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The Cauchy-Dirichlet Problem for Singular Nonlocal Diffusions on Bounded Domains

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Dedicated to Juan Luis Vázquez for his 75th birthday.

“Con ammirazione ed affetto per Juan Luis, Maestro della diffusione nonlineare.”

Abstract

We study the homogeneous Cauchy-Dirichlet Problem (CDP) for a nonlinear and nonlocal diffusion equation of singular type of the form $\partial_t u = -\mathcal{L}u^m$ posed on a bounded Euclidean domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and $N \geq 1$. The linear diffusion operator \mathcal{L} is a sub-Markovian operator, allowed to be of nonlocal type, while the nonlinearity is of singular type, namely $u^m = |u|^{m-1}u$ with $0 < m < 1$. The prototype equation is the Fractional Fast Diffusion Equation (FFDE), when \mathcal{L} is one of the three possible Dirichlet Fractional Laplacians on Ω .

Our main results shall provide a complete basic theory for solutions to (CDP): existence and uniqueness in the biggest class of data known so far, both for nonnegative and signed solutions; sharp smoothing estimates: besides the classical $L^p - L^\infty$ smoothing effects, we provide new weighted estimates, which represent a novelty also in well studied local case, i.e. for solutions to the FDE $u_t = \Delta u^m$. We compare two strategies to prove smoothing effects: Moser iteration VS Green function method.

Due to the singular nonlinearity and to presence of nonlocal diffusion operators, the question of how solutions satisfy the lateral boundary conditions is delicate. We answer with quantitative upper boundary estimates that show how boundary data are taken.

Once solutions exist and are bounded we show that they extinguish in finite time and we provide upper and lower estimates for the extinction time, together with explicit sharp extinction rates in different norms.

The methods of this paper are constructive, in the sense that all the relevant constants involved in the estimates are computable.

KEYWORDS. Fast diffusion equation; nonlocal operators; singular parabolic equations; a priori estimates; smoothing effects; extinction time; existence and uniqueness; Green functions.

MSC2020 CLASSIFICATION. Pri: 35K55, 35B65, 35A01, 35B45, 35R11. Sec: 35K67, 35K61, 35A02.

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1 Introduction

We study the homogeneous Cauchy Dirichlet Problem for a nonlinear and nonlocal diffusion equation of singular type of the form

$$\partial_t u = -\mathcal{L}u^m \tag{FFDE}$$

posed on a bounded Euclidean domain $\subset \mathbb{R}^N$ with smooth boundary and $N \geq 1$. The linear diffusion operator \mathcal{L} is a densely defined sub-Markovian operator, allowed to be of nonlocal type, while the nonlinearity is of singular type, namely $u^m = |u|^{m-1}u$ with $0 < m < 1$.

On the whole space, the prototype equation is the so-called Fractional Fast Diffusion Equation on the whole space, studied in [3, 17, 53, 54, 55],

$$u_t = -(-\Delta_{\mathbb{R}^N})^s u^m \tag{1}$$

where the Fractional Laplacian is commonly represented via the hypersingular kernel

$$(-\Delta_{\mathbb{R}^N})^s f(x) := P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy,$$

but other equivalent definitions are possible on \mathbb{R}^N , see e.g. [23, 46].

These models have received a lot of attention in the last years, especially because of their natural interplay with probability and stochastic processes, but also for the numerous applications, for instance to anomalous diffusions in physics and biology, cf. [10, 47, 59]. These equations appear in boundary heat control problems [3], and also as hydrodynamic limits of interacting particle systems with long-range dynamics, cf. [38, 39]. We refer to [3, 17, 18, 54, 59, 64, 66, 67] for further details about possible applications of these useful nonlinear and nonlocal diffusion models.

A quite complete theory for the Cauchy problem on \mathbb{R}^N for (1) has been developed for all $m > 0$, covering the Fractional Porous Medium Equation (FPME, $m > 1$) and the Fractional Fast Diffusion Equation (FFDE, $m < 1$), see [3, 17, 36, 51, 52, 53, 54, 55, 58, 60, 65, 68], including numerical methods [29, 61]. Of course, when $m = 1$ we have the Fractional Heat Equation (FHE) which has a very rich literature, see [7, 16] and references therein.

This kind of problems has been extensively studied when $\mathcal{L} = -\Delta$, in which case the equation becomes the classical Porous Medium Equation ($m > 1$) or Fast Diffusion Equation ($m < 1$), [30, 32, 62, 63].

Even if *the results of this paper apply also in the local case* ($s = 1$), and provide some *novelties also for the classical FDE* when $\mathcal{L} = -\Delta$, we are mainly interested here in treating nonlocal diffusion operators, in particular fractional Laplacian operators. Since we are working on a bounded domain, the situation dramatically changes and there are several non-equivalent versions of fractional Laplacian operators¹: the Restricted Fractional Laplacian (RFL), the Spectral Fractional Laplacian (SFL), and the Censored Fractional Laplacian (CFL); see Section 2.2 for more details.

The basic theory for the FFDE on bounded domains is only partially understood, [15, 50, 54], and not in the generality nor in the unified framework that we present here. We find here the biggest class of data - known so far - to which the basic theory, existence, uniqueness and boundedness of solutions applies. We shall complement the theory with quantitative estimates about the finite extinction time, together with sharp extinction rates.

¹We use these names because they already appeared in some of our previous works [12, 13, 14, 15, 18, 19], but we point out that RFL is usually known as the Standard Fractional Laplacian, or plainly Fractional Laplacian, and CFL is often called Regional Fractional Laplacian.

The Cauchy-Dirichlet problem for the classical FDE, $u_t = \Delta u^m$, has attracted the attention of prominent researchers since the celebrated paper of Berryman and Holland [8]. The basic theory, local and global Harnack inequalities and optimal interior regularity are well understood [30, 33, 34]. The asymptotic behaviour is a delicate issue, started with the pioneering paper [8] and followed by progressive advances [1, 34, 35]. The sharp asymptotic results were found only recently in [11]. Recent results appeared about optimal boundary regularity [40] and asymptotic behaviour [41].

The Cauchy-Dirichlet Problem for singular nonlocal diffusions. Consider the problem

$$\begin{cases} \partial_t u(t, x) = -\mathcal{L}u^m(t, x) & \text{on } (0, +\infty) \times \Omega, \\ u(t, \cdot) = 0 & \text{on the lateral boundary, } \forall t > 0, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \end{cases} \quad (\text{CDP})$$

where $0 < m < 1$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, which we assume at least of class $C^{2,\alpha}$, even most of the results indeed hold for $C^{1,1}$ domains, but we choose not to address delicate questions of boundary regularity here. Throughout the paper we assume $N > 2s$, but most of the results can be easily adapted to the case when $N = 1$ and $s \in [1/2, 1)$.

When $m = 1$, we have the fractional heat equation, whose theory can be considered nowadays classical, since Fourier analysis allows to obtain a complete basic theory analogous to the case $s = 1$ of the local heat equation. However many basic questions have been solved only recently, like sharp bounds for the heat kernel by Zhang in 2002, and more recently by many authors often by means of probabilistic techniques, we refer to Section 2.2 for more details and references. Many fine properties of solutions are based on the representation formula, and become quite simple once good (or better, sharp) bounds on the heat kernel are known.

An “almost representation formula”. In the nonlinear case there is not a representation formula, however we shall show how to produce a nonlinear analogous through the Green function method, see section 3.2. This “fundamental pointwise formula”, see Lemma 3.4, was proven for the first time by Vázquez and one of the authors in 2015, [18] in the case $m > 1$, here for the first time when $m < 1$. It was later extended to more general operators in [12, 13, 14, 15, 18, 19]. The Green function method allows to construct a solid theory for the Porous Medium on Riemannian Manifolds, [6].

A short reminder about nonlocal degenerate equation on bounded domains. When $m > 1$ we have the FPME for which a quite complete theory was developed by Figalli, Sire, Ros-Oton, Vázquez and one of the authors [12, 13, 15, 18, 19]. The concept of Weak Dual Solutions (WDS) that we use here, together with the “Green function method” and the “almost representation formula” were introduced by Vázquez and one of the authors in [18] in the case where \mathcal{L} is the Spectral Fractional Laplacian. The methods turned out to be flexible and extended to a large class of operators, [13, 15, 19]. In these papers existence and uniqueness of (minimal) WDS is proven, together with weighted $L^1 - L^\infty$ smoothing effects, which constitute what we call the basic theory. In clear contrast with the local case, that exhibits the peculiar phenomenon of finite speed of propagation, in [12, 13] it is shown that the FPME propagates with infinite speed. Together with the Global Harnack Principle, (global optimal explicit upper and lower estimates, which imply more classical form of parabolic Harnack inequalities) allow to prove higher regularity results, [12, 13], up to $C_{t,x}^{1+\alpha,\infty}$ when the operator allows it. The most intriguing thing however is the appearance of an anomalous boundary behavior, that only happens in some parameter range, and shows how the nonlinear fractional world can present unexpected behaviours, that do not happen in the local case. Also the sharp asymptotic behaviour has been proven in [13, 15].

The goal of this paper is to extend the basic theory developed for the FPME to the fast diffusive case, and to mark the qualitative and quantitative differences between local and nonlocal versions and degenerate versus singular diffusions.

Main results: a basic theory for FFDE. Few partial results appear in literature dealing with the theory of FFDE, essentially only the papers [15, 50, 54], contain some advances in this direction, essentially only when \mathcal{L} is either the SFL or the RFL. We shall build *a basic theory*, which includes *existence and uniqueness in the largest (known so far) class of data*, namely H^* for signed solutions and $L^1_{\Phi_1}$ for nonnegative ones. We also prove different *energy estimates and $L^p - L^\infty$ smoothing effects*. For the proof of the smoothing estimates, we *compare the classical Moser iteration with the Green function method*, providing two different proofs of the smoothing effects, under different assumptions, in order to cover as many operators as possible. Once solutions are bounded we see that they extinguish in finite time and we show explicit sharp extinction rates. We detail our main results in the next section. *All our proofs are constructive*, meaning that all the constants in the inequalities are computable.

About the lateral boundary condition, dual formulation of the problem. Since we are allowing the linear operator \mathcal{L} to be nonlocal, the lateral boundary condition is tricky to define in general, since it depends on the operator itself. Let us make an example with the three most common Dirichlet Fractional Laplacians on bounded domains, the Spectral (SFL), the Restricted (or the “standard”, RFL) and the Censored (or Regional, CFL). The first one is defined as the spectral power of the Dirichlet Laplacian, hence the boundary condition needs only to be imposed on the topological boundary $\partial \Omega$. In the case of the RFL, since the operator needs to be defined on the whole \mathbb{R}^N , we need to impose that our solutions vanish on the complement of Ω , or we need to “restrict” the set of admissible functions to the ones supported in $\overline{\Omega}$, hence the name. As for the CFL the boundary condition is even more subtle: the Dirichlet boundary condition holds only for $s \in (1/2, 1)$, while when $s \in (0, 1/2]$ a kind of Neumann condition holds. This can be understood in terms of the underlying α -stable process, see the original paper of Bogdan, Burdzy and Chen [9], for more details.

We overcome this difficulty by means of *a dual formulation of the problem*, through the *Green function* of the linear operator, that *encodes the boundary conditions*. Indeed, we shall prove that solutions are zero at the boundary in a quantitative way, namely for a.e. $x \in \partial \Omega$ we have that

$$0 \leq u(t, x) \lesssim \text{dist}(x, \partial \Omega)^\beta \quad \text{for some } \beta > 0$$

See Theorem 9 for a more precise and quantitative statement.

Degenerate VS singular diffusions: a problem at the boundary. To better understand the issue, let us consider the local case,

$$u_t = \Delta u^m = m \nabla \cdot (u^{m-1} \nabla u)$$

it is clear that when $m > 1$ the (time dependent) diffusion matrix $u^{m-1} I$ becomes zero at the boundary, as u does. This is one of the causes of the finite speed of propagation and, in the nonlocal case, of the appearance of the anomalous boundary behaviour. On the other hand, when $m < 1$, the diffusion matrix becomes infinite, as $u = 0$ on the lateral boundary. This causes serious regularity issues, that are understood here by the condition $2s > \gamma$, in the smoothing effects or on the conditions that guarantees that WDS are strong. We shall explain with more details later on. We foresee the appearance of an anomalous boundary behaviour also in this case, but for different reasons, and this will be investigated in a forthcoming paper where we shall analyze the sharp boundary behaviour, an impossible task without the results contained in this paper.

2 Preliminaries and main results

2.1 Main Assumptions

In order to understand the main results, it is important to discuss our main hypotheses, their connections and basic implications.

Assumptions on \mathcal{L} . The linear operator $\mathcal{L} : \text{dom}(\mathcal{L}) \subseteq L^1(\Omega) \rightarrow L^1(\Omega)$ is densely defined and sub-Markovian, more precisely, it satisfies:

- (A1) \mathcal{L} is m -accretive in $L^1(\Omega)$
- (A2) If $0 \leq f \leq 1$ then $0 \leq e^{-t\mathcal{L}}f \leq 1$, or equivalently,
- (A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \text{dom}(\mathcal{L})$, $\mathcal{L}u \in L^q(\Omega)$, $1 \leq q \leq \infty$, $v \in L^{q/(q-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e., then

$$\int v(x)\mathcal{L}u(x) dx \geq 0.$$

We recall that the assumptions on \mathcal{L} are the same as in [14], where detailed proofs of the claims below can be found as well as a rich set of examples of operators. See also [13, 19] and Section 2.2.

Assumptions on \mathcal{L}^{-1} . In order to develop the theory that follows in this paper we will need to deal with the dual form of (CDP). Therefore, we assume that \mathcal{L} has an inverse $\mathcal{L}^{-1} : L^1(\Omega) \rightarrow L^1(\Omega)$ that can be represented by a kernel \mathbb{G} as

$$\mathcal{L}^{-1}[f](x) := \int \mathbb{G}(x, y)f(y)dy,$$

which satisfies at least one of the next assumptions, for some $s \in (0, 1]$ and for a.e. $x, y \in \Omega$:

- There exists a constant $c_1, c_2 > 0$ such that

$$0 \leq \mathbb{G}(x, y) \leq \frac{c_1}{|x - y|^{N-2s}}. \quad (\text{K1})$$

According to Proposition 5.1 of [14], this assumption guarantees that \mathcal{L}^{-1} is compact in $L^2(\Omega)$, hence that it has a discrete set of eigen-elements for \mathcal{L} , namely (λ_k, Φ_k) , with $\Phi_k \in L^\infty(\Omega)$. In particular there exists a ground state $\Phi_1 \geq 0$, and the Poincaré inequality holds with $\lambda_1 > 0$

$$\lambda_1 \int \mathcal{L}^{-1}f dx \leq \int f^2 dx \quad \text{or equivalently} \quad \lambda_1 \int f^2 dx \leq \int f\mathcal{L}f dx.$$

The eigenfunctions are always Hölder continuous in the interior, see Theorem 7.1 of [14].

(K1) implies Hardy-Littlewood-Sobolev and Sobolev type inequalities: It has been shown in Theorem 7.5 of [15] that assumption (K1) implies the Hardy-Littlewood-Sobolev inequality:

$$\|f\|_{H^*(\Omega)} \leq \mathcal{H}_{\mathcal{L}}\|f\|_{(2^*)'} \quad \text{with} \quad (2^*)' = \frac{2N}{N+2s}. \quad (2)$$

Moreover, following Lieb's duality argument [48], as in Proposition 7.4 of [15], we have that the above inequality is equivalent to the fractional Sobolev inequality:

$$\|f\|_{2^*} \leq \mathcal{S}_{\mathcal{L}}\|f\|_{H(\Omega)} \quad \text{with} \quad 2^* = \frac{2N}{N-2s}. \quad (3)$$

We refer to the Appendix of [15] for further details.

- The second assumption is needed when we want to take into account boundary behaviour. Let

$$\delta_\gamma(x) := \text{dist}(x, \partial \Omega)^\gamma \quad \text{for some } \gamma \in (0, 1].$$

There exist $c_0, c_1, \kappa > 0$ such that

$$c_0 \delta_\gamma(x) \delta_\gamma(y) \leq \mathbb{G}(x, y) \leq \frac{c_1}{|x - y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x - y|^\gamma} \wedge 1 \right) \quad (\text{K2})$$

Proposition 5.3 of [14], guarantees that under this assumption, the boundary behaviour of eigenfunction is dictated by the power γ , namely

$$\underline{\kappa} \delta_\gamma(x) \leq \Phi_1(x) \leq \bar{\kappa} \delta_\gamma(x) \quad \text{and} \quad |\Phi_n(x)| \leq \bar{\kappa}_n \delta_\gamma(x)$$

In this case, eigenfunction are classical in the interior, i.e. $C^{2s+\alpha}$, whenever the operator allows it, and have a sharp boundary regularity $C^\gamma(\bar{\Omega})$, see Theorems 7.1 and 7.2 of [14] for more details.

It is therefore convenient to use the ground state as a smooth extension of the distance to the boundary (to the power γ), and this allows us to rephrase the assumption (K2) as follows:

$$c_0 \Phi_1(x) \Phi_1(y) \leq \mathbb{G}(x, y) \leq \frac{c_1}{|x - y|^{N-2s}} \left(\frac{\Phi_1(x)}{|x - y|^\gamma} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x - y|^\gamma} \wedge 1 \right) \quad (\text{K3})$$

- In many examples, the Green function satisfies an even stronger estimate

$$\mathbb{G}(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x - y|^\gamma} \wedge 1 \right) \quad (\text{K4})$$

This is the case for instance of the three most common Fractional Laplacians, namely for the SFL $\gamma = 1$, while for RFL $\gamma = s$, and CFL $\gamma = 2s - 1$). See Section 2.2 for more details.

Notations. We will write the L^p norm in Ω as $\|f\|_p := (\int_\Omega |f|^p dx)^{1/p}$, but for weighted norms with the first eigenvalue Φ_1 we use $\|f\|_{L^p_{\Phi_1}(\Omega)} := (\int_\Omega |f|^p \Phi_1 dx)^{1/p}$ in the space $L^p_{\Phi_1}(\Omega) := \{f \in L^1_{\text{loc}}(\Omega) : \|f\|_{L^p_{\Phi_1}(\Omega)} < +\infty\}$. The Heaviside function will be denoted by $\text{sign}_+(f) := \max\{0, \text{sign} f\}$, and positive and negative part as $[f]_+ := \max\{0, f\}$ and $[f]_- = \max\{0, -f\}$, respectively. We also use the following notation for maxima and minima: $f \wedge g := \max\{f, g\}$, $f \vee g := \min\{f, g\}$.

2.2 Different fractional Laplacian operators on domains and other examples

We briefly exhibit a number of examples to which our results apply. These include a wide class of local and nonlocal operators. We just sketch the essential points, we shall give the values γ, s in each example, and specify the lateral boundary conditions. See [13, 19] for a more detailed exposition.

The Restricted Fractional Laplacian (RFL). We define the fractional Laplacian operator acting on bounded domain by using the integral representation on the whole space in terms of a hypersingular kernel, namely

$$(-\Delta_\Omega)^s f(x) = c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy$$

where we restrict the operator to functions that are zero outside Ω . The initial and boundary conditions associated to (CDP) are $u(t, x) = 0$ in $(0, \infty) \times \mathbb{R}^N \setminus \Omega$ and $u(0, \cdot) = u_0$. As explained in [15], such boundary condition can be understood via the Caffarelli-Silvestre extension [23]. The sharp expression of the boundary behaviour for the RFL was investigated in [56]. The sharp expression of the boundary behavior for the RFL was investigated in [57]. We refer to [15] for a careful construction of the RFL in the framework of fractional Sobolev spaces, and [7] for a probabilistic interpretation. See also [37]. For this operator, assumptions (A1), (A2) and (K4) are satisfied with $\gamma = s$, cf. [45].

Other RFL-type integral operators “with Hölder coefficients” can be considered

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} dy,$$

where a is a measurable symmetric function, bounded between two positive constants, and satisfying

$$|a(x, y) - a(x, x)| \chi_{|x-y|<1} \leq c|x - y|^\sigma, \quad \text{with } 0 < s < \sigma \leq 1,$$

for some $c > 0$. Actually, we can allow even more general kernels, cf. [18, 42]. Then, for all $s \in (0, 1]$, the Green function $\mathbb{G}(x, y)$ of \mathcal{L} satisfies (K4) with $\gamma = s$, cf. Corollary 1.4 of [42].

Spectral Fractional Laplacian (SFL). Consider the so-called *spectral definition* of the fractional power of the classical Dirichlet Laplacian Δ_Ω on Ω defined by a formula in terms of semigroups, namely

$$(-\Delta_\Omega)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_\Omega} f(x) - f(x)) \frac{dt}{t^{1+s}} = \sum_{j=1}^\infty \lambda_j^s \hat{f}_j \phi_j,$$

where (λ_j, ϕ_j) , $j = 1, 2, \dots$ are the eigenvalues and eigenfunctions of the classical Dirichlet Laplacian on Ω , $\hat{f} = \int_\Omega f(x) \phi_j(x) dx$, and $\|\phi_j\|_2 = 1$. The initial and boundary conditions associated to (CDP) are $u(t, x) = 0$ on $(0, \infty) \times \partial\Omega$ and $u(0, \cdot) = u_0$. For this operator, assumptions (A1), (A2), and (K2) are satisfied with $\gamma = 1$. Assumption (K2) and also (K4) can be obtained by the Heat kernel estimates valid for the case $s = 1$, cf. [31], as explained in [18, 19].

Spectral powers of linear operators in divergence form. Consider any operator $\mathcal{A} = -\sum_{i,j=1}^N \partial_i (a_{ij} \partial_j)$ with uniformly elliptic C^1 coefficients and with discrete spectrum (λ_j, ϕ_j) (consequence of uniform ellipticity). We can build the following spectral power of the operator: for any $s \in (0, 1]$

$$\mathcal{L}f(x) := \mathcal{A}^s f(x) := \sum_{j=1}^\infty \lambda_j^s \hat{f}_j \phi_j(x), \quad \text{where} \quad \hat{f}_j = \int_\Omega f(x) \phi_j dx.$$

The Green function of this operator satisfies (K2) with $\gamma = 1$, see [31].

Censored Fractional Laplacian (CFL) with general kernels. This third kind of Fractional Laplacian was introduced in [9], in connection with censored stable processes. The operators takes the form:

$$\mathcal{L}f(x) = \text{P.V.} \int (f(x) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} dy, \quad \text{with } \frac{1}{2} < s < 1,$$

where $a(x, y)$ is a symmetric function of class C^1 bounded between two positive constants. Actually, not only \mathcal{L} satisfies (A1), (A2), but the Green function $\mathbb{G}(x, y)$ satisfies (K4) with $\gamma = 2s - 1$, cf. Corollary 1.2 of [25]. The boundary condition is a bit mysterious here: notice that when $s \in (0, 1/2]$, the boundary condition transitions from Dirichlet to a Neumann type, as explained in [9]. In this

case it is clear the advantage of the weak dual formulation (**WDS**) which encodes the boundary condition in the Green function. In literature this is also known as Regional Fractional Laplacian.

Other examples. As it is shown in [13, 19], this assumptions hold for many other examples:

(i) Sums of two fractional operators: $\mathcal{L} = (-\Delta_{| \cdot |})^s + (-\Delta_{| \cdot |})^\sigma$, with $0 < \sigma < s \leq 1$, where $(-\Delta_{| \cdot |})^s$ is the RFL. Here, (**K4**) holds with $s \in (0, 1)$ and $\gamma = s$, see [24].

(ii) Sum of the Laplacian and a nonlocal operator kernels

$$\mathcal{L} = a(-\Delta_{| \cdot |}) + A_s, \quad \text{with } 0 < s < 1 \quad \text{and} \quad a \geq 0,$$

with

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \leq 1}) \chi_{|y| \leq 1} d\nu(y),$$

where the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies suitable integrability conditions at zero and infinity. Here, (**K4**) holds with $s = 1$, $\gamma = 1$.

(iii) Schrödinger equations for non-symmetric diffusions $\mathcal{L} = A + \mu \cdot \nabla + \nu$, where A is a uniformly elliptic operator with C^1 coefficients both in divergence and non-divergence form, more details can be found in [43]. Here, (**K4**) holds with $s = 1$, $\gamma = 1$.

(iv) Gradient perturbation of restricted fractional Laplacians, $\mathcal{L} = (-\Delta_{| \cdot |})^s + b \cdot \nabla$, where b is a vector valued function belonging to a suitable Kato class. Here, (**K4**) holds with $\gamma = s$, see [27].

(v) Relativistic stable processes,

$$\mathcal{L} = (c^{1/s} - \Delta)^s - c, \quad \text{with } c > 0, \quad \text{and } 0 < s \leq 1.$$

Here, (**K4**) holds with $\gamma = s$, see [26].

2.3 The dual formulation of the problem. Different concepts of solutions

We can reformulate problem (**CDP**) in an equivalent dual form, by means of the inverse operator \mathcal{L}^{-1} :

$$\begin{cases} \mathcal{L}^{-1} u_t(t, x) = -u^m(t, x) & \text{on } (0, +\infty) \times \Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases} \quad (\text{CDP}^*)$$

A clear advantage of this formulation is that the lateral boundary conditions are encoded in the inverse \mathcal{L}^{-1} . Now we define a concept of weak solutions suitable for the above formulation, firstly introduced in [18, 19].

