

Non-Kähler Calabi-Yau geometry and pluriclosed flow [☆]Mario Garcia-Fernandez ^{a,*}, Joshua Jordan ^b, Jeffrey Streets ^b^a *Universidad Autónoma de Madrid and Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Ciudad Universitaria de Cantoblanco, 28049 Madrid, Spain*^b *Rowland Hall, University of California, Irvine, Irvine, CA 92617, United States of America*

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ABSTRACT

Hermitian, pluriclosed metrics with vanishing Bismut-Ricci form give a natural extension of Calabi-Yau metrics to the setting of complex, non-Kähler manifolds, and arise independently in mathematical physics. We reinterpret this condition in terms of the Hermitian-Einstein equation on an associated holomorphic Courant algebroid, and thus refer to solutions as Bismut Hermitian-Einstein. This implies Mumford-Takemoto slope stability obstructions, and using these we exhibit infinitely many topologically distinct complex manifolds in every dimension with vanishing first Chern class which do not admit Bismut Hermitian-Einstein metrics. This reformulation also leads to a new description of pluriclosed flow in terms of Hermitian metrics on holomorphic Courant algebroids, implying new global existence results, in particular on all complex non-Kähler surfaces of Kodaira dimension $\kappa \geq 0$. On complex manifolds which admit Bismut-flat metrics we show global existence and convergence of pluriclosed flow to a Bismut-flat metric, which in turn gives a classification of generalized Kähler structures on these spaces.

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R É S U M É

Les métriques Hermitiennes pluriformées avec forme de Bismut-Ricci nulle donnent une extension naturelle des métriques de Calabi-Yau au cadre des variétés complexes non-Kähler, et apparaissent indépendamment en physique mathématique. Nous réinterprétons cette condition en termes de l'équation d'Hermite-Einstein sur un algébroid de Courant holomorphe associé, et nous appellerons donc ces solutions des métriques de Bismut Hermite-Einstein. Cela donne des obstructions de stabilité de pente de Mumford-Takemoto, et en utilisant celles-ci, nous mettons en évidence, pour toute dimension, une infinité de variétés complexes topologiquement distinctes avec première classe de Chern nulle qui n'admettent pas de métrique Bismut Hermite-Einstein. Cette reformulation conduit également à une nouvelle description du flot pluriformé en termes de métriques hermitienne sur les algébroides de Courant holomorphes, impliquant de nouveaux résultats d'existence globale, en particulier sur toutes les surfaces complexes non-Kähler de dimension de Kodaira $\kappa \geq 0$.

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Sur les variétés complexes qui admettent une métrique Bismut-plate, nous montrons l'existence globale et la convergence d'un flux plurifermé vers une métrique Bismut-plate, qui à son tour donne une classification des structures de Kähler généralisées sur ces espaces.

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1. Introduction

The Calabi-Yau Theorem [73] gives a definitive answer to the questions of existence and uniqueness of Kähler, Ricci-flat metrics on a given complex manifold, showing that when $c_1(M) = 0$, there exists a unique such metric in every Kähler class. Later Cao [19] proved this theorem using Kähler-Ricci flow, showing global existence and convergence to a Ricci-flat metric with arbitrary initial data. These profound results have applications throughout mathematics and physics, and in recent years there has been interest in extending this theory beyond Kähler manifolds to more general settings in complex geometry. Given a complex manifold (M^{2n}, J) let g denote a Hermitian metric with Kähler form $\omega = gJ$. We say that the metric is pluriclosed if

$$H := -d^c\omega, \quad dH = 0.$$

This is a natural generalization of the Kähler condition $d\omega = 0$, and for instance pluriclosed metrics exist on all compact complex surfaces [32]. These metrics appear naturally in the index theory of non-Kähler manifolds [11], and in the physics of supersymmetry. There are many natural Hermitian connections in this setting, and here we focus on the Bismut connection ([11], [65])

$$\nabla^B = \nabla + \frac{1}{2}g^{-1}H,$$

where ∇ denotes the Levi-Civita connection of g . This is a Hermitian connection, and as such determines a representative of $c_1(M)$ via

$$\rho_B = \frac{1}{2} \operatorname{tr} R^B J \in \Lambda^2(M).$$

This is a natural Ricci-type curvature which coincides with the usual Ricci form in case g is Kähler. We say that a pluriclosed metric is *Bismut Hermitian-Einstein* (the reason for this terminology will be explained below) if

$$\rho_B \equiv 0.$$

Expressed in terms of the usual Ricci curvature, this equation is equivalent to

$$\begin{aligned} \operatorname{Rc} - \frac{1}{4}H^2 + \frac{1}{2}L_{\theta^\sharp}g &= 0, \\ d^*H - d\theta + i_{\theta^\sharp}H &= 0, \end{aligned}$$

where $H^2(X, Y) = \langle i_X H, i_Y H \rangle$, and $\theta = Jd^*\omega$ is the Lee form. This equation is thus a generalization of the Calabi-Yau condition to complex, non-Kähler geometry. Furthermore, in the compact case it follows that Bismut Hermitian-Einstein metrics automatically satisfy a *generalized Ricci soliton* equation (cf. Proposition 2.6)

$$\begin{aligned}\mathrm{Rc} - \frac{1}{4}H^2 + \nabla^2 f &= 0, \\ d^*H + i_{\nabla f}H &= 0.\end{aligned}$$

These match the equations of motion (for the metric and the three-form flux) in compactifications of type II supergravity [18].

Taking inspiration from the Calabi-Yau Theorem, a natural initial question is whether the condition $c_1(M) = 0$ guarantees the existence of a Bismut Hermitian-Einstein metric.¹ Already in the first nontrivial dimension $n = 2$ we see a number of new issues arise, and the answer is emphatically no. First, it follows from a theorem of Gauduchon-Ivanov [30] (cf. [29, Theorem 8.26]) that the only non-Kähler solutions occur on *standard Hopf surfaces*, where (up to finite quotients) the metric is isometric to the product metric $g_{S^3} \times g_{S^1}$. Since all minimal non-Kähler complex surfaces of Kodaira dimension $\kappa \geq 0$ have vanishing first Chern class, they are thus examples showing that one cannot guarantee existence in general. Furthermore, there exist pluriclosed metrics which are generalized Ricci solitons on non-standard Hopf surfaces ([58,63]), which fit into a smooth family with the example just described. A complete existence theory should naturally incorporate such solitons.

Our first step in tackling these subtleties is to reformulate the Bismut Hermitian-Einstein condition using the tools of *generalized geometry*. Note that every pluriclosed metric determines a class $[\partial\omega]$ in the first Čech cohomology group $H^1(\Lambda_{cl}^{2,0})$ of the sheaf $\Lambda_{cl}^{2,0}$ of closed $(2,0)$ -forms on M . Such a class corresponds to an isomorphism class of exact holomorphic Courant algebroids [38], which are holomorphic vector bundles fitting into an exact sequence

$$0 \longrightarrow T_{1,0}^* \xrightarrow{\pi^*} \mathcal{Q} \xrightarrow{\pi} T^{1,0} \longrightarrow 0$$

and satisfying further axioms (cf. Definition 3.1). Furthermore, the structure of the holomorphic vector bundle is given by a twisted $\bar{\partial}$ -operator determined by $\partial\omega$ [11]. Pluriclosed metrics with a fixed torsion class $[\sqrt{-1}\tau] \in H^1(\Lambda_{cl}^{2,0})$ can be identified with pairs (ω, β) , where $\tau - \partial\omega = d\beta$, and using this description we determine a generalized Hermitian metric on \mathcal{Q} via (see Lemma 3.16)

$$G = \begin{pmatrix} g_{i\bar{j}} + \beta_{ik}\bar{\beta}_{j\bar{l}}g^{\bar{l}k} & \sqrt{-1}\beta_{ip}g^{\bar{l}p} \\ -\sqrt{-1}\bar{\beta}_{j\bar{p}}g^{\bar{p}k} & g^{\bar{l}k} \end{pmatrix}.$$

Building on this, we give an adapted presentation of ([11, Theorem 2.9]), which shows that the *Chern curvature* of the Hermitian metric G is equivalent in a canonical way to the *Bismut curvature* of g , yielding explicit relationships between their Ricci tensors. Specifically, letting Ω^G denote the Chern curvature of G , we set

$$S_g^G = \sqrt{-1} \mathrm{tr}_\omega \Omega^G \in \mathrm{End}(\mathcal{Q}).$$

As it turns out, the tensor S_g^G is in a natural sense equivalent to the Bismut-Ricci form, and in particular

$$S_g^G \equiv 0 \quad \Longleftrightarrow \quad \rho_B \equiv 0.$$

It follows that the existence of Bismut Hermitian-Einstein metrics requires Mumford-Takemoto slope stability of the associated holomorphic Courant algebroid. The precise statement below is Corollary 4.5. Going further we use this abstract slope stability obstruction to give a concrete obstruction in terms of holomorphic maps in Theorem 4.7 and Corollary 4.9. As a consequence we give the first examples in complex

¹ We note that we mean here and throughout vanishing as a de Rham class, as opposed to a Bott-Chern class. See [67] for a review of non-Kähler Calabi-Yau geometries assuming $c_1 = 0$ in Bott-Chern sense.

dimension $n \geq 3$ of compact complex manifolds with vanishing first Chern class which do not admit Bismut Hermitian-Einstein metrics, in fact we give infinite families in every dimension (cf. Examples 4.10, 4.11). We state this for emphasis as our first main theorem:

Theorem 1.1. *In every dimension there exist infinitely many complex manifolds with vanishing first Chern class which do not admit a Bismut Hermitian-Einstein metric.*

Returning to the question of constructibility of solutions, and recalling the construction of Calabi-Yau metrics using Kähler-Ricci flow, it is natural to construct Bismut Hermitian-Einstein metrics by means of a geometric flow. With the goal of constructing canonical geometries on complex manifolds in mind, the third author and Tian introduced a natural geometric flow of pluriclosed metrics, called pluriclosed flow [60]. This equation is expressed in terms of a pair (ω, β) as above via

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}, \quad \frac{\partial}{\partial t}\beta = -\rho_B^{2,0}.$$

This equation is strictly parabolic, and admits short-time solutions with arbitrary initial data on compact manifolds. Global existence and convergence results in various geometric settings have appeared in [6–8, 13, 25, 55–57]. Using the discussion above, we see that it is possible to equivalently formulate pluriclosed flow in terms of a family of generalized Hermitian metrics G on \mathcal{Q} , which satisfy

$$G^{-1} \frac{\partial}{\partial t} G = -S_g^G.$$

Viewed this way, pluriclosed flow bears a formal similarity to the Hermitian-Yang-Mills flow [23], although here there is further nonlinearity due to the fact that the time-dependent metric g is used to take the trace defining S_g^G , whereas a background metric is used in HYM flow. Precursors of this formulation of pluriclosed flow appeared in [54, 43], where the equation was shown to hold locally. The key point in producing a general, global formula is the use of the twisted $\bar{\partial}$ -operator on \mathcal{Q} . Restricting to the setting of vanishing first Chern class, and in view of the nonexistence phenomena discussed above, two natural questions emerge. First, *if a Bismut Hermitian-Einstein metric exists, does the flow always converge to it?* Second, *if no Bismut Hermitian-Einstein metric exists, what geometric singularities does the flow encounter?*

To address the first question, let us restrict to what so far gives the only known examples of non-Kähler Bismut Hermitian-Einstein structures, namely when the associated Bismut connection is *flat*. First note that by the classical results of Cartan-Schouten ([20, 21] cf. also [1]), it follows that the universal cover of such a manifold is isometric as a Riemannian manifold to a simply connected Lie group equipped with bi-invariant metric and flat Cartan connection. The low-dimensional compact examples give some of the first known examples of complex, non-Kähler manifolds. The classic Hopf/Boothby metric on standard Hopf surfaces is Bismut-flat (cf. Example 2.7), as are the standard metrics on Calabi-Eckmann threefolds (see Example 2.8). These are special cases of the construction of Samelson [52], who showed generally that any compact Lie group of even dimension admits left-invariant complex structures compatible with a bi-invariant metric, and that these structures are Bismut-flat (cf. also [2]). It was shown recently in [71] that all compact Bismut-flat Hermitian manifolds are covered by these ‘Samelson spaces’.

Our second main theorem gives a definitive picture of the global existence and convergence of pluriclosed flow on such backgrounds:

Theorem 1.2. *Let (M^{2n}, ω_F, J) be a compact Bismut-flat manifold. Given ω_0 a pluriclosed metric so that $[\partial\omega_0] = [\partial\omega_F] \in H_{\bar{\partial}}^{2,1}$, the solution to pluriclosed flow with initial data ω_0 exists on $[0, \infty)$ and converges to a Bismut-flat metric ω_∞ .*

In view of our description of pluriclosed flow in terms of metrics on holomorphic Courant algebroids, the hypothesis $[\partial\omega_0] = [\partial\omega_F] \in H_{\bar{\partial}}^{2,1}$ is natural, as then the solution to pluriclosed flow exists on a fixed holomorphic Courant algebroid with the same underlying holomorphic vector bundle as our given background flat metric. In many concrete cases, however, we obtain a complete description of the global existence and convergence by exploiting the structure of the underlying cohomologies. This is the case for instance in Examples 2.7 and 2.8 below, where we obtain a complete description of the global existence and convergence of the flow with arbitrary initial data on standard Hopf surfaces and Calabi-Eckmann threefolds, in particular confirming part (1) of Conjecture 7.6 of [59]. Furthermore, we illustrate several higher dimensional examples of compact Lie groups where Theorem 1.2 can be applied.

From a PDE perspective Theorem 1.2 illustrates interesting behavior. Pluriclosed metrics are described locally by a $(1,0)$ -form, and pluriclosed flow admits a reduction to a quasilinear degenerate parabolic system of equations for a $(1,0)$ -form (cf. [54]). Such systems tend to exhibit local singularities, such as the classic neckpinch singularities of Ricci flow and mean curvature flow, and yet Theorem 1.2, describing the global behavior with arbitrary initial data on certain backgrounds, shows that such behaviors cannot occur for pluriclosed flow. From a more geometric point of view, we note that while recently there have been some results on geometric flows of non-Kähler metrics converging to rigid, Kähler metrics (cf. [44,54,68]), or converging to interesting non-Kähler metrics assuming a certain symmetric ansatz ([48,53]), Theorem 1.2 seems to be the first result showing that a natural class of *non-Kähler* metrics is globally attractive for a geometric flow with arbitrary initial data.

The proof of Theorem 1.2 relies principally on the formulation of pluriclosed flow in terms of the associated generalized Hermitian metric G described above. Using this we derive extensions of the parabolic Schwarz Lemma for Kähler-Ricci flow to the setting of pluriclosed flow, which now have reaction terms expressed in terms of the Chern curvature of a chosen background metric on \mathcal{Q} , which in turn is equivalent to the Bismut curvature of a background pluriclosed metric g . Assuming the background metric has vanishing Bismut curvature, this leads to uniform equivalence of the time-dependent metrics G_t , and furthermore to a Calabi/Yau type estimate for the Chern connection of G . After this uniform C^∞ estimates follow from Schauder theory, leading to global existence of the flow. The a priori estimates imply subconvergence of the flow to a Bismut-flat metric along some sequence of times, and further maximum principle arguments set up against the background of this subsequential limit show convergence of the flow line.

Turning to the second question above, we can ask what geometric singularities pluriclosed flow encounters on manifolds with vanishing first Chern class but which do not admit a Bismut Hermitian-Einstein metric. As discussed above, the first such examples occur for minimal non-Kähler complex surfaces of Kodaira dimension $\kappa \geq 0$. We use our methods above to prove the definitive long-time existence result in this setting:

Theorem 1.3. *Let (M^4, J) be a minimal non-Kähler surface of Kodaira dimension $\kappa \geq 0$. Given ω_0 a pluriclosed metric on M , the solution to pluriclosed flow with initial condition ω_0 exists on $[0, \infty)$.*

The proof follows similar lines to Theorem 1.2 as described above, relying on a priori estimates for the metric derived from the Schwarz lemma. It is known [15,70] that such manifolds are finitely covered by holomorphic principal T^2 -fibrations over Riemann surfaces, and after passing to the relevant cover we use the fibration structure to obtain an estimate for the horizontal component of the metric. Using then special properties of the Bismut curvature tensor of a background T^2 -invariant metric, we are able to derive maximum principles leading to uniform control over the whole metric. This confirms the ‘cone conjecture’ [62, Conjecture 5.2] for these manifolds. It is expected that blowdown limits at infinity for this flow converge in Gromov-Hausdorff sense to the base of the fibration equipped with a constant curvature metric (resp. to a point) when $\kappa = 1$ (resp. $\kappa = 0$). This has been confirmed for T^2 -invariant initial data in [53] (cf. Remark 5.12).

