



Universidad Autónoma
de Madrid

Biblos-e Archivo
Repositorio Institucional UAM

Repositorio Institucional de la Universidad Autónoma de Madrid

<https://repositorio.uam.es>

Esta es la **versión de autor** del artículo publicado en:
This is an **author produced version** of a paper published in:

Proceedings of the American Mathematical Society 151.9 (2023): 3845-3854

DOI: <https://doi.org/10.1090/proc/16382>

Copyright: © 2023 American Mathematical Society

El acceso a la versión del editor puede requerir la suscripción del recurso
Access to the published version may require subscription

ESTIMATES FOR TRUNCATED AREA FUNCTIONALS ON THE BLOCH SPACE

IASON EFRAIMIDIS, ALEJANDRO MAS, AND DRAGAN VUKOTIĆ

ABSTRACT. Recently, Kayumov [4] obtained a sharp estimate for the n -th truncated area functional for normalized functions in the Bloch space for $n \leq 5$ and then, together with Wirths [5], extended the result for $n = 6$. We prove that for the functions with non-negative Taylor coefficients, the same sharp estimate is valid for all n . For arbitrary functions, we obtain an estimate that is asymptotically of the same order but slightly larger (roughly by a factor of $4/e$). We also consider related weighted estimates for functionals involving the powers n^t , $t > 0$, and show that the exponent $t = 1$ represents the critical case for the expected sharp estimate.

1. INTRODUCTION

Let \mathbb{D} denote the unit disc in the complex plane. The *Bloch space* \mathcal{B} (see [1]) is the Banach space of all analytic functions in \mathbb{D} with the finite Bloch norm:

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|.$$

For a function in \mathcal{B} that vanishes at the origin, written as a power series $f(z) = \sum_{n=1}^{\infty} b_n z^n$, the following sharp estimate for the n -th Taylor coefficient is known [8]:

$$B_n = \sup\{|b_n| : \|f\|_{\mathcal{B}} \leq 1\} = \frac{n+1}{2n} \left(\frac{n+1}{n-1} \right)^{(n-1)/2}, \quad n \geq 2.$$

In the case $n = 1$ this should be interpreted as $B_1 = 1$ (and is easily proved directly). It can be checked by elementary means that the sequence $(B_n)_n$ is strictly increasing.

The Dirichlet space \mathcal{D} is the set of all analytic functions in the unit disc \mathbb{D} with the finite area integral

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) = \sum_{n=1}^{\infty} n |b_n|^2.$$

It is easy to see that $\mathcal{D} \subset \mathcal{B}$ but not the other way around. Thus, for a function $f \in \mathcal{B}$, the area functional $\sum_{k=1}^{\infty} k |b_k|^2$ need not be finite. However, any truncated area functional is bounded:

$$\Phi_n(f) = \sum_{k=1}^n k |b_k|^2 \leq C_n \|f\|_{\mathcal{B}}^2,$$

Date: 08 December, 2022.

2010 Mathematics Subject Classification. 30H30, 30C50.

Key words and phrases. Bloch space, coefficient problems.

All authors are partially supported by PID2019-106870GB-I00 from MICINN, Spain. The first author is supported by a María Zambrano contract, reference number CA3/RSUE/2021-00386, from UAM and Ministerio de Universidades, Spain (Plan de Recuperación, Transformación y Resiliencia).

where the constant C_n depends only on n but not on f . This has led Kayumov [4] to show that, for small values of n , the above truncated area functional is dominated by the supremum of its last term only:

$$(1) \quad \sum_{k=1}^n k|b_k|^2 \leq nB_n^2, \quad 2 \leq n \leq 5,$$

for all $\|f\|_{\mathcal{B}} \leq 1$ with $f(0) = 0$, with equality if and only if f is an appropriate constant multiple of z^n . Kayumov and Wirths [5] then showed that the estimate also holds for $n = 6$. Related extremal problems have also been considered recently in other papers (cf. [3] and [7]).

