[55] W. Kinnersley, *Type D Vacuum Metrics*, J. Math. Phys. **10** (1969) 1195–1203.

[56] B. Carter, *Axisymmetric Black Hole Has Only Two Degrees of Freedom*, Phys. Rev. Lett. **26** (1971) 331–333.

[57] D. C. Robinson, *Uniqueness of the Kerr Black Hole*, Phys. Rev. Lett. **34** (Apr., 1975) 905–906.

[58] R. Ruffini and J. A. Wheeler, *Introducing the black hole*, Phys. Today **24** (1971) 30.

[59] R. Emparan and H. S. Reall, *Black Holes in Higher Dimensions*, Living Rev. Rel. **11** (2008) 6, [0801.3471].

[60] G. T. Horowitz, ed., *Black holes in higher dimensions*. Cambridge Univ. Pr., Cambridge, UK, 2012.

[61] R. C. Myers and M. J. Perry, *Black Holes in Higher Dimensional Space-Times*, Annals Phys. **172** (1986) 304.

[62] R. Emparan and H. S. Reall, *A Rotating black ring solution in five-dimensions*, Phys. Rev. Lett. **88** (2002) 101101, [hep-th/0110260].

[63] R. Emparan, *Rotating circular strings, and infinite nonuniqueness of black rings*, JHEP **03** (2004) 064, [hep-th/0402149].

[64] B. S. Kay and R. M. Wald, *Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon*, Phys. Rept. **207** (1991) 49–136.

[65] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231–252, [hep-th/9711200].

[66] T. W. Baumgarte and S. L. Shapiro, *Binary black hole mergers*, Phys. Today **64N10** (2011) 32–37.

[67] C. V. Vishveshwara, *Scattering of Gravitational Radiation by a Schwarzschild Black-hole*, Nature **227** (1970) 936–938.

[68] W. H. Press, *Long Wave Trains of Gravitational Waves from a Vibrating Black Hole*, *Astrophys. J.* **170** (Dec., 1971) L105.

[69] E. Berti, V. Cardoso and A. O. Starinets, *Quasinormal modes of black holes and black branes*, *Class. Quant. Grav.* **26** (2009) 163001, [0905.2975].

[70] S. Detweiler, *Black holes and gravitational waves. III - The resonant frequencies of rotating holes*, *Astrophys. J.* **239** (July, 1980) 292–295.

[71] O. J. C. Dias, M. Godazgar, J. E. Santos, G. Carullo, W. Del Pozzo and D. Laghi, *Eigenvalue repulsions in the quasinormal spectra of the Kerr-Newman black hole*, *Phys. Rev. D* **105** (2022) 084044, [2109.13949].

[72] O. J. C. Dias, M. Godazgar and J. E. Santos, *Eigenvalue repulsions and quasinormal mode spectra of Kerr-Newman: an extended study*, 2205.13072.

[73] G. Carullo, D. Laghi, N. K. Johnson-McDaniel, W. Del Pozzo, O. J. C. Dias, M. Godazgar et al., *Constraints on Kerr-Newman black holes from merger-ringdown gravitational-wave observations*, *Phys. Rev. D* **105** (2022) 062009, [2109.13961].

[74] M. Giesler, M. Isi, M. A. Scheel and S. Teukolsky, *Black Hole Ringdown: The Importance of Overtones*, *Phys. Rev. X* **9** (2019) 041060, [1903.08284].

[75] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel and S. A. Teukolsky, *Testing the no-hair theorem with GW150914*, *Phys. Rev. Lett.* **123** (2019) 111102, [1905.00869].

[76] M. Isi, W. M. Farr, M. Giesler, M. A. Scheel and S. A. Teukolsky, *Testing the Black-Hole Area Law with GW150914*, *Phys. Rev. Lett.* **127** (2021) 011103, [2012.04486].

[77] E. E. Flanagan and T. Hinderer, *Constraining neutron star tidal Love numbers with gravitational wave detectors*, *Phys. Rev. D* **77** (2008) 021502, [0709.1915].

[78] T. Hinderer, *Tidal Love numbers of neutron stars*, *Astrophys. J.* **677** (2008) 1216–1220, [0711.2420].

[79] T. Damour and A. Nagar, *Relativistic tidal properties of neutron stars*, *Phys. Rev. D* **80** (2009) 084035, [0906.0096].

[80] T. Binnington and E. Poisson, *Relativistic theory of tidal Love numbers*, *Phys. Rev. D* **80** (2009) 084018, [0906.1366].

[81] A. Le Tiec, M. Casals and E. Franzin, *Tidal Love Numbers of Kerr Black Holes*, *Phys. Rev. D* **103** (2021) 084021, [2010.15795].

[82] H. S. Chia, *Tidal deformation and dissipation of rotating black holes*, *Phys. Rev. D* **104** (2021) 024013, [2010.07300].

[83] V. Cardoso, M. Kimura, A. Maselli and L. Senatore, *Black Holes in an Effective Field Theory Extension of General Relativity*, *Phys. Rev. Lett.* **121** (2018) 251105, [1808.08962].

[84] R. Emparan, A. Fernandez-Pique and R. Luna, *Geometric polarization of plasmas and Love numbers of AdS black branes*, *JHEP* **09** (2017) 150, [1707.02777].

[85] B. Kol and M. Smolkin, *Black hole stereotyping: Induced gravito-static polarization*, *JHEP* **02** (2012) 010, [1110.3764].

[86] V. Cardoso, L. Gualtieri and C. J. Moore, *Gravitational waves and higher dimensions: Love numbers and Kaluza-Klein excitations*, *Phys. Rev. D* **100** (2019) 124037, [1910.09557].

[87] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Static response and Love numbers of Schwarzschild black holes*, *JCAP* **04** (2021) 052, [2010.00593].

[88] L. Hui, A. Joyce, R. Penco, L. Santoni and A. R. Solomon, *Ladder symmetries of black holes. Implications for love numbers and no-hair theorems*, *JCAP* **01** (2022) 032, [2105.01069].

[89] V. Cardoso and F. Duque, *Environmental effects in gravitational-wave physics: Tidal deformability of black holes immersed in matter*, *Phys. Rev. D* **101** (2020) 064028, [1912.07616].

[90] M. Bianchi, D. Consoli, A. Grillo, J. F. Morales, P. Pani and G. Raposo, *Distinguishing fuzzballs from black holes through their multipolar structure*, *Phys. Rev. Lett.* **125** (2020) 221601, [2007.01743].

[91] D. Pereñiguez and V. Cardoso, *Love numbers and magnetic susceptibility of charged black holes*, *Phys. Rev. D* **105** (2022) 044026, [2112.08400].

[92] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837–2840, [hep-th/9301052].

[93] V. Cardoso and O. J. C. Dias, *Rayleigh-Plateau and Gregory-Laflamme instabilities of black strings*, *Phys. Rev. Lett.* **96** (2006) 181601, [hep-th/0602017].

[94] R. Emparan and R. C. Myers, *Instability of ultra-spinning black holes*, *JHEP* **09** (2003) 025, [hep-th/0308056].

[95] N. Tanahashi and T. Tanaka, *Black holes in braneworld models*, *Prog. Theor. Phys. Suppl.* **189** (2011) 227–268, [1105.2997].

[96] H. S. Reall, *Classical and thermodynamic stability of black branes*, *Phys. Rev. D* **64** (2001) 044005, [hep-th/0104071].

[97] R. Brito, V. Cardoso and P. Pani, *Superradiance: New Frontiers in Black Hole Physics*, *Lect. Notes Phys.* **906** (2015) pp.1–237, [1501.06570].

[98] T. Regge and J. A. Wheeler, *Stability of a schwarzschild singularity*, *Phys. Rev.* **108** (Nov, 1957) 1063–1069.

[99] F. J. Zerilli, *Effective potential for even-parity regge-wheeler gravitational perturbation equations*, *Phys. Rev. Lett.* **24** (Mar, 1970) 737–738.