Theorem 1.2 also has applications to generalized Kähler (GK) geometry [38]. A GK structure is a triple (g, I, J) consisting of a Riemannian metric g compatible with two integrable complex structures I and J , with Kähler forms ω_I and ω_J satisfying

$$d_I^c \omega_I = H = -d_J^c \omega_J, \quad dH = 0.$$

A key example of generalized Kähler structure occurs for Bismut-flat structures. In particular, if we fix a compact Lie group with bi-invariant metric g , and let I and J denote left- and right-invariant complex structures compatible with g , then the triple (g, I, J) is generalized Kähler [38]. Furthermore, there always exist infinite dimensional families of local deformations of a given generalized Kähler structure using natural classes of Hamiltonian flows [10,35]. These families are analogous to Kähler classes, and it is natural to seek classification of generalized Kähler structures up to equivalence via these deformations. Given this, it is natural to ask if a left-invariant complex structure I as above can be part of an ‘exotic’ generalized Kähler structure, where J does not arise via the construction above, up equivalence by Hamiltonian flow. We note that already on the torus this question is nontrivial due to the construction of nonstandard orthogonal complex structures on flat tori in complex dimension $n \geq 3$ [14].

From this perspective, the question of the topology of this space is loosely analogous to questions on the uniqueness of symplectic structures, in particular conjectures on whether the space of symplectic structures on a hyperKähler four-manifold is connected, for which geometric flow methods have been proposed (cf. [24]). More generally, the flow method has been employed in various settings to understand the global structure of natural classes of geometric structures (e.g. [9,17,47]). It turns out that the pluriclosed flow provides a very natural approach to understanding the nonlinear space of generalized Kähler structures. As shown in [61], the pluriclosed flow furthermore preserves the generalized Kähler condition, when coupled to an evolution equation for one of the complex structures. This is called *generalized Kähler-Ricci flow*, and is expressed in terms of the Kähler form ω_J , a one-parameter family $\beta_J \in \Lambda_J^{2,0}$, and a one-parameter family of complex structures I as

$$\frac{\partial}{\partial t} \omega_J = -(\rho_B^J)^{1,1}, \quad \frac{\partial}{\partial t} \beta_J = -(\rho_B^J)^{2,0}, \quad \frac{\partial}{\partial t} I = L_{\theta_I^\sharp - \theta_J^\sharp} I,$$

where $\theta_J = Jd^* \omega_J$ is the Lee form of the Hermitian structure (g, J) , with θ_I defined similarly. In particular, the metrics ω_J are evolving by pluriclosed flow, and so Theorem 1.2 applies to give a global description of the space of generalized Kähler structures on Bismut-flat backgrounds.

Corollary 1.4. *Let (M^{2n}, ω_F, J) be a compact Bismut-flat manifold. If (g, I, J) is a generalized Kähler structure on M such that $[\partial \omega_J] = [\partial \omega_F] \in H_{\bar{\partial}}^{2,1}$, then the solution to generalized Kähler-Ricci flow with this initial condition exists on $[0, \infty)$ and converges to a Bismut-flat structure (g_∞, I_∞, J) . In particular, I is biholomorphic to a complex structure compatible with a Bismut-flat metric.*

As a concrete example, we use Corollary 1.4 to classify all generalized Kähler structures on standard Hopf surfaces (cf. Example 2.7).

Here is an outline of the rest of the paper. We begin in §2 by recalling background results from Hermitian geometry, and discuss general results on Bismut Hermitian-Einstein metrics. In §3 we expound upon ideas of Bismut (cf. [11] §IIb), describing pluriclosed metrics in a fixed Aeppli class on M in terms of generalized Hermitian metrics G on the canonically associated holomorphic Courant algebroid, with twisted $\bar{\partial}$ -operator. Furthermore, we derive explicit formulas relating the Chern curvature of G to the Bismut curvature tensor of g . This provides an explicit connection to Hermitian-Yang-Mills theory, and we use this in §4 to provide slope-stability obstructions to the existence of Bismut Hermitian-Einstein metrics, and the existence of manifolds with $c_1 = 0$ which do not admit Bismut Hermitian-Einstein metrics. We turn in §5 to the

implications for pluriclosed flow. We first observe an extension of the Schwarz Lemma to the setting of holomorphic Courant algebroids. Using this, we obtain natural parabolic differential inequalities for the generalized Hermitian metric along solutions to pluriclosed flow, leading then to the proofs of the global existence and convergence results.

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2. Background

2.1. Pluriclosed metrics

Let (M^{2n}, J) be a complex manifold. A Riemannian metric on (M, J) is Hermitian if $g(JX, JY) = g(X, Y)$ for $X, Y \in TM$. Given a Hermitian metric g we define the associated Kähler form $\omega \in \Lambda_{\mathbb{R}}^{1,1}$ by

$$\omega(X, Y) = g(JX, Y).$$

A Hermitian metric g on (M, J) is called *pluriclosed* if

$$2\sqrt{-1}\partial\bar{\partial}\omega = dd^c\omega = 0,$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$ is the conjugate differential, and in particular $d^c\omega = -d\omega(J, J, J)$. Notice that a pluriclosed metric canonically defines a pair of cohomological classes

$$[\omega] \in H_A^{1,1}, \quad [\partial\omega] \in H_{\bar{\partial}}^{2,1}$$

which play a significant role in the present work. Here $H_{\bar{\partial}}^{p,q}$ and $H_A^{p,q}$ denote the Dolbeault and Aeppli cohomology groups of (M, J) , respectively, the latter being defined by

$$H_A^{p,q} := \frac{\{\text{Ker } \partial\bar{\partial} : \Lambda^{p,q} \rightarrow \Lambda^{p+1,q+1}\}}{\{\partial\alpha + \bar{\partial}\gamma \mid \alpha + \gamma \in \Lambda^{p-1,q} \oplus \Lambda^{p,q-1}\}}.$$

A Hermitian manifold (M, g, J) has a canonical one-parameter family of Hermitian connections [34], i.e. connections for which g and J are parallel. This line of canonical connections is uniquely determined by the following two classical connections.

Definition 2.1. Let (M^{2n}, g, J) be a Hermitian manifold. Let ∇ denote the Levi-Civita connection associated to g . The *Chern connection* is defined by

$$\langle \nabla_X^C Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \frac{1}{2}d\omega(JX, Y, Z).$$

The *Bismut connection* is defined by

$$\langle \nabla_X^B Y, Z \rangle = \langle \nabla_X Y, Z \rangle - \frac{1}{2}d^c\omega(X, Y, Z).$$

Given a Hermitian connection ∇' on (M^{2n}, g, J) , we let $R^{\nabla'}$ denote the associated $(3, 1)$ -curvature tensor. That is,

$$R^{\nabla'}(X, Y)Z = \nabla'_X(\nabla'_Y Z) - \nabla'_Y(\nabla'_X Z) - \nabla'_{[X, Y]}Z.$$

We furthermore adopt the standard notation for the $(4, 0)$ -curvature tensor,

$$R^{\nabla'}(X, Y, Z, W) = \left\langle R^{\nabla'}(X, Y)Z, W \right\rangle.$$

Definition 2.2. Let (M^{2n}, g, J) be a Hermitian manifold endowed with the Bismut connection ∇^B , and set $R^B = R^{\nabla^B}$. Define the *first Ricci curvature* ρ_B via

$$\rho_B(X, Y) := \frac{1}{2} \left\langle R^B(X, Y)J e_i, e_i \right\rangle,$$

where $\{e_i\}$ is any orthonormal basis for the tangent space at a given point.

To finish this section, we recall a classical curvature identity which is crucial to our discussion. In particular, there is a relationship between the curvature tensor of the Bismut connection ∇^B , and that of the metric compatible connection ∇^- , often referred to as *Hull connection* after [41], defined by

$$\langle \nabla_X^- Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \frac{1}{2} d^c \omega(X, Y, Z). \quad (2.1)$$

Proposition 2.3. (cf. [11, Theorem 1.6]) Let (M^{2n}, g, J) be a pluriclosed manifold. Then, setting $R^- = R^{\nabla^-}$, one has

$$R^B(X, Y, Z, W) = R^-(Z, W, X, Y).$$

2.2. Bismut Hermitian-Einstein metrics

In this subsection we record some fundamental points regarding Bismut Hermitian-Einstein metrics. We begin by formally stating the definition.

Definition 2.4. Given a complex manifold (M^{2n}, J) , a pluriclosed metric g on M is *Bismut Hermitian-Einstein* if

$$\rho_B \equiv 0.$$

To better understand this condition, we first give a reformulation in terms of the Riemannian Ricci tensor, using an identity for the Bismut curvature.

Proposition 2.5. [29, Proposition 8.10] Let (M^{2n}, g, J) be a pluriclosed manifold, with $H = -d^c \omega$, and $\theta = -d^* \omega \circ J$ is the Lee form. Then

$$\begin{aligned} \rho_B^{1,1}(\cdot, J\cdot) &= \text{Rc} - \frac{1}{4} H^2 + \frac{1}{2} L_{\theta^\sharp} g, \\ \rho_B^{2,0+0,2}(\cdot, J\cdot) &= -\frac{1}{2} d^* H + \frac{1}{2} d\theta - \frac{1}{2} i_{\theta^\sharp} H. \end{aligned}$$

From this proposition it follows that Bismut Hermitian-Einstein structures satisfy a natural coupling of the classic Einstein equation and the equation for a harmonic three-form. Moreover, it follows that they are automatically *steady generalized Ricci solitons* (cf. [29, Ch. 4, Proposition 8.14]). We next record a further consequence arising from the variational theory of the generalized scalar curvature functional (Perelman \mathcal{F} -functional for generalized Ricci flow).

Proposition 2.6. Suppose (M^{2n}, J) is a compact complex manifold, and g is a pluriclosed Bismut Hermitian-Einstein metric. Let $u = e^{-f/2} \in C^\infty(M)$ be the first eigenfunction of the Schrödinger operator $-4\Delta + R - \frac{1}{12} |H|^2$. Then

$$\begin{aligned} \operatorname{Rc} - \frac{1}{4}H^2 + \nabla^2 f &= 0 \\ d^*H + i_{\nabla f}H &= 0. \end{aligned} \quad (2.2)$$

Consequently, the vector fields $V = \frac{1}{2}(\theta^\sharp - \nabla f)$ and IV are both Killing and holomorphic and V satisfies

$$\frac{1}{2}d\theta - i_V H = 0. \quad (2.3)$$

Furthermore, if V vanishes identically then g is Kähler Ricci-flat.

Proof. Given g as in the statement, define $H = -d^c\omega$ and $\theta = -d^*\omega \circ J$ as above. Let (g_t, H_t) denote the solution to generalized Ricci flow with initial data (g, H) . By the identities of Proposition 2.5, one has $(g_t, H_t) = (\phi_t^*g, \phi_t^*H)$, where ϕ_t is the one-parameter family of diffeomorphisms generated by θ^\sharp . Due to the monotonicity formula for the \mathcal{F} functional along generalized Ricci flow (cf. [29, Proposition 6.8, Corollary 6.11]), the existence of f satisfying (2.2) as claimed follows. Combining now equation (2.2) with Proposition 2.5, V must be Killing and furthermore it satisfies (2.3). Arguing now as in the proof of [64, Proposition 4.1], the second equation in (2.2) implies

$$(L_V\omega)^{2,0} = 0 = (L_{IV}\omega)^{1,1}.$$

Using that V is Killing, it follows that $L_V I = 0$, and hence $L_{IV}\omega = (L_{IV}\omega)^{1,1} = 0$. Thus, since IV is also holomorphic, we conclude that $L_{IV}g = 0$.

Finally, if V vanishes identically then one has $\theta = -df$. Since g is pluriclosed, if we assume $n > 2$ it follows from [42, Theorem 1.3] that g is Kähler. In the case $n = 2$, by the conformal transformation law for the Lee form the conformally related Hermitian metric $e^f g$ is Kähler. But by the uniqueness of the Gauduchon metric in a fixed conformal class, it follows that, after possibly modifying f by a constant, $g = e^f g$ and so $f \equiv 0$. Thus for any choice of n the metric g is Kähler, and Ricci flat. \square

As noted in the introduction, the system of equations (2.2) is the same as the equations of motion (for the metric and the three-form flux) in compactifications of type II supergravity [18]. Classification results for solutions to this more general soliton system on complex surfaces have appeared recently in [58,63], with a new class of complete solutions arising from an extension of the Gibbons-Hawking ansatz recently appearing [64]. The starting point is the observation that the vector field $V = \frac{1}{2}(\theta^\sharp - \nabla f)$ is real holomorphic, and JV is Killing (cf. [64, Proposition 4.1]). It is interesting to notice that the Killing condition $L_V g = 0$ combined with (2.3) yields a *generalized isometry* on an exact Courant algebroid determined by the torsion H (cf. [29, Proposition 2.53]).

2.3. Bismut-flat structures

An important class of Bismut Hermitian-Einstein structures occurs when the associated Bismut connection is actually flat. As a matter of fact, Bismut-flat manifolds are the only known examples of non-Kähler Bismut Hermitian-Einstein metrics to the present day. In this section we recall fundamental results on the structure of compact pluriclosed Bismut-flat manifolds. We will also provide concrete examples where Theorem 1.2 and Corollary 1.4 apply. To begin we recall two basic examples, namely the standard Hopf surfaces, and Calabi-Eckmann threefolds.

Example 2.7. Given complex numbers α, β satisfying $|\alpha| \leq |\beta| < 1$, we obtain a *Hopf surface* via

$$M_{\alpha\beta} = \mathbb{C}^2 \setminus \{\mathbf{0}\} / \langle (z_1, z_2) \rightarrow (\alpha z_1, \beta z_2) \rangle.$$

For any α, β this is a complex manifold diffeomorphic to $S^3 \times S^1$. We furthermore say that it is *standard* if $|\alpha| = |\beta|$. In this case there is a natural pluriclosed metric known as the Hopf/Boothby metric defined by the invariant Kähler form on $\mathbb{C}^2 \setminus \{0\}$ by

$$\omega_{\text{Hopf}} = \frac{\sqrt{-1}}{|z|^2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

This metric is the unique (up to scaling) bi-invariant Hermitian metric for $S^3 \times S^1 \cong SU(2) \times U(1)$, and the associated Bismut connection is flat. Recall also that $h_A^{1,1}(M_{\alpha\beta}) = 1$ (see the proof of [3, Theorem 3.3]) and that the Aeppli class of a pluriclosed metric on a compact complex surface is non-zero (see e.g. [49]). Therefore, given any pluriclosed metric ω on $M_{\alpha\beta}$ one has $[\omega] = \lambda[\omega_{\text{Hopf}}]$ for $0 < \lambda \in \mathbb{R}$, where the positivity of λ follows e.g. by integration on a holomorphic curve (cf. below). Using now that $[\partial\omega] \in H_{\bar{\partial}}^{2,1}$ factorizes through the natural map

$$H_A^{1,1} \rightarrow H_{\bar{\partial}}^{2,1}: [\omega] \mapsto [\partial\omega],$$

we have that Theorem 1.2 applies giving convergence of the pluriclosed flow to a Bismut-flat structure for any initial data. Due to the classification by Gauduchon-Ivanov [30] (cf. also [29, Theorem 8.26]), it follows that ω_{Hopf} is the unique Bismut-flat metric in its Aeppli class, thus the limiting metric is a scalar multiple of the Hopf metric.

The Hopf metric above is part of two distinct types generalized Kähler structures. In the first case, the second compatible complex structure I is obtained by modifying the complex structure on \mathbb{C}^2 such that (z_1, \bar{z}_2) are holomorphic coordinates. Note that, so defined, I is also Hermitian with respect to $g_{\text{Hopf}} = \omega_{\text{Hopf}}(, J)$. Furthermore, I and J have opposite orientations and they satisfy $[I, J] = 0$. The possible generalized Kähler structures of the form (g_{Hopf}, I, J) where I and J induce opposite orientations are classified in [5, Theorem 3]. In the second case [38], consider the compatible complex structure I' defined by the holomorphic $(2, 0)$ -form

$$\Omega_{I'} = (\bar{z}_1 dz_1 + z_2 d\bar{z}_2) \wedge (\bar{z}_1 dz_2 - z_2 d\bar{z}_1).$$

Then (g_{Hopf}, I', J) is a generalized Kähler structure whose associated Poisson tensor $\sigma = \frac{1}{2}[I, J]$ is generically nondegenerate but degenerates along the pair of elliptic curves $\{z_1 = 0\}, \{z_2 = 0\}$. The possible generalized Kähler structures of the form (g_{Hopf}, I, J) where I and J induce the same orientation form a finite dimensional family and are classified by twistor constructions in [33] (cf. also [51]).