A standard argument involving normal families can be used to show that, for each fixed $n \geq 1$, there exists an extremal function for the problem

$$(2) \quad M_n = \sup \left\{ \Phi_n(f) = \sum_{k=1}^n k|b_k|^2 : \|f\|_{\mathcal{B}} \leq 1, f(z) = \sum_{k=1}^{\infty} b_k z^k \right\}.$$

Since $\|B_n z^n\|_{\mathcal{B}} = 1$, it is clear that $M_n \geq nB_n^2$. A natural question is whether estimate (1) extends for arbitrary $n \geq 2$. The present note is devoted to this question.

We first display some evidence towards the correctness of the expected bound (1). Specifically, as the main result of Section 2, we prove this inequality for all n for the functions with non-negative coefficients. This is the content of Theorem 2.

Next, in Section 3 we consider a more general weighted functional $\Phi_n^t(f) = \sum_{k=1}^n k^t |b_k|^2$ for $t \geq 0$ and the corresponding question as to whether $\Phi_n^t(f) \leq n^t B_n^2$ for all functions f with $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{k=1}^{\infty} b_k z^k$. It is easy to see that this inequality holds for all $t \geq 2$. Other results obtained here imply that it is true for all $t \geq 1$ for the functions with non-negative Taylor coefficients. However, we show that, when $t < 1$, this estimate fails for all sufficiently large n , even for functions with non-negative coefficients (see Theorem 3). Thus, the exponent $t = 1$ considered by Kayumov and Wirths is the critical value, which provides further motivation for studying generalizations of their results.

In Section 4 of the paper we observe that $\sum_{k=1}^n k|b_k|^2 \leq \frac{n^n}{(n-1)^{n-1}} \leq \frac{4}{e} nB_n^2$, for all $n \geq 2$ (Proposition 4). Although larger, this bound is asymptotically of the same order as nB_n^2 . We also note that a typical Marty-type relation for the coefficients of extremal functions shows that at least one of the coefficients b_n and b_{n+1} must be zero, as expected. However, obtaining more detailed information from a number of possible variational methods has proved surprisingly difficult for us. Even proving that for any extremal function we must have $b_n \neq 0$ has eluded us so far.

In the final Subsection 4.3 we include further discussion and remarks, showing that our Theorem 2 does not imply directly the desired estimate in the general case. We include a relevant example in this respect.

2. FUNCTIONS WITH NON-NEGATIVE COEFFICIENTS

In this section, we will prove that the bound obtained in [4, 5] is correct for all n for the functions with non-negative Taylor coefficients.

2.1. On extremal functions in the general setting. The following observations will be useful. For any $f \in \mathcal{B}$ and any constant c we have $\Phi_n(cf) = |c|^2 \Phi_n(f)$. Thus, if $\|f\|_{\mathcal{B}} < 1$, since the function $f/\|f\|_{\mathcal{B}}$ has unit norm and $\Phi_n(f/\|f\|_{\mathcal{B}}) > \Phi_n(f)$, it is clear that f cannot be an extremal function. In other words, if F is an extremal function for (2), we must have $\|F\|_{\mathcal{B}} = 1$.

Extremal functions are clearly not unique since for a function $f \in \mathcal{B}$ its rotation f_λ , given by $f_\lambda(z) = f(\lambda z)$, $|\lambda| = 1$, has the same norm and $\Phi_n(f_\lambda) = \Phi_n(f)$. Another issue is whether such a function is unique up to a rotation, which is what is expected by the findings from [4] and [5] for $2 \leq n \leq 6$.

