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ORIGINAL ARTICLE

On the Controllability of Entropy Solutions of Scalar Conservation Laws at a Junction via Lyapunov Methods

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Abstract

In this note, we prove a controllability result for entropy solutions of scalar conservation laws on a star-shaped graph. Using a Lyapunov-type approach, we show that, under a monotonicity assumption on the flux, if u and v are two entropy solutions corresponding to different initial data and same in-flux boundary data (at the exterior nodes of the star-shaped graph), then $u \equiv v$ for a sufficiently large time. In order words, we can drive u to the target profile v in a sufficiently large control time by inputting the trace of v at the exterior nodes as in-flux boundary data for u. This result can also be shown to hold on tree-shaped networks by an inductive argument. We illustrate the result with some numerical simulations.

Keywords Scalar conservation laws · Entropy solutions · Controllability · Lyapunov · Networks · Star-shaped graphs · Tree-shaped graphs

Mathematics Subject Classification (2010) 35L65 · 90B20

1 Introduction

Hyperbolic models on networks are extensively used to describe applied problems related to blood circulation [43, 68], gas pipelines [27, 28], vehicular traffic [44], irrigation channels [41], supply chains [38], etc. (see [22] and the references therein for further information).

Dedicated to Alfio Quarteroni on the occasion of his 70th brithday.

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We focus on a network composed by a single junction with n incoming and m outgoing edges (see Fig. 1). Following the notations of [44], we describe the junction by a finite set of incoming edges, labeled by $i \in \mathcal{I}_{in} := \{1, \ldots, n\}$ and parameterized by the segments $I_i := (-L_i, 0)$, and by a finite set of outgoing edges, labeled by $j \in \mathcal{I}_{out} := \{n+1, \ldots, n+m\}$ and parameterized by the segments $I_j := (0, L_j)$, with $L_i, L_j > 0$. In both cases the junction is at $x = 0$. We shall also use the notation $\mathcal{G} = (0, \mathcal{E})$, where $\mathcal{E} = \{I_\ell\}_{\ell \in \{1,\ldots,n+m\}}$, to denote the star-shaped graph described above.

For each edge of the graph, we consider the dynamics given by a scalar hyperbolic conservation laws with flux f_ℓ (with $\ell \in \{1, \ldots, n+m\}$) satisfying the following conditions:

(F1) $f_\ell \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}_+)$;
(F2) f_ℓ is *non-degenerate*, i.e. for all (ξ, ζ) in $\mathbb{R} \times \mathbb{R} \setminus (0, 0)$, we have $\mathcal{L}\left(\left\{z \in \mathbb{R} : \xi + \zeta f_\ell'(z) = 0\right\}\right) = 0$, where $\mathcal{L}$ denotes the Lebesgue measure;
(F3) $\inf_{\xi \in \mathbb{R}} f_\ell'(\xi) \geq c_\ell > 0$.

With these assumptions, we study the system

$$\begin{cases} \partial_t u_i + \partial_x f_i(u_i) = 0, & t > 0,\ x \in I_i, \\ \partial_t u_j + \partial_x f_j(u_j) = 0, & t > 0,\ x \in I_j, \\ u_i(0, x) = u_{0,i}(x), & x \in I_i, \\ u_j(0, x) = u_{0,j}(x), & x \in I_j, \\ u_i(t, -L_i) = u_{b,i}(t), & t > 0, \\ \sum_{i=1}^{n} f_i(u_i(t, 0-)) = \sum_{j=n+1}^{n+m} f_j(u_j(t, 0+)), & t > 0, \end{cases} \tag{1.1}$$

for all $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, n+m\}$. Here, for every $\ell \in \{1, \ldots, n+m\}$, the initial data satisfies $u_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ and, at the entry points of the network, we prescribed an *in-flux boundary condition* with data $u_{b,i} \in L^\infty((0, \infty); \mathbb{R}_+)$ for $i \in \{1, \ldots, n\}$. The condition in the last line is a modellistic choice: it imposes the conservation of the total density at the junction.

Let us comment on the assumptions needed for the flux functions. Hypothesis (F2) is a technical condition needed to guarantee the existence of traces for L^∞ entropy solutions of the conservation laws (see [64] and also [65, 71]): i.e., the non-degeneracy of the flux yields a regularization effect at the boundary (see [36]). See also [19] for further information on trace theorems in the linear case.

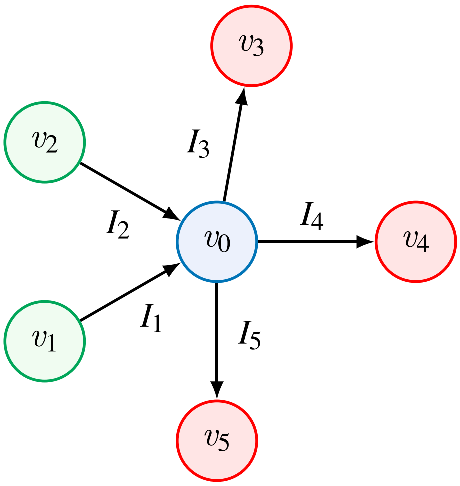

Fig. 1 Junction with $n = 2$ incoming and $m = 3$ outgoing edges

Hypothesis (F3) corresponds, roughly speaking, to the requirement that all generalized characteristics (see [37]) exiting from $(t, x) \in \{0\} \times I_\ell$ leave the cylinder $(0, T) \times I_\ell$ before time $T_\ell = L_\ell / c_\ell$, so that the dynamics on each edge only depends on the boundary data at the exterior nodes and not on the initial data for a sufficiently large time horizon. Under hypotheses (F3), the effective boundary condition is imposed only at $x = -L_i$ for $i \in \{1, \dots, n\}$ (i.e. at the incoming boundary); moreover, the junction conditions is given more explicitly (see Remark 2.1 below).

We remark that (F3) makes (1.1) not suitable for traffic flow models as they usually consider bell-shaped flux functions (see [44] and references therein). On the other hand, in supply chain production models, a typical flux function is given by the *M/M/1 queuing model with capacity one*[1] (see [38]), i.e. $f(u) = \frac{u}{u+1}$, which satisfies $f'(u) = \frac{1}{(u+1)^2} \geq c > 0$.

The well-posedness of a suitable notion of entropy solutions for conservation laws at a junction has been subject of intensive investigation. We refer to [22, 44] for a survey of this research area and various definitions of admissible solutions at a junction. In the present paper, we focus on the (unique) entropy-admissible solution that is obtained by the vanishing viscosity approximation process (see [12, 63]), whose definition is recalled in Section 2.

The main result in our contribution concerns the controllability to trajectories of entropy solutions of (1.1), i.e. steering the solution to a given target entropy-admissible profile using the boundary datum as control.

Theorem 1.1 (Controllability of entropy solutions on star-shaped graphs) *Let us assume that hypotheses (F1) – (F3)*[2] *are satisfied and let* $\mathbf{v} = (v_1, \dots, v_{n+m})$ *be the entropy solution of* (1.1) *(in the sense of Definition 2.1) with initial data* $v_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ *for* $\ell \in \{1, \dots, n+m\}$ *and boundary data* $v_{b,i} \in L^\infty((0, T); \mathbb{R}_+)$ *for* $i \in \{1, \dots, n\}$ *(*$\mathbf{v}$ *is a target profile). Let us consider any other initial data* $u_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ *for* $\ell \in \{1, \dots, n+m\}$. *Then, the entropy solution* $\mathbf{u} = (u_1, \dots, u_{n+m})$ *of* (1.1) *corresponding to initial data* $u_{0,\ell}$ *and the in-flux boundary data of* $\mathbf{v}$, *i.e.* $u_{b,i} \equiv v_{b,i}$ *for all* $i \in \{1, \dots, n\}$, *satisfies*

$$u_\ell(t, x) = v_\ell(t, x), \quad t > \hat{T},\ a.e. x \in I_\ell,\ \forall \ell \in \{1, \dots, n+m\},$$

where the control time $\hat{T}$ *is given by* $\hat{T} := \max_{i \in \mathcal{I}_{in}} \{L_i / c_i\} + \max_{j \in \mathcal{I}_{out}} \{L_j / c_j\}$.