Definition 2.1 (Weak dual solutions). Let $T > 0$, we say that $u \in C((0, T) : L^1_{\Phi_1}(\Omega))$ is a *weak dual solution* of (**CDP**) if $u^m \in L^1([\emptyset, T] : L^1_{\Phi_1}(\Omega))$ and

$$\int_0^T \int_{\Omega} \mathcal{L}^{-1} u \partial_t \psi \, dx \, dt = \int_0^T \int_{\Omega} u^m \psi \, dx \, dt \quad \forall \psi / \Phi_1 \in C^1_c((0, T) : L^\infty(\Omega)). \quad (\text{WDS})$$

We say that u is a WDS of the Cauchy-Dirichlet Problem (**CDP**), corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$, if moreover $u \in C([0, T) : L^1_{\Phi_1}(\Omega))$, and

$$\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}(\Omega)} = 0.$$

A WDS is called *strong* if in addition $t \partial_t u \in L^\infty((0, T) : L^1_{\Phi_1}(\cdot))$. A WDS is called *Minimal Weak Dual Solution* (MWDS) if it is obtained as the non-decreasing limit of a sequence of semigroup (mild, gradient flow) solutions.

We will construct our WDS, more precisely the MWDS, as the monotone limit of gradient flow solutions in a suitable Hilbert space, modeled ad hoc for the operator \mathcal{L} . The class of nonnegative WDS contains the nonnegative semigroup solutions that will form our approximating sequence, see Lemma 8. We shall construct such semigroup solutions using the celebrated theory of Brezis on Maximal Monotone Operators [20, 21], see also Komura [44] and the excellent lecture notes [2]: we define the free energy (or entropy functional) as follows

$$\mathcal{E}(u) = \begin{cases} \frac{1}{1+m} \int |u|^{1+m}(x) dx & \text{if } u \in L^{1+m}(\cdot), \\ \infty & \text{otherwise,} \end{cases}$$

and show that \mathcal{L} is the subdifferential of the convex and lower-semicontinuous energy functional \mathcal{E} on the Hilbert space $H^*(\cdot)$. Note that $H^*(\cdot)$ is defined as the (topological) dual space of $H(\cdot)$, the domain of the quadratic form associated to \mathcal{L} :

$$H(\cdot) = \left\{ u \in L^2(\cdot) : \int u \mathcal{L} u dx < \infty \right\}$$

We can endow $H^*(\cdot)$ with the natural scalar product and the associated norm

$$\langle u, v \rangle_{H^*(\cdot)} = \int u \mathcal{L}^{-1} v dx \quad \text{and} \quad \|u\|_{H^*(\cdot)}^2 = \langle u, u \rangle_{H^*(\cdot)} = \int u \mathcal{L}^{-1} u dx.$$

We refer to [15] for further details about the spaces $H(\cdot)$ and $H^*(\cdot)$ and their relation with the fractional Sobolev spaces $H_0^s(\cdot)$, $H_0^{1/2}(\cdot)$, $H^s(\cdot)$ and their duals $H^{-s}(\cdot)$. Following [2, 20, 21], we recall the definition of semigroup (or gradient flow) solutions adapted to our setting:

Definition 2.2 (Gradient flow and EVI solutions).

- We say that $u : (0, +\infty) \rightarrow L^{1+m}(\cdot)$ is a *gradient flow (GF)* of the functional \mathcal{E} defined above, if $u(t) \in \text{AC}_{\text{loc}}((0, \infty) : H^*(\cdot))$ and

$$-\partial_t u(t) \in \partial \mathcal{E}(u(t)) \quad \text{for a.e. } t \in (0, \infty).$$

We say that $u(t)$ starts from $u_0 \in H^*(\cdot)$ if $\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{H^*(\cdot)} = 0$.

- A curve $u(t) \in \text{AC}_{\text{loc}}((0, \infty) : H^*(\cdot))$ is called an *EVI solution* starting from $u_0 \in H^*(\cdot)$ if for any $w \in H^*(\cdot)$ we have that $\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{H^*(\cdot)} = 0$ and that

$$\frac{1}{2} \frac{d}{dt} \|u(t) - w\|_{H^*(\cdot)}^2 \leq \mathcal{E}(w) - \mathcal{E}(u(t)) \quad \text{for a.e. } t \in (0, \infty).$$

It is well-known that if \mathcal{E} is convex and lower-semicontinuous then a locally absolutely continuous curve $u(t) \in H^*(\cdot)$ is a GF if and only if it is a EVI solution, see for instance Theorem 11.15 of [2].

We are now in the position to state the celebrated Brezis-Komura Theorem, the nonlinear analogue of the Hille-Yosida or Lumer-Phillips Theorem. The statement, adapted to our setting, reads:

Theorem 2.3 (Brezis-Komura [20, 44]). *For every $u_0 \in H^*(\Omega)$, there exists a unique gradient flow starting from u_0 that we denote by $u(t) = S_t u_0$. This defines a continuous semigroup $S_t : H^*(\Omega) \rightarrow L^{1+m}(\Omega)$ for $t > 0$ with the T -contraction property in $H^*(\Omega)$*

$$\|(S_t u_0 - S_t v_0)_\pm\|_{H^*(\Omega)} \leq \|(u_0 - v_0)_\pm\|_{H^*(\Omega)}, \quad \forall t > 0, \quad \forall u_0, v_0 \in H^*(\Omega). \quad (4)$$

Moreover, GF solutions are H^* -strong, namely for all $T > 0$ we have

$$t u^m, t \partial_t u \in L^\infty((0, T) : H^*(\Omega)).$$

Indeed, GF solutions are H -energy solution, i.e. the following energy estimate for every $t > t_1 \geq 0$ we have

$$\|u^m(t_1)\|_{H(\Omega)}^2 \leq \frac{(1+m)\mathcal{E}(u(t))}{2m(t_1-t)} \leq \frac{1}{2m(1+m)} \frac{\|u(t_0)\|_{H^*(\Omega)}^2}{(t_1-t)(t-t_0)}.$$

Remark 1. (i) Note that in this class of GF solutions, the comparison principle holds, namely by (4) it is clear that if $u_0 \leq v_0$ a.e. in Ω , then $u \leq v$ a.e. in $(0, \infty) \times \Omega$. A formal proof of (4) is the following: by Kato inequality we know that $\text{sign}_+(u^m - v^m)\mathcal{L}(u^m - v^m) \geq \mathcal{L}(u^m - v^m)_+$ so that

$$\begin{aligned} \frac{d}{dt} \|(u-v)_+\|_{H^*}^2 &= -2 \int \text{sign}_+(u^m - v^m)\mathcal{L}(u^m - v^m)\mathcal{L}^{-1}(u-v)_+ dx \\ &\leq -2 \int (u^m - v^m)_+(u-v)_+ dx. \end{aligned}$$

Indeed, this is rigorous since solutions are H^* -strong, i.e. $\partial_t u(t, \cdot) \in H^*(\Omega)$.

(ii) Notice that uniqueness in H^* follows by (4), indeed we have the H^* -contraction (just summing the positive and negative part estimates)

$$\|u(t) - v(t)\|_{H^*(\Omega)} \leq \|u_0 - v_0\|_{H^*(\Omega)} \quad \text{for all } t \geq 0.$$

(iii) Note that here *we do not assume any sign condition on u_0* , and this is to the best of our knowledge *the largest class of data without sign restrictions, for which existence and uniqueness hold*. The above theorem generalizes the existence and uniqueness result of [15], Theorem 2.2, where the case of SFL and RFL were thoroughly investigated in the framework of fractional Sobolev Spaces. Most of that theory applies also in the present case, but we have preferred to simplify the setup aiming at a larger class of nonnegative solution, which are the WDS in the $L^1_{\Phi_1}$ framework.

(iv) This Theorem follows by adapting Brezis' proof [20] originally in the H^{-1} framework, to the present H^* setting, as it has been done in [15], where more general nonlinearities and further details are given; in particular, we recall that under assumption (K1) it is possible to identify H^* and H^{-s} , when $s \neq \frac{1}{2}$.

The first of our main results concerns existence and uniqueness of nonnegative solutions.

Theorem 2.4 (Existence and uniqueness of nonnegative MWDS in $L^1_{\Phi_1}$). *Let (A1) and (A2) hold. Then, for every $0 \leq u_0 \in L^1_{\Phi_1}(\Omega)$, there exist a unique minimal weak dual solution u of (CDP) with*

$$\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}(\Omega)} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{u(t+h) - u(t)}{h} \Big|_{L^1_{\Phi_1}(\Omega)} \leq \frac{2\|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t}.$$

Moreover, the T -contraction estimates hold: let $0 \leq u_0, v_0 \in L^1_{\Phi_1}(\cdot)$ and $u(t), v(t)$ be the corresponding MWDS, then, for all $t \geq 0$ we have

$$\|(u(t) - v(t))_{\pm}\|_{L^1_{\Phi_1}(\cdot)} \leq \|(u_0 - v_0)_{\pm}\|_{L^1_{\Phi_1}(\cdot)}.$$

Remark 2. (i) As for the previous Theorem, the T -contraction implies comparison and uniqueness in $L^1_{\Phi_1}(\cdot)$ (via contractivity), but we stress on the fact that our proof only guarantees the validity of these properties for the MWDS, a priori we can not exclude that $u_0 \in L^1_{\Phi_1}(\cdot)$ produces other WDS.

(ii) A closer inspection of the proof reveals that the MWDS does not depend on the particular choice of the approximating sequence, we only require the approximating sequence to be monotone increasing, see Step 4 of the proof of Theorem 2.4 for more details.

(iii) The question of uniqueness is a delicate issue in this $L^1_{\Phi_1}(\cdot)$ -setting, since the energy functional is

$$\mathcal{F}[u] = \int u^m \Phi_1 \, dx$$

which is clearly *not convex* when $m \in (0, 1)$. A full uniqueness result for WDS is still missing. This is an intriguing open problem and it would imply that all WDS are minimal (hence a priori more regular). This lack of convexity makes it impossible to apply Crandall-Liggett type Theorems, as done in the L^1 -setting in [28]. See [36, 60] for related uniqueness results on \mathbb{R}^N .

(iv) A larger “existence” class of nonnegative solutions. When we deal with nonnegative functions, $H^*(\cdot) \subset L^1_{\Phi_1}(\cdot)$, indeed by Cauchy-Schwartz inequality and the normalization $\|\Phi_1\|_2 = 1$, we get

$$\begin{aligned} \|u\|_{L^1_{\Phi_1}(\cdot)} &= \int u \Phi_1 \, dx = \int \mathcal{L}^{-\frac{1}{2}}(u) \mathcal{L}^{\frac{1}{2}}(\Phi_1) \, dx \\ &\leq \mathcal{L}^{-\frac{1}{2}}u \Big|_2 \mathcal{L}^{\frac{1}{2}}\Phi_1 \Big|_2 = \lambda_1^{\frac{1}{2}} \|u\|_{H^*(\cdot)}. \end{aligned} \tag{5}$$

To the best of our knowledge, the largest class of nonnegative data for which existence of solutions to the (CDP) is guaranteed, is precisely $L^1_{\Phi_1}(\cdot)$.

In Section 5.5, we will proof that under some assumptions MWDS are indeed $L^1_{\Phi_1}(\cdot)$ -strong.

Theorem 2.5 ($L^1_{\Phi_1}(\cdot)$ -strong solutions). *Let (A1) and (A2) hold. Then, the MWDS u corresponding to the initial datum $0 \leq u_0 \in L^1_{\Phi_1}(\cdot)$, is moreover a $L^1_{\Phi_1}(\cdot)$ -strong solution, with the bound*

$$\|\partial_t u(t)\|_{L^1_{\Phi_1}(\cdot)} \leq \frac{2 \|u_0\|_{L^1_{\Phi_1}(\cdot)}}{(1-m)t} \quad \forall t > 0,$$

whenever one of the following additional conditions hold:

1. Let $N > 2s > \gamma$ and either (K1) or (M1) hold. Moreover, assume $u_0 \in L^p(\cdot)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$.
2. Let $N, \gamma > 2s$, $m \in (0, \frac{2s}{\gamma})$ and either (K1) or (M1) hold. Assume moreover that $u_0 \in L^p(\cdot)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$.
3. Let $N > 2s > \gamma$ and (K2) hold. Assume moreover that $u_0 \in L^p_{\Phi_1}(\cdot)$ with $p \geq 1$ if $m \in (m_{c,\gamma}, 1)$ or $p > p_{c,\gamma}$ if $m \in (0, m_{c,\gamma}]$.

Remark 3. Essentially the above theorem says that MWDS are strong when they are bounded. Since the $L^1 - L^\infty$ smoothing effects are not true for $L^1_{\Phi_1}$ data when m is close to zero, it is therefore quite natural to expect that we need some extra L^p integrability on the initial datum to obtain bounded -hence strong- solutions, as it will be made precise below.

2.4 Smoothing effects

In this paper we compare two different methods for proving smoothing effects for solutions of (CDP) in order to be able to cover a larger class of operators. First, we show how Moser iteration can be used to obtain the boundedness of weak dual solutions with initial data $u_0 \in L^p(\cdot)$ and p depending on $0 < m < 1$. This first part will require some assumption on the operator together with the validity of functional inequality of Sobolev and Stroock-Varopoulos type, which we call (M1), see Section 3.1 for a careful explanation. Sometime these ingredients are not at hand, but we can find results about the inverse of the operator. This is the key tool needed in order to use the Green function method, introduced in [18], based on the dual formulation of the problem, that allows to prove an “almost representation formula”, i.e. pointwise bounds essential to prove the smoothing effects.

(K1) Assumption and unweighted Smoothings. When dealing with the question of boundedness of solutions, it is convenient to introduce the following exponents

$$m_c := \frac{N - 2s}{N} \quad \text{and} \quad p_c := \frac{N(1 - m)}{2s}.$$

As in the local case $s = 1$, when $m < m_c$ we have that L^1 data do not necessarily produce bounded solutions, as firstly shown by Brezis and Friedman [22], see also a thorough discussion in [62, 17].

We present now our main results about $L^p - L^\infty$ estimates: with and without weights.

Theorem 2.6 ($L^p - L^\infty$ smoothing). *Let $N > 2s$, $m \in (0, 1)$ and assume (A1), (A2) and either (K1) or (M1). Let u be a nonnegative WDS of (CDP) corresponding to the initial datum $u_0 \in L^p(\cdot)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, for every $t > t_0 \geq 0$ we have*

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t - t_0)^{N\vartheta_p}} \quad \text{with} \quad \vartheta_p = \frac{1}{2sp - N(1 - m)}, \quad (6)$$

where $0 < \bar{\kappa}$ only depends on N, m, s, p and t_0 .

Moser Iteration VS Duality and Green function method. In Section 3 we will provide two different proofs of this Theorem, one based on a nonlinear variant of the classical Moser iteration which will require only assumptions on the operator \mathcal{L} , namely (M1), the other based on dual formulation of the problem and we will require only assumption on \mathcal{L}^{-1} , namely (K1). In the first case, we shall see that (M1) involves the validity of Sobolev and Stroock-Varopoulos type inequalities. In the second case, the assumption (K1) compares the nonnegative Green function -the kernel of \mathcal{L}^{-1} - from above with the Green function of the Fractional Laplacian on \mathbb{R}^N . Indeed this proves that the smoothing (6) is valid for all $p \in \left(\frac{N(1-m)}{2s}, \frac{N}{2s}\right)$. (By Hölder inequality, (6) with $p < \frac{N}{2s}$ implies $L^{p_1} - L^\infty$ smoothing effects for all $p_1 > p > p_c$, but the exponents may not be sharp (for small times)

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t - t_0)^{N\vartheta_p}} \leq \bar{\kappa}_1 \frac{\|u(t_0)\|_{p_1}^{2sp\vartheta_p}}{(t - t_0)^{N\vartheta_p}}.$$

In order to obtain sharp exponents as in (6), we need Kato inequality (66), see Section 3.2.1.

New $H^ - L^\infty$ smoothings.* We also proof $H^* - L^\infty$ estimates using Theorem 2.6 along with the energy estimate below, Lemma 3.6. For this last estimate, we have to introduce another critical exponent

$$m_s = \frac{N - 2s}{N + 2s}.$$

Theorem 2.7 ($H^* - L^\infty$ smoothing). *Let $N > 2s$, $m \in (m_s, 1)$ and assume (A1), (A2) and (K1). Let u be a nonnegative WDS of (CDP) corresponding to the initial datum $u_0 \in H^*(\cdot)$. Then, for every $t > t_0 \geq 0$ we have*

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_{H^*(\cdot)}^{4s\vartheta_{1+m}}}{(t - t_0)^{(N+2s)\vartheta_{1+m}}} \quad \text{with} \quad \vartheta_{1+m} = \frac{1}{2s(1+m) - N(1-m)}, \quad (7)$$

where $0 < \bar{\kappa}$ only depends on N, m, s and \cdot .

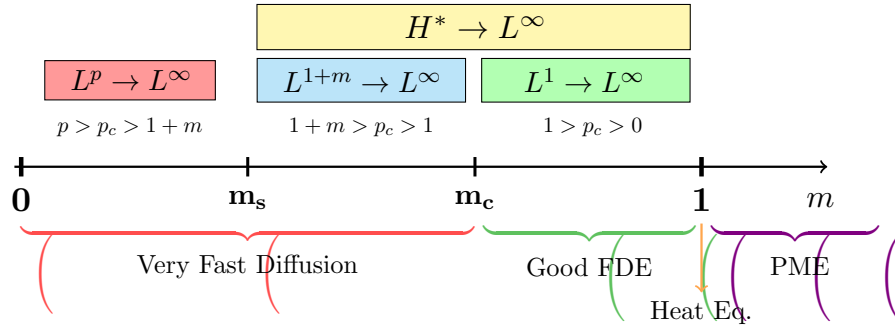


Figure 1: $L^p - L^\infty$ and $H^* - L^\infty$ smoothing effects in the different fast diffusion regimes in relation with the critical exponents:

$$p_c = \frac{N(1-m)}{2s} \quad m_c = \frac{N-2s}{N} \quad m_s = \frac{N-2s}{N+2s}$$

(K2) Assumption: weighted smoothing effects and upper boundary estimates. The Green function method is somehow more flexible and allows to prove more general smoothing effects, for data in $L^p_{\Phi_1}$, as follows, but we shall introduce first two new exponents that naturally appear in this weighted setting

$$m_{c,\gamma} := \frac{N + \gamma - 2s}{N} \quad \text{and} \quad p_{c,\gamma} := \frac{N(1-m)}{2s - \gamma}.$$

Theorem 2.8 ($L^p_{\Phi_1} - L^\infty$ smoothing). *Let $N > 2s > \gamma$, $m \in (0, 1)$ and assume (A1), (A2) and (K2). Let u be a nonnegative WDS of (CDP) corresponding to the initial datum $u_0 \in L^p_{\Phi_1}(\cdot)$ with $p \geq 1$ if $m \in (m_{c,\gamma}, 1)$ or $p > p_{c,\gamma}$ if $m \in (0, m_{c,\gamma}]$. Then, for every $t > t_0 \geq 0$ we have*

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_{L^p_{\Phi_1}(\cdot)}^{(2s-\gamma)p\vartheta_p}}{(t - t_0)^{N\vartheta_p}} \quad \text{with} \quad \vartheta_{p,\gamma} = \frac{1}{(2s - \gamma)p - N(1 - m)}. \quad (8)$$

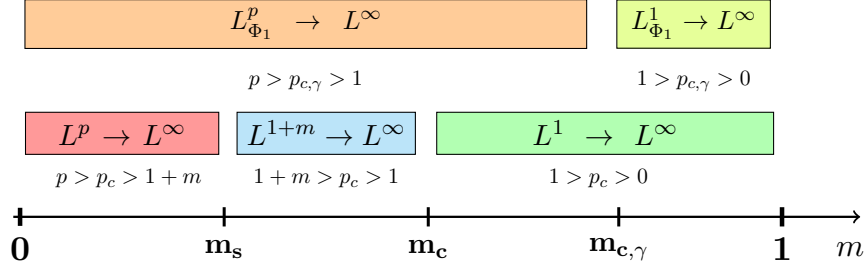


Figure 2: Weighted and unweighted smoothing effects in the different fast diffusion regimes in relation with the critical exponents:

$$\mathbf{m}_c = \frac{N - 2s}{N} \quad \mathbf{m}_{c,\gamma} = \frac{N + \gamma - 2s}{N} \quad \mathbf{m}_s = \frac{N - 2s}{N + 2s} \quad \mathbf{p}_c = \frac{N(1 - m)}{2s} \quad \mathbf{p}_{c,\gamma} = \frac{N(1 - m)}{2s - \gamma}$$

Remark 4. Boundedness of signed solutions. All our smoothing effects are stated for non-negative solutions, but they can easily be extended to signed solutions, since a closer inspection of the proof reveals that they hold for nonnegative subsolutions, hence they hold for the positive and negative part (both subsolutions) hence for $|u|$, just by summing the bounds (and possibly paying a 2 in $\bar{\kappa}$).

2.5 Upper boundary behaviour

Under assumption **(K2)** we can prove quantitative upper bounds of the solutions in terms of the first eigenfunction that we recall to satisfy

$$\Phi_1(x) \asymp \delta_\gamma(x) = \text{dist}(x, \partial \)^\gamma.$$

The following result provides a strong quantitative information of how the lateral boundary conditions are satisfied.

Theorem 2.9 (Upper boundary estimate). *Let $N > 2s$, $m \in (0, 1)$ and assume **(A1)**, **(A2)** and **(K2)**. Let u be a nonnegative WDS of **(CDP)** corresponding to the initial datum $u_0 \in L^p(\)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$. Then, $u(t)$ satisfies boundary condition and we have*

$$u^m(t, x_0) \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t - t_0)^{1+N\vartheta_p}} \mathcal{B}_1(\Phi_1(x_0)) \quad \text{for a.e. } x_0 \in \quad \text{and } \forall t > t_0 \geq 0, \quad (9)$$

with $\bar{\kappa}$ depending on p, m, s, γ, N and \quad , and \mathcal{B}_1 is defined as in Lemma 4.1 of [14],

$$\mathcal{B}_1(\Phi_1(x_0)) := \begin{cases} \Phi_1(x_0), & \text{for } 2s > \gamma, \\ \Phi_1(x_0)(1 + |\log \Phi_1(x_0)|), & \text{for } 2s = \gamma, \\ \Phi_1(x_0)^{\frac{2s}{\gamma}}, & \text{for } 2s < \gamma. \end{cases} \quad (10)$$

Moreover, let $N > 2s > \gamma$ and $u(t)$ a nonnegative WDS where $u_0 \in L^p_{\Phi_1}(\)$ with $p \geq 1$ if $m \in (m_{c,\gamma}, 1)$ or $p > p_{c,\gamma}$ if $m \in (0, m_{c,\gamma}]$. Then $u(t)$ also satisfies boundary condition and we have

$$u^m(t, x_0) \leq \bar{\kappa} \frac{\|u(t_0)\|_{L^p_{\Phi_1}(\)}^{(2s-\gamma)p\vartheta_p}}{(t - t_0)^{1+N\vartheta_p}} \Phi_1(x_0) \quad \text{for a.e. } x_0 \in \quad \text{and all } t > t_0 \geq 0.$$

2.6 Finite Extinction Time and extinction rates

Once we have that solutions are bounded, we will show that they extinguish in finite time.

Proposition 1 (*L^p estimates and Extinction Time*). *Assume (A1), (A2), (M1) and let $m \in (0, 1)$. Let u be a nonnegative WDS corresponding to the initial datum $u_0 \in L^p(\cdot)$ with $p > p_c$. Then, there exists a finite extinction time $T > 0$ such that*

$$0 \leq T = T(u_0) \leq c_p \|u_0\|_p^{1-m},$$

where $c_p > 0$ only depends on $p, m, s, N, \lambda_1, \mathcal{S}_{\mathcal{L}}, \cdot$. In addition, for every $0 \leq t_0 \leq t \leq T$ we have

$$c_p(T - t) \leq \|u(t)\|_p^{1-m} \leq \|u(t_0)\|_p^{1-m} - c_p(t - t_0). \quad (11)$$

Proposition 2 (*H^* estimates and Extinction Time*). *Assume (A1), (A2), (K1) and let $m \in (0, 1)$. Let u be a nonnegative WDS with $u_0^\alpha \in H^*(\cdot)$ and let us define $\alpha_c = \min\{1, \frac{(N+2s)(1-m)}{4s}\}$. Then, there exists a finite extinction time $T > 0$ such that*

$$0 \leq T = T(u_0) \leq c_\alpha \|u_0^\alpha\|_{H^*(\cdot)}^{\frac{1-m}{\alpha}} \quad \forall \alpha > \alpha_c,$$

with c_α depending on α, m, s, N and $\mathcal{H}_{\mathcal{L}}$. Moreover, for every $0 \leq t_0 \leq t \leq T$ we have

$$c_\alpha(T - t) \leq \|u^\alpha(t)\|_{H^*(\cdot)}^{\frac{1-m}{\alpha}} \leq \|u^\alpha(t_0)\|_{H^*(\cdot)}^{\frac{1-m}{\alpha}} - c_\alpha(t - t_0). \quad (12)$$

Proposition 3 (*$L^1_{\Phi_1}$ estimates*). *Assume (A1), (A2), (K1) and let $m \in (0, 1)$. Let u be a nonnegative WDS with $u_0 \in L^1_{\Phi_1}(\cdot)$, and let $T > 0$, be its extinction time. Then*

$$\|u(t)\|_{L^1_{\Phi_1}(\cdot)} \leq c_1(T - t)^{\frac{1}{1-m}} \quad \forall 0 \leq t < T,$$

with $c_1 = \lambda_1^{\frac{1}{1-m}} \|\Phi_1\|_1$. As a consequence, we have the following lower bound for the extinction time,

$$T \geq \lambda_1 \left(\|u_0\|_{L^1_{\Phi_1}} \|\Phi_1\|_1 \right)^{1-m}.$$

In the ‘‘Sobolev’’ regime of fast diffusion $m \in (m_s, 1)$, we show that the above time decay is optimal.