2.2. Functions with non-negative coefficients. We now turn to the restricted extremal problem for functions with non-negative coefficients:

$$(3) \quad \sup \left\{ \Phi_n(f) = \sum_{k=1}^n k|b_k|^2 : \|f\|_{\mathcal{B}} \leq 1, f(z) = \sum_{k=1}^{\infty} b_k z^k, b_j \geq 0 \text{ for all } j \geq 1 \right\}.$$

A few simple observations are in order. If $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{j \geq 1} b_j z^j$ with $b_j \geq 0$ for all $j \geq 1$, it is obvious that $|f'(z)| \leq f'(|z|)$, hence

$$\|f\|_{\mathcal{B}} = \sup_{0 \leq r < 1} (1 - r^2) |f'(r)|.$$

Next, by the same argument as in Subsection 2.1, all extremal functions for the restricted problem (3) have this property as well. If f is such a function, it is clear that the truncated polynomial $p_n(z) = \sum_{j=1}^n b_j z^j$ (the Taylor polynomial of f of order n) has the property that

$$\Phi_n(p_n) = \Phi_n(f) \quad \text{and} \quad \|p_n\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}} = 1.$$

Moreover, if there exists at least one $k > n$ such that $b_k > 0$, then clearly $\|p_n\|_{\mathcal{B}} < 1$. But then f cannot be extremal for Φ_n since $p_n/\|p_n\|_{\mathcal{B}}$ and f both have norm one and

$$\Phi_n\left(\frac{p_n}{\|p_n\|_{\mathcal{B}}}\right) = \frac{1}{\|p_n\|_{\mathcal{B}}^2} \Phi_n(f) > \Phi_n(f).$$

Thus, the only candidates for extremal functions for (3) are polynomials of degree at most n . Note, however, that this consideration is not strictly necessary in the proof of the following lemma.

LEMMA 1. *Let $f(z) = \sum_{j \geq 1} b_j z^j$ be an extremal function for the restricted extremal problem (3), with $b_j \geq 0$ for all $j \geq 1$. If $1 \leq k < m \leq n$ and $b_k > 0$, then*

$$b_m \leq \frac{m-k}{2m} b_k.$$

Proof. Let

$$g(z) = f(z) - b_k z^k + \frac{k}{m} b_k z^m.$$

It is again an analytic function with non-negative coefficients. Since

$$g'(r) = f'(r) + k b_k (r^{m-1} - r^{k-1}) \leq f'(r)$$

for $0 \leq r < 1$, it follows that $\|g\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}}$. Hence g is in competition with f and therefore

$$\Phi_n(g) = 2kb_k b_m + \frac{k^2}{m} b_k^2 - kb_k^2 + \Phi_n(f) \leq \Phi_n(f).$$

Since $b_k > 0$ by assumption, we obtain $2b_m + \frac{k}{m} b_k \leq b_k$, and the desired conclusion follows. \square

We are now ready to prove the main result of this section.

THEOREM 2. *If $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{j \geq 1} b_j z^j$ with $b_j \geq 0$ for all $j \geq 1$, then*

$$\Phi_n(f) = \sum_{k=1}^n k|b_k|^2 \leq nB_n^2, \quad \text{for all } n \geq 2.$$

Moreover, equality is only achieved for $f(z) = B_n z^n$.

Proof. First of all, we observe that it suffices to prove the statement only for extremal functions. We proceed by induction on n . As is well known, the statement is true for $n = 2$. Let $n \geq 3$ and assume the inequality is true for all $k < n$. We will now show that it also holds for n .

If f is extremal for (3), as observed before, f is a polynomial of degree at most n . Suppose that $b_j > 0$ for some j with $1 \leq j < n$, and let $k = \max\{j : 1 \leq j < n \text{ and } b_j > 0\}$. Recalling that the sequence $(B_n)_n$ is increasing, by our inductive assumption, we clearly have

$$nB_k^2 \leq nB_n^2 \leq \Phi_n(f) = \sum_{j=1}^k j b_j^2 + n b_n^2 = \Phi_k(f) + n b_n^2 \leq k B_k^2 + n b_n^2.$$

It follows that $n b_n^2 \geq (n - k) B_k^2$ and therefore

$$b_n \geq \sqrt{\frac{n-k}{n}} B_k > \frac{1}{2} \frac{n-k}{n} B_k \geq \frac{1}{2} \frac{n-k}{n} b_k \geq b_n$$

in view of Lemma 1 (with $m = n$), which is clearly absurd. Thus, it follows that $b_j = 0$ for all j with $1 \leq j < n$. As was already observed, only functions of norm one can be extremal and it follows immediately that $B_n z^n$ is the only possible extremal function in this case. \square

It should be noted that Theorem 2 is also true for functions f with $f(z) = \sum_{k \geq 1} b_k z^k$ with $b_k = |b_k| e^{i(a+kb)}$, for some $a, b \in \mathbb{R}$. This is clear since, given a function f of this kind, the function $e^{-ia} f(e^{-ib} z)$ also vanishes at the origin and has the same norm as f and non-negative coefficients.

