



On the coexistence of competing memes in the same social network

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ABSTRACT

We present a model of double viral spreading on a multi-network suitable for modeling the diffusion of competitive memes. The main characteristics of the model is that it allows the possibility of changing from one opinion to another without having to go through an agnostic (susceptible) phase.

We analyze the survival of two conflicting memes in the limit of $t \rightarrow \infty$ as a function of the virilities of the two memes as well as the rate at which an agent can change from believing one meme to believing the other. The parameter space is divided into four survival regions, depending on which meme survives (none, one of the two, both). We derive exact equations for the boundaries between the regions in the case of regular graphs, and, in the general case, we characterize the behavior for weak and strong memes.

One relevant finding is that, contrary to the viral spread model in Sahneh and Scoglio (2014), the crossover allows the co-existence of two memes in the same network. We present simulations to validate the theoretical results.

1. Introduction

Multiple viral spreading on one network, or on multiple networks defined on the same nodes, has attracted substantial attention thanks to the rich dynamics it generates [1–3]. The idea and the models of viral spreading originated in the study of the diffusion of pathogens [4,5], and from there they spreaded to the analysis of many other problems that exhibit a similar dynamics, from product diffusion [6], to worm propagation in computer networks [7].

In this paper we use viral spread models to analyze the spread of two contrasting memes, which we indicate as P and Q . We assume that the two are contradictory, so that an agent cannot reasonably hold both true at the same time. Our model is based on the two-virus propagation model of [8], but to that model we add a possibility typical of memes: that of changing opinion. In other word, we assume that an agent in state Q (viz., that believes meme Q to be true) may move in a single step to state P (viz., may believe that P is true and Q false) without going through a susceptible state in which it believes none of the memes (a state that we indicate as S). This possibility has never been contemplated in virus models, as no model considers a virus that, upon infecting an agent, cures it from another virus.

Work on information diffusion in social networks can be roughly divided in two groups [9]. On the one hand, there are models of opinion formation [10,11]. Opinion is normally modeled as a stochastic process with a continuous range. In general, for individual i , the opinion at time t is $o_i(t) \in [-1, 1]$, varying continuously between two “extremist” positions with $o_i(t) = 0$ representing either a centrist or an indifferent position. The main techniques in this area are game-theoretic [12], studying in particular the existence of Nash equilibria [13], or kinetic exchange models [14,15], which build on equilibria of gas dynamic models [16]. Different approaches consider $o_i \in \{0, 1\}$ and apply Ising, voting or Sznadaj [17] models. Variations may be introduced, such as the presence of a recommender system [10] or the presence of information sources of fixed opinion with confirmation bias [18].

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¹ Simone Santini was supported in part by the Spanish Ministerio de Ciencia e Innovacion under the grant N. PID2019-108965GB-I00 *Más allá de la recomendación estática: equidad, interacción y transparencia*.

On the other hand, we have the study of meme spreading [19,20]. In this case, meme acceptance is assumed to be a binary stochastic process: at any time, each individual either believes a meme or does not. Several techniques have been used in this area, from game-theoretic models [21], to experimental studies [22], to viral models [23]. One of the most significant and general results for a single virus spread is the existence of a spreading threshold, which depends on the epidemiological constants as well as on the structure of the network. If β is the infection rate of the virus, δ the recovery rate, and $\tau = \beta/\delta$ is the effective infection rate, then the virus spreads if $\tau > 1/\lambda_1$, where λ_1 is the largest eigenvalue of the adjacency matrix of the network [23]. This result shows that vulnerability to viral spread is a characteristic of the network: λ_1 is never smaller than the mean degree of the network, limiting the resiliency of networks with many links. For example, the Erdős-Renyi graph [24] have a larger spectral radius than scale-free graphs [25] with the same average connectivity, and are therefore, *ceteris paribus*, more vulnerable to viral spread.

The model in which we are interested in this paper considers the spread of *two* contradictory memes. Here, as in most two-virus models, we assume that the two spread on different networks defined on the same agents. That is, the model is a multigraph $G = (V, E_P, E_Q)$, with $E_P \in V \times V$ and $E_Q \in V \times V$.

Several techniques have been used for the characterization of competitive diffusion. *Bond percolation* [26] assumes that an infected person has only one chance to infect its neighbors and that once infected it never becomes susceptible again. In this way, one can model the problem as a random removal of graph edges, and the spread of the virus can be modeled as percolation in the resulting graph. The original model [26] assumed that the viruses spread one at the time on the same network, but the model has been extended to simultaneous spread [1] as well as to two-layer networks [2].

A second class of models – among which the one that we consider in this paper – considers viral spreading as a Markov model in which the state is given by the epidemiological status of each agent: susceptible (S), infected by one virus (P) or infected by the other (Q). The complete model is intractable: on a network of N individuals, the model has 3^N states [27,28]. In order to make it tractable, approximations such as mean field are used [8,23].

Accurately modeling multiple viral diffusion is a complex problem due to the different manners in which the spreading elements may interact: they may reinforce each other [29], weaken each other [30], or exclude each other [26]; the relation between them may be symmetrical or asymmetrical [3,31]. The most common simplifying assumption is that of the $SPSQSS$ model [8]²: two viruses, P and Q , act independently on susceptible (healthy) individuals, and they are synchronically exclusive (an agent infected by P cannot at the same time be infected by Q and vice versa), as well as diachronically (an agent infected by a virus must go back to the susceptible state before it can be infected by the other). The main result of these methods are related to the possible coexistence of the two viruses vis-à-vis the dominance of one of them. Wang et al. [32] proved that competitive, exclusive viruses cannot coexist in scale-free networks, and Prakash et al. [33] extended the result to arbitrary networks. Darabi and Scoglio [8] proved that the two viruses may coexist if they spread on different networks ($E_P \neq E_Q$) but coexistence on the same network is impossible.

The model presented in this paper can be described as $SPQS$, as it allows direct transition between the states P and Q without going through S . This model is consistent with the fact that an agent may instantly change its mind from believing P to believing Q and vice-versa, without having to go through the state S . This model is an extension of the *fact checking* one proposed in [19]. We show that the extinction threshold is the same as in $SPSQS$, but in $SPQS$ the two memes can coexist in the same network.

A final observation. Our model is intended to model the interaction between agents that leads to possible changes of opinion. Consequently, we are interested in the act of proselytizing. That is, an agent if, for example, in state P it believes P and makes this belief public, so that its neighbors may be influenced. This is consistent with behavior on social networks: what spreads a meme is not the personal belief of their participants, but the fact that these beliefs are made public [34–37].

2. The model

We use a model based on competitive, exclusive memes spreading over a two-layer network, originally proposed in [38].