Now suppose we have an arbitrary generalized Kähler structure (M, g, I, J) where $(M, J) \cong M_{\alpha\beta}$. Applying now Corollary 1.4 and the discussion above, the generalized Kähler Ricci flow with initial data (g, I, J) exists for all times and converges to a generalized Kähler structure $(g_{\text{Hopf}}, I_{\infty}, J)$ with I_{∞} biholomorphic to I . Thus the generalized Kähler-Ricci flow deforms all generalized Kähler structures on $M_{\alpha\beta}$ to the space of GK structures of the form (g_{Hopf}, I, J) . Therefore, a classification of generalized Kähler structures of the form (g, I, J) where $(M, J) \cong M_{\alpha\beta}$ follows from [5, 33].

Example 2.8. Consider $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}$ endowed with the \mathbb{C} -action given by

$$\gamma(z, w) = (e^{\gamma}z, e^{\sqrt{-1}\gamma}w).$$

This action is free and proper, the quotient space (M, J) is diffeomorphic to $S^3 \times S^3$, and this is an example of a Calabi-Eckmann space. Furthermore M is the total space of a holomorphic T^2 fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1$. This is the product of the standard Hopf fibrations $\pi_i: S^3 \rightarrow \mathbb{CP}^1$. Let ξ_i denote the canonical vector fields associated to this fibration on the two factors, with μ_i the associated canonical connections satisfying

$d\mu_i = \pi_i^* \omega_{FS}$. Note that the complex structure further satisfies $J\xi_1 = \xi_2$. We consider the Kähler form associated to the product of round metrics, namely

$$\omega = \pi_1^* \omega_{FS} + \pi_2^* \omega_{FS} + \mu_1 \wedge \mu_2.$$

It follows that

$$d\omega = d\mu_1 \wedge \mu_2 - \mu_1 \wedge d\mu_2 = \pi_1^* \omega_{FS} \wedge \mu_2 - \mu_1 \wedge \pi_2^* \omega_{FS},$$

hence

$$H = -d^c \omega = d\omega(J, J, J) = \pi_1^* \omega_{FS} \wedge \mu_1 - \pi_2^* \omega_{FS} \wedge \mu_2.$$

It is clear then that the geometric structure is that of a product of two copies of the S^3 factor of the Hopf metric, and furthermore that $dH = 0$. It can be checked that (M, J, ω) is indeed Bismut-flat, and that θ^\sharp , where $\theta = -\frac{1}{2}(\mu_1 + \mu_2)$, is a Killing field which further preserves H . We refer to [29, Example 8.35] for further details on the geometry.

For the Calabi-Eckmann threefold one has $h_A^{1,1} = 1$ (see the proof of [3, Proposition 3.4]). Furthermore, it follows by integrating over the T^2 fibers that for an arbitrary pluriclosed metric on M one has

$$0 \neq [\omega'] \in H_A^{1,1},$$

and hence there exists $\lambda > 0$ so that $[\omega'] = \lambda[\omega]$. Arguing now as in Example 2.7, we obtain that

$$[\partial\omega'] = \lambda[\partial\omega] \in H_{\bar{\partial}}^{2,1},$$

and hence Theorem 1.2 applies giving global existence and convergence of the pluriclosed flow to a Bismut-flat structure for any initial data.

These examples are both covered by the general construction of Samelson [52], which we recall next. Let G denote a compact connected real Lie group of dimension $2n$. We fix a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g} := \text{Lie } G$ inducing a bi-invariant metric g on G . Any choice of integrable left-invariant complex structure J_L on G compatible with g determines a Bismut-flat structure on G , with Bismut torsion given by the Cartan three-form

$$H(X, Y, Z) = -d_{J_L}^c \omega(X, Y, Z) = g([X, Y], Z),$$

for any triple of left-invariant vector fields X, Y, Z on G . Furthermore, denote by J_R the right-invariant extension of the almost complex structure induced by J_L on \mathfrak{g} . Then, the triple (g, J_L, J_R) is a natural generalized Kähler structure on G [38]. In particular, Example 2.7 can be recovered from this construction for $G = SU(2) \times U(1)$, while Example 2.8 can be recovered taking $G = SU(2) \times SU(2)$ (for suitable choices of J_L).

To the knowledge of the authors, in general there is no available calculation of the cohomology groups $H_{\bar{\partial}}^{2,1}$ and $H_A^{1,1}$ in the construction of Samelson. Hence our argument in the previous two examples, which grants the global existence of the pluriclosed flow for any initial data, does not apply directly. Note that, for G compact and simply connected, a calculation of the Dolbeault cohomology for an arbitrary left-invariant complex structure was undertaken in [50, Proposition 5.9]. In order to discuss a pair of concrete examples, we need a more explicit description of Samelson's construction. An integrable left-invariant complex structure on G compatible with g is equivalent to a choice of complex Lie subalgebra

$$\mathfrak{s} \subset \mathfrak{g}^c := \mathfrak{g} \otimes \mathbb{C}$$

satisfying

$$\mathfrak{s} \cap \mathfrak{g} = \{0\}, \quad \dim_{\mathbb{C}} \mathfrak{s} = n, \quad \langle \mathfrak{s}, \mathfrak{s} \rangle = 0,$$

where the last condition means that \mathfrak{s} is isotropic with respect to the \mathbb{C} -linear extension of \langle, \rangle . This data is actually equivalent to a choice of maximal torus $T \subset G$ with Lie algebra \mathfrak{t} , and a choice of isotropic subspace

$$\mathfrak{a} \subset \mathfrak{t}^c := \mathfrak{t} \otimes \mathbb{C}$$

such that $\mathfrak{a} \cap \mathfrak{t} = \{0\}$ (see e.g. [50]).

Example 2.9. Consider $G = SU(3)$ endowed with the bi-invariant metric g given by minus the Killing form. Let J_L be a g -compatible integrable left-invariant complex structure on G with associated maximal torus $T^2 \subset G$ and (isotropic) one-dimensional abelian subalgebra

$$\mathfrak{a} \subset \mathfrak{t}^c.$$

Then, there is an isomorphism of bigraded algebras [50, Section 4]

$$H_{\bar{\partial}}^{*,*}(G, J_L) \cong \Lambda^* \langle u_{0,1}, u_{2,1} \rangle \otimes \mathbb{C}[u_{1,1}] / \langle u_{1,1}^3 \rangle$$

where $u_{p,q}$ has bi-degree (p, q) . In particular

$$H_{\bar{\partial}}^{2,1}(G, J_L) \cong \mathbb{C} \langle u_{2,1} \rangle \cong \mathbb{C}.$$

Therefore, Theorem 1.2 applies giving convergence of the pluriclosed flow to a Bismut-flat structure for any initial data ω_0 such that

$$[\partial\omega_0] \in \mathbb{R}_{>0} \cdot [\partial\omega] \subset \mathbb{C}. \quad (2.4)$$

Similarly, applying Corollary 1.4, the generalized Kähler Ricci flow with such initial data (g_0, I_0, J_L) exists for all times and converges to a generalized Kähler structure $(g_\infty, I_\infty, J_L)$ with Bismut flat metric g_∞ and I_∞ biholomorphic to I_0 .

Example 2.10. Consider $G = SO(9)$ endowed with the bi-invariant metric g given by minus the Killing form. Choose a maximal torus $T^4 \subset G$ and an orthonormal basis for its Lie algebra

$$\mathfrak{t} = \langle e_1, e_2, e_3, e_4 \rangle.$$

Consider the union of twelve lines in $\mathbb{P}(\mathfrak{t}^c)$ defined by the equations

$$\sum_j x_j^2 = 0, \quad \sum_j x_j^6 = 0,$$

where $x = \sum_j x_j e_j \in \mathfrak{t}^c$. A choice of such line determines an isotropic subspace $\mathfrak{a} \subset \mathfrak{t}^c$ with no real points, and hence it defines a g -compatible integrable left-invariant complex structure J_L on G . Then, there is an isomorphism of bigraded algebras [50, p. 149]

$$H_{\bar{\partial}}^{*,*}(G, J_L) \cong \Lambda^* \mathfrak{a}^* \otimes \Lambda^*(u_{2,1}, u_{5,6}) \otimes \mathbb{C}[u_{1,1}, v_{1,1}] / \langle u_{1,1}^4 + v_{1,1}^4, u_{1,1}^4 v_{1,1}^4 \rangle$$

where $u_{p,q}$ has bi-degree (p, q) and the elements in \mathfrak{a}^* have bi-degree $(0, 1)$. In particular

$$H_{\bar{\partial}}^{2,1}(G, J_L) \cong \mathbb{C} \langle u_{2,1} \rangle \cong \mathbb{C}.$$

Therefore, Theorem 1.2 and Corollary 1.4 apply also in this example, for any initial data ω_0 such that (2.4) is satisfied.

3. Pluriclosed metrics and holomorphic Courant algebroids

In this section we show that a pluriclosed structure yields a natural notion of Hermitian metric G on a holomorphic Courant algebroid. We then use different methods to rederive a striking relationship—originally observed in [11]—between the curvature of the Chern connection associated to G and the curvature of the Bismut connection associated to the underlying pluriclosed structure. Our results extend the study of metrics on holomorphic Courant algebroids initiated in [27,28,54].

3.1. Holomorphic Courant algebroids

Let (M, J) be a complex manifold. We denote by \mathcal{O}_M and $\underline{\mathbb{C}}$ the sheaves of holomorphic functions and \mathbb{C} -valued constant functions on M , respectively.

Definition 3.1. A *holomorphic Courant algebroid* is a holomorphic vector bundle $\mathcal{Q} \rightarrow (M, J)$ together with a nondegenerate holomorphic symmetric bilinear form $\langle \cdot, \cdot \rangle$, a holomorphic vector bundle morphism $\pi : \mathcal{Q} \rightarrow T^{1,0}$ called anchor map, and a Dorfman bracket on holomorphic sections of \mathcal{Q} , that is, a homomorphism of sheaves of $\underline{\mathbb{C}}$ -modules

$$[\cdot, \cdot] : \mathcal{Q} \otimes_{\underline{\mathbb{C}}} \mathcal{Q} \rightarrow \mathcal{Q},$$

satisfying, for $u, v, w \in \mathcal{Q}$ and $\phi \in \mathcal{O}_M$,

- (D1): $[u, [v, w]] = [[u, v], w] + [v, [u, w]],$
- (D2): $\pi([u, v]) = [\pi(u), \pi(v)],$
- (D3): $[u, \phi v] = \pi(u)(\phi)v + \phi[u, v],$
- (D4): $\pi(u) \langle v, w \rangle = \langle [u, v], w \rangle + \langle v, [u, w] \rangle,$
- (D5): $[u, v] + [v, u] = 2\pi^* d \langle u, v \rangle.$

We will denote a holomorphic Courant algebroid $(\mathcal{Q}, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ simply by \mathcal{Q} . Using the isomorphism $\langle \cdot, \cdot \rangle : \mathcal{Q} \rightarrow \mathcal{Q}^*$ we obtain a holomorphic sequence

$$T_{1,0}^* \xrightarrow{\pi^*} \mathcal{Q} \xrightarrow{\pi} T^{1,0}. \quad (3.1)$$

Definition 3.2. We will say that \mathcal{Q} is *exact* if (3.1) induces an exact sequence of holomorphic vector bundles

$$0 \longrightarrow T_{1,0}^* \xrightarrow{\pi^*} \mathcal{Q} \xrightarrow{\pi} T^{1,0} \longrightarrow 0. \quad (3.2)$$

A complete classification of exact holomorphic Courant algebroids was obtained by Gualtieri in [38]. Let us summarize the result which we will use.

Theorem 3.3. *Let (M, J) be a complex manifold. Denote by $\Lambda_{cl}^{2,0}$ the sheaf of closed $(2, 0)$ -forms on M . Then, the set of isomorphism classes of exact holomorphic Courant algebroids on M is bijective to the first Čech cohomology $H^1(\Lambda_{cl}^{2,0})$. Furthermore, there is a vector space isomorphism*

$$H^1(\Lambda_{cl}^{2,0}) \cong \frac{\text{Ker } d: \Lambda^{3,0+2,1} \rightarrow \Lambda^{4,0+3,1+2,2}}{\text{Im } d: \Lambda^{2,0} \rightarrow \Lambda^{3,0+2,1}}. \quad (3.3)$$

A one-cocycle for the sheaf $\Lambda_{cl}^{2,0}$ determines local data for gluing the model $T^{1,0} \oplus T_{1,0}^*$ by means of holomorphic B -field transformations. On the other hand, a representative of a cohomology class in the right hand side of (3.3) yields a convenient global description of a holomorphic Courant algebroid.

Definition 3.4. Let (M, J) be a complex manifold. Given $\tau \in \Lambda^{3,0+2,1}$, $d\tau = 0$, we denote by

$$\mathcal{Q}_\tau = T^{1,0} \oplus T_{1,0}^*$$

the exact holomorphic Courant algebroid with Dolbeault operator

$$\bar{\partial}^\tau(X + \xi) = \bar{\partial}X + \bar{\partial}\xi - i_X\tau^{2,1},$$

symmetric bilinear form

$$\langle X + \xi, X + \xi \rangle = \xi(X), \quad (3.4)$$

bracket on holomorphic sections given by

$$[X + \xi, Y + \eta]_\tau = [X, Y] + \partial(\eta(X)) + i_X\partial\eta - i_Y\bar{\partial}\xi + i_Yi_X\tau^{3,0},$$

and anchor map $\pi(X + \xi) = X$. It is not difficult to check that \mathcal{Q}_τ so defined satisfies the axioms in Definition 3.1.

The holomorphic Courant algebroids of our main interest arise via a reduction mechanism from an exact Courant algebroid in the smooth category. Recall that a smooth exact Courant algebroid E over a smooth manifold M is given by an exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0,$$

satisfying a set of axioms analogous to those in Definition 3.2. A choice of isotropic splitting of the previous sequence determines uniquely a closed real three-form $H \in \Lambda^3$ and an isomorphism

$$E \cong TM \oplus T^*M,$$

such that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is as in (3.4), the anchor map is $\pi(X + \xi) = X$, and the bracket is given by

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - i_Yd\xi + i_Yi_XH.$$

Consider the smooth complex Courant algebroid $E \otimes \mathbb{C}$, with Courant structure given by the \mathbb{C} -linear extensions of the symmetric bilinear form $\langle \cdot, \cdot \rangle$, the bracket $[\cdot, \cdot]_H$, and the anchor map π .

Definition 3.5. Let E be a smooth exact Courant algebroid over a complex manifold (M, J) . A *lifting* of $T^{0,1}$ to $E \otimes \mathbb{C}$ is an isotropic, involutive subbundle $\ell \subset E \otimes \mathbb{C}$ mapping isomorphically to $T^{0,1}$ under the \mathbb{C} -linear extension of the anchor map $\pi: E \otimes \mathbb{C} \rightarrow T \otimes \mathbb{C}$.

A lifting relates to the complex Courant algebroid $E \otimes \mathbb{C}$ as a Dolbeault operator relates to a smooth complex vector bundle, in the sense that it enable us to construct a Courant algebroid in the holomorphic category out of $E \otimes \mathbb{C}$. More precisely, following [38] we consider the reduction of $E \otimes \mathbb{C}$ by ℓ given by the orthogonal bundle

$$\mathcal{Q}_\ell := \ell^\perp / \ell,$$

where ℓ^\perp is the orthogonal complement of ℓ with respect to the symmetric pairing on $E \otimes \mathbb{C}$. Since ℓ is a lifting of $T^{0,1}$ the kernel of $\pi|_{\ell^\perp}$ is $T_{1,0}^*$, and therefore \mathcal{Q}_ℓ is an extension of the form (3.2). The Dolbeault operator on \mathcal{Q}_ℓ is defined as follows: given s a smooth section of \mathcal{Q}_ℓ , we define

$$\bar{\partial}_X^\ell s = [\tilde{X}, \tilde{s}] \mod \ell$$

where $X \in T^{0,1}$, \tilde{X} is the unique lift of X to ℓ , and \tilde{s} is any lift of s to a section of ℓ^\perp . The Jacobi identity for the Dorfman bracket on $E \otimes \mathbb{C}$ implies that $\bar{\partial}^\ell \circ \bar{\partial}^\ell = 0$ and that it induces a Dorfman bracket on the holomorphic sections of \mathcal{Q}_ℓ .

Our next goal is to make the previous construction more explicit by choosing an isotropic splitting of E . For a proof of the next result we refer to [29].

Lemma 3.6. Let (M, J) be a complex manifold. Given $H_0 \in \Lambda^3$ a closed real three-form, $dH_0 = 0$, consider the exact Courant algebroid $(T \oplus T^*, \langle, \rangle, [,]_{H_0}, \pi)$ as above. Then, a lifting

$$\ell \subset (T \oplus T^*) \otimes \mathbb{C}$$

of $T^{0,1}$ is equivalent to a pair (ω, b) , where $\omega \in \Lambda_{\mathbb{R}}^{1,1}$ and $b \in \Lambda^2$, satisfying

$$H_0 = -d^c \omega - db. \quad (3.5)$$

More explicitly, given (ω, b) satisfying (3.5) the lifting is

$$\ell = \ell(\omega, b) := \{e^{b+\sqrt{-1}\omega}(X^{0,1}), X^{0,1} \in T^{0,1}\}, \quad (3.6)$$

and, conversely, any lifting is uniquely expressed in this way.