Theorem 1.1 adapts a result proved by Donadello and Perrollaz in [39, Proposition 4] (for multi-dimensional scalar conservation laws) to the case of a networked system. The strategy of our proof is also similar: we employ a Lyapunov functional consisting of an exponentially-weighted L^1 norm. The added difficulty of the case of a networked system, compared to [39], consists in the handling of the transmission condition at the junction: we need to carefully consider an adapted entropy-admissibility (see Section 2) in order to

[1] The model name is written in *Kendall's notation* (see [56]): M/M/1 means that the system has a Poisson arrival process, an exponential service time distribution, and one server (M here stands for Markovian).

[2] Or, more precisely, the infimum in (F3) should be taken over the convex hull of the ranges of u_i and v_i (note that the system satisfies the ordering property, cf. [12]).

propagate information across the junction. This is similar in spirit to the considerations of [23] about the controllability of a linear hyperbolic first-order problem.

Remark 1.1 (Controllability of entropy solutions on tree-shaped graphs) We can prove a similar result on a *tree-shaped network*, i.e. a network without loops, arguing by induction as in [23]. In the case of a tree, in the statement of Theorem 1.1, the control time is given by the *maximal propagation time* required for information to flow out of the tree as defined in [23, Definition 3.1]. For networks with loops, difficulties arise, as already pointed out in the case of linear transport problems in [23, Section 3.4].

Finally, let us consider a viscous regularization of (1.1):

$$\begin{cases} \partial_t u_{\varepsilon,i} + \partial_x f_i(u_{\varepsilon,i}) = \varepsilon \partial^2_{xx} u_{\varepsilon,i}, & t > 0,\ x \in I_i, \\ \partial_t u_{\varepsilon,j} + \partial_x f_j(u_{\varepsilon,j}) = \varepsilon \partial^2_{xx} u_{\varepsilon,j}, & t > 0,\ x \in I_j, \\ u_{\varepsilon,i}(0,x) = u_{0,\varepsilon,i}(x), & x \in I_i, \\ u_{\varepsilon,j}(0,x) = u_{0,\varepsilon,j}(x), & x \in I_j, \\ u_{\varepsilon,i}(t,-L_i) = u_{b,i}(t), & t > 0, \\ u_{\varepsilon,j}(t,L_j) = u_{b,j}(t), & t > 0, \\ \sum_{i=1}^{n} \big(f_i(u_{\varepsilon,i}(t,0-)) - \varepsilon \partial_x u_{\varepsilon,i}(t,0-) \big) & \\ \quad = \sum_{j=n+1}^{n+m} \big(f_j(u_{\varepsilon,j}(t,0+)) - \varepsilon \partial_x u_{\varepsilon,j}(t,0+) \big), & t > 0, \\ u_{\varepsilon,i}(t,0-) = u_{\varepsilon,j}(t,0+), & t > 0, \end{cases} \tag{1.2}$$

for all $i \in \{1,\dots,n\}$ and $j \in \{n+1,\dots,n+m\}$. We remark that, due to the effect of viscosity, at time $T = \hat{T}$, a small exponential tail remains as an error when we consider the evolution of the difference of two solutions $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$ with different initial condition and same boundary data. This is summarized in the following stabilization result.

Theorem 1.2 (Exponential stabilization for the viscous problem) *Let us assume that hypotheses (F1) and (F3) are satisfied and* $n \le m$. *Let* $\mathbf{u}_\varepsilon = (u_{\varepsilon,1},\dots,u_{\varepsilon,n+m})$ *and* $\mathbf{v}_\varepsilon = (v_{\varepsilon,1},\dots,v_{\varepsilon,n+m})$ *be classical solutions of* (1.2) *(in the sense of* [25, Theorem 1.2]*) with initial data* $u_{0,\varepsilon,\ell} \in C^\infty(I_\ell;\mathbb{R}_+)$ *and* $v_{0,\varepsilon,\ell} \in C^\infty(I_\ell;\mathbb{R}_+)$ *respectively and same boundary data* $u_{b,\ell} \equiv v_{b,\ell} \in L^\infty((0,T);\mathbb{R}_+)$ *for all* $\ell \in \{1,\dots,n+m\}$. *Then,*

$$\sum_{\ell=1}^{n+m} \|u_{\varepsilon,\ell}(t,\cdot) - v_{\varepsilon,\ell}(t,\cdot)\|_{L^1(I_\ell)} \le e^{-\frac{c\alpha}{2\varepsilon}((1-\frac{\alpha}{2})ct-L)} \sum_{\ell=1}^{n+m} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)}, \quad t > 0,$$

for any $\alpha \in (0,1]$, $c = \min_{\ell \in \{1,\dots,n+m\}} c_\ell$, *and* $L := \max_{i\in\{1,\dots,n\}} L_i + \max_{j \in \{n+1,\dots,n+m\}} L_j$.

This kind of stabilization result provides a robustness estimate for Theorem 1.1 and is the first step towards the analysis of the cost of controllability for conservation laws on networks in the vanishing viscosity singular limit, which will be tackled in forthcoming works (cf. [23, Proposition 7.1] for the corresponding result in the linear setting). We remark that the role of the assumption $n \le m$ in the energy dissipation mechanism for viscous conservation laws at a junction is also discussed in [24].

We refer to [35, 47–50] for the study of the problem of uniform controllability of linear or nonlinear transport problems in Euclidean domains in the vanishing viscosity limit or in the zero diffusion-dispersion singular limit; and to [23] for linear advection-diffusion equations on networks.

1.1 Literature on Well-posedness and Controllability of Scalar Conservation Laws on Networks

The study of conservation laws on networks goes back to [26, 53] and has received much attention over the last two decades. We refer the reader to [22, 44] for an extensive survey. We emphasize that, recently, a well-posedness result for a suitable notion of *entropy-admissible solution* has been obtained in [12, 63]. However, no controllability results seem to have been obtained in the framework of entropy solutions. There are only results on some optimization problems (see [6, 7]); stabilization issues [40]; and on the related topic of conservation laws on the real line with space-discontinuous flux (see [1, 8]).

On the other hand, the controllability and stabilization of (systems of) conservation laws on networks have been widely studied in the context of smooth solutions (see, e.g., [51, 52] and the references therein).

For the IBVP associated with (systems of) conservation laws, the study of controllability has a much longer history. In the framework of classical solutions, the controls, in addition to driving the state to the target, also prevent the formation of singularities (see [17, 32, 58, 59] and the references therein). In the context of entropy solutions, the set of admissible target states has been investigated in [9, 10, 13, 14, 29] and several controllability results have been obtained by relying on the method of generalized characteristics introduced by Dafermos in [37] (see [10, 14, 29, 55, 66]); on the Lax–Oleinik representation formula (see [2, 13]); or on the return method introduced by Coron in [30] (see [47, 55, 57]). We remark, however, that the Lax–Oleinik formula is applicable only when the flux function is strictly convex/concave; the theory of generalized characteristics includes flux functions with one inflection point; and the return method was used in [57] to cover the case of a finite number of inflection points. More recently, in [39], Donadello and Perrollaz used the classical ideas of Lyapunov functionals coming from the study of asymptotic stabilization (see [17, 18, 31, 33, 67]) to prove a null-controllability result for multi-dimensional conservation laws without convexity/concavity assumptions, but under an assumption analogous to (F3).

On the other hand, for systems of conservation laws, the only available tool for investigating the exact controllability of entropy solutions in one space dimension is wave front tracking algorithm (see [20]), which was employed in [9, 21, 45, 46, 60, 72]. The asymptotic stabilization of entropy solutions of systems of conservation laws has also been subject to investigation in [11, 17, 34].

In this note, we use the Lyapunov-type approach from [39] to give a short proof of the null-controllability result in Theorem 1.1. This establishes the first controllability results for scalar conservation laws on networks in the context of entropy solutions and complements the existing literature on the controllability of smooth solutions for hyperbolic systems on networks.

2 Entropy Admissible Solutions for Scalar Conservation Laws on Networks

In this section, following [12], we review some known results on the entropy formulation for conservation laws at a junction. We remark that the theory of [12] was developed in the case of bell-shaped fluxes; however, the results still apply under the assumption (F3), which is the setting of the more recent works [42, 63].

