The measurement of profit, profitability, cost and revenue efficiency through data envelopment analysis: A comparison of models using BenchmarkingEconomicEfficiency.jl☆

Javier Barbero [a,b], José L. Zofío [a,c,*]

[a] *Department of Economics, Universidad Autónoma de Madrid, Spain*
[b] *Oviedo Efficiency Group, Universidad de Oviedo, Spain*
[c] *Erasmus Research Institute of Management, Erasmus University, Rotterdam, Netherlands*

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ABSTRACT

We undertake a systematic comparison of existing models measuring and decomposing the economic efficiency of organizations. For this purpose we introduce the package **BenchmarkingEconomicEfficiency.jl** for the open-source Julia language including a set of functions to be used by scholars and professionals working in the fields of economics, management science, engineering, and operations research. Using mathematical programming methods known as Data Envelopment Analysis, the software develops code to decompose economic efficiency considering alternative definitions: profit, profitability, cost and revenue. Economic efficiency can be decomposed, multiplicative or additively, into a technical (productive) efficiency term and a residual term representing allocative (or price) efficiency. We include traditional decompositions like the radial efficiency measures associated with the input (cost) and output (revenue) approaches, as well as new ones corresponding to the Russell measures, the directional distance function, DDF (including novel extensions like the reverse DDF, modified DDF, or generalizations based on Hölder norms), the generalized distance function, and additive measures like the slack based measure, their weighted variants, etc. Moreover, regardless the underlying economic efficiency model, many of these technical inefficiency measures are available for calculation in a computer software for the first time. This article details the theoretical methods and the empirical implementation of the functions, comparing the obtained results using a common dataset on Taiwanese Banks.

1. Introduction

The measurement and decomposition of the economic performance of organizations (firms, branches, departments, etc.), is receiving increasing attention from theoretical and applied scholars interested in identifying the technical and allocative causes underlying suboptimal market behavior, i.e., why organizations fail to achieve an economic goal. For the individual firm, and given market prices for inputs and outputs, economic efficiency analysis compares its observed profit, profitability, cost or revenue, with the optimal benchmark within the industry or corporation, e.g., maximum profit. Efficiency analysis, pioneered by Farrell [1], is a helpful analytical tool. It represents a systematic method of comparing your performance to that of your rivals, and so contributes with meaningful quantity and price indicators that can supplement common financial measurements such as,

for example, return-on-assets (ROA). It may also be used to create awareness of internal and external processes as monitoring tool. Internally, it provides objective information on the relative performance of individual units that allow a better allocation of incentives (e.g., among individual retail locations within a chain store, specific branches of a bank, etc.). Externally, it assists in identifying areas where businesses fall behind their rivals, contributing to the decision-making process aimed at improving their position in the marketplace (e.g., failure to introduce new processes, favor more lucrative products markets, etc.).

Profit is defined as observed revenue minus observed cost, whereas profit inefficiency is defined as the gap between maximal profit and observed profit. Alternatively, one can define multiplicatively the concept of profitability as revenue divided by cost. Then, the ratio of observed

profitability to maximum profitability is defined as profitability efficiency. One can also analyze economic performance from the partial cost and revenue perspectives. In this respect, cost efficiency can be multiplicatively defined as the ratio of minimum cost to observed cost, or additively defined as observed cost less minimum cost. Similarly, revenue efficiency can be expressed as a ratio of observed to maximum revenue, or as maximum revenue minus observed revenue. Considering these economic definitions, it follows that economic efficiency measures find their maximum at one, while additive economic *in*efficiency measures find their minimum at zero. At these upper and lower values the firm is economically efficient. In the multiplicative case, the larger the efficiency score, the greater the efficiency level. Contrarily, in the additive approach, the larger the score the greater the inefficiency, thus the difference in name.

Economic theory shows that, based on the duality between a supporting economic function like maximum profit, and the production technology, represented by a technical inefficiency measure, TI, profit efficiency is additively decomposed into the efficiency score plus a (residual) factor capturing allocative inefficiency, AI; i.e. $EI = TI + AI$. Technical inefficiency measures the profit loss that the organization experiences by not using the production technology at its full potential, thereby failing to reach the production frontier. Once the organization solves its technical inefficiency, any extra profit loss is due to suboptimal outputs' supply or inputs' demand at their market prices, corresponding to allocative efficiency definition—[2,3]. Alternatively, for a multiplicative definition of economic inefficiency, such as profitability efficiency, it can be consistently disaggregated into technical inefficiency times a (once again) residual factor capturing allocative inefficiency: $EE = TE \times AE$, as shown by Zofío and Prieto [4]. While it is only possible to decompose profit inefficiency additively and profitability efficiency multiplicatively, cost and revenue (in)efficiency may be decomposed both ways. **BenchmarkingEconomicEfficiency.jl** solves all these models, additive or multiplicative, recalling the most popular measures of technical (in)efficiency.

Given data on a set of observed firms, it is possible to measure and decompose economic efficiency empirically by resorting to mathematical programming techniques known as Data Envelopment Analysis (DEA). DEA approximates the technology by identifying the supporting hyperplanes (or facets) that make the best-practice frontier. On the one hand, this allows calculating a wide range of efficiency measures; i.e., the first component of the economic efficiency decomposition. On the other hand, we can also identify the reference economic benchmarks maximizing profit, profitability or revenue, and minimizing cost, which are necessary to calculate the different measures of economic efficiency. Because DEA identifies economical and technological reference benchmarks for each firm, it offers managers real-world peers that serve as role models to improve performance.

As we show in this article, the observational orientation and non-parametric nature of DEA have facilitated its application in many studies of economic efficiency across different sectors—for an authoritative introduction including a review of applications see [5]. The increasing use of DEA has resulted in numerous monographs introducing distinct approaches to economic efficiency analysis; among others [6–8], and [3].

DEA methods to measure economic efficiency can be found in standard software packages like Stata, [9]—which includes user-written commands by Lee [10], and LIMDEP, [11]. There are also dedicated commercial software by Emrouznejad and Cabanda [12]; non-commercial software accompanying academic handbooks—[13,14]; stand-alone packages such as [15,16] (programmed in R, [17], and MATLAB, [18], respectively)[1]; free-ware programs—[19]; and tutorials

for spreadsheets: [20,21]. Recently, several web-based applications have been created using the Shiny package of R, where users can upload their data and solve DEA models interactively: [22,23]. Finally [24] carry out an extensive survey of the existing software, while [25] present and compare eleven relevant packages, each designed with a different purpose, and discuss their pros and cons.

A general drawback of the previous contributions is that they just implement the classical models related to the basic decomposition of cost and revenue efficiency using the multiplicative approach *à la* Farrell, i.e., using the radial input and output measures introduced by Charnes et al. [26]. When calculating profit inefficiency, only the approach based on the directional distance function is available, while newest proposals like those decomposing profitability inefficiency have never been included in any of the above software. Therefore, while these software include the basic DEA economic efficiency analysis, a comprehensive package that implements the most recent and state-of-the-art models is missing.

Moreover, none of the previous proposals have been made available in the open-source Julia language [27]. Julia is a high-level programming language, with features well suited for numerical analysis and computational science. To solve the DEA mathematical problems, our toolbox makes use of the **JuMP.jl** package for linear and nonlinear optimization [28]. **JuMP.jl** can be combined with numerous solvers (open-source and commercial) like Ipopt, GLPK, Gurobi, CPLEX, among others.

BenchmarkingEconomicEfficiency.jl implements the measurement of economic efficiency from all perspectives: profit, profitability, cost and revenue. All these measures of economic efficiency can be solved relying on a large set of technical efficiency models. Following the literature, we start with the classic multiplicative decompositions of cost and revenue efficiency following [1,29]. For the additive approach we consider: (i) the Russell measures–[30], (ii) the weighted additive measure [31], (iii) the slack based measure [32]—equivalent to the enhanced Russell graph measure previously proposed by Pastor et al. [33], (iv) the directional distance function, DDF, [34], along with derivatives like the modified DDF [35], the reverse DDF [36], or the generalization represented by the efficiency measures corresponding to Hölder norms [37], and, finally, (v) the new generalized direct approach that does not rely on duality theory, see [38]. Regarding the decomposition of profit inefficiency we consider all these possibilities, while we rely on the generalized distance function to decompose profitability efficiency [4].

BenchmarkingEconomicEfficiency.jl presents a comprehensive collection of baseline functions that cover all models described in the literature for measuring and decomposing economic efficiency. It is freely accessible under the MIT License and may be downloaded via the Julia package management. All supplemental information (source code, data and examples) for replicating all the results is available at https://benchmarkingeconomicefficiency.com, including a series of Jupyter notebooks that ease the implementation of the models and learning process.

As for the structure of this article, we present in the following section how the quantity and price data is organized in Julia and provide a brief description of the empirical data used to illustrate the package. We then present in Section 3 the additive decomposition of profit inefficiency considering the previous technical inefficiency measures. Section 4 is devoted to profitability efficiency, which is decomposed multiplicatively. Afterwards, because measuring and decomposing cost and revenue *in*efficiencies represent particular cases of the profit analysis when considering either the input (cost) or output (revenue) dimensions, we briefly sketch in Section 5 their multiplicative and additive decomposition. In all sections we illustrate the different models using a common dataset of Taiwanese banks, while providing a summary of the empirical results in Section 6. Conclusions are drawn in Section 7.

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[1] Although the functions written by Álvarez et al. [16] are for MATLAB, they can be easily adapted for the open source language Octave, whose syntax is largely compatible.

2. Data structures

To measure the economic performance of organizations relying on data envelopment analysis we require information on a set of $j = 1, \ldots, J$ firms as well as on input and output market prices. A firm produces the vector of $n = 1, \ldots, N$ output quantities $\mathbf{y}_j \in \mathbb{R}_{++}^N$ using the vector of $m = 1, \ldots, M$ input quantities $\mathbf{x}_j \in \mathbb{R}_{++}^M$ according to the following technology characterized by variable returns to scale: $T = \{(\mathbf{x}, \mathbf{y}) \, | \mathbf{x} \geq X\lambda, \ \mathbf{y} \leqslant Y\lambda, \ \mathbf{e}\lambda = 1 \ \lambda \geq \mathbf{0}\}$. Here $X \in \mathbb{R}^{N \times J}$ and $Y \in \mathbb{R}^{M \times J}$ are the observed matrices of input and output quantities for all firms, $\lambda = \left(\lambda_1, \ldots, \lambda_J\right)^\top$ is a semipositive vector, and $\mathbf{e} = (1, \ldots, 1)^T$ is a row vector of dimension J with all elements equal to one. The characterization of the production technology under constant returns to scale, denoted by T_{CRS}, drops the condition that $\mathbf{e}\lambda = \sum_{j=1}^{J} \lambda_j = 1$, see [39]. Bringing together quantities and prices results in the following data structure: $(\boldsymbol{x}_j, \boldsymbol{w}, \boldsymbol{y}_j, \boldsymbol{p})$.

2.1. Installing the package

BenchmarkingEconomicEfficiency.jl is available through the Julia package manager and can be installed using these commands:

```
julia> using Pkg
julia> Pkg.add("BenchmarkingEconomicEfficiency")
```

Additional packages that will be used in the examples — **DataFrames.jl** and **CSV.jl** — can be installed following the same procedure:

```
julia> Pkg.add("DataFrames")
julia> Pkg.add("CSV")
```

Then, we can load the three packages with:

```
julia> using BenchmarkingEconomicEfficiency
julia> using DataFrames
julia> using CSV
```

2.2. Dataset and statistical sources

We illustrate the economic efficiency models using data of 31 Taiwanese banks constructed by Juo et al. [40].[2] These authors study profit change and decompose it through the so-called Profit-Luenberger indicator. Balk [41] qualifies their methodology by showing that their Profit-Luenberger indicator is equivalent to a Bennet quantity indicator, while [42] propose an improved ('complete') decomposition of profit change that does not include residual price terms as in [40] model. A discussion of the technology, the statistical sources and variables specification can be found in their article. The characterization of the production technology corresponds to the so-called intermediation approach, see [43].[3] According with this approach, banks use labor and capital to collect deposits from savers and produce loans and other earning assets for borrowers. There are three inputs: financial funds (x_1), labor (x_2), and physical capital (x_3). Outputs are financial investments (y_1) and loans (y_2). Table 1 reports the main statistics for quantities and prices. The same data set has been used by Balk and Zofío [45] to illustrate symmetric decompositions of cost variation and by Balk [46] to decompose total factor productivity growth.

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[2] We thank the authors for providing the data.

[3] See [44] for a recent study modeling the intermediation approach through a multi-stage production process focused on non-performing loans.

2.3. Importing the data in julia

Example data is provided in the accompanying file `DataBanks.csv`. The CSV file can be imported in Julia into a data frame with the following code:

```
julia> df = DataFrame(CSV.File('DataBanks.csv'))
```

Let us explore the contents of the data frame:

```
julia> describe(df)
```

```
11x7 DataFrame
 Row | variable  mean       min          median    max         nmissing  eltype
     | Symbol    Union...   Any          Union...  Any         Int64     DataType
-----|--------------------------------------------------------------------------
   1 | Name                 Bank SinoPac           Union Bank         0  String31
   2 | X1        7.95536e5  25019        428995.0  3171493            0  Int64
   3 | X2        3826.19    201.999      3146.0    9537.98            0  Float64
   4 | X3        13393.2    505          8721.0    76576              0  Int64
   5 | W1        0.00639582 0.00249864   0.0061253 0.0185859          0  Float64
   6 | W2        1.25863    0.716977     1.21676   2.2963             0  Float64
   7 | W3        0.317114   0.0729602    0.301688  0.762522           0  Float64
   8 | Y1        1.96808e5  1681         157870.0  904580             0  Int64
   9 | Y2        6.09489e5  66947        328574.0  2091100            0  Int64
  10 | P1        0.0349242  0.00589346   0.0146744 0.304364           0  Float64
  11 | P2        0.021117   0.0125006    0.0189938 0.0686822          0  Float64
```

By rows, the first variable, `Name`, contains the name of the banks, whereas the rest of the variables contain data for inputs, outputs, and prices.

Data are handled as regular Julia vectors and matrices, representing the inputs of the different functions that we described below. We can extract the data from the data frame to regular Julia matrices running the following code:

```
julia> X = [df.X1 df.X2 df.X3];
julia> W = [df.W1 df.W2 df.W3];
julia> Y = [df.Y1 df.Y2];
julia> P = [df.P1 df.P2];
julia> banks = df.Name;
```

3. Measuring and decomposing profit inefficiency

Profit inefficiency aims at analyzing if firms are capable of maximizing profit. Consequently, when firms maximize the difference between revenue and cost by supplying and demanding the optimal quantities of outputs and inputs at their market prices, they are profit efficient. The profit function is given by $\Pi(\mathbf{w}, \mathbf{p}) = \max_{\mathbf{x}, \mathbf{y}} \left\{ \mathbf{p} \cdot \mathbf{y} - \mathbf{w} \cdot \mathbf{x} \, | \mathbf{x} \geq X\lambda, \ \mathbf{y} \leqslant Y\lambda, \ \mathbf{e}\lambda = 1, \lambda \geq \mathbf{0} \right\}, \mathbf{w} \in \mathbb{R}_{++}^M, \ \mathbf{p} \in \mathbb{R}_{++}^N$. The DEA program calculating maximum profit and the optimal quantities of outputs and inputs is:

$$\Pi(\mathbf{w}, \mathbf{p}) = \max_{\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}} \quad \sum_{n=1}^{N} p_n y_n - \sum_{m=1}^{M} w_m x_m$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq x_m, \ m = 1, \ldots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_n, \quad n = 1, \ldots, N,$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda_j \geq 0, \qquad j = 1, \ldots, J.$$

(1)

Denoting the firm being evaluated by $\left(\mathbf{x}_o, \mathbf{y}_o\right) \in \mathbb{R}_+^{M+N}, \mathbf{x}_o \neq 0_M, \mathbf{y}_o \neq 0_N$, *Profit inefficiency* is defined as maximum profit minus observed profit; i.e., $\Pi I(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}) = \Pi(\mathbf{w}, \mathbf{p}) - \Pi_o = \Pi(\mathbf{w}, \mathbf{p}) - (\mathbf{p} \cdot \mathbf{y}_o - \mathbf{w} \cdot \mathbf{x}_o) = \Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om}\right) \geq 0$. The standard approach decomposing profit inefficiency into a technical inefficiency measure, generally denoted by $TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)$—where the subscript

Table 1
Descriptive statistics, Taiwanese Banks, 2010.