Consider a population of N agents, and two memes that propagate along different routes (e.g., one meme may propagate through broadcast radio and the other through word of mouth). This corresponds to a multi-network $G = (V, E_P, E_Q)$, where $V = \{1, \dots, N\}$ is the set of individuals, $E_P \subseteq V \times V$ is the set of links along which the first meme propagates (henceforth: meme 1 or meme p), and $E_Q \subseteq V \times V$ is the set of links along which the second meme propagates (henceforth: meme 2 or meme q) (Fig. 1).

We shall on occasion use the “projections” of the network on its two component graphs $G_p = (V, E_p)$ and $G_q = (V, E_q)$. Define the adjacency matrices $\mathbf{A} = \{a_{ij}\}$ and $\mathbf{B} = \{b_{ij}\}$ with elements

$$a_{ij} = \begin{cases} 1 & (i, j) \in E_p \\ 0 & \text{otherwise} \end{cases} \quad b_{ij} = \begin{cases} 1 & (i, j) \in E_q \\ 0 & \text{otherwise} \end{cases}$$

We assume that the graphs are undirected ($\mathbf{A} = \mathbf{A}'$ and $\mathbf{B} = \mathbf{B}'$). Let $d_i^A = \sum_j a_{ij}$ be the \mathbf{A} -degree of node i , set $\mathbf{d}^A = [d_1^A, \dots, d_N^A]'$, $\mathbf{D}_A = \text{diag}(d_1^A, \dots, d_N^A)$. Let $\lambda_A = \lambda_1(\mathbf{A})$ be the largest eigenvalue of \mathbf{A} , and $\mathbf{a} = [a_1, \dots, a_N]'$ be the corresponding unitary eigenvector. The corresponding values for \mathbf{B} : d_i^B , \mathbf{d}^B , \mathbf{D}_B , λ_B , and \mathbf{b} are defined analogously.

The individuals are modeled using a $SPQS$ model: each node of the network is, at any time, in one of three possible states (S , P , Q) that, in the meme scenario, are interpreted as follows:

² In [8], the model is called SI_1SI_2S , I_1 and I_2 being the two viruses. We have changed the denomination here to make it consistent with that used in this paper.

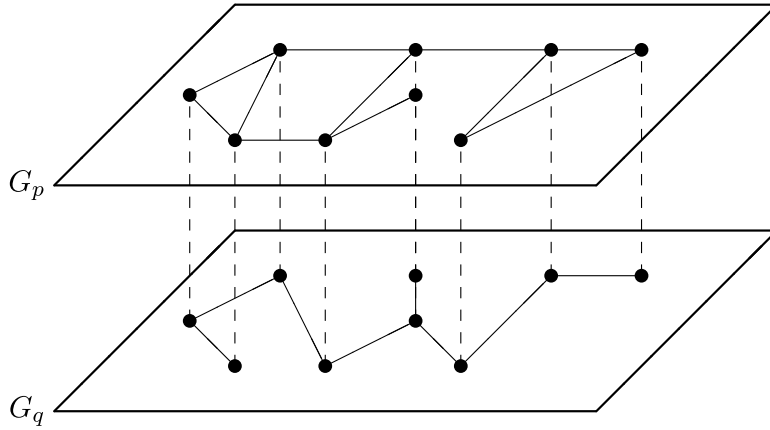


Fig. 1. Schematic view of the multi-network model. The model is composed of a set of nodes (V) and two independent sets of edges between them (E_p and E_q). The model can be projected on either set of edges obtaining the propagation graphs $G_p = (V, E_p)$ and $G_q = (V, E_q)$.

S susceptible. The agent is not actively spreading any of the two memes.

P p -infected. The agent spreads meme p (and, consequently, because of synchronic exclusivity, does not spread meme q).

Q q -infected. The symmetric relation: the agent spreads meme q and not p .

No agent can spread p and q simultaneously. In our model, belief is active, and includes proselytism, so agents that (actively) believe a meme will try to convince their neighbors. Agents that privately believe the meme but do not communicate it and do not proselytize will be in state S . The basic process is similar to that used in [8] for the case of viruses and [19] for memes. The infection and curing processes for the two memes are characterized by the parameters (β_p, δ_p) and (β_q, δ_q) respectively. The process is represented schematically in Fig. 2: Let an agent have, at time t , $N_p(t)$ neighbors that believe p and $N_q(t)$ that believe q . Then the agent can change state according to the following processes:

From state S

- a p -believing process: Poisson process with rate $\beta_p N_p(t)$ leading to state P ;
- a q -believing process: Poisson process with rate $\beta_q N_q(t)$ leading to state Q .

From state P

- a disbelieving process, with rate δ_p , leading to state S ;
- a change-to- Q process with rate $\beta_{pq} N_q(t)$ leading to state Q .

From state Q

- a disbelieving process, with rate δ_q , leading to state S ;
- a change-to- p process with rate $\beta_{qp} N_p(t)$ leading to state P .

Consistently with our interpretation, the β s are the instantaneous probabilities that an agent begin to spread an opinion, while the δ s those that an agent will cease to proselytize about the opinion.

We shall consider that β_{qp} is proportional to β_p and that β_{pq} is proportional to β_q : $\beta_{qp} = \alpha_q \beta_p$, and $\beta_{pq} = \alpha_p \beta_q$. The rationale here is that if a meme is “strong”, in the sense that it may convince agents without any specific belief, it is also “strong” in the sense that it can convince agents to change their mind. The parameters α_p and α_q will be assumed to be constant.

We shall consider $\alpha_p \leq 1$ and $\alpha_q \leq 1$, implying that it is harder to change from one meme to another than to become susceptible. One case of special interest will be that in which one of the two parameters α_p or α_q vanishes. If we assume, for example, that p is truthful information and q a piece of fake news, setting $\alpha_q = 0$ entails that agents who believe the fake news can verify it and change their mind, while agents who believe the truth will not find any factual support for, and never switch directly to, falsehood.

The situation is symmetric so, for the sake of simplification, we shall always assume that p is the truth, so $\alpha_p > 0$ always, while α_q can be equal to zero.

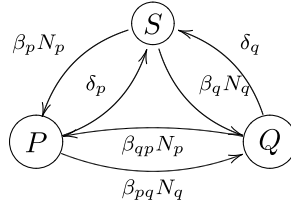


Fig. 2. The transition diagram for an agent in the *SPQS* model. The agent can be in one of three states: *susceptible* (*S*: not propagating either meme), propagating meme *P*, or propagating meme *Q*.