[100] F. J. Zerilli, *Gravitational field of a particle falling in a schwarzschild geometry analyzed in tensor harmonics*, *Phys. Rev. D* **2** (Nov, 1970) 2141–2160.

[101] V. Moncrief, *Gravitational perturbations of spherically symmetric systems. I. The exterior problem*, Annals of Physics **88** (Dec., 1974) 323–342.

[102] A. Ishibashi and H. Kodama, *Perturbations and Stability of Static Black Holes in Higher Dimensions*, Prog. Theor. Phys. Suppl. **189** (2011) 165–209, [1103.6148].

[103] H. Kodama, R. A. Konoplya and A. Zhidenko, *Gravitational instability of simply rotating AdS black holes in higher dimensions*, Phys. Rev. D **79** (2009) 044003, [0812.0445].

[104] H. K. Kunduri, J. Lucietti and H. S. Reall, *Gravitational perturbations of higher dimensional rotating black holes: Tensor perturbations*, Phys. Rev. D **74** (2006) 084021, [hep-th/0606076].

[105] S. A. Teukolsky, *Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations*, Astrophys. J. **185** (1973) 635–647.

[106] R. M. Wald, *Construction of solutions of gravitational, electromagnetic, or other perturbation equations from solutions of decoupled equations*, Phys. Rev. Lett. **41** (Jul, 1978) 203–206.

[107] V. Frolov, P. Krtous and D. Kubiznak, *Black holes, hidden symmetries, and complete integrability*, Living Rev. Rel. **20** (2017) 6, [1705.05482].

[108] E. Poisson, *Gravitational radiation from a particle in circular orbit around a black hole. i. analytical results for the nonrotating case*, Phys. Rev. D **47** (Feb, 1993) 1497–1510.

[109] O. J. C. Dias and J. E. Santos, *Boundary Conditions for Kerr-AdS Perturbations*, JHEP **10** (2013) 156, [1302.1580].

[110] S. Chandrasekhar, *The Gravitational Perturbations of the Kerr Black Hole. I. The Perturbations in the Quantities which Vanish in the Stationary State*, Proceedings of the Royal Society of London Series A **358** (Jan., 1978) 421–439.

[111] S. Chandrasekhar, *The Gravitational Perturbations of the Kerr Black Hole. II. The Perturbations in the Quantities which are Finite in the Stationary State*, Proceedings of the Royal Society of London Series A **358** (Jan., 1978) 441–465.

[112] J. M. Cohen and L. S. Kegeles, *Electromagnetic fields in curved spaces: A constructive procedure*, Phys. Rev. D **10** (Aug, 1974) 1070–1084.

[113] P. L. Chrzanowski, *Vector potential and metric perturbations of a rotating black hole*, Phys. Rev. D **11** (Apr., 1975) 2042–2062.

[114] G. 't Hooft and M. J. G. Veltman, *One loop divergencies in the theory of gravitation*, *Ann. Inst. H. Poincare Phys. Theor. A* **20** (1974) 69–94.

[115] S. Deser and P. van Nieuwenhuizen, *One-loop divergences of quantized einstein-maxwell fields*, Phys. Rev. D **10** (Jul, 1974) 401–410.

[116] S. Deser, H.-S. Tsao and P. van Nieuwenhuizen, *One-loop divergences of the einstein-yang-mills system*, Phys. Rev. D **10** (Nov, 1974) 3337–3342.

[117] S. W. Hawking and J. B. Hartle, *Energy and angular momentum flow into a black hole*, *Commun. Math. Phys.* **27** (1972) 283–290.

[118] J. M. Bardeen, B. Carter and S. W. Hawking, *The Four laws of black hole mechanics*, *Commun. Math. Phys.* **31** (1973) 161–170.

[119] S. W. Hawking, *Gravitational radiation from colliding black holes*, *Phys. Rev. Lett.* **26** (May, 1971) 1344–1346.

[120] D. Sudarsky and R. M. Wald, *Extrema of mass, stationarity, and staticity, and solutions to the einstein-yang-mills equations*, *Phys. Rev. D* **46** (Aug, 1992) 1453–1474.

[121] W. Israel, *Third law of black-hole dynamics: A formulation and proof*, *Phys. Rev. Lett.* **57** (Jul, 1986) 397–399.

[122] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7** (Apr, 1973) 2333–2346.

[123] S. W. Hawking, *Black hole explosions*, *Nature* **248** (1974) 30–31.

[124] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[125] J. D. Bekenstein, *Generalized second law of thermodynamics in black-hole physics*, *Phys. Rev. D* **9** (Jun, 1974) 3292–3300.

[126] A. C. Wall, *Proof of the generalized second law for rapidly changing fields and arbitrary horizon slices*, *Phys. Rev. D* **85** (May, 2012) 104049.

[127] L. Susskind, *Some speculations about black hole entropy in string theory*, `hep-th/9309145`.

[128] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99–104, [`hep-th/9601029`].

[129] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063, [`1905.08762`].

[130] G. Penington, *Entanglement Wedge Reconstruction and the Information Paradox*, *JHEP* **09** (2020) 002, [`1905.08255`].

[131] A. Almheiri, R. Mahajan, J. Maldacena and Y. Zhao, *The Page curve of Hawking radiation from semiclassical geometry*, *JHEP* **03** (2020) 149, [`1908.10996`].

[132] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *Replica Wormholes and the Entropy of Hawking Radiation*, *JHEP* **05** (2020) 013, [`1911.12333`].

[133] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, *Replica wormholes and the black hole interior*, *JHEP* **03** (2022) 205, [`1911.11977`].

[134] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *The entropy of Hawking radiation*, *Rev. Mod. Phys.* **93** (2021) 035002, [`2006.06872`].

[135] J. Maldacena, *Black holes and quantum information*, Nature Rev. Phys. **2** (2020) 123–125.

[136] G. Barnich and F. Brandt, *Covariant theory of asymptotic symmetries, conservation laws and central charges*, Nucl. Phys. B **633** (2002) 3–82, [hep-th/0111246].

[137] G. Compère and A. Fiorucci, *Advanced Lectures on General Relativity*, 1801.07064.

[138] K. Prabhu, *The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom*, Class. Quant. Grav. **34** (2017) 035011, [1511.00388].

[139] R. M. Wald, *Black hole entropy is the Noether charge*, Phys. Rev. D **48** (1993) R3427–R3431, [gr-qc/9307038].

[140] V. Iyer and R. M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D **50** (1994) 846–864, [gr-qc/9403028].

[141] T. Ortín, *Komar integrals for theories of higher order in the Riemann curvature and black-hole chemistry*, JHEP **08** (2021) 023, [2104.10717].

[142] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism. 1. General theorems*, Commun. Math. Phys. **174** (1995) 57–92, [hep-th/9405109].

[143] T. Padmanabhan, *Some aspects of field equations in generalised theories of gravity*, Phys. Rev. D **84** (2011) 124041, [1109.3846].

[144] I. Racz and R. M. Wald, *Global extensions of space-times describing asymptotic final states of black holes*, Class. Quant. Grav. **13** (1996) 539–553, [gr-qc/9507055].

[145] T. Jacobson, G. Kang and R. C. Myers, *On black hole entropy*, Phys. Rev. D **49** (1994) 6587–6598, [gr-qc/9312023].

[146] P. A. Cano and A. Murcia, *Electromagnetic Quasitopological Gravities*, JHEP **10** (2020) 125, [2007.04331].

[147] K. Copsey and G. T. Horowitz, *The Role of dipole charges in black hole thermodynamics*, Phys. Rev. D **73** (2006) 024015, [hep-th/0505278].

[148] D. Marolf, *Chern-Simons terms and the three notions of charge*, in *International Conference on Quantization, Gauge Theory, and Strings: Conference Dedicated to the Memory of Professor Efim Fradkin*, pp. 312–320, 6, 2000. hep-th/0006117.