Proposition 4 (*Sharp L^{1+m} extinction rate*). *Let $0 < m < 1$ and assume (A1) and (A2). Let u be a nonnegative WDS with $u_0 \in L^{1+m}(\cdot)$. If there exists an extinction time T , then for every $0 \leq t < T$ we have*

$$\|u(t)\|_{1+m}^{1-m} \leq (1 - m) \mathcal{Q}[u_0] (T - t). \quad (13)$$

In addition, if $m \in (m_s, 1)$ and we assume (M1), then for every $0 \leq t < T$ we have

$$c_m(T - t) \leq \|u(t)\|_{1+m}^{1-m} \leq (1 - m) \mathcal{Q}[u_0] (T - t). \quad (14)$$

Proposition 5 (Sharp H^* extinction rate). *Let $0 < m < 1$ and u be a nonnegative WDS with initial data $u_0 \in L^{1+m} \cap H^*(\cdot)$. Assume (A1) and (A2). If there exists an extinction time $T > 0$ and (K1) holds, then*

$$\|u(t)\|_{H^*(\cdot)} \leq c_1 \mathcal{Q}^*[u_0]^{\frac{1}{1-m}} (T-t)^{\frac{1}{1-m}} \quad \text{for every } 0 \leq t < T,$$

with $c_1 = (1-m)^{\frac{1}{1-m}}$. In addition, if $m \in (m_s, 1)$, then we also have the lower estimate

$$c_0 (T-t)^{\frac{1}{1-m}} \leq \|u(t)\|_{H^*(\cdot)} \leq c_1 \mathcal{Q}^*[u_0]^{\frac{1}{1-m}} (T-t)^{\frac{1}{1-m}} \quad \text{for every } 0 \leq t < T,$$

with c_0 depending on m, s, N , and $\mathcal{H}_{\mathcal{L}}$.

3 Smoothing effects. Moser iteration VS Green function method

The aim of this section is to present two different strategies to obtain $L^p - L^\infty$ estimates. The first is based on the ‘‘Green function method’’, the second is the more classical approach through Moser iteration. For these two methods, we use different assumptions, as we shall explain carefully below. The advantage of the Green function method is that the proof is simpler and it allows to obtain weighted estimates, which are new also in the case of the classical FDE, i.e. $\mathcal{L} = -\Delta$. As we shall see, the smoothing effect does not always hold for merely L^1 or $L^1_{\Phi_1}$ functions, as in the degenerate case, cf. [13, 18, 19]. As it happens in the local case, more integrability is required to obtain bounded solutions to FDE, and new exponents appear: m_c which can be characterized as the first m such that L^1 data do not produce bounded solutions, and $p_c = p_c(m)$ as the minimal integrability required to have bounded solutions. The function $p_c(m)$ creates a ‘‘green line’’² in the (m, p) -plane that identifies the zone of validity of the smoothing, see Figure 3 (with $s = \gamma = 1$ in the local case). This is carefully explained in the monograph [62] and in the Appendix of [17]. Here we present new critical exponents in the weighted and unweighted case, that allow to extend the ‘‘green line’’ of validity of the smoothing effects, to the nonlocal setting.

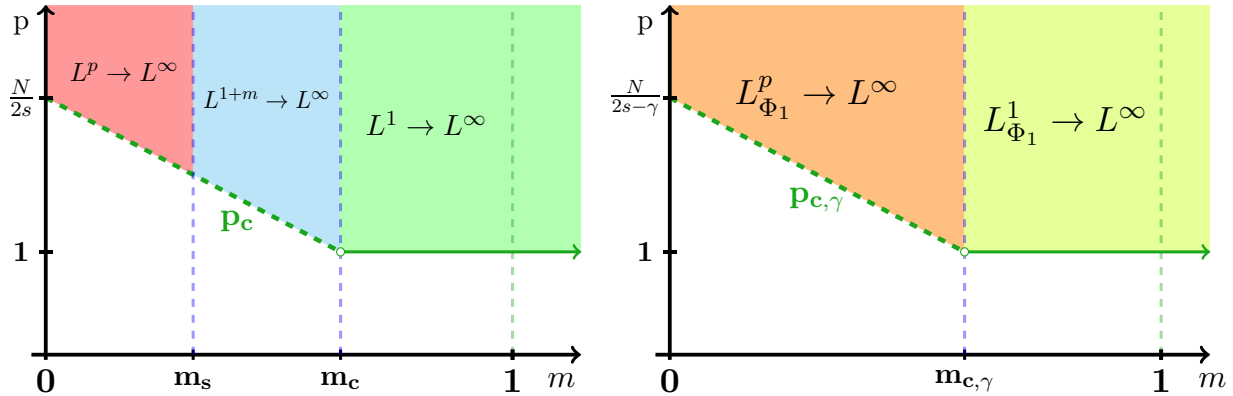


Figure 3: On the left side, the green line in the (m, p) -plane. On the right side, the weighted green line in the (m, p) -plane.

²Juan Luis Vázquez, to whom we dedicate this work, explained the ‘‘green line’’ to M.B. in 2005, as something he wanted to add in the wonderful monographs that he was writing at the time, [62, 63], to clarify the ‘‘mess of exponents’’.

3.1 Moser iteration

Functional setting and idea of the proof. One possible strategy to prove smoothing effects is to use a variant of the celebrated Moser iteration, following the main steps developed in [54] in the case of the fractional Porous medium equation. Moser iteration relies on two main ingredients: a suitable Sobolev inequality for the quadratic form associated to the operator, and a Stroock-Varopoulos type inequality. This was firstly used in [54], in the case of the Fractional Laplacian on \mathbb{R}^N and of the standard Fractional Sobolev inequality on $H^s(\mathbb{R}^N)$, together with the relation $\int f \mathcal{L}f = \|f\|_H^2 \asymp \|f\|_{H^s(\mathbb{R}^N)}^2$. In our setting, we need the following Sobolev inequality: there exists $2^* > 2$ such that for all $f \in H(\cdot)$

$$\|f\|_{2^*}^2 \leq \mathcal{S}_{\mathcal{L}}^2 \|f\|_{H(\cdot)}^2 = \mathcal{S}_{\mathcal{L}}^2 \int f \mathcal{L}f \, dx = \mathcal{S}_{\mathcal{L}}^2 \|\mathcal{L}^{1/2}u\|_2^2. \quad (15)$$

We also deduce a family of Gagliardo-Nirenberg-Sobolev (GNS) inequality: let $p > q > 0$ and

$$\frac{2q}{q+m-1} \leq 2^*, \quad \text{and} \quad \frac{q+m-1}{2q} = \frac{\theta}{2^*} + (1-\theta)\frac{q+m-1}{2p},$$

then, by interpolation of L^p norms, we have

$$\|f\|_{\frac{2q}{q+m-1}} \leq \|f\|_{2^*}^\theta \|f\|_{\frac{2p}{q+m-1}}^{1-\theta} \leq \mathcal{S}_{\mathcal{L}}^\theta \|f\|_{H(\cdot)}^\theta \|f\|_{\frac{2p}{q+m-1}}^{1-\theta}. \quad (\text{GNS})$$

However, GNS are not sufficient to run Moser iteration in the nonlocal setting. We need the Stroock-Varopoulos inequality: that there exists a constant $c_{m,p} > 0$

$$\int u^{q-1} \mathcal{L}u^m \, dx \geq c_{m,q} \int u^{\frac{q+m-1}{2}} \mathcal{L}u^{\frac{q+m-1}{2}} \, dx = c_{m,q} \|\mathcal{L}^{1/2}u^{\frac{q+m-1}{2}}\|_{H(\cdot)}^2 = c_{m,q} \|u^{\frac{q+m-1}{2}}\|_{H(\cdot)}^2. \quad (16)$$

So that, combining the two above inequalities, one gets

$$\int u^{q-1} \mathcal{L}u^m \, dx \geq c_{m,q} \|u^{\frac{q+m-1}{2}}\|_{H(\cdot)}^2 \geq c_{m,q} \mathcal{S}_{\mathcal{L}}^{-2} \frac{\|u\|_q^{\frac{q+m-1}{2\theta}}}{\|u\|_p^{\frac{1-\theta}{\theta} \frac{q+m-1}{2}}}. \quad (\text{M1})$$

This implies the decay of the L^q -norm, i.e. $\|u(t)\|_q \leq \|u(t_0)\|_q$ for all $t \geq t_0 \geq 0$ and all $q \in [1, \infty]$.

The latter inequality is the minimal assumption to run the Moser iteration. We shall discuss under which assumption this is true. Somehow, the Stroock-Varopoulos inequality is a quantitative version of (A2'), which only asserts that the above integral is nonnegative.

In the local case, the validity of this (M1) is usually a consequence of the classical ellipticity condition or of assumptions on the kernel of the operator, in the nonlocal case: there exist $0 < \lambda < \Lambda$ such that for all $\xi \in \mathbb{R}^N$

$$\text{if } \mathcal{L}[f] = - \sum_{i,j=1}^N \partial_i(a_{i,j} \partial_j f) \left(\text{then we need } \lambda |\xi|^2 \leq \sum_{i,j=1}^N \xi_i a_{i,j} \xi_j \leq \Lambda |\xi|^2, \right. \quad (17)$$

and the Stroock-Varopoulos inequality follows by integration by parts and chain rule:

$$\int u^{p-1} \mathcal{L}u^m \, dx \geq \lambda \int \nabla(u^{p-1}) \cdot \nabla(u^m) \, dx = c_{m,p} \lambda \int |\nabla u^{\frac{p+m-1}{2}}|^2 \, dx$$

and we obtain (M1) just by using the classical Sobolev inequality on $H_0^1(\cdot)$, with $2^* = \frac{2N}{N-2} > 2$.

In the nonlocal case, the situation gets more involved: define the operator as

$$\text{if } \mathcal{L}[f](x) = \text{P.V.} \int_{\mathbb{R}^N} [f(x) - f(y)] K(x, y) dy \quad (18)$$

restricted to functions supported in $\bar{\Omega}$, and impose the ellipticity condition on the kernel:

$$\frac{\lambda}{|x - y|^{N+2s}} \leq K(x, y) \leq \frac{\Lambda}{|x - y|^{N+2s}}. \quad (19)$$

Under this assumptions, using the Stroock-Varopoulos inequality (6.2), and the Sobolev inequality on $H_0^s(\cdot)$, one can prove (M1), as it has been done for the first time in [53]. See also [15] for more details about Sobolev spaces on bounded domains.

Unfortunately, the above assumptions do not cover the case of the Spectral Fractional Laplacian, and more generally “spectral type operators”, i.e. powers of elliptic operators of type (17) or even of type (18). This is because spectral operators can be represented with a kernel as follows:

$$\mathcal{L}^s[f] = \text{P.V.} \int [f(x) - f(y)] K(x, y) dy + B(x)f(x) \quad (20)$$

and there exist $0 < \lambda < \Lambda$ such that for a.e. $x, y \in \Omega$ we have

$$\lambda \frac{\delta_\gamma(x)\delta_\gamma(y)}{|x - y|^{N+2s}} \leq K(x, y) \leq \Lambda \frac{\delta_\gamma(x)\delta_\gamma(y)}{|x - y|^{N+2s}} \quad \text{and} \quad 0 \leq B(x) \leq \frac{c}{\delta(x)^{2s}}, \quad (L1)$$

and having a kernel supported in $\bar{\Omega}$ and zero at the boundary, they clearly do not satisfy the ellipticity condition (19). We will show how to deal with these operators more easily, with the Green function method in Section 3.2, but we can also take into account the following remark.

Assumption (K1) and (L1) imply (M1). We have seen that assumption (K1) implies the validity of a Hardy-Littlewood (HLS) inequality: for all $f \in H^*(\cdot)$

$$\|f\|_{H^*}^2 = \|\mathcal{L}^{-1/2}f\|_2^2 \leq \mathcal{H}_\mathcal{L} \|f\|_{(2^*)'} \quad \text{with} \quad (2^*)' = \frac{2N}{N + 2s}, \quad (21)$$

which is equivalent to the desired Sobolev inequality,

$$\|f\|_{2^*}^2 \leq \mathcal{S}_\mathcal{L}^2 \|f\|_{H(\cdot)}^2 \quad \text{with} \quad 2^* = \frac{2N}{N - 2s}. \quad (22)$$

by using Legendre duality, as carefully explained in Appendix 7.8 of [15].

On the other hand, the Stroock-Varopoulos follows by the fact that $K(x, y) \geq 0$, see Lemma 6.2 for a proof. Hence, (M1) holds under the assumption (K1), an upper bound on the Green function, the kernel of the inverse \mathcal{L}^{-1} .

We refer to [19, 14, 13] for more details and further examples of operators, briefly summarized in Section 2.2.

We begin by proving how (M1) easily implies smoothing effects from L^p to L^q with $p_c < p < q < +\infty$.

Lemma 3.1 (L^p - L^q smoothing). *Let \mathcal{L} satisfy (A1), (A2), and (M1) with $2^* = 2N/(N - 2s)$. Let u be a GF solution with $0 \leq u_0 \in L^\infty(\cdot)$. Then, for all $\max\{1, p_c\} < p < q < +\infty$ there exists $\bar{\kappa}_{p,q} > 0$ such that for all $t > t_0 \geq 0$:*

$$\|u(t)\|_q \leq \bar{\kappa}_{p,q} \frac{\|u(t_0)\|_p^{\frac{p\vartheta_p}{q\vartheta_q}}}{(t - t_0)^{\frac{N(q-p)\vartheta_p}{q}}}, \quad \text{with } \vartheta_r = \frac{1}{2sr - N(1 - m)} \quad (23)$$

where $\bar{\kappa}_{p,q}$ only depends on p, q, N, m, s , and is given in (24).

Proof. Let us begin by deriving the L^q -norm and using (M1), recalling that in this case the constant in the Stroock-Varopoulos inequality is given by

$$c_{m,q} = \frac{4(q-1)m}{(q+m-1)^2},$$

so that

$$\frac{d}{dt} \int u^q dx = q \int u^{q-1} \partial_t u dx = -q \int u^{q-1} \mathcal{L} u^m dx \leq -\mathcal{S}_{\mathcal{L}}^{-2} \frac{4q(q-1)m}{(q+m-1)^2} \frac{\|u\|_q^{\frac{q+m-1}{2\theta}}}{\|u\|_p^{\frac{1-\theta}{\theta} \frac{q+m-1}{2}}}.$$

Few remarks are in order: first, we have to ensure that

$$\frac{2p}{p+m-1} \leq 2^* = \frac{2N}{N-2s}, \quad \text{that is } p \geq p_c = \frac{N(1-m)}{2s}.$$

Hence, we have

$$\frac{d}{dt} \int u^q dx \leq -\bar{\kappa}_0 \left(\int u^q dx \right)^{\frac{q+m-1}{q\theta}} \quad \text{with } \bar{\kappa}_0 = \frac{4q(q-1)m}{(q+m-1)^2} \frac{\mathcal{S}_{\mathcal{L}}^{-2}}{\|u_0\|_p^{\frac{(1-\theta)(q+m-1)}{\theta}}}.$$

Integrating the above inequality on $[t_0, t]$, we get

$$\left(\int u^q dx \right)^{-\frac{q+m-1}{q\theta} + 1} \geq \bar{\kappa}_0 \left(\frac{q+m-1}{q\theta} - 1 \right) (t - t_0)$$

recalling that

$$\frac{q+m-1}{q\theta} - 1 = \frac{2sp - N(1-m)}{N(q-p)}$$

we obtain

$$\begin{aligned} \int u(t)^q dx &\leq \left(\bar{\kappa}_0 \frac{2sp - N(1-m)}{N(q-p)} (t - t_0) \right)^{-\frac{N(q-p)}{2sp - N(1-m)}} \\ &= \left(\left(\frac{N \mathcal{S}_{\mathcal{L}}^2 (q+m-1)^2 (q-p)}{4q(q-1)m(2sp - N(1-m))} \cdot \frac{\|u_0\|_p^{\frac{2sq - N(1-m)}{N(q-p)}}}{t - t_0} \right)^{\frac{N(q-p)}{2sp - N(1-m)}} \right), \end{aligned}$$

which gives exactly (23) with the constant

$$\bar{\kappa}_{p,q} := \left(\frac{N\mathcal{S}_{\mathcal{L}}^2(q-p)(q+m-1)^2\vartheta_p}{4q(q-1)m} \right)^{\frac{N(q-p)\vartheta_p}{q}}. \quad (24)$$

□

Notice that we cannot let $q \rightarrow \infty$ in the Lemma above, since the constant blows up. Hence, we use Moser iteration to prove from Lemma 23 that solutions are bounded. The following Theorem is a slightly different version of Theorem 2.6, since we ask only for assumptions on the operator \mathcal{L} . Theorem 2.6 is proved recalling that (K1) and (L1) implies (M1).

Theorem 3.2 ($L^p - L^\infty$ smoothing). *Let \mathcal{L} satisfy (A1), (A2), and (M1) with $2^* = 2N/(N - 2s)$. Let u be a nonnegative WDS with $u_0 \in L^p(\cdot)$ and $p > p_c$ if $m \in (0, m_c]$ or $p \geq 1$ if $m \in (m_c, 1)$. Then, there exists a constant $\bar{\kappa} = \bar{\kappa}(N, m, s, p, \mathcal{S}_{\mathcal{L}}, \cdot) > 0$ such that for any $t > t_0 \geq 0$*

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \quad \text{with} \quad \vartheta_p = \frac{1}{2sp - N(1-m)}. \quad (25)$$

Proof. We split the proof in three steps: first we establish the bound (25) for nonnegative bounded GF solutions for $p > \max\{1, p_c\}$, then we establish it for $p = 1$ when $m \in (m_c, 1)$ in Step 2, then we will deduce the final result by approximation in Step 3.

• **STEP 1. Smoothing estimates for bounded GF solutions.** We begin by establishing the bound (25) for nonnegative bounded GF solutions, more precisely we will prove the following

Claim: Let \mathcal{L} satisfy (A1), (A2) and (M1) with $2^* = 2N/(N - 2s)$ and let u be a GF solution with $0 \leq u_0 \in L^\infty(\cdot)$. Then, for any $p > \max\{1, p_c\}$ there exists a constant $\bar{\kappa} = \bar{\kappa}(N, m, s, p, \mathcal{S}_{\mathcal{L}}, \cdot) > 0$ such that (25) holds for any $t > t_0 \geq 0$.

Proof of the claim. Let us rewrite (23) for each $k \geq 1$ with $p_k = 2^k p$ and t_k such that $t_k - t_{k-1} = \frac{t-t_0}{2^k}$,

$$\|u(t_k)\|_{p_k} \leq \left(\frac{c_k}{t_k - t_{k-1}} \right)^{\frac{N(p_k - p_{k-1})\vartheta_{k-1}}{p_k}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\vartheta_{k-1}}{p_k}} \quad (26)$$

where $\vartheta_k := \vartheta_{p_k} = (2sp_k - N(1-m))^{-1}$ and

$$c_k := \frac{N\mathcal{S}_{\mathcal{L}}^2(p_k - p_{k-1})(p_k + m - 1)^2\vartheta_{k-1}}{4p_k(p_k - 1)m}.$$

Then, let us bound c_k uniformly in k using the definition of p_k :

$$\begin{aligned} c_k &\leq \frac{N\mathcal{S}_{\mathcal{L}}^2}{8m} \frac{(p_k + m - 1)^2}{(p_k - 1)(2sp_{k-1} - N(1-m))} = \frac{N\mathcal{S}_{\mathcal{L}}^2}{8m} \frac{(1 - \frac{1-m}{p_k})^2}{(1 - \frac{1}{p_k})(s - \frac{N(1-m)}{p_k})} \\ &\leq \frac{N\mathcal{S}_{\mathcal{L}}^2}{2m} \frac{p^2}{(p-1)(2sp - N(1-m))} =: \bar{c} \end{aligned}$$

which is bounded since $p > p_c$. Hence, we can iterate in (26)

$$\|u(t_k)\|_{p_k} \leq \left(\frac{\bar{c}}{t_k - t_{k-1}} \right)^{\frac{N(p_k - p_{k-1})\vartheta_{k-1}}{p_k}} \|u(t_{k-1})\|_{p_{k-1}}^{\frac{p_{k-1}\vartheta_{k-1}}{p_k}}$$

$$\begin{aligned}
&= \left(\frac{\bar{c}}{t_k - t_{k-1}} \right)^{\frac{N(p_k - p_{k-1})}{p_k} \vartheta_{k-1}} \left(\frac{\bar{c}}{t_{k-1} - t_{k-2}} \right)^{\frac{N(p_{k-1} - p_{k-2})}{p_{k-1}} \vartheta_{k-2}} \frac{p_{k-1} \vartheta_{k-1}}{p_k \vartheta_k} \|u(t_{k-2})\|_{p_{k-2}}^{\frac{p_{k-2} \vartheta_{k-2}}{p_k \vartheta_k}} \\
&\quad \vdots \\
&= \prod_{j=1}^k \left(\frac{\bar{c}}{t_j - t_{j-1}} \right)^{\frac{N(p_j - p_{j-1})}{p_k} \frac{\vartheta_j \vartheta_{j-1}}{\vartheta_k}} \|u(t_0)\|_p^{\frac{p \vartheta_p}{p_k \vartheta_k}} \\
&= \left[\prod_{j=1}^k \left(2^j \frac{\bar{c}}{t - t_0} \right)^{\frac{N(\vartheta_{j-1} - \vartheta_j)}{2s}} \right]^{\frac{1}{p_k \vartheta_k}} \|u(t_0)\|_p^{\frac{p \vartheta_p}{p_k \vartheta_k}},
\end{aligned}$$

where in the last inequality we have used that $(p_j - p_{j-1})\vartheta_{j-1}\vartheta_j = \frac{(\vartheta_{j-1} - \vartheta_j)}{2s}$.

Now, we see that the product is uniformly bounded with respect to k :

$$\begin{aligned}
\prod_{j=1}^k \left(2^j \frac{\bar{c}}{t - t_0} \right)^{\frac{N(\vartheta_{j-1} - \vartheta_j)}{2s}} &= \exp \left[\frac{N \log(2)}{2s} \sum_{j=1}^k j(\vartheta_{j-1} - \vartheta_j) + \frac{N}{2s} \log \left(\frac{\bar{c}}{t - t_0} \right) \sum_{j=1}^k (\vartheta_{j-1} - \vartheta_j) \right] \left(\right. \\
&\leq \exp \left[\frac{N \log(2)}{2s} \sum_{j=1}^k \frac{j}{2s p_{j-1}} + \frac{N}{2s} \log \left(\frac{\bar{c}}{t - t_0} \right) (\vartheta_p - \vartheta_k) \right] \left(\right. \\
&\leq \exp \left[\frac{N \log(2)}{4s^2 p} \sum_{j=1}^k \frac{j}{2^j} + \frac{N}{2s} \log \left(\frac{\bar{c}}{t - t_0} \right) (\vartheta_p - \vartheta_k) \right] \left(\right. \\
&= 2^{\frac{N}{2s^2 p}} \left(\frac{\bar{c}}{t - t_0} \right)^{\frac{N(\vartheta_p - \vartheta_k)}{2s}}.
\end{aligned}$$

Notice that $\lim_{k \rightarrow \infty} (p_k \vartheta_k)^{-1} = 2s$ and $\lim_{k \rightarrow \infty} \vartheta_k = 0$. Therefore, since $t \geq t_k$ for all $k \geq 1$ and decay of the L^p -norm holds, we conclude that

$$\begin{aligned}
\|u(t)\|_\infty &= \lim_{k \rightarrow \infty} \|u(t)\|_{p_k} \leq \lim_{k \rightarrow \infty} \|u(t_k)\|_{p_k} \\
&\leq \lim_{k \rightarrow \infty} \left(2^{\frac{N}{2s^2 p}} \left(\frac{\bar{c}}{t - t_0} \right)^{\frac{N(\vartheta_p - \vartheta_k)}{2s}} \right)^{\frac{1}{p_k \vartheta_k}} \|u(t_0)\|_p^{\frac{p \vartheta_p}{p_k \vartheta_k}} \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2s p \vartheta_p}}{(t - t_0)^{N \vartheta_p}}.
\end{aligned}$$

This is exactly (25) when $p > \max\{1, p_c\}$, and the proof of the claim is complete.

• **STEP 2.** *The case $p = 1$ and $m \in (m_c, 1)$.* We extend the claim of Step 1 to the case $p = 1$. Let u be a nonnegative gradient flow solution with $u_0 \in L^\infty(\cdot)$ and $m \in (m_c, 1)$. Take any $p_1 > 1 > p_c$, hence $u(t)$ satisfies the claim for p_1 ,

$$\begin{aligned}
\|u(t)\|_\infty &\leq \frac{c_1}{(t - t_0)^{N \vartheta_{p_1}}} \|u(t_0)\|_{p_1}^{2s p_1 \vartheta_{p_1}} = \frac{c_1}{(t - t_0)^{N \vartheta_{p_1}}} \left(\int u(t_0)^{(p_1-1)+1} dx \right)^{2s \vartheta_{p_1}} \\
&\leq \frac{c_1 \|u(t_0)\|_\infty^{2s \vartheta_{p_1} (p_1-1)}}{(t - t_0)^{N \vartheta_{p_1}}} \|u(t_0)\|_1^{2s \vartheta_{p_1}} \leq \frac{1}{2} \|u(t_0)\|_\infty + 2^{2s \vartheta_{p_1} (p_1-1)} \frac{c_1 \|u(t_0)\|_1^{2s \vartheta_{p_1}}}{(t - t_0)^{N \vartheta_{p_1}}} \left. \right)^{\frac{\vartheta_1}{\vartheta_{p_1}}}
\end{aligned}$$

where we have used Young inequality, $ab \leq \frac{1}{2}a^{\frac{1}{\alpha}} + 2^{1-\alpha} b^{\frac{1}{1-\alpha}}$, with $\alpha = 2s\vartheta_{p_1}(p_1 - 1) < 1$. We can eliminate the term $\frac{1}{2}\|u(t_0)\|_{\infty}$ by applying De Giorgi's Lemma 6.3 and conclude the proof of the claim also in this case.