	Inputs						Outputs			
	x_1	x_2	x_3	w_1	w_2	w_3	y_1	y_2	p_1	p_2
Average	795,536	3826	13,393	0.0064	1.25866	0.3171	196,8086	609,489	0.0349	0.0211
Median	428,995	3146	8721	0.0061	1.2168	0.3017	157,870	328,574	0.0147	0.0190
Max.	3,171,493	9538	76,576	0.0186	2.2963	0.7625	904,580	2,091,100	0.3044	0.0687
Min.	25,019	202	505	0.0025	0.7170	0.0730	1681	66,947	0.0059	0.0125
St.Dev.	768,008	2729	15,185	0.0026	0.3963	0.1697	215,063	582,854	0.0668	0.0095

$EM(G)$ represents a specific measure, and an allocative term, follows the same methodology common to all technical measures.

Technical inefficiency measures the distance between the production frontier and the firm. If a firm is technically efficient, its value is null, i.e, $TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o) = 0$. Otherwise, the firm is technically inefficient: $TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o) > 0$. After calculating technical inefficiency, and relying on the duality between the profit function and each technical inefficiency measure, we can establish a Fenchel-Mahler inequality by which *normalized* profit inefficiency: $N\Pi I(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}) = \Pi I(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})/NF_{EM(G)}$, is larger or equal to technical efficiency, i.e. $\Pi I(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})/NF_{EM(G)} \geq TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)$, where the divisor $NF_{EM(G)}$ is a normalizing scalar derived from the duality relationship.[4]

Afterwards, by closing the inequality, we can recover a scalar representing normalized allocative inefficiency as a residual. Allocative inefficiency measures the profit loss due to the fact that the projected benchmark of the firm on the frontier – through the technical inefficiency measure – does not supply the optimal output and input quantities that jointly maximize profit.

This implies that normalized profit inefficiency can be decomposed as follows:

$$\underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om} \right)}{NF_{EM(G)}}}_{\text{Norm. Profit Inefficiency}} = \qquad (2)$$

$$= \underbrace{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

We now present different decompositions of profit inefficiency considering the most relevant technical inefficiency measures found in the literature. Besides the common property of commensurability, each of these measures results in a particular decomposition whose strengths and weaknesses in terms of a set of desirable of properties are inherited from those of the underlying inefficiency measure: $TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)$—as discussed in [3, Chap.14]. For each of these measures we comment on whether they satisfy the two most relevant properties discussed in the literature: (1) the indication property, implying that the measure is consistent with the definition of Pareto-Koopmans efficiency, and (2) the essential property, by which allocative inefficiency vanishes when the evaluated firm is projected to a benchmark maximizing profit. We briefly present both properties.

As for the *indication property* concerning the technical efficiency measure $EG(M)$ in (2), it is satisfied if $EG(M)$ ensures that the evaluated firm is projected to a benchmark belonging to the strongly efficient subset of the technology. This subset is defined as:

$$\partial^S(T) = \{(\mathbf{x}, \mathbf{y}) \in T : (\mathbf{x}', -\mathbf{y}') \leq (\mathbf{x}, -\mathbf{y}), (\mathbf{x}', \mathbf{y}') \neq (\mathbf{x}, \mathbf{y}) \Rightarrow (\mathbf{x}', \mathbf{y}') \notin T\}. \qquad (3)$$

Intuitively, this subset includes all feasible production plans that are not dominated. This implies that a firm (or projection) belonging to $\partial^S(T)$ is efficient in the sense of Pareto-Koopmans: An observed firm is efficient if increasing one of its outputs implies decreasing at least one other output or increasing the use of at least one input. Likewise, if decreasing the use of any input entails increasing at least the use of another input or decreasing at least one output. As a result, an inefficient producer might generate the same outputs with fewer inputs or use the same inputs to generate more of at least one product.

Several efficiency measures like, for example, the directional distance function *DDF* or the multiplicative generalized (hyperbolic) distance function *GDF*, fail to satisfy this property because the projected benchmarks belong to the weakly efficient subset, implying that *individual* output increases and input reductions might be possible. The weakly efficient subset is defined as follows:

$$\partial^W(T) = \{(\mathbf{x}, \mathbf{y}) \in T : (\mathbf{x}', -\mathbf{y}') < (\mathbf{x}, -\mathbf{y}) \Rightarrow (\mathbf{x}', \mathbf{y}') \notin T\}. \qquad (4)$$

When choosing a weakly efficient measure to assess technical efficiency it is recommended to use of a two-stage strategy. After calculating the technical efficiency measure in the first stage, a second additive model looking for individual slacks is solved, [47]. **BenchmarkingEconomicEfficiency.jl** solves both stages and reports if these slacks exist.

The *essential property* is related to the value and meaning of allocative efficiency when decomposing profit inefficiency (2), see [48]. The property states that the allocative efficiency of an observation that is projected to a benchmark that maximizes profit, denoted by $(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)})$, should be zero; that is, $AI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}) = AI_{EM(G)}(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}) = 0$. The intuition behind this property is self-evident, since the projection maximizes profit, no allocative inefficiency may exist. These authors refined this property by considering that firms' allocative inefficiency must be equal to the allocative efficiency of its projection, regardless of whether this projection is a profit maximizing benchmark or not, i.e., $AI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}) = AI_{EM(G)}(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})$. Unfortunately, many additive measures fail to comply with the essential property or its extension, and we identify this drawback when commenting on each *EM(G)*.

3.1. The Russell inefficiency measure

Taking the Russell measure proposed by Färe and Lovell [30] as reference, Aparicio et al. [49] introduced the Fenchel-Mahler inequality for oriented Russell measures. The existence of a dual correspondence between the profit function and the Russell graph measure of technical inefficiency was introduced in [50]. A motivation for defining the Russell graph measure is that it projects the firm to $\partial^S(T)$, thereby complying with the indication property. Unfortunately, it does not satisfy the essential property [48, p. 120], and therefore a trade off exist between both properties. The Russell graph measure quantifying the technical inefficiency of a firm can be calculated through the following

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[4] The tilde '~' over prices denotes the normalization of profit efficiency and its accompanying allocative term. Normalizing profit inefficiency makes it satisfy the desirable property of commensurability, i.e., it is units' invariant. Note that the normalizing factor in the denominator, $NF_{EM(G)}$, can act as divisor of the prices. Thus, $(\tilde{\mathbf{w}}, \tilde{\mathbf{p}}) = (\mathbf{w}/NF_{EM(G)}, \mathbf{p}/NF_{EM(G)})$.

program:

$$TE_{RM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) = \min_{\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\lambda}} \quad \frac{1}{M+N}\left(\sum_{m=1}^{M} \theta_m + \sum_{n=1}^{N} \frac{1}{\phi_n}\right)$$

$$s.t.$$

$$\sum_{j=1}^{J} \lambda_j x_{jm} = \theta_m x_{om}, \qquad m = 1, \dots, M$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} = \phi_n y_{on}, \qquad n = 1, \dots, N$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\theta_m \leq 1, \qquad m = 1, \dots, M$$

$$\phi_n \geq 1, \qquad n = 1, \dots, N$$

$$\lambda_j \geq 0, \qquad j = 1, \dots, J$$

(5)

The optimal solutions of (5), θ_m^* and ϕ_m^*, calculate any possible proportional decrease in input usage, and proportional expansion of output production, respectively. The above program takes the average of these contraction and expansion rates. DEA finds the supporting hyperplane that is used as benchmark for $(\mathbf{x}_o, \mathbf{y}_o)$. This hyperplane is constructed as the linear combination of the firms whose λ_j multipliers are not null, thereby defining the enveloping surface. As model (5) is non-linear, several authors have proposed different transformations to obtain approximate values. Halická and Trnovská [50] showed how to reformulate (5) as a semidefinite programming (SDP) model, including its dual program.

Departing from $TE_{RM(G)}(\mathbf{x}_o, \mathbf{y}_o)$ we can define its technical inefficiency counterpart as $TI_{RM(G)}(\mathbf{x}_o, \mathbf{y}_o) = 1 - TE_{RM(G)}(\mathbf{x}_o, \mathbf{y}_o)$. Afterwards one can split normalized profit inefficiency into its corresponding technical and allocative terms: $N\Pi_{RM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = TI_{RM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) + AI_{RM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right)$; i.e.,

$$\underbrace{\frac{\Pi\left(\mathbf{w}, \mathbf{p}\right) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om}\right)}{(M+N)\min\left\{w_1 x_{o1}, \dots, w_M x_{oM}, p_1 y_{o1}, \dots, p_N y_{oN}\right\}}}_{\text{Norm. Profit Inefficiency}} =$$

$$\underbrace{\left[1 - \frac{1}{M+N}\left(\sum_{m=1}^{M} \theta_m^* + \sum_{n=1}^{N} \frac{1}{\phi_n^*}\right)\right]}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{RM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

(6)

BenchmarkingEconomicEfficiency.jl decomposes profit inefficiency based on the Russell graph measure (5) by using the **JuMP.jl** software written by Dunning et al. [28], in conjunction with the 'Ipopt' solver, [51]. The code is:

```julia
julia> deaprofitrussell(X, Y, W, P, names = banks)
```

```
Russell Profit DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Orientation = Graph; Returns to Scale = VRS
-------------------------------------------------------------
                           Profit      Technical   Allocative
-------------------------------------------------------------
Export-Import Bank         3.03911e-5  4.81347e-6  2.55776e-5
Bank of Taiwan             0.179835    2.71764e-9  0.179835
Taipei Fubon Bank          0.220019    2.34592e-7  0.220019
Bank of Kaohsiung          4.95721     0.319545    4.63766
Land Bank                  1.3703e-7   5.04573e-8  8.65722e-8
...
Hwatai Bank                136.01      0.54445     135.466
Cota Bank                  43.6129     0.528602    43.0843
Industrial Bank of Taiwan  1.56449     1.7079e-5   1.56447
Bank SinoPac               0.422128    0.125224    0.296904
Shin Kong Bank             9.67215     0.435376    9.23677
-------------------------------------------------------------
```

Then, we can recover the information about the inefficiency measure using the following syntax:

```julia
julia> dearussell(X, Y, orient = :Graph, rts = :VRS,
names = banks)
```

We illustrate the model by commenting on the profit inefficiency of the ten banks reported in the previous table. The first and fifth banks, Export-Import Bank and Land Bank, maximize profit, implying that they are both technical an allocative efficient. Hence, the numerical values of these components, reported in scientific notation, are effectively equal to zero. Subsequently we observe that the second bank (Bank of Taiwan) and the third bank (Taipei Fubon Bank) are profit inefficient. Since both banks are technically efficient, i.e. the values of the technical component are zero once again, we learn that they define the production frontier and, therefore, their profit inefficiency is only allocative, implying that, given their market prices, they demand and supply suboptimal quantities of inputs and outputs, respectively. The fourth bank (Bank of Kaohsiung) is both technical and allocative inefficient, because both components are greater than zero: 0.32 and 4.64, respectively. Summing both values we obtain the magnitude of total profit inefficiency: 4.96. The performance of the last five banks in the sample regarding their inefficiencies may be categorized in the same way and therefore we do not comment the results further. Moreover, the discussion of the results of the following models is equivalent to this one, only differing in the numerical values of the inefficiencies, but not in the categorization of the bank as efficient or inefficient, whose status remain the same. Therefore we do not discuss the results of the different models individually. However, in the results Section 6 we compare all of them through box-plots and determine the compatibility of their distribution by calculation their ranking correlation.

3.2. The weighted additive inefficiency measure

The weighted additive technical inefficiency measure introduced by Cooper et al. [31] can be also used to decompose of profit inefficiency. This technical inefficiency measure considers individual input and output slacks, denoted by: $\mathbf{s}^- \in \mathbb{R}^M$ and $\mathbf{s}^+ \in \mathbb{R}^N$. These slacks measure the quantity gaps to the projected benchmark on the frontier: i.e., $s_m^- = x_{om} - \hat{x}_{om}, s_m^- \geq 0, m = 1, \dots, M$, and $s_n^+ = \hat{y}_{on} - y_{on}, s_n^+ \geq 0, n = 1, \dots, N$. The DEA graph model for measuring technical inefficiency is:

$$TI_{WADF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+\right) = \max_{\mathbf{s}^-, \mathbf{s}^+, \boldsymbol{\lambda}} \quad \sum_{m=1}^{M} \rho_m^- s_m^- + \sum_{n=1}^{N} \rho_n^+ s_n^+$$

$$s.t.$$

$$\sum_{j=1}^{J} \lambda_j x_{jm} + s_m^- \leq x_{om}, \qquad m = 1, \dots, M$$

$$-\sum_{j=1}^{J} \lambda_j y_{jr} + s_n^+ \leq -y_{on}, \qquad n = 1, \dots, N$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$s_m^- \geq 0, \qquad m = 1, \dots, M$$

$$s_n^+ \geq 0, \qquad n = 1, \dots, N$$

$$\lambda_j \geq 0, \qquad j = 1, \dots, J$$

(7)

where the input and output weights: $\boldsymbol{\rho}^- = (\rho_1^-, \dots, \rho_M^-) \in R_{++}^M$ and $\boldsymbol{\rho}^- = (\rho_1^+, \dots, \rho_N^+) \in R_{++}^N$ indicate their relative importance when measuring technical inefficiency—hence the name of the measure. Assigning unit values program (7) results in the standard additive model proposed by Charnes et al. [47]. However, by choosing alternative weights the *WA* inefficiency measure nests a range of DEA models known as general efficiency measures (GEMs). As we show below,

Benchmarking Economic Efficiency.jl allows to choose among a wide range of models.

Given the evaluated firm $(\mathbf{x}_o, \mathbf{y}_o)$, program (7) looks for the maximum possible input reduction and output expansion consistent with the technology. A firm is technically efficient if the optimal slacks are zero: $\mathbf{s}^{-*} = \mathbf{s}^{+*} = 0$, so $TI_{WA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+\right) = 0$. If any of the slacks is greater than zero, then individual input reductions or output expansions are possible. Consequently, the larger the slacks, the greater the inefficiency. Program (7) ensures that the projected benchmark on the production frontier $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$, defined as $\hat{x}_{om} = \sum_{j=1}^{J} \lambda_j^* x_{jm}, m = 1, \ldots, M$, and $\hat{y}_{on} = \sum_{j=1}^{J} \lambda_j^* y_{jn}, n = 1, \ldots, N$, is strongly efficient. This implies that the *WA* model satisfies the indication property, although it does not comply with the essential property [48, p. 120].