[149] G. W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, Phys. Rev. Lett. **77** (1996) 4992–4995, [hep-th/9607108].

[150] T. Jacobson and A. Mohd, *Black hole entropy and Lorentz-diffeomorphism Noether charge*, Phys. Rev. D **92** (2015) 124010, [1507.01054].

[151] T. Ortin, *A Note on Lie-Lorentz derivatives*, *Class. Quant. Grav.* **19** (2002) L143–L150, [hep-th/0206159].

[152] T. Ortin, *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2nd ed. ed., 7, 2015, 10.1017/CBO9781139019750.

[153] L. Fatibene and M. Francaviglia, *General theory of lie derivatives for lorentz tensors*, 2009. 10.48550/ARXIV.0904.0258.

[154] A. Lichnerowicz, *Spineurs harmoniques*, *C. R. Acad. Sci. Paris* **257** (1963) 7–9.

[155] Y. Kosmann, *Dérivées de Lie des spineurs*, *C. R. Acad. Sci. Paris Sér. A* **262** (1966) A289–A292.

[156] Y. Kosmann, *Dérivées de Lie des spineurs*, *Annali Mat. Pura Appl.* **(IV) 91** (1972) 317–395.

[157] D. J. Hurley and M. A. Vandyck, *On the concepts of Lie and covariant derivatives of spinors. Part 1*, *J. Phys. A* **27** (1994) 4569–4580.

[158] M. A. J. Vandyck, *ON THE PROBLEM OF SPACE-TIME SYMMETRIES IN THE THEORY OF SUPERGRAVITY*, *Gen. Rel. Grav.* **20** (1988) 261–277.

[159] M. A. Vandyck, *ON THE PROBLEM OF SPACE-TIME SYMMETRIES IN THE THEORY OF SUPERGRAVITY. 2: N=2 SUPERGRAVITY AND SPINORIAL LIE DERIVATIVES*, *Gen. Rel. Grav.* **20** (1988) 905–925.

[160] J. M. Figueroa-O'Farrill, *On the supersymmetries of Anti-de Sitter vacua*, *Class. Quant. Grav.* **16** (1999) 2043–2055, [hep-th/9902066].

[161] G. Compere, *Note on the First Law with p-form potentials*, *Phys. Rev. D* **75** (2007) 124020, [hep-th/0703004].

[162] Z. Elgood, P. Meessen and T. Ortín, *The first law of black hole mechanics in the Einstein-Maxwell theory revisited*, *JHEP* **09** (2020) 026, [2006.02792].

[163] T. Ortín and D. Pereñiguez, *Magnetic Charges and Wald Entropy (To appear)*, .

[164] D. Kastor, *Komar Integrals in Higher (and Lower) Derivative Gravity*, *Class. Quant. Grav.* **25** (2008) 175007, [0804.1832].

[165] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, *Class. Quant. Grav.* **26** (2009) 195011, [0904.2765].

[166] R. B. Mann, *Black Holes: Thermodynamics, Information, and Firewalls*. SpringerBriefs in Physics. Springer, 2015, 10.1007/978-3-319-14496-2.

[167] D. Kubiznak, R. B. Mann and M. Teo, *Black hole chemistry: thermodynamics with Lambda*, *Class. Quant. Grav.* **34** (2017) 063001, [1608.06147].

[168] P. Meessen, D. Mitsios and T. Ortín, *Black hole chemistry, the cosmological constant and the embedding tensor*, 2203.13588.

[169] A. C. Wall, *A Second Law for Higher Curvature Gravity*, *Int. J. Mod. Phys. D* **24** (2015) 1544014, [1504.08040].

[170] X. Dong, *Holographic Entanglement Entropy for General Higher Derivative Gravity*, *JHEP* **01** (2014) 044, [1310.5713].

[171] A. C. Wall, *A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices*, *Phys. Rev. D* **85** (2012) 104049, [1105.3445].

[172] S. Sarkar and A. C. Wall, *Generalized second law at linear order for actions that are functions of Lovelock densities*, *Phys. Rev. D* **88** (2013) 044017, [1306.1623].

[173] M. B. Green and J. H. Schwarz, *Covariant Description of Superstrings*, *Phys. Lett. B* **136** (1984) 367–370.

[174] M. B. Green, J. H. Schwarz and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION*. Cambridge Monographs on Mathematical Physics. 7, 1988.

[175] M. B. Green, J. H. Schwarz and E. Witten, *SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY*. 7, 1988.

[176] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007, 10.1017/CBO9780511618123.

[177] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys. Lett. B* **103** (1981) 207–210.

[178] L. Brink, P. Di Vecchia and P. S. Howe, *A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String*, *Phys. Lett. B* **65** (1976) 471–474.

[179] S. Deser and B. Zumino, *A Complete Action for the Spinning String*, *Phys. Lett. B* **65** (1976) 369–373.

[180] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions*, *Nucl. Phys. B* **31** (1971) 86–112.

[181] P. Ramond, *Dual Theory for Free Fermions*, *Phys. Rev. D* **3** (1971) 2415–2418.

[182] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[183] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, *Strings in Background Fields*, *Nucl. Phys. B* **262** (1985) 593–609.

[184] A. Sen, *The Heterotic String in Arbitrary Background Field*, *Phys. Rev. D* **32** (1985) 2102.

[185] C. G. Callan, Jr., I. R. Klebanov and M. J. Perry, *String Theory Effective Actions*, *Nucl. Phys. B* **278** (1986) 78–90.

[186] M. T. Grisaru, A. E. M. van de Ven and D. Zanon, *Four Loop beta Function for the N=1 and N=2 Supersymmetric Nonlinear Sigma Model in Two-Dimensions*, *Phys. Lett. B* **173** (1986) 423–428.

[187] M. T. Grisaru and D. Zanon, *σ Model Superstring Corrections to the Einstein-hilbert Action*, *Phys. Lett. B* **177** (1986) 347–351.

[188] M. B. Green and J. H. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, *Phys. Lett. B* **149** (1984) 117–122.

[189] E. A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, *Nucl. Phys.* **B328** (1989) 439–468.

[190] J. L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, *Nucl. Phys. B* **270** (1986) 186–204.

[191] G. Barnich and G. Compere, *Conserved charges and thermodynamics of the spinning Godel black hole*, *Phys. Rev. Lett.* **95** (2005) 031302, [hep-th/0501102].

[192] E. Frodden and D. Hidalgo, *Surface Charges for Gravity and Electromagnetism in the First Order Formalism*, *Class. Quant. Grav.* **35** (2018) 035002, [1703.10120].

[193] E. Frodden and D. Hidalgo, *Surface Charges Toolkit for Gravity*, *Int. J. Mod. Phys. D* **29** (2020) 2050040, [1911.07264].

[194] P. B. Aneesh, S. Chakraborty, S. J. Hoque and A. Virmani, *First law of black hole mechanics with fermions*, *Class. Quant. Grav.* **37** (2020) 205014, [2004.10215].

[195] S. Gao, *The First law of black hole mechanics in Einstein-Maxwell and Einstein-Yang-Mills theories*, *Phys. Rev. D* **68** (2003) 044016, [gr-qc/0304094].

[196] A. Salam and E. Sezgin, eds., *Supergravities in Diverse Dimensions: Commentary and Reprints (In 2 Volumes)*. World Scientific, Singapore, 1989, 10.1142/0277.

[197] Y. Tachikawa, *Black hole entropy in the presence of Chern-Simons terms*, *Class. Quant. Grav.* **24** (2007) 737–744, [hep-th/0611141].

[198] Z. Elgood and T. Ortin, *T duality and Wald entropy formula in the Heterotic Superstring effective action at first order in α'*, 2005.11272.

