



Deformation theory of orthogonal and symplectic sheaves

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ABSTRACT

We show that the deformation functor of an orthogonal (*resp.* symplectic) sheaf over a smooth projective scheme admits a miniversal pro-family, identifying its space of first-order deformations with the first hypercohomology space of a complex which is naturally constructed out of the orthogonal (*resp.* symplectic) sheaf. We also provide an obstruction theory of these objects whose target is the second hypercohomology space of this complex.

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1. Introduction

1.1. Motivation

Moduli spaces of principal bundles usually carry interesting geometric structures, being a powerful, and often unique, source of examples of varieties with prescribed properties and characteristics. Nevertheless, these spaces might be non-compact whenever the base (smooth) scheme has dimension higher than 1. The construction of moduli spaces of principal

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sheaves or singular principal bundles achieved by Gomez, Sols, Schmitt and Langer [20,21,32,33,18,19] provides a natural compactification of the moduli space of principal bundles for a connected complex reductive structure group over a smooth projective scheme. If one is interested on a particular property enjoyed by the moduli of principal bundles, it is natural to study whether or not it extends to the compactification that the moduli space of principal sheaves provides. For such task, one needs a local description of the moduli space precisely over the locus where the principal sheaves fail to be principal bundles. Such description would naturally derive from deformation theory of principal sheaves, which is our goal. As a first step, we focus in the deformation theory of orthogonal and symplectic sheaves (see Section 1.2).

Let us highlight some open questions that fall within the previous framework:

- Moduli spaces of principal sheaves on symplectic surfaces (*i.e.* K3 or abelian) are natural candidates for new examples of compact hyperKähler varieties as all currently known examples appear as (desingularization of) moduli spaces of sheaves on these surfaces [6]. G -bundles on a K3 surface were intensively studied by Friedman, Morgan and Witten [15–17] and Donagi [14] for a general G , and, when $G = \mathrm{SO}(n, k)$, by Bershadsky, Johansen, Pantev and Sadov [7]. The associated moduli spaces are hyperKähler (hence holomorphically symplectic) and play a central role in string theory and F-theory, specially when the second Chern class is non trivial. The case of abelian surfaces was addressed by Bryan, Donagi and Leung [10] who studied the moduli space of G -bundles with trivial Chern classes. In this case, the principal sheaves are indeed principal bundles and the associated moduli spaces either can not be desingularized or, in the case of $\mathrm{SL}(n, k)$ and $\mathrm{Sp}(2m, k)$, provide already known examples of compact hyperKähler varieties. Less is known in the case of non trivial second Chern classes, where as in the K3 case, the bundle moduli space is not compact. Out of the previous discussion, an interesting open question, both for K3 or abelian surfaces, is whether or not the holomorphic symplectic form extends from the bundle moduli space to the compactification, and, in the singular locus, if there exists a symplectic desingularization. For both tasks one needs a local description of the moduli spaces precisely over the locus where the principal sheaves fail to be principal bundles. When $G = \mathrm{SO}(n, k)$ or $\mathrm{Sp}(2m, k)$, such description would require a good understanding of the deformation theory of orthogonal or symplectic sheaves.
- The Yang-Mills equation is of great importance in physics as it describes the behavior of classical interaction fields. The ADHM construction by Atiyah, Drinfeld Hitchin and Manin [2] associates solutions to the Yang-Mills equation to the so-called instanton bundles in \mathbb{P}^3 , holomorphic bundles with trivial first Chern class and satisfying certain vanishing conditions in cohomology. This construction of instanton bundles which makes use of monads, ultimately relying in linear algebraic data, was generalized to odd dimensional projective spaces \mathbb{P}^{2n+1} by Spindler and Okonek [36], and to even dimensional projective spaces by Jardim [23], who took a step further in the generalization, defining instanton sheaves as torsion free sheaves with trivial first Chern class and satisfying the already mentioned vanishing in cohomology. The monad formalism allowed the study of local questions like smoothness or regularity of the moduli space of instanton bundles [11,25,26] (as well as some global questions like afineness or irreducibility). As one can study the Yang-Mills equations associated to other groups, it is therefore natural to consider instanton bundles equipped with a symplectic or orthogonal structure. This has been addressed by several authors [3,13,24] and some local properties, such as smoothness or reducibility of the moduli spaces of orthogonal or symplectic instanton bundles have been described [1,8,9,12,28].

It was considered by Henni, Jardim and Martins in [22] a generalization of these objects to the notion of orthogonal or symplectic instanton sheaves. Hence, it is natural to study whether or not smoothness extends to the moduli space of the later. Since the vanishing in cohomology condition is open, the study of deformation theory of symplectic and orthogonal sheaves would provide tools to attack this problem from an alternative point of view to that provided by the monad formalism.

- K. Uhlenbeck [37] constructed a canonical compactification of the moduli space of instanton bundles by allowing them to degenerate into ideal instantons, giving rise to the so-called Uhlenbeck space. When the base variety is a 4-manifold, Donaldson introduced an invariant of it by considering a class of polynomials defined by self-intersection of certain subsets of the associated Uhlenbeck space. Li [27] considered an algebraic structure on the Uhlenbeck space, obtained (when the base variety is an algebraic surface) by blowing-down some closed subschemes of the moduli space of (torsion free) sheaves on the surface. As this moduli space of sheaves is known to be smooth [29] in this case, it provides a modular resolution of the Uhlenbeck space. Most importantly, Li's work provides an algebraic interpretation of the Donaldson polynomials in terms of the mentioned modular resolution. Balaji [4] extended Uhlenbeck's compactification to the case of principal bundles. This opens the door for the definition of new invariants analogous to Donaldson polynomials, associated to groups other than $\mathrm{GL}(n, k)$. A necessary requirement for this goal is the existence of a modular desingularization of these Balaji-Uhlenbeck spaces, which is still missing at the time. It has been predicted by Balaji itself [4] and others [35,5] that Balaji-Uhlenbeck spaces could be obtained by blowing down moduli spaces of principal sheaves and some evidence of this has been provided by Scalise [34,35] in the framed case. Hence, if these spaces turn up to be smooth, they would naturally provide a modular desingularization of the Balaji-Uhlenbeck spaces. Again, this requires a local description of the moduli spaces of principal sheaves which would derive from a deformation theory of these objects.

1.2. Main results and structure of the paper

As a first step towards the study of the deformation theory of principal sheaves, we consider in this article orthogonal and symplectic sheaves. These are pairs consisting on a torsion-free sheaf E equipped with a morphism $E \otimes E \rightarrow \mathcal{O}_X$, invariant or anti-invariant under permutation, giving a symmetric or symplectic form on the restriction of our sheaf E to the locus where it is locally-free. Using these data, one can naturally construct a morphism in the derived category, and we refer to its cone shifted by -1 as the deformation complex Δ^\bullet associated to the orthogonal or symplectic sheaf (E, ϕ) . Note that the hypercohomology of Δ^\bullet fits in the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{i-1}(E, E) \longrightarrow V^{i-1} \longrightarrow \mathbb{H}^i(\Delta^\bullet) \longrightarrow \mathrm{Ext}^i(E, E) \longrightarrow V^i \longrightarrow \cdots$$

where the V^i are related to certain subspaces of $\mathrm{Ext}^{i-1}(E \otimes E, \mathcal{O}_X)$.

A close version of this deformation complex appears in [34] where a preliminary study of the deformation theory of quadratic sheaves is presented, along with a beautiful study of framed symplectic sheaves and their moduli spaces (see also [35]).

Associated to any orthogonal and symplectic sheaf (E, ϕ) we define its deformation functor $\mathrm{Def}_{(E, \phi)}$. The main result of this paper is the description of the properties of $\mathrm{Def}_{(E, \phi)}$ [Theorems 4.1 and 5.5].

Theorem. *Given an orthogonal or symplectic sheaf (E, ϕ) , its deformation functor $\mathrm{Def}_{(E, \phi)}$ admits a miniversal pro-family which is universal when (E, ϕ) is simple. If Δ^\bullet denotes the deformation complex associated to (E, ϕ) , one has that*

- the infinitesimal automorphisms of (E, ϕ) are classified by $\mathbb{H}^0(\Delta^\bullet)$,
- the first order deformations of (E, ϕ) are parametrized by $\mathbb{H}^1(\Delta^\bullet)$,
- and one can construct an obstruction theory for $\mathrm{Def}_{(E, \phi)}$ modeled in $\mathbb{H}^2(\Delta^\bullet)$.

When (E, ϕ) is locally free representing an $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{Sp}(2m, \mathbb{C})$ -bundle, $\mathbb{H}^i(\Delta^\bullet)$ is identified with the cohomology of the endomorphism bundle of (E, ϕ) [Remark 3.1] and we recover the usual deformation theory of $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{Sp}(2m, \mathbb{C})$ -bundles.

Let us briefly sketch the structure of our paper. After recalling the basic definitions of deformation and obstruction theory in 2.1, we review in Section 2.2 the classical case of coherent sheaves achieved by Grothendieck. We finish the preliminaries by presenting orthogonal and symplectic sheaves in Section 2.3. In Section 3 we introduce the deformation complex, providing a description of its zero, first and second hypercohomology spaces. We define the deformation functor associated to a orthogonal and symplectic sheaf in Section 4 and we verify that it satisfies the first, second and third Schlessinger conditions. In the way to check the third condition we show that the space of first order deformations of orthogonal (resp. symplectic) sheaves coincide with the first hypercohomology space of the deformation complex, which is finite dimensional. In the case of simple orthogonal and symplectic sheaves, we verify as well that the forth Schlessinger condition holds. Finally, we construct, in Section 5, an obstruction theory for orthogonal and symplectic sheaves whose target is the second hypercohomology space of the deformation complex.

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2. Preliminaries

2.1. Deformation and obstruction theory

See [30] for an introduction to deformation theory. Let k be an algebraically closed field, (\mathbf{Art}) the category of all finite Artin local k -algebras with residue field k and denote by (\mathbf{Sets}) the category of all sets. Denote by $(\mathbf{FinVect})$ the category of finite dimensional k -vector spaces. We construct the functor $k(\bullet) : (\mathbf{FinVect}) \rightarrow (\mathbf{Art})$ by setting $k(V) = k \oplus V$ as k -vector spaces, and ring structure given by $(k, v) \cdot (k', v') = (kk', k'v + kv')$. Note that $k(V)$ is the Artin local algebra whose maximal ideal is the vector space $\mathfrak{m} = V$, satisfying $\mathfrak{m}^2 = 0$, and its residue field is k . Note that one naturally has that $k(k) \cong k[\epsilon]/(\epsilon^2)$. We say that $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$ is a *small extension* in (\mathbf{Art}) if $\mathfrak{m}_B H = 0$.

Given a deformation functor $\mathbf{F} : (\mathbf{Art}) \rightarrow (\mathbf{Sets})$, we list below the so-called *Schlessinger conditions* for \mathbf{F} .

S1: For any homomorphism $C \rightarrow A$ and any small extension $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$ in (\mathbf{Art}) , the induced morphism

$$\mathbf{F}(B \times_A C) \longrightarrow \mathbf{F}(B) \times_{\mathbf{F}(A)} \mathbf{F}(C)$$

is surjective.

S2: For any $B \in (\mathbf{Art})$ and any $V \in (\mathbf{FinVect})$, the induced morphism

$$\mathbf{F}(B \times_k k\langle V \rangle) \longrightarrow \mathbf{F}(B) \times \mathbf{F}(k\langle V \rangle)$$

is bijective.

S3: The space of first-order deformations $\mathbf{F}(k[\epsilon]/(\epsilon^2))$ is a finite dimensional k -vector space.

S4: For any $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$ small extension in (\mathbf{Art}) , the induced morphism

$$\mathbf{F}(B \times_A B) \longrightarrow \mathbf{F}(B) \times_{\mathbf{F}(A)} \mathbf{F}(B)$$

is bijective.

A pro-family is a family r parametrized by a complete local k -algebra R with residue field k . A pro-family is versal if any family parametrized by $A \in (\mathbf{Art})$ is the pull-back of r by a morphism $f : R \rightarrow A$. It is a miniversal pro-family if furthermore the induced map on first order infinitesimal deformations

$$\mathrm{Hom}_{k\text{-alg}}(R, k[\epsilon]/(\epsilon^2)) \longrightarrow \mathbf{F}(k[\epsilon]/(\epsilon^2))$$

is an isomorphism. A functor is pro-representable if there is a universal pro-family (i.e., the morphism $f : R \rightarrow A$ above is unique).

Schlessinger [31] proved that a deformation functor admits a miniversal pro-family if and only if it satisfies **S1**, **S2** and **S3**. Moreover, it is pro-representable if and only if it satisfies **S4** along with the previous conditions.

An *obstruction theory* for the deformation functor \mathbf{F} consists on a k -vector space $\mathrm{Obs}(\mathbf{F})$ and, for any small extension $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$ in (\mathbf{Art}) , a morphism

$$\Omega_\tau : \mathbf{F}(A) \longrightarrow H \otimes_k \mathrm{Obs}(\mathbf{F}) \quad (2.1)$$

satisfying the conditions listed below:

O1: The sequence of sets

$$\mathbf{F}(B) \xrightarrow{\mathbf{F}(\tau)} \mathbf{F}(A) \xrightarrow{\Omega_\tau} H \otimes_k \mathrm{Obs}(\mathbf{F})$$

is exact in the middle.