• **STEP 3. Approximation with GF solutions.** We approximate the initial data $0 \leq u_0 \in L^p(\cdot)$ by truncation, $u_{0,n} = \min\{u_0, n\}$, so that $u_{0,n} \rightarrow u_0$ in L^p . Since $u_{0,n} \in L^{\infty}(\cdot) \subset H^*(\cdot)$, there exists a gradient flow solution $u_n(t)$ given by Theorem 2.3, that is also bounded and satisfies the Claim of Step 1. Hence, by lower semicontinuity of the L^{∞} norm we obtain

$$\|u(t)\|_{\infty} \leq \liminf_{n \rightarrow \infty} \|u_n(t)\|_{\infty} \leq \liminf_{n \rightarrow \infty} \bar{\kappa}(t - t_0)^{-N\vartheta_p} \|u_n(t_0)\|_p^{2sp\vartheta_p} = \bar{\kappa}(t - t_0)^{-N\vartheta_p} \|u(t_0)\|_p^{2sp\vartheta_p}.$$

This concludes the proof. \square

3.2 Green function method

In this section we will show the advantage of the Weak Dual Formulation (WDS) and the power of the Green function method: we shall obtain a fundamental pointwise estimates (31) of Lemma 3.4, the ‘‘almost representation formula’’, the nonlinear analogous of the representation formula in the linear case, proven in [18, 19] in the case $m > 1$. Another advantage, is that we only need assumption (K1) to obtain the same results obtained with the Moser iteration in the previous section, with a much simpler proof. On the other hand, the Green function method allows also to prove weighted smoothing effects, and boundary estimates: since in the nonlocal setting it is quite rare to have weighted Sobolev type inequalities, essential to run a Moser Iteration, here we just need to know the boundary behavior of the Green function, namely (K2).

Let us recall the assumptions on \mathcal{L}^{-1} that we will use, in this Section:

- There exists a constant $c_1, \gamma > 0$ such that

$$0 \leq \mathbb{G}(x, y) \leq \frac{c_1}{|x - y|^{N-2s}}. \quad (\text{K1})$$

- Let $\Phi_1 \geq 0$ be the first eigenfunction of \mathcal{L} . Then, $\Phi_1(x) \asymp \text{dist}(x, \partial \Omega)^{\gamma}$, and we assume

$$c_0 \Phi_1(x)\Phi_1(y) \leq \mathbb{G}(x, y) \leq \frac{c_1}{|x - y|^{N-2s}} \left(\frac{\Phi_1(x)}{|x - y|^{\gamma}} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x - y|^{\gamma}} \wedge 1 \right). \quad (\text{K3})$$

As we have already explained in Section 2.1, assumption (K1) on guarantees existence of a positive and bounded eigenfunction Φ_1 , and the validity of HLS, Sobolev and Poincaré inequalities. If we assume (K2) then $\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)^{\gamma}$, cf. [14] for further details. Also, (K2) implies (K3), which is more practical in the computations: Φ_1 turns out to be a smart&smooth extension of $\text{dist}(\cdot, \partial \Omega)^{\gamma}$.

In what follow, we shall use the following useful Lemma of [14] that allows to estimate the L^q norm of the Green function, under (K1) and (K2) assumptions.

Lemma 3.3 (Green function estimates I, [14]). *Let \mathbb{G} be the kernel of \mathcal{L}^{-1} , and assume that (K1) holds. Then, for all $0 < q < N/(N - 2s)$, there exist a constant $c_2, \kappa(q) > 0$ such that*

$$\sup_{x_0 \in \Omega} \int \mathbb{G}^q(x, x_0) dx \leq c_2, \kappa(q). \quad (27)$$

Moreover, if (K2) holds, then for the same range of q there exists a constant $c_3, c_4(q) > 0$ such that, for all $x_0 \in \mathbb{R}^N$,

$$c_3(q)\Phi_1(x_0) \leq \left(\int \mathbb{G}^q(x, x_0) dx \right)^{\frac{1}{q}} \leq c_4(q)\mathcal{B}_q(\Phi_1(x_0)), \quad (28)$$

where $\mathcal{B}_q : [0, \infty) \rightarrow [0, \infty)$ is defined as follows:

$$\mathcal{B}_q(\Phi_1(x_0)) := \begin{cases} \Phi_1(x_0), & \text{for } 0 < q < \frac{N}{N-2s+\gamma}, \\ \Phi_1(x_0) \left(1 + \log \Phi_1(x_0)\right)^{\frac{1}{q}}, & \text{for } q = \frac{N}{N-2s+\gamma}, \\ \Phi_1(x_0)^{\frac{N+q(N-2s)}{q}}, & \text{for } \frac{N}{N-2s+\gamma} < q < \frac{N}{N-2s}. \end{cases}$$

Finally, for all $0 \leq f \in L^1_{\Phi_1}(\mathbb{R}^N)$, (K2) implies that

$$\int f(x)\mathbb{G}(x, x_0) dx \geq c_0 \Phi_1(x_0) \|f\|_{L^1_{\Phi_1}(\mathbb{R}^N)} \quad \text{for all } x_0 \in \mathbb{R}^N. \quad (29)$$

The constants c_i , $i = 0, \dots, 5$, depend only on s, N, γ, q , and have an explicit expression, cf. [14].

Lemma 3.4 (Fundamental pointwise estimates). *Assume (A1) and (A2). Let u be a bounded and nonnegative GF solution, then for all $x_0 \in \mathbb{R}^N$ and $0 \leq t_0 \leq t_1$ we have*

$$\frac{u^m(t_1, x)}{t_1^{\frac{m}{1-m}}} \leq \frac{1}{1-m} \int \frac{u(t_0, y) - u(t_1, y)}{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}} \mathbb{G}(x, y) dy \leq \frac{u^m(t_0, x)}{t_0^{\frac{m}{1-m}}}. \quad (30)$$

The proof presented here is an adaptation to the FDE of the proof of Proposition 4.2 of [18], or Proposition 5.1 of [19].

Proof. Let us give first a formal proof: test the weak formulation (WDS) of the equation with (the a priori not admissible test function) $\chi_{[t_0, t_1]} \delta_{x_0}$, in order to obtain on the left-hand side

$$\int_{t_0}^{t_1} \int u_t(t, x) \mathcal{L}^{-1}[\delta_{x_0}] dx dt = \int (u(t_1, x) - u(t_0, x)) \mathbb{G}(x_0, x) dx,$$

and on the right-hand side

$$- \int_{t_0}^{t_1} \int u^m(t, x) \delta_{x_0}(x) dx dt = - \int_{t_0}^{t_1} u^m(t, x_0) dt.$$

Using the time monotonicity, namely that $\left(\frac{t}{t_1}\right)^{\frac{1}{1-m}} u(t_1) \leq u(t) \leq u(t_0) \left(\frac{t}{t_0}\right)^{\frac{1}{1-m}}$, we can estimate the time integral from above and below, and obtain (30).

For a rigorous proof it suffices to approximate δ_{x_0} with $\varphi_n = \frac{\chi_{B_{1/n}(x_0)}}{|B_{1/n}(x_0)|}$ which is admissible in the (WDS) formulation, and obtain that

$$\int_{t_0}^{t_1} \int u^m(t, x) \varphi_n(x) dx dt \xrightarrow{n \rightarrow \infty} \int_{t_0}^{t_1} u^m(t, x_0) dt$$

at every Lebesgue point of $u(t, \cdot)$. Also, we shall approximate $\chi_{[t_0, t_1]}$ with χ_n as in Lemma 6.4. \square

In what follows we will only need the lower bound of (31), in the following form:

Proposition 6 (Fundamental upper bounds). *Assume (A1), (A2) and Kato's inequality (66). Let u be a bounded and nonnegative GF solution, then for all $x_0 \in \Omega$ and $0 \leq t_0 \leq \tau < t \leq t_1$ we have*

$$u^{p+m-1}(t, x_0) \leq c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int u(\tau, x)^p \mathbb{G}(x_0, x) dx. \quad (31)$$

where $c_{p,m} = \frac{p+m-1}{m(1-m)}$. Note that when $p = 1$ the above inequality follows from (30), and there is no need of assuming Kato's inequality (66).

Proof. First, we multiply the equation by $pu^{p-1}\mathbb{G}$ and integrate on $[0, T] \times \Omega$. On one hand, we obtain

$$\int \int_{t_0}^{t_1} pu^{p-1}(t, x) \partial_t u(t, x) \mathbb{G}(x_0, x) dt dx = \int (u(t_1, x)^p - u(t_0, x)^p) \mathbb{G}(x_0, x) dx.$$

On the other hand, we use Kato's inequality with $v = u^m$ and $f(v) = \frac{m}{p+m-1} v^{\frac{p+m-1}{m}}$

$$\begin{aligned} -p \int_{t_0}^{t_1} \int u^{p-1} \mathcal{L}(u(t, x)^m) \mathbb{G}(x_0, x) dx dt &\leq -\frac{pm}{p+m-1} \int_{t_0}^{t_1} \int \mathcal{L}(u^{p+m-1}(t, x)) \mathbb{G}(x_0, x) dx dt \\ &= -\frac{pm}{p+m-1} \int_{t_0}^{t_1} u^{p+m-1}(t, x_0) dt. \end{aligned}$$

Using the time monotonicity, namely that $\left(\frac{t}{t_1}\right)^{\frac{1}{1-m}} u(t_1) \leq u(t)$, we can estimate the time integral from above and obtain (6). \square

Lemma 3.5 ($L^p_{\Phi_1}$ -stability). *Let u be a nonnegative GF solution with $u_0 \in L^p_{\Phi_1}(\Omega)$ and $p \geq 1$, then*

$$\|u(t_1)\|_{L^p_{\Phi_1}(\Omega)} \leq \|u(t_0)\|_{L^p_{\Phi_1}(\Omega)} \quad \text{for every } 0 < t_0 < t_1 < T.$$

Proof. Without loss of generality, we assume $u(t) \in L^\infty(\Omega)$. Let us multiply the equation pointwise with $pu^{p-1}\Phi_1\chi_{[t_0, t_1]}$ and integrate:

$$\int_{t_0}^{t_1} \int pu^{p-1} \partial_t u \Phi_1 dx dt = \int_0^T \int \partial_t (u^p \Phi_1) \chi_{[t_0, t_1]} dx dt = \int u^p(t_1) \Phi_1 dx - \int u^p(t_0) \Phi_1 dx.$$

On the other hand, by Kato's inequality we get

$$\begin{aligned} -\int_{t_0}^{t_1} \int pu^{p-1} \mathcal{L}u^m \Phi_1 dx dt &\leq -\frac{p(p+m-1)}{m} \int_{t_0}^{t_1} \int \mathcal{L}u^{p+m-1} \Phi_1 dx dt \\ &= -\frac{\lambda_1 p(p+m-1)}{m} \int_{t_0}^{t_1} \int u^{p+m-1} \Phi_1 dx dt \leq 0. \end{aligned}$$

Finally, we shall approximate $\chi_{[t_0, t_1]}$ with χ_n as in Lemma 6.4. \square

3.2.1 Proof of Theorems 2.6 and 2.8

We shall prove simultaneously Theorems 2.6 and 2.8, indeed the only difference in their proof is one step (Step 2A and 2B below) that involve Green function estimates which hold under assumptions (K1) or (K2), respectively. This shows how the Green function method allows to deal with both weighted and unweighted smoothing effects essentially in the same way.

It is enough to prove the result for bounded nonnegative GF solutions u and then approximate the WDS by means of GF solutions $u_n(t)$ starting at $u_{0,n}(t) = \min\{u_0, n\}$, see Step 3.

• STEP 1. *Fundamental pointwise estimate and De Giorgi Lemma.* Let us assume that u is a nonnegative GF solution with $u_0 \in L^\infty(\cdot)$. Then the following estimate holds for all $0 \leq t_0 < t_1$

$$\begin{aligned} \|u(t_1)\|_\infty^{p+m-1} &\leq 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau \in [t_0, t_1] \\ x_0 \in \cdot}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx \right]^{\frac{p+m-1}{\varepsilon}} \\ &\quad + c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau \in [t_0, t_1] \\ x_0 \in \cdot}} \int_{\setminus B_R(x_0)} u(\tau, x)^p \mathbb{G}(x_0, x) dx. \end{aligned} \quad (32)$$

To prove the above inequality, we apply the Fundamental upper bounds (31) of Proposition 6 to obtain that for all $x_0 \in \cdot$ and $0 \leq t_0 \leq \tau < t \leq t_1$,

$$u^{p+m-1}(t, x_0) \leq c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int u(\tau, x)^p \mathbb{G}(x_0, x) dx. \quad (33)$$

Next, we split the last integral in two parts: fix $R > 0$ to be determined later and let $\varepsilon \in (0, p+m-1)$

$$\begin{aligned} u^{p+m-1}(t, x_0) &\leq c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \|u(\tau)\|_\infty^{p+m-1-\varepsilon} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx \\ &\quad + c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{\setminus B_R(x_0)} u^p(\tau, x) \mathbb{G}(x_0, x) dx \\ &\leq \frac{1}{2} \|u(\tau)\|_\infty^{p+m-1} + 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx \right]^{\frac{p+m-1}{\varepsilon}} \\ &\quad + c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \int_{\setminus B_R(x_0)} u(\tau, x)^p \mathbb{G}(x_0, x) dx \end{aligned}$$

by using Young's inequality, $ab \leq \frac{1}{2}a^\sigma + 2^{\frac{1}{\sigma-1}} b^{\frac{\sigma}{\sigma-1}}$, with $\sigma = \frac{p+m-1}{p+m-1-\varepsilon} > 1$.

Taking supremum on both sides we obtain for all $0 \leq t_0 \leq \tau < t \leq t_1$,

$$\begin{aligned} \|u(t)\|_\infty^{p+m-1} &\leq \frac{1}{2} \|u(\tau)\|_\infty^{p+m-1} \\ &+ 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau \in [t_0, t_1] \\ x_0 \in B_R(x_0)}} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx \right]^{\frac{p+m-1}{\varepsilon}} \\ &+ c_{p,m} \frac{t_1^{\frac{p+m-1}{1-m}}}{(t-\tau)^{\frac{p}{1-m}}} \sup_{\substack{\tau \in [t_0, t_1] \\ x_0 \in \setminus B_R(x_0)}} \int_{\setminus B_R(x_0)} u(\tau, x)^p \mathbb{G}(x_0, x) dx \end{aligned}$$

We can conclude by De Giorgi's Lemma 6.3, with the function $Z(t) := \|u(t)\|_\infty^{p+m-1}$ and obtain (32). *Important:* The next step is the only point where we shall distinguish between weighted (Step 2B) and unweighted smoothing effects (Step 2A).

• STEP 2.A. $L^p - L^\infty$ smoothing effects via (K1). Given any p as in Theorem 2.6, we shall prove the following estimate for bounded GF solutions: for every $t > t_0 \geq 0$ we have

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_p^{2sp\vartheta_p}}{(t-t_0)^{N\vartheta_p}} \quad \text{with} \quad \vartheta_p = \frac{1}{2sp - N(1-m)}, \quad (34)$$

where $\bar{\kappa} > 0$ only depends on N, m, s, p and .

The proof consists in carefully estimating the two terms in the right-hand side of (32) separately, using assumption (K1), namely $\mathbb{G}(x_0, x) \leq c_1 |x - x_0|^{-(N-2s)}$. First, we observe that for all exponent $0 < q < \frac{N}{N-2s}$ and $q' = \frac{q}{q-1} > \frac{N}{2s}$ we can use Hölder's inequality

$$\begin{aligned} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx &\leq c_1 \int_{B_R(x_0)} \frac{u^{1-m+\varepsilon}(\tau, x)}{|x - x_0|^{N-2s}} dx \\ &\leq c_1 \|u(\tau)\|_{q'(1-m+\varepsilon)}^{1-m+\varepsilon} \left[\int_{B_R(x_0)} \frac{1}{|x - x_0|^{q(N-2s)}} dx \right]^{\frac{1}{q}} \\ &\leq c_2 \|u(t_0)\|_{q'(1-m+\varepsilon)}^{1-m+\varepsilon} R^{\frac{N-q(N-2s)}{q}} = c_2 \|u(t_0)\|_{q'(1-m+\varepsilon)}^{1-m+\varepsilon} R^{2s - \frac{N}{q'}} \end{aligned}$$

where $c_2 > 0$ depends on q, N, s and c_1 . We have also used the L^p -norm decay since if we fix $\varepsilon \in \left(0, \frac{2sp - N(1-m)}{N}\right) \subset (0, p+m-1)$ we can choose $q' := \frac{p}{(1-m+\varepsilon)}$ as

$$\frac{N}{2s} < q' = \frac{p}{1-m+\varepsilon}.$$

On the other hand, using the L^p -norm decay we have

$$\int_{\setminus B_R(x_0)} u(\tau, x)^p \mathbb{G}(x_0, x) dx \leq c_1 \int_{\setminus B_R(x_0)} \frac{u(\tau, x)^p}{|x - x_0|^{N-2s}} dx \leq c_1 \frac{\|u(t_0)\|_p^p}{R^{N-2s}}.$$

Plugging the above estimates in (32) and choosing $t_0 = 0$, we obtain

$$\|u(t_1)\|_\infty^{p+m-1} \leq 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[c_2 \frac{\|u_0\|_p^{1-m+\varepsilon}}{t_1} R^{\frac{2sp - N(1-m+\varepsilon)}{p}} \right]^{\frac{p+m-1}{\varepsilon}} + c_3 \frac{\|u_0\|_p^p}{t_1} \frac{1}{R^{N-2s}}.$$

Finally, we choose

$$R = \left(\frac{t_1}{\|u_0\|_p^{1-m}} \right)^{\frac{p}{2s - N(1-m)}}$$

and, by the time-shift invariance of the equation, we obtain the desired smoothing effects (34) for every $p \geq 1$ if $m \in (m_c, 1)$ and $p > p_c$ if $m \in (0, m_c)$.

• STEP 2.B. $L_{\Phi_1}^p - L^\infty$ smoothing effects via (K2). Under assumption (K2), we shall prove the following smoothing effect for bounded GF solutions and any p as in Theorem 2.8: for every $t > t_0 \geq 0$ we have

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_{L_{\Phi_1}^p(\cdot)}^{(2s-\gamma)p\vartheta_p}}{(t-t_0)^{N\vartheta_p}}, \quad \text{with } \vartheta_{p,\gamma} = \frac{1}{(2s-\gamma)p - N(1-m)}. \quad (35)$$

The proof follows analogously to STEP 2.A, that is, we have to estimate properly the two terms of (32), but in this case we have to use assumption (K2), namely $\mathbb{G}(x_0, x) \leq c_1 |x-x_0|^{-(N+\gamma-2s)} \Phi_1(x)$.

First, observe that for every $0 < q < \frac{N}{N+\gamma-2s}$ and $q' = \frac{q}{q-1} > \frac{N}{2s-\gamma}$ we can define $p := q'(1-m+\varepsilon)$ to obtain by Hölder's inequality that

$$\begin{aligned} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx &\leq c_1 \int_{B_R(x_0)} \frac{u^{1-m+\varepsilon}(\tau, x) \Phi_1(x)}{|x-x_0|^{N+\gamma-2s}} dx \\ &\leq c_1 \|\Phi_1\|_\infty^{1-\frac{1}{q'}} \|u(\tau)\|_{L_{\Phi_1}^p(\cdot)}^{1-m+\varepsilon} \left[\int_{B_R(x_0)} \frac{1}{|x-x_0|^{q(N+\gamma-2s)}} dx \right]^{\frac{1}{q}} \\ &\leq c_2 \|u(t_0)\|_{L_{\Phi_1}^p(\cdot)}^{1-m+\varepsilon} R^{\frac{N-q(N+\gamma-2s)}{q}} = c_2 \|u(t_0)\|_{L_{\Phi_1}^p(\cdot)}^{1-m+\varepsilon} R^{\frac{(2s-\gamma)p-N(1-m+\varepsilon)}{p}} \end{aligned} \quad (36)$$

where $c_2 > 0$ depends on q_0, N, s, Φ_1 and c_1 , and we have also used the $L_{\Phi_1}^p$ -norm decay. Notice that we have to choose $\varepsilon \in \left(0, \frac{(2s-\gamma)p-N(1-m)}{N}\right) \subset (0, p+m-1)$ to ensure

$$\frac{N}{2s-\gamma} < q' = \frac{p}{1-m+\varepsilon}.$$

On the other hand, we use the $L_{\Phi_1}^p$ -norm decay to obtain

$$\begin{aligned} \int_{\setminus B_R(x_0)} u(\tau, x)^p \mathbb{G}(x_0, x) dx &\leq c_1 \int_{\setminus B_R(x_0)} \frac{u(\tau, x)^p \Phi_1(x)}{|x-x_0|^{N+\gamma-2s}} dx \\ &\leq c_1 \frac{\|u(t_0)\|_{L_{\Phi_1}^p(\cdot)}^p}{R^{N+\gamma-2s}}. \end{aligned}$$

Plugging the above estimates in (32) and choosing $t_0 = 0$, we obtain

$$\|u(t_1)\|_\infty^{p+m-1} \leq 2^{\frac{p+m-1-\varepsilon}{\varepsilon}} \left[c_2 \frac{\|u_0\|_{L_{\Phi_1}^p(\cdot)}^{1-m+\varepsilon}}{t_1} R^{\frac{(2s-\gamma)p-N(1-m+\varepsilon)}{p}} \right]^{\frac{p+m-1}{\varepsilon}} + c_3 \frac{\|u_0\|_{L_{\Phi_1}^p(\cdot)}^p}{t_1} \frac{1}{R^{N+\gamma-2s}}. \quad (37)$$

with $c_2, c_3 > 0$ depending on p, N, s, γ, Φ_1 and c_1 . Finally, we choose in (37)

$$R = \left(\frac{t_1}{\|u_0\|_p^{1-m}} \right)^{\frac{p}{(2s-\gamma)p - N(1-m)}}$$

and, by the time-shift invariance of the equation, we obtain the desired smoothing effects (35) for every admissible p .

• STEP 3. *Approximation with GF solutions.* We approximate the initial data $0 \leq u_0 \in L^p(\cdot)$ by truncation, $u_{0,n} = \min\{u_0, n\}$, so that $u_{0,n} \rightarrow u_0$ in L^p . Since $u_{0,n} \in L^\infty(\cdot) \subset H^*(\cdot)$, there exists a gradient flow solution $u_n(t)$ given by Theorem 2.3, that is also bounded and satisfies the Claim of Step 1. Hence, by lower semicontinuity of the L^∞ norm we obtain

$$\begin{aligned} \|u(t)\|_\infty &\leq \lim_{n \rightarrow \infty} \|u_n(t)\|_\infty \leq \lim_{n \rightarrow \infty} \bar{\kappa}(t - t_0)^{-N\vartheta_p} \|u_n(t_0)\|_p^{2sp\vartheta_p} \\ &= \bar{\kappa}(t - t_0)^{-N\vartheta_p} \|u(t_0)\|_p^{2sp\vartheta_p}. \end{aligned}$$

This concludes the proof of the smoothing effects of Theorems 2.6 and 2.8 assuming Kato inequality.

• STEP 4. *Smoothing without using Kato inequality.* We notice that when $p = 1$, the fundamental upper bound (31) of Proposition 3.4 does not require the validity of Kato inequality, hence we can repeat Step 1 with $p = 1$, and obtain inequality (32) with $p = 1$ without using Kato inequality. We can then repeat Steps 2.A and 2.B respectively, and this is where the restriction $p < N/(2s - \gamma)$ comes from, as we shall see. The unweighted case corresponds to $\gamma = 0$ and follows with minor modifications.

The first integral of (32) can be estimated as in (36)

$$\begin{aligned} \int_{B_R(x_0)} u^{1-m+\varepsilon}(\tau, x) \mathbb{G}(x_0, x) dx &\leq c_1 \int_{B_R(x_0)} \frac{u^{1-m+\varepsilon}(\tau, x) \Phi_1(x)}{|x - x_0|^{N+\gamma-2s}} dx \\ &\leq c_2 \|u(t_0)\|_{L_{\Phi_1}^{q'(1-m+\varepsilon)}(\cdot)}^{1-m+\varepsilon} R^{\frac{N-q(N+\gamma-2s)}{q}} = c_2 \|u(t_0)\|_{L_{\Phi_1}^{p_0}(\cdot)}^{1-m+\varepsilon} R^{\frac{(2s-\gamma)p_0 - N(1-m+\varepsilon)}{p_0}} \end{aligned} \quad (38)$$

where we have set $p_0 := q'(1 - m + \varepsilon) > \frac{N(1-m+\varepsilon)}{2s-\gamma}$.

We next have to estimate the second integral of (32) as follows

$$\begin{aligned} \int_{\setminus B_R(x_0)} u(\tau, x) \mathbb{G}(x_0, x) dx &\leq c_1 \int_{\setminus B_R(x_0)} \frac{u(\tau, x) \Phi_1(x)}{|x - x_0|^{N+\gamma-2s}} dx \\ &\leq c_2 \frac{\|u(t_0)\|_{L_{\Phi_1}^q(\cdot)}}{R^{q(N+\gamma-2s)-N}}. \end{aligned} \quad (39)$$

with $q' > \frac{N}{N+\gamma-2s}$, so that $q < \frac{N}{2s-\gamma}$. This is where the restriction on p comes from.