Following [31,52] split normalized profit inefficiency into technical and allocative inefficiencies: $NII_{WA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = TI_{WA(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) + AI_{WA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$:

$$
\underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om}\right)}{\min\left\{\frac{w_1}{\rho_1^-}, \ldots, \frac{w_M}{\rho_M^-}, \frac{p_1}{\rho_1^+}, \ldots, \frac{p_N}{\rho_N^+}\right\}}}_{\text{Norm. Profit Inefficiency}} =
$$

$$
= \underbrace{\sum_{m=1}^{M} \rho_m^- s_m^{-*} + \sum_{n=1}^{N} \rho_n^+ s_n^{+*}}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{WA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)}_{\text{Norm. Allocative Inefficiency}} \geq 0,
$$

(8)

Calculating profit inefficiency according to (8) requires researchers to choose the weights for the inputs and the outputs. When running the corresponding function in Julia, it is possible to directly choose among the following programmed options:

- `:Ones`. The weights of inputs and outputs are equal to one, $\left(\boldsymbol{\rho}^-, \boldsymbol{\rho}^+\right) = (\mathbf{1}, \mathbf{1})$, thereby solving the standard additive model proposed by Charnes et al. [47];
- `:MIP`. Uses the so-called Measure of Inefficiency Proportions (MIP) measure, [53]. The weights are: $\left(\boldsymbol{\rho}^-, \boldsymbol{\rho}^+\right) = \left(1/\mathbf{x}_o, 1/\mathbf{y}_o\right)$, where $1/\mathbf{x}_o = \left(1/x_{1o}, \ldots, 1/x_{Mo}\right)$ and $1/\mathbf{y}_o = \left(1/y_{1o}, \ldots, 1/y_{No}\right)$;
- `:RAM`. Uses the Range Adjusted Measure of Inefficiency (RAM) (see [53]). The weights are: $\left(\boldsymbol{\rho}^-, \boldsymbol{\rho}^+\right) = \left(1/(M+N)\mathbf{R}^-, 1/(M+N)\mathbf{R}^+\right)$, where $\mathbf{R}^- = \left(R_1^-, \ldots, R_M^-\right)$ with $R_m^- = \max_{1 \leq j \leq J}\{x_{jm}\} - \min_{1 \leq j \leq J}\{x_{jm}\}$, and $\mathbf{R}^+ = \left(R_1^+, \ldots, R_N^+\right)$ with $R_n^+ = \max_{1 \leq j \leq J}\{y_{jn}\} - \min_{1 \leq j \leq J}\{y_{jn}\}$;
- `:BAM`. Uses the Bounded Adjusted Measure (BAM) [54]. The weights are: $\boldsymbol{\rho}^- = 1/\left[(M+N)\left(x_o - \underline{x}\right)\right]$, where $\underline{\mathbf{x}} = \left(x_1, \ldots, x_M\right)$ with $\underline{x}_m = \min_{1 \leq j \leq J}\{x_{jm}\}, m = 1, \ldots, M$, and $\boldsymbol{\rho}^+ = 1/\left[(M+N)\left(\bar{y} - y_o\right)\right]$, where $\bar{\mathbf{y}} = \left(\bar{y}_1, \ldots, \bar{y}_N\right)$ with $\bar{y}_n = \max_{1 \leq j \leq J}\{y_{jn}\}, n = 1, \ldots, N$; and, finally,
- `:Normalized`. Uses the normalized weighted additive model that considers the standard deviations of inputs $\sigma^- = \left(\sigma_1^-, \ldots, \sigma_M^-\right)$ and outputs $\sigma^+ = \left(\sigma_1^+, \ldots, \sigma_N^+\right)$ to make it units' invariant, [55]. In this case the weights are: $\left(\rho^-, \rho^+\right) = \left(1/\sigma^-, 1/\sigma^+\right)$.

Researchers may also pass their own weights using the option `:Custom` and supplying a vector or matrix of weights for inputs and outputs using `rhoX` and `rhoY`. The default weights are `:Ones` when none of the above possibilities is chosen.

To illustrate the weighted additive profit inefficiency model we choose the `:MIP` option:

```julia
julia> deaprofitadd(X, Y, W, P, :MIP, names = banks)
```

```
Profit Additive DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Weights = MIP; Returns to Scale = VRS
--------------------------------------------------------
                           Profit    Technical    Allocative
--------------------------------------------------------
Export-Import Bank          0.0      -1.78195e-16  1.78195e-16
Bank of Taiwan              0.899175 -2.52669e-16  0.899175
Taipei Fubon Bank           1.1001    0.0          1.1001
Bank of Kaohsiung          24.786     7.07646     17.7096
Land Bank                   1.59211e-14 0.0        1.59211e-14
...
Hwatai Bank               680.052    41.9682     638.084
Cota Bank                 218.064    24.8209     193.243
Industrial Bank of Taiwan   7.82245   0.0          7.82245
Bank SinoPac                2.11064   0.652049     1.45859
Shin Kong Bank             48.3607   15.488       32.8728
--------------------------------------------------------
```

Then, we can recover the information about the quantity slacks comprising the inefficiency measure by running the code:

```julia
julia> deaadd(X, Y, :MIP, names = banks)
```

3.3. The enhanced Russell graph (or slack-based) inefficiency measure

The enhanced Russell graph measure *ERG* was proposed by Pastor et al. [33], with the objective of overcoming the non-linear nature of the Russell Graph Measure presented in (5).[5] [33] introduced a non-radial model that accounts for both inputs and outputs (graph or non-oriented) as the Russell proposal, but that is easier to compute through linear programming—as opposed to the standard Russell model. Consequently, they specified the new measure resorting to the same variables as the Russell Graph Measure, including the vector of the 'lambdas' (λ's) defining the benchmark hyperplanes, the 'thetas' (θ's) measuring individual inputs reductions, and the 'phys' (ϕ's) measuring the proportional output increments. The technical efficiency model is given by:

$$
TE_{ERG(SBM)(G)}\left(x_o, y_o\right) = \min_{\theta, \phi, \lambda} \frac{\frac{1}{M}\sum_{m=1}^{M} \theta_m}{\frac{1}{N}\sum_{n=1}^{N} \phi_n}
$$

$$
s.t.
$$

$$
\sum_{j=1}^{J} \lambda_j x_{mj} = \theta_m x_{om}, \quad m = 1, \ldots, M
$$

$$
\sum_{j=1}^{J} \lambda_j y_{nj} = \phi_n y_{on}, \quad n = 1, \ldots, N
$$

$$
\sum_{j=1}^{J} \lambda_j = 1,
$$

$$
\theta_m \leq 1, \quad\quad m = 1, \ldots, M
$$

$$
\phi_n \geq 1, \quad\quad n = 1, \ldots, N
$$

$$
\lambda_j \geq 0, \quad\quad j = 1, \ldots, J
$$

(9)

Comparing model (9) with program (5) defining the Russell graph measure in the previous section, we note that the only difference is the objective function, which was formulated as $\frac{1}{M+N}\left(\sum_{m=1}^{M} \theta_m + \sum_{n=1}^{N} \frac{1}{\phi_n}\right)$. This change is critical however for the resolution of the model since the new objective function in (9) is now fractional; specifically, it is a fraction of two linear expressions, which makes it simpler to transform and solve.

After several transformations and algebraic steps we obtain the final linear program calculating the *ERG(SMB)* measure of technical

---

[5] Two years later [32] proposed the exact same measure, which he termed the 'slack-based measure', SBM—hence we denote the enhanced Russell graph as '*ERG(SBM)*'

efficiency:

$$
TE_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) = \min_{\mathbf{t}^-, \mathbf{t}^+, \boldsymbol{\mu}, \boldsymbol{\beta}} \beta - \frac{1}{M}\sum_{m=1}^{M}\frac{t_m^-}{x_{om}}
$$

$$
s.t.
$$

$$
\beta + \frac{1}{N}\sum_{n=1}^{N}\frac{t_n^+}{y_{on}} = 1
$$

$$
\sum_{j=1}^{J}\mu_j x_{jm} = \beta x_{om} - t_m^-, \quad m = 1, \dots, M
$$

$$
\sum_{j=1}^{J}\mu_j y_{jn} = \beta y_{on} + t_n^+, \quad n = 1, \dots, N
$$

$$
\sum_{j=1}^{J}\mu_j = \beta,
$$

$$
\beta \geq 0,
$$

$$
t_m^- \geq 0, t_n^+ \geq 0, \qquad \forall m, n,
$$

$$
\mu_j \geq 0, \qquad j = 1, \dots, J,
$$

$$(10)$$

where

$$
\beta = \left(1 + \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^+}{y_{on}}\right)^{-1},
$$

$$
t_m^- = \beta s_m^-, \, m = 1, \dots, M,
$$

$$
t_n^+ = \beta s_n^+, \, n = 1, \dots, N,
$$

$$
\mu_j = \beta \lambda_j, \, j = 1, \dots, J.
$$

$$(11)$$

From the solution to (10) we can recover the following measure of technical inefficiency,

$$
TI_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) = 1 - TE_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) = \tag{12}
$$

$$
1 - \frac{1 - \frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{-*}}{x_{om}}}{1 + \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{on}}} = \frac{\frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{on}} + \frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{-*}}{x_{om}}}{1 + \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{no}}}.
$$

Resorting to the duality between this expression and the profit function presented in [56], we can establish the corresponding decomposition of profit inefficiency: $N\Pi I_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$ $= TI_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) + AI_{ERG(SBM)(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$, i.e.,

$$
\underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N}p_n y_{on} - \sum_{m=1}^{M}w_m x_{om}\right)}{\delta_{(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}, \mathbf{w})}\left(1 + \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{on}}\right)}}_{\text{Norm. Profit Inefficiency}} =
$$

$$
= \underbrace{\left(\frac{\frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{on}} + \frac{1}{M}\sum_{m=1}^{M}\frac{s_m^{-*}}{x_{om}}}{\left(1 + \frac{1}{N}\sum_{n=1}^{N}\frac{s_n^{+*}}{y_{on}}\right)}\right)}_{\text{Graph Techncial Inefficiency}} + \underbrace{AI_{ERG=SBM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{p}}, \tilde{\mathbf{w}}\right)}_{\text{Norm. Allocative Inefficency}} \geq 0,
$$

$$(13)$$

where $\delta_{(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}, \mathbf{w})} = \min\left\{N p_n y_{on}, n = 1, \dots, N, M w_m x_{om}, m = 1, \dots, M\right\}$ in the normalization factor of the *ERG(SBM)*. As most additive measures, the *ERG=SBM* satisfies the indication property, but fails to meet the essential property [48, p.120]. **BenchmarkingEconomicEfficiency.jl** allows calculation and decomposition of profit inefficiency using the *ERG(SBM)* technical measure by typing this code:

```julia
julia> deaprofiterg(X, Y, W, P, names = banks)
```

```
Enhanced Russell Graph Slack Based Measure Profit DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
--------------------------------------------------------
                        Profit    Technical   Allocative
--------------------------------------------------------
Export-Import Bank      0.0       0.0         0.0
Bank of Taiwan          0.299725  1.11022e-16 0.299725
```

```
Taipei Fubon Bank          0.550048      2.22045e-16    0.550048
Bank of Kaohsiung          2.75099       0.78295        1.96804
Land Bank                  7.96057e-15   -2.22045e-16   8.18261e-15
...
Hwatai Bank                15.7188       0.958676       14.7601
Cota Bank                  8.36232       0.942322       7.42
Industrial Bank of Taiwan  3.91122       -3.10862e-15   3.91122
Bank SinoPac               0.763301      0.226657       0.536644
Shin Kong Bank             2.86958       0.906448       1.96313
--------------------------------------------------------
```

Regarding the information about the underlying technical inefficiency: $TE_{ERG=SBM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)$, it can be recovered by running:

```julia
julia> deaerg(X, Y, rts = :VRS, names = banks)
```

3.4. The directional distance function

Chambers et al. [57] proposed the so-called directional distance function, *DDF*, projecting observation $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ to the production frontier in the direction given by the nonnegative vector $\mathbf{g} = \left(\mathbf{g}_o^-, \mathbf{g}_o^+\right) \neq \mathbf{0}_{M+N}$, $\mathbf{g}_o^- \in \mathbb{R}^M$ and $\mathbf{g}_o^+ \in \mathbb{R}^N$. Inputs and outputs are reduced and increased according to the scalar β, identifying as reference projection $(\hat{\mathbf{y}}_o, \hat{\mathbf{x}}_o) = (\mathbf{y}_o + \beta^*\mathbf{g}_o^+, \mathbf{x}_o - \beta^*\mathbf{g}_o^-)$. Because this projection may belong to the weakly efficient frontier of the technology, $\partial^W(T)$, the *DDF* does not comply with the indication property, while it meets the essential property, see Proposition 1 in [48].

The *DDF* technical inefficiency measure is defined as follows:

$$
TI_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+\right) = \max\left\{\beta : \left(\mathbf{x}_o - \beta\mathbf{g}_o^-, \mathbf{y}_o + \beta\mathbf{g}_o^+\right) \in \partial^W(T), \, \beta \geq 0\right\}.
$$

$$(14)$$

Its associated DEA linear program is:

$$
TI_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+\right) = \max_{\beta, \lambda} \beta
$$

$$
s.t.
$$

$$
\sum_{j \in J}\lambda_j x_{jm} \leq x_{om} - \beta g_{om}^-, \quad m = 1, \dots, M \tag{15}
$$

$$
\sum_{j \in J}\lambda_j y_{jn} \geq y_{on} + \beta g_{on}^+, \quad n = 1, \dots, N
$$

$$
\sum_{j \in J}\lambda_j = 1
$$

$$
\lambda_j \geq 0, \, j \in J.
$$

If the optimal solution is null, $\beta^* = 0$, then the firm is technically efficient. Conversely, if $\beta^* > 0$, the firm is inefficient, with the projection $(\hat{\mathbf{y}}_o, \hat{\mathbf{x}}_o)$ dominating $\left(\mathbf{x}_o, \mathbf{y}_o\right)$. Inspecting the usual input and output inequalities in program (15), it may be possible that further input excesses and output shortfalls exist in the form of slacks: $\mathbf{s}^- > 0$ and $\mathbf{s}^+ > 0$, respectively. In this model, the inputs slacks are equal to $\mathbf{s}^- = \mathbf{x}_o - \beta\mathbf{g}_o^- - X\lambda$, while the output slacks are $\mathbf{s}^+ = Y\lambda - \mathbf{y}_o + \beta\mathbf{g}_o^+$. Consequently, after calculating the *DDF*, **BenchmarkingEconomicEfficiency.jl** performs a second stage to determine if these slacks exist. Researchers should consider that the decomposition of profit inefficiency into technical and allocative terms maybe misrepresented due to the existence of slacks.

The decomposition of the normalized (so-called *Nerlovian*) profit inefficiency based on the directional distance function was proposed by Chambers et al. [34]: $N\Pi I_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = TI_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+\right) + AI_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$. This corresponds to:

$$
\underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N}p_n y_{on} - \sum_{m=1}^{M}w_m x_{om}\right)}{\sum_{m=1}^{M}w_m g_{om}^- + \sum_{n=1}^{N}p_n g_{on}^+}}_{\text{Norm. Profit Inefficiency}} = \tag{16}
$$

$$
= \underbrace{\beta_{DDF(G)}^*}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{DDF(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)}_{\text{Norm. Allocative Inefficiency}} \geq 0.
$$

The directional vector **g** is exogenoulsy chosen, although the majority of empirical studies use the observed amounts of inputs and outputs; i.e., $\mathbf{g} = \left(\mathbf{g}_o^-, \mathbf{g}_o^+\right) = \left(\mathbf{x}_o, \mathbf{y}_o\right)$. In this case the normalization factor in (16) equals the sum of observed cost and revenue: $\frac{\Pi(\mathbf{w}, \mathbf{p}) - \Pi_o}{C_o + R_o}$.