O2: For any morphism of small extensions, that is, a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & B & \xrightarrow{\tau} & A \longrightarrow 0 \\ & & \downarrow h & & \downarrow \beta & & \downarrow \alpha \\ 0 & \longrightarrow & H' & \longrightarrow & B' & \xrightarrow{\tau'} & A' \longrightarrow 0, \end{array}$$

the induced diagram,

$$\begin{array}{ccc} \mathbf{F}(A) & \xrightarrow{\Omega_\tau} & H \otimes_k \mathrm{Obs}(\mathbf{F}) \\ \mathbf{F}(\alpha) \downarrow & & \downarrow \mathbf{1}_{\mathrm{Obs}(\mathbf{F})} \otimes_k h \\ \mathbf{F}(A') & \xrightarrow{\Omega_{\tau'}} & H' \otimes_k \mathrm{Obs}(\mathbf{F}), \end{array}$$

commutes.

2.2. Coherent sheaves

Let X be a projective scheme over k . Denote by $\mathrm{Coh}(X)$ the category of coherent sheaves on X , and write $\mathcal{D}^b(X)$ for the bounded derived category of quasi-coherent sheaves with coherent cohomology. Given $E \in \mathrm{Coh}(X)$, by abuse of notation, we denote by $E \in \mathcal{D}^b(X)$ the complex supported on 0-degree given by E .

For any coherent sheaf E one defines its *dual sheaf* by setting $E^\vee := \mathrm{Hom}_{\mathrm{Coh}(X)}(E, \mathcal{O}_X)$ and, for any complex $F^\bullet \in \mathcal{D}^b(X)$, its associated *dual complex* is $\mathbf{D}F^\bullet := \mathrm{Hom}_{\mathcal{D}^b(X)}(F^\bullet, \mathcal{O}_X)$. Every torsion free coherent sheaf injects naturally into its double dual, $E \hookrightarrow E^{\vee\vee}$, but unless E is reflexive, $E^{\vee\vee}$ is not isomorphic to E . On the other hand $\mathbf{D} \circ \mathbf{D} = \mathbf{1}$, so \mathbf{D} is an autoequivalence of $\mathcal{D}^b(X)$. If E is locally free, $\mathbf{D}E \cong E^\vee$, but this does not hold for a general coherent sheaf.

For any coherent sheaf E over a projective scheme X , we define its deformation functor

$$\mathrm{Def}_E : (\mathbf{Art}) \longrightarrow (\mathbf{Sets})$$

by associating to any $A \in (\mathbf{Art})$ the set of isomorphism classes of pairs (\mathcal{E}, γ) , where \mathcal{E} is a coherent sheaf on $X_A := X \times \text{Spec}(A)$, flat over $\text{Spec}(A)$, and $\gamma : \mathcal{E}|_X \rightarrow E$ is an isomorphism. We say that two pairs (\mathcal{E}, γ) and (\mathcal{E}', γ') are isomorphic if there exists an isomorphism $f : \mathcal{E} \rightarrow \mathcal{E}'$ such that $\gamma' \circ (f|_X) = \gamma$. Every morphism of Artin algebras $a : A \rightarrow A'$, induces naturally a morphism $p_a : X_{A'} \rightarrow X_A$. Functoriality of Def_E follows from applying pull-backs under p_a .

Grothendieck showed that the cohomology of the complex $\mathbf{DE} \otimes^L E$ rules the deformation and obstruction theory of E . In particular Def_E admits a miniversal pro-family and its space of first-order deformations is

$$\text{Def}_E(k[\epsilon]/(\epsilon^2)) \cong \text{Ext}_X^1(E, E) \cong H^1(\mathbf{DE} \otimes^L E).$$

If further, E is simple, then Def_E is pro-representable. Also, Def_E admits a deformation theory with vector space

$$\text{Obs}(\text{Def}_E) \cong \text{Ext}_X^2(E, E) \cong H^2(\mathbf{DE} \otimes^L E).$$

In particular, when $\text{Ext}_X^2(E, E) = 0$, the deformation functor Def_E is formally smooth.

2.3. Orthogonal and symplectic sheaves

An *orthogonal sheaf* (resp. a *symplectic sheaf*) on the projective scheme X is a pair (E, ϕ) , where E is a torsion-free coherent sheaf on X and $\phi : E \otimes E \rightarrow \mathcal{O}_X$ is a homomorphism which is symmetric, $\phi \circ \theta_E = \phi$ (resp. anti-symmetric, $\phi \circ \theta_E = -\phi$), under the permutation $\theta_E : E \otimes E \rightarrow E \otimes E$, and such that its restriction $\phi|_{U_E}$ to the open subset U_E where E is locally free is non-degenerate. Recalling that the center of $\text{O}(2m, k)$ (resp. $\text{Sp}(2m, k)$) is $\{1, -1\}$ and the center of $\text{O}(2m+1, k)$ is $\{1\}$, we say that an orthogonal (resp. symplectic) sheaf (E, ϕ) is *simple* if its automorphism group is $\text{Aut}(E, \phi) = \{1_E, -1_E\}$ when it has even rank, or $\text{Aut}(E, \phi) = \{1_E\}$ when it has odd rank.

A family of orthogonal (resp. symplectic) sheaves parametrized by S is a pair (\mathcal{E}, Φ) such that $\mathcal{E} \rightarrow X \times S$ is a torsion-free coherent sheaf, flat over S and such that $\mathcal{E}_s := \mathcal{E}|_{X \times \{s\}}$ is torsion-free for each closed point $s \in S$, and $\Phi : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{O}_{X \times S}$ is a symmetric (resp. anti-symmetric) homomorphism whose restriction $\Phi|_{U_{\mathcal{E}}}$ to the open set where \mathcal{E} is locally free is non-degenerate. Two families (\mathcal{E}, Φ) and (\mathcal{E}', Φ') of orthogonal (resp. symplectic) sheaves are *isomorphic* if there exists an isomorphism $f : \mathcal{E} \rightarrow \mathcal{E}'$ such that $\Phi|_{U_{\mathcal{E}}} = \Phi'|_{U_{\mathcal{E}}} \circ (f|_{U_{\mathcal{E}}} \otimes f|_{U_{\mathcal{E}}})$.

In the remaining of the section we will see how orthogonal and symplectic sheaves provide a well defined geometrical object living in the derived category.

Given any complex $F^\bullet \in \mathcal{D}^b(X)$, denote by θ_{F^\bullet} the derived permutation of $F^\bullet \otimes^L F^\bullet$. Adjunction gives

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}^b(X)}(F^\bullet \otimes F^\bullet, \mathcal{O}_X) & \cong & \text{Hom}_{\mathcal{D}^b(X)}(F^\bullet, \mathbf{DF}^\bullet) \\ \phi^\bullet & \longmapsto & \phi_{\text{ad}}^\bullet \end{array} \quad (2.2)$$

As in the case of sheaves, permutation composed with adjunction dualizes morphisms of the form $\psi^\bullet : F^\bullet \rightarrow \mathbf{DF}^\bullet$,

$$(\psi^\bullet \circ \theta_{F^\bullet})_{\text{ad}} = \mathbf{D}\psi_{\text{ad}}^\bullet. \quad (2.3)$$

One can give a description of orthogonal and symplectic sheaves in terms of the derived category.

Proposition 2.1. *Every orthogonal (resp. symplectic) sheaf determines (up to isomorphism) a pair (E, ϕ_{ad}) , where E is a complex supported on 0 determined by a torsion-free sheaf, and $\phi_{\text{ad}} \in \text{Hom}_{\mathcal{D}^b(X)}(E, \mathbf{DE})$ such that $\phi_{\text{ad}} = \mathbf{D}\phi_{\text{ad}}$ (resp. $\phi_{\text{ad}} = -\mathbf{D}\phi_{\text{ad}}$) and $\mathbb{H}^0(\phi_{\text{ad}})|_{U_E} : E|_{U_E} \rightarrow E^\vee|_{U_E}$ is an isomorphism.*

Proof. This follows immediately after (2.2), (2.3). \square

3. The deformation complex

Consider a coherent sheaf E over X projective and a morphism $\phi : E \otimes E \rightarrow \mathcal{O}_X$ such that $\phi \circ \theta_E = \phi$ and recall from Proposition 2.1 that it defines a morphism $\phi_{\text{ad}} : E \rightarrow \mathbf{DE}$ in $\mathcal{D}^b(X)$. Inspired by J. Scalise [34], we define

$$\Delta_{(E, \phi)}^+ := (\mathbf{1}_{\mathbf{DE} \otimes^L \mathbf{DE}} + \theta_{\mathbf{DE}}) \circ (\mathbf{1}_{\mathbf{DE}} \otimes \phi_{\text{ad}}) : \mathbf{DE} \otimes^L E \longrightarrow (\mathbf{DE} \otimes^L \mathbf{DE})^+.$$

Analogously, when $\phi \circ \theta_E = -\phi$, set

$$\Delta_{(E, \phi)}^- := (\mathbf{1}_{\mathbf{DE} \otimes^L \mathbf{DE}} - \theta_{\mathbf{DE}}) \circ (\mathbf{1}_{\mathbf{DE}} \otimes \phi_{\text{ad}}) : \mathbf{DE} \otimes^L E \longrightarrow (\mathbf{DE} \otimes^L \mathbf{DE})^-.$$

Let us also consider their associated mapping cone shifted by -1 ,

$$\Delta_{(E, \phi)}^{+, \bullet} := \text{Cone}(\Delta_{(E, \phi)}^+)[-1],$$

and

$$\Delta_{(E,\phi)}^{-,\bullet} := \text{Cone}(\Delta_{(E,\phi)}^-)[-1].$$

These complexes fit in the distinguished triangle

$$\Delta_{(E,\phi)}^{\pm,\bullet} \longrightarrow \mathbf{D}E \otimes^L E \xrightarrow{\Delta_{(E,\phi)}^{\pm}} (\mathbf{D}E \otimes^L \mathbf{D}E)^{\pm} \longrightarrow \Delta_{(E,\phi)}^{\pm,\bullet}[1],$$

giving a long-exact sequence in hypercohomology.

$$\begin{aligned} \dots \longrightarrow \mathbb{H}^{i-1}(\mathbf{D}E \otimes^L E) &\longrightarrow \mathbb{H}^{i-1}\left((\mathbf{D}E \otimes^L \mathbf{D}E)^{\pm}\right) \longrightarrow \mathbb{H}^i(\Delta_{(E,\phi)}^{\pm,\bullet}) \longrightarrow \\ &\longrightarrow \mathbb{H}^i(\mathbf{D}E \otimes^L E) \longrightarrow \mathbb{H}^i\left((\mathbf{D}E \otimes^L \mathbf{D}E)^{\pm}\right) \longrightarrow \dots \end{aligned}$$

Remark 3.1. When (E, ϕ) is locally trivial representing an $\text{O}(n, \mathbb{C})$ or $\text{Sp}(2m, \mathbb{C})$ -bundle, ϕ is an isomorphism between E and $E^{\vee} = \mathbf{D}E$. In that case, $\Delta_{(E,\phi)}^{\pm}$ is a morphism of locally free sheaves and $\mathbb{H}^i(\Delta_{(E,\phi)}^{\pm,\bullet})$ is identified with the cohomology of its kernel.

One can give a description of the first groups of hypercohomology of the deformation complex.

Proposition 3.2. Let E be a coherent sheaf over X projective and consider $\phi : E \otimes E \rightarrow \mathcal{O}_X$ such that $\phi \circ \theta_E = \pm \phi$. Consider a finite dimensional k -vector space V . Then,

$$V \otimes_k \mathbb{H}^0(\Delta_{(E,\phi)}^{\pm,\bullet}) = \{\lambda \in \text{Hom}_{\text{Coh}}(E, E) \text{ such that } \phi \circ (\mathbf{1}_E \otimes \lambda) + \phi \circ (\lambda \otimes \mathbf{1}_E) = 0\},$$

is the space of infinitesimal endomorphisms of (E, ϕ) .