Define the *effective infection rates* as

$$\tau = \frac{\beta_p}{\delta_p} \quad \theta = \frac{\beta_q}{\delta_q} \quad (1)$$

and the *recovery ratio* as

$$\xi = \frac{\delta_p}{\delta_q}. \quad (2)$$

In all the following, we shall consider situations in which τ and θ are variables, but they change in such a way that ξ remains constant. The effective infection rate of a meme is the average number of agents a proselytizer convinces before ceasing proselytism [39], that is, it is a measure of the penetration power of a meme.

The model is a Markov process with a state-space given by the set of states of all the nodes. The model quickly becomes intractable for large networks, since the state space grows as $\Theta(3^N)$. The problem can be simplified with suitable approximations. In particular, let $s_i(t)$, $p_i(t)$ and $q_i(t)$ are the probabilities that node i be in state *S*, *P*, *Q* (respectively) at time t . For future use, we define the vectors $s = [s_1, \dots, s_N]'$, $p = [p_1, \dots, p_N]'$ and $q = [q_1, \dots, q_N]'$.

In the mean field approximation [8,40], we can write the evolution equations for p and q as

$$\begin{aligned} \dot{p}_i &= [\beta_p s_i + \beta_{qp} q_i] \sum_{j=1}^N a_{ij} p_j - \beta_{pq} p_i \sum_{j=1}^N b_{ij} q_j - \delta_p p_i \\ \dot{q}_i &= [\beta_q s_i + \beta_{pq} p_i] \sum_{j=1}^N b_{ij} q_j - \beta_{qp} q_i \sum_{j=1}^N a_{ij} p_j - \delta_q q_i \end{aligned} \quad (3)$$

or, with the definitions in (1) and (2),

$$\begin{aligned} \frac{1}{\delta_p} \dot{p} &= \tau [s_i + \alpha_q q_i] \sum_{j=1}^N a_{ij} p_j - \theta \frac{\alpha_p}{\xi} p_i \sum_{j=1}^N b_{ij} q_j - p_i \\ \frac{1}{\delta_q} \dot{q} &= \theta [s_i + \alpha_p p_i] \sum_{j=1}^N b_{ij} q_j - \tau \alpha_q \xi q_i \sum_{j=1}^N a_{ij} p_j - q_i \end{aligned} \quad (4)$$

The derivation of the mean field equation for a Markov model on a graph is detailed in Appendix A.

The two equations are symmetric: one can be transformed into the other via the transformation

$$p_i \leftrightarrow q_i \quad \alpha_p \leftrightarrow \alpha_q \quad \tau \leftrightarrow \theta \quad \xi \leftrightarrow \frac{1}{\xi}. \quad (5)$$

This transformation will be useful in the following to extend properties and results from one meme to the other.

3. Types of equilibria

The equilibrium solution for the two equations is given by

$$\begin{aligned} p^* &= \tau [(1 - p_i - q_i) + \alpha_q q_i^*] \sum_{j=1}^N a_{ij} p_j^* - \theta \frac{\alpha_p}{\xi} p_i^* \sum_{j=1}^N b_{ij} q_j^* \\ q^* &= \theta [(1 - p_i - q_i) + \alpha_p p_i^*] \sum_{j=1}^N b_{ij} q_j^* - \tau \alpha_q \xi q_i^* \sum_{j=1}^N a_{ij} p_j^*, \end{aligned} \quad (6)$$

where we have used $s_i + p_i + q_i = 1$. We are interested in studying the properties of these equilibrium solutions. We consider four classes of equilibria:

I meme extinction equilibrium: none of the two memes survives in the long term and they both die out as $t \rightarrow \infty$, that is, in term of the stable solution, $p_i^* = q_i^* = 0$;

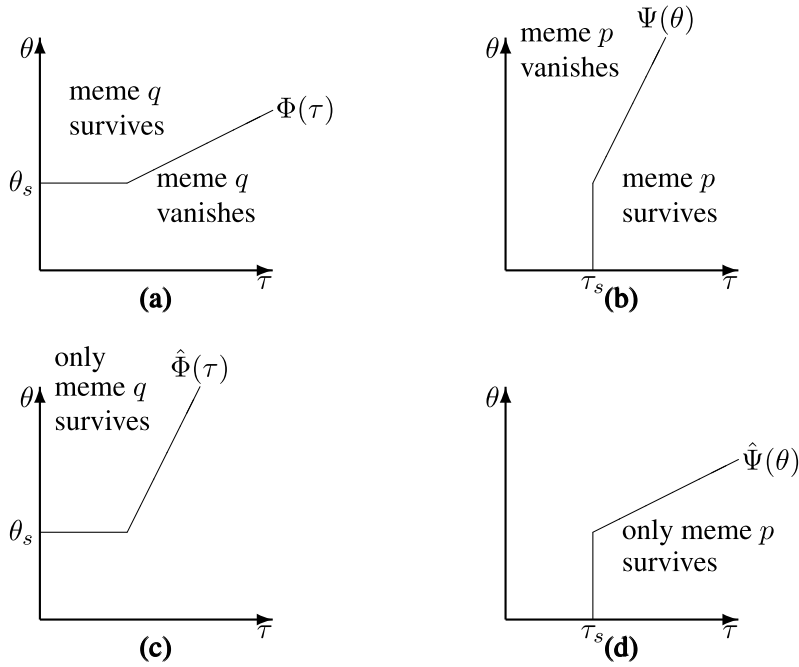


Fig. 3. Qualitative illustration of survival and absolute dominance. In the area $\tau < \tau_s$ and $\theta < \theta_s$ both memes vanish. In (a), meme q survives if its effective infection rate is $\theta > \Phi(\tau)$, and vanishes otherwise. Similarly, in (b), meme p survives if $\tau > \Psi(\theta)$ and vanishes otherwise. In (c), whenever $\theta > \hat{\Phi}(\tau)$, meme q dominates absolutely, that is, meme p vanishes and, similarly in (d) if $\tau > \hat{\Psi}(\theta)$, meme p dominates and meme q vanishes. The figure illustrates how the four curves Ψ , $\hat{\Psi}$, Φ , and $\hat{\Phi}$ are not independent, as the region of absolute dominance of a meme corresponds to the region in which the other vanishes.