The observed input and output matrices can be passed as directional vectors by adding Gx = X, Gy = Y to the syntax of the function. Nevertheless this is not the only possibility. **BenchmarkingEconomicEfficiency.jl** includes a set of options for choosing a directional vector. The following options are available:

- :Observed. Gx = :Observed, Gy = :Observed. Considers as directional vector the observed input and output quantities of the firm: $\mathbf{g} = \left(\mathbf{g}_o^-, \mathbf{g}_o^+\right) = \left(\mathbf{x}_o, \mathbf{y}_o\right)$;
- :Ones. Gx = :Ones, Gy = :Ones. Sets the directional vectors to one: $\mathbf{g} = \left(\mathbf{g}_o^-, \mathbf{g}_o^+\right) = \left(\mathbf{1}_M, \mathbf{1}_N\right)$;
- :Mean. Gx = :Mean, Gy = :Mean. Sets the directional vector to $\mathbf{g} = \left(\bar{x}_M, \bar{y}_N\right)$, using the mean of each different input and output: $\bar{x}_m = \sum_{j=1}^J x_{mj}/J$ and $\bar{y}_n = \sum_{j=1}^J y_{nj}/J$, respectively;
- :Zeros. Either Gx = :Zeros or Gy = :Zeros. This allows to set either the input or output direction to zero. This is useful when calculating: (1) input oriented *DDFs*, for example $\mathbf{g} = \left(\mathbf{x}_o, \mathbf{0}_N\right)$, $\mathbf{g} = \left(\mathbf{1}_M, \mathbf{0}_N\right)$ or $\mathbf{g} = \left(\bar{\mathbf{x}}_M, \mathbf{0}_N\right)$, or (2) output oriented *DDFs*, for instance $\mathbf{g} = \left(\mathbf{0}_M, \mathbf{y}_o\right)$, $\mathbf{g} = \left(\mathbf{0}_M, \mathbf{1}_N\right)$, or $\mathbf{g} = \left(\mathbf{0}_M, \bar{\mathbf{y}}_N\right)$. These partially oriented *DDFs* are used to decompose cost and revenue inefficiency in forthcoming Section 5.2. Finally,
- :Monetary. Gx = :Monetary, Gy = :Monetary. This option ensures that profit inefficiency, as well as its technical and allocative components are valued in monetary units. To achieve this results the normalizing factor in (16) must satisfy the following constraint: $\sum_{m=1}^M w_m g_{om}^- + \sum_{n=1}^N p_n y_{on}^+ = 1$, see [58].

We note that the adoption of a common direction for all firms—therefore excluding the observed quantities—ensures that the decomposition complies with the extended essential property, see Proposition 5 in [48]. We now present the decomposition of profit inefficiency using the average of the input and output quantities as directional vector. The code in the package **BenchmarkingEconomicEfficiency.jl** is the following:

```
julia> deaprofit(X, Y, W, P, Gx = :Mean, Gy = :Mean,
names = banks)


Profit DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
Gx = Mean; Gy = Mean
-------------------------------------------------------------
                          Profit    Technical    Allocative
-------------------------------------------------------------
Export-Import Bank        0.0          0.0          0.0
Bank of Taiwan            0.210065    -7.08067e-17  0.210065
Taipei Fubon Bank         0.201694     0.0          0.201694
Bank of Kaohsiung         0.36845      0.0418304    0.32662
Land Bank                 1.93201e-15  0.0          1.93201e-15
...
Hwatai Bank               0.670465     0.0476872    0.622777
Cota Bank                 0.754719     0.0287997    0.725919
Industrial Bank of Taiwan 0.0930958    0.0          0.0930958
Bank SinoPac              0.258314     0.060563     0.197751
Shin Kong Bank            1.49562      0.0863343    1.40929
-------------------------------------------------------------
```

Then, we can recover the information about the *DDF* and any possible slacks by running the code:

```
julia> deaddf(X, Y, rts = :VRS, Gx = :Mean, Gy = :Mean,
names = banks)
```

3.5. The Hölder distance function

Another relevant option to decompose profit inefficiency is the use of different Hölder norms when defining distance functions as

proposed by Briec [37], Briec and Lemaire [59], and Briec and Lesourd [60]. As opposed to previous measures of technical inefficiency, the Hölder distance functions conform with the principle of least action by searching for the nearest or 'least distance' to the production frontier. This is in contrast to some of the previous models, particularly, those based on slacks such as the weighted additive measures in Section 3.2, that maximize the distance to the production frontier.

The Hölder norms ℓ_h ($h \in [1, \infty]$) are defined over a g-dimensional real normed space:

$$\| \cdot \|_h : z \to \|z\|_h = \begin{cases} \left(\sum_{j=1}^g \left| z_j \right|^h \right)^{1/h} & \text{if } h \in [1, \infty[\\ \max_{j=1,\ldots,g} \left\{ \left| z_j \right| \right\} & \text{if } h = \infty \end{cases} \qquad (17)$$

where $z = \left(z_1, \ldots, z_g\right) \in R^g$. When determining the technical inefficiency of firms, Briec [37] considered the weakly efficient reference set (4), i.e,

$$TI_{WH\ddot{o}lder(G)}\left(\mathbf{x}_o, \mathbf{y}_o, h\right) = \inf_{\mathbf{u}, \mathbf{v}} \left\{ \left\| \left(\mathbf{x}_o, \mathbf{y}_o\right) - \left(\mathbf{u}, \mathbf{v}\right) \right\|_h : \left(\mathbf{u}, \mathbf{v}\right) \in \partial^W (T) \right\}. \qquad (18)$$

This optimization model finds the minimum distance between the firm $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ and the weakly efficient subset of the technology.

The choice of a meaningful norm h corresponds to the researcher. **BenchmarkingEconomicEfficiency.jl** implements the Hölder function under the following norms: unit ($h = 1$), infinitum ($h = \infty$), and $h = 2$, corresponding to the Euclidean distance. Computationally, the Hölder distance functions require solving non-linear formulations. However, for $h = 1$ and $h = \infty$ it is possible to resort to linear DEA models because their topological geometries correspond to polyhedral sets. For these two norms it can be shown that their corresponding Hölder distance functions are equivalent to the directional distance function *DDF* previously presented with a concrete specification of the directional vector. For $h = 1$, $TI_{WH\ddot{o}lder(G)}\left(\mathbf{x}_o, \mathbf{y}_o, 1\right)$ is computed by finding the minimum of the $M + N$ values $TI_{DDF(I)}\left(\mathbf{x}_o, \mathbf{y}_o, \left(0, \ldots, 1_{(m')}, \ldots, 0\right), \mathbf{0}_N\right)$, $m' = 1, \ldots, M$—an input oriented *DDF*, and $TI_{DDF(O)}\left(x_o, y_o, \mathbf{0}_M, \left(0, \ldots, 1_{(n')}, \ldots, 0\right)\right)$, $n' = 1, \ldots, N$—an output oriented *DDF*. In the case of $h = \infty$, $TI_{WH\ddot{o}lder(G)}\left(x_o, y_o, \infty\right)$ is equivalent to the *DDF* model (15), fixing the directional vector at $\mathbf{g} = \left(\mathbf{g}_o^-, \mathbf{g}_o^+\right) = \left(\mathbf{1}_M, \mathbf{1}_N\right)$.

As for Euclidean norm $h = 2$, determining its value is more complex because calculating the shortest distance involves the minimization of a convex function considering the complement of a convex set. This distance can be calculated by solving a quadratic optimization problem coupled with Special Ordered Sets (SOS). This constitutes a particular mathematical approach to account for complementarity conditions. Pastor et al. [3, Chap. 8] propose a bi-level linear model to find the Euclidean distance from the firm $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ to the weakly efficient frontier (See Box I). There are alternative ways to solve bi-level mathematical programs. One possibility is to use the KKT (Karush–Kuhn–Tucker) conditions of the linear model embedded in (19), corresponding to $\max_{\beta, \gamma} \beta$ and its associated constraints.

The package **BenchmarkingEconomicEfficiency.jl** solves model (19) with quadratic second order sets (SOS) constraints resorting to Gurobi, [61]. This commercial optimizer, which is also available under a free license for academic use only, can be easily added to Julia.[6]

Following Briec and Lesourd (1999) it is possible to resort to duality theory to decompose profit inefficiency: $N\Pi I_{WH\ddot{o}lder(G)}\left(\mathbf{x}_o, \mathbf{y}_o, h, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$

$$TI_{W\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,2\right) = \min_{\mathbf{x},\mathbf{y},\boldsymbol{\lambda},\beta,\boldsymbol{\gamma}} \sqrt{\sum_{m=1}^{M}\left(x_{om}-x_m\right)^2 + \sum_{n=1}^{N}\left(y_n-y_{on}\right)^2}$$

$$s.t. \quad \sum_{j=1}^{J}\lambda_j x_{jm} \leq x_m, \qquad m=1,\dots,M$$

$$\sum_{j=1}^{J}\lambda_j y_{jn} \geq y_n, \qquad n=1,\dots,N$$

$$\sum_{j=1}^{J}\lambda_j = 1,$$

$$\beta = 0,$$

$$\max_{\beta,\boldsymbol{\gamma}} \quad \beta \tag{19}$$

$$s.t. \quad \sum_{j=1}^{J}\gamma_j x_{jm} \leq x_m - \beta, \qquad m=1,\dots,M$$

$$\sum_{j=1}^{J}\gamma_j y_{jn} \geq y_n + \beta, \qquad n=1,\dots,N$$

$$\sum_{j=1}^{J}\gamma_j = 1,$$

$$\lambda_j,\gamma_j \geq 0, \qquad j=1,\dots,J$$

$$x_m \geq 0, \qquad m=1,\dots,M$$

$$y_n \geq 0, \qquad n=1,\dots,N$$

Box I.

$$= TI_{W\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h\right) + AI_{W\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h,\tilde{\mathbf{w}},\tilde{\mathbf{p}}\right).$$ For the Euclidean norm $h=2$, this yields the following expression:

$$\underbrace{\frac{\Pi(\mathbf{w},\mathbf{p})-\left(\sum_{n=1}^{N}p_n y_{on}-\sum_{m=1}^{M}w_m x_{om}\right)}{\|(w,p)\|_q}}_{\text{Norm. Profit Inefficiency}} =$$

$$\underbrace{\sqrt{\sum_{m=1}^{M}\left(x_{om}-x_m^*\right)^2 + \sum_{n=1}^{N}\left(y_n^*-y_{on}\right)^2}}_{\text{Graph Technnical Inefficiency}} + \underbrace{AI_{W\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,\tilde{\mathbf{w}},\tilde{\mathbf{p}},h\right)}_{\text{Norm. Profit Inefficiency}} \geq 0, \tag{20}$$

where x_m^* and y_n^* are the solution to problem (19). For the remaining norms: $h=1$ and $h=\infty$, the decompositions are equal to the *DDF* (16), with directional vectors set to the values previously mentioned.

As for the properties satisfied by these decompositions, one drawback of using Hölder distance functions is that they are not units invariant; i.e. their value would change if the units of measurement of inputs and outputs are transformed. Aware of this limitation, Briec [37, p. 125] proposed to weight the Hölder distance function. This results into a weighted weakly (WW) efficiency measure based on Hölder norms:

$$TI_{WW\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h\right) =$$
$$\inf_{\mathbf{u},\mathbf{v}}\left\{\left\|\left(\frac{x_{o1}-v_1}{x_{o1}},\dots,\frac{x_{oM}-v_M}{x_{oM}},\frac{y_{o1}-u_1}{y_{o1}},\dots,\frac{y_{oN}-u_N}{y_{oN}}\right)\right\|_h : (\mathbf{v},\mathbf{u}) \in \partial^w(T)\right\}. \tag{21}$$

Based on this distance function, one decomposes profit inefficiency into technical and allocative terms that are units' invariant:
$$N\Pi_{WW\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h,\tilde{\mathbf{w}},\tilde{\mathbf{p}}\right) = TI_{WW\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h\right) +$$

$AI_{WW\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h,\tilde{\mathbf{w}},\tilde{\mathbf{p}}\right)$:

$$\underbrace{\frac{\Pi(\mathbf{w},\mathbf{p})-\left(\sum_{n=1}^{N}p_n y_{on}-\sum_{m=1}^{M}w_m x_{om}\right)}{\left\|\left(w_1 x_{o1},\dots,w_M x_{oM},p_1 y_{o1},\dots,p_N y_{oN}\right)\right\|_q}}_{\text{Norm. Profit Inefficiency}} =$$

$$\underbrace{\sqrt{\sum_{m=1}^{M}\left(\frac{x_{om}-x_m^*}{x_{om}}\right)^2 + \sum_{n=1}^{N}\left(\frac{y_m^*-y_o}{y_{om}}\right)^2}}_{\text{Graph Technnical Inefficiency}} + \underbrace{AI_{WW\,Hölder(G)}\left(\mathbf{x}_o,\mathbf{y}_o,h,\tilde{\mathbf{w}},\tilde{\mathbf{p}}\right)}_{\text{Norm. Profit Inefficiency}} \geq 0. \tag{22}$$

The syntax used in **BenchmarkingEconomicEfficiency.jl** includes the possibility of decomposing profit inefficiency under the previous norms: ℓ_1, ℓ_∞, and ℓ_2, considering the unweighted model, (20), or its weighted version, (22). Table 2 summarizes the models available to the researcher.

We illustrate the weighted Hölder profit inefficiency model using the ℓ_∞ norm. This requires the use of next syntax:

```julia
julia> deaprofitholder(X, Y, W, P, l = Inf, weight =
true, names = banks)
```

```
Profit Holder LInf DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
Weighted (weakly) Holder distance function
----------------------------------------------------------------
                            Profit      Technical    Allocative
----------------------------------------------------------------
Export-Import Bank          0.0         -9.77224e-17  9.77224e-17
Bank of Taiwan              0.0609693   -2.04769e-17  0.0609693
Taipei Fubon Bank           0.146089    0.0           0.146089
Bank of Kaohsiung           1.72965     0.178596      1.55106
Land Bank                   9.00745e-16 0.0           9.00745e-16
...
Hwatai Bank                 4.69325     0.354964      4.33829
Cota Bank                   6.1504      0.284206      5.8662
Industrial Bank of Taiwan   0.870515    0.0           0.870515
Bank SinoPac                0.235837    0.0544932     0.181344
Shin Kong Bank              3.69483     0.160121      3.53471
----------------------------------------------------------------
```

Table 2
Directional vectors and weights corresponding to Hölder norms.

Norm	Unweighted[a]	Weighted: 'weight=true'
ℓ_1	$\mathbf{g}_o^- = (0, \ldots, 1_{(m')}, \ldots 0)$, $m' = 1, \ldots, M$ $\mathbf{g}_o^+ = (0, \ldots, 1_{(n')}, \ldots 0)$, $n' = 1, \ldots, N$	$\mathbf{g}_o^- = (0, \ldots, x_{(m'o)}, \ldots 0)$, $m' = 1, \ldots, M$ $\mathbf{g}_o^+ = (0, \ldots, y_{(n'o)}, \ldots 0)$, $n' = 1, \ldots, N$
ℓ_∞	$\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+) = (\mathbf{1}_M, \mathbf{1}_N)$	$\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+) = (\mathbf{x}_{oM}, \mathbf{y}_{oN})$
ℓ_2	$(\mathbf{1}_M, \mathbf{1}_N)$	$\left((1/\mathbf{x}_{oM})^2, (1/\mathbf{y}_{oN})^2 \right)$

[a]Unweighted results are obtained by default when omitting 'weight = true' from the syntax.

Then, we can recover the information about the Hölder distance function along with any possible slacks by running the code:

```
julia> deaholder(X, Y, l = Inf, weight = true, orient = :Graph, rts = :VRS, names = banks)
```

3.6. The modified directional distance function

Profit inefficiency can be decomposed using the modified directional distance function *MDDF*. The *MDDF* was proposed by Aparicio et al. [35] and presents two distinctive advantages over the standard *DDF* discussed in Section 3.4. First, as previously remarked, under the customary directional vector $\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+) = (\mathbf{x}_o, \mathbf{y}_o)$, the normalization factor in the decomposition of profit inefficiency (16) is $\sum_{m=1}^{M} w_m x_{om} + \sum_{n=1}^{N} p_n y_{on} = C_o + R_o$. This expression has no apparent economic interpretation because it corresponds to the monetary sum of the firm's cost and revenue (dollars spent and dollars earned). In turn, the duality of the *MDDF* with the profit function results in a normalization factor that is either cost or revenue, but not the sum of the two, which results in a meaningful interpretation of profit inefficiency as 'lost profit on outlay' (cost) or 'lost profit on earnings' (revenue). Second, while the *DDF* aims at reducing inputs and increasing outputs in the same proportion β—see (14), the *MDDF* adds flexibility to the evaluation process by permitting different variations in inputs and outputs when attaining technical efficiency.