We would like to choose now $p_0 = q$, and this is possible only when $\frac{N(1-m+\varepsilon)}{2s-\gamma} < p_0 = q < \frac{N}{2s-\gamma}$ for all $\varepsilon \in (0, m)$, which means in fact for all $p_0 \in \left(\frac{N(1-m)}{2s-\gamma}, \frac{N}{2s-\gamma} \right)$. (Hence, for all such p_0 we have

$$\|u(t_1)\|_\infty^m \leq 2^{\frac{m-\varepsilon}{\varepsilon}} \left[c_2 \frac{\|u_0\|_{L_{\Phi_1}^{p_0}(\cdot)}^{1-m+\varepsilon}}{t_1} R^{\frac{(2s-\gamma)p_0 - N(1-m+\varepsilon)}{p_0}} \right]^{\frac{m}{\varepsilon}} \left(+ \frac{c_4}{t_1} \frac{\|u(t_0)\|_{L_{\Phi_1}^{p_0}(\cdot)}}{R^{\frac{p_0}{p_0-1}(N+\gamma-2s)-N}} \right).$$

with $c_2, c_4 > 0$ depending on $p_0, N, s, \gamma, \Phi_1$ and c_1 . Letting $R^{p_0\vartheta_{p_0}} = \|u_0\|_{L_{\Phi_1}^{p_0}(\cdot)}^{m-1} t_1$ gives the desired bounds. This concludes the proof of Theorems 2.6 and 2.8. \square

3.3 $H^* - L^\infty$ smoothing effects. Proof of Theorem 2.7

It is possible to use the $L^p - L^\infty$ estimate obtained above to get $H^* - L^\infty$ smoothing effects, but it is required to be in the regime $m \in (m_s, 1)$, since we need the energy estimates below.

Lemma 3.6 (GF solutions are Energy solutions). *Assume (A1) and (A2) and let u be a GF solution with $u_0 \in H^*(\cdot)$. Then, for every $t_1 > t > t_0 \geq 0$ we have*

$$\|u^m(t_1)\|_{H(\cdot)}^2 \leq \frac{1}{2m} \frac{\|u(t)\|_{1+m}^{1+m}}{(t_1 - t)} \leq \frac{1}{2m(1+m)} \frac{\|u(t_0)\|_{H^*(\cdot)}^2}{(t_1 - t)(t - t_0)}. \quad (40)$$

Proof. We provide a formal proof in order to explain the main ideas, for a rigorous proof see Proposition 11.9 of [2].

First inequality. Let us derive the H^* -norm as in (48) and use the decay of the L^{1+m} -norm

$$\frac{d}{dt} \int u^m \mathcal{L}u^m dx \leq -2m \frac{(\int u^m \mathcal{L}u^m dx)^2}{\int u^{1+m} dx} \leq -2m \frac{(\int u^m \mathcal{L}u^m dx)^2}{\int u(t_0)^{1+m} dx}.$$

The result follows by integrating on $[t, t_1]$.

Second inequality. We derive the L^{1+m} -norm (the energy) as in (46) to obtain the following ODE,

$$\frac{d}{dt} \int u^{1+m} dx \leq -\frac{1+m}{\|u(t_0)\|_{H^*(\cdot)}^2} \left(\int u^{1+m} dx \right)^2.$$

Then, the upper estimate follows by integrating on $[t_0, t]$. \square

From the above energy estimate Theorem 2.7 easily follows, let us recall the statement: *Let $N > 2s$, $m \in (m_s, 1)$ and assume (A1), (A2) and (K1). Let u be a nonnegative WDS of (CDP) corresponding to the initial datum $u_0 \in H^*(\cdot)$. Then, for every $t > t_0 \geq 0$ we have*

$$\|u(t)\|_\infty \leq \bar{\kappa} \frac{\|u(t_0)\|_{H^*(\cdot)}^{4s\vartheta_{1+m}}}{(t - t_0)^{(N+2s)\vartheta_{1+m}}} \quad \text{with} \quad \vartheta_{1+m} = \frac{1}{2s(1+m) - N(1-m)}.$$

Proof of Theorem 2.7. We combine the energy estimate of Lemma 3.6 with $L^{1+m} - L^\infty$ smoothing effect. Given $t > t_0 \geq 0$, let us choose $\tilde{t} = \frac{t+t_0}{2}$ so that

$$\|u(t)\|_\infty \leq c \frac{\|u(\tilde{t})\|_{1+m}^{(1+m)2s\vartheta_{1+m}}}{(t - \tilde{t})^{N\vartheta_{1+m}}} \leq \frac{c}{(1+m)} \frac{\|u(t_0)\|_{H^*(\cdot)}^{4s\vartheta_{1+m}}}{(t - \tilde{t})^{N\vartheta_{1+m}} (\tilde{t} - t_0)^{2s\vartheta_{1+m}}} \leq \bar{\kappa} \frac{\|u(t_0)\|_{H^*(\cdot)}^{4s\vartheta_{1+m}}}{(t - t_0)^{(N+2s)\vartheta_{1+m}}}. \quad \square$$

3.4 Upper boundary estimates.

Once we have proved unweighted and weighted smoothing effects, we are able to show that bounded WDS satisfy the lateral boundary conditions as explained in the Introduction and in Section 2.5.

We are in the position to prove the upper boundary estimate of Theorem 2.9.

Proof of Theorem 2.9. From Theorem 2.6 it follows that $u(t)$ is bounded for every $t > 0$. Indeed, for a.e. $x_0 \in \cdot$ and $t > 0$ we have (recall that \mathcal{B}_1 is defined in (10))

$$\int u(t/2, x) \mathbb{G}(x_0, x) dx \leq \|u(t/2)\|_\infty \|\mathbb{G}(x_0, \cdot)\|_{L^1(\cdot)} \leq c_1 \|u(t/2)\|_\infty \mathcal{B}_1(\Phi_1(x_0)), \quad (41)$$

where in the last step we have used Lemma 4.2 of [14] which requires (K2). We remark that constant c_1 depends on m, s, γ and N . Then, we use (41) in estimate (31) with $p = 1$, $t_1 = t$ and $\tau = t/2$,

$$u^m(t, x_0) \leq \frac{2^{\frac{1}{1-m}}}{(1-m)t} \int u(t/2, x) \mathbb{G}(x_0, x) dx \leq \frac{2^{\frac{1}{1-m}} c_1}{(1-m)t} \|u(t/2)\|_\infty \mathcal{B}_1(\Phi_1(x_0)).$$

Now, let us apply the smoothing estimate of Theorem 2.6,

$$u^m(t, x_0) \leq \frac{2^{\frac{1}{1-m}} c_1}{(1-m)t} \|u(t/2)\|_\infty \mathcal{B}_1(\Phi_1(x_0)) \leq \frac{2^{\frac{2sp\vartheta_p}{1-m}} c_1}{(1-m)t^{1+N\vartheta_p}} \|u_0\|_p^{2sp\vartheta_p} \mathcal{B}_1(\Phi_1(x_0)).$$

Estimate (9) follows from the property of semigroups (time-shift invariance). For the weighted result with $2s > \gamma$, we repeat this last step with the smoothing estimate of Theorem 2.8,

$$u^m(t, x_0) \leq \frac{2^{\frac{1}{1-m}} c_1}{(1-m)t} \|u(t/2)\|_\infty \Phi_1(x_0) \leq \frac{2^{\frac{(2s-\gamma)p\vartheta_p}{1-m}} c_1}{(1-m)t^{1+N\vartheta_p}} \|u_0\|_{L_{\Phi_1}^p(\cdot)}^{(2s-\gamma)p\vartheta_p} \Phi_1(x_0). \quad \square$$

4 Energy estimates and Finite Extinction Time

Once we have that solutions are bounded for sufficiently smooth initial data, we will prove that they vanish in finite time, with estimates from above and below for the extinction time in different norms. Also we will show sharp time decay estimates as $t \rightarrow T^-$ that show how bounded solution extinguish in finite time, and constitute the first step towards the understanding of the asymptotic behaviour, which is a really delicate issue already in the local case, cf. for instance [11].

4.1 Norm estimates and bounds for the extinction time

The first estimates that we present about the L^p -norm requires Sobolev and Stroock-Varopoulos inequalities which can be deduced from assumption (K1) and (M1) as we show in Section 3.1.

L^p estimates and Extinction Time: *Assume (A1), (A2), (K1) and (L1). Let u be a nonnegative WDS corresponding to the initial datum $u_0 \in L^p(\cdot)$ with $p > p_c$. Then, there exist an extinction time $T > 0$ and for every $0 \leq t_0 \leq t \leq T$ we have*

$$c_p(T-t) \leq \|u(t)\|_p^{1-m} \leq \|u(t_0)\|_p^{1-m} - c_p(t-t_0), \quad (42)$$

where $c_p > 0$ only depends on $p, m, s, N, \lambda_1, \mathcal{S}_{\mathcal{L}}$.

Proof of Proposition 1. Recall that (K1) and (L1) implies (M1) and Sobolev inequality, see Section 3.1. Hence,

$$\begin{aligned} \frac{d}{dt} \int u^p dx &= -p \int u^{p-1} \mathcal{L}u^m dx \leq -c_{m,p} \|\mathcal{L}^{\frac{1}{2}} u^{\frac{p+m-1}{2}}\|_2^2 \\ &\leq -c_{m,p} \|\mathcal{S}_{\mathcal{L}}^{-1} u^{\frac{p+m-1}{2}}\|_{\frac{2p}{p+m-1}}^2 = -c_{m,p} \|\mathcal{S}_{\mathcal{L}}^{-1} u\|_p^{p+m-1} \end{aligned} \quad (43)$$

where in the last inequality we have use Sobolev in the following form,

$$\|u\|_{\frac{2p}{p+m-1}} \leq \left| \frac{p+m-1}{2p} - \frac{1}{2^*} \right| \|u\|_{2^*} \leq \left| \frac{p+m-1}{2p} - \frac{1}{2^*} \right| \mathcal{S}_{\mathcal{L}} \|\mathcal{L}^{-1} u\|_2,$$

since $p > p_c$ implies $\frac{2p}{p+m-1} < 2^* = \frac{2N}{N-2s}$. Now, integrating the ODE that $\|u(t)\|_p^p$ satisfies on the intervals $[t_0, t]$ and $[t, T]$ we obtain (42). For the upper bound on T , just let $t = 0$ in (42). \square

It is also possible to prove upper bounds of the extinction time in terms of the H^* -norm using the HLS inequality, which is provided by assumption (K1), see (2).

H* estimates and Extinction Time: Assume (A1), (A2) and (K1). Let u be a nonnegative WDS with $0 < m < 1$ and let $\alpha_c = \min\{1, \frac{(N+2s)(1-m)}{4s}\}$. Then, there exists an extinction time $T > 0$ and for every $0 \leq t_0 \leq t \leq T$ we have

$$c_\alpha(T-t) \leq \|u^\alpha(t)\|_{H^*(\cdot)}^{\frac{1-m}{\alpha}} \leq \|u^\alpha(t_0)\|_{H^*(\cdot)}^{\frac{1-m}{\alpha}} - c_\alpha(t-t_0), \quad (44)$$

with $c_\alpha > 0$ depending on α, m, s, N and $\mathcal{H}_\mathcal{L}$.

Proof of Proposition 2. Let us derive in time the H^* -norm of u^α for any $\alpha > \alpha_c$,

$$\begin{aligned} \frac{d}{dt} \int u^\alpha \mathcal{L}^{-1} u^\alpha dx &= 2\alpha \int u^{\alpha-1} \partial_t u \mathcal{L}^{-1} u^\alpha dx = -2\alpha \int u^{\alpha-1} \mathcal{L} u^m \mathcal{L}^{-1} u^\alpha dx \\ &\leq -\frac{2\alpha m}{\alpha+m-1} \int \mathcal{L} u^{\alpha+m-1} \mathcal{L}^{-1} u^\alpha dx = -\frac{2\alpha m}{\alpha+m-1} \int u^{2\alpha+m-1} dx, \end{aligned}$$

where we have applied Kato's inequality (66) with $f(v) = \frac{m}{\alpha+m-1} v^{\frac{\alpha+m-1}{m}}$ and $v = u^m$. Assumption (K1), ensures the validity of the HLS inequality for the H^* -norm (2), that applied to $f = u^\alpha$ with $q = \frac{2\alpha+m-1}{\alpha}$, gives

$$\frac{d}{dt} \int u^\alpha \mathcal{L}^{-1} u^\alpha dx \leq -\frac{2\alpha m}{\alpha+m-1} \int (u^\alpha)^{\frac{2\alpha+m-1}{\alpha}} dx \leq -\frac{2\alpha m \mathcal{H}_\mathcal{L}^{-\frac{2\alpha+m-1}{\alpha}}}{\alpha+m-1} \left(\int u^\alpha \mathcal{L}^{-1} u^\alpha dx \right)^{\frac{2\alpha+m-1}{2\alpha}}.$$

Integrating this ODE on the intervals $[t_0, t]$ and $[t, T]$ we conclude (44). The upper bound of the extinction time is provided by choosing $t = 0$ in the lower bound of (44). \square

Once we have proven the extinction of the solution, we can study the rate of extinction. For this purpose, we provide the decay of the $L_{\Phi_1}^1$ -norm in terms of the extinction time. This result is deduced directly from the formulation of the WDS without assumptions on \mathcal{L}^{-1} .

$L_{\Phi_1}^1$ estimates: Assume (A1) and (A2). Let u be a nonnegative WDS with $0 < m < 1$. Given the extinction time $T > 0$, there exists $c_1 > 0$ depending on λ_1 and Φ_1 such that

$$\|u(t)\|_{L_{\Phi_1}^1(\cdot)} \leq c_1(T-t)^{\frac{1}{1-m}} \quad \forall 0 \leq t < T.$$

Proof of Proposition 3. Let us consider the test function $\psi(\tau, x) = \lambda_1 \Phi_1(x) \chi_{[t, T]}(\tau)$ in the weak dual formulation (we approximate $\chi_{[t, T]}$ as in Lemma 6.4) and use that Φ_1 satisfies $\mathcal{L}^{-1} \Phi = \lambda_1^{-1} \Phi_1$,

$$\begin{aligned} \|u(t)\|_{L_{\Phi_1}^1} - \|u(T)\|_{L_{\Phi_1}^1} &= \int_0^T \int u \partial_\tau \chi_{[t, T]} \Phi_1 dx d\tau = \lambda_1 \int_0^T \int \mathcal{L}^{-1} u \partial_\tau \chi_{[t, T]} \Phi_1 dx d\tau \\ &= \int_0^T \int \mathcal{L}^{-1} u \partial_\tau \psi dx d\tau = \int_0^T \int u^m \psi dx d\tau = \lambda_1 \int_t^T \int u^m \Phi_1 dx d\tau \\ &\leq \lambda_1 \|\Phi\|_1^{1-m} \int_t^T \|u(\tau)\|_{L_{\Phi_1}^1}^m d\tau \leq \lambda_1 \|\Phi_1\|_1^{1-m} \|u(t)\|_{L_{\Phi_1}^1}^m (T-t) \end{aligned}$$

where we have use Hölder and the $L_{\Phi_1}^1$ -decay in the last inequalities. The result follows from the fact that T is the extinction time, hence $\|u(T)\|_{L_{\Phi_1}^1} = 0$. \square

4.2 Nonlinear Rayleigh Quotients and extinction rates

We now restrict to the exponent range $m \in (m_s, 1)$. Let us consider the “Dual” Nonlinear Rayleigh Quotient

$$\mathcal{Q}^*[f] = \frac{\|f\|_{1+m}^{1+m}}{\|f\|_{H^*}^{1+m}} = \frac{\int |f|^{1+m} dx}{\left(\int f \mathcal{L}^{-1} f dx\right)^{\frac{1+m}{2}}}.$$

We shall show that $\mathcal{Q}^*[u(t)]$ is decreasing along the FFDE flow, namely that

$$\frac{d}{dt} \mathcal{Q}^*[u(t)] \leq - (1+m) \frac{\left(\int u^{1+m} dx\right)^2}{\left(\int u \mathcal{L}^{-1} u dx\right)^{\frac{1+m}{2}+1}} + (1+m) \frac{\left(\int u \mathcal{L}^{-1} u dx\right)^{\frac{m-1}{2}} \left(\int u^{1+m} dx\right)}{\left(\int u \mathcal{L}^{-1} u dx\right)^{1+m}} \leq 0, \quad (45)$$

since we have that

$$\frac{d}{dt} \left(\left(\int u \mathcal{L}^{-1} u dx \right)^{\frac{1+m}{2}} \right) = -(1+m) \left(\int u \mathcal{L}^{-1} u dx \right)^{\frac{m-1}{2}} \left(\int u^{1+m} dx \right)$$

and also that

$$\begin{aligned} \frac{d}{dt} \int \frac{u^{1+m}}{1+m} dx &= \int u^m \partial_t u dx = - \int u^m \mathcal{L} u^m dx = - \int \frac{(\mathcal{L}^{1/2} u^m)^2 (\mathcal{L}^{-1/2} u)^2}{(\mathcal{L}^{-1/2} u)^2} dx \\ &\leq - \frac{\|(\mathcal{L}^{1/2} u^m)(\mathcal{L}^{-1/2} u)\|_1^2}{\|(\mathcal{L}^{-1/2} u)^2\|_1} \leq - \frac{1}{\|u(t_0)\|_{H^*}^2} \left(\int u^{1+m} dx \right)^2, \end{aligned} \quad (46)$$

where we have used the decay of the H^* -norm and the Cauchy-Schwarz inequality in the following form

$$\int \frac{f^2}{g} dx \geq \frac{\|f\|_1^2}{\|g\|_1}. \quad (47)$$

Let us consider the Nonlinear Rayleigh Quotient

$$\mathcal{Q}[f] = \frac{\|f^m\|_{H(\cdot)}^2}{\|f\|_{1+m}^{2m}} = \frac{\int f^m \mathcal{L} f^m dx}{\left(\int |f|^{1+m} dx\right)^{\frac{2m}{1+m}}}.$$

As before, $\mathcal{Q}[u(t)]$ decays along the FFDE flow,

$$\frac{d}{dt} \mathcal{Q}[u(t)] = \frac{\frac{d}{dt} \left(\int u^m \mathcal{L} u^m dx\right)}{\left(\int u^{1+m} dx\right)^{\frac{2m}{1+m}+1}} + 2m \frac{\left(\int u^m \mathcal{L} u^m dx\right)^2}{\left(\int u^{1+m} dx\right)^{\frac{2m}{1+m}+1}} \leq 0$$

since (47) implies that

$$\frac{d}{dt} \int u^m \mathcal{L} u^m dx = -2m \int \frac{u^{2m} (\mathcal{L} u^m)^2}{u^{1+m}} dx \leq -2m \frac{\left(\int u^m \mathcal{L} u^m dx\right)^2}{\int u^{1+m} dx}. \quad (48)$$

The decay of the Nonlinear Rayleigh Quotient $\mathcal{Q}[u(t)]$ provides another proof of the upper estimate of the L^{1+m} -norm without using a Sobolev-type inequality.

Sharp L^{1+m} decay rate: Let $m \in (m_s, 1)$ and assume (A1), (A2) and (M1). Let u be a nonnegative WDS with $u_0 \in L^{1+m}(\cdot)$. If there exists a extinction time T , then for every $T > t \geq 0$ we have

$$c_m(T-t) \leq \|u(t)\|_{1+m}^{1-m} \leq (1-m) \mathcal{Q}[u_0] (T-t). \quad (49)$$

In addition, the upper bound holds for every $m \in (0, 1)$.

Proof of Proposition 4. Let us derive the L^{1+m} -norm of $u(t)$,

$$\begin{aligned} \frac{d}{dt} \int u^{1+m} dx &= -(1+m) \int u^m \mathcal{L}u^m dx = -(1+m) \mathcal{Q}[u(t)] \left(\int u^{1+m} dx \right)^{\frac{2m}{1+m}} \\ &\geq -(1+m) \mathcal{Q}[u_0] \left(\int u^{1+m} dx \right)^{\frac{2m}{1+m}}, \end{aligned}$$

since $\mathcal{Q}[u(t)] \leq \mathcal{Q}[u_0]$. Hence, the result follows from integrating on $[t, T]$. For the lower bound we apply Proposition 1 with $p = 1 + m$. \square

On the other hand, we prove the extinction rate of the H^* -norm presented in Proposition 5
Sharp H^* decay rate: Let $m \in (m_s, 1)$ and u a nonnegative WDS with initial data $u_0 \in L^{1+m} \cap H^*(\cdot)$. Assume (A1), (A2) and (K1). If there exists and extinction time $T > 0$, then

$$c_0(T-t)^{\frac{1}{1-m}} \leq \|u(t)\|_{H^*(\cdot)} \leq c_1 \mathcal{Q}^*[u_0]^{\frac{1}{1-m}} (T-t)^{\frac{1}{1-m}} \quad \text{for every } 0 \leq t < T,$$

with $c_0 = c_0(m, s, N, \cdot, \mathcal{H}_{\mathcal{L}})$ and $c_1 = (1-m)^{\frac{1}{1-m}}$. If $m \in (0, m_s)$, then we only have the upper bound.

Proof Proposition 5. Upper bound. We derive the H^* -norm and use the monotonicity of $\mathcal{Q}^*[u(t)]$,

$$\begin{aligned} \frac{d}{dt} \int u \mathcal{L}^{-1} u dx &= -2 \int u^{1+m} dx = -2 \mathcal{Q}^*[u(t)] \left(\int u \mathcal{L}^{-1} u dx \right)^{\frac{1+m}{2}} \\ &\geq -2 \mathcal{Q}^*[u_0] \left(\int u \mathcal{L}^{-1} u dx \right)^{\frac{1+m}{2}}. \end{aligned}$$

The upper estimate follows from integrating in $[t, T]$ the ODE for $\|u(t)\|_{H^*}$.

Lower bound. Since $1+m > (2^*)' = \frac{2N}{N+2s}$, we have that (K1) implies HLS inequality. Therefore,

$$\|f\|_{H^*} \leq \mathcal{H}_{\mathcal{L}} \|f\|_{\frac{2N}{N+2s}} \leq \mathcal{H}_{\mathcal{L}} \left| \frac{N+2s}{2N} - \frac{1}{1+m} \right| \|f\|_{1+m}.$$

Now, we derive in time the H^* -norm of the solution $u(t)$ and we apply the estimate above,

$$\frac{d}{dt} \int u \mathcal{L}^{-1} u dx = -2 \int u^{1+m} dx \leq -2(c \mathcal{H}_{\mathcal{L}})^{-(1+m)} \left(\int u \mathcal{L}^{-1} u dx \right)^{\frac{1+m}{2}}.$$

The lower estimate follows from integrating on $[t, T]$ the ODE for $\|u(t)\|_{H^*}$. \square

5 Existence and uniqueness

In this section, we prove the existence and uniqueness of MWDS. For this, in the first section, we prove the existence of GF solutions, adapting to our setting the celebrated theorem of Brezis-Komura [20, 44]. This class of solutions enjoys useful properties and we use them to build a monotone approximating sequence, needed to build the MWDS. In Section 5.2, we prove the time monotonicity of the norm and the fact that GF solutions are strong in $L^1_{\Phi_1}(\cdot)$, and we also give an explicit bound for $\|\partial_t u\|_{L^1_{\Phi_1}(\cdot)}$. Then, we use the strong formulation of the problem to prove the T-contractivity in $L^1_{\Phi_1}(\cdot)$. In Section 5.3, we prove that nonnegative GF solutions are indeed WDS. Finally, in Section 5.4 we give the proof of Theorem 2.4, and in Section 5.5 we prove Theorem 2.7.

5.1 Proof of Theorem 2.3. Existence of GF solutions

In order to obtain existence of solutions of (CDP), we shall adapt the Brezis-Komura Theorem to our $H^*(\cdot)$ setting. We recall that the energy functional is defined as

$$\mathcal{E}(u) = \begin{cases} \frac{1}{1+m} \int |u|^{1+m} dx & \text{if } u \in L^{1+m}(\cdot), \\ \infty & \text{otherwise.} \end{cases}$$

First, let us check if conditions to apply Brezis-Komura Theorem are satisfied, i.e. if the functional $\mathcal{E}(u)$ is convex and lower semicontinuous in $H^*(\cdot)$.

Lemma 5.1. *The functional $\mathcal{E}(u)$ is convex and lower semicontinuous in $H^*(\cdot)$.*

We prove the above lemma in Section 6.2. This is enough to prove Theorem 2.3, indeed:

Proof of Theorem 2.3. Once we know by Lemma 5.1 that \mathcal{E} is convex and lower semicontinuous in $H^*(\cdot)$, the rest of the proof follows as in Theorem 2.2 of [15], which is the adaptation to our setting of the proof of Brezis [20] for the classical H^{-1} case. See also [2]. \square

5.2 Time Monotonicity and $L^1_{\Phi_1}$ -contractivity

In this section, we prove the Benilan-Crandall estimates, i.e. the time monotonicity of solutions. We shall also see that nonnegative GF solutions are strong in $L^1_{\Phi_1}(\cdot)$, and we show different $L^1_{\Phi_1}$ -contractivity of solutions. All these properties hold for GF solutions, this makes them a very good choice to build the approximating sequence needed to construct the MWDS, that will inherit most of these properties.

In order to prove that GF solutions are strong in $L^1_{\Phi_1}(\cdot)$ we need the following theorem of Benilan and Gariépy [5], that we restate here in our notations:

Theorem 5.2 (Theorem 1.1 of [5]). *Let $w = u^{\frac{1+m}{2}} \in W^{1,1}((0,T) : L^1_{\Phi_1}(\cdot))$, $p(r) = \frac{2}{1+m} r^{\frac{1-m}{1+m}}$ and*

$$v = \int_0^w p(r) dr \in BV((0,T) : L^1_{\Phi_1}(\cdot)),$$

where BV is the class of functions of bounded variation. Then $v \in W^{1,1}((0,T) : L^1_{\Phi_1}(\cdot))$ and the chain rule holds for the time derivative:

$$\partial_t v(t) = p(w(t)) \partial_t w(t) = \partial_t u(t).$$

We recall that $u \in L^1_{\Phi_1}(\cdot)$ is a BV function if its distributional derivative $\partial_t u$ can be represented by a finite Radon measure. In our case, the distributional derivative $\partial_t u$ must be a Radon measure, as the limit of incremental quotients of u that are uniformly bounded in $L^1_{\Phi_1}(\cdot)$. Our purpose is to prove that, indeed, this limit is in $\partial_t u$ is a function of $L^1_{\Phi_1}(\cdot)$.