3. ON MORE GENERAL WEIGHTED ESTIMATES

As was noted in [4, p. 467], it is quite simple to show that for any function with $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{k \geq 1} b_k z^k$ the sharp estimate

$$\sum_{k=1}^n k^2 |b_k|^2 \leq n^2 B_n^2$$

holds for all $n \geq 1$. This suggests the idea of studying the generalized functional

$$\Phi_n^t(f) = \sum_{k=1}^n k^t |b_k|^2$$

with $t \geq 0$. A natural question is whether $\Phi_n^t(f) \leq n^t B_n^2$, under the same hypotheses as above, and whether the inequality is sharp. We already know that this is true when $t = 2$ and the problem studied previously constitutes the case $t = 1$.

It is easy to see that if the above weighted estimate is valid for all n for the functional Φ_n^t , then the analogous estimate is also satisfied for the functional Φ_n^s , whenever $s \geq t$. Indeed, if $\|f\|_{\mathcal{B}} \leq 1$, $f(z) = \sum_{k \geq 1} b_k z^k$, the inequality

$$\Phi_n^t(f) = \sum_{k=1}^n k^t |b_k|^2 \leq n^t B_n^2$$

is satisfied and $s \geq t$, then clearly

$$\Phi_n^s(f) = \sum_{k=1}^n k^s |b_k|^2 \leq n^{s-t} \sum_{k=1}^n k^t |b_k|^2 \leq n^{s-t} n^t B_n^2 = n^s B_n^2.$$

Thus, for all functions f such that $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{k \geq 1} b_k z^k$, it follows that

$$(4) \quad \sum_{k=1}^n k^t |b_k|^2 \leq n^t B_n^2,$$

whenever $t \geq 2$. Additionally, if f has non-negative Taylor coefficients, inequality (4) actually holds whenever $t \geq 1$ in view of Theorem 2.

The following result will show that for all t with $t < 1$ the above estimate (4) fails in a strong way. In fact, it cannot even be true for the functions with non-negative Taylor coefficients when n is sufficiently large. This shows that the exponent $t = 1$ (the basic case studied here and in [4, 5]) is really critical.

THEOREM 3. *Given $t < 1$, there exist a positive integer N that depends only on t and such that for all $n \geq N$ there exists a function $f_n \in \mathcal{B}$ with non-negative Taylor coefficients, $f_n(0) = 0$, $\|f_n\|_{\mathcal{B}} \leq 1$, and with the property that*

$$\Phi_n^t(f_n) > n^t B_n^2.$$

Proof. Let $\varepsilon > 0$ such that $t + \varepsilon < 1$, for example, $\varepsilon = (1 - t)/2$. Choose an integer $N \geq (2e^2)^{2/(1-t)}$ and, for $n \geq N$, consider the functions

$$f_n(z) = z + b_n z^n, \quad b_n = \sqrt{B_n^2 - \frac{1}{n^{t+\varepsilon}}}.$$

Clearly,

$$\Phi_n^t(f_n) = 1 + n^t \left(B_n^2 - \frac{1}{n^{t+\varepsilon}} \right) > n^t B_n^2,$$

hence it is only left to check that $\|f_n\|_{\mathcal{B}} \leq 1$ for all $n \geq N$. To this end, note that

$$\|f_n\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |1 + n b_n z^{n-1}| = \sup_{0 \leq r < 1} (1 - r^2) (1 + n b_n r^{n-1}).$$