Let us start by considering an IBVP on the half-line for a scalar conservation law with Lipschitz continuous flux:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t > 0, x > 0, \\ u(0,x) = u_0(x), & x > 0, \\ u(t,0) = u_b(t), & t > 0. \end{cases} \tag{2.1}$$

We say that u is an entropy solution of (2.1) if it is a *Kružkov entropy solution* in the interior of the half-plane $\mathbb{R}_+ \times \mathbb{R}_+$, i.e.

$$\partial_t |u - c| + \partial_x (\text{sign}(u - c)(f(u) - f(c))) \leq 0$$

holds in the sense of distributions, and if it satisfies the boundary condition *in the sense of Bardos–LeRoux–Nédélec* (see [15, 16]), i.e. the trace $u(t, 0+)$ satisfies

$$f(u(t, 0+)) = G(u_b(t), u(t, 0+)),$$

where G denotes the *Godunov numerical flux associated to* f (see [54, Eq. (3.8)]), which is given by

$$G(a,b) = \begin{cases} \min_{\xi \in [a,b]} f(\xi) & \text{if } a \leq b, \\ \max_{\xi \in [b,a]} f(\xi) & \text{if } a \geq b. \end{cases}$$

Due to the results in [65, 71], for a Lipschitz continuous flux f such that f' is not identically zero on any interval (cf. assumptions (F1)–(F2)), the function $u(t, \cdot)$ admits one-sided limits; in particular, we can define the strong trace of u on $\mathbb{R}_+ \times \{0\}$ which is mentioned above. The Bardos–LeRoux–Nédélec condition is generally recognized as the correct interpretation of the Dirichlet boundary condition for hyperbolic conservation laws. This is justified in particular by convergence of vanishing viscosity or numerical approximations of the boundary value problem: indeed, it may happen that the limit (hyperbolic) problem satisfies an effective boundary condition that may differ from the formal boundary condition prescribed for the approximation level due to viscous or numerical boundary layer effects (see [15, 69] for a more detailed discussion of boundary conditions for hyperbolic conservation laws).

After these preliminaries, let us now present the notions of entropy- admissible solutions for conservation laws on networks studied in [12]. We remark that there the authors considered $I_i = \mathbb{R}_-$ and $I_j = \mathbb{R}_+$, so we need to slightly extend [12, Definition 1.2] to deal with the case of I_ℓ being segments.

Definition 2.1 (Entropy admissible solution: formulation using Godunov fluxes at the junction) Let $u_{0,\ell} \in L^\infty(I_\ell; \mathbb{R}_+)$ and $u_{b,i} \in L^\infty(\mathbb{R}_+)$, we say that $\mathbf{u} = (u_1, \dots, u_{n+m})$ is an *entropy solution* of (1.1) if $u_\ell \in L^\infty((0,\infty) \times I_\ell)$ for all $\ell \in \{1, \dots, n+m\}$ and the following conditions are satisfied:

1. For all $\ell \in \{1, \dots, n+m\}$, the function u_ℓ is an entropy solution of the conservation law in the interior of I_ℓ, i.e. for all non-negative test functions $\varphi_\ell \in C_c^\infty([0,\infty) \times I_\ell)$

and for any constant $c \geq 0$, there holds

$$\int_0^{\infty} \int_{I_\ell} (\eta(u_\ell, c)\partial_t \varphi_\ell + q_\ell(u_\ell, c)\partial_x \varphi_\ell) \, \mathrm{d}x\mathrm{d}t + \int_{I_\ell} \eta(u_{0,\ell}, c)\varphi_\ell(0, x)\mathrm{d}x \geq 0,$$

where $\eta(u_\ell, c) := |u_\ell - c|$ and $q_\ell(u_\ell, c) := \mathrm{sign}(u_\ell - c)(f_\ell(u_\ell) - f_\ell(c))$ are called the *Kružkov's entropy-entropy flux pairs*;

2. The boundary conditions in the exterior vertices of the network are satisfied in the sense of Bardos–LeRoux–Nédélec, i.e.

$$f_i(u_i(t, -L_i+)) = G_i(u_{b,i}(t), u_i(t, -L_i+)), \quad \text{a.e. } t > 0, \quad i \in \{1, \ldots, n\},$$

where G_i and G_j are the Godunov fluxes associated to f_i and f_j respectively;

3. The junction condition is satisfied in the following sense: there exists a function $p \in L^{\infty}((0, \infty); \mathbb{R}_+)$ such that

$$\begin{aligned} f_i(u_i(t, 0-)) &= G_i(u_i(t, 0-), p(t)), \quad \text{a.e. } t > 0, \quad i \in \{1, \ldots, n\}, \\ f_j(u_j(t, 0+)) &= G_j(p(t), u_j(t, 0+)), \quad \text{a.e. } t > 0, \quad j \in \{n+1, \ldots, n+m\}, \end{aligned}$$

and the conservativity condition

$$\sum_{i=1}^{n} G_i(u_i(t, 0-), p(t)) = \sum_{j=n+1}^{n+m} G_j(p(t), u_j(t, 0+)), \quad \text{for a.e. } t > 0$$

holds.

Remark 2.1 (The case of monotone fluxes) Under hypothesis (F3), our flux is strictly increasing. In this particular case, we remark that the Godunov flux is given by $G(a, b) = f(a)$. As a consequence, as already remarked from the beginning of the paper, in Point (2) of Definition 2.1, we cannot impose a boundary condition at $x = L_j-$, i.e.

$$f_j(u_j(t, L_j-)) = G_i(p(t), u_j(t, L_j-)), \quad j \in \{n+1, \ldots, n+m\};$$

on the contrary, the in-flux boundary condition can be imposed only at $x = -L_i$ for $i \in \{1, \ldots, n\}$:

$$f_i(u_i(t, -L_i+)) = f_i(u_{b,i}(t)), \quad i \in \{1, \ldots, n\}.$$

We also note that, being the flux invertible, we can equivalently write

$$u_i(t, -L_i+) = u_{b,i}(t), \quad i \in \{1, \ldots, n\}.$$

Moreover, Point (3) reduces to

$$\begin{aligned} f_j(u_j(t, 0+)) &= f_{n+1}(u_{n+1}(t, 0+)), \quad j \in \{n+1, \ldots, n+m\}, \\ \sum_{i=1}^{n} f_i(u_i(t, 0-)) &= \sum_{j=n+1}^{n+m} f_j(u_j(t, 0+)). \end{aligned}$$

The second line indicates the conservation of mass; the first one, indicates that the entropy-admissibility condition amounts to requiring an equi-distribution of the flux coming out of the junction.

Definition 2.1 can be equivalently reformulated in terms of an *adapted entropy inequality* that accounts for the admissibility of $\mathbf{u}$ at the junction (see [12, Definition 2.10]).

Definition 2.2 (Entropy admissible solution: formulation using adapted entropies at the junction) Let $u_{0,\ell} \in L^{\infty}(I_\ell; \mathbb{R}_+)$ and $u_{b,i} \in L^{\infty}(\mathbb{R}_+)$, we say that $\mathbf{u} = (u_1, \ldots, u_{n+m})$ is an *entropy solution* of (1.1) if $u_\ell \in L^\infty((0,\infty)\times I_\ell)$ for all $\ell \in \{1,\dots,n+m\}$ and the following conditions are satisfied:

1. Points (1) and (2) of Definition 2.1 hold;
2. For any $\mathbf{c} = (c_1,\dots,c_{n+m}) \in \mathcal{G}_{VV}$, $\mathbf{u}$ satisfies the *adapted entropy inequality* on the network, i.e. for all non-negative test functions $\varphi_\ell \in C_c^\infty((0,\infty)\times \bar{I}_\ell)$ such that $\varphi_\ell(t,0) = \varphi_1(t,0)$, there holds
$$\sum_{\ell=1}^{n+m} \int_0^\infty \int_{I_\ell} \big(\eta(u_\ell,c_\ell)\partial_t\varphi_\ell + q_\ell(u_\ell,c_\ell)\partial_x\varphi_\ell\big)\mathrm{d}x\mathrm{d}t \geq 0,$$
where $\eta(u_\ell,c_\ell) = |u_\ell - c_\ell|$ and $q_\ell(u_\ell,c) = \mathrm{sign}(u_\ell - c_\ell)(f_\ell(u_\ell) - f_\ell(c_\ell))$. Here $\mathcal{G}_{VV}$ denotes the *vanishing viscosity germ*, defined as follows (see [12, Definition 2.1]):
$$\mathcal{G}_{VV} = \left\{ \begin{array}{l} \mathbf{u} = (u_1,\dots,u_{m+n}) : \exists p \geq 0 \text{ such that} \\ \sum_{i=1}^n G_i(u_i,p) = \sum_{j=m+1}^{m+n} G_j(p,u_j) \text{ and} \\ G_i(u_i,p) = f_i(u_i), \qquad G_j(p,u_j) = f_j(u_j), \\ \forall i \in \{1,\dots,n\},\ j \in \{n+1,\dots,n+m\} \end{array} \right\}$$