Proof. By definition $\mathbb{H}^{i+1}(\Delta_{(E,\phi)}^{\pm,\bullet}) = \mathbb{H}^i(\text{Cone}(\Delta_{(E,\phi)}^{\pm}))$, so we focus on the description of the complex $\text{Cone}(\Delta_{(E,\phi)}^{\pm})$. Picking the locally free resolution $W^{\bullet} \xrightarrow{\pi} E \rightarrow 0$, one can see that this complex is 0 for $H < -1$, and in degrees -1 and 0 amounts to

$$0 \longrightarrow \text{Hom}_X(W_0, E) \xrightarrow{\delta_{-1}} \begin{pmatrix} \text{Hom}_X(W_{-1}, E) \\ \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^{\pm} \end{pmatrix},$$

with

$$\delta_{-1} = \begin{pmatrix} -(\bullet) \circ \partial_{-1} \\ \phi \circ (\pi \otimes (\bullet)) \circ (\mathbf{1}_{W_0 \otimes W_0} \pm \theta_{W_0}) \end{pmatrix},$$

where π denotes the projection $W_0 \xrightarrow{\pi} E \rightarrow 0$ and its dual, π^{\vee} , the inclusion $0 \rightarrow E^{\vee} \xrightarrow{\pi^{\vee}} W_0^{\vee}$. The first statement follows from the fact that $\mathbb{H}^{-1}(\text{Cone}(\Delta_{(E,\phi)}^{\pm})) = \ker(\delta_{-1})$ and

$$\begin{aligned} \phi \circ (\pi \otimes \lambda) \pm \phi \circ (\pi \otimes \lambda) \circ \theta_{W_0} &= \phi \circ (\pi \otimes \lambda) \pm \phi \circ \theta_E \circ (\lambda \otimes \pi) \\ &= \phi \circ (\pi \otimes \lambda) + \phi \circ (\lambda \otimes \pi). \end{aligned}$$

Then $\mathbb{H}^0(\Delta_{(E,\phi)}^{\pm,\bullet})$ amounts to the space of infinitesimal automorphisms of (E, ϕ) and the result follows. \square

Proposition 3.3. Let E be a coherent sheaf over X projective and consider $\phi : E \otimes E \rightarrow \mathcal{O}_X$ such that $\phi \circ \theta_E = \pm \phi$. Consider a finite dimensional k -vector space V . Then, $V \otimes_k \mathbb{H}^1(\Delta_{(E,\phi)}^{\pm,\bullet})$ is the finite dimensional vector space classifying isomorphism classes of 1-extensions

$$0 \longrightarrow V \otimes_k E \xrightarrow{i} F \xrightarrow{j} E \longrightarrow 0, \quad (3.1)$$

equipped with

$$\Phi : F \otimes F / I \rightarrow V \otimes_k \mathcal{O}_X,$$

where $I \subset F \otimes F$ is the subsheaf generated by $(f_1, (i \circ j)(f_2)) - ((i \circ j)(f_1), f_2)$ for all $f_i \in F$, and Φ is such that

$$\Phi \circ \theta_F = \pm \Phi \quad (3.2)$$

and

$$\Phi \circ (\mathbf{1}_F \otimes i) = (\mathbf{1}_V \otimes_k \phi) \circ (\pi \otimes \mathbf{1}_{(V \otimes_k E)}). \quad (3.3)$$

Proof. We have to describe $V \otimes_k \mathbb{H}^0 \left(\text{Cone} \left(\Delta_{(E, \phi)}^\pm \right) \right)$, which is finite dimensional since V is and so are all the cohomology spaces of $\mathbf{D}E \otimes^L E$ and $\mathbf{D}E \otimes^L \mathbf{D}E$. Taking the locally free resolution $W^\bullet \xrightarrow{\pi} E \rightarrow 0$, the complex $\text{Cone} \left(\Delta_{(E, \phi)}^\pm \right)$ can be described in degrees 0 and 1 as

$$\left(\begin{array}{c} \text{Hom}_X(W_{-1}, E) \\ \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm \end{array} \right) \xrightarrow{\delta_0} \left(\begin{array}{c} \text{Hom}_X(W_{-2}, E) \\ \text{Hom}_X(W_0 \otimes W_{-1}, \mathcal{O}) \end{array} \right),$$

with

$$\delta_0 = \begin{pmatrix} -(\bullet) \circ \partial_{-2} & 0 \\ \phi \circ (\pi \otimes (\bullet)) & (\bullet) \circ (\mathbf{1}_{W_0} \otimes \partial_{-1}) \end{pmatrix}.$$

We have

$$\ker(\delta_0) = \left\{ \begin{array}{l} (\eta, \Psi) \in \text{Hom}_X(W_{-1}, E) \oplus \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm \\ \eta \circ \partial_{-2} = 0 \\ \phi \circ (\pi \otimes \eta) + \Psi \circ (\mathbf{1}_{W_0} \otimes \partial_{-1}) = 0 \end{array} \right\}$$

and $\mathbb{H}^0 \left(\text{Cone} \left(\Delta_{(E, \phi)}^\pm \right) \right) = H^0(\ker(\delta_0))$. For any pair $(\overline{\eta}, \overline{\Psi}) \in V \otimes_k H^0(\ker(\delta_0))$, with $\overline{\eta} \in \text{Hom}_X(W_{-1}, V \otimes_k E)$ and $\overline{\Psi} \in \text{Hom}_X(W_0 \otimes W_0, V \otimes_k \mathcal{O}_X)^\pm$, one can naturally construct an extension (3.1) setting

$$F := (V \otimes_k E) \oplus W_0/(\overline{\eta} \oplus \partial_{-1})(W_{-1}),$$

with the injection

$$\begin{array}{ccc} i: E & \longrightarrow & F = (V \otimes_k E) \oplus W_0/(\overline{\eta} \oplus \partial_{-1})(W_{-1}) \\ \overline{e} & \longmapsto & [(\overline{e}, 0)] \end{array}$$

and the projection

$$\begin{array}{ccc} j: F = (V \otimes_k E) \oplus W_0/(\overline{\eta} \oplus \partial_{-1})(W_{-1}) & \longrightarrow & W_0/\partial_{-1}(W_{-1}) \cong E \\ & \longmapsto & [w]. \end{array}$$

Let $\overline{\phi}$ denote $\mathbf{1}_V \otimes_k \phi$. We have that $\overline{\phi} \circ (\mathbf{1}_V \otimes_k \theta_E) = \pm \overline{\phi}$ by hypothesis on ϕ . Observe also that we pick $\overline{\Psi}$ in $\text{Hom}_X(W_0 \otimes W_0, V \otimes_k \mathcal{O}_X)^\pm$, hence $\overline{\Psi} \circ \theta_{W_0} = \pm \overline{\Psi}$. Therefore,

$$\overline{\phi} + \overline{\Psi}: (V \otimes E \otimes E) \oplus (W_0 \otimes W_0) \otimes (E \oplus W_0) \rightarrow V \otimes_k \mathcal{O}_X$$

satisfies

$$(\overline{\phi} + \overline{\Psi}) \circ ((\mathbf{1}_V \otimes_k \theta_E) \oplus \theta_{W_0}) = \pm(\overline{\phi} + \overline{\Psi}). \quad (3.4)$$

Recalling that $(\overline{\eta}, \overline{\Psi}) \in V \otimes_k \ker(\delta_0)$, we have $\overline{\phi} \circ (\pi \otimes \overline{\eta}) + \overline{\Psi} \circ (\mathbf{1}_{W_0} \otimes \partial_{-1}) = 0$, we observe that

$$(\overline{\phi} + \overline{\Psi})|_{(E \oplus W_0) \otimes (\overline{\eta} \oplus \partial_{-1})(W_{-1})} = 0 \quad (3.5)$$

As a direct consequence of (3.4) and (3.5), one has

$$(\overline{\phi} + \overline{\Psi})|_{(\overline{\eta} \oplus \partial_{-1})(W_{-1}) \otimes (E \oplus W_0)} = 0 \quad (3.6)$$

Then, $\overline{\phi} + \overline{\Psi}$ defines $\Phi: F \otimes F \rightarrow \mathcal{O}_X$. Since $\overline{\phi} + \overline{\Psi}$ satisfies (3.4), it follows that Φ satisfies (3.2). Obviously, $\overline{\phi} + \overline{\Psi}$ restricted to $E \oplus 0$ coincides with $\overline{\phi}$, hence Φ satisfies (3.3) as well.

We now study the action of $\text{Im}(\delta_{-1})$ on $\ker(\delta_0)$. For any $(\overline{\eta}, \overline{\Psi}) \in \text{Hom}_X(W_{-1}, E) \oplus \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)$ and any $\lambda \in \text{Hom}_X(W_0, E)$, set

$$(\overline{\eta}', \overline{\Psi}') = (\overline{\eta}, \overline{\Psi}) + \delta_{-1}(\lambda) = (\overline{\eta} - \lambda \circ \partial_{-1}, \overline{\Psi} + \overline{\phi} \circ (\pi \otimes \lambda) + \overline{\phi} \circ (\lambda \otimes \pi)), \quad (3.7)$$

Let F' be $E \oplus W_0/((- \overline{\eta}') \oplus \partial_{-1})(W_{-1})$. Consider the isomorphism

$$\left(\begin{array}{cc} \mathbf{1}_E & \lambda \\ 0 & \mathbf{1}_{W_0} \end{array} \right): E \oplus W_0 \xrightarrow{\cong} E \oplus W_0. \quad (3.8)$$

Since the image under (3.8) of $(\overline{\eta} \oplus \partial_{-1})(W_{-1})$ is precisely $(\overline{\eta}' \oplus \partial_{-1})(W_{-1})$, this descends to an isomorphism

$$\lambda_1: F \xrightarrow{\cong} F'.$$

Let $\Phi' \in \text{Hom}_X(F' \otimes F', \mathcal{O}_X)^\pm$ defined by $\overline{\phi} + \overline{\Psi}'$. We can check that

$$\Phi = \Phi' \circ (\lambda_1 \otimes \lambda_1), \quad (3.9)$$

so $(\overline{\eta}, \overline{\Psi})$ and $(\overline{\eta}', \overline{\Psi}')$ define isomorphic extensions, with this isomorphism relating the corresponding quadratic form.

Conversely, suppose we are given an extension of the form (3.1) and $\Phi : F \otimes F \rightarrow \mathcal{O}_X$ satisfying (3.2). Picking a locally free resolution $W^\bullet \xrightarrow{\pi} E \rightarrow 0$, the extension (3.1) determines $\overline{\eta} \in \text{Hom}_X(W_{-1}, E)$ such that $\overline{\eta}(\partial_{-2}(W_{-2})) = 0$. Taking the pull-back of Φ under $E \oplus W_0 \rightarrow F$ and restricting to W_0 , we obtain $\overline{\Psi} \in \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm$. Since it comes from Φ defined over F , it follows that $\overline{\Psi}$ satisfies (3.5) and (3.6). Therefore $(\overline{\eta}, \overline{\Psi})$ lies in $\ker(\delta_0)$. Suppose further that we are given two isomorphic extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & E \longrightarrow 0 \\ & & \parallel & & \downarrow \lambda_1 & & \parallel \\ 0 & \longrightarrow & E & \longrightarrow & F' & \longrightarrow & E \longrightarrow 0, \end{array}$$

and $\Phi \in \text{Hom}_X(F \otimes F, \mathcal{O}_X)^\pm$ and $\Phi' \in \text{Hom}_X(F' \otimes F', \mathcal{O}_X)^\pm$ satisfying (3.9). Let $(\overline{\eta}, \overline{\Psi})$ be the element of $\ker(\delta_0)$ associated to the first extension equipped with Φ , and let $(\overline{\eta}', \overline{\Psi}') \in \ker(\delta_0)$ be the pair associated to the second extension and Φ' . Since λ_1 defines an isomorphism of extensions, it then comes from some isomorphism $E \oplus W_0 \rightarrow E \oplus W_0$ of the form (3.8) for some $\lambda \in \text{Hom}_X(W_0, W_0)$. It then follows that, λ is such that (3.7) holds, so both $(\overline{\eta}, \overline{\Psi})$ and $(\overline{\eta}', \overline{\Psi}')$ are related by the action of $\text{Im}(\delta_{-1})$. \square

Proposition 3.4. *Let E be a coherent sheaf over X smooth and projective and consider $\phi : E \otimes E \rightarrow \mathcal{O}_X$ such that $\phi \circ \theta_E = \pm \phi$. Consider a finite dimensional k -vector space V . Then, $V \otimes_k \mathbb{H}^2(\Delta_{(E, \phi)}^{\pm, \bullet})$ is the space classifying equivalence classes of 2-extensions*

$$0 \longrightarrow V \otimes_k E \xrightarrow{i} F \xrightarrow{f} G \xrightarrow{j} E \longrightarrow 0, \quad (3.10)$$

together with a class

$$[\mu] \in \text{Hom}_X(G \otimes F, V \otimes_k \mathcal{O}_X) / (\text{Hom}_X(G \otimes G, V \otimes_k \mathcal{O}_X)^\pm \circ (\mathbf{1}_G \otimes f))$$

whose elements satisfy

$$\mu \circ (\mathbf{1}_G \otimes i) = \mathbf{1}_V \otimes \phi(j \otimes \mathbf{1}_E), \quad (3.11)$$

and

$$\mu \circ (f \otimes \mathbf{1}_F) = \pm \mu \circ (f \otimes \mathbf{1}_F) \circ \theta_F. \quad (3.12)$$

The zero element in $V \otimes_k \mathbb{H}^2(\Delta_{(E, \phi)}^{\pm, \bullet})$ corresponds to a 2-extension that splits

$$0 \longrightarrow V \otimes_k E \xrightarrow{i} F = (V \otimes_k E) \oplus \ker j \xrightarrow{f} G \xrightarrow{j} E \longrightarrow 0, \quad (3.13)$$

and $[\mu]$ such that

$$[\mu] \circ (\mathbf{1}_G \otimes q) = 0 \quad (3.14)$$

in $\text{Ext}^1(E \otimes E, V \otimes_k \mathcal{O}_X)^\pm$, where q denotes the projection $F \rightarrow \ker j$.