Thus, in order to finish the proof, we need only show that for $n \geq N$ and for all $r \in (0, 1)$ the inequality $(1 - r^2)(1 + nb_n r^{n-1}) \leq 1$ holds. The last statement is easily seen to be equivalent to

$$nb_n r^{n-3}(1 - r^2) \leq 1, \quad \text{for all } r \in (0, 1) \quad \text{and } n \geq N.$$

If we define $h(r) = nb_n r^{n-3}(1 - r^2)$, it is clear that

$$\max_{0 \leq r \leq 1} h(r) = \max \left\{ h(0), h\left(\sqrt{\frac{n-3}{n-1}}\right), h(1) \right\} = nb_n \left(\frac{n-3}{n-1}\right)^{\frac{n-3}{2}} \frac{2}{n-1}.$$

It is, thus, only left to show that $nb_n \left(\frac{n-3}{n-1}\right)^{\frac{n-3}{2}} \frac{2}{n-1} \leq 1$ for $n \geq N$. The following chain of equivalences is clear:

$$\begin{aligned} nb_n \left(\frac{n-3}{n-1}\right)^{\frac{n-3}{2}} \frac{2}{n-1} \leq 1 &\iff b_n^2 \leq \frac{(n-1)^{n-1}}{4n^2(n-3)^{n-3}} \iff B_n^2 - \frac{1}{n^{t+\varepsilon}} \leq \frac{(n-1)^{n-1}}{4n^2(n-3)^{n-3}} \\ &\iff \frac{(n+1)^{n+1}}{4n^2(n-1)^{n-1}} - \frac{(n-1)^{n-1}}{4n^2(n-3)^{n-3}} \leq \frac{1}{n^{t+\varepsilon}} \\ &\iff \frac{(n+1)^2 \left(1 + \frac{2}{n-1}\right)^{n-1} - (n-1)^2 \left(1 + \frac{2}{n-3}\right)^{n-3}}{n} \leq 4n^{1-t-\varepsilon}. \end{aligned}$$

Obviously, the right-hand side in the last inequality tends to infinity when n tends to infinity. On the other hand, using the classical inequalities $\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}$ for all $x > 0$, we see that the left-hand side in the last inequality in the chain above satisfies

$$\begin{aligned} \frac{(n+1)^2 \left(1 + \frac{2}{n-1}\right)^{n-1} - (n-1)^2 \left(1 + \frac{2}{n-3}\right)^{n-3}}{n} &\leq e^2 \frac{(n+1)^2 - (n-1)^2 \left(1 + \frac{2}{n-3}\right)^{-2}}{n} \\ &= e^2 \frac{(n+1)^2 - (n-3)^2}{n} \\ &= e^2 \frac{8n-8}{n} \\ &\leq 8e^2. \end{aligned}$$

Since $N \geq (2e^2)^{2/(1-t)}$ and $\varepsilon = (1-t)/2$ (by our choices), it is clear that $4n^{1-t-\varepsilon} \geq 8e^2$ and the desired conclusion follows. \square

4. DISCUSSION OF THE GENERAL CASE

4.1. An estimate of correct asymptotic order. If the conjectured bound on the truncated functional is true for all n , the following estimate cannot be sharp. However, it may still be useful so we mention it here.

PROPOSITION 4. *If $\|f\|_{\mathcal{B}} \leq 1$ and $f(z) = \sum_{k=1}^{\infty} b_k z^k$, then*

$$(5) \quad \sum_{k=1}^n k|b_k|^2 \leq \frac{n^n}{(n-1)^{n-1}}, \quad \text{for all } n \geq 2.$$

This estimate is of the same asymptotic order as the conjectured estimate since

$$\frac{32}{27}nB_n^2 \leq \frac{n^n}{(n-1)^{n-1}} \leq \frac{4}{e}nB_n^2, \quad \text{for all } n \geq 2.$$