Remark 2.2 (On the vanishing viscosity germ and Oleinik-type inequalities) We can characterize the vanishing viscosity germ $\mathcal{G}_{VV}$ by a set of *Oleinik-type inequalities* (see [12, Lemma 2.2]) observing that the following equivalences hold for all $i \in \{1,\dots,n\}$ and $j \in \{n+1,\dots,n+m\}$:
$$\forall \xi \in [\min\{u_i,p\},\max\{u_i,p\}] : G_i(u_i,p) = f_i(u_i) \Longleftrightarrow (u_i - p)(f_i(\xi) - f_i(u_i)) \geq 0,$$
$$\forall \xi \in [\min\{u_j,p\},\max\{u_j,p\}] : G_j(p,u_j) = f_j(u_j) \Longleftrightarrow (p - u_j)(f_j(u_j) - f_j(\xi)) \geq 0.$$

Under assumptions (F1)–(F3), it can be proved that such entropy solutions exist and are the limit of a vanishing viscosity approximation process (see [12, Theorem 4.1]) and Godunov-type numerical schemes (see [12, Theorem 3.3] and also [70] for a more explicit implementation of the scheme). Moreover, with this entropy formulation, the following uniqueness result can be proved ([12, Proposition 3.1]).

Theorem 2.1 (L^1-stability of entropy solutions) *Let us assume that (F1)–(F3) hold and let* $\mathbf{u}$ *and* $\mathbf{v}$ *be entropy solutions of* (1.1) *in the sense of Definition 2.1 with initial data* $u_{0,\ell}, v_{0,\ell} \in L^\infty((0,\infty)\times I_\ell)$ *respectively and same boundary data* $u_{b,i} \in L^\infty((0,\infty);\mathbb{R}_+)$. *Then, for* ℓ *in* $1,\dots,n+m$
$$\sum_{i=1}^n \|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)} + \sum_{j=n+1}^{n+m} \|u_j(t,\cdot) - v_j(t,\cdot)\|_{L^1(I_j)}$$
$$\leq \sum_{i=1}^n \|u_i(0,\cdot) - v_i(0,\cdot)\|_{L^1(I_i)} + \sum_{j=n+1}^{n+m} \|u_j(0,\cdot) - v_j(0,\cdot)\|_{L^1(I_j)}$$
for every $t > 0$. *In particular, there exists at most one entropy solution for given initial and boundary data.*

Due to the finite speed of propagation of the waves of hyperbolic conservation laws, these existence and uniqueness results can be extended inductively to more general networks (see [44]).

3 Proof of the Controllability and Exponential Stabilization Results via a Lyapunov Approach

3.1 Controllability of the Hyperbolic Problem

Before going into the proof of Theorem 1.1, we shall outline the strategy with the following toy problem.[3]

Remark 3.1 (A case study: the IBVP for the linear transport equation) We consider

$$\begin{cases} \partial_t u + c\partial_x u = 0, & t > 0, \ x \in (0, L), \\ u(0, x) = u_0(x), & x \in (0, L), \\ u(t, 0) = 0, & t > 0, \end{cases} \tag{3.1}$$

where $L > 0$, $c > 0$, and $u_0 \in L^2(0, L)$. Let us define the Lyapunov functional

$$\forall t \geq 0, \quad J_\nu(t) = \int_0^L u^2 e^{-\nu x} \mathrm{d}x, \tag{3.2}$$

with $\nu > 0$, and compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} J_\nu(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^L u^2 e^{-\nu x} \mathrm{d}x \\ &= \int_0^L 2u\partial_t u e^{-\nu x} \mathrm{d}x = -\int_0^L 2cu\partial_x u e^{-\nu x} \mathrm{d}x \\ &= -\nu c \int_0^L u^2 e^{-\nu x} \mathrm{d}x \underbrace{-[u^2 e^{-\nu x}]_0^L}_{\leq 0} \\ &\leq -\nu c J_\nu(t). \end{aligned}$$

Gronwall's lemma yields

$$J_\nu(t) \leq e^{-c\nu t} J_\nu(0).$$

We then observe that

$$e^{-\nu} \|u(t, \cdot)\|^2_{L^2(0,L)} \leq J_\nu(t) \leq \|u(t, \cdot)\|^2_{L^2(0,L)}.$$

Putting these together, we have

$$e^{-L\nu} \|u(t, \cdot)\|^2_{L^2(0,L)} \leq e^{-c\nu t} \|u_0\|_{L^2(0,L)},$$

i.e.

$$\|u(t, \cdot)\|^2_{L^2(0,L)} \leq e^{-c\nu t + L\nu} \|u_0\|_{L^2(0,L)} = e^{-c\nu(t - \frac{L}{c})} \|u_0\|_{L^2(0,L)}.$$

Therefore, letting $\nu \to \infty$, we conclude $\|u(t, \cdot)\|_{L^2(0,L)} = 0$ for $t > L/c$.

In order to make the proof of Remark 3.1 rigorous for the conservation laws, we need to rely on the entropy formulation (see [39]). Moreover, to adapt the argument to the case of networked systems, we need to take particular care of the transmission of information at the junction.

[3] This example was presented by V. Perrollaz in the conference "VIII Partial Differential Equations, Optimal Design and Numerics", 2019.

Proof of Theorem 1.1 Following the strategy in [39], we define, for each edge $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$, the Lyapunov functional

$$\forall t \geq 0, \quad J_{\nu,i}(t) := \int_{-L_i}^{0} |u_i(t,x) - v_i(t,x)| e^{-\nu x} \, \mathrm{d}x,$$
$$J_{\nu,j}(t) := \int_{0}^{L_j} |u_j(t,x) - v_j(t,x)| e^{-\nu x} \, \mathrm{d}x,$$

for a fixed $\nu > 0$.

Step 1: *Analysis of the in-coming edges.* Given $\bar{t} \geq 0$, for any $i \in \{1, \dots, n\}$, the edge-wise entropy condition (see Point (1) of Definition 2.1) yields, by a "doubling of variables"-type argument (see [39]),

$$\begin{aligned} 0 \leq & \int_0^{\bar{t}} \int_{-L_i}^{0} |u_i(t,x) - v_i(t,x)| \partial_t \varphi_i(t,x) \mathrm{d}x \mathrm{d}t \\ & + \int_0^{\bar{t}} \int_{-L_i}^{0} \operatorname{sign}\big(u_i(t,x) - v_i(t,x)\big)\big(f_i(u_i(t,x)) - f_i(v_i(t,x))\big) \partial_x \varphi_i(t,x) \mathrm{d}x \mathrm{d}t, \\ & + \int_{-L_i}^{0} |u_i(0,x) - v_i(0,x)| \varphi_i(0,x) \mathrm{d}x \end{aligned}$$

with $0 \leq \varphi_i \in C_c^\infty([0,\infty) \times (-L_i, 0))$. Here, we used the existence of a strong trace at the boundary to use point (2) of Definition 2.1—namely, that there exist $\gamma_i \in L^\infty(0,T)$ and a negligible set $P_i \subset I_i$ such that

$$\lim_{\substack{x \to -L_i^+ \\ x \notin P_i}} \int_0^T |u_i(t,x) - \gamma_i(t)| \mathrm{d}t = 0.$$

We consider a sequence $\{\varphi_{i,k}\}_{k \in \mathbb{N}} \subset C_c^\infty([0,\infty) \times (-L_i, 0))$ such that

$$\varphi_{i,k} \to \chi_{(-\infty,\bar{t}]} e^{-\nu x} \quad \text{strongly in } L^1 \text{ as } k \to \infty.$$

Then, letting $k \to \infty$, we obtain

$$J_{\nu,i}(\bar{t}) \leq J_{\nu,i}(0) - \nu \int_0^{\bar{t}} \int_{-L_i}^{0} e^{-\nu x} \operatorname{sign}\big(u_i(t,x) - v_i(t,x)\big)\big(f_i(u_i(t,x)) - f_i(v_i(t,x))\big) \, \mathrm{d}x \, \mathrm{d}t. \tag{3.3}$$

Here, we needed to use the existence of strong traces at the boundary guaranteed by (F2).