The *MDDF* technical inefficiency measure is defined as follows:

$$TI_{MDDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+ \right)$$
$$= \max \left\{ \beta_{\mathbf{x}} + \beta_{\mathbf{y}} : \left(\mathbf{x}_o - \beta_o \mathbf{g}_o^-, \mathbf{y}_o + \beta_{\mathbf{y}} \mathbf{g}_o^+ \right) \in \partial^W (T), \ \beta_{\mathbf{x}}, \beta_{\mathbf{y}} \geq 0 \right\}. \quad (23)$$

where, once again as in (14), the nonnegative vector $\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+) \neq \mathbf{0}_{M+N}$, $\mathbf{g}_o^- \in \mathbb{R}^M$ and $\mathbf{g}_o^+ \in \mathbb{R}^N$, which is specified by the researcher, sets the direction towards the production frontier. The *MDDF* function is calculated through DEA methods with the next model:

$$TI_{MDDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+ \right) = \max_{\beta_{\mathbf{x}}, \beta_{\mathbf{y}}, \boldsymbol{\lambda}} \quad \beta_{\mathbf{x}} + \beta_{\mathbf{y}}$$
$$s.t.$$
$$\sum_{j=1}^{J} \lambda_j x_{jm} \leq x_{om} - \beta_{\mathbf{x}} g_{om}, \quad m = 1, \ldots, M$$
$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_{on} + \beta_{\mathbf{y}} g_{on}, \quad n = 1, \ldots, N$$
$$\sum_{j=1}^{J} \lambda_j = 1,$$
$$\lambda_j \geq 0, \qquad j = 1, \ldots, J$$
$$\beta_x, \beta_y \geq 0$$
$$(24)$$

Aparicio et al. [35] show that if the researcher chooses as directional vector the observed input and output quantities: $\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+) = (\mathbf{x}_o, \mathbf{y}_o)$, and observed profit is non negative: $\Pi_o \geq 0$, then the normalization factor relating profit and the *MDDF* through duality, is equal to observed cost. Therefore we obtain the following decompositions of economic inefficiency into the *MDDF* and allocative inefficiency: $N\Pi_{MDDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}} \right) = TI_{MDDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o \right) + AI_{MDDF(G)}$

$(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})$. That is,

$$\underbrace{\frac{\Pi (\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om} \right)}{C_o}}_{\text{Norm. Profit Inefficiency}} =$$
$$= \underbrace{\beta_{\mathbf{x}}^* + \beta_{\mathbf{y}}^*}_{\text{Technical Inefficiency}} + \underbrace{AI_{DDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}} \right)}_{\text{Norm. Allocative Inefficiency}} \geq 0. \quad (25)$$

where $\beta_{\mathbf{x}}^* + \beta_{\mathbf{y}}^*$ are the solutions to model (24). In case the observed firms incurs in economic losses presenting a negative profit, $\Pi_o < 0$, the corresponding duality can be established in terms of a normalization factor equivalent to observed revenue. Consequently, a measure of 'profit loss to earnings' can be defined and decomposed:

$$\underbrace{\frac{\Pi (\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om} \right)}{R_o}}_{\text{Norm. Profit Inefficiency}} =$$
$$= \underbrace{\beta_{\mathbf{x}}^* + \beta_{\mathbf{y}}^*}_{\text{Graph Technical Inefficiency}} + \underbrace{AI'_{DDF(G)} \left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}} \right)}_{\text{Norm. Allocative Inefficiency}} \geq 0. \quad (26)$$

BenchmarkingEconomicEfficiency.jl allows calculating profit inefficiency using the *MDDF* through the following syntax:

```
julia> deaprofitmddf(X, Y, W, P, Gx = :Observed, Gy = :Observed, names = banks)
```

```
Profit Modified DDF DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
Gx = Observed; Gy = Observed
------------------------------------------------------------
                            Profit      Technical   Allocative
------------------------------------------------------------
Export-Import Bank          0.0         0.0         0.0
Bank of Taiwan              0.596709    3.42411e-18 0.596709
Taipei Fubon Bank           1.1001      0.0         1.1001
Bank of Kaohsiung           9.75109     0.361837    9.38925
Land Bank                   4.15667e-15 0.0         4.15667e-15
...
Hwatai Bank                 27.9857     0.731325    27.2544
Cota Bank                   36.7423     0.638023    36.1043
Industrial Bank of Taiwan   7.82245     0.0         7.82245
Bank SinoPac                1.62129     0.11124     1.51005
Shin Kong Bank              25.8171     0.346674    25.4705
------------------------------------------------------------
```

The underlying inefficiency information can be obtained through the corresponding function.

```
julia> deamddf(X, Y, rts = :VRS, Gx = :Observed, Gy= :Observed, names = banks)
```

3.7. The reverse directional distance function

Another option found in the literature to decompose profit inefficiency uses the reverse directional distance function *RDDF* as measure of technical inefficiency. The advantage of the *RDDF*, proposed by Pastor et al. [36], is that it relates existing additive measures of technical inefficiency to the popular directional distance function. The *RDDF* is capable of transforming any measure of graph technical inefficiency,

EM(G), such as the enhanced Russell measure or the weighted additive *WA* measure previously presented, into a single scalar measure corresponding to a standard *DDF*. Therefore, given the set of *J* firms under study, F_J, and their projections on the frontier, denoted by $\hat{F}_J$, the *RDDF* assigns a new *DDF* score β to the original *EM(G)*, compatible with the projections $\hat{F}_J$.

To calculate the $RDDF\left(EM(G), F_J, \hat{F}_J\right)$ for firm $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ we need to determine the direction $\mathbf{g} = (\mathbf{g}_o^-, \mathbf{g}_o^+)$ linking the observation to its projection under *EM(G)*, $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o) \in \hat{F}_J$. This directional vector can be obtained by subtracting the coordinates of the firm under evaluation from its projected benchmark. Afterwards the value of the *RDDF* is calculated.

We now present the decomposition of profit inefficiency based on the *RDDF*, which is equivalent to that presented in Section 3.4:

$NII_{RDDF(EM(G),F_J,\hat{F}_J)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = TI_{RDDF(EM(G),F_J,\hat{F}_J)}$ $\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+\right) + AI_{RDDF(EM(G),F_J,\hat{F}_J)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$, i.e.,

$$\underbrace{\frac{\Pi\left(\mathbf{w}, \mathbf{p}\right) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om}\right)}{\sum_{m=1}^M w_m g_{om}^- + \sum_{n=1}^N p_n g_{on}^+}}_{\text{Norm. Profit Inefficiency}} =$$

$$= \underbrace{\beta^*_{RDDF(EM(G),F_J,\hat{F}_J)}}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{RDDF(EM(G),F_J,\hat{F}_J)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_o^-, \mathbf{g}_o^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

(27)

where the efficiency score $\beta^*_{RDDF(EM(G),F_J,\hat{F}_J)}$ for observations that are technically inefficient is calculated by solving the *DDF* model (14) with the associated directional vectors. We note that the whether the *RDDF* considers projections belonging to the strongly or weakly efficient subset of the technology is inherited from the technical inefficiency measure *EM(G)* of choice. Regarding the essential property, it complies with it as the *DDF* on which it is based. However, it cannot comply with the extended version of the essential property because the directional vectors, obtained from the difference between projected and observed input and output quantities, differ among observations [48, p. 123].

We now show the implementation of the profit inefficiency decomposition considering the enhanced Russell measure presented in Section 3.3 as the technical inefficiency measure of choice; i.e., $EMG(SBM)(G)$. The syntax for the normalized decomposition is as follows:

```
julia> deaprofitrddf(X, Y, W, P, :ERG, names = banks)
```

```
Profit Reverse DDF DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
Associated efficiency measure = ERG
-----------------------------------------------------------
                          Profit      Technical   Allocative
-----------------------------------------------------------
Export-Import Bank        0.0         0.0          0.0
Bank of Taiwan            0.299725    -4.82974e-13 0.299725
Taipei Fubon Bank         0.550048    0.0          0.550048
Bank of Kaohsiung         2.72869     0.78295      1.94574
Land Bank                 7.96057e-15 0.0          7.96057e-15
...
Hwatai Bank               13.5607     0.958676     12.602
Cota Bank                 7.01996     0.942322     6.07764
Industrial Bank of Taiwan 3.91122     0.0          3.91122
Bank SinoPac              0.536348    0.226657     0.30969
Shin Kong Bank            2.71129     0.906448     1.80485
-----------------------------------------------------------
```

We can recover the information about the underlying efficiency measures through the following code:

```
julia> dearddf(X, Y, :ERG, rts = :VRS, names = banks)
```

3.8. The general direct approach

The most recent model to decompose profit inefficiency has been proposed by Pastor et al. [38]. In contrast to the standard approach that

relies on duality theory to obtain a suitable Fenchel-Mahler inequality that allows recovering allocative efficiency as a residual—see (2), the so-called *Generalized Direct Approach, GDA* offers a new framework that, based on equalities, simplifies the process of decomposing profit efficiency. The new approach is *general* because it can be developed for any of the technical efficiency measure, *EM(G)*, previously discussed. Also, it is also *easier* to develop, because one does not need to solve for Fenchel-Mahler inequalities through duality methods. Finally the results are *exact*, because being based on equalities, the allocative inefficiency values cannot be overestimated as happens with the standard approaches when failing to comply with the essential property, see [38].

To implement the new approach, ones needs two elements: the value of the technical inefficiency of the firm under evaluation using the preferred efficiency measure: $TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)$, and the benchmark input and output values on the frontier: $\left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right)$. Then, profit inefficiency of firm $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ is decomposed into the sum of two terms. The first term corresponds to the scalar product of the optimal slack vector, $\left(\mathbf{s}_{oEM(G)}^{-*}, \mathbf{s}_{oEM(G)}^{+*}\right) = \left(\mathbf{x}_o - \hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)} - \mathbf{y}_o\right)$—i.e., the L_1-path between the observation and its benchmark, and the vector of market prices $(\mathbf{w}, \mathbf{p})$. This measures the profit loss due to technical inefficiency: $\mathbf{p} \cdot \mathbf{s}_{oEM(G)}^{+*} + \mathbf{w} \cdot \mathbf{s}_{oEM(G)}^{-*}$. The second term of the decomposition is the remaining profit inefficiency, which is the profit loss of the projection on the production frontier: $\left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right)$. We see that the these two terms are expressed in monetary values and comply with the notions of technical and allocative inefficiency, respectively.

Therefore, relating the profit technological gap, $\left(\mathbf{p} \cdot \mathbf{s}_{oEM(G)}^{+*} + \mathbf{w} \cdot \mathbf{s}_{oEM(G)}^{-*}\right)$, with the technical inefficiency measure $TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)$, we can obtain the following decomposition:

$$\Pi\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right) =$$
$$= TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) \times \left(\frac{\mathbf{p} \cdot \mathbf{s}_{oEM}^{+*} + \mathbf{w} \cdot \mathbf{s}_{oEM}^{-*}}{TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)}\right) + \Pi\left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}, \mathbf{w}, \mathbf{p}\right).$$

(28)

Here, the profit loss associated with the technological gap corresponds to the technical inefficiency of the firm itself, $TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)$, times a normalizing factor $NF_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$, which measures the profit loss per unit of technical inefficiency. This correspond to the technical profit inefficiency of the firm and, consequently, the remaining profit inefficiency captured in the last term, effectively corresponds to the allocative inefficiency of $\left(\mathbf{x}_o, \mathbf{y}_o\right)$. As already shown, this last inefficiency is the profit inefficiency of the firm's benchmark projection $\left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right)$.

The use of the normalizing factor, $NF_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$ in the above decomposition of profit inefficiency ensures that it is units' invariant. Note that the three terms of the equality are divided by this factor. When defining the normalization factor we consider whether the observation is technically inefficient: $TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) > 0$, with $\left(\mathbf{x}_o, \mathbf{y}_o\right) \neq \left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right)$, or technically efficient: $TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) = 0$, with $\left(\mathbf{x}_o, \mathbf{y}_o\right) = \left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right)$. Hence,

$$NF_{EM(G)}^{GDA}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right) =$$
$$= \begin{cases} \frac{\mathbf{p} \cdot \mathbf{s}_{EM(G)}^{+*} + \mathbf{w} \cdot \mathbf{s}_{EM}^{-*}}{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}, & \left(\mathbf{x}_o, \mathbf{y}_o\right) \neq \left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right), \\ k_{o\$}, \ k_o > 0, & \left(\mathbf{x}_o, \mathbf{y}_o\right) = \left(\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}\right). \end{cases}$$

(29)

where $k_{o\$}$ is a scalar, also expressed in monetary units, that simply translates the null technical efficiency score into the same currency values. With this qualification in mind we introduce the normalized expression of the *GDA*: $NII_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = TI_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right) +$

$AI_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)$. Specifically,

$$\underbrace{\frac{\Pi\left(\mathbf{w}, \mathbf{p}\right) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om}\right)}{NF_{EM(G)}^{GDA}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right)}}_{\text{Norm. Profit Inefficiency}} =$$

$$= \underbrace{TI_{EM(G)}\left(\mathbf{x}_o, \mathbf{y}_o\right)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right)}_{\text{Norm. Allocative Inefficiency}} \geq 0. \qquad (30)$$

where normalized allocative inefficiency corresponds to: $AI_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}\right) = AI_{GDA(G)}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right)/NF_{EM(G)}^{GDA}\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right)$.

BenchmarkingEconomicEfficiency.jl implements the profit inefficiency decomposition associated with the *GDA* relying on the enhanced Russell graph measure. This decomposition can be calculated both in normalized terms, (30), and monetary terms, (28). The syntax for the normalized (units' invariant) decompositions is the following:

```julia
julia> deaprofitgda(X, Y, W, P, :ERG, names = banks)
```

```
General Direct Approach Profit DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
Returns to Scale = VRS
Associated efficiency measure = ERG
-------------------------------------------------------------
                          Profit      Technical    Allocative
-------------------------------------------------------------
Export-Import Bank        0.0         0.0          0.0
Bank of Taiwan            0.299725    1.11022e-16  0.299725
Taipei Fubon Bank         0.550048    2.22045e-16  0.550048
Bank of Kaohsiung         2.72869     0.78295      1.94574
Land Bank                 7.96057e-15 -2.22045e-16 7.96057e-15
...
Hwatai Bank               13.5607     0.958676     12.602
Cota Bank                 7.01996     0.942322     6.07764
Industrial Bank of Taiwan 3.91122     -3.10862e-15 3.91122
Bank SinoPac              0.536348    0.226657     0.30969
Shin Kong Bank            2.71129     0.906448     1.80485
-------------------------------------------------------------
```

while the monetary decomposition can be obtained by including `monetary = true` in the above functions.

Finally, we can recover the information about the inefficiency measures by running the code:

```julia
julia> deaerg(X, Y, rts = :VRS, names = banks)
```

4. Measuring and decomposing profitability efficiency

The performance of firms from an economic perspective can be assessed through their *Profitability inefficiency*, considering as economic goal the maximization of revenue to cost. The profitability function is defined as the maximum observed value of that ratio considering the technology and the prices of inputs and outputs, i.e., $\Gamma\left(\mathbf{w}, \mathbf{p}\right) = \max_{\mathbf{x}, \mathbf{y}}\left\{\mathbf{p}\cdot\mathbf{y}/\mathbf{w}\cdot\mathbf{x} \mid \mathbf{x} \geq X\lambda, \mathbf{y} \leqslant Y\lambda, \lambda \geq \mathbf{0}\right\}, \mathbf{w} \in \mathbb{R}_{++}^{M}, \mathbf{p} \in \mathbb{R}_{++}^{N}$—see [3, Chap.4] for a discussion of its properties considering minimal regularity conditions.