[199] T. Ortin, *O(n,n) invariance and Wald entropy formula in the Heterotic Superstring effective action at first order in alpha'*, 2005.14618.

[200] T. Azeyanagi, R. Loganayagam, G. S. Ng and M. J. Rodriguez, *Covariant Noether Charge for Higher Dimensional Chern-Simons Terms*, *JHEP* **05** (2015) 041, [1407.6364].

[201] I. A. Bandos and T. Ortin, *On the dualization of scalars into $(d-2)$-forms in supergravity. Momentum maps, R-symmetry and gauged supergravity*, *JHEP* **08** (2016) 135, [1605.05559].

[202] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-simons Terms in Ten-dimensions*, *Phys. Lett. B* **218** (1989) 210–215.

[203] T. Regge and C. Teitelboim, *Role of Surface Integrals in the Hamiltonian Formulation of General Relativity*, *Annals Phys.* **88** (1974) 286.

[204] L. F. Abbott and S. Deser, *Stability of Gravity with a Cosmological Constant*, *Nucl. Phys.* **B195** (1982) 76–96.

[205] G. Barnich, *Boundary charges in gauge theories: Using Stokes theorem in the bulk*, *Class. Quant. Grav.* **20** (2003) 3685–3698, [hep-th/0301039].

[206] D. Astefanesei and E. Radu, *Quasilocal formalism and black-ring thermodynamics*, *Phys. Rev. D* **73** (Feb, 2006) 044014.

[207] M. Rogatko, *Black rings and the physical process version of the first law of thermodynamics*, *Phys. Rev. D* **72** (2005) 074008, [hep-th/0509150].

[208] M. Rogatko, *First law of black ring thermodynamics in higher dimensional dilaton gravity with $p+1$ strength forms*, *Phys. Rev. D* **73** (Jan, 2006) 024022.

[209] Z. Elgood, T. Ortín and D. Pereñíguez, *The first law and Wald entropy formula of heterotic stringy black holes at first order in α'*, *JHEP* **05** (2021) 110, [2012.14892].

[210] P. A. Cano, S. Chimento, R. Linares, T. Ortín and P. F. Ramírez, *α' corrections of Reissner-NordstrÃ¶m black holes*, *JHEP* **02** (2020) 031, [1910.14324].

[211] R. R. Khuri and T. Ortin, *A Nonsupersymmetric dyonic extreme Reissner-Nordstrom black hole*, *Phys. Lett. B* **373** (1996) 56–60, [hep-th/9512178].

[212] F. Faedo and P. F. Ramirez, *Exact charges from heterotic black holes*, *JHEP* **10** (2019) 033, [1906.12287].

[213] H. Elvang, R. Emparan and P. Figueras, *Non-supersymmetric black rings as thermally excited supertubes*, *JHEP* **02** (2005) 031, [hep-th/0412130].

[214] J. Lee and R. M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725–743.

[215] J. M. Maldacena, *Black holes in string theory*. PhD thesis, Princeton U., 1996. hep-th/9607235.

[216] D. Astefanesei, R. Ballesteros, D. Choque and R. Rojas, *Scalar charges and the first law of black hole thermodynamics*, *Phys. Lett. B* **782** (2018) 47–54, [1803.11317].

[217] K. Hajian and M. M. Sheikh-Jabbari, *Solution Phase Space and Conserved Charges: A General Formulation for Charges Associated with Exact Symmetries*, *Phys. Rev.* **D93** (2016) 044074, [1512.05584].

[218] Z. Elgood, D. Mitsios, T. Ortín and D. Pereñíguez, *The first law of heterotic stringy black hole mechanics at zeroth order in α'*, *JHEP* **07** (2021) 007, [2012.13323].

[219] H. Nicolai and P. K. Townsend, *N=3 Supersymmetry Multiplets with Vanishing Trace Anomaly: Building Blocks of the N>3 Supergravities*, *Phys. Lett. B* **98** (1981) 257–260.

[220] P. A. Cano, P. Meessen, T. Ortín and P. F. Ramírez, *α'-corrected black holes in String Theory*, *JHEP* **05** (2018) 110, [1803.01919].

[221] P. A. Cano, S. Chimento, P. Meessen, T. Ortín, P. F. Ramírez and A. Ruipérez, *Beyond the near-horizon limit: Stringy corrections to Heterotic Black Holes*, *JHEP* **02** (2019) 192, [1808.03651].

[222] A. Fontanella and T. Ortín, *On the supersymmetric solutions of the Heterotic Superstring effective action*, *JHEP* **06** (2020) 106, [1910.08496].

[223] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, 10.7208/chicago/9780226870373.001.0001.

[224] J. D. Edelstein, K. Sfetsos, J. A. Sierra-Garcia and A. Vilar López, *T-duality equivalences beyond string theory*, *JHEP* **05** (2019) 082, [1903.05554].

[225] P. A. Cano, T. Ortín, A. Ruipérez and M. Zatti, *Non-supersymmetric black holes with α' corrections*, *JHEP* **03** (2022) 103, [2111.15579].

[226] T. Ortín, A. Ruipérez and M. Zatti, *Extremal stringy black holes in equilibrium at first order in* α', 2112.12764.

[227] P. Breitenlohner, D. Maison and G. W. Gibbons, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, *Commun. Math. Phys.* **120** (1988) 295.

[228] L. Smarr, *Mass formula for Kerr black holes*, *Phys. Rev. Lett.* **30** (1973) 71–73.

[229] D. Kastor, S. Ray and J. Traschen, *Smarr Formula and an Extended First Law for Lovelock Gravity*, *Class. Quant. Grav.* **27** (2010) 235014, [1005.5053].

[230] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys. B* **500** (1997) 75–93, [hep-th/9702103].

[231] A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008, [hep-th/0508042].

[232] A. Komar, *Covariant conservation laws in general relativity*, *Phys. Rev.* **113** (1959) 934–936.

[233] S. Liberati and C. Pacilio, *Smarr Formula for Lovelock Black Holes: a Lagrangian approach*, *Phys. Rev.* **D93** (2016) 084044, [1511.05446].

[234] E. Cremmer, J. Scherk and S. Ferrara, *SU(4) Invariant Supergravity Theory*, *Phys. Lett. B* **74** (1978) 61–64.

[235] R. Kallosh and T. Ortin, *Charge quantization of axion - dilaton black holes*, *Phys. Rev. D* **48** (1993) 742–747, [hep-th/9302109].

[236] P. Meessen and T. Ortin, *The Supersymmetric configurations of N=2, D=4 supergravity coupled to vector supermultiplets*, *Nucl. Phys. B* **749** (2006) 291–324, [hep-th/0603099].

[237] P. Meessen, T. Ortin, J. Perz and C. S. Shahbazi, *H-FGK formalism for black-hole solutions of N=2, d=4 and d=5 supergravity*, *Phys. Lett. B* **709** (2012) 260–265, [1112.3332].

[238] P. Galli, T. Ortin, J. Perz and C. S. Shahbazi, *Non-extremal black holes of N=2, d=4 supergravity*, *JHEP* **07** (2011) 041, [1105.3311].

[239] E. Lozano-Tellechea and T. Ortin, *The General, duality invariant family of nonBPS black hole solutions of N=4, D = 4 supergravity*, *Nucl. Phys. B* **569** (2000) 435–450, [hep-th/9910020].

[240] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet and A. Van Proeyen, *Supersymmetry as a cosmic censor*, *Phys. Rev. D* **46** (1992) 5278–5302, [hep-th/9205027].

[241] T. Ortin, *Electric - magnetic duality and supersymmetry in stringy black holes*, *Phys. Rev. D* **47** (1993) 3136–3143, [hep-th/9208078].

[242] R. Kallosh, D. Kastor, T. Ortin and T. Torma, *Supersymmetry and stationary solutions in dilaton axion gravity*, *Phys. Rev. D* **50** (1994) 6374–6384, [hep-th/9406059].