In order to estimate the last term of (3.3), we observe that, for all $(a,b) \in \mathbb{R}^2$,

$$\begin{aligned} \operatorname{sign}(a-b)(f_\ell(a) - f_\ell(b)) &= \operatorname{sign}(a-b) \left(\int_0^1 f_\ell'(b + s(a-b))(a-b) \, \mathrm{d}s \right) \\ &= |a-b| \int_0^1 f_\ell'(b + s(a-b)) \, \mathrm{d}s \\ &\geq |a-b| \int_0^1 c_\ell \, \mathrm{d}s \geq c_\ell |a-b|, \end{aligned}$$

where we used assumption (F3) to bound f_ℓ' from below. Therefore, we obtain

$$J_{\nu,i}(\bar{t}) \leq J_{\nu,i}(0) - \nu c_i \int_0^{\bar{t}} J_{\nu,i}(s) \, \mathrm{d}s,$$

which yields, by Gronwall's lemma,

$$J_{\nu,i}(\bar{t}) \leq e^{-c_i \nu \bar{t}} J_{\nu,i}(0). \tag{3.4}$$

As $\bar{t}$ was arbitrarily chosen, we can write, for all $t \geq 0$,

$$\|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)} \leq J_{\nu,i}(t) \leq e^{\nu L_i} \|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)}. \tag{3.5}$$

Thus, plugging (3.5) into (3.4), we compute

$$\|u_i(t,\cdot) - v_i(t,\cdot)\|_{L^1(I_i)} \leq J_{\nu,i}(t) \leq e^{\nu L_i - \nu c_i t} J_{\nu,i}(0) \leq e^{-\nu c_i \left(t - \frac{L_i}{c_i}\right)} \|u_{0,i} - v_{0,i}\|_{L^1(I_i)}$$

and, letting $\nu \to \infty$, we conclude that $u_i(t,\cdot) - v_i(t,\cdot) = 0$ for $t > L_i/c_i$. Therefore, $u_i(t,\cdot) = v_i(t,\cdot)$ for all $i \in \{1,\ldots,n\}$ if $t > \max_{i \in \mathcal{I}_{in}}\{L_i/c_i\}$.

Step 2: *Analysis of the out-going edges.* By Definition 2.1 (and Remark 2.1), the traces of $\mathbf{u}$ and $\mathbf{v}$ at the junction satisfy

$$f_j(u_j(t,0+)) - f_j(v_j(t,0+)) = f_{n+1}(u_{n+1}(t,0+)) - f_{n+1}(v_{n+1}(t,0+)), \quad \forall j \in \{n+1,\ldots,n+m\}, \tag{3.6}$$

$$\sum_{i=1}^{n} f_i(u_i(t,0-)) - f_i(v_i(t,0-)) = \sum_{j=1}^{n+m} f_j(u_j(t,0+)) - f_j(v_j(t,0+)). \tag{3.7}$$

From Step 1, for all $i \in \{1,\ldots,n\}$, we have $u_i(t,0-) - v_i(t,0-) = 0$ for $t > \max_{i \in \mathcal{I}_{in}}\{L_i/c_i\}$. Then, from (3.7), we have

$$\sum_{j=1}^{n+m} f_j(u_j(t,0+)) - f_j(v_j(t,0+)) = 0.$$

By (3.6), this yields $u_j(t,0+) = v_j(t,0+)$ for $t > \max_{i \in \mathcal{I}_{in}}\{L_i/c_i\}$ for all $j \in \{n+1,\ldots,n+m\}$. Then, we can repeat the argument of Step 1: we consider the Lyapunov functional

$$J_{\nu,j}(t) = \int_0^{L_j} |u_j(t,x) - v_j(t,x)| e^{-\nu x}\,\mathrm{d}x$$

and prove that $u_j(t,\cdot) = v_j(t,\cdot)$ for all $j \in \{n+1,\ldots,n+m\}$ if $t > \max_{i \in \mathcal{I}_{out}}\{L_j/c_j\}$.

Step 3: *Conclusion of the argument.* Putting Step 1 and Step 2 together, we conclude that, for any

$$t > \hat{T} := \max_{i \in \mathcal{I}_{in}}\{L_i/c_i\} + \max_{j \in \mathcal{I}_{out}}\{L_j/c_j\},$$

it holds

$$u_\ell(t,x) = v_\ell(t,x) \quad \text{for almost every } x \text{ in } I_\ell,\ \forall \ell \in \{1,\ldots,n+m\}.$$

□

3.2 Exponential Stabilization of the Viscous System

In this section, we prove the stabilization result for the viscous problem. As in the previous section, we first illustrate the strategy with a toy problem.

Remark 3.2 (The effect of viscosity in the toy problem) Let us consider a viscous regularization of the toy problem (3.1):

$$\begin{cases} \partial_t u_\varepsilon + c\partial_x u_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0,\ x \in (0, L), \\ u_\varepsilon(0, x) = u_0(x), & x \in (0, L), \\ u_\varepsilon(t, 0) = u_\varepsilon(t, 1) = 0, & t > 0, \end{cases}$$

with $\varepsilon > 0$, $L > 0$, and $c > 0$. Then, we can estimate the Lyapunov functional in (3.2) as follows:

$$J_\nu(t) \le e^{-(c\nu - \varepsilon\nu^2)t} J_\nu(0).$$

This yields

$$\|u_\varepsilon(t, \cdot)\|_{L^2(0,L)} \le e^{-(c\nu-\varepsilon\nu^2)t + L\nu} \|u_0\|_{L^2(0,L)},$$

which only implies an *exponential stabilization* result:

$$\|u_\varepsilon(t, \cdot)\|_{L^2(0,L)} \le e^{-\nu(c-\varepsilon\nu)(t-\frac{L}{c-\varepsilon\nu})} =: C_1 e^{-C_2 t} \|u_0\|_{L^2(0,L)},$$

with $C_1 = e^{L\nu}$ and $C_2 = c\nu - \varepsilon\nu^2$ ($C_2 > 0$ for $\nu\varepsilon < c$).

As expected, the effect of viscosity prevents from controlling exactly the state to zero by simply using null boundary data; instead, at the time $t \ge L/c$, still a small exponential tail remains. More precisely, we let $\alpha \in (0, 1)$ and $\nu = -\frac{c\alpha}{2\varepsilon}$ and compute

$$\|u_\varepsilon(t, \cdot)\|_{L^2(0,L)} \le e^{-\frac{c\alpha}{2\varepsilon}((1-\frac{\alpha}{2})ct-1)} \|u_0\|_{L^2(0,L)}.$$

For $t > \frac{1}{c(1-\alpha)}$, we deduce

$$\|u_\varepsilon(t, \cdot)\|_{L^2(0,L)} \le e^{-\frac{c\alpha^2}{4\varepsilon(1-\alpha)}} \|u_0\|_{L^2(0,L)}.$$

This estimate is motivated by [4, Lemma 2.1]: it is consistent with the decay of the free solution of advection-diffusion equations first used in [35] to prove a uniform controllability result.

The same point can be made when considering the controllability/stabilization of numerical approximations of (2.1) that introduce artificial viscosity.