Remark 3.7. Notice that the inclusion $E \subset E \otimes \mathbb{C}$ defines a *compact form* in the sense of [27, Definition 5.4]. Hence, the analogy between liftings of $T^{0,1}$ and Dolbeault operators suggests that the previous result can be regarded as a *Chern correspondence* in our setting. We refer to [27] for further insights on this relation in a more general setup.

Our next result gives an explicit formula for the exact holomorphic Courant algebroid associated to a lifting $\ell(\omega, b)$.

Lemma 3.8. Let (M, J) be a complex manifold endowed with an exact Courant algebroid $(T \oplus T^*, \langle, \rangle, [,]_{H_0}, \pi)$ as above. Let $\ell(\omega, b)$ be a lifting of $T^{0,1}$ as in Lemma 3.6. Then, using the notation in Definition 3.4, there is a canonical isomorphism

$$\mathcal{Q}_{\ell(\omega, b)} \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}.$$

Proof. We have

$$\ell(\omega, b)^\perp = e^{b+\sqrt{-1}\omega}(T^{1,0}) \oplus T_{1,0}^* \oplus \ell$$

and therefore there is a smooth bundle isomorphism

$$\begin{aligned} \mathcal{Q}_{\ell(\omega, b)} &\rightarrow T^{1,0} \oplus T_{1,0}^* \\ [e^{b+\sqrt{-1}\omega}Y + \eta] &\mapsto Y + \eta. \end{aligned} \tag{3.7}$$

The agreement of the pairing and the anchor map with the ones on $\mathcal{Q}_{2\sqrt{-1}\partial\omega}$ is straightforward. Let us now express the Dolbeault operator in terms of (3.7). Given $X \in T^{0,1}$ and $Y + \eta \in T^{1,0} \oplus T_{1,0}^*$, we have

$$\begin{aligned} [e^{b+\sqrt{-1}\omega}X, e^{b+\sqrt{-1}\omega}Y + \eta] &= e^{b+\sqrt{-1}\omega}[X, Y]^{1,0} + L_X\eta + i_Y i_X(H_0 + db + \sqrt{-1}d\omega) \quad \text{mod } \ell \\ &= e^{b+\sqrt{-1}\omega}\bar{\partial}_X Y + i_X \bar{\partial}\eta - i_X i_Y(2\sqrt{-1}\partial\omega), \quad \text{mod } \ell \end{aligned}$$

which recovers the Dolbeault operator in Definition 3.4 when $\tau = 2\sqrt{-1}\partial\omega$. Similarly, for $X + \xi, Y + \eta$ holomorphic sections of $T^{1,0} \oplus T_{1,0}^*$, we find

$$\begin{aligned} [e^{b+\sqrt{-1}\omega}X + \xi, e^{b+\sqrt{-1}\omega}Y + \eta] &= e^{b+\sqrt{-1}\omega}[X, Y] + L_X\eta - i_Y d\xi + i_Y i_X(2\sqrt{-1}\partial\omega) \\ &= e^{b+\sqrt{-1}\omega}[X, Y] + L_X\eta - i_Y d\xi + i_Y \bar{\partial}\xi \\ &= e^{b+\sqrt{-1}\omega}[X, Y] + \partial(\eta(X)) + i_X \partial\eta - i_X \partial\xi \end{aligned}$$

as claimed. \square

Remark 3.9. With the notation of Lemma 3.6 one has that $dd^c\omega = 0$ and hence a choice of lifting $\ell \subset (T \oplus T^*) \otimes \mathbb{C}$ yields a well-defined Aeppli class

$$[\omega] \in H_A^{1,1}.$$

In fact, if ℓ and ℓ' are liftings such that $[\omega] = [\omega'] \in H_A^{1,1}$, then it follows that $\mathcal{Q}_\ell \cong \mathcal{Q}_{\ell'}$. To see this, notice that

$$\omega' = \omega + \bar{\partial}\xi^{1,0} + \partial\eta^{0,1}$$

implies $\partial\omega' = \partial\omega - d\bar{\partial}\xi^{1,0}$. Our claim follows combining Lemma 3.8 with Theorem 3.3.

3.2. Generalized Hermitian metrics

We introduce next a new ingredient, namely, generalized metrics, which will lead us naturally to the study of Hermitian metrics on exact holomorphic Courant algebroids. Recall that a generalized metric on a smooth exact Courant algebroid E is given by an orthogonal decomposition

$$E = V_+ \oplus V_-$$

such that the restriction of the ambient metric to V_+ (resp. V_-) is positive definite (resp. negative definite). Recall also that a generalized metric determines uniquely a Riemann metric g on M and an isotropic splitting

of E . In particular, it has an associated isomorphism $E \cong (T \oplus T^*, \langle, \rangle, [,]_H, \pi)$ for a uniquely determined closed three-form H , such that

$$V_{\pm} = \{X \pm g(X), X \in T\}. \quad (3.8)$$

The basic interaction between generalized metrics and complex geometry is provided by the following definition.

Definition 3.10. Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E . We say that a generalized metric $E = V_+ \oplus V_-$ is *compatible with J* if

$$\ell = \{e \in V_+ \otimes \mathbb{C}, \pi(e) \in T^{0,1}\} \subset E \otimes \mathbb{C}$$

is a lifting of $T^{0,1}$.

Using the splitting of E determined by the generalized metric, it is not difficult to see that Definition 3.10 implies that g is Hermitian and furthermore

$$\ell = e^{\sqrt{-1}\omega} T^{0,1}$$

where $\omega = gJ$ is the associated Hermitian form. Applying now Lemma 3.6 we obtain the following.

Lemma 3.11. Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E . A generalized metric $E = V_+ \oplus V_-$ is compatible with J if and only if the associated Riemannian metric g is Hermitian and furthermore

$$H = -d^c \omega. \quad (3.9)$$

In particular g is pluriclosed.

Given a compatible generalized metric, we can find an alternative presentation of the associated holomorphic Courant algebroid $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ (see Lemma 3.8) which will naturally endow this bundle with a Hermitian metric. To see this, note that $V_+^\perp = V_-$ implies that

$$\ell^\perp = (V_- \otimes \mathbb{C}) \oplus \ell.$$

Therefore, as a smooth orthogonal bundle \mathcal{Q}_ℓ is canonically isomorphic to

$$\mathcal{Q}_\ell := \ell^\perp / \ell \cong V_- \otimes \mathbb{C}.$$

Definition 3.12. Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the induced *generalized Hermitian metric G* on \mathcal{Q}_ℓ is defined by

$$G([s_1], [s_2]) = -2 \langle \pi_- s_1, \overline{\pi_- s_2} \rangle$$

for $[s_j] \in \ell^\perp / \ell$ and $\pi_- : \ell^\perp \rightarrow V_- \otimes \mathbb{C}$ the orthogonal projection.

We are ready to prove the main result of this section, where we calculate the Chern connection of the induced generalized Hermitian metric G in terms of the connection ∇^- associated to the underlying pluriclosed structure (see (2.1)). This result provides an interpretation of [11, Theorem 2.9] in the language of holomorphic Courant algebroids.

Proposition 3.13. *Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the Chern connection of the associated generalized Hermitian metric G on \mathcal{Q}_ℓ is given by*

$$i_X \nabla_G^C s = \pi_- [\sigma_+ X, \pi_- s] \quad (3.10)$$

via the isomorphism $\mathcal{Q}_\ell \cong V_- \otimes \mathbb{C}$. Here, $\sigma_+ X = X + g(X)$ is the inverse of the isomorphism $\pi|_{V_+}: V_+ \rightarrow T$. More explicitly, via the identification $V_- \cong T$, the Chern connection is given by

$$\nabla_X^- Y = \nabla_X Y + \frac{1}{2} g^{-1} d^c \omega(X, Y, \cdot). \quad (3.11)$$

Proof. Observe that the right hand side of (3.10) defines an orthogonal connection on V_- , which can be identified with ∇^- via the isomorphism $\pi|_{V_-}: V_- \rightarrow T$ (see e.g. [29, Proposition 3.14]). Therefore, ∇^- extends \mathbb{C} -linearly to a G -unitary connection on $V_- \otimes \mathbb{C}$. By the abstract definition of the Dolbeault operator on \mathcal{Q}_ℓ , we immediately see that $(\nabla^-)^{0,1}$ coincides with $\bar{\partial}^\ell$. \square

In our next result we calculate an explicit formula for the generalized Hermitian metric G in terms of the isomorphism $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ in Lemma 3.8.

Lemma 3.14. *Let (M, J) be a complex manifold endowed with a smooth exact Courant algebroid E and a compatible generalized metric $E = V_+ \oplus V_-$. Then, the orthogonal isomorphism $\psi: \mathcal{Q}_{2\sqrt{-1}\partial\omega} \rightarrow V_- \otimes \mathbb{C}$ induced by Lemma 3.8 is given by*

$$\psi(X + \xi) = e^{\sqrt{-1}\omega} X - \frac{1}{2} e^{-\sqrt{-1}\omega} g^{-1} \xi.$$

Consequently,

$$\psi^* G(X + \xi, X + \xi) = 2g(X, \bar{X}) + (2g)^{-1}(\xi, \bar{\xi}).$$

Proof. The first part follows from

$$\psi(X + \xi) = e^{\sqrt{-1}\omega} X + \pi_- \xi = e^{\sqrt{-1}\omega} X + \frac{1}{2}(\xi - g^{-1}\xi) = e^{\sqrt{-1}\omega} X - \frac{1}{2} e^{-\sqrt{-1}\omega} g^{-1} \xi.$$

The second part is a straightforward calculation and is left to the reader. \square

For our applications it will be convenient to fix a background exact holomorphic Courant algebroid and generalized Hermitian metric. This motivates the following definition, which is inspired by [26, 27].

Definition 3.15. Let (M, J) be a complex manifold endowed with an exact holomorphic Courant algebroid \mathcal{Q} . A generalized Hermitian metric on \mathcal{Q} is given by a triple (E, V_+, φ) , where

- (1) E is an exact Courant algebroid over M ,
- (2) $V_+ \subset E$ is a generalized metric compatible with J ,
- (3) $\varphi: \mathcal{Q}_\ell \rightarrow \mathcal{Q}$ is an isomorphism of holomorphic Courant algebroids inducing the identity on M .

Observe that a generalized Hermitian metric (E, V_+, φ) on \mathcal{Q} induces a generalized Hermitian metric G' on \mathcal{Q}_ℓ as in Definition 3.12. Therefore, via the isomorphism φ we obtain a Hermitian metric

$$G = \varphi_* G'$$

on \mathcal{Q} compatible with the orthogonal structure. By abuse of notation, we will also call G a *generalized Hermitian metric*. We next unravel the previous definition in terms of the model in Definition 3.4.

Lemma 3.16. *Let (M, J) be a complex manifold, and $\tau_0 \in \Lambda^{3,0+2,1}$, $d\tau_0 = 0$. Then, there is a one to one correspondence between the set of generalized Hermitian metrics on $\mathcal{Q}_{2\sqrt{-1}\tau_0}$ and*

$$\{\omega + \beta \mid \omega > 0, d\beta = \tau_0 - \partial\omega\} \subset \Lambda_{\mathbb{R}}^{1,1} \oplus \Lambda^{2,0}.$$

Furthermore, the generalized Hermitian metric $G = \varphi_* G'$ is given by

$$G(X + \xi, X + \xi) = 2g(X, \bar{X}) + (2g)^{-1}(\xi + 2\sqrt{-1}\beta(X), \bar{\xi} - 2\sqrt{-1}\beta(X)).$$

Proof. The pair (E, V_+) determines $\omega > 0$ and an isomorphism $E \cong (T \oplus T^*, \langle, \rangle, [,]_H)$ for $H = -d^c\omega$. By Lemma 3.8, $\mathcal{Q}_\ell \cong \mathcal{Q}_{2\sqrt{-1}\partial\omega}$ and, by Theorem 3.3, the isomorphism $\varphi: \mathcal{Q}_\ell \rightarrow \mathcal{Q}_{\tau_0}$ corresponds to

$$\varphi = e^{-2\sqrt{-1}\beta}: T^{1,0} \oplus T_{1,0}^* \rightarrow T^{1,0} \oplus T_{1,0}^*$$

for $\beta \in \Lambda^{2,0}$ satisfying $2\sqrt{-1}d\beta = 2\sqrt{-1}\tau_0 - 2\sqrt{-1}\partial\omega$. The last part of the statement is now straightforward from Lemma 3.14. We refer to [27] for further details. \square

Remark 3.17. For our applications to the pluriclosed flow, we will need to fix an initial pluriclosed metric g and a background pluriclosed metric g' . We will not require that the associated holomorphic Courant algebroids $\mathcal{Q}_{2\sqrt{-1}\partial\omega}$ and $\mathcal{Q}_{2\sqrt{-1}\partial\omega'}$ are isomorphic, but the weaker condition of being isomorphic as holomorphic orthogonal bundles. In practice, this boils down to the explicit condition

$$\bar{\partial}\beta = \partial\omega' - \partial\omega \tag{3.12}$$

for some $\beta \in \Lambda^{2,0}$ or, equivalently,

$$[\partial\omega] = [\partial\omega'] \in H_{\bar{\partial}}^{2,1}(X).$$

One can easily see that a pair (ω, β) as in (3.12) defines a generalized Hermitian metric in the exact holomorphic Courant algebroid $\mathcal{Q}_{2\sqrt{-1}\partial\omega' + 2\sqrt{-1}\partial\beta}$, that is, the twisting of $\mathcal{Q}_{2\sqrt{-1}\partial\omega'}$ by the d -closed $(3,0)$ -form $2\sqrt{-1}\partial\beta$ (see Definition 3.4).

3.3. Bismut's identity from the classical viewpoint

As originally observed in [11, Theorem 2.9] in a different context, there is a striking relationship between the curvature of the Chern connection associated to the generalized Hermitian metric G in Definition 3.12 and the curvature of the Bismut connection associated to the underlying pluriclosed structure. In the setup of generalized geometry this is a straightforward consequence of Proposition 3.13 combined with Proposition 2.3. Since this relation plays an important role in our main results, in the present and the next section we discuss at length this interesting link from a classical point of view. In the sequel, the Dorfman bracket in the holomorphic Courant algebroids of our interest will play no role, and hence we will focus

on the underlying holomorphic orthogonal bundle given by the sequence (3.2). We will also use a different normalization for $\partial\omega$ in the Dolbeault operator which simplifies the notation.

Let (M, g, J) be a pluriclosed manifold. The complexification $TM \otimes \mathbb{C}$ of the tangent bundle TM is endowed with two natural geometric structures: a complex-valued metric induced by the \mathbb{C} -linear extension of g

$$\langle X, Y \rangle_0 := -\frac{1}{2}g(X, Y)$$

and a real structure, that is, a \mathbb{C} -antilinear involution $\sigma_0: TM \otimes \mathbb{C} \rightarrow TM \otimes \mathbb{C}$, induced by complex conjugation

$$\sigma_0(X) = \overline{X}.$$

The involution σ_0 is orthogonal with respect to $\langle \cdot, \cdot \rangle_0$, and the combination of both yields the Hermitian metric

$$G_0(X, Y) := -2 \langle X, \sigma_0(Y) \rangle_0 = g(X, \overline{Y}).$$

Consider the isomorphism of smooth complex vector bundles $\psi_g: TM \otimes \mathbb{C} \rightarrow T^{1,0} \oplus T_{1,0}^*$ induced by the pluriclosed structure (g, J) , defined by

$$\psi_g(X) = X^{1,0} - \sqrt{-1}\omega(X^{0,1}, \cdot) = X^{1,0} - g(X^{0,1}). \quad (3.13)$$

In the next result we give a formula for the structure $(\langle \cdot, \cdot \rangle_0, \sigma_0, G_0)$ transported to $T^{1,0} \oplus T_{1,0}^*$ via ψ_g .

Lemma 3.18. *The isomorphism (3.13) induces a complex-valued metric $\langle \cdot, \cdot \rangle = (\psi_g)_* \langle \cdot, \cdot \rangle_0$, an orthogonal real structure $\sigma = (\psi_g)_* \sigma_0$, and a Hermitian metric $G' = (\psi_g)_* G_0$ on $T^{1,0} \oplus T_{1,0}^*$, given by*

$$\langle X + \xi, X + \xi \rangle = \xi(X), \quad \sigma(X + \xi) = -g^{-1}\overline{\xi} - g\overline{X},$$

and

$$G'(X + \xi, X + \xi) := -2 \langle X + \xi, \sigma(X + \xi) \rangle = g(X, \overline{X}) + g^{-1}(\xi, \overline{\xi}).$$

In matrix notation, we have

$$G' = \begin{pmatrix} g_{i\overline{j}} & 0 \\ 0 & g^{\overline{l}k} \end{pmatrix}. \quad (3.14)$$

Proof. We have $\psi_g^{-1}(X + \xi) = X - g^{-1}\xi$ where $g: T_{1,0}^* \rightarrow T^{0,1}$ is the isomorphism induced by the \mathbb{C} -linear extension of the metric g . The proof is now straightforward. \square

Notice that the formula for the transported complex-valued metric $\langle \cdot, \cdot \rangle$ is independent of g and coincides with the bilinear form of the holomorphic Courant algebroid $\mathcal{Q}_{\sqrt{-1}\partial\omega}$ in Definition 3.4. Furthermore, (3.14) coincides up to normalization with the formula for the generalized Hermitian metric in Lemma 3.14.