Proof. We study $V \otimes_k \mathbb{H}^1(\text{Cone}(\Delta_{(E, \phi)}^\pm))$. Using the locally free resolution $W^\bullet \xrightarrow{\pi} E \rightarrow 0$, the mapping cone of $\Delta_{(E, \phi)}^\pm$ in degrees 1 and 2 is given by

$$\left(\begin{array}{c} \text{Hom}_X(W_{-2}, E) \\ \text{Hom}_X(W_0 \otimes W_{-1}, \mathcal{O}_X) \end{array} \right) \xrightarrow{\delta_1} \left(\begin{array}{c} \text{Hom}_X(W_{-3}, E) \\ \text{Hom}_X(W_{-1} \otimes W_{-1}, \mathcal{O}_X)^\mp \\ \text{Hom}_X(W_0 \otimes W_{-2}, \mathcal{O}_X) \end{array} \right),$$

where

$$\delta_1 = \left(\begin{array}{cc} -(\bullet) \circ \partial_{-3} & 0 \\ 0 & (\bullet) \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) \circ (\mathbf{1}_{W_{-1}} \mp \theta_{W_{-1}}) \\ \phi \circ (\pi \otimes (\bullet)) & -(\bullet) \circ (\mathbf{1}_{W_0} \otimes \partial_{-2}) \end{array} \right).$$

Then,

$$\ker(\delta_1) = \left\{ \begin{array}{l} (\chi, \Xi) \in \text{Hom}_X(W_{-2}, E) \oplus \text{Hom}_X(W_0 \otimes W_{-1}, \mathcal{O}_X) \\ \chi(\partial_{-3}(W_{-3})) = 0 \\ \Xi \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) = \pm \Xi \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) \circ \theta_{W_{-1}} \\ \phi \circ (\pi \otimes \chi) - \Xi \circ (\mathbf{1}_{W_0} \otimes \partial_{-2}) = 0 \end{array} \right\}.$$

Using $(\overline{\chi}, \overline{\Xi}) \in V \otimes_k \ker(\delta_1)$, where $\overline{\chi} \in \text{Hom}_X(W_{-1}, V \otimes_k E)$ and $\overline{\Xi} \in \text{Hom}_X(W_0 \otimes W_{-1}, V \otimes_k \mathcal{O}_X)$ consider the projection $j: G \rightarrow E$ to be $\pi: W_0 \rightarrow E$ and let us construct

$$F := (V \otimes_k E) \oplus W_{-1} / (\overline{\chi} \oplus \partial_{-2})(W_{-2}) \quad (3.15)$$

and consider the injection

$$\begin{array}{ccc} i: E & \longrightarrow & F = (V \otimes_k E) \oplus W_{-1} / (\overline{\chi} \oplus \partial_{-2})(W_{-2}) \\ \overline{e} & \longmapsto & [(\overline{e}, 0)] \end{array}$$

and the morphism

$$\begin{array}{ccc} f: F = (V \otimes_k E) \oplus W_{-1} / (\overline{\chi} \oplus \partial_{-2})(W_{-2}) & \longrightarrow & G = W_0 \\ [(\overline{e}, w)] & \longmapsto & \partial_{-1}(w), \end{array}$$

which is well defined since $\partial_{-1} \circ \partial_{-2} = 0$. Note that we have obtained a 2-extension of the form of (3.10).

Consider now

$$\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}: W_0 \otimes ((V \otimes_k E) \oplus W_{-1}) \longrightarrow V \otimes_k \mathcal{O}_X.$$

Since $(\overline{\chi}, \overline{\Xi}) \in V \otimes_k \ker(\delta_1)$ satisfy $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \overline{\chi}) - \overline{\Xi} \circ (\mathbf{1}_{W_0} \otimes \partial_{-2}) = 0$, it follows that $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}$ vanishes at $W_0 \otimes ((\overline{\chi} \oplus \partial_{-2})(W_{-2}))$, hence it descends to

$$\mu: G \otimes F \rightarrow V \otimes_k \mathcal{O}_X,$$

where we recall (3.15) and that $G = W_0$. Since $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}$ restricted to $G \otimes (V \otimes_k E \oplus 0)$ amounts to $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E)$, (3.11) follows naturally. Note also that

$$(\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}) \circ (\partial_{-1} \otimes \mathbf{1}_{(V \otimes_k E) \oplus W_{-1}}) = \overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}).$$

Then (3.12) follows from the identity $\overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) = \pm \overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) \circ \theta_{W_{-1}}$ that any $(\overline{\chi}, \overline{\Xi}) \in V \otimes_k \ker(\delta_1)$ satisfies.

For any $\overline{\eta} \in V \otimes_k \text{Hom}_X(W_{-2}, E)$, we set

$$(\overline{\chi}', \overline{\Xi}') = (\overline{\chi}, \overline{\Xi}) + (\mathbf{1}_V \otimes_k \delta_0) \cdot (\overline{\eta}, 0) = (\overline{\chi} - \overline{\eta} \circ \partial_{-2}, \overline{\Xi} + \mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \eta)),$$

and define

$$F' = (V \otimes_k E) \oplus W_{-1} / (\overline{\chi}' \oplus \partial_{-2})(W_{-2}).$$

Observe that the isomorphism

$$\begin{pmatrix} \mathbf{1}_{V \otimes_k E} & -\overline{\eta} \\ 0 & \mathbf{1}_{W_0} \end{pmatrix}: (V \otimes_k E) \oplus W_{-1} \xrightarrow{\cong} (V \otimes_k E) \oplus W_{-1}. \quad (3.16)$$

sends $(\overline{\chi} \oplus \partial_{-2})(W_{-2})$ to $(\overline{\chi}' \oplus \partial_{-2})(W_{-2})$, hence (3.16) provides an isomorphism

$$\eta_F: F \xrightarrow{\cong} F'.$$

One obtains a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V \otimes_k E & \longrightarrow & F & \longrightarrow & G = W_0 & \longrightarrow & E & \longrightarrow & 0 \\ & & \parallel & & \downarrow \eta_F & & \parallel & & \parallel & & \\ 0 & \longrightarrow & V \otimes_k E & \longrightarrow & F' & \longrightarrow & G' = W_0 & \longrightarrow & E & \longrightarrow & 0, \end{array}$$

so both $\overline{\chi}$ and $\overline{\chi}'$ define the same class of extensions. Note also that $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}'$ vanishes in $(\overline{\chi}' \oplus \partial_{-2})(W_{-2})$, defining $\mu': G' \otimes F' \rightarrow V \otimes_k \mathcal{O}_X$ satisfying (3.11) and (3.12). Note also that

$$\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}' = \mathbf{1}_V \otimes_k (\phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}) \circ \begin{pmatrix} \mathbf{1}_{V \otimes_k E} & -\overline{\eta} \\ 0 & \mathbf{1}_{W_0} \end{pmatrix},$$

so one gets

$$\mu' \circ (\mathbf{1}_G \otimes \eta_1) = \mu.$$

Therefore, the 2-extension and the morphism that we obtain from $(\overline{\chi}, \overline{\Xi})$ are equivalent to the 2-extension and the morphism that we obtain from $(\overline{\chi}', \overline{\Xi}')$.

Take now $\overline{\Psi} \in V \otimes_k \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm$ and define

$$(\overline{\chi}, \overline{\Xi}') = (\overline{\chi}, \overline{\Xi}) + (\mathbf{1}_V \otimes_k \delta_0) \cdot (0, \overline{\Psi}) = (\overline{\chi}, \overline{\Xi} + \overline{\Psi} \circ (\mathbf{1}_{W_0} \otimes \partial_{-1})).$$

Since $\overline{\chi}$ does not change, we get F and G as before. We observe that $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi}'$ descends to $\mu'' = \mu + \overline{\Psi} \circ (\mathbf{1}_G \otimes f)$, so $[\mu''] = [\mu]$. Observe that μ'' also satisfies (3.11) and (3.12).

Conversely, choose representants (3.10) and $\mu : G \otimes F \rightarrow V \otimes_k \mathcal{O}_X$ of a given 2-extension class of E by E , and of a certain class in $\text{Hom}_X(G \otimes F, V \otimes_k \mathcal{O}_X) / \text{Hom}_X(G \otimes G, V \otimes_k \mathcal{O}_X)^\pm \circ (\mathbf{1}_{W_0} \otimes f)$ satisfying (3.11) and (3.12). Using the universal property of projective modules (recall that locally free sheaves are projective) one can always complete to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_{-2}/\partial_{-3}(W_{-3}) & \longrightarrow & W_{-1} & \longrightarrow & W_0 & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow \overline{\chi} & & \downarrow \chi_F & & \downarrow \chi_G & & \parallel & & \\ 0 & \longrightarrow & V \otimes_k E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 0. \end{array} \quad (3.17)$$

This defines a morphism $\overline{\chi} : W_{-2} \rightarrow V \otimes_k E$ with $\overline{\chi}(\partial_{-3}(W_{-3})) = 0$. Let us denote the composition of $\overline{\Xi} := \mu \circ (\chi_G, \chi_F)$. Note that the commutativity of (3.17) together with (3.11) and (3.12) imply, respectively, that

$$\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \overline{\chi}) - \overline{\Xi} \circ (\mathbf{1}_{W_0} \otimes \partial_{-2}) = 0$$

and

$$\overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) = \pm \overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) \circ \theta_{W_{-1}}.$$

This completes the proof of the first statement.

To describe the 0 element in $V \otimes_k \mathbb{H}^1(\text{Cone}(\Delta_{(E,\phi)}^\pm))$, we first note that any $(0, \overline{\Xi}) \in V \otimes_k \ker(\delta_1)$ gives

$$F = (V \otimes_k E) \oplus W_{-1}/(0 \oplus \delta_{-2})(W_{-2}) = (V \otimes_k E) \oplus \ker \pi,$$

with $\overline{\Xi}$ satisfying

$$\overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) = \pm \overline{\Xi} \circ (\partial_{-1} \otimes \mathbf{1}_{W_{-1}}) \circ \theta_{W_{-1}}$$

and

$$\overline{\Xi} \circ (\mathbf{1}_{W_0} \otimes \partial_{-2}) = 0.$$

Therefore, $(0, \overline{\Xi}) \in \ker(\delta_1)$ determines a short exact sequence of the form (3.13) together with an element $[\overline{\Xi}] \in \text{Ext}_X^1(E \otimes E, V \otimes_k \mathcal{O}_X)^\pm$. If μ is the descent of $\mathbf{1}_V \otimes_k \phi \circ (\pi \otimes \mathbf{1}_E) - \overline{\Xi} : W_0 \otimes (V \otimes_k E \oplus W_{-1}) \rightarrow V \otimes_k \mathcal{O}_X$ to a morphism in $\text{Hom}_X(W_0 \otimes \ker(\pi), V \otimes_k \mathcal{O}_X)^\pm$, observe that $\mu \circ (\mathbf{1}_G \otimes q)$ coincides with $\overline{\Xi}$. This concludes the proof. \square

4. Deformation of orthogonal and symplectic sheaves

Given an orthogonal sheaf (E, ϕ) over the projective scheme X , we define its deformation functor

$$\text{Def}_{(E,\phi)}^+ : (\mathbf{Art}) \longrightarrow (\mathbf{Sets})$$

by associating to any $A \in (\mathbf{Art})$ the set of isomorphism classes of triples $(\mathcal{E}, \Phi, \gamma)$, where (\mathcal{E}, Φ) is a family of orthogonal sheaves on X_A , (so \mathcal{E} is a torsion-free coherent sheaf on X_A flat over $\text{Spec}(A)$), and $\gamma : (\mathcal{E}, \Phi)|_X \rightarrow (E, \phi)$ is an isomorphism of orthogonal (resp. symplectic) sheaves. Two triples $(\mathcal{E}, \Phi, \gamma)$ and $(\mathcal{E}', \Phi', \gamma')$ are isomorphic if there exists an isomorphism $f : (\mathcal{E}, \Phi) \rightarrow (\mathcal{E}', \Phi')$ such that $\gamma' \circ (f|_X) = \gamma$. As in the case of the deformation functor of coherent sheaves, functoriality of $\text{Def}_{(E,\phi)}^+$ follows from applying pull-backs under the morphisms $p_a : X_{A'} \rightarrow X_A$ for any $a : A \rightarrow A'$.

Analogously, associated to any symplectic sheaf (E, ϕ) over X , its deformation functor

$$\text{Def}_{(E,\phi)}^- : (\mathbf{Art}) \longrightarrow (\mathbf{Sets})$$

is constructed by associating to every $A \in (\mathbf{Art})$ the set of isomorphism classes of triples $(\mathcal{E}, \Phi, \gamma)$, where (\mathcal{E}, Φ) is a family of symplectic sheaves on X_A , and $\gamma : (\mathcal{E}, \Phi)|_X \rightarrow (E, \phi)$ is an isomorphism of symplectic sheaves. The notion of isomorphism of triples is analogous to the case of orthogonal sheaves. As before, functoriality under pull-backs holds in this case as well.