Proof of Theorem 1.2 Let $\mathbf{u}_\varepsilon = (u_{\varepsilon,1}, \dots, u_{\varepsilon,n+m})$ and $\mathbf{v}_\varepsilon = (v_{\varepsilon,1}, \dots, v_{\varepsilon,n+m})$ be classical solutions of (1.2) and let us consider the following Lyapunov functional:

$$\begin{aligned} \forall t \ge 0, \quad J_\nu(t) &:= \sum_{i=1}^{n} \int_{-L_i}^{0} |u_{\varepsilon,i}(t, x) - v_{\varepsilon,i}(t, x)| e^{-\nu x}, \mathrm{d}x \\ &\quad + \sum_{j=n+1}^{n+m} \int_{0}^{L_j} |u_{\varepsilon,j}(t, x) - v_{\varepsilon,j}(t, x)| e^{-\nu x}, \mathrm{d}x, \end{aligned}$$

for a fixed $\nu > 0$

Then, as in the proof of Theorem 1.1, but using the junction condition of (1.2) similarly to [12, Eq. (89)], we compute, for $\bar{t} > 0$,

$$
\begin{aligned}
0 \le & -\sum_{i=1}^{n}\int_{-L_i}^{0} |u_{\varepsilon,i}(\bar{t},x)-v_{\varepsilon,i}(\bar{t},x)|e^{-\nu x}\mathrm{d}x - \sum_{j=n+1}^{n+m}\int_{0}^{L_j} |u_{\varepsilon,j}(\bar{t},x)-v_{\varepsilon,j}(\bar{t},x)|e^{-\nu x}\mathrm{d}x \\
& + \sum_{i=1}^{n}\int_{-L_i}^{0} |u_{\varepsilon,i}(0,x)-v_{\varepsilon,i}(0,x)|e^{-\nu x}\mathrm{d}x + \sum_{j=n+1}^{n+m}\int_{0}^{L_j} |u_{\varepsilon,j}(0,x)-v_{\varepsilon,j}(0,x)|e^{-\nu x}\mathrm{d}x \\
& -\nu\sum_{i=1}^{n}\int_0^{\bar{t}}\int_{-L_i}^{0} \mathrm{sign}\big(u_{\varepsilon,i}(t,x)-v_{\varepsilon,i}(t,x)\big)\big(f_i(u_{\varepsilon,i}(t,x)) - f_i(v_{\varepsilon,i}(t,x))\big)e^{-\nu x}\mathrm{d}x\mathrm{d}t \\
& -\nu\sum_{j=n+1}^{n+m}\int_0^{\bar{t}}\int_{0}^{L_j} \mathrm{sign}\big(u_{\varepsilon,j}(t,x)-v_{\varepsilon,j}(t,x)\big)\big(f_j(u_{\varepsilon,j}(t,x)) - f_j(v_{\varepsilon,j}(t,x))\big)e^{-\nu x}\mathrm{d}x\mathrm{d}t \\
& -\varepsilon\nu^2\sum_{i=1}^{n}\int_0^{\bar{t}}\int_{-L_i}^{0} |u_{\varepsilon,i}(t,x)-v_{\varepsilon,i}(t,x)|e^{-\nu x}\mathrm{d}x\mathrm{d}t \\
& -\varepsilon\nu^2\sum_{j=n+1}^{n+m}\int_0^{\bar{t}}\int_{0}^{L_j} |u_{\varepsilon,j}(t,x)-v_{\varepsilon,j}(t,x)|e^{-\nu x}\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where we got rid of an extra boundary term

$$\varepsilon\nu(n-m)\int_0^{\bar{t}} |u_{\varepsilon,1}(t,0)-v_{\varepsilon,1}(t,0)|\mathrm{d}t$$

thanks to the assumption $n \le m$.

This yields

$$
\begin{aligned}
J_\nu(\bar{t}) \le\ & J_\nu(0) + \varepsilon\nu^2\int_0^{\bar{t}} J_\nu(t)\mathrm{d}t \\
& -\nu\int_0^{\bar{t}}\sum_{i=1}^{n} c_i\int_{-L_i}^{0} |u_{\varepsilon,i}(t,x)-v_{\varepsilon,i}(t,x)|e^{-\nu x}\mathrm{d}x\mathrm{d}t \\
& -\nu\int_0^{\bar{t}}\sum_{j=n+1}^{n+m} c_j\int_{0}^{L_j} |u_{\varepsilon,j}(t,x)-v_{\varepsilon,j}(t,x)|e^{-\nu x}\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Taking $c = \min_{\ell\in\{1,\dots,n+m\}} c_\ell$, we get

$$J_\nu(\bar{t}) \le J_\nu(0) - (c\nu - \varepsilon\nu^2)\int_0^{\bar{t}} J_\nu(t)\mathrm{d}t,$$

which, by Gronwall's inequality, gives, for all $t \ge 0$,

$$J_\nu(t) \le e^{-(c\nu-\varepsilon\nu^2)t} J_\nu(0).$$

This implies the claimed exponential stabilization result for a sufficiently small $\nu > 0$. More precisely, it gives

$$\sum_{i=1}^{n} \|u_{\varepsilon,\ell}(t,\cdot) - v_{\varepsilon,\ell}(t,\cdot)\|_{L^1(I_\ell)} \le e^{-(c\nu-\varepsilon\nu^2)t+L\nu}\sum_{i=1}^{n} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)},$$

where $L := \max_{i\in\{1,\dots,n\}} L_i + \max_{j\in\{n+1,\dots,n+m\}} L_j$. Therefore, by choosing $\nu = -\frac{c\alpha}{2\varepsilon}$ for any $\alpha \in (0, 1]$, we compute

$$\sum_{\ell=1}^{n+m} \|u_{\varepsilon,\ell}(t,\cdot) - v_{\varepsilon,\ell}(t,\cdot)\|_{L^1(I_\ell)} \leq e^{-\frac{c\alpha}{2\varepsilon}((1-\frac{\alpha}{2})ct-L)} \sum_{\ell=1}^{n+m} \|u_{\varepsilon,0,\ell} - v_{\varepsilon,0,\ell}\|_{L^1(I_\ell)}.$$

□

4 Numerical Illustrations

In this section, we present some numerical simulations to illustrate our main result. We consider a star-shaped graph with $n = 2$ incoming edges of length 1 and $m = 3$ outgoing edges of length 1 and let $f_\ell(\xi) := \frac{\xi}{1+\xi}$ for $\ell \in \{1, \dots, 5\}$. We shall apply the Godunov numerical scheme proposed in [63] (and implemented by M. Musch in [62]). We simulate the evolution of the dynamics corresponding to the following sets of initial and boundary data.

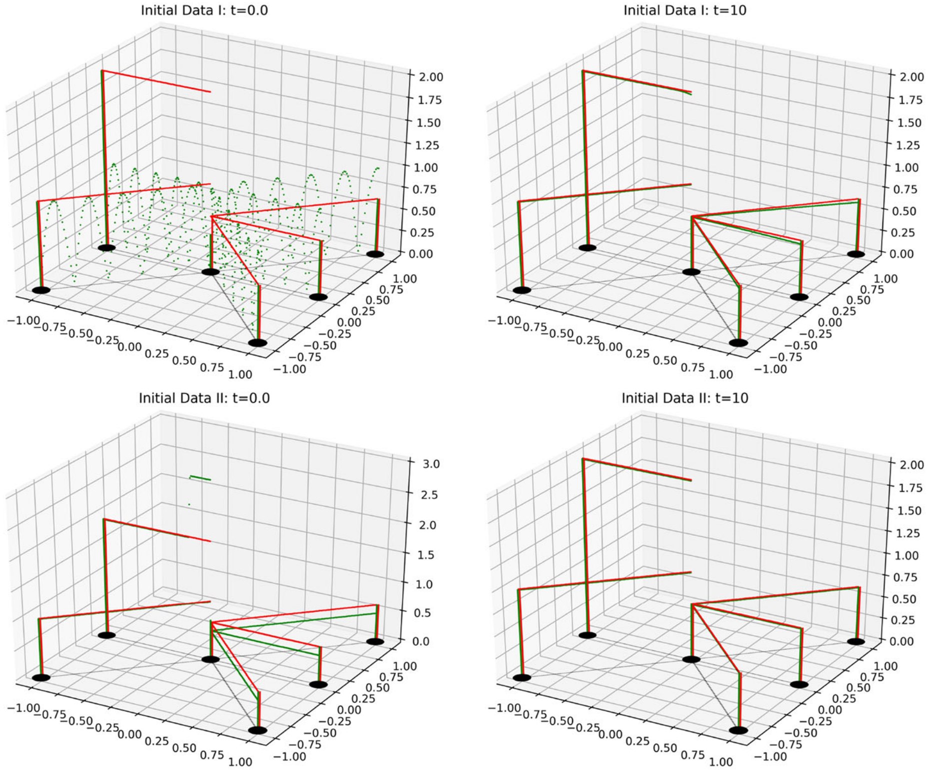

Fig. 2 **First row:** Simulation of Example 1 at times $t = 0$ and $t = 10$. **Second row:** Simulation of Example 2 at times $t = 0$ and $t = 10$. In both cases, the CFL (Courant–Friedrichs–Lewy) number is $C = 0.5$ and the space mesh size is $\Delta x = 2^{-6}$ (for each edge). We refer to [62] for the code which can be used to produce the figures and videos of the evolution