In the next result we show that the holomorphic structure on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$ has also a natural interpretation in terms of the complexified tangent bundle. For this, note that the \mathbb{C} -linear extension of the metric connection ∇^- to $TM \otimes \mathbb{C}$ is Hermitian (see (2.1)). Furthermore, using the fact that $R^B \in \Lambda^2 \otimes \Lambda^{1,1}$ combined with Proposition 2.3 it follows that

$$R^- \in \Lambda^{1,1} \otimes \text{End}(TM \otimes \mathbb{C}),$$

and therefore ∇^- induces a structure of holomorphic vector bundle on $TM \otimes \mathbb{C}$, with integrable Dolbeault operator $(\nabla^-)^{0,1}$.

Lemma 3.19. *The map $\psi_g: TM \otimes \mathbb{C} \rightarrow T^{1,0} \oplus T_{1,0}^*$ defined by (3.13) induces an isomorphism of orthogonal holomorphic vector bundles*

$$\psi_g: (TM \otimes \mathbb{C}, (\nabla^-)^{0,1}) \rightarrow \mathcal{Q}_{\sqrt{-1}\partial\omega}.$$

Proof. By Lemma 3.18 we have that ψ_g is a complex isometry. A straightforward calculation using the definitions of Bismut and Chern connections shows that

$$\langle \nabla_X^B Y, Z \rangle = \langle \nabla_X^C Y, Z \rangle + \frac{1}{2} d\omega(JX, Y, Z) + \frac{1}{2} d\omega(JX, JY, JZ),$$

which in turn implies

$$\nabla_{X^{0,1}}^- Y = \nabla_{X^{0,1}}^C Y - \sqrt{-1} g^{-1} \partial\omega(X^{0,1}, Y^{0,1}, \cdot).$$

Therefore, using that the Chern connection is metric compatible, we obtain

$$\psi_g(\nabla_{X^{0,1}}^- \psi_g^{-1}(Y^{1,0} + \xi^{1,0})) = \nabla_{X^{0,1}}^C (Y^{1,0} + \xi^{1,0}) + \sqrt{-1} \partial\omega(X^{0,1}, Y^{1,0}, \cdot),$$

which corresponds to the Dolbeault operator in Definition 3.4 for $\tau = \sqrt{-1}\partial\omega$. \square

In the next result we recover Proposition 3.13 in terms of classical geometry. This observation is originally due to Bismut [11, Theorem 2.9].

Proposition 3.20 (*Bismut's identity*). *Let (M, g, J) be a pluriclosed manifold. Consider the associated holomorphic orthogonal bundle $\mathcal{Q}_{\sqrt{-1}\partial\omega}$, as in Definition 3.4, endowed with the Hermitian metric G' (see (3.14)). Then, the Chern connection $\nabla_{G'}^C$ of G' satisfies*

$$\nabla_{G'}^C = (\psi_g)_* \nabla^-, \quad \Omega_{G'}^C = (\psi_g)_* R^-.$$

Proof. The statement follows from Lemma 3.19 combined with $\psi_* G_0 = G'$, and the uniqueness of the Chern connection on a Hermitian holomorphic vector bundle. \square

Similarly as in the proof of Lemma 3.16, given a background pluriclosed metric g_0 and a pair (ω, β) satisfying

$$\bar{\partial}\beta = \partial\omega_0 - \partial\omega \tag{3.15}$$

one has an induced isomorphism of holomorphic orthogonal bundles

$$e^{-\sqrt{-1}\beta}: \mathcal{Q}_{\sqrt{-1}\partial\omega} \rightarrow \mathcal{Q}_{\sqrt{-1}\partial\omega_0}.$$

Thus, (ω, β) induces in a natural way a Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ compatible with the orthogonal structure, defined by $G = (e^{\sqrt{-1}\beta})^* G'$ (see Remark 3.17). We close this section with a formula for the generalized Hermitian metric G and its Chern connection. In particular, the next lemma combined with Proposition 3.20 recovers the calculation in [43, Lemma 2.1] in a more conceptual way. The proof is straightforward from Proposition 3.20.

Proposition 3.21. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$, as in Definition 3.4. Given (ω, β) as in (3.15), the induced generalized Hermitian metric $G = (e^{\sqrt{-1}\beta})^* G'$ on \mathcal{Q} is given by*

$$G(X + \xi, X + \xi) = g(X, \bar{X}) + g(\xi + \sqrt{-1}\beta(X), \bar{\xi} - \sqrt{-1}\beta(\bar{X})).$$

In matrix notation, we have

$$G = \begin{pmatrix} g_{i\bar{j}} + \beta_{ik}\bar{\beta}_{j\bar{l}}g^{\bar{l}k} & \sqrt{-1}\beta_{ip}g^{\bar{l}p} \\ -\sqrt{-1}\bar{\beta}_{j\bar{p}}g^{\bar{p}k} & g^{\bar{l}k} \end{pmatrix}. \quad (3.16)$$

Furthermore, the Chern connection ∇_G^C of G satisfies

$$(e^{\sqrt{-1}\beta})_* \nabla_G^C = (\psi_g)_* \nabla^-, \quad (e^{\sqrt{-1}\beta})_* \Omega_G^C = (\psi_g)_* R^-.$$

3.4. Explicit formulas

Proposition 3.21, in conjunction with Proposition 2.3, gives a complete description of the Chern connection of the generalized metric G in terms of the Bismut curvature of g . In this subsection we give a completely explicit proof of Proposition 3.21, extending the approach of [43, Lemma 2.1]. These identities provide some further insight into the relationship of these curvatures, and furthermore play a role in deriving certain analytic estimates for pluriclosed flow below.

We first give an explicit computation of the Chern connection for the generalized Hermitian metric associated to a given pluriclosed metric as in Proposition 3.21. In the statement below we choose $\{z^i\}$ local complex coordinates with associated coordinate basis $\{\frac{\partial}{\partial z^i}\}$ and cobasis $\{dz^i\}$. Together these form a basis for $T^{1,0} \oplus T_{1,0}^*$, referred to as

$$Z^i = \frac{\partial}{\partial z^i}, \quad W^i = dz^i.$$

Lemma 3.22. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 3.4, and a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 3.21. Then the Chern connection ∇_G^C of G is expressed in local complex coordinates as*

$$\begin{aligned} \Gamma_{iZ^a}^{Z^b} &= g^{\bar{c}b} g_{a\bar{c},i} - \sqrt{-1} g^{\bar{c}b} g^{\bar{q}p} \beta_{ap} \bar{T}_{\bar{q}\bar{c}i}, \\ \Gamma_{iZ^a}^{W^b} &= \nabla_i^C \beta_{ab} + g^{\bar{q}p} g^{\bar{c}m} \beta_{ap} \beta_{bm} \bar{T}_{\bar{q}\bar{c}i}, \\ \Gamma_{iW^a}^{Z^b} &= -g^{\bar{c}b} g^{\bar{q}a} \bar{T}_{\bar{q}\bar{c}i}, \\ \Gamma_{iW^a}^{W^b} &= -g^{\bar{q}a} g_{b\bar{q},i} - \sqrt{-1} g^{\bar{q}a} g^{\bar{c}m} \beta_{bm} \bar{T}_{\bar{q}\bar{c}i}. \end{aligned}$$

Proof. Choose some local coordinates $\{z^i\}$ on M , with associated basis $\{e_A\} = \{\frac{\partial}{\partial z^i}, dz^i\}$. We first define a section of $T_{0,1}^* \otimes \text{End}(T^{1,0} \oplus T_{1,0}^*)$ by

$$i_{\bar{\partial}_i} \mathcal{T}^0(X + \xi) = \begin{pmatrix} 0 & 0 \\ T^0(\cdot, \bar{\partial}_i, \cdot) & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = T^0(X, \bar{\partial}_i, \cdot).$$

Note that for a section s of $T^{1,0} \oplus T_{1,0}^*$ one has

$$\bar{\partial}^{\sqrt{-1}\partial\omega_0} s = \bar{\partial}s + \mathcal{T}^0(s).$$

Thus we apply metric compatibility of the Chern connection to derive

$$\begin{aligned}\partial_i G(e_A, \bar{e}_B) &= G((\nabla_G^C)_i e_A, \bar{e}_B) + G(e_A, i_{\partial_i} \partial \bar{e}_B) \\ &= \Gamma_{iA}^P G_{P\bar{B}} + G(e_A, i_{\partial_i} \bar{T}^0(\bar{e}_B)) = \Gamma_{iA}^P G_{P\bar{B}} + (\bar{T}^0)_{i\bar{B}}^{\bar{P}} G_{A\bar{P}}.\end{aligned}$$

Rearranging this and using the explicit form of G yields the lemma. \square

Next we use Lemma 3.22 to give a direct proof of the identification of the Bismut curvature of g with the Chern curvature of G .

Proposition 3.23. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 3.4, and a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 3.21. Then the Chern curvature Ω_G^C of the Hermitian metric G has coefficients*

$$\begin{aligned}e^{-\sqrt{-1}\beta}(\Omega_G^C)_{i\bar{j}}(e^{\sqrt{-1}\beta}\partial_k) &= R_{k\bar{m}i\bar{j}}^B g^{\bar{m}l} \partial_l - R_{kli\bar{j}}^B dz^l, \\ e^{-\sqrt{-1}\beta}(\Omega_G^C)_{i\bar{j}}(e^{\sqrt{-1}\beta}dz^k) &= g^{\bar{c}l} g^{\bar{q}k} R_{\bar{q}\bar{c}i}^B \partial_l - g^{\bar{q}k} R_{l\bar{q}i\bar{j}}^B dz^l.\end{aligned}$$

Proof. Fix $p \in M$, and choose local complex coordinates near p . Since the computation is local, without loss of generality we can conjugate by a locally defined holomorphic $(2, 0)$ form to assume that $\beta(p) = 0$. We first compute the curvature operator acting on sections of the tangent bundle. We will use the following identities relating the Bismut curvature R^B , and the Chern torsion T and Chern curvature R^C , whose proof is left as an exercise:

$$R_{i\bar{j}k\bar{l}}^B = R_{k\bar{l}i\bar{j}}^C - T_{i\bar{p}l} g^{\bar{q}p} \bar{T}_{\bar{j}\bar{q}k}, \quad R_{ij\bar{k}\bar{l}}^B = \nabla_k^C T_{i\bar{j}\bar{l}}.$$

Using these together with Lemma 3.22, we obtain

$$\begin{aligned}(\Omega^G)_{i\bar{j}} \partial_k &= -\nabla_i^G (T_{k\bar{l}\bar{j}}^0 dz^l) - \nabla_{\bar{j}}^G (\Gamma_{iZ^l}^Z \partial_l + \Gamma_{iZ^k}^{W^l} dz^l) \\ &= [(R^C)_{i\bar{j}k}^l - \sqrt{-1}\beta_{kr,\bar{j}} g^{\bar{s}r} g^{\bar{p}l} \bar{T}_{\bar{p}\bar{s}i} - g^{\bar{s}l} g^{\bar{r}p} \bar{T}_{\bar{s}\bar{r}i} T_{kp\bar{j}}^0] \partial_l - [\nabla_i^C T_{k\bar{l}\bar{j}}] dz^l \\ &= [(R^C)_{i\bar{j}k}^l - g^{\bar{s}r} g^{\bar{p}l} \bar{T}_{\bar{p}\bar{s}i} T_{kr\bar{j}}] \partial_l - [\nabla_i^C T_{k\bar{l}\bar{j}}] dz^l \\ &= R_{k\bar{m}i\bar{j}}^B g^{\bar{m}l} \partial_l - R_{kli\bar{j}}^B dz^l,\end{aligned}$$

as claimed. We proceed similarly for the curvature operator acting on sections of the cotangent bundle:

$$\begin{aligned}(\Omega^G)_{i\bar{j}} dz^k &= -\nabla_{\bar{j}}^G (\Gamma_{iW^k}^{Z^l} \partial_l + \Gamma_{iW^k}^{W^l} dz^l) \\ &= [-\Gamma_{iW^k,\bar{j}}^{Z^l}] \partial_l + [\Gamma_{iW^k}^{Z^p} T_{p\bar{l}\bar{j}}^0 - \Gamma_{iW^k,\bar{j}}^{W^l}] dz^l \\ &= [g^{\bar{c}l} g^{\bar{q}k} \nabla_{\bar{j}}^C \bar{T}_{\bar{q}\bar{c}i}] \partial_l + [-(R^C)_{i\bar{j}l}^k + g^{\bar{q}k} g^{\bar{c}p} \bar{T}_{\bar{c}\bar{q}i} (\sqrt{-1}\partial\beta_{lp\bar{j}} + T_{lp\bar{j}}^0)] dz^l \\ &= [g^{\bar{c}l} g^{\bar{q}k} \nabla_{\bar{j}}^C \bar{T}_{\bar{q}\bar{c}i}] \partial_l - [(R^C)_{i\bar{j}l}^k - g^{\bar{q}k} g^{\bar{c}p} \bar{T}_{\bar{c}\bar{q}i} T_{p\bar{l}\bar{j}}] dz^l \\ &= g^{\bar{c}l} g^{\bar{q}k} R_{\bar{q}\bar{c}i}^B \partial_l - g^{\bar{q}k} R_{l\bar{q}i\bar{j}}^B dz^l,\end{aligned}$$

where again we applied the curvature identities above. \square

4. Slope stability criteria for Bismut Hermitian-Einstein metrics

In this section we discuss necessary conditions for the existence of a Bismut Hermitian-Einstein metric in relation to slope stability for the associated holomorphic Courant algebroid. First in §4.1 we recall fundamental points of Hermitian-Einstein metrics on holomorphic vector bundles. Then in §4.2 we explain how Bismut Hermitian-Einstein metrics are in fact Hermitian-Einstein metrics on the relevant holomorphic Courant algebroid, leading to slope stability obstructions to existence. Finally in §4.3 we derive concrete obstructions in terms of holomorphic maps from the underlying complex manifold, yielding explicit families in all dimensions of manifolds with vanishing first Chern class but no Bismut Hermitian-Einstein metric.

4.1. Hermitian-Einstein metrics and slope stability

Definition 4.1. Let (M^{2n}, g, J) be a Hermitian manifold and suppose $(\mathcal{E}, h) \rightarrow M$ denote a holomorphic vector bundle with Hermitian metric h and associated Chern connection ∇^h . The *second Ricci curvature* is

$$S_g^h := \sqrt{-1} \operatorname{tr}_\omega \Omega^h \in \operatorname{End}(E).$$

The existence of metrics with vanishing second Ricci curvature, or more generally Hermitian-Einstein metrics, is governed by slope stability criteria as in the Donaldson-Uhlenbeck-Yau Theorem [23,69] and its extensions to Hermitian manifolds (see [16,45,46]). To state the precise result which we will use, let us recall first some basic definitions. Given a coherent sheaf \mathcal{F} of \mathcal{O}_M -modules over M , the determinant $\det \mathcal{F} := ((\Lambda^r \mathcal{F})^*)^*$, where r denotes the rank of \mathcal{F} , is a holomorphic line bundle over M . Given now an Aepli class $a \in H_A^{n-1, n-1}$, we can define the slope of \mathcal{F} by

$$\mu_a(\mathcal{F}) = \frac{c_1(\det \mathcal{F}) \cdot a}{r},$$

where $c_1(\det \mathcal{F}) \in H_{BC}^{1,1}(M)$ is the first Chern class of $\det \mathcal{F}$, regarded an element in the Bott-Chern cohomology of M . Here we use the standard duality pairing

$$H_{BC}^{1,1} \otimes H_A^{n-1, n-1} \rightarrow \mathbb{C}.$$

Definition 4.2. Let (M, J) be a compact complex manifold endowed with an Aepli class $a \in H_A^{n-1, n-1}$. A holomorphic vector bundle \mathcal{E} over M is *a-stable* if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ one has

$$\mu_a(\mathcal{F}) < \mu_a(\mathcal{E}).$$

We say that \mathcal{E} is *a-polystable* if it is a direct sum of *a-stable* bundles with the same slope.