We see that the complex $\Delta_{(E,\phi)}^{\pm, \bullet}$ governs the deformation theory of $\text{Def}_{(E,\phi)}^\pm$.

Theorem 4.1. Let (E, ϕ) be an orthogonal (resp. symplectic) sheaf over the smooth projective scheme X . Then the deformation functor $\text{Def}_{(E, \phi)}^+$ (resp. $\text{Def}_{(E, \phi)}^-$) admits a miniversal pro-family and the associated space of first-order deformations is

$$\text{Def}_{(E, \phi)}^\pm(k[\epsilon]/(\epsilon^2)) \cong \mathbb{H}^1(\Delta_{(E, \phi)}^{\pm, \bullet}). \quad (4.1)$$

If, further, (E, ϕ) is simple, $\text{Def}_{(E, \phi)}^\pm$ is pro-representable.

Proof. Given a torsion-free sheaf E with $\phi : E \otimes E \rightarrow \mathcal{O}_X$ satisfying $\phi = \pm \phi \circ \theta_E$, we have to check whether $\text{Def}_{(E, \phi)}^\pm$ satisfies the Schlessinger conditions **S1**, **S2** and **S3**. Let us denote by $b : B \times_A C \rightarrow B$ the projection to the first factor and by $c : B \times_A C \rightarrow C$ the projection to the second. Consider the associated morphisms $p_b : X_B \rightarrow X_{B \times_A C}$ and $p_c : X_{B \times_A C} \rightarrow X_C$. Condition **S1** holds if for any homomorphism $B \rightarrow A$ and any small extension $0 \rightarrow H \rightarrow C \rightarrow A \rightarrow 0$ in **(Art)**, the morphism induced by taking pull-backs under p_b and p_c ,

$$\text{Def}_{(E, \phi)}^\pm(B \times_A C) \longrightarrow \text{Def}_{(E, \phi)}^\pm(B) \times_{\text{Def}_{(E, \phi)}^\pm(A)} \text{Def}_{(E, \phi)}^\pm(C),$$

is surjective. Consider $(\mathcal{E}_B, \Phi_B, \gamma_B) \in \text{Def}_{(E, \phi)}^\pm(B)$ and $(\mathcal{E}_C, \Phi_C, \gamma_C) \in \text{Def}_{(E, \phi)}^\pm(C)$ for which there exists an isomorphism $f : \mathcal{E}_B|_{X_A} \rightarrow \mathcal{E}_C|_{X_A}$ satisfying that $\gamma_B = \gamma_C \circ f|_X$ and $\Phi_B|_{X_A} = \Phi_C|_{X_A} \circ (f \otimes f)$. Thanks to deformation theory of sheaves we know that Def_E verifies **S1**, so

$$\text{Def}_E(B \times_A C) \longrightarrow \text{Def}_E(B) \times_{\text{Def}_E(A)} \text{Def}_E(C),$$

is surjective and there exists $(\mathcal{E}', \gamma) \in \text{Def}_E(B \times_A C)$ such that $g_B : (\mathcal{E}', \gamma)|_{X_B} \xrightarrow{\cong} (\mathcal{E}_B, \gamma_B)$ and $g_C : (\mathcal{E}', \gamma)|_{X_C} \xrightarrow{\cong} (\mathcal{E}_C, \gamma_C)$ satisfy that $g_C|_{X_A} = f \circ g_B|_{X_A}$. It remains to construct a quadratic form on \mathcal{E}' compatible with Φ_B and Φ_C under pull-backs.

Write $\mathcal{E}'_B, \mathcal{E}'_C$ and \mathcal{E}'_A for $\mathcal{E}'|_{X_B}, \mathcal{E}'|_{X_C}$ and $\mathcal{E}'|_{X_A}$ respectively. One trivially has that

$$\mathcal{E}' = \mathcal{E}'_B \times_{\mathcal{E}'_A} \mathcal{E}'_C, \quad (4.2)$$

so

$$\mathcal{E}' \otimes \mathcal{E}' = (\mathcal{E}'_B \otimes \mathcal{E}'_B) \times_{(\mathcal{E}'_A \otimes \mathcal{E}'_A)} (\mathcal{E}'_C \otimes \mathcal{E}'_C).$$

Since $g_C|_{X_A} = f \circ g_B|_{X_A}$ and $\Phi_B|_{X_A} = \Phi_C|_{X_A} \circ (f \otimes f)$, it follows that

$$(g_B \otimes g_B)^* \Phi_B|_{X_A} = (g_C \otimes g_C)^* \Phi_C|_{X_A}.$$

Then, they define

$$\Phi : \mathcal{E}' \otimes \mathcal{E}' \longrightarrow \mathcal{O}_{X_{B \times_A C}},$$

which naturally satisfies $p_b^* \Phi = \Phi_B$ and $p_c^* \Phi = \Phi_C$. By hypothesis, one has that $p_b^* \Phi_B = \pm p_b^* \Phi_B \circ \theta_{\mathcal{E}'_B}$ and similarly for C and A . It then follows that

$$\Phi = \pm \Phi \circ \theta_{\mathcal{E}'}.$$

Recall that $U_{\mathcal{E}'}$ the open subset of $X_{(B \times_A C)}$ where \mathcal{E}' is locally free. Thanks to (4.2), one has that

$$U_{\mathcal{E}'} = U_{\mathcal{E}'_B} \times_{U_{\mathcal{E}'_A}} U_{\mathcal{E}'_C}.$$

By hypothesis, $(g_B \times g_B)^* \Phi_B$ and $(g_C \times g_C)^* \Phi_C$ are non-degenerate over $U_{\mathcal{E}'_B} = U_{\mathcal{E}'_B}$ and $U_{\mathcal{E}'_C} = U_{\mathcal{E}'_C}$. Since Φ is constructed by gluing $(g_B \times g_B)^* \Phi_B$ and $(g_C \times g_C)^* \Phi_C$, it then follows that Φ is non-degenerate over $U_{\mathcal{E}'}$. This proves that $\text{Def}_{(E, \phi)}^\pm$ satisfies **S1**.

Since Def_E satisfies the condition **S2**, to prove that $\text{Def}_{(E, \phi)}^\pm$ also full-fills this condition it is enough to see that, when $A = k$, given a quadratic form Φ' on \mathcal{E}' compatible with γ and such that $\Phi'|_{\mathcal{E}'_B \otimes \mathcal{E}'_B} = (h_B \otimes h_B)^* \Phi|_{\mathcal{E}'_B \otimes \mathcal{E}'_B}$ for an automorphism h_B of $(\mathcal{E}'_B, \gamma|_{\mathcal{E}'_B})$ and $\Phi'|_{\mathcal{E}'_C \otimes \mathcal{E}'_C} = (h_C \otimes h_C)^* \Phi|_{\mathcal{E}'_C \otimes \mathcal{E}'_C}$ under an automorphism h_C of $(\mathcal{E}'_C, \gamma|_{\mathcal{E}'_C})$, one can construct an automorphism h of (\mathcal{E}', γ) sending Φ' to Φ . Since $h_B|_X = \gamma|_X = h_C|_X$, it follows from (4.2) that one can construct $h : \mathcal{E}' \rightarrow \mathcal{E}'$ by gluing h_B and h_C along X . It is straight-forward that h satisfies the required condition so $\text{Def}_{(E, \phi)}^\pm$ satisfies **S2**.

We address **S3** now. First, note that one can endow the space of first-order deformations $\text{Def}_{(E, \phi)}^\pm(k\langle k \rangle)$ with a k -vector space structure. Using the inverse of the bijective map of **S2** when $B = k\langle k \rangle$, and the morphism $\langle + \rangle : k\langle k \oplus k \rangle \rightarrow k\langle k \rangle$ induced by the sum of the elements in the maximal ideal, one can define the sum within the space of first-order deformations,

$$\text{Def}_{(E, \phi)}^\pm(k\langle k \rangle) \times \text{Def}_{(E, \phi)}^\pm(k\langle k \rangle) \xrightarrow{1:1} \text{Def}_{(E, \phi)}^\pm(k\langle k \oplus k \rangle) \xrightarrow{\text{Def}_{(E, \phi)}^\pm(\langle + \rangle)} \text{Def}_{(E, \phi)}^\pm(k\langle k \rangle).$$

One can easily check that $\text{Def}_{(E,\phi)}^\pm(k\langle k \rangle)$ equipped with this sum satisfies the axioms of a k -vector space.

Recall now that $k\langle k \rangle = k[\epsilon]/(\epsilon^2)$. Suppose that M is a $\mathcal{O}_X \times_k k[\epsilon]/(\epsilon^2)$ -module. Since $\mathcal{O}_X \times_k k[\epsilon]/(\epsilon^2) = \mathcal{O}_X + \epsilon \mathcal{O}_X$ with $\epsilon^2 = 0$, we see that a $\mathcal{O}_X \times_k k[\epsilon]/(\epsilon^2)$ -module structure is determined by a \mathcal{O}_X -module structure and the action of ϵ on it, $\epsilon k \times_k M \rightarrow k \times_k M$. Thanks to Proposition 3.3, one can then consider the map

$$\mathbb{H}^1(\Delta_{(E,\phi)}^\pm) \longrightarrow \text{Def}_{(E,\phi)}^\pm(k[\epsilon]/(\epsilon^2)) \quad (4.3)$$

that sends the short exact sequence of \mathcal{O}_X -modules $0 \rightarrow E \xrightarrow{i} F \xrightarrow{j} E \rightarrow 0$ and $\Phi_1 : F \otimes F \rightarrow \mathcal{O}_X$ to the triple $(\mathcal{E}, \Phi, \gamma)$ where \mathcal{E} is F endowed with a $\mathcal{O}_X \times_k k[\epsilon]/(\epsilon^2)$ -module structure determined by the usual \mathcal{O}_X -module structure on F and the action of ϵ on \mathcal{E} defined by the composition $j \circ i : E \rightarrow E$. The isomorphism $\gamma : \mathcal{E}|_X \rightarrow E$ is determined by the projection $j : F \rightarrow E$ and Φ is naturally determined by Φ_1 .

Conversely, given the isomorphism class of $(\mathcal{E}, \Phi, \gamma)$ in $\text{Def}_{(E,\phi)}^\pm(k[\epsilon]/(\epsilon^2))$, we construct an exact sequence $0 \rightarrow E \xrightarrow{i} F \xrightarrow{j} E \rightarrow 0$. Tensorize \mathcal{E} with $0 \rightarrow k \rightarrow k[\epsilon]/(\epsilon^2) \rightarrow k \rightarrow 0$ to obtain

$$0 \longrightarrow k \otimes_{k[\epsilon]/(\epsilon^2)} \mathcal{E} \cong E \xrightarrow{i'} \mathcal{E} \xrightarrow{j'} k \otimes_{k[\epsilon]/(\epsilon^2)} \mathcal{E} \cong E \longrightarrow 0.$$

Then consider the push-forward under the natural projection $\pi : X \times_k \text{Spec}(k[\epsilon]/(\epsilon^2)) \rightarrow X$ gives a short exact sequence given by $F = \pi_* \mathcal{E}$, the projection $j : F \rightarrow E$ is determined by the composition $\gamma \circ \pi_* j'$ and $\Phi_1 = (\pi \otimes \pi)^* \Phi$. This provides an inverse for (4.3) so **S3** is satisfied and (4.1) holds.

Finally, we address **S4** for (E, ϕ) simple. Observe that it follows naturally from **S1** and Lemma 4.2. \square

The following result provides a characterization of simple orthogonal and symplectic sheaves on X_A in terms of their restriction to the closed subset $X \subset X_A$.

Lemma 4.2. *For any $A \in (\mathbf{Art})$ and any orthogonal (resp. symplectic) sheaf (\mathcal{E}_A, Φ_A) over X_A , we have that (\mathcal{E}_A, Φ_A) is simple if and only if $(E, \phi) := (\mathcal{E}_A|_X, \Phi_A|_X)$ is simple.*

Proof. The A -module $H^0(X_A, \text{End}(\mathcal{E}_A))$ is finitely generated. Note that $H^0(X_A, \text{End}(\mathcal{E}_A)) \otimes_A k = H^0(X, \text{End}(E))$ and $\mathbf{1}_{\mathcal{E}_A} \otimes_A 1 = \mathbf{1}_E$.

By Nakayama's lemma, if $\{\mathbf{1}_{\mathcal{E}_A}, b_2, \dots, b_n\} \subset H^0(X_A, \text{End}(\mathcal{E}_A))$ are such that $\{\mathbf{1}_E, (b_2 \otimes_A 1), \dots, (b_n \otimes_A 1)\}$ is a basis of $H^0(X, \text{End}(E))$, then $\{\mathbf{1}_{\mathcal{E}_A}, b_2, \dots, b_n\}$ generate $H^0(X_A, \text{End}(\mathcal{E}_A))$. Then, there is no $b' \in H^0(X_A, \text{End}(\mathcal{E}_A))$ such that $b' \neq \mathbf{1}_{\mathcal{E}_A}$ with restriction $b' \otimes_A 1 = \mathbf{1}_E$. Otherwise, as $b' = a_1 \otimes_A \mathbf{1}_{\mathcal{E}_A} + a_2 \otimes_A b_2 + \dots + a_n \otimes_A b_n$, we would obtain a contradiction with the linear independence of $\{\mathbf{1}_E, (b_2 \otimes_A 1), \dots, (b_n \otimes_A 1)\}$.