- **Example 1.** *Oscillatory initial data vs. edge-wise constant entropy solution:*

$$\begin{aligned} \mathbf{u}_0 &= (|\sin(16x)|, |\sin(16x)|, |\cos(16x)|, |\cos(16x)|, |\cos(16x)|); \\ \mathbf{v}_0 &= (2, 1, 7/11, 7/11, 7/11); \\ u_{b,1} &= v_{b,1} = 2, \quad u_{b,2} = v_{b,2} = 1. \end{aligned}$$

- **Example 2.** *Initial data containing one shock in an incoming edge vs. edge-wise constant entropy solution:*

$$\begin{aligned} \mathbf{u}_0 &= (2\chi_{(-1,-0.2)} + 3\chi_{(-0.2,0)}, 1, 1/2, 1/2, 1/2); \\ \mathbf{v}_0 &= (2, 1, 7/11, 7/11, 7/11); \\ u_{b,1} &= v_{b,1} = 2, \quad u_{b,2} = v_{b,2} = 1. \end{aligned}$$

As discussed in the previous section, the effect of numerical viscosity prevents finite-time exact controllability with these boundary controls; but, for sufficiently refined meshes, the exponential error tail is not distinguishable in Fig. 2. After sufficiently long time, we get $\mathbf{u}(T, \cdot) = \mathbf{v}(T, \cdot) = \mathbf{v}_0$ for both examples ($\mathbf{v}_0$ being, in both examples, an edge-wise constant entropy-admissible solution, i.e. $\mathbf{v}_0 \in \mathcal{G}_{VV}$).

5 Conclusions

In the present contribution, we extended the result in [39] to the case of hyperbolic conservation laws on a network (without loops) and remarked on the effect of viscosity. Interesting questions for forthcoming works include:

- the study of the cost of controllability in the vanishing viscosity singular limit (see [23] for the linear case)—possibly also replacing assumption (F3) with a convexity/concavity condition as in [57], which would require acting on all boundary nodes with a control;
- the study of the competing effect of dissipation and dispersion for the cost of controllability in the singular limit (cf. [49]);
- a detailed numerical estimation of the cost of boundary controls in the vanishing viscosity limit (see [3–5, 61]);
- the study of different types of entropy condition at the junction (namely, not the one arising from the vanishing viscosity approximation.

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References

1. Adimurthi, Ghoshal, S.S.: Exact and optimal controllability for scalar conservation laws with discontinuous flux. Commun. Contemp Math. https://doi.org/10.1142/S0219199722500249 (2022)
2. Adimurthi, Ghoshal, S.S., Veerappa Gowda, G.D.: Exact controllability of scalar conservation laws with strict convex flux. Math. Control Relat. Fields **4**, 401–449 (2014)
3. Amirat, Y., Münch, A.: Asymptotic analysis of an advection-diffusion equation and application to boundary controllability. Asymptot. Anal. **112**, 59–106 (2019)
4. Amirat, Y., Münch, A.: On the controllability of an advection-diffusion equation with respect to the diffusion parameter: asymptotic analysis and numerical simulations. Acta Math. Appl. Sin. Engl. Ser. **35**, 54–110 (2019)
5. Amirat, Y., Münch, A.: Asymptotic analysis of an advection-diffusion equation involving interacting boundary and internal layers. Math. Methods Appl. Sci. **43**, 6823–6860 (2020)
6. Ancona, F., Cesaroni, A., Coclite, G.M., Garavello, M.: On the optimization of conservation law models at a junction with inflow and flow distribution controls. SIAM J. Control Optim. **56**, 3370–3403 (2018)
7. Ancona, F., Cesaroni, A., Coclite, G.M., Garavello, M.: On optimization of traffic flow performance for conservation laws on networks. Minimax Theory Appl. **6**, 205–226 (2021)
8. Ancona, F., Chiri, M.T.: Attainable profiles for conservation laws with flux function spatially discontinuous at a single point. ESAIM Control Optim. Calc. Var. **26**, 124 (2020)
9. Ancona, F., Coclite, G.M.: On the attainable set for temple class systems with boundary controls. SIAM J. Control Optim. **43**, 2166–2190 (2005)
10. Ancona, F., Marson, A.: On the attainable set for scalar nonlinear conservation laws with boundary control. SIAM J. Control Optim. **36**, 290–312 (1998)
11. Ancona, F., Marson, A.: Asymptotic stabilization of systems of conservation laws by controls acting at a single boundary point. In: Ancona, F. et al. (eds.) Control Methods in PDE-Dynamical Systems. Contemporary Mathematics, vol. 426, pp. 1–43. Amer. Math. Soc., Providence, RI (2007)
12. Andreianov, B., Coclite, G.M., Donadello, C.: Well-posedness for vanishing viscosity solutions of scalar conservation laws on a network. Discrete Contin. Dyn. Syst. **37**, 5913–5942 (2017)
13. Andreianov, B., Donadello, C., Ghoshal, S.S., Razafison, U.: On the attainable set for a class of triangular systems of conservation laws. J. Evol. Equ. **15**, 503–532 (2015)
14. Andreianov, B., Donadello, C., Marson, A.: On the attainable set for a scalar nonconvex conservation law. SIAM J. Control Optim. **55**, 2235–2270 (2017)
15. Andreianov, B., Sbihi, K.: Well-posedness of general boundary-value problems for scalar conservation laws. Trans. Amer. Math. Soc. **367**, 3763–3806 (2015)
16. Bardos, C., LeRoux, A.Y., Nédélec, J.C.: First order quasilinear equations with boundary conditions. Commun. Partial Differ. Equ. **4**, 1017–1034 (1979)
17. Bastin, G., Coron, J.-M.: Stability and Boundary Stabilization of 1-D Hyperbolic Systems. Progress in Nonlinear Differential Equations and Their Applications, vol. 88. Springer, Cham (2016)
18. Blandin, S., Litrico, X., Delle Monache, M.L., Piccoli, B., Bayen, A.: Regularity and Lyapunov stabilization of weak entropy solutions to scalar conservation laws. IEEE Trans. Autom. Control **62**, 1620–1635 (2017)
19. Boyer, F.: Trace theorems and spatial continuity properties for the solutions of the transport equation. Differ. Integral Equ. **18**, 891–934 (2005)
20. Bressan, A.: Hyperbolic Systems of Conservation Laws. The One-dimensional Cauchy Problem. Oxford Lecture Series in Mathematics and its Applications, vol. 20. Oxford University Press, Oxford (2000)
21. Bressan, A., Coclite, G.M.: On the boundary control of systems of conservation laws. SIAM J. Control Optim. **41**, 607–622 (2002)
22. Bressan, A., Čanić, S., Garavello, M., Herty, M., Piccoli, B.: Flows on networks: recent results and perspectives. EMS Surv. Math. Sci. **1**, 47–111 (2014)
23. Bárcena-Petisco, J.A., Cavalcante, M., Coclite, G.M., De Nitti, N., Zuazua, E.: Control of hyperbolic and parabolic equations on networks and singular limits. Preprint (2021)
24. Cazacu, C.M., Ignat, L.I., Pazoto, A.F., Rossi, J.D.: A convection–diffusion model on a star-shaped graph. Nonlinear Differ. Equ. Appl. **29**, 17 (2022)
25. Coclite, G.M., Garavello, M.: Vanishing viscosity for traffic on networks. SIAM J. Math. Anal. **42**, 1761–1783 (2010)
26. Coclite, G.M., Garavello, M., Piccoli, B.: Traffic flow on a road network. SIAM J. Math. Anal. **36**, 1862–1886 (2005)
27. Colombo, R.M., Garavello, M.: On the Cauchy problem for the p-system at a junction. J. Math. Anal. **39**, 1456–1471 (2008)