The relation between slope stability and the Hermitian-Einstein equation is provided by the following important result (cf [46]):

Theorem 4.3. Let (M, J) be a compact complex manifold. Let $\tilde{\omega}$ be a Gauduchon Hermitian metric on M with Aepli class $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1, n-1}$. A holomorphic vector bundle \mathcal{E} over (M, J) admits a Hermitian metric h solving the Hermitian-Einstein equation

$$S_g^h = 2\pi\mu_a(\mathcal{E}) \operatorname{Id}_{\mathcal{E}}$$

if and only if \mathcal{E} is *a-polystable*.

4.2. Bismut Hermitian-Einstein metrics and slope stability of \mathcal{Q}

The theory of Hermitian-Einstein metrics implies obstructions to the existence of Bismut Hermitian-Einstein metrics. The fundamental point in our discussion is that the Hermitian Yang-Mills curvature of a generalized metric G on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$ is completely determined by the Bismut Ricci curvature of the associated Hermitian metric g on M . We make this concrete below.

Proposition 4.4. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 3.4. Consider a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 3.21. Then the Chern connection ∇_G^C of the Hermitian metric G , defined as in (3.16) satisfies*

$$S_g^G = \sqrt{-1}(e^{\sqrt{-1}\beta})^* \begin{pmatrix} -g^{-1}\rho_B^{1,1} & g^{-1}\rho_B^{0,2}g^{-1} \\ \rho_B^{2,0} & -\rho_B^{1,1}g^{-1} \end{pmatrix}.$$

Consequently, g is Bismut Hermitian-Einstein if and only if G is Hermitian-Einstein with respect to g , that is, if and only if

$$S_g^G = 0.$$

Proof. This follows by taking the trace of the formula for Ω_G^C in Proposition 3.23. Alternatively, this follows from Proposition 3.20, since

$$\begin{aligned} S_g^G &= \sqrt{-1} \operatorname{tr}_\omega \Omega_G^C \\ &= \sqrt{-1}(e^{\sqrt{-1}\beta})^* \psi_* \operatorname{tr}_\omega R^- \\ &= -\sqrt{-1}(e^{\sqrt{-1}\beta})^* \psi_* g^{-1} \rho_B, \end{aligned}$$

where for the last identity we have used Proposition 2.3 to conclude

$$(\operatorname{tr}_\omega R^-)X = \frac{1}{2} \sum_{j=1}^{2n} R^-(e_i, J e_i)X = -\frac{1}{2} \sum_{j=1}^{2n} g^{-1} R^B(X, \cdot, J e_i, e_i) = -g^{-1} \rho_B(X).$$

Finally, given $X + \xi \in T^{1,0} \oplus T_{1,0}^*$, we calculate

$$\begin{aligned} (\psi_* g^{-1} \rho_B)(X) &= \psi(g^{-1} \rho_B(X)) = g^{-1} \rho_B^{1,1}(X) - \rho_B^{2,0}(X) \\ (\psi_* g^{-1} \rho_B)(\xi) &= -\psi(g^{-1} \rho_B g^{-1} \xi) = \rho_B^{1,1} g^{-1} \xi - g^{-1} \rho_B^{0,2} g^{-1} \xi. \quad \square \end{aligned}$$

As a straightforward consequence of Theorem 4.3 and Proposition 4.4 we obtain the following necessary condition to the existence of a pluriclosed Bismut Hermitian-Einstein metric.

Corollary 4.5. *Let (M, g, J) be a pluriclosed manifold. Denote by $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1, n-1}$ the Aeppli class of the unique Gauduchon metric $\tilde{\omega}$ in the conformal class of ω , such that $\int_X \tilde{\omega}^n = \int_X \omega^n$. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega}$ as in Definition 3.4. Assume that the metric g is Bismut Hermitian-Einstein. Then, for any subsheaf $\mathcal{F} \subset \mathcal{Q}$ one has*

$$\mu_a(\mathcal{F}) \leq 0, \tag{4.1}$$

with equality only if \mathcal{Q} splits holomorphically.

Proof. The proof is an easy consequence of Proposition 4.4 and Theorem 4.3, after noting that \mathcal{Q} satisfies $c_1(\mathcal{Q}) = 0 \in H_{BC}^{1,1}(M)$, since $\det \mathcal{Q}$ admits a canonical holomorphic trivialization induced by the holomorphic pairing. \square

Remark 4.6. The holomorphic Courant algebroid \mathcal{Q} has a natural orthogonal structure. Thus, by general theory, in the previous result we can restrict to isotropic subsheaves, that is $\mathcal{F} \subset \mathcal{Q}$ such that $\langle \mathcal{F}, \mathcal{F} \rangle = 0$, in order to measure the slope inequality (4.1) (see [12]).

As observed in Remark 3.9, the holomorphic structure on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$ is determined up to isomorphism by the Aeppli class of the pluriclosed metric $[\omega] \in H_A^{1,1}$. In general, the stability condition depends in a intricate way on the Bismut Hermitian-Einstein pluriclosed metric. This is due to the fact that the map

$$\omega \mapsto a = [\tilde{\omega}^{n-1}] \in H_A^{n-1, n-1}(M)$$

is typically a complicated function in the space of pluriclosed metrics. In the special case of complex surfaces $n = 2$, this map only depends on the Aeppli class $[\omega] \in H_A^{1,1}$, and is just the identity map.

4.3. Obstructions to Bismut Hermitian-Einstein metrics

In the following we will see that we can obtain some more concrete implications of Corollary 4.5 for the existence of pluriclosed Bismut Hermitian-Einstein metrics. In particular, our next result provides a clean obstruction to the existence of such metrics on a compact complex manifold. For this, we exploit the fact that any exact holomorphic Courant algebroid has a canonical isotropic subsheaf, given by the holomorphic cotangent bundle

$$T_{1,0}^* \xrightarrow{\pi^*} \mathcal{Q}.$$

We will say that an Aeppli class $a \in H_A^{n-1, n-1}(M)$ is *positive* if $a = [\tilde{\omega}^{n-1}]$, for some Gauduchon metric $\tilde{\omega}$ on M .

Theorem 4.7. *Let (M, J) be a compact connected complex manifold. Assume that M admits a pluriclosed metric g which is Bismut Hermitian-Einstein. Then, there exists a positive Aeppli class $a \in H_A^{n-1, n-1}(M)$ such that, for any complex manifold Z and any holomorphic map $f: M \rightarrow Z$ such that df is surjective at one point, one has*

$$f^*c_1(Z) \cdot a \geq 0$$

Furthermore, for $Z = (M, J)$ and $f = \text{Id}$, $c_1(M) \cdot a > 0$ unless g is Kähler.

Proof. Let g as in the statement and consider the holomorphic Courant algebroid $\mathcal{Q} := \mathcal{Q}_{\sqrt{-1}\partial\omega}$. Let $a = [\tilde{\omega}^{n-1}]$ be the Aeppli class of the associated (normalized) Gauduchon metric $\tilde{\omega}$. By Corollary 4.5, the holomorphic vector bundle underlying \mathcal{Q} is a -polystable. Let $f: M \rightarrow Z$ be as in the statement. Then, the differential $df: T_{1,0} \rightarrow T^{1,0}Z$ induces a morphism

$$f^*T_{1,0}^*Z \longrightarrow T_{1,0}^* \xrightarrow{\pi^*} \mathcal{Q}.$$

Since df is surjective at one point, by Sard's Theorem there exists a dense open subset $R \subset Z$ of regular values of f . Using that M is connected and that f is holomorphic, it follows that $f^{-1}(R) \subset M$ is also

dense (in fact, the complement is an analytic subspace of codimension ≥ 1). Hence, the previous morphism induces a subsheaf

$$f^*T_{1,0}^*Z \hookrightarrow \mathcal{Q}.$$

The inequality of slopes (4.1) now gives

$$0 \geq c_1(f^*T^*Z) \cdot a = -f^*c_1(Z) \cdot a.$$

As for the last part, if $c_1(M) \cdot a = 0$ then by Proposition 4.4 and Corollary 4.5 we have a holomorphic splitting $\mathcal{Q} = T^{1,0} \oplus T_{1,0}^*$ and hence $\partial\omega = 0$. \square

Remark 4.8. An alternative proof of the last part of Theorem 4.7 can be obtained as a consequence of a result of Gauduchon [31]. For instance, $\rho_B = 0$ implies that the Chern scalar is non-negative and bigger than zero at some point if g is non-Kähler (see Proposition 2.5). From this, it follows that $c_1(M) \cdot a \geq 0$ with equality only if g is Kähler (see [29, Proposition 8.30]).

We next obtain a more concrete criteria derived from Theorem 4.7.

Corollary 4.9. *Let $f: (M, J) \rightarrow Z$ be a holomorphic map of compact connected complex manifolds. Assume that df is surjective at some point and that Z is Kähler with $c_1(Z) < 0$. Then M does not admit a Bismut Hermitian-Einstein pluriclosed metric.*

Proof. By Aubin-Yau's Theorem there exists a Kähler-Einstein metric ω_Z on Z with negative scalar curvature, that is, such that $\rho_Z = -\omega_Z$. Let $a = [\tilde{\omega}^{n-1}] \in H_A^{n-1, n-1}(M)$ be a positive Aeppli class on M . Then

$$f^*c_1(Z) \cdot a = - \int_X f^*\omega_Z \wedge \tilde{\omega}^{n-1}.$$

By hypothesis there exists $x \in M$ such that f is a submersion, and hence, arguing as in the proof of Theorem 4.7, the preimage of the set of regular values is open and dense. On this locus $f^*\omega_Z \wedge \tilde{\omega}^{n-1} > 0$, and hence $f^*c_1(Z) \cdot a < 0$. \square

As a consequence of Corollary 4.9 we obtain examples of compact pluriclosed manifolds (M^{2n}, J) with $c_1(M) = 0 \in H^2(M, \mathbb{Z})$ which do not admit a Bismut Hermitian-Einstein metric. To the knowledge of the authors, this is the first class of such examples in the literature for dimension $n \geq 3$ (the case $n = 2$ is settled by [30]). In order to present our examples we start with some general discussion of principal bundles over complex manifolds. Let Z be a Kähler manifold. Let $T = \mathbb{C}^n/\Lambda$ be an n -dimensional complex torus. Let

$$\delta: H^1(T, \mathbb{Z}) \rightarrow H^2(Z, \mathbb{Z})$$

be an homomorphism of \mathbb{Z} -modules such that $c_1(Z) \in \text{Im } \delta$. We can identify δ with $c \in H^2(Z, \mathbb{Z}) \otimes \Lambda$, and hence it determines a topologically non-trivial principal T -bundle $\pi: M \rightarrow Z$ with characteristic class c . Assuming further that $\text{Im } \delta \subset H^{1,1}(Z)$, M can be endowed with a holomorphic structure. The first Chern class satisfies $c_1(M) = \pi^*c_1(Z) \in H^2(M, \mathbb{Z})$ and hence it vanishes because $c_1(Z) \in \text{Im } \delta$ [40]. Furthermore, M is non-Kähler by Blanchard's Theorem.

Example 4.10. Consider the case that Z is a compact connected Riemann surface with genus ≥ 2 , and hence $c_1(Z) < 0$. Let $\pi: M \rightarrow Z$ be a non-trivial principal T -bundle over M . By dimensional reasons the condition $c_1(Z) \in \text{Im } \delta \subset H^{1,1}(Z)$ is always satisfied, and hence $c_1(M) = 0 \in H^2(M, \mathbb{Z})$. Choose a principal connection $\theta = (\theta_1, \dots, \theta_{2n})$ on M and define a T -invariant complex structure on M by $J\theta_{2j-1} = \theta_{2j}$. Choose a Kähler metric ω_Z on Z and consider the Hermitian form

$$\omega = \pi^*\omega_Z + \sum_{j=1}^n \theta_{2j-1} \wedge \theta_{2j}.$$

Then, we have

$$dd^c\omega = -d^c\left(\sum_{j=1}^n \pi^*F_{\theta_{2j-1}} \wedge \theta_{2j} - \theta_{2j-1} \wedge p^*F_{\theta_{2j}}\right) = 0$$

by dimensional reasons, where F_{θ_j} denotes the curvature of θ_j . Therefore, M is a pluriclosed manifold with vanishing first Chern class. Applying now Corollary 4.9, we conclude that M does not admit a Bismut Hermitian-Einstein metric.

Example 4.11. Let Z be an algebraic complex surface with $c_1(Z) < 0$ and let $T = \mathbb{C}/\Lambda$. By the Aubin-Yau Theorem we have ω_Z a Kähler-Einstein metric on Z with negative scalar curvature and $[\omega_Z] \in H^2(Z, \mathbb{Z})$. Choose α a primitive $(1, 1)$ -form with $[\alpha] \in H^2(Z, \mathbb{Z})$, and define δ so that its image is spanned by

$$\delta_1 = -[\omega_Z] = c_1(Z), \quad \delta_2 = [\alpha].$$

On the corresponding T -bundle we can choose a connection $\theta = (\theta_1, \theta_2)$ with curvature

$$\frac{\sqrt{-1}}{2\pi}F_\theta = (-\omega_Z, \alpha),$$

and a T -invariant complex structure such that $J\theta_1 = \theta_2$. For any $u \in C^\infty(Z)$ we define the Hermitian metric $\omega = \pi^*e^u\omega_Z + \theta_1 \wedge \theta_2$. Then, a direct calculation shows that [37]

$$dd^c\omega = \pi^*(\Delta(e^u)\omega_Z^2 - \alpha^2 - \omega_Z^2),$$

and hence the existence of a pluriclosed metric reduces to solve

$$[\omega_Z]^2 = -[\alpha] \cdot [\alpha] \in \mathbb{N}.$$

Taking $\iota: Z \hookrightarrow \mathbb{P}^3$ a degree $d \geq 5$ projective hypersurface, we have $c_1(Z) = (4-d)\iota^*H$, for H the hyperplane class, and we obtain the condition

$$(d-4)^2 = -[\alpha] \cdot [\alpha]$$

for a primitive $(1, 1)$ -class $[\alpha]$. We have $H^2(Z, \mathbb{Z}) \cong \mathbb{Z}^{d(d(d-4)+6)-2}$ and, assuming that d is odd, the intersection pairing is given by the standard symmetric bilinear form with signature $-(d-2)d(d+2)/3$. Using Hirzebruch's formula for the Hodge numbers of projective hypersurfaces to calculate $h^{1,1}(M)$, one can prove that such a class always exists. For example, taking $d = 5$ one has $H^2(Z, \mathbb{Z}) \cong \mathbb{Z}^{53}$, $h^{1,1}(M) = 45$, and signature -35 . Therefore, there exists a 36-dimensional subspace of primitive $(1, 1)$ -classes, and hence there is a primitive $(1, 1)$ -class $[\alpha]$ with $[\alpha] \cdot [\alpha] = -1$.

5. Pluriclosed flow

In this section we turn to understanding the implications of Sections 3 and 4 for pluriclosed flow. As explained in the introduction, a solution to pluriclosed flow is a one-parameter family of pairs (ω_t, β_t) on a complex manifold solving

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}, \quad \frac{\partial}{\partial t}\beta = -\rho_B^{2,0}.$$

As a preliminary step we discuss an adaptation of the Schwarz Lemma to the setting of holomorphic Courant algebroids. Given this, our first main task is to reinterpret pluriclosed flow as an evolution equation for the associated generalized Hermitian metrics G_t (cf. Proposition 5.5). We use this to derive natural evolution equations for quantities measuring G_t against some background choice of Hermitian metric on the associated holomorphic Courant algebroid, which is achieved using the Schwarz Lemma. Next, in §5.3 we give the proof of Theorem 1.2 using a priori estimates derived from the evolution equations for G_t adapted to the case of a Bismut flat background. Building on this we derive the consequences for generalized Kähler geometry in §5.4. In §5.5 we prove Theorem 1.3. The key point is to use that such surfaces are finitely covered by holomorphic principal T^2 -bundles. Using this structure we obtain partial estimates for G_t by applying the Schwarz Lemma to the projection map of the fibration. Using this partial estimate and then exploiting special properties of the Bismut curvature of a choice of invariant background metric yields the global existence.

5.1. Schwarz Lemma

In this section we prove an extension of Yau’s Schwarz Lemma [72] to the setting of generalized metrics on holomorphic Courant algebroids. To begin, let \mathcal{W} be a holomorphic vector bundle over a complex manifold (M, J) . We fix a Hermitian metric g on M and define the Chern Laplacian on functions $f \in C^\infty(M)$ by

$$\Delta f := \frac{n\sqrt{-1}\partial\bar{\partial}f \wedge \omega^{n-1}}{\omega^n} = \text{tr}_\omega(\sqrt{-1}\partial\bar{\partial}f).$$

Given a holomorphic section $w \in H^0(M, \mathcal{W})$, as a basic consequence of the Weitzenböck formula (cf. [22]) one has

$$\Delta|w|_h^2 = |\nabla_h^C w|_{g,h}^2 - \langle S_g^h w, w \rangle_h \quad (5.1)$$

for any choice of Hermitian metric h on \mathcal{W} with Chern connection ∇_h^C . Note that $|\nabla_h^C w|_{g,h}^2$ is calculated using the background Hermitian metric g jointly with the given Hermitian metric on the bundle.