Then, the only element in $H^0(X_A, \text{End}(\mathcal{E}_A))$ that restrict to $\mathbf{1}_E$ is $\mathbf{1}_{\mathcal{E}_A}$ itself. If $\text{Aut}(E, \phi) = \{\mathbf{1}_E, -\mathbf{1}_E\}$, it then follows that $\text{Aut}(\mathcal{E}_A, \Phi_A)$ is $\{\mathbf{1}_{\mathcal{E}_A}, -\mathbf{1}_{\mathcal{E}_A}\}$. \square

5. Obstruction theory for orthogonal and symplectic sheaves

In this section we will see that the second cohomology space of the deformation complex defined in Section 3 provides an obstruction theory for the deformation functors of orthogonal and symplectic sheaves. We begin by the construction of the morphism (2.1) in this case.

Let (E, ϕ) be an orthogonal (resp. symplectic) sheaf over X projective and let $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$ be a small extension of Artin algebras with residue field k . We want to construct a morphism from $\text{Def}_{(E,\phi)}^\pm(A)$ to $H \otimes_k \mathbb{H}^2(\Delta_{(E,\phi)}^\pm)$. Then, after Proposition 3.4, given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}_{(E,\phi)}^\pm(A)$, we should construct a 2-extension of the form (3.10) equipped with a class of morphisms $[\mu] \in \text{Hom}_X(G \otimes F, \mathcal{O}_X) / (\text{Hom}_X(G \otimes G, \mathcal{O}_X)^\pm \circ (\mathbf{1}_G \otimes f))$ satisfying (3.11) and (3.12).

From the small extension $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$, one naturally obtains the short exact sequence of \mathcal{O}_{X_B} -modules

$$0 \longrightarrow H \otimes_k \mathcal{O}_X \xrightarrow{\sigma_0} \mathcal{O}_{X_B} \xrightarrow{\rho_0} p_* \mathcal{O}_{X_A} \longrightarrow 0, \quad (5.1)$$

where we denote by $p : X_A \hookrightarrow X_B$ the morphism associated to $B \twoheadrightarrow A$.

Let us consider a locally free resolution $\mathcal{W}_A^\bullet \xrightarrow{\pi_A} \mathcal{E}_A \rightarrow 0$ such that, for $i > 0$,

$$H^i(X_A, \mathcal{W}_{A,0}) = 0.$$

Along with the above locally free resolution, consider a locally free sheaf $\mathcal{W}_{B,0}$ satisfying $\mathcal{W}_{B,0}|_{X_A} \cong \mathcal{W}_{A,0}$ and

$$H^i(X_B, \mathcal{W}_{B,0}) = 0, \quad (5.2)$$

for $i > 0$. It can be easily verified that such a choice exists. Set $W_i := \mathcal{W}_{A,i}|_X$, with differentials $\partial_i := \partial_{A,i}|_X$ and $\pi := \gamma_A \circ \pi_A|_X : W_0 \rightarrow E$. Note that $W^\bullet \xrightarrow{\pi} E \rightarrow 0$ is a locally free resolution.

Inspired by the classical approach to obstruction theory of coherent sheaves, we consider the short exact sequences of \mathcal{O}_{X_B} -modules

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & (5.3) \\
 & \downarrow & & \downarrow & & & \\
 & H \otimes_k \partial_{-1}(W_{-1}) & & p_* \partial_{A,-1}(\mathcal{W}_{A,-1}) & & & \\
 & \downarrow & & \downarrow & & & \\
 0 \longrightarrow & H \otimes_k W_0 & \xrightarrow{\sigma} & \mathcal{W}_{B,0} & \xrightarrow{\rho} & p_* \mathcal{W}_{A,0} & \longrightarrow 0 \\
 & \downarrow & & & & \downarrow \pi_A & \\
 & H \otimes_k E & & & & p_* \mathcal{E}_A & \\
 & \downarrow & & & & \downarrow & \\
 & 0 & & & & 0, &
 \end{array}$$

where ρ is induced by the restriction $X_B \rightarrow X_A$ and σ is the inclusion of the kernel of ρ . Denote

$$\mathcal{F} := \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1})) / \sigma(H \otimes_k \partial_{-1}(W_{-1})). \quad (5.4)$$

We then see that, out of (5.3), one can naturally construct an extension

$$0 \longrightarrow H \otimes_k E \xrightarrow{i_0} \mathcal{F} \xrightarrow{f_0} p_* \partial_{A,-1}(\mathcal{W}_{A,-1}) \longrightarrow 0, \quad (5.5)$$

which defines naturally $\zeta \in \text{Ext}_{X_B}^1(p_* \partial_{A,-1}(\mathcal{W}_{A,-1}), H \otimes_k E)$.

Having (5.1) in mind, observe that any \mathcal{O}_{X_B} -module \mathcal{M} such that $(H \otimes_k \mathcal{O}_X) \otimes \mathcal{M} = 0$ is naturally equipped with the inherited $\mathcal{O}_{X_B}/(H \otimes_k \mathcal{O}_X) \cong p_* \mathcal{O}_{X_A}$ -module structure. Note that there exists an equivalence of categories sending the $p_* \mathcal{O}_{X_A}$ -module \mathcal{M} to the \mathcal{O}_{X_A} -module \mathcal{M}^A . Under this equivalence, $p_* \partial_{A,-1}(\mathcal{W}_{A,-1})$ gives naturally the \mathcal{O}_{X_A} -module $\partial_{A,-1}(\mathcal{W}_{A,-1})$. Since $H^2 = 0$, we see that $H \otimes_k E$ is annihilated by $H \otimes_k \mathcal{O}_X$ producing the \mathcal{O}_{X_A} -module $H \otimes_k p_{A,*} E$, where $p_A : X \hookrightarrow X_A$ is the inclusion of the closed reduced subscheme associated to the structural projection to the residue field, $A \rightarrow k$. Hence, after (5.5) and right-exactness of tensor product, one has that $(H \otimes_k \mathcal{O}_X) \otimes \mathcal{F} = 0$ so \mathcal{F} gives rise to the \mathcal{O}_{X_A} -module \mathcal{F}^A . Therefore, from (5.5) we obtain the extension of \mathcal{O}_{X_A} -modules

$$0 \longrightarrow H \otimes_k p_{A,*} E \xrightarrow{i_0^A} \mathcal{F}^A \xrightarrow{f_0^A} \partial_{A,-1}(\mathcal{W}_{A,-1}) \longrightarrow 0, \quad (5.6)$$

associated to $\zeta^A \in \text{Ext}_{X_A}^1(\partial_{A,-1}(\mathcal{W}_{A,-1}), H \otimes_k p_{A,*} E)$.

Composing (5.6) with the projection $\mathcal{W}_{A,0} \xrightarrow{\pi_A} \mathcal{E}_A$, one gets an element $\pi_A^* \zeta^A$ of $\text{Ext}_{X_A}^2(\mathcal{E}_A, H \otimes_k p_{A,*} E)$ associated to the 2-extension

$$0 \longrightarrow H \otimes_k p_{A,*} E \xrightarrow{i_0^A} \mathcal{F}^A \xrightarrow{f_0^A} \mathcal{W}_{A,0} \xrightarrow{\pi_A} \mathcal{E}_A \longrightarrow 0. \quad (5.7)$$

One can prove that $\text{Ext}_{X_A}^2(\mathcal{E}_A, H \otimes_k p_{A,*} E) \cong \text{Ext}_X^2(E, H \otimes_k E)$, so (5.7) is completely determined by its restriction to X ,

$$0 \longrightarrow H \otimes_k E \xrightarrow{i} F := \mathcal{F}^A|_X \xrightarrow{f} W_0 \xrightarrow{\pi} E \longrightarrow 0. \quad (5.8)$$

Note that i and f are given respectively by $i_0^A|_X$, $f_0^A|_X$ and we recall that $\pi = \gamma_A \circ \pi_A|_X$. Setting $G := W_0$ and $j := \pi$, we see that (5.8) gives a 2-extension of the form (3.10).

Take now $\Phi_A : \mathcal{E}_A \otimes \mathcal{E}_A \rightarrow \mathcal{O}_{X_A}$, and consider $(\pi_A \otimes \pi_A)^* \Phi_A \in \text{Hom}_{X_A}(\mathcal{W}_{A,0} \otimes \mathcal{W}_{A,0}, \mathcal{O}_{X_A})^\pm$. Recalling (5.2), one naturally has that $\text{Ext}_{X_B}^1(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, \mathcal{O}_{X_B}) = 0$, hence the functor $\text{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, \bullet)^\pm$ applied to the short exact sequence (5.1) returns the following short exact sequence in cohomology,

$$\begin{aligned}
 0 \longrightarrow \text{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, H \otimes_k \mathcal{O}_X)^\pm &\longrightarrow \text{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, \mathcal{O}_{X_B})^\pm \\
 &\longrightarrow \text{Hom}_{X_A}(\mathcal{W}_{A,0} \otimes \mathcal{W}_{A,0}, \mathcal{O}_{X_A})^\pm \longrightarrow 0.
 \end{aligned} \quad (5.9)$$

It follows that $(\pi_A \otimes \pi_A)^* \Phi_A$ determines a class $[\Upsilon]$ in

$$\text{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, \mathcal{O}_{X_B})^\pm / \sigma_0 \circ \text{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, H \otimes_k \mathcal{O}_X)^\pm, \quad (5.10)$$

where

$$\begin{aligned} \mathrm{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, H \otimes_k \mathcal{O}_X)^\pm &\cong \mathrm{Hom}_{X_B}(\mathcal{W}_{B,0}|_X \otimes \mathcal{W}_{B,0}|_X, H \otimes_k \mathcal{O}_X)^\pm \\ &\cong H \otimes_k \mathrm{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm. \end{aligned} \quad (5.11)$$

Pick any representant $\Upsilon \in \mathrm{Hom}_{X_B}(\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}, \mathcal{O}_{X_B})^\pm$ of the class $[\Upsilon]$ in (5.10) fixed by $(\pi_A \otimes \pi_A)^* \Phi_A$. We obviously have that $\Upsilon|_{X_A} = (\pi_A \otimes \pi_A)^* \Phi_A$. Therefore, the restriction to X_A of the image under Υ of the subspace $\mathcal{W}_{B,0} \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1})) \subset \mathcal{W}_B \otimes \mathcal{W}_B$ vanishes,

$$\Upsilon(\mathcal{W}_{B,0} \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1})))|_{X_A} = (\pi_A \otimes \pi_A)^* \Phi_A(\mathcal{W}_A \otimes \partial_{A,-1}(\mathcal{W}_{A,-1})) = 0.$$

It then follows that,

$$\Upsilon(\mathcal{W}_{B,0} \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1}))) \subset \sigma(H \otimes_k \mathcal{O}_X). \quad (5.12)$$

Since $H^2 = 0$ one has that the intersection of $(H \otimes_k W_0) \otimes \mathcal{W}_{B,0}$ and $\mathcal{W}_{B,0} \otimes (H \otimes_k W_0)$ is $0 \cong (H \otimes_k W_0) \otimes (H \otimes_k W_0)$. Thanks to this, and recalling that $H \cdot \mathfrak{m}_A = 0$, one can then consider the subspaces of $\mathcal{W}_{B,0} \otimes \mathcal{W}_{B,0}$

$$V_1 := (H \otimes_k W_0) \otimes \mathcal{W}_{B,0} \cong (H \otimes_k W_0) \otimes \mathcal{W}_{A,0} \cong H \otimes_k (W_0 \otimes W_0),$$

and

$$V_2 := \mathcal{W}_{B,0} \otimes (H \otimes_k W_0) \cong \mathcal{W}_{A,0} \otimes (H \otimes_k W_0) \cong H \otimes_k (W_0 \otimes W_0).$$

By construction $\Upsilon|_X = (\pi \otimes \pi)^* \phi$. Then, by continuity, we have that

$$\Upsilon|_{V_i} \cong \mathbf{1}_H \otimes (\pi \otimes \pi)^* \phi. \quad (5.13)$$

It is a consequence of (5.13) that

$$\Upsilon(\mathcal{W}_{B,0} \otimes (H \otimes_k \partial_{-1}(W_{-1}))) = 0 \quad (5.14)$$

since $\mathcal{W}_B \otimes (H \otimes_k \partial_{-1}(W_{-1})) \cong W_0 \otimes (H \otimes_k \partial_{-1}(W_{-1}))$. Also, as $(H \otimes_k W_0) \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1})) \cong (H \otimes_k W_0) \otimes \partial_{A,-1}(\mathcal{W}_{A,-1}) \cong (H \otimes_k W_0) \otimes \partial_{-1}(W_{-1})$, we have

$$\Upsilon((H \otimes_k W_0) \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1}))) = 0. \quad (5.15)$$

It follows from (5.12), (5.14) and (5.15) that Υ applied to $\mathcal{W}_{B,0} \otimes \rho^{-1}(p_* \partial_{A,-1}(\mathcal{W}_{A,-1}))$ descends to the morphism of \mathcal{O}_{X_B} -modules

$$\Upsilon_{\mathcal{F}}: \mathcal{W}_{A,0} \otimes \mathcal{F} \rightarrow H \otimes_k \mathcal{O}_X.$$

Consider $\Upsilon_{\mathcal{F}}^A: \mathcal{W}_{A,0} \otimes \mathcal{F}^A \rightarrow H \otimes_k \mathcal{O}_X$ to be the morphism of \mathcal{O}_{X_A} -modules associated to $\Upsilon_{\mathcal{F}}$ under the equivalence mentioned above. Since we defined $F_1 = \mathcal{F}^A|_X$, let us set accordingly

$$\mu := \Upsilon_{\mathcal{F}}^A|_X: W_0 \otimes F_1 \rightarrow H \otimes_k \mathcal{O}_X. \quad (5.16)$$

Recall that Υ is defined up to the additive action of (5.11). We note that this action corresponds to the additive action of $\mathrm{Hom}_X(W_0 \otimes W_0, H \otimes_k \mathcal{O}_X)^\pm \circ (\mathbf{1}_{W_0} \otimes f)$ over μ . It follows from (5.13) that μ satisfies (3.11). Also, as $\Upsilon = \pm \Upsilon \circ \theta_{\mathcal{W}_{B,0}}$ by construction, one naturally has that μ satisfies (3.12) as well. Note that for any other locally free resolution satisfying (5.2), we would obtain a 2-extension in the same equivalence class as (5.8) and the corresponding class of morphism as (5.16).