II p dominance: the meme p dominates until, at $t \rightarrow \infty$, q disappear. In this case, $q_i^* = 0$ and p_i^* is given by

$$\frac{p_i^*}{1 - p_i^*} = \tau \sum_{j=1}^N a_{ij} p_j^*; \quad (7)$$

III q dominance: symmetric to the previous one. Now $p_i^* = 0$ and q_i^* is given by

$$\frac{q_i^*}{1 - q_i^*} = \theta \sum_{j=1}^N b_{ij} q_j^*; \quad (8)$$

IV coexistence: in this case both memes remain active as $t \rightarrow \infty$, that is $p_i^* \geq 0$ and $q_i^* \geq 0$

For each value of τ , there are two critical values of θ , which we indicate as $\theta' = \Phi(\tau)$ and $\theta'' = \hat{\Phi}(\tau)$. The first value is the *survival threshold*: given an effective infection rate τ for meme p , if $\theta < \Phi(\tau)$, meme q vanishes. The second is the *absolute dominance threshold*: given a value of τ for p , if $\theta > \hat{\Phi}(\tau)$ q dominates and p vanishes. Similar thresholds are defined for τ : $\tau' = \Psi(\theta)$, $\tau'' = \hat{\Psi}(\theta)$.

In addition to these, there are the *absolute survival thresholds* τ_s and θ_s , defined by $\tau_s = \Psi(\theta_s)$, $\theta_s = \Phi(\tau_s)$ below which a meme vanishes independently of the presence and the behavior of the other. If $\tau < \tau_s$, then p vanishes unconditionally, therefore

$$\tau < \tau_s \implies \Phi(\tau) = \hat{\Phi}(\tau) = \theta_s$$

and, similarly,

$$\theta < \theta_s \implies \Psi(\theta) = \hat{\Psi}(\theta) = \tau_s$$

The values $\Phi(\tau)$, $\hat{\Phi}(\tau)$, $\Psi(\theta)$, and $\hat{\Psi}(\theta)$ are not independent: beyond the absolute survival threshold, q survives if and only if p does not dominate, and vice-versa. That is, given a value $\tau > \tau_s$, the condition $\theta < \Phi(\tau)$ determines at the same time the vanishing of q and the dominance of p , and the same holds for $\tau < \Psi(\theta)$. The situation is depicted schematically in Fig. 3. The figure shows that the region of absolute dominance of one meme corresponds to the region (beyond the absolute survival threshold) where the other vanishes. That is, for $\tau > \tau_s$ and $\theta > \theta_s$, we have

$$\hat{\Psi}(\theta) = \Phi^{-1}(\theta)$$

$$\hat{\Phi}(\tau) = \Psi^{-1}(\tau).$$

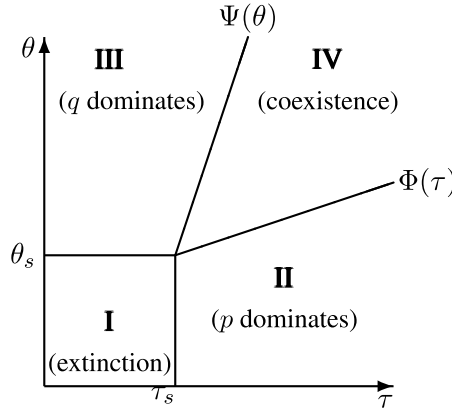


Fig. 4. The four regions describing the possible outcomes of the interaction between the two memes. In region I both memes die out, in region II meme p dominates and meme q dies out, in region III the opposite is true: q dominates and p dies out; finally, in region IV there is coexistence: both memes survive. The situation can be described completely once we determine the values θ_s and τ_s and the curves $\Phi(\tau)$ and $\Psi(\theta)$.

In the following, we shall consider only the curves $\Phi(\tau)$ and $\Psi(\theta)$. These considerations also give us a qualitative schema for the relation between the four regions I–IV mentioned above (Fig. 4).

3.1. Extinction of both memes

The analysis of region I is quite similar to the analysis carried out in [8]. Region I is the region of stability of the solution $p_i = q_i = 0$. Consider first $q_i = 0$. In this case the solution for p_i is the solution of

$$\dot{p}_i = \tau(1 - p_i) \sum_j a_{ij} p_j - p_i,$$

that is,

$$\dot{p} = [\tau \text{diag}(1 - p_i) \mathbf{A} - \mathbf{I}] p.$$

The solution $p = 0$ is stable if all the eigenvalues of $[\tau \text{diag}(1 - p_i) \mathbf{A} - \mathbf{I}]|_{p=0}$ are negative,³ that is if $\lambda_1(\tau \mathbf{A} - \mathbf{I}) < 0$, where λ_1 is the largest eigenvalue of the matrix. This leads to

$$\tau_s = \frac{1}{\lambda_{\mathbf{A}}}.$$

Reasoning in the same way with $p = 0$, the equation for q leads to

$$\theta_s = \frac{1}{\lambda_{\mathbf{B}}}.$$

3.2. Border of regions II and III

Consider region II, in which $p_i \geq 0$, $q_i = 0$. Since $q_i = 0$, the equilibrium solution of p is determined by (7). The border between region II and IV is given by the stability of the solution $q_i = 0$ under the condition that p be given by (7). We write the equation for q in the form $\dot{q} = G(q)$, with, from (4), and writing $s_i = 1 - p_i - q_i$, we have

$$G_i(q) = \theta [1 - q_i - (1 - \alpha_p) p_i] \sum_{j=1}^N b_{ij} q_j - \alpha_q \xi \tau \sum_{j=1}^N a_{ij} p_j - q_i.$$

The stability is given in this case by the eigenvalues of ∇G computed in $q = 0$. To determine $\nabla G_i|_0$ we compute

$$\begin{aligned} \left. \frac{\partial G_i}{\partial q_j} \right|_0 &= \theta [1 - (1 - \alpha_p) p_i] b_{ij} - \alpha_q \xi \tau \sum_{j=1}^N a_{ij} p_j \delta_{ij} - \delta_{ij} \\ &= \theta [1 - (1 - \alpha_p) p_i] b_{ij} - \alpha_q \xi \frac{p_i}{1 - p_i} \delta_{ij} - \delta_{ij}, \end{aligned}$$

³ Technically, if they have negative real part. However, the matrices involved are symmetric and therefore all their eigenvalues are real.

from which

$$\nabla G|_0 = \theta [\mathbf{I} - (1 - \alpha_p) \text{diag}(p_i)] \mathbf{B} - \alpha_q \xi \text{diag}\left(\frac{p_i}{1 - p_i}\right) - \mathbf{I}.$$

The stability condition $\lambda_1(\nabla G) < 0$ leads to the implicit equation for the boundary $\Phi(\tau)$

$$\lambda_1 [\Phi(\tau) (\mathbf{I} - (1 - \alpha_p) \text{diag}(p_i)) \mathbf{B} - \alpha_q \xi \text{diag}\left(\frac{p_i}{1 - p_i}\right)] = 1. \quad (9)$$

Symmetry and the application of (5) lead to an implicit equation for $\Psi(\theta)$:

$$\lambda_1 [\Psi(\theta) (\mathbf{I} - (1 - \alpha_q) \text{diag}(q_i)) \mathbf{A} - \frac{\alpha_p}{\xi} \text{diag}\left(\frac{q_i}{1 - q_i}\right)] = 1. \quad (10)$$

4. Guiding example: regular graphs

Eqs. (9) and (10) cannot in general be solved analytically, and in the following section we shall do a qualitative study for $\tau \sim \frac{1}{\lambda_A}$, $\theta \sim \frac{1}{\lambda_B}$ as well as for the cases $\tau \rightarrow \infty$ and $\theta \rightarrow \infty$.