[243] E. Bergshoeff, R. Kallosh and T. Ortin, *Stationary axion / dilaton solutions and supersymmetry*, *Nucl. Phys. B* **478** (1996) 156–180, [hep-th/9605059].

[244] A. D. Shapere, S. Trivedi and F. Wilczek, *Dual dilaton dyons*, *Mod. Phys. Lett. A* **6** (1991) 2677–2686.

[245] G. W. Gibbons, *Antigravitating Black Hole Solitons with Scalar Hair in N=4 Supergravity*, *Nucl. Phys. B* **207** (1982) 337–349.

[246] G. W. Gibbons and K.-i. Maeda, *Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields*, *Nucl. Phys. B* **298** (1988) 741–775.

[247] D. Garfinkle, G. T. Horowitz and A. Strominger, *Charged black holes in string theory*, *Phys. Rev. D* **43** (1991) 3140.

[248] K. P. Tod, *More on supercovariantly constant spinors*, *Class. Quant. Grav.* **12** (1995) 1801–1820.

[249] M. Rogatko, *Stationary axisymmetric axion - dilaton black holes: Mass formulae*, *Class. Quant. Grav.* **11** (1994) 689–693.

[250] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, *Nucl. Phys. B* **193** (1981) 221–244.

[251] S. Deser and P. van Nieuwenhuizen, *One Loop Divergences of Quantized Einstein-Maxwell Fields*, *Phys. Rev.* **D10** (1974) 401.

[252] S. Deser, H.-S. Tsao and P. van Nieuwenhuizen, *One Loop Divergences of the Einstein Yang-Mills System*, *Phys. Rev.* **D10** (1974) 3337.

[253] D. J. Gross and J. H. Sloan, *The Quartic Effective Action for the Heterotic String*, *Nucl. Phys.* **B291** (1987) 41–89.

[254] D. J. Gross and E. Witten, *Superstring Modifications of Einstein's Equations*, *Nucl. Phys.* **B277** (1986) 1.

[255] M. B. Green and C. Stahn, *D3-branes on the Coulomb branch and instantons*, *JHEP* **09** (2003) 052, [hep-th/0308061].

[256] S. Frolov, I. R. Klebanov and A. A. Tseytlin, *String corrections to the holographic RG flow of supersymmetric SU(N) x SU(N + M) gauge theory*, *Nucl. Phys.* **B620** (2002) 84–108, [hep-th/0108106].

[257] R. C. Myers and J. Z. Simon, *Black Hole Thermodynamics in Lovelock Gravity*, *Phys. Rev.* **D38** (1988) 2434–2444.

[258] R.-G. Cai, *Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev.* **D65** (2002) 084014, [hep-th/0109133].

[259] D. Lovelock, *Divergence-free tensorial concomitants*, *aequationes mathematicae* **4** (1970) 127–138.

[260] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498–501.

[261] J. T. Wheeler, *Symmetric Solutions to the Gauss-Bonnet Extended Einstein Equations*, *Nucl. Phys.* **B268** (1986) 737–746.

[262] J. T. Wheeler, *Symmetric Solutions to the Maximally Gauss-Bonnet Extended Einstein Equations*, *Nucl. Phys.* **B273** (1986) 732.

[263] D. G. Boulware and S. Deser, *String Generated Gravity Models*, *Phys. Rev. Lett.* **55** (1985) 2656.

[264] M. H. Dehghani and R. Pourhasan, *Thermodynamic instability of black holes of third order Lovelock gravity*, *Phys. Rev.* **D79** (2009) 064015, [0903.4260].

[265] J. de Boer, M. Kulaxizi and A. Parnachev, *Holographic Lovelock Gravities and Black Holes*, *JHEP* **06** (2010) 008, [0912.1877].

[266] X. O. Camanho and J. D. Edelstein, *A Lovelock black hole bestiary*, *Class. Quant. Grav.* **30** (2013) 035009, [1103.3669].

[267] R. C. Myers and B. Robinson, *Black Holes in Quasi-topological Gravity*, *JHEP* **08** (2010) 067, [1003.5357].

[268] J. Oliva and S. Ray, *A new cubic theory of gravity in five dimensions: Black hole, Birkhoff's theorem and C-function*, *Class. Quant. Grav.* **27** (2010) 225002, [1003.4773].

[269] M. H. Dehghani, A. Bazrafshan, R. B. Mann, M. R. Mehdizadeh, M. Ghanaatian and M. H. Vahidinia, *Black Holes in Quartic Quasitopological Gravity*, *Phys. Rev.* **D85** (2012) 104009, [1109.4708].

[270] A. Cisterna, L. Guajardo, M. Hassaine and J. Oliva, *Quintic quasi-topological gravity*, *JHEP* **04** (2017) 066, [1702.04676].

[271] A. de la Cruz-Dombriz, A. Dobado and A. L. Maroto, *Black Holes in f(R) theories*, *Phys. Rev.* **D80** (2009) 124011, [0907.3872].

[272] Y.-Z. Li, H.-S. Liu and H. Lu, *Quasi-Topological Ricci Polynomial Gravities*, *JHEP* **02** (2018) 166, [1708.07198].

[273] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, *Spherically Symmetric Solutions in Higher-Derivative Gravity*, *Phys. Rev.* **D92** (2015) 124019, [1508.00010].

[274] H. Lu, A. Perkins, C. N. Pope and K. S. Stelle, *Black Holes in Higher-Derivative Gravity*, *Phys. Rev. Lett.* **114** (2015) 171601, [1502.01028].

[275] R. A. Hennigar, D. Kubiznak and R. B. Mann, *Generalized quasitopological gravity*, *Phys. Rev.* **D95** (2017) 104042, [1703.01631].

[276] P. Bueno and P. A. Cano, *On black holes in higher-derivative gravities*, *Class. Quant. Grav.* **34** (2017) 175008, [1703.04625].

[277] J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, *Quintessential Quartic Quasi-topological Quartet*, *JHEP* **05** (2017) 134, [1703.11007].

[278] P. Bueno, P. A. Cano and R. A. Hennigar, *(Generalized) quasi-topological gravities at all orders*, 1909.07983.

[279] P. Bueno, P. A. Cano, J. Moreno and A. Murcia, *All higher-curvature gravities as Generalized quasi-topological gravities*, 1906.00987.

[280] P. Bueno and P. A. Cano, *Universal black hole stability in four dimensions*, *Phys. Rev.* **D96** (2017) 024034, [1704.02967].

[281] M. Mir, R. A. Hennigar, J. Ahmed and R. B. Mann, *Black hole chemistry and holography in generalized quasi-topological gravity*, *JHEP* **08** (2019) 068, [1902.02005].

[282] M. Mir and R. B. Mann, *On generalized quasi-topological cubic-quartic gravity: thermodynamics and holography*, *JHEP* **07** (2019) 012, [1902.10906].

[283] P. Bueno and P. A. Cano, *Einsteinian cubic gravity*, *Phys. Rev.* **D94** (2016) 104005, [1607.06463].

[284] B. Kleihaus, J. Kunz, S. Mojica and E. Radu, *Spinning black holes in Einstein-Gauss-Bonnet-dilaton theory: Nonperturbative solutions*, *Phys. Rev.* **D93** (2016) 044047, [1511.05513].

[285] T. Delsate, C. Herdeiro and E. Radu, *Non-perturbative spinning black holes in dynamical Chern–Simons gravity*, *Phys. Lett.* **B787** (2018) 8–15, [1806.06700].

[286] A. Anabalon, N. Deruelle, Y. Morisawa, J. Oliva, M. Sasaki, D. Tempo et al., *Kerr-Schild ansatz in Einstein-Gauss-Bonnet gravity: An exact vacuum solution in five dimensions*, *Class. Quant. Grav.* **26** (2009) 065002, [0812.3194].