Proof. Starting with the standard estimate based on Parseval's equality [1, 4]:

$$(6) \quad \sum_{k=1}^n k^2 |b_k|^2 \rho^{k-1} \leq \frac{1}{(1-\rho)^2}, \quad 0 \leq \rho < 1,$$

and integrating with respect to ρ from 0 to x , we obtain:

$$\sum_{k=1}^n k |b_k|^2 x^{k-1} \leq \frac{1}{1-x}, \quad 0 \leq x < 1.$$

Since $x^{k-1} \geq x^{n-1}$ whenever $1 \leq k \leq n$, it follows that

$$\sum_{k=1}^n k |b_k|^2 \leq \frac{1}{x^{n-1}(1-x)}$$

for all $x \in (0, 1)$. The right-hand side attains its minimum value

$$(7) \quad \frac{n^n}{(n-1)^{n-1}}$$

at the point $x = (n-1)/n$. This yields the desired estimate (5).

As for the order of the estimate obtained, this is easily seen by elementary calculus since the sequence

$$\frac{n^n}{(n-1)^{n-1}} / (nB_n^2) = \frac{4}{\left(\frac{n+1}{n}\right)^{n+1}}$$

is increasing and convergent to $\frac{4}{e} \simeq 1.4715$. □

4.2. Remarks on the n -th coefficient of an extremal function. It is desirable to obtain further partial information on the Taylor coefficients of extremal functions for the functional Φ_n . For example, one would like to show that we must always have $b_n \neq 0$. However, this seemingly simple point has eluded us so far and we have not been able to find a satisfactory proof.

It is, however, relatively simple to show that for every extremal function either $b_{n+1} = 0$ or $b_n = 0$ by working out an analogue of the Marty variation as in the theory of univalent functions (cf. [2, pp. 59–60]). This can be proved by using conformal invariance; more specifically, if F is extremal, the function

$$F_\lambda(z) = F\left(\frac{z+\lambda}{1+\bar{\lambda}z}\right) - F(\lambda) = \sum_{k=1}^{\infty} B_k(\lambda) z^k$$

has Bloch norm one and satisfies $F_\lambda(z) = 0$, hence it competes with f : $\sum_{k=1}^n k |B_k(\lambda)|^2 \leq \sum_{k=1}^n k |b_k|^2$. By considering the asymptotic expansion:

$$|B_k(\lambda)|^2 = |b_k|^2 + 2\operatorname{Re} \left\{ \left((k+1)b_{k+1}\bar{b}_k - (k-1)\overline{b_{k-1}}b_k \right) \lambda \right\} + O(|\lambda|^2), \quad k \geq 1,$$

for λ close to zero, where $b_0 = 0$, after some algebra, this leads to

$$\operatorname{Re} \left\{ \left(n(n+1)b_{n+1} \overline{b_n} \right) \lambda \right\} + O(|\lambda|^2) \leq 0$$

and, upon dividing by $|\lambda|$ and letting $|\lambda| \rightarrow 0$, since the argument of λ can be arbitrary, we conclude that $n(n+1)b_{n+1} \overline{b_n} = 0$, which proves the statement.

As a way of restating the desired property of the n -th coefficients of extremal functions for problem (2), it is not difficult to show that the sequence $(M_n)_n$ of numbers defined by (2) is strictly increasing if and only if for each extremal function $F(z) = \sum_{k=1}^{\infty} b_k z^k$ we have $b_n \neq 0$.

4.3. Concluding remarks. We end this note by pointing out some limitations of the method employed in Subsection 2.2.

Regarding Theorem 2, one may ask whether in the general case we may restrict our attention to polynomials only. However, knowing only that $\Phi_n(p) \leq nB_n^2$ for all polynomials p of unit norm does not imply in a direct way the same inequality for general functions of norm one. This can be seen as follows. While for a function f in the unit ball and its n -th Taylor polynomial p_n we have $\Phi_n(f) = \Phi_n(p_n)$, it can be seen that there are functions f in the unit ball for which $\|p_n\|_{\mathcal{B}} > 1$, hence for the normalized function $p_n/\|p_n\|_{\mathcal{B}}$ we obtain $\Phi_n(p_n/\|p_n\|_{\mathcal{B}}) = \Phi_n(f)/\|p_n\|_{\mathcal{B}}^2 < \Phi_n(f)$, showing that the maximum of the functional over the normalized polynomials could in theory be strictly smaller than the global maximum for problem (2). Of course, showing that the conjectured bound is correct would amount to proving that such functions f cannot be extremal; hence, some additional work is required.