Thus, we have completed the construction of a morphism associated to the deformation functor $\mathrm{Def}_{(E,\phi)}^+$ and an small extension of Artin algebras $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$,

$$\Omega_{\tau}^+: \mathrm{Def}_{(E,\phi)}^+(A) \longrightarrow H \otimes_k \mathbb{H}^2\left(\Delta_{(E,\phi)}^{+,\bullet}\right), \quad (5.17)$$

sending $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \mathrm{Def}_{(E,\phi)}^\pm(A)$ to the point of $H \otimes_k \mathbb{H}^2\left(\Delta_{(E,\phi)}^{+,\bullet}\right)$ given by the 2-extension (5.8) and the class of morphisms given by (5.16). Similarly, associated to $\mathrm{Def}_{(E,\phi)}^-$, we construct

$$\Omega_{\tau}^-: \mathrm{Def}_{(E,\phi)}^-(A) \longrightarrow H \otimes_k \mathbb{H}^2\left(\Delta_{(E,\phi)}^{-,\bullet}\right). \quad (5.18)$$

We now see that these maps provide an obstruction theory for orthogonal and symplectic sheaves. Some results of obstruction theory of sheaves are needed and, for the reader's convenience, we include them instead of just cite them. We start by checking condition **01**.

Proposition 5.1. Consider the small extension of Artin algebras $0 \rightarrow H \rightarrow B \xrightarrow{\tau} A \rightarrow 0$. Given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}_{(E, \phi)}^{\pm}(A)$, one has $\Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A) = 0$ if and only if there exists $(\mathcal{E}_B, \Phi_B, \gamma_B) \in \text{Def}_{(E, \phi)}^{\pm}(B)$ such that $(\mathcal{E}_B, \Phi_B, \gamma_B)|_{X_A} \cong (\mathcal{E}_A, \Phi_A, \gamma_A)$.

Proof. From obstruction theory of sheaves, $(\mathcal{E}_A, \gamma_A)$ lifts to $(\mathcal{E}_B, \gamma_B)$ if and only if one can give an exact filler of (5.3). It is a standard result of abelian categories (see [30, Lemma 3.10] for instance) that exact fillers of (5.3) exist if and only if the short exact sequence (5.5) splits. In that case there exists a splitting morphism,

$$s : p_* \partial_{A, -1}(\mathcal{W}_{A, -1}) \longrightarrow \mathcal{F},$$

whose composition with ρ is the identity. Hence

$$s(p_* \partial_{A, -1}(\mathcal{W}_{A, -1})) \cong \ker \pi_A. \quad (5.19)$$

Fixing an splitting morphism, we can define $\mathcal{V}_B \subset \mathcal{W}_{B, 0}$ as the preimage of $s(p_* \partial_{A, -1}(\mathcal{W}_{A, -1}))$ under the projection

$$\rho^{-1}(p_* \partial_{A, -1}(\mathcal{W}_{A, -1})) \longrightarrow \mathcal{F} = \rho^{-1}(p_* \partial_{A, -1}(\mathcal{W}_{A, -1})) / H \otimes_k \partial_{-1}(W_{-1}).$$

Then, set

$$\mathcal{E}_B := \mathcal{W}_{B, 0} / \mathcal{V}_B.$$

One can easily check that this construction provides a coherent sheaf \mathcal{E}_B over X_B satisfying $\mathcal{E}_B|_{X_A} \cong \mathcal{E}_A$ and, furthermore, \mathcal{E}_B is flat over B (see for instance [30, Lemma 3.14]). One trivially has that $\mathcal{E}_B|_X = \mathcal{E}_A|_X$, so we pick γ_B to be $\gamma_A : \mathcal{E}_B|_X = \mathcal{E}_A|_X \xrightarrow{\cong} E$.

Up to this point, we have just reproduced the classical theory of sheaves, seeing $(\mathcal{E}_A, \gamma_A)$ lifts to $(\mathcal{E}_B, \gamma_B)$ if and only if (5.5) splits. Pick $[\Upsilon]$ to be the class in (5.10) given by $(\pi_A \otimes \pi_A)^* \Phi_A$. If (5.5) splits, we have that a representant Υ of $[\Upsilon]$ defines $\Phi_B \in \text{Hom}_{X_B}(\mathcal{E}_B \otimes \mathcal{E}_B, \mathcal{O}_{X_B})^{\pm}$ if and only if

$$\Upsilon(\mathcal{W}_{B, 0} \otimes \mathcal{V}_B) = 0. \quad (5.20)$$

It is a consequence of (5.12), (5.14) and (5.15) that (5.20) holds whenever

$$\Upsilon_{\mathcal{F}}(\mathcal{W}_{A, 0} \otimes s(p_* \partial_{A, -1}(\mathcal{W}_{A, -1}))) = 0. \quad (5.21)$$

It remains to show that (5.5) splits and (5.21) holds for some representant of $[\Upsilon]$ in (5.10) if and only if the image of $\Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A) = 0$. By the second statement of Proposition 3.4, the later is equivalent to the fact that the 2-extension given in (5.8) has the form (3.13) and the class $[\mu]$ in $\text{Hom}_X(W_0 \otimes F, H \otimes_k \mathcal{O}_X)^{\pm} / \text{Hom}_X(W_0 \otimes W_0, H \otimes_k \mathcal{O}_X)^{\pm} \circ (\mathbf{1}_{W_0} \otimes f)$ defined in (5.16) satisfies (3.14). Note that (5.5) splits if and only if (5.6) splits, and further, (5.6) splits if and only if the 2-extension (5.7) splits. Since $\text{Ext}_{X_A}^2(\mathcal{E}_A, H \otimes_k p_{A,*} E) \cong \text{Ext}_X^2(E, H \otimes_k E)$, we have that (5.7) splits if and only if (5.8) splits giving rise to a 2-extension of the form (3.13). In this case and recalling (5.19), the equation (5.21) holds if and only if

$$\mu(W_0 \otimes \ker \pi) = 0. \quad (5.22)$$

This is the case whenever the class $[\mu]$ satisfies (3.14), so Ω_{τ}^{\pm} the proof is complete. \square

We check now that the morphisms Ω_{τ}^{\pm} satisfy condition **02**. Observe that any morphism of small extension decomposes into

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & B & \xrightarrow{\tau} & A \longrightarrow 0 \\ & & \downarrow \bar{h} & & \downarrow \bar{\beta} & & \parallel \\ 0 & \longrightarrow & \bar{H} := H / \ker \bar{h} & \longrightarrow & \bar{B} := B / \ker \bar{h} & \xrightarrow{\bar{\tau}} & A \longrightarrow 0, \end{array} \quad (5.23)$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & B & \xrightarrow{\tau} & A \longrightarrow 0 \\ & & \downarrow h & & \downarrow \beta & & \downarrow \alpha \\ 0 & \longrightarrow & H' & \longrightarrow & B' & \xrightarrow{\tau'} & A' \longrightarrow 0, \end{array} \quad (5.24)$$

where h is an injection. Therefore, to check that **02** holds, it is enough to check it for morphisms of small extensions of the form (5.23) and (5.24). We start by the first type.

Proposition 5.2. Consider the morphism of small extensions (5.23). Given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}_{(E, \phi)}^{\pm}(A)$, one has that

$$\Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A) = (\bar{h} \otimes \mathbf{1}_{\mathbb{H}^2}) \circ \Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A).$$

Proof. Since one obviously has that \bar{h} sends $\ker \bar{h}$ to 0, observe that $\bar{h} \otimes \mathbf{1}_{\mathbb{H}^2}$ applied to the 2-extension (5.8) gives

$$0 \rightarrow \bar{H} \otimes_k E \rightarrow (F / \ker \bar{h} \otimes_k E) \rightarrow W_0 \rightarrow E \rightarrow 0. \quad (5.25)$$

Pick now $\mu : W_0 \otimes F \rightarrow H \otimes_k \mathcal{O}_X$ given in (5.16) and note that

$$(\bar{h} \otimes \mathbf{1}_{\mathcal{O}_X}) \circ \mu(W_0 \otimes (\ker \bar{h} \otimes_k E)) = 0.$$

Then, $(\bar{h} \otimes \mathbf{1}_{\mathcal{O}_X}) \circ \mu$ descends to

$$\bar{\mu} : W_0 \otimes (F / (\ker \bar{h} \otimes_k E)) \rightarrow \bar{H} \otimes_k E, \quad (5.26)$$

which is the image of μ under $\bar{h} \otimes \mathbf{1}_{\mathbb{H}^2}$. We have seen that $(\bar{h} \otimes \mathbf{1}_{\mathbb{H}^2}) \circ \Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A)$ is determined by (5.25) and (5.26).

We now study $\Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A)$. Since $X_A \xrightarrow{p} X_B$ is the composition $X_A \xrightarrow{\bar{p}} X_{\bar{B}} \xrightarrow{p_{\bar{B}}} X_B$ one can consider in this case $\mathcal{W}_{\bar{B},0} := \mathcal{W}_{B,0}|_{X_{\bar{B}}}$ and the morphism $\bar{\rho} : \mathcal{W}_{\bar{B},0} \rightarrow \bar{\rho}_* \mathcal{W}_{A,0}$ being the restriction $\rho|_{X_{\bar{B}}}$. Pick also $\bar{\sigma} : \bar{H} \otimes_k W_0 \rightarrow \mathcal{W}_{\bar{B},0}$ corresponding to σ on $X_{\bar{B}}$. Recall \mathcal{F} from (5.4) and construct $\bar{\mathcal{F}}$ accordingly. It follows from the previous discussion that

$$\mathcal{F}|_{X_{\bar{B}}} = \bar{\mathcal{F}}.$$

Observing that $\ker \bar{h} \otimes_k E$ is the kernel of $\mathcal{F} \rightarrow \bar{\mathcal{F}} = \mathcal{F}|_{X_{\bar{B}}}$, it then follows that

$$\bar{\mathcal{F}}^A \cong \mathcal{F}^A / \ker \bar{h} \otimes_k E. \quad (5.27)$$

Up to here, we have been dealing with obstruction theory of sheaves. We now address the quadratic form. Observe that (5.9) also holds for $X_{\bar{B}}$. Therefore $(\pi_A \otimes \pi_A)^* \Phi_A$ defines a class in the space (5.10) adapted to $X_{\bar{B}}$ from which can pick a representant $\bar{\Upsilon} \in \text{Hom}_{X_{\bar{B}}}(\mathcal{W}_{\bar{B},0} \otimes \mathcal{W}_{\bar{B},0}, \mathcal{O}_{\bar{B}})^{\pm}$ that satisfies

$$\Upsilon|_{X_{\bar{B}}} = \bar{\Upsilon}. \quad (5.28)$$

It is a direct consequence of (5.27) and (5.28) that $\Omega_{\tau}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A)$ is determined by (5.25) and (5.26). This concludes the proof. \square

Since h on (5.24) is injective, H defines naturally a subspace of H' so one can give (non-canonically) a decomposition of the vector spaces

$$H' = H \oplus H'' \quad (5.29)$$

and we can assume

$$h = \mathbf{1}_H \oplus 0. \quad (5.30)$$

Out of (5.24) and the (non-canonical) decomposition (5.29), one can always construct a small extension $\bar{\tau}'$ and a morphism of small extensions making the following diagram commutative,