[287] H.-C. Kim and R.-G. Cai, *Slowly Rotating Charged Gauss-Bonnet Black holes in AdS Spaces*, *Phys. Rev.* **D77** (2008) 024045, [0711.0885].

[288] R. Yue, D. Zou, T. Yu, P. Li and Z. Yang, *Slowly rotating charged black holes in anti-de Sitter third order Lovelock gravity*, *Gen. Rel. Grav.* **43** (2011) 2103–2114, [1011.5293].

[289] C. Adair, P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann (in preparation), *Slowly-rotating black holes in Einsteinian cubic gravity*, .

[290] B. Chen and L. C. Stein, *Deformation of extremal black holes from stringy interactions*, *Phys. Rev.* **D97** (2018) 084012, [1802.02159].

[291] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis and E. Winstanley, *Dilatonic black holes in higher curvature string gravity*, *Phys. Rev.* **D54** (1996) 5049–5058, [hep-th/9511071].

[292] T. Torii, H. Yajima and K.-i. Maeda, *Dilatonic black holes with Gauss-Bonnet term*, *Phys. Rev.* **D55** (1997) 739–753, [gr-qc/9606034].

[293] S. O. Alexeev and M. V. Pomazanov, *Black hole solutions with dilatonic hair in higher curvature gravity*, *Phys. Rev.* **D55** (1997) 2110–2118, [hep-th/9605106].

[294] S. Alexander and N. Yunes, *Chern-Simons Modified General Relativity*, *Phys. Rept.* **480** (2009) 1–55, [0907.2562].

[295] R. A. Hennigar and R. B. Mann, *Black holes in Einsteinian cubic gravity*, *Phys. Rev.* **D95** (2017) 064055, [1610.06675].

[296] P. Bueno and P. A. Cano, *Four-dimensional black holes in Einsteinian cubic gravity*, *Phys. Rev.* **D94** (2016) 124051, [1610.08019].

[297] R. A. Hennigar, M. B. J. Poshteh and R. B. Mann, *Shadows, Signals, and Stability in Einsteinian Cubic Gravity*, *Phys. Rev.* **D97** (2018) 064041, [1801.03223].

[298] M. B. J. Poshteh and R. B. Mann, *Gravitational Lensing by Black Holes in Einsteinian Cubic Gravity*, *Phys. Rev.* **D99** (2019) 024035, [1810.10657].

[299] A. Dey, P. Roy and T. Sarkar, *On holographic Rényi entropy in some modified theories of gravity*, *JHEP* **04** (2018) 098, [1609.02290].

[300] P. Bueno, P. A. Cano and A. Ruiperez, *Holographic studies of Einsteinian cubic gravity*, *JHEP* **03** (2018) 150, [1802.00018].

[301] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *Universality of Squashed-Sphere Partition Functions*, *Phys. Rev. Lett.* **122** (2019) 071602, [1808.02052].

[302] G. Arciniega, J. D. Edelstein and L. G. Jaime, *Towards purely geometric inflation and late time acceleration*, 1810.08166.

[303] A. Cisterna, N. Grandi and J. Oliva, *On four-dimensional Einsteinian gravity, quasitopological gravity, cosmology and black holes*, 1811.06523.

[304] G. Arciniega, P. Bueno, P. A. Cano, J. D. Edelstein, R. A. Hennigar and L. G. Jaime, *Geometric Inflation*, 1812.11187.

[305] X.-H. Feng, H. Huang, Z.-F. Mai and H. Lu, *Bounce Universe and Black Holes from Critical Einsteinian Cubic Gravity*, *Phys. Rev.* **D96** (2017) 104034, [1707.06308].

[306] P. Bueno, P. A. Cano, R. A. Hennigar and R. B. Mann, *NUTs and bolts beyond Lovelock*, *JHEP* **10** (2018) 095, [1808.01671].

[307] M. R. Mehdizadeh and A. H. Ziaie, *Traversable wormholes in Einsteinian cubic gravity*, 1903.10907.

[308] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058, [hep-th/0606244].

[309] A. M. Ghezelbash, R. B. Mann and R. D. Sorkin, *The Disjointed thermodynamics of rotating black holes with a NUT twist*, *Nucl. Phys.* **B775** (2007) 95–119, [hep-th/0703030].

[310] J. M. Bardeen and G. T. Horowitz, *The Extreme Kerr throat geometry: A Vacuum analog of AdS(2) x S**2*, *Phys. Rev.* **D60** (1999) 104030, [hep-th/9905099].

[311] S. Aretakis, *Horizon Instability of Extremal Black Holes*, *Adv. Theor. Math. Phys.* **19** (2015) 507–530, [1206.6598].

[312] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200].

[313] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[314] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[315] D. T. Son and A. O. Starinets, *Viscosity, Black Holes, and Quantum Field Theory*, *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95–118, [0704.0240].

[316] S. S. Gubser and A. Karch, *From gauge-string duality to strong interactions: A Pedestrian's Guide*, *Ann. Rev. Nucl. Part. Sci.* **59** (2009) 145–168, [0901.0935].

[317] S. A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, *Class. Quant. Grav.* **26** (2009) 224002, [0903.3246].

[318] C. P. Herzog, *Lectures on Holographic Superfluidity and Superconductivity*, *J. Phys. A* **42** (2009) 343001, [0904.1975].

[319] R. Konoplya and A. Zhidenko, *Quasinormal modes of black holes: From astrophysics to string theory*, *Rev. Mod. Phys.* **83** (2011) 793–836, [1102.4014].

[320] D. Birmingham, I. Sachs and S. N. Solodukhin, *Conformal field theory interpretation of black hole quasinormal modes*, *Phys. Rev. Lett.* **88** (2002) 151301, [hep-th/0112055].

[321] D. T. Son and A. O. Starinets, *Minkowski space correlators in AdS / CFT correspondence: Recipe and applications*, *JHEP* **09** (2002) 042, [hep-th/0205051].

[322] A. O. Starinets, *Quasinormal modes of near extremal black branes*, *Phys. Rev. D* **66** (2002) 124013, [hep-th/0207133].

[323] G. Policastro, D. T. Son and A. O. Starinets, *From AdS / CFT correspondence to hydrodynamics*, JHEP **09** (2002) 043, [hep-th/0205052].

[324] C. P. Herzog, *The Hydrodynamics of M theory*, JHEP **12** (2002) 026, [hep-th/0210126].

[325] A. Nunez and A. O. Starinets, *AdS / CFT correspondence, quasinormal modes, and thermal correlators in N=4 SYM*, Phys. Rev. D **67** (2003) 124013, [hep-th/0302026].

[326] P. K. Kovtun and A. O. Starinets, *Quasinormal modes and holography*, Phys. Rev. D **72** (2005) 086009, [hep-th/0506184].

[327] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance, and holography*, JHEP **04** (2008) 100, [0712.2451].

[328] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP **10** (2008) 091, [0806.1218].

[329] G. T. Horowitz and V. E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, Phys. Rev. D **62** (2000) 024027, [hep-th/9909056].

[330] V. Cardoso and J. P. Lemos, *Quasinormal modes of Schwarzschild anti-de Sitter black holes: Electromagnetic and gravitational perturbations*, Phys. Rev. D **64** (2001) 084017, [gr-qc/0105103].

[331] V. Cardoso, R. Konoplya and J. P. Lemos, *Quasinormal frequencies of Schwarzschild black holes in anti-de Sitter space-times: A Complete study on the asymptotic behavior*, Phys. Rev. D **68** (2003) 044024, [gr-qc/0305037].

[332] S. Musiri and G. Siopsis, *Asymptotic form of quasinormal modes of large AdS black holes*, Phys. Lett. B **576** (2003) 309–313, [hep-th/0308196].

[333] V. Cardoso and J. P. Lemos, *Quasinormal modes of toroidal, cylindrical and planar black holes in anti-de Sitter space-times*, Class. Quant. Grav. **18** (2001) 5257–5267, [gr-qc/0107098].