Another natural question is whether from the estimate obtained only for functions with non-negative coefficients in Theorem 2 we can immediately deduce the result for arbitrary functions. However, this does not seem so simple since, employing the usual terminology, the unit ball of \mathcal{B} is not *solid*. In other words, if $f(z) = \sum_{k=1}^{\infty} b_k z^k$ is in the ball of \mathcal{B} , the function $F(z) = \sum_{k=1}^{\infty} |b_k| z^k$ need not belong to the ball of \mathcal{B} . It is clear that F is also analytic in the disc and $\|f\|_{\mathcal{B}} \leq \|F\|_{\mathcal{B}}$ by crude estimates based on the triangle inequality, but there are examples showing that this inequality can actually be strict. More generally speaking, a change of signs in some of the coefficients of a function could enlarge its norm. Thus, if f and F are as above (hence F has non-negative coefficients) and also $1 = \|F\|_{\mathcal{B}} > \|f\|_{\mathcal{B}}$, then $f/\|f\|_{\mathcal{B}}$ is also of unit norm and

$$\Phi_n \left(\frac{f}{\|f\|_{\mathcal{B}}} \right) = \frac{\Phi_n(F)}{\|f\|_{\mathcal{B}}^2} > \Phi_n(F),$$

so F cannot be extremal. Hence, a proof of the desired estimate in the general case would show in particular that functions F with the above property cannot be extremal.

One single example can be used to illustrate both phenomena (after the appropriate normalization).

EXAMPLE 5. Let $n \geq 2$ and let ε be sufficiently small, say $0 < \varepsilon \leq 1/5$. Consider the function $f(z) = z + B_n z^n - \frac{\varepsilon z^{2n-1}}{2n-1}$. The associated functions discussed above are $F(z) = z + B_n z^n + \frac{\varepsilon z^{2n-1}}{2n-1}$ and $p_n(z) = z + B_n z^n$ and we have the inequality

$$(8) \quad \|F\|_{\mathcal{B}} > \|p_n\|_{\mathcal{B}} > \|f\|_{\mathcal{B}}.$$

To check the first inequality in (8), recall from Subsection 2.2 that for every function with positive coefficients in \mathcal{B} the supremum defining its norm is attained on the radius $[0, 1)$. In particular, this shows that

$$\|F\|_{\mathcal{B}} = \sup_{0 \leq r < 1} (1 - r^2)(1 + nB_n r^{n-1} + \varepsilon r^{2n-2}) > \sup_{0 \leq r < 1} (1 - r^2)(1 + nB_n r^{n-1}) = \|p_n\|_{\mathcal{B}}.$$

The above inequality holds because we have the strict inequality

$$(1 - r^2)(1 + nB_n r^{n-1} + \varepsilon r^{2n-2}) > (1 - r^2)(1 + nB_n r^{n-1}), \quad \text{for } 0 < r < 1,$$

equality holds for $r = 0$ and at $r = 1$, and also because the smaller quantity is strictly bigger than 1 (the value of both sides at $r = 0$) for at least one $r \in (0, 1)$. This last observation follows in view of the fact that $\|B_n z^n\|_{\mathcal{B}} = 1$ and this norm is attained at the value $r_n = \sqrt{\frac{n-1}{n+1}} \in (0, 1)$, hence

$$\|p_n\|_{\mathcal{B}} = \sup_{0 \leq r < 1} (1 - r^2)(1 + nB_n r^{n-1}) \geq (1 - r_n^2)(1 + nB_n r_n^{n-1}) > (1 - r_n^2)nB_n r_n^{n-1} = 1.$$