The following result holds for the whole class of nonnegative WDS, and will be essential later.

Lemma 5.3. *Let $0 \leq u \leq v$ be two ordered WDS of (CDP) with initial data $0 \leq u_0 \leq v_0$. Then, for all $0 \leq t_0 \leq t_1 < \infty$, the following property holds*

$$\int [v(t_1) - u(t_1)] \Phi_1 dx \leq \int [v(t_0) - u(t_0)] \Phi_1 dx.$$

Proof. We apply the weak formulation (WDS) to the difference $v - u$, using the admissible test function $\psi_n = \Phi_1 \chi_n$, where χ_n is the same as in (6.4). Then, the left-hand side of the equality (WDS) reads

$$\begin{aligned} \int_0^T \int \mathcal{L}^{-1}(v - u) \partial_t \psi_n dx dt &= \int_0^T \int (v - u) \partial_t \chi_n \mathcal{L}^{-1} \Phi_1 dx dt \\ &\xrightarrow{n \rightarrow \infty} \lambda_1^{-1} \int [(v(t_0) - u(t_0)) - (v(t_1) - u(t_1))] \Phi_1 dx. \end{aligned}$$

As for the other side of the equality, we have

$$\int_0^T \int \mathcal{L}^{-1}(v - u) \partial_t \psi_n dx dt = \int_0^T \int (v^m - u^m) \psi_n dx dt \xrightarrow{n \rightarrow \infty} \int_{t_0}^{t_1} \int (v^m - u^m) \Phi_1 dx dt \geq 0,$$

since $u \leq v$ implies $u^m \leq v^m$ whenever $m > 0$, and this proves the lemma. \square

The proofs of the next results, Lemmata 5.4, 5.5 and 5.6, follow the ideas in [4, 5], adapted in [53] to the case of the RFL on \mathbb{R}^N and on domains. However we offer few interesting twists and variants.

Lemma 5.4 (Time Monotonicity). *Let u be a GF solution to (CDP), then*

$$u_t \leq \frac{u}{(1 - m)t}, \tag{50}$$

in the sense of distributions. This is the weak formulation of the fact that $t \mapsto t^{-\frac{1}{1-m}} u$ is non increasing on t for a.e. $x \in \cdot$.

We will adapt the classical proof of [4] for the L^1 case, see also [63].

Proof. First of all, notice that GF solutions satisfy T-contractivity property in H^* , hence, given two ordered solutions $u \leq v$, we have that for every $0 \leq t_0 \leq t_1 \leq \infty$,

$$\begin{aligned} 0 \leq \|(u(t_1) - v(t_1))_+\|_{L^1_{\Phi_1}(\cdot)} &\leq \lambda_1^{1/2} \|(u(t_1) - v(t_1))_+\|_{H^*(\cdot)} \\ &\leq \lambda_1^{1/2} \|(u(t_0) - v(t_0))_+\|_{H^*(\cdot)} = 0, \end{aligned}$$

which implies comparison in $L^1_{\Phi_1}(\cdot)$ and, particularly, almost everywhere in \cdot .

Consider the rescaled function

$$u_\lambda(t, x) = \lambda^{-\frac{1}{1-m}} u(\lambda t, x),$$

which is also a solution of (CDP). Now, for a fixed t , let us choose $\lambda = \frac{t+h}{t}$ with $h \geq 0$ and observe that, $\lambda \geq 1$ and $\lambda^{-\frac{1}{1-m}} \leq 1$, hence, $u_\lambda(0) \leq u_0$ and $u_\lambda(t) \leq u(t)$ by comparison. Then,

$$\begin{aligned} \frac{u(t+h, x) - u(t, x)}{h} &= \frac{1}{h} \left[\left(\frac{t+h}{t} \right)^{\frac{1}{1-m}} u_\lambda(t, x) - u_\lambda(t, x) \right] \left(\frac{u_\lambda(t, x) - u(t, x)}{h} \right) \\ &\leq \frac{(t+h)^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}}{h} \frac{u_\lambda(t, x)}{t^{\frac{1}{1-m}}}. \end{aligned} \quad (51)$$

The above formula has to be intended in the distributional sense. Since we know by Theorem 2.3 that GF solutions are H^* -strong, namely $t \partial_t u \in L^\infty((0, T) : H^*(\cdot))$, this implies convergence of the right-hand side to the distributional time derivative $\partial_t u$ as $h \rightarrow 0^+$, and concludes the proof. \square

Lemma 5.5. *Let u be a GF solution of (CDP), then $\partial_t u^{\frac{1+m}{2}} \in L^2_{\text{loc}}((0, T) : L^2(\cdot))$, more precisely, we have that for all $0 < t_0 < t_1 < T$*

$$\|\partial_t u^{\frac{1+m}{2}}\|_{L^2((t_0, t_1) : L^2(\cdot))}^2 \leq \frac{c(1+m)\|u(t_0/2)\|_{H^*(\cdot)}^2}{4m^2} \left(\frac{t_1}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{t_1}{t_0} - \frac{1}{t_1} \right). \quad (52)$$

Moreover, the bound is uniform in $t_0 \geq 0$, since $\|u(t_0/2)\|_{H^*(\cdot)}^2 \leq \|u_0\|_{H^*(\cdot)}^2$.

Proof. We follow the strategy of Lemma 8.1. of [54] who, in turn, give a generalized result of the the original one in [5]. We would like to use $\xi(t)\partial_t u^m(t, x)$ as test function in the $H^*(\cdot)$ formulation of the problem (see (5.7)), where $\xi(t) \in C_c^\infty(0, \infty)$. However, a priori we do not know if $\partial_t u^m(t, x)$ is sufficiently regular for this purpose so, following the idea in [5], we will use the Steklov averages.

For any $g \in L^1_{\text{loc}}(\mathbb{R})$ we define the Steklov average as

$$g^h(t, x) = \frac{1}{h} \int_t^{t+h} g(\tau, x) d\tau, \quad \text{so that} \quad \partial_t g^h(t, x) = \frac{g(t+h, x) - g(t, x)}{h}.$$

We already know that a GF solution satisfies the equation pointwise in the $H^*(\cdot)$ sense and, consequently, so does its Steklov average, i.e. for every $t \in (0, T)$

$$\partial_t u^h = -\mathcal{L}[(u^m)^h] \quad \text{a.e. in}$$

with $\partial_t u^h, \mathcal{L}[(u^m)^h] \in H^*(\cdot)$. Now, we multiply the above equation by a suitable test function ψ and we integrate in time and space to get the following expression

$$\int_0^T \int u_t^h \psi dx dt = - \int_0^T \int \mathcal{L}(u^m)^h \psi dx dt \quad \forall \psi \in C_c^1([0, T], H(\cdot)).$$

At this point, notice that $\partial_t (u^m)^h$ has the same regularity as u^m , namely, by Lemma 3.6 we have that $u^m \in H(\cdot)$ so, therefore, we can take $\psi = \xi \partial_t (u^m)^h$ as test function in the equation above,

where $\xi \in C_c^\infty((t_0/2, T))$, $0 \leq \xi \leq 1$ and $\xi = 1 \in [t_0, t_1]$ for $0 \leq t_0 \leq t_1 \leq T$. Since everything is well defined now, we have

$$\begin{aligned} \int_0^T \int \partial_t(u^h) \partial_t(u^m)^h \xi \, dx \, dt &= -\frac{1}{2} \int_0^T \int \xi \frac{\partial}{\partial t} |\mathcal{L}^{1/2}[(u^m)^h]|^2 \, dx \, dt \\ &= \frac{1}{2} \int_0^T \int \partial_t \xi |\mathcal{L}^{1/2}[(u^m)^h]|^2 \, dx \, dt. \end{aligned}$$

Then, we apply numerical inequality (68) on the left hand side to get

$$\begin{aligned} \partial_t(u^h) \partial_t(u^m)^h &= \frac{1}{h^2} [u(t+h) - u(t)] [u^m(t+h) - u(t)^m] \\ &\geq \frac{4m}{(1+m)^2} \frac{1}{h^2} \left[u^{\frac{1+m}{2}}(t+h) - u(t)^{\frac{1+m}{2}} \right]^2 = \frac{4m}{(1+m)^2} |\partial_t(u^{\frac{1+m}{2}})^h|^2. \end{aligned}$$

Now, notice that $\|\partial_t \xi\|_{L^\infty(\frac{t_0}{2}, t_1)} \leq c(\frac{2}{t_0} + \frac{1}{T-t_1})$ for every $t_0 < t_1$ in $(0, T)$, so that

$$\begin{aligned} \int_{t_0}^{t_1} \int |\partial_t(u^{\frac{1+m}{2}})^h|^2 \, dx \, dt &\leq \frac{(1+m)^2}{8m} \|\partial_t \xi\|_{L^\infty(\frac{t_0}{2}, t_1)} \int_{t_0/2}^{t_1} \int \partial_t |\mathcal{L}^{1/2}[(u^m)^h]|^2 \, dx \, dt \\ &= \frac{(1+m)^2}{8m} \|\partial_t \xi\|_{L^\infty(\frac{t_0}{2}, t_1)} \frac{1}{h^2} \int_{t_0/2}^{t_1} \int_t^{t+h} \int_t^{t+h} \int u^m(\tau) \mathcal{L} u^m(\zeta) \, dx \, d\tau \, d\zeta \, dt \\ &\leq \frac{(1+m)^2}{8m} \|\partial_t \xi\|_{L^\infty(\frac{t_0}{2}, t_1)} \frac{1}{h^2} \int_{t_0/2}^{t_1} \int_t^{t+h} \int_t^{t+h} \|u(\tau)^m\|_{H(\cdot)} \|u(\zeta)^m\|_{H(\cdot)} \, d\tau \, d\zeta \, dt \\ &\leq \frac{(1+m)^2}{8m} \|\partial_t \xi\|_{L^\infty(\frac{t_0}{2}, t_1)} \int_{t_0/2}^{t_1} \frac{2}{m(1+m)} \frac{\|u(t_0/2)\|_{H^*(\cdot)}^2}{t^2} \, dt \\ &\leq \frac{c(1+m)\|u(t_0/2)\|_{H^*(\cdot)}^2}{4m^2} \left(\frac{2}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\right) \end{aligned}$$

where we have used Cauchy-Schwarz inequality, in the third line, and Lemma 3.6. Finally, the result follows by taking the limit as $h \rightarrow 0$ on the left hand side and using Fatou's Lemma. \square

Lemma 5.6. *Let $u \in H^*(\cdot)$ be a GF solution to the problem (CDP) with initial data $0 \leq u_0 \in H^*(\cdot)$. Then, u is a strong solution in $L^1_{\Phi_1}(\cdot)$ and for every $t \in (0, T)$,*

$$\|\partial_t u\|_{L^1_{\Phi_1}(\cdot)} \leq \frac{2\|u_0\|_{L^1_{\Phi_1}(\cdot)}}{(1-m)t} \leq \frac{2\lambda_1^{1/2}\|u_0\|_{H^*(\cdot)}}{(1-m)t}.$$

We will adapt the classical proof of [4] for the L^1 case, see also [63]. The main difference in the $L^1_{\Phi_1}$ case, is that we will only use the ‘‘half’’ contractivity for ordered solutions of Lemma 5.3, instead of the full contractivity as in original proof of [4].

Proof. We follow the same idea of the proof of Lemma 5.4 to get

$$\frac{1}{h} \|u(t+h) - u(t)\|_{L^1_{\Phi_1}(\cdot)} \leq \frac{1}{h} \left(\frac{t+h}{t} \right)^{\frac{1}{1-m}} - 1 \|u_\lambda(t)\|_{L^1_{\Phi_1}(\cdot)} + \frac{1}{h} \|u_\lambda(t) - u(t)\|_{L^1_{\Phi_1}(\cdot)}.$$

Here, using Lemma 5.3 and the decay of $L_{\Phi_1}^1(\cdot)$ -norm we arrive to

$$\frac{1}{h} \|u(t+h) - u(t)\|_{L_{\Phi_1}^1(\cdot)} \leq \frac{1}{h} \left(\frac{t+h}{t} \right)^{\frac{1}{1-m}} - 1 \|u_{\lambda,0}\|_{L_{\Phi_1}^1(\cdot)} + \frac{1}{h} \|u_{\lambda,0} - u_0\|_{L_{\Phi_1}^1(\cdot)}.$$

Then, passing to the limit as in (5.4) we get that the time increments are bounded in $L_{\Phi_1}^1(\cdot)$,

$$\lim_{h \rightarrow 0} \int \frac{u(t+h, x) - u(t, x)}{h} \Phi_1 dx \leq \frac{2\|u_0\|_{L_{\Phi_1}^1(\cdot)}}{(1-m)t} \leq \frac{2\lambda_1^{1/2}\|u_0\|_{H^*(\cdot)}}{(1-m)t}, \quad (53)$$

by (5), thus, we have that $u \in BV((0, T), L_{\Phi_1}^1(\cdot))$. With this and Lemma 5.5, applying Theorem 5.2 with $v = u$, we conclude that $\partial_t u \in L_{\Phi_1}^1(\cdot)$. \square

The above result implies that the equation is satisfied pointwise as functions of $L_{\Phi_1}^1(\cdot)$. We use this fact to prove the following proposition:

Proposition 7 (T-contractivity in $L_{\Phi_1}^1$). *Let u and v be two GF solutions of (CDP) with initial data $u_0, v_0 \in H^*(\cdot)$. Then, for all $0 \leq t_0 \leq t_1 < \infty$ we have the following property*

$$\int [u(t_1) - v(t_1)]_{\pm} \Phi_1 dx \leq \int [u(t_0) - v(t_0)]_{\pm} \Phi_1 dx.$$

Notice that T-contractivity implies usual contractivity,

$$\|u(t_1) - v(t_1)\|_{L_{\Phi_1}^1(\cdot)} \leq \|u(t_0) - v(t_0)\|_{L_{\Phi_1}^1(\cdot)}.$$

Proof. It is sufficient to prove the result for $[\cdot]_+$, the case of $[\cdot]_-$ being completely analogous. We start from the strong formulation of (CDP). Let us multiply the equation by

$$\psi(t, x) = \text{sign}_+(u - v) \Phi_1 \chi_{[t_0, t_1]}(t)$$

and integrate in time and space. Then, notice that on one hand

$$\begin{aligned} \int_0^T \int (\partial_t u - \partial_t v) \psi dx dt &= - \int_0^T \int (u - v) \partial_t (\text{sign}_+(u - v) \chi_{[t_0, t_1]}) \Phi_1 dx dt \\ &= - \int [u(t_0) - v(t_0)]_+ \Phi dx + \int [u(t_1) - v(t_1)]_+ \Phi dx. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} \int_0^T \int (\partial_t u - \partial_t v) \psi dx dt &= \int_0^T \int -\mathcal{L}(u^m - v^m) \psi dx dt \\ &= - \int_{t_0}^{t_1} \int \mathcal{L}(u^m - v^m) \text{sign}_+(u - v) \Phi_1 dx dt \\ &\leq - \int_{t_0}^{t_1} \int \mathcal{L}([u^m - v^m]_+) \Phi_1 dx dt = -\lambda_1 \int_{t_0}^{t_1} \int [u^m - v^m]_+ \Phi_1 dx dt \leq 0 \end{aligned}$$

where we have used Kato inequality (66) for the convex function $[\cdot]_+$. \square

5.3 Nonnegative GF solutions are WDS

In this Section, we prove a technical fact needed in Section 5.4, in the proof of the existence of WDS, Theorem 2.4. We will show in Lemma 5.8, that GF solutions, in the sense of Def. 2.2, satisfy a suitable weak formulation, that characterizes an “intermediate” class of solutions that we will call H^* -solutions. Finally, Proposition 8, shows that nonnegative H^* -solutions are WDS.

We need to introduce an intermediate concept of solution, a weak formulation suitable for GF solutions.

Definition 5.7. (H^* -solutions). $u \in C([0, T] : H^*(\cdot))$ is an H^* -solution of (CDP) if $u^m \in L^1([0, T] : H(\cdot))$ such that

$$\int_0^T \int u \psi_t \, dx \, dt = \int_0^T \int u^m \mathcal{L} \psi \, dx \, dt \quad \forall \psi \in C_c^1([0, T], H(\cdot)),$$

or equivalently,

$$\int_0^T \int u \partial_t (\mathcal{L}^{-1} \varphi) \, dx \, dt = \int_0^T \int u^m \varphi \, dx \, dt \quad \forall \varphi \in C_c^1([0, T], H^*(\cdot)).$$

Notice that both statements are equivalent due to the fact that \mathcal{L} is an isomorphism between $H(\cdot)$ and $H^*(\cdot)$, as remarked in [15] where this concept of solutions has been previously introduced.

We show next that GF solutions satisfy the $H^*(\cdot)$ formulation, which constitutes the link between the two important class of solutions of the paper, i.e. GF and WDS.

Lemma 5.8. Let $u \in AC_{\text{loc}}((0, \infty); H^*(\cdot))$ be a GF solution of (CDP) given by Theorem 2.3, then, it is an H^* -solution.

Proof. Let $\epsilon \in \mathbb{R} \setminus \{0\}$ and $\varphi \in C_c^1([0, T], H^*(\cdot))$, then, from the definition of gradient flow and subdifferential we have that

$$\mathcal{E}(u + \epsilon \varphi) \geq \mathcal{E}(u) + \epsilon \langle -u_t, \varphi \rangle_{H^* \times H^*} \quad \text{a.e. in } t \in (0, \infty), \quad \forall \varphi \in C_c^1([0, T], H^*(\cdot)).$$

Now, if we compute the Taylor expansion of $|u + \epsilon \varphi|^{m+1}$ around u we get

$$\epsilon \int |u|^{m-1} u \varphi \, dx + o(\epsilon) \geq -\epsilon \int u_t \mathcal{L}^{-1} \varphi \, dx.$$

Here, we divide by ϵ and reparing to the fact that ϵ can be either positive and negative, we find

$$\int |u|^{m-1} u \varphi \, dx = - \int u_t \mathcal{L}^{-1} \varphi \, dx \quad \text{a.e. in } t \in (0, \infty), \quad \forall \varphi \in C_c^1([0, T], H^*(\cdot)).$$

with $u_t \in H^*(\cdot)$ and therefore $|u|^{m-1} u \in H(\cdot)$. As $x : (0, \infty) \rightarrow D(\mathcal{E}) \subset H^*$ is absolutely continuous we have that $u_t, |u|^{m-1} u \in L^1((0, T); H^*)$. Finally, since \mathcal{L} is an isomorphism between H and H^* we have that $\varphi = \mathcal{L} \psi$ for $\psi \in H$, so integrating by parts in time we get

$$\int_0^T \int |u|^{m-1} u \mathcal{L} \psi \, dx \, dt = \int_0^T \int u \psi_t \, dx \, dt \quad \text{for all } \psi \in C_c^1([0, T], H(\cdot)).$$

Hence u is a H^* solution. □

Lemma 5.9. *Let u be a nonnegative GF solution with $u_0 \in L^1_{\Phi_1}(\cdot)$. Then, for every $t_1 \geq t_0 \geq 0$ we have*

$$0 \leq \int (u(t_0) - u(t_1))\Phi_1 dx \leq c_m \|u_0\|_{L^1_{\Phi_1}(\cdot)}^m (t_1 - t_0), \quad (54)$$

with $c_m = \lambda_1 \|\Phi_1\|_1^{1-m}$. Moreover, letting $T = T(u_0)$ be the extinction time, we get

$$\|u(t)\|_{L^1_{\Phi_1}(\cdot)} \leq c_1 (T - t)^{\frac{1}{1-m}}. \quad (55)$$

Proof. Let us derive the $L^1_{\Phi_1}$ -norm of the solution and use Hölder inequality,

$$\frac{d}{dt} \int u(t)\Phi_1 dx = - \int \mathcal{L}u^m \Phi_1 dx = -\lambda_1 \int u^m \Phi_1 dx \geq -\lambda_1 \|\Phi_1\|_1^{1-m} \left(\int u(t)\Phi_1 dx \right)^m. \quad (56)$$

We apply the decay of $L^1_{\Phi_1}(\cdot)$ on the right hand side and we integrate in (t_0, t_1) to get

$$\|u(t_1)\|_{L^1_{\Phi_1}(\cdot)} - \|u(t_0)\|_{L^1_{\Phi_1}(\cdot)} \geq -\lambda_1 \|\Phi_1\|_1^{1-m} \|u_0\|_{L^1_{\Phi_1}(\cdot)}^m (t_1 - t_0)$$

(54) follows by the nonnegativity of $u(t)$. For the extinction rate, we just integrate (56) on $[t, T]$. \square

Proposition 8 (Nonnegative GF solutions are WDS). *Let u be a GF solution of (CDP) with initial datum $0 \leq u_0 \in H^*(\cdot)$. Then, $u(t)$ is a WDS, according to Definition 2.1.*

Proof. By Lemma (5.8), we have that $u(t)$ is a H^* solution. Therefore, let us prove that the set of test functions in H^* solutions is bigger than in the case of weak dual solution. Since $\|\Phi_1\|_{L^2} = 1$, we have

$$\|\psi\|_{H^*(\cdot)}^2 = \int \psi \mathcal{L}^{-1} \psi dx \leq \lambda_1^{-1} \|\psi/\Phi\|_{\infty}^2.$$

Moreover, by (5) we have that non negative solutions in H^* are indeed in the natural space for WDS:

$$\|u\|_{L^1_{\Phi_1}(\cdot)} \leq \lambda_1^{1/2} \|u\|_{H^*(\cdot)}.$$

Then, we have to verify that the next equality is true

$$\int_0^T \int u \partial_t (\mathcal{L}^{-1} \psi) dx dt = \int_0^T \int \mathcal{L}^{-1} u \partial_t (\psi) dx dt \quad \forall \frac{\psi}{\Phi_1} \in C_c^1([0, T], L^\infty(\cdot)). \quad (57)$$

Indeed, we have

$$\|\partial_t \psi\|_{H^*(\cdot)}^2 = \int \partial_t \psi \mathcal{L}^{-1} (\partial_t \psi) dx \leq \|\partial_t \psi / \Phi_1\|_{\infty} \int \Phi_1 \mathcal{L}^{-1} (\partial_t \psi) dx \leq \lambda_1^{-1} \|\partial_t \psi / \Phi_1\|_{\infty}^2$$

and (57) holds. It remains to prove that $u \in C((0, T] : L^1_{\Phi_1}(\cdot))$. From the time monotonicity of Lemma 5.4, we deduce that

$$u(t_1) - u(t_0) \leq \left[1 - \left(\frac{t_0}{t_1} \right)^{\frac{1}{1-m}} \right] u(t_1) \quad \forall t_1 \geq t_0 > 0 \text{ and a.e. } x \in \cdot,$$

and hence we have the following estimate for the positive part,

$$[u(t_1) - u(t_0)]_+ \leq \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} u(t_1) \quad \forall t_1 \geq t_0 > 0 \text{ and a.e. } x \in \mathbb{R}. \quad (58)$$

Recalling that $|f| = 2[f]_+ - f$ for any function, let us bound $\|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\cdot)}$ as follows

$$\begin{aligned} \|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\cdot)} &= 2 \int [u(t_1) - u(t_0)]_+ \Phi_1 \, dx - \int (u(t_1) - u(t_0)) \Phi_1 \, dx \\ &\leq \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \int u(t_1) \Phi_1 \, dx + c_m \|u_0\|_{L^1_{\Phi_1}(\cdot)}^m (t_1 - t_0), \end{aligned} \quad (59)$$

where we have used (58) and Lemma 5.9 to conclude. \square

5.4 Proof of Theorem 2.4. Existence of nonnegative MWDS

In this section we prove Theorem 2.4 by approximating WDS from below with bounded GF solutions. We recall the statement of Theorem 2.4 for the reader's convenience:

For every $0 \leq u_0 \in L^1_{\Phi_1}(\cdot)$ there exist a unique MWDS $u \in C((0, T], L^1_{\Phi_1}(\cdot))$ of (CDP) such that

$$\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}(\cdot)} = 0 \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{u(t+h) - u(t)}{h} \Big|_{L^1_{\Phi_1}(\cdot)} \leq \frac{2\|u_0\|_{L^1_{\Phi_1}(\cdot)}}{(1-m)t}.$$

Moreover, the T -contraction estimates hold: let $0 \leq u_0, v_0 \in L^1_{\Phi_1}(\cdot)$ and $u(t), v(t)$ be the corresponding MWDS, then, for all $t \geq 0$ we have

$$\|(u(t) - v(t))_{\pm}\|_{L^1_{\Phi_1}(\cdot)} \leq \|(u_0 - v_0)_{\pm}\|_{L^1_{\Phi_1}(\cdot)}. \quad (60)$$

Proof of Theorem 2.4. The proof is divided into several steps.

• **STEP 1.** *Existence of the minimal limit solution.* We approximate $0 \leq u_0 \in L^1_{\Phi_1}(\cdot)$ from below with the non-decreasing and bounded sequence $0 \leq u_{0,n} = u_0 \wedge n \in L^\infty(\cdot)$. Recall that the integrability of \mathbb{G} implies that $L^\infty(\cdot) \subset H^*(\cdot)$, therefore $u_{0,n} \in H^*(\cdot)$ for all $n \geq 1$. By monotone convergence, we have that $u_{0,n} \rightarrow u_0$ in the strong $L^1_{\Phi_1}$ topology, namely

$$0 \leq \|u_0 - u_{0,n}\|_{L^1_{\Phi_1}(\cdot)} = \int (u_0 - u_{0,n}) \Phi_1 \, dx \xrightarrow{n \rightarrow +\infty} 0. \quad (61)$$

Now, we are in the position to apply Theorem 2.3 which provides us the existence of the nonnegative GF solution $u_n(t)$ in $H^*(\cdot)$ starting at $u_{0,n}$. Since the L^∞ norm is non-increasing in time (just let $p \rightarrow \infty$ in (43)), we know that $u_n(t) \in L^\infty(\cdot)$. Next, Lemma 8 tells us that u_n is also a WDS corresponding to the initial datum $u_{0,n} \in L^\infty(\cdot)$. Finally, since GF solutions satisfy the comparison principle, we have that $u_n(t, x) \leq u_{n+1}(t, x)$ for a.e. $(t, x) \in (0, \infty) \times \mathbb{R}$.