Before we do that, however, in this section, we do a quantitative study of a special case, namely that of *regular graphs*. This example will allow us to do some qualitative considerations of general validity, and will work as a special case on which we shall build the more general considerations of the following section.

Assume that the two sets of edges A and B are regular, so that in A each node has m_a neighbors, and in B each node has m_b . In terms of the matrices \mathbf{A} and \mathbf{B} , this entails that $\sum_j a_{ij} = m_a$ and $\sum_j b_{ij} = m_b$ for all i .

Let $p = p\mathbf{1}$, with $p > 0$ and $\mathbf{1} = [1, 1, \dots, 1]'$ and, similarly, $q = q\mathbf{1}$. Then

$$\mathbf{A}p = p\mathbf{A}\mathbf{1} = pm_a\mathbf{1} = m_ap$$

$$\mathbf{B}q = q\mathbf{B}\mathbf{1} = qm_b\mathbf{1} = m_bq,$$

that is, m_a is an eigenvalue of \mathbf{A} and m_b of \mathbf{B} . The Perron–Frobenius theorem [41] entails that $m_a = \lambda_A$ and $m_b = \lambda_B$.

For $\tau < \frac{1}{m_a}$, $\theta < \frac{1}{m_b}$ we are in region I and both memes vanish. Consider now region II. From (7) we have

$$p = 1 - \frac{1}{\tau m_a} = 1 - \frac{1}{\tau \lambda_A},$$

therefore

$$\frac{p_i}{1 - p_i} = \tau \lambda_A - 1,$$

and

$$\text{diag}(p_i) = \left(1 - \frac{1}{\tau \lambda_A}\right) \mathbf{I},$$

$$\text{diag}\left(\frac{p_i}{1 - p_i}\right) = (\tau \lambda_A - 1) \mathbf{I}.$$

With this we rewrite (9) as

$$\lambda_1 [\Phi(\tau) \frac{1}{\tau \lambda_A} [1 + \alpha_p(\tau \lambda_A - 1)] \mathbf{B} - \alpha_q \xi (\tau \lambda_A - 1) \mathbf{I}] = 1,$$

which leads to

$$\Phi(\tau) = \frac{1}{1 + \alpha_p(\tau \lambda_A - 1)} \left[\tau \frac{\lambda_A}{\lambda_B} [1 + \alpha_q \xi (\tau \lambda_A - 1)] \right] \quad (11)$$

and, through (5):

$$\Psi(\theta) = \frac{1}{1 + \alpha_q(\theta \lambda_B - 1)} \left[\theta \frac{\lambda_B}{\lambda_A} \left[1 + \frac{\alpha_p}{\xi} (\theta \lambda_B - 1) \right] \right]. \quad (12)$$

The derivatives of Φ and Ψ at the survival critical point ($\tau = \tau_s$ and $\theta = \theta_s$) determine the existence of a critical region. We have

$$\frac{d}{d\tau} \Phi = \frac{-\alpha_p \lambda_A}{[1 + \alpha_p(\tau \lambda_A - 1)]^2} \tau \frac{\lambda_A}{\lambda_B} [1 + \alpha_q \xi (\tau \lambda_A - 1)] + \frac{1}{1 + \alpha_p(\tau \lambda_A - 1)} \frac{\lambda_A}{\lambda_B} [1 + 2\alpha_1 \xi \lambda_A \tau - \alpha_q \xi]$$

and

$$\left. \frac{d\Phi}{d\tau} \right|_{\tau=1/\lambda_A} = \frac{\lambda_A}{\lambda_B} [\alpha_q \xi - \alpha_p + 1]. \quad (13)$$

Finally, applying (5):

$$\left. \frac{d\Psi}{d\theta} \right|_{\theta=1/\lambda_B} = \frac{\lambda_B}{\lambda_A} \left[\frac{\alpha_p}{\xi} - \alpha_q + 1 \right].$$

We use these results to determine the existence of a zone of coexistence (zone IV) in two cases: *weak* memes (memes near the vanishing points, viz., $\tau \sim 1/\lambda(A)$ and $\theta \sim 1/\lambda(B)$) and *aggressive* memes ($\tau \rightarrow \infty$ and $\theta \rightarrow \infty$).

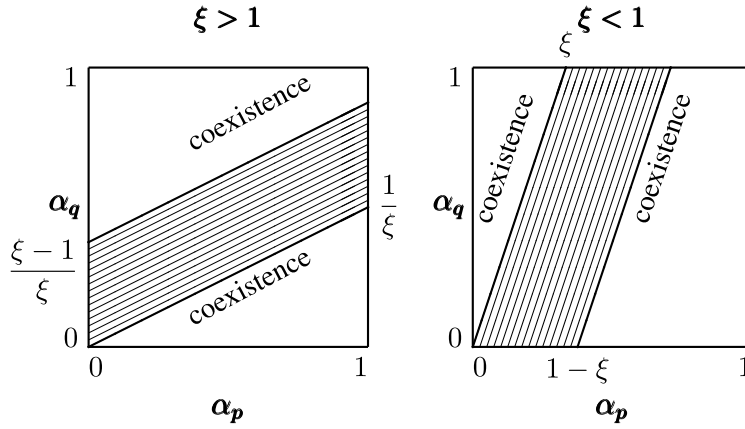


Fig. 5. Regions in the space α_p, α_q that allow the coexistence of two “weak” memes ($\tau \sim 1/\lambda(\mathbf{A})$, $\theta \sim 1/\lambda(\mathbf{B})$) in a regular graph. In the shaded region, no coexistence is possible: one of the two memes will dominate and the other will vanish. The first figure refers to the case $\xi > 1$, viz, $\delta_p > \delta_q$, the second to the case $\xi < 1$.