A relevant technological characteristic of the profitability function is the existence of constant returns to scale at the maximizing benchmark, [4]. Georgescu-Roegen [62] argued in favor of the profitability function as "… an *economic* criterion on which to base the choice between two linear processes..".., which "*must be independent of the scale of production*, whereas $\mathbf{p}\cdot\mathbf{y}$, $\mathbf{w}\cdot\mathbf{x}$, and $\mathbf{p}\cdot\mathbf{y} - \mathbf{w}\cdot\mathbf{x}$ are not"—his italics and our notation. Note that having a measure of economic performance that is independent of returns to scale is also desirable when relating the generalized distance function to productivity indexes, since they should comply with the so-called proportionality property, which is verified when the technology satisfies constant returns to scale, [46].

Maximum profitability can be computed through the following program:

$$\Gamma\left(\mathbf{w}, \mathbf{p}\right) = \max_{\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}} \quad \sum_{n=1}^{N} p_n y_n / \sum_{m=1}^{M} w_m x_m$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq x_m, \ m = 1, \dots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_n, \quad n = 1, \dots, N, \qquad (31)$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda_j \geq 0, \qquad j = 1, \dots, J.$$

Note that this program is defined under the assumption of variable returns to scale, while at the optimal solution, represented by $\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*$, the technology is characterized by constant returns. For firm $\left(\mathbf{x}_o, \mathbf{y}_o\right) \in \mathbb{R}_+^{M+N}$, $\mathbf{x}_o \neq 0_M, \mathbf{y}_o \neq 0_N$, profitability inefficiency is the ratio of the firms' observed profitability divided by maximum profitability; i.e.,

$$\Gamma E\left(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}\right) = \frac{\Gamma_o}{\Gamma(\mathbf{w}, \mathbf{p})} = \frac{\mathbf{p}\cdot\mathbf{y}_o/\mathbf{w}\cdot\mathbf{x}_o}{\Gamma(\mathbf{p}, \mathbf{w})} =$$

$$= \frac{\sum_{n=1}^{N} p_n y_{on}/\sum_{m=1}^{M} w_m x_{om}}{\Gamma(\mathbf{p}, \mathbf{w})} \leq 1. \qquad (32)$$

Based on duality, profitability efficiency can be decomposed through graph multiplicative measures that simultaneously contract inputs and expand outputs. The hyperbolic measure introduced by Färe et al. [63, Chap. 5] evaluates firms' efficiency considering those changes. Later on, this measure was generalized by Chavas and Cox [64] who called it generalized distance function *GDF*. The *GDF*, whose properties are discussed by Pastor et al. [3, Chap.4], is defined as follows:

$$TE_{GDF}^{CRS}\left(\mathbf{x}, \mathbf{y}, \alpha\right)$$
$$= \min\left\{\delta^{CRS} > 0 : \left(\left(\delta^{CRS}\right)^{1-\alpha}\mathbf{x}, \left(\delta^{CRS}\right)^{-\alpha}\mathbf{y}\right) \in \partial^W\left(T^{CRS}\right), \ \alpha \in [0, 1]\right\}, \qquad (33)$$

where the parameter α sets the direction to the frontier. It is immediately clear that the *GDF* generalizes existing measures. When $\alpha = 0$ one obtains the input distance function, while if $\alpha = 1$ one obtains the output distance function. Both distance functions are considered in the next section when presenting the multiplicative decomposition of cost and revenue efficiency. Finally, the *GDF* also nests the hyperbolic measure when $\alpha = 0.5$

One can resort to Data Envelopment Analysis to calculate the efficiency of firm $\left(\mathbf{x}_o, \mathbf{y}_o\right)$ in terms of the *GDF*. This requires solving next program:

$$TE_{GDF}^{CRS}\left(\mathbf{x}_o, \mathbf{y}_o, \alpha\right) = \min_{\delta, \boldsymbol{\lambda}} \delta^{CRS}$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq \left(\delta^{CRS}\right)^{1-\alpha} x_{om}, \ m = 1, \dots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_{on}/\left(\delta^{CRS}\right)^{\alpha}, \ n = 1, \dots, N, \qquad (34)$$

$$\lambda_j \geq 0.$$

The fact that the reference benchmark should produce under constant returns to scale implies that any firm producing under decreasing or increasing returns to scale incurs scale inefficiencies. Scale inefficiency is then another source of profitability that must be considered in the model. Consequently, the sources of productive inefficiency can be technical, i.e., the firm lays inside the technology set, or related to a suboptimal scale. This implies that technical efficiency under constant returns to scale can be decomposed into the usual technical efficiency under variables returns to scale times a factor representing scale inefficiency; i.e $TE_{GDF}^{CRS}\left(\mathbf{x}, \mathbf{y}, \alpha\right) = TE_{GDF}\left(\mathbf{x}, \mathbf{y}, \alpha\right) \times SE_{GDF}\left(\mathbf{x}, \mathbf{y}, \alpha\right) = \delta \times \left(\delta^{CRS}/\delta\right) \leq 1$. In this expression, $TE_{GDF}\left(\mathbf{x}, \mathbf{y}, \alpha\right) = \delta$ is the generalized distance function under the variable returns to scale technology T, rather than

its constant returns specification in (33). Its calculation requires solving program (34) with the additional constraint: $\sum_{j=1}^{J} \lambda_j = 1$. Since both programs calculating maximum profitability and the generalized distance function are non-linear, **BenchmarkingEconomicEfficiency.jl** uses the **JuMP.jl** environment, coupled with the 'Ipopt' solver.

Once both technical efficiency measures have been calculated, we rely on the duality theory relating the profitability function and the *GDF*, [4]. These authors establish the Fenchel-Mahler inequality by which profitability efficiency (32), is greater or equal in value to the technical efficiency measure under CRS, i.e., $\Gamma E(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}) \geq TE_{\mathrm{GDF}}^{\mathrm{CRS}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)$. Considering the decomposition of $TE_{\mathrm{GDF}}^{\mathrm{CRS}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)$ into variables returns efficiency and scale efficiency, and closing the inequality with the addition of a residual term capturing allocative inefficiency, yields:

$$\underbrace{\frac{\sum_{n=1}^{N} p_n y_{on} / \sum_{m=1}^{M} w_m x_{om}}{\Gamma(\mathbf{p}, \mathbf{w})}}_{\text{Profitability Efficiency}} = \underbrace{TE_{\mathrm{GDF}}^{\mathrm{CRS}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}_{\text{Graph Technical Efficiency } CRS} \times \underbrace{AE_{\mathrm{GDF}}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})}_{\text{Allocative Efficiency}} =$$

$$\underbrace{TE_{\mathrm{GDF}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}_{\text{Graph Technical Efficiency } VRS} \times \underbrace{SE_{\mathrm{GDF}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}_{\text{Scale Efficiency}} \times \underbrace{AE_{\mathrm{GDF}}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})}_{\text{Allocative Efficiency}} \geq 0. \quad (35)$$

As for the relevant properties of the profitability decomposition using the *GDF*, it fails to comply with the indication property by projecting the observation to the weakly efficient set of the technology. This implies that, when calculating $TE_{GDF}(\mathbf{x}_o, \mathbf{y}_o, \alpha)$, there may exist further individual reductions of inputs or expansion of outputs—both under CRS or VRS. Consequently, **BenchmarkingEconomicEfficiency.jl** performs a subsequent additive model to determine if these slacks exist. Finally, it is relevant to note that since the profit efficiency measure is multiplicative, the decomposition of profitability efficiency satisfies both the essential property and its extension, see Propositions 1 and 4 in [48].

We now show how to implement the decomposition of profitability efficiency and its decomposition into technical efficiency, scale efficiency and allocative efficiency. The syntax is as follows:

```
julia> deaprofitability(X, Y, W, P, alpha = 0.5, names
= banks)
```

```
Profitability DEA Model
DMUs = 31; Inputs = 3; Outputs = 2
alpha = 0.5; Returns to Scale = VRS
-------------------------------------------------------------
                      Profitability    CRS      VRS     Scale    Allocative
-------------------------------------------------------------
Export-Import Bank      0.999999    0.999997  0.999999  0.999998   1.0
Bank of Taiwan          0.393451    1.0       1.0       1.0        0.393451
Taipei Fubon Bank       0.444591    0.614278  1.0       0.614278   0.723762
Bank of Kaohsiung       0.400458    0.48107   0.700667  0.686588   0.832432
Land Bank               0.501915    0.830479  1.0       0.830479   0.604368
...
Hwatai Bank             0.272076    0.328527  0.496978  0.661049   0.828168
Cota Bank               0.26831     0.475605  0.565756  0.840654   0.564144
Industrial Bank of Taiwan 0.448944  0.947522  0.999997  0.947525   0.473809
Bank SinoPac            0.461054    0.647851  0.896737  0.722453   0.711667
Shin Kong Bank          0.15606     0.287427  0.725546  0.396153   0.542955
-------------------------------------------------------------
```

We can learn about the technical efficiency measures under constant and variable returns by calling the corresponding functions, which include the calculation of the input and output slacks in a second stage:

```
julia> deagdf(X, Y, alpha = 0.5, rts = :CRS, names =
banks)
```

```
julia> deagdf(X, Y, alpha = 0.5, rts = :VRS, names =
banks)
```

5. Measuring and decomposing cost and revenue efficiency

Here we summarize how to decompose economic efficiency from the partial perspectives represented by firms' cost and revenue. Under these two alternative dimensions it is assumed that firms either aim at minimizing the cost of producing a vector of outputs, or maximize the

revenue they can obtain from using a given level of inputs. Economic efficiency from a revenue or cost perspective can be measured and decomposed multiplicatively or additively. The functional form depends on the chosen (in)efficiency measure. The multiplicative approach is most well-known as it corresponds to Farrell's original proposal. For this reason we initiate the presentation of these models with this approach and then follow with the many additive decompositions that have been proposed in the literature, based on the different inefficiency measures *EM(G)* already presented when decomposing profit inefficiency, but with a partial input our output orientation, i.e., *EM(I)* or *EM(O)*.

As before, measuring economic efficiency requires the definition of the optimal economic goal to be achieved by the firms. In this case, minimizing the cost of producing output $\mathbf{y}_o$, represented by the input set $L(\mathbf{y}_o)$, or the maximizing the revenue obtainable from input $\mathbf{x}_o$, represented by the output set $P(\mathbf{x}_o)$. These functions are expressed as follows: $C(\mathbf{y}, \mathbf{w}) = \min_{\mathbf{x}} \{\mathbf{w} \cdot \mathbf{x} \mid \mathbf{x} \geq X\lambda, \mathbf{y}_o \leqslant Y\lambda, \mathbf{e}\lambda = 1, \lambda \geq \mathbf{0}\}$, $\mathbf{w} \in \mathbb{R}_{++}^{M}, \mathbf{y}_o \geq \mathbf{0}_N$, and $R(\mathbf{x}, \mathbf{p}) = \max_{\mathbf{y}} \{\mathbf{p} \cdot \mathbf{y} \mid \mathbf{x}_o \geq Y\lambda, \mathbf{e}\lambda = 1, \lambda \geq \mathbf{0}\}$, $\mathbf{p} \in \mathbb{R}_{++}^{N}, \mathbf{x}_o \geq \mathbf{0}_N$, whose properties under minimal regularity conditions are discussed in [3, Chap.2].

Minimum cost along with the optimal input quantities can be calculated through DEA by solving the following model

$$C(\mathbf{y}_o, \mathbf{w}) = \min_{\mathbf{x}, \lambda} \sum_{m=1}^{M} w_m x_m$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq x_m, \ m = 1, \ldots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_{on}, \ n = 1, \ldots, N, \quad (36)$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda \geq 0,$$

while the program determining maximum revenue and its associated optimal output quantities is:

$$R(\mathbf{x}_o, \mathbf{p}) = \max_{\mathbf{y}, \lambda} \sum_{n=1}^{N} p_n y_n$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq x_{om}, \ m = 1, \ldots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_n, \ n = 1, \ldots, N, \quad (37)$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda \geq 0.$$

5.1. Multiplicative models based on the radial input-and output-oriented measures

It is now possible to define cost efficiency and revenue efficiency multiplicatively as follows. For firm $(\mathbf{x}_o, \mathbf{y}_o)$ *cost efficiency* corresponds to minimum cost over observed cost, i.e., $CE(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}) = C(\mathbf{y}_o, \mathbf{w})/C_o = C(\mathbf{y}_o, \mathbf{w})/\mathbf{w} \cdot \mathbf{x}_o = C(\mathbf{y}_o, \mathbf{w})/\sum_{m=1}^{M} w_m x_{om} \leq 1$. Alternatively, *revenue efficiency* corresponds to observed revenue to maximum revenue: $RE(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}) = R_o/R(\mathbf{x}_o, \mathbf{p}) = \mathbf{p} \cdot \mathbf{y}_o/R(\mathbf{x}_o, \mathbf{p}) = \sum_{n=1}^{N} p_n y_{on}/R(\mathbf{x}_o, \mathbf{p}) \leq 1$.

5.1.1. Cost efficiency

Färe and Primont [2] present the duality results that allow to relate numerically the value of cost efficiency with that of Farrell's radial input measure $R(I)$. This last measure represents the maximum

equiproportional reduction in the observed input vector necessary to reach the production frontier. For the firm under evaluation $(\mathbf{x}_o, \mathbf{y}_o)$, this can be computed through the following DEA model:[7]

$$TE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o) = \min_{\theta, \boldsymbol{\lambda}} \quad \theta$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq \theta x_{om}, \ m = 1, \dots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq y_n, \ n = 1, \dots, N, \quad (38)$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda \geq 0.$$

Given the optimal value θ^*, the set of constraints ensure that the projection $(\theta^* \mathbf{x}_o, \mathbf{y}_o)$ belongs to the technology $L(\mathbf{y}_o)$. The program searches for the value of θ that projects $\mathbf{x}_o$ radially to its frontier benchmark represented by $\hat{\mathbf{x}}_o = \theta^* \mathbf{x}_o$. A value of $\theta^* = 1$ indicates that the firm is technically efficient, while if $\theta^* < 1$, then the firm is inefficient and its benchmark projection on the frontier, given by $(\boldsymbol{\lambda} X, \boldsymbol{\lambda} Y)$, dominates $(\mathbf{x}_o, \mathbf{y}_o)$.

We now show the duality result relating cost efficiency and Farrell's radial input efficiency measure $R(I)$, allowing the decomposition of cost efficiency: $CE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}) = TE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o) \times AE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w})$, i.e.,

$$\underbrace{\frac{C(\mathbf{y}_o, \mathbf{w})}{\sum_{m=1}^{M} w_m x_{om}}}_{\text{Cost Efficiency}} = \underbrace{\theta^*}_{\text{Technical Efficiency}} \times \underbrace{AE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w})}_{\text{Allocative Efficiency}} \leq 1. \quad (39)$$

Cost efficiency may be calculated through the following syntax:

```julia
julia> deacost(X, Y, W, names = banks)
```

while the input efficiency measure can be recalled through the following function:

```julia
julia> dea(X, Y, orient = :Input, rts = :VRS, names = banks)
```

One drawback of the multiplicative decomposition of cost efficiency using Farrell's radial input measure is its inability to comply with the property of indication, implying that the benchmark projection $\hat{\mathbf{x}}_o = \theta^* \mathbf{x}_o$ may not be Pareto-Koopmans efficient, and therefore there may exist input and output slacks. Accordingly, decomposition (39) overestimates the value of technical efficiency and, correspondingly, underestimates that of allocative efficiency. One the other hand, the multiplicative decompositions of cost and revenue efficiencies satisfy the essential property and its extension, see Propositions 1 and 4 in [48], so allocative efficiency is measured correctly. As before, **BenchmarkingEconomicEfficiency.jl** performs a two-stage analysis to check if these slacks exist. The results are automatically obtained when running the function calculating the input technical efficiency measure above.