Since for any fixed $t > 0$ and a.e. $x \in \mathbb{R}$ the sequence $u_n(t, x)$ is monotone increasing in n , the limit exists (can be $+\infty$) and we can define our candidate limit solution as follows

$$u(t, x) := \liminf_{n \rightarrow \infty} u_n(t, x) \quad \text{for a.e. } (t, x) \in (0, \infty) \times \mathbb{R}.$$

We shall see that this limit solution defines a continuous path in $t \mapsto u(t) \in L^1_{\Phi_1}(\cdot)$.

Convergence in $L^1_{\Phi_1}$. Notice that for all $t > 0$ we have that (take $k > n$ so that $u_k \geq u_n$)

$$\begin{aligned} \|u(t) - u_n(t)\|_{L^1_{\Phi_1}(\cdot)} &\leq \liminf_{k \rightarrow \infty} \|u_k(t) - u_n(t)\|_{L^1_{\Phi_1}(\cdot)} \leq \liminf_{k \rightarrow \infty} \|u_k(0) - u_n(0)\|_{L^1_{\Phi_1}(\cdot)} \\ &= \|u_0 - u_{0,n}\|_{L^1_{\Phi_1}(\cdot)} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned} \quad (62)$$

where the first inequality follows by Fatou's lemma, in the second step we have used the $L^1_{\Phi_1}$ contraction for ordered WDS, and finally that $u_{0,n}$ and $u_k(0) = u_{0,k}$ tend to u_0 strongly in $L^1_{\Phi_1}(\cdot)$ by (61).

• **STEP 2.** $u \in C((0, T], L^1_{\Phi_1}(\cdot))$. By Proposition 8, we know that $u_n \in C((0, T] : L^1_{\Phi_1}(\cdot))$, indeed, by (59) and the decay of $\|u_n(t)\|_{L^1_{\Phi_1}(\cdot)}$, we have that for every $0 < t_0 \leq t_1 \leq T$

$$\|u_n(t_1) - u_n(t_0)\|_{L^1_{\Phi_1}(\cdot)} \leq \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \|u_{0,n}\|_{L^1_{\Phi_1}(\cdot)} + c_m(t_1 - t_0)^{\frac{1}{1-m}}.$$

Then, since $u_{0,n} \rightarrow u_0$ strongly in $L^1_{\Phi_1}(\cdot)$ as explained in Step 1, we have that by Fatou's Lemma

$$\|u(t_1) - u(t_0)\|_{L^1_{\Phi_1}(\cdot)} \leq \lim_{n \rightarrow \infty} \|u_n(t_1) - u_n(t_0)\|_{L^1_{\Phi_1}(\cdot)} \leq \frac{t_1^{\frac{1}{1-m}} - t_0^{\frac{1}{1-m}}}{t_1^{\frac{1}{1-m}}} \|u_0\|_{L^1_{\Phi_1}(\cdot)} + c_m(t_1 - t_0)^{\frac{1}{1-m}}, \quad (63)$$

hence $u \in C((0, T], L^1_{\Phi_1}(\cdot))$ as desired. Notice that the above estimate just gives uniform boundedness of the limiting Radon measure when $t_0 = 0$, and it is not sufficient to conclude continuity on $[0, T]$.

• **STEP 3.** *The limit solution is a WDS.* Let us prove that the limit solution u constructed above is a WDS in the sense of Definition 2.1. Indeed, we shall show that satisfies the weak formulation (WDS), which amounts to check that

$$\lim_{n \rightarrow \infty} \underbrace{\int_0^T \int \mathcal{L}^{-1}(u - u_n) \partial_t \psi \, dx \, dt}_{(I)} = \lim_{n \rightarrow \infty} \underbrace{\int_0^T \int (u^m - u_n^m) \psi \, dx \, dt}_{(II)} = 0.$$

First, notice that (I) vanishes as $n \rightarrow \infty$ as a consequence of (62)

$$(I) \leq \frac{\partial_t \psi}{\Phi_1} \int_0^T \int \mathcal{L}^{-1}(u - u_n) \Phi_1 \, dx \, dt \leq \lambda_1^{-1} \frac{\partial_t \psi}{\Phi_1} \int_0^T \int (u(t) - u_n(t)) \Phi_1 \, dx \, dt \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, by Proposition 7, we have that following the same argument as for (I),

$$\begin{aligned} (II) &\leq \frac{\psi}{\Phi_1} \int_0^T \int (u^m - u_n^m) \Phi_1 \, dx \, dt \leq \frac{\psi}{\Phi_1} \int_0^T \int (u - u_n)^m \Phi_1 \, dx \, dt \\ &\leq \frac{\psi}{\Phi_1} \|\Phi_1\|_1^{1-m} \int_0^T \left(\int (u(t) - u_n(t)) \Phi_1 \, dx \right)^m dt \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

We conclude that $u \in C((0, T], L^1_{\Phi_1}(\cdot))$ is a MWDS.

• **STEP 4.** *The initial datum is taken strongly in $L^1_{\Phi_1}$.* Once we have shown that our limit solution is a WDS according to Definition 2.1, we need to show that it takes the initial datum strongly in $L^1_{\Phi_1}$, that is $u \in C([0, T], L^1_{\Phi_1}(\cdot))$ satisfying the limit $\lim_{t \rightarrow 0^+} \|u(t) - u_0\|_{L^1_{\Phi_1}} = 0$. Actually it is enough to check the latter which in turn implies the $L^1_{\Phi_1}$ -continuity at zero. Let us add and subtract in $\|u(t) - u_0\|_{L^1_{\Phi_1}(\cdot)}$ the truncation of the initial data $u_{0,n} \in L^\infty(\cdot) \cap H^*(\cdot)$ and the associated GF solutions $u_n(t) \in L^\infty(\cdot) \cap H^*(\cdot)$:

$$\begin{aligned} \int |u(t) - u_0|_{\Phi_1} dx &\leq \int |u(t) - u_n(t)|_{\Phi_1} dx + \int |u_n(t) - u_{n,0}|_{\Phi_1} dx + \int |u_{0,n} - u_0|_{\Phi_1} dx \\ &= (I) + (II) + (III). \end{aligned}$$

Now let us analyse each addend in order to pass to the limit. Since $u_n(t) \leq u(t)$, by Lemma 5.3 we have that

$$(I) = \int (u(t) - u_n(t))_{\Phi_1} dx \leq \int (u_0 - u_{0,n})_{\Phi_1} dx.$$

Moreover, by construction it holds that $u_{0,n} \leq u_0$, therefore

$$(III) = \int (u_0 - u_{0,n})_{\Phi_1} dx.$$

On the other hand, let us estimate (II) using that $u_n(t) \rightharpoonup u_{n,0}$ in $H^*(\cdot)$ as $t \rightarrow 0^+$ by definition of GF solution and $\|u_n(t)\|_2 \leq \|u_{n,0}\|_2$ for all $t > 0$.

First, we prove that weak convergence in $H^*(\cdot)$ implies weak convergence in $L^2(\cdot)$ using that $H(\cdot)$ is dense in $L^2(\cdot)$. Given any function $\psi \in L^2(\cdot)$ consider a sequence $\{\varphi_k\}_{k \geq 1} \subset H(\cdot)$ satisfying $\varphi_k \rightarrow \psi$ in $L^2(\cdot)$. Then,

$$\begin{aligned} \int (u_n(t) - u_{n,0})\psi dx &= \int (u_n(t) - u_{n,0})\varphi_k dx + \int (u_n(t) - u_{n,0})(\psi - \varphi_k) dx \\ &\leq \int (u_n(t) - u_{n,0})\varphi_k dx + \|u_n(t) - u_{n,0}\|_2 \|\psi - \varphi_k\|_2 \\ &\leq \int (u_n(t) - u_{n,0})\varphi_k dx + 2\|u_{n,0}\|_2 \|\psi - \varphi_k\|_2. \end{aligned}$$

If we take limit when $t \rightarrow 0^+$ we obtain

$$\lim_{t \rightarrow 0^+} \int (u_n(t) - u_{n,0})\psi dx \leq 2\|u_{n,0}\|_2 \|\psi - \varphi_k\|_2 \quad \forall k \geq 1,$$

since $u_n(t) \rightharpoonup u_{n,0}$ in $H^*(\cdot)$. Letting $k \rightarrow \infty$ above gives the weak- $L^2(\cdot)$ convergence of $u_n(t)$ to $u_{n,0}$ as $t \rightarrow 0^+$. Next we prove convergence of $L^2(\cdot)$ -norms, namely that $\|u_n(t)\|_2 \rightarrow \|u_{n,0}\|_2$ as $t \rightarrow 0^+$, using the weak lower semicontinuity of the norm: since $u_n(t) \rightharpoonup u_{n,0}$ in $L^2(\cdot)$ and $\|u_n(t)\|_2 \leq \|u_{n,0}\|_2$, we have

$$\|u_{n,0}\|_2 \leq \liminf_{t \rightarrow 0^+} \|u_n(t)\|_2 \leq \limsup_{t \rightarrow 0^+} \|u_n(t)\|_2 \leq \|u_{n,0}\|_2.$$

This implies strong- $L^2(\cdot)$ convergence of $u_n(t)$ to $u_{n,0}$ as $t \rightarrow 0^+$, since in any Hilbert space, weak convergence plus convergence of norms implies strong convergence. Then, we estimate (II) as follows

$$(II) = \int |u_n(t) - u_{n,0}| \Phi_1 \, dx \leq \|u_n(t) - u_{n,0}\|_2 \|\Phi_1\|_2 = \|u_n(t) - u_{n,0}\|_2 \xrightarrow{t \rightarrow 0^+} 0. \quad (64)$$

Finally, putting together the estimates of (I), (II) and (III) we have shown that

$$\int |u(t) - u_0| \Phi_1 \, dx \leq 2 \int (u_0 - u_{n,0}) \Phi_1 \, dx + \|u_n(t) - u_{n,0}\|_2,$$

which implies by (64)

$$\lim_{t \rightarrow 0^+} \int |u(t) - u_0| \Phi_1 \, dx \leq 2 \int (u_0 - u_{n,0}) \Phi_1 \, dx \quad \forall n \geq 1.$$

We conclude taking limit when $n \rightarrow +\infty$ and applying (61).

• STEP 5. *Independence of approximating sequence and uniqueness of MWDS.* Let us see that MWDS do not depend on the particular choice of the approximating sequence, we only require the approximating sequence to be monotone increasing. We adapt here the proof of Theorem 3.2 of [17] for reader's convenience. Let us choose another nonnegative monotone sequence $\{v_{0,k}\} \in L^\infty(\cdot)$ which converges to u_0 from below. Repeating the construction of the previous steps, there exists a weak dual solution $v(t, x) \in C([0, T] : L^1_{\Phi_1}(\cdot))$ generated by the GF solutions starting from $v_{0,k}$. We want to prove that $u = v$, i.e. that the minimal weak dual solution is unique. For this, we start by proving that $u \geq v$ and then we prove the opposite inequality. First, let us prove that $u \geq v$. By Proposition 7, we have that

$$\lim_{n \rightarrow \infty} \int [v_k - u_n]_+ \Phi_1 \leq \lim_{n \rightarrow \infty} \int [v_{0,k} - u_{0,n}]_+ \Phi_1 \leq \int [v_{0,k} - u_0]_+ \Phi_1 = 0,$$

as $v_{0,k} \leq u_0$ by construction. This gives that $v_k \leq u$ for all k , so taking the limit we get that $v \leq u$. The opposite inequality follows by exchanging the roles of u and v .

Remark. Notice that we have proven uniqueness of MWDS, which does not exclude a priori that there can be other WDS starting from u_0 which are not obtained as limit of a nondecreasing sequence of GF solutions. Strong uniqueness is known so far only for strong WDS solutions, c.f. Theorem 2.7.

• STEP 6. *T-contractivity for MWDS. Comparison and Contractivity.* We approximate again $u_0 \in L^1_{\Phi_1}(\cdot)$ by the same sequence as in the uniqueness proof u_n so by Lemma 7 we have that

$$\lim_{n \rightarrow \infty} \int [u_n(t_1) - v_n(t_1)]_{\pm} \Phi_1 \leq \lim_{n \rightarrow \infty} \int [u_{0,n} - v_{0,n}]_{\pm} \Phi_1.$$

The strong L^1 convergence of $u_{0,n}, v_{0,n}$ implies the fact that $\int [u_{0,n} - v_{0,n}]_+ \Phi_1 \rightarrow \int [u_0 - v_0]_+ \Phi_1$, and Fatou's Lemma applied on the left-hand side allow to conclude. Consequently, we have contractivity for MWDS, hence, uniqueness in $L^1_{\Phi_1}$ of MWDS, as solutions get closer as t increases. Moreover, by T-contractivity we have that comparison holds for MWDS, i.e. if $u_0(x) \leq v_0(x)$ a.e $x \in \cdot$ then

$$0 \leq \int [u - v]_+ \Phi_1 \leq \int [u_0 - v_0]_+ \Phi_1 = 0,$$

which implies that $u(t) \leq v(t)$ a.e $x \in \Omega$, as $\Phi_1 > 0$ in the interior of Ω .

• **STEP 7.** *Boundedness of incremental quotient of u .* Again, for the approximating sequence, notice that before passing to the limit $h \rightarrow 0$ in (53) we have

$$\int \frac{u_n(t+h, x) - u_n(t, x)}{h} \Phi_1 dx \leq \frac{1}{h} [(t+h)^{\frac{1}{1-m}} - t^{\frac{1}{1-m}}] \frac{2 \|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}}{t^{\frac{1}{1-m}}}.$$

Now, we pass to the limit as $n \rightarrow \infty$ on the right hand side and on the left by Fattou's lemma. The rest follows by the same argument as in the last part of the proof of Lemma 5.6. \square

5.5 Proof of Theorem 2.7. $L^1_{\Phi_1}$ -strong solutions and uniqueness

We prove now in which cases are MWDS actually $L^1_{\Phi_1}$ -strong solutions and satisfies the equation $u_t = -\mathcal{L}u^m$ pointwise as function of $L^1_{\Phi_1}$. Indeed, this also provides T-contractivity hence strong uniqueness in $L^1_{\Phi_1}$. Let us recall Theorem 2.7:

Assume (A1) and (A2). Then, for every $u_0 \in L^1_{\Phi_1}(\Omega)$ there exist a unique $L^1_{\Phi_1}$ -strong WDS u , hence it is the MWDS corresponding to u_0 , whenever one of the following additional conditions is satisfied:

- i) $N > 2s > \gamma$ and either (K1) or (M1) hold. Moreover, $u_0 \in L^p(\Omega)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$.
- ii) $N, \gamma > 2s$, $m \in (0, \frac{2s}{\gamma})$ and either (K1) or (M1) hold. Moreover, $u_0 \in L^p(\Omega)$ with $p \geq 1$ if $m \in (m_c, 1)$ or $p > p_c$ if $m \in (0, m_c]$.
- iii) $N > 2s > \gamma$ and (K2) hold. Moreover, $u_0 \in L^p_{\Phi_1}(\Omega)$ with $p \geq 1$ if $m \in (m_{c,\gamma}, 1)$ or $p > p_{c,\gamma}$ if $m \in (0, m_{c,\gamma}]$.

Proof of Theorem 2.7. In the three cases we follow the same steps. We use Theorem 5.2 for the MWDS in order to prove that $\partial_t u$ is a integrable function, more precisely that $t\partial_t u \in L^\infty((0, T) : L^1_{\Phi_1}(\Omega))$. We have to check if the assumptions of Theorem 5.2 are satisfied, namely, if the solution satisfies $u \in BV((0, T) : L^1_{\Phi_1}(\Omega))$ and $u^{\frac{1+m}{2}} \in W^{1,1}((0, T) : L^1_{\Phi_1}(\Omega))$.

• **STEP 1.** $u \in BV((0, T) : L^1_{\Phi_1}(\Omega))$: The former condition follows by applying Lemma 5.6 to the monotone increasing GF solutions $u_n(t)$ that approximate the MWDS $u(t)$ and passing to the limit,

$$\|\partial_t u\|_{L^1_{\Phi_1}(\Omega)} \leq \lim_{n \rightarrow \infty} \|\partial_t u_n\|_{L^1_{\Phi_1}(\Omega)} \leq \lim_{n \rightarrow \infty} \frac{2 \|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t} = \frac{2 \|u_0\|_{L^1_{\Phi_1}(\Omega)}}{(1-m)t}$$

since we use Fatou's Lemma and $u_{0,n} \rightarrow u_0$ in $L^1_{\Phi_1}(\Omega)$.

• **STEP 2.** $u^{\frac{1+m}{2}} \in W^{1,1}((0, T) : L^1_{\Phi_1}(\Omega))$: On the other hand, the latter condition will be satisfied by Lemma 5.5, but first we have to bound the $\|u_n(t_0/2)\|_{H^*(\Omega)}$ in terms of $\|u_{0,n}\|_{L^1_{\Phi_1}(\Omega)}$ in order to pass to the limit. Thus, let study the three cases of Theorem 2.7:

- i) Let $u_n(t)$ be a GF solution, then by Theorem 2.9, recalling the fact that $\mathcal{L}^{-1}\Phi_1 = \lambda_1^{-1}\Phi_1$, we

have

$$\begin{aligned} \|u_n(t)\|_{H^*(\cdot)}^2 &= \int u_n \mathcal{L}^{-1} u_n \, dx \leq \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}(\cdot)} \int (\mathcal{L}^{-1} u_n) \Phi_1 \, dx = \lambda_1^{-1} \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}(\cdot)} \|u_n(t)\|_{L^1_{\Phi_1}(\cdot)} \\ &\leq \bar{\kappa} \lambda_1^{-1} \|u_{0,n}\|_{L^1_{\Phi_1}(\cdot)} \frac{\|u_{0,n}\|_p^{\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}} \leq \bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\frac{p}{p-1}} \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}} \end{aligned}$$

where we have used the decay of the $L^1_{\Phi_1}$ -norm and Hölder's inequality. Hence, combining the above estimate with inequality (52) of Lemma 5.5, we obtain

$$\begin{aligned} \|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0, t_1); L^2(\cdot))} &\leq c_m \left(\frac{\vartheta}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\|u_n(t_0/2)\|_{H^*(\cdot)}^2 \right) \\ &\leq c_m \left(\frac{\vartheta}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\frac{p}{p-1}} \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{(t_0/2)^{\frac{1+N\vartheta p}{m}}} \right). \end{aligned}$$

We can let $n \rightarrow \infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n} \rightarrow u_0$ in $L^p(\cdot)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}} \in W^{1,1}((0, T) : L^1_{\Phi_1}(\cdot))$.

- ii) Let $u_n(t)$ be a GF solution, then by Theorem 2.9 with $m < \frac{2s}{\gamma} < 1$ and $\mathcal{L}^{-1}\Phi_1 = \lambda_1^{-1}\Phi_1$, we have

$$\begin{aligned} \|u_n(t)\|_{H^*(\cdot)}^2 &= \int u_n \mathcal{L}^{-1} u_n \, dx \leq \frac{u_n(t)}{\Phi_1^{\frac{2s}{m}}(\cdot)} \int \Phi_1^{\frac{2s}{m}} \mathcal{L}^{-1} u_n \, dx \\ &\leq \|\Phi_1\|_{\infty}^{\frac{2s}{m}-1} \frac{u_n(t)}{\Phi_1^{\frac{2s}{m}}(\cdot)} \int \Phi_1 \mathcal{L}^{-1} u_n \, dx \leq \lambda_1^{-1} \|\Phi_1\|_{\infty}^{\frac{2s}{m}-1} \|u_n(t)\|_{L^1_{\Phi_1}(\cdot)} \frac{u_n(t)}{\Phi_1^{\frac{2s}{m}}(\cdot)} \\ &\leq \bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\infty}^{\frac{2s}{m}-1} \|u_{0,n}\|_{L^1_{\Phi_1}(\cdot)} \frac{\|u_{0,n}\|_p^{\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}} \leq \bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\infty}^{\frac{2s}{m}-1} \|\Phi_1\|_{\frac{p}{p-1}} \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}} \\ &\leq \bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\infty}^{\frac{2s}{m}} \|\Phi_1\|_{\frac{p}{p-1}} \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{t^{\frac{1+N\vartheta p}{m}}}, \end{aligned}$$

where we have used the decay of the $L^1_{\Phi_1}$ -norm and Hölder's inequality. Hence, combining the above estimate with inequality (52) of Lemma 5.5, we obtain

$$\begin{aligned} \|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0, t_1); L^2(\cdot))} &\leq c_m \left(\frac{\vartheta}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\|u_n(t_0/2)\|_{H^*(\cdot)}^2 \right) \\ &\leq c_m \left(\frac{\vartheta}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_{\infty}^{\frac{2s}{m}} \|\Phi_1\|_{\frac{p}{p-1}} \frac{\|u_{0,n}\|_p^{1+\frac{2sp\vartheta p}{m}}}{(t_0/2)^{\frac{1+N\vartheta p}{m}}} \right). \end{aligned}$$

We can let $n \rightarrow \infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n} \rightarrow u_0$ in $L^p(\cdot)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}} \in W^{1,1}((0, T) : L^1_{\Phi_1}(\cdot))$.

iii) Let $u_n(t)$ be a GF solution, then by Theorem 2.9 and recalling the fact that $\mathcal{L}^{-1}\Phi_1 = \lambda_1^{-1}\Phi_1$, we have

$$\begin{aligned} \|u_n(t)\|_{H^*(\cdot)}^2 &= \int u_n \mathcal{L}^{-1}u_n \, dx \leq \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}(\cdot)} \int (\mathcal{L}^{-1}u_n)\Phi_1 \, dx = \lambda_1^{-1} \frac{u_n(t)}{\Phi_1^{\frac{1}{m}}(\cdot)} \|u_n(t)\|_{L_{\Phi_1}^1(\cdot)} \\ &\leq \bar{\kappa} \lambda_1^{-1} \|u_{0,n}\|_{L_{\Phi_1}^1(\cdot)} \frac{\|u_{0,n}\|_{L_{\Phi_1}^p(\cdot)}^{\frac{(2s-)\rho p}{m}}}{t^{\frac{1+N\rho p}{m}}} \leq \bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_1^{\frac{p}{p-1}} \frac{\|u_{0,n}\|_{L_{\Phi_1}^p(\cdot)}^{1+\frac{(2s-)\rho p}{m}}}{t^{\frac{1+N\rho p}{m}}} \end{aligned}$$

where we have used the decay of the $L_{\Phi_1}^1$ -norm and Hölder's inequality. Notice that in the case $p = 1$ we just stop at the second inequality and the Theorem is proven. In the rest of the cases, we combine the above estimate with inequality (52) of Lemma 5.5, to obtain

$$\begin{aligned} \|\partial_t u_n^{\frac{1+m}{2}}\|_{L^2((t_0, t_1); L^2(\cdot))} &\leq c_m \left(\frac{2}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \|u_n(t_0/2)\|_{H^*(\cdot)}^2 \\ &\leq c_m \left(\frac{2}{t_0} + \frac{1}{T-t_1} \right) \left(\frac{2}{t_0} - \frac{1}{t_1} \right) \left(\bar{\kappa} \lambda_1^{-1} \|\Phi_1\|_1^{\frac{p}{p-1}} \frac{\|u_{0,n}\|_{L_{\Phi_1}^p(\cdot)}^{1+\frac{(2s-)\rho p}{m}}}{(t_0/2)^{\frac{1+N\rho p}{m}}} \right). \end{aligned}$$

We can let $n \rightarrow \infty$ and using Fatou's Lemma on the left-hand side, and that $u_{0,n} \rightarrow u_0$ in $L^p(\cdot)$ on the right-hand side, therefore we conclude that $u^{\frac{1+m}{2}} \in W^{1,1}((0, T) : L_{\Phi_1}^1(\cdot))$. \square

6 Appendix

6.1 Useful inequalities

Operators with nonnegative kernels. Let \mathcal{L} be an operator of the form

$$\mathcal{L}[v](x) := \text{P.V.} \int [v(x) - v(y)] \mathbb{K}(x, y) \, dy + B(x)v(x) \quad \text{with} \quad \mathbb{K}, B \geq 0. \quad (65)$$

We show next a simple formal proof of Kato and Stroock-Varopoulos inequalities, without any claim of originality, indeed different proofs appear in literature for operators like \mathcal{L} . Our proof of Stroock-Varopoulos is inspired from [54], and from personal communications with the authors of that paper. Of course, justifying that all the integrals involved are finite would make this proofs rigorous. Throughout the paper, we pay attention to this fact and all the involved quantities are finite.