[334] A. S. Miranda and V. T. Zanchin, *Quasinormal modes of plane-symmetric anti-de Sitter black holes: A Complete analysis of the gravitational perturbations*, Phys. Rev. D **73** (2006) 064034, [gr-qc/0510066].

[335] A. S. Miranda, J. Morgan and V. T. Zanchin, *Quasinormal modes of plane-symmetric black holes according to the AdS/CFT correspondence*, JHEP **11** (2008) 030, [0809.0297].

[336] M. Giammatteo and I. G. Moss, *Gravitational quasinormal modes for Kerr anti-de Sitter black holes*, Class. Quant. Grav. **22** (2005) 1803–1824, [gr-qc/0502046].

[337] A. Taub, *Empty space-times admitting a three parameter group of motions*, Annals Math. **53** (1951) 472–490.

[338] E. Newman, L. Tamburino and T. Unti, *Empty space generalization of the Schwarzschild metric*, *J. Math. Phys.* **4** (1963) 915.

[339] S. W. Hawking, C. J. Hunter and D. N. Page, *Nut charge, anti-de Sitter space and entropy*, *Phys. Rev.* **D59** (1999) 044033, [hep-th/9809035].

[340] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, *Large N phases, gravitational instantons and the nuts and bolts of AdS holography*, *Phys. Rev.* **D59** (1999) 064010, [hep-th/9808177].

[341] Y. Imamura and D. Yokoyama, *N=2 supersymmetric theories on squashed three-sphere*, *Phys. Rev.* **D85** (2012) 025015, [1109.4734].

[342] D. Martelli, A. Passias and J. Sparks, *The supersymmetric NUTs and bolts of holography*, *Nucl. Phys. B* **876** (2013) 810–870, [1212.4618].

[343] N. Bobev, T. Hertog and Y. Vreys, *The NUTs and Bolts of Squashed Holography*, *JHEP* **11** (2016) 140, [1610.01497].

[344] N. Bobev, P. Bueno and Y. Vreys, *Comments on Squashed-sphere Partition Functions*, *JHEP* **07** (2017) 093, [1705.00292].

[345] P. Bueno, P. A. Cano, R. A. Hennigar, V. A. Penas and A. Ruipérez, *Partition functions on slightly squashed spheres and flux parameters*, *JHEP* **04** (2020) 123, [2001.10020].

[346] C. W. Misner, *The Flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space*, *J. Math. Phys.* **4** (1963) 924–938.

[347] V. Manko and E. Ruiz, *Physical interpretation of NUT solution*, *Class. Quant. Grav.* **22** (2005) 3555–3560, [gr-qc/0505001].

[348] D. Astefanesei, R. B. Mann and E. Radu, *Breakdown of the entropy/area relationship for NUT-charged spacetimes*, *Phys. Lett.* **B620** (2005) 1–8, [hep-th/0406050].

[349] G. Clément, D. Gal'tsov and M. Guenouche, *Rehabilitating space-times with NUTs*, *Phys. Lett. B* **750** (2015) 591–594, [1508.07622].

[350] G. Clément, D. Gal'tsov and M. Guenouche, *NUT wormholes*, *Phys. Rev. D* **93** (2016) 024048, [1509.07854].

[351] R. A. Hennigar, D. Kubizňák and R. B. Mann, *Thermodynamics of Lorentzian Taub-NUT spacetimes*, *Phys. Rev. D* **100** (2019) 064055, [1903.08668].

[352] A. B. Bordo, F. Gray, R. A. Hennigar and D. Kubizňák, *Misner Gravitational Charges and Variable String Strengths*, *Class. Quant. Grav.* **36** (2019) 194001, [1905.03785].

[353] A. Ballon Bordo, F. Gray, R. A. Hennigar and D. Kubizňák, *The First Law for Rotating NUTs*, *Phys. Lett. B* **798** (2019) 134972, [1905.06350].

[354] L. Ciambelli, C. Corral, J. Figueroa, G. Giribet and R. Olea, *Topological Terms and the Misner String Entropy*, *Phys. Rev. D* **103** (2021) 024052, [2011.11044].

[355] R. G. Leigh, A. C. Petkou and P. Petropoulos, *Holographic Three-Dimensional Fluids with Nontrivial Vorticity*, *Phys. Rev. D* **85** (2012) 086010, [1108.1393].

[356] R. G. Leigh, A. C. Petkou and P. Petropoulos, *Holographic Fluids with Vorticity and Analogue Gravity*, *JHEP* **11** (2012) 121, [1205.6140].

[357] G. Kalamakis, R. G. Leigh and A. C. Petkou, *Aspects of Holography of Taub-NUT-AdS4*, 2009.08022.

[358] D. Astefanesei, R. B. Mann and E. Radu, *Nut charged space-times and closed timelike curves on the boundary*, *JHEP* **01** (2005) 049, [hep-th/0407110].

[359] D. Brecher, P. DeBoer, D. Page and M. Rozali, *Closed time - like curves and holography in compact plane waves*, *JHEP* **10** (2003) 031, [hep-th/0306190].

[360] O. L. Petersen, *Wave equations with initial data on compact cauchy horizons*, *arXiv: Analysis of PDEs* (2018) .

[361] O. L. Petersen, *Extension of Killing vector fields beyond compact Cauchy horizons*, *Adv. Math.* **391** (2021) 107953, [1903.09135].

[362] K. Godel, *An Example of a new type of cosmological solutions of Einstein's field equations of graviation*, *Rev. Mod. Phys.* **21** (1949) 447–450.

[363] M. M. Som and A. K. Raychaudhuri, *Cylindrically symmetric charged dust distributions in rigid rotation in general relativity*, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **304** (1968) 81–86.

[364] D. Leahy, *SCALAR AND NEUTRINO FIELDS IN THE GODEL UNIVERSE. (TALK)*, *Int. J. Theor. Phys.* **21** (1982) 703–753.

[365] M. Novello, N. Svaiter and M. Guimaraes, *Synchronized frames for Godel's universe*, *Gen. Rel. Grav.* **25** (1993) 137–164.

[366] E. Radu and D. Astefanesei, *Quantum effects in a rotating space-time*, *Int. J. Mod. Phys. D* **11** (2002) 715–732, [gr-qc/0112029].

[367] M. Kontsevich and G. Segal, *Wick Rotation and the Positivity of Energy in Quantum Field Theory*, *Quart. J. Math. Oxford Ser.* **72** (2021) 673–699, [2105.10161].

[368] E. Witten, *A Note On Complex Spacetime Metrics*, 2111.06514.

[369] M. Visser, *Feynman's $i\epsilon$ prescription,almost real spacetimes, and acceptable complex spacetimes*, 2111.14016.

[370] R. Young, *SEMICLASSICAL STABILITY OF ASYMPTOTICALLY LOCALLY FLAT SPACES*, *Phys. Rev. D* **28** (1983) 2420–2435.

[371] C. Warnick, *Semi-classical stability of AdS NUT instantons*, *Class. Quant. Grav.* **23** (2006) 3801–3817, [hep-th/0602127].

[372] G. Holzegel, *A Note on the instability of Lorentzian Taub-NUT-space*, *Class. Quant. Grav.* **23** (2006) 3951–3962, [gr-qc/0602045].

[373] P. Krtouš, V. P. Frolov and D. Kubizňák, *Separation of Maxwell equations in Kerr–NUT–(A)dS spacetimes*, *Nucl. Phys. B* **934** (2018) 7–38, [1803.02485].

[374] S. Chandrasekhar, *The mathematical theory of black holes*. 1985.

[375] O. J. C. Dias, H. S. Reall and J. E. Santos, *Kerr-CFT and gravitational perturbations*, *JHEP* **08** (2009) 101, [0906.2380].