5.1.2. Revenue efficiency

Regarding the multiplicative decomposition of revenue efficiency, the standard duality results relate the revenue function with Farrell's radial output measure $R(O)$. This measure represents the maximum equiproportional expansion in the observed output vector necessary to reach the production frontier. For the firm under evaluation $(\mathbf{x}_o, \mathbf{y}_o)$, the following DEA program allows its measurement:

$$TE_{R(O)}(\mathbf{x}_o, \mathbf{y}_o) = \max_{\xi, \boldsymbol{\lambda}} \quad \xi$$

$$\text{s.t.} \quad \sum_{j=1}^{J} \lambda_j x_{jm} \leq x_{om}, \ m = 1, \dots, M,$$

$$\sum_{j=1}^{J} \lambda_j y_{jn} \geq \xi y_{on}, \ n = 1, \dots, N, \quad (40)$$

$$\sum_{j=1}^{J} \lambda_j = 1,$$

$$\lambda \geq 0.$$

where ξ^* denotes now the optimal solution. Again, the constraints ensure that the firm $(\mathbf{x}_o, \xi^* \mathbf{y}_o)$ belong to the technology T. The program searches for the maximum value of ξ that projects radially the output vector $\mathbf{y}_o$ to its frontier benchmark represented by $\hat{\mathbf{y}}_o = \xi^* \mathbf{y}_o$. When $\xi^* = 1$ the firm is technically technically efficient, while if $1/\xi^* < 1$ the firm is technically inefficient. In this case, its benchmark projection on the frontier, given by $(\boldsymbol{\lambda} X, \boldsymbol{\lambda} Y)$ dominates $(\mathbf{x}_o, \mathbf{y}_o)$.

We now present the duality result describing the relationship between the revenue function and Farrell's radial output efficiency measure. This allows decomposing revenue efficiency into technical and allocative terms: $RE_{R(O)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}) = TE_{R(O)}(\mathbf{x}_o, \mathbf{y}_o) \times AE_{R(O)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p})$, i.e.,

$$\underbrace{\frac{\sum_{n=1}^{N} p_n y_{on}}{R(\mathbf{x}_o, \mathbf{p})}}_{\text{Revenue Efficiency}} = \underbrace{1/\xi^*}_{\text{Technical Efficiency}} \times \underbrace{AE_{R(O)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p})}_{\text{Allocative Efficiency}} \leq 1 \quad (41)$$

Revenue efficiency can be calculated through the following syntax:

```julia
julia> dearevenue(X, Y, P, names = banks)
```

while Farrell's output technical efficiency measure, along with any slacks, can be recalled though the following function:

```julia
julia> dea(X, Y, orient = :Output, rts = :VRS, names = banks)
```

Again, the Farrell's output radial measure fails to satisfy the indication property, implying that the benchmark projection $\hat{\mathbf{y}}_o = \xi^* \mathbf{y}_o$ may not be Pareto-Koopmans efficient. Consequently there may exist input and output slacks. Accordingly, decomposition (41) overestimates the level of technical efficiency and underestimates that of allocative efficiency. Again, let us define the corresponding vectors of inputs and outputs slacks: $\mathbf{s}^- \in \mathbb{R}^M$ and $\mathbf{s}^+ \in \mathbb{R}^N$. Then $\mathbf{s}^- = \mathbf{x}_o - \mathbf{X}\boldsymbol{\lambda} \geq 0$, and $\mathbf{s}^+ = \mathbf{Y}\boldsymbol{\lambda} - \xi^* \mathbf{y}_o \geq 0$ for the optimal values $(\xi, \boldsymbol{\lambda})$. Running the Julia function above performs a two-stage analysis that includes the calculation of the slacks, which are reported along with the output technical efficiency measure.

5.2. Additive models based on inefficiency measures

Cost inefficiency can be defined also additively by subtracting minimum cost from observed cost: $CI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}) = C_o - C(\mathbf{y}_o, \mathbf{w}) = \mathbf{w} \cdot \mathbf{x}_o - C(\mathbf{y}_o, \mathbf{w}) = \sum_{m=1}^{M} w_m x_{om} - C(\mathbf{y}_o, \mathbf{w}) \geq 0$. Alternatively, *revenue inefficiency* is defined as the difference between maximum revenue and observed revenue: $RI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}) = R(\mathbf{x}_o, \mathbf{p}) - R_o = R(\mathbf{x}_o, \mathbf{p}) - \mathbf{p} \cdot \mathbf{y}_o = R(\mathbf{x}_o, \mathbf{p}) - \sum_{n=1}^{N} p_n y_{on} \geq 0$. Therefore if $CI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}) = 0$ the firm minimizes cost, and the greater the cost inefficiency value, the greater the excess in cost of the firm with respect to the economic benchmark. In the same vein, if $RI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}) = 0$ the firm maximizes revenue, so the greater the revenue inefficiency value, the greater the revenue loss.

Cost and revenue inefficiency can be decomposed using the different technical inefficiency measures already presented when decomposing profit efficiency, but defined either on the input or output production

---

[7] This program represents the 'envelopment form' introduced by Charnes et al. [26] under constant returns to scale. Dual programs corresponding to the 'multipliers form' are presented in [13]. Also, as in the previous programs identifying minimum cost and maximum revenue, the technology is characterized by variables returns to scale, with the sum of the lambdas equal to one: $\sum_{j=1}^{J} \lambda_j = 1$, see [39].

possibility sets, $L(\mathbf{y}_o)$ or $P(\mathbf{x}_o)$, respectively. Regarding cost inefficiency, it is decomposed into an input oriented inefficiency measure, generally denoted by $TI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o)$—where the subscript $EM(I)$ represents a specific measure, plus the allocative term. Analogously, revenue inefficiency is decomposed into an output oriented inefficiency measure, denoted by $TI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o)$, plus the corresponding allocative term. Regarding efficiency measurement, firms with null efficiency scores, $EM(I) = EM(O) = 0$, are technically efficient. On the contrary, the firm is technically inefficient when the inefficiency scores are positive. After calculating input and output technical inefficiencies, and relying on their duality with the cost and revenue functions, we can establish two Fenchel-Mahler inequalities by which:

- *Normalized* cost inefficiency: $NCI(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}) = CI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w})$ $/NF_{EM(I)}$, is greater or equal in value to the input-oriented technical inefficiency measure, i.e., $NCI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w})/NF_{EM(I)} \geq TI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o)$, $CI(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}})$ and, correspondingly,

- *Normalized* revenue inefficiency $NRI(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{p}}) = RI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p})$ $/NF_{EM(O)}$, is greater or equal in value to the output-oriented technical inefficiency measure, i.e., $NRI(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p})/NF_{EM(O)} \geq TI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o)$.

In both cases the divisors $NF_{EM(I)}$ and $NF_{EM(O)}$ are normalizing factors derived from the duality relationship. Afterwards, a residual representing normalized allocative inefficiency is obtained by closing the inequalities. Cost allocative inefficiency is the costs excess incurred by the firm (or its projection) by not demanding the optimal quantities of inputs quantities. Revenue allocative inefficiency is the revenue loss incurred by the firm (or its projection) by not supplying the optimal output quantities.

Then, normalized cost and revenue inefficiencies can be decomposed as follows:

$$\underbrace{\frac{\sum_{m=1}^{M} w_m x_{om} - C(\mathbf{y}_o, \mathbf{w})}{NF_{EM(I)}}}_{\text{Norm. Cost Inefficiency}}$$

$$= \underbrace{TI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Input Technical Inefficiency}} + \underbrace{AI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}})}_{\text{Norm. Allocative Inefficiency}} \geq 0, \qquad (42)$$

$$\underbrace{\frac{R(\mathbf{x}_o, \mathbf{p}) - \sum_{n=1}^{N} p_n y_{on}}{NF_{EM(O)}}}_{\text{Norm. Revenue Inefficiency}}$$

$$= \underbrace{TI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Output Technical Inefficiency}} + \underbrace{AI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0. \qquad (43)$$

As with profit inefficiency, the additive decomposition of cost and revenue inefficiency can be done using the previous eight technical inefficiency models included in **BenchmarkingEconomicEfficiency.jl**: Russell, weighted additive (WA), enhanced Russell graph (ERG=SBM), directional distance function (DDF), Hölder distance function (HDF), modified directional distance fucntion (MDDF), reverse directional distance fucntion (RDDF) and the general direct approach (GDA). Again, since all decompositions are normalized they satisfy the property of commensurability (or units' invariance), and therefore do not depend on the measurement units of quantities and prices. Additionally, their strengths and weaknesses regarding the indication and essential properties are equivalent to their profit inefficiency counterparts already discussed. Due to space limitations and since these additive decompositions of cost and revenue inefficiency are particular cases of the profit model already presented, we do not report the syntax or numerical results for the Taiwanese banks. This exercise is left to the readers, who can run the associated Jupyter notebooks accompanying this paper on its dedicated webpage: https://benchmarkingeconomicefficiency.com/notebooks/.

6. Empirical results: Profit inefficiency of Taiwanese banks

We now illustrate the empirical applications of economic efficiency using a real-life dataset of financial institutions. Earlier studies resorted to the radial multiplicative model presented by Farrell to decompose cost efficiency, while most recent analyses of profit inefficiency rely on the directional distance function model. One appealing feature of this study is that all previous decompositions are applied to the same dataset of Taiwanese banks. Hence, we can determine to what extent the various profit, technical and allocative inefficiency measures are empirically different. Here we focus on the decomposition of profit inefficiency and leave to the reader the comparison of economic performance considering the profitability, or the cost or revenue models.

While the value of profit inefficiency in monetary units is the same for all decompositions—i.e., the numerator in the left hand side of (2): $\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^{N} p_n y_{on} - \sum_{m=1}^{M} w_m x_{om} \right)$, its *normalized* values $N\Pi I(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})$ differ because each decomposition has its own normalization factor, $NF_{EM(G)}$, i.e., the denominator in the left hand side of (2). This implies that there is a trade-off between the commensurability property (units' invariance) of the profit inefficiency measures, and their numerical comparability. Therefore, the profit, technical, and allocative inefficiency values are not directly comparable *vis-à-vis* between models.

Fig. 1 depicts box-plots of the values of the normalized profit inefficiencies and their technical and allocative components for the eight alternative models reported in previous sections: Russell, weighted additive (WA), enhanced Russell graph (ERG=SBM), directional distance function (DDF), Hölder distance function (HDF), Modified DDF (MDDF), Reverse DDF (RDDF) and generalized direct approach (GDA). Each box depicts the distance between the 1st and 3rd quartiles of the profit inefficiency distribution, while the horizontal lines correspond to the medians. The variability of the distributions within these interquartile ranges appears to be rather small for most of the models, except for the WA (considering the measure of inefficiency proportions, MIP, as weights) and the MDDF. Finally, the distance between the upper and lower whiskers represent one and half times the interquartile range, while the dots identify outliers laying beyond those values. In all cases, three banks (Export-Import Bank, Land bank and Mega Bank) maximize profit, thereby being profit, technical and allocative efficient. The rest of the sample banks are profit inefficient. Nevertheless, ten out of these banks are technically efficient thereby defining the frontier, which implies that the only source of profit inefficiency is allocative. The remaining banks are both technical and allocative inefficient.

Since the numerical values of *normalized* profit inefficiency cannot be directly compared across models, we determine whether the different rankings based on the inefficiency scores are compatible or not. Table 3 displays Kendall's τ rank correlation coefficients. The results for all bilateral comparisons show positive and significant correlations. The lower correlations are found between the directional distance function (DDF)—a weak efficiency measure that may project observations to facets of the frontier that are dominated (i.e., slacks may exist), and strong efficiency measures like the Russell, weighted additive (WA) and enhanced Russell graph (ERG). Finally, some correlations between measures are the same, showing that some profit inefficiency measures yield the exact same ranking. This is the case of the DDF with the Hölder and modified DDF (MDDF), 0.66, or the DDF with the reverse DDF and general direct approach (GDA), 0.76. This is an expected result since these pairs of measures are related. For example, the measures obtained under the RDDF and the GDA approaches consider the ERG as the initial efficiency measure projecting the observations to the frontier, which are subsequently used to calculate and decompose profit inefficiency. That is why the rank correlation between the RDDF and the GDA is one. In general, it is reassuring that the ranking of banks remains stable given the range of models proposed in the literature.
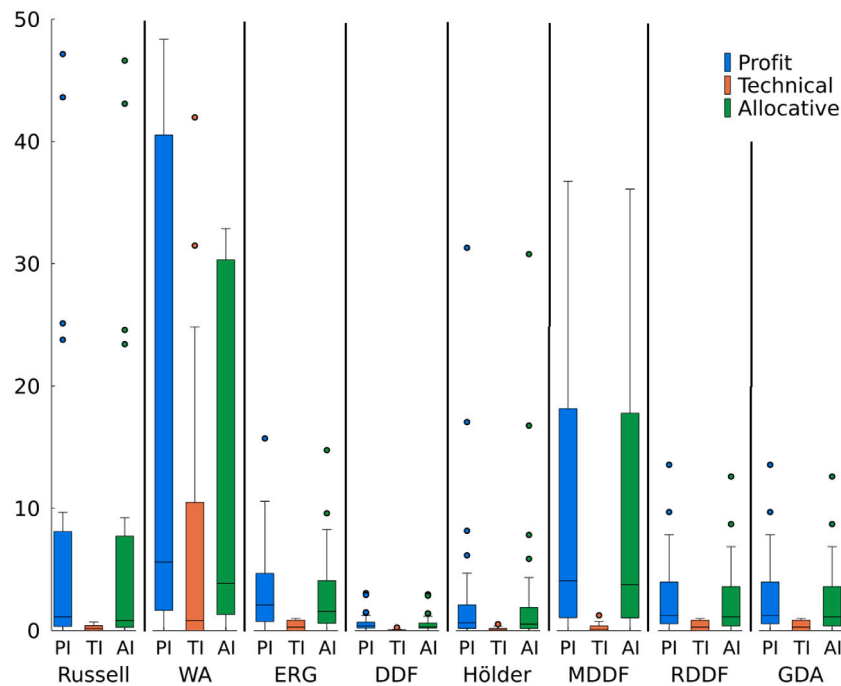
Fig. 1. Box-plots of profit inefficiency.

Table 3
Kendall's τ rank correlation coefficients of inefficiency scores.

	Russell	WA	ERG	DDF	Holder	MDDF	RDDF	GDA
Russell	1							
WA	0.9957*	1						
ERG	0.9140*	0.9183*	1					
DDF	0.6387*	0.6430*	0.5613*	1				
Holder	0.8839*	0.8882*	0.8151*	0.6602*	1			
MDDF	0.8409*	0.8452*	0.7978*	0.6602*	0.9312*	1		
RDDF	0.8280*	0.8323*	0.8968*	0.5011*	0.7892*	0.7634*	1	
GDA	0.8280*	0.8323*	0.8968*	0.5011*	0.7892*	0.7634*	1.0000*	1

*Denotes significance at the 0.01 level.