We are interested in the application of formula (5.1) to the following general setup: Let $\pi_{\mathcal{E}}: \mathcal{E} \rightarrow M$ and $\pi_{\mathcal{F}}: \mathcal{F} \rightarrow N$ denote holomorphic vector bundles over complex manifolds M and N respectively. Suppose $\Phi: \mathcal{E} \rightarrow \mathcal{F}$ is a morphism of holomorphic vector bundles covering $\phi: M \rightarrow N$, i.e. there is a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\Phi} & \mathcal{F} \\ \pi_{\mathcal{E}} \downarrow & & \downarrow \pi_{\mathcal{F}} \\ M & \xrightarrow{\phi} & N. \end{array}$$

Given such a map, there is a tautological holomorphic section of $\mathcal{E}^* \otimes \phi^* \mathcal{F}$ which, by abuse of notation, we denote also

$$\Phi \in H^0(M, \mathcal{E}^* \otimes \phi^* \mathcal{F}).$$

Furthermore, any pair of Hermitian metrics G and \tilde{G} on \mathcal{E} and \mathcal{F} , respectively, induce a Hermitian metric $G^{-1} \otimes \tilde{G}$ on $\mathcal{E}^* \otimes \phi^* \mathcal{F}$. Observe that, for any choice of frames on \mathcal{E} and \mathcal{F} , one has

$$|\Phi|_{G^{-1}, \tilde{G}}^2 = (\phi^* \tilde{G}_{\alpha\bar{\gamma}}) \Phi_j^\alpha \overline{\Phi_k^\gamma} G^{j\bar{k}}.$$

Furthermore, the induced Chern connection $\nabla^{C, G^{-1}, \tilde{G}}$ acts on Φ , defining a tensor $A(G, \tilde{G}, \Phi)$ by

$$A(e) = (\nabla^{C, G^{-1}, \tilde{G}} \Phi)(e) = \phi^* \nabla^{C, \tilde{G}}(\Phi e) - \Phi(\nabla^{C, G} e), \quad (5.2)$$

for any smooth section e of \mathcal{E} . As a direct application of (5.1) we obtain the following:

Lemma 5.1. *Let $\mathcal{E} \rightarrow M$ and $\mathcal{F} \rightarrow N$ denote holomorphic vector bundles over complex manifolds M and N respectively, and suppose $\Phi : \mathcal{E} \rightarrow \mathcal{F}$ is a holomorphic map of vector bundles covering $\phi : M \rightarrow N$. Given g a Hermitian metric on M , G a Hermitian metric on \mathcal{E} and \tilde{G} a metric on \mathcal{F} , one has*

$$\Delta |\Phi|_{G^{-1}, \tilde{G}}^2 = |A|_{g, G^{-1}, \tilde{G}}^2 + \left\langle \Phi \circ S_g^G - \phi^* S_g^{\tilde{G}} \circ \Phi, \Phi \right\rangle_{G^{-1}, \tilde{G}}.$$

We next apply Lemma 5.1 to various situations of our interest. The simplest case is to apply it to the identity map of a fixed holomorphic vector bundle. We note that in this case the tensor $A = \nabla^{C, G^{-1}, \tilde{G}} \Phi$ is the difference of two Chern connections, and we use some more common notation for this:

Definition 5.2. Given G, \tilde{G} Hermitian metrics on a holomorphic vector bundle \mathcal{E} over M , let

$$\Upsilon(G, \tilde{G}) := \nabla_G^C - \nabla_{\tilde{G}}^C \in T_{1,0}^* \otimes \text{End}(\mathcal{E})$$

denote the difference of the associated Chern connections. When taking the norm of Υ , we require a metric on $T_{1,0}^*$ as well as one on \mathcal{E} and \mathcal{E}^* . These choices will be denoted explicitly as subscripts, using possibly a given metric and its inverse on both \mathcal{E} and \mathcal{E}^* if it is not explicitly indicated.

Lemma 5.3. *Let $\mathcal{E} \rightarrow M$ be a holomorphic vector bundle over a complex manifold M . Given g a Hermitian metric on M and G and \tilde{G} Hermitian metrics on \mathcal{E} , one has that*

$$\Delta(\text{tr}_G \tilde{G}) = |\Upsilon(G, \tilde{G})|_{g, G^{-1}, \tilde{G}}^2 + \text{tr}_G \left\langle \left(S_g^G - S_g^{\tilde{G}} \right) \cdot, \cdot \right\rangle_{\tilde{G}}.$$

Proof. It follows from Lemma 5.1, setting $M = N$, $\mathcal{E} = \mathcal{F}$, and $\Phi = \text{Id}$. \square

We next consider specifically the case of generalized Hermitian metrics on exact holomorphic Courant algebroids, as in Proposition 3.21.

Lemma 5.4. *Let (M, g_0, J) be a pluriclosed manifold. Consider the associated orthogonal holomorphic vector bundle $\mathcal{Q} = \mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ as in Definition 3.4, and a generalized Hermitian metric $G = G(\omega, \beta)$ as in Proposition 3.21. Choose an arbitrary Hermitian metric \tilde{G} on \mathcal{Q} . Then, one has that*

$$\Delta(\text{tr}_G \tilde{G}) = |\Upsilon(G, \tilde{G})|_{g, G^{-1}, \tilde{G}}^2 + \text{tr}_G \left\langle \left(S_g^G - S_g^{\tilde{G}} \right) \cdot, \cdot \right\rangle_{\tilde{G}}.$$

Furthermore, provided that we take $\tilde{G} = G(\omega_0, 0)$, one also has that

$$\begin{aligned}\psi_{g_0}^{-1} \circ \Upsilon(G, \tilde{G}) \circ \psi_g &= (\varphi_{g_0, g, \beta}) \cdot \nabla_g^- - \nabla_{g_0}^- \\ \mathrm{tr}_G \tilde{G} &= \mathrm{tr}_{\tilde{G}} G = \mathrm{tr}_g g_0 + \mathrm{tr}_{g_0} g + |\beta|_{g, g_0}^2\end{aligned}$$

where ψ_g, ψ_{g_0} are as in Section 3.3, and $(\varphi_{g_0, g, \beta}) \cdot \nabla_g^-$ denotes the action on ∇_g^- of the complex gauge transformation $\varphi_{g_0, g, \beta} \in \mathrm{End}(TM \otimes \mathbb{C})$, given by

$$\varphi_{g_0, g, \beta}(X) = X^{1,0} + g_0^{-1}(gX^{0,1} + \sqrt{-1}\beta X^{1,0}).$$

Proof. The first part of the statement is a special case of Lemma 5.3. Assuming now $\tilde{G} = G(\omega_0, 0)$ and setting $G' = G(\omega, 0)$, one has that

$$\mathrm{tr}_G \tilde{G} = |e^{-\sqrt{-1}\beta}|_{G'^{-1}, \tilde{G}}^2 = (g_0)_{j\bar{k}} g^{j\bar{k}} + g_{j\bar{k}} g_0^{j\bar{k}} + g_0^{l\bar{m}} g^{j\bar{k}} \beta_{jl} \overline{\beta_{km}} = \mathrm{tr}_g g_0 + \mathrm{tr}_{g_0} g + |\beta|_{g, g_0}^2.$$

Finally, using Proposition 3.21, one has that

$$\begin{aligned}\Upsilon(G, \tilde{G}) &= (e^{\sqrt{-1}\beta})^* \nabla_{G'}^C - \nabla_{\tilde{G}}^C \\ &= (e^{-\sqrt{-1}\beta} \circ \psi_g)_* \nabla_g^- - (\psi_{g_0})_* \nabla_{g_0}^- \\ &= \psi_{g_0} \circ ((\psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta} \circ \psi_g) \cdot \nabla_g^- - \nabla_{g_0}^-) \circ \psi_{g_0}^{-1}\end{aligned}$$

and also that

$$\begin{aligned}\psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta} \circ \psi_g(X) &= \psi_{g_0}^{-1} \circ e^{-\sqrt{-1}\beta}(X^{1,0} - gX^{0,1}) \\ &= \psi_{g_0}^{-1}(X^{1,0} - gX^{0,1} - \sqrt{-1}\beta X^{1,0}) \\ &= X^{1,0} + g_0^{-1}(gX^{0,1} + \sqrt{-1}\beta X^{1,0}). \quad \square\end{aligned}$$

5.2. Pluriclosed flow and holomorphic Courant algebroids

Proposition 5.5. *Given $(M^{2n}, \omega_t, \beta_t, J)$ a solution to pluriclosed flow, the associated family of generalized Hermitian metrics G_t on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$ satisfies*

$$G^{-1} \frac{\partial}{\partial t} G = -S_g^G.$$

Proof. Fix $p \in M$ and $t > 0$. Conjugating by a constant B -field transformation we may assume without loss of generality that $\beta_t(p) = 0$. Given this setup, using Proposition 3.21 and Proposition 4.4 we can compute that

$$\begin{aligned}\partial_t G(X + \xi, X + \xi) &= \partial_t g(X, \overline{X}) - \partial_t g(g^{-1}\xi, \overline{g^{-1}\xi}) + \sqrt{-1}g^{-1}(\partial_t \beta(X), \overline{\xi}) - \sqrt{-1}g^{-1}(\xi, \overline{\partial_t \beta(X)}) \\ &= -\sqrt{-1}\partial_t \omega(X, \overline{X}) - \sqrt{-1}\partial_t \omega(g^{-1}\xi, \overline{g^{-1}\xi}) \\ &\quad + \sqrt{-1}\partial_t \beta(X, g^{-1}\overline{\xi}) - \sqrt{-1}\partial_t \overline{\beta}(\overline{X}, g^{-1}\xi), \\ G(S_g^G(X + \xi), X + \xi) &= \sqrt{-1}G(-g^{-1}\rho_B^{1,1}X + g^{-1}\rho_B^{0,2}g^{-1}\xi + \rho_B^{2,0}X - \rho_B^{1,1}g^{-1}\xi, X + \xi) \\ &= \sqrt{-1}(-\rho_B^{1,1}(X, \overline{X}) + \rho_B^{0,2}(g^{-1}\xi, \overline{X}) \\ &\quad + \rho_B^{2,0}(X, g^{-1}\overline{\xi}) - \rho_B^{1,1}(g^{-1}\xi, g^{-1}\overline{\xi}))\end{aligned}$$

for any $X + \xi \in T^{1,0} \oplus T_{1,0}^*$, where we have used that $\partial_t g = \partial_t \omega(\cdot, J)$. Therefore, the family satisfies $G^{-1} \frac{\partial}{\partial t} G = -S_g^G$ if and only if

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}, \quad \frac{\partial}{\partial t}\beta = -\rho_B^{2,0}, \quad \frac{\partial}{\partial t}\bar{\beta} = -\rho_B^{0,2}.$$

Using now that ρ_B is a real two-form, the statement follows. \square

The Schwarz Lemma computations from Section 5.1 will help us derive useful evolution equations for measuring the metric along pluriclosed flow.

Proposition 5.6. *Fix $(M^{2n}, \omega_t, \beta_t, J)$ a solution to pluriclosed flow, with G_t the associated family of generalized Hermitian metrics on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$. Given \tilde{G} a Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega_0}$, we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}_G \tilde{G} = -\left|\Upsilon(\tilde{G}, G)\right|_{g, G^{-1}, \tilde{G}}^2 + \text{tr}_G \left\langle S_g^{\tilde{G}} \cdot, \cdot \right\rangle_{\tilde{G}}.$$

Proof. Applying Lemma 5.4 we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \text{tr}_G \tilde{G} &= \text{tr} G^{-1} S_g^G \tilde{G} \\ &= \text{tr}_G \left\langle S_g^G \cdot, \cdot \right\rangle_{\tilde{G}} \\ &= \Delta \text{tr}_G \tilde{G} - \left|\Upsilon(\tilde{G}, G)\right|_{g, G^{-1}, \tilde{G}}^2 + \text{tr}_G \left\langle S_g^{\tilde{G}} \cdot, \cdot \right\rangle_{\tilde{G}}, \end{aligned}$$

as required. \square

Proposition 5.7. *Fix $(M^{2n}, \omega_t, \beta_t, J)$ a solution to pluriclosed flow, with G_t the associated family of generalized metrics on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$. Given \tilde{G} a Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega}$, we have*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \left|\Upsilon(G, \tilde{G})\right|_{g, G^{-1}, G}^2 &= -|\nabla \Upsilon|_{g, G^{-1}, G}^2 - |\bar{\nabla} \Upsilon + \bar{T} \cdot \Upsilon|_{g, G^{-1}, G}^2 \\ &\quad + T \star \Upsilon \star \Omega^{\tilde{G}} + \Upsilon \star \bar{T} \star \Omega^{\tilde{G}} + \Upsilon \star \nabla^{\tilde{G}} \Omega^{\tilde{G}}, \end{aligned}$$

where

$$(\bar{T} \cdot \Upsilon)_{\bar{i}jA}^B = g^{\bar{k}l} \bar{T}_{\bar{i}\bar{k}j} \Upsilon_{lA}^B.$$

Proof. As we have established the evolution equation $G^{-1} \frac{\partial}{\partial t} G = -S_g^G$ in the general setting here of twisted Courant algebroids in Proposition 5.5, the result follows formally exactly as in [43, Proposition 3.3]. \square

5.3. Global existence and convergence on Bismut-flat backgrounds

In this subsection we will finish the proof of Theorem 1.2. The proof relies principally on the evolution equations for G derived above. Using the assumption of a background metric which is Bismut-flat, it is possible to remove the reaction terms in the heat equations of Propositions 5.6 and 5.7, leading directly to a priori estimates on G . These will then lead to full regularity and convergence of the pluriclosed flow. Before giving the proof we prove a technical lemma which clarifies that a priori estimates for G translate in a natural way to various a priori estimates for the associated pluriclosed metric ω and torsion potential β .

Lemma 5.8. *Suppose (M^{2n}, J) is a complex manifold, and suppose (ω, β) and $(\tilde{\omega}, \tilde{\beta})$ are pluriclosed metrics on M such that the associated generalized Hermitian metrics G and \tilde{G} are defined on the same holomorphic Courant algebroid \mathcal{Q} and satisfy*

$$\Lambda^{-1}\tilde{G} \leq G \leq \Lambda\tilde{G}, \quad |\Upsilon(G, \tilde{G})|_{g,G} < \Lambda.$$

Then there exists a constant $A = A(n, \Lambda)$ such that

$$A^{-1}\tilde{g} \leq g \leq A\tilde{g}, \quad |\beta|_{\tilde{g}} \leq A, \quad |\Upsilon(g, \tilde{g})|_{\tilde{g}} \leq A.$$

Proof. By the assumed uniform equivalence of G and \tilde{G} and their explicit expressions from Proposition 3.21, it follows that for $\xi \in \Lambda^{1,0}$ we have

$$\Lambda^{-1}\tilde{g}^{-1}(\xi, \bar{\xi}) = \Lambda^{-1}\tilde{G}(\xi, \bar{\xi}) \leq G(\xi, \bar{\xi}) = g^{-1}(\xi, \bar{\xi}) \leq \Lambda\tilde{G}(\xi, \bar{\xi}) = \Lambda\tilde{g}^{-1}(\xi, \bar{\xi}).$$

This implies the claimed uniform equivalence of g and \tilde{g} . A similar argument using sections of the tangent bundle then yields the upper bound for $|\beta|_{\tilde{g}}$.