[376] V. Cardoso, O. J. C. Dias, G. S. Hartnett, L. Lehner and J. E. Santos, *Holographic thermalization, quasinormal modes and superradiance in Kerr-AdS*, *JHEP* **04** (2014) 183, [1312.5323].

[377] S. A. Teukolsky and W. H. Press, *Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnet ic radiation*, *Astrophys. J.* **193** (1974) 443–461.

[378] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [1602.03837].

[379] LIGO Scientific, Virgo collaboration, R. Abbott et al., *GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run*, 2010.14527.

[380] GRAVITY collaboration, R. Abuter et al., *Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole*, *Astron. Astrophys.* **636** (2020) L5, [2004.07187].

[381] V. Cardoso and P. Pani, *Testing the nature of dark compact objects: a status report*, *Living Rev. Rel.* **22** (2019) 4, [1904.05363].

[382] G. Bertone and T. Tait, M. P., *A new era in the search for dark matter*, *Nature* **562** (2018) 51–56, [1810.01668].

[383] N. Bar, K. Blum, T. Lacroix and P. Panci, *Looking for ultralight dark matter near supermassive black holes*, *JCAP* **07** (2019) 045, [1905.11745].

[384] E. Berti, A. Sesana, E. Barausse, V. Cardoso and K. Belczynski, *Spectroscopy of Kerr black holes with Earth- and space-based interferometers*, *Phys. Rev. Lett.* **117** (2016) 101102, [1605.09286].

[385] P. A. Seoane et al., *The Effect of Mission Duration on LISA Science Objectives*, 2107.09665.

[386] P. T. Chrusciel, J. Lopes Costa and M. Heusler, *Stationary Black Holes: Uniqueness and Beyond*, *Living Rev. Rel.* **15** (2012) 7, [1205.6112].

[387] V. Cardoso and L. Gualtieri, *Testing the black hole 'no-hair' hypothesis*, *Class. Quant. Grav.* **33** (2016) 174001, [1607.03133].

[388] E. Barausse, V. Cardoso and P. Pani, *Can environmental effects spoil precision gravitational-wave astrophysics?*, *Phys. Rev. D* **89** (2014) 104059, [1404.7149].

[389] V. Cardoso, K. Destounis, F. Duque, R. P. Macedo and A. Maselli, *(F)hairy tails: black holes in galaxies*, 2109.00005.

[390] V. Cardoso and P. Pani, *Tests for the existence of black holes through gravitational wave echoes*, *Nature Astron.* **1** (2017) 586–591, [1709.01525].

[391] C. Murray and S. Dermott, *Solar System Dynamics.* Cambridge University Press, Cambridge, UK, 2000.

[392] E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic.* Cambridge University Press, Cambridge, UK, 2014.

[393] N. Gürlebeck, *No-hair theorem for Black Holes in Astrophysical Environments*, *Phys. Rev. Lett.* **114** (2015) 151102, [1503.03240].

[394] A. Le Tiec and M. Casals, *Spinning Black Holes Fall in Love*, *Phys. Rev. Lett.* **126** (2021) 131102, [2007.00214].

[395] R. A. Porto, *The effective field theorist's approach to gravitational dynamics*, *Phys. Rept.* **633** (2016) 1–104, [1601.04914].

[396] R. A. Porto, *The Tune of Love and the Nature(ness) of Spacetime*, *Fortsch. Phys.* **64** (2016) 723–729, [1606.08895].

[397] I. Rothstein*Analytic Calculations of Gravitational Wave Signatures During Early Stages of Inspirals* (talk presented at the Simons Center meeting "GW161212: The Universe through gravitational waves," 2017) .

[398] M. Dine and W. Fischler, *The Not So Harmless Axion*, *Phys. Lett. B* **120** (1983) 137–141.

[399] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper and J. March-Russell, *String Axiverse*, *Phys. Rev. D* **81** (2010) 123530, [0905.4720].

[400] D. J. E. Marsh, *Axion Cosmology*, *Phys. Rept.* **643** (2016) 1–79, [1510.07633].

[401] V. Cardoso, E. Franzin, A. Maselli, P. Pani and G. Raposo, *Testing strong-field gravity with tidal Love numbers*, *Phys. Rev. D* **95** (2017) 084014, [1701.01116].

[402] S. S. Seahra, C. Clarkson and R. Maartens, *Detecting extra dimensions with gravity wave spectroscopy: the black string brane-world*, *Phys. Rev. Lett.* **94** (2005) 121302, [gr-qc/0408032].

[403] H. Kodama and A. Ishibashi, *Master equations for perturbations of generalized static black holes with charge in higher dimensions*, *Prog. Theor. Phys.* **111** (2004) 29–73, [hep-th/0308128].

[404] F. Tangherlini, *Schwarzschild field in n dimensions and the dimensionality of space problem*, *Nuovo Cim.* **27** (1963) 636–651.

[405] G. Kristensson, *Second Order Differential Equations.* Springer New York, 2010.

[406] "*NIST Digital Library of Mathematical Functions.*" http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15.

[407] H. Fang and G. Lovelace, *Tidal coupling of a Schwarzschild black hole and circularly orbiting moon*, *Phys. Rev. D* **72** (2005) 124016, [gr-qc/0505156].

[408] S. E. Gralla, *On the Ambiguity in Relativistic Tidal Deformability*, *Class. Quant. Grav.* **35** (2018) 085002, [1710.11096].

[409] H. Bateman and A. Erdélyi, *Higher transcendental functions*. California Institute of technology. Bateman Manuscript project. McGraw-Hill, New York, NY, 1955.

[410] G. Bonelli, C. Iossa, D. P. Lichtig and A. Tanzini, *Exact solution of Kerr black hole perturbations via CFT_2 and instanton counting. Greybody factor, Quasinormal modes and Love numbers*, 2105.04483.

[411] G. Bonelli, C. Iossa, D. P. Lichtig and A. Tanzini, *Irregular Liouville correlators and connection formulae for Heun functions*, 2201.04491.

[412] P. A. Cano and D. Pereñiguez, *Quasinormal modes of NUT-charged black branes in the AdS/CFT correspondence*, 2101.10652.

[413] R. A. Hennigar, D. Kubizňák and R. B. Mann, *Entropy Inequality Violations from Ultraspinning Black Holes*, *Phys. Rev. Lett.* **115** (2015) 031101, [1411.4309].

[414] L. S. Kegeles and J. M. Cohen, *Constructive procedure for perturbations of spacetimes*, *Phys. Rev. D* **19** (Mar, 1979) 1641–1664.

[415] P. L. Chrzanowski, *Vector potential and metric perturbations of a rotating black hole*, *Phys. Rev. D* **11** (Apr, 1975) 2042–2062.

[416] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, *All supersymmetric solutions of minimal supergravity in five- dimensions*, *Class. Quant. Grav.* **20** (2003) 4587–4634, [hep-th/0209114].

[417] J. P. Gauntlett and J. B. Gutowski, *General concentric black rings*, *Phys. Rev. D* **71** (2005) 045002, [hep-th/0408122].

[418] J. B. Gutowski and W. Sabra, *General supersymmetric solutions of five-dimensional supergravity*, *JHEP* **10** (2005) 039, [hep-th/0505185].

[419] J. Bellorin and T. Ortin, *Characterization of all the supersymmetric solutions of gauged N=1, d=5 supergravity*, *JHEP* **08** (2007) 096, [0705.2567].

[420] J. Bellorin, *Supersymmetric solutions of gauged five-dimensional supergravity with general matter couplings*, *Class. Quant. Grav.* **26** (2009) 195012, [0810.0527].

[421] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *$N = 2$ supergravity in five-dimensions revisited*, *Class. Quant. Grav.* **21** (2004) 3015–3042, [hep-th/0403045].

[422] J. Bellorin, P. Meessen and T. Ortin, *All the supersymmetric solutions of N=1,d=5 ungauged supergravity*, *JHEP* **01** (2007) 020, [hep-th/0610196].