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & B & \xrightarrow{\tau} & A \longrightarrow 0 \\ & & \downarrow h & & \downarrow \beta & & \downarrow \alpha \\ 0 & \longrightarrow & H \oplus H'' & \longrightarrow & B' & \xrightarrow{\tau'} & A' \longrightarrow 0 \\ & & \downarrow \bar{h}' & & \downarrow \bar{\beta}' & & \parallel \\ 0 & \longrightarrow & H & \longrightarrow & \bar{B}' := B'/H'' & \xrightarrow{\bar{\tau}'} & A' \longrightarrow 0, \end{array} \quad (5.31)$$

where \bar{h}' and $\bar{\beta}'$ are the obvious projections. If we set $\beta' := \bar{\beta}' \circ \beta$ and note that $\bar{h}' \circ h = \mathbf{1}_H$, we obtain the morphism of small extensions

$$\begin{array}{ccccccc}
0 & \longrightarrow & H & \longrightarrow & B & \xrightarrow{\tau} & A \longrightarrow 0 \\
& & \parallel & & \downarrow \beta' & & \downarrow \alpha \\
0 & \longrightarrow & H & \longrightarrow & \bar{B}' & \xrightarrow{\bar{\tau}'} & A' \longrightarrow 0.
\end{array} \quad (5.32)$$

Let us consider the morphisms of schemes associated to the morphism of algebras appearing in (5.31) and (5.32). Thanks to the commutativity of (5.31) one has the following commuting diagram of morphisms between schemes,

$$\begin{array}{ccccc}
X_{A'} & \xrightarrow{p_\alpha} & X_A & & \\
\downarrow p' & \searrow \bar{p}' & \downarrow p & & \\
& X_{\bar{B}'} & & & \\
\downarrow p_{\bar{B}'} & \swarrow p_{\beta'} & \downarrow p & & \\
X_{B'} & \xrightarrow{p_\beta} & X_B & &
\end{array}$$

whose right-upper subdiagram

$$\begin{array}{ccc}
X_{A'} & \xrightarrow{p_\alpha} & X_A \\
\downarrow \bar{p}' & & \downarrow p \\
X_{\bar{B}'} & \xrightarrow{p_{\beta'}} & X_B
\end{array} \quad (5.33)$$

is Cartesian.

Before checking **O2** restricted to morphisms of small extensions of the form (5.24), we will study its compatibility with those of the form (5.32).

Proposition 5.3. Consider the morphism of small extensions (5.32). Given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}_{(E, \phi)}^\pm(A)$, one has that

$$\Omega_{\bar{\tau}'}^\pm(p_\alpha^*(\mathcal{E}_A, \Phi_A, \gamma_A)) = \Omega_{\bar{\tau}}^\pm(\mathcal{E}_A, \Phi_A, \gamma_A).$$

Proof. We study $\Omega_{\bar{\tau}'}^\pm(p_\alpha^*(\mathcal{E}_A, \Phi_A, \gamma_A))$. Consider the locally free resolution $\mathcal{W}_A^\bullet \xrightarrow{\pi_A} \mathcal{E}_A \rightarrow 0$ and take its pull-back $p_\alpha^* \mathcal{W}_A^\bullet \xrightarrow{p_\alpha^* \pi_A} p_\alpha^* \mathcal{E}_A \rightarrow 0$ which is obviously a locally free resolution of $p_\alpha^* \mathcal{E}_A$. One can choose $\mathcal{W}_{B,0}$ and $\mathcal{W}_{\bar{B}',0} := p_{\beta'}^* \mathcal{W}_{B,0}$ satisfying in both cases the cohomology vanishing (5.2) for $i > 0$.

Recall that the morphism

$$\bar{p}' : \mathcal{W}_{\bar{B}',0} = p_{\beta'}^* \mathcal{W}_{B,0} \longrightarrow \bar{p}'_* \mathcal{W}_{A',0} = \bar{p}'_* p_\alpha^* \mathcal{W}_{A,0},$$

is given by the restriction to $X_{A'}$. Since $p_{\beta'}|_{X_{A'}} = p_\alpha$, $\bar{p}'|_{X_{A'}} = \mathbf{1}_{X_{A'}}$ and $p|_{X_A} = \mathbf{1}_{X_A}$, one has that

$$\bar{p}'_* p_\alpha^* \mathcal{W}_{A,0}|_{X_{A'}} = p_{\beta'}^* p_* \mathcal{W}_{A,0}|_{X_{A'}}.$$

By the Cartesianity of (5.33), it follows that $\bar{p}'_* p_\alpha^* \mathcal{W}_{A,0}$ is supported over $X_{A'}$. By all of the above, one has that $\bar{p}' \cong p_{\beta'}^* \rho$ and this implies that

$$\bar{\mathcal{F}}' \cong p_{\beta'}^* \mathcal{F}. \quad (5.34)$$

Hence $\bar{F}' := (\bar{\mathcal{F}}')^{A'}|_X$ is isomorphic to F and fits in the 2-extension (5.8).

We now move forward from the obstruction theory of sheaves. Recall that we chose $\mathcal{W}_{B,0}$ and $\mathcal{W}_{\bar{B}',0} := p_{\beta'}^* \mathcal{W}_{B,0}$ in such a way that both satisfy the cohomology vanishing (5.2) for $i > 0$. Let $[\Upsilon]$ be the class in (5.10) determined by $(\pi_A \otimes \pi_A)^* \Phi_A$ and note that $p_{\beta'}^* \Upsilon$ is a representant of the corresponding class determined to $(\pi_{A'} \otimes \pi_{A'})^* p_\alpha^* \Phi_A$. After this and (5.34), it follows that

$$\bar{\mu}' := (p_{\beta'}^* \Upsilon)_{p_{\beta'}^* \mathcal{F}}^{A'}|_X \cong p_\alpha^* (\Upsilon_{\mathcal{F}}^A)|_X \cong \mu,$$

where μ is defined in (5.16). This concludes the proof, as μ and the 2-extension (5.8) determine $\Omega_{\bar{\tau}}^\pm(\mathcal{E}_A, \Phi_A, \gamma_A)$ as well. \square

We now check that the set of maps defined in (5.17) and (5.18) satisfy condition **02** restricted to morphisms of small extensions of the form (5.24).

Proposition 5.4. Consider the morphism of small extensions (5.24). Given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}_{(E, \phi)}^{\pm}(A)$, one has that

$$\Omega_{\tau'}^{\pm}(p_{\alpha}^*(\mathcal{E}_A, \Phi_A, \gamma_A)) = (h \otimes \mathbf{1}_{\mathbb{H}^2}) \circ \Omega_{\tau'}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A).$$

Proof. We first describe $(h \otimes \mathbf{1}_{\mathbb{H}^2}) \circ \Omega_{\tau'}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A)$, where we recall that $\Omega_{\tau'}^{\pm}(\mathcal{E}_A, \Phi_A, \gamma_A)$ is determined by the 2-extension (5.8) and μ given in (5.16). Recall as well that H' and h decompose as indicated in (5.29) and (5.30). In that case, observe that $h \otimes \mathbf{1}_{\mathbb{H}^2}$ applied to (5.8) gives

$$0 \rightarrow (H \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow (F \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow W_0 \rightarrow E \rightarrow 0. \quad (5.35)$$

Also, note that the image of μ under $(h \otimes \mathbf{1}_{\mathbb{H}^2})$ decomposes in two direct summands. The first summand is μ and the second is the zero element, which corresponds to $\mathbf{1}_{H''} \otimes_k \phi(\pi \otimes \mathbf{1}_E) : W_0 \otimes (H'' \otimes_k E) \rightarrow H'' \otimes_k \mathcal{O}_X$ after (3.12) and (3.14). Then, the image of μ under $h \otimes \mathbf{1}_{\mathbb{H}^2}$ is

$$\mu \oplus (\mathbf{1}_{H''} \otimes_k \phi(\pi \otimes \mathbf{1}_E)) : W_0 \otimes ((F \otimes_k E) \oplus (H'' \otimes_k E)) \longrightarrow (H \oplus H'') \otimes_k \mathcal{O}_X. \quad (5.36)$$

Let us now study $\Omega_{\tau'}^{\pm}(p_{\alpha}^*(\mathcal{E}_A, \Phi_A, \gamma_A))$. We consider $\mathcal{W}_{A',i}$, $\mathcal{W}_{B',0}$ and the morphisms $\rho' : \mathcal{W}_{B',0} \rightarrow p'_* \mathcal{W}_{A',0}$ as we did in the beginning of this section. Set as well

$$\mathcal{W}_{B',0}|_{X_{\overline{B}'}} = \mathcal{W}_{\overline{B}',0}. \quad (5.37)$$

and

$$\rho'|_{X_{\overline{B}'}} = \overline{\rho}' : \mathcal{W}_{\overline{B}',0} \rightarrow \overline{p}'_* \mathcal{W}_{A',0} = p'_* \mathcal{W}_{A',0}|_{X_{\overline{B}'}}. \quad (5.38)$$

Defining \mathcal{F}' and $\overline{\mathcal{F}}'$ as in (5.4), it follows from (5.37) and (5.38) that

$$\mathcal{F}'|_{X_{\overline{B}'}} = \overline{\mathcal{F}}'. \quad (5.39)$$

From the description of \mathcal{F}' that we obtain from (5.5) one can obtain the following short exact sequence of $\mathcal{O}_{X_{B'}}$ -modules

$$0 \longrightarrow H'' \otimes_k E \longrightarrow \mathcal{F}' \longrightarrow p_{\overline{B}',*} \overline{\mathcal{F}}' \longrightarrow 0,$$

that gives rise to the short exact sequence of $\mathcal{O}_{X_{A'}}$ -modules

$$0 \longrightarrow H'' \otimes_k p_{A',*} E \longrightarrow (\mathcal{F}')^{A'} \longrightarrow p_{\overline{B}',*} (\overline{\mathcal{F}}')^{A'} \longrightarrow 0. \quad (5.40)$$

We see that (5.39) provides naturally a splitting of (5.40), so

$$(\mathcal{F}')^{A'} \cong p_{\overline{B}',*} (\overline{\mathcal{F}}')^{A'} \oplus (H'' \otimes_k p_{A',*} E). \quad (5.41)$$

Hence, the 2-extension determined by $\Omega_{\tau'}^{\pm}(p_{\alpha}^*(\mathcal{E}_A, \Phi_A, \gamma_A))$ is

$$0 \rightarrow (H \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow (\overline{\mathcal{F}}' \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow W'_0 \rightarrow E \rightarrow 0, \quad (5.42)$$

where $\overline{\mathcal{F}}'$ denotes $(\overline{\mathcal{F}}')^{A'}|_X$ and W'_0 is the restriction to X of $\mathcal{W}_{A',0}$.

Only at this point, we find ourselves in a position to give a step forward from the classical case of sheaves. One has that (5.9) also holds over $X_{B'}$ and $X_{\overline{B}'}$. Therefore $(\pi_A \otimes \pi_A)^* \Phi_A$ defines classes in the corresponding spaces (5.10) defined over $X_{B'}$ and $X_{\overline{B}'}$. Note also that one can choose representants $\Upsilon' \in \text{Hom}_{X_{B'}}(\mathcal{W}_{B',0} \otimes \mathcal{W}_{B',0}, \mathcal{O}_{B'})^{\pm}$ and $\overline{\Upsilon}' \in \text{Hom}_{X_{\overline{B}'}}(\mathcal{W}_{\overline{B}',0} \otimes \mathcal{W}_{\overline{B}',0}, \mathcal{O}_{\overline{B}'})^{\pm}$ satisfying

$$\Upsilon'|_{X_{\overline{B}'}} = \overline{\Upsilon}'. \quad (5.43)$$

Denote by μ' and $\overline{\mu}'$ the maps defined in (5.16) out of Υ' and $\overline{\Upsilon}'$. It then follows from (5.13), (5.41) and (5.43) that

$$\mu' = \overline{\mu}' \oplus (\mathbf{1}_{H''} \otimes_k \phi(\pi \otimes \mathbf{1}_E)). \quad (5.44)$$

The result follows from Proposition 5.3 after comparing (5.35) with (5.42) and (5.36) with (5.44). \square

The following summarizes all the previous results in this section.

Theorem 5.5. *Let (E, ϕ) be an orthogonal (resp. symplectic) sheaf over the projective scheme X . Then the deformation functor $\text{Def}_{(E, \phi)}^+$ (resp. symplectic) admits an obstruction theory with vector space*

$$\text{Obs}\left(\text{Def}_{(E, \phi)}^{\pm}\right) = \mathbb{H}^2\left(\Delta_{(E, \phi)}^{\pm, \bullet}\right).$$

Therefore, $\text{Def}_{(E, \phi)}^{\pm}$ are formally smooth when $\mathbb{H}^2\left(\Delta_{(E, \phi)}^{\pm, \bullet}\right) = 0$.

Proof. The theorem is a consequence of Propositions 5.1, 5.2 and 5.4. \square

Data availability

No data was used for the research described in the article.

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