The second inequality in (8) requires more work. In order to determine $\left\|z + B_n z^n - \frac{\varepsilon z^{2n-1}}{2n-1}\right\|_{\mathcal{B}}$, we need to obtain an upper estimate on $|1 + nB_n z^{n-1} - \varepsilon z^{2n-2}|$. Writing $z^{n-1} = R e^{it}$ and then $R = r^{n-1}$, $\cos t = x$, one easily computes

$$\begin{aligned} |1 + nB_n z^{n-1} - \varepsilon z^{2n-2}|^2 &= |1 + nB_n R e^{it} - \varepsilon R^2 e^{2it}|^2 \\ &= 1 + n^2 B_n^2 R^2 + \varepsilon^2 R^4 + 2nB_n R x - 2nB_n \varepsilon R^3 x - 2\varepsilon R^2 (2x^2 - 1) \\ &= -4\varepsilon R^2 x^2 + 2nB_n R (1 - \varepsilon R^2) x + 1 + n^2 B_n^2 R^2 + 2\varepsilon R^2 + \varepsilon^2 R^4 \\ &= u(x). \end{aligned}$$

Then $u'(x) = 2R(nB_n(1 - \varepsilon R^2) - 4\varepsilon R x) \geq 0$ in view of the obvious inequalities $nB_n(1 - \varepsilon R^2) \geq 1 - \varepsilon R^2 \geq 1 - \varepsilon \geq 4\varepsilon$. Thus, u attains its maximum at $x = 1$ and the maximum value is

$$u(1) = 2nB_n R (1 - \varepsilon R^2) + 1 + n^2 B_n^2 R^2 - 2\varepsilon R^2 + \varepsilon^2 R^4 = (1 + nB_n R)^2 + \varepsilon R^2 (\varepsilon R^2 - 2 - 2nB_n R).$$

The second summand in the last expression is negative, hence $u(x) \leq u(1) < (1 + nB_n R)^2$. This yields

$$\left\|z + B_n z^n - \frac{\varepsilon z^{2n-1}}{2n-1}\right\|_{\mathcal{B}} \leq \sup_{0 \leq r < 1} (1 - r^2)(1 + nB_n r^{n-1}) = \|p_n\|_{\mathcal{B}}.$$

Again, the inequality is easily seen to be strict since $\|p_n\| > 1$, hence this norm is attained for some $r \in (0, 1)$, while the inequality in $u(x) < (1 + nB_n R)^2 = (1 + nB_n r^{n-1})^2$ is strict for $r \in (0, 1)$.

REFERENCES

- [1] J.M. Anderson, J. Clunie, Ch. Pommerenke, On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12—37.
- [2] P.L. Duren, *Univalent Functions*, Springer-Verlag, New York 1983.
- [3] O.V. Ivrii, I.R. Kayumov, Makarov's principle for the unit ball in Bloch space, *Mat. Sb.* **208** (2017), no. 3, 96—110; translation in *Sb. Math.* **208** (2017), no. 3-4, 399—412.
- [4] I.R. Kayumov, A note on an area-type functional of Bloch functions, *Lobachevskii J. Math.* **38** (2017), no. 3, 466—468.

- [5] I.R. Kayumov, K.-J. Wirths, Coefficient inequalities for Bloch functions, *Lobachevskii J. Math.* **40** (2019), No. 9, pp. 1319–1323.
- [6] I.R. Kayumov, K.-J. Wirths, On the sum of squares of the coefficients of Bloch functions, *Monatsh. Math.* **190** (2019), no. 1, 123–135.
- [7] I.R. Kayumov, K.-J. Wirths, Coefficients problems for Bloch functions, *Anal. Math. Physics* **9** (2019), 1069–1085.
- [8] K.-J. Wirths, Über holomorphe Funktionen, die einer Wachstumsbeschränkung unterliegen (German), *Arch. Math. (Basel)* **30** (1978), no. 6, 606–612.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN
Email address: iason.efraimidis@uam.es

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE MÁLAGA, 29071 MÁLAGA, SPAIN
Email address: alejandro.mas@uma.es

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN
Email address: dragan.vukotic@uam.es