Consider weak memes first. There is coexistence if

$$\frac{d\Phi}{d\tau} \frac{d\Psi}{d\theta} < 1,$$

that is

$$(\alpha_q \xi - \alpha_p + 1) \left(\frac{\alpha_p}{\xi} - \alpha_q + 1 \right) < 1, \quad (14)$$

which gives:

$$\left[\frac{1}{\xi} + \left(\alpha_q - \frac{\alpha_p}{\xi} \right) \right] \left[1 - \left(\alpha_q - \frac{\alpha_p}{\xi} \right) \right] < \frac{1}{\xi},$$

leading to

$$\left(\alpha_q - \frac{\alpha_p}{\xi} \right) \left[\frac{\xi - 1}{\xi} - \left(\alpha_q - \frac{\alpha_p}{\xi} \right) \right] < 0. \quad (15)$$

If $\xi > 1$, the solutions are

$$\begin{aligned} \alpha_q &< \frac{\alpha_p}{\xi} \\ \alpha_q &> \frac{\alpha_p}{\xi} + \frac{\xi - 1}{\xi} \end{aligned} \quad (16)$$

while for $\xi < 1$

$$\begin{aligned} \alpha_q &> \frac{\alpha_p}{\xi} \\ \alpha_q &< \frac{\alpha_p}{\xi} - \frac{1 - \xi}{\xi}. \end{aligned} \quad (17)$$

The two coexistence regions are illustrated in Fig. 5. Note that if $\xi = 1$ all values of α_p and α_q permit coexistence.

For aggressive memes, if $\tau \lambda_A \gg 1$ and $\theta \lambda_B \gg 1$ we can approximate (11) as

$$\Phi(\tau) \sim \begin{cases} \frac{\lambda_A}{\lambda_B} \tau + \alpha_q \xi \frac{\lambda_A^2}{\lambda_B} \tau^2 & \alpha_p = 0 \\ \frac{1}{\alpha_p \lambda_B} + \frac{\alpha_q}{\alpha_p} \frac{\lambda_A}{\lambda_B} \xi \tau & \alpha_p > 0 \end{cases}$$

and (12) as

$$\Psi(\tau) \sim \begin{cases} \frac{\lambda_B}{\lambda_A} \theta + \frac{\alpha_p}{\xi} \frac{\lambda_B}{\lambda_A} \theta^2 & \alpha_q = 0 \\ \frac{1}{\alpha_q \lambda_A} + \frac{\alpha_p}{\alpha_q} \frac{\lambda_B}{\lambda_A} \frac{1}{\xi} \theta & \alpha_q > 0. \end{cases}$$

We have to consider four cases.

$$\alpha_p = \alpha_q = 0$$

This reduces our model to that in [8]. In this case

$$\begin{aligned}\Phi(\tau) &\sim \frac{\lambda_A}{\lambda_B} \tau \\ \Psi(\theta) &\sim \frac{\lambda_B}{\lambda_A} \theta.\end{aligned}$$

The two lines are parallel, therefore there is a coexistence region only if there was one for weak memes. In particular, we find again the result in [8]: if the two memes spread on the same graph ($\mathbf{A} = \mathbf{B}$) there can be no coexistence.

$$\alpha_p = 0, \alpha_q > 0$$

In this case

$$\begin{aligned}\Phi(\tau) &\sim \frac{\lambda_A}{\lambda_B} \tau + \alpha_q \xi \frac{\lambda_A^2}{\lambda_B} \tau^2 \\ \Psi(\theta) &\sim \frac{1}{\alpha_q \lambda_A}.\end{aligned}$$

and there is always a coexistence region

$$\alpha_q = 0, \alpha_p > 0$$

This case is analogous to the previous one: there is always coexistence.

$$\alpha_p > 0, \alpha_q > 0$$

In this case

$$\begin{aligned}\Phi(\tau) &\sim \frac{1}{\alpha_p \lambda_B} + \frac{\alpha_q}{\alpha_p} \frac{\lambda_A}{\lambda_B} \xi \tau \\ \Psi(\theta) &\sim \frac{1}{\alpha_q \lambda_A} + \frac{\alpha_p}{\alpha_q} \frac{\lambda_B}{\lambda_A} \frac{1}{\xi} \theta.\end{aligned}$$

Here too we have two parallel lines but, since the constant terms are always positive, there is always a coexistence region.

5. The general case

We now consider the case of a general multi-network, beginning with weak memes, near the critical point. Considering $\Phi(\tau)$, near the critical point we have

$$\Phi(\tau) = \frac{1}{\lambda_B} + \frac{d}{d\tau} \Phi \Big|_{1/\lambda_A} \left(\tau - \frac{1}{\lambda_A} \right) + O \left(\left(\tau - \frac{1}{\lambda_A} \right)^2 \right). \quad (18)$$

and similarly for $\Psi(\theta)$. The derivation in Appendix 6 yields

$$\frac{d\Phi}{d\tau} = \frac{\lambda_A}{\lambda_B} [1 - \alpha_p + \xi \alpha_q] \frac{\sum_{i=1}^N a_i b_i^2}{\sum_{i=1}^N b_i^3}. \quad (19)$$

and, through (5),

$$\frac{d\Psi}{d\theta} = \frac{\lambda_B}{\lambda_A} [1 - \alpha_q + \frac{\alpha_p}{\xi}] \frac{\sum_{i=1}^N b_i a_i^2}{\sum_{i=1}^N a_i^3}. \quad (20)$$

where $\mathbf{a} = [a_1, \dots, a_N]'$ and $\mathbf{b} = [b_1, \dots, b_N]'$ are the normal eigenvectors associated to $\lambda(\mathbf{A})$ (in \mathbf{A}) and $\lambda(\mathbf{B})$ (in \mathbf{B}), respectively.

The condition for the presence of a coexistence zone for weak memes is

$$\frac{d\Phi}{d\tau} \frac{d\Psi}{d\theta} = [1 - \alpha_p + \xi \alpha_q] [1 - \alpha_q + \frac{\alpha_p}{\xi}] \frac{\left(\sum_{i=1}^N b_i a_i^2 \right) \left(\sum_{i=1}^N a_i b_i^2 \right)}{\left(\sum_{i=1}^N a_i^3 \right) \left(\sum_{i=1}^N b_i^3 \right)} < 1.$$

If the two meme spread in the same graph, this condition reduces to (14)

Consider now the case of aggressive memes ($\tau \rightarrow \infty, \theta \rightarrow \infty$), beginning with $\Phi(\tau)$. For $\tau \rightarrow \infty$, we have, from (7), $p_i \rightarrow 1$. Write $p_i \sim 1 - k_i/\tau$, with k_i to be determined. Plugging this in (7), we have

$$\frac{\tau}{k_i} \left(1 - \frac{k_i}{\tau} \right) = \tau \sum_{j=1}^N a_{ij} \left(1 - \frac{k_j}{\tau} \right) \sim \tau \sum_{j=1}^N a_{ij} = \tau d_i^A,$$

where d_i^A is the degree of node i in graph A . This leads to $k_i = 1/d_i^A$ and

$$p_i \sim 1 - \frac{1}{\tau d_i^A}.$$

With this approximation we have

$$\begin{aligned} \text{diag}(p_i) &\sim \mathbf{I} - \frac{1}{\tau} \mathbf{D}_A^{-1}, \\ \text{diag}\left(\frac{p_i}{1-p_i}\right) &\sim \tau \mathbf{D}_A \end{aligned}$$

and (9) can be rewritten as

$$\lambda_1 \left[\frac{\Phi(\tau)}{\tau} \mathbf{D}_A^{-1} (\mathbf{I} + \tau \alpha_p \mathbf{D}) \mathbf{B} - \alpha_q \tau \xi \mathbf{D} \right] = 1.$$