7. Conclusions

BenchmarkingEconomicEfficiency.jl is a Julia package that offers, for the first time, a complete set of functions to calculate and decompose the most relevant measures of economic efficiency using data envelopment analysis. The package unifies the presentation of the different methods proposed in the literature in the last two decades within an integrated framework that uses a standardized notation. The package covers profit inefficiency, profitability efficiency, cost (in)efficiency and revenue (in)efficiency. The models implemented include the classical multiplicative approaches decomposing cost efficiency and revenue efficiency using Farrell's radial measures, and the popular approach to decompose profit inefficiency considering the directional distance function. The latest developments regarding economic efficiency measurement are also incorporated by considering the multiplicative decomposition of profitability efficiency using the generalized distance function, and several options for the additive decomposition of profit inefficiency. In this last case we consider the Russell measures, the weighted additive measures, the enhanced Russell measures (or slack-based measures), the Hölder distance functions, and the newest proposals based on the modified and reverse directional distance functions, as well as the generalized direct approach.

Each decomposition is characterized by a set of features. The desirable properties that the economic efficiency measures should satisfy, along with those of their technical and allocative components, are discussed by [3, Chap. 14]. They conclude that there is a trade-off between the alternative models in terms of the indication and essentiality properties discussed here. Each technical efficiency measure has its pros and cons that are passed to the decomposition of economic efficiency. Therefore they conclude that there is not a superior approach *per se*. Nevertheless, they explore the set of properties for each measure and, based on their interpretability, comparability and consistency regarding the measurement of allocative efficiency (the so-called essential properties), conclude that the radial approach for multiplicative decompositions and the directional distance functions for additive decompositions (including its variations) would be preferred. In this regard, choosing a specific technical efficiency measure ultimately depends on the preferences of the researcher and the characteristics of the benchmarking exercise. Finally, we have shown the relationship between the alternative measures of economic efficiency, the characteristics of the technology, and how they can be decomposed into technical and allocative terms. For this purpose we exemplify all the models considering a common dataset of banks.

Our goal is that the package can be used as a guide for those involved in economic efficiency measurement, by highlighting its potential applicability. We also show how to structure the data, the syntax required to run the different functions and how to interpret the results. This makes the new package a relevant self-contained software for these benchmarking techniques programmed in the Julia language. An advantage of open software is that users can delve into how the

different DEA models are programmed. Moreover, they can modify the code so as to suit their needs. This will allow the scientific community to revise it and contribute to its improvement. We thank our readers in advance for providing such valuable input and enhancing the options available to benchmark economic efficiency.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

All supplemental information (source code, data and examples) for replicating all the results is available at https://benchmarkingeconomicefficiency.com, including a series of Jupyter notebooks that ease the implementation of the models and learning process.

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References

[1] Farrell M. The measurement of productive efficiency. J Roy Statist Soc Ser A 1957;120(3):253–81. http://dx.doi.org/10.2307/2343100.

[2] Färe R, Primont D. Multi-output production and duality: theory and applications. Boston/London/Dordrecht: Kluwer Academic Publishers; 1995.

[3] Pastor J, Aparicio J, Zofío JL. Benchmarking economic efficiency: technical and allocative fundamentals. Springer, Cham; 2022.

[4] Zofío JL, Prieto AM. Return to dollar, generalized distance function and the Fisher productivity index. Span Econ Rev 2006;8:113–38. http://dx.doi.org/10.1007/s10108-006-9004-0.

[5] Fried HO, Lovell CAK, Schmidt SS. In: Fried HO, Lovell CAK, Schmidt SS, editors. The measurement of productifve efficiency and productivity change. Oxford University Press; 2008, p. 3–91.

[6] Grifell-Tatjé E, Lovell C. Productity accounting: the economics of business performance. Cambridge, Mass: Cambridge University Press; 2015.

[7] Paradi J, Sherman HD, Tam FK. Data envelopment analysis in the financial services industry. a guide for practitioners and analysts. Springer-Verlag; 2018.

[8] Silva E, Stefanou SE, Oude Lansink A. Dynamic efficiency and productivity measurement. Oxford University Press; 2020.

[9] StataCorp. Stata statistical software: release 17. College Station, TX: StataCorp LLC; 2021, URL http://www.stata.com.

[10] Lee C. Allocative efficiency analysis using DEA in stata. Stata Users Group; 2012, SAN12 Stata Conference 1.

[11] Econometric Software, Inc. LIMDEP. Plainview, NY: Econometric Software, Inc; 2009, URL http://www.limdep.com/.

[12] Emrouznejad A, Cabanda E, editors. Managing service productivity: using frontier efficiency methodologies and multicriteria decision making for improving service performance. Berlin: Springer-Verlag; 2014.

[13] Cooper WW, Seiford LM, Tone K. Data envelopment analysis: a comprehensive text with models, applications, references and DEA-Solver software. Princeton: Princeton University Press; 2007.

[14] Bogetoft P, Otto L. Benchmarking with DEA, SFA, and R. New York: Springer-Verlag; 2011.

[15] Wilson PW. FEAR: A software package for frontier efficiency analysis with R. Socio-Econ Plan Sci 2008;42(4):247–54. http://dx.doi.org/10.1016/j.seps.2007.02.001.

[16] Álvarez IC, Barbero J, Zofío JL. A data envelopment analysis toolbox for MATLAB. J Stat Softw 2020;95(3):1–49. http://dx.doi.org/10.18637/jss.v095.i03.

[17] R Core Team. R: A language and environment for statistical computing. Vienna, Austria: R Foundation for Statistical Computing; 2020, URL https://www.R-project.org/.

[18] The MathWorks, Inc. MATLAB — *the language of technical computing*, version R2017b (8.5). Natick, Massachusetts: The MathWorks, Inc; 2017, URL http://www.mathworks.com/products/matlab/.

[19] Sheel H. EMS: efficiency measurement system user's manual. Dortmund: University of Dortmund; 2000.

[20] Sherman HD, Zhu J. Service productivity management: improving service productivity using data envelopment analysis. New York: Springer-Verlag; 2006, http://dx.doi.org/10.1007/0-387-33231-6.

[21] Zhu J. Quantitative models for performance evaluation and benchmarking: data envelopment analysis with spreadsheets and DEA excel solver. third ed.. Springer-Verlag; 2014, http://dx.doi.org/10.1007/978-1-4757-4246-6.

[22] Aparicio Baeza J, Ortiz Henarejos L, Pastor Ciurana J. Design and implementation of a Shiny interactive web application by RStudio for estimating data envelopment analysis efficiency measures. Boletín de Estadística e Investigación Operativa 2018;34(1):25–50.

[23] Benitez R, Coll-Serrano V, Bolós V. deaR-Shiny: An interactive web app for data envelopment analysis. Sustainability 2021;13(12):6774. http://dx.doi.org/10.3390/su13126774.

[24] Daraio C, Kerstens K, Nepomuceno T, Sickles RC. Productivity and efficiency analysis software: An exploratory bibliographical survey of the options. J Econ Surv 2019;33(1):85–100. http://dx.doi.org/10.1111/joes.12270.

[25] Alinezhad A, Khalili J, Mirtaleb S. Introduction and comparison of data envelopment analysis software packages. New York: Nova Science Publishers, Inc.; 2021.

[26] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision making units. European J Oper Res 1978;2(6):429–44. http://dx.doi.org/10.1016/0377-2217(78)90138-8.

[27] Bezanson J, Edelman A, Karpinski S, Shah VB. Julia: A fresh approach to numerical computing. SIAM Rev 2017;59(1):65–98. http://dx.doi.org/10.1137/141000671.

[28] Dunning I, Huchette J, Lubin M. JuMP: A modeling language for mathematical optimization. SIAM Rev 2017;59(2):295–320. http://dx.doi.org/10.1137/15M1020575.

[29] Shephard R. Theory of cost and production functions. Princeton, NJ: Princeton University Press; 1970.

[30] Färe R, Lovell CK. Measuring the technical efficiency of production. J Econom Theory 1978;19(1):150–62. http://dx.doi.org/10.1016/0022-0531(78)90060-1.

[31] Cooper W, Pastor J, Aparicio J, Borras F. Decomposing profit inefficiency in DEA through the weighted additive model. European J Oper Res 2011;212:411–6. http://dx.doi.org/10.1016/j.ejor.2011.01.054.

[32] Tone K. A slacks-based measure of efficiency in data envelopment analysis. European J Oper Res 2001;130(3):498–509. http://dx.doi.org/10.1016/S0377-2217(99)00407-5.

[33] Pastor JT, Ruiz JL, Sirvent I. An enhanced DEA russell graph efficiency measure. European J Oper Res 1999;115(3):596–607. http://dx.doi.org/10.1016/S0377-2217(98)00098-8.

[34] Chambers RG, Chung Y, Färe R. Profit, directional distance functions, and Nerlovian efficiency. J Optim Theory Appl 1998;98(2):351–64. http://dx.doi.org/10.1023/a:1022637501082.

[35] Aparicio J, Pastor JT, Ray SC. An overall measure of technical inefficiency at the firm and at the industry level: The 'lost profit on outlay'. European J Oper Res 2013;226(1):154–62. http://dx.doi.org/10.1016/j.ejor.2012.10.028.

[36] Pastor J, Aparicio J, Alcaraz J, Vidal F, Pastor D. In: Aparicio J, Lovell C, Pastor J, editors. Advances in efficiency and productivity. Cham: Springer; 2016.

[37] Briec W. Hölder distance function and measurement of technical efficiency. J Product Anal 1998;11:111–31.

[38] Pastor J, Zofío JL, Aparicio J, Pastor D. A general direct approach for decomposing profit inefficiency. Omega 2023;102889. http://dx.doi.org/10.1016/j.omega.2023.102889.

[39] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Manage Sci 1984;30(9):1078–92. http://dx.doi.org/10.1287/mnsc.30.9.1078.

[40] Juo J-C, Fu T-T, Yu M-M, Lin Y-H. Profit-oriented productivity change. Omega 2015;57:176–87. http://dx.doi.org/10.1016/j.omega.2015.04.013.

[41] Balk B. Profit-oriented productivity change: A comment. Omega 2018;78:176–8. http://dx.doi.org/10.1016/j.omega.2017.06.011.

[42] Aparicio A, Zofío J. Decomposing profit change: Konüs, bennet and luenberger indicators. Socio-Econ Plan Sci 2023;87(B):101573. http://dx.doi.org/10.1016/j.seps.2023.101573.

[43] Sealey CW, Lindley JT. Inputs, outputs, and a theory of production and cost at depository financial institutions. J Finance 1977;32(4):1251–66. http://dx.doi.org/10.1111/j.1540-6261.1977.tb03324.x.

[44] Shen Z, Li J, Vardanyan M, Wang B. Nonparametric shadow pricing of non-performing loans: A study of the Chinese banking sector. Ann Oper Res 2022. http://dx.doi.org/10.1007/s10479-022-05088-2.

[45] Balk BM, Zofío JL. Symmetric decompositions of cost variation. European J Oper Res 2020;285(3):1189–98. http://dx.doi.org/10.1016/j.ejor.2020.02.034.

[46] Balk BM. Productivity. concepts, measurement, aggregation and decomposition. Springer, Cham, Switzerland; 2021.

[47] Charnes A, Cooper W, Golany B, Seiford L, Stutz J. Foundations of data envelopment analysis for Pareto-Koopmans efficient empirical productionfunctions. J Econometrics 1985;30(1):91–107. http://dx.doi.org/10.1016/0304-4076(85)90133-2.

[48] Aparicio J, Zofío JL, Pastor JT. Decomposing economic efficiency into technical and allocative components: An essential property. J Optim Theory Appl 2023;197:98–129. http://dx.doi.org/10.1007/s10957-023-02188-2.

[49] Aparicio J, Borras F, Pastor JT, Vidal F. Measuring and decomposing firm's revenue and cost efficiency: The russell measures revisited. Int J Prod Econ 2015;165:19–28. http://dx.doi.org/10.1016/j.ijpe.2015.03.018.

[50] Halická M, Trnovská M. The russell measure model: Computational aspects, duality, and profit efficiency. European J Oper Res 2018;268(1):386–97. http://dx.doi.org/10.1016/j.ejor.2018.01.012.

[51] Wächter A, Biegler L. On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. Math Program 2006;106:25–57.

[52] Aparicio J, Pastor JT, Vidal F. The weighted additive distance function. European J Oper Res 2016;254(1):338–46. http://dx.doi.org/10.1016/j.ejor.2016.04.006.

[53] Cooper W, Park K, Pastor J. RAM: A range adjusted measure of inefficiency for use with additive models, and relations to others models and measures in DEA. J Product Anal 1999;11:5–42.

[54] Cooper W, Pastor J, Borras F, Aparicio J, Pastor D. BAM: A bounded adjusted measure of efficiency for use with bounded additive models. J Product Anal 2011;35(2):85–94.

[55] Lovell C, Pastor J. Units invariant and translation invariant DEA models. Oper Res Lett 1995;18(3):147–51. http://dx.doi.org/10.1016/0167-6377(95)00044-5.

[56] Aparicio J, Ortiz L, Pastor JT. Measuring and decomposing profit inefficiency through the slacks-based measure. European J Oper Res 2017;260(2):650–4. http://dx.doi.org/10.1016/j.ejor.2016.12.038.

[57] Chambers RG, Chung Y, Färe R. Benefit and distance functions. J Econom Theory 1996;70(2):407–19. http://dx.doi.org/10.1006/jeth.1996.0096.

[58] Zofío JL, Pastor JT, Aparicio J. The directional profit efficiency measure: On why profit pnefficiency is either technical or allocative. J Product Anal 2013;40(3):257–66. http://dx.doi.org/10.1007/s11123-012-0292-0.

[59] Briec W, Lemaire B. Technical efficiency and distance to a reverse convex set. European J Oper Res 1999;114(1):178–87. http://dx.doi.org/10.1016/S0377-2217(98)00089-7.

[60] Briec W, Lesourd J. Metric distance function and profit: Some duality results. J Optim Theory Appl 1999;101(1):15–33. http://dx.doi.org/10.1023/A:1021762809393.

[61] Gurobi. Gurobi optimizer reference manual: version 9.1. Beaverton, OR: Gurobi Optimization, LLC; 2021, URL https://www.gurobi.com/.

[62] Georgescu-Roegen. In: Koopmans T, editor. Activity analysis of production and allocation. New York: Wiley; 1951, p. 116–31.

[63] Färe R, Grosskopf S, Lovell C. The measurement of efficiency of production. Boston: Kluwer; 1985.

[64] Chavas J-P, Cox T. A generalized distance function and the analysis of production efficiency. Southern Econ J 1999;66(2):294–318. http://dx.doi.org/10.1002/j.2325-8012.1999.tb00248.x.

Ph.D. Javier Barbero: Javier Barbero holds a Ph.D. in Economics from Universidad Autónoma de Madrid in 2016. He has visited several universities including the London School of Economic and Political Science (LSE) and the Université du Québec à Montréal (ESG-UQAM). Currently he is Assistant professor in the Department of Economics at Universidad Autónoma de Madrid. Previously, he was an Economic Analyst at the Directorate for Growth & Innovation in the European Commission Joint Research Centre (JRC), located in Seville (Spain), as a member of the Territorial Data Analysis and Modeling (TEDAM) team. His research interests are spatial economics, regional economic development, innovation, and efficiency and productivity analysis. He is also interested in computational economics and has published several articles in top journals developing code for efficiency and productivity measurement.

Ph.D. José Luis Zofío: José Luis Zofío is a professor of Economics at the Universidad Autónoma de Madrid, and former Chair of the Department of Economics. He is also Visiting Professor at Erasmus University and Visiting Fellow to the Erasmus Research Institute of Management, ERIM, where he collaborates with academics from the Rotterdam School of Management. His research interests are related to measurement theory in economics, in particular the use of index numbers for efficiency and productivity analysis, as well as spatial economics and trade theory. From an empirical perspective he undertakes multidisciplinary research, publishing numerous articles in top fields journals and book chapters related to environmental economics, transportation, innovation, and regional science. Finally, he has made several contributions in the field computational economics, with several packages for Matlab and Julia that focus on data envelopment analysis and regression methods.