Lemma 6.1. (Kato's inequality). *Let \mathcal{L} be an operator as in (65), and let $f \in C^0(\mathbb{R})$ be a convex function with $f(0) \leq 0$. Then, if $\mathcal{L}[v] \in L_{\text{loc}}^1(\cdot)$, Kato inequality holds in the sense of distributions:*

$$\mathcal{L}f(v) \leq f'(v) \mathcal{L}v. \quad (66)$$

Proof. Recall that a convex function lies above its tangent:

$$f(v) - f(w) \leq f'(v)(v - w) \quad \forall v, w \in \mathbb{R}.$$

Then, since $\mathbb{K}(x, y) > 0$ and $B(x) \geq 0$, it holds that

$$\begin{aligned}\mathcal{L}f(v(x)) &= \int_{\mathbb{R}^N} [f(v(x)) - f(v(y))] \mathbb{K}(x, y) dy + B(x)f(v(x)) \\ &\leq \int_{\mathbb{R}^N} [f(v(x)) - f(v(y))] \mathbb{K}(x, y) dy + B(x) [f(v(x)) - f(0)] \\ &\leq \int_{\mathbb{R}^N} f'(v(x)) [v(x) - v(y)] \mathbb{K}(x, y) dy + B(x)f'(v(x))v(x) \\ &= f'(v(x))\mathcal{L}f(v(x)).\end{aligned}$$

□

Lemma 6.2. (Stroock-Varopoulos inequality). *Let \mathcal{L} be an operator as in (65) with $s \in (0, 1)$. Then for any $q > 1$*

$$\int v^{q-1} \mathcal{L}v dx \geq \frac{4(q-1)}{q^2} \int \mathcal{L}^{1/2} v^{q/2}{}^2 dx \quad (67)$$

for all $v \in L^q(\cdot)$ such that $\mathcal{L}^{s/2}v \in L^q(\cdot)$.

Proof. In this proof we will use the following inequality: let $0 \leq \alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta = 2$, then

$$(a^\alpha - b^\alpha)(a^\beta - b^\beta) \geq \alpha\beta(a - b)^2, \quad \forall a, b \in \mathbb{R} \quad (68)$$

Which can be proved just by Cauchy-Schwarz inequality as follows

$$(a - b)^2 = \left(\int_b^a dt \right)^2 = \left(\int_b^a t^{-\frac{1}{2}} t^{-\frac{1}{2}} dt \right)^2 \leq \left(\int_b^a t^{\alpha-1} dt \right) \left(\int_b^a t^{\beta-1} dt \right) \left(\frac{a^\alpha - b^\alpha}{\alpha} \frac{a^\beta - b^\beta}{\beta} \right).$$

Using the symmetry of the kernel, Fubini's theorem and inequality (68) we get

$$\begin{aligned}\int v^{q-1} \mathcal{L}v dx &= \int \left[\int v^{q-1}(x)(v(x) - v(y)) \mathbb{K}(x, y) dy + B(x)v^q(x) \right] dx \\ &= \frac{1}{2} \int \int (v^{q-1}(x) - v^{q-1}(y))(v(x) - v(y)) \mathbb{K}(x, y) dy dx + \int B(x)v^q(x) dx \\ &\geq \frac{1}{2} \frac{4(q-1)}{q^2} \int \int (v^{q/2}(x) - v^{q/2}(y))^2 \mathbb{K}(x, y) dy dx + \int B(x)v^q(x) dx \\ &= \frac{4(q-1)}{q^2} \int v^{q/2} \mathcal{L}v^{q/2}.\end{aligned}$$

Now, we can use the assumption (L1) to get

$$\int v^{q/2} \mathcal{L}v^{q/2} \asymp \int \int \frac{(v^{q/2}(x) - v^{q/2}(y))^2}{|x - y|^{N+2s}} dy dx + \int B(x)v^q(x) dx,$$

which tells us that $\int v^{q/2} \mathcal{L}v^{q/2}$ is a Dirichlet form (symmetric and closed). This and the fact that $-\mathcal{L}$ is a non-positive self-adjoint operator allows us to write

$$\int v^{q/2} \mathcal{L}v^{q/2} = \int \mathcal{L}^{1/2} v^{q/2}{}^2 dx.$$

□

We recall a variant of a brilliant Lemma due to De Giorgi, often used in Elliptic equations that we adapt here for our purposes.

Lemma 6.3. (De Giorgi). *Let $Z(t)$ be a bounded nonnegative function in the interval $[t_0, t_1]$. Assume that for $t_0 \leq \tau < t \leq t_1$ we have*

$$Z(t) \leq \theta Z(\tau) + \frac{A}{(t-\tau)^\alpha} + \frac{B}{(t-s)^\beta} + C \quad (69)$$

with $A, B, C \geq 0$, $\alpha > 0$ and $0 \leq \theta \leq 1$. Then

$$Z(t_1) \leq \frac{Ac(\alpha, \lambda, \theta)}{(t_1 - t_0)^\alpha} + \frac{Bc(\beta, \lambda, \theta)}{(t_1 - t_0)^\beta} + \frac{C}{(1 - \theta)} \quad (70)$$

where

$$c(\alpha, \lambda, \theta) = \frac{1}{(1 - \lambda)^\alpha \left(1 - \frac{\theta}{\lambda}\right)} \quad \text{for any } \lambda \in (\theta^{\frac{1}{\alpha}}, 1).$$

Proof. We define the sequence $\{\tau_i\}$ as follows

$$\tau_0 = t_1, \quad \tau_{i+1} = \tau_i - (1 - \lambda)\lambda^i(t_1 - t_0)$$

so $\tau_k = t_1 - \sum_{i=0}^{k-1} (1 - \lambda)\lambda^i(t_1 - t_0)$ and $\lim_{k \rightarrow \infty} \tau_k = t_0$. Now, we can iteratively write (69) as

$$\begin{aligned} Z(\tau_0) &\leq \theta Z(\tau_1) + \frac{A}{(\tau_0 - \tau_1)^\alpha} + \frac{B}{(\tau_0 - \tau_1)^\beta} + C \\ &\leq \theta^2 Z(\tau_2) + \frac{\theta A}{(\tau_1 - \tau_2)^\alpha} + \frac{\theta B}{(\tau_1 - \tau_2)^\beta} + \theta C + \frac{A}{(\tau_0 - \tau_1)^\alpha} + \frac{B}{(\tau_0 - \tau_1)^\beta} + C \\ &\quad \vdots \\ &\leq \theta^k Z(\tau_k) + \frac{A}{(1 - \lambda)^\alpha (t_1 - t_0)^\alpha} \sum_{i=0}^{k-1} \left(\frac{\theta}{\lambda^\alpha}\right)^i + \frac{B}{(1 - \lambda)^\beta (t_1 - t_0)^\beta} \sum_{i=0}^{k-1} \left(\frac{\theta}{\lambda^\beta}\right)^i + C \sum_{i=0}^{k-1} \theta^i. \end{aligned}$$

Finally, as the series on the right hand side converge we take the limit as $k \rightarrow \infty$ and we get (70). \square

6.2 Technical proofs

Lemma 6.4. *(Decay of $L^1_{\Phi_1}$ -norm). Let $u(t)$ be a nonnegative WDS, then*

$$\|u(t_1)\|_{L^1_{\Phi_1}} \leq \|u(t_0)\|_{L^1_{\Phi_1}} \quad \text{for every } 0 < t_0 < t_1 < T.$$

Proof. We want to use the inadmissible test function $\psi(t, x) = \chi_{[t_0, t_1]}(t)\Phi_1(x)$ in the equation, therefore we consider the C^1_c approximation in time $\psi_n(t, x) = \chi_n(t)\Phi_1(x)$ where

$$\chi_n(t) = \begin{cases} \frac{n^2}{2}(t - t_0 + \frac{1}{n})^2 & \text{if } t \in [t_0 - \frac{1}{n}, t_0) \\ 1 - \frac{n^2}{2}(t - t_0 - \frac{1}{n})^2 & \text{if } t \in [t_0, t_0 + \frac{1}{n}) \\ 1 & \text{if } t \in [t_0 + \frac{1}{n}, t_1 - \frac{1}{n}) \\ 1 - \frac{n^2}{2}(t - t_1 + \frac{1}{n})^2 & \text{if } t \in [t_1 - \frac{1}{n}, t_1) \\ \frac{n^2}{2}(t - t_1 - \frac{1}{n})^2 & \text{if } t \in [t_1, t_1 + \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

Using the LHS of (WDS), we obtain

$$\int_0^T \int \mathcal{L}^{-1} u \partial_t \psi \, dx \, dt = \int_0^T \int u \partial_t \chi_n \mathcal{L}^{-1} \Phi_1 \, dx \, dt = \lambda_1^{-1} \int_0^T \|u(t)\|_{L^1_{\Phi_1}} \partial_t \chi_n(t) \, dt.$$

Let us show that $\partial_t \chi_n$ is an approximation of $\delta_{t_0} - \delta_{t_1}$. We know that

$$\partial_t \chi_n = \begin{cases} n^2(t - t_0 + \frac{1}{n}) & \text{if } t \in [t_0 - \frac{1}{n}, t_0) \\ -n^2(t - t_0 - \frac{1}{n}) & \text{if } t \in [t_0, t_0 + \frac{1}{n}) \\ n^2(t - t_1 + \frac{1}{n}) & \text{if } t \in [t_1 - \frac{1}{n}, t_1) \\ -n^2(t - t_1 - \frac{1}{n}) & \text{if } t \in [t_1, t_1 + \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us consider the following approximation of the identity:

$$h_n(t) = \begin{cases} n^2(t + \frac{1}{n}) & \text{if } t \in [-\frac{1}{n}, 0) \\ -n^2(t - \frac{1}{n}) & \text{if } t \in [0, \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen that for any $f \in C(\mathbb{R})$ it holds that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(t) h_n(t) \, dt = f(0)$, since we can take limits in

$$\inf_{t \in [-\frac{1}{n}, \frac{1}{n}]} \{f(t)\} \leq \int_{\mathbb{R}} f(t) h_n(t) \, dt \leq \sup_{t \in [-\frac{1}{n}, \frac{1}{n}]} \{f(t)\} \quad \forall n > 0.$$

Notice that the $\inf\{f\}$ and the $\sup\{f\}$ above are attain due to Weierstrass Theorem.

Then, as we can rewrite $\partial_t \chi_n(t) = h_n(t - t_0) - h_n(t - t_1)$, we conclude that

$$\begin{aligned} \|u(t_0)\|_{L^1_{\Phi_1}} - \|u(t_1)\|_{L^1_{\Phi_1}} &= \lim_{n \rightarrow \infty} \int_0^T \|u(t)\|_{L^1_{\Phi_1}} (h_n(t - t_0) - h_n(t - t_1)) \, dt \\ &= \lim_{n \rightarrow \infty} \int_0^T \|u(t)\|_{L^1_{\Phi_1}} \partial_t \chi_n(t) \, dt = \lim_{n \rightarrow \infty} \lambda_1 \int_0^T \int u^m \Phi_1 \chi_n \, dx \, dt \geq 0, \end{aligned}$$

since using ψ_n as a test function in (WDS), it follows that

$$\int_0^T \|u(t)\|_{L^1_{\Phi_1}} \partial_t \chi_n(t) \, dt = \lambda_1 \int_0^T \int u^m \Phi_1 \chi_n \, dx \, dt \geq 0 \quad \forall n > 0.$$

□

Proof of Lemma 5.1. We define $j(r) = \frac{1}{m+1} |r|^{m+1}$ and we notice that it satisfies $\lim_{|r| \rightarrow +\infty} \frac{j(r)}{|r|} = +\infty$.

Convexity condition follows from the convexity of j , thus, let us check the l.s.c condition. We have two different convergence settings:

- Let $u_n \rightarrow u$ in $H^*(\cdot)$ and assume $\forall \lambda > 0 \exists n \in \mathbb{N}$ such that $\int j(u_n) > \lambda$. Then, it holds that

$$\liminf_{n \rightarrow \infty} \mathcal{E}(u_n) = +\infty \geq \mathcal{E}(u).$$

- Let $u_n \rightarrow u$ in $H^*(\Omega)$ and suppose $\exists \lambda > 0$ such that $\forall n \in \mathbb{N} \int_{\Omega} j(u_n) \leq \lambda$.

In this second case, we follow the same strategy as Brezis does for $H^{-1}(\Omega)$ in [20]. First, we show that $(u_n)_{n \in \mathbb{N}}$ is uniformly integrable, i.e.

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } |E| < \delta \text{ then } \int_E |u_n| < \epsilon \quad \forall n \in \mathbb{N}.$$

Let $A > \frac{2\lambda}{\epsilon}$ and R be such that $\frac{j(r)}{|r|} \geq A$ for $|r| > R$. If we choose $\delta < \frac{\epsilon}{2R}$, then

$$\int_E |u_n| dx = \int_{\{x \in E: u_n(x) \geq R\}} |u_n| dx + \int_{\{x \in E: u_n(x) < R\}} |u_n| dx \leq \int \frac{j(u_n)}{A} dx + \int_E R dx \leq \frac{\lambda}{A} + R\delta < \epsilon.$$

Then, we apply Dunford-Pettis theorem, which states that in bounded domain uniformly integrable implies weakly precompact in $L^1(\Omega)$. Hence, $(u_n)_{n \in \mathbb{N}}$ has a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ which converge weakly in $L^1(\Omega)$, that is,

$$\exists \tilde{u} \in L^1(\Omega) \text{ such that } u_{n_k} \rightharpoonup \tilde{u} \text{ in } L^1(\Omega).$$

Further, $u = \tilde{u}$ in $H^*(\Omega)$, as $\{u_n\}$ already converges weakly to u in $H^*(\Omega)$ and $C_c^\infty(\Omega) \subset L^\infty(\Omega) \cap H(\Omega)$, which is dense in $H(\Omega)$.

Now, to conclude the proof, we just have to show that $\mathcal{E}(u)$ is weakly lower semicontinuous in $L^1(\Omega)$. So, let $u_n \rightarrow u$ in $L^1(\Omega)$, then, there exists a subsequence $(u_{n_l})_{l \in \mathbb{N}}$ which converge almost everywhere to u . Since j is non negative, we can apply Fatou's Lemma,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} j(u_n) dx &\geq \int_{\Omega} \liminf_{n \rightarrow \infty} j(u_n) dx = \int_{\Omega} \lim_{n \rightarrow \infty} \inf_{m \geq n} \{j(u_m)\} dx \\ &\geq \int_{\Omega} \lim_{l \rightarrow \infty} \inf_{m \geq n_l} \{j(u_{n_l})\} dx = \int_{\Omega} \liminf_{l \rightarrow \infty} j(u_{n_l}) dx = \int_{\Omega} j(u) dx, \end{aligned}$$

where we have used in the last equality that j is continuous (notice that it suffices to be l.s.c.). Finally, since $\mathcal{E}(u)$ is convex and l.s.c. in $L^1(\Omega)$, it holds that $\mathcal{E}(u)$ is weakly l.s.c. in $L^1(\Omega)$ by Mazur's Lemma. □

References

- [1] G. Akagi, *Stability of non-isolated asymptotic profiles for fast diffusion*. Comm. Math. Phys. **345** (2016), no. 1, 77–100.
- [2] L. Ambrosio, E. Brué, D. Semola. “*Lectures on Optimal Transport*”, vol. 130, Unitext, Springer, 2021.
- [3] I. Athanasopoulos, L. A. Caffarelli. *Continuity of the temperature in boundary heat control problems* Adv. Math. **224** (2010), no. 1, 293-315.
- [4] P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry (suppl. to Amer. Jour. Math.), Johns Hopkins Univ. Press, Baltimore, Md., (1981), 23–39.
- [5] P. Bénilan, R. F. Gariepy, *Strong solutions in L^1 of degenerate parabolic equations*, Journal of differential equation, **119**, (1995), no. 2, 473-502
- [6] E. Berchio, M. Bonforte, D. Ganguly, G. Grillo, *The fractional porous medium equation on the hyperbolic space*. Calc. Var. **59**, (2020) n. 169.

- [7] R. M. Blumenthal, R. K. Gettoor. *Some theorems on stable processes*. Trans. Amer. Math. Soc. **95**, (1960), no. 2, 263–273
- [8] J. G. Berryman, C. J. Holland, *Stability of the separable solution for fast diffusion*. Arch. Rational Mech. Anal. **74** (1980), no. 4, 379–388.
- [9] K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields **127** (2003), 89–152.
- [10] M. Bologna, P. Grigolini, C. Tsallis, *Anomalous diffusion associated with nonlinear fractional derivative Fokker-Planck-like equation: Exact time-dependent solutions*, Physical Review E **62**, (2000).
- [11] M. Bonforte, A. Figalli. *Sharp extinction rates for fast diffusion equations on generic bounded domains*. Comm. Pure Appl. Math **74** (2021), 744–789.
- [12] M. Bonforte, A. Figalli, X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*. Comm. Pure Appl. Math. **70**, (2017), n. 8, 1472–1508
- [13] M. Bonforte, A. Figalli, J.L. Vázquez. *Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains*. Anal. PDE, **11**, (2018), 945–982.
- [14] M. Bonforte, A. Figalli, J.L. Vázquez. *Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations*, Calc. Var. Partial Differential Equations **57** (2018), no. 2, Art. 57, 34 pp.
- [15] M. Bonforte, Y. Sire, J. L. Vázquez, *Existence, Uniqueness and Asymptotic behaviour for fractional porous medium on bounded domains*. Discr. Cont. Dyn. Sys. **35** (2015), 5725–5767.
- [16] M. Bonforte, Y. Sire, J. L. Vázquez, *Optimal Existence and Uniqueness Theory for the Fractional Heat Equation*. Nonlin. Anal. TMA **153**, (2017) 142–168.
- [17] M. Bonforte, J. L. Vázquez. *Quantitative Local and Global A Priori Estimates for Fractional Nonlinear Diffusion Equations*, Advances in Math **250** (2014), 242–284.
- [18] M. Bonforte, J. L. Vázquez. *A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains*, Arch. Rat. Mech. Anal. **218** (2015), no. 1, 317–362.
- [19] M. Bonforte and J. L. Vázquez. *Fractional nonlinear degenerate diffusion equations on bounded domains, I: Existence, uniqueness and upper bounds*, Nonlinear Anal. **131** (2016), 363–398
- [20] H. Brezis. *Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations*, Proc. Symp. Nonlinear Funct. Anal., Madison, Acad. Press (1971), 101–156.
- [21] H. Brezis. “Opérateurs Maximaux Monotones et Semi-groupes de Contractions Dans Les Espaces de Hilbert”, North-Holland, (1973).
- [22] H. Brezis, A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl. (9) **62** (1983), no.1, 73–97.
- [23] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*. Comm. Partial Diff. Eq. **32** (2007), no. 7-9, 1245–1260.
- [24] Z.-Q. Chen, P. Kim, R. Song, *Dirichlet heat kernel estimates for $\Delta^{-1/2} + \Delta^{-1/2}$* . Illinois J. Math. **54** (2010), no. 4, 1357–1392.
- [25] Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes*. Probab. Theory Relat. Fields **146** (2010) 361–399
- [26] Z.-Q. Chen, P. Kim, R. Song, *Green function estimates for relativistic stable processes in half-space-like open sets*. Stochastic Process. Appl. **121** (2011), no. 5, 1148–1172.
- [27] Z.-Q. Chen, P. Kim, R. Song, *Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation*. Ann. Probab. **40** (2012), no. 6, 2483–2538.
- [28] M. G. Crandall, M. Pierre, *Regularizing Effect for $u_t = A\varphi(u)$ in L^1* , J. Funct. Anal. **45**, (1982), 194–212
- [29] N. Cusimano, F. del Teso, L. Gerardo-Giorda. *Numerical approximations for fractional elliptic equations VIA the method of semigroups*. ESAIM Math. Model. Numer. Anal. **54** (2020), no. 3, 751–774.
- [30] P. Daskalopoulos, C. E. Kenig, “Degenerate diffusions. Initial value problems and local regularity theory”. EMS Tracts in Mathematics, **1**. European Mathematical Society (EMS), Zürich, 2007. x+198 pp. ISBN: 978-3-03719-033-3
- [31] E. B. Davies. “Heat kernels and spectral theory”, Cambridge Tracts in Mathematics, 92. Cambridge

- University Press, Cambridge, 1990. x+197 pp. ISBN: 0-521-40997-7
- [32] E. DiBenedetto. “*Degenerate parabolic equations*”. Universitext. Springer-Verlag, New York, 1993. xvi+387 pp. ISBN: 0-387-94020-0
- [33] E. DiBenedetto, U. Gianazza, V. Vespri. “*Harnack’s inequality for degenerate and singular parabolic equations*”, Springer Monographs in Mathematics, Springer 2011.
- [34] E. DiBenedetto, Y. C. Kwong, V. Vespri, *Local Space-Analiticity of Solutions of Certain Singular Parabolic Equations*, Indiana University Math. J. **40**, n.2 (1991), 741–765.
- [35] E. Feireisl, F. Simondon, *Convergence for Semilinear Degenerate Parabolic Equations in several Space Dimension*, J. Din. and Diff. Eq. **12**, (2000), 647–673.
- [36] G. Grillo, M. Muratori, F. Punzo. *Fractional porous media equations: existence and uniqueness of weak solutions with measure data*. Calc. Var. Partial Differential Equations **54** (2015), no. 3, 3303–3335.
- [37] G. Grubb, *Fractional Laplacians on domains, a development of Hörmander’s theory of mu-transmission pseudodifferential operators*, Adv. in Math. **268** (2015), 478–528
- [38] M. Jara, *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math., **62** (2009), 198–214.
- [39] M. D. Jara, T. Komorowski, S. Olla, *Limit theorems for additive functionals of a Markov chain*. Ann. Appl. Probab. **19** (2009), no. 6, 2270—2300.
- [40] T. Jin, J. Xiong, *Optimal boundary regularity for fast diffusion equations in bounded domains*. Amer. J. Math. to appear. <https://arxiv.org/abs/1910.05160>
- [41] T. Jin, J. Xiong, *Bubbling and extinction for some fast diffusion equations in bounded domains*. Preprint (2022). <https://arxiv.org/abs/2008.01311>
- [42] K.-Y. Kim, P. Kim. *Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets*, Stochastic Processes and their Applications **124** (2014) 3055–3083.
- [43] P. Kim, R. Song. *Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts*. J. Math. Anal. Appl. **332** (2007), no. 1, 57–80.
- [44] Komura Y. *Nonlinear semi-groups in Hilbert space*. J. Math. Soc. Japan **19**, 493-507 (1967)
- [45] T. Kulczycki. *Properties of Green function of symmetric stable process*. Probab. Math. Statist. **17** (2) (1997) 339-364. Acta Univ. Wrastislaw, no. 2029.
- [46] M. Kwasnicki. *Ten equivalent definitions of the fractional Laplace operator*. Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51.
- [47] E. K. Lenzi, R. S. Mendes, C. Tsallis, *Crossover in diffusion equation: Anomalous and normal behaviors*, Physical Review E **67**, 031104 (2003).
- [48] E. H. Lieb. *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*. Ann. of Math. (2) **118** (1983), no. 2, 349–374.
- [49] E.H. Lieb, M. Loss. “*Analysis*”, American Mathematical Soc. (14) (2001)
- [50] Q.-H. Nguyen, J. L. Vázquez. *Porous medium equation with nonlocal pressure in a bounded domain*. Comm. Partial Differential Equations **43** (2018), no. 10, 1502–1539.
- [51] A. de Pablo, F. Quirós, A. Rodríguez. *Nonlocal filtration equations with rough kernels.*, Nonlinear Anal. **137** (2016), 402–425.
- [52] A. de Pablo, F. Quirós, A. Rodríguez. *Regularity theory for singular nonlocal diffusion equations.*, Calc. Var. Partial Differential Equations **57** (2018), no. 5, Paper No. 136, 14 pp.
- [53] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez. *A fractional porous medium equation*. Adv. Math. **226** (2011), no. 2, 1378-1409.
- [54] A. de Pablo, F. Quirós, A. Rodríguez, J.L. Vázquez, *A general fractional porous medium equation*, Comm. Pure Appl. Math. **65** (2012) 1242-1284.
- [55] A. de Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, *Classical solutions and higher regularity for nonlinear fractional diffusion equations*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 7, 1949–1975.
- [56] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*. Publ. Mat. **60** (2016), no. 1, 3–26.

- [57] X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275–302.
- [58] D. Stan, F. del Teso, J. L. Vazquez, *Transformations of self-similar solutions for porous medium equations of fractional type*. Nonlinear Anal. **119** (2015), 62–73.
- [59] D. Stan, J. L. Vázquez, *The Fisher-KPP equation with nonlinear fractional diffusion*, SIAM J. Math. Anal. **46** (2013) (5), 3241–3276
- [60] F. del Teso, J. Endal, E.R. Jakobsen. *Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type*. Adv. Math. **305** (2017), 78–143.
- [61] F. del Teso, J. Endal, E.R. Jakobsen. *Robust numerical methods for nonlocal (and local) equations of porous medium type. Part I: Theory*. SIAM J. Numer. Anal. **57** (2019), no. 5, 2266–2299.
- [62] J. L. Vázquez. “*Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*”. Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.
- [63] J. L. Vázquez. “*The Porous Medium Equation. Mathematical Theory*”, vol. Oxford Mathematical Monographs, Oxford University Press, Oxford, (2007).
- [64] J. L. Vázquez. *Nonlinear Diffusion with Fractional Laplacian Operators*. in “Nonlinear partial differential eq.: the Abel Symposium 2010”, Holden, Helge & Karlsen, Kenneth H. eds., Springer, (2012), 271–298.
- [65] J. L. Vázquez. *Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type*. J. Eur. Math. Soc. **16** (2014), 769–803.
- [66] J. L. Vázquez. *Recent progress in the theory of Nonlinear Diffusion with Fractional Laplacian Operators*. In “Nonlinear elliptic and parabolic differential eq.”, Disc. Cont. Dyn. Syst. - S **7**, no. 4 (2014), 857–885.
- [67] J. L. Vázquez. *The Mathematical Theories of Diffusion: Nonlinear and Fractional Diffusion*. In “Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions: Cetraro, Italy 2016”, Springer Lecture Notes in Mathematics, ISBN 978-3-319-61493-9. C.I.M.E. Foundation Subseries (2017), 205–278.
- [68] J. L. Vázquez, B. Volzone. *Optimal estimates for Fractional Fast diffusion equations*. J. Math. Pures Appl. **103** (2015), 535–556.