That is, $\rho(\mathbf{M}) = 1$, where

$$\mathbf{M} \triangleq \mathbf{M}_1 - \mathbf{M}_2 = \frac{\Phi(\tau)}{\tau} \mathbf{D}_A^{-1} (\mathbf{I} + \tau \alpha_p \mathbf{D}) \mathbf{B} - \alpha_q \tau \xi \mathbf{D}$$

and ρ is the spectral radius. Since $\rho(\mathbf{M})$ is finite and is a norm, we can assume that there is C such that

$$\rho(\mathbf{M}) - \rho(\mathbf{M}_2) = C$$

(viz., either both radii are finite or they are both infinite). Noting that $\rho(\mathbf{D}) = d_+^A \triangleq \max\{d_i^A\}$, we have

$$\frac{\Phi(\tau)}{\tau} \rho(\mathbf{D}_A^{-1} (\mathbf{I} + \tau \alpha_p \mathbf{D}_A) \mathbf{B}) - \alpha_q \tau \xi d_+^A = C.$$

With this approximation we have

(i) $\alpha_p = 0$

$$\Phi(\tau) \sim \frac{\alpha_q \xi d_+^A}{\rho(\mathbf{D}_A^{-1} \mathbf{B})} \tau^2,$$

(ii) $\alpha_p > 0$: setting $\mathbf{I} + \tau \alpha_p \mathbf{D}_A \sim \tau \alpha_p \mathbf{D}_A$:

$$\Phi(\tau) \sim \frac{\alpha_q}{\alpha_p} \frac{\xi d_+^A}{\lambda_B} \tau.$$

For $\Psi(\theta)$, we apply the transformation (5) obtaining

$$\lambda_1 \left[\frac{\Psi(\theta)}{\theta} \mathbf{D}_B^{-1} (\mathbf{I} + \theta \alpha_q \mathbf{D}_B) \mathbf{A} - \frac{\alpha_p}{\xi} \theta \mathbf{D}_B \right] = 1,$$

that is, under the same approximation

$$\frac{\Psi(\theta)}{\theta} \rho(\mathbf{D}_B^{-1} (\mathbf{I} + \theta \alpha_q \mathbf{D}_B) \mathbf{A}) - \frac{\alpha_p d_+^B}{\xi} \theta = C.$$

In this case

$\alpha_p = 0$

$$\Psi(\theta) \sim \frac{C}{\alpha_q \lambda_A}$$

$\alpha_p > 0$

$$\Psi(\theta) \sim \frac{\alpha_p}{\alpha_q} \frac{d_+^B}{\xi \lambda_A} \theta$$

The same considerations as in Section 4 suggest that there is always a coexistence region for aggressive memes (remember that we consider always $\alpha_q > 0$).

6. Simulations

We simulate meme diffusion in the case $\alpha_p = 0$. This case, in addition to allowing a clearer verification of the theoretical results, is of considerable practical interest for the diffusion of fake news among limited information rational agents.

The qualitative behavior predicted for Φ and Ψ is shown in Fig. 6; $\Phi(\tau)$ shows a quadratic behavior for large τ , while $\Psi(\theta)$ tends to a limit $1/(\alpha_q \lambda_A)$.

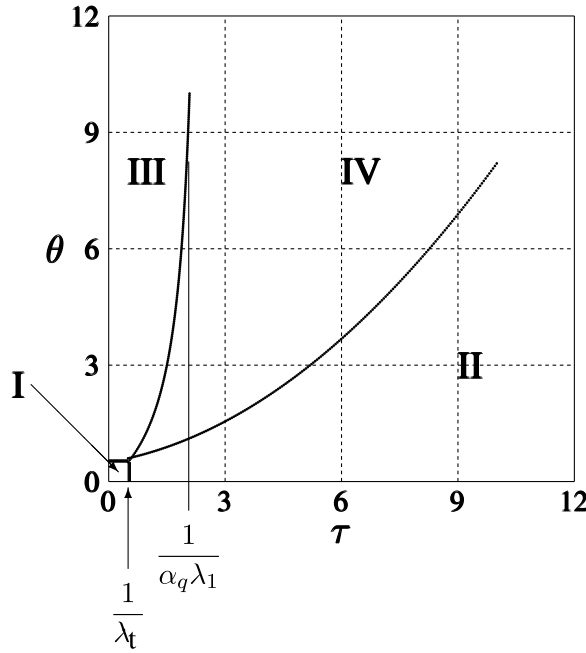


Fig. 6. Qualitative behavior of the coexistence thresholds Φ and Ψ for the case $\alpha_p = 0$ (rational verification: see text). The square at the bottom-left of the figure is the region of non-diffusion: both memes will vanish. The curve Ψ tends to a limit equal to $1/(\alpha_q \lambda_A)$ for $\theta \rightarrow \infty$, while $\Phi(\tau)$ goes to infinity like $O(\tau^2)$. The region labels (I–IV) correspond to the regions defined in Section 3.

The simulations were carried out on a random graph with $N = 5000$ nodes generated using the Barabasi–Albert algorithm [42]. This model was chosen because it exhibit a power-law distribution of degree, and a clustering coefficient that is typical of most social networks [43]. The stability of the results was checked by repeating the simulation on Barabasi–Albert [25] and Erdős–Renyi [24] graphs, which exhibit different characteristics.

The graph exhibits a power-law distribution of the type $P[d_i = k] \sim k^{-\gamma}$. With the parameters used, $\gamma = 2.75$, and the average number of neighbors per node is $\bar{d} = 7.8$, while $\lambda_A = 19.95$. Note that we limit $\alpha_q < 0.05$ because with high values of α_q the simulation becomes unstable for high τ due to the finite number of nodes and of time steps, giving virtually always a dominance of p and making it hard to estimate $\Phi(\tau)$. The mean field approximation that we use is derived in the hypothesis of asynchronous updates [44]. In particular, given $D \in \mathbb{N}_+$, one defines $\epsilon = 1/D$ and considers the enumerable time set

$$T_\epsilon = \{0, \epsilon, 2\epsilon, \dots\}$$

D is chosen large enough so that the probability of two nodes updating in the same time interval is negligible, and the mean field approximation is obtained for $D \rightarrow \infty$ (see [44] for details). In our simulations, we divide each time step in $3N$ clicks (N being the number of nodes), and at each click we update a randomly picked node. This corresponds to approximating the mean field with $\epsilon = 1/3N$.