28. Colombo, R.M., Garavello, M.: Comparison among different notions of solution for the p-system at a junction. Discrete Contin. Dyn. Syst. (Dynamical systems, differential equations and applications. 7th AIMS Conference, suppl.), 181–190 (2009)
29. Corghi, M., Marson, A.: On the attainable set for scalar balance laws with distributed control . ESAIM Control Optim. Calc. Var. **22**, 236–266 (2016)
30. Coron, J.-M.: Global asymptotic stabilization for controllable systems without drift. Math. Control Signals Syst. **5**, 295–312 (1992)
31. Coron, J.-M.: On the null asymptotic stabilization of the two-dimensional incompressible Euler equations in a simply connected domain . SIAM J. Control Optim. **37**, 1874–1896 (1999)
32. Coron, J.-M.: Control and Nonlinearity. Mathematical Surveys and Monographs, vol. 136. American Mathematical Society, Providence, RI (2007)
33. Coron, J.-M., Bastin, G., d'Andréa-Novel, B.: Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. SIAM J. Control Optim. **47**, 1460–1498 (2008)
34. Coron, J.-M., Ervedoza, S., Ghoshal, S.S., Glass, O., Perrollaz, V.: Dissipative boundary conditions for 2×2 hyperbolic systems of conservation laws for entropy solutions in BV. J. Differ. Equ. **262**, 1–30 (2017)
35. Coron, J.-M., Guerrero, S.: Singular optimal control: a linear 1-D parabolic-hyperbolic example. Asymptot. Anal. **44**, 237–257 (2005)
36. Crippa, G., Otto, F., Westdickenberg, M.: Regularizing effect of nonlinearity in multidimensional scalar conservation laws. In: Ambrosio, L. et al. (eds.) Transport Equations and Multi-D Hyperbolic Conservation Laws. Lecture Notes of the Unione Matematica Italiana, vol. 5, pp. 77–128. Springer, Berlin (2008)
37. Dafermos, C.M.: Generalized characteristics and the structure of solutions of hyperbolic conservation laws. Indiana Univ. Math. J. **26**, 1097–1119 (1977)
38. D'Apice, C., Göttlich, S., Herty, M., Piccoli, B.: Modeling, Simulation, and Optimization of Supply Chains. A Continuous Approach. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2010)
39. Donadello, C., Perrollaz, V.: Exact controllability to trajectories for entropy solutions to scalar conservation laws in several space dimensions. C. R. Math. Acad. Sci. Paris **357**, 263–271 (2019)
40. Dus, M.: BV exponential stability for systems of scalar conservation laws using saturated controls. SIAM J. Control Optim. **59**, 1656–1679 (2021)
41. Elshobaki, M., Valiani, A., Caleffi, V.: Junction Riemann problem for one-dimensional shallow water equations with bottom discontinuities and channels width variations. J. Hyperbolic Differ. Equ. **15**, 191–217 (2018)
42. Fjordholm, U.S., Musch, M., Risebro, N.H.: Well-posedness and convergence of a finite volume method for conservation laws on networks. SIAM J. Numer. Anal. **60**, 606–630 (2022)
43. Formaggia, L., Quarteroni, A., Veneziani, A. (eds.): Cardiovascular Mathematics. Modeling and Simulation of the Circulatory System. MS&A, vol. 1. Springer, Milan (2009)
44. Garavello, M., Piccoli, B.: Traffic Flow on Networks. Conservation Laws Models. AIMS Series on Applied Mathematics, vol. 1. American Institute of Mathematical Sciences (AIMS). Springfield, MO (2006)
45. Glass, O.: On the controllability of the 1-D isentropic Euler equation. J. Eur. Math. Soc. **9**, 427–486 (2007)
46. Glass, O.: On the controllability of the non-isentropic 1-D Euler equation. J. Differ. Equ. **257**, 638–719 (2014)
47. Glass, O., Guerrero, S.: On the uniform controllability of the Burgers equation. SIAM J. Control Optim. **46**, 1211–1238 (2007)
48. Glass, O., Guerrero, S.: Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit. Asymptot. Anal. **60**, 61–100 (2008)
49. Glass, O., Guerrero, S.: Uniform controllability of a transport equation in zero diffusion-dispersion limit. Math. Models Methods Appl. Sci. **19**, 1567–1601 (2009)
50. Guerrero, S., Lebeau, G.: Singular optimal control for a transport-diffusion equation. Commun. Partial Differ. Equ. **32**, 1813–1836 (2007)
51. Gugat, M., Herty, M., Schleper, V.: Flow control in gas networks: exact controllability to a given demand. Math. Methods Appl. Sci. **34**, 745–757 (2011)
52. Gugat, M., Herty, M., Schleper, V.: Erratum: flow control in gas networks: exact controllability to a given demand [2815765]. Math. Methods Appl. Sci. **38**, 1001–1004 (2015)
53. Holden, H., Risebro, N.H.: A mathematical model of traffic flow on a network of unidirectional roads. SIAM J. Math. Anal. **26**, 999–1017 (1995)

54. Holden, H., Risebro, N.H.: Front Tracking for Hyperbolic Conservation Laws, 2nd edn. Applied Mathematical Sciences, vol. 152. Springer, Heidelberg (2015)
55. Horsin, T.: On the controllability of the Burgers equation. ESAIM Control Optim. Calc. Var. **3**, 83–95 (1998)
56. Kendall, D.G.: Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain. Ann. Math. Stat. **24**, 338–354 (1953)
57. Léautaud, M.: Uniform controllability of scalar conservation laws in the vanishing viscosity limit. SIAM J. Control Optim. **50**, 1661–1699 (2012)
58. Li, T.: Controllability and Observability for Quasilinear Hyperbolic Systems. AIMS Series on Applied Mathematics, vol. 3. American Institute of Mathematical Sciences (AIMS). Springfield, MO; Higher Education Press, Beijing (2010)
59. Li, T., Rao, B.: Exact controllability and exact observability for quasilinear hyperbolic systems: Known results and open problems. In: Li, T.-T., Peng, Y.-J., Rao, P. (eds.) Some Problems on Nonlinear Hyperbolic Equations and Applications. Series in Contemporary Applied Mathematics, vol. 15, pp. 374–385. Higher Ed. Press, Beijing (2010)
60. Li, T., Yu, L.: One-sided exact boundary null controllability of entropy solutions to a class of hyperbolic systems of conservation laws. J. Math. Pures Appl. **107**(9), 1–40 (2017)
61. Münch, A.: Numerical estimations of the cost of boundary controls for the equation $Y_t - \varepsilon Y_{xx} + M y_x = 0$ with respect to ε. In: Doubova, A. et al. (eds.) Recent Advances in PDEs: Analysis, Numerics and Control. SEMA SIMAI Springer Series, vol. 17, pp. 159–191. Springer, Cham (2018)
62. Musch, M.: CLoNe GitHub repository. https://github.com/MarkusMusch/CLoNe
63. Musch, M., Fjordholm, U.S., Risebro, N.H.: Well-posedness theory for nonlinear scalar conservation laws on networks. Netw. Heterog. Media **17**, 101–128 (2022)
64. Neves, W., Panov, E., Silva, J.: Strong traces for conservation laws with general nonautonomous flux. SIAM J. Math. Anal. **50**, 6049–6081 (2018)
65. Panov, E.Y.: Existence of strong traces for generalized solutions of multidimensional scalar conservation laws. J. Hyperbolic Differ. Equ. **2**, 885–908 (2005)
66. Perrollaz, V.: Exact controllability of scalar conservation laws with an additional control in the context of entropy solutions. SIAM J. Control Optim. **50**, 2025–2045 (2012)
67. Perrollaz, V.: Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law. Ann. Inst. H. Poincaré, Anal. Non Linéaire **30**, 879–915 (2013)
68. Quarteroni, A.: Cardiovascular mathematics. In: International Congress of Mathematicians, vol. I, pp. 479–512. European Mathematical Society, Zürich (2007)
69. Rossi, E.: Definitions of solutions to the IBVP for multi-dimensional scalar balance laws. J. Hyperbolic Differ. Equ. **15**, 349–374 (2018)
70. Towers, J.D.: An explicit finite volume algorithm for vanishing viscosity solutions on a network. Netw. Heterog. Media **17**, 1–13 (2022)
71. Vasseur, A.: Strong traces for solutions of multidimensional scalar conservation laws. Arch. Rational Mech. Anal. **160**, 181–193 (2001)
72. Yu, L.: One-sided exact boundary null controllability of entropy solutions to a class of hyperbolic systems of conservation laws with characteristics with constant multiplicity. Chin. Ann. Math. Ser. B **39**, 947–962 (2018)