To estimate the connection, we first recall the computation of the connection coefficients in Lemma 3.22

$$(\Upsilon^G)_{iW^a}^{Z^b} = g^{\bar{c}b}g^{\bar{q}a}\overline{T}_{\bar{q}\bar{c}i} - \tilde{g}^{\bar{c}b}\tilde{g}^{\bar{q}a}\overline{T}_{\bar{q}\bar{c}i}.$$

Taking norms and using the uniform equivalence of g and \tilde{g} and the estimate for $\Upsilon(G, \tilde{G})$ we obtain

$$|T|_{\tilde{g}} \leq C.$$

Now turning to the tangent-tangent component of Υ^G from Lemma 3.22 we can express

$$(\Upsilon^g)_{ia}^b = (\Upsilon^G)_{iZ^a}^{Z^b} - \sqrt{-1}g^{\bar{c}b}g^{\bar{q}p}\beta_{ap}\overline{T}_{\bar{q}\bar{c}i} + \sqrt{-1}\tilde{g}^{\bar{c}b}\tilde{g}^{\bar{q}p}\tilde{\beta}_{ap}\overline{T}_{\bar{q}\bar{c}i}.$$

We have estimated all terms on the right hand side of this equation, thus the estimate for $\Upsilon(g, \tilde{g})$ follows. \square

Proof of Theorem 1.2. Let ω_F denote the given Bismut-flat metric, and let $\mathcal{Q}_{\sqrt{-1}\partial\omega_F}$ denote the holomorphic Courant algebroid associated to $[\partial\omega_F]$. Furthermore, let G_F denote the Hermitian metric on $\mathcal{Q}_{\sqrt{-1}\partial\omega_F}$ associated to ω_F via Proposition 3.21. Now given ω_0 another pluriclosed metric satisfying $[\partial\omega_0] = [\partial\omega_F] \in H_{\bar{\partial}}^{2,1}$, we can choose $\beta \in \Lambda^{2,0}$ such that

$$\bar{\partial}\beta = \partial\omega_F - \partial\omega_0.$$

Let G_0 denote the metric associated to (ω_0, β_0) as in Proposition 3.21. By [60], there exists $\epsilon > 0$ and a solution G_t to pluriclosed flow with initial data G_0 on $[0, \epsilon)$. Since ω_F is Bismut-flat, it follows from Proposition 3.20 that the Chern curvature of G_F vanishes, and thus we obtain from Proposition 5.6 the evolution equations

$$\left(\frac{\partial}{\partial t} - \Delta\right) \text{tr}_G G_F = -|\Upsilon(G_F, G)|_{g, G^{-1}, G_F}^2. \quad (5.3)$$

It follows from the maximum principle that, for any interval $[0, T]$ on which the solution exists,

$$\sup_{M \times [0, T]} \text{tr}_G G_F \leq \sup_{M \times \{0\}} \text{tr}_G G_F.$$

We note that for two generalized Hermitian metrics G, \tilde{G} on a fixed holomorphic Courant algebroid it follows that $\text{tr}_G \tilde{G} = \text{tr}_{\tilde{G}} G$ (see Lemma 5.4). Thus there exists a uniform constant $\Lambda > 0$ so that for any time t one has

$$\Lambda^{-1}G_F \leq G_t \leq \Lambda G_F. \quad (5.4)$$

Furthermore, let $\Upsilon = \Upsilon(G_t, G_F)$ as in Definition 5.2. Again using that the Chern curvature of G_F vanishes, it follows from Proposition 5.7 that

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\Upsilon|_{g,G}^2 = -|\nabla \Upsilon|_{g,G}^2 - |\bar{\nabla} \Upsilon + \bar{T} \cdot \Upsilon|_{g,G}^2. \quad (5.5)$$

It follows from the maximum principle that for any interval $[0, T]$ on which the solution exists,

$$\sup_{M \times [0, T]} |\Upsilon(G, G_F)|_{g,G}^2 \leq \sup_{M \times \{0\}} |\Upsilon(G, G_F)|_{g,G}^2.$$

Using Lemma 5.8 we thus obtain uniform equivalence and a C^1 bound for the classical objects (ω_t, β_t) . We can now argue as in [43, Theorem 1.2], there are uniform C^∞ estimates for G_t and g_t for all times. Thus the flow exists for all time, finishing the claim of long-time existence.

To show convergence we first note that by putting together (5.3) and (5.5), and using the uniform equivalence estimate of (5.4), it follows that there is a constant $A > 0$ so that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \left(t |\Upsilon(G, G_F)|_{g,G}^2 + A \operatorname{tr}_{G_F} G\right) \leq 0.$$

It follows from the maximum principle that for any time $t > 0$ one has

$$\sup_{M \times \{t\}} |\Upsilon(G, G_F)|_{g,G}^2 \leq \frac{A \sup_{M \times \{0\}} \operatorname{tr}_{G_F} G}{t}. \quad (5.6)$$

Using the uniform C^∞ estimates for G_t , every sequence of times $\{t_j\} \rightarrow \infty$ contains a subsequence which converges to a limiting metric G_∞ . By (5.6) it follows that $\Upsilon(G_\infty, G_F) = 0$, and thus G_∞ is Chern-flat. Choosing such a flat limit G_∞ , we can repeat the above analysis with G_F replaced by G_∞ . In particular, for a large time t such that $|G_t - G_\infty|_{G_\infty} \leq \epsilon$, we have

$$2n \leq \sup_{M \times \{t\}} \operatorname{tr}_{G_\infty} G \leq 2n + C\epsilon$$

for some uniform constant C . These inequalities will be preserved for all times larger than t by (5.3), and then the convergence to G_∞ follows. \square

5.4. Contractibility of the space of generalized Kähler structures

In this subsection we prove Corollary 1.4, regarding generalized Kähler structures on Bismut flat manifolds. Key to our argument is the Poisson tensor

$$\sigma = \frac{1}{2}g^{-1}[I, J]$$

associated to a generalized Kähler structure (g, I, J) . The tensor σ was discovered by Apostolov-Gauduchon-Grantcharov and Hitchin [4, 39] (cf. also [51]), and was shown to be constant along a solution to generalized Kähler-Ricci flow in [35, Corollary 1.5].

Proof of Corollary 1.4. Let (M^{2n}, ω_F, J) be a Bismut-flat manifold, and fix (g, I, J) a generalized Kähler structure on M such that $[\partial\omega_J] = [\partial\omega_F] \in H_{\bar{\partial}}^{2,1}$. By Theorem 1.2, the solution (g_t, I_t, J) to generalized

Kähler-Ricci flow with initial condition (g, I, J) exists for all time, and g_t converges to a Bismut-flat metric. Using [29, Lemma 9.27] we can express the evolution equation for I as

$$\frac{\partial}{\partial t} I = L_{\theta_I^\# - \theta_J^\#} I = \rho_B^J \cdot \sigma.$$

As the evolution of G_t is determined by ρ_B and G_t is converging to a limit G_∞ , it follows that I will also converge to a smooth limit I_∞ as claimed. As $I_t = \phi_t^* I$ and I_t has uniform C^∞ estimates, it follows that the diffeomorphisms ϕ_t also converge to a limit ϕ_∞ , and thus I_∞ is biholomorphic to I , as claimed. \square

5.5. Global existence on complex surfaces with $\kappa \geq 0$

In this subsection we prove Theorem 1.3. A key point is to find a background metric with certain curvature properties for every choice of \mathcal{Q} . To begin, we show that every class in $H_{BC}^{1,1}$ is represented by an invariant form. We build on an observation of Teleman, which gives an explicit characterization of the failure of the $\partial\bar{\partial}$ -lemma on compact complex surfaces.

Lemma 5.9 ([66]). *Let (M^4, J) be a compact complex surface, and let*

$$B_{\mathbb{R}}^{1,1} = \{\mu \in \Lambda_{\mathbb{R}}^{1,1} \mid \exists a \in \Lambda_{\mathbb{R}}^1 = da\}.$$

Then there exists an exact sequence

$$0 \rightarrow \sqrt{-1}\partial\bar{\partial}\Lambda_{\mathbb{R}}^0 \rightarrow B_{\mathbb{R}}^{1,1} \rightarrow \mathbb{R},$$

where the final map is the L^2 inner product with a pluriclosed metric.

Lemma 5.10. *Let (M^4, J) be a compact complex surface which is the total space of a holomorphic principal T^2 -bundle. Given ω a pluriclosed metric on M there exists a T^2 -invariant metric in $[\omega] \in H_A^{1,1}$.*

Proof. First choose $\hat{\omega}$ a T^2 -invariant pluriclosed metric on M , which always exists by averaging an arbitrary pluriclosed metric over the T^2 -action. We first use this to show that every class in $H_{BC}^{1,1}$ admits T^2 -invariant representatives. Now fix $[\phi] \in H_{BC}^{1,1}$. We can define

$$\hat{\phi} := \int_{g \in T^2} g^* \phi.$$

The form $\hat{\phi}$ is T^2 -invariant, and to show that $[\hat{\phi}] = [\phi] \in H_{BC}^{1,1}$, it suffices by a standard chain-homotopy argument to show that for X any vector field tangent to the T^2 -action, one has $L_X \phi \in \sqrt{-1}\partial\bar{\partial}\Lambda_{\mathbb{R}}^0$. By Lemma 5.9, it suffices to show that the L^2 inner product with $\hat{\omega}$ vanishes. Using Stokes Theorem and the Cartan formula we compute

$$\langle L_X \phi, \hat{\omega} \rangle_{\hat{\omega}} = \int_M L_X \phi \wedge \hat{\omega} = \int_M L_X (\phi \wedge \hat{\omega}) = \int_M di_X (\phi \wedge \hat{\omega}) = 0,$$

as required. Now knowing this, we assume that $\hat{\omega}$ is the average of ω over the T^2 -action, and show that $[\hat{\omega}] = [\omega] \in H_A^{1,1}$. It suffices to prove that the infinitesimal action preserves Aeppli cohomology classes, and for this we use that integration gives a perfect pairing between $H_A^{1,1}$ and $H_{BC}^{1,1}$. Thus we fix $[\phi] \in H_{BC}^{1,1}$ with a T^2 -invariant representative $\hat{\phi}$, and integrate by parts to conclude

$$\int_M L_X \omega \wedge \hat{\phi} = - \int_M \omega \wedge d(i_X \hat{\phi}) = - \int_M \omega \wedge L_X \hat{\phi} = 0,$$

as required. \square

Next we record a key lemma computing the Bismut curvature tensor of a T^2 -invariant pluriclosed metric. Such invariant metrics are described by the Kaluza-Klein ansatz (cf. [58, Definition 5.1]), and the curvature computation below is implicit in [58, Proposition 5.12].

Lemma 5.11. *Let (M^4, J) be a compact complex surface which is the total space of a holomorphic principal T^2 -bundle over a Riemann surface Σ . Let ω denote a T^2 -invariant pluriclosed metric on M , expressed as*

$$\omega = \pi^* \omega_\Sigma + \text{tr}_h \mu \wedge J\mu,$$

where ω_Σ is a metric on Σ , $\mu + J\mu$ is a Hermitian connection, and h is an inner product on \mathfrak{t}^2 . Then

$$R^B = \frac{1}{2} \left(R_{\omega_\Sigma} - |F_\mu|_{g_\Sigma, h}^2 \right) \pi^* \omega_\Sigma \otimes \pi^* \omega_\Sigma + h(d \text{tr}_{\omega_\Sigma} F_\mu, \cdot) \otimes \pi^* \omega_\Sigma.$$

Proof of Theorem 1.3. Fix (M^4, J) a minimal compact complex non-Kähler surface of Kodaira dimension $\kappa \geq 0$. It follows from the Kodaira classification that the universal cover of M is either $\mathbb{H} \times \mathbb{C}$ (if $\kappa = 1$) or $\mathbb{C} \times \mathbb{C}$ (if $\kappa = 0$). Furthermore, by [15, Lemmas 1, 2] or [70, §7], M admits a finite unramified covering which is an elliptic fiber bundle, and moreover admits a holomorphic principal T^2 action. Given a pluriclosed metric on M we can lift the initial data to this finite cover. As solutions to pluriclosed flow are unique for given initial data [60], the flow line preserves invariance under the group of deck transformations and thus to obtain regularity of flow lines on M it is equivalent to analyze the equation on this finite cover. Thus we only consider the case where $\pi : M^4 \rightarrow \Sigma$ is already a holomorphic T^2 -bundle over a compact Riemann surface Σ , where $\chi(\Sigma) < 0$ if $\kappa = 1$ and $\chi(\Sigma) = 0$ if $\kappa = 0$.

Fix ω_0 a pluriclosed metric on M . By Lemma 5.10, there exists a T^2 -invariant metric $\hat{\omega} = \pi^* \omega_\Sigma + \text{tr}_h \mu \wedge J\mu \in [\omega_0]$. We can modify the metric on Σ so that R_Σ is constant, and by Hodge theory further modify the principal connection μ to assume that $\text{tr}_{\omega_\Sigma} F_\mu$ is constant. These changes preserve the associated Aeppli cohomology class on M and so we assume without loss of generality that $\hat{\omega}$ satisfies these conditions. The metric $\hat{\omega}$ defines a holomorphic Courant algebroid $\mathcal{Q}_{\sqrt{-1}\partial\hat{\omega}}$ together with a generalized Hermitian metric \hat{G} . Now by construction we can choose $\beta \in \Lambda^{2,0}$ such that

$$\bar{\partial}\beta = \partial\hat{\omega} - \partial\omega_0.$$

Let G_0 denote the metric associated to (ω_0, β_0) as in Proposition 3.21. By [60], there exists $\epsilon > 0$ and a solution G_t to pluriclosed flow with initial data G_0 on $[0, \epsilon)$.

We first obtain a partial estimate on the metric using the fibration structure and the Schwarz Lemma. The holomorphic Courant algebroid \mathcal{Q} comes equipped with a natural holomorphic projection map onto $T_M^{1,0}$ which we denote $\pi_{\mathcal{Q}}$. We furthermore obtain from the fibration structure the holomorphic map $d\pi : T_M^{1,0} \rightarrow T_\Sigma^{1,0}$. Composing these yields the holomorphic map of vector bundles $\Phi = d\pi \circ \pi_{\mathcal{Q}} : \mathcal{Q} \rightarrow T_\Sigma^{1,0}$. It follows from the construction and Proposition 3.21 that

$$|\Phi|_{G, g_\Sigma}^2 = \text{tr}_\omega \pi^* \omega_\Sigma.$$

Furthermore, using Lemma 5.1 (where $A = A(G, g_\Sigma, \Phi)$ is defined by (5.2)) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{tr}_\omega \pi^* \omega_\Sigma &= \frac{\partial}{\partial t} |\Phi|_{G, g_\Sigma}^2 \\ &= \langle \Phi \circ S_g^G, \Phi \rangle_{G^{-1}, g_\Sigma} \\ &= \Delta_g |\Phi|_{G^{-1}, g_\Sigma}^2 - |A|_{g, G^{-1}, g_\Sigma}^2 + \langle S_g^{g_\Sigma} \circ \Phi, \Phi \rangle_{G^{-1}, g_\Sigma} \\ &= \Delta_g \operatorname{tr}_\omega \pi^* \omega_\Sigma - |A|_{g, G^{-1}, g_\Sigma}^2 + \frac{1}{2} R_\Sigma (\operatorname{tr}_\omega \pi^* \omega_\Sigma)^2. \end{aligned}$$

Note that by construction R_Σ is constant, either -2 or 0 depending on whether $\kappa = 1$ or 0 . By the maximum principle we conclude for any smooth existence time $T > 0$ the estimate

$$\sup_{M \times \{T\}} \operatorname{tr}_\omega \pi^* \omega_\Sigma \leq (C + \frac{1}{2} |R_\Sigma| T)^{-1}. \quad (5.7)$$

We next establish the uniform equivalence of the metrics G_t along the flow. Combining Proposition 5.6 with Proposition 3.20, the curvature computation of Lemma 5.11, and the estimate (5.7) we obtain for a topological constant λ ,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \operatorname{tr}_G \hat{G} &= - \left| \Upsilon(G, \hat{G}) \right|_{g, G^{-1}, \hat{G}}^2 + \operatorname{tr}_G \left\langle S_g^{\hat{G}} \cdot, \cdot \right\rangle_{\tilde{G}} \\ &= - \left| \Upsilon(G, \hat{G}) \right|_{g, G^{-1}, \hat{G}}^2 + \operatorname{tr}_G \langle \operatorname{tr}_g (\lambda \pi^* \omega_\Sigma \otimes \psi_* \pi^* \omega_\Sigma) \cdot, \cdot \rangle_{\hat{G}} \\ &\leq C (\operatorname{tr}_g \pi^* \omega_\Sigma) \operatorname{tr}_G \hat{G} \\ &\leq C \operatorname{tr}_G \hat{G}. \end{aligned}$$

By the maximum principle we conclude

$$\sup_{M \times \{T\}} \operatorname{tr}_G \hat{G} \leq e^{CT}.$$

This implies that G_t and \hat{G} are uniformly equivalent on any compact time interval, and from Proposition 5.7 we conclude

$$\left(\frac{\partial}{\partial t} - \Delta \right) \left| \Upsilon(G, \hat{G}) \right|_{g, G^{-1}, G}^2 \leq C \left(1 + \left| \Upsilon(G, \hat{G}) \right|_{g, G^{-1}, G}^2 \right).$$

By the maximum principle we obtain a uniform estimate for $\left| \Upsilon(G, \hat{G}) \right|_{g, G^{-1}, G}^2$ on any finite time interval, and the proof of long-time existence now concludes as in Theorem 1.2. \square

Remark 5.12. For ω_t a pluriclosed flow as in Theorem 1.3 one expects the blowdown limits $\frac{\omega_t}{2t}$ to converge with bounded curvature to either a point in the case $\kappa = 0$, or to (Σ, ω_Σ) in the case $\kappa = 1$, where ω_Σ denotes the unique conformal metric of curvature -1 . A basic fact in this direction is that pluriclosed flow preserves the area of the T^2 fibers, and thus the area of the fibers goes to zero along any blowdown sequence. Furthermore, assuming the solution satisfies type III curvature and diameter bounds, this kind of limiting behavior can be derived by the use of an expanding entropy functional [36]. Assuming the initial metric is T^2 -invariant this behavior was shown in [53].

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