The values of τ and θ were obtained by manipulating β_p , β_q , δ_p , and δ_q in such as way that $\delta_p = \delta_q$ always, so that $\xi = 1$. The predicted behavior is approximated using the Eqs. (11) and (12) valid for regular graphs, which, in this case, become

$$\begin{aligned} \Phi(\tau) &= \tau(1 + \alpha_q \xi(\tau \lambda_A - 1)) \\ \Psi(\theta) &= \frac{\theta}{1 + \frac{\alpha_q}{\xi}(\theta \lambda_A - 1)}. \end{aligned}$$

These curves predict quite well the simulated results, well within the standard deviation, for low α_q (Figs. 7 and 8), but the predicted behavior is quite off for higher α_q and intermediate values of τ (Fig. 9). In this case, the effects of the topology of the network are more evident, and the regular graph approximation is no longer valid.

7. Conclusions

We have extended the model of two-virus spreading to model the possibility of direct change of opinion in meme spread (the possibility of moving from one opinion to another without going through an agnostic state). In particular, we have studied the characteristics of the probability of holding either opinion in the equilibrium when $t \rightarrow \infty$.

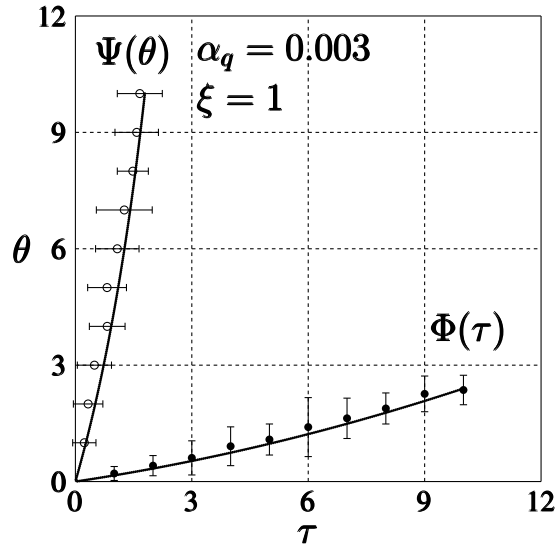


Fig. 7. The predicted Φ and Ψ versus simulation results for $\alpha_q = 0.003$. Region I, in which both memes vanish is defined by $\tau, \theta < 1/\lambda_A \approx 0.05$ and is not visible in the diagram. The simulation data are obtained from 20 simulations on the same graph; the figure shows average and standard deviation.

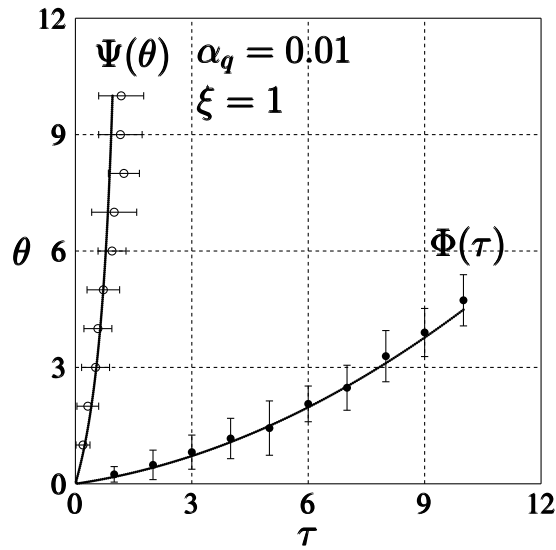


Fig. 8. The predicted Φ and Ψ versus simulation results for $\alpha_q = 0.01$. Region I, in which both memes vanish is defined by $\tau, \theta < 1/\lambda_A \approx 0.05$ and is not visible in the diagram. The simulation data are obtained from 20 simulations on the same graph; the figure shows average and standard deviation.

The steady-state equilibria of the two meme depends on their “epidemiological” characteristics through the parameters τ , θ , α_p , α_q , ξ , as well as on those of the network through the eigenvalues $\lambda(\mathbf{A})$, $\lambda(\mathbf{B})$ and the associated eigenvectors \mathbf{a} and \mathbf{b} .

In the case of regular graphs, the presence of a coexistence zone in the space of the parameters τ , θ is independent on the neighborhood size (m_a and m_b) of the two networks, being given by (15), while the point at which this region fades into the extinction zone does depend on these values, being given by $\tau = 1/m_a$, $\theta = 1/m_b$. Note that, as expected, the more connected the graph, the lower the threshold is: memes spread better in a heavily connected graph.

The situation for general graphs is more complex: the coexistence for weak memes depends on the eigenvectors \mathbf{a} and \mathbf{b} , which makes a complete analysis hard and heavily dependent on the structure of the graph. The results of the simulation, however, suggest that the same qualitative conclusions as in the case of regular graphs can be reached.

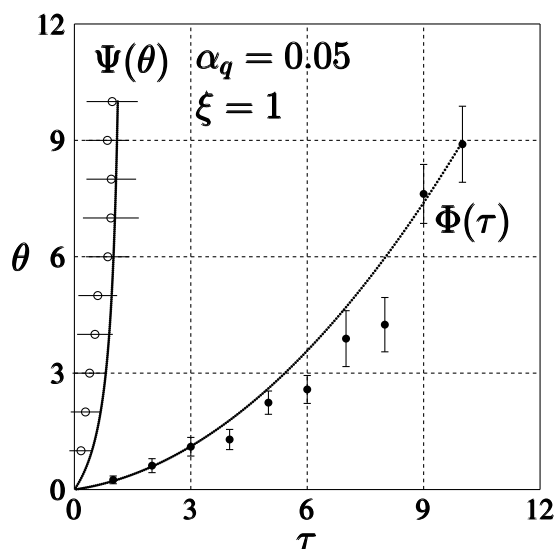


Fig. 9. The predicted Φ and Ψ versus simulation results for $\alpha_q = 0.05$. Region I, in which both memes vanish is defined by $\tau, \theta < 1/\lambda_A \approx 0.05$ and is not visible in the diagram. The simulation data are obtained from 20 simulations on the same graph; the figure shows average and standard deviation.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Simone Santini reports administrative support and equipment, drugs, or supplies were provided by Spain Ministry of Science and Innovation. No relationship to declare

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.physa.2023